I. Tripolar Coordinates
(Straight Line and Circle).

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Thesis: D. Sc., March 1924

Def. The distances or the ratios of the distances of a point from the vertices of a given triangle of reference are called the tripolar coordinates of the point.

In this paper ΔABC is taken to be the triangle of reference. The tripolar coordinates are $(\rho_1 \rho_2 \rho_3)$, the trilinear $(\alpha\ \beta\ \gamma)$ and the barycentric or areal $(x\ y\ z)$. O is the circumcentre of ΔABC.

1. Given the ratios of the distances of a point $(\rho_1 \rho_2 \rho_3)$, to determine the point or points.

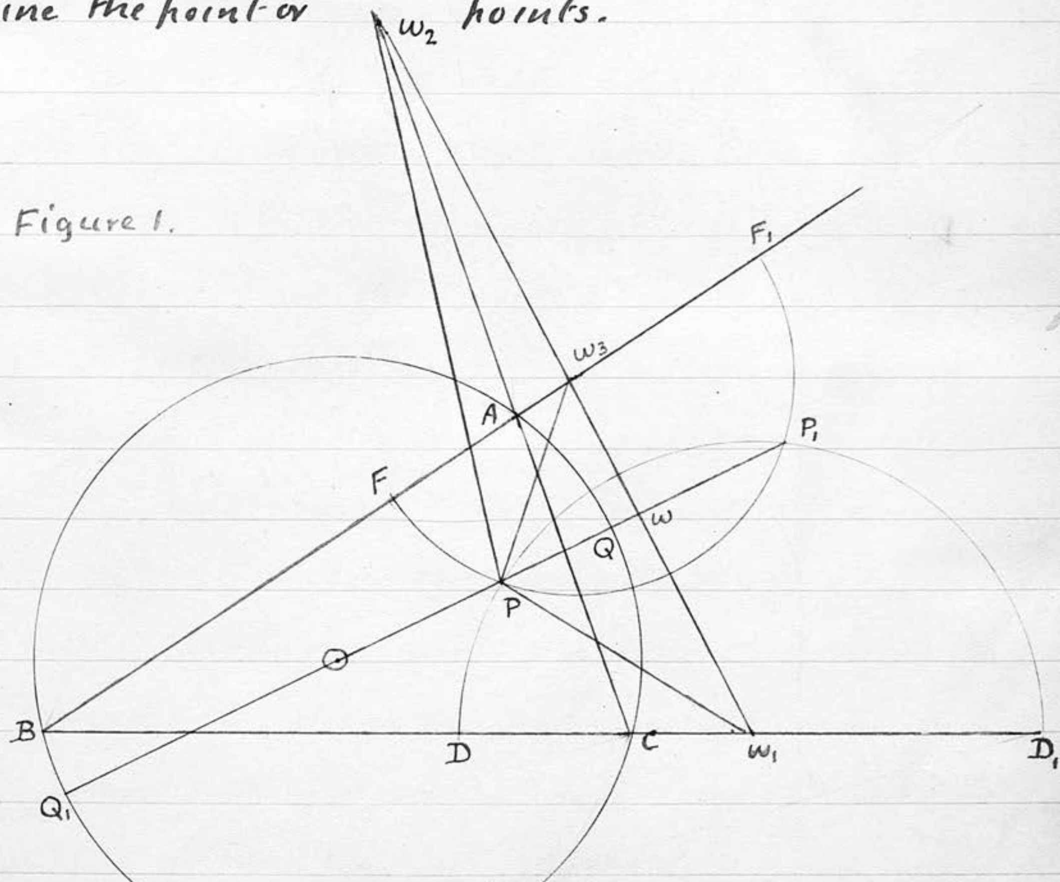


Figure 1.

Divide BC internally and externally in D and D_1 so that

$BD : CD = \rho_2 : \rho_3 = BD_1 : CD_1$ (see figure 1.)

and similarly divide CA and AB in E and E_1, and in F and F_1 so that the ratios of division are as $\rho_3 : \rho_1$ and as $\rho_1 : \rho_2$.

Let $\omega_1\ \omega_2\ \omega_3$ be the centres of the circles described on DD_1, EE_1, FF_1 as diameters, and let the circles (DD_1) and (FF_1) cut in P and P_1

Since $(BDCD_1)$ is harmonic, $BP : CP = BP_1 : CP_1 = \rho_2 : \rho_3$

and since $(BFAF_1)$ is harmonic $AP : BP = AP_1 : BP_1 = \rho_1 : \rho_2$

$\therefore\ AP : BP : CP = \rho_1 : \rho_2 : \rho_3 = AP_1 : BP_1 : CP_1$

The symmetry shows that P and P_1 lie also on the circle (EE_1)

Hence there are in general two points whose tripolar coordinates are as $\rho_1 : \rho_2 : \rho_3$ and these points lie on the circles (DD_1), (EE_1) and (FF_1). These are called "corresponding" points.

The circum circle of ΔABC cuts each of the circles (DD_1) etc orthogonally, and hence PP_1 passes through O, and

$OP.OP_1 = R^2$ (R = radius of circum circle of ΔABC)

$\therefore$ P and P_1 are inverse points, the circumcircle being the circle of inversion. Hence any point on the circumcircle is its own corresponding point, and if the locus of P be known, the locus of its corresponding point can be determined by the theory of inversion. (vide § 34)

Again $\omega_1\ \omega_2\ \omega_3$ are collinear, and the straight line $\omega_1\,\omega_3\,\omega_2$

bisects PP_1 at right angles (say at ω), and OPP_1 is the Radical Axis of the circles (DD_1) etc., so that $O\omega^2 = \omega P^2 + R^2$. (vide §7)

Suppose OPP_1 cuts the circumcircle in Q and Q_1, then $(Q_1 PQP_1)$ is harmonic and $\because$ A lies on the circle on QQ_1 as diameter,

$$\frac{AP}{AP_1} = \frac{PQ}{P_1Q} = \frac{R-OP}{OP_1-R} = \frac{R-OP}{\frac{R^2}{OP}-R} = \frac{OP(R-OP)}{R(R-OP)} = \frac{OP}{R}$$

By the theory of inversion, if M and N be any 2 points, then the distance between their corresponding points M_1 and N_1 equals $\frac{R^2}{OM.ON}\cdot MN$

2. Given the ratios of the distances $(\rho_1 \rho_2 \rho_3)$, to find condition that there may be 2 points P and P_1.

$$\frac{BD}{CD} = \frac{BD_1}{CD_1} = \frac{\rho_2}{\rho_3} \qquad \therefore BD = \frac{\rho_2 a}{\rho_2+\rho_3} \quad \text{and } BD_1 = \frac{\rho_2 a}{\rho_2-\rho_3} \quad \text{(vide Fig. 1)}$$

$$\therefore B\omega_1 = \frac{\rho_2^2 a}{\rho_2^2-\rho_3^2} \qquad \text{and } D\omega_1 = \frac{\rho_2\rho_3 a}{\rho_2^2-\rho_3^2}$$

Similarly $B\omega_3 = \frac{\rho_2^2 c}{\rho_2^2-\rho_1^2}$ and $F\omega_3 = \frac{\rho_1\rho_2 c}{\rho_2^2-\rho_1^2}$

$$(\omega_1\omega_3)^2 = B\omega_1^2 + B\omega_3^2 - 2B\omega_1 . B\omega_3 \cos B.$$

$$= \frac{\rho_2^4 a^2}{(\rho_2^2-\rho_3^2)^2} + \frac{\rho_2^4 c^2}{(\rho_2^2-\rho_1^2)^2} - \frac{2\rho_2^4 ac\cos B.}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}$$

$$(\omega_1 D + \omega_3 F)^2 = \left(\frac{\rho_2\rho_3 a}{\rho_2^2-\rho_3^2} + \frac{\rho_1\rho_2 c}{\rho_2^2-\rho_1^2}\right)^2$$

Now circles (DD_1) and (FF_1) cut in 2 points, touch at 1 point, or do not meet according as $\omega_1\omega_3 <=> \omega_1 D + \omega_3 F$

i.e. according as

$$\rho_2^2\left(\frac{\rho_3 a}{\rho_2^2-\rho_3^2}+\frac{\rho_1 c}{\rho_2^2-\rho_1^2}\right)^2 \;> \;=\; <\; \rho_2^4\left(\frac{a^2}{(\rho_2^2-\rho_3^2)^2}+\frac{c^2}{(\rho_2^2-\rho_1^2)^2}-\frac{2ac\cos B}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}\right)$$

i.e. acc. as $$\frac{\rho_3^2 a^2}{(\rho_2^2-\rho_3)^2}+\frac{\rho_1^2 c^2}{(\rho_2^2-\rho_1^2)^2}+\frac{2\rho_3\rho_1 ac}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)} \;> \;=\; <$$

$$\rho_2^2\left(\frac{a^2}{(\rho_2^2-\rho_3)^2}+\frac{c^2}{(\rho_2^2-\rho_1^2)^2}-\frac{2ac\cos B}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}\right)$$

i.e as $$\frac{2\rho_1\rho_3 ac}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)}+\frac{2\rho_2^2 ac\cos B}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)} \;> \;=\; <\; \frac{a^2}{\rho_2^2-\rho_3^2}+\frac{c^2}{\rho_2^2-\rho_1^2}$$

i.e as $$2\rho_1\rho_3 ac + 2\rho_2^2 ac\cos B \;> \;=\; <\; a^2\rho_2^2 - a^2\rho_1^2 + c^2\rho_2^2 - c^2\rho_3^2$$

i.e as $$a^2\rho_1^2 + c^2\rho_3^2 + 2\rho_1\rho_3 ac \;> \;=\; <\; \rho_2^2(a^2+c^2-2ac\cos B)$$

$$> \;=\; <\; b^2\rho_2^2$$

i.e as $$a\rho_1 + c\rho_3 \;> \;=\; <\; b\rho_2$$

or in any figure according as two of the quantities $a\rho_1$, $b\rho_2$, $c\rho_3$ are together greater, equal to or less than the third.

3. To find the ratios of the distances $\omega_1\omega_3$, $\omega_1\omega_2$, $\omega_2\omega_3$ in terms of ρ_1, ρ_2, ρ_3.

$$(\omega_1\omega_3)^2 = \rho_2^4\left(\frac{a^2}{(\rho_2^2-\rho_3^2)^2}+\frac{c^2}{(\rho_2^2-\rho_1^2)^2}-\frac{a^2+c^2-b^2}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}\right) \quad \text{(vide § 2)}$$

$$= \rho_2^4\left[\frac{a^2}{\rho_2^2-\rho_3^2}\left(\frac{1}{\rho_2^2-\rho_3^2}-\frac{1}{\rho_2^2-\rho_1^2}\right)+\frac{c^2}{\rho_2^2-\rho_1^2}\left(\frac{1}{\rho_2^2-\rho_1^2}-\frac{1}{\rho_2^2-\rho_3^2}\right)+\frac{b^2}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}\right]$$

$$\cancel{= \rho_2^4\left[\frac{a^2(\rho_3^2-\rho_1^2)+c^2}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}\right.}$$

$$= \rho_2^4\left[\frac{a^2(\rho_3^2-\rho_1^2)}{(\rho_2^2-\rho_3^2)^2(\rho_2^2-\rho_1^2)}+\frac{c^2(\rho_1^2-\rho_3^2)}{(\rho_2^2-\rho_1^2)^2(\rho_2^2-\rho_3^2)}+\frac{b^2}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)}\right]$$

$$= \frac{\rho_2^4\sum a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}{(\rho_2^2-\rho_1^2)^2(\rho_2^2-\rho_3^2)^2}$$

$$\therefore\ \omega_1\omega_3 = \frac{\rho_2^2\sqrt{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)}$$ with similar expressions for $\omega_1\omega_2$ and $\omega_2\omega_3$.

$$\therefore\ \omega_2\omega_3 : \omega_3\omega_1 : \omega_1\omega_2 = \frac{\rho_1^2}{(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)} : \frac{\rho_2^2}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)} : \frac{\rho_3^2}{(\rho_3^2-\rho_1^2)(\rho_3^2-\rho_2^2)}$$

$$= \rho_1^2(\rho_2^2-\rho_3^2) : \rho_2^2(\rho_3^2-\rho_1^2) : \rho_3^2(\rho_1^2-\rho_2^2)$$

4. To find the angles $\omega_1 P\omega_3$, etc.

$$\omega_1 P = \omega_1 D = \frac{\rho_2\rho_3 a}{\rho_2^2-\rho_3^2}, \qquad \omega_3 P = \omega_3 F = \frac{\rho_2\rho_1 c}{\rho_2^2-\rho_1^2}$$

$$(\omega_1\omega_3)^2 = \omega_1 P^2 + \omega_3 P^2 - 2\omega_1 P.\,\omega_3 P \cos\omega_1 P\omega_3$$

$$= \frac{\rho_2^2\rho_3^2 a^2}{(\rho_2^2-\rho_3^2)^2} + \frac{\rho_2^2\rho_1^2 c^2}{(\rho_2^2-\rho_1^2)^2} - \frac{2\rho_2^2\rho_1\rho_3\, ac\cos\omega_1 P\omega_3}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}$$

$$\therefore\ \frac{\rho_3^2 a^2}{(\rho_2^2-\rho_3^2)^2} + \frac{\rho_1^2 c^2}{(\rho_2^2-\rho_1^2)^2} - \frac{2\rho_1\rho_3\, ac\cos\omega_1 P\omega_3}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)} = \frac{\rho_2^2 a^2}{(\rho_2^2-\rho_3^2)^2} + \frac{\rho_2^2 c^2}{(\rho_2^2-\rho_1^2)^2} - \frac{2\rho_2^2 ac\cos B}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}$$

$$\therefore\ \frac{a^2}{\rho_2^2-\rho_3^2} + \frac{c^2}{\rho_2^2-\rho_1^2} + \frac{2\rho_1\rho_3\, ac\cos\omega_1 P\omega_3}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)} = \frac{2\rho_2^2 ac\cos B}{(\rho_2^2-\rho_3^2)(\rho_2^2-\rho_1^2)}$$

$$\therefore\ a^2(\rho_2^2-\rho_1^2) + c^2(\rho_2^2-\rho_3^2) + 2\rho_1\rho_3\, ac\cos\omega_1 P\omega_3 = 2\rho_2^2 ac\cos B.$$

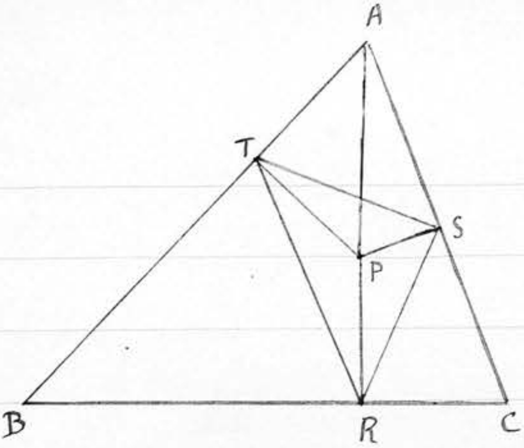
$$\therefore\ \rho_2^2(a^2+c^2-2ac\cos B) = a^2\rho_1^2 + c^2\rho_3^2 - 2ac\rho_1\rho_3\cos\omega_1 P\omega_3$$

$$\therefore\ b^2\rho_2^2 = a^2\rho_1^2 + c^2\rho_3^2 - 2ac\rho_1\rho_3\cos\omega_1 P\omega_3$$

Similarly $a^2\rho_1^2 = b^2\rho_2^2 + c^2\rho_3^2 - 2bc\rho_2\rho_3\cos\omega_2 P\omega_3$

and $c^2\rho_3^2 = a^2\rho_1^2 + b^2\rho_2^2 - 2ab\rho_1\rho_2\cos\omega_1 P\omega_2$

Let the perpendiculars from P to the sides BC, CA, AB be PR, PS, PT. (vide Fig 2, next page), and let AP, BP, CP equal $K\rho_1$, $K\rho_2$, $K\rho_3$ respectively. Then $ST = AP\sin A$, =



$= \dfrac{K_1 \rho_1 a}{2R}$ $\therefore$ the sides of ΔRST are proportional to $a\rho_1, b\rho_2, c\rho_3$

$\therefore b^2\rho_2^2 = a^2\rho_1^2 + c^2\rho_3^2 - 2ac\,\rho_1\rho_3 \cos TSR.$

Hence the angles $\omega_2 P \omega_3$, $\omega_3 P \omega_1$, $\omega_1 P \omega_2$ are equal respectively to $\angle$s TRS, TSR, STR.

Similarly if $\perp$s be dropped from P_1 on the sides viz PR_1, PS_1, PT_1, $\Delta R_1S_1T_1$ has its sides proportional to $a\rho_1, b\rho_2, c\rho_3$ and ΔRST and $\Delta R_1S_1T_1$ are similar Δs.

5. The tripolar coordinates of $\omega_1, \omega_2, \omega_3$.

$$B\omega_1 = \frac{\rho_2^2 a}{\rho_2^2 - \rho_3^2}, \quad C\omega_1 = \frac{\rho_3^2 a}{\rho_2^2 - \rho_3^2}$$

$$A\omega_1^2 = B\omega_1^2 + BA^2 - 2B\omega_1 . BA\cos B.$$

$$= \frac{\rho_2^4 a^2}{(\rho_2^2-\rho_3^2)^2} + c^2 - \frac{2\rho_2^2 ac\cos B}{(\rho_2^2-\rho_3^2)}$$

$$= \frac{\rho_2^4 a^2 + c^2(\rho_2^2-\rho_3^2)^2 - 2\rho_2^2 ac\cos B(\rho_2^2-\rho_3^2)}{(\rho_2^2-\rho_3^2)^2}$$

$$= \frac{a^2\rho_2^4 + c^2\rho_2^4 - 2c^2\rho_2^2\rho_3^2 + c^2\rho_3^4 - \rho_2^2(\rho_2^2-\rho_3^2)(a^2+c^2-b^2)}{(\rho_2^2-\rho_3^2)^2}$$

$$= \frac{b^2\rho_2^4 + c^2\rho_3^4 - \rho_2^2\rho_3^2(b^2+c^2-a^2)}{(\rho_2^2-\rho_3^2)^2} =$$

$$= \left(b^2\rho_2^4 + c^2\rho_3^4 - 2bc\rho_2^2\rho_3^2\cos A\right)\Big/\left(\rho_2^2-\rho_3^2\right)^2$$

$$\therefore A\omega_1^2 : B\omega_1^2 : C\omega_1^2 = b^2\rho_2^4+c^2\rho_3^4-2bc\rho_2^2\rho_3^2\cos A : a^2\rho_2^4 : a^2\rho_3^4$$

with similar expressions for tripolar coordinates of ω_2 and ω_3

(These are used in § 23 to find the tripolar equation to the line $\omega_1\omega_3\omega_2$)

6. Given the ratios of the distances (ρ_1, ρ_2, ρ_3), to find the actual lengths.

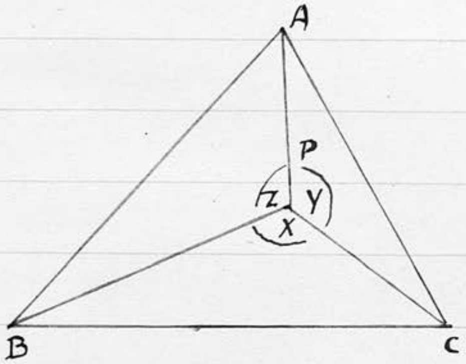


Fig. 3

Let $AP = K\rho_1$, $BP = K\rho_2$, $CP = K\rho_3$, and the angles at P be X, Y, Z (see Fig. 3).

Then $\quad 1 + 2\cos X.\cos Y.\cos Z = \cos^2 X + \cos^2 Y + \cos^2 Z$

$$\therefore\ 1 + 2\left(\frac{K^2\rho_2^2+K^2\rho_3^2-a^2}{2K^2\rho_2\rho_3}\right)\left(\frac{K^2\rho_3^2+K^2\rho_1^2-b^2}{2K^2\rho_1\rho_3}\right)\left(\frac{K^2\rho_1^2+K^2\rho_2^2-c^2}{2K^2\rho_1\rho_2}\right) =$$

$$\Sigma\left(\frac{K^2\rho_2^2+K^2\rho_3^2-a^2}{2K^2\rho_2\rho_3}\right)^2$$

$$\therefore\ 4K^6\rho_1^2\rho_2^2\rho_3^2 + (K^2\rho_2^2+K^2\rho_3^2-a^2)(K^2\rho_3^2+K^2\rho_1^2-b^2)(K^2\rho_1^2+K^2\rho_2^2-c^2) =$$

$$\Sigma\, K^2\rho_1^2(K^2\rho_2^2+K^2\rho_3^2-a^2)^2$$

This equation on multiplication and simplification becomes

$$K^4\left[\Sigma\, a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\right] - K^2\left[\Sigma\, a^2b^2(\rho_1^2+\rho_2^2) - \Sigma\, a^4\rho_1^2\right] + a^2b^2c^2 = 0$$

or $K^4\left[\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\right] - 2abcK^2(\Sigma a\cos A.\rho_1^2) + a^2b^2c^2 = 0$

$$\therefore K^2 = \frac{2abc(\Sigma a\cos A.\rho_1^2) \mp \sqrt{4a^2b^2c^2(\Sigma a\cos A.\rho_1^2)^2 - 4a^2b^2c^2\left[\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\right]}}{2\left[\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\right]}$$

Note $\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2) = \Sigma(a^2\rho_1^4) - 2\Sigma(ab\cos C\,\rho_1^2\rho_2^2)$

$$K^2 = \frac{abc\left\{\Sigma a\cos A.\rho_1^2 \mp \sqrt{2\Sigma ab\sin A\sin B\rho_1^2\rho_2^2 - \Sigma a^2\sin^2 A.\rho_1^4}\right\}}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

Let K_1 and K_2 be the two values of K

$$K_1^2.K_2^2 = \frac{a^2b^2c^2}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)} \quad \text{and}$$

$$\frac{K_1^2}{K_2^2} = \frac{\Sigma a\cos A.\rho_1^2 - \sqrt{2\Sigma ab\sin A\sin B\,\rho_1^2\rho_2^2 - \Sigma a^2\sin^2 A.\rho_1^4}}{\Sigma a\cos A.\rho_1^2 + \sqrt{2\Sigma ab\sin A\sin B\rho_1^2\rho_2^2 - \Sigma a^2\sin^2 A.\rho_1^4}}$$

It may be noted that $\dfrac{\text{area of }\Delta RST}{\text{area of }\Delta R_1S_1T_1}$ (vide §4) $= \dfrac{AP^2}{AP_1^2} = \dfrac{K_1^2}{K_2^2}$

The distance between two corresponding points P and P_1 may be found as follows:- $OP = \dfrac{R.AP}{AP_1} = R\dfrac{K_1}{K_2}$ (vide §1)

$OP_1 = R\dfrac{K_2}{K_1}$ $\quad\therefore PP_1 = R\left(\dfrac{K_2}{K_1} - \dfrac{K_1}{K_2}\right) = R\left(\dfrac{K_2^2-K_1^2}{K_1K_2}\right)$ (vide §11)

If P and P_1 and φ and φ_1 be pairs of corresponding points then $\dfrac{P_1\varphi_1}{P\varphi} = \dfrac{R^2}{OP.O\varphi}$ (vide §1)

Let the coordinates of P be $K_1\rho_1$, $K_1\rho_2$, $K_1\rho_3$, and of P_1 be $K_2\rho_1$, $K_2\rho_2$, $K_2\rho_3$ and those of φ and φ_1 be $\lambda_1\sigma_1$, $\lambda_1\sigma_2$, $\lambda_1\sigma_3$ and $\lambda_2\sigma_1$, $\lambda_2\sigma_2$, $\lambda_2\sigma_3$, then $OP = R\dfrac{K_1}{K_2}$ and

$O\varphi = R\dfrac{\lambda_1}{\lambda_2}$ $\quad\therefore \dfrac{P_1\varphi_1}{P\varphi} = \dfrac{R^2.K_2}{R.K_1}\cdot\dfrac{\lambda_2}{R\lambda_1} = \dfrac{K_2\lambda_2}{K_1\lambda_1}$

Examples (1) For I (the incentre of ΔABC) $\rho_1 : \rho_2 : \rho_3 = \frac{1}{\sin\frac{A}{2}} : \frac{1}{\sin\frac{B}{2}} : \frac{1}{\sin\frac{C}{2}}$

$$2\Sigma \sin^2 A \sin^2 B\, \rho_1^2\rho_2^2 - \Sigma \sin^4 A\, \rho_1^4 = 2\Sigma \frac{\sin^2 A \sin^2 B}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}} - \Sigma \frac{\sin^4 A}{\sin^4\frac{A}{2}} =$$

$$16\left(2\Sigma \cos^2\frac{A}{2}\cos^2\frac{B}{2} - \Sigma\cos^4\frac{A}{2}\right) = \frac{16 s^2}{a^2b^2c^2}\left\{2\Sigma ab(s-a)(s-b) - \Sigma a^2(s-a)^2\right\}$$

$$= \frac{16 s^2}{a^2b^2c^2} \times 4\Delta^2 = \frac{64 s^2\Delta^2}{a^2b^2c^2}$$

$$\therefore OI^2 = R^2 \frac{K_1^2}{K_2^2} = R^2\left(\Sigma\frac{\sin A\cos A}{\sin^2\frac{A}{2}} - \frac{8s\Delta}{abc}\right)\Big/\left(\Sigma\frac{\sin A\cos A}{\sin^2\frac{A}{2}} + \frac{8s\Delta}{abc}\right) =$$

$$R^2\left(\frac{4abcs - 16\Delta^2}{8R(s-a)(s-b)(s-c)} - \frac{8s\Delta}{abc}\right)\Big/\left(\frac{4abcs\Delta - 16\Delta^2}{8R(s-a)(s-b)(s-c)} + \frac{8s\Delta}{abc}\right) =$$

$$R^2\left(\frac{abcs^2}{2R\Delta^2} - \frac{2\Delta^2 s}{R\Delta^2} - \frac{8s\Delta}{abc}\right)\Big/\left(\frac{abcs^2}{2R\Delta^2} - \frac{2\Delta^2 s}{R\Delta^2} + \frac{8s\Delta}{abc}\right) =$$

$$R^2\left(\frac{2}{r} - \frac{4}{R}\right)\Big/\left(\frac{2}{r}\right) = R^2 - 2Rr. \quad \text{(vide §11)}$$

Let I' be corresponding point to I, then $(OI')^2 = \frac{R^4}{OI^2} = \frac{R^3}{R-2r}$. and $II' = OI' - OI = \frac{R\sqrt{R}}{\sqrt{R-2r}} - \sqrt{R(R-2r)} = \frac{2r\sqrt{R}}{\sqrt{R-2r}}$.

(2) In exactly the same way, if $I_1\, I_2\, I_3$ (the excentres) have corresponding points $I_1'\, I_2'\, I_3'$, then

$OI_1' = \frac{R\sqrt{R}}{\sqrt{R+2r_1}}$, $OI_2' = \frac{R\sqrt{R}}{\sqrt{R+2r_2}}$, $OI_3' = \frac{R\sqrt{R}}{\sqrt{R+2r_3}}$, and the respective values of $\frac{K_2}{K_1}$ are $\sqrt{\frac{R}{R+2r_1}}, \sqrt{\frac{R}{R+2r_2}}, \sqrt{\frac{R}{R+2r_3}}$.

$\therefore$ the distances of the points corresponding to $I\, I_1\, I_2\, I_3$ from O are proportional to $\frac{1}{\sqrt{R-2r}}$, $\frac{1}{\sqrt{R+2r_1}}$, $\frac{1}{\sqrt{R+2r_2}}$, $\frac{1}{\sqrt{R+2r_3}}$

$$I_1 I_1' = OI_1 - OI_1' = \sqrt{R(R+2r_1)} - \frac{R\sqrt{R}}{\sqrt{R+2r_1}} = \frac{2r_1\sqrt{R}}{\sqrt{R+2r_1}}.$$

and $$\frac{I_1'I_2'}{I_1 I_2} = \frac{R^2}{OI_1 . OI_2} = \frac{R^2}{\sqrt{R^2+2Rr_1}.\sqrt{R^2+2Rr_2}} = \frac{R\sqrt{R+2r_3}}{\sqrt{(R+2r_1)(R+2r_2)(R+2r_3)}}$$

But $I_1 I_2 = 4R\cos\frac{C}{2}$ $\therefore I_1'I_2' = \dfrac{4R^2\sqrt{R+2r_3}.\cos\frac{C}{2}}{\sqrt{(R+2r_1)(R+2r_2)(R+2r_3)}}$

$\therefore$ sides of $\Delta I_1'I_2'I_3'$ are proportional to $\cos\frac{A}{2}\sqrt{R+2r_1}$, $\cos\frac{B}{2}\sqrt{R+2r_2}$, $\cos\frac{C}{2}\sqrt{R+2r_3}$, while those of $\Delta I_1 I_2 I_3$ are proportional to $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$.

(3) The tripolar coordinates of orthocentre (H) are as $\cos A : \cos B : \cos C$.

$$\therefore \frac{K_2^2}{K_1^2} = \frac{\Sigma\cos^2 A\sin A\cos A + \sqrt{2\Sigma\cos^2 A\sin^2 A\cos^2 B\sin^2 B - \Sigma\cos^4 A\sin^4 A}}{\Sigma\cos^2 A\sin A\cos A - \sqrt{2\Sigma\cos^2 A\sin^2 A\cos^2 B\sin^2 B - \Sigma\cos^4 A\sin^4 A}}$$

$$= \frac{\Sigma(\cos^2 A\sin A\cos A) + 4\sin A\cos A\sin B\cos B\sin C\cos C}{\Sigma(\cos^2 A\sin A\cos A) - 4\sin A\cos A\sin B\cos B\sin C\cos C}$$

$$= \frac{\Sigma(\cos^2 A\sin 2A) + \sin 2A.\sin 2B.\sin 2C}{\Sigma(\cos^2 A\sin 2A) - \sin 2A.\sin 2B.\sin 2C}$$

$$= \frac{2\sin A\sin B\sin C}{2\sin A\sin B\sin C - 2\sin 2A\sin 2B\sin 2C}$$

$$= \frac{1}{1 - 8\cos A\cos B\cos C}.$$

$\therefore OH^2 = R^2\dfrac{K_1^2}{K_2^2} = R^2(1 - 8\cos A\cos B\cos C)$ (vide § 11)

and if H_1 be corresponding point to H, $OH_1 = \dfrac{R^2}{1-8\cos A\cos B\cos C}$

$$HH_1 = OH_1 - OH = \frac{8\cos A\cos B\cos C \cdot R}{\sqrt{1-8\cos A\cos B\cos C}}.$$

7. A proof that $O\omega^2 = \omega P^2 + R^2$ (vide Fig 1).

$$O\omega_1^2 = B\omega_1^2 + BO^2 - 2B\omega_1 . BO . \cos(90-A)$$

$$= \frac{\rho_2^4 a^2}{(\rho_2^2-\rho_3^2)^2} + R^2 - \frac{2\rho_2^2 a}{\rho_2^2-\rho_3^2} . R\sin A.$$

$$= \frac{\rho_2^4 a^2}{(\rho_2^2-\rho_3^2)^2} - \frac{\rho_2^2 a^2}{\rho_2^2-\rho_3^2} + R^2$$

$$= \frac{a^2\rho_2^2\rho_3^2}{(\rho_2^2-\rho_3^2)^2} + R^2 = D\omega_1^2 + R^2 = \omega_1 T^2 + R^2$$

$$\therefore\ O\omega^2 + \omega\omega_1^2 = \omega P^2 + \omega\omega_1^2 + R^2$$

$$\therefore\ O\omega^2 = \omega P^2 + R^2.$$

8 Coordinates of ω

$$AP^2 + AP_1^2 = 2A\omega^2 + 2P\omega^2$$

$$\therefore\ 4A\omega^2 = 2AP^2 + 2AP_1^2 - PP_1^2 = 2\rho_1^2(K_1^2+K_2^2) - R^2\left(\frac{K_2^2-K_1^2}{K_1K_2}\right)^2$$

$$= 2\rho_1^2(K_1^2+K_2^2) - R^2\left\{\frac{(K_1^2+K_2^2) - 4K_1^2K_2^2}{K_1^2K_2^2}\right\} =$$

$$2\rho_1^2\left(\frac{2abc\,\Sigma(a\cos A.\rho_1^2)}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}\right) - 4R^2\left(\frac{(\Sigma a\cos A.\rho_1^2)^2 - \Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}\right)$$

$$= \frac{4\rho_1^2\left(abc(\Sigma a\cos A.\rho_1^2)\right) - \left(2\Sigma a^2b^2\rho_1^2\rho_2^2 - \Sigma a^4\rho_1^4\right)}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

$$= \frac{\left\{\begin{array}{l} a^2\rho_1^4(a^2+4bc\cos A) + b^2\rho_2^4 + c^2\rho_3^4 + 2ab\rho_1^2\rho_2^2(2bc\cos B - ab) \\ + 2ac\rho_1^2\rho_3^2(2bc\cos C - ac) - 2b^2c^2\rho_2^2\rho_3^2 \end{array}\right\}}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

$$= \frac{\rho_1^4\left[(b^2+c^2)^2 - 4b^2c^2\cos^2 A\right] + (b^2\rho_2^2 - c^2\rho_3^2)^2 - 2\rho_1^2(b^2-c^2)(b^2\rho_2^2 - c^2\rho_3^2)}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

$$= \frac{4b^2c^2\sin^2 A.\rho_1^4 + (b^2-c^2)^2\rho_1^4 - 2\rho_1^2(b^2\rho_2^2 - c^2\rho_3^2) + (b^2\rho_2^2 - c^2\rho_3^2)^2}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

$$= \frac{16\Delta^2.\rho_1^4 + \{b^2(\rho_1^2-\rho_2^2) - c^2(\rho_1^2-\rho_3^2)\}^2}{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}$$

$$\therefore A\omega^2 : B\omega^2 : C\omega^2 = 16\Delta^2\rho_1^4 + \{b^2(\rho_1^2-\rho_2^2) - c^2(\rho_1^2-\rho_3^2)\}^2 : 16\Delta^2\rho_2^4 + \{c^2(\rho_2^2-\rho_3^2) - a^2(\rho_2^2-\rho_1^2)\}^2 : 16\Delta^2\rho_3^4 + \{a^2(\rho_3^2-\rho_1^2) - b^2(\rho_3^2-\rho_2^2)\}^2$$

Example: For othocentre H, $\rho_1 : \rho_2 : \rho_3 = \cos A : \cos B : \cos C$.

$$\therefore A\omega^2 \propto 16\Delta^2.\cos^4 A + \{b^2(\cos^2 A - \cos^2 B) - c^2(\cos^2 A - \cos^2 C)\}^2$$

$$\propto 4b^2c^2\sin^2 A\cos^4 A + \{b^2(\sin^2 B - \sin^2 A) - c^2(\sin^2 C - \sin^2 A)\}^2$$

$$\propto 4b^2c^2\sin^2 A\cos^4 A + \left\{\frac{b^2(b^2-a^2) - c^2(c^2-a^2)}{4R^2}\right\}^2$$

$$\propto 4b^2c^2\sin^2 A\cos^4 A + \left\{\frac{(b^2-c^2).2bc\cos A}{4R^2}\right\}^2$$

$$\propto 4b^2c^2\cos^2 A\{\sin^2 A\cos^2 A + (\sin^2 B - \sin^2 C)^2\}$$

$$\propto 4b^2c^2\sin^2 A\cos^2 A\{\cos^2 A + \sin^2 \overline{B-C}\}$$

$$\propto 16\Delta^2.\cos^2 A\{\cos^2 A + \sin^2\overline{B-C}\}$$

$$\propto \cos^2 A\{1 - \sin 2B.\sin 2C\}$$

$$\therefore A\omega^2 : B\omega^2 : C\omega^2 = \cos^2 A(1-\sin 2B.\sin 2C) : \cos^2 B(1-\sin 2C.\sin 2A) : \cos^2 C(1-\sin 2A\sin 2B)$$

9. Relations between trilinear or areal coordinates and tripolar coordinates of a point

(a) Given trilinear coordinates of a point, to determine the tripolar coordinates.

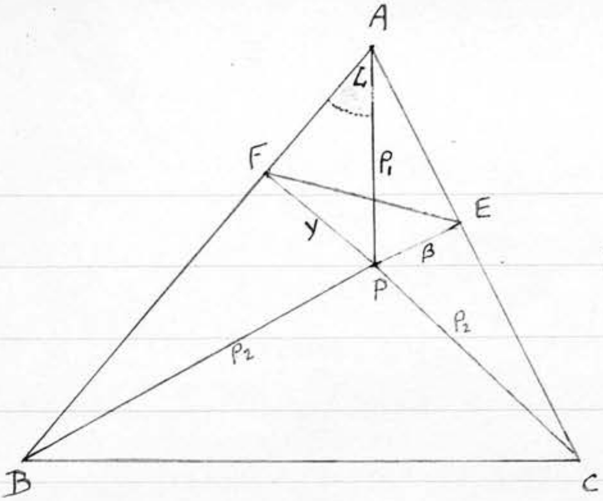


Fig. 4

$$EF^2 = AP^2 \sin^2 A \quad = \rho_1^2 \sin^2 A$$

$$EF^2 = EP^2 + FP^2 - 2\,EP.FP \cos FPE$$

$$= \beta^2 + y^2 + 2\beta y \cos A \quad = (\beta+y)^2 \cos^2\frac{A}{2} + (\beta-y)^2 \sin^2\frac{A}{2}$$

$\therefore \rho_1^2 = \dfrac{\beta^2 + y^2 + 2\beta y \cos A}{\sin^2 A}$ with corresponding relations for ρ_2^2 and ρ_3^2.

For areal coordinates $\rho_1^2 = \dfrac{\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} + \dfrac{2yz}{bc}\cos A}{\sin^2 A}$

$$\therefore \rho_1^2 = \frac{c^2y^2 + b^2z^2 + 2bcyz\cos A}{4\Delta^2}$$

Examples. Incentre (I); Trilinear $(1,1,1)$; Tripolar $\left(\dfrac{1}{\sin\frac{A}{2}}, \dfrac{1}{\sin\frac{B}{2}}, \dfrac{1}{\sin\frac{C}{2}}\right)$

Excentre opposite A; Trilinear $(-1, 1, 1)$; Tripolar $\left(\dfrac{1}{\sin\frac{A}{2}}, \dfrac{1}{\cos\frac{B}{2}}, \dfrac{1}{\cos\frac{C}{2}}\right)$

Circumcentre; Trilinear $(\cos A, \cos B, \cos C)$: Tripolar $(1, 1, 1)$

Centroid; Trilinear $\left(\dfrac{1}{\sin A}, \dfrac{1}{\sin B}, \dfrac{1}{\sin C}\right)$: Tripolar $\left(\sqrt{b^2+c^2+2bc\cos A}, \sqrt{c^2+a^2+2ca\cos B}, \sqrt{a^2+b^2+2ab\cos C}\right)$

Symmedian Point; Trilinear $(\sin A, \sin B, \sin C)$; Tripolar $\left(\dfrac{\sqrt{b^2+c^2+2bc\cos A}}{a}, \dfrac{\sqrt{c^2+a^2+2ca\cos B}}{b}, \dfrac{\sqrt{a^2+b^2+2ab\cos C}}{c}\right.$ or

$$\left\{\left(\cot\frac{A}{2}.\cos\frac{B-C}{2}\right)^2+\left(\tan\frac{A}{2}.\sin\frac{B-C}{2}\right)^2\right\}^{\frac{1}{2}} : \left\{\left(\cot\frac{B}{2}.\cos\frac{C-A}{2}\right)^2+\left(\tan\frac{B}{2}.\sin\frac{C-A}{2}\right)^2\right\}^{\frac{1}{2}} :$$
$$\left\{\left(\cot\frac{C}{2}\cos\frac{A-B}{2}\right)^2+\left(\tan\frac{C}{2}.\sin\frac{A-B}{2}\right)^2\right\}^{\frac{1}{2}}$$

Orthocentre ; Trilinear $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$: Tripolar $(\cos A, \cos B, \cos C)$

Nine-point-centre; Trilinear $[\cos(B-C), \cos(C-A), \cos(A-B)]$;

Tripolar $\left((1+8\cos A\sin B\sin C)^{\frac{1}{2}}, (1+8\cos B\sin A\sin C)^{\frac{1}{2}}, (1+8\cos C\sin A\sin B)^{\frac{1}{2}}\right)$

(vide §25 Ex. 3.)

(b) Let (ρ_1, ρ_2, ρ_3) and $(\sigma_1, \sigma_2, \sigma_3)$ be isogonal points

$$\rho_1^2 = \frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}, \quad \sigma_1^2 = \frac{\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{2\cos A}{\beta\gamma}}{\sin^2 A} =$$

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\beta^2\gamma^2\sin^2 A} = \frac{\rho_1^2}{\beta^2\gamma^2} = \frac{\alpha^2\rho_1^2}{\alpha^2\beta^2\gamma^2}$$

$$\therefore \sigma_1 : \sigma_2 : \sigma_3 = \alpha\rho_1 : \beta\rho_2 : \gamma\rho_3$$

Examples : Circumcentre and Orthocentre

Circumcentre $\rho_1 : \rho_2 : \rho_3 = 1:1:1$ and $\alpha:\beta:\gamma = \cos A:\cos B:\cos C$

Orthocentre $\sigma_1 : \sigma_2 : \sigma_3 = \cos A : \cos B : \cos C$.

Two Brocard Points

1st Brocard Point Ω; $\rho_1 : \rho_2 : \rho_3 = \frac{b}{a} : \frac{c}{b} : \frac{a}{c}$ and $\alpha : \beta : \gamma =$
$\frac{c}{b} : \frac{a}{c} : \frac{b}{a}$ $\quad \left(\rho_1^2 : \rho_2^2 : \rho_3^2 = \frac{\Sigma a^2b^2}{a^4c^2} : \frac{\Sigma a^2b^2}{b^4a^2} : \frac{\Sigma a^2b^2}{c^4b^2}\right)$

2nd Brocard Point Ω'; $\sigma_1 : \sigma_2 : \sigma_3 = \frac{b}{a}\times\frac{c}{b} : \frac{a}{c}\times\frac{c}{b} : \frac{a}{c}\times\frac{b}{a} = \frac{c}{a} : \frac{a}{b} : \frac{b}{c}$.

It may be noted that $\frac{\rho_2}{\rho_3} = \frac{\sin(C-\omega)}{\sin\omega} = \frac{\sigma_1}{\sigma_3}$

$$\therefore \cot\omega = \frac{\rho_2}{\rho_3\sin C} + \cot C = \frac{\sigma_1}{\sigma_3\sin C} + \cot C.$$

$$\therefore \cot\omega = R\left(\frac{\rho_2}{c\rho_3} + \frac{\rho_3}{a\rho_1} + \frac{\rho_1}{b\rho_2}\right) = R\left(\frac{\sigma_1}{c\sigma_3} + \frac{\sigma_3}{b\sigma_2} + \frac{\sigma_2}{a\sigma_1}\right)$$

(c) Given tripolar coordinates, to find trilinear and areal coordinates.

In Fig. 4, let $\angle FAP = L$, and $\angle EAP = A - L$

& let $\rho_1 \rho_2 \rho_3$ be the actual coordinates of P.

$$\cos L = \frac{\rho_1^2 + c^2 - \rho_2^2}{2\rho_1 c}, \quad \cos(A-L) = \frac{\rho_1^2 + b^2 - \rho_3^2}{2\rho_1 b}$$

$$\therefore \frac{\rho_1^2 + b^2 - \rho_3^2}{2\rho_1 b} = \cos A \cos L + \sin A \sin L$$

$$= \cos A . \frac{\rho_1^2 + c^2 - \rho_2^2}{2\rho_1 c} + \sin A . \frac{y}{\rho_1}$$

$$\therefore 2y \sin A = \frac{\rho_1^2 + b^2 - \rho_3^2}{b} - \frac{\rho_1^2 + c^2 - \rho_2^2}{c} . \cos A$$

$$= \frac{\rho_1^2(c - b\cos A) + bc(b - c\cos A) - c\rho_3^2 + b\rho_2^2 \cos A}{bc}$$

$$= \frac{a\cos B . \rho_1^2 + b\cos A . \rho_2^2 - c\rho_3^2 + abc\cos C}{bc.}$$

$$y = \frac{a\cos B . \rho_1^2 + b\cos A . \rho_2^2 - c\rho_3^2 + abc\cos C}{4\Delta}$$

Hence $$Z = \frac{ac\cos B . \rho_1^2 + bc\cos A . \rho_2^2 - c^2\rho_3^2 + abc^2\cos C}{4\Delta}$$

$$\therefore \alpha : \beta : y = b\cos C . \rho_2^2 + c\cos A . \rho_3^2 - a\rho_1^2 + abc\cos A :$$
$$c\cos A . \rho_3^2 + a\cos C . \rho_1^2 - b\rho_2^2 + abc\cos B :$$
$$a\cos B . \rho_1^2 + b\cos A . \rho_2^2 - c\rho_3^2 + abc\cos C.$$

These relations are largely used in the remainder of the paper, and especially when used $\rho_1 \rho_2 \rho_3$ are actual coordinates of the point.

Examples. $\rho_1 = \rho_2 = \rho_3 = R$. $\therefore \alpha : \beta : y = abc\cos A : abc\cos B :$
$abc\cos C = \cos A : \cos B : \cos C$. (Circumcentre)

The equations to the sides of ΔABC are

BC, $\alpha = 0$ i.e. $b\cos C\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 + abc\cos A = 0$

CA, $\beta = 0$ $\quad c\cos A.\rho_3^2 + a\cos C.\rho_1^2 - b\rho_2^2 + abc\cos B = 0$

AB, $\gamma = 0$ $\quad a\cos B.\rho_1^2 + b\cos A.\rho_2^2 - c\rho_3^2 + abc\cos C = 0$

(vide § 34 b)

10. In the equation

$$K^4\left[\Sigma\, a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\right] - 2abcK^2(\Sigma a\cos A.\rho_1^2) + a^2b^2c^2 = 0 \text{ (vide § 6)}$$

let $K = 1$, so that $\rho_1\ \rho_2\ \rho_3$ are actual coordinates, then

$$\Sigma\, a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2) = 2abc(\Sigma a\cos A.\rho_1^2) - a^2b^2c^2$$

$$= 8R^2\Delta(\Sigma \sin 2A.\rho_1^2) - 16R^2\Delta^2$$

$$\therefore \frac{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}{16\Delta^2} = \frac{\Sigma(\sin 2A.\rho_1^2)}{2\Delta} - R^2.$$

This relation is used in next paragraph. (11).

11. Distance between two points $(\rho_1\,\rho_2\,\rho_3)$ and $(\sigma_1\,\sigma_2\,\sigma_3)$

Let corresponding trilinear coordinates be $(\alpha_1\ \beta_1\ \gamma_1)$ and $(\alpha_2\ \beta_2\ \gamma_2)$ and let the distance between points be d

$$d^2 = -\Sigma\left[\frac{(\beta_1-\beta_2)(\gamma_1-\gamma_2)\sin A}{\sin A\sin B\sin C}\right]$$

$$= -\Sigma\frac{\left[\begin{array}{l}\sin A\{a\cos C(\rho_1^2-\sigma_1^2) + c\cos A(\rho_3^2-\sigma_3^2) - b(\rho_2^2-\sigma_2^2)\}\times \\ \qquad\{a\cos B(\rho_1^2-\sigma_1^2) + b\cos A(\rho_2^2-\sigma_2^2) - c(\rho_3^2-\sigma_3^2)\}\end{array}\right]}{16\Delta^2\sin A\sin B\sin C}$$

Substituting for $\beta_1 = (a\cos C\,\rho_1^2 + c\cos A\,\rho_3^2 - b\rho_2^2 + abc\cos B)/4\Delta$. etc.

$$d^2 = -\frac{\left[\begin{array}{l}\Sigma\{(\rho_1^2-\sigma_1^2)^2(a^2\cos B\cos C\sin A - a^2\cos B\sin B - a^2\cos C\sin C)\} \\ + \Sigma\left\{\begin{array}{l}(\rho_1^2-\sigma_1^2)(\rho_2^2-\sigma_2^2)ab(\cos A\cos C\sin A + \cos B\cos C\sin B \\ - \cos^2 C\sin C - \cos B\sin A - \cos A\sin B + \sin C\end{array}\right\}\end{array}\right]}{16\Delta^2\sin A\sin B\sin C}$$

$$d^2 = -\frac{\left[\Sigma\left\{(\rho_1^2-\sigma_1^2)^2 a^2 \sin A \sin B \sin C\right\} + \Sigma\left\{\frac{(\rho_1^2-\sigma_1^2)(\rho_2^2-\sigma_2^2)\, ab\cos C}{2\sin A\sin B\sin C}\right\}\right]}{16\Delta^2 \sin A \sin B \sin C}$$

$$= \frac{-\Sigma\left\{2ab\cos C(\rho_1^2-\sigma_1^2)(\rho_2^2-\sigma_2^2)\right\} - \Sigma\left\{a^2(\rho_1^2-\sigma_1^2)^2\right\}}{16\Delta^2}$$

$$= -\Sigma\left[a^2\left\{(\rho_1^2-\sigma_1^2)(\rho_2^2-\sigma_2^2) + (\rho_1^2-\sigma_1^2)(\rho_3^2-\sigma_3^2) - (\rho_2^2-\sigma_2^2)(\rho_3^2-\sigma_3^2) - (\rho_1^2-\sigma_1^2)^2\right\}\right] / 16\Delta^2$$

$$= \frac{\Sigma\left[a^2\left\{(\rho_1^2-\rho_2^2)-(\sigma_1^2-\sigma_2^2)\right\}\left\{(\rho_1^2-\rho_3^2)-(\sigma_1^2-\sigma_3^2)\right\}\right]}{16\Delta^2}$$

This may be changed into a different form, making use of relation in §10

$$d^2 = \frac{\Sigma a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)}{16\Delta^2} + \frac{\Sigma a^2(\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_3^2)}{16\Delta^2}$$

$$- \frac{\Sigma a^2(\rho_1^2-\rho_2^2)(\sigma_1^2-\sigma_3^2)}{16\Delta^2} - \frac{\Sigma a^2(\rho_1^2-\rho_3^2)(\sigma_1^2-\sigma_2^2)}{16\Delta^2}$$

$$= R^2 \frac{\Sigma(\sin 2A.\,\rho_1^2)}{2\Delta} - R^2 + R^2 \frac{\Sigma(\sin 2A.\,\sigma_1^2)}{2\Delta} - R^2$$

$$+ \frac{2\left(\Sigma\, ab\cos C\,\rho_1^2\sigma_2^2 - \Sigma a^2\rho_1^2\sigma_1^2\right)}{16\Delta^2}$$

$$= R^2\left[\frac{\Sigma \sin 2A(\rho_1^2+\sigma_1^2)}{2\Delta} - 2\right] +$$

$$\frac{\Sigma\left\{a\rho_1^2(b\cos C\,\sigma_2^2 + c\cos B\,\sigma_3^2 - a\sigma_1^2)\right\}}{16\Delta^2} +$$

$$\frac{\Sigma\left\{a\sigma_1^2(b\cos C\,\rho_2^2 + c\cos B.\,\rho_3^2 - a\rho_1^2)\right\}}{16\Delta^2}$$

$$\text{or } = \frac{\sum\{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + 2abc\cos A)\}}{16\Delta^2} +$$

$$\frac{\sum\{a\sigma_1^2(b\cos C.\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 + 2abc\cos A)\}}{16\Delta^2} - 2R^2.$$

When $\sigma_1 = \sigma_2 = \sigma_3 = R$ i.e. $(\sigma_1\sigma_2\sigma_3)$ is circumcentre

$$d^2 = \frac{\sum\{a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\}}{16\Delta^2} = R^2\left\{\frac{\sum \sin 2A.\rho_1^2}{2\Delta}\right\} - R^2.$$

Examples: (a) I is $\left(\frac{r}{\sin\frac{A}{2}}, \frac{r}{\sin\frac{B}{2}}, \frac{r}{\sin\frac{C}{2}}\right)$

$$\therefore OI^2 = \frac{\sum\left\{a^2r^2\left(\frac{1}{\sin^2\frac{A}{2}} - \frac{1}{\sin^2\frac{B}{2}}\right)\left(\frac{1}{\sin^2\frac{A}{2}} - \frac{1}{\sin^2\frac{C}{2}}\right)\right\}}{16\Delta^2}$$

$$= \sum\left[a^2\left\{\frac{bc(s-a)}{s} - \frac{ac(s-b)}{s}\right\}\left\{\frac{bc(s-a)}{s} - \frac{ab(s-c)}{s}\right\}\right]\Big/16\Delta^2$$

$$= \sum\{a^2bc(b-a)(c-a)\}\Big/16\Delta^2$$

$$= \frac{abc}{16\Delta^2}\left\{abc - 8(s-a)(s-b)(s-c)\right\}$$

$$= R^2 - \frac{abc}{4\Delta}.\frac{2.(s-a)(s-b)(s-c)}{\Delta}$$

$$= R^2 - 2Rr.$$

H is $(2R\cos A, 2R\cos B, 2R\cos C)$

$$\text{(b) } OH^2 = \frac{\sum\{a^2(4R^2\cos^2A - 4R^2\cos^2B)(4R^2\cos^2A - 4R^2\cos^2C)\}}{16\Delta^2}$$

$$= \sum\left\{\frac{a^2R^4}{\Delta^2}(\cos^2A - \cos^2B)(\cos^2A - \cos^2C)\right\}$$

$$= R^2\sum\left\{\frac{(\cos^2A - \cos^2B)(\cos^2A - \cos^2C)}{\sin^2A\sin^2B\sin^2C}\right\}$$

$$= R^2\left\{1 - \frac{\sum(\sin^4A\sin^2B) - 2\sin^2A\sin^2B\sin^2C - \sum(\sin^6A)}{\sin^2A\sin^2B\sin^2C}\right\}$$

$$OH^2 = R^2\left\{1 - \frac{\Sigma a^4b^2 - 2a^2b^2c^2 - \Sigma a^6}{a^2b^2c^2}\right\}$$

$$= R^2\left\{1 - \frac{(b^2+c^2-a^2)}{bc}\cdot\frac{(c^2+a^2-b^2)}{ac}\cdot\frac{(a^2+b^2-c^2)}{ab}\right\}$$

$$= R^2(1 - 8\cos A\cos B\cos C).$$

(c) $O\omega^2$ (vide Fig 1 and § 7) $= \left(\dfrac{OP_1+OP_2}{2}\right)^2 = \dfrac{OP_1^{\,2}+OP_2^{\,2}+2R^2}{4}$

$$= \frac{R^2\,\dfrac{\Sigma\{\sin 2A\,(AP_1^{\,2}+AP_2^{\,2})\}}{2\Delta} - \cancel{2R^2} + \cancel{2R^2}}{4}$$

$$= \frac{R^2\,\Sigma\{\sin 2A.\rho_1^{\,2}(k_1^{\,2}+k_2^{\,2})\}}{8\Delta} = \frac{R\,\Sigma a\cos A.\rho_1^{\,2}(2abc\,\Sigma a\cos A.\rho_1^{\,2})}{8\Delta\,\Sigma a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})}$$

$$= \frac{2abcR\{\Sigma a\cos A.\rho_1^{\,2}\}^2}{8\Delta\,\Sigma\{a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})\}}$$

$$= \frac{R^2\{\Sigma a\cos A.\rho_1^{\,2}\}^2}{\Sigma\{a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})\}}$$

$$\therefore O\omega = \frac{R\,\Sigma(a\cos A.\rho_1^{\,2})}{\sqrt{\Sigma\{a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})\}}}$$

$$\therefore \omega P^2 = \frac{R^2\{\Sigma a\cos A.\rho_1^{\,2}\}^2}{\Sigma\{a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})\}} - R^2$$

$$\therefore PP_1^{\,2} = \frac{4R^2\{\Sigma a\cos A.\rho_1^{\,2}\}^2}{\Sigma a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})} - 4R^2.$$ (ρ_1 ρ_2 ρ_3 being ratios of coordinates)

Now $\Sigma\{a\cos A.\rho_1^{\,2}\}^2 - \Sigma\{a^2(\rho_1^{\,2}-\rho_2^{\,2})(\rho_1^{\,2}-\rho_3^{\,2})\} =$

$$= \Sigma(a^2\cos^2 A.\rho_1^{\,4}) + 2\Sigma(ab\cos A\cos B.\rho_1^{\,2}\rho_2^{\,2})$$
$$- \Sigma(a^2\rho_1^{\,4}) + 2\Sigma(ab\cos C.\rho_1^{\,2}\rho_2^{\,2}) =$$

$$= 2\Sigma(ab\sin A\sin B\,\rho_1^2.\rho_2^2) - \Sigma(a^2\sin^2 A.\rho_1^4) \quad \text{(vide §6)}$$

$$= (a\sin A.\rho_1 + b\sin B\,\rho_2 + c\sin C.\rho_3)(-a\sin A.\rho_1' + b\sin B.\rho_2 + c\sin C.\rho_3) \times$$
$$(a\sin A.\rho_1 - b\sin B.\rho_2 + c\sin C.\rho_3)(a\sin A.\rho_1 + b\sin B.\rho_2 - c\sin C.\rho_3)$$

$$\therefore PP_1^2 = \frac{(a\rho_1 + b\rho_2 + c\rho_3)(-a\rho_1 + b\rho_2 + c\rho_3)(a\rho_1 - b\rho_2 + c\rho_3)(a\rho_1 + b\rho_2 - c\rho_3)}{2\Sigma(ab\cos C.\rho_1^2.\rho_2^2) - \Sigma(a^2.\rho_1^4)}$$

where ρ_1, ρ_2, ρ_3 are the ratios of the coordinates of P and P_1

$ST = \dfrac{K_1\rho_1 a}{2R}$. Let $\Delta RST = \Delta_1$ (vide §4)

Then $16\Delta_1^2 = \dfrac{K_1^4}{16R^4}(a\rho_1 + b\rho_2 + c\rho_3)(-a\rho_1 + b\rho_2 + c\rho_3)(a\rho_1 - b\rho_2 + c\rho_3)(a\rho_1 + b\rho_2 - c\rho_3)$

and $OP^2 = \dfrac{K_1^4}{16\Delta^2}\Sigma a^2(\rho_1^2 - \rho_2^2)(\rho_1^2 - \rho_3^2)$

$$= \frac{K_1^4}{16\Delta^2}\left[2\Sigma(ab\cos C\,\rho_1^2.\rho_2^2) - \Sigma a^2\rho_1^4\right]$$

$$\therefore \frac{16\Delta_1^2}{OP^2} = \frac{K_1^4}{16R^4}.\frac{16\Delta^2}{K_1^4}.PP_1^2$$

$$\therefore \Delta_1 = \frac{\Delta.\,OP.\,PP_1}{4R^2} = \frac{OP.PP_1}{2}\sin A\sin B\sin C.$$

Similarly if $\Delta R_1S_1T_1$ be Δ_2, $\Delta_2 = \dfrac{OP_1.PP_1\sin A\sin B\sin C}{2}$

$$\therefore \frac{\Delta_1}{\Delta_2} = \frac{OP}{OP_1} = \frac{K_1^2}{K_2^2} \quad \text{(vide §6)}$$

and $PP_1^2 = 16R^2.\dfrac{\Delta_1 \times \Delta_2}{\Delta^2}$ or $PP_1 = 4R.\dfrac{\sqrt{\Delta_1.\Delta_2}}{\Delta}$.

(d) when $r_1 = r_2 = r_3 = R$, $d^2 = R^2\left\{\frac{\Sigma \sin 2A.\rho_1^2}{2\Delta}\right\} - R^2$.

$\therefore$ if $d = R$

$$R^2\left(\frac{\Sigma \sin 2A.\rho_1^2}{2\Delta}\right) - R^2 = R^2$$

$\therefore \Sigma \sin 2A.\rho_1^2 = 4\Delta$ is the equation to the circumcircle

Similarly the equation to any circle whose centre is O

is $\Sigma \sin 2A.\rho_1^2 = 2\Delta\left(\frac{d^2+R^2}{R^2}\right)$

e.g. circle with O as centre and OI as radius is

$$\Sigma \sin 2A.\rho_1^2 = 2\Delta\left(\frac{2R^2 - 2Rr}{R^2}\right) = 4\Delta\left(\frac{R-2r}{R}\right).$$

and with centre O and OH as radius is

$$\Sigma \sin 2A.\rho_1^2 = 2\Delta\left(\frac{2R^2 - 8R^2\cos A\cos B\cos C}{R^2}\right) = 4\Delta(1 - 4\cos A\cos B\cos C)$$

The Straight Line

2. Let the line be $l\alpha + m\beta + n\gamma = 0$

Substitute for α, β, γ, the values given in § 9c. viz:

$\alpha = (b\cos C.\rho_2^2 + c\cos B\rho_3^2 - a\rho_1^2 + abc\cos A)/4\Delta$ etc. and the equation becomes

$$a\rho_1^2(l - m\cos C - n\cos B) + b\rho_2^2(m - l\cos C - n\cos A) + c\rho_2^2(n - l\cos B - m\cos A)$$
$$= abc(l\cos A + m\cos B + n\cos C)$$

Note that the sum of the coefficients of $\rho_1^2, \rho_2^2, \rho_3^2 =$

$$l(a - b\cos C - c\cos B) + m(b - a\cos C - c\cos A) + n(c - a\cos B - b\cos A)$$
$$= 0$$

Let the equation be $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = k$.

$$\therefore \Sigma\left\{\lambda\left(\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}\right)\right\} = k.$$

$$\therefore \Sigma\{\lambda\sin^2 B\sin^2 C(\beta^2+\gamma^2+2\beta\gamma\cos A)\} = k\sin^2 A\sin^2 B\sin^2 C$$

$$\therefore \Sigma\{\lambda b^2c^2(\beta^2+\gamma^2+2\beta\gamma\cos A)\} = 4\Delta^2 k$$
$$= k(a\alpha + b\beta + c\gamma)^2$$

$$\therefore \Sigma\{\alpha^2 a^2(\mu c^2 + \nu b^2 - k)\} + 2\Sigma\{\beta\gamma bc(\lambda bc\cos A - k)\} = 0 \quad (1)$$

This equation factorizes into

$$\left[\Sigma\{a\alpha(\mu c^2 + \nu b^2 - k)\}\right]\left[\Sigma a\alpha\right] = 0 \quad (2)$$ if coefficients of $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$ of (1) equal respectively those of (2)

Take $\beta\gamma$, then $bc(\lambda c^2 + \nu a^2 - k + \lambda b^2 + \mu a^2 - k) = 2bc(\lambda bc - k)$

ie $\lambda(b^2 + c^2 - 2bc\cos A) + \mu a^2 + \nu a^2 = 0$

i.e. $a^2(\lambda + \mu + \nu) = 0$ i.e. $\lambda + \mu + \nu = 0$

Similarly it can be shown on like condition for $\gamma\alpha$, and $\alpha\beta$, that $\lambda+\mu+\nu=0$

now $\Sigma a\alpha \neq 0 \quad \therefore \quad \Sigma\left[a\alpha(\mu c^2+\nu b^2-K)\right]=0$

i.e $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=K$ represents a straight line, if $\lambda+\mu+\nu=0$

Here $l : m : n = a(\mu c^2+\nu b^2-K) : b(\lambda c^2+\nu a^2-K) : c(\lambda b^2+\mu a^2-K)$

This can be shown easily by solving the equations

$$\frac{\lambda}{l-m\cos C-n\cos B}=\frac{\mu}{m-l\cos C-n\cos A}=\frac{\nu}{n-l\cos B-m\cos A}=\frac{K}{abc(l\cos A+m\cos B+n\cos C)}$$

It may be here noted that $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=K$ in general is the equation to a circle, for the conditions that (1) represents a circle are

$$b^2c^2(\lambda c^2+\nu a^2-K)+b^2c^2(\lambda b^2+\mu a^2-K)-2b^2c^2(\lambda bc\cos A-K)=$$
$$a^2c^2(\lambda b^2+\mu a^2-K)+a^2c^2(\mu c^2+\nu b^2-K)-2a^2c^2(\mu ac\cos B-K)=$$
$$a^2b^2(\mu c^2+\nu b^2-K)+a^2b^2(\lambda c^2+\nu a^2-K)-2a^2b^2(\nu ab\cos C-K)$$

or
$$b^2c^2(\lambda c^2+\nu a^2-\cancel{K}+\lambda b^2+\mu a^2-\cancel{K}-2\lambda bc\cos A+\cancel{2K})=$$
$$a^2c^2(\lambda b^2+\mu a^2-\cancel{K}+\mu c^2+\nu b^2-\cancel{K}-2\mu ac\cos B+\cancel{2K})=$$
$$a^2b^2(\mu c^2+\nu b^2-\cancel{K}+\lambda c^2+\nu a^2-\cancel{K}-2\nu ab\cos C+\cancel{2K})$$

or $\quad a^2b^2c^2(\lambda+\mu+\nu)=a^2b^2c^2(\lambda+\mu+\nu)=a^2b^2c^2(\lambda+\mu+\nu).$

If the line $l\alpha + m\beta + n\gamma = 0$ pass through the circumcentre $(\alpha : \beta : \gamma = \cos A : \cos B : \cos C)$, then $l\cos A + m\cos B + n\cos C = 0$

$\therefore \Sigma[a\rho_1^2(l - m\cos C - n\cos B)] = abc.0 = 0$

$\therefore$ the equation of a circumdiameter is $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = 0$ where, of course, $\lambda + \mu + \nu = 0$.

The following proof of the equation to a circumdiameter is well known. (I have not managed to trace the author).

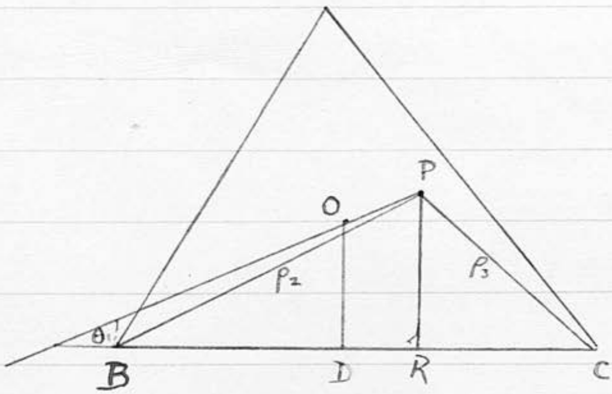


Fig. 5.

Let θ_1 θ_2 θ_3 be the direction angles of the circumdiameter OP, then $\rho_2^2 - \rho_3^2 = a.2DR = a.2OP\cos\theta_1$ etc.

and $(\rho_2^2 - \rho_3^2)\rho_1^2 + (\rho_3^2 - \rho_1^2)\rho_2^2 + (\rho_1^2 - \rho_2^2)\rho_3^2 = 0$

$\therefore$ $a\cos\theta_1 . \rho_1^2 + b\cos\theta_2 . \rho_2^2 + c\cos\theta_3 . \rho_3^2 = 0$ where

$a\cos\theta_1 + b\cos\theta_2 + c\cos\theta_3 = 0$

$\therefore$ $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = 0$ is a circumdiameter where $\lambda + \mu + \nu = 0$.

Mr. William Gallatly gives the following proof that $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = K$ represents a straight line.

" To find the equation to a straight line with direction angles $\theta_1, \theta_2, \theta_3$ and at a distance d from O.

Transferring to Cartesian coordinates we see that the equation differs by a constant from that of the parallel circumdiameter. It must therefore be of the form

$$a\cos\theta_1 . \rho_1^2 + b\cos\theta_2 . \rho_2^2 + c\cos\theta_3 . \rho_3^2 = k_1$$

Let $A'B'C'$ be the mid points of the sides, and let $A'O$ meet the line in A''

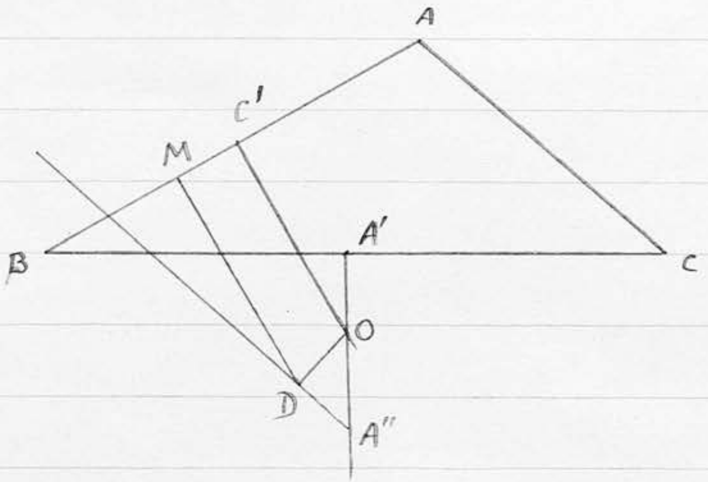


Fig. 6

Then if $(\sigma_1, \sigma_2, \sigma_3)$ be the coordinates of A'',

$$K_1 = a\cos\theta_1 . \sigma_1^2 + b\cos\theta . \sigma_2^2 + c\cos\theta_3 . \sigma_2^2 \quad (\sigma_2 = \sigma_3)$$

$$= a\cos\theta_1 (\sigma_1^2 - \sigma_2^2)$$

$$= a\cos\theta_1 (AM^2 - MB^2)$$

$$= a\cos\theta_1 . 2c . C'M$$

$$= a(OD/OA'') . 2c . OA'' \sin B$$

$$= d . 2ac \sin B$$

$$= 4d\Delta$$

Hence the required equation is $a\cos\theta_1 . \rho_1^2 + \cdots\cdots = 4d\Delta$.

13. To find the perpendicular distance from the point $(\sigma_1, \sigma_2, \sigma_3)$ to the line $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = K$.

The line in trilinear coordinates is $\Sigma[a(\mu c^2 + \nu b^2 - K)\alpha] = 0$

and the point $(\sigma_1 \sigma_2 \sigma_3)$ is

$$\alpha' = (b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)/4\Delta \quad \text{etc.}$$

The length of $\perp^r$ is $\dfrac{l\alpha' + m\beta' + n\gamma'}{\sqrt{l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C}} =$

$$\frac{\Sigma\{a(\mu c^2+\nu b^2-K)(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)\}}{4\Delta\sqrt{\Sigma\{a^2(\mu c^2+\nu b^2-K)^2\} - 2\Sigma\{ab(\mu c^2+\nu b^2-K)(\nu a^2+\lambda c^2-K)\cos C\}}}$$

The numerator $= \sigma_2^2[ab\cos C(\mu c^2+\nu b^2-K) - b^2(\nu a^2+\lambda c^2-K) + bc\cos A(\lambda b^2+\mu a^2-K)]$

$$+ \sigma_3^2[ac\cos B(\mu c^2+\nu b^2-K) + bc\cos A(\nu a^2+\lambda c^2-K) - c^2(\lambda b^2+\mu a^2-K)]$$

$$+ \sigma_1^2[-a^2(\mu c^2+\nu b^2-K) + ab\cos C(\nu a^2+\lambda c^2-K) + ac\cos B(\lambda b^2+\mu a^2-K)]$$

$$+ abc[a\cos A(\mu c^2+\nu b^2-K) + b\cos B(\nu a^2+\lambda c^2-K) + c\cos C(\lambda b^2+\mu a^2-K)]$$

$$= b\sigma_2^2[(ac^2\cos C + ca^2\cos A)\mu + (ab^2\cos C - a^2b)\nu + (b^2c\cos A - bc^2)\lambda]$$

$$+ c\sigma_3^2[(ac^2\cos B - a^2c)\mu + (ab^2\cos B + a^2b\cos A)\nu + (bc^2\cos A - b^2c)\lambda]$$

$$+ a\sigma_1^2[(a^2c\cos B - ac^2)\mu + (a^2b\cos C - ab^2)\nu + (bc^2\cos C + b^2c\cos B)\lambda]$$

$$+ abc[(ac^2\cos A + a^2c\cos C)\mu + (ab^2\cos A + a^2b\cos B)\nu + (bc^2\cos B + b^2c\cos C)\lambda]$$

$$- Kabc(a\cos A + b\cos B + c\cos C)$$

$$= \Sigma\{abc\sigma_1^2[(c\cos C + a\cos A)\mu - b\cos B(\nu+\lambda)]\}$$

$$+ a^2b^2c^2(\lambda+\mu+\nu) - Kabc(a\cos A + b\cos B + c\cos C)$$

$$= abc(a\cos A + b\cos B + c\cos C)(\lambda\sigma_1^2 + \mu\sigma_2^2 + \nu\sigma_3^2 - K)$$

for $\lambda+\mu+\nu = 0$.

$\Sigma\{a^2(\mu c^2+\nu b^2-K)^2\} - 2\Sigma\{ab(\mu c^2+\nu b^2-K)(\nu a^2+\lambda c^2-K)\cos C\} =$

$$= \Sigma\{a^2(\mu^2c^4+\nu^2b^4+k^2+2\mu\nu b^2c^2-2k\mu c^2-2k\nu b^2)\}$$
$$-2\Sigma\{abc\cos C(\mu\nu a^2c^2+\lambda\mu c^4+\nu^2a^2b^2+\lambda\nu b^2c^2-k\nu a^2-k\lambda c^2$$
$$-k\mu c^2-k\nu b^2+k^2)\}$$
$$= \Sigma\{\lambda^2(b^2c^4+b^4c^2-2b^3c^3\cos A)\} +$$
$$2\Sigma\{\lambda\mu(a^2b^2c^2-abc^4\cos C-a^2bc^3\cos A-ab^2c^3\cos B)\}$$
$$+k^2(a^2+b^2+c^2-2ab\cos C-2bc\cos A-2ca\cos B)$$
$$-2\Sigma\{k\lambda(2b^2c^2-abc^2\cos C-b^3c\cos A-bc^3\cos A-ab^2c\cos B)\}$$
$$= \Sigma\lambda^2a^2b^2c^2+2\Sigma\{\lambda\mu abc^2(ab-c^2\cos C-ac\cos A-bc\cos B)\}$$
$$-2\Sigma\{k\lambda bc(2bc-ac\cos C-b^2\cos A-c^2\cos A-ab\cos B)\}$$
$$= \Sigma\lambda^2a^2b^2c^2+2\Sigma\lambda\mu a^2b^2c^2\cos 2C.$$

$\therefore$

$$\perp^r = \frac{(a\cos A+b\cos B+c\cos C)(\lambda\sigma_1^2+\mu\sigma_2^2+\nu\sigma_3^2-k)}{4\Delta\sqrt{\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C}.}$$
$$= \frac{\lambda\sigma_1^2+\mu\sigma_2^2+\nu\sigma_3^2-k}{2R\sqrt{\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C}}.$$

$\therefore$ $\perp^r$ from circumcentre on line $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=k$

is $\dfrac{k}{2R\sqrt{\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C}}$ for $\rho_1=\rho_2=\rho_3=R$, and $\lambda+\mu+\nu=0$.

$$= \frac{k}{2RD} \quad \text{where } D=\sqrt{\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C}.$$

Reverting to M^r. Gallatly's equation at end of p12, we have

$$\frac{\lambda}{a\cos\theta_1} = \frac{\mu}{b\cos\theta_2} = \frac{\nu}{c\cos\theta_3} = \frac{k}{4\Delta.\frac{k}{2RD}} = \frac{RD}{2\Delta}.$$

$$\therefore \cos\theta_1 = \frac{2\Delta.\lambda}{aRD} = \frac{4R^2\sin A\sin B\sin C.\lambda}{2R^2\sin A.D} = \frac{2\sin B\sin C.\lambda}{D}$$

Note that $d = \frac{K}{2RD}$. and to convert $\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = K$ multiply each term by $\frac{2\Delta}{RD} = \frac{4R\sin A\sin B\sin C}{D} = \frac{\Sigma a\cos A}{D}$.

4. Condition that 2 straight lines may be parallel.

Let the lines be $\lambda_1\rho_1^2 + \mu_1\rho_2^2 + \nu_1\rho_3^2 = K_1$ and $\lambda_2\rho_1^2 + \mu_2\rho_2^2 + \nu_2\rho_3^2 = K_2$

The condition that the lines $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $l_2\alpha + m_2\beta + n_2\gamma = 0$ are parallel is that

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ a, & b, & c \end{vmatrix} = 0$$

Substituting $l_1 = a(\mu_1c^2 + \nu_1b^2 - K_1)$ etc., the required condition is

$$\begin{vmatrix} a(\mu_1c^2 + \nu_1b^2 - K_1), & b(\lambda_1c^2 + \nu_1a^2 - K_1), & c(\lambda_1b^2 + \mu_1a^2 - K_1) \\ a(\mu_2c^2 + \nu_2b^2 - K_2), & b(\lambda_2c^2 + \nu_2a^2 - K_2), & c(\lambda_2b^2 + \mu_2a^2 - K_2) \\ a, & b, & c \end{vmatrix} = 0$$

that is
$$\begin{vmatrix} \mu_1c^2 + \nu_1b^2 - K_1, & \lambda_1c^2 + \nu_1a^2 - K_1, & \lambda_1b^2 + \mu_1a^2 - K_1 \\ \mu_2c^2 + \nu_2b^2 - K_2, & \lambda_2c^2 + \nu_2a^2 - K_2, & \lambda_2b^2 + \mu_2a^2 - K_2 \\ 1, & 1, & 1, \end{vmatrix} = 0$$

or $\Sigma\{(\mu_1 c^2+\nu_1 b^2-k_1)(\lambda_2 c^2+\nu_2 a^2-\cancel{k_2}-\lambda_2 b^2-\mu_2 a^2+\cancel{k_2})\}=0$

or $\Sigma\{(\mu_1 c^2+\nu_1 b^2)(\lambda_2 c^2+\nu_2 a^2-\lambda_2 b^2-\mu_2 a^2)\}=0$ (k_1's cancel out)

or $\Sigma\{(\mu_1 c^2+\nu_1 b^2)(\lambda_2 c^2-\lambda_2 b^2+\overline{\nu_2-\mu_2}\,a^2)\}=0$

or $\Sigma\{a^4(\mu_2\nu_1-\mu_1\nu_2)\}+\Sigma\{a^2b^2(\cancel{\nu_1\nu_2}-\nu_1\mu_2+\nu_1\lambda_2-\cancel{\nu_1\nu_2}-\lambda_1\nu_2+\mu_1\nu_2)\}=0$

or $\Sigma\{a^2(\mu_2\nu_1-\mu_1\nu_2)(a^2-b^2-c^2)\}=0$

or $2\Sigma\{a^2bc\cos A(\mu_1\nu_2-\mu_2\nu_1)\}=0$

or $a\cos A(\mu_1\nu_2-\mu_2\nu_1)+b\cos B(\nu_1\lambda_2-\nu_2\lambda_1)+c\cos C(\lambda_1\mu_2-\lambda_2\mu_1)=0$

i.e. $\Sigma\{(\lambda_1\mu_2-\lambda_2\mu_1)\sin 2C\}=0$

15. Equation to the straight line through $(\sigma_1,\sigma_2,\sigma_3)$ parallel to the line $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=k$

Let the line be $L\rho_1^2+M\rho_2^2+N\rho_3^2-\overline{K}=0$

then $$L\sigma_1^2+M\sigma_2^2+N\sigma_3^2-\overline{K}=0$$

and $L(\mu\sin 2C-\nu\sin 2B)+M(\nu\sin 2A-\lambda\sin 2C)+N(\lambda\sin 2C-\mu\sin 2A)=0$

(Condition for parallelism, vide §14)

and $L+M+N=0$

$$\therefore\quad\begin{vmatrix}\rho_1^2, & \rho_2^2, & \rho_3^2, & 1\\ \sigma_1^2, & \sigma_2^2, & \sigma_3^2, & 1\\ \mu\sin 2C-\nu\sin 2B, & \nu\sin 2A-\lambda\sin 2C, & \lambda\sin 2C-\mu\sin 2A, & 0\\ 1, & 1, & 1, & 0\end{vmatrix}=0$$

This becomes

$$\Sigma\{((\nu+\mu)\sin 2A-2\sin A\cos\overline{B-C}\,\lambda)(\rho_1^2-\sigma_1^2)\}=0$$

and placing $\nu+\mu=-\lambda$ etc, the equation is

$$-\Sigma\{\lambda(\sin 2A+\sin 2B+\sin 2C)(\rho_1^2-\sigma_1^2)\}=0$$

or $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=\lambda\sigma_1^2+\mu\sigma_2^2+\nu\sigma_3^2$.

The same result can be got by using the fact that the equation to the line parallel to $l\alpha+m\beta+n\gamma=0$ through (α',β',γ') is $\dfrac{l\alpha+m\beta+n\gamma}{l\alpha'+m\beta'+n\gamma'}=\dfrac{a\alpha+b\beta+c\gamma}{a\alpha'+b\beta'+c\gamma'}$

and substituting for $l=a(\mu c^2+\nu b^2-K)$ etc, and for

$\alpha=b\cos C\,\rho_2^2+c\cos B\,\rho_3^2-a\rho_1^2+abc\cos A$ etc, and for

$\alpha'=b\cos C.\sigma_2^2+c\cos B.\sigma_3^2-a\sigma_1^2+abc\cos A$ etc.

(It is unnecessary to put in 4Δ in denominators. They cancel each other)

The equation to the straight line O parallel to $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=K$ will be

$\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=(\lambda+\mu+\nu)R^2=0$ (vide § 12)

Example. Let A', B', C' be the medial Δ., then the coefficients of B' are $\left(\frac{a}{2},\frac{\sqrt{a^2+c^2+2ac\cos B}}{2},\frac{c}{2}\right)$

$\therefore$ equation to $B'C'$, the straight-line parallel to BC through B

is $b\cos C.\rho_2^2+c\cos B.\rho_3^2-a\rho_1^2=b\cos C\left(\frac{a^2+c^2+2ac\cos B}{4}\right)+c\cos B.\frac{c^2}{4}-a.\frac{a^2}{4}$

$=b\cos C.\frac{(a^2+c^2+2ac\cos B)}{4}+c\cos B.\frac{c^2}{4}-\frac{a(b\cos C+c\cos B)^2}{4}$

$$\therefore\ b\cos C.\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 = \frac{b^2}{4}\cos C(b - a\cos C) + \frac{c^2}{4}\cos B(c - a\cos B)$$
$$+ \frac{abc\cos B\cos C}{2}$$
$$= \frac{b^2c}{4}\cos A\cos C + \frac{c^2b}{4}\cos A\cos B + \frac{abc\cos B\cos C}{2}$$
$$= \frac{abc\cos A}{4} + \frac{abc\cos B\cos C}{2}$$
$$= \frac{abc}{4}.\cos \overline{B-C}.$$

$\therefore$ the sides of the medial triangle are represented by

$$B'C',\quad b\cos C\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 = \frac{abc}{4}\cos\overline{B-C}$$
$$C'A',\quad c\cos A.\rho_3^2 + a\cos C.\rho_1^2 - b\rho_2^2 = \frac{abc}{4}\cos\overline{C-A}$$
$$A'B',\quad a\cos B.\rho_1^2 + b\cos A.\rho_2^2 - c\rho_3^2 = \frac{abc}{4}\cos\overline{A-B}.$$

The length of the $\perp^r$ from O to the line $B'C'$ =

$$\frac{\frac{abc}{4}\cos\overline{B-C}}{R\sqrt{b^2\cos^2C + c^2\cos^2B + a^2 + 2bc\cos B\cos C\cos 2A - 2ab\cos C.\cos 2C - 2ac\cos B.\cos 2B}}.$$

Substituting for $a = b\cos C + c\cos B$, the part under the square root becomes $2b^2\cos^2C(1-\cos 2C) + 2c^2\cos^2B(1-\cos 2B) +$
$2bc\cos B\cos C(1+\cos 2A - \cos 2B - \cos 2C) =$

$$4b^2\cos^2C\sin^2C + 4c^2\cos^2B\sin^2B + 8bc\cos B\cos C\sin B\sin C\cos A$$
$$= 4b^2\sin^2C\left(\cos^2C + \cos^2B + 2\cos A\cos B\cos C\right)$$
$$= 4b^2\sin^2C\sin^2A.$$

$$\therefore\ \perp^r = \frac{abc\cos\overline{B-C}}{8Rb\sin A\sin C} = \frac{4R\Delta\cos\overline{B-C}}{8Ra\sin B\sin C} =$$

$$= \frac{2R^2 \sin A \sin B \sin C . \cos \overline{B-C}}{2R \sin A \sin B \sin C} = R \cos \overline{B-C}$$

$\therefore$ $\perp^{rs}$ from O to the sides of $\Delta A'B'C'$ are proportional to $\cos \overline{B-C}$, $\cos \overline{C-A}$, $\cos \overline{A-B}$.

16. Condition that 2 straight lines are at right angles to each other.

Let the lines be $\lambda_1 \rho_1^2 + \mu_1 \rho_2^2 + \nu_1 \rho_3^2 = K_1$ and $\lambda_2 \rho_1^2 + \mu_2 \rho_2^2 + \nu_2 \rho_3^2 = K_2$

The condition that the lines $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $l_2\alpha + m_2\beta + n_2\gamma = 0$ are perpendicular to each other is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1)\cos A - (n_1 l_2 + n_2 l_1)\cos B - (l_1 m_2 + l_2 m_1)\cos C = 0$$

Substituting as before, we have

$$\Sigma\{a^2(\mu_1 c^2 + \nu_1 b^2 - K_1)(\mu_2 c^2 + \nu_2 b^2 - K_2)\} -$$

$$\Sigma\{bc\cos A[(\lambda_1 c^2 + \nu_1 a^2 - K_1)(\lambda_1 b^2 + \mu_2 a^2 - K_2) + (\lambda_1 b^2 + \mu_1 a^2 - K_1)(\lambda_2 c^2 + \nu_2 a^2 - K_2)]\} = 0$$

This on multiplication and simplification becomes

$$\Sigma\{a^2[\mu_1\mu_2 c^4 + \nu_1\nu_2 b^4 + b^2c^2(\mu_1\nu_2 + \mu_2\nu_1)]\}$$

$$- \Sigma\{bc\cos A[2\lambda_1\lambda_2 b^2c^2 + (\nu_1\lambda_2 + \lambda_1\nu_2)a^2b^2 + (\lambda_1\mu_2 + \lambda_2\mu_1)a^2c^2 + (\nu_1\mu_2 + \mu_1\nu_2)a^4]\} = 0$$

i.e. to $\Sigma\{\lambda_1\lambda_2(b^2c^4 + b^4c^2 - 2b^3c^3\cos A)\} +$

$$\Sigma\{(\lambda_1\mu_2 + \lambda_2\mu_1)(a^2b^2c^2 - a^2c^2 . bc\cos A - b^2c^2 . ca\cos B - c^4 . ab\cos C)\} = 0$$

or $\Sigma(\lambda_1\lambda_2 . a^2b^2c^2) + \Sigma\{(\lambda_1\mu_2 + \lambda_2\mu_1) . abc^2 . (ab - ac\cos A - bc\cos B - c^2\cos C)\} = 0$

or $\Sigma(\lambda_1\lambda_2 . a^2b^2c^2) + \Sigma\{(\lambda_1\mu_2+\lambda_2\mu_1).a^2b^2c^2.(1-2\sin^2 C)\} = 0$

$\therefore$ Condition for perpendicularity is

$$\Sigma(\lambda_1\lambda_2) + \Sigma\{(\lambda_1\mu_2+\lambda_2\mu_1).\cos 2C\} = 0$$

These pairs of conditions for parallelism and perpendicularly may be noted.

Parallelism: Trilinear Coords. $\Sigma\{(m_1n_2-m_2n_1)\sin A\}=0$

Tripolar ---- $\Sigma\{(\mu_1\nu_2-\mu_2\nu_1)\sin 2A\} = 0$

Perpendicularity: Trilinear ..... $\Sigma l_1 l_2 + \Sigma\{(m_1n_2+m_2n_1)\cos A\} = 0$

Tripolar ..... $\Sigma\lambda_1\lambda_2 + \Sigma\{(\mu_1\nu_2+\mu_2\nu_1)\cos 2A\} = 0$

7. Equation to the straight line through $(\sigma_1, \sigma_2, \sigma_3)$ perpendicular to the line $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2 = K$.

Let the straight line perpendicular to $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=0$ through the circumcentre be $L\rho_1^2+M\rho_2^2+N\rho_3^2=0$

Then $L(\lambda+\nu\cos 2B+\mu\cos 2C)+M(\mu+\lambda\cos 2C+\nu\cos 2B) +$

$N(\nu+\lambda\cos 2B+\mu\cos 2A) = 0$

and $L+M+N=0$

$$\therefore \begin{vmatrix} \rho_1^2 , & \rho_2^2 , & \rho_3^2 \\ \lambda+\nu\cos 2B+\mu\cos C , & \mu+\lambda\cos 2C+\nu\cos 2A , & \nu+\lambda\cos 2B+\mu\cos 2A \\ 1 , & 1 , & 1 \end{vmatrix} = 0$$

$\therefore \Sigma\{\rho_1^2(\mu + \lambda\cos 2C + \nu\cos 2A - \nu - \lambda\cos 2B - \mu\cos 2A)\} = 0$

$\therefore \Sigma\{\rho_1^2(2\mu\sin^2 A - 2\nu\sin^2 A + 2\lambda\sin A.\sin\overline{B-C})\} = 0$

$\therefore \Sigma\{a\rho_1^2(\overline{\mu-\nu}\sin A + \lambda\sin\overline{B-C})\} = 0$

$\therefore \Sigma\{a\rho_1^2(\overline{\mu-\nu}\sin\overline{B+C} - \overline{\mu+\nu}.\sin\overline{B-C})\} = 0 \quad (\lambda+\mu+\nu=0)$

$\therefore \Sigma\{a\rho_1^2(\mu\cos B.c - \nu\cos C.b)\} = 0$

$\therefore$ line parallel to this through $(\sigma_1, \sigma_2, \sigma_3)$ is represented by

$$\Sigma\{a\rho_1^2(c\cos B.\mu - b\cos C.\nu)\} = \Sigma\{a\sigma_1^2(c\cos B.\mu - b\cos C.\nu)\}$$

Example. Equation to altitude from A to BC.

A is $(0, c, b)$ and BC is $b\cos C.\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 = -abc\cos A$

$\therefore$ equation is $a\rho_1^2(c\cos B.b\cos C - b\cos C.c\cos B) + b\rho_2^2(a\cos C.c\cos B + c\cos A.a) + c\rho_3^2(-ab\cos A - a\cos B.b\cos C) =$

$bc^2(a\cos C\,c\cos B + c\cos A.a) + cb^2(-ab\cos A - a\cos B.b\cos C)$

i.e. $b\rho_2^2\,ac\sin B\sin C + c\rho_3^2\,ab\sin B\sin C = \quad = 0$

$bc^2.ac\sin B\sin C - cb^2.ab\sin B\sin C$

or $\rho_2^2 - \rho_3^2 = c^2 - b^2$. (as is easily seen to be true.)

18. Equation to the straight line passing through the points $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3)

Let the line through $(\sigma_1, \sigma_2, \sigma_3)$ be

$$L(\rho_1^2 - \sigma_1^2) + M(\rho_2^2 - \sigma_2^2) + N(\rho_3^2 - \sigma_3^2) = 0 \text{ (vide § 15)}$$

Since it passes through (τ_1, τ_2, τ_3)

$$L(\tau_1^2-\sigma_1^2)+M(\tau_2^2-\sigma_2^2)+N(\tau_3^2-\sigma_3^2)=0$$

and $\quad L + M + N = 0$

$$\therefore \begin{vmatrix} \rho_1^2-\sigma_1^2, & \rho_2^2-\sigma_2^2, & \rho_3^2-\sigma_3^2 \\ \tau_1^2-\sigma_1^2, & \tau_2^2-\sigma_2^2, & \tau_3^2-\sigma_3^2 \\ 1, & 1, & 1, \end{vmatrix} = 0$$

or $\quad \Sigma\{(\rho_1^2-\sigma_1^2)(\tau_2^2-\sigma_2^2-\tau_3^2+\sigma_3^2)\} = 0$

or $\quad \Sigma\{\rho_1^2(\overline{\tau_2^2-\tau_3^2} - \overline{\sigma_2^2-\sigma_3^2})\} = \Sigma\{\sigma_1^2(\overline{\tau_2^2-\tau_3^2} - \overline{\sigma_2^2-\sigma_3^2}\}$

$$= \Sigma\{\sigma_1^2(\tau_2^2-\tau_3^2)\}$$

Examples $\quad$ The median AA'

Here A is $(0, c, b)$ and $A'\left(\sqrt{bc\cos A+\frac{a^2}{4}}, \frac{a}{2}, \frac{a}{2}\right)$

$\therefore \rho_1^2\left(c^2-b^2-\cancel{\frac{a^2}{4}}+\cancel{\frac{a^2}{4}}\right)+\rho_2^2\left(b^2-0^2-\cancel{\frac{a^2}{4}}+bc\cos A+\cancel{\frac{a^2}{4}}\right)+$

$\rho_3^2\left(0^2-c^2-bc\cos A-\cancel{\frac{a^2}{4}}+\cancel{\frac{a^2}{4}}\right) = \left(bc\cos A+\frac{a^2}{4}\right)(c^2-b^2)+\frac{a^2}{4}.b^2+\frac{a^2}{4}(0^2-c^2)$

$\therefore \rho_1^2(c^2-b^2)+\rho_2^2(b^2+bc\cos A)-\rho_3^2(c^2+bc\cos A) = (bc\cos A)(c^2-b^2)$

or $(b+c)\rho_1^2 - \dfrac{b(b+c\cos A)}{b-c}.\rho_2^2 - \dfrac{c(c+b\cos A)}{c-b}.\rho_3^2 = (bc\cos A)(b+c)$

The line joining the Brocard Points Ω & Ω'

Ω is $\left(2R\sin\omega.\frac{b}{a}, 2R\sin\omega.\frac{c}{b}, 2R\sin\omega.\frac{a}{c}\right)$

Ω' is $\left(2R\sin\omega.\frac{c}{a}, 2R\sin\omega.\frac{a}{b}, 2R\sin\omega.\frac{b}{c}\right)$

Hence the ratios of the coordinates are only required

$\therefore \rho_1^2\left(\frac{a^2}{b^2}-\frac{b^2}{c^2}-\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)+\rho_2^2\left(\frac{b^2}{c^2}-\frac{c^2}{a^2}-\frac{a^2}{c^2}+\frac{b^2}{a^2}\right)+$

$\rho_3^2\left(\frac{c^2}{a^2}-\frac{a^2}{b^2}-\frac{b^2}{a^2}+\frac{c^2}{b^2}\right) = \frac{b^2}{a^2}\left(\frac{a^2}{b^2}-\frac{b^2}{c^2}\right)+\frac{c^2}{b^2}\left(\frac{b^2}{c^2}-\frac{c^2}{a^2}\right)+\frac{a^2}{c^2}\left(\frac{c^2}{a^2}-\frac{a^2}{b^2}\right)$

$\therefore \Sigma\{a^2(b^2.\overline{a^2-b^2} - c^2.\overline{c^2-a^2})\}\rho_1^2 = \Sigma\{a^2(b^2c^2-a^2)\}$

The equation to the straight line joining O to a point $(\sigma_1, \sigma_2, \sigma_3)$ is $\Sigma\{\rho_1^2(\sigma_2^2-\sigma_3^2)\} = 0$ for here $\tau_1 = \tau_2 = \tau_3 = R$. (i.e. the equation to OPP_1 for ratios $\sigma_1\sigma_2\sigma_3$ are only required)

Example OI. I is $\left(\frac{r}{\sin\frac{A}{2}}, \frac{r}{\sin\frac{B}{2}}, \frac{r}{\sin\frac{C}{2}}\right)$

$\therefore OI$ is $\Sigma\rho_1^2\left(\frac{1}{\sin^2\frac{B}{2}} - \frac{1}{\sin^2\frac{C}{2}}\right) = 0$

i.e. $\Sigma\{\rho_1^2 a.(b-c)\} = 0$

OH. H is $(2R\cos A, 2R\cos B, 2R\cos C)$

$\therefore OH$ is $\Sigma\{\rho_1^2(\cos^2 B - \cos^2 C)\} = 0$

i.e. $\Sigma\rho_1^2(b^2 - c^2) = 0$

19. Condition that three points are collinear

Let the points be (ρ_1, ρ_2, ρ_3) $(\sigma_1, \sigma_2, \sigma_3)$ (τ_1, τ_2, τ_3)

Condition is by last paragraph

$$\Sigma\{\rho_1^2(\overline{\tau_2^2-\tau_3^2} - \overline{\sigma_2^2-\sigma_3^2})\} = \Sigma\{\sigma_1^2(\tau_2^2-\tau_3^2)\}$$

Example O, G (centroid) and H are collinear

O is (R, R, R), G is $\left(\frac{\sqrt{b^2+c^2+2bc\cos A}}{3}, \frac{\sqrt{c^2+a^2+2ac\cos B}}{3}, \frac{\sqrt{a^2+b^2+2ab\cos C}}{3}\right)$, and H is $(2R\cos A, 2R\cos B, 2R\cos C)$

$\therefore$ Condition is

$$\Sigma\left\{\frac{b^2+c^2+2bc\cos A}{9}(-4R^2\cos^2 B + 4R^2\cos^2 C)\right\} = 0$$

or $\Sigma(b^2+c^2+2bc\cos A)(\cos^2 C - \cos^2 B) = 0$

or $\Sigma\,(b^2+c^2+2bc\cos A)(b^2-c^2)=0$

or $\Sigma\,(b^4-c^4+2b^2c\cos A-2bc^2\cos A)=0$

or $\Sigma\,(b^2+c^2-a^2)(b^2-c^2)=0$, which is true.

20 To find the angle ϕ between 2 lines

$$\lambda_1\rho_1^2+\mu_1\rho_2^2+\nu_1\rho_3^2=K_1 \quad\text{and}\quad \lambda_2\rho_1^2+\mu_2\rho_2^2+\nu_2\rho_3^2=K_2$$

The angle between the lines equals the angle between lines parallel to them through O, i.e between

$$\lambda_1\rho_1^2+\mu_1\rho_2^2+\nu_1\rho_3^2=0 \quad\text{and}\quad \lambda_2\rho_1^2+\mu_2\rho_2^2+\nu_2\rho_3^2=0$$

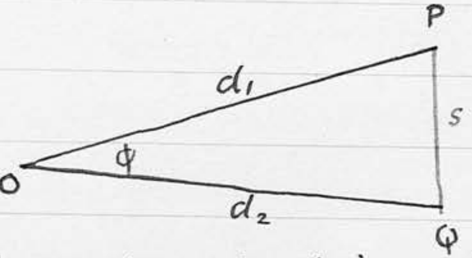


Fig. 7.

Let P and Q on the lines $\lambda_1\rho_1^2+\mu_1\rho_2^2+\nu_1\rho_3^2=0$ and $\lambda_2\rho_1^2+\mu_2\rho_2^2+\nu_2\rho_3^2=0$ respectively and let the distances OP, OQ, PQ be d_1, d_2, s. Let coordinates of P be $(\sigma_1,\sigma_2,\sigma_3)$ and of Q be (τ_1,τ_2,τ_3).

Then
$$\cos\phi=\frac{d_1^2+d_2^2-s^2}{2d_1d_2}$$

$$=\frac{\left[\begin{aligned}&\Sigma\{a^2\cancel{(\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_3^2)}\}+\Sigma\{a^2\cancel{(\tau_1^2-\tau_2^2)(\tau_1^2}-\tau_3^2)\}\\&-\Sigma\{a^2\cancel{(\sigma_1^2-\sigma_2^2)}(\sigma_1^2-\sigma_3^2)\}-\Sigma\{a^2\cancel{(\tau_1^2-\tau_2^2)}(\tau_1^2-\tau_3^2)\}\\&+\Sigma\{a^2(\sigma_1^2-\sigma_2^2)(\tau_1^2-\tau_3^2)\}+\Sigma\{a^2(\sigma_1^2-\sigma_3^2)(\tau_1^2-\tau_2^2)\}\end{aligned}\right]}{2\sqrt{\Sigma\{a^2(\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_3^2)\}.\Sigma\{a^2(\tau_1^2-\tau_2^2)(\tau_1^2-\tau_3^2)\}}}$$

(vide § 11)

The equation to the straight line joining O to P is also

$\Sigma\{\rho_1^2(\sigma_2^2-\sigma_3^2)\}=0$ and joining O to Q is

$\Sigma\{\rho_1^2(\tau_2^2-\tau_3^2)\}=0$ (vide §18)

$$\therefore \lambda_1 : \mu_1 : \nu_1 = \sigma_2^2-\sigma_3^2 : \sigma_3^2-\sigma_1^2 : \sigma_1^2-\sigma_2^2$$

$$\text{and } \lambda_2 : \mu_2 : \nu_2 = \tau_2^2-\tau_3^2 : \tau_3^2-\tau_1^2 : \tau_1^2-\tau_2^2$$

$$\therefore \cos\phi = \frac{-\left[a^2(\mu_1\nu_2+\mu_2\nu_1)+b^2(\nu_1\lambda_2+\nu_2\lambda_1)+c^2(\lambda_1\mu_2+\lambda_2\mu_1)\right]}{2\sqrt{(a^2\mu_1\nu_1+b^2\nu_1\lambda_1+c^2\lambda_1\mu_1)(a^2\mu_2\nu_2+b^2\nu_2\lambda_2+c^2\lambda_2\mu_2)}}$$

$$= \frac{-\Sigma\{a^2(\mu_1\nu_2+\mu_2\nu_1)\}}{2\sqrt{\{\Sigma a^2\mu_1\nu_1\}.\{\Sigma a^2.\mu_2\nu_2\}}}$$

Example. angle between OI and OH

OI is $\Sigma\{a(b-c).\rho_1^2\}=0$ and OH is $\Sigma\{(b^2-c^2).\rho_1^2\}=0$

$$\cos\phi = \frac{-\left[a^2\{b(c-a)(a^2-b^2)+c(a-b)(c^2-a^2)\}+b^2\{a(b-c)(a^2-b^2)+c(a-b)(b^2-c^2)\}+c^2\{a(b-c)(c^2-a^2)+b(c-a)(b^2-c^2)\}\right]}{2\sqrt{\begin{array}{l}\{a^2bc(c-a)(a-b)+c^2ab(b-c)(c-a)+b^2ac(b-c)(a-b)\}\times\\ \{a^2(c^2-a^2)(a^2-b^2)+b^2(b^2-c^2)(a^2-b^2)+c^2(b^2-c^2)(c^2-a^2)\}\end{array}}}$$

$$= \cancel{2\Sigma\{s.ab.(a^2-b^2)\}-2\Sigma\{c(a-b)^2(a+b)\}}$$

$$= \frac{\Sigma a^5b-2\Sigma a^4bc-2\Sigma a^3b^3+\Sigma a^3b^2c}{2\sqrt{\{8abc(s-a)(s-b)(s-c)-a^2b^2c^2\}}\times\sqrt{\{a^2b^2c^2(8\cos A\cos B\cos C-1)\}}}$$

$$= \frac{16\Delta^2(2R^2-2Rr-2r^2-4R^2\cos A\cos B\cos C)}{32\Delta^2\sqrt{(R^2-2Rr).R^2(1-8\cos A\cos B\cos C)}}$$

$$= \frac{OI^2+OH^2-IH^2}{2OI.OH} \quad \text{for } OI^2=R^2-2Rr,$$

$OH^2=R^2(1-8\cos A\cos B\cos C)$ and $IH^2=2r^2-4R^2\cos A\cos B\cos C$.

21. Coordinates of Point of intersection of the two lines, the equations of which are $\lambda_1\rho_1^2+\mu_1\rho_2^2+\nu_1\rho_3^2=K_1$ and $\lambda_2\rho_1^2+\mu_2\rho_2^2+\nu_2\rho_3^2=K_2$.

$$\lambda_1\rho_1^2+\mu_1\rho_2^2-(\lambda_1+\mu_1)\rho_3^2=K_1$$

$$\lambda_2\rho_1^2+\mu_2\rho_2^2-(\lambda_2+\mu_2)\rho_3^2=K_2$$

$$\therefore\ \lambda_1(\lambda_2+\mu_2)\rho_1^2+\mu_1(\lambda_2+\mu_2)\rho_2^2-(\lambda_1+\mu_1)(\lambda_2+\mu_2)\rho_3^2=K_1(\lambda_2+\mu_2)$$

$$\lambda_2(\lambda_1+\mu_1)\rho_1^2+\mu_2(\lambda_2+\mu_2)\rho_3^2-(\lambda_1+\mu_1)(\lambda_2+\mu_2)\rho_3^2=K_2(\lambda_2+\mu_2)$$

$$\therefore\ (\lambda_1\mu_2-\lambda_2\mu_1)\rho_1^2-(\lambda_1\mu_2-\lambda_2\mu_1)\rho_2^2=K_1(l_2+\mu_2)-K_2(\lambda_1+\mu_1)$$

$$=\nu_1K_2-\nu_2K_1$$

$$\therefore\ \rho_1^2-\rho_2^2=\frac{\nu_1K_2-\nu_2K_1}{\lambda_1\mu_2-\lambda_2\mu_1}$$

Similarly $\rho_1^2-\rho_3^2=\dfrac{\mu_1K_2-\mu_2K_1}{\lambda_1\nu_2-\lambda_2\nu_1}$

Now $\Sigma\{a^2(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)\}=2\Sigma(a^2bc\cos A.\rho_1^2)-a^2b^2c^2$

$$=abc\left\{2a\rho_1^2.\cos A+2b.\cos B\left(\rho_1^2-\frac{\nu_1K_2-\nu_2K_1}{\lambda_1\mu_2-\lambda_2\mu_1}\right)+2c.\cos C\left(\rho_1^2-\frac{\mu_1K_2-\mu_2K_1}{\lambda_1\nu_2-\lambda_2\nu_1}\right)-abc\right\}$$

$$\therefore\ (2abc.\Sigma a\cos A)\rho_1^2=\Sigma\left\{\frac{a^2(\nu_1K_2-\nu_2K_1)(\mu_1K_2-\mu_2K_1)}{(\lambda_1\mu_2-\lambda_2\mu_1)(\lambda_1\nu_2-\lambda_2\nu_1)}\right\}+$$

$$2ab^2c\cos B\left(\frac{\nu_1K-\nu_2K_1}{\lambda_1\mu_2-\lambda_2\mu_1}\right)+2abc^2\cos C\left(\frac{\mu_1K_2-\mu_2K_1}{\lambda_1\nu_2-\lambda_2\nu_1}\right)+a^2b^2c^2$$

Now placing for $a^2=b^2+c^2-2bc\cos A$, and remembering that $\lambda_1\mu_2-\lambda_2\mu_1=\mu_1\nu_2-\mu_2\nu_1=\nu_1\lambda_2-\nu_2\lambda_1$, and that $2abc\,\Sigma(a\cos A)=16\Delta^2$, this becomes.

$$16\Delta^2.\rho_1^2 = b^2\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\lambda_2-\nu_2\lambda_1}\right)^2 + c^2\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\lambda_2-\mu_2\lambda_1}\right)^2$$

$$-2bc\cos A\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\lambda_2-\nu_2\lambda_1}\right)\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\lambda_2-\mu_2\lambda_1}\right)$$

$$+2ab^2c\cos B\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\lambda_2-\nu_2\lambda_1}\right)+2abc^2\cos C\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\lambda_2-\mu_2\lambda_1}\right)+a^2b^2c^2$$

or $$16\Delta^2.\rho_1^2 = \left[b\sin B\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\lambda_2-\nu_2\lambda_1}\right) - c\sin C\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\lambda_2-\mu_2\lambda_1}\right)\right]^2 +$$

$$\left[b\cos B\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\lambda_2-\nu_2\lambda_1}\right) + c\cos C\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\lambda_2-\mu_2\lambda_1}\right) + abc\right]^2$$

Similarly $$16\Delta^2.\rho_2^2 = \left[c\sin C\left(\frac{\lambda_1 k_2-\lambda_2 k_1}{\lambda_1\mu_2-\lambda_2\mu_1}\right) - a\sin A\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\mu_2-\nu_2\mu_1}\right)\right]^2 +$$

$$\left[c\cos C.\left(\frac{\lambda_1 k_2-\lambda_2 k_1}{\lambda_1\mu_2-\lambda_2\mu_1}\right) + a\cos A\left(\frac{\nu_1 k_2-\nu_2 k_1}{\nu_1\mu_2-\nu_2\mu_1}\right) + abc\right]^2$$

and $$16\Delta^2.\rho_3^2 = \left[a\sin A\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\nu_2-\mu_2\nu_1}\right) - b\sin B\left(\frac{\lambda_1 k_2-\lambda_2 k_1}{\lambda_1\nu_2-\lambda_2\nu_1}\right)\right]^2 +$$

$$\left[a\cos A\left(\frac{\mu_1 k_2-\mu_2 k_1}{\mu_1\nu_2-\mu_2\nu_1}\right) + b\cos B\left(\frac{\lambda_1 k_2-\lambda_2 k_1}{\lambda_1\nu_2-\lambda_2\nu_1}\right) + abc\right]^2$$

Example.

OH is $(b^2-c^2)\rho_1^2+(c^2-a^2)\rho_2^2+(a^2-b^2)\rho_3^2=0$

and median AA' is $(b+c)\rho_1^2 - b\dfrac{(b+c\cos A)}{b-c}.\rho_2^2 - c\dfrac{(c+b\cos A)}{c-b}.\rho_3^2 = bc(b+c)\cos A$

(vide §18)

Here $k_1=0$

$$\therefore 16\Delta^2.\rho_1^2 = \left[b\sin B\left(\frac{(a^2-b^2)(b+c)\,bc\cos A}{(a^2-b^2)(b+c)-(b+c).c(c+b\cos A)}\right) - c\sin C\left(\frac{(c^2-a^2)(b+c)bc\cos A}{(c^2-a^2)(b+c)+(b+c)b(b+c\cos A)}\right)\right]^2$$

$$+\left[b\cos B\left(\frac{(a^2-b^2)(b+c)bc\cos A}{(a^2-b^2)(b+c)-(b+c).c.(c+b\cos A)}\right) + c\cos B\left(\frac{(c^2-a^2)(b+c)bc\cos A}{(c^2-a^2)(b+c)+(b+c).b(b+c\cos A)}\right) + abc\right]^2$$

$$16\Delta^2\cdot\rho_1^2 = \left[b\sin B\left(\frac{(a^2-b^2)bc\cos A}{a^2-b^2-c^2-bc\cos A}\right) - c\sin C\left(\frac{(c^2-a^2)bc\cos A}{c^2-a^2+b^2+bc\cos A}\right)\right]^2 +$$

$$\left[b\cos B\left(\frac{(a^2-b^2)bc\cos A}{a^2-b^2-c^2-bc\cos A}\right) + c\cos C\left(\frac{(c^2-a^2)bc\cos A}{c^2-a^2+b^2+bc\cos A}\right) + abc\right]^2$$

$$= \left[b\sin B\left(\frac{a^2-b^2}{-3}\right) - c\sin C\left(\frac{c^2-a^2}{3}\right)\right]^2 + \left[b\cos B\left(\frac{a^2-b^2}{-3}\right) + c\cos C\left(\frac{c^2-a^2}{3}\right) + abc\right]^2$$

$$= \frac{b^2(a^2-b^2)^2}{9} + \frac{c^2(c^2-a^2)^2}{9} + a^2b^2c^2 + \frac{2bc\cos A(a^2-b^2)(c^2-a^2)}{9} -$$

$$2ab^2c\cos B\left(\frac{a^2-b^2}{3}\right) + 2abc^2\cos C\left(\frac{c^2-a^2}{3}\right)$$

$$= \left(a^6 - 2b^6 - 2c^6 - 4a^4b^2 + 5a^2b^4 - 4a^4c^2 + 5a^2c^4 + 2b^4c^2 + 2b^2c^4 + 6a^2b^2c^2\right) \div 9$$

$$= 16\Delta^2\left(\frac{2b^2+2c^2-a^2}{9}\right) = 16\Delta^2\frac{(b^2+c^2+2bc\cos A)}{9}.$$

$$\therefore \rho_1^2 : \rho_2^2 : \rho_3^2 = b^2+c^2+2bc\cos A : c^2+a^2+2ca\cos B : a^2+b^2+2ab\cos C.$$

22. Equation of the line joining O to point of intersection of $\Sigma\lambda_1\rho_1^2 = K_1$ and $\Sigma\lambda_2\rho_1^2 = K_2$

Consider the equation $\Sigma\left(\frac{\lambda_1}{K_1} - \frac{\lambda_2}{K_2}\right)\cdot\rho_1^2 = 0$

The sum of the coefficients of $\rho_1^2, \rho_2^2, \rho_3^2$ vis. $\Sigma\left(\frac{\lambda_1}{K_1} - \frac{\lambda_2}{K_2}\right) = \frac{\lambda_1+\mu_1+\nu_1}{K_1} - \frac{\lambda_2+\mu_2+\nu_2}{K_2} = 0$ $\therefore$ it is the equation to a straight line. It passes through the circumcentre because its absolute term (K) is zero. It passes through the point of intersection of $\Sigma\lambda_1\rho_1^2 = K_1$ and $\Sigma\lambda_2\cdot\rho_1^2 = K_2$, for $\Sigma\left(\frac{\lambda_1}{K_1} - \frac{\lambda_2}{K_2}\right)\cdot\rho_1^2 = 0$ is $\Sigma\frac{\lambda_1}{K_1}\cdot\rho_1^2 = \Sigma\frac{\lambda_2}{K_2}\cdot\rho_1^2$ or

$\Sigma\left(\frac{\lambda_1}{K_1}\cdot\rho_1^2\right) - 1 = \Sigma\left(\frac{\lambda_2}{K_2}\cdot\rho_1^2\right) - 1$, and hence if a

coordinates of a point satisfy both $\Sigma \frac{\lambda_1}{k_1}.\rho_1^2 - 1 = 0$ or $\Sigma \lambda_1.\rho_1^2 = 0$ and
$\Sigma \frac{\lambda_2}{k_2}.\rho_1^2 - 1 = 0$ or $\Sigma \lambda_2.\rho_1^2 = k_2$, they satisfy the equation $\Sigma\left(\frac{\lambda_1}{k_1} - \frac{\lambda_2}{k_2}\right).\rho_1^2 = 0$ or $\Sigma\{(\lambda_1 k_2 - \lambda_2 k_1).\rho_1^2\} = 0$.

Example. Line joining O to intersection of AD (altitude) and BC, i.e. O to D.

AD is $\rho_2^2 - \rho_3^2 = c^2 - b^2$ and BC is $a\rho_1^2 - b\cos C.\rho_2^2 - c\cos B.\rho_3^2 = abc\cos A$.

OD is $-a(c^2-b^2).\rho_1^2 + [abc\cos A + b\cos C(c^2-b^2)].\rho_2^2 + [-abc\cos A + c\cos B(c^2-b^2)].\rho_3^2 = 0$

i.e. $a(b^2-c^2)\rho_1^2 + b(ac\cos A + \cos C(c^2-b^2)).\rho_2^2 - c\{ab\cos A + \cos B(b^2-c^2)\}\rho_3^2 = 0$

By substituting for cos A, cos B, cos C, this can be transformed into $(b^2-c^2).\rho_1^2 - b^2\cos 2C.\rho_2^2 + c^2\cos 2B.\rho_3^2 = 0$.

23. To find the equation to the line $\omega_1\omega_3\omega_2$ (vide Fig 1)

In §5, the ratios of the coordinates of ω_1 are found to be

$\frac{b^2\rho_2^4 + c^2\rho_3^4 - 2bc\cos A.\rho_2^2\rho_3^2}{(\rho_2^2-\rho_3^2)^2} : \frac{a^2\rho_2^4}{(\rho_2^2-\rho_3^2)^2} : \frac{a^2\rho_3^4}{(\rho_2^2-\rho_3^2)^2}$ or if $(\sigma_1\sigma_2\sigma_3)$ be the ratios of the coordinates of P and P_1, those of ω_1 are

$\frac{b^2\sigma_2^4 + c^2\sigma_3^4 - 2bc\cos A.\sigma_2^2\sigma_3^2}{(\sigma_2^2-\sigma_3^2)^2} : \frac{a^2\sigma_2^4}{(\sigma_2^2-\sigma_3^2)^2} : \frac{a^2\sigma_3^4}{(\sigma_2^2-\sigma_3^2)^2}$ and those of ω_3

$\frac{c^2\sigma_1^4}{(\sigma_1^2-\sigma_2^2)^2} : \frac{c^2\sigma_2^4}{(\sigma_1^2-\sigma_2^2)^2} : \frac{a^2\sigma_1^4 + b^2\sigma_2^4 - 2ab\cos C\sigma_1^2.\sigma_2^2}{(\sigma_1^2-\sigma_2^2)^2}$.

Now utilizing the equation to a line passing through two points, given in §18, we find the equation to $\omega_1\omega_3$ to be

$$\rho_1^2\left[\frac{a^2(\sigma_2^4-\sigma_3^4)}{(\sigma_2^2-\sigma_3^2)^2} - \frac{c^2\sigma_2^4-a^2\sigma_1^4-b^2\sigma_2^4+2ab\cos C\,\sigma_1^2\sigma_2^2}{(\sigma_1^2-\sigma_2^2)^2}\right] +$$

$$\rho_2^2\left[\frac{a^2\sigma_3^4-b^2\sigma_2^4-c^2\sigma_3^4+2bc\cos A.\sigma_2^2\sigma_3^2}{(\sigma_2^2-\sigma_3^2)^2} - \frac{a^2\sigma_1^4+b^2\sigma_2^4-2ab\cos C\,\sigma_1^2\sigma_2^2-c^2\sigma_1^4}{(\sigma_1^2-\sigma_2^2)^2}\right]$$

$$+\rho_3^2\left[\frac{b^2\sigma_2^4+c^2\sigma_3^4-2bc\cos A.\sigma_2^2\sigma_3^2-a^2\sigma_2^4}{(\sigma_2^2-\sigma_3^2)^2} - \frac{c^2\sigma_1^4-c^2\sigma_2^4}{(\sigma_1^2-\sigma_2^2)^2}\right] =$$

$$\frac{c^2\sigma_1^4}{(\sigma_1^2-\sigma_2^2)^2}\left\{\frac{a^2(\sigma_2^4-\sigma_3^4)}{(\sigma_2^2-\sigma_3^2)^2}\right\} + \frac{c^2\sigma_2^4}{(\sigma_1^2-\sigma_2^2)^2}\left\{\frac{a^2\sigma_3^4-b^2\sigma_2^4-c^2\sigma_3^4+2bc\cos A.\sigma_2^2\sigma_3^2}{(\sigma_2^2-\sigma_3^2)^2}\right\}$$

$$+\left(\frac{a^2\sigma_1^4+b^2\sigma_2^4-2ab\cos C\,\sigma_1^2\sigma_2^2}{(\sigma_1^2-\sigma_2^2)^2}\right)\left(\frac{b^2\sigma_2^4+c^2\rho_3^4-2bc\cos A\,\sigma_2^2\sigma_3^2-c^2\sigma_2^4}{(\sigma_2^2-\sigma_3^2)^2}\right)$$

i.e. $\rho_1^2\left[\cancel{\frac{a^2(\sigma_2^2+\sigma_3^2)}{\sigma_2^2-\sigma_3^2}}+c\right.$

$$\rho_1^2\left[\frac{a^2(\sigma_2^2+\sigma_3^2)}{\sigma_2^2-\sigma_3^2}+\frac{c^2\sigma_2^2+a^2\sigma_1^2-b^2\sigma_2^2}{\sigma_1^2-\sigma_2^2}\right]+\rho_2^2\left[\frac{-a^2\sigma_3^2-b^2\sigma_2^2+c^2\sigma_3^2}{\sigma_2^2-\sigma_3^2}-\right.$$

$$\left.\frac{a^2\sigma_1^2-b^2\sigma_2^2-c^2\sigma_1^2}{\sigma_1^2-\sigma_2^2}\right]+\rho_3^2\left[\frac{b^2\sigma_2^2-c^2\sigma_3^2-a^2\sigma_2^2}{\sigma_2^2-\sigma_3^2}-\frac{c^2(\sigma_1^2+\sigma_2^2)}{\sigma_1^2-\sigma_2^2}\right]=$$

$$\frac{c^2\sigma_1^4}{(\sigma_1^2-\sigma_2^2)^2}\left(\frac{a^2(\sigma_2^2+\sigma_3^2)}{\sigma_2^2-\sigma_3^2}\right)+\left(\frac{c^2\sigma_2^4}{(\sigma_1^2-\sigma_2^2)}\right)\cdot\left(\frac{c^2\sigma_3^2-a^2\sigma_3^2-b^2\sigma_2^2}{\sigma_2^2-\sigma_3^2}\right)+$$

$$\left\{\frac{a^2\sigma_1^4+b^2\sigma_2^4-2ab\cos C\,\sigma_1^2\sigma_2^2}{(\sigma_1^2-\sigma_2^2)^2}\right\}\cdot\left\{\frac{b^2\sigma_2^2-c^2\sigma_3^2-a^2\sigma_2^2}{\sigma_2^2-\sigma_3^2}\right)$$

By multiplication and simplification it can be simplified to

$$\rho_1^2\left[2a^2\sigma_1^2-2ab\cos C\,\sigma_2^2-2ac\cos B.\sigma_3^2\right)+\rho_2^2\left(-2ab\cos C.\sigma_1^2-2bc\cos A.\sigma_3^2\right.$$

$$\left.+2b^2\sigma_2^2\right)+\rho_3^2\left(-2ac\cos B.\sigma_1^2-2bc\cos A.\sigma_2^2+2c^2\sigma_3^2\right)=$$

$$a^2c^2\sigma_1^2-c^4\sigma_3^2+a^2c^2\sigma_3^2+b^2c^2\sigma_2^2+a^2b^2\sigma_1^2-b^4\sigma_2^2+b^2c^2\sigma_3^2$$

$$-a^4\sigma_1^2 \quad = \Sigma\{a^2\sigma_1^2(b^2+c^2-a^2)\} = 2\Sigma(a^2bc\cos A.\sigma_1^2)$$

$\therefore \Sigma\{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2)\} = -abc.\Sigma(a\cos A.\sigma_1^2)$

or $\Sigma\{a\sigma_1^2(b\cos C.\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 + abc\cos A)\} = 0$

or $\Sigma(\sigma_1^2.x) = 0$ (x, y, z barycentric coords)

$\therefore \Sigma(\sigma_1^2 x) = 0$ is the barycentric equation to $\omega_1 \omega_3 \omega_2$, where

$\sigma_1 : \sigma_2 : \sigma_3$ = ratios of coordinates of P or P_1.

(This equation has been derived with great simplicity by Mr. Gallatly, vide § 33. It can also be derived by using the equation in § 17 $\Sigma\{a\rho_1^2(c\cos B.u - b\cos C.r)\} = \Sigma\{a\sigma_1^2(c\cos B.u - b\cos C.r)\}$ and the coordinates of ω found in § 8).

The Circle

24. It has already been shown in §12 that the equation $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=k$ in general (i.e. when $\lambda+\mu+\nu \neq 0$) represents a circle. and in §11d that the equation to the circumcircle is $\Sigma(\sin 2A.\rho_1^2) = 4\Delta$ and the equation to any circle with centre O and radius d is $\Sigma(\sin 2A.\rho_1^2) = 2\Delta\left(\frac{d^2+R^2}{R^2}\right)$

The equation to the circumcircle can simply be derived from its trilinear equation $\Sigma(\sin A.\beta.\gamma)=0$ by substitution for α, β, γ ($\alpha = \frac{b\cos C.\rho_1^2+c\cos B.\rho_2^2-a\rho_1^2+abc\cos A}{4\Delta}$ etc).
or conversely it may be shown to be the required equation thus.

$$\Sigma(\sin 2A.\rho_1^2)=4\Delta$$

$$\therefore\ \Sigma\left\{\sin 2A.\left(\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}\right)\right\}=4\Delta$$

$$\text{i.e. } \Sigma\{\alpha^2(\cot B+\cot C)\}+2\Sigma\left\{\frac{\cos^2 A}{\sin A}.\beta\gamma\right\}=2\Delta$$

$$\text{i.e. } \Sigma\{\alpha^2\sin^2 A\}+2\Sigma\{\cos^2 A.\sin B\sin C.\beta\gamma\}=2\Delta\sin A\sin B\sin C$$

$$\text{i.e. } \Sigma\{\alpha^2\sin^2 A\}+2\Sigma\{(\sin B\sin C-\sin^2 A\sin B\sin C).\beta\gamma\}=2\Delta\sin A\sin B\sin C$$

$$\text{i.e. } \Sigma a^2\alpha^2+2\Sigma\{bc.\beta\gamma\}-2\Sigma\{bc\sin^2 A.\beta\gamma\}=8R^2\Delta\sin A\sin B\sin C$$
$$=4\Delta^2$$

$$\text{or } \Sigma(a\alpha+b\beta+c\gamma)^2-2\Sigma(bc\sin^2 A.\beta\gamma)=4\Delta^2$$

$$\text{i.e. } 2\Sigma(bc\sin^2 A.\beta\gamma)=0$$

$$\text{or } \Sigma(a.\beta\gamma)=0.$$

25 Equation to the circle, whose centre Q is $(\sigma_1 \sigma_2 \sigma_3)$ and radius is q. Let P be any point (ρ_1, ρ_2, ρ_3) on the circle

In § 11, it was shown that

$$q^2 \text{ or } (PQ^2) = R^2 \Sigma\left(\frac{\sin 2A . \rho_1^2}{2\Delta}\right) + 2\Sigma\left\{\frac{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B \sigma_3^2 - a\sigma_1^2)}{16\Delta^2}\right\}$$

$$+ R^2\Sigma\left(\frac{\sin 2A . \sigma_1^2}{2\Delta}\right) - 2R^2$$

$$= \Sigma\left\{\frac{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)}{8\Delta^2}\right\} +$$

$$d^2 - R^2 \qquad (\text{where } d = OQ)$$

$\therefore$ the required equation is

$$\Sigma\{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)\} = 8\Delta^2(q^2 - d^2 + R^2)$$

Let l, m, n be the barycentric coordinates of Q, so that

$$\sigma_1 = \frac{a(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)}{4\Delta},$$ then this equation becomes

$$l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = (l+m+n)(q^2 - d^2 + R^2)$$

for $l + m + n = \dfrac{abc(a\cos A + b\cos B + c\cos C)}{4\Delta} = \dfrac{4R\Delta . 4R\sin A\sin B\sin C}{4\Delta}$

$$= \frac{8\Delta^2}{4\Delta} = 2\Delta.$$

If this circle cuts the circumcircle orthogonally $d^2 = q^2 + R^2$

$$\therefore l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = 0 \text{ or } \Sigma\{a\rho_1^2(b\cos C.\sigma_2^2 + c\cos B.\sigma_3^2 - a\sigma_1^2 + abc\cos A)\} = 0$$

Examples (1) Circumcircle.

$$\sigma_1 = \sigma_2 = \sigma_3 = R, \quad q = R, \quad d = 0$$

i.e. $\Sigma(a^2bc\cos A.\rho_1^2) = 16\Delta^2 . R^2$ or $\Sigma(a\cos A.\rho_1^2) = 4R\Delta,$

or. $\Sigma(\sin 2A \cdot \rho_1^2) = 4\Delta$.

Incircle (2) $q = r$, $d^2 = R^2 - 2Rr$, $\sigma_1 = \frac{r}{\sin\frac{A}{2}}$, $\sigma_2 = \frac{r}{\sin\frac{B}{2}}$, $\sigma_3 = \frac{r}{\sin\frac{C}{2}}$.

Equation is $\Sigma\left\{a\rho_1^2\left(\frac{b\cos C \cdot r^2}{\sin^2\frac{B}{2}} + \frac{c\cos B \cdot r^2}{\sin^2\frac{C}{2}} - \frac{ar^2}{\sin^2\frac{A}{2}} + abc\cos A\right)\right\} =$

$$8\Delta^2(r^2 - R^2 + 2Rr + R^2)$$
$$= 8\Delta^2 r(r+2R)$$

Making use of $r = R(\cos A + \cos B + \cos C - 1)$ and $abc = 4R\Delta$.

this becomes $\Sigma(a\rho_1^2) = 2\Delta(r+2R)$

Nine-point Circle (3)

First to find coordinates of its centre (N)

$$OA^2 + AH^2 = 2NA^2 + \frac{OH^2}{2}$$

$$\therefore 2R^2 + 8R^2\cos^2 A = 4\sigma_1^2 + R^2(1 - 8\cos A\cos B\cos C)$$

$$\therefore 4\sigma_1^2 = R^2 + 8R^2(\cos^2 A + \cos A\cos B\cos C)$$

$$= R^2 + 8R^2\cos A\sin B\sin C$$

$\therefore \sigma_1^2 = \frac{R^2}{4} + 2R^2\cos A\sin B\sin C$. with similar expressions for σ_2^2 and σ_3^2

$$\ell = \left\{abc\cos C\left(\frac{R^2}{4} + 2R^2\cos B\sin A\sin C\right) + ac\cos B\left(\frac{R^2}{4} + 2R^2\sin A\sin B\cos C\right) - a^2\left(\frac{R^2}{4} + 2R^2\cos A\sin B\sin C\right) + a^2bc\cos A\right\} \div 4\Delta$$

$$= \left[2aR^2(a\sin B\sin C\cos B\cos C + a\sin C\sin B\cos B\cos C - a\cos A\sin B\sin C) + a^2bc\cos A\right] \div 4\Delta$$

$$= \left[2a^2R^2\sin B\sin C(2\cos B\cos C + \cos A)\right] \div 4\Delta$$

$$= \frac{2aR \cdot 2R^2\sin A\sin B\sin C\cos\overline{B-C}}{4\Delta} = \frac{aR\cos\overline{B-C}}{2}$$

$$= R^2\sin A\cos\overline{B-C} = \frac{R^2(\sin 2B + \sin 2C)}{2}$$

$\therefore$ nine point circle is

$$\Sigma\left\{\frac{R^2}{2}(\sin 2B+\sin 2C).\rho_1^2\right\} = R^2(\sin 2A+\sin 2B+\sin 2C)\left(\frac{R^2}{4}-\frac{R^2}{4}+2R^2\cos A\cos B\cos C+R^2\right)$$

$$= 2\Delta R^2(2\cos A\cos B\cos C+1)$$

or $\Sigma\{(\sin 2B+\sin 2C).\rho_1^2 = 4\Delta(1+2\cos A\cos B\cos C)$.

(Of course in this and in many other cases the values of l, m, n can be derived more simply by other methods)

26. Proof of the equation to circle i.e. $l\rho_1^2+m\rho_2^2+n\rho_3^2=(l+m+n)(q^2-d^2+R^2)$

by Mr. R. F. Davis.

Take rectangular axes at Q, and let $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$ $(x\,y)$ be the Cartesian coordinates of A, B, C, P.

Then $l\rho_1^2 = l(a_1-x)^2+l(a_2-y)^2$

$$= l.AQ^2+l.PQ^2-2x.la_1-2y.la_2$$

But, since Q is the mean centre for masses l, m, n

$\therefore\ la_1+mb_1+nc_1=0$; $la_2+mb_2+nc_2=0$;

$\therefore\ l\rho_1^2+m\rho_2^2+n\rho_3^2 = l.AQ^2+m.BQ^2+n.CQ^2+(l+m+n).PQ^2$

This being true for any point P, is true for O

$\therefore\ l.R^2+m.R^2+n.R^2 = \Sigma l.AQ^2+(l+m+n).OQ^2$

$\therefore\ l\rho_1^2+m\rho_2^2+n\rho_3^2 = (l+m+n)(QP^2-QO^2+R^2)$

$$= (l+m+n)(q^2-d^2+R^2)$$

27. The circle (DD_1), centre ω, cuts the circumcircle orthogonally
$\therefore$ its equation is of the form $l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = 0$ (vide §1)
The barycentric coordinates of ω_1 are such that

$$l : m : n = 0 : \sigma_3^2(\sigma_3^2 - \sigma_2^2) : \sigma_2^2(\sigma_2^2 - \sigma_3^2), \text{ where}$$

$(\sigma_1 \sigma_2 \sigma_3)$ are the tripolar coordinates of P and P_1
$\therefore$ this circle has for its equation

$$\sigma_3^2 \cdot \rho_2^2 - \sigma_2^2 \cdot \rho_3^2 = 0$$ with similar equations for circles (EE_1) and (FF_1).

28. Given the equation to a circle, to find the tripolar coordinates of its centre.

Suppose the equation to the circle to be $l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = (l+m+n)k^2$

Then the ratios of the barycentric coordinates of the centre are ~~as~~ $l : m : n$.

In §9 it was shown that if the tripolar coordinates of a point are ρ_1, ρ_2, ρ_3 and barycentric are x, y, z

$$\rho_1^2 = \frac{c^2y^2 + b^2z^2 + 2bcxy\cos A}{4\Delta^2} \text{ etc.}$$

Hence if the tripolar coordinates of the centre be $(\sigma_1, \sigma_2, \sigma_3)$

$$\sigma_1 : \sigma_2 : \sigma_3 = (m^2c^2 + n^2b^2 + 2mnbc\cos A)^{\frac{1}{2}} : (na^2 + l^2c^2 + 2nlca\cos B)^{\frac{1}{2}}$$
$$: (m^2a^2 + l^2b^2 + 2lmab\cos C)^{\frac{1}{2}}.$$

$$\text{or } \sigma_1^2 = \frac{m^2c^2 + n^2b^2 + 2mnbc\cos A}{(l+m+n)^2} \text{ etc.}$$

29. Given the equation to a circle, to find its radius.

Let the equation be $l\rho_1^2+m\rho_2^2+n\rho_3^2=(l+m+n)k^2$;

q = radius, d = distance of centre $(\sigma_1\sigma_2\sigma_3)$ from O.

$$d^2 = R^2\,\Sigma\frac{(\sin 2A.\sigma_1^2)}{2\Delta} - R^2$$

$$= \frac{R}{2\Delta}\left\{\frac{\Sigma\{a\cos A.(m^2c^2+n^2b^2+2mnbc\cos A)\}}{(l+m+n)^2}\right\} - R^2$$

$$= \frac{R}{2\Delta}\left\{\frac{\Sigma(l^2abc)+2\Sigma(abcmn\cos^2 A)}{(l+m+n)^2}\right\} - R^2$$

$$= 2R^2\left\{\frac{(l+m+n)^2-2\Sigma(mn\sin^2 A)}{(l+m+n)^2}\right\} - R^2.$$

$$= R^2 - \frac{\Sigma(a^2mn)}{(l+m+n)^2} = R^2\frac{(\Sigma l^2+2\Sigma lm\cos 2C)}{(l+m+n)^2}$$

now $q^2 = K^2+d^2-R^2$

$$= K^2 - \frac{\Sigma(a^2mn)}{(l+m+n)^2}$$

30 Equation to the tangent to a circle at a point on the circle.

Let the circle be $l\rho_1^2+m\rho_2^2+n\rho_3^2=(l+m+n)k^2$, where (l,m,n) are the barycentric coordinates of the centre

Let the tripolar coordinates of the centre be $(\sigma_1\sigma_2\sigma_3)$, the tripolar coordinates of the point be $(T_1T_2T_3)$ and the barycentric coordinates of the point be $(l_1 m_1 n_1)$

Then $lT_1^2+mT_2^2+nT_3^2=(l+m+n)k^2$.

The equation to the line joining the centre to $(T_1T_2T_3)$ is

$$\Sigma\{\rho_1^2(\overline{T_2^2-T_3^2}-\overline{\sigma_2^2-\sigma_3^2})\} = \Sigma\{\sigma_1^2(T_2^2-T_3^2)\} \qquad \text{(vide § 18)}$$

The line through $(T_1 T_2 T_3)$ perpendicular to this has for its equation

$$\Sigma\left[a\rho_1^2\left\{c\cos B.\left(\overline{T_3^2-T_1^2}-\overline{\sigma_3^2-\sigma_1^2}\right)-b\cos C\left(\overline{T_1^2-T_2^2}-\overline{\sigma_1^2-\sigma_2^2}\right)\right\}\right]=$$

$$\Sigma\left[aT_1^2\left\{c\cos B\left(\overline{T_3^2-T_1^2}-\overline{\sigma_3^2-\sigma_1^2}\right)-b\cos C\left(\overline{T_1^2-T_2^2}-\overline{\sigma_1^2-\sigma_2^2}\right)\right\}\right] \text{ (vide § 17)}$$

or $$\Sigma\left[\rho_1^2 a\left\{\left(c\cos B.T_3^2+b\cos C.T_2^2-aT_1^2\right)-\left(c\cos B.\sigma_3^2+b\cos C\sigma_2^2-a\sigma_1^2\right)\right\}\right]=$$

$$\Sigma\left[T_1^2 a\left\{\left(\cos B.\,T_3^2+b\cos C.\,T_2^2-aT_1^2\right)-\left(c\cos B.\sigma_3^2+b\cos C.\sigma_2^2-a\sigma_1^2\right)\right\}\right]$$

or $$\Sigma(l_1-l).\rho_1^2 = \Sigma l_1 T_1^2-\Sigma l T_1^2$$

$$= \Sigma l_1 T_1^2-(l+m+n)K^2.$$

(Note that $l_1-l+m_1-m+n_1-n = (l_1+m_1+n_1)-(l+m+n) = 2\Delta-2\Delta=0$)

If (l, m, n) are the ratios of the barycentric coordinates of the centre and $(l_1\ m_1\ n_1)$ similarly of the point, the equation becomes.

$$\Sigma\left(\frac{l_1}{l_1+m_1+n_1}-\frac{l}{l+m+n}\right).\rho_1^2=\Sigma\left(\frac{l_1}{l_1+m_1+n_1}.T_1^2\right)-K^2. \quad \text{or}$$

$$\Sigma\left\{l_1(l+m+n)-l(l_1+m_1+n_1)\right\}.\rho_1^2=\Sigma l_1(l+m+n).T_1^2-(l+m+n)(l_1+m_1+n_1)K^2.$$

or $$\Sigma\left\{(l_1m-lm_1)-(n_1l-nl_1)\right\}.\rho_1^2=(l+m+n)\left\{\Sigma l_1.T_1^2-(l_1+m_1+n_1)K^2\right\}.$$

Example. Equation to tangent to circumcircle at A.

Circumcircle is $\Sigma\sin 2A.\rho_1^2=4\Delta=\frac{R^2}{2}(\sin 2A+\sin 2B+\sin 2C)$

Barycentric coordinates of A $(1,0,0)$, tripolar $(0, c, b)$

$\therefore$ Equation is

$$(\sin 2B+\sin 2C).\rho_1^2-\sin 2B.\rho_2^2-\sin 2C.\rho_3^2=$$

$$(\sin 2A+\sin 2B+\sin 2C)\left(0-1.\frac{R^2}{2}\right)=-4\Delta \quad \text{or}$$

$$\sin 2B.\rho_2^2+\sin 2C.\rho_3^2-(\sin 2B+\sin 2C).\rho_1^2=4\Delta.$$

31. The length of the tangent from a point to a circle.

Let circle be $l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = (l+m+n)k^2$, and let the point be $(\tau_1 \tau_2 \tau_3)$. Let the centre of circle be $(\sigma_1 \sigma_2 \sigma_3)$, radius q, and distance between $(\sigma_1 \sigma_2 \sigma_3)$ and $(\tau_1 \tau_2 \tau_3)$ be p.

$$p^2 = R^2\left[\frac{\sum \sin 2A(\sigma_1^2+\tau_1^2)}{2\Delta} - 2\right] + \frac{\sum\{\tau_1^2(ab\cos C\,\sigma_2^2 + ac\cos B.\sigma_3^2 - a\sigma_1^2)\}}{8\Delta^2}$$

(vide § 11)

$$= \frac{R^2}{2\Delta}.\sum(\sin 2A.\sigma_1^2) + \cancel{\frac{R^2}{2\Delta}.\sum(\sin 2A.\tau_1^2)} - 2R^2$$

$$+ \frac{\sum\{\tau_1^2(ab\cos C\sigma_2^2 + ac\cos B.\sigma_3^2 - a\sigma_1^2 + a^2bc\cos A)}{8\Delta^2} - \cancel{\frac{abc.\sum(q_1^2 a\cos A)}{8\Delta^2}}$$

$$= \frac{R^2}{2\Delta}.\sum(\sin 2A.\sigma_1^2) - 2R^2 + \frac{\sum l\tau_1^2}{2\Delta}$$

$$= \frac{\sum l\tau_1^2}{2\Delta} - \frac{\sum(a^2mn)}{(l+m+n)^2} \qquad \text{(vide § 29)}$$

$$(\text{Tangent})^2 = p^2 - q^2 = \left(\frac{\sum l\tau_1^2}{2\Delta} - \frac{\sum(a^2mn)}{(l+m+n)^2}\right) - \left(k^2 - \frac{\sum(a^2mn)}{(l+m+n)^2}\right)$$

$$= \frac{\sum l\tau_1^2}{2\Delta} - k^2.$$

$$= \frac{\sum l\tau_1^2}{l+m+n} - k^2.$$

The square of the tangent from O to this circle $= R^2 - k^2$

for $k^2 = q^2 - d^2 + R^2 \quad \therefore d^2 - q^2 = R^2 - k^2.$

32. Radical Axis of two Circles.

Let the circles be $l_1\rho_1^2+m_1\rho_2^2+n_1\rho_3^2=(l_1+m_1+n_1)k_1^2$ and

$$l_2\rho_1^2+m_2\rho_2^2+n_2\rho_3^2=(l_2+m_2+n_2)k_2^2.$$

From previous paragraph, it at once follows that the required radical axis $\dfrac{\Sigma l_1\rho_1^2}{l_1+m_1+n_1}-k_1^2=\dfrac{\Sigma l_2\rho_2^2}{l_2+m_2+n_2}-k_2^2$

or $\Sigma\left(\dfrac{l_1}{l_1+m_1+n_1}-\dfrac{l_2}{l_2+m_2+n_2}\right).\rho_1^2=k_1^2-k_2^2$

or if l_1 etc be the actual barycentric coordinates

$$\frac{\Sigma(l_1-l_2).\rho_1^2}{2\Delta}=k_1^2-k_2^2 \quad \text{or } \Sigma(l_1-l_2).\rho_1^2=2\Delta(k_1^2-k_2^2)$$

Examples (1) Radical axis of the three circles (DD_1), (EE_1), (FF_1) in Fig. 1. viz OPP_1

Circle (DD_1) is $\sigma_2^2.\rho_3^2-\sigma_3^2.\rho_2^2=0$

" (EE_1) is $\sigma_3^2\rho_1^2-\sigma_1^2.\rho_3^2=0$

Radical axis $\dfrac{-\sigma_3^2}{\sigma_3^2-\sigma_1^2}.\rho_1^2-\dfrac{\sigma_3^2}{\sigma_2^2-\sigma_3^2}.\rho_2^2+\left(\dfrac{\sigma_2^2}{\sigma_2^2-\sigma_3^2}+\dfrac{\sigma_1^2}{\sigma_3^2-\sigma_1^2}\right).\rho_3^2=0$

$$\text{or } (\sigma_2^2-\sigma_3^2).\rho_1^2+(\sigma_3^2-\sigma_1^2).\rho_2^2+(\sigma_1^2-\sigma_2^2)\rho_3^2=0 \text{ (vide § 18)}$$

(2) Circumcircle and nine point circle.

Radical Axis is $\Sigma R^2\dfrac{\left(\sin 2A-\dfrac{\sin 2B+\sin 2C}{2}\right)\rho_1^2}{2\Delta}=R^2(2-1-2\cos A\cos B\cos C)$

$$\text{or } \Sigma\left\{\left(2\sin 2A-\sin 2B-\sin 2C\right).\rho_1^2\right\}=4\Delta(1-2\cos A\cos B\cos C)$$

(3) Circumcircle and Incircle.

Radical axis is $\Sigma\dfrac{(R^2\sin 2A-ar).\rho_1^2}{2\Delta}=2R^2-r^2-2Rr.$

or $\Sigma\{a(R\cos A - r).\rho_1^2\} = 2\Delta(2R^2 - r^2 - 2Rr)$

33. The Tripolar Coordinates of Limiting Points

At the end of § 23. it was shown that the barycentric equation to $\omega_1\omega_2\omega_3$ (Fig. 1) is $\Sigma(\sigma_1^2 . x) = 0$, where $(\sigma_1\sigma_2\sigma_3)$ = ratio of coordinates of P and P_1, the limiting points of the coaxal system, of which $\omega_1\omega_2\omega_3$ is the radical axis.

Of this Mr. Gallatly gives the following proof
"Since $OP.OP_1 = R^2$, the circle ABC belongs to the coaxal system which has P and P_1 for Limiting Points, and therefore $\omega_1\omega_2\omega_3$ for Radical axis; so that, if π_1, π_2, π_3 are the perpendiculars from A, B, C on $\omega_1\omega_2\omega_3$, we have by coaxal theory

$$2\,OP.\pi_1 = AP^2 \text{ or } \pi_1 \propto \sigma_1^2$$

Hence the equation to the radical axis $\omega_1\omega_2\omega_3$ is

$$\sigma_1^2 . x + \sigma_2^2 . y + \sigma_3^2 . z = 0$$

And conversely, if the radical axis be

$$\lambda x + \mu y + \nu z = 0$$

then the tripolar coordinates of P and P_1 are

$$\sqrt{\lambda},\ \sqrt{\mu},\ \sqrt{\nu}."$$

Suppose $\sigma_1\sigma_2\sigma_3$ are the tripolar coordinates of the limiting points, then $\Sigma\sigma_1^2 . x = 0$ is the radical axis.

i.e. $\Sigma\left\{\sigma_1^2 \dfrac{a(b\cos C.\rho_2^2 + c\cos B.\rho_3^2 - a\rho_1^2 + abc\cos A)}{4\Delta}\right\} = 0$

i.e. $\Sigma\left\{\rho_1^2\left(-a^2\sigma_1^2 + ab\cos C.\sigma_2^2 + ac\cos B.\sigma_3^2\right)\right\} = -abc\,\Sigma(a\cos A.\sigma_1^2)$

Let now this radical axis be $\dfrac{l\rho_1^2 + m\rho_2^2 + n\rho_3^2}{2\Delta} = k^2$

then $\dfrac{-a^2\sigma_1^2 + ab\cos C.\sigma_2^2 + ac\cos B.\sigma_3^2}{l} = \dfrac{-b\sigma_2^2 + ab\cos C.\sigma_1^2 + bc\cos A.\sigma_3^2}{m}$

$$= \frac{-c^2\sigma_3^2 + ac\cos B.\sigma_1^2 + bc\cos A.\sigma_2^2}{n} = \frac{-abc.\Sigma(a\cos A.\sigma_1^2)}{2\Delta k^2}$$

$$= \frac{-2R\Sigma(a\cos A.\sigma_1^2)}{k^2}$$

Solving these equations we have

$$\frac{\sigma_1^2}{2\Delta k^2 - nb^2 - mc^2} = \frac{\sigma_2^2}{2\Delta k^2 - lc^2 - na^2} = \frac{\sigma_3^2}{2\Delta k^2 - lb^2 - ma^2}$$

Examples. (1) Incircle

$$\sigma_1^2 \propto 2\Delta(2R^2 - r^2 - 2Rr) - c^2(bR\cos B - br) - b^2(cR\cos C - cr)$$

$$\propto bc\left[(2R^2 - r^2 - 2Rr)\frac{a}{2R} - aR + (b+c)r\right]$$

$$\propto bc\left[\cancel{aR} - \frac{ar^2}{2R} - ar - \cancel{aR} + (b+c)r\right]$$

$$\propto bc\left[-\frac{a(s-a)(s-b)(s-c)}{2Rs} + 2(s-a)r\right]$$

$$\propto \frac{bc(s-a)}{2Rs}\left[-a(s-b)(s-c) + 4Rrs\right]$$

$$\propto \frac{abc(s-a)}{2Rs}\left[-(s-b)(s-c) + bc\right]$$

$$\propto \frac{abc(s-a)^2}{2R}.$$

$\therefore\ \sigma_1 : \sigma_2 : \sigma_3 = s-a : s-b : s-c.$

(2) Nine point Circle

$$\sigma_1^2 \propto 2\Delta(1-2\cos A\cos B\cos C) - b^2\left(\sin 2C - \frac{\sin 2A+\sin 2B}{2}\right)$$
$$- c^2\left(\sin 2B - \frac{\sin 2A+\sin 2C}{2}\right)$$
$$\propto -4\Delta\cos A\cos B\cos C + \tfrac{1}{2}b^2(\sin 2A+\sin 2B-\sin 2C)$$
$$+\tfrac{1}{2}c^2(\sin 2A+\sin 2C-\sin 2B)$$
$$\propto -4\Delta\cos A\cos B\cos C + 2b^2\sin C\cos A\cos B + 2c^2\sin B\cos A\cos C$$
$$\propto -4\Delta\cos A\cos B\cos C + 2bc\sin B\cos A\cos B + 2bc\sin C\cos A\cos C$$
$$\propto 2\cos A(-2\Delta\cos B\cos C + bc\sin B\cos B + bc\sin C\cos C)$$
$$\propto 2bc\cos A(-\sin A\cos B\cos C + \sin A\cos\overline{B-C})$$
$$\propto 2bc\sin A\cos A(\sin B\sin C)$$
$$\propto 4\Delta\cos A\sin B\sin C.$$

$\therefore\ \sigma_1^2 : \sigma_2^2 : \sigma_3^2 = \cot A : \cot B : \cot C$

or $\sigma_1 : \sigma_2 : \sigma_3 = \sqrt{\cot A} : \sqrt{\cot B} : \sqrt{\cot C}.$

4. Given the equation to the locus of a point, to find the equation to the locus of its corresponding point (Inversion Theory)

Let P_1 be the corresponding point to P, and let the coordinates of P be $(\rho_1\rho_2\rho_3)$ and of P_1 $(r_1r_2r_3)$

Then $\rho_1^2 = \dfrac{r_1^2.R^2}{OP_1^2} = \dfrac{r_1^2.R^2}{\left(\dfrac{R^2\Sigma\sin 2A.r_1^2}{2\Delta} - R^2\right)} = \dfrac{2\Delta.r_1^2}{\Sigma\sin 2A.r_1^2 - 2\Delta}$

(a) Suppose the equation to the locus of P be

$l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = 0$, then the equation to the locus of P_1 will be $\frac{R}{OP_1^2}\left(lr_1^2+mr_2^2+nr_3^2\right)=0$ or

$lr_1^2+mr_2^2+nr_3^2=0$

i.e. if locus of P be a circumdiameter or a circle cutting circumcircle orthogonally, the locus of P_1 is the same circumdiameter or the same circle orthogonal to the circumcircle.

(b) Suppose equation to the locus of P be $\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2=K$ where $\lambda+\mu+\nu=0$ i.e locus of P is a straight line;

then the equation to the locus of P_1 will be

$$\Sigma(K.\sin 2A-2\Delta.\lambda).r_1^2=2\Delta K.$$

Here the $\Sigma(K.\sin 2A-2\Delta\lambda) \neq 0$ $\therefore$ locus of P_1 is a circle. The point (R,R,R) satisfies the equation $\therefore$ locus of P_1 is a circle passing through O.

The equation can be put in the form

$$\Sigma(K.\sin 2A-2\Delta.\lambda)r_1^2=\left\{\Sigma(K\sin 2A-2\Delta\lambda)\right\}.R^2$$
$$=R^2\Sigma K\sin 2A.$$

The centre of this circle is

$$\sigma_1^2=\Big[(K.\sin 2B-2\Delta\mu)^2c^2+(K\sin 2C-2\Delta.\nu)b^2+$$
$$2(K.\sin 2B-2\Delta\mu).(K\sin 2C-2\Delta\nu)bc\cos A)\Big]\div\left(K.\Sigma\sin 2A\right)^2$$

(vide §28)

$$=\frac{R^2}{4K^2}\left[(2K-\mu c^2-\nu b)^2+4R^2(\mu c\cos C-\nu.b\cos B)^2\right]$$

with similar expressions for σ_2^2 and σ_3^2.

and its radius $q=\frac{R^2}{2d}$ (where d is the perpendicular distance of O from the straight line)

$$=R^2\div\frac{K}{R}\sqrt{\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C}$$ (vide § 13)

$$=\frac{R^3}{K.\left(\Sigma\lambda^2+2\Sigma\lambda\mu\cos 2C\right)^{\frac{1}{2}}}.$$

Example. Take the line BC, $a\rho_1^2 - b\cos C.\rho_2^2 - c\cos B.\rho_3^2 = abc\cos A$.

Equation to the locus of P_1 is

$(abc\cos A.\sin 2A - 2\Delta.a)r_1^2 + (abc\cos A.\sin 2B + 2\Delta b\cos C)r_2^2 +$

$(abc\cos A.\sin 2C + 2\Delta c\cos B).r_3^2 = 2\Delta.abc\cos A$ or

$a\cos 2A.r_1^2 + b.\cos\overline{A-B}.r_2^2 + c.\cos\overline{C-A}.r_3^2 = abc\cos A$.

and $\sigma_1^2 = \dfrac{R^2}{4a^2b^2c^2\cos^2 A}\left[(2abc\cos A + b\cos C.c^2 + c\cos B.b^2)^2 + 4R^2(-bc\cos^2 C + bc\cos^2 B)^2\right]$

$= \dfrac{R^2}{4\cos^2 A}\left[(\cos A + 2\sin B\sin C)^2 + \sin^2\overline{B-C}\right]$

$= \dfrac{R^2}{4\cos^2 A}.(1 + 8\cos A\sin B\sin C)$

$\sigma_2^2 = \dfrac{R^2}{4a^2b^2c^2\cos^2 A}\left[(2abc\cos A + c\cos B a^2 - ac^2)^2 + 4R^2\left(\begin{matrix}-ac\cos A\cos B\\ -ac\cos C\end{matrix}\right)^2\right]$

$= \dfrac{R^2}{4b^2\cos^2 A}\left[(2b\cos A + a\cos B - c)^2 + 4R^2(\sin A\sin B)^2\right]$

$= \dfrac{R^2}{4\cos^2 A} = \sigma_3^2$

(These can easily be verified by a figure)

$q = \dfrac{R^2}{2d} = \dfrac{R^2}{2R\cos A} = \dfrac{R}{2\cos A}$.

(c) Let the equation to the locus of P be

$$l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = (l+m+n).R^2 \quad \text{where } l+m+n \neq 0$$

The equation to the locus of P_1 is

$\Sigma\{(l+m+n).R^2\sin 2A - 2\Delta l\}.r_1^2 = 2\Delta(l+m+n).R^2$

Here $\Sigma\{(l+m+n).R^2\sin 2A - 2\Delta l\} = 0$

$\therefore$ locus of P_1 is a straight line and its $\perp^r$ distance

(d) from O is $\dfrac{R^2}{2q} = \dfrac{R^2}{2\sqrt{R^2 - \frac{\Sigma a^2 mn}{(l+m+n)^2}}} =$ (vide §29)

$$d = \frac{(l+m+n)R}{2\left(\Sigma l^2 + 2\Sigma lm\cos 2C\right)^{\frac{1}{2}}}.$$

Example. Take circle centre I, radius IO viz

$$a\rho_1^2 + b\rho_2^2 + c\rho_3^2 = (a+b+c).R^2$$

Equation to locus of P_1 is

$$\Sigma\left\{(a+b+c)R^2.\sin 2A - 2\Delta.a\right\}.r_1^2 = 2\Delta(a+b+c).R^2.$$

or $\Sigma(R^2.\sin 2A - ar).r_1^2 = 2\Delta.R^2$

or $\Sigma a(R\cos A - r).r_1^2 = 2\Delta.R^2.$

and the distance of this line from O is

$$d = \frac{R^2}{2q} = \frac{R^2}{2\sqrt{R^2-2Rr}} = \frac{R}{2}\sqrt{\frac{R}{R-2r}}.$$

(d) Suppose the equation to the locus of P is

$$l\rho_1^2 + m\rho_2^2 + n\rho_3^2 = (l+m+n).K^2, \text{ where } l+m+n \neq 0.$$

then the equation to the locus of P_1 is

$$\Sigma\left\{(l+m+n).K^2\sin 2A - 2\Delta.l\right).r_1^2 = 2\Delta(l+m+n).K^2$$

Let coefficients of $r_1^2\ r_2^2\ r_3^2$ be respectively L, M, N.

then $L+M+N = (l+m+n).K^2(\Sigma\sin 2A) - (l+m+n).2\Delta$

$$= 4(l+m+n)\sin A\sin B\sin C.(K^2-R^2)$$

and the equation to locus of P_1 becomes.

$$L r_1^2 + M.r_2^2 + N.r_3^2 = (L+M+N)\frac{R^2K^2}{K^2-R^2}.$$

Its radius can be calculated by means of §29,

$$Q^2 = \frac{R^2K^2}{K^2-R^2} - \frac{\Sigma a^2MN}{(L+M+N)^2}$$ and substituting values of L, M, N in terms of l, m, n, K.

It may be found more simply as follows.

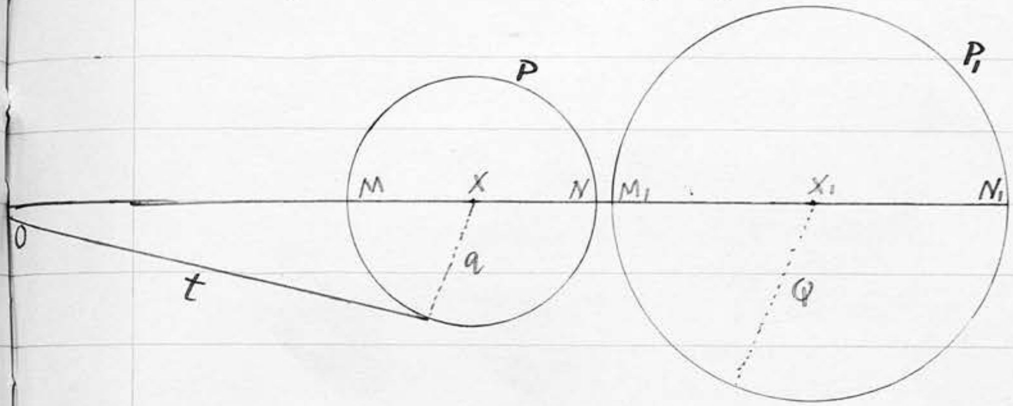


Fig. 7.

Let P and P_1 be the circles, radii q and Q; and let t be tangent from O to circle P, then $\dfrac{Q}{q} = \dfrac{R^2}{t^2}$. But $t^2 = R^2 - K^2$ (vide § 31)

$$\therefore Q = q \times \frac{R^2}{R^2 - K^2} = \frac{R^2}{R^2 - K^2} \cdot \sqrt{K^2 - \frac{\Sigma a^2 mn}{(l+m+n)^2}}.$$

$$\sigma_1^2 = \frac{M^2c^2 + N^2b^2 + 2MNbc\cos A}{(L+M+N)^2}.$$

$$= \frac{(Mc\sin C + Nb\sin B)^2 + (Mc\cos C - Nb\cos B)^2}{(L+M+N)^2}$$

$$= \Big\{\Big[\{(l+m+n)K^2\sin 2B - 2\Delta m\}c\sin C + \{(l+m+n).K^2\sin 2C - 2\Delta n\}.b\sin B\Big]^2$$
$$+ \Big[\{(l+m+n).K^2\sin 2B - 2\Delta m\}.c\cos C - \{(l+m+n).K^2\sin 2C - 2\Delta n\}.b\cos B\Big]^2\Big\}$$
$$\div \Big[16\,(l+m+n)^2\sin^2 A\sin^2 B\sin^2 C\,(K^2 - R^2)^2\Big]$$

$$= \frac{R^2\Big[\{(l+m+n)\cdot K^2 - mc^2 - nb^2\}^2 + 4R^2(mc\cos C - nb\cos B)^2\Big]}{4(l+m+n)^2(R^2-K^2)^2}$$

with similar expressions for σ_2^2 and σ_3^2.

Example. Incircle. Equation is

$$a\rho_1^2 + b\rho_2^2 + c\rho_3^2 = 2\Delta(r+2R) = (a+b+c)(r^2+2Rr).$$

The equation to its inverse is

$$\Sigma\{2s(r^2+2Rr).\sin 2A - 2\Delta.a\}.r_1^2 = 2s.2\Delta.(r^2+2Rr)$$

or $$\Sigma\{(r+2R)\sin 2A - a\}.r_1^2 = 2\Delta.(r+2R).$$

The radius of this circle $= r \times \dfrac{R^2}{R^2-(r^2+2Rr)} = \dfrac{R^2 r}{R^2-r^2-2Rr} = Q$ (say)

and $$\sigma_1^2 = \frac{R^2.}{16s^2(R^2-r^2-2Rr)^2}.\left[\{4s(r^2+2Rr)-bc^2-b^2c\}^2 + 4R^2(bc\cos C - bc\cos B)^2\right]$$

$$= \frac{4b^2c^2R^2}{16s^2(R^2-r^2-2Rr)^2}.\left[\{(r+2R)\sin A - R(\sin C+\sin B)\}^2 + \{R(\cos C-\cos B)\}^2\right]$$

$$= \frac{R^2r^2}{(R^2-r^2-2Rr)^2\sin^2 A}.\left[\{(r+2R)\sin A - R(\sin B+\sin C)\}^2 + \{R(\cos C-\cos B)\}^2\right]$$

$$= \frac{R^2r^2.4\cos^2\frac{A}{2}}{(R^2-r^2-2Rr)^2\sin^2 A}.\left[(r+2R)^2\sin^2\tfrac{A}{2} - 2R(r+2R)\sin\tfrac{A}{2}\cos\tfrac{B-C}{2} + R^2\right]$$

$$= \frac{R^2r^2}{(R^2-r^2-2Rr)^2\sin^2\frac{A}{2}}.\left[R^2 + (r+2R)\sin\tfrac{A}{2}\left(r\sin\tfrac{A}{2} - 4R\sin\tfrac{B}{2}\sin\tfrac{C}{2}\right)\right]$$

$$= \frac{R^2r^2}{(R^2-r^2-2Rr)^2\sin^2\frac{A}{2}}.\left[R^2 - r(r+2R)\cos^2\tfrac{A}{2}\right]$$

$$= \frac{R^2r^2}{(R^2-r^2-2Rr)^2\sin^2\frac{A}{2}}.\left[R^2\sin^2\tfrac{A}{2} + (R^2-r^2-2Rr)\cos^2\tfrac{A}{2}\right]$$

$$= \frac{R^2r^2}{R^2-r^2-2Rr}.\left[\frac{R^2}{R^2-r^2-2Rr} + \cot^2\tfrac{A}{2}\right] = Q^2 + Qr\cot^2\tfrac{A}{2}$$

with similar expressions for σ_2^2 and σ_3^2

Distance of O from centre of corresponding (inverse) circle, viz OX_1 (vide Fig 7)

$$OX_1 = \frac{OM_1+ON_1}{2} = \tfrac{1}{2}\left(\frac{R^2}{OM} + \frac{R^2}{ON}\right) = \tfrac{1}{2}\left(\frac{R^2}{d-q_1} + \frac{R^2}{d+q_1}\right)$$

$$= \frac{R^2 d}{d^2-q^2} = R^2.\left\{R^2 - \frac{\Sigma a^2mn}{(l+m+n)^2}\right\}^{\frac{1}{2}} \div (R^2-K^2)$$

$$= \frac{R^3}{R^2-K^2}.\frac{(\Sigma l^2 + 2\Sigma lm\cos 2C)^{\frac{1}{2}}}{(l+m+n)}$$

In the example given on incircle, this distance becomes equal to $\frac{R^2\sqrt{R^2-2Rr}}{R^2 2Rr-r^2}$. i.e. $\frac{R^2}{OI^2-r^2} \cdot OI$.

II. Concurrency of lines joining vertices of a triangle to opposite vertices of triangles on its sides.

Concurrency of lines joining vertices of a triangle to opposite vertices of triangles on its sides.

By A. G. BURGESS, M.A., F.R.S.E.

[Extracted from the *Proceedings of the Edinburgh Mathematical Society*, Vol. XXXII., Session 1913–1914.]

(1). Let ABC be the given triangle; A′BC, B′CA, C′BA triangles described externally on its sides, and let the angles of these triangles be $A'BC = \mu_1$, $A'CB = \nu_1$, $B'AC = \lambda_2$, $B'CA = \nu_2$, $C'AB = \lambda_3$, $C'BA = \mu_3$ (Fig. 1).

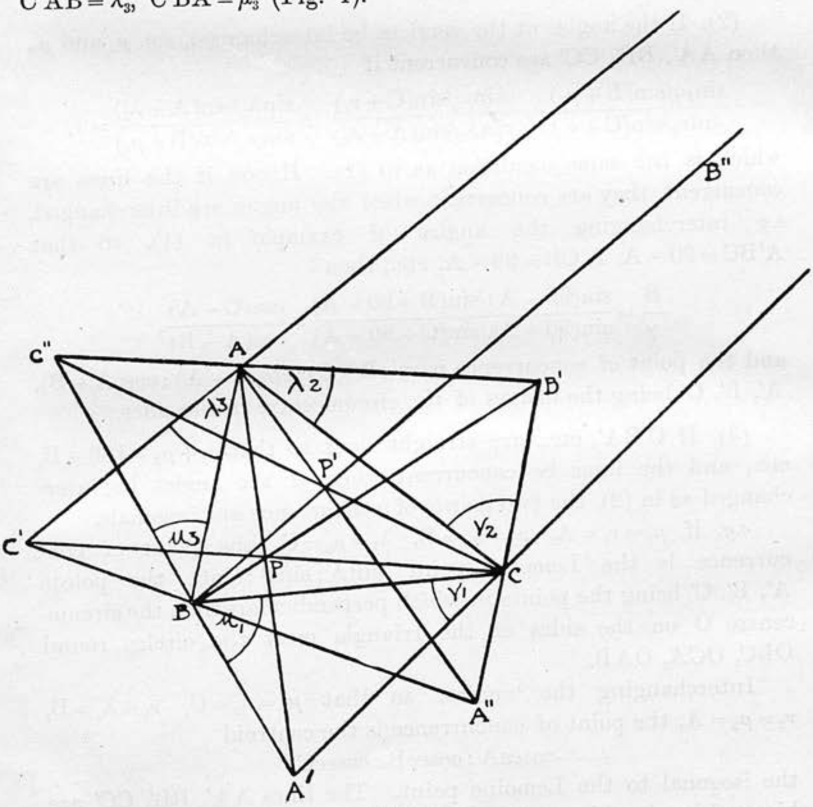


Fig. 1.

Using normal coordinates, then for AA′, $\dfrac{\beta}{\gamma}=\dfrac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}$,

∴ the lines AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}\cdot\frac{\sin\nu_2/\sin(C+\nu_2)}{\sin\lambda_2/\sin(A+\lambda_2)}\cdot\frac{\sin\lambda_3/\sin(A+\lambda_3)}{\sin\mu_3/\sin(B+\mu_3)}=1.$$

e.g. if $\mu_1=\lambda_2=90-C$, $\nu_1=\lambda_3=90-B$, $\nu_2=\mu_3=90-A$,

$$\frac{\beta}{\gamma}=\frac{\sin(90-C)/\sin(B+90-C)}{\sin(90-B)/\sin(C+90-B)}=\frac{\sec B}{\sec C},$$

and the lines are concurrent at the orthocentre $\sec A:\sec B:\sec C$. (The points A′, B′, C′ are the points in which the altitudes intersect the circumcircle of △ABC).

(2). If the angles at the vertices be interchanged, *e.g.* μ_1 and μ_3, then AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_3/\sin(B+\mu_3)}{\sin\nu_2/\sin(C+\nu_2)}\cdot\frac{\sin\nu_1/\sin(C+\nu_1)}{\sin\lambda_3/\sin(A+\lambda_3)}\cdot\frac{\sin\lambda_2/\sin(A+\lambda_2)}{\sin\mu_1/\sin(B+\mu_1)}=1,$$

which is the same condition as in (1). Hence if the lines are concurrent, they are concurrent when the angles are interchanged, *e.g* interchanging the angles of example in (1), so that A′BC = 90 − A, A′CB = 90 − A, etc., then

$$\frac{\beta}{\gamma}=\frac{\sin(90-A)/\sin(B+90-A)}{\sin(90-A)/\sin(C+90-A)}=\frac{\cos(C-A)}{\cos(A-B)},$$

and the point of concurrence is $\cos(B-C):\cos(C-A):\cos(A-B)$, A′, B′, C′ being the images of the circumcentre in the sides.

(3). If C′BA′, etc., are straight lines, so that $\mu_1+\mu_3=180-B$, etc., and the lines be concurrent, then if the angles be interchanged as in (2), the two points of concurrence are isogonals.

e.g. if $\mu_1=\nu_1=A$, $\nu_2=\lambda_2=B$, $\lambda_3=\mu_3=C$, the point of concurrence is the Lemoine point $\sin A:\sin B:\sin C$, the points A′, B′, C′ being the points in which perpendiculars from the circumcentre O on the sides of the triangle meet the circles round OBC, OCA, OAB.

Interchanging the angles, so that $\mu_1=\lambda_2=C$, $\nu_1=\lambda_3=B$, $\nu_2=\mu_3=A$, the point of concurrence is the centroid

$$\operatorname{cosec}A:\operatorname{cosec}B:\operatorname{cosec}C,$$

the isogonal to the Lemoine point. The lines AA′, BB′, CC′ are bisected by the sides of the triangle.

(4). If $\mu_1 = \nu_1 = A \pm \theta$, $\nu_2 = \lambda_2 = B \pm \theta$, $\mu_3 = \lambda_3 = C \pm \theta$,

then $$\frac{\beta}{\gamma} = \frac{\sin(A \pm \theta)/\sin(B + A \pm \theta)}{\sin(A \pm \theta)/\sin(C + A \pm \theta)} = \frac{\sin(B \mp \theta)}{\sin(C \mp \theta)},$$

and the lines are concurrent at the point

$$\sin(A \mp \theta) : \sin(B \mp \theta) : \sin(C \mp \theta).$$

This point lies on the line joining the Lemoine point $\sin A : \sin B : \sin C$ to the circumcentre $\cos A : \cos B : \cos C$, for $\Sigma \sin(A \mp \theta)\sin(B - C) = 0$.

e.g. If $\theta = 0$, the point is the Lemoine point, and if $\theta = 60°$, the points are $\sin(A \mp 60) : \sin(B \mp 60) : \sin(C \mp 60)$, the two Isodynamic points.

(5). If all six angles $= \theta$, the point of concurrence is $\operatorname{cosec}(A + \theta) : \operatorname{cosec}(B + \theta) : \operatorname{cosec}(C + \theta)$, the isogonal to the point derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

e.g. If $\theta = 60°$, the point is the Inner Isogonic point, the isogonal of one of the Isodynamic points.

(6). If $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, the lines AA′, BB′, CC′ are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A + \lambda)} : \frac{\sin\mu}{\sin(B + \mu)} : \frac{\sin\nu}{\sin(C + \nu)}.$$

e.g. If $\lambda = 90 - A$, $\mu = 90 - B$, $\nu = 90 - C$, the point of concurrence is the circumcentre $\cos A : \cos B : \cos C$, A′, B′, C′ lying on the circumcircle of $\triangle ABC$.

If $\lambda = 90 - C$, $\mu = 90 - A$, $\nu = 90 - B$, the point is

$$\frac{\cos C}{\cos(C - A)} : \frac{\cos A}{\cos(A - B)} : \frac{\cos B}{\cos(B - C)}.$$

(7). If P be a point within $\triangle ABC$, and if AP, BP, CP be produced to meet the circumcircles of $\triangle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB are $180 - \lambda$, $180 - \mu$, $180 - \nu$, for $A'BC = A'PC = C'PA = C'BA$, etc., and $BA'C = CPB = CAB = \lambda$, etc., and $BPC = 180 - \lambda$, etc.

e.g. If $\lambda = 90 - \dfrac{A}{2}$, $\mu = 90 - \dfrac{B}{2}$, $\nu = 90 - \dfrac{C}{2}$, the point P is the incentre 1 : 1 : 1, and the $\angle$s at P are $90 + \dfrac{A}{2}$, etc., A′, B′, C′ being the three excentres of $\triangle ABC$.

If $\lambda = A$, $\mu = B$, $\nu = C$, the point is the orthocentre.

(8). If the lines C′B, B′C be produced to meet in A″, A′C, C′A in B″, A′B, B′A in C″, and AA′ BB′, CC′ be concurrent, then AA″, BB″, CC″ are concurrent, for the condition for concurrence is

$$\frac{\sin(B+\mu_3)/\sin\mu_3}{\sin(C+\nu_2)/\sin\nu_2} \cdot \frac{\sin(C+\nu_1)/\sin\nu_1}{\sin(A+\lambda_3)/\sin\lambda_3} \cdot \frac{\sin(A+\lambda_2)/\sin\lambda_2}{\sin(B+\mu_2)/\sin\mu_1} = 1,$$

the same condition as in (1); the point of concurrence is the isogonal to the point got by interchanging the angles μ_1 and μ_3, etc., as in (2).

e.g. If as in (1) $\mu_1 = \lambda_2 = 90 - C$, $\nu_1 = \lambda_3 = 90 - B$, $\nu_2 = \mu_3 = 90 - A$, the point of concurrence is $\sec(B - C) : \sec(C - A) : \sec(A - B)$, the isogonal to the point $\cos(B - C) : \cos(C - A) : \cos(A - B)$ derived in (2) by interchanging the angles.

(9). If A′B′C′ be regarded as the original triangle with triangles C′BA′, A′CB′, B′AC′ described on the sides, then if the lines AA′, BB′, CC′ are concurrent, B′B″, C′C″, A′A″ are concurrent, the points A″, B″, C″ being the points derived as in (8).

(10). If $\mu_1 = \mu_3 = \mu$, etc., the point of concurrence of AA″, BB″, CC″ is

$$\frac{\sin(A+\lambda)}{\sin\lambda} : \frac{\sin(B+\mu)}{\sin\mu} : \frac{\sin(C+\nu)}{\sin\nu},$$

and the lines AA″, BB″, CC″ are the isogonals of AA′, BB′, CC′.

e.g. If $\lambda = C$, $\mu = A$, $\nu = B$, AA′, BB′, CC′ meet in

$\frac{\sin C}{\sin B} : \frac{\sin A}{\sin C} : \frac{\sin B}{\sin A}$ and AA″, BB″, CC″ in $\frac{\sin B}{\sin C} : \frac{\sin C}{\sin A} : \frac{\sin A}{\sin B}$, the two Brocard points.

If $\lambda = \mu = \nu = \theta$, AA′, BB′, CC′ meet in $\operatorname{cosec}(A+\theta) : \operatorname{cosec}(B+\theta) : \operatorname{cosec}(C+\theta)$ and AA″, BB″, CC″ in $\sin(A+\theta) : \sin(B+\theta) : \sin(C+\theta)$, the same point as was derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

(11). If $\lambda + \mu + \nu = 180°$, and AA″, BB″, CC″ meet in P′, BP′CA″, etc., are concyclic, for $BP'C = A + \lambda$, and $BA''C = 180 - A - \lambda$.

Again, $BAB' = A + \lambda$ and $BA''B' = BA''C = 180 - A - \lambda$, ∴ BAB′A″ and the other five sets of corresponding points are concyclic.

e.g. If $\lambda = \mu = \nu = 60°$, P′ is the isogonal of the Inner Isogonic point and $BP'C = A + 60$.

(12). If the triangles BA′C, CB′A, AC′B be described internally on the sides, AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B-\mu_1)}{\sin\nu_1/\sin(C-\nu_1)}\cdot\frac{\sin\nu_2/\sin(C-\nu_2)}{\sin\lambda_2/\sin(A-\lambda_2)}\cdot\frac{\sin\lambda_3/\sin(A-\lambda_3)}{\sin\mu_3/\sin(B-\mu_3)}=1.$$

As in the case of the external triangles, the lines are concurrent if μ_1 and μ_3, etc., be interchanged, and if $\mu_1=\mu_3=\mu$, $\nu_1=\nu_2=\nu$, $\lambda_2=\lambda_3=\lambda$, the lines are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A-\lambda)}:\frac{\sin\mu}{\sin(B-\mu)}:\frac{\sin\nu}{\sin(C-\nu)}.$$

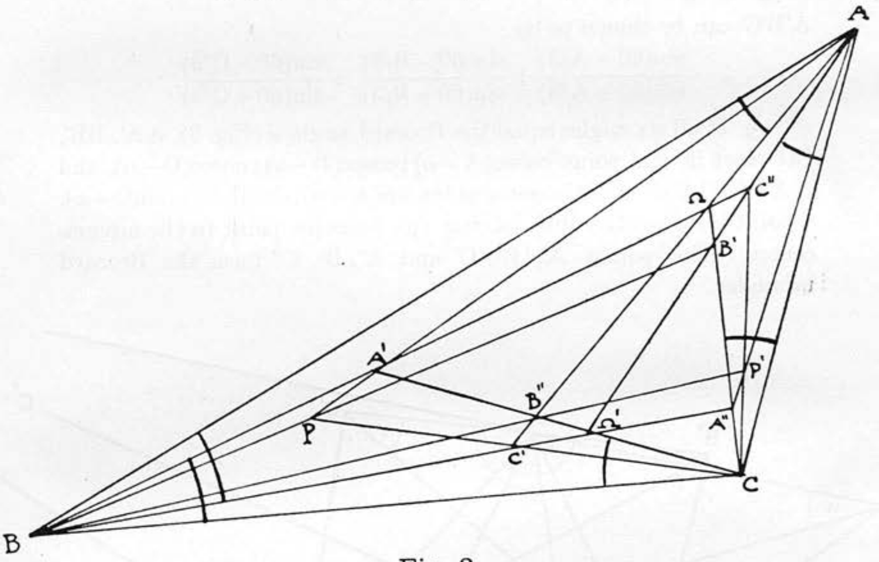


Fig. 2.

e.g. If $\mu_1=\mu_3=90-A$, $\nu_1=\nu_2=90-B$, $\lambda_2=\lambda_3=90-C$, the point of concurrence is $\frac{\cos C}{\cos B}:\frac{\cos A}{\cos C}:\frac{\cos B}{\cos A}$.

If $\lambda+\mu+\nu=180$, then as in the case of the external triangles, A′BPC, etc., are concyclic points, the angles between the lines AP, BP, CP being λ, μ, ν.

e.g. If $\lambda=\mu=\nu=60°$, the point is

$$\operatorname{cosec}(A-60):\operatorname{cosec}(B-60):\operatorname{cosec}(C-60),$$

the Outer Isogonic point.

(13). If CB′, BC′ be produced to meet in A″, etc., then if AA′, BB′, CC′ be concurrent, so are AA″, BB″, CC″, and the point of concurrence is the isogonal of the point got by interchanging the angles μ_1 and μ_3, etc., just as in the case of the external mentioned in (8). A′A″, B′B″, C′C″ are also concurrent as in (9).

e.g. If $\mu_1=\mu_3=\frac{B}{3}$, $\nu_1=\nu_2=\frac{C}{3}$, $\lambda_2=\lambda_3=\frac{A}{3}$, AA′, BB′, CC′ meet in the point $\sec\frac{A}{3}:\sec\frac{B}{3}:\sec\frac{C}{3}$ and AA″, BB″, CC″ in the point $\cos\frac{A}{3}:\cos\frac{B}{3}:\cos\frac{C}{3}$. The first point with reference to triangle A′B′C′ can be shown to be

$$\frac{\sin(60-A/3)}{\sin(60+A/3)}:\frac{\sin(60-B/3)}{\sin(60+B/3)}:\frac{\sin(60-C/3)}{\sin(60+C/3)}.$$

e.g. If all six angles equal the Brocard angle ω (Fig. 2), AA′, BB′, CC′ meet in the point $\operatorname{cosec}(A-\omega):\operatorname{cosec}(B-\omega):\operatorname{cosec}(C-\omega)$, and AA″, BB″, CC″ in the isogonal point $\sin(A-\omega):\sin(B-\omega):\sin C-\omega)$, a point lying on the line joining the Lemoine point to the circumcentre. The points A′, B′, C′ and A″, B″, C″ form the Brocard triangles.

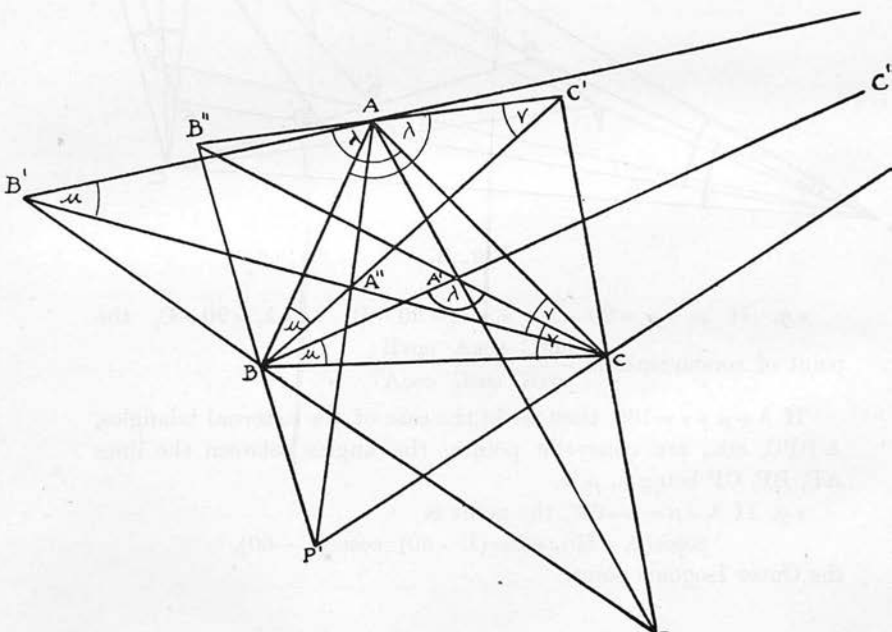


Fig. 3.

(14). If (Fig. 3) $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, and if $\lambda + \mu + \nu = 180°$, and AA′, BB′, CC′ meet in P, and AA″, BB″, CC″ in P′, then as in case of the external triangles mentioned in (10), the following sets of four points are concyclic, BP′CA″, CP′AB″, AP′BC″, CAC′A″, CBC′B″, ABA′B″, ACA′C″, BCB′C″, BAB′A″.

e.g. If $\lambda = \mu = \nu = 60$, P′ is $\sin(A - 60) : \sin(B - 60) : \sin(C - 60)$, one of the Isodynamic points; see (4).

Determinants connected with the Periodic Solutions of Mathieu's Equation.

[Extracted from the *Proceedings of the Edinburgh Mathematical Society*, Vol. XXXIII. (Part 2), Session 1914–1915.]

Determinants connected with the Periodic Solutions of Mathieu's Equation.

By A. G. Burgess, Research Student, Edinburgh University Mathematical Laboratory.

(*Read 11th June 1915. Received 28th June 1915.*)

§ 1. *Introduction.*

Various solutions of Mathieu's equation,* or the equation of the elliptic cylinder functions

$$\frac{d^2y}{dz^2} + (a + 16\,q \cos 2\,z)\,y = 0$$

have recently been discussed in an elegant series of papers in these *Proceedings* † These papers have dealt with both the periodic and the quasi-periodic solutions, but the present paper merely considers determinants which give the infinite series of relations which exist between a and q when the solutions are purely periodic, *i.e.* are the solutions denoted by Professor Whittaker ‡

$$ce_0(z),\ ce_1(z)\ \ldots\ ce_n(z)$$
$$se_1(z)\ \ldots\ se_n(z).$$

The first set of determinants are derived from Lamé's differential equation, which is denoted in the Riemann notation by

$$y = P\left\{\begin{array}{ccccc} 0 & b^2 & c^2 & \infty & \\ 0 & 0 & 0 & -\dfrac{n}{2} & x \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2}+\dfrac{n}{2} & \end{array}\right\}$$

and the other set by means of G. W. Hill's method.§

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* *Liouville's Journal*, sér 2, t. xiii., pp. 137–203.

† *Proceedings of Edinburgh Mathematical Society*, Vols. XXXII. and XXXIII. (Part 1.)

‡ *Proceedings of the Mathematical Congress*, 1912, Vol. I.

§ Hill: *Acta Mathematica*, Vol. VIII., pp. 1–36, 1886.

Afterwards it is shown how the equations which give one set of determinants can be got from the equations corresponding to the other set.

§ 2. *By Lamé's Equation.*

Lamé's equation is

$$x(x-b^2)(x-c^2)\frac{d^2y}{dx^2}+\tfrac{1}{2}(3x^2-2b^2x-2c^2x+b^2c^2)\frac{dy}{dx}-\tfrac{1}{4}\{n(n+1)x+A\}y=0.$$

Divide throughout by c^2, and make $c\rightarrow\infty$, and also n and A, but so that $\frac{n(n+1)}{c^2}$ and $\frac{A}{c^2}$ remain finite, and the equation becomes

$$x(x-b^2)\frac{d^2y}{dx^2}+\tfrac{1}{2}(2x-b^2)\frac{dy}{dx}-(Cx+D)y=0 \qquad (1)$$

where $C=-\frac{n(n+1)}{c^2}$ and $D=-\frac{A}{4c^2}$.

Substituting $b^2\cos^2 z$ for x, we have

$$b^2\cos^2 z\,(b^2\cos^2 z-b^2)\left\{\frac{-2b^2\sin z\cos z\,\frac{d^2y}{dz^2}+2b^2(\cos^2 z-\sin^2 z)\frac{dy}{dz}}{-8b^6\sin^3 z\cos^3 z}\right\}$$
$$+\tfrac{1}{2}\left\{\frac{2b^2\cos^2 z-b}{-2b^2\sin z\cos z}\right\}\frac{dy}{dz}-(Cb^2\cos^2 z+D)y=0.$$

This simplifies to $\frac{d^2y}{dz^2}+(4b^2C\cos^2 z+4D)y=0.$

Comparing this equation with Mathieu's equation,

$$\frac{d^2y}{dz^2}+(a+16q\cos 2z)y=0,\ \text{ or }\ \frac{d^2y}{dz^2}+(a+16q+32q\cos^2 z)y=0,$$

we see that they are the same equation, if $a-16q=4D$ and $32q=4b^2C$. There is no loss of generality in taking $b^2=1$, in which case the equation (1) becomes

$$(-x^2+x)y''+\tfrac{1}{2}(-2x+1)y'+(Cx+D)y=0 \qquad (2)$$

where $a-16q=4D$ and $32q=4C$.

If certain relations exist between C and D, the solution of equation (2) can be expressed as a polynomial in x, such as

$y = a_0 + a_1 x + a_2 x^2 \ldots$ Now, as the substitution was $x = \cos^2 z$, the relations between C and D which give the polynomial solutions of equation (2) will give relations under which Mathieu's equation can be solved in the form $y = a_0 + a_1 \cos^2 z + a_2 \cos^4 z \ldots$ *i.e.* the $ce_0(z)$, $ce_2(z)$, $ce_4(z)$ solutions, for these can be expressed in terms of $\cos^2 z$.

By repeated differentiation of the simplified form of Lamé's equation the following series of equations are obtained :—

$$(-x^2+x)\,y'' + (-\ x + \tfrac{1}{2})\,y' + (Cx + D)\,y = 0$$
$$(-x^2+x)\,y''' + (-3x + \tfrac{3}{2})\,y'' + (Cx + D - 1)\,y' + C\,y = 0$$
$$(-x^2+x)\,y^{\text{iv}} + (-5x + \tfrac{5}{2})\,y''' + (Cx + D - 4)\,y'' + 2C\,y' = 0$$
$$(-x^2+x)\,y^{\text{v}} + (-7x + \tfrac{7}{2})\,y^{\text{iv}} + (Cx + D - 9)\,y''' + 3C\,y'' = 0$$
etc.

In these equations substitute $x = 0$, $C = 8q$, $D = \dfrac{a - 16q}{4}$, and eliminate y, y', y'' ..., and the following infinite determinant is derived.

$$\begin{vmatrix} \frac{a-16q}{4} & \frac{1}{2} & 0 & 0 & 0 \ldots\ldots \\ 8q & \frac{a-16q}{4} - 1^2 & \frac{3}{2} & 0 & 0 \ldots\ldots \\ 0 & 16q & \frac{a-16q}{4} - 2^2 & \frac{5}{2} & 0 \ldots\ldots \\ 0 & 0 & 24q & \frac{a-16q}{4} - 3^2 & \frac{7}{2} \ldots\ldots \\ 0 & 0 & 0 & 32q & \frac{a-16q}{4} - 4^2 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

If each row be multiplied by 4, the determinant becomes

$$\begin{vmatrix} a-16q & 2 & 0 & 0 & 0 \ldots\ldots \\ 32q & a-16q-2^2 & 6 & 0 & 0 \ldots\ldots \\ 0 & 64q & a-16q-4^2 & 10 & 0 \ldots\ldots \\ 0 & 0 & 96q & a-16q-6^2 & 14 \ldots\ldots \\ 0 & 0 & 0 & 128q & a-16q-8^2 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

The leading diagonal shows that when $q=0$, $a=0, 2^2, 4^2, 6^2 \ldots$, which is what was to be expected, as the relations between a and q corresponding to the solutions $ce_0(z)$, $ce_2(z)$, $ce_4(z)$ ... reduce to these when $q=0$.

The value of the determinant

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 & \ldots \\ c_2 & a_2 & b_2 & 0 & \ldots \\ 0 & c_3 & a_3 & b_3 & \ldots \\ 0 & 0 & c_4 & a_4 & \ldots \end{vmatrix} = \begin{vmatrix} a_1 & 1 & 0 & 0 & \ldots \\ d_2 & a_2 & 1 & 0 & \ldots \\ 0 & d_3 & a_3 & 1 & \ldots \\ 0 & 0 & d_4 & a_4 & \ldots \end{vmatrix},$$

in which the leading diagonals are the same and one of the side diagonals is 1, 1, 1 ... provided $d_{r+1} = b_r c_{r+1}$.

Accordingly the determinant is equivalent to the continuant

$$\begin{vmatrix} a-16q & 1 & 0 & 0 \ldots\ldots\ldots \\ 64q & a-16q-2^2 & 1 & 0 \ldots\ldots\ldots \\ 0 & 384q & a-16q-4^2 & 1 \ldots\ldots\ldots \\ 0 & 0 & 960q & a-16q-6^2 \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0 \quad (Cel)$$

I have adopted the notation *Cel*, *Soh'* for the determinants derived, so that they may be easily referred to, *c* and *s* signifying the determinants corresponding to the $ce(z)$ and $se(z)$ functions respectively, *e* and *o* the even and odd suffixes of $ce(z)$ and $se(z)$, and *l* and *h* denoting whether they were derived ~~my~~ means of Lamé's equation or Hill's method.

This determinant as it stands is divergent. It can be made convergent by dividing each row by certain factors, but our purpose is to truncate the determinant so as to get approximate relations between a and q, and as the rows could be multiplied by these factors again after truncation, the same determinant is ultimately derived as would be when *Cel* is truncated directly.*

The determinant *Cel* was truncated to 8 columns and 8 rows. The method of reversion of series was applied and the relation corresponding to the $ce_0(z)$ solution was found to be

$$a = -32q^2 + 224q^4 - \tfrac{29696}{9} q^6 + \ldots \dagger$$

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* This remark applies to all the determinants derived.

† *Cf.* Young: *Proceedings of Edinburgh Mathematical Society*, Vol. XXXII. Lindsay Ince: *Proceedings of Edinburgh Mathematical Society*, Vol. XXXIII. (Part 1.)

In order to get the relation corresponding to $ce_2(z)$, $a+2^2$ was substituted for a. On truncating to 5 columns and 5 rows

$$a = \tfrac{80}{3}q^2 - \tfrac{6104}{27}q^4 \ldots$$

Similarly by substituting $a+4^2$, $a+6^2$, ... for a, the relations between a and q corresponding to $ce_4(z)$, $ce_6(z)$ ... can be got.

§ 3. *Determinant corresponding to* $ce_1(z)$, $ce_3(z)$, etc.

The foregoing suggests that by a different substitution an equation might be derived from which, by a similar method, the relations between a and q corresponding to $ce_1(z)$, $ce_3(z)$ ... might be got.

$ce_1(z)$, $ce_3(z)$, ... can be expressed in terms of $\cos z$. Accordingly the substitution $x = \cos z$ should give determinants corresponding to all the $ce(z)$ solutions.

If $x = \cos z$ be substituted in Mathieu's equation, it becomes

$$(1-x^2)\,y'' - xy' + (a - 16\,q + 32\,qx^2)\,y = 0.$$

By repeated differentiation and substitution of $x = 0$, the following sets of equations are got :—

$$y'' + (a-16q)\,y = 0 \qquad y''' + (a-16q-1^2)\,y' = 0$$

$$y^{\text{iv}} + (a-16q-2^2)\,y'' + 64q\,y = 0 \qquad y^{\text{v}} + (a-16q-3^2)\,y''' + 192q\,y' = 0$$

$$y^{\text{vi}} + (a-16q-4^2)y^{\text{iv}} + 384qy'' = 0 \qquad y^{\text{vii}} + (a-16q-5^2)y^{\text{v}} + 640qy''' = 0$$

....

$ce_0(z)$, $ce_2(z)$, etc., are functions of even powers of $\cos z$; when $\cos z = 0$, y', y''', y^{v} ... are each zero, whereas $ce_1(z)$, $ce_3(z)$, etc., are functions of odd powers of $\cos z$, and y, y'', y^{iv}, y^{vi}, ... are each zero on a like substitution. Accordingly the first set of equations give relations corresponding to $ce_0(z)$, $ce_2(z)$, etc., and the second set corresponding to $ce_1(z)$, $ce_3(z)$, etc.

The first set gives on elimination the same determinant *Cel*, and the second the determinant

$$\begin{vmatrix} a-16q-1^2 & 1 & 0 & 0 \;\ldots\ldots \\ 192q & a-16q-3^2 & 1 & 0 \;\ldots\ldots \\ 0 & 640q & a-16q-5^2 & 1 \;\ldots\ldots \\ 0 & 0 & 1344q & a-16\,q-7^2 \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0 \quad (Col)$$

The leading diagonal shows that this corresponds to $ce_1(z)$, $ce_3(z)$, etc. On substituting $a+1^2$ for a and truncating to 4 rows and columns the relation corresponding to $ce_1(z)$,

$$a = -8q - 8q^2 + 8q^3 \ldots$$

was found.

In like manner the relations for $ce_3(z)$, $ce_5(z)$ ... can be obtained.

§ 4. *Determinant corresponding to* $se_1(z)$, $se_3(z)$, etc.

These are functions of $\sin z$, and, accordingly, the substitution $x = \sin z$ will give an equation from which determinants may be derived corresponding to these solutions and such others as can be expressed as polynomials in $\sin z$, viz. $ce_0(z)$, $ce_2(z)$, etc.

The substitution of $x = \sin z$ in Mathieu's equation gives

$$(1-x^2)\,y'' - x\,y' + (a + 16q - 32q\,x^2)\,y = 0.$$

Now $ce_0(z)$, $ce_2(z)$ ... can be expressed in terms of even powers of $\sin z$, whereas $se_1(z)$, $se_3(z)$, ... only involve the odd powers. Therefore when $\sin z = 0$ (*i.e.* $x = 0$), y', y''', y^{v} ... are each zero for $ce_0(z)$, etc., and y, y'', y^{iv} ... are each zero for $se_1(z)$, etc. Hence two determinants are again derived,

$$\begin{vmatrix} a+16q & 1 & 0 & 0 & \ldots\ldots \\ -64q & a+16q-2^2 & 1 & 0 & \ldots\ldots \\ 0 & -384q & a+16q-4^2 & 1 & \ldots\ldots \\ 0 & 0 & -960q & a+16q-6^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0 \quad (Cel') \text{ corresponding to } ce_0(z), \text{ etc.}$$

and

$$\begin{vmatrix} a+16q-1^2 & 1 & 0 & 0 & \ldots\ldots \\ -192q & a+16q-3^2 & 1 & 0 & \ldots\ldots \\ 0 & -640q & a+16q-5^2 & 1 & \ldots\ldots \\ 0 & 0 & -1344q & a+16q-7^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0 \quad (Sol') \text{ corresponding to } se_1(z), \text{ etc.}$$

The dash at *Cel* and *Sol* signifies that $a + 16q$ is the principal term instead of $a - 16q$.

Determinant *Cel'* when truncated produced the same relations as *Cel*, whereas *Sol'*, corresponding to $se_1(z)$, gave

$$a = 8q - 8q^2 - 8q^3 \ldots.$$

§ 5. *Determinant corresponding to* $se_2(z)$, $se_4(z)$, etc.

$se_2(z)$, $se_4(z)$, etc., cannot be expressed as polymonials in $\sin z$, $\cos z$, $\sin^2 z$, or $\cos^2 z$, hence none of the foregoing substitutions will give the relations corresponding to these solutions, but, as they can be expressed in terms of $\sin 2z$, $\sin 4z$, etc., if they be divided by $\sin z$, $\frac{se_2(z)}{\sin z}$, $\frac{se_4(z)}{\sin z}$, etc., can be expressed in terms of $\cos z$. Accordingly, the substitution $y = \xi \sin z$ was made in Mathieu's equation, and the equation

$$(1 - x^2)\,\xi'' - 3x\xi' + (a - 16q - 1^2 + 32qx^2)\,\xi = 0$$

was derived, in which $x = \cos z$. Now, $se_1(z)$, $se_3(z)$, etc., may also be expressed in terms of $\cos z$, when divided by $\sin z$, and hence this equation will give a determinant corresponding to these solutions as well. The two determinants derived were

$$\begin{vmatrix} a-16q-1^2 & 1 & 0 & 0 & \cdots\cdots \\ 64q & a-16q-3^2 & 1 & 0 & \cdots\cdots \\ 0 & 384q & a-16q-5^2 & 1 & \cdots\cdots \\ 0 & 0 & 960q & a-16q-7^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

(Sol) corresponding to $se_1(z)$, $se_3(z)$, etc.

and

$$\begin{vmatrix} a-16q-2^2 & 1 & 0 & 0 & \cdots\cdots \\ 192q & a-16q-4^2 & 1 & 0 & \cdots\cdots \\ 0 & 640q & a-16q-6^2 & 1 & \cdots\cdots \\ 0 & 0 & 1344q & a-16q-8^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

(Sel) corresponding to $se_2(z)$, $se_4(z)$, etc.

Determinant Sol when truncated produced the same relations as Sol', whereas Sel, corresponding to $se_2(z)$, gave

$$a = -\tfrac{16}{3}q^2 + \tfrac{40}{27}q^4 \ldots.$$

§ 6. *Notes on the foregoing Determinants.*

Determinants Cel and Col were got from the equation

$$(1 - x^2)\,y'' - xy' + (a - 16q + 32qx^2)\,y = 0,$$

and Cel' and Sol' from the equation

$$(1 - x^2)\,y'' - xy' + (a + 16q - 32qx^2)\,y = 0.$$

It will be noticed that these equations differ only in the sign of q. As the relations corresponding to $ce_0(z)$, $ce_2(z)$, etc., were got from both of these equations, these relations must involve only even powers of q, so that the interchange of the sign of q may leave them unaltered. This agrees with the values found for q,

$$a = -32\,q^2 + 224\,q^4 \dots \text{ etc.}$$

Hence determinant Cel = determinant Cel'.

Again, the first of these equations gave determinant Col corresponding to $ce_1(z)$, etc., and the second determinant Sol' corresponding to $se_1(z)$, etc. Therefore the relation corresponding to $se_1(z)$ may be got from that corresponding to $ce_1(z)$ by changing the sign of q. This agrees with the values found,

$$ce_1(z) = -8\,q - 8\,q^2 + 8\,q^3 \dots \text{ and } se_1(z) = +8\,q - 8\,q^2 - 8\,q^3 \dots .$$

Similarly in § 5 $y = \xi \cos z$ might have been substituted, and the equation

$$(1-x^2)\,\xi'' - 3\,x\,\xi' + (a + 16\,q - 1^2 - 32\,q\,x^2)\,\xi = 0$$

have been got instead of

$$(1-x^2)\,\xi'' - 3\,x\,\xi' + (a + 16\,q - 1^2 + 32\,q\,x^2)\,\xi = 0,$$

and determinants corresponding to $ce_1(z)$, $ce_3(z)$, etc., and $se_2(z)$, $se_4(z)$, etc., derived. These would be

$$\left|\begin{array}{ccccc} a+16q-1^2 & 1 & 0 & 0 & \dots\dots \\ -64q & a+16q-3^2 & 1 & 0 & \dots\dots \\ 0 & -384q & a+16q-5^2 & 1 & \dots\dots \\ 0 & 0 & -960q & a+16q-7^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}\right| = 0$$

(Col') corresponding to $ce_1(z)$, etc.

and

$$\left|\begin{array}{ccccc} a+16q-2^2 & 1 & 0 & 0 & \dots\dots \\ -192q & a+16q-4^2 & 1 & 0 & \dots\dots \\ 0 & -640q & a+16q-6^2 & 1 & \dots\dots \\ 0 & 0 & -1344q & a+16q-8^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}\right| = 0$$

(Sel') corresponding to $se_2(z)$, $se_4(z)$, etc.

Hence the relation corresponding to $se_2(z)$, $se_4(z)$, etc., must involve only even powers of q. This agrees with the relation found, $a = -\frac{16}{3}\,q^2 + \frac{40}{27}\,q^4 \dots .$

It will be noticed that

(1) *Cel′* and *Col′* have another diagonal $-64\,q$, $-384\,q$, etc.
Sel′ and *Sol′* - - - - - $-192\,q$, $-640\,q$, etc.

(2) *Cel* and *Sol* - - - - $64\,q$, $384\,q$, etc.
Col and *Sel* - - - - $192\,q$, $640\,q$, etc.

The coefficients of q are $\pm 64\,(r-1)\,(2\,r-3)$ for one set of determinants, and $\pm 64\,(r-1)\,(2\,r-1)$ for the other, where r is the number of the row in which the quantity occurs.

§ 7. *Hill's Method.*

Assuming for the solution $y=\sum_{n=-\infty}^{\infty} b_n\, e^{2niz}$, and substituting Mathieu's equation becomes

$$\sum_{n=-\infty}^{\infty}(2\,ni)^2\,b_n\,e^{2niz}+(a+8\,q\,e^{2niz}+8\,q\,e^{-2iz})\sum_{n=-\infty}^{\infty} b_x\,e^{2niz}=0.$$

Equating to zero the coefficients of e^{2niz}, the relation (which holds true for all integral values of n)

$$8\,q\,b_{n-1}+[(2\,ni)^2+a]\,b_n+8\,q\,b_{n-1}=0$$

is obtained.

If in this equation values of n from $-\infty$ to $+\infty$ are inserted, a series of equations is derived, the central ones of which are

$$\begin{aligned}
&8\,q\,b_{-3}+(a-4^2)\,b_{-2}+8\,q\,b_{-1}=0 && (n=-2)\\
&8\,q\,b_{-2}+(a-2^2)\,b_{-1}+8\,q\,b_0=0 && (n=-1)\\
&8\,q\,b_{-1}+\quad a\,b_0\quad +8\,q\,b_1=0 && (n=0)\\
&8\,q\,b_0+(a-2^2)\,b_1+8\,q\,b_2=0 && (n=1)\\
&8\,q\,b_1+(a-4^2)\,b_2+8\,q\,b_3=0 && (n=2)
\end{aligned}$$

Eliminating the b's we get the doubly infinite determinant

$$\left|\begin{matrix}
\cdots\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots\\
\cdots\cdots 8q & a-4^2 & 8q & 0 & 0 & 0\cdots\cdots\\
\cdots\cdots 0 & 8q & a-2^2 & 8q & 0 & 0\cdots\cdots\\
\cdots\cdots 0 & 0 & 8q & a & 8q & 0\cdots\cdots\\
\cdots\cdots 0 & 0 & 0 & 8q & a-2^2 & 8q\cdots\cdots\\
\cdots\cdots 0 & 0 & 0 & 0 & 8q & a-4^2\cdots\\
\cdots\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots
\end{matrix}\right|=0$$

This infinite determinant of order $2r+1$ can be split up into two determinants—

$$\left|\begin{array}{ccccc} a & 16q & 0 & 0 & 0\ldots\ldots \\ 8q & a-2^2 & 8q & 0 & 0\ldots\ldots \\ 0 & 8q & a-4^2 & 8q & 0\ldots\ldots \\ 0 & 0 & 8q & a-6^2 & 8q\ldots\ldots \\ 0 & 0 & 0 & 8q & a-8^2\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right| = 0 \quad (Ceh)$$

and

$$\left|\begin{array}{cccc} a-2^2 & 8q & 0 & 0\ldots\ldots \\ 8q & a-4^2 & 8q & 0\ldots\ldots \\ 0 & 8q & a-6^2 & 8q\ldots\ldots \\ 0 & 0 & 8q & a-8^2\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right| = 0 \quad (Seh)$$

Assuming the expansion for $ce_0(z)$ and retaining terms up to q^4,

$$\begin{aligned} ce_0(z) &= 1 + (4q - 28q^3 \ldots)\cos 2z + (2q^2 - \tfrac{160}{9}q^4 \ldots)\cos 4z + \ldots \\ &= b_0 + b_1 e^{2iz} + b_2 e^{4iz} + \ldots \\ &\quad\; + b_{-1} e^{-2iz} + b_{-2} e^{-4iz} + \ldots \end{aligned}$$

we see that $b_1 = b_{-1}$, $b_2 = b_{-2}$, etc.

Returning to the equations, substitute these, and the equation for $n=0$ becomes $a\,b_0 + 16\,q\,b_1 = 0$, while the others become identical in pairs, *i.e.* equation for $n=r$ is the same as that for $n=-r$. If the b's be eliminated in the equations, determinant *Ceh* is derived. If *Ceh* be truncated to 4 columns and rows,

$$a = -32\,q^2 + 224\,q^4 \ldots \text{ corresponding to } ce_0(z),$$

the same relation as was derived from *Cel*.

The leading diagonal, when $q=0$, gives $a = 2^2, 4^2, 6^2, \ldots$ and hence the values of a corresponding to $ce_2(z)$, $ce_4(z)$, may be derived.

$$\begin{aligned} se_2(z) &= \sin 2z + (\tfrac{2}{3}q - \tfrac{5}{3}q^3)\sin 4z + \tfrac{1}{6}q^2 \sin 6z + \ldots \\ &= b_0 + b_1 e^{2iz} + b_2 e^{4iz} + b_3 e^{6iz} + \ldots \\ &\quad\; + b_{-1} e^{-2iz} + b_{-2} e^{-4iz} + b_{-3} e^{-6iz} + \ldots \end{aligned}$$

Hence $b_0 = 0$, $b_1 = -b_{-1}$, $b_2 = -b_{-2}$, etc.

Substituting in the equations and eliminating the b's, we derive determinant *Seh*. As the leading diagonal is $a-2^2$, $a-4^2$, etc., this determinant will give the values of a corresponding to $se_2(z)$, $se_4(z)$, etc. If it be truncated to 4 columns and rows, and $a+2^2$ substituted for a, $a=-\frac{16}{3}q^2+\frac{40}{27}q^4 \ldots$, the value derived from *Sel.*

§ 8. *Determinants corresponding to* $ce_1(z)$, $ce_3(z)$, etc., $se_1(z)$, $se_3(z)$, etc.

Assuming for the solution $y=\sum_{n=-\infty}^{n=\infty} b_n e^{(2n-1)iz}$, and substituting in Mathieu's equation, we find it becomes

$$\sum_{n=-\infty}^{\infty}\{(2n-1)i\}^2 b_n e^{(2n-1)iz}+(a+8qe^{2iz}+8qe^{-2iz})\sum_{n=-\infty}^{\infty} b_n e^{(2n-1)iz}=0.$$

If the coefficients of $e^{(2n-1)iz}$ be equated to zero, the relation (which holds true for all integral values of n)

$$8qb_{n-1}+[\{(2n-1)i\}^2+a]b_n+8qb_{n+1}=0$$

is obtained.

In this equation values of n from $n=-\infty$ to $+\infty$ are inserted, and a series of equations is obtained, the central ones of which are

$$\begin{aligned}
&8qb_{-3}+(a-5^2)b_{-2}+8qb_{-1}=0 &&(n=-2)\\
&8qb_{-2}+(a-3^2)b_{-1}+8qb_0=0 &&(n=-1)\\
&8qb_{-1}+(a-1^2)b_0+8qb_1=0 &&(n=0)\\
&8qb_0+(a-1^2)b_1+8qb_2=0 &&(n=1)\\
&8qb_1+(a-3^2)b_2+8qb_3=0 &&(n=2)
\end{aligned}$$

If the b's be eliminated the doubly infinite determinant of order $2r$ is derived.

$$\begin{vmatrix}
\cdots\cdots & & & & & \cdots\cdots\\
\ldots\ldots 8q & a-5^2 & 8q & 0 & 0 & 0\ldots\ldots\\
\ldots\ldots 0 & 8q & a-3^2 & 8q & 0 & 0\ldots\ldots\\
\ldots\ldots 0 & 0 & 8q & a-1^2 & 8q & 0\ldots\ldots\\
\ldots\ldots 0 & 0 & 0 & 8q & a-1^2 & 8q\ldots\ldots\\
\ldots\ldots 0 & 0 & 0 & 0 & 8q & a-3^2\ldots\\
\cdots\cdots & & & & & \cdots\cdots
\end{vmatrix}=0$$

This can be broken up into two determinants

$$\begin{vmatrix} a-1^2+8q & 8q & 0 & 0\ \ldots\ldots \\ 8q & a-3^2 & 8q & 0\ \ldots\ldots \\ 0 & 8q & a-5^2 & 8q\ \ldots\ldots \\ 0 & 0 & 8q & a-7^2\ \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0 \quad (Coh)$$

and

$$\begin{vmatrix} a-1^2-8q & 8q & 0 & 0\ \ldots\ldots \\ 8q & a-3^2 & 8q & 0\ \ldots\ldots \\ 0 & 8q & a-5^2 & 8q\ \ldots\ldots \\ 0 & 0 & 8q & a-7^2\ \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0 \quad (Soh)$$

$$\begin{aligned} ce_1(z) &= \cos z \qquad + (q-q^2)\cos 3z + \ldots \\ &= b_1 e^{iz} \qquad + b_2 e^{3iz} + \ldots \\ &\quad + b_0 e^{-iz} \qquad + b_{-1} e^{-3iz} + \ldots \end{aligned}$$

Hence $b_0 = b_1,\ b_2 = b_{-1},\ b_3 = b_{-2}$, etc.

Substituting in the equations and eliminating the b's, we derive determinant *Coh*. Hence this determinant gives the relation between a and q corresponding to $ce_1(z)$, $ce_3(z)$, etc. When truncated to 3 columns and rows, $a = -8q - 8q^2 + 8q^3 \ldots$ for $ce_1(z)$, the value found from *Col.*

$$\begin{aligned} se_1(z) &= \sin z \qquad + (q+q^2)\sin 3z + \ldots \\ &= b_1 e^{iz} \qquad + b_2 e^{3iz} + \ldots \\ &\quad + b_0 e^{-iz} \qquad + b_{-1} e^{-3iz} + \ldots \end{aligned}$$

Hence $b_0 = -b_1,\ b_{-1} = -b_2,\ b_{-2} = -b_3$, etc.

If these substitutions be made in the equations and the b's eliminated, determinant *Soh* is got. Accordingly this determinant gives the relation between a and q corresponding to $se_1(z)$, $se_3(z)$, etc. When truncated to 3 columns and rows, $a = 8q - 8q^2 - 8q^3 \ldots$ for $se_1(z)$, the value found from *Sol.*

If it be remembered that instead of $8q$ and $8q$ in the two diagonals 1 and $64q^2$ may be substituted, it will be seen that a change of sign in q will leave the determinants corresponding to $ce_0(z)$ and $se_2(z)$ unaltered, whereas it will convert the determinant corresponding to $ce_1(z)$ into that of $se_1(z)$ and *vice versa.*

It may be remarked that in order to get the value of a correct to q^n, $n+1$ rows and columns had to be taken in the determinants derived by the first method, whereas by Hill's method n rows and columns were sufficient, and that the calculations were far easier.

§ 9. *Equivalence of the Equations and Determinants.*

In the first method $ce_0(z)$ was taken as equal to

$$A + B\cos^2 z + C\cos^4 z \dots$$

and in the second method equal to

$$a + b\cos 2z + c\cos 4z \dots$$

We can therefore, by considering such equations as

$$\cos 2z = 2\cos^2 z - 1,\ \cos 4z = 8\cos^4 z - 8\cos^2 z + 1,\ \text{etc.}$$

get a series of relations connecting $A, B, C, \dots$ with $a, b, c, \dots$. Call these relations *I.* Now in the first method we get linear equations for $A, B, C, \dots$. Call these equations *II.* In the second method we get linear equations for $a, b, c, \dots$. Let these be called *III.* Equations *III.* must be transformable into equations *II.* by means of relations *I.*, and hence the determinants derived from these equations equivalent.

As an example take *Cel* and *Ceh.*

$$y = A + B\cos^2 z + C\cos^4 z + D\cos^6 z + \dots$$

or $$y = A + Bx^2 + Cx^4 + Dx^6 + \dots \quad \text{where } x = \cos z.$$

$$\therefore\ y = A,\ y' = 0,\ y'' = 2B,\ y''' = 0,\ y^{\text{iv}} = 24C,\ y^{\text{v}} = 0,\ y^{\text{vi}} = 720D,$$
$$\text{when } x = 0.$$

Again, $$y = b_0 + 2b_1\cos 2z + 2b_2\cos 4z + 2b_3\cos 6z \dots$$
$$= b_0 + 2b_1(2\cos^2 z - 1) + 2b_2(8\cos^4 z - 8\cos^2 z + 1)$$
$$+ 2b_3(32\cos^6 z - 48\cos^4 z + 18\cos^2 z - 1) + \dots$$

$$\therefore\ y = b_0 - 2b_1 + 2b_2 - 2b_3 + \dots$$
$$y'' = 8b_1 - 32b_2 + 72b_3 - \dots$$
$$y^{\text{iv}} = 384b_2 - 2304b_3 + \dots \qquad \text{etc.}$$

The equations from which *Ceh* was derived were:—

$$ab_0 + 16qb_1 = 0, \quad 8qb_0 + (a-4)b_1 + 8qb_2 = 0,$$
$$8qb_1 + (a-16)b_2 + 8qb_3 = 0, \quad \text{etc.}$$

To derive the first equation from which *Cel* was got, multiply the first of these equations by 1, the second by -2, the third by $+2$, etc., *i.e.* by the coefficients of $b_0, b_1, b_2, \ldots$ in the relation connecting them with y. Add these and equate to zero.

$$\therefore \quad (ab_0 + 16\,qb_1) - 2\,[8\,q\,b_0 + (a-4)\,b_1 + 8\,q\,b_2]$$
$$2\,[8\,q\,b_1 + (a-16)\,b_2 + 8\,q\,b_3] + \ldots = 0.$$

Rearranging this equation we have

$$(a - 16\,q)\,(b_0 - 2\,b_1 + 2\,b_2 - \ldots) + (8\,b_1 - 32\,b_2 + 72\,b_3 - \ldots) = 0,$$

i.e. $(a-16\,q)\,y + y'' = 0$, the first of the equations from which *Cel* was got.

Again, multiplying the second of the equations by 8, the third by -32, the fourth by $+72$, ... etc., *i.e.* by the coefficients of $b_1, b_2, b_3, \ldots$ in the relation connecting them and y'', and adding and equating to zero, we have

$$8\,[8\,q\,b_0 + (a-4)\,b_1 + 8\,q\,b_2] - 32\,[8\,q\,b_1 + (a-16)\,b_2 + 8\,q\,b_3]$$
$$+ 72\,[8\,q\,b_2 + (a-36)\,b_3 + 8\,q\,b_4] - \ldots = 0.$$

Rearranging this equation, it becomes

$$64\,q\,(b_0 - 2\,b_1 + 2\,b_2 - 2\,b_3 \ldots) + (a - 16\,q - 4)\,(8b_1 - 32\,b_2 + 72\,b_3 \ldots) = 0,$$

i.e. $64\,q\,y + (a - 16\,q - 4)\,y'' = 0$, the second of the equations from which *Cel* was got.

Continuing in an exactly similar manner we can derive the other equations from which *Cel* was got.

Hence the determinant *Cel* is equivalent to *Ceh*.

It will be noticed that the multipliers of the equations

$$a\,b_0 + 16\,q\,b_1 = 0, \text{ etc.,}$$

were the coefficients of b_0, b_1, etc., in the relations connecting them and y, y'', etc., and that these coefficients were the coefficients of $\cos^r z$ in the expansion of $\cos 2\,n\,z$ in terms of $\cos z$, multiplied by certain factors.

Similarly it can be shown that

$$Cel' = Ceh,\ Col = Col' = Coh,\ Sel = Sel' = Seh,\ Sol = Sol' = Soh.$$

The following table shows the various multipliers to convert the equations of the second method into the first two equations of the first method.

Determinant.		Multipliers for 1st Equation.	Multipliers for 2nd Equation.
Ceh	*Cel*	$+1, -2, +2,$	$+8, -32, +72,$ ....
	Cel′	$+1, +2, +2,$	$-8, -32, -72,$ ....
Coh	*Col*	$+2, -6, +10,$	$+48, -240, +672,$ ....
	Col′	$+2, +2, +2,$	$-16, -48, -96,$ ....
Soh	*Sol*	$+2, -2, +2,$	$+16, -48, +96,$ ....
	Sol′	$+2, +6, +10,$	$-48, -240, -672,$ ....
Seh	*Sel*	$+4, -8, +12,$	$+96, -384, +960,$ ....
	Sel′	$+4, +8, +12,$	$-96, -384, -960,$ ....

or thus

Determinant.		Multipliers for 1st Equation.	Multipliers for 2nd Equation.
Ceh	*Cel*	1, 2 (coefficients of $\cos^{\circ} z$ in $\cos 2nz$)	4 (coefficients of $\cos^2 z$ in $\cos 2nz$)
	Cel′	1, 2 (coefficients of $\sin^{\circ} z$ in $\cos 2nz$)	4 (coefficients of $\sin^2 z$ in $\cos 2nz$)
Coh	*Col*	2 (coefficients of $\cos z$ in $\cos(2n-1)z$)	12 (coefficients of $\cos^3 z$ in $\cos(2n-1)z$)
	Col′	$2\left(\text{coefficients of } \sin^{\circ} z \text{ in } \dfrac{\cos(2n-1)z}{\cos z}\right)$	$4\left(\text{coefficients of } \sin^2 z \text{ in } \dfrac{\cos(2n-1)z}{\cos z}\right)$
Soh	*Sol*	$2\left(\text{coefficients of } \cos^{\circ} z \text{ in } \dfrac{\sin(2n-1)z}{\sin z}\right)$	$4\left(\text{coefficients of } \cos^2 z \text{ in } \dfrac{\sin(2n-1)z}{\sin z}\right)$
	Sol′	2 (coefficients of $\sin z$ in $\sin(2n-1)z$)	12 (coefficients of $\sin^3 z$ in $\sin(2n-1)z$)
Seh	*Sel*	$2\left(\text{coefficients of } \cos z \text{ in } \dfrac{\sin 2nz}{\sin z}\right)$	$12\left(\text{coefficients of } \cos^3 z \text{ in } \dfrac{\sin 2nz}{\sin z}\right)$
	Sel′	$2\left(\text{coefficients of } \sin z \text{ in } \dfrac{\sin 2nz}{\sin z}\right)$	$12\left(\text{coefficients of } \sin^3 z \text{ in } \dfrac{\sin 2nz}{\sin z}\right)$

When it is remembered that changing the sign of q converts *Cel* into *Cel′*, and *Sel* into *Sel′*, but *Col* into *Sol′* and *Col′* into *Sol* a glance at the above tables shows that they possess, in addition to others, the same interesting features. It will be noted, for example, *Cel* and *Cel′* have the same multipliers, differing only in the sign of every second multiplier, and from the second table that the alteration is merely the substitution of sine for cosine. The same applies to *Sel* and *Sel′*. But in the case of the odd functions, *Col* and *Sol′* correspond in this way, and *Col′* and *Sol*. A reference to the transformation of *Ceh* into *Cel*, and the corresponding one of *Ceh* into *Cel′* at once supplies the reason.

In conclusion, I desire to thank Professor Whittaker, at whose suggestion this investigation was begun, for much useful advice during its progress.

Theorems in connection with lines drawn thro[ugh a]
pair of points parallel and antiparallel to the s[ides]
of a triangle.

Theorems in connection with lines drawn through a pair of points parallel and antiparallel to the sides of a triangle.

By A. G. Burgess, M.A.

Figure 4. p. 20.

1. *Parallel to sides.*

(a) Let O and O′ be two reciprocal points,

x, y, z, the $\perp^{\text{rs}}$ from O on the sides,

x', y', z', " " " O′ " " " ;

DE(α), FG(β), HI(γ) the intercepts made on the sides by lines drawn parallel to the sides through O ;

D′E′(α'), F′G′ (β'), H′I′ (γ') similar intercepts for O′.

Then the six triangles

OD′E′, O′DE, OF′G′, O′FG, OH′I′, O′HI are equal.

$\alpha^2 : a^2 :: \triangle\text{ODE} : \triangle\text{ABC}$ or 2S, and $\alpha . x = 2\triangle\text{ODE}$.

$$\therefore \quad \alpha = \frac{a^2x}{4\text{S}}, \quad \beta = \frac{b^2y}{4\text{S}}; \quad \alpha' = \frac{a^2x'}{4\text{S}}, \text{ etc.}$$

Let AO cut BC in M, AO′ in M′,

then BM = M′C, BM′ = MC (by definition of reciprocal points).

$$\text{BM} : \text{MC} = \triangle\text{AOB} : \triangle\text{AOC}$$

$$= cz \qquad : by.$$

$$\text{BM}' : \text{M}'\text{C} = cz' \qquad : by'.$$

$$\therefore \quad cz : by \quad :: \quad by' : cz'.$$

$$\therefore \quad c^2zz' = b^2yy' = a^2xx'.$$

But $\alpha x' = \dfrac{a^2xx'}{4\text{S}}$, etc.

$$\therefore \quad \alpha x' = \alpha' x = \beta y' = \beta' y = \gamma z' = \gamma' z.$$

But $\alpha x' = 2\triangle\text{O}'\text{DE}$, etc.

$\therefore$ the six triangles are equal.

(b) Since the perpendiculars from the incentre are equal, the intercepts cut off by parallels through the reciprocal point to the incentre must be equal.

7

FIGURE 5. p. 20.

2. *Antiparallel to sides.*

(*a*) Let O and O′ be two points, and let the intercepts cut off be by lines drawn antiparallel to the sides.

$$\triangle\text{ODE is isosceles.}$$

$$\therefore \quad \frac{a}{2x} = \cot\text{ODE} = \cot\text{A}.$$

$$\therefore \quad \alpha = 2x\cot\text{A}, \quad \beta = 2y\cot\text{B}, \quad \text{etc.}$$

$$\alpha' = 2x'\cot\text{A}, \quad \text{etc.}$$

$$\therefore \quad \alpha x' = 2xx'\cot\text{A} = \alpha' x \quad \text{or} \quad \triangle\text{O}'\text{DE} = \triangle\text{OD}'\text{E}',$$

$$\beta y' = 2yy'\cot\text{B} = \beta' y \quad \text{or} \quad \triangle\text{O}'\text{FG} = \triangle\text{OF}'\text{G}',$$

$$\gamma z' = 2zz'\cot\text{C} = \gamma' z \quad \text{or} \quad \triangle\text{O}'\text{HI} = \triangle\text{OH}'\text{I}';$$

and the six triangles will be equal if the conditions

$$xx'\cot\text{A} = yy'\cot\text{B} = zz'\cot\text{C} \text{ are fulfilled,}$$

$$\text{or if} \quad xx' : yy' : zz' \quad :: \quad \tan\text{A} : \tan\text{B} : \tan\text{C}.$$

In analogy to parallels and antiparallels such a pair of points might be called antireciprocal points.

Now if O be the orthocentre $x : y : z \quad :: \quad \frac{1}{\cos\text{A}} : \frac{1}{\cos\text{B}} : \frac{1}{\cos\text{C}}$,

and if O′ be the symmedian point $x' : y' : z' \quad :: \sin\text{A} : \sin\text{B} : \sin\text{C}$. Hence for these two points the six triangles are equal.

(*b*) The intercepts made on the sides by lines drawn antiparallel to the sides will be equal, if the point through which they are drawn has its perpendiculars on the sides in the ratio

$$\tan\text{A} : \tan\text{B} : \tan\text{C}.$$

If $\alpha = \beta = \gamma$, then $x\cot\text{A} = y\cot\text{B} = z\cot\text{C}$,

$$\therefore \quad x : y : z = \frac{1}{\cot\text{A}} : \frac{1}{\cot\text{B}} : \frac{1}{\cot\text{C}}$$

$$= \tan\text{A} : \tan\text{B} : \tan\text{C}.$$

Hence such a point might be called antireciprocal point to the incentre.

On ~~an instrument~~ for ~~trisecting any~~ angle.*

By ~~Jas. N. Miller~~.

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* *Vide Proc.* R.S.E., Vol. XXIV., pp. 7-8.

Fig. 4

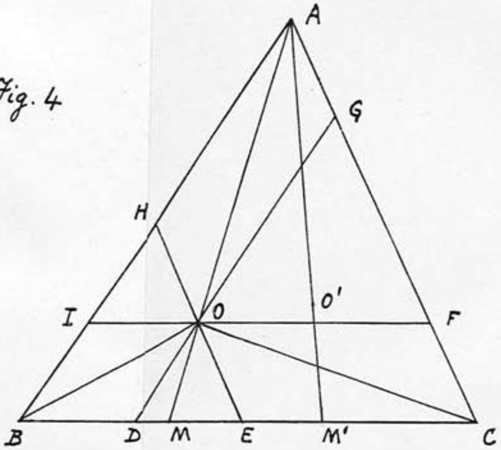
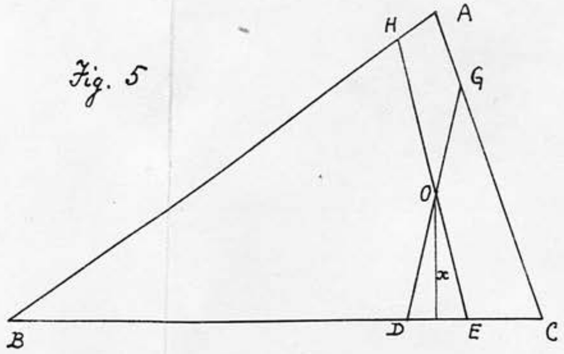


Fig. 5



Notes on Antireciprocal Points.

(see previous paper, page 18)

Notes on Antireciprocal Points.

By A. G. Burgess, M.A.

[Extracted from the *Proceedings of the Edinburgh Mathematical Society*, Vol. XXI., Session 1902–1903.]

Definition. If x, y, z and ξ, η, ζ be the perpendiculars on the sides BC, CA, AB of the $\triangle$ ABC from points O and O′, then O and O′ are antireciprocal points if $x\xi : y\eta : z\zeta :: \tan A : \tan B : \tan C$.

I. Construction to find a point antireciprocal to O (Fig. 4). p 29.

Draw through O a line MN antiparallel to BC. Draw OY perpendicular to AC, and OZ perpendicular to AB. Draw lines parallel to AB and AC, and at distances from them respectively equal to YN and MZ, and let them cut in P. Join AP. Find a similar line BQ, and let AP and BQ cut in O′. O′ is the required point. Let the perpendiculars from O be x, y, z and those from O′, ξ, η, ζ.

$$\begin{aligned} \eta : \zeta &= \mathrm{MZ} : \mathrm{YN} \\ &= \mathrm{OZ}/\tan\mathrm{OMZ} : \mathrm{OY}/\tan\mathrm{ONY} \\ &= \tan B/\mathrm{OY} : \tan C/\mathrm{OZ} \\ &= \tan B/y : \tan C/z \end{aligned}$$

$$\therefore \quad y\eta : z\zeta = \tan B : \tan C.$$

Similarly $$x\xi : z\zeta = \tan A : \tan C$$

$$\therefore \quad x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C.$$

$\therefore$ O′ is the antireciprocal of O.

II. Construction to find a point antireciprocal to itself (Fig. 5). p. 29.

Draw AD perpendicular to BC, and produce it to meet the semicircle described on BC as diameter in E. Draw lines parallel to AB and CA, and at distances from them respectively equal to BE and CE. Let them cut in P. Join AP. Find a similar line BQ. Let AP and BQ cut in O. O is the required point.

$$\begin{aligned} y^2 : z^2 = \mathrm{CE}^2 : \mathrm{BE}^2 &= \mathrm{CD} : \mathrm{BD} \\ &= \mathrm{CD}/\mathrm{AD} : \mathrm{BD}/\mathrm{AD} \\ &= \mathrm{AD}/\mathrm{BD} : \mathrm{AD}/\mathrm{CD} \\ &= \tan \mathrm{B} : \tan \mathrm{C}. \end{aligned}$$

Similarly $\quad x^2 : z^2 = \tan \mathrm{A} : \tan \mathrm{C}$

$$\therefore \quad x^2 : y^2 : z^2 = \tan \mathrm{A} : \tan \mathrm{B} : \tan \mathrm{C}$$

$$\text{or} \quad x\xi : y\eta : z\zeta = \tan \mathrm{A} : \tan \mathrm{B} : \tan \mathrm{C}$$

where $\quad x = \xi,\ y = \eta,\ z = \zeta.$

$\therefore$ O is the required point.

$$x : y : z = \sqrt{\tan \mathrm{A}} : \sqrt{\tan \mathrm{B}} : \sqrt{\tan \mathrm{C}}$$

so that the point whose trilinear coordinates are

$$\sqrt{\tan \mathrm{A}}, \quad \sqrt{\tan \mathrm{B}}, \quad \sqrt{\tan \mathrm{C}}$$

is the antireciprocal of itself.

The three triangles formed by drawing through this point lines antiparallel to the sides of the $\triangle$ ABC will be equal. The intercepts cut off on the sides by these antiparallels are proportional to

$$\sqrt{\cot \mathrm{A}}, \quad \sqrt{\cot \mathrm{B}}, \quad \sqrt{\cot \mathrm{C}}.$$

There are four such points, one internal and three external. Their coordinates are given by $\sqrt{\tan \mathrm{A}},\ \pm \sqrt{\tan \mathrm{B}},\ \pm \sqrt{\tan \mathrm{C}}$.

Definition. The antireciprocal of a line is the locus of the antireciprocals of all points in the line.

1. The antireciprocal of a line is a conic passing through the vertices of the triangle.

Let $lx+my+nz=0$ be the equation of the line expressed in trilinear coordinates. Then since $x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C$, the antireciprocal to $lx+my+nz=0$ is

$$\frac{l\tan A}{\xi}+\frac{m\tan B}{\eta}+\frac{n\tan C}{\zeta}=0,$$

or $$\eta\zeta l\tan A+\zeta\xi m\tan B+\xi\eta n\tan C=0. \qquad (1)$$

This represents a conic passing through the vertices of the triangle.

2. The antireciprocal of the circumcircle is the axis of homology of the triangle and its orthic triangle.

$\eta\zeta\sin A+\zeta\xi\sin B+\xi\eta\sin C=0$ is the equation of the circumcircle. Its antireciprocal is $x\cos A+y\cos B+z\cos C=0$, and this is the equation of the said axis of homology.

3. The antireciprocal of a line through a vertex consists of another line through that same vertex, and the opposite side of the triangle.

Let $lx+my=0$ be a line through C.

The equation of its antireciprocal is

$$\eta\zeta l\tan A+\zeta\xi m\tan B=0,$$

or $$\zeta=0(\mathrm{AB}),\ \eta l\tan A+\xi m\tan B=0 \text{ (a line through C)}.$$

The vertex C is the antireciprocal of any point in the opposite side AB.

4. The antireciprocal of a tangent to the circumcircle is a conic touching the line $\Sigma x\cos A=0$ at the antireciprocal of the point of contact of the tangent and the circumcircle.

The condition that $\Sigma lx=0$ touch the circumcircle $\Sigma\eta\zeta\sin A=0$ is that $\Sigma\sqrt{l\sin A}=0$, and the condition that $\Sigma x\cos A=0$ touch the conic $\Sigma\eta\zeta l\tan A=0$ is that $\Sigma\sqrt{l\tan A\,.\,\cos A}=0$, the same condition.

If x, y, z are the coordinates of the point in which $\Sigma x\cos A = 0$ touches $\Sigma \eta\zeta l\tan A = 0$,

$$\begin{aligned} x:y:z \quad = \quad & l\tan A(\ \ l\sin A - m\sin B - n\sin C) \\ & : m\tan B(-l\sin A + m\sin B - n\sin C) \\ & : n\tan C\,(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

if ξ, η, ζ are the coordinates of the point in which the line $\Sigma lx = 0$ touches the circumcircle

$$\begin{aligned} \xi:\eta:\zeta \quad = \quad & 1/(\ \ l\sin A - m\sin B - n\sin C) \\ & : 1/(-l\sin A + m\sin B - n\sin C) \\ & : 1/(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

and these two are antireciprocals since $x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C$. The equation of the tangent at C to the conic $\Sigma \eta\zeta l\tan A = 0$ is $\eta l\tan A + \xi m\tan B = 0$; or $\eta l\tan A + \xi m\tan B = 0$ is a tangent to a series of conics $\eta\zeta l\tan A + \zeta\xi m\tan B + \xi\eta n\tan C = 0$, where n has different values. But $\zeta(\eta l\tan A + \xi m\tan B) = 0$ is the antireciprocal of $lx + my = 0$, and the antireciprocal of the conic is $lx + my + nz = 0$. These lines are concurrent in a point in AB, no matter what n may be. Hence, if a number of lines are concurrent in a point in a side of a triangle, their antireciprocals have a common tangent at the opposite vertex, namely that part, passing through the vertex, of the antireciprocal of the line joining the point of concurrence of the lines to the opposite vertex.

Figure 6. p.30.

5. The antireciprocal of the tangent at a vertex to the circumcircle consists partly of the line joining the vertex to the point of concurrence of the opposite side and the line $\Sigma x\cos A = 0$.

The tangent at the vertex C is $x\sin B + y\sin A = 0$. The antireciprocal of this line is $\xi\cos A + \eta\cos B = 0$ and $\zeta = 0$ (CH_3 and BA). Now the lines $\xi\cos A + \eta\cos B = 0$, $\zeta = 0$, $\Sigma x\cos A = 0$ (H_2H_3) are concurrent. The side DE of the orthic triangle ($x\cos A + y\cos B - z\cos C = 0$) also passes through H_3.

6. If three lines passing through the vertices be concurrent, then their antireciprocals must also be concurrent, for they pass through the antireciprocal of the point of concurrence of the first three.

For example, the lines joining the vertices to the opposite excentres pass through the incentre, and the lines joining the vertices to the antireciprocals of the excentres pass through the antireciprocal of the incentre. The lines joining the vertices to the opposite exsymmedian points pass through the insymmedian point, and the lines joining the vertices to the antireciprocals of the exsymmedian points pass through the antireciprocal of the insymmedian point, *i.e.*, the orthocentre. The three antireciprocals of the exsymmedian points can thus be easily found, for if the points H_1, H_2, H_3 be found, the intersection of BH_2 and AD gives L_1, the antireciprocal of K_1, the exsymmedian point opposite A. The triangles $L_1L_2L_3$ and ABC form the antireciprocal of triangle $K_1K_2K_3$. The two triangles ABC, $L_1L_2L_3$ have a common centre of homology O, and a common axis of homology H_2H_3.

7. If O, L_1, L_2, L_3 be the points $(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C})$ found as in Construction II., then

$$\text{the line } OCL_3 \text{ is } \frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0,$$

$$L_1CL_2 \text{ is } \frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0, \text{ etc.}$$

The line $\quad H_2H_3$ is $$\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0.$$

Each of the lines AOL_1 AL_2L_3 is with the side opposite the vertex through which the line passes, its own antireciprocal. The antireciprocal of

$$\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$$

is $$\zeta\eta\sqrt{\tan A} + \xi\zeta\sqrt{\tan B} + \xi\eta\sqrt{\tan C} = 0.$$

The following table shows the connection between the lines.

Triangles.	Centre of homology.	Axis of homology.	Tangents at vertices to antireciprocal of axis of homology.
ABC, $L_1L_2L_3$	O_1, $\sqrt{\tan A}$, $\sqrt{\tan B}$, $\sqrt{\tan C}$	H_2H_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$
			BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$
			AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$
ABC, OL_1L_2	L_3, $\sqrt{\tan A}$, $\sqrt{\tan B}$, $-\sqrt{\tan C}$	DH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$
			BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$
			CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$
ABC, OL_1L_3	L_2, $\sqrt{\tan A}$, $-\sqrt{\tan B}$, $\sqrt{\tan C}$	FH_2, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$
			BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$
			CF, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$
ABC, OL_2L_3	L_1, $-\sqrt{\tan A}$, $\sqrt{\tan B}$, $\sqrt{\tan C}$	EH_1, $-\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$
			BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$
			CF, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$

There are four conics, corresponding to the four points, and each of the six lines passing through the vertices is a tangent to two of the conics. Thus each conic touches the other three conics at different vertices.

The line $px+qy+rz=0$ will touch the conic

$$\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$$

if $$\sqrt{pl\tan A}\pm\sqrt{qm\tan B}\pm\sqrt{rn\tan C}=0.$$

Hence the line $lx+my+nz=0$ will touch its own antireciprocal if $l\sqrt{\tan A}\pm m\sqrt{\tan B}\pm n\sqrt{\tan C}=0$, that is if the line $lx+my+nz=0$ pass through one of the four points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

Since $$x:y:z=\frac{\tan A}{\xi}:\frac{\tan B}{\eta}:\frac{\tan C}{\zeta}$$

$$\therefore\quad x\xi\tan B-y\eta\tan A=0,\ \text{and}\ x\xi\tan C-z\zeta\tan A=0.$$

Hence (x, y, z) is the point of intersection of the polars of (ξ, η, ζ) with respect to two degenerate conics,

$$x^2\tan B-y^2\tan A=0,\ \ x^2\tan C-z^2\tan A=0.$$

Since a line corresponds to a conic, and to a point corresponds the intersection of its polars with respect to two fixed conics, this quadric transformation is a Beltrami one, for a discussion of the difference between which and the Hirst transformation see Mr Charles Tweedie's paper read before the Royal Society of Edinburgh on 15th July 1901.

The conics $x^2\tan B-y^2\tan A=0$, etc., break up into the lines

$$\frac{x}{\sqrt{\tan A}}-\frac{y}{\sqrt{\tan B}}=0,\quad \frac{x}{\sqrt{\tan A}}+\frac{y}{\sqrt{\tan B}}=0,\ \text{etc.},$$

or the lines joining the vertices to the points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

8. The conic $\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$ will be a rectangular hyperbola, if

$$l\sin A+m\sin B+n\sin C=0.$$

If the line $lx+my+nz=0$ pass through the insymmedian point (sinA, sinB, sinC), then the condition for a rectangular hyperbola is fulfilled. Hence the antireciprocals of all lines passing through the insymmedian point are rectangular hyperbolas. This is otherwise seen ; for if the line pass through the insymmedian point, its antireciprocal must pass through the orthocentre, and is therefore a rectangular hyperbola. In particular, the antireciprocal of the line joining the orthocentre and insymmedian point is the rectangular hyperbola passing through the vertices and these two points. Since five points on it are known, it can easily be drawn by Pascal's theorem. (Fig. 7.) page 31.

The coordinates of its centre are

$$\frac{\sin(B-C)}{\cos A}, \quad \frac{\sin(C-A)}{\cos B}, \quad \frac{\sin(A-B)}{\cos C}.$$

This point lies on the nine-point circle.

The equation of this rectangular hyperbola is

$$\eta\zeta\sin 2A\sin(B-C)+\xi\zeta\sin 2B\sin(C-A)+\xi\eta\sin 2C\sin(A-B)=0.$$

The line joining the orthocentre and insymmedian point is

$$x\cos^2 A\sin(B-C)+y\cos^2 B\sin(C-A)+z\cos^2 C\sin(A-B)=0.$$

Fig. 4

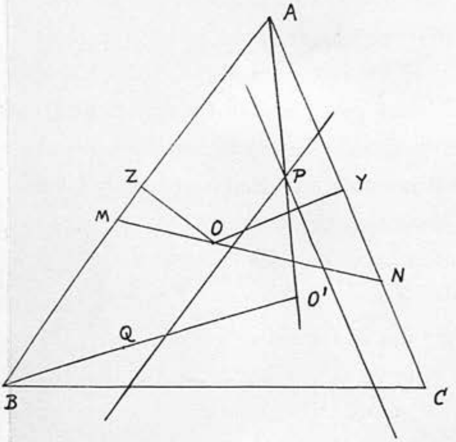


Fig. 5

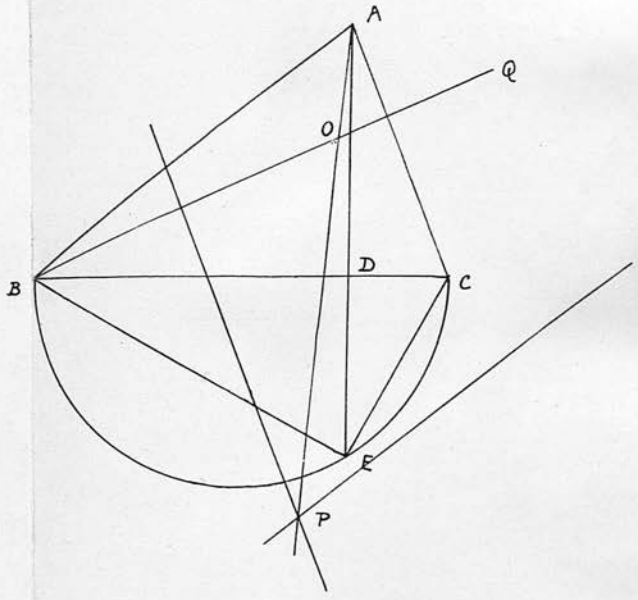


Fig. 6

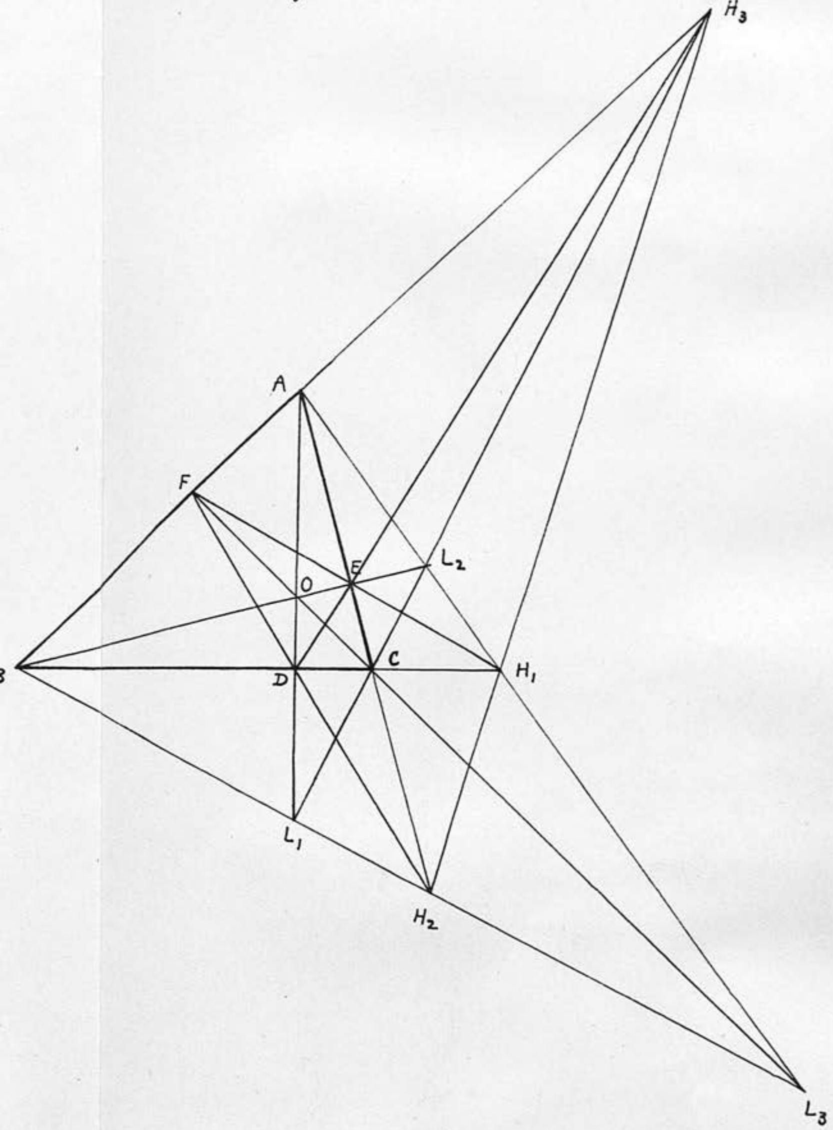
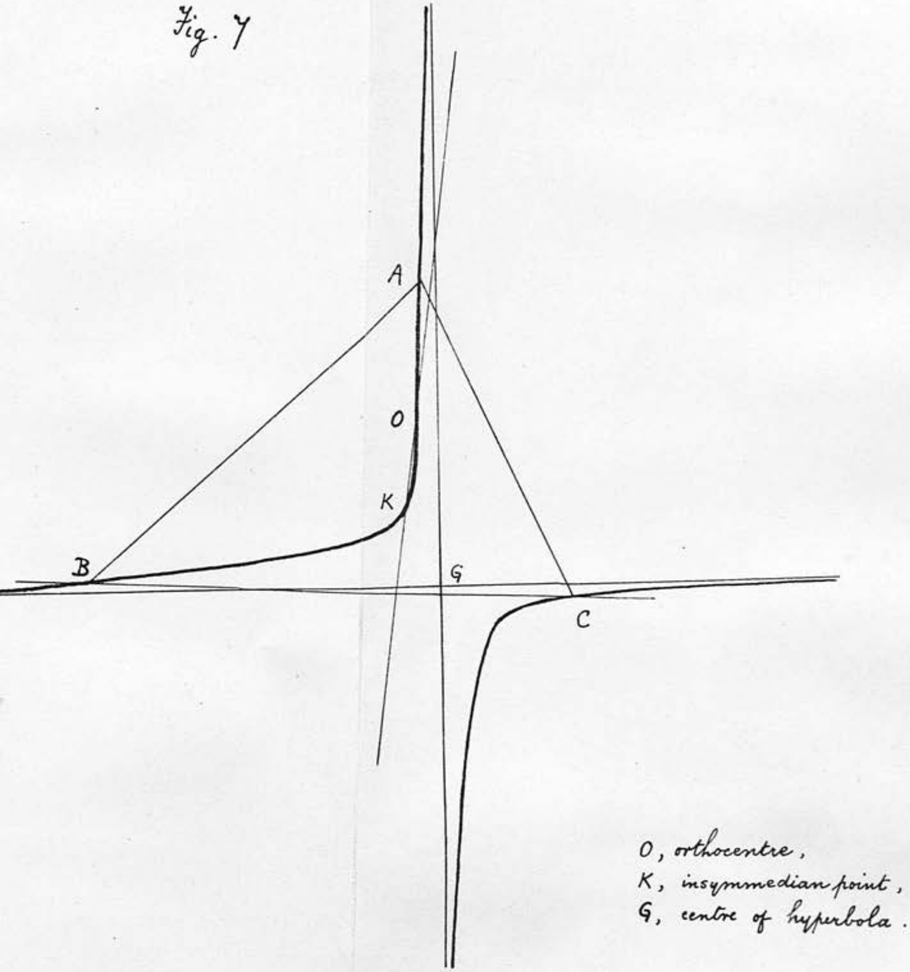


Fig. 7


A
B
C
O
K
G
O, orthocentre,
K, insymmedian point,
G, centre of hyperbola.

Theorems connected with Simson / Wallace / Pedal Line

[Extracted from Vol XXIV. Proceedings Edinburgh Math. Soc.].

Figure 16.

1. (i) If XYZ is the Simson Line $P(ABC)$, and PM is perpendicular to XYZ, and cuts the sides BC, CA, AB in $\bar{U}, \bar{V}, \bar{W}$, then

$$P\bar{U}.\,P\bar{V}.\,P\bar{W} = PA.PB.PC = 4R^2.PM.$$

$$\frac{PM}{PX} = \sin PXM = \sin PBA = \frac{PA}{2R}. \text{ and } \frac{PX}{PB} = \sin PBC = \frac{PC}{2R}.$$

$$\therefore\ 4R^2.PM = PA.PB.PC.$$

$$\therefore\ PM = 2R.\frac{PA}{2R}.\frac{PB}{2R}.\frac{PC}{2R} = 2R\sin PCA.\sin PAB.\sin PBC.$$

$$\therefore\ \frac{PM}{PX}.\frac{PM}{PY}.\frac{PM}{PZ} = \frac{PA}{2R}.\frac{PB}{2R}.\frac{PC}{2R} = \frac{4R^2.PM}{8R^3}$$

$$\therefore\ 2R.PM^2 = PX.PY.PZ$$

$\because\ \angle PX\bar{U}$ and $\angle PMX$ are right angles, $PX^2 = PM.P\bar{U}.$

$$\therefore\ PX^2.PY^2.PZ^2 = PM^3.P\bar{U}.P\bar{V}.P\bar{W}$$

$$\therefore\ P\bar{U}.P\bar{V}.P\bar{W} = 4R^2.PM. = PA.PB.PC.$$

(ii) If PX_1, PY_1, PZ_1; PX_2, PY_2, PZ_2 are two sets of three straight lines which make angle α with the sides of triangle ABC, then $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are straight lines. Let them intersect in Q.

The following sets of points are concyclic:— $PBXZ$, $PXCY$, $PZAY$, PX_1CY_1, PX_1BZ_1, PZ_1Y_1A, PX_2CY_2, PBX_2Z_2, PAZ_2Y_2.

Now $\angle PX_2Z_2 = \angle PBZ_2 = \angle PX_1Z_1$

$\therefore\ P, X_1, X_2, Q$ are concyclic and similarly so are P, Z_1, Q, Z_2 and P, Y_1, Q, Y_2

$$\therefore\ \angle PQZ_1 = \angle PX_2B = \alpha = \angle PX_1B = \angle PQZ_2$$

and $\angle BZX = \angle BPX = \angle X_1PX - \angle X_1PB = 90^\circ - \alpha - \angle X_1Z_1B =$

$\angle ZPZ_2 - \angle Z_2Z_1Q = \angle ZPZ_2 - \angle Z_2PQ = \angle ZPQ.$

PZ is perpendicular to AB $\therefore$ PQ is perpendicular to AB.

$\therefore$ Q lies on PM.

Again $\angle MPZ = 90^\circ - \angle PBC$. and $\angle PZ_1Q = 180^\circ - \angle PAY_1 = \angle PBC$

$$\therefore \frac{PM}{PZ} = \cos MPZ = \sin PBC$$

$$\frac{PZ}{PZ_1} = \sin\alpha$$

$$\frac{PZ_1}{PQ} = \frac{\sin PQZ_1}{\sin PZ_1Q} = \frac{\sin\alpha}{\sin PBC}$$

$$\therefore \frac{PM}{PQ} = \sin^2\alpha \qquad \text{or } PQ = PM.\operatorname{cosec}^2\alpha$$

$$\therefore MQ = PM.\cot^2\alpha$$

Let XYZ meet $X_1Y_1Z_1$ in L_1 and $X_2Y_2Z_2$ in L_2

then $\frac{ML_1}{MQ} = \tan\alpha$ $\therefore ML_1 = PM.\cot\alpha$

$\therefore \angle PL_1M = \alpha$ $\quad \therefore \angle PL_1Q = 90^\circ.$

$\therefore$ since P, Q, X_1, X_2 are concyclic, the line $P(ABC)$ is the line $P(QX_1X_2)$ and hence also $P(QY_1Y_2)$ and $P(QZ_1Z_2)$

(iii) When $\alpha = 45^\circ$

M is the mid-point of PQ; and if O is the orthocentre of ABC, OQ is therefore parallel to XYZ (since XYZ bisects OP).

The locus of Q, as P moves on the circle, is given by the equation

$$\rho = 2R\left[\cos A\cos\theta - \sin\theta\sin\{2\theta-(B-C)\}\right]$$

with reference to O as pole and OA as initial line.

The curve has three loops of different sizes and can easily be traced from the fact that OQ is at right angles to PQ.

If, in particular, ABC is equilateral (so that O is the circumcentre), the locus of Q is given by $\rho = R\cos 3\theta$, a hypotrochoid with three loops each of which is in area one-twelfth of the circle.

Fig. 17.

2. (i) If PX_1, PY_1, PZ_1; PX_2, PY_2, PZ_2 make angle α_1 with the sides of ABC; and PX_3, PY_3, PZ_3; PX_4, PY_4, PZ_4 - - - - - - - α_2 - - - - - - - - ; if Q_1, Q_2, are the two corresponding positions of Q and if the four lines X, Y, Z, etc., intersect one another besides in T_1, T_2, T_3, T_4 and intersect the Simson Line $P(ABC)$ in L_1, L_2, L_3, L_4:

then $\angle PX_4Q_2 = \angle PBA = \angle PX_2Q_1$ $\therefore P, X_4, X_2, T_2$ are concyclic.

$\therefore \angle PT_2Q_2 = \angle PQ_1Q_2$ $\therefore PT_2$ is a tangent to the circumcircle of $\triangle T_2Q_2Q_1$

$$\therefore PT_2^2 = PQ_2 . PQ_1 = PM^2 . \operatorname{cosec}^2\alpha_1 . \operatorname{cosec}^2\alpha_2$$

$$\therefore PT_2 = PM . \operatorname{cosec}\alpha_1 . \operatorname{cosec}\alpha_2$$

Similarly PT_1, PT_3, PT_4 equal the same quantity

$\therefore T_1, T_2, T_3, T_4$ lie on a circle which has P as centre and cuts orthogonally the circles $T_1Q_1Q_2$, $T_2Q_1Q_2$, $T_3Q_1Q_2$, $T_4Q_1Q_2$.

(ii) PL_2 is perpendicular to X_2Q_1

$\therefore L_2$ is the middle point of $T_2 T_4$.

Similarly L_1, L_3, L_4 are the middle points of $T_1 T_3$, $T_1 T_4$, $T_2 T_3$

$\therefore T_3 T_4$ and $T_1 T_2$ are parallel to XYZ and equidistant from it. The distance equals $PM.\cot\alpha_1.\cot\alpha_2$

(iii) $\angle T_4 P Q_2 = \angle T_4 Q_2 Q_1 - \angle P T_4 Q_2 = \alpha_2 - \alpha_1 = \angle Q_2 T_2 T_4$

$\therefore P, Q_2, T_4, T_2$ are concyclic

Similarly P, Q_2, T_3, T_1; P, T_4, Q_1, T_1 : P, T_3, Q_1, T_2 are concyclic.

The Simson Line $P(ABC)$ is also the following 34 Simson Lines:

$P(AY_1Z_1)$, etc ; $P(BZ_1X_1)$, etc : $P(CX_1Y_1)$ etc :

$P(Q_1 X_1 X_2)$, $P(Q_2 X_3 X_4)$, etc: etc. $P(Q_1 T_1 T_4)$, $P(Q_1 T_2 T_3)$,

$P(Q_2 T_1 T_3)$, $P(Q_2 T_2 T_4)$; $P(T_1 X_1 X_3)$, $P(T_2 X_2 X_4)$,

$P(T_3 X_1 X_4)$, $P(T_4 X_2 X_3)$, etc., etc.

3. If A_1, A_2, A_3, A_4 be four concyclic points; $O_1, \text{----}, O_4$ the orthocentres of the four triangles formed by them:

the quadrilaterals $A_1 A_2 A_3 A_4$, $O_1 O_2 O_3 O_4$ are equal in all respects, and if C, F be their circumcentres, the four Simson Lines of A_1, A_2, A_3, A_4 are concurrent at the mid-point P of CF, which is also the mid-point of $A_1 O_1, \text{----}, A_4 O_4$.

Fig 18.

If $A_1, \text{------}, A_5$ are points on a circle whose centre is C; $O_1, \text{----}, O_5$ the orthocentres of the five triangles formed by sets of three consecutive vertices of the pentagon $A_1 A_2 A_3 A_4 A_5$; $Q_1, \text{-----}, Q_5$ the orthocentres of the five triangles formed each by one side and the opposite vertex; if $B_1, \text{-----}, B_5$, are the mid-points of the sides, and $G_1, \text{-----} G_5$ are the mid-points of the diagonals.; if $F_1, \text{-----}; F_5$ are the circumcentres of the five cyclic quadrilaterals (see last page) formed by orthocentres of triangles whose vertices are chosen from $A_1, \text{-----}, A_5$; and $P_1, \text{---}, P_5$ are the mid-points of $CF_1, \text{-------}, CF_5$: it is clear that the pentagon $P_1 P_2 P_3 P_4 P_5$ has its sides parallel and half the length of the sides of the original pentagon; and if D is the circumcentre of this cyclic pentagon, and S the point of trisection such that $CS = 2SD$, the five straight lines $A_1P_1, \text{-----}, A_5P_5$ are concurrent at S which is a point of trisection of each, as also of the other ten lines $O_1B_1, \text{-----}, O_5B_5$, $Q_1G_1, \text{-----} Q_5G_5$; S being the centre of homology of the two similar and similarly-situated pentagons.

Again the pentagon $F_1 F_2 F_3 F_4 F_5$ is equal in all respects to $A_1 A_2 A_3 A_4 A_5$ and similarly situated to it; and if E be its circumcentre, D bisects CE and is the centre of homology of $A_1 A_2 A_3 A_4 A_5$, $F_1 F_2 F_3 F_4 F_5$.

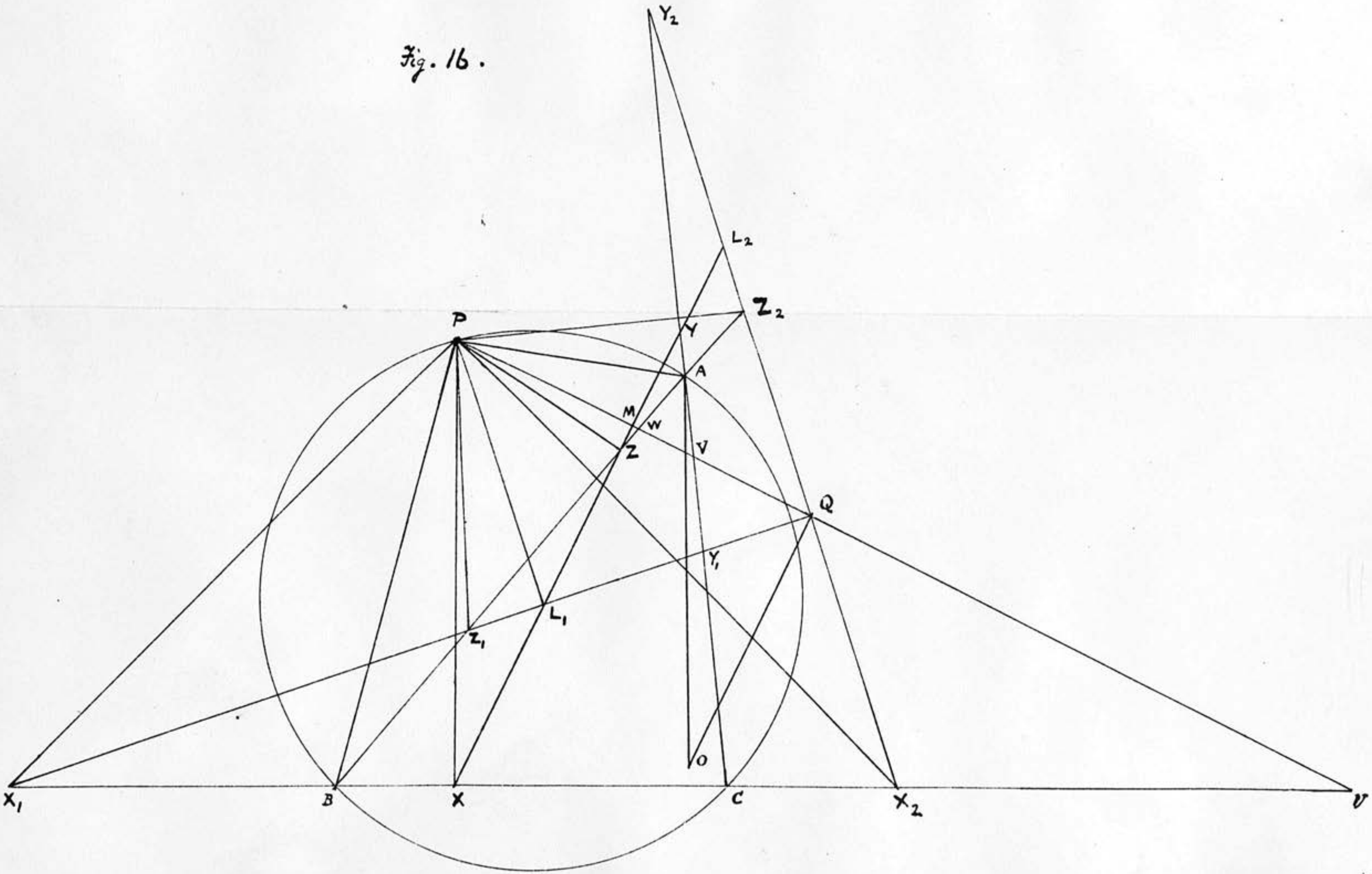
The theory can be extended according to the following table: —

Number of given points $A_1, A_2, \ldots$	Specification of polygon.	Ratio of linear dimensions to original polygon.	Circumcentre denoted by	Centre of homology with original polygon.
4	$O_1O_2O_3O_4$	1	F	P
5	$F_1F_2F_3F_4F_5$	1	E	D
	$P_1P_2P_3P_4P_5$	$\frac{1}{2}$	D	S
6	$E_1E_2E_3E_4E_5E_6$	1	H	K
	$D_1D_2D_3D_4D_5D_6$	$\frac{1}{2}$	K	L
	$S_1S_2S_3S_4S_5S_6$	$\frac{1}{3}$	L	M
7	$H_1H_2H_3H_4H_5H_6H_7$	1	etc.	etc.
	$K_1 \ldots\ldots\ldots\ldots K_7$	$\frac{1}{2}$		
	$L_1 \ldots\ldots\ldots\ldots L_7$	$\frac{1}{3}$		
	$M_1 \ldots\ldots\ldots\ldots M_7$	$\frac{1}{4}$		
		etc.		

This last paragraph is the one referred to in Professor Coolidge's "Treatise on the Circle and the Sphere" p94. Theorem 166.

"This is contained implicitly in an elaborate theorem due to Burgess, 'Theorems connected with Simson's Line', Proceedings Edinburgh Math. Soc., Vol XXIV, 1906, p126."

Fig. 16.



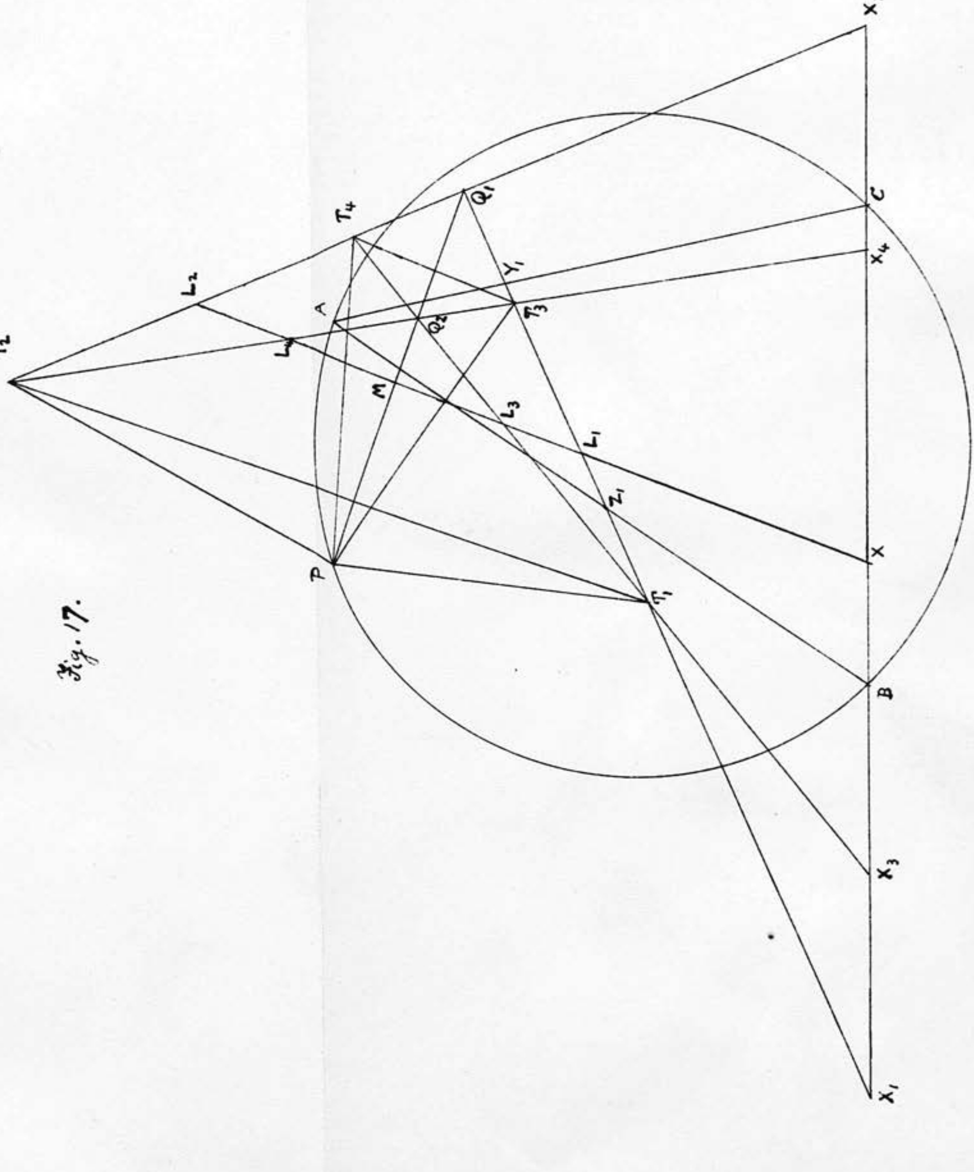


Fig. 17.

Fig. 18.