Mechanising Euler's use of Infinitesimals in the Proof of the Basel Problem

Imogen I. Morris

Doctor of Philosophy

Artificial Intelligence and its Applications Institute
School of Informatics
University of Edinburgh

2023

Abstract

In 1736 Euler published a proof of an astounding relation between π and the reciprocals of the squares.

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} \dots$$

Until this point, π had not been part of any mathematical relation outside of geometry. This relation would have had an almost supernatural significance to the mathematicians of the time. But even more amazing is Euler's proof. He factorises a transcendental function as if it were a polynomial of infinite degree. He discards infinitely-many infinitely-small numbers. He substitutes 1 for the ratio of two distinct infinite numbers.

Nowadays Euler's proof is held up as an example of both genius intuition and flagrantly unrigorous method. In this thesis we describe how, with the aid of nonstandard analysis, which gives a consistent formal theory of infinitely-small and large numbers, and the proof assistant Isabelle, we construct a partial formal proof of the Basel problem which follows the method of Euler's proof from his 'Introductio in Analysin Infinitorum'. We use our proof to demonstrate that Euler was systematic in his use of infinitely-large and infinitely-small numbers and did not make unjustified leaps of intuition. The concept of 'hidden lemmas' was developed by McKinzie and Tuckey based on Lakatos and Laugwitz to represent general principles which Euler's proof followed. We develop a theory of infinite 'hyperpolynomials' in Isabelle in order to formalise these hidden lemmas and we find that formal reconstruction of his proof using hidden lemmas is an effective way to discover the nuances in Euler's reasoning and demystify the controversial points. In conclusion, we find that Euler's reasoning was consistent and insightful, and yet has some distinct methodology to modern deductive proof.

Acknowledgements

I would like to thank my supervisor Jacques Fleuriot who introduced me to nonstandard analysis, interactive theorem proving and Euler's mathematics, and has been an inspiration to me in exploring this wonderful topic. I thank also the friends and fellow students who have taken the time to read parts of my draft and give me feedback. In particular, I thank Robin Carter for checking and helping to correct my pen-and-paper proofs.

I would also like to thank my parents, my brother Edmund and especially my best friend and sister-in-law Xue. I would not have had the strength and motivation to write this without your support.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Imogen I. Morris*)

Contents

Chapter 1

Introduction

In 1644 Mengoli proposes the following problem: what is the sum of the reciprocals of the squares? This series converges exceptionally slowly: to find the result to six decimal places would require summing 1000 terms. Almost a century later, after many mathematicians, including Jacob, Johann and Daniel Bernoulli, Leibniz, Stirling, and de Moivre, had tried and failed, Euler proves that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=\frac{\pi^2}{6}$$

His proof is met with simultaneous admiration and skepticism from his contemporaries [33, p.1-4]. In this proof, at a time when the Fundamental Theorem of Algebra is not yet proven, he derives the sum by factorising the sine function as though it were an infinite polynomial. Euler revisits his proof four times [70, p.5], finally writing it out in a textbook for the analysis of the infinite [29]. This last proof induces similar feelings of wonderment and incredulity in the modern reader. Euler seems at once careless and masterful: discarding infinitely-many infinitesimals, equating infinitesimally-different polynomials, and extending theorems beyond the finite to the infinite.

Pen-and-paper reconstructions of Euler's arguments using nonstandard analysis seem to demonstrate that Euler's reasoning could be made consistent and that Euler appreciated the subtleties of infinitesimals [57, 55, 76, 5]. A formalisation of Euler's arguments in the proof-assistant Isabelle can take this further since it would ensure the concepts involved are perfectly compatible and precisely expressed. It would also highlight any intuitive leaps or misunderstandings that humans might not notice. For this research, we have chosen to formalise Euler's proof of the Basel problem, as he stated it in his 'Introductio in Analysin Infinitorum'(henceforth the 'Introductio') [29].

1.1 Motivation

In the 18th century, mathematicians were exploring the infinite: finding the sums of infinite series, evaluating infinite products, taking the ratios of infinitely-small quantities and expressing quantities as infinitely continued fractions. Not every mathematician was comfortable using or manipulating infinitely-large and infinitely-small numbers. They could easily achieve absurd results. Euler seemed to have a genius intuition for working with these quantities. In the 'Introductio' he tried to explain their use to any student of mathematics. He felt, although subtle, working with these numbers could be done using 'ordinary algebra' [29, p. v]. However, although this textbook is still lauded as one of the best and most illuminating textbooks, even to learn from today [10], he did not manage to convince the world that infinite and infinitesimal numbers should have a place in mathematics. When limits were invented as a tool, mathematicians gradually considered these as safer and more rigorous and they replaced the more intuitive infinitesimals. But even Cauchy, popularly regarded as rejecting this old method, still relied on infinitely-small numbers in his arguments [53, p. 202].

In the 1960s Abraham Robinson created nonstandard analysis which gave a formal basis to infinitely-large and infinitely-small numbers as an extension of the real numbers [15, p. 78]. There is reasonable doubt that the hyperreals are exactly like the infinitesimals and infinities that Euler used in an ontological sense [34, 5]. Nonetheless, we believe his arguments can faithfully be reproduced with the use of hyperreals. They can also be rehabilitated in terms of standard analysis e.g. limits but these do not follow the form of the argument so closely. There are several pen-and-paper reconstructions of Euler's arguments using nonstandard analysis [76, 57, 55]. These reconstructions argue that in his apparent leaps of intuition he was obeying general principles and these principles are called 'hidden lemmas' by McKinzie and Tuckey [55] and Laugwitz [52].

1.2 The Basel Problem

The Basel Problem was first thought of by Mengoli in 1644 [7]. He wanted to know the sum of the reciprocals of the squares:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\cdots.$$

He did not know the answer, not even a good estimate of it, since the series converges extremely slowly. It was not until 1665 that Wallis was able to compute it to three decimal places. Jakob Bernoulli, Leibniz and other leading mathematicians of the time were unable to solve it [57]. Euler heard about it from his mentor Johann Bernoulli (Jakob Bernoulli's brother) and was the first to discover that the sum of the series was:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\cdots=\frac{\pi^2}{6}.$$

This was all the more amazing to the 18th century mathematicians because π, the number that describes a circle, was turning up in a context which had no apparent connections to circles or geometry. Euler was able to calculate the sum to 17 decimal places before he managed to prove it, so he may have guessed the true sum even before he produced his first proof. [70, p. 5]

Euler's first published 'proof' [31] was doubted by other mathematicians of the time. However, it certainly provided justification for the result, and Euler had shown that the method could be used to derive other known results, so it was reasonable to assume it had some meaning. The proof involved the factorisation of the sine function by its roots, just as we factorise polynomials. Bernoulli asked how it was known that all the roots had been found. [26, §4 - §6]. Whether or not it was justified to treat the sine function as an infinite polynomial, this proof had a gap, which Euler knew about and tried to fill. He found a way to complete the proof, and this also led to his discovery of a shorter demonstration which is he presented in his 'Introductio' [29].

Euler later found solutions to more generalisations of the Basel problem, and a further proof of the original problem using l'Hôpital's rule [70]. However, he never managed to generalise it to odd numbers and this remains unsolved to this day. In fact, one generalisation of the Basel problem is the Zeta function [4, p. 1067], which is the fundamental concept behind the Riemann Hypothesis [9].

1.3 Aims of this thesis

This thesis includes a partial formalisation in Isabelle of Euler's proof of the Basel problem from the 'Introductio'. However, the formalisation is the means by which we accomplish the aims of thesis, rather than the aim itself. Our aims are as follows

1. To improve our understanding of the concepts used by Euler in the 'Introductio', in particular, 'infinitesimal', 'infinite number' and 'polynomial', by using the analogous concepts in nonstandard analysis and to follow Euler's reasoning using formal proof in Isabelle

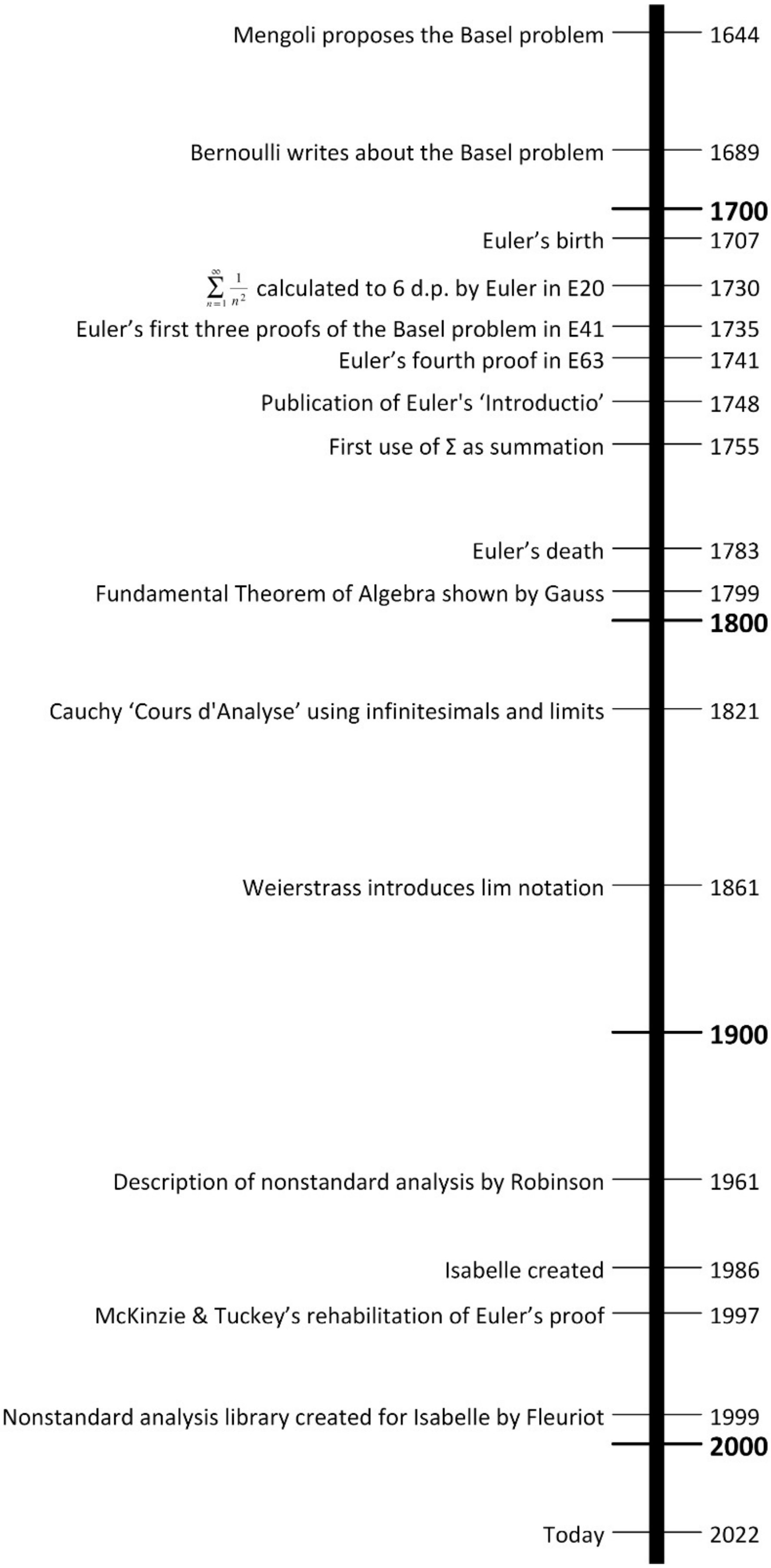

Figure 1.1: Timeline of Euler's proof of the Basel problem: Euler's papers are referred to by their Eneström numbers E20 [25], E41 [27] and E63 [28].

2. To examine the validity of the logic of Euler's reasoning and the use of hidden lemmas (see Section 4.4.3) to explain the 'leaps of intuition' in Euler's proof

3. To provide insight into the reasoning of Euler's proof of the Basel problem from the 'Introductio' and its relationship to the rehabilitation given by McKinzie and Tuckey [57].

Some parts of the thesis are a reconstruction of Euler's work directly, some parts are reconstructions of Euler's reasoning through McKinzie and Tuckey's interpretation, and other parts are reconstructions of McKinzie and Tuckey's reasoning that does not directly correspond to parts of Euler's proof.

To approach McKinzie and Tuckey's reasoning which does not directly correspond to Euler, we have used a combination of formal proof for discovery of gaps or show validity and we have often used pen-and-paper proofs to plug the gaps we have discovered. It would be ideal to have plugged the gaps also using formal proof, however, we have used formal proof as a tool to break down the proof layers and force us to examine the details the proof and how the structure of the proof fits together. Once this additional level of detail is achieved, there are diminishing returns from breaking the proof down a further level of detail. The proofs are less mathematically-complex at the lower level: they do not involve as great leaps of reasoning and to formalise them would raise fewer points of interest, although it could still be long-winded. In this thesis we have used formal proof as a tool for understanding and analysing proofs rather than considering it to be a necessary replacement for mathematical proof.

As for our approach to Euler's reasoning, we have focused on the parts which are considered to be controversial. We have not made great changes to McKinzie and Tuckey's interpretation of Euler's reasoning as we have found it well-thought out and, by comparison with Euler's original text, generally faithful. However, we have reanalysed their concept of determinacy/Euler-convergence since this is the greatest liberty and arguably the greatest discovery they have made in their interpretation: they claim that it transforms Euler's reasoning into 'a model of deductive reasoning'. We would not entirely agree with this statement, however, we do find that it allows us to formalise Euler's reasoning with minimal deviations.

1.3.1 Categories of unformalised reasoning

There are many parts of Euler's reasoning which have not been formalised by us. These fall into the following categories:

1. They do not correspond to modern deductive reasoning
2. They are uncontroversial, but long-winded in Isabelle.
3. They are not relevant to the main body of the proof of the Basel Problem (and the truth of the statement is not under question)

Parts of McKinzie and Tuckey's reasoning also fit into the second two categories. In Section 6.1 we will describe which parts of Euler's proof have been formalised and how those which have not been fit into these categories. In Figure 6.1 we show the structure and scope of the formalised proof. Accompanying this we give Table 6.1 which summarises the categories the unformalised reasoning falls into along with Table 6.2 which summarises the formalised reasoning with the reason we chose them as candidates for formalisation.

Now we justify the main choices made to accomplish our research aims: why we focus on the proof of the Basel Problem from the 'Introductio', why we use non-standard analysis to represent Euler's reasoning, and why we use Isabelle as our proof assistant.

1.4 Versions of Euler's proof of the Basel problem

There is no single account of Euler's proof of the Basel problem, not even in terms of the basic structure. There is even some dispute as to how many proofs of the Basel problem Euler published. This can probably be explained by the fact that the proofs are not entirely distinct from each other and some have more unique features than others. As related by Sandifer [70], Euler published four distinct proofs: three in 1735 [27] and a fourth in 1741 [28]. However, as Sandifer acknowledges, Euler published versions of his proofs of the Basel problem in other papers and books. There is a paper published by Euler in 1743 entitled 'Another dissertation on the sums of the series of reciprocals arising from the powers of the natural numbers, in which the same summations are derived from a completely different source' [26] (title translated from the Latin by Aycock [33]) and from the title alone we could deduce this is yet another proof of the Basel problem. There is, of course, also a version of the proof given in the 'Introductio' [29].

Euler's proofs of the Basel problem were not distinct proofs, rather, they were different iterations of, or variations on, the same proof. The 1741 proof is nearly identical to that given in the 'Introductio': the main difference between them is that as the 'Introductio' was intended as a textbook, his presentation is more detailed. Even within the 'Introductio', Euler's proof is spread across several chapters and is intertwined with related work on power series, logarithms and polynomial factors. So it is a subjective task to separate out parts of the 'Introductio' and call them together 'the proof of the Basel Problem'. Several of the papers which we draw on for their analysis of this proof are in fact analysing only Euler's factorisation of the sine function. However, the factorisation overlaps so thoroughly with the proof of the Basel problem, that the two derivations are simply different perspectives on the same proof.

A key decision in our work was to select a version of the proof for mechanisation. The proof as related in the 'Introductio' has two significant advantages over the others. First, it is the easiest to follow and most rigorous account of all his proofs since Euler wrote the 'Introductio' as a textbook for beginners. Second, McKinzie and Tuckey give a thorough pen-and-paper rehabilitation of this proof of the Basel problem using nonstandard analysis. This makes it the most feasible choice [57] and we base our formalisation of the proof on their rehabilitation.

1.5 Theories of infinitesimals: nonstandard analysis

It is necessary to find a suitable theory of analysis and justify that it captures Euler's reasoning. Some mathematicians do not find that infinitesimals represent accurately how Euler reasoned [35]. Other methods have been used to understand his work, for example, protolimits [35, p. 17]. We do believe Euler explicitly uses infinitesimals and discuss his conception of them in Section 2.3.1.2. Even if we accept that infinitesimals best describe Euler's reasoning, we must deduce which properties his infinitesimals obeyed. It is possible that Euler's theory was inconsistent, if every step is interpreted literally, but we should look for the closest consistent theory that we are aware of. The two most famous theories of infinitesimals are synthetic differential geometry [6] and nonstandard analysis. The latter was Robinson's attempt at giving the infinitesimal analysis of the 18th century a rigorous and consistent basis [40, 43]. Both have been used to understand early analysis[1]. However we know that Euler does not use nilpotent infinitesimals in the Introductio since his infinitesimals were invertible, which therefore rules out synthetic differential geometry. Besides this, there are several rehabilitations of Euler's arguments in terms of nonstandard analysis. Nonstandard analysis

[1] See Bell's textbook [6] for a description of nilpotent infinitesimals and their use in early analysis.

also has a well-developed library in Isabelle on which to base this work.

1.5.1 Different approaches to nonstandard analysis

Nonstandard analysis is concerned with the *hyperreals* which are an extension of the reals to include infinitesimals and infinitely-large numbers. Different approaches can be taken to nonstandard analysis. Robinson's original construction of the hyperreals (the closure of the reals with the infinitesimal and infinite numbers) used the Compactness theorem to form a model of the hyperreals [40]. Alternatively, the hyperreals can be defined as certain equivalence classes of sequences of reals numbers via the ultrapower construction [40], or, now that we know the hyperreals are consistent, we could take an axiomatic approach [43]. However, all these approaches agree on the properties of the hyperreals, and as Bair et al. argue [5], for understanding Euler's work, it is the methods of proof that we are interested in rather than the ontology of the concepts. Thus we can consider them in terms of their properties since we wish to observe whether they behave analogously to Euler's infinitely-large and infinitely-small numbers. We describe the relevant parts of nonstandard analysis and compare them in detail to Euler's concepts in Section 2.3.

1.6 Interactive theorem provers: Isabelle

There are several suitable interactive theorem provers within which a proof of the Basel problem could be formalised. Notable among them are Coq [8], Lean [60] and Isabelle [65]. Although each have advantages, Isabelle has a well-developed nonstandard analysis library [45] which meant that there would be less work required to develop the mathematical framework for our proof. Nonetheless we found that there were many additional theorems in nonstandard analysis that we needed to prove to progress our proof of the Basel problem (see Sections 3.1- 3.4). Isabelle also has a language for structured proof called Isar [62] (see Section 2.2.1), which was useful for representing Euler's reasoning more faithfully. Worth mentioning is another formalisation of nonstandard analysis by Ruben Gamboa in the automated theorem prover ACL2 [38]. ACL2 has a first-order logic with only limited support for quantification therefore it is not suitable for our purposes.

1.6.1 Current nonstandard analysis library in Isabelle

The current nonstandard analysis library in Isabelle uses the ultrapower construction of the hyperreals (see Section 2.3.1.3) [45]. Major results that have been formalised include the Transfer Principle (see Section 1.5), which has been formalised as a method; properties of hypernatural numbers: these are the ordinary natural numbers together with some infinite numbers that share similar attributes including

- an induction principle
- hypercomplex numbers, which are an extension of the hyperreals using the square root of minus 1 in exactly the way that the complex numbers are an extension of the real numbers
- limits
- continuity
- hyperreal powers, including infinite and infinitesimal powers.

1.7 Previous work

Here we motivate the use of McKinzie and Tuckey's and to a lesser extent other mathematicians' work to guide our proof-plan. We also describe prior formalisations of historical mathematics, Newton's Principia and Hilbert's Geometry, and we compare these endeavours to this project. Finally, we discuss different rigorous theories of infinitesimals.

1.7.1 Previous formalisations of historical mathematics

There have been other formalisations which have attempted to reconstruct the mathematics of the past in proof assistants. The most directly similar is a mechanisation of Newton's 'Principia Mathematica' in Isabelle by Fleuriot and Paulson [36, 37]. This also uses nonstandard analysis to represent infinitesimals, along with geometrical reasoning, and Newton was nearly a contemporary of Euler: he died when Euler was 19. Much of the nonstandard analysis library in Isabelle was originally developed for that project. Newton discusses 'ultimate' situations and 'evanescent' quantities which Fleuriot and Paulson choose to represent using the infinitesimals from nonstandard analysis. They are able to reconstruct Newton's proofs as formal proofs, however, many points had to be decided by their interpretation, for example, to represent time geometrically as distance or area [37, p. 2]. Euler has more clarity in his prose and his proof of the Basel problem from the 'Introductio' has no directly geometrical reasoning. Therefore less interpretation is required, although this also means that we might wish to avoid using geometry as a tool when it comes to filling any potential gaps in his argument.

Meikle [58] and Scott [72, 74, 73], along with Fleuriot, consecutively worked on formalising parts of Hilbert's 'Grundlagen der Geometrie' in Isabelle. Hilbert's 'Grundlagen' gives a set of axioms for geometry along with theorems derived from them. Hilbert was a driving force behind the formalisation of mathematics and advocated for developing firm foundations from a finite, complete set of axioms [80, §1.1]. The aims of his programme, although doomed by Gödel's Incompleteness Theorems [80, §1.4], fit themselves well to formalisation in a proof assistant, in contradistinction with the axiomless mathematics of Euler's time. Yet, Scott and Fleuriot discovered that the Polygonal Jordan Curve Theorem, whose proof was claimed by Hilbert to be 'without serious difficulty' [42, p. 6], was extremely involved to prove and required the help of automated proof search [72]. Meikle and Fleuriot conducted careful formalisation of Hilbert's axioms and some of his theorems, and found that Hilbert still relied on intuition and diagrams for some points of his reasoning [58, p. 334]. Scott and Fleuriot found some subtle errors in Meikle and Fleuriot's formalised axioms [71]. The difficulties in formalising mathematics which seems designed for the purpose illustrates how much interpretation and precision is required in reconstructing historical, and even modern, mathematics.

1.7.2 Previous formalisations of the Basel problem

Several formalisations of the Basel problem have previously been carried out, however all of them use standard analysis. In Isabelle, the proof of the Basel problem has been carried out by Eberl [19]. The origin of the proof being formalised is unclear to us. It uses the language of modern standard analysis, and it uses concepts such as continuity which were not fully formulated in Euler's time. The proof uses the Weierstrass product form of sine. This is the lemma used in Euler's 1735 proof, but made rigorous, so it is likely that Eberl's proof relates to this proof by Euler although it does not directly match the reasoning of any of Euler's proofs. In mathematical notation, Eberl proves

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.$$

A proof of the Basel problem has also been formalised in Coq by Madiot [54]. It is unclear which proof is being used in this formalisation: however one step in the proof is to link the even terms of the $1/n^2$ series to the series itself. This suggests the proof is similar to one by Eberlein [20] in which at one point he simplifies by only considering *odd* terms. Another proof of the Basel problem was formalised by Pąk and Korniłowicz in Mizar [64]. They state that their proof is elementary, however it does use many concepts that Euler and his contemporaries would have been unaware of including general algebraic structures such as commutative rings, loop structures, integral domains and a sieve functor. Nor does it appear to correspond in form to any of the 18th century proofs. They find the value of the sum by bounding the series to m above and below and using the Squeeze Theorem. A proof given by Cauchy in 1821 has been formalised by Carneiro in Metamath [12]. Due to Cauchy's reputation for rigour and as one of the creators of modern analysis, some may consider Cauchy's proof the first rigorous proof of the Basel problem. Finally, a formalisation is given in HOL-Light by John Harrison based on Taylor Series [41] and in Lean by Masdeu based on a proof by Daners [13].

1.7.3 Use of nonstandard analysis in formalisation

Nonstandard analysis has been used by Fleuriot to formalise some of Newton's reasoning from his 'Principia' in the proof assistant Isabelle [36]. The reasoning in the 'Principia' is geometric and includes some infinitely-small geometric quantities. Newton was an older contemporary of Euler, and Euler of course used the calculus that Newton and Leibniz had invented, although he avoided using it in the 'Introductio'. That Newton's reasoning can be faithfully represented by a formalisation using nonstandard analysis provides support that Euler's reasoning could be.

1.7.4 Reconstructions of Euler's work in nonstandard analysis

There are many commentaries on Euler's work, especially the Basel problem, which interpret his mathematics using standard analysis. See Varadarajan [77] for example who uses modern notation and concepts, e.g. limits, and ascribes the arguments to Euler without even noting the differences or Eberlein's [20] derivation of Euler's infinite product for sine. Eberlein acknowledges that although his proof follows Euler's basic idea, it uses standard 19th century analysis and he therefore calls it a new proof. However commentaries on Euler's work using nonstandard analysis are by their nature more likely to analyse Euler's use of infinitesimals. Even so, we should not assume that Euler's infinitesimals are identical with those described by nonstandard analysis. Bair et al. [5] argue that even if their properties are identical, they are constructed with a different ontological basis. As we also noted in Section 1.5, there are other theories of infinitesimals, e.g. synthetic differential geometry which uses nilpotent infinitesimals. However nonstandard analysis is a good fit with many concepts that parallel Euler's own. We compare Euler's concepts to nonstandard analysis throughout the thesis, see for example Section 2.3.1.2. The closest prior rehabilitations of Euler's proof of the Basel problem have been done using nonstandard analysis. McKinzie and Tuckey [57] and Kanovei's [76] reconstructions of Euler's proof of the Basel problem and infinite product for sine closely follow Euler's arguments and concentrate on filling in gaps that they identify in his arguments. For this reason, they are a good basis for formalising Euler's proofs faithfully. McKinzie and Tuckey have also reconstructed some of Euler's other work by using nonstandard analysis [55, 57]. We will give a brief description here of their reconstruction of some of the results from the 'Introductio' apart from the Basel problem (as we reference their reconstruction of the Basel problem throughout our thesis, we do not describe that here). Their methods for reconstructing the other results are also useful for this project: for example we use their notion of *determinacy* (see Section 4.4.2).

McKinzie and Tuckey define e by the expression $(1+1/K)^K$ for infinite hypernatural K following Euler. This includes showing that the expression is determinate, that is if the value chosen for K is changed to another infinite hypernatural, the expression will not change by more than an infinitesimal (see Section 4.4.2). They derive, as Euler did, the power series for sine and cosine. Interestingly, their argument in one step of this proof appeals to a diagram to demonstrate that $0 < \sin\theta < \theta < \tan\theta$ for hyperreal $0 \leq \theta \leq \pi/2$. This trigonometric estimate is also used in their rehabilitation of the Basel problem but without justification. Implicitly, they assume that usual geometric diagrams describe a metric geometry where the metric takes values in the hyperreals. This appeal to geometry is consistent with Euler's conception of sine and cosine as defined via geometry. However, whatever 'proof' this diagram forms is not readily formalisable: it is based on triangle geometry and perhaps a degree of intuition. In order to formalise it, we would need an alternative way to obtain this result from trigonometry. Unfortunately Isabelle's standard definition of sine and tangent is in terms of their power series, and thus using these as definitions would make the argument circular. Elsewhere, we formalised another definition of sine and cosine in terms of three simple characterising equations including the angle-sum identity for cosine [59]. However, it was not possible to directly prove that there did in fact exist a unique pair of functions satisfying these equations. So for this, we would need to look to geometry.

After reconstructing Euler's proof of the power series for sine and cosine, McKinzie and Tuckey follow Euler's reasoning to derive the power series for the natural logarithm ln. As support for their rehabilitation of Euler's proofs, they also prove and formulate many of the lemmas that Euler used both explicitly and implicitly. They prove the Binomial Theorem for fractional (hyperreal) exponents since Euler used this theorem in his proofs. Similarly, they prove the Summation Comparison Theorem. They also carefully define any concepts that they believe Euler used such as determinacy. According to McKinzie and Tuckey, determinacy provides a criterion for judging whether infinitely many infinitesimals may be ignored in an infinite sum, which was a question Euler considered on multiple occasions. Hence it is an important concept for formalising Euler's work. We discuss determinacy and the related concept of Euler-convergence in full detail in Section 4.4.2.

1.7.5 Other approaches to Euler's work

There are interpretations of Euler's work which do not use infinitesimals. Ferraro argues that interpreting Euler's arguments by using any kind of modern mathematics, standard or nonstandard analysis, 'is only possible if elements are added which are essentially alien to it' [35, p. 2]. In particular, he makes the case that nonstandard analysis could not be the correct infinitesimal theory in which to interpret Euler's work because he believes that Euler meant to use exact equality where nonstandard analysis could only use the infinitely-close relation (see Section 1.5). However, for the purpose of this research it is necessary to interpret Euler's work within some formal theory so we consider it sufficient to find the closest one and acknowledge the differences between it and Euler's ideas. Also, we are focused on Euler's methods rather than the ontology of his concepts. Thus the theory in which we explore Euler's arguments need not be identical to Euler's theory provided the methods of proof closely agree.

Nonetheless, the commentaries on Euler's work and that of his contemporaries that do not explicitly interpret their arguments in terms of any specific formal theory are still useful for understanding Euler's ideas and informing the formalisation. Laugwitz argues that what appear to be gaps in the arguments of Euler and his contemporaries are actually instances of lemmas which seem obvious if we remember that they are using infinitesimals [51]. This is a useful idea for understanding Euler's arguments and was applied by McKinzie and Tuckey [57].

1.7.6 Using modern mathematics to represent Euler's reasoning

Once a modern theory to formalise Euler's analysis has been decided upon, there remains the question of how it can be matched-up with Euler's work. Bair et al.[5] and McKinzie and Tuckey [57, p. 36] argue that Euler had two distinct concepts of zero, and thus two concepts of equality. One of these is of course, the usual concept of equality. The second is of being 'relatively zero', i.e. infinitesimal relative to the other numbers in the context. For example, supposing ε to be infinitesimal, and a to be finite, Euler might write $a^2 + a\varepsilon = a^2$ since $a\varepsilon$ is infinitesimal compared to a^2. Euler called these two concepts of equality 'arithmetic' and 'geometric' comparison [5]. We can match up one version of comparison with actual equality and another with the infinitely-close relation.

Euler occasionally substitutes infinitely many infinitesimals for zero, even though it is possible for the sum of infinitely many infinitesimals to be non-infinitesimal. For example, see the omission of the term x^2/N^2 in an infinite product in Section 4.2.3. This is an example of the general concept of a 'hidden lemma' which Laugwitz claims is used in infinitesimal analysis by mathematicians such as Euler and Cauchy [53]. There are parts of Euler's reasoning where an operation is performed which is not justified in general, but in that instance there is a hidden lemma which justifies it [57]. If hidden lemmas are indeed used in places where infinitesimals can be discarded only under certain conditions, their implicit nature may be a reason infinitesimal analysis was regarded with suspicion for so long and yet was so successful for developing early results in analysis. Sometimes Euler and his contemporaries seemed unaware that a particular scenario required such a lemma. Sometimes they were aware, but gave empirically-inductive arguments rather than the theorem we would expect for modern standards of rigour. Sometimes they did state principles which would supply the lemma, but were misunderstood. Possibly sometimes they were aware of such lemmas even if they did not state them anywhere, but we may never know.

1.7.7 Placing Euler's analysis in context

An important consideration is to understand Euler's analysis within context, and refrain from viewing his arguments through a modern lens. In the 18th century, they used different notation. Euler used i for infinite numbers rather than the square root of -1, and ω for infinitesimals. Notation was also not as standardised as it is now: Euler introduced $\sum$ notation for sums and $f(x)$ notation for functions but neither were immediately adopted [11]. They did not have the same concept of the methodology of mathematics that we do now. Pólya argues that Euler used empirically-inductive reasoning (in the spirit of the scientific process) [67]. Euler's contemporaries still interpreted axioms as describing indubitable truths since non-Euclidean geometry, the main reason for the overthrow of this idea, was not yet discovered. Mathematicians did not even have the vocabulary or concepts to express Gödel's Incompleteness Theorems. Many mathematicians still did not conceive of imaginary numbers existing with the same validity as real numbers, including Euler himself [46, p. 714]. In order to understand Euler within his context, we should avoid using in our proofs 'any principle or notion that could not be stated in the language of Euler's time' as McKinzie and Tuckey say [57].

1.8 Organisation of the thesis

This thesis is organised into 6 chapters:

> **This chapter** forms our introduction. It gives the context and motivation for the work, the aims of the thesis, describes previous work and outlines a summary of contributions.
>
> **Chapter 2** gives background information on nonstandard analysis and Isabelle which the reader will need to understand the rest of the thesis.
>
> **Chapter 3** describes our mechanisation of the mathematical context which was required for formalising Euler's proof of the Basel problem. We chose to place in this chapter those parts of the formalisation which were general or outside the main proof of the Basel problem.
>
> **Chapter 4** gives our initial analysis of Euler's proof and describes relevant analysis and reconstruction of Euler's proof by others, including McKinzie and

Tuckey. Here we identify the gaps in Euler's proof and describe important proof-specific concepts that we will use to mechanise the proof.

Chapter 6 describes our partial mechanisation of Euler's proof of the Basel problem. We also describe gaps which we have discovered both in Euler's and McKinzie and Tuckey's proofs and give proposed solutions for filling those gaps.

Chapter 7 contains the conclusion in which we reflect on the decisions taken in the mechanisation process and discuss the extent to which we have accomplished our aims. We place our work into the context of the other research on this topic and we discuss potential future work.

Each chapter except the Introduction and Conclusion ends with a summary of the contents. This provides a reminder to the reader of what was covered and gives an overview of the contents of that chapter.

1.9 Proof scripts

The accompanying proof scripts containing our mechanisation can be downloaded at the following link: `https://aiml.inf.ed.ac.uk/wp-content/uploads/2022/10/basel_problem.zip`. They are intended to be run in the 2021-1 version of Isabelle within the HOL-Nonstandard Analysis session.

Chapter 2

Mathematical and system context

This thesis is based on concepts in mathematics which, as they are not part of the mathematical mainstream, may be outside the reader's expertise. It is also based on the use of a specific theorem-prover, Isabelle [65], which comes with its own jargon and methodology. To make the thesis as self-contained as possible, we aim to include here the relevant mathematical and system context. We also aim to make each following chapter and section as self-contained as possible by presenting specific information in the context where it is used. However, this chapter is useful to present those concepts which are fundamental or are used so often that to explain them in context would become repetitive. We also take the opportunity to explain how we have related and derived each concept from the ones used by Euler. We begin by examining Euler's concept of proof.

2.1 Euler's concept of proof

Euler's concept of proof may have been wider than the current well-developed and restrictive idea of proof. At this moment in history, we imagine that mathematics consists of theorems built up from axioms by combining them according to logical rules. This may not be how it really works in practice, but it informs our modern idea of rigorous proof. Although Euclid's Elements, written around 2000 years before, introduced the axiomatic method, this only provided axioms for Euclidean geometry. Euler was certainly familiar with deductive reasoning from geometry. However, Pólya analyses Euler's most criticised proof, which we discuss in Section 4.1, and based on this argues that Euler also used empirically-inductive reasoning, as is used in the experimental sciences. According to Pólya, Euler reasons by analogy to discover the factorisation of sine as an infinite polynomial and thus prove the Basel problem [67]. Euler tries his method of proof out on other examples. He compares his exact solutions to numerical values which he computes, and he also rediscovers known results using his method. Eventually, he is lead by his exploration to find a new rigorous proof, which he gives a version of in the 'Introductio'. McKinzie and Tuckey argue that, in this way, Euler established that his method could both explain already known phenomena and predict new phenomena, which confirms it as a scientific hypothesis [57, p. 32]. Euler considered the validity of his method to be an important result in its own right: not just interesting because it gives him the value of the series $1+1/4+1/9+1/16+\cdots$ [26, p. 4].

Euler saw one of the functions of proof as being its power to convince others. He took the doubts of his fellow mathematicians seriously, and they were a driving force behind his further explorations [26, pp. 3-4]. Euler's marvellous intuition is often described. He was also a master of exploring mathematics and coming up with new concepts, results and ways of describing existing mathematics. Mathematicians are able to discover and reasonably justify much more than they can prove. Euler would have felt limited if he had constrained himself to only publishing or writing about ideas that he could give deductive arguments for. But when the empirically-inductive arguments failed to convince, he could put greater effort into converting his intuitions into a deductive proof, as we see with the history of his proofs of the Basel problem.

2.2 Proofs and concepts in Isabelle

First, we introduce the methodology and general concepts we have used to formalise Euler's reasoning in Isabelle. This forms a useful background to Chapter 6.

2.2.1 Using structured proof to represent Euler's reasoning

This thesis aims to represent Euler's reasoning as faithfully as possible, in order to to better understand it and discover the gaps in his argument. Thus our formal proof needs to follow the structure of mathematical proof, which will also have the benefit of improving its readability. To accomplish this, we use the structured proof language Isar, which allows prose-style proof within Isabelle. We give now as an example, the Isar proof in Listing 2.1 which represents the top-level case-split in our representation of Euler's proof of the Basel problem (see Section 6.10). Before the Isar proof, we give a rewriting of it in mathematical style. The concepts used (infinite hypernatural numbers, the relation $\simeq$, the hypersum ${}^*\sum$) may not yet be familiar to the reader and are given in Section 2.3. We also refer to the theorems Basel_problem_odd and Basel_problem_even_is_odd (which essentially states that the even case can be reduced to the odd case) among others. These are given in Section 6.7. However, their contents are not relevant at the moment as this proof is given only to demonstrate formal proof in Isabelle.

Theorem (Basel problem)**.** *Let N be an infinite hypernatural. Then*

$$ {}^*\sum_{n=1}^{N} \frac{1}{n^2} \simeq \frac{\pi^2}{6}. $$

Proof. Assume first that N is odd. Then we can obtain a k such that $N = 2k+1$. We can deduce that k is also an infinite hypernatural. Thus by Theorem Basel_problem_odd we have ${}^*\sum_{n=1}^{2k+1} 1/n^2 \simeq \pi^2/6$ which shows this case.

Next assume that N is not odd. Then we can obtain a k such that $N = 2k$. Because N is an infinite hypernatural, we can rewrite it as $N = 2(k-1)+2$ where $k-1$ is itself an infinite hypernatural and thus $k \geq 1$. By Theorem Basel_problem_even_is_odd we can deduce ${}^*\sum_{n=1}^{2(k-1)+2} 1/n^2 = {}^*\sum_{n=1}^{2(k-1)+1} 1/n^2$. Again, by Theorem Basel_problem_odd we also have ${}^*\sum_{n=1}^{2k+1} 1/n^2 \simeq \pi^2/6$ and thus ${}^*\sum_{n=1}^{N} 1/n^2 \simeq \pi^2/6$. □

Listing 2.1: Basel problem case-split

```
theorem Basel_problem:
assumes "N ∈HNatInfinite"
shows "hypsum (λn. 1/(n∈ℕ pow 2)) {1..< N+1} ≈(star_of pi pow 2) / 6"
proof (cases "odd N")
  assume "odd N"
  then obtain k where k_def:"N=2·k+1"
    using oddE by blast
  from this assms have "k ∈HNatInfinite"
    by (rule odd_HNatInfinite)
  then have "hypsum (λn. 1/(n∈ℕ pow 2)) {1..< 2·k+1+1} ≈(star_of pi pow 2) / 6"
    by (rule Basel_problem_odd)
  then show ?thesis using k_def by simp
next
  assume "¬odd N"
```

```
  from this hypnat_odd_or_even2 obtain k where k_def:"N = 2·k"
    by blast
  from k_def assms have k_minus_one_inf: "k−1 ∈HNatInfinite"
    using HNatInfinite_half HNatInfinite_diff Nats_1 by blast
  have "k≥1"
    using HNatInfinite_half assms k_def one_le_HNatInfinite by blast
  from this k_def have N_as_k:"N = 2·(k−1) +2"
    by (rule extract_factor_of_2_hypnat)
  have "hypsum (λn. 1/(n∈ℕ pow 2)) {1..< 2 · (k−1)+2+1} ≈
          hypsum (λn. 1/(n∈ℕ pow 2)) {1..< 2 · (k−1)+1+1}"
    by (rule Basel_problem_even_is_odd)
  also from k_minus_one_inf have "\ldots ≈ (star_of pi pow 2) / 6"
    by (rule Basel_problem_odd)
  finally show ?thesis
    using N_as_k by simp
qed
```

Statements encased in " " are literals and can be thought of as the equations and formulas that litter pen-and-paper mathematical proofs. Any statement in the proof may be named e.g. k_def or N_as_k. These names are referred to in later steps of the proof along with the names of previously proven theorems such as Basel_problem_odd. The **theorem** *name* **assumes** *assumptions* **shows** *statement* set up what is to be proved. The proof itself is done by a case-split on odd N. The beginning of the proof is indicated by **proof**, division between cases is indicated by **next** and the conclusion of the proof is marked by **qed**. Intermediate statements in the proof are prefaced with the keyword **have** and concluding statements are prefaced with the keyword **show**. If these follow from the previous line, we use **then** or **from** this. Below each statement, except for assumptions, is the proof which consists of **by** *method*. This *method* may be the use of a previously proven theorem e.g. **by** (rule Basel_problem_odd) or it may be an automatic proof method e.g. **by** blast. The automatic proof methods are used to fill in small steps that Isabelle is able to discover a proof by itself. On two occasions, we **obtain** a variable k. This can be thought of as the step where we give a name to the witness from the existential statement on the previous line, allowing us to use this variable in the rest of the proof. All of these keywords correspond with steps usually undertaken by mathematicians in their proofs.

One strategy which this proof demonstrates is breaking down each proof into smaller lemmas. This makes each part of the proof easier to find, because it is labelled with a lemma name and has its own separate section. More importantly, it reduces redundancy since lemmas can be reused. For example, here a mathematician would likely say ‘we can, without loss of generality’ assume that N is odd, if they mention it at all. Since this is a formal proof, in Isabelle, we must consider every case, so here we case-split on whether N is odd or even. However, we reduce the case where N is even to an intermediate statement which can be proved by using the same lemma Basel_problem_odd as in the odd case. If we had not extracted this as a lemma, the proof would have been partially duplicated.

2.2.2 Locales

We have not proven all the theorems necessary for mechanising Euler’s proof in Isabelle. Some of these (e.g. the determinacy of some series - see Section 5.2) would be relevant to the context of Euler’s proof but will be left for future work. Some of those are well-known mathematically and outside the direct reasoning of the proof, and proving them would be tangential to the aim of this thesis. We explain these choices individually in our description of the mechanisation: see, for example, the Fundamental Theorem of Algebra (Section 3.3) and the Trinomial Lemma (Section 6.4.1). Rather than adding these statements as axioms, which theoretically could introduce inconsistency to the theory, we add them via a *locale*. This is an Isabelle structure allowing the formalisation of a set of statements which are then are then attached as assumptions to any theorems proven within the *context* of the locale. The locale assumptions can be proven later and the assumptions are then discharged from all the theorems proven within the locale context. In Listing 2.2, we give the locale formalising the Fundamental Theorem of Algebra which we need for the proof of the Basel Problem. This locale consists of a single fact. We will explain statement given in the locale in Section 3.3 once we have introduced the notion of hyperfinite sums and products Section 3.2.4 and the type complex star of hypercomplex numbers.

Listing 2.2: Fundamental Theorem of Algebra locale

```
locale Fundamental_theorem_of_algebra =
assumes FTA:
"( *fn* b) N ≠ 0 ⟹
    ∃a. (λx. (( *fn* b) N) · hyprod (λn. x − ( *fn* a) n) {1..< N+1}) =
    (λx. hypsum (λn. ( *fn* b) n · (x :: complex star) pow n) {0 ..< N+1})"
```

2.2.3 The Hilbert choice operator

Functions in Isabelle are required to be total [48, p.2]. Furthermore, the domain of a function in Isabelle must be a type, not simply a set. In order to make composition and application of functions as simple as possible in our theory, we use as few types as possible for the domain and range of our functions. For example, it is useful to use types which are closed under algebraic operations, such as $\mathbb{R}$, ${}^*\mathbb{R}$, $\mathbb{N}$ and ${}^*\mathbb{N}$, as domains and ranges, rather than defining new types based on subsets of the sets associated with those types. However, some functions are naturally only defined on intervals or subsets, e.g. the inverses of sine and cosine. With these kinds of functions, it is possible to pick arbitrary values for them to take outside of the domain on which they are naturally defined, in order to extend the domain to the whole type. A useful Isabelle concept for handling this is the Hilbert choice operator [49, p. 727], SOME in Isabelle.

The following definition gives a real number r which corresponds to a given hyperreal number h.

```
definition unstarreal
  where "unstarreal h = (SOME r. h = star_of (r :: real))"
```

In this definition we use the function star_of which, given a real number r, constructs the corresponding hyperreal number star_of r in Isabelle. Since the hyperreals contain the reals as a subset, this construction is always possible. But it may not be possible to give a corresponding real number for every hyperreal: e.g. a nonzero infinitesimal would have no equivalent in the reals. Thus SOME r. h = star_of r refers to the real number corresponding to hyperreal h when such a number exists (see Section 2.3.1.3), otherwise it refers to an arbitrary element of the correct type, which allows it to fulfil the requirement that it is a total function. We can also use SOME to define a partial inverse (as we do in Section 3.1). For example, SOME r. r^2= x would give an arbitrary number which is the square-root of x. However, it is not possible to prove that this is either specifically the positive square root or specifically the negative square root.

2.3 Using nonstandard analysis in Isabelle to represent Euler's analysis

In this section we will introduce the reader to the essential concepts of nonstandard analysis as they are given in the nonstandard analysis library in Isabelle/HOL. We will also discuss how the concepts can be used to approximate the ones used by Euler, and their limitations when used to describe Euler's reasoning.

The current nonstandard analysis library in Isabelle uses the ultrapower construction of the hyperreals (see Section 1.5) [36, 45]. Major results formalised in this library and elsewhere include

1. the Transfer Principle (Section 2.3.2.5), has been formalised as a proof method;

2. properties of *hypernatural* numbers: these are the ordinary natural numbers together with infinite numbers that share similar attributes including an induction principle;

3. *hypercomplex* numbers, which are an extension of the hyperreals using the square root of minus 1 in exactly the way that the complex numbers are an extension of the real numbers;

4. limits: continuity and hyperreal powers, including infinite and infinitesimal powers.

Nonstandard analysis is concerned with the hyperreals which are an extension of the real numbers with infinitesimals and infinitely-large numbers, and by algebraic closure, many additional numbers. Here we give a brief description of their properties to demonstrate that the infinitesimal and infinitely-large hyperreals behave in a similar way to Euler's infinitesimals and infinitely-large numbers. They obtain these properties from their construction which we will describe concisely in Section 2.3.1.3. We compare them to Euler's numbers in Section 2.3.1.2. We also hope this will give the reader a sufficient intuitive understanding of the hyperreals so that they can follow the reasoning we lay out in Chapters 4 and 6. Our presentation is based on Goldblatt's [40].

The hyperreals are denoted by ${}^*\mathbb{R}$. We define the set of infinitesimals $\mathbb{E}$ as $\mathbb{E} = \{x. \forall r \in \mathbb{R}.\ r > 0 \rightarrow \mid x \mid < \mid r \mid\}$. Notice zero itself is infinitesimal. We define the set of infinite numbers $\mathbb{L}$ as $\mathbb{L} = \{x.\ \forall r \in \mathbb{R}.\ \mid x \mid > \mid r \mid\}$. Imagine that we begin by adding some infinitesimals and infinitely-large numbers to the ordinary reals. We now have a cluster of infinitesimal numbers around zero. Each real number e.g. π will similarly have a cluster of numbers that are infinitely-close to it, called a *halo* [40, p. 52]. The numbers in the halo of a real number r are not themselves real numbers, except for r itself. Even the infinitely-large numbers will have a halo. Also, each of the hyperreal numbers has a negative counterpart. They obey all the usual laws of the reals (such as commutativity and associativity), but not the Archimedean principle which states that given two positive numbers x and y, there is an integer n such that $nx > y$. This gives us an initial picture of the hyperreals. We can describe their properties more precisely as follows.

Let ε and δ be infinitesimals, let a and b be hyperreal numbers that are not infinitesimal or infinite, which we call *appreciable* [40, p. 50], let L and M be infinite and let n be a finite natural number.

- Zero: zero is the only infinitesimal real number.

- Addition: $\varepsilon+\delta$ is infinitesimal, $\varepsilon+a$ is appreciable and finite, $\varepsilon+L$ and $a+L$ are infinite. We cannot determine in general $a+b$ and $M+L$. However, if a and b are either both positive or both negative, we know $a+b$ is appreciable and if L and M are either both positive or both negative, we know $L+M$ is infinite.

- Negatives: $-\varepsilon$ is infinitesimal, $-a$ is appreciable and $-L$ is infinite.

- Multiples: $\varepsilon\delta$ is infinitesimal, $a\varepsilon$ is infinitesimal, ab is appreciable, ML is infinite and aM is infinite. However εL may be infinitesimal (e.g. $1/L^2 \cdot L$), appreciable (e.g. $1/L \cdot L$), or infinite (e.g. $1/L \cdot L^2$).

- Reciprocals: Given ε and b nonzero, $1/\varepsilon$ is infinite, $1/L$ is infinitesimal and $1/b$ is appreciable. Quotients can be deduced from the rules for multiples.

- Powers. ε^n is infinitesimal, a^n is appreciable and L^n is infinite.

- Roots. Given positive ε, a and L, $\sqrt[n]{\varepsilon}$ is infinitesimal, $\sqrt[n]{a}$ is appreciable and $\sqrt[n]{L}$ is infinite.

In nonstandard analysis it is useful to have the concept of numbers being infinitely close to one another. Thus we define the *infinitely-close relation* as $a \simeq b$ if $a - b \in \mathbb{E}$ i.e. a and b are infinitely close if their difference is infinitesimal. It is an equivalence relation and shares many properties with equality. This makes it a good representation of Euler's = when dealing with infinitesimals (see Section 2.3.1.2).

A significant result supporting the validity of nonstandard analysis is the *transfer principle* [40, Chap. 4]. This is a theorem that allows us to take certain statements in standard analysis, obtain their transfer, and get a corresponding result in nonstandard analysis. It also allows us to transfer statements from nonstandard analysis to standard analysis under certain conditions. It is useful for the formalisation since it avoids the need to reprove some results that are already available in standard analysis. In Section 2.3.2.5 we describe its formalisation as a method in Isabelle.

2.3.1 Numbers, quantities and equality

In this section we discuss Euler's understanding of numbers, quantities and equality. We are then able to describe the relevant concepts from nonstandard analysis in this context.

2.3.1.1 Euler's concept of a number

Our present concept of a number is quite flexible, and can be given to anything which we can perform arithmetical operations on. The number systems which first occur to us are the natural numbers, rationals, reals and complex numbers, but specific examples of any kind of algebraic structure, such as a ring or field, could be called number systems in the right context. In the 18th century, this widening of the concept of number was still in progress. Naturals and rationals were universally accepted as numbers. Irrationals, which were famously rejected by the Pythagoreans [1, p. 35], were used in abundance, but they derived their validity from being described in terms of rationals. 'Imaginary' values had been given their derogatory epithet less than a century before Euler by Descartes [14], but they were also beginning to be regarded as useful tools. In fact, Euler is regarded as one of the first mathematicians to popularise their usage because of his formula $e^{ix} = \cos x + i \sin x$ [61]. The concept of transcendental values has also been credited in part to Euler [22], who first described the number e and detailed the concept of transcendence in his 'Introductio'.

Euler used different words to refer to 'number' and 'quantity' (in his original Latin, 'numerus' and 'quantitas'). It is relevant to know why he made this distinction, and what he meant by each of these concepts. Petrie claims that Euler used the word 'number' only for naturals and rationals and 'quantity' for irrationals, transcendentals and imaginary numbers [66, p. 281, pp. 285-286]. However, this was not the case at the point that Euler wrote the 'Introductio', since in §3 of the first chapter he states

> ‘a variable quantity encompasses within itself absolutely all numbers, both positive and negative, integers and rationals, irrationals and transcendentals. Even zero and imaginary numbers are not excluded...

Euler even uses the word number to refer to infinite magnitudes in §138 of the ‘Introductio’. Later on, Euler gave a description of ‘number’ that makes it clear he regarded them as ratios

> ...a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.’ [32]

As Petrie points out, this would appear to exclude complex numbers from Euler’s definition of number [66, p. 284]. In the first chapter of the ‘Introductio’ Euler usefully explains what he means by such concepts as constants, variables and functions, although it is hard to interpret what he meant without letting our understanding be coloured by the present meanings of each term. In his own words from the ‘Introductio’ §§1 − 2, ‘A constant quantity is a determined quantity which always keeps the same value. ... A variable quantity is one which is not determined or is universal, which can take on any value.’ Although these definitions are tautological in the same vein as the primitive notions from Euclid’s ‘Elements’, they still tell us something about his concepts. This makes it clear that he regarded quantities as being either constant – ‘determined’ – or variable. Functions themselves, he states, are ‘a function itself of a variable quantity will be a variable quantity’. We can deduce that Euler’s concept of ‘quantity’ is inextricably linked and overlaps with his concept of function (see Section 2.3.2.1).

We could interpret Euler’s concept of quantity as one explanation for why he was comfortable using infinitesimals. Although infinitesimal quantities and quantities that include infinitesimal increments are never fully calculated or ‘determined’ they are not excluded from being quantities, since they might be considered variable quantities, similar to the constructivist idea of conceiving of infinity as a process. This would pleasingly allow Euler’s concept of infinitesimals to be considered as a precursor to limits as Ferraro does [34, p. 45]. However, McKinzie and Tuckey have demonstrated that treating infinitesimals as numbers using the theory of nonstandard analysis allows us to follow Euler’s reasoning, whether or not this exactly matches Euler’s ontology [55].

Although Euler may not have given the same footing to transcendentals and irrationals as he did to natural and rational numbers, he was more open minded than many modern mathematicians about what mathematical tools he allowed himself to use in his published work. Some of his proofs (e.g. his first proof of the Basel problem) were clearly intended as narrative or empirically-inductive derivations rather than watertight arguments (although in Section 2.1 we argue that he also had proofs which he intended to fall into the latter category). In the next section, we describe how he conceived of and used infinite and infinitesimal quantities, which are rarely used in contemporary mathematics and were seen as mathematical pariahs from the time of Cauchy to Robinson.

2.3.1.2 Euler's concept of infinite and infinitesimal numbers

Purely from an analysis of their properties, Euler's infinities and infinitesimals behave similarly to the infinite and infinitesimal numbers from nonstandard analysis. The main point of difference is that Euler does not explicitly distinguish between true equality and equality up to an infinitesimal, which gives rise to the infinitely-close relation of nonstandard analysis. Both are denoted by = in his proofs and in fact he often uses $\varepsilon = 0$ to mean that ε is infinitesimal. However, he is actually aware of the distinction between infinitesimally close and equal. In several places Euler states that he can omit infinitesimal increments, with justification (see for example §156 of the 'Introductio' [29]). Euler in fact gives an example where it is not justified to substitute in a quantity which differs by an infinitesimal (see §§155-156 of the 'Introductio'). Since Euler uses the distinction, and since it would otherwise be difficult to create a consistent theory, we formalise the two concepts of equality distinctly in our Isabelle proofs.

2.3.1.3 Construction of the hyperreals in Isabelle

In the Isabelle Nonstandard Analysis Library the hyperreals are constructed using the ultrapower construction, which is described by Goldblatt [40, Section 2.3]. Given this construction, the hyperreals are a consistent number system if and only if the reals are consistent, assuming the Axiom of Choice. In essence, each hyperreal is given by an equivalence class of sequences of real numbers. The equivalence relation is defined by a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, which is a subset of $\mathcal{P}(\mathbb{N})$ with certain extra properties:

1. $\mathcal{U} \neq \varnothing$
2. $\varnothing \notin \mathcal{U}$
3. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$
4. if $A \in \mathcal{U}$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in \mathcal{U}$

5. for any $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} - A \in \mathcal{U}$.

A nonprincipal ultrafilter exists on any infinite set, but its existence relies on the axiom of choice, so we are unable to constructively specify the ultrafilter. Let $\{r_n\}$ be a sequence $r : \mathbb{N} \to \mathbb{R}$. Then the hyperreal r is the equivalence class of sequences $[\{r_n\}]$ defined by the congruence relation $\{A_n\} \sim \{B_n\} \leftrightarrow \{n.\, A_n = B_n\} \in \mathcal{U}$. In fact, $\mathbb{N}$ itself is in the ultrafilter and so is any set that is different from $\mathbb{N}$ by only finitely many values. Thus the sequence $\{\pi, \pi, \pi \ldots\}$ with every value equal to π and any sequence distinct from π in only finitely many positions, e.g. $\{\pi, 1, \pi, \pi, \pi, \pi \ldots\}$, will be in the same equivalence class, and will correspond to the hyperreal $[\{\pi\}]$. It is possible to prove that the hyperreal $[\{\pi\}]$ has the same properties as the standard real π. We can express the congruence relation between the two sequences as $\{A_n\}$ and $\{B_n\}$ are congruent if $A_n = B_n$ *for almost all n* or equivalently, if it is *eventually true* that $A_n = B_n$. These two modes of expression have equivalent formulations in the Isabelle Nonstandard Analysis Library.

1. $\forall_F$n in $\mathcal{U}$. (A n = B n)

2. eventually (λn. A n = B n) $\mathcal{U}$

The first formulation $\forall_F$ corresponds to 'almost all', the second eventually corresponds to 'eventually'. Our sequences are represented by functions A and B from the naturals to the reals. Hence A n stands for the usual sequence $\{A_n\}$. In the Isabelle Nonstandard Analysis Library, we write star_n A to denote the equivalence class which is defined by the sequence A. Each such equivalence class is a hyperreal number.

2.3.1.4 Hyperreal types in Isabelle

In fact, it is not just the hyperreals that are defined in the Isabelle Nonstandard Analysis Library, but the nonstandard models of every type universe. More theorems exist about the hyperreals and hypernaturals, but some theorems have been shown for a general nonstandard type. The nonstandard type corresponding to the type 'a is given by 'a star. The types for hyperreal and hypernatural are thus real star and nat star respectively, however there are also the synonyms hypreal and hypnat.

2.3.1.5 Hyperreal sets in Isabelle

The set of infinite hypernatural numbers is denoted HNatInfinite in Isabelle. The set of infinite hyperreals is denoted HInfinite and the set of finite hyperreals is denoted HFinite. Finally, UNIV is the set of all elements of some type: this may be any type. Thus UNIV::hypreal set is the set of all hyperreals.

2.3.2 Sets and functions

The now-familiar concepts of sets and functions were still being developed in the 18th century. Indeed, Euler is credited with coming-up with the modern concept of a function, although we will see that his concept is not entirely equivalent to our own. We analyse how Euler conceived of these notions so that we can evaluate our own representations of his reasoning about them.

2.3.2.1 Euler's concept of a function

Euler begins the 'Introductio' with a discussion of his notion of function, which is not quite as broad as the modern one. This places some emphasis on the importance of functions in his work, and also helps us understand how we might faithfully represent the functions in the proof of the Basel problem. Euler states briefly:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant qualities. [29, §4, p. 3]

This is somewhat vague. We can wonder if by stating that the function is an analytic expression he means that it is an analytic function in the modern sense that it has a convergent power series at every point. However 'convergence' was not yet a fully developed concept at the time. If Euler meant that he defined functions to be expressions with power series that are 'convergent', surely he would state that explicitly. In Chapter 4 of the 'Introductio' he does make it clear that he believes all functions can be represented by an infinite polynomial which 'gives the value of the function', and thus has a numerical limit, although to some value which may be infinite. Even more unusually, he allows his infinite polynomials to have exponents of any real value so these are not power series. This is similar but not equivalent to the modern notion of 'analytic'. But from his discussion of functions in Chapter 1, it seems instead that he thinks of functions as being built up from basic operations including algebraic operations and composition with various common transcendental functions, and that this is what he means by 'analytic expression'. He later states

> Functions are divided into algebraic and transcendental. The former are those made up from only algebraic operations, the latter are those which involve transcendental operations. [29, §4, p. 3]

We may wonder what kind of transcendental functions Euler conceives of. Certainly, sine, cosine, tangent, logarithms and exponentials are included. At the beginning of Chapter 6, he says 'the concept of a transcendental function depends on integral calculus'. We may conclude that Euler allows any kind of transcendental function defined in terms of algebraic operations and integrals[1] and presumably derivatives. Interestingly, Euler considered the function z^π algebraic, which nowadays would be considered transcendental, so his understanding of the distinction is not the same as the modern one. Euler also discusses functions of several variables and allows multivalued functions such as square roots. Petrie describes Euler's concept of a function as being closer to the modern idea of a formula [66, p.282]. Despite these nuances, 'Introductio' was in fact the first book to place the concept of a function so centrally and to outline it with such clarity and closeness to the modern picture of a function [47, p.405].

2.3.2.2 Internal functions

In the proof of the Basel problem, the only transcendental functions used are the exponential, cosine and, implicitly, sine. All of these can be represented in standard real analysis, as can all algebraic functions, at least up to some infinitesimal or infinite constant. In nonstandard analysis, functions which depend upon nonstandard numbers, but which are otherwise defineable in the reals, e.g. $\sin Nx$, x/N or x^N for some nonstandard N, are called 'internal' and they have the additional property that they are defined by sequences of real functions, due to how nonstandard numbers are defined. In the Isabelle Nonstandard Analysis Library, the internal function defined by the sequence f of real functions is written as $\mathsf{{}^*fn^*\ f}$ where

$$(\ {}^*\mathsf{fn}^*\ \mathsf{f})\ \mathsf{x} = [\{\mathsf{f}_n\ \mathsf{x}_n\}]$$

for any hyperreal argument $\mathsf{x} = [\{\ \mathsf{x}_n\}]$.

[1] Euler predated Cauchy, and was a continental mathematician, so presumably used Leibniz's notion of an integral.

A hyperreal N is defined by the underlying sequence of hyperreals $[\{N_n\}]$ as we explained in Section 2.3.1.3. We can thus define the internal functions mentioned as examples above e.g.

$$\frac{x}{N} = \left[\left\{\frac{x_n}{N_n}\right\}\right].$$

When the sequence $\{f_n x_n\}$ is constant, we call the function the *nonstandard extension* of the corresponding real function. E.g. $\sin x$ can be extended to the hyperreals in this way. Internal functions are far more manageable than general nonstandard functions because their properties can be defined in terms of standard functions. They are not directly covered by the transfer principle, as it is mechanised in Isabelle, but we can still prove properties about internal functions first for the real functions whose sequence defines them. This allows us to use proof by induction to a certain extent to prove properties about these functions.

2.3.2.3 Euler's concept of sets

Euler spent several chapters of the 'Introductio' outlining the concept of a function, which was influential in the history of mathematics [47, p.405]. By contrast, at least in the 'Introductio', he does not refer to sets explicitly, but instead implicitly describes relevant numbers by the conditions they satisfy. For example, rather than referring to the set or collection of all square numbers, he might say 'all square numbers', or define a sequence with N terms, each of which are squares. Euler is credited with the invention of Euler diagrams, a version of Venn diagrams [69]. However, he did not conceive of them as representing sets but rather the domain of truth of logical propositions. Hence, we cannot conclude that Euler had a notion of a set as a mathematical object in the same way he considered a function or number to be. Since in modern mathematics, sets and logical propositions are often used interchangeably, we can closely follow Euler's reasoning whilst using the modern representation of sets. Sets are also necessary in Isabelle so that we can use the nonstandard analysis library and represent Euler's numbers as hyperreals.

2.3.2.4 Internal sets

Similarly to internal functions, there are internal sets which are defined in terms of sequences of subsets of the reals. Again, if the sequence is constant, we call them extensions of the real set. In the proof of the Basel problem, we only need quite simple sets to represent Euler's reasoning, many of which are internal. For example, the set of hypernatural numbers, hypernatural intervals (Section 3.2.3) and the set of odd numbers (Section 3.2.5.1). However, we must also consider some sets which are not internal such as the set of all infinite numbers, or the set of infinitesimals.

In the Isabelle Nonstandard Analysis Library, the internal set defined by the sequence E n of real subsets is written as *sn* E. We denote the set of internal sets by InternalSets.

2.3.2.5 The Transfer Principle

The Transfer Principle in nonstandard analysis allows us to take statements about real functions and sets, and formulate equivalent statements about their extensions which have the same truth value as the original statements.

The transfer method in Isabelle [44] allows a goal to be transformed to a logically-equivalent goal involving different types. We apply it to transform goals about terms of type 'a star to goals about terms of type 'a. Goals in terms of types hypreal and hypnat (which are abbreviations for real star and nat star respectively) result in goals about terms of type real and nat respectively. The transfer method is possible to apply in statements that involve only extensions and internal predicates because the Transfer Principle states that the transformed goals have the same truth value. This transformation forms part of a valid proof and enables us to use the corresponding statements in terms of reals and naturals, which have often been proven already in the Isabelle libraries. We give an example of its usage in Listing 2.3.

Listing 2.3: Example of usage of transfer method

```
lemma pow_zero:"⋀a. a pow (0::hypnat) = 1"
  by transfer simp
```

The function pow : $^*\mathbb{R} \times {}^*\mathbb{N} \rightarrow {}^*\mathbb{R}$ is an extension of the function ^ : $\mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ (See Section 2.3.2.2). Applying the transfer method leads to the new goal

```
⋀a. a ^ (0::nat) = 1
```

which is easily proved by the automatic proof method simp.

2.4 Summary

In this chapter we outlined the mathematical background in nonstandard analysis necessary to understand our formalisation along with a description of proof and concepts in Isabelle. At each point we connected various notions – e.g. the Transfer Principle, infinitesimals, internal sets and functions – to Euler's concepts and proof in order to contextualise them. We also discussed Isabelle's structured proof, locales and the Hilbert choice operator.

Chapter 3

Mechanisation of the mathematical context

In this chapter we will describe our formalisation of concepts and theorems from non-standard analysis and our formalisation of polynomials as functions. The choices we have made are guided by representing Euler's concepts and methods, following McKinzie and Tuckey's rehabilitation of Euler's proof, and developing a consistent mathematical context within which to formalise our proof of the Basel problem. Some concepts overlap with Isabelle's standard libraries or theories already in the Archive of Formal Proofs [3], however, we have taken a different approach which works better for representing Euler's methods. An example of this is our theory of polynomials which represents them concretely as functions rather than as abstract algebraic entities. This mechanisation could form a useful background for other formalisations of Euler's work or that of his contemporaries.

3.1 Extending the Isabelle theory on internal sets and functions

In Chapter 2 we discussed how we can use internal sets and functions to follow Euler's reasoning. Some useful and basic theorems on internal sets and functions, and some useful defined concepts, are not included in Isabelle's Nonstandard Analysis library. We will describe their formalisation here since these properties and definitions will be useful for the reader when we discuss our mechanisation of Euler's proof of the Basel problem in Chapter 6.

It is useful to define an operator unstarfun which obtains the underlying sequence defining any internal function.

```
definition unstarfun
  where "unstarfun X = (SOME F. X = *fn* F)"
```

The SOME operator used in the definition returns a fixed but arbitrary element F satisfying the property X = *fn* F if such an element can be proved to exist, which may not be possible (see Section 2.2.3 for an explanation of this operator).

We prove various properties of internal functions. Some properties relate the internal functions to their underlying sequences of real functions. Others are intrinsic properties. We prove, among other things, that the identity function is internal and that multiplication, addition, raising to a power etc. are internal operators. We prove that the composition of an internal function with another internal function produces an internal function and similarly that composition with an internal operator such as multiplication produces an internal function.

In our theory we also find it useful to define a partial inverse to star_n. The construct unstarnum, given a hyperreal, returns a representative of the equivalence class of real sequences defining that hyperreal.

```
definition unstarnum
  where "unstarnum X = (SOME A. X = star_n A)"
```

Again, the SOME operator used in the definition returns a fixed but arbitrary element A satisfying the property X = star_n A, if such an element can be proved to exist, which happily in this case is true for any X.

We have defined starfun2_n which allows us to express functions that are internal with respect to two variables. We proved the following theorems about it, among others, which express it in terms of internal functions of single variables.

```
lemma starfun2_n_starfun_n:
"(*f2n* f) (star_n X) (star_n Y) = (*fn* (λn x. f n (X n) x)) (star_n Y)"
and
"(*f2n* f) (star_n X) (star_n Y) = (*fn* (λn x. f n x (Y n))) (star_n X)"
```

We use starfun2_n in Section 5.8 where it helps us to express the notion of multiplying-out.

We also prove some basic properties of internal sets. Some properties relate the internal sets to their underlying sequences of real subsets, for example, that two internal sets are equal if their underlying sequences are equal for almost all terms in the sequence and that an internal bijection between internal sets is equivalent to the corresponding relation holding between their underlying sequences for almost all terms.

lemma eventually_bij_betw:
"bij_betw (*fn* f) (*sn* A) (*sn* B)
= ($\forall_F$ n in $\mathcal{U}$. bij_betw (f n) (A n) (B n))"

In the Isabelle Nonstandard Analysis library there are theorems stating that the intersections and unions of two internal sets are internal. This gives us an algebra of internal sets. Given a small number of fundamental internal sets, we can prove that a greater variety of sets are internal by writing them as intersections and unions of sets that we already know to be internal. We have shown that the empty set is internal and that hypernatural intervals are internal (Section 3.2.3), and thus we have obtained many internal sets by combining these with other known internal sets such as the hypernaturals and the hyperreals.

We have defined some notation for coercion functions. In pen-and-paper mathematics, we often combine a hypernatural number with a hyperreal number, for example 2π. However, multiplication in Isabelle takes two terms of the same type, and many other functions behave in this way. Therefore, we use coercion functions to transform elements of one type to that of a type that includes the first. For example hypreal_of_hypnat takes an element of type hypnat and returns the corresponding element of type hypreal. Thus 2::hypnat becomes 2::hypreal.

notation
hcomplex_of_hypreal ("_$_{\in\mathbb{R}}$" [100] 100)

notation
hypreal_of_hypnat ("_$_{\in\mathbb{N}}$" [100] 100)

3.2 Hyperpolynomials

Euler's 'Introductio' involves what a modern mathematician would call power series, as well as what would be considered polynomials, except that they are explicitly considered to have infinite degree. We formalise both these notions as *hyperpolynomials*. These are a direct generalisation of real polynomials to nonstandard analysis, but are allowed to have an infinite number of terms, as long as that number is a hypernatural. In the next sections, we will discuss the prerequisite notions of hypercardinality, hyperfiniteness and hyperfinite sums, and their formalisation in Isabelle, before combining these notions to give our definition of a hyperpolynomial as it is given in our formal theory.

3.2.1 Euler's concept of series, products and polynomials

Euler writes that 'the nature of polynomial functions is very well understood' [29, p. 50]. This is at a time when the Fundamental Theorem of Algebra was not yet proven. Euler clearly regards it as an indisputable fact nonetheless: at the beginning of his discussion of trinomial factors of polynomials he says 'It is clear now that the number of these linear factors is determined by the greatest power of z' (see Section 3.3).

Many of the 'polynomials' Euler deals with in the 'Introductio' are infinite series and products: at least, they are polynomials which have a degree which Euler explicitly states to be an infinite number (we give an example near the end of this section). One of Euler's most controversial arguments is his factorisation of the power series for sine as though it were an infinite polynomial. Euler was aware that infinite series and products, such as power series, do not always behave like true finite polynomials: he listened to the doubts of his contemporaries about his factorisation of the sine function and tried to provide additional justification for his reasoning [26, §3]. However, he certainly blurred the distinction between power series and polynomials, and we will soon see he was often vague about the length (or degree) of the polynomials he was discussing.

It is commonly claimed that there was no concept of the convergence of a series at the time of Euler [16, p. 367]. However, as McKinzie and Tuckey point out, Euler did have a similar notion of when a series takes a 'clear and definite value'. McKinzie and Tuckey develop this into the notion of Euler convergence (see Section 4.4.2) which they use to rehabilitate his proof of the Basel problem. Euler regards his infinite series as functions [29, p. 50] as opposed to formal power series. Just as we do now, he viewed functions as having definite values, stating 'a function itself of a variable quantity will be a variable quantity' [29, p. 3] Thus, some notion similar to convergence is important and, we argue, necessary to his reasoning.

In our modern way of thinking, sums and products are indexed by sets. The particular objects within these sets do not usually matter, only the cardinality of the sets, and sometimes whether or not they are ordered sets. The natural numbers are used for most indexing purposes. The summation sign $\sum$ was Euler's idea, however he did not use it until after he published the 'Introductio'. When he used it in 'Institutiones Calculi Differentialis' [23, §26] he used it without indices. Cauchy later used the summation sign with natural number indices [11, p. 61]. Within the 'Introductio', in the original Latin text, Euler writes '&c' to indicate the continuation of the series or product[1]. He rarely gives the general term of a series, instead he gives the first several terms, and it is sometimes nontrivial to deduce the general term from this. The length of the sum or product usually has to be worked out from previous steps, for example, Euler asks us to let N be an infinitely-large number and expands $(1+x/N)^N$ as $1+x/1+x^2/1\cdot 2+x^3/(1\cdot 2\cdot 3)+x^4/(1\cdot 2\cdot 3\cdot 4)+\&c$ [29, p. 94].

In the next section, we describe how we use modern indexing along with series and product notation to represent Euler's sums and products. The natural numbers are not sufficient to represent Euler's sums and products directly, since, as we have seen, Euler often uses infinitely-large numbers as the degrees of his polynomials. Hence we use hypernatural numbers for the indices.

3.2.2 Hypercardinality

Euler did not explicitly mention sets, nor did he mention cardinality. However, cardinality, taking some hypernatural value, is a useful way to define sets which can index the series and products of infinite natural number length which Euler defines. In Isabelle, cardinality is defined only for finite sets, i.e. those in bijection with a natural interval, and for all other sets it takes the value 0. In order to formalise Euler's proof of the Basel problem, we would like to define a version of cardinality which would give a value to any hyperfinite sets, i.e. those in bijection with a hypernatural interval. Due to the construction of the hyperreals as sequences of reals, and internal sets as sequences of real subsets, the notions of hyperfiniteness and hypercardinality are defined in our theory as follows.

```
definition hyperfinite :: "'a star set ⇒ bool"
where "hyperfinite A = (eventually (λn. finite (unstarset A n)) 𝓤)"

definition hypcard::"('a star) set ⇒ nat star"
where "hypcard A = star_n (λn. card ((unstarset A) n))"
```

If A is not internal, then the sequence unstarset A is an arbitrary element of the correct type, and nothing can be proven about it. But if A is internal, then card ((

[1] Blanton transcribes this as 'etc. ' [29].

unstarset A) n) becomes itself a sequence of the cardinalities of the sets which define A (see SubSection 2.3.2.4). Finally star_n transforms that sequence back into a nonstandard number, a hypernatural. Since hypcard is defined in terms of card, it inherits analogous properties to card and similarly for hyperfinite and finite . However, these properties still required some nontrivial proof in Isabelle since they do not follow from the existing transfer method in Isabelle which is defined only for nonstandard extensions (see Section 7.4). We prove, for example, that hyperfinite sets are in bijection with hypernatural intervals (see Section 3.2.3). This would allow us to index our hyperfinite sums with sets other than hypernatural intervals, although this has not been necessary in our theory.

3.2.3 Hyperintervals

The half-open interval $[a,b)$ is written {a ..< b} in Isabelle. We can directly use the Isabelle construction for an interval to define intervals of hypernaturals and hyperreals. However, in almost all situations that we use hyperintervals, we also need to use the property that the hyperinterval is an internal set. Thus we prove the following lemma which describes a hyper-interval in terms of its underlying sequence of standard intervals.

lemma interval_starset_n:
"*sn* (λn.{ (unstarnum a n) ..< (unstarnum b n)}) = {a..<b}"

Our theory of hyperintervals allows us to state and prove the following theorem which gives the characterising property of hypercardinality and hyperfiniteness. In the theorem below, {0 ..<(hypcard S)} is a hypernatural interval since hypercardinality is of hypernatural type. The predicate bij_betw f S1 S2 takes a function f and two sets S1 and S2. It is true if f forms a bijection between S1 and S2 and it is false otherwise.

lemma hyperfinite_imp_bij_betw_hypnat_int:
assumes "hyperfinite S" "S ∈ InternalSets"
shows "∃f. bij_betw f S {0 ..<(hypcard S)}"

The proof relies on the following lemma eventually_bij_betw proved in our theory characterising internal bijections between internal sets in terms of the ultrafilter.

lemma eventually_bij_betw:
"bij_betw (*fn* f) (*sn* A) (*sn* B) = ($\forall_F$ n in $\mathcal{U}$. bij_betw (f n) (A n) (B n))"

We have now defined and linked the concepts of hyperfiniteness and hypernatural intervals and we are ready to define hyperfinite sums and products.

3.2.4 Hypernatural indexed sums and products

Recall that Euler does not write out the *n*th term of a sum or product with an infinite natural number of terms and that the length of the sum or product usually has to be worked out from previous steps. It is not convenient to be intentionally vague with the degree of the polynomials in our Isabelle formalisation, thus we use a more precise notation. We define a hyperfinite sum as follows.

```
definition hypsum ::
"('a star ⇒ 'b star) ⇒ ('a star) set ⇒ 'b star"
where "hypsum f S = star_n (λn. (sum (unstarfun f n) (unstarset S n)))"
```

Here 'a and 'b denote polymorphic variable types, which stand in place of whichever concrete type, real, nat etc. which we may wish to instantiate them to. star is the type constructor for nonstandard numbers e.g. real star would be type hyperreal. This allows hypsum to be indexed by a set of any nonstandard type and have terms of any nonstandard type. Notice that we define hypsum in terms of the already existing sum defined for standard types. To do this, we use unstarfun and unstarset to obtain the underlying standard sequences defining the nonstandard function and index set. Such sequences only exist for internal functions and sets, thus hypsum is only well-defined for internal functions and sets. Similarly to the definitions of hyperfiniteness and hypercardinality, this will only be meaningful if the function and set are internal. Thus far it does not seem obvious why we describe hypsum as a hyperfinite sum, but, in the same way that sum relies on its index set being finite for its characterising properties to hold, so does hypsum rely on its index set being hyperfinite for us to be able to prove useful properties of hypsum. Hyperfinite products are defined analogously as follows.

```
definition hyprod ::
"('a star ⇒ 'b star) ⇒ ('a star) set ⇒ 'b star"
where "hyprod f S = star_n (λn. (prod (unstarfun f n) (unstarset S n)))"
```

With these notions of hyperfinite sums and products, we can now represent hyperpolynomials as functions, which gives us a way to imitate Euler's infinite polynomials in Isabelle. In the next sections, we describe our hyperpolynomials and use them to formalise a version of the Fundamental Theorem of Algebra.

3.2.5 Hyperpolynomials

With these notions of hyperfinite sums and products, we can now represent hyperpolynomials as functions, which gives us a way to imitate Euler's infinite polynomials in Isabelle. Euler helpfully tells us that the general form of a polynomial is.

> In a polynomial function the variable z has no negative exponents whatsoever, nor does it contain fractional expressions in which the variable z enters into the denominators ... This then is the general formula for polynomial functions:
>
> $$a+bz+cz^2+dz^3+ez^4+fz^5+\cdots$$
>
> etc. It is impossible to think up a polynomial function which is not included in this expression. [29, p. 6][2]

Euler considers his polynomials to have real coefficients, although he does acknowledge the possibility of complex roots [29, p. 116]. He discusses 'polynomials' of infinite degrees, and treats them mostly in the same way. Thus we consider a hyperpolynomial to be a function that can be written in the canonical form

$$\sum_{n=N}^{M} a_n x^n \tag{3.1}$$

where a_n is an internal function of n, and N and M are hypernaturals. This is the general form of the polynomial given by Euler expressed in nonstandard analysis, although we have allowed the polynomial to start at exponent N rather than 0, and we have specified the degree of the polynomial. We initially formalised hypercomplex hyperpolynomials, i.e. hyperpolynomials of a hypercomplex variable with hypercomplex coefficients. This was because we planned to use the Fundamental Theorem of Algebra (FTA) which in its most general form must be expressed in terms of polynomials with complex coefficients. The following definition gives a relation which is defined generally and can be applied to both hypercomplex and hyperreal hyperpolynomials.

```
definition isHypfactorOf ::
"(('a ::{power,comm_monoid_add}) star ⇒ 'a star)⇒ ('a star ⇒ 'a star) ⇒ bool"
 ("_ isHypfactorOf _" [60, 60] 60) where
"A isHypfactorOf B = (∃N M cf. B =
  (λ x. A x · hypsum (λn. (*fn* cf) n · x pow n) {(N::nat star) ..< M + 1}))"
```

[2]The ellipsis is added.

The idea behind this definition is that A is a hyperfactor of B whenever there is a hyperpolynomial C such that $B = AC$. We judge that C is a hyperpolynomial if it can be written in the form hypsum (λn. (*fn* cf) n · x pow n) {(N::nat star) ..< M + 1}. In Section 5.3.1 we explain why we take the interval {N ..< M + 1} rather than {N .. M}. This is our Isabelle representation of (3.1). We have proven that isHypfactorOf is a reflexive relation. We have partially proved it is a transitive relation. The proof is complete apart from the lemma stating that a hyperpolynomial multiplied by a hyperpolynomial is itself a hyperpolynomial. This could be proved by internal induction. We also prove some other general properties including

lemma linear_factor_gives_root:
assumes "(λx.(x − (a::hypreal))) isHypfactorOf B"
shows "B a = 0"

and

lemma isHypfactorOf_multiple:
assumes "a isHypfactorOf b"
shows "a isHypfactorOf (λx. (k::'a:: field star) · b x)".

However, for the main part of the Basel problem, we used hyperpolynomials with hyperreal coefficients of a hyperreal variable. We also mechanised many properties of these which were necessary in proving the hidden lemmas (see Section 4.4). The main challenges were proving that the functions involved were internal, which was time-consuming but not of particular interest. Here is a typical example of a useful property of hyperpolynomials. In mathematical notation this would be

Lemma 1. *Let a, k and N be hypernaturals with $k \leq a \leq N$. Then*

$$\sum_{i=a}^{N} c_i x^n = x^k \sum_{i=a}^{N} c_i x^{n-k}$$

and its Isabelle representation is below.

lemma polynomial_extract_factor_of_x_pow_k:
assumes "a≤N" "k ≤a"
shows "hypsum (λn. (*fn* c) n · (x :: hypreal) pow n) {a..<(N::nat star)} =
x pow k · hypsum (λn. (*fn* c) n · x pow (n − k)) {a..<N}"

We now describe some concepts that we defined in order to index our hyperpolynomials appropriately in our proof of the Basel problem.

3.2.5.1 Even and odd indexing of hyperpolynomials

Odd and even functions are defined by odd and even power series. Hyperbolic sine has an odd power series and in Euler's proof of the Basel problem, he expands $(e^x - e^{-x})/2 = 1/2\left[(1+x/N)^N - (1-x/N)^N\right]$ (i.e. hyperbolic sine) as

$$x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots$$

We defined generally the concept of odd and even intervals in Isabelle to give a convenient and expressive way to describe this reasoning. We defined strict and non-strict intervals with both or single bounds. Here is an example.

definition
  lessThanOdd :: "'b ::{comm_monoid_add,comm_monoid_mult,minus,ord,
    semiring_parity} => 'b set" ("(1{..<odd_})")
  where "{..<odd M} ≡{x. x<M ∧odd x}"

We also proved characterising theorems of these intervals, for example:

lemma hypsum_even_characterisation_cases:
"hypsum (*fn* a) {N..<even M} =
 hypsum (λn. if ∃k. n = 2 · k then (*fn* a) n
   else (0:: hypreal)) {N..<(M::hypnat)}".

Later in McKinzie and Tuckey's reconstruction of Euler's proof, they describe the polynomial as a product of half of its degree (interpreted as a hypernatural number). The parity of the degree affects what half its degree would be. So we have proven relations between even intervals and intervals up to 'half degrees'. We use a transferred version of the ceiling function $^*\lceil \mathsf{N} \rceil^1{}_2$ in our representation, defined as follows.

definition upper_half_nat :: "nat ⇒ nat" ("$\lceil$_$\rceil^1{}_2$")
  where "$\lceil$N$\rceil^1{}_2$ = (if ∃k. N = 2 · k then N div 2 else (N+1) div 2)"

definition upper_half_nat_star :: "nat star ⇒ nat star" ("$^*\lceil$_$\rceil^1{}_2$")
  where "$^*\lceil$N$\rceil^1{}_2$ = star_n (λn. $\lceil$unstarnum N n$\rceil^1{}_2$)"

We can then characterise even intervals using $^*\lceil \mathsf{N} \rceil^1{}_2$ as follows.

lemma even_interval_upper_half_nat_star:
"⋀i. (2 · i ∈ {0..<even N}) = (i ∈ {0..<$^*\lceil$N$\rceil^1{}_2$})"

3.3 Fundamental Theorem of Algebra

Euler's proof of the Basel problem uses a trinomial (in modern terminology, quadratic) factorisation of an infinite polynomial (Section 4.2.2). This relies on a nonstandard version of the statement of the Fundamental Theorem of Algebra (FTA) which we mechanise using our theory of hyperpolynomials. Euler's treatment of infinite-polynomials as though they were finite was one of the aspects of his proof of the Basel problem that was remarked upon by his contemporaries.

3.3.1 The role of the FTA in our mechanisation

In the following discussion and formalisation of the statement of the FTA, we explore Euler's understanding of the FTA and how we could represent that in our theory of hyperpolynomials. This is important since it clarifies Euler's understanding of his infinite polynomials and demonstrates that he did not consider them entirely separately from the usual polynomials of finite-degree. With our nonstandard representation of polynomials, we were able to represent that aspect of his thinking since the hyperpolynomials can take finite or infinite degree. However, in our mechanisation, we did not prove the Trinomial Lemma which is based on the FTA (see Section 6.4.1 for our discussion of the Trinomial Lemma) and we consider both of these (the FTA and the Trinomial Lemma) to be external to the reasoning that makes up the proof of the Basel Problem, and only of interest with regards to Euler's concepts. In Figure 6.1 there is a visual representation of the role of the FTA in our mechanisation.

3.3.2 Euler and the Fundamental Theorem of Algebra

Euler had an interesting relationship with the Fundamental Theorem of Algebra (FTA). In 1742, he claimed to have a proof of it for polynomials of degree up to 6. In 1748, within the 'Introductio' at the beginning of his discussion of trinomial factors of polynomials, he states:

> It is clear now that the number of these linear factors (of a polynomial with real coefficients) is determined by the greatest power of z. [29, p. 116]

In 1749, the year after he published the 'Introductio' he published a proof of it: a proof which assumed that a polynomial of degree n will have n complex roots. This fact seems almost equivalent to the FTA to a modern reader. However, Dunham states that in the mid 18^{th} century the FTA was considered to be as follows:

> Any polynomial with real coefficients can be factored into the product of real linear and/or real quadratic factors [17, p. 283].

Euler calls these real quadratic factors ‘trinomials’ and devotes Chapter 9 of the ‘Introductio’ to outlining their form both in the general case and for particular polynomials: including the polynomial at the heart of the Basel problem. His derivation of the trinomial factors rests on the modern version of the FTA:

> Any non-constant polynomial of degree n with complex coefficients has, counted with multiplicity, exactly n complex roots.

Therefore we find it necessary to mechanise this version of the FTA (Section 3.3.3). Gauss later gave the first reasonably complete proof of the FTA (both in its modern and 18th century forms) in his PhD dissertation of 1799 [17, p. 292][39]. He gives his own analysis of Euler’s proof, making several criticisms, including pointing to Euler’s assumption that the polynomial will have as many roots as its degree. He says:

> ...it can certainly not be understood with that clarity which must always be insisted upon in mathematics...

However, Gauss’ own proof rests on an assumption which he considered to be obvious, yet was only proven in 1920 [75, p. 290][3]. This demonstrates a common theme in 18th century proofs: if assumptions were convincing, it was not always considered necessary to prove them, and many of them were out of the reach of the contemporary mathematical knowledge.

In the proof of the Basel problem, Euler applied his FTA to a polynomial of infinite degree. This was a point which was criticised by his contemporaries. However, in nonstandard analysis, the FTA can be extended via the transfer principle to hyperpolynomials including those with infinite hypernatural degree. In the next section we describe such a version of the FTA.

3.3.3 Mechanisation of the Fundamental Theorem of Algebra

Euler thought of the FTA as finding real linear and real quadratic factors of polynomials with real coefficients. Euler might not recognise the modern version of the FTA as the 18th century one, however, he would recognise it as an obvious fact: so obvious that he thought it was an acceptable assumption to base his proof of the 18th century FTA

[3] This assumption is that two particular algebraic curves meet inside a particular circle. There is some background setup before the assumption can be stated precisely so we will not summarise it here but it is well explained in Stillwell’s book [75, p. 290].

on. There is a formalisation and proof of the FTA in Isabelle/HOL/Computational-Algebra. This uses the formulation that every nonconstant single-variable polynomial with complex coefficients has a complex root.

lemma fundamental_theorem_of_algebra:
assumes nc: "¬ constant (poly p)"
shows "∃z::complex. poly p z = 0"

However, we cannot directly apply this theorem to our polynomials. The term poly refers to abstract algebraic objects. It would be nontrivial to prove the equivalence of our functional representation of polynomials with the algebraic representation, and so we directly formulate the version of the FTA which is useful to us. Via the transfer principle, we can and have extended the FTA to all hyperpolynomials including those with infinite hypernatural degree.

Theorem 2 (Fundamental Theorem of Algebra for Hyperpolynomials)**.** *Given a hyperpolynomial* $^*\sum_{n=0}^{N} b_n x^n$ *with internal coefficients* $b_n : {}^*\mathbb{N} \to {}^*\mathbb{C}$ *for which the Nth coefficient* b_N *is nonzero, we can write it as a product of linear factors* $(x - a_n)$ *given by the hyperfinite sequence of internal coefficients* $a_n : {}^*\mathbb{N} \to {}^*\mathbb{C}$ *such that*

$$^*\sum_{n=0}^{N} b_n x^n = b_N {}^*\prod_{n=1}^{N} (x - a_n)$$

As we stated in Section 2.2.2, we formalise this in a locale. It will be possible to instantiate the locale if our functional polynomials are unified with the algebraic representation. We did make some progress towards this by proving ‘equating coefficients’ (see Section 6.3) which shows that polynomials are uniquely defined by their coefficients in their canonical representation $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^a + \cdots$.

locale Fundamental_theorem_of_algebra =
assumes FTA:
"(*fn* b) N ≠ 0 ⟹
∃a. (λx. ((*fn* b) N) · hyprod (λn. x − (*fn* a) n) {1..< N+1})
= (λx. hypsum (λn. (*fn* b) n · (x :: complex star) pow n) {0 ..< N+1})"

It would be possible to derive another version of the FTA as a lemma. This version gives a degree and lowest exponent, and appears to be more general but in fact it is equivalent, since we can define a polynomial to have some coefficients that are zero, and that way include a coefficient for every power of the variable up to the degree.

lemma FTA_upper_and_lower_degree:

```
assumes " N ≥1" "M ≤N" "( *fn* b) N ≠ 0"
shows
"∃a. (λx. (( *fn* b) N) · hyprod (λn. x − ( *fn* a) n) {1..< N})
= (λx. hypsum (λn. ( *fn* b) n · (x::complex star) pow n) {N ..< (M + 1)})"
```

3.4 Overspill and Sequential Theorems

Euler's proof deals with what modern mathematicians consider to be convergent sequences of elementary functions. As we have seen in Section 2.3.2.2, our nonstandard analysis interpretation of these is as internal functions of the hypernaturals. If we subsequently make a statement about these sequences, as long as that statement uses only internal operations, the statement will itself be an internal predicate. By definition, internal predicates can only be true on internal sets. We can use this fact to our advantage and derive the truth of the predicate on wider sets than we originally proved it for. E.g. if we prove an internal predicate true for all infinite hypernaturals, since the infinite hypernaturals do not form an internal set, the predicate must also be true for some finite hypernaturals. We can actually formulate a stronger statement called the Overspill Theorem by McKinzie and Tuckey [57] or Overflow Principle by Goldblatt [40, p. 129]. We shall call it 'overspill' since that is the most common terminology for this theorem. McKinzie and Tuckey give a version of the Overspill Theorem used in the proofs of their Hidden Lemmas.

Theorem 3 (McKinzie and Tuckey's Overspill)**.** *Let $\phi(n)$ be an equation or inequality of elementary sequences. Then $\phi(j)$ holds for all finite j greater than some finite m iff $\phi(J)$ holds for all infinite J less than some infinite M.*

McKinzie and Tuckey find it sufficient to specify that it holds for equations and inequalities between elementary sequences. A more general version for internal sets is given in Goldblatt's textbook [40, p. 129]. The theorem is given here as Goldblatt wrote it apart from changes in notation.

Theorem 4 (Goldblatt's Overspill)**.** *Let S be an internal subset of ${}^*\mathbb{N}$ and $k \in \mathbb{N}$. If $n \in S$ for all $n \in \mathbb{N}$ with $k \leq n$, then there is an infinite $K \in {}^*\mathbb{N}$ with $n \in S$ for all $n \in {}^*\mathbb{N}$ with $k \leq n \leq K$.*

Goldblatt only gives the implication in one direction but trivially it would hold in the other direction (since all $n \in \mathbb{N}$ are less than any infinite K). The version of this theorem that we formalise in Isabelle is similar to the way that the way McKinzie and Tuckey phrase their theorem, yet it is as general as Goldblatt's version. The mathematical version of our theorem can be given as follows.

Theorem 5 (Our Overspill)**.** *Let S be an internal subset of ${}^*\mathbb{N}$. If for all $n \in \mathbb{N}$, we have $n \in S$, then there is an infinite $K \in {}^*\mathbb{N}$ with $n \in S$ for all $n \in {}^*\mathbb{N}$ with $n \le K$.*

Although this seems at first glance less general than Goldblatt's version, we can argue that it is equivalent with it. First assume our version and try to prove Goldblatt's. Since S is a variable, it can be instantiated to any internal set and any internal predicate $P(n)$ defines an internal set $S = \{x. P(x)\}$. Given an internal subset T of ${}^*\mathbb{N}$, then $P(n) = k \le n \to n \in T$ would be one such internal predicate. Plugging this instantiation into our version gives us Goldblatt's version. Then for the other direction, just choose the $k \in \mathbb{N}$ from Goldblatt's version to be 0.

The following lemma is the basic principle upon which the Overspill Theorem rests: an internal predicate can only be true over an internal set. Our proof in Isabelle uses the definition of an internal predicate.

```
lemma internal_predicate_true_on_internal_set:
"{x. ( *pn* P) x} ∈ InternalSets"
```

The Overspill Theorem is related to a corollary: the Sequential Theorem.

Theorem 6 (McKinzie and Tuckey's Sequential Theorem)**.** *Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be elementary sequences. If $a_n \simeq b_n$ for all finite n, then there is an infinite N such that for all n smaller than N, $a_n \simeq b_n$.*

At first it looks like a simple application of the Overspill Theorem to a predicate, until we recall that the infinitely-close relation $\simeq$ is not an internal operator. Thus the proof involves rewriting the relation $a_n \simeq b_n$ between terms of the sequence as an internal predicate which is provably equivalent to the original relation (see Section 3.4.2).

The Sequential Theorem is used to prove the Hidden Lemmas, as formulated by McKinzie and Tuckey. We discuss the Hidden Lemmas in the context of Euler's proof in Section 4.4 and we describe their mechanisation in Isabelle in Chapter 6. However, as the Overspill and Sequential Theorems are general results in nonstandard-analysis we will describe their mechanisations here.

3.4.1 Mechanisation of the overspill theorem

McKinzie and Tuckey provide a proof only of the first direction.

> For the first direction, assume that m is finite and that $\phi(j)$ holds for all finite j greater than m. If $\phi(j)$ holds for all j greater than m then we are done. Otherwise, there is a least counterexample; that is, there is an M greater than m such that $\phi(M)$ fails but $\phi(j)$ holds for all j greater than m and less than M. This M cannot be finite. The other direction is left to the reader.

The following lemma formalises the first direction of the proof for our Overspill Theorem.

```
lemma Nats_overspill_interval:
assumes "∀n∈ℕ. (*pn* p) n"
shows "∃N∈HNatInfinite. ∀n<N. (*pn* p) n"
```

McKinzie and Tuckey's proof does not explicitly rely on internal propositions or sets. However, when formalising this in Isabelle we were able to determine that the step where they obtain a 'least counterexample' does rely on the concepts involved being internal. The Isabelle proof of this lemma is similar to McKinzie and Tuckey's mathematical proof. Our internal proposition (*pn* p) n corresponds to their $\phi(j)$. Instead of considering whether $\phi(j)$ holds for all j greater than m we make a case-split on the proposition {n. (*pn* p) n } = (UNIV::nat star set). If it is true, then (*pn* p) n holds for all hypernaturals. Thus ∃N∈HNatInfinite. ∀n<N. (*pn* p)n as long as there exists an infinite hypernatural. If not, we construct the internal set S which describes the values on which (*pn* p) n does not hold.

The two following lemmas show the converse direction of the Overspill Theorem which is much easier to prove. Note that we can prove this direction also for the hyperreals.

```
lemma internal_predicates_on_HInfinite_also_on_finite_values:
assumes "∀(x::real star)∈HInfinite. ( *pn* P) x"
shows "∃k∈HFinite. ( *pn* P) k"
```

```
lemma internal_predicates_on_HNatInfinite_also_on_finite_values:
assumes "∀(x::nat star)∈HNatInfinite. ( *pn* P) x"
shows "∃k∈HNatFinite. ( *pn* P) k"
```

Hidden Sublemma (ii) (see Section 5.8) uses the converse direction of the Overspill Theorem, however, the first direction of the Overspill Theorem is used in the proof of the Sequential Theorem (Section 3.4.2) and the rest of the Hidden Lemmas depend on the Sequential Theorem (see Sections 5.3- 5.5). Hence the Hidden Lemmas require both directions of the Overspill Theorem.

3.4.2 The Sequential Theorem

We use the Sequential Theorem in our proofs of the Hidden Lemmas in Sections 5.3-5.5. Our mechanised proof of the Sequential Theorem follows that of McKinzie and Tuckey [55]. It is an application of the Overspill Theorem, but as McKinzie and Tuckey say, not a direct application since the relation $\simeq$ is not internal. We replace $a_n \simeq b_n$ with $|a_n - b_n| < 1/n$, with $n \in {}^*\mathbb{N}$, which is internal so that we can apply the Overspill Theorem. We formalise McKinzie and Tuckey's argument by a case-split on whether n is infinite.

```
lemma sequential_theorem:
assumes "∀n∈ℕ. ( *fn* a) n ≈((( *fn* b) n) :: hypreal)"
shows "∃N∈HNatInfinite. ∀n<N. ( *fn* a) n ≈( *fn* b) n"
```

We prove the Sequential Theorem using the following lemma which is an intermediate step between the Overspill Theorem and the Sequential Theorem.

```
lemma Nats_overspill:
assumes "∀n∈ℕ. (*pn* p) n"
shows "∃N∈HNatInfinite. (*pn* p) N"
```

Our proof of this theorem relies on showing that the infinite hypernaturals are an external set in the hypernaturals. We describe external sets and their formalisation in the next section.

3.4.3 External sets

External sets are exactly those which are not internal (Section 2.3.2.4). Examples of external sets in the hyperreals include the infinite hypernaturals, the finite hyperreals and the infinitesimals. Clearly external sets can still be useful sets to discuss. We give the following definition of external sets in our formalisation.[4]

```
definition ExternalSets where "ExternalSets = {E. ¬(∃e. E = *sn* e)}"
```

[4]Alternatively we could have given the definition ExternalSets = − InternalSets which would have been a simpler expression. Instead we prove this from the definition.

Externality and internality of sets are relative to the type in which those sets are embedded. For example, UNIV is the set of all elements of some type and UNIV is always internal, with respect to its own type. Yet UNIV may be any set provided that set forms the entire type. Even the infinite hypernaturals are internal relative to the type that they define. Thus we have proven separately in our formalisation that the infinite hypernaturals are external relative to the hyperreals and relative to the hypernaturals. Mathematicians e.g. Goldblatt, do not usually think in terms of types and Goldblatt simply states that any set of interest e.g. $\mathbb{N}$ is external, but he does make it clear at the beginning of the section that he is considering them as subsets of the hyperreals [40, p. 131].

To prove a set is external, we typically use the algebra of internal sets and argument by contradiction to show that it cannot be internal. If it were internal, then combining it with an internal set (either by union, intersection, difference or other means) would produce an internal set. However, to begin this chain of reasoning, it is necessary to start with one set which we know to be external. Following Goldblatt [40, p.132] we first show that the infinite hypernaturals of type hypnat are external. This follows from the internal least number principle which states that every internal subset of the hypernaturals contains a smallest member. In our formalisation we prove the following fundamental sets are external:

1. the infinite hypernaturals of type hypnat i.e. $\mathbb{L} \cap {}^*\mathbb{N} \subset {}^*\mathbb{N}$ (in our Isabelle theory as HNatInfinite_ExternalSets),

2. the finite hypernaturals of type hypnat i.e. $\mathbb{N} \subset {}^*\mathbb{N}$ (in our Isabelle theory as HNatFiniteInHypnats_ExternalSets),

3. the finite hypernaturals of type hypreal i.e. $\mathbb{N} \subset {}^*\mathbb{R}$ (in our Isabelle theory as HNatFiniteInHypreals_ExternalSets),

4. the hypernaturals of type hypreal i.e. ${}^*\mathbb{N} \subset {}^*\mathbb{R}$ (in our Isabelle theory as HNatInHypreals_internal),

5. the infinite hyperreals i.e. $\mathbb{L} \subset {}^*\mathbb{R}$ (in our Isabelle theory as HInfinite_ExternalSets),

6. the finite hyperreals i.e. $\mathbb{L} \cap {}^*\mathbb{R} \subset {}^*\mathbb{R}$ (in our Isabelle theory as HFinite_ExternalSets).

Each hypernatural set is proved to be external twice: once as a set of type hypnat set, and once as a set of type hypreal set. The hypnat proofs follow Goldblatt's reasoning. The hypreal proofs use coercion functions which force some difference between formal proof in Isabelle and standard mathematical proof. For an Isabelle proof, we need to prove that the inverse of the coercion function is internal, which makes the reasoning more involved.

3.5 Summary

In this chapter we described the mechanisation of some of the results from nonstandard analysis necessary for the proof of the Basel problem:

- definition of unstarfun (Section 3.1): an operator that, for nonstandard functions, allow us to find a representative of the equivalence-class of the underlying sequences of standard functions;
- definition of unstarnum (Section 3.1): an operator that, for nonstandard numbers, allow us to find a representative of the equivalence-class of the underlying sequences of standard numbers;
- further properties of internal sets (Section 3.1);
- the Overspill Theorem (Section 3.4) with its proof in Isabelle (Section 3.4.1);
- the Sequential Theorem (Section 3.4.2) with its proof in Isabelle;
- External sets (Section 3.4.3), their definition and proofs that some of the fundamental sets are external.

We also described the mechanisation of a theory of polynomials in Isabelle designed to more directly capture Euler's reasoning about these objects.

- hyperfiniteness, hypercardinality, hyperintervals and their characterising theorems with proofs in Isabelle (Sections 3.2.2- 3.2.3);
- hyperfinite sums and products and their characterising theorems with proofs in Isabelle (Section 3.2.4);
- The Fundamental Theorem of Algebra (Section 3.3.3) as a locale.

In order to place these in context and compare them to Euler's concepts, we described

- Euler's notion of series, product, and polynomial (Section 3.2.1);
- Euler's version of the Fundamental Theorem of Algebra (Section 3.3.2).

In the next chapter we introduce the reader to Euler's proof and the concepts of *hidden lemma* and *E-convergence* which will be necessary to the formalisation.

Chapter 4

Analysis of the Basel Problem

The purpose of this chapter is to outline Euler's proof of the Basel problem, and make note of the points of interest, thereby allowing the reader to understand what is going on from a mathematical perspective. This is particularly important since we intend our mechanisation of the Basel Problem to improve our understanding of Euler's proof. We have informed our mechanisation using McKinzie and Tuckey's rehabilitation of the proof [57], as well as Kanovei's analysis of Euler's factorisation of sine [76], and occasionally we have looked to standard-analysis rewritings of the proof (e.g. Eberlein [21]) for interpretation of Euler's reasoning.

Therefore, in this chapter, we will discuss Euler's proof with reference to the approaches and insights we have taken from these prior interpretations. We give an overview of two of Euler's proofs of the Basel problem. First, his most criticised proof in which he factorised sine as an infinite polynomial [27], and second, his proof given in the 'Introductio' [29]. On occasion, we point out how modern mathematicians interpret the various parts of the proof. We give special attention to the interpretations in terms of nonstandard analysis, as these are most relevant to us, but we also mention standard analysis accounts of the proof where appropriate. We do not seek to discuss Euler's proof, independently and objectively, but instead build upon the prior work. We identify new points of interest as we proceed and seek to examine these in Euler's terms.

We also introduce the concepts of hidden lemmas, Euler-convergence and determinacy which were introduced by McKinzie and Tuckey to rehabilitate Euler's proof. These are used centrally in our formalisation.

4.1 Overview of Euler's most criticised proof

In the introduction we mentioned some of the criticisms of Euler's proof of the Basel problem. Most of the modern criticisms are directed at what could be called his 'third' proof, from 1736, which was a refinement of his argument from 1734 that the sum of the reciprocals of the squares add up to $\frac{\pi^2}{6}$ [27].

In this third proof, Euler states that the power series for sine can be obtained by 'a well-known method' [27, §3, p. 3]. This method is to write sine as a product of linear factors given by its roots

$$\sin s = s\left(1-\frac{s^2}{\pi^2}\right)\left(1-\frac{s^2}{4\pi^2}\right)\left(1-\frac{s^2}{9\pi^2}\right)\left(1-\frac{s^2}{16\pi^2}\right)\cdots$$

and multiplying-out this product to obtain a power series for sine.

$$\sin s = s - \left(\frac{1}{\pi^2}+\frac{1}{(2\pi)^2}+\frac{1}{(3\pi)^2}+\cdots\right)s^3+\cdots \tag{4.1}$$

He says '...if our method should appear to some as not reliable enough...' [27, §10, p. 4] indicating that he considered this 'proof' as a method of calculating a result, rather than an indisputable chain of reasoning. Pólya argues that he meant this argument in the spirit of scientific reasoning [67]. McKinzie and Tuckey summarise Pólya's points neatly.

> A scientific hypothesis gains confirmation (or corroboration) when it is shown to lead to (i) the *explanation* of an already known phenomenon, and (ii) the *prediction* of some unexpected new phenomenon. In 1734, Euler established both of these points for [his method of factorising sine by its roots to obtain a power series].

Euler obtains the sum of the reciprocals of the squares by equating the second coefficient of (4.1) with the second coefficient of the other power series for sine $s-\frac{1}{6}s^3+\frac{1}{24}s^5-\cdots$.

In a paper written eight years later, Euler gives another proof of the Basel problem and writes a description and criticism of his own earlier proof [26]. It is interesting to look at what he says about his much criticised proof. He says he regards the theorem as 'already established truth' [33, §1, p. 2], implying that he was satisfied with his earlier derivation of the theorem. He says that '...the argument pleased the smartest Geometers so much that they did not only consider it to be correct but also invested a lot of work to find the same summations using methods familiar to them' [33, pp. 1-2]. It is interesting that he does not seem to think it was the rigour of the argument which convinced them, otherwise he might have said 'there was no step they could argue with' but rather the brilliance of it: it 'pleased them'. Yet he also acknowledges that the method could 'justly be in doubt' [33, §3, p. 2]. He states that his method of considering sine as an infinite polynomial and factoring it according to its roots was novel. He also justifies why he thought it was appropriate to factor it that way in this instance. According to Euler, Bernoulli and Cramer originally doubted his method but now these doubts are 'almost forgotten'. In this paper [26] Euler proves that the power series of sine has no imaginary roots and claims that it was therefore valid to factor it using its real roots.

4.2 Overview of Euler's proof from the Introductio

Euler's proof from the 'Introductio' was given over several chapters of that book and is intertwined with other results.

The strategy of Euler's proof of the Basel problem from the 'Introductio' is to write the polynomial $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$, with infinite hypernatural N, in two different ways. This is in fact hyperbolic sine, up to an infinitesimal. This is not mathematically essential to Euler's argument, when interpreted with a formalist understanding, but it did influence his method of discovery and also provides a convenient name for the polynomial. Because of the way his proof is presented in the 'Introductio', we could choose to interpret the first stage of Euler's proof either as using the MacLaurin series for hyperbolic sine, or we could interpret it as expanding $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ directly by the Binomial Theorem. In Chapter 7, Euler writes a^x as a polynomial which can be expanded into its MacLaurin series using the Binomial Theorem, and thereby he writes $\frac{e^x-e^{-x}}{2}$ as $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ which he then expands using the Binomial Theorem. Since the rest of the proof of the Basel problem can be reduced to expanding $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ in a different way, it would be unnecessary to include rewriting into $\frac{e^x-e^{-x}}{2}$ as part of the formal proof, even though Euler took this expression as his starting point. This should explain why some mathematicians describe this proof as factorising the sine function (really hyperbolic sine, but of course they are easily obtained from each other by a substitution of ix as Euler noted) as an infinite polynomial.

We note the following points of interest in the structure of the reasoning of Euler's proof

1. The expansion of hyperbolic sine using the Binomial Theorem

2. The application of the Trinomial Lemma and subsequent multiplication of the factors with discarded increments (see Section 4.6)

3. The substitution of a cosine term

4. The apparent equating of coefficients to obtain the final result

These all correspond to 'gaps' or points of interest identified by McKinzie and Tuckey, Kanovei or other mathematicians. It is difficult to neatly separate these points of interest into distinct and singular parts of the proof. This is evidenced by the different ways they have been split up by modern commentators. Also, when we begin to analyse and mechanise each of point of interest in Euler's proof, we realise that there is an intricate web of reasoning behind each of these steps. Some may be the application of more than one significant result, depending on how Euler's reasoning is interpreted.

4.2.1 Binomial expansion

First Euler uses the Binomial Theorem to expand $\left(1+\frac{x}{N}\right)^N$ and $\left(1-\frac{x}{N}\right)^N$. The expansion of $(1+\frac{x}{N})^N$ by the Binomial Theorem is

$$1+\frac{1}{1}x+\frac{1(N-1)}{1\cdot 2N}x^2+\frac{1(N-1)(N-2)}{1\cdot 2N\cdot 3N}x^3+\frac{1(N-1)(N-2)(N-3)}{1\cdot 2N\cdot 3N\cdot 4N}x^4+\cdots$$

Euler says (ellipses are added)

> Since N is infinitely large, $\dfrac{N-1}{N}=1\ldots\dfrac{N-2}{N}=1$, $\dfrac{N-3}{N}=1$, and so forth. It follows that $\dfrac{N-1}{2N}=\dfrac{1}{2}$, $\dfrac{N-2}{3N}=\dfrac{1}{3}$, $\dfrac{N-3}{4N}=\dfrac{1}{4}$, and so forth. ... we substitute these values ... [29, §116]

He thus obtains

$$\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]=\frac{x}{1}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\ldots \tag{4.2}$$

We describe how we mechanise the nonstandard Binomial Theorem in Section 6.5. Notice that if we suppose N to be equal to some odd hypernatural K, this polynomial is the same for K or for $K+1$. Thus, the degree N is equal either to such a K, or such a $K+1$, depending on whether N is odd or even. If $N=K+1$ is even, we may take K to be the degree instead. Hence we may assume N is odd. In Isabelle, we may formalise this using a case split (discussed in Section 2.2). Euler does not make this reasoning explicit, but in the next step of the proof it becomes relevant.

4.2.2 Trinomial factors

In Chapter 9 of the 'Introductio, Euler discusses 'trinomial factors': these are polynomial factors which we would nowadays call 'quadratic'.[1] He discovered that

Theorem 7 (Trinomial Lemma)**.** *The factors of $a^n - z^n$ are $a^2 - 2az\cos\frac{2k\pi}{n} + z^2$ where we let $2k$ range through all even integers not exceeding n.*

The 'Trinomial Lemma' is our name for this theorem and it is a special case of Euler's more general exploration of the trinomial factors of polynomials. We discuss our formalisation choices relating to it in Section 6.4.1. However, Euler states that for any such factors which are perfect squares, we instead take only the square-root of the trinomial as the factor [29, §151]. Euler applies this lemma with $a = 1 + \frac{x}{N}$, $z = 1 - \frac{x}{N}$ and $n = N$ for an infinitely-large natural number N, and as we may assume N is odd, we find that $2az\cos\frac{2k\pi}{n}$ is only equal to $2az$ when $k = 0$, due to the properties of cosine. Hence, $a^2 - 2az\cos\frac{2k\pi}{n} + z^2$ is only a perfect square when $k = 0$, in which case, the factor is $a - z = \frac{2x}{N}$, however Euler discards the constant factor $\frac{2}{N}$ and only says 'the resulting first term will be x'.

4.2.3 Rewriting the factors

In next step of the proof Euler rewrites the general factor in a simpler form. We have

$$a^2 - 2az\cos\frac{2k\pi}{n} + z^2 = 2 + \frac{2x^2}{N^2} - 2\left(1 - \frac{x^2}{N^2}\right)\cos\frac{2k}{N}\pi. \tag{4.3}$$

Now we rewrite $\cos 2k/N\pi$ using the rules Euler listed earlier in the 'Introductio' as being 'known from trigonometry'. Note that Euler does not give this derivation explicitly. We find

$$\begin{aligned}\cos\frac{2k}{N}\pi &= \cos^2\frac{k}{N}\pi - \sin^2\frac{k}{N}\pi \\ &= 1 - 2\sin^2\frac{k}{N}\pi \quad \textit{(using the Pythagorean identity)}\end{aligned}$$

and now, as Euler would put it, we use that $\sin\varepsilon = \varepsilon$ for infinitesimal ε (taken literally this is not quite true: in terms of nonstandard analysis we would write it instead as $\sin\varepsilon \simeq \varepsilon$ for infinitesimal ε, where $\simeq$ is the infinitely-close relation). Hence,

$$\cos\frac{2k}{N}\pi = 1 - 2\frac{k^2}{N^2}\pi^2. \tag{4.4}$$

[1] Euler is not wrong to call these trinomials however: they are indeed polynomials consisting of three terms.

Euler substitutes (4.4) into (4.3) and obtains after rearrangement a simpler form of the general factor:

$$\frac{4k^2\pi^2}{N^2}\left(1+\frac{x^2}{k^2\pi^2}-\frac{x^2}{N^2}\right). \tag{4.5}$$

This factor is itself divisible by $(1+x^2/k^2\pi^2-x^2/N^2)$ so Euler now argues that our original polynomial has this as a factor, however he says 'we omit the term x^2/N^2 since even when multiplied by N it remains infinitely small' [2] (recall that we have $(N-1)/2$ trinomial factors plus a factor of x according to the Trinomial Lemma). Notice that this claim is not valid if $k=0$, but that case has been dealt with separately.

4.2.4 Multiplication of factors

Now Euler puts the factors together to get

$$\begin{aligned}\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right] = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\\ \left(1+\frac{x^2}{16\pi^2}\right)\left(1+\frac{x^2}{25\pi^2}\right)\cdots\end{aligned} \tag{4.6}$$

Euler has thus far proved that $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ is divisible by the factors on the right hand side of (4.6). However, since he discarded many constant factors we do not know what the leading coefficient should be. We can use a version of the Fundamental Theorem of Algebra, to compare the expansion of this product with the leading coefficient in the previous expansion (4.2) and conclude that the leading coefficient must be 1 as Euler has given it. At this point Euler has rewritten $1/2\left[(1+x/N)^N-(1-x/N)^N\right]$ as (4.2) and (4.6).

4.2.5 Equating coefficients

Euler now argues that we are able to equate coefficients in the following sense, which we discuss further in Section 4.4.7.

> **165.** If $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}=(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)$ etc., then these factors, whether they be finite or infinite in number, must produce the expression $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}$, when they are actually multiplied. It follows then that the coefficient A is equal to the sum $\alpha+\beta+\gamma+\delta+\text{etc.}$ [29, §165]

[2] Euler originally used j to represent the infinite number which we have changed to N to be consistent with the modern convention e.g. McKinzie and Tuckey's notation.

Euler reminds us that he has expanded $\frac{e^x - e^{-x}}{2}$ in two different ways, and thus the two expansions can be 'equated'.

$$\frac{x}{1} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$
$$= x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\left(1+\frac{x^2}{16\pi^2}\right)\left(1+\frac{x^2}{25\pi^2}\right)\cdots$$

He then states 'it follows that'

$$1 + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots \tag{4.7}$$
$$= \left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\left(1+\frac{x^2}{16\pi^2}\right)\left(1+\frac{x^2}{25\pi^2}\right)\cdots \quad .$$

Recall that in Euler's proof '=' means equality up to an infinitesimal. In terms of nonstandard analysis, we would often be using the infinitely-close relation rather than true equality, and the 'equation' in (4.7) is one of these cases[3]. Euler now applies the reasoning from §165 as though (4.7) uses true equality. Notice this appears to be a gap in his proof which we discuss further in Section 4.4.7 and Section 6.9. Before applying §165, he first lets $x^2 = \pi^2 z$. This is simply a substitution which will make his subsequent reasoning easier to follow. For $z > 0$ that gives us

$$1 + \frac{\pi^2}{1\cdot 2\cdot 3}z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}z^2 + \ldots = (1+z)\left(1+\frac{z}{4}\right)\left(1+\frac{z}{9}\right)\left(1+\frac{z}{16}\right)\left(1+\frac{z}{25}\right)\ldots \tag{4.8}$$

Now he applies §165 to equate the second coefficient $\frac{\pi^2}{1\cdot 2\cdot 3}$ with $A = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \ldots$. This gives our beautiful result

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \ldots \quad .$$

[3] See Section 5.5.1.4 to see an example where it is invalid to divide by x on both sides of the infinitely-close relation. In the current case, however, there is not truly any issue since we can restrict the domain of x to be finite and noninfinitesimal without spoiling Euler's chain of reasoning. In Section 6.9.1 we will give McKinzie and Tuckey's version of this, where they equate coefficients of the odd powers, thus eliminating the division by x.

However, there are more beautiful results shortly to follow. We did not mention this above, but §165 was followed by §166 which gave the expressions

$$\begin{aligned} P &= \alpha+\beta+\gamma+\delta+\varepsilon+\dots \\ Q &= \alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2+\dots \\ R &= \alpha^3+\beta^3+\gamma^3+\delta^3+\varepsilon^3+\dots \\ S &= \alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+\dots \\ T &= \alpha^5+\beta^5+\gamma^5+\delta^5+\varepsilon^5+\dots \\ V &= \alpha^6+\beta^6+\gamma^6+\delta^6+\varepsilon^6+\dots \end{aligned} \tag{4.9}$$

which give powers of the coefficients of the binomial product from §165[4]. Euler states that these can be found in terms of A, B, C, D etc. by $P = A$, $Q = AP - 2B$, $R = AQ - BP + 3C$ etc. . He says 'The truth of these formulas is intuitively clear, but a rigorous proof will be given in the differential calculus'. Now, from (4.9), and (4.8) he can obtain the summation of the reciprocals of any power of the squares.

$$\begin{aligned} \frac{\pi^4}{90} &= 1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}\cdots \quad , \\ \frac{\pi^6}{945} &= 1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}\cdots \quad , \\ \frac{\pi^8}{9450} &= 1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}\cdots \quad , \\ \frac{\pi^{10}}{93555} &= 1+\frac{1}{4^5}+\frac{1}{9^5}+\frac{1}{16^5}+\frac{1}{25^5}\cdots \quad . \end{aligned}$$

and in fact, the result $\pi^2/6 = 1 + 1/4 + 1/9 + 1/16 + 1/25\dots$ was a special case of this reasoning since $P = A$ is the only formula which could be discovered without the help of Euler's 'rigorous proof' using differential calculus. We will not cover these further summations in this thesis apart from a mention in Section 7.4.

In Section 6.9.1 we give McKinzie and Tuckey's version of Euler's reasoning, and explain how they attempt to fill the gap we mentioned above, which they call Gap 3. Their method is a minor simplification of Euler's and works just for proving that $\sum_{n=1}^{N} 1/n^2 = \pi^2/6$ for infinite N. The way they fill Euler's gap also raises further questions which we address. But first, we must become familiar with the concepts of Euler-convergence and hidden lemmas since these are integral to McKinzie and Tuckey's reasoning.

[4]Euler was using the Latin alphabet so there was no 'U'!

4.3 Gaps identified by McKinzie and Tuckey

We briefly state the gaps that McKinzie and Tuckey identify in Euler's proof (by paraphrasing their words) since we will refer to these on several occasions [57, pp. 37-38].

Gap 1. The substitution of 1 for $\frac{N-1}{N}, \frac{N-2}{N}, \frac{N-3}{N}$, and so forth.

Gap 2. The substitution of $\frac{1}{2}\left(\frac{2k\pi}{N}\right)^2$ for $1-\cos\frac{2k\pi}{N}$ and the substitution of 0 for x^2/N^2 in each factor.

Gap 3. The equation of the cubic terms in the first and second expansion of hyperbolic sine.

4.4 Using the concept of hidden lemmas to rehabilitate Euler's proof

In their papers on Euler's analysis of infinities [57, 55], McKinzie and Tuckey identify several hidden lemmas. Of course, it may be a matter of opinion whether

1. these gaps in Euler's proof amount to a lemma

2. Euler recognised that he was using some proposition or law in these gaps.

However, McKinzie and Tuckey give a convincing account of how these lemmas may be used to rehabilitate Euler's proofs in a way which is sympathetic to his original reasoning, and the proofs of the lemmas they state do not use any principle or notion which could not have been expressed in the language of Euler's time. They do introduce a key concept, that of Euler-convergence/determinacy, but they intend this as a description of a distinction that Euler himself appeared to make. They also, of course, use a rigorous theory of infinities and infinitesimals i.e. nonstandard analysis.

4.4.1 Euler's concept of convergence

In his 1734 paper on the harmonic series [24], Euler expressed a principle under which a series has a finite sum.

> A series which, after its continuation to the infinite, has a finite sum, does not accept increase, even if it is continued twice as far, but that which is added in thought after an infinite will actually be infinitely small. For, otherwise, the sum of the series continued to an infinite would not be determinate and, consequently, not finite. From this follows that, if what

> originates from the continuation beyond an infinitesimal term be of finite magnitude, then the sum of the series must necessarily be infinite. Thus, from that principle we shall be able to judge whether the sum of any given series be infinite or finite. [5]

This has similarities to the Cauchy convergence criterion. We can remove the ambiguity from Euler's principle by seeking counterexamples. Finding a 'counterexample' does not necessarily indicate that Euler's criterion is wrong, but rather qualifies the sense in which he meant it.

Laugwitz points out that Euler must have meant the principle to apply to series with positive terms since he states that the series will be either finite or infinite. He also argues that 'duplo longius' (twice as far) is not meant literally as twice, but some arbitrary natural number of potentially infinite size [53, p. 207].

Euler's condition is stated by Laugwitz to be both necessary and sufficient. Mathematicians have attempted to give counterexamples to it being a sufficient condition. Most notably, Pringsheim gives the series $\sum_{n=2}^{\infty} 1/(n \log n)$ as a counterexample [53, p. 207]. He takes Euler's 'duplo longius' literally and thus wishes us to consider $\sum_{n=i+1}^{n=2i} 1/(n \log n)$, where i is an infinite number. In 1735 Euler gave the sum from $i+1$ to ni as being infinitely close to $\log(1+\log n/\log i)$ [30]. Thus the sum from $i+1$ to $2i$ will evaluate as $\log(1+\varepsilon)$, where ε is an infinitesimal. This is infinitely close to 0 and so even though this series diverges, the sum from $i+1$ to $2i$ is infinitesimal. Pringsheim thus believes he has found a counterexample to the sufficiency of Euler's criterion. Laugwitz argues that this only proves that 'duplo longius' was not meant literally by Euler, since, when we consider the sum from $i+1$ to ni, we obtain $\sum_{n=i}^{n=2i} 1/(n \log n)$ where if $n = i$ this is no longer infinitely close to 0. In this way Laugwitz dismisses the counterexample and uses it to qualify the principle. McKinzie and Tuckey follow Laugwitz's interpretation here.

We can also give a 'counterexample' to it being a necessary condition. Take some series which when continued to an infinite i is finite. Then define the $(i+1)^{\text{th}}$ term to be some non-infinitesimal finite number, and let all the terms thereafter be 0. From this we can deduce that either

- Euler meant that his series cannot depend on infinite numbers, or was otherwise placing restrictions on the functions which he was using to define the terms of the series. For example, that they are in some sense continuous.
- Euler meant that the sum must be finite for all infinite indices.

[5] Euler's original statement is in Latin: this English version is taken from Laugwitz. [53, p. 206]

McKinzie and Tuckey take the latter interpretation to formulate their notion of Euler-convergence.

Within the principle, Euler also says what is required for a sum to have a determined value, which, as he says, is necessary for it to be finite. Determinacy by itself is a useful property, and is used by McKinzie and Tuckey.

Euler did not mention this principle in any other work [53]. However, he appears to have used it implicitly in the 'Introductio'. We can only speculate on his reasons for avoiding any explicit mention. Perhaps he was not himself convinced of its truth in the general case, but could more easily see its usefulness and validity in specific examples. McKinzie speculates that Euler became interested in series which take on infinite sums and did not wish to exclude them from consideration [56, pp. 116-117]. His single explanation of the convergence criterion seems to have been unnoticed by later mathematicians. It was not mentioned by Bolzano or Cauchy when they formulated their own principles, and thus Euler's principle did not leave its mark on the history of mathematics.

Laugwitz uses the Eulerian convergence criterion to define a hidden lemma in Euler's 1748 expansion of the exponential series. McKinzie and Tuckey define two separate notions which parallel the convergence criterion: determinacy [55, p. 349] and Euler-convergence [57, p. 44]. They then extend Laugwitz's work by formulating three hidden lemmas and two hidden sublemmas based on these. They use them to reconstruct Euler's 1748 proof of the Basel Problem with his reasoning intact and the 'gaps' filled or explained. Following Laugwitz and McKinzie and Tuckey, we use the convergence criterion and the hidden lemmas based on it to partially reconstruct Euler's proof in Isabelle as described in Chapter 6. In the following sections we describe McKinzie and Tuckey's hidden lemmas and the concepts they are based on.

4.4.2 Determinacy and Euler-convergence

In their rehabilitation of the proof of the Basel problem, McKinzie and Tuckey express *Euler-convergence* as follows [57].

Definition 1 (Euler-convergence)**.** *A sum $a_1 + a_2 + a_3 + \cdots$ is Euler-convergent iff (i) a_k is defined, by an elementary function, (ii) for all infinite J, the sum $a_1 + a_2 + a_3 + \cdots + a_J$ is finite, and (iii) for all infinite J and K the sum $a_J + a_{J+1} + \cdots + a_K$ is infinitesimal.*

Similarly, a product $(1+b_1)(1+b_2)(1+b_3)\cdots$ is Euler-convergent iff (i) b_k is defined by an elementary function, (ii) for all infinite J, the product $(1+b_1)(1+b_2)\dots(1+b_J)$

is finite, and (iii) for all infinite J and K the product $(1+b_J)(1+b_{J+1})\cdots(1+b_K)$ *differs infinitesimally from* 1.

They use 'elementary' function to represent Euler's concept of a function, however, they state than an internal function is one possible broad interpretation [57, p. 43].

McKinzie and Tuckey define the related notion of *determinacy* in another of their papers where they reconstruct some of Euler's reasoning from the 'Introductio' in non-standard analysis. They express determinacy as follows [55].

Definition 2 (Determinacy)**.** *A hypersequence* $s_0,\ s_1,\ s_2,\ \ldots$ *is said to be* determinate *iff* $s_M \simeq s_N$ *for all infinite M and N. If* $a_0,\ a_1,\ a_2,\ \ldots$ *is a hypersequence, then a series* $a_0 + a_1 + a_2 + \cdots$ *is said to be determinate iff the hypersequence of partial sums defined by* $s_n = a_0 + a_1 + a_2 + \cdots + a_n$ *is determinate.*

Similarly, a product is determinate iff the hypersequence of partial products is determinate. Although Euler-convergence and determinacy are expressed in quite different terms by McKinzie and Tuckey, we can think of Euler-convergence as simply being determinacy plus finiteness. We explain the nuances more thoroughly in Section 4.4.10.

4.4.2.1 Relation to nonstandard definition of convergence

In nonstandard analysis, convergence of a sequence is defined by [40]

Definition 3 (Convergence of a sequence)**.** *A real-valued sequence* $\langle s_n : n \in \mathbb{N}\rangle$ *converges to* $L \in \mathbb{R}$ *if and only if* $s_n \simeq L$ *for all infinite n.*

We can also define Cauchy convergence of a sequence [40].

Definition 4 (Cauchy convergence of a sequence)**.** *A real-valued sequence* $\langle s_n\rangle$ *is Cauchy in* $\mathbb{R}$ *if and only if all its extended terms are infinitely-close to each other.*

They are equivalent to the definitions of convergence and Cauchy-convergence in standard analysis. Euler-convergence and determinacy are variations on convergence. The difference is that we work with hypersequences $s_n : {}^*\mathbb{N} \to {}^*\mathbb{R}$. As McKinzie and Tuckey define it, Euler-convergence could be called Cauchy Euler-convergence if we were using terminology similar to the convention for convergence.

4.4.3 Hidden lemmas

Following Laugwitz [51] and Lakatos [50], McKinzie and Tuckey formulate 'hidden lemmas' in places where Euler appears to have used his convergence criterion to conclude results that are not generally true, but would be justified for Euler-convergent series or products. In their words, such hidden lemmas are 'principles under which the steps from one equation to the next in Euler's arguments are simple and obvious.' [57, p. 44]. The intention behind a hidden lemma is to express formally a principle which Euler can be assumed to have known, at least intuitively, and used often. From hints in the way Euler expresses his reasoning, we can also try to argue that he was conscious of these general principles, and that he felt them to be obvious. It would be sufficient to use only the notion of Euler-convergence to follow Euler's reasoning in his proof of the Basel problem, and McKinzie and Tuckey use only this notion in their rehabilitation of Euler's proof [57]. However, some of their Hidden Lemmas from this proof can in fact hold for the weaker notion of determinacy as well (see Section 4.4.4). It is unnecessary to additionally prove finiteness, and thus obtain Euler-convergence, in every case and proving finiteness is convoluted in Isabelle even if it can be trivially observed by a mathematician. We prefer to formalise the Hidden Lemmas and gaps with determinacy where possible, and Euler-convergence where necessary. This also has the advantage of allowing us to see the required conditions for each step of the proof. We describe how we use McKinzie and Tuckey's concepts of Euler-convergence, determinacy and Hidden Lemmas in Section 5.1.

4.4.4 Hidden lemmas and the choice between determinacy and Euler-convergence

We initially attempted to formalise the hidden lemmas using only the concept of determinacy which is simpler than Euler-convergence. However, it became clear when proving the Second Hidden Lemma that Euler-convergence is truly needed: i.e. we need the finiteness property along with determinacy. The infinitely-close relation begins to behave less like equality when we use products or multiplication, unless the quantities involved are appreciable.

Equality allows for substitution and cancellation without any special conditions on the quantities involved. We may compare this to the infinitely-close relation. Cancellation of addition is also valid for the infinitely-close relation

$$a + c \simeq b + c \longrightarrow a \simeq b$$

No conditions are on a, b or c other than that they are hyperreal numbers. The analogous theorem for multiplication is

$$ac \simeq bc \longrightarrow a \simeq b$$

and this requires c is both finite and not infinitesimal. As this theorem is used in the proof of the Second Hidden Lemma, this means that the products (which are used as c is here) must be proved to be finite and non-infinitesimal. Thus, the Second Hidden Lemma requires Euler-convergence. We discuss this more fully in Section 5.4.

However, some of the hidden lemmas can be formalised using only determinacy. We prefer determinacy to Euler-convergence where possible, because it is a weaker condition. We summarise the minimal conditions for the various hidden lemmas and sublemmas, along with the justification, in Table 4.4. In the following section we will give the hidden lemmas as McKinzie and Tuckey stated them as we will need to understand their versions before comparing with the versions formalised in Isabelle.

4.4.5 The First Hidden Lemma

The First Hidden Lemma is also named the Summation Comparison Theorem by McKinzie and Tuckey [55, p. 349]. They state it for Euler-convergent polynomials as follows.

First Hidden Lemma. *If the sums $a_1 + a_2 + a_3 + \cdots$ and $b_1 + b_2 + b_3 + \cdots$ are Euler-convergent, and if for each finite n, $a_n \simeq b_n$, then for all N, $a_1 + a_2 + a_3 + \cdots + a_N \simeq b_1 + b_2 + b_3 + \cdots + b_N$.*

We saw in Section 4.2.1 that Euler appears to set equal two polynomials whose coefficients differ infinitesimally from each other: he substitutes 1 for $(N-1)/N$, $(N-2)/N$, $(N-3)/N$ etc. , where N is infinitely large so that

$$1 + \frac{1}{1}x + \frac{1(N-1)}{1 \cdot 2N}x^2 + \frac{1(N-1)(N-2)(N-3)}{1 \cdot 2N \cdot 3N \cdot 4N}x^4 + \cdots$$

can be rewritten as

$$\frac{x}{1} + \frac{x^3}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots$$

We know that Euler often wrote '=' when he was aware that the results differ by an infinitesimal, so in this case we can reinterpret this using '$\simeq$', however, this is not enough to make it universally true that any two series whose terms are infinitely-close will themselves be infinitely-close. Here is a counterexample. Let N be an infinite hypernatural. Then $1/N \simeq 1/2N$ but

$$\sum_{i=1}^{N} \frac{1}{N} \not\simeq \sum_{i=1}^{N} \frac{1}{2N}. \tag{4.10}$$

The First Hidden Lemma is meant to qualify when such substitution is allowed and bridge this gap.

4.4.6 The Second Hidden Lemma

The Second Hidden Lemma, also called the Product Comparison Theorem, is stated by McKinzie and Tuckey as follows

Second Hidden Lemma. *If the products* $(1+a_1)(1+a_2)(1+a_3)\dots$ *and* $(1+b_1)(1+b_2)(1+b_3)\dots$ *are Euler-convergent, and if for each finite n,* $a_n \simeq b_n$*, then for all N,* $(1+a_1)(1+a_2)(1+a_3)\dots(1+a_N) \simeq (1+b_1)(1+b_2)(1+b_3)\dots(1+b_N)$.

They use it to fill the second gap that they identify in Euler's proof by setting:

$$a_k = \frac{x^2}{(k\pi)^2} \text{ and } b_k = \frac{x^2}{(k\pi)^2} - \frac{x^2}{N^2} \tag{4.11}$$

4.4.7 The Third Hidden Lemma

McKinzie and Tuckey identify the third gap in Euler's proof as the equating of the cubic terms from his two separate expansions of hyperbolic sine. If this was equating coefficients between two *equal* polynomials, there would be no gap. However, both in Euler's proof and McKinzie and Tuckey's rehabilitation, the polynomials differ by an infinitesimal[6].

The Third Hidden Lemma, which is a partial converse to the First Hidden Lemma (when $x = 1$), is a way to match the coefficients of polynomials which are only infinitely-close. [7]

[6]Euler does use the symbol = nonetheless: see Section 2.3.1.2 for our analysis of what Euler meant by this relation.

[7]For the purposes of our Isabelle theory, we can only know a function is a polynomial if it has a representation of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_Nx^N$ for hypernatural N. To be precise, the function is of two variables, x and N, and this sum is expressed using hypsum. Multiples of polynomials

Third Hidden Lemma. *Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$. If for all finite x, $f(x)$ and $g(x)$ are Euler-convergent and $f(x) \simeq g(x)$, then $a_n \simeq b_n$ for all n.*

The Third Hidden Lemma under the infinitely-close relation corresponds to 'equating coefficients' under equality.

McKinzie and Tuckey also describe this as the Polynomial Comparison Theorem [57, p. 52]. Their reasoning for how this lemma fills their third gap is given in the context of their proof [57, pp. 47-48]. They take the polynomials

$$f(x) = x + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N$$

and

$$g(x) = x + \left(\sum_{k=1}^{(N-1)/2} \frac{1}{(k\pi)^2}\right)x^3 + \left(\cdots\right)x^5 + \cdots$$

and apply the Third Hidden Lemma to these. The polynomial $g(x)$ is obtained by multiplying-out and gathering-terms from

$$h(x) = x \prod_{k=1}^{(N-1)/2} \left(1 + \frac{x^2}{(k\pi)^2}\right). \tag{4.12}$$

This is the origin of $g(x)$, but note that this does not mean $g(x) = h(x)$ unless it has been proven that multiplying-out and gathering infinitely-many terms does not change the value of the polynomial. Thus they have proven $h(x) = f(x)$ but not $g(x) = h(x)$ or $g(x) \simeq h(x)$. Therefore, this is a gap in their transitive reasoning, since the antecedent $f(x) \simeq g(x)$ of the Third Hidden Lemma is not proved explicitly by them.[8]. We discuss this further in Section 6.9.1.

4.4.8 The Hidden Sublemmas

The Hidden Sublemma (i) given by McKinzie and Tuckey describes the relation of Euler-convergence between sums and products. Hidden Sublemma (ii) tells us that Euler-convergence is preserved by multiplying out. They combine these two lemmas into a single Hidden Sublemma which they state as follows.

are not easily proved to be polynomials in Isabelle. We have not proven this in Isabelle since this was not a priority for formalisation as it is outside the main reasoning of the proof. Mathematically, it is straightforward and can be done by induction. Sums of polynomials are easily proved to be polynomials, but this must be proved rather than known implicitly by closure.

[8] Kanovei, even though he quotes Euler multiplying-out (and implicitly gathering-terms of) the same polynomial, does not note that as one of his 'controversial points'. [76, pp. 68-69]

Hidden Sublemma. *Assume that b_n is defined by a nonnegative elementary function.*

(i) The product $(1+b_1)(1+b_2)(1+b_3)\dots$ is Euler-convergent if and only if the sum $b_1+b_2+b_3+\dots$ is Euler-convergent.

(ii) If for a given x the product $(1+b_1x)(1+b_2x)(1+b_3x)\dots$ is Euler-convergent, then the sum $1+(b_1+b_2+b_3+\dots)x+(b_1b_2+b_1b_3+b_2b_3+\dots)x^2+(b_1b_2b_3+\dots)x^3+\dots$ obtained by multiplying-out the products, is Euler-convergent as well.

Since x is assumed to be given, the product $(1+b_1x)(1+b_2x)(1+b_3x)\dots$ could be equally well represented as simply $(1+c_1)(1+c_2)(1+c_3)\dots$. Thus both of the Sublemmas give series whose Euler-convergence is equivalent to the Euler-convergence of a binomial product of the form $(1+c_n)$. The Hidden Sublemma (i) expresses a symmetry between Euler-convergence of sums and products: the degree of closeness to the identity that is required for the tail of a sum to converge is the same as the degree of closeness to the identity that is required for the tail of a product to converge. In Section 6.2 and Section 5.8 we discuss Hidden Sublemma (ii) further and our formalisation of its statement. In Section 6.9.1 we point out that Hidden Sublemma (ii) may not actually be necessary for rehabilitating Euler's proof.

4.4.9 Use of the hidden lemmas, sublemmas and Euler-convergence

So far in this chapter we have given a description of Euler's proof, and we have introduced McKinzie and Tuckey's perspective of Euler's proof by giving the hidden lemmas and describing Euler-convergence/determinacy. They provide a rehabilitated proof based on Euler's, but using their hidden lemmas to 'complete' the proof. We refer the reader to McKinzie and Tuckey's paper to see in more detail how they use the hidden lemmas in their rehabilitation of Euler's proof [57]. We give brief descriptions of their method in context.

Our mechanisation, although incomplete, provided the language to formulate and specify the gaps to an even higher level of precision and how the hidden lemmas are used to fill these gaps. In Chapter 6 we explain how we use the hidden lemmas and sublemmas to fill the gaps in Euler's proof, which has some differences from McKinzie and Tuckey's approach. The mechanisation also lead us to discover places where even McKinzie and Tuckey's rehabilitated proof has unclear steps. We will discuss these in the next chapter since discovery and understanding of the additional gaps is intertwined with the mechanisation.

4.4.10 Relations between the different notions of Euler-convergence and determinacy, and their use in the Hidden Lemmas

We wish to be clear about the relative strengths of each notion of convergence. This is needed since we formalise the hidden lemmas with minimal assumptions. A necessary step to clarifying this is explained now.

4.4.10.1 Euler-determinacy

In Section 4.4.1 we stated that Euler's convergence criterion is similar to Cauchy's. The similarity is because it is expressed in terms of closeness between sufficiently-large terms of the series rather than absolute closeness over the entire series. For the usual definition of convergence in nonstandard-analysis, we can have either convergence or Cauchy-convergence. McKinzie and Tuckey express Euler-convergence with the Cauchy property but they express determinacy differently. Their reason for doing this is because they wished to extend the notion of determinacy to sequences, and capture the determinacy of both sums and products with a single notion. We would like to make the same distinction for determinacy, but it would be unfair to Euler to use the term 'Cauchy' as the qualifier. Therefore we will define Euler-determinacy. This notion is also closer to what Euler called 'determinate' than McKinzie and Tuckey's determinacy, thus the name is appropriate.

Definition 5 (Euler-determinacy)**.** *We will say a hypersum $\sum_{n=a}^{b} x_n$ is Euler-determinate if*

$$\sum_{n=N}^{M} x_n \simeq 0$$

for all infinite hypernatural N, M and similarly a hyperproduct $\prod_{n=a}^{b} x_n$ is Euler-determinate if

$$\prod_{n=N}^{M} x_n \simeq 1$$

for all infinite hypernatural N, M.

Note that finite Euler-determinacy, i.e. Euler-determinacy with the additional property that the hypersum/hyperroduct is finite when taken from index 0 to an infinite hypernatural index, is simply Euler-convergence as defined by McKinzie and Tuckey. In order for the reader to develop an intuition for the concepts of Euler-convergence, finite determinacy, Euler-determinacy and determinacy, we have provided Table 4.1 and Table 4.2 which give examples of each type of sum and product.

Table 4.1: Examples of the hierarchy of determinate/Euler-convergent sums

Type of determinacy	Example	Justification
Finite determinate Euler-convergent	$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{N^2}$ (the Basel series)	We proved this in Section 5.2.1
Determinate Euler-determinate Not finite determinate	$L+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{N^2}$ for some $L\in\mathbb{L}\cup{}^*\mathbb{N}$	Follows from the determinacy of the Basel series
Not determinate	$1+4+9+16+\cdots+N^2$	The distance between successive terms grows

Table 4.2: Examples of the hierarchy of determinate/Euler-convergent products. Take $L\in\mathbb{L}\cup{}^*\mathbb{N}$.

Type of determinacy	Example	Justification
Euler-convergent	$\prod_{k=1}^{N}\left(1+\frac{1}{(k\pi)^2}\right)$ (infinite factorisation of $\sinh 1$)	See Section 5.2.3
Finite determinate Not Euler-convergent	$\prod_{k=1}^{N}(1/k)$	The terms become infinitesimal
Euler-determinate Not finite determinate	$L\cdot\prod_{k=1}^{N}\left(1+\frac{1}{(k\pi)^2}\right)$	Follows from the Euler-convergence of the infinite factorisation of $\sinh 1$
Determinate Not Euler-determinate	$\prod_{k=0}^{N}a_k$ with $a_k=1$ when $k\neq 2,L$, $a_2=\frac{1}{L^2}$ and $a_L=\frac{1}{L}$	See Proposition 4
Not determinate	$1\cdot 2\cdot 3\cdot 4\cdots N$	The distance between successive terms grows

4.4.10.2 Relations between the convergence properties

Given the many variations on Euler-convergence and determinacy, we define here abbreviations for these properties which we use to formally express the relations between them. The following notation is only used within this section and in Table 4.3 and Table 4.4 and should not be read literally as sums and products or numbers .

E	for Euler-convergence
F	for finiteness
d	for determinacy
$\sum$	for sums
$\prod$	for products
$\substack{M \\ N}$	to indicate Eulerness (i.e. the Euler property from Section 5.1.2)

For example, Euler-convergence for sums is $\mathrm{E}\sum$ (equivalently $\mathrm{Fd}\sum_N^M$) and determinacy for products is $\mathrm{d}\prod$. The following equivalences hold:

$$\mathrm{Fd}\sum_N^M \equiv \mathrm{E}\sum$$

and

$$\mathrm{d}\sum_N^M \equiv \mathrm{d}\sum$$

The following implication holds

$$\mathrm{E}\prod \Longrightarrow \mathrm{Fd}\prod_N^M$$

and since Euler-convergence is determinacy plus finiteness the following implications also hold

$$\mathrm{E}\sum \Longrightarrow \mathrm{d}\sum_N^M$$
$$\mathrm{E}\prod \Longrightarrow \mathrm{d}\prod_N^M$$

and of course

$$\mathrm{Fd}\sum \Longrightarrow \mathrm{d}\sum$$
$$\mathrm{Fd}\prod \Longrightarrow \mathrm{d}\prod .$$

Table 4.3: Types of determinacy required to be proved for each polynomial

Polynomial	Type of determinacy required	Antecedent for	Consequent of	Proof
$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{N^2}$	Fd$\sum$	Hidden Lemma 3 Hidden Sublemma (i)	–	Isabelle (Section 5.2.1)
$1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^N}{N!}$	Fd$\sum$	Hidden Lemma 1 Corollary	–	Pen-and-paper (Section 5.2.2)
$x+\frac{N(N-1)(N-2)}{N^3}\frac{1}{3!}x^3+\cdots+\frac{N!}{N^N}\frac{x^N}{N!}$	d$\sum$	Hidden Lemma 1	–	Incomplete
$x+\frac{x^3}{3!}+\cdots+\frac{x^N}{N}$	d$\sum$	Hidden Lemma 1 Hidden Lemma 3	–	Pen-and-paper (Section 5.2.2.3)
$x\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\frac{1/2\,(2k\pi/N)^2}{1-\cos 2k\pi/N}-\frac{x^2}{N^2}\right)$	E$\prod$	Hidden Lemma 2	Hidden Sublemma (i)	Incomplete (Section 5.2.3)
$x\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right)$	E$\prod$	Hidden Lemma 2	Hidden Sublemma (i)	Incomplete (Section 5.2.3)
$x+x^3\left(\sum_{k=1}^{(N-1)/2}\right)+x^5(\cdots)+\cdots$	d$\sum$	Hidden Lemma 3	Hidden Sublemma (ii)	Incomplete

Table 4.4: Types of determinacy required in the Hidden Lemmas and Sublemmas

Lemma	Weakest type of determinacy required in antecedent	Strongest type of determinacy given in consequent	Proof complete in Isabelle
Hidden Lemma 1	d$\sum$	–	Complete
Hidden Lemma 2	Fd$\prod$	–	Complete with extra assumptions
Hidden Lemma 3	d$\sum$	–	Complete except one lemma with extra assumptions
Hidden Sublemma (i)	Fd$\sum$ or E$\prod$	Fd$\sum$ or E$\prod$	Incomplete
Hidden Sublemma (ii)	E$\prod$	Fd$\sum$	Incomplete

Using our abbreviations, we can notate Table 4.3 which gives the polynomials in the proof of the Basel problem that require proving some kind of determinacy or Euler-convergence. We have columns for the Hidden Lemmas and Sublemmas which the determinacy of these polynomials is used in: either as the antecedent or as the consequent. We also give Table 4.4 which shows the requirements of Euler-convergence or determinacy for each of the Hidden Lemmas and Sublemmas.

For the rest of this chapter, we will look at some of Euler's reasoning in more detail, focusing on parts which are not analysed in McKinzie and Tuckey's paper [57].

4.5 Euler's §165 and equating coefficients

Euler's intuition for why §165 can be taken seems to be based on equating coefficients. It also seems to be based on Vieta's formulas [78]. We described his depiction of this step in Section 4.2.5. However, he applies §165 in a case where true equality does not hold: McKinzie and Tuckey use the 3rd Hidden Lemma to replace this instance. We discuss *true* equating coefficients and the relation between Vieta's formulas and §165 in this section.

4.5.1 Equating coefficients

Equating-coefficients is so natural to most mathematicians that it is not immediately obvious where this property of polynomials comes from. It is in fact part of the accepted definition of a polynomial: polynomials are uniquely determined by their coefficients and their degree. Thus proving that coefficients may be equated is the same as proving that polynomials are well-defined. We can prove this, when considering polynomials as functions[9], by successive differentiation of a generalised polynomial. This is how we formalise our proof of equating coefficients in Section 6.3. It is necessary to equate coefficients in the true sense at another point in Euler's proof: when he writes hyperbolic sine as the product of its trinomial factors, he simplifies the constant factor, presumably by comparing with the leading coefficient of the other expansion of hyperbolic sine [29, §156, p. 126].

[9]When polynomials are considered as elements of a polynomial ring, they exist as lists of coefficients. So equating coefficients truly arises from their definition, and it is not necessary to prove it. However, if we wish to show that there are some functions which represent the properties defined by a polynomial ring, then it is necessary to prove that equating coefficients is possible.

In Gap 3, Euler merely matches coefficients of two polynomials which are infinitely-close. In other places, e.g. where he determines the constant factor in the trinomial expansion of hyperbolic sine, he truly equates coefficients of polynomials which are equal. By comparing the language he used, we can find out whether he made any distinction between the two situations. There does not in fact appear to be much difference: in both cases he provides no additional justification. The two situations must be formalised differently, but he may not have considered them to be different.

4.5.2 Euler's version of Vieta's formulas

In this section, we describe, and justify with mathematical reasoning and proof, our understanding of the origin of Euler's method described in §165. We conduct this process literally and consider Euler's '=' as equality here, since the intention is merely to understand where Euler's §165 came from. Recall that §165 is the proposition in the 'Introductio' which Euler uses to multiply-out the factors obtained by the Trinomial Lemma. We have discussed and quoted it in Section 4.2.5, and we state it here again for convenience since we refer to Euler's notation.

> **165.** If $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.} = (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)$ etc., then these factors, whether they be finite or infinite in number, must produce the expression $1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}$, when they are actually multiplied. It follows then that the coefficient A is equal to the sum $\alpha+\beta+\gamma+\delta+\text{etc.}$

Euler's assumption, i.e. that

$$\begin{aligned}1+Az+Bz^2+Cz^3+Dz^4+\text{etc.}\\ =(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ \text{etc.}\end{aligned} \tag{4.13}$$

should always be satisfied in the sense that $(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)$ etc. may always be expanded in this form. We can show this as follows. If we use the nonstandard version of the Fundamental Theorem of Algebra as we have formalised it in Isabelle, then we find

$$\begin{aligned}1+a_1z+a_2z^2+a_3z^3+\cdots+a_Nz^N\\ =a_N(z-r_1)(z-r_2)(z-r_3)\cdots(z-r_N)\end{aligned} \tag{4.14}$$

where we have used the notation a_i for the i^{th} coefficient of Euler's polynomial, r_i for the i^{th} root of the polynomial and N for the hypernatural degree of the polynomial. It is justified to assume that the polynomial has a degree, because although Euler writes the polynomial here without a degree, he will later use the property outlined in §165 on a polynomial with a degree. Note also that the first antecedent of §165 implies 0 is not a root. So we can now define:

$$b_i = -\frac{1}{r_i}. \tag{4.15}$$

Then

$$\begin{aligned}(z-r_1)(z-r_2)(z-r_3)\cdots(z-r_N)\\ = \frac{1}{b_1 b_2 b_3 \cdots b_N}(1+b_1 z)(1+b_2 z)(1+b_3 z)\cdots(1+b_N z).\end{aligned}$$

Hence

$$\begin{aligned}1+a_1 z+a_2 z^2+a_3 z^3+\cdots+a_N z^N\\ = \frac{a_N}{b_1 b_2 b_3 \cdots b_N}(1+b_1 z)(1+b_2 z)(1+b_3 z)\cdots(1+b_N z)\end{aligned}$$

and by equating the leading coefficient we see $a_N = b_1 b_2 b_3 \cdots b_N$ so we derive

$$\begin{aligned}1+a_1 z+a_2 z^2+a_3 z^3+\cdots+a_N z^N\\ = (1+b_1 z)(1+b_2 z)(1+b_3 z)\cdots(1+b_N z)\end{aligned} \tag{4.16}$$

which is equivalent to Euler's assumption (4.13).

We can show that the property which Euler describes in §165 is equivalent to Vieta's formulas. We shall give here a pen-and-paper proof. So far we have been intentionally agnostic about whether N is a natural or hypernatural number: our reasoning is valid for both. We will continue to be agnostic in our equivalence proof.

First we state Vieta's formulas and Euler's §165 mathematically. One aspect which deserves some discussion is the notation used for the symmetric sum. Vieta's formulas are commonly expressed as follows.

Proposition 1 (Vieta's formulas with sum over ordered indices)**.** *Given a polynomial with degree N, coefficients a_i and roots r_i*

$$\sum_{1\le i_1<i_2<\cdots<i_k\le N} \prod_{j=1}^{k} r_{i_j} = (-1)^k \frac{a_{N-k}}{a_N}.$$

for $k = 1, 2, \ldots N$.

However, in the only current formalisation of Vieta's formulas in Isabelle, Eberl has an AFP entry on 'Symmetric Polynomials' in which he proves Vieta's formulas [18], and he uses k-subsets to formalise the sum of ordered indices. We cannot easily reuse Eberl's mechanisation of this result since the formalisation of polynomials used by him is in terms of lists, and cannot easily be translated to our functional definition. However, his formalisation gives us an idea of the best way to express Vieta's formulas in Isabelle, as opposed to mathematically.

Definition 6 (Set of k-subsets)**.** *The set of k-subsets of A is given by* $[A]^k = \{X \mid X \subseteq A \wedge |X| = k\}$.

We find k-subsets simple to formalise, compared to the sum with ordered indices, which requires indices indexed with indices, Besides this, k-subsets have some well-known properties that we can make use of in our equivalence proof. Therefore we will express the two propositions using sums over these sets, and conduct our proof using this notation. We will describe our formalisations of these results in Section 6.2.1.

Proposition 2 (Vieta's formulas with sum over k-subsets)**.** *Given a polynomial with degree N, coefficients a_i and roots r_i*

$$\sum_{X \in [\{1,2,\ldots N\}]^k} \prod_{x \in X} r_x = (-1)^k \frac{a_{N-k}}{a_N}$$

for $k = 1, 2, \ldots n$.

Proposition 3 (Euler §165 with sum over k-subsets)**.** *Given a polynomial as in* (4.16)*, then its coefficients a_n are given by*

$$a_n = \sum_{X \in [\{1,2,\ldots N\}]^n} \prod_{x \in X} b_x$$

for $n \in \mathbb{N}$.

Theorem 8. *Propositions* (2) *and* (3) *are equivalent.*

Proof. Let us rewrite Vieta's formulas to give the nth coefficient a_n as Euler's does. As k takes any value $1, 2, \ldots N$ we can let $n = N - k$. Then $k = N - n$ and after this substitution and rearrangement, we obtain an equivalent of Vieta's formulas:

$$a_n = (-1)^{N-n} a_N \sum_{X \in [\{1,2,\ldots N\}]^{N-n}} \prod_{x \in X} r_x \tag{4.17}$$

We can now show that Euler and Vieta's formulas are equivalent. First recall,

$a_N = b_1 b_2 b_3 \cdots b_N$ and $b_i = -1/r_i$, $i \in \{1, 2, \ldots N\}$,

$$\begin{aligned}
a_n &= (-1)^{N-n} b_1 b_2 b_3 \cdots b_N \sum_{X \in [\{1,2,\ldots N\}]^{N-n}} \prod_{x \in X} -\frac{1}{b_x} \\
&= b_1 b_2 b_3 \cdots b_N \sum_{X \in [\{1,2,\ldots N\}]^{N-n}} \prod_{x \in X} \frac{1}{b_x} \\
&= \sum_{X \in [\{1,2,\ldots N\}]^{N-n}} b_1 b_2 b_3 \cdots b_N \prod_{x \in X} \frac{1}{b_x} \\
&= \sum_{X \in [\{1,2,\ldots N\}]^{N-n}} \prod_{x \in \{1,2,\ldots N\} - X} b_x
\end{aligned}$$

As by definition $\binom{N}{n}$ gives the number of n-subsets of $\{1, 2, \ldots N\}$, and $\binom{N}{N-k} = \binom{N}{k}$, thus $\sum_{X \in [\{1,2,\ldots N\}]^{N-n}}$ has the same number of terms as $\sum_{X \in [\{1,2,\ldots N\}]^{n}}$. Further, the complements of the $(N-n)$-subsets of $\{1, 2, \ldots N\}$ in $\{1, 2, \ldots N\}$ are exactly the n-subsets of $\{1, 2, \ldots N\}$. So

$$a_n = \sum_{X \in [\{1,2,\ldots N\}]^{n}} \prod_{x \in X} b_x$$

which is exactly Euler §165 as we described it in Proposition 3. □

In Section 6.2 we discuss approaches to formalising Euler's §165. Euler's only words towards a proof are that 'the factors must produce the [same] expression . . . when they are actually multiplied'.

4.6 Euler's Trinomial Lemma

Euler says he is interested in 'the quadratic factors of the form $p - qz + rz^2$ which are real, but whose linear factors are complex.' He calls these 'trinomial factors', and deduces their form. Euler's derivation of the trinomial factors is impressive in its generality, but he also discusses the two particular examples $a^n + z^n$ and $a^n - z^n$. The latter gives us the factors he uses in his proof of the Basel problem.

4.6.1 Euler's statement of the Fundamental Theorem of Algebra

The Trinomial Lemma is based on the Fundamental Theorem of Algebra (FTA). In §143 Euler uses some apparently erroneous reasoning to 'deduce' the FTA. He claims

> If the polynomial is $\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \ldots$ and a linear factor is of the form $p - qz$, then it is clear that whenever $p - qz$ is a factor of the function $\alpha + \beta z + \gamma z^2 + \ldots$, and when we substitute p/q for z, then the factor $p - qz$

> becomes zero and the proposed function vanishes. It follows that $p-qz$ is a factor or divisor of the polynomial $\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\ldots$ whenever $\alpha+\beta p/q+\gamma p^2/q^2+\delta p^3/q^3+\varepsilon p^4/q^4+\ldots=0$. Conversely, if all the roots p/q of this equation have been extracted, they will give all of the linear factors of the proposed polynomial $\alpha+\beta z+\gamma z^2+\delta z^3+\ldots$, that is $p-qz$. It is clear now that the number of these linear factors is determined by the greatest power of z.

If we assume that Euler is using material implication here, we could understand the form of his reasoning as follows:

$$p-qz \text{ is a factor of the polynomial} \rightarrow \text{the polynomial vanishes at } z=\frac{p}{q} \tag{4.18}$$

hence

$$\text{the polynomial vanishes at } z=\frac{p}{q} \rightarrow p-qz \text{ is a factor of the polynomial.} \tag{4.19}$$

This is clearly not a valid form of reasoning. The conclusion of this argument is the Factor Theorem. He then claims this shows a version of the FTA to be true. The Factor Theorem and the most simple statement of the FTA (every nonconstant single-variable polynomial with complex coefficients has at least one complex root: let us label this FTA1) can be combined to show that the number of complex factors of the polynomial is equal to its degree (let us call this FTA2). However, this corollary deserves proof and does depend on FTA1 which Euler does not state.

It would be unfair, and improbable, to conclude that Euler does not understand the basic properties of implication from this example of his reasoning. A more reasonable interpretation is that he is describing observed properties of polynomials. When he states that (4.18) thus (4.19) he could be hinting that the two properties go hand-in-hand (which they do) rather than they follow from each other (which they do not). Previous analyses of Euler's work [67, 57] observe that Euler often uses empirically-inductive reasoning in the style of the physical sciences, as we mentioned in Section 2.1. As for the deduction of FTA2, he is likely to consider FTA1 a self-evident fact or at least, since it was commonly used without proof in the 18th century, something which can be assumed.

4.6.2 Overview of Euler's reasoning behind the Trinomial Lemma

Now we give a brief description of Euler's reasoning behind the trinomial lemma. We do not intend to explain the reasoning in detail, but instead to simply indicate some of the tools and concepts which Euler is using.

Euler used the Factor Theorem to begin his line of reasoning in §143. He claims that complex factors (which he calls imaginary) come in pairs (complex conjugates) that are multiplied to make a factor of the polynomial of the form $p - qz + rz^2$ where p, q and r are real. He uses the discriminant to decide when factors of the form $p - qz + rz^2$ have complex factors themselves and when they have real factors: the factors are complex when $4pr < q^2$. He states this is equivalent to asking if $q/(2\sqrt{pr}) < 1$. This allows him to say that the factors are complex linear if $q/(2\sqrt{pr})$ is equal to the cosine or sine of some angle. This reframing allows Euler to use trigonometry as a replacement for complex numbers: in the previous chapter of the 'Introductio' he proved many trigonometric identities so he is well-armed to deal with them. Euler can then express the roots of the polynomial in polar-coordinate form. The polynomial of course vanishes at its roots and he uses this to obtain simultaneous equations which can be solved for the unknowns p, q and r.

4.6.3 Challenges in our partial mechanisation of the Trinomial Lemma

McKinzie and Tuckey do not prove the trinomial lemma in their rehabilitation of Euler's proof, although they still use a version of it. The trinomial argument is one of the pillars on which the proof of the Basel problem rests, however, the reasoning used in it is different from the rest of the Basel proof in that Euler uses no infinitesimals. Yet the reasoning in it is equally controversial. He apparently believes that this line of reasoning could be applied to any power series and hence any 'whole function'[10]. He will later apply the reasoning to 'polynomials' of infinite degree, as he does in the Basel proof. We explored formalising the Trinomial Lemma and mechanised some of the necessary background: the beginnings of a theory of hyperpolynomials. In Section 6.4.1 we describe how we mechanise the statement of the Trinomial Lemma in locale. We have not mechanised a proof of the Trinomial Lemma. This is partially because it is not a central part of Euler's proof of the Basel Problem and indeed McKinzie and Tuckey did not reconstruct the proof of this lemma. However, Euler's justification for the Trinomial Lemma is also more similar to an exploration and less like a proof. Therefore his discussion is not as directly formalisable and as the formal proof would necessarily diverge significantly from his reasoning, there would be less to gain from comparing his reasoning to any formalisation. Finally, the truth of the Trinomial Lemma is not

[10]In modern terms he may mean analytic real function.

under any doubt and neither was it criticised by contemporary mathematicians since it did not involve any controversial reasoning with infinitesimals.

In the following discussion we outline our understanding of Euler's exploration and which parts of it could be formalisable. In Section 7.4 we also discuss future steps that could be taken to give a proof of the lemma if desired.

Euler's argument includes meta-mathematical reasoning. When he discards real factors whose trinomials are perfect squares, he argues that this is because the pair of equations he gives in §148 are in fact the same equation in that case. From this we can observe that Euler considers the argument to generate the theorem, along with all its exceptions and counterexamples. He does not state the theorem in full at the beginning followed by a proof, as is the way in modern mathematical textbooks, but instead allows the reader to deduce what theorem he has derived by following his line of reasoning.

Meta-mathematics is possible to formalise in Isabelle, but since the object-level and meta-level must be clearly distinguished between, it is inconvenient at best. However, there are options for proving that perfect squares are an exception to the trinomial Factor Theorem without using meta-mathematical reasoning.

Suppose for simplicity we are considering the trinomial factors $a^2 - 2az\cos 2k\pi/n + z^2$, where $0 \leq 2k \leq n$, of $a^n - z^n$. This is all we require for the proof of the Basel problem. Assume that we know the factors of the trinomials are factors of the polynomial. We can justify Euler's remark that the factors come in pairs except for $2k = 0$ by using the FTA. We find that for $2k = 0$, the expression $a^2 - 2az\cos 0 + z^2 = (a - z)^2$ is a perfect square, and, if n is even, for $2k = n$, the expression $a^2 - 2az\cos\pi + z^2 = (a + z)^2$ is also a perfect square. We can deduce that these are the only perfect square trinomials.

Proof. All perfect square trinomials are, by definition, squares of factors of a polynomial in a and z, thus they must be of the form $(c \pm z)^2$, where c is any quantity. By symmetry, they must also be of the form $(c \pm a)^2$, where c is any quantity. Thus they must be $(a - z)^2$ or $(a + z)^2$ (or $(z - a)^2$ or $(-a - z)^2$ but these are equivalent). □

For each $k = 1 \ldots n-1$, ($k = 1 \ldots n$ if n is odd) the expression $a^2 - 2az\cos 2k\pi/n + z^2$ is not a perfect square, since cosine is not equal to 1 or -1 in the interval $(0, \pi)$. Thus for these values of k the trinomials each have two distinct roots. Using combinatorics, we observe that when n is odd, there are $n - 1$ root pairs, and thus by the FTA the perfect square trinomial $(a - z)^2$ can only give one factor $(a - z)$. When n is even, there are $n - 2$ root pairs and the trinomials $(a - z)^2$ and $(a + z)^2$ are distinct. Thus by the FTA, we obtain $(a-z)$ and $(a+z)$ as factors of multiplicity one.

There are other potential difficulties in writing a formal proof of the trinomial Factor Theorem in Isabelle, in general or even just for the special case $a^n - z^n$. Euler seems to be intentionally vague about the degree of the polynomial. As stated earlier, he intends any 'whole function' to be representable by a polynomial. Thus he is considering infinite power series with the same validity as polynomials of finite degree.

4.7 Summary

In this chapter we gave an overview of Euler's reasoning in two of his proofs of the Basel problem.

- His most often criticised and widely-discussed proof, which involves factorising sine into an infinite product of its factors (Section 4.1).
- His proof from the 'Introductio' (Section 4.2). The reasoning of this proof is based on his earlier proof, however he has adapted it so that it provides deductive justification. This is the proof which we mechanise and wherever we mention 'Euler's proof of the Basel problem' hereafter, we are referring to this proof.

We went into further depth on two parts of Euler's proof of the Basel problem

- Euler's §165 (Section 4.5). The application of Euler's §165 to the factorisation of hyperbolic sine forms the third gap in his proof of the Basel problem identified by McKinzie and Tuckey.
- Euler's Trinomial Lemma (Section 4.6).

Furthermore we discussed the concepts used by McKinzie and Tuckey to rehabilitate Euler's proof.

- Euler-convergence (Section 4.4.2)
- determinacy (Section 4.4.2)
- We expanded on Euler-convergence and determinacy by adding the related notions of Euler-determinacy and finite determinacy.
- Their hidden lemmas which they use to fill the three gaps that they identify in Euler's proof Sections 4.4.5- 4.4.7

- Their Hidden Sublemma parts (i) and (ii) which is used to help bridge the gaps and show Euler-convergence so that they may apply the hidden lemmas (Section 4.4.8).

In this chapter we also gave Table 4.3 and Table 4.4 summarising the interaction between the hidden lemmas and the concepts of Euler-convergence and determinacy. We base our mechanisation on McKinzie and Tuckey's previous work, and thus we will mechanise Euler-convergence, determinacy and the hidden lemmas in Chapter 6

Chapter 5

Mechanising determinacy and Euler-convergence

The purpose of this chapter is to describe our mechanisation of the concepts of Euler-convergence and determinacy introduced by McKinzie and Tuckey based on the Eulerian convergence criterion which they identified in his work. These concepts of determinate and Euler-convergent series and products are the key to their rehabilitated proof of the Basel problem. They are necessary to at least attempt to give a formal representation of the leaps of reasoning in Euler's proof that still remains faithful to his argument and his knowledge. Thus we begin by explaining how we mechanise the concepts of determinacy and Euler-convergence which we use throughout our formalisation. We then apply these concepts to specific series and products which are used in the proof of the Basel problem. Finally, in this chapter, we describe the extent of our mechanisation of McKinzie and Tuckey's Hidden Lemmas and how our formalisation lead to the discovery of gaps in McKinzie and Tuckey's proofs. Our description of the mechanisation of the statement of Hidden Sublemma (ii) (see Section 5.8) depends on a concept that we introduce in Section 6.2.1 so the reader may wish to read that section beforehand although we also give an informal explanation in context. In Chapter 6 we will integrate these results into our partial mechanisation and exploration of Euler's proof of the Basel Problem.

5.1 Mechanising determinacy and Euler-convergence

We discussed in Section 4.4.2 McKinzie and Tuckey's hidden lemmas, which are used to fill the gaps in Euler's proof, and how they rely on the concepts of Euler-convergence and determinacy, which we also outlined in Section 4.4.2. Here we will describe our formalisation of Euler-convergence and how we break it down into the three separate properties

- finiteness
- Eulerness
- determinacy

as we discussed in Section 4.4.10. This allows us to discuss the assumptions and proofs of the hidden lemmas with greater detail and nuance.

5.1.1 Determinacy, finite determinacy and Euler-convergence

We formalise the notion of a determinate sequence as follows.

```
definition determinate where
"determinate b = (∀N∈HNatInfinite. ∀M ∈HNatInfinite. b M ≈ b N)"
```

We can now state in Isabelle that a series is determinate by writing

```
determinate (λM. hypsum ( *fn* b) {0..< M})
```

and similarly for products. The determinacy condition for series (products) is the same as insisting that the tail of the series (product) is infinitely close to the additive (multiplicative) identity.

Isabelle is not as flexible as pen-and-paper mathematics when it comes to formulating definitions. In this case, in Isabelle, Euler-convergence must be defined separately for sums and products, despite the fact that it is really the same notion for both. This is because the overarching notion of inductively-applied (or hypernatural–inductively-applied) group product has not been abstracted far enough in Isabelle/HOL. In the conclusion Section 7.2 we revisit this along with related difficulties for expressing definitions in Isabelle.

```
definition EconvergentSum where
"EconvergentSum d ≡
(∀N∈HNatInfinite. ∀M∈HNatInfinite. hypsum (*fn* d) {N..< M} ≈0 ∧ hypsum (*fn* d)
    {0..< N} ∈ HFinite)"
```

```
definition EconvergentProd where
"EconvergentProd d ≡
(∀N∈HNatInfinite. ∀M∈HNatInfinite. hypsum (*fn* d) {N..< M} ≈0 ∧ hypsum (*fn* d)
    {0..< N} ∈ HFinite)"
```

We must also define it for coefficient functions, rather than polynomials, and due to this restriction, we have to state, as part of the definition, that the polynomial starting from 0 is finite. Hence if we are given a polynomial with coefficient function b_n where $b_0 = L$ for some infinitely-large L, and is zero otherwise, we cannot express that $*\sum_{n=1}^{N} b_n$ is Euler-convergent using these Isabelle definitions.[1] We made the choice not to use these definitions in our Isabelle theory, and where we must use the property of Euler-convergence, we write it out in full (see Listing 5.1 and Listing 5.2 in the next section).

Ultimately, we mainly replace Euler-convergence with the related notion of finite determinacy Fdeterminate which can be much more elegantly expressed. This notion is equivalent to Euler-convergence under certain conditions which we explore in Section 5.1.2.

```
definition Fdeterminate where
  "Fdeterminate b ≡(∀N∈HNatInfinite. ∀M ∈HNatInfinite. b M ≈ b N) ∧
                   (∀M ∈HNatInfinite. b M ∈ HFinite)"
```

Finite determinacy is generally a weaker notion than Euler-convergence, as is determinacy, and as we aim to have minimal assumptions to the hidden lemmas, we find these notions useful in our formalisation.

5.1.2 Equivalences between Euler-determinacy, determinacy, finite determinacy and Euler-convergence

We claimed in Section 4.4.2 that Euler-convergence is just determinacy plus finiteness. In this section, we show that McKinzie and Tuckey's definition of Euler-convergence is equivalent to finite determinacy under certain conditions.

As we stated in Section 4.4.10.1, it is possible to represent determinacy of hyperseries and hyperproducts in a way analogous to the Cauchy convergence criterion for sequences in standard analysis. That is, instead of saying that the summation or

[1] Given a polynomial which is Euler-convergent after a certain finite index k, we could of course redefine the coefficient function so that the k^{th} index is instead the 0^{th} index.

product to an infinite N must be infinitely-close to the summation or product to any other infinite M, we can say that the sum between some infinite N, M is infinitesimal, or that the product between N, M is infinitesimally-close to 1. Recall that $\mathbb{L}$ is the set of infinitely-large hyperreals. In mathematical notation,

$$\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \sum_{i=0}^{N} a_i \approx \sum_{i=0}^{M} a_i$$

and

$$\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{i=0}^{N} a_i \approx \prod_{i=0}^{M} a_i$$

express determinacy for sums and products respectively and

$$\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \sum_{i=N}^{M} a_i \approx 0$$

and

$$\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{i=N}^{M} a_i \approx 1$$

express Euler-determinacy.

We prove relations between finite determinacy with Euler-convergence for both hyperseries and hyperproducts. Recall the relations we stated in Section 4.4.10. For hyperseries, it is possible to prove both that determinacy is equivalent to Euler-determinacy and that finite determinacy is equivalent to Euler-convergence. However for hyperproducts, we discovered through our proof that Euler-convergence is only equivalent to finite determinacy if we also assume that the hyperproduct does not contain infinitesimal terms. Furthermore for hyperproducts, Euler-determinacy cannot be proven equivalent to determinacy since finiteness is an essential element of the equivalency proof.

We expressed Hidden Sublemma (ii) using finite determinacy rather than Euler-convergence, and thus we have additional proof-steps in that proof (see Section 5.7.1). We formalise the equivalence between Euler-determinacy and determinacy of hypersum as follows. We do not have an Isabelle abbreviation for Euler-determinacy, but it is represented by the right-hand side.

```
lemma Euler_determinacy_hypsum: "determinate (λM. hypsum ( *fn* b) {0..< M})
= (∀N ∈HNatInfinite. (∀M ∈HNatInfinite. hypsum ( *fn* b) {N..< M} ≈ 0)"
```

This version of the lemma is the most faithful to McKinzie and Tuckey's use of determinacy, as they always consider their series and products starting from zero. Using pen-and-paper, this is without loss of generality, since the series or product could always be relabelled in this way. However, we find that a generalised version is necessary for some proofs in Isabelle, due to the way that hypsum and hyprod are defined. This version does not start the sum at zero and allows for the upper indices of the hypersums to be a finite amount greater than the variable M.

Listing 5.1: Finite determinacy is equivalent to Euler-convergence for sums

```
lemma Euler_determinacy_mismatched_shift_hypsum:
assumes "i ∈ℕ" and "c ∈ ℕ" and "k ∈ ℕ"
shows "determinate (λM. hypsum (λn. ( *fn* b) n) {i..< M + k}) =
      (∀J∈HNatInfinite. ∀K∈HNatInfinite. hypsum (*fn* b) {J..<K + c} ≈ 0)"
```

Finally, we show Euler-convergence is equivalent to finite determinacy for hyperproducts. We allow the product to begin at a finite index i which need not be zero. As we stated above, the finiteness is an essential assumption to the proof, so the analogous theorem for determinacy cannot be shown. We cannot show that Euler-determinacy is equivalent to determinacy for hyperproducts unless we make the further assumption that the hyperproduct is not infinitesimal for any upper index.

Proposition 4. *Determinacy does not imply Euler-determinacy for hyperproducts*

Proof. Suppose k is a finite hypernatural and K is an infinite hypernatural. Let ε be a nonzero infinitesimal. Define $a_i = 1$ for $i \neq k, K$, $a_k = \varepsilon^2$ and $a_K = 1/\varepsilon$. Then ${}^*\prod_{i=0}^{K-1} a_i = \varepsilon^2 \approx \varepsilon = {}^*\prod_{i=0}^{K} a_i$ thus the product is determinate. However, given some $M > K$, then ${}^*\prod_{i=K}^{M} a_i = 1/\varepsilon$, which is infinite, thus the product is not Euler-convergent. □

If we assume that the product is not infinitesimal for any finite index, we rule out such counterexamples.

Listing 5.2: Finite determinacy is equivalent to Euler-convergence for products given an assumption

```
lemma Econvergent_Fdeterminate_variable_initial_value_hyprod:
assumes "c∈ℕ" "i∈ℕ" "∀N ∈HNatInfinite. hyprod (*fn* b) {i ..<N} ∉ Infinitesimal "
shows "Fdeterminate (λM. hyprod ( *fn* b) {i ..< M+c}) =
(∀N ∈HNatInfinite . (∀M ∈HNatInfinite . hyprod ( *fn* b) {N..< M+c} ≈ (1::hypreal)
∧ hyprod (*fn* b) {i ..<M + c} ∈ HFinite))"
```

McKinzie and Tuckey's hidden lemmas are slightly less general than our versions, since we make minimal assumptions. However, their statements follow from our formalisation.

The conditions on the theorems on the relations between determinacy and Euler-convergence have implications for the hidden lemmas, whose proofs they are used in, and therefore for the entire formalisation of the Basel problem. All the series and products involved have conditions on them which must be proven.

5.1.3 Characterising properties of determinacy and Euler-convergence

We can think of determinate series and products as having a periodicity property[2]. For some hypernatural n, let $S(m) = \sum_{i=n}^{m} s_i$ be a determinate series. Then

$$S(M) \simeq S(M+c)$$

for infinite hypernatural M and finite hyperinteger c. And thus $S(m+c)$ is also determinate. Similarly, if $S(m+c)$ is determinate this implies $S(m)$ is determinate. If c is assumed hypernatural (i.e. if we restrict it to positive values), then we do not need to restrict it to being finite. This periodicity property allows us to express lemmas on determinacy with 'mismatched shifts'. We have not proved this property in general, however, we have only incorporated it into the statement of Listing 5.1 in Section 5.1.2. The other special property of determinate series is that we only need to consider the tail of the series. Therefore, it does not matter from which index we begin the series, so long as it is finite. We have proven this property in full generality as the following lemma.

Listing 5.3: *A series is determinate regardless whether it begins at* 0 *or* $i \in \mathbb{N}$.

```
lemma determinate_variable_initial_value_hypsum:
assumes "i ∈ℕ"
shows
   "determinate (λM. hypsum ( *fn* b) {i ..<M + c}) =
    determinate (λM. hypsum ( *fn* b) {0..<M + c})"
```

Notice also that we prove this for M+c rather than simply M. Of course the expression $\sum_{n=i}^{M+c} a_n$ can be rewritten as $\sum_{n=i}^{N} a_n$ by setting $N = M + c$. However, the function λM. hypsum (*fn* b) { i ..< M + c}) cannot be directly rewritten as a function of the form

[2] McKinzie and Tuckey describe this as 'the value ... cannot depend perceptibly on the particular infinite value of N' [55, p. 348].

λN. hypsum (*fn* b) {i ..<N}). The best we could achieve would be λM. hypsum (*fn* (λx n. b (x + unstarnum c n) n) {i − c..<M}) but this is tedious to prove equivalent to λM. hypsum (*fn* b) {i ..<M + c}) (and in fact, is only equivalent if we assume $i \geq c$). But λN. hypsum (*fn* b) {i ..<N}) can be written as function of the form λM. hypsum (*fn* b) {i ..<M + c}) by writing λN. hypsum (*fn* b) {i ..<N + 0}). Therefore, representing the series in the lemma as λM. hypsum (*fn* b) {i ..<M + c} makes it far easier to reuse the lemma.

For determinate products, it only takes a single term of the product with finite index to be zero for the whole product to disappear and to be trivially determinate. So we require additional assumptions to prove that the determinacy of a product does not depend on the initial index: in short, the product from zero to the initial index must be finite and non-infinitesimal. This is formalised as follows with proof.

```
lemma determinate_variable_initial_value_hyprod:
fixes b::"nat ⇒ nat ⇒ real"
assumes "i ∈ℕ" and
"hyprod (*fn* b) {0..< i} ∈ HFinite" and
"hyprod (*fn* b) {0..< i} ∉ Infinitesimal "
shows
"determinate (λM. hyprod ( *fn* b) {i ..<M + c}) =
 determinate (λM. hyprod ( *fn* b) {0..<M + c})"
```

We can prove similar theorems for finite determinacy. These combine the determinacy theorems with theorems that allow hyperfiniteness of sums and products to be shifted.

```
lemma hypsum_HFinite_tail:
assumes "∀M∈HNatInfinite. hypsum (*fn* b) {0..<M + c} ∈HFinite" and
"∀x. 0 ≤ (*fn* b) x" and "i ∈ ℕ" and "c ∈ ℕ"
shows "∀M∈HNatInfinite. hypsum (*fn* b) {i..<M + c} ∈ HFinite"
```

The finite determinacy theorems need to be separated into ‘gain’, where the summation index interval is extended, and ‘loss’, where the summation index interval is shortened.

```
lemma Fdeterminate_shift_hypsum_gain:
assumes "Fdeterminate (λM. hypsum ( *fn* b) {i..<M + c})" and "i ∈ ℕ" and "c ∈ ℕ"
and "hypsum ( *fn* b) {0..< i} ∈ HFinite"
shows "Fdeterminate (λM. hypsum ( *fn* b) {0..<M + c})"
```

```
lemma Fdeterminate_shift_hypsum_loss:
```

```
assumes "Fdeterminate (λM. hypsum ( *fn* b) {0..<M + c})" "i ∈ ℕ" "c ∈ ℕ"
and "∀x. ( *fn* b) x ≥ 0"
shows  "Fdeterminate (λM. hypsum ( *fn* b) {i..<M + c})"
```

Listing 5.4: Finite determinacy of hyperproduct with widened initial bound

```
lemma Fdeterminate_initial_value_hyprod_gain:
fixes b::"nat ⇒ nat ⇒ real"
assumes "Fdeterminate (λM. hyprod ( *fn* b) {i..<M + c})"
and "i ∈ ℕ" and "c ∈ ℕ"
and "hyprod (*fn*  b) {0..<i} ∉  Infinitesimal "
and "hyprod ( *fn*  b)  {0..< i} ∈ HFinite"
shows "Fdeterminate (λM. hyprod ( *fn* b) {0..<M + c})"
```

The lemma in Listing 5.4 is used to prove a generalised Second Hidden Lemma which inherits additional assumptions from it.

```
lemma Fdeterminate_initial_value_hyprod_loss:
fixes b::"nat ⇒ nat ⇒ real"
assumes "Fdeterminate (λM. hyprod ( *fn* b) {0..<M + c})"
and "i ∈ ℕ" and "c ∈ ℕ"
and "hyprod (*fn*  b) {0..<i} ∈ HFinite"
and "hyprod (*fn*  b) {0..<i} ∈ HFinite − Infinitesimal "
and "hyprod (*fn*  b) {0..<i} ∉  Infinitesimal "
shows  "Fdeterminate (λM. hyprod ( *fn* b) { i ..<M + c})"
```

5.1.4 Comparison Test for determinacy

This lemma is a slight generalisation of McKinzie and Tuckey's Comparison Test for Determinacy. This is a version of the Squeeze Theorem from standard analysis.

```
lemma comparison_test_determinacy_variable_initial_value:
assumes "∀n≥j. ( *fn* a) n ≥ 0"
and "k∈ℕ" "j∈ℕ" "c∈ℕ" "i∈ℕ"
and "determinate (λM. hypsum ( *fn* b) {i..<  M + c})"
and "∀n≥k. ( *fn* a)  n ≤   ( *fn* b)  n"
shows "determinate (λM. hypsum ( *fn* a) {j..< M + c})"
```

The proof relies on the lemma Euler_determinacy_variable_initial_value_hypsum which is another variation on the lemmas described in the previous section, and that lemma allows this one to index the series from any natural number. This lemma will be used to show that $\sum_{i=0}^{N} 1/i!$ and $\sum_{i=0}^{N} x^i/i!$ are determinate (Section 5.2.2).

5.1.5 Proving additional general properties of determinacy

Determinate series can be combined in certain ways to preserve their determinacy. Some lemmas are also true for determinate products. There are also more rules for combining determinacy of series and products than we list or prove, since we have only proved those we found useful in our proof of the Basel problem. In each of the lemmas below we have assigned the most general types for which the lemma is still true, however we have omitted the types here for ease of reading.

Listing 5.5: *The constant sequence is determinate.*

```
lemma constant_seq_determinate:
  "determinate (λn. c)"
```

Listing 5.6: *The series consisting only of zeros $\sum_{n=i}^{M} 0$ is determinate.*

```
lemma determinate_hypsum_zero:
 "determinate (λM. hypsum (*fn*  (λn x. 0)) {i..<M})"
```

Listing 5.7: *Given two determinate series $\sum_{n=i}^{M} a_n$ and $\sum_{n=i}^{M} b_n$, the series formed by termwise addition $\sum_{n=i}^{M} (a_n + b_n)$ is also determinate.*

```
lemma determinate_hypsum_add:
assumes "determinate (λM. hypsum ( *fn* a) {i..< M})" and
             "determinate (λM. hypsum ( *fn* b) {i..<  M})"
shows "determinate (λH. hypsum (λn. (*fn*  a) n + ( *fn* b) n) {i..<H})"
```

Listing 5.8: *Given two determinate series $\sum_{n=i}^{M} a_n$ and $\sum_{n=i}^{M} b_n$, the series formed by taking the difference termwise $\sum_{n=i}^{M} (a_n - b_n)$ is also determinate.*

```
lemma determinate_hypsum_diff:
assumes "determinate (λM. hypsum ( *fn* a) {i..< M})" and
             "determinate (λM. hypsum ( *fn* b) {i..<  M})"
shows "determinate (λH. hypsum (λn. (*fn*  a) n − ( *fn* b) n) {i..<H})"
```

Listing 5.9: *Given two determinate hyperpolynomials (power series) $\sum_{n=i}^{M} a_n x^n$ and $\sum_{n=i}^{M} b_n x^n$, the polynomial whose coefficients are given by the termwise difference between the coefficients of the original polynomials $\sum_{n=i}^{M} (a_n - b_n) x^n$ is also determinate.*

```
lemma determinate_diff_polynomial:
assumes
  "∀x∈HFinite. determinate (λM. hypsum (λn. ( *fn* a) n · x pow n) {i ..< M})" and
  "∀x∈HFinite. determinate (λM. hypsum (λn. ( *fn* b) n · x pow n) {i ..< M})"
shows
  "∀x∈HFinite.
     determinate (λH. hypsum (λn. ((*fn* a) n − ( *fn* b) n) · x pow n) {i ..<H})"
```

Listing 5.10: *If a series $\sum_{n=i}^{M} a_n$ is determinate, then if $k \in \mathbb{N}$ so must be $\sum_{n=i}^{M} a_{n+k}$.*

```
lemma determinate_shift_argument:
assumes "k∈ℕ" "i∈ℕ" "determinate (λM. hypsum (λn. (*fn* a) n) {i..<M})"
shows "determinate (λM. hypsum (λn. (*fn* a) (n+k)) {i..<M})"
```

These general properties can now be used, along with the Hidden Sublemmas (Section 4.4.8), to help show that the series and products in Euler's proof are determinate.

5.2 Proving determinacy of series and products

There are specific series and products which need to be proved determinate and Euler-convergent so that the hidden lemmas can be applied. We have formalised their statements in Isabelle and some of their proofs. The reasoning in their proofs does not correspond to any reasoning given by Euler. We also do not consider their truth to be under question as determinacy is an analogue of convergence which is well-established for these series and products. Therefore they fall under the third category of reasoning described in Section 1.3 and we have not found it necessary to mechanise all their proofs. We still explore the reasoning of some their proofs out of interest. For the series $1 + x + x^2/2! + x^3/3! + \dots$ we analyse the justification of its Euler-convergence given by McKinzie and Tuckey and find it insufficient. We give our own pen-and-paper proof of its Euler-convergence in Section 5.2.2.2.

5.2.1 The Basel series is Euler-convergent

The proof that the Basel series $1+1/4+1/9+1/16+1/25+\cdots$ is Euler-convergent is completely formalised. This is a useful result since the Euler-convergent of several other polynomials follows from this. First we will outline McKinzie and Tuckey's proof since we follow their proof closely with our mechanisation.

5.2.1.1 McKinzie and Tuckey's proof that the Basel series is Euler-convergent

They first show that all the partial sums to an infinite degree are finite. They then show Euler-determinacy and the two combined give Euler-convergence. We give their proof here providing some additional explanation. We can bound above the sequence of partial sums from n to m with $1 < n < m$:

$$\begin{aligned} 0 < \sum_{k=n}^{m} \frac{1}{k^2} < \sum_{k=n}^{m} \frac{1}{k^2 - \frac{1}{4}} &= \sum_{k=n}^{m} \left(\frac{1}{k-\frac{1}{2}} \frac{1}{k+\frac{1}{2}} \right) \\ &= \sum_{k=n}^{m} \left(\frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}} \right) \\ &= \frac{1}{n-\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}. \end{aligned}$$

The series $\sum_{k=n}^{m} \left(\frac{1}{k-1/2} - \frac{1}{k+1/2} \right)$ is telescoping, so all terms cancel except the first and last. Observe that if $n = N$ and $m = M$ are infinite hypernatural numbers, and we assume $M > N$ without loss of generality, then $\frac{1}{n-1/2} - \frac{1}{m+1/2}$ is infinitesimal, so we have shown that $\sum_{k=n}^{m} 1/k^2$ is Euler-determinate. Furthermore, using the same line of reasoning, we can deduce that

$$\sum_{k=1}^{N} \frac{1}{k^2} < 1 + \sum_{k=2}^{N} \frac{1}{k^2 - \frac{1}{4}} = 1 + \frac{1}{2-\frac{1}{2}} - \frac{1}{N+\frac{1}{2}} < \frac{5}{3}$$

if N is infinite, thus this is finite and we have now shown that $\sum_{k=n}^{m}$ is Euler-convergent.

5.2.1.2 Mechanisation of the proof that the Basel series is Euler-convergent

We split the proof of Euler-convergence into a lemma showing finiteness (Listing 5.11) and a lemma showing Euler-determinacy (Listing 5.12) just as McKinzie and Tuckey do.

Listing 5.11: The Basel series is finite

```
lemma Basel_sum_finite:
assumes "N >1"
shows "hypsum (λn. 1/(n∈ℕ pow 2)) {(1::hypnat)..< N +1} < 1+ 1/(2 − 1/2)"
```

We show the series is less than $1 + \frac{1}{2-1/2}$ rather than $5/3$ since simplifying the quantity was unnecessary for showing that it is finite. In fact, showing that $1 + \frac{1}{2-1/2}$ is finite requires its own proof in Isabelle. We argue that since it is a combination of finite non-infinitesimal quantities it is itself finite.

Listing 5.12: The Basel series is determinate

```
lemma Basel_sum_determinate:
"determinate (λN. hypsum (λn. 1/(n∈ℕ pow 2)) {(1::hypnat)..< N +1})"
```

To show that the series is determinate we make essentially the same argument as McKinzie and Tuckey, however in Isabelle we cannot 'assume $M > N$ without loss of generality': this requires a case-split. To prove both the lemmas above, it is useful to separately prove the following lemmas.

```
lemma Basel_sum_ineq:
assumes "N≥m" and "m>0"
shows "hypsum (λn. 1/(n∈ℕ pow 2)) {m..< N +1}
        < hypsum (λn. (1/(n∈ℕ − 1/2))− (1/(n∈ℕ + 1/2))) {m..< N +1}"

lemma Basel_telescoping_series:
assumes "N ≥m"
shows "hypsum (λn. 1/(n∈ℕ − 1/2)) {m..< N +1}
        − hypsum (λn. 1/(n∈ℕ + 1/2)) {m..< N +1} =
        1/(m∈ℕ − 1/2) − 1/(N∈ℕ + 1/2)"
```

A significant proportion of these proofs is spent in proving that all the functions involved are internal. Recall that hypsum and hyprod are only defined for internal functions (see Section 3.2.4), which makes this a necessary step in the proofs.

5.2.1.3 Use of the Euler-convergence of the Basel series

The Euler-convergence of the Basel series is not used directly in McKinzie and Tuckey's proof of the Basel problem, but it is used to show that other sums are Euler-convergent (see Section 5.2.2.1) [57, p. 45].

By Hidden Sublemma (ii), they show that the sum

$$x + x^3\left(\sum_{k=1}^{(N-1)/2} \frac{1}{(k\pi)^2}\right) + x^5\left(\cdots\right) + \cdots \tag{5.1}$$

is Euler-convergent for all finite x. Thus the Basel series is a coefficient in an Euler-convergent sum. They then apply the Third Hidden Lemma to (5.1) but explicitly this uses the Euler-convergence of (5.1) not of the Basel series. However, when we formalised the Third Hidden Lemma, we realised that it was necessary to assume that the differences between the coefficients of the two polynomials (that we apply the lemma to) were finite (Section 5.5.2.1). Therefore, technically the finiteness of the Basel series is needed for the application of the Third Hidden Lemma. However, each of the infinitely-many coefficients of the polynomial would also need to be proven finite. This would be difficult to show, and this can be avoided by making a stronger assumption than strictly necessary for the Third Hidden Lemma: namely, by assuming Euler-convergence rather than determinacy.

5.2.2 The power series of e^x is determinate

McKinzie and Tuckey state that the series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..$ (the power series of e^x) is determinate for finite x. They justify this [57, p. 45]. However, we examine their justification and find it insufficient.

5.2.2.1 McKinzie and Tuckey's justification that the power series of e^x is determinate

In their paper, McKinze and Tuckey first show that the Basel series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$ is Euler-convergent. We formalised this proof in Section 5.2.1. They use this to show that the series $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$ is also Euler-convergent. They implicitly use the First Hidden Lemma (Comparison Test) for Euler-convergence:

> Since for $k>3$, we have $1/k! < 1/k^2$, we may also conclude that $1+1/2!+1/3!+1/4!+1/5!+\cdots$ is Euler-convergent, and furthermore that for all finite x, the sum $1+x+x^2/2!+x^3/3!\cdots$ is Euler-convergent. [57, p. 45]

We are most interested in showing the determinacy part of Euler-convergence, since this will be what is needed in the proof of the Basel problem (to apply the First Hidden Lemma in Gap 1 Section 6.8.1).[3] There is no theorem that says if a series $\sum_{k=1}^{n} a_k$ is Euler-convergent then the series $\sum_{k=1}^{n} a_k x^k$ is also Euler-convergent for all finite x, nor can there be such a theorem. Perhaps they mean to argue that, just as $\frac{1}{k!}<\frac{1}{k^2}$, also $\frac{x^k}{k!}<\frac{x^k}{k^2}$. This is true for $x>0$. But again, they have not shown that $1+x+\frac{x^2}{4}+\frac{x^3}{9}+\frac{x^4}{16}+\cdots$ is Euler-convergent, so this argument does not entirely work.

[3]However, finiteness is used for the Corollary (Section 5.6) and thus indirectly in proof of Hidden Sublemma(i) (Section 5.7).

5.2.2.2 Our pen-and-paper proof and formalisation of the statement

Since McKinzie and Tuckey's proof is flawed, we provide a pen-and-paper proof of the determinacy of $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..$ for finite $x \geq 0$. They do not directly use the determinacy of this series in their proof of the Basel problem. In Section 5.2.2.3 we explain how the determinacy of this series is nonetheless useful in our formal proof of the Basel problem. We have not extended our pen-and-paper proof of this to a formal proof since it is not part of the main proof which we have sought to understand through formalisation and its truth is not under question.

We formalise the statement of the determinacy of $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..$ as follows.

```
lemma power_over_Hfact_sum_determinate:
assumes "x ∈HFinite" "x≥0"
shows "determinate (λN. hypsum (λn. x pow n/Hfact n {1..< N +1})"
```

The term `Hfact n` is the extension of the factorial function $n!$.

For the pen-and-paper proof that we have been able to provide, it is necessary to add the assumption that $x \geq 0$. This has implications throughout the formalised proof of the Basel problem, but it does not affect the validity of the proof or the final statement of the problem (see Section 6.8).

Theorem 9. *The series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ is Euler-determinate (and thus determinate) for finite $x \geq 0$.*

Proof. By the definition of Euler-determinacy, the series will be determinate if for all infinite N and M with $M \geq N$ we have

$$\frac{x^N}{N!}+\frac{x^{N+1}}{(N+1)!}+\cdots+\frac{x^M}{M!} \simeq 0.$$

We can rewrite the LHS so that

$$\frac{x^N}{N!}+\frac{x^{N+1}}{(N+1)!}+\cdots+\frac{x^M}{M!}=\frac{x^N}{N!}\left(1+\frac{x}{N+1}+\frac{x^2}{(N+1)(N+2)}+\cdots+\frac{x^{M-N}}{(N+1)\cdots M}\right).$$

Since $x^N/N!$ is infinitesimal, it would be sufficient to show that

$$1+\frac{x}{N+1}+\frac{x^2}{(N+1)(N+2)}+\cdots+\frac{x^{M-N}}{(N+1)\cdots M} \tag{5.2}$$

is finite. We can bound this sum above using the assumption that $x \geq 0$

$$\begin{aligned}
1+\frac{x}{N+1}+\frac{x^2}{(N+1)(N+2)}+\cdots+\frac{x^{M-N}}{(N+1)\cdots M} &= \sum_{k=1}^{M-N} \frac{x^k}{\prod_{i=N+1}^{N+k} i} \\
&< \sum_{k=1}^{M-N} \left(\frac{x}{N}\right)^k \\
&= \frac{x}{N}\frac{1-\left(\frac{x}{N}\right)^{M-N}}{1-\frac{x}{N}} \\
&\simeq 0
\end{aligned} \tag{5.3}$$

by using the properties of geometric partial sums. The First Hidden Lemma (Comparison Test for determinacy of series) (Section 4.4.5) with (5.3) implies that our sum (5.2) is infinitesimal and therefore finite. □

5.2.2.3 Use of the determinacy of the power series of e^x

To fill Gap 1 of the Basel problem it is necessary to apply the First Hidden Lemma. This requires us to use the fact that

$$x+\tfrac{x^3}{3!}+\cdots+\tfrac{x^N}{N!}$$

is determinate. We have not mechanised the proof of its determinacy, however it follows from the determinacy of the power series of e since

$$\sum_{k=1}^{N} \frac{x^k}{k!} - \sum_{k=1}^{N} \frac{(-x)^k}{k!} = x+\frac{x^3}{3!}+\cdots+\frac{x^N}{N!} \tag{5.4}$$

for odd N. We must also use the lemmas about the algebra of determinate series we proved in Section 5.1.5. However, the determinacy of the power series of e^x is also used directly in the proof of Hidden Sublemma (i) (see Section 5.7), which we have not fully mechanised. We explain why the mechanisation of this is incomplete in Section 6.1.2.2.

5.2.3 Finite determinacy of the products

The finite determinacy of the two products

$$x \prod_{k=1}^{(N-1)/2} \left(1+\frac{x^2}{(k\pi)^2}\frac{\frac{1}{2}\left(\frac{2k\pi}{N}\right)^2}{1-\cos\frac{2k\pi}{N}}-\frac{x^2}{N^2}\right) \text{ and } x \prod_{k=1}^{(N-1)/2} \left(1+\frac{x^2}{(k\pi)^2}\right)$$

results from the finite determinacy of the Basel series (see Section 5.2.1) and Hidden Sublemma (ii), however we have not yet mechanised this.

5.3 First Hidden Lemma

The First Hidden Lemma is by far the most straightforward to mechanise in Isabelle and also has the shortest proof as given by McKinzie and Tuckey [57, p. 50]. The short proof supports McKinzie and Tuckey's (and Laugwitz's) claim that it is a principle which Euler tacitly used. McKinzie also argues that Halley used this principle before Euler [56]. We give a detailed explanation of the formalisation of its statement since it is the first of the hidden lemmas that we are describing.

5.3.1 Statement of the First Hidden Lemma in Isabelle

As we described in Section 4.4.5, McKinzie and Tuckey state the First Hidden Lemma, which is more descriptively named the Summation Comparison Theorem, as

> If the sums $a_1+a_2+a_3+\cdots$ and $b_1+b_2+b_3+\cdots$ are Euler-convergent, and if for each finite n, $a_n \simeq b_n$, then for all N, $a_1+a_2+\cdots a_N \simeq b_1+b_2+\cdots b_N$ [57, p. 50]

in their paper on reconstructing the proof of the Basel problem in nonstandard analysis. They consistently use Euler-convergence throughout this paper. However, they also give the First Hidden Lemma in their subsequent paper where they rely only on the more general notion of determinacy (see Section 4.4.2 and Section 5.1.1) [55, p. 349]. We were able to formalise and prove the lemma using the weaker assumption of determinacy. This allows the lemma to be more easily applied in the full proof of the Basel problem and elsewhere. Our Isabelle statement of the lemma:

```
lemma hidden_lemma1:
assumes "determinate (λM. hypsum (*fn* a) {0..< M})"
        "determinate (λM. hypsum (*fn* b) {0..< M})"
        "⋀n. n ∈ ℕ ⟹(*fn* a) ≈(*fn* b) n"
shows "hypsum (*fn* a) {k..< N + 1} ≈ hypsum (*fn* b) {k..< N + 1} "
```

generalises the summation indices in the conclusion. They begin at k rather than 0. This increases the complexity of the proof of the lemma but turns out to be necessary for fitting the lemma into the overall proof. The summation indices in the assumption can be generalised using one of the lemmas proved in Section 5.1.3. We also have a choice between using the Isabelle interval {k.. N}, which corresponds to $\{x.\, k \le x \le N\}$, and {k..< N + 1}, which corresponds to $\{x.\ k \le x < N+1\}$, to define the bounds of the hypersum. The latter might seem to be a strange choice, but our proof has to rely on the theorem hypsum_split_interval which, represented in mathematical notation is this theorem:

Theorem (Splitting hypernatural sums)**.** *Let $n \le k \le m$ be hypernaturals. Then*

$$\sum_{i\in[n,m)} a_i = \sum_{i\in[n,k)} a_i + \sum_{i\in[k,m)} a_i$$

The same theorem cannot be as easily expressed using closed intervals. The final formalisation choice in the First Hidden Lemma is to use internal functions to represent what McKinzie and Tuckey implicitly assume to be 'elementary sequences'[4]. We make the choice to represent sets and functions internally throughout and we will not discuss it further apart from rare exceptions to this rule (see Section 2.3.2 for an explanation of why we use internal functions and sets to represent Euler's concepts). Apart from the formalisation choices between determinacy/Euler-convergence and strict/non-strict intervals, the statement of the lemma in Isabelle closely follows the mathematical statement given by McKinzie and Tuckey.

5.3.2 Proof of the First Hidden Lemma

For ease of reference, we first quote McKinzie and Tuckey's proof of the First Hidden Lemma.

> If $a_n \simeq b_n$ for all finite n, then by the Finite Induction Principle, $a_1 + \ldots + a_n \simeq b_1 + \ldots + b_n$ for all finite n as well. By the Sequential Theorem, there is an infinite J such that for all n less than J, $a_1 + \ldots + a_n \simeq b_1 + \ldots + b_n$. Let N be greater than J. If the sums are Euler-convergent, then by definition, $a_J + \ldots + a_N$ and $b_J + \ldots + b_N$ are both infinitesimal, and hence for all $N, a_1 + \ldots + a_N \simeq b_1 + \ldots + b_N$. [57, p. 50]

Since we begin our hypersum at k rather than 0, we have extra cases to consider in our mechanised proof. First we consider the case where k is infinite. If N is also infinite, then the hypersums we are considering, i.e. hypsum (*fn* a) {k..< N + 1} and hypsum (*fn* b) {k..< N + 1}, are the tails of determinate series. Therefore we can use the lemma Euler_determinacy_hypsum (see Section 5.1.2) to conclude that they are

[4]We deduce this because they refer to 'elementary sequences' in earlier lemmas. They also state that the internal functions of Robinson's theory are a 'particularly broad' interpretation of 'elementary' in this context [57, p. 43]

both infinitesimals. Then they must be infinitely-close to each other. If N is finite then the interval {k..<N+1} is just the empty set, and thus the hypersums we are considering are both zero and of course infinitely-close to each other.

The case which corresponds to McKinzie and Tuckey's proof is where k, like 0, is finite. We might think that we also need to consider whether k is less than N. However, the construction of their proof makes this unnecessary, since both k and N are compared to an intermediate J. They use finite induction to add up infinitely-close terms of the two sequences to get two sums of finitely many terms which are infinitely-close to each other. They state

> If $a_n \simeq b_n$ for all finite n, then by the Finite Induction Principle, $a_1 + \cdots + a_n \simeq b_1 + \cdots + b_n$ for all finite n as well.

This works for any finite number of terms. So they can then use the Sequential Theorem to extend this to the infinite case, almost magically. As they say

> By the Sequential Theorem, there is an infinite J such that for all n less than J, $a_1 + \cdots + a_n \simeq b_1 + \cdots + b_n$.

The use of finite induction is easily reproduced in Isabelle using the lemma below.

```
lemma finite_induction_principle_hypnat_type:
assumes "(n::hypnat)∈ℕ" "P 0"
        "⋀k. ⟦k<n;P k⟧ ⟹P (k+1)"
shows "P n"
```

This is a corollary of the standard induction rule in Isabelle which we proved so that it can be applied to hypernatural intervals. However, we have to compare our initial value k to n since it is not always less than or equal to n. We formalise this as a case-split with an easily proven degenerate case. Similarly to McKinzie and Tuckey we use the Sequential Theorem to obtain an infinite J for which the following proposition holds.

```
∀n < J. hypsum ( *fn* b) {k..< n+1} ≈ hypsum ( *fn* a) {k..< n+1}
```

At this point in the proof, McKinzie and Tuckey state 'Let N [the degree of the polynomial] be greater than J'. In Isabelle we must represent this with a case-split on $N+1 > J$. They then continue

> If the sums are Euler-convergent, then by definition, $a_J + \cdots + a_N$ and $b_J + \cdots + b_N$ are both infinitesimal, and hence for all N, $a_1 + a_2 + \cdots a_N$ and $b_1 + b_2 + \cdots b_N$.

We are able to follow this in Isabelle using $N+1$ instead of N. The Isabelle proof is 96 lines compared to McKinzie and Tuckey's 6 line proof. The greater length of the Isabelle proof is partially due to the addition of all the trivial and degenerate cases which needed to be considered and then discharged; partially due to the steps which combine and split hypersums, where the proof assistant required more guidance than a human might to follow the reasoning; and clear formatting requires more lines than would be used in a pen-and-paper proof.

5.4 Second Hidden Lemma

The Second Hidden Lemma (or Product Comparison Theorem [55, p. 349]) is intended by McKinzie and Tuckey to be exactly analogous to the First Hidden Lemma (Summation Comparison Theorem), but stated for products rather than sums. However, we find that there is a significant increase in complexity, especially if we generalise the lemma to allow for a different initial index.

5.4.1 Statement of the Second Hidden Lemma

They state it identically to the First Hidden Lemma, with no additional assumptions and say only 'The proof is similar to (the First Hidden Lemma)'.

> If the products $(1+a_1)(1+a_2)(1+a_3)\cdots$ and $(1+b_1)(1+b_2)(1+b_3)\cdots$ are Euler-convergent, and if for each finite n, $a_n \simeq b_n$, then for all N, $(1+a_1)(1+a_2)\cdots(1+a_N) \simeq (1+b_1)(1+b_2)\cdots(1+b_N)$ [57, p. 50]

The only difference the reader might observe is that they multiply $(1+a_i)$ rather than summing a_i. In fact, this is not a real difference since it is the increments from the multiplicative identity (i.e. 1) in the case of products, and the increments from the additive identity (i.e. 0) in the case of sums, which must be infinitely-close. So they state the first two Hidden Lemmas entirely analogously. In fact, they are both instances of the same general theorem which could be stated at the level of group theory.

In Isabelle, we state the lemma without writing the products as increments from the identity. For this reason, one could either consider our statement to be a generalisation of theirs, since they often assume their terms a_n and b_n to be non-negative,[5] or it could be considered a simplified way of stating the same fact. Our lemma requires one more

[5]They did not explicitly state a_n and b_n to be non-negative in this case, but if the Second Hidden Lemma was generalised to any initial index, this assumption would be needed.

assumption than McKinzie and Tuckey explicitly state. We justify its necessity in Section 5.4.5.

```
lemma hidden_lemma2_initial_index_zero:
assumes
Fdeterminate_b: "Fdeterminate (λM. hyprod ( *fn* b) {0..< M})" and
Fdeterminate_a: "Fdeterminate (λM. hyprod ( *fn* a) {0..< M})" and
a_b_inf_close: "∀n ∈ ℕ. ( *fn* a) n ≈ ( *fn* b) n" and
a_HFinite: "∀n ∈ ℕ. (*fn* a) n ∈ HFinite" and
shows "hyprod ( *fn* b) {0..< N+1} ≈hyprod ( *fn* a) {0..< N+1}"
```

This statement of the Second Hidden Lemma starts the product from the initial index 0, which corresponds to McKinzie and Tuckey's statement[6]. In fact, the Second Hidden Lemma should be true no matter which finite initial index is chosen. We have mechanised a second version which allows for any finite initial index, but this version has several more assumptions, some of which are required, and some which could potentially be removed. We will not go into detail on this second version. Mechanising the proof of Second Hidden Lemma in Isabelle led to the discovery that it has greater complexity than the First Hidden Lemma, both in its proof and in its statement. Informally we can explain this increased complexity by the interaction between the infinitely-close relation and products. The infinitely-close relation behaves like equality when addition is concerned but with the introduction of multiplication it behaves differently. In the next section, we consider whether finite determinacy is required for the lemma, or if it is sufficient to use determinacy.

5.4.2 Finite determinacy

In Section 4.4.2 we noted the difference between the related concepts of Euler-convergence, finite determinacy and determinacy. Since determinacy is the weakest assumption, we use it wherever possible, for example, in our statement of the First Hidden Lemma (see Section 5.3.1). However, for the Second Hidden Lemma, we do need to assume the finiteness of the product, since we will eventually divide by its tail and will then wish to conclude that the infinitely-close relation is preserved (see Section 5.4.4). When using the infinitely-close relation to make conclusions about multiplication or division, the quantities need to be well-behaved in some way: i.e. non-infinitesimal

[6]Of course, they start from 1, but this difference is not significant. The significance is that a single number is chosen.

or non-infinite. If the quantities are not restricted, then the infinitely-close relation behaves very differently from equality. Thus we assume it is finite determinate.

5.4.3 Additional assumption

As mentioned, there is an additional assumption a_HFinite which we had to make for the Second Hidden Lemma. We can state this in mathematical notation as $k \in \mathbb{N} \longrightarrow a_k$ is finite. In Section 5.4.5 we prove it is necessary in order to guarantee the truth of the lemma, so it can rightly be called a missing assumption. The analogous result for the function b easily follows from ∀n ∈ ℕ. (*fn* a) n ≈ (*fn* b) n. We can obtain the fact that if $k \in \mathbb{N}$ then $\prod_{n=0}^{k} a_n$ is finite from a_HFinite by finite induction. We call this fact a_hyprod_HFinite. Both a_hyprod_HFinite and a_HFinite are necessary to make McKinzie and Tuckey's proof work, but were not stated by them.

5.4.4 Structure of mechanised proof

McKinzie and Tuckey give no proof, but they do give a proof of the analogous First Hidden Lemma (see Section 5.3.2). We base our proof of the Second Hidden Lemma on this. As they do, we obtain that $a_1 \cdots a_{n-1} a_n \simeq b_1 \cdots b_{n-1} b_n$ for all finite n by finite induction. This step requires an additional assumption. We have to split the products $a_1 \cdots a_{n+1}$ and $b_1 \cdots b_{n+1}$ so that we can use our induction hypothesis and conclude that $a_1 \cdots a_{n+1} \simeq b_1 \cdots b_{n+1}$. In the process we must use this property of the infinitely-close relation

$$\text{If } c \simeq d \text{ and } e \simeq f, \text{ for } d \text{ and } f \text{ finite, then } ce \simeq df.$$

This ends up requiring that if $k \in \mathbb{N}$ then $\prod_{n=0}^{k} a_n$ is finite and, further, if $k \in \mathbb{N}$ then the coefficient a_k is finite. Of course, the first can be proved from the latter and we thus add the latter fact to the statement of the Second Hidden Lemma as an assumption. Due to the simplicity of the proof, and the rigidity of the properties of the infinitely-close relation, it seems unlikely that there could be any way to avoid making this assumption.

5.4.5 Necessity of the additional assumption

The assumption can easily be discharged when the Second Hidden Lemma is used in Euler's proof of the Basel problem. But as a critique of the proof of the Second Hidden Lemma given by McKinzie and Tuckey, we will show now that the assumption is necessary and should have been provided by McKinzie and Tuckey[7]. As this is a detour from the main focus of the thesis, this is only a pen-and-paper example provided for the interest of the reader.

Our additional assumption is

Assumption 2 (A). *If $k \in \mathbb{N}$ then the coefficient a_k is finite.*

The other assumptions are

Assumption 3 (B).

1. $\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{k=0}^{N} a_k \simeq \prod_{k=0}^{M} a_k$ *(i.e. a_k is determinate)*
2. $\forall N \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{k=0}^{N} a_k$ *is finite.*
3. $\forall N, M \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{k=0}^{N} b_k \simeq \prod_{k=0}^{M} b_k$ *(i.e. b_k is determinate).*
4. $\forall N \in \mathbb{L} \cap {}^*\mathbb{N}.\ \prod_{k=0}^{N} b_k$ *is finite.*
5. $\forall k \in \mathbb{N}.\ b_k \simeq a_n$

Our proof is of

Statement 1 (C).

$$\forall N \in {}^*\mathbb{N}.\ \prod_{k=0}^{N} b_k \simeq \prod_{k=0}^{N} a_k$$

Our proof of the Second Hidden Lemma in Isabelle has shown A $\wedge$ B $\longrightarrow$ C. If we can find some a_k, b_k such that $\mathrm{B}(a_k, b_k) \wedge \neg\mathrm{A}(a_k, b_k) \wedge \neg\mathrm{C}(a_k, b_k)$ then we would have an example demonstrating that B alone is not enough to guarantee C and hence A is a necessary assumption. Trivially, products which contain a coefficient of zero but would otherwise not be determinate would provide such an example, but we can also provide an example of a product with non-zero coefficients. Consider the following definition of the coefficients:

Define $a_0 = b_0 = L$, for some infinite L. Define $a_1 = 1/L$ and $b_1 = 1/(2L)$. The rest of the coefficients are given by $a_k = b_k = 1$ for $k \neq 0, 1$.

To show $\neg A(a_k, b_k)$ is satisfied:

[7] However, we could have chosen instead the following assumption which would also allow our above proof to work:

Assumption 1 (A′). *If $k \in \mathbb{N}$ then the coefficient a_k is not infinitesimal.*

By construction, the coefficients at $i = 0$ are infinite and hence $\neg\mathrm{A}(a_k, b_k)$ is satisfied.

To show $B(a_k, b_k)$ is satisfied:

For B.1 and B.3 both products are determinate since their tails are products of ones and thus not depended on the particular infinite value they are taken until. For B.2 and B.4 the values of the products are 1 and $1/2$ respectively if the products are continued beyond their first coefficient. For B.5 the only coefficient at which they differ is $a_1 = 1/L$ and $b_1 = 1/(2L)$ and these values are both infinitesimal so they are infinitely-close.

To show $\neg C(a_k, b_k)$ is satisfied:

For some $N \in {}^*\mathbb{N}$ we have that

$$\prod_{k=0}^{N} a_k = L\frac{1}{L} = 1 \not\approx \frac{1}{2} = L\frac{1}{2L} = \prod_{k=0}^{N} b_k.$$

5.5 Third Hidden Lemma

The Third Hidden Lemma (or Polynomial Comparison Theorem) masquerades as the principle of 'equating coefficients' in Euler's proof. In this section we will explain McKinzie and Tuckey's proof and the gaps and points of interest which we have identified in it. We then describe how these issues can be addressed to obtain a complete proof and how we formalise this proof in Isabelle (see Section 5.5.3), with the exception of one lemma which was not explicitly proved by McKinzie and Tuckey and for which we have provided a pen-and-paper proof in Section 5.5.2.3. We believe this lemma forms a significant gap in their proof since it requires a general property of determinacy which McKinzie and Tuckey did not mention or prove. We formalised the proof of the Third Hidden Lemma in order to understand the points of interest in it and identify gaps in its reasoning. Our mechanisation of the Lemma achieved this goal. We did not deem it particularly useful to mechanise our proof of the gap identified since it uses more elementary reasoning compared with McKinzie and TuckeyâĂŹs proof.

5.5.1 McKinzie and Tuckey's proof of the Third Hidden Lemma

In Euler's proof of the Basel problem, he appears to equate coefficients to conclude that $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$. However, since the polynomials involved are only infinitely-close rather than actually equal, McKinzie and Tuckey instead fill this gap with the Third Hidden Lemma. In Section 4.5.1 we compared the Third Hidden Lemma to the method of equating coefficients. We proved the method in Section 6.3 using finite induction. The structure of their proof of the Third Hidden Lemma also relies on finite induction, however this is where the similarity ends. At several points, we find that it is not obvious how the next step is justified in their proof, as we indicate in Sections 5.5.1.3- 5.5.1.4.

5.5.1.1 Statement

Recall the statement of the Third Hidden Lemma which we discussed in Section 4.4.7.

Third Hidden Lemma. *Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$. If for all finite x, $f(x)$ and $g(x)$ are Euler-convergent and $f(x) \simeq g(x)$, then $a_n \simeq b_n$ for all n.*

They assume that $f(x) \simeq g(x)$. If we took this to mean that $\sum_{i=0}^{n} a_i x^i \simeq \sum_{i=0}^{n} b_i x^i$ for all degrees $n \in {}^*\mathbb{N}$ some steps in the proof of the Third Hidden Lemma, such as concluding $a_n \simeq b_n$ for all finite n, would become trivial, so instead we take it to mean that they are infinitely-close for all infinite degrees. There is an argument to be made that they should be assumed infinitely-close for only a single infinite degree H. In fact, this would be equivalent since they are Euler-convergent and the tails of both polynomials are infinitesimal.[8] See the reasoning in Section 5.5.1.5.

5.5.1.2 Structure of the proof

We quote here McKinzie and Tuckey's proof since we reference it extensively in the following discussion.

> We first show that if any polynomial $p(x)$ given by $\sum_{k=0}^{L} c_k x^k$ is Euler-convergent for all finite x, and if $p(x) \simeq 0$ for all finite, noninfinitesimal x, then $c_0 \simeq 0$. To this end, first observe that $p(1/m) \simeq 0$ for all finite, nonzero m, and hence by the Sequential Theorem, $p(1/M) \simeq 0$ for all sufficiently small infinite M. Let M be infinite and sufficiently small. Next observe that $c_1 \cdot 1/M + c_2 \cdot 1/M^2 + \ldots + c_k \cdot 1/M^k \simeq 0$ for all finite k, and hence by a second application of the Sequential Theorem, $\sum_{k=1}^{N} c_k/M^k \simeq 0$ for all sufficiently small infinite N. Let N be infinite and sufficiently small, and less than H. Then since $p(1/M) = c_0 + \sum_{k=1}^{N} c_k/M^k + \sum_{k=N+1}^{H} c_k/M^k$, and since this last summand is infinitesimal by Euler-convergence, we conclude that $c_0 \simeq 0$.
>
> Now let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots$, and assume that for all finite $x, g(x)$ and $f(x)$ are Euler-convergent and $f(x) \simeq$

[8] Even if we allowed $f(x)$ and $g(x)$ to take different infinite degrees, this would be equivalent by Euler-convergence.

> $g(x)$. Then the reasoning of the previous paragraph implies that $a_0 \simeq b_0$. Now assume that n is finite and that for all k less than $n, a_k \simeq b_k$. Then for all finite x,
>
> $$x^n \left(a_n + a_{n+1}x + \ldots + a_H x^{H-n}\right) \simeq x^n \left(b_n + b_{n+1}x + \ldots + b_H x^{H-n}\right)$$
>
> This implies that the polynomial $p(x)$ given by $\sum_{k=n}^{H} (a_k - b_k) x^{k-n}$ is Euler-convergent for all finite x, and is infinitesimal for all finite, noninfinitesimal x. By the argument in the preceding paragraph, we conclude that $a_n \simeq b_n$ as well. By the Finite Induction Principle, we conclude that $a_n \simeq b_n$ for all finite n. That $a_N \simeq b_N$ holds for all infinite N follows from the Euler-convergence of $f(x)$ and $g(x)$. [57, p. 52]

McKinzie and Tuckey use strong induction over the natural numbers (as opposed to the hypernatural numbers). They are inducting over the proposition $a_n \simeq b_n$, and so they obtain $a_n \simeq b_n$ for all finite n. They then use Euler-convergence of f and g to show that $a_N \simeq b_N$ for all infinite N (see Section 5.5.1.4).

5.5.1.3 Base case and main lemma

Their base-case is to show $a_0 \simeq b_0$. However, they do not prove this directly. Instead they prove a lemma stating '...if any polynomial $p(x)$ given by $\sum_{i=0}^{L} c_i x^i$ is Euler-convergent for all finite x, and if $p(x) \simeq 0$ for all finite, noninfinitesimal x, then $c_0 \simeq 0$.' They later implicitly apply this lemma to $p(x) = \sum_{i=0}^{L}(a_i - b_i)x^i$ to conclude $a_0 - b_0 \simeq 0$. They also use it to show the induction step, thus we will call this the 'main lemma' of the proof.

McKinzie and Tuckey's proof of their lemma involves two successive applications of the Sequential Theorem. They do this by discretising the second assumption and they state 'observe that $p(1/m) \simeq 0$ for all finite, nonzero m'. Implicitly, they assume m to be a hypernatural number. In our formalisation of this proof, we need to rewrite m as $m+1$ in order to explicitly avoid dividing by zero. They then use the Sequential Theorem to apply the proposition to infinite values which results in $p(1/M) \simeq 0$ for all sufficiently small infinite M. Hence they have now extended $p(x)$ to some infinitesimal values of x. McKinzie and Tuckey stated their Sequential Theorem as applying to 'n smaller than N' and yet they apply it as though it applies to 'n smaller than *or equal to* N'.The latter version is a corollary of the former, so although we formalised the Sequential Theorem using $<$, this does not present a problem.

Thus far we have described McKinzie and Tuckey's proof that $\sum_{i=0}^{L} c_i/m^i \simeq 0$ for all nonzero m less than some infinite M. They next tell us to 'observe' that $\sum_{i=1}^{k} c_i/M^i \simeq 0$

for all finite k without any further justification: we provide a justification in Section 5.5.2. McKinzie and Tuckey then apply the Sequential Theorem a second time to obtain $\sum_{i=1}^{N} c_i/M^i \simeq 0$. They now introduce H and they write it as the degree of the polynomial, so we could deduce $H = L$. They claim H is larger than N. So according to them L can be taken to be larger than N. Yet there is no way of knowing if L is larger or smaller than N. We formalise this as a case-split:

If $L > N$ then we can follow their original reasoning. They write

$$p\left(\frac{1}{M}\right) = c_0 + \sum_{i=1}^{N} \frac{c_i}{M^i} + \sum_{i=N+1}^{L} \frac{c_i}{M^i}$$

where all the terms except c_0 have been shown to be infinitesimal (the last term is infinitesimal by Euler-convergence and it would be infinitesimal even if only determinacy had been assumed), hence $c_0 \simeq 0$.

If $L \leq N$ then we instead write

$$p\left(\frac{1}{M}\right) = c_0 + \sum_{i=1}^{L} \frac{c_i}{M^i}$$

We use the second application of the Sequential Theorem to justify that L is sufficiently small, thus $\sum_{i=1}^{L} c_i/M^i \simeq 0$ and hence $c_0 \simeq 0$.

5.5.1.4 Induction step

Once McKinzie and Tuckey have shown the base-case, they assume that n is finite and for all $k < n$ we have $a_k \simeq b_k$ (corresponding to the hypothesis for finite strong induction). They then conclude that for all finite x,

$$x^n(a_n + a_{n+1}x + \cdots + a_H x^{H-n}) \simeq x^n(b_n + b_{n+1}x + \cdots + b_H x^{H-n}) \tag{5.8}$$

It is not mentioned what H is: recall H was also introduced with no explanation in the main lemma. Although $f(x)$ and $g(x)$ are presented as degreeless, we deduce from context that H is their infinite hypernatural degree. Their reasoning for obtaining (5.8) may not be obvious so we provide further details here. McKinzie and Tuckey assume that $f(x) \simeq g(x)$. We took this to mean that they are infinitely-close for all infinite degrees. They then claim that the polynomial $\sum_{k=n}^{H} (a_k - b_k)x^{k-n}$ is Euler-convergent for all finite x, and is infinitesimal for all finite, noninfinitesimal x. This will allow them to fulfill the explicit requirements of the main lemma. However, again, we need to understand how to derive these claims. If x is noninfinitesimal, then so will be x^n and we can thus divide both sides of (5.8) by x^n to show that $\sum_{k=n}^{H} a_k x^{k-n} \simeq \sum_{k=n}^{H} b_k x^{k-n}$. Thus

$\sum_{k=n}^{H}(a_k - b_k)x^{k-n}$ is infinitesimal for all finite, noninfinitesimal x. However, Euler-convergence must be obtained for all finite x, even those which are infinitesimal. It is not obvious how to show this as we cannot divide through by x^n when x is infinitesimal. The main lemma cannot be applied without Euler-convergence; so this appears to form a gap in the reasoning and we will provide our own justification in Section 5.5.2. After application of the main lemma, McKinzie and Tuckey deduce that $a_n \simeq b_n$, concluding the induction step.

5.5.1.5 Concluding the proof

McKinzie and Tuckey have shown that $a_n \simeq b_n$ for all finite n, by finite induction. They state that $a_N \simeq b_N$ also holds for all infinite N, since $f(x)$ and $g(x)$ are Euler-convergent for all x. We will spell out the reasoning further for clarity: let $x = 1$ and consider the infinite number N. By the definition of Euler-convergence, we have that $\sum_{k=N}^{N} a_k 1^n = a_N \simeq 0$ and $\sum_{k=N}^{N} b_k 1^n = b_N \simeq 0$. In fact, determinacy would be sufficient to show this since we do not use the finiteness property here. Thus $a_N \simeq b_N$ for infinite N. Hence they have shown $a_n \simeq b_n$ for all n.

5.5.2 Addressing the potential issues

Now we will explain how we understand and fill the parts of McKinzie and Tuckey's reasoning which are not obvious.

5.5.2.1 Justification of the missing step from the main lemma

We pointed out that in the main lemma McKinzie and Tuckey 'observe' that $\sum_{i=1}^{k} c_i/M^i \simeq 0$ for all finite k. We must justify this step. We might think to prove $\sum_{i=1}^{k} c_i/M^i \simeq 0$ for all finite k simply from the Euler-convergence of $\sum_{i=0}^{L} c_i x^i$. However, this is not possible. Suppose $c_1 = M^2$, $c_2 = -M^2$ and thereafter $c_i = 0$. This series is clearly Euler-convergent, and yet $\sum_{i=1}^{k} c_i/M^i = M^2/M - M^2/M^2 = M - 1$ which is infinite.

Later in the proof, the c_i are defined to be $a_i - b_i$ and our aim is to show that $a_i \simeq b_i$ for all i. In Section 5.5.1.4 we considered and rejected the possibility that $f(x) \simeq g(x)$ should be taken to mean $\sum_{i=0}^{n} a_i x^i \simeq \sum_{i=0}^{n} b_i x^i$ for all n. However, if this were the case, then we could easily derive $\sum_{i=1}^{k}(a_i - b_i)/M^i \simeq 0$ for all finite k.

By contrast, it would seem very reasonable to assume that the c_i must be finite for all finite i. In this case, $\sum_{i=1}^{k} c_i/M^i$ is a finite sum of infinitesimals, and so is itself infinitesimal, as desired. [9] In Section 5.5.3.1 we describe how we use the assumption that the c_i are finite for all finite i to mechanise the proof of the main lemma. Since we have added this assumption to the main lemma, we will discuss whether it can be discharged wherever the main lemma is used in the proof of the Third Hidden Lemma.

5.5.2.2 Inherited assumption

We have not yet proved the assumption that the c_is are finite in the main lemma. The main lemma is used twice in the rest of the proof: once in the base-case and once in the induction step. Thus we need to either discharge the additional assumption or pass it on to the statement of the Third Hidden Lemma.

The first time that McKinzie and Tuckey apply the main lemma is to conclude $a_0 \simeq b_0$. However, this is easily obtained from $f(x) \simeq g(x)$ when $x = 0$. Therefore, we can avoid the first application of the main lemma. Second, McKinzie and Tuckey apply the main lemma to obtain $a_n \simeq b_n$. In this case, $c_i = a_{i+n} - b_{i+n}$ and so we must show that $a_i - b_i$ is finite for $i \in \mathbb{N}$. We have not found a way to show this.[10] So, for now, this has to be an assumption added to the Third Hidden Lemma.

5.5.2.3 Euler-convergence for infinitesimal x

When x is non-infinitesimal, we can deduce Euler-convergence of $\sum_{k=n}^{H}(a_k - b_k)x^{k-n}$ from (5.8) by dividing both sides by x^n and preserving the infinitely-close relation. However, when x is infinitesimal, we must take another approach. Our argument is based on a geometric sum. We prove determinacy, rather than Euler-convergence, because determinacy is sufficient for the reasoning of the main lemma. First we prove a general condition for the determinacy of polynomials with finite coefficients.

Theorem 10. *If c_I is finite for all infinite I then $q(\varepsilon) = \sum_{i=J}^{H} c_i \varepsilon^i \simeq 0$ for all infinite J, H and infinitesimal ε with $J \le H$. Hence, a polynomial with finite coefficients will be determinate for all infinitesimal arguments.*

Proof. First split $q(\varepsilon)$ into two sums:

$$q(\varepsilon) = \sum_{i \in S_-} d_i |\varepsilon|^i + \sum_{i \in S_+} d_i |\varepsilon|^i$$

[9] When we apply the third Hidden Lemma to the proof of the Basel problem, the a_is and b_is are finite. Thus the c_is are also finite, so assuming finiteness should be unproblematic in theory. However, we do find it is not easy to prove.

[10] We know that $f(x) \simeq g(x)$ although we only assumed this to hold for infinite degrees. If it held for finite degrees we could easily prove our assumption.

where $d_i = c_i(-1)^i$ if ε is negative and $d_i = c_i$ if ε is positive, and where the set S_- is the values of $i \in [J,H]$ for which d_i is negative, and the set S_+ is the values of $i \in [J,H]$ for which d_i is positive. We now define m_- and m_+ to be the minima of $\{d_i : i \in S_-\}$ and $\{d_i : i \in S_+\}$ respectively. We analogously define M_- and M_+ to be the maxima. Since S_- and S_+ are hyperfinite and d_i is finite for all $i \in [J,H]$, these quantities exist and are finite. Thus

$$m_- \sum_{i \in S_-} |\varepsilon|^i + m_+ \sum_{i \in S_+} |\varepsilon|^i \leq q(\varepsilon) \leq M_- \sum_{i \in S_-} |\varepsilon|^i + M_+ \sum_{i \in S_+} |\varepsilon|^i$$

and further

$$-\left(|m_-| \sum_{i \in S_-} |\varepsilon|^i + m_+ \sum_{i \in S_+} |\varepsilon|^i \right) \leq q(\varepsilon) \leq |M_-| \sum_{i \in S_-} |\varepsilon|^i + M_+ \sum_{i \in S_+} |\varepsilon|^i.$$

Now let $M_L = \max(|m_-|, m_+)$ and $M_R = \max(|M_-|, M_+)$. They are both positive and finite. Then

$$-M_L \sum_{i=J}^{H} |\varepsilon|^i \leq q(\varepsilon) \leq M_R \sum_{i=J}^{H} |\varepsilon|^i. \tag{5.9}$$

Since $\sum_{i=J}^{H} |\varepsilon|^i$ is a geometric sum, we may deduce

$$\sum_{i=J}^{H} |\varepsilon|^i = |\varepsilon|^J \frac{1 - |\varepsilon^{H-J+1}|}{1 - |\varepsilon|} \simeq 0$$

thus in (5.9), $q(\varepsilon)$ is squeezed between two infinitesimals and so must itself be infinitesimal. □

Theorem 10 may initially seem obvious. However the sum $\sum_{i=J}^{H} c_i$ could potentially be infinite and so we cannot assume that multiplying term-by-term with infinitesimals must make it infinitesimal. Indeed this is not even always true for a finite sum which we now demonstrate.

Theorem 11. *$\sum_{i=1}^{N} c_i$ is finite and ε is infinitesimal does not imply $\sum_{i=1}^{N} c_i \varepsilon^i$ is also finite.*

Proof. Let N be an infinite hypernatural. Consider the sum where $c_1 = N^2$ and $c_i = -N$ for $i \in [2, N+1]$. Then $\sum_{i=1}^{N} c_i = N^2 + \sum_{i=2}^{N+1} -N = N^2 - N^2 = 0$ which is finite. Let $\varepsilon = 1/N$

which is infinitesimal. Then

$$\begin{aligned}
\sum_{i=1}^{N} c_i \varepsilon^i =& N^2 \frac{1}{N} + \sum_{i=2}^{N+1} -N\left(\frac{1}{N}\right)^i \\
=& N + \sum_{i=1}^{N} -\left(\frac{1}{N}\right)^i \\
=& N - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\right)^{i-1} \\
>& N - \frac{1}{N}\sum_{i=1}^{N} 1 \\
=& N - \frac{N}{N} \\
=& N - 1
\end{aligned}$$

which is infinite. □

This should convince the reader that reasoning about sums of infinitesimals has pitfalls for intuition and so statements should require careful proof[11]. We can now deduce the determinacy of $\sum_{k=n}^{H}(a_k - b_k)x^{k-n}$ for infinitesimal x. [12]

Lemma 12 (Proof of gap in Third Hidden Lemma)**.** *Let ε be infinitesimal. Suppose $n \in \mathbb{N}$, and let J and H be infinite hypernaturals with $J \leq H$. Let $q(\varepsilon) = \sum_{i=J}^{H}(a_i - b_i)\varepsilon^{i-n}$. Then $q(\varepsilon) \simeq 0$ i.e. $\sum_{i=n}^{H}(a_i - b_i)\varepsilon^{i-n}$ is determinate.*

Proof. From the Euler-convergence of $f(x)$ and $g(x)$ we know that $a_H \simeq 0$ and $b_H \simeq 0$ for all infinite H (see Section 5.5.1.5). Hence $a_H - b_H$ must be infinitesimal and is certainly finite. Let $c_i = a_{i+n} - b_{i+n}$. Then we have

$$q(\varepsilon) = \sum_{i=J}^{H}(a_i - b_i)\varepsilon^{i-n} = \sum_{i=J-n}^{H-n} c_i \varepsilon^i$$

By Theorem 10, $q(\varepsilon)$ is infinitesimal. □

5.5.2.4 Summary of McKinzie and Tuckey's proof of the Third Hidden Lemma

The Sequential Theorem is used twice in this proof. It provides the flexibility to move between the finite and the infinite for discrete propositions. Observe that nowhere in

[11] This may initially seem to contradict the Comparison Test for Determinacy given in Section 5.1.4. However, the example we have given does not satisfy the assumptions of that theorem: most importantly, the c_is in the example take both negative and positive values.

[12] Another approach could be to use the continuity of polynomials. This leads to a concept which is beyond those used in the rest of McKinzie and Tuckey's paper. We could make the argument that Euler would have understood this property in an intuitive sense. However, trying to make the continuity of the polynomial precise using nonstandard analysis quickly becomes complicated.

the proof has the finiteness property of Euler-convergence been used. Thus we can instead make the more general assumption of determinacy, and this is the approach we take in our mechanised proof.

5.5.3 Isabelle proof of the Third Hidden Lemma

We now describe the Isabelle formalisation of the proof. We already described the line of reasoning of the formal proof in juxtaposition with McTuckey's reasoning in Section 5.5.1.

5.5.3.1 Statement

We propose the following modified version of the Third Hidden Lemma based on our analysis of the potential issues in its proof.

Third Hidden Lemma (Version for formalisation)**.** *Let* $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ *and* $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$. *If for all finite x,* $f(x)$ *and* $g(x)$ *are determinate,* $\sum_{n=0}^{N} a_nx^n \simeq \sum_{n=0}^{N} b_nx^n$ *for all infinitely-large N, and* $a_k - b_k$ *is finite for all* $k \in \mathbb{N}$*, then* $a_n \simeq b_n$ *for all n.*

Our statement of the Third Hidden Lemma in Isabelle is

```
lemma hidden_lemma3:
assumes
"∀x∈HFinite. determinate (λM. hypsum (λn. ( *fn* a) n · x pow n) {0..< M})"
and
"∀x∈HFinite. determinate (λM. hypsum (λn. ( *fn* b) n · x pow n) {0..< M})"
and
"∀x∈HFinite. ∀M. (hypsum (λn. ( *fn* a) n · (x :: real star) pow n) {0..< M}) ≈
                  (hypsum (λn. ( *fn* b) n · x pow n) {0..< M})"
and
"∀k∈ℕ. (*fn* a) k − (*fn* b) k ∈ HFinite"
shows "∀n. ( *fn* a) n ≈ ( *fn* b) n"
```

Recall that we formalise it using determinacy rather than Euler-convergence. Another point of difference is that McKinzie and Tuckey write the relation between the polynomials $f(x)$ and $g(x)$ in a way that is ambiguous about their degrees. Finally, we formalised our Third Hidden Lemma with the additional assumption

```
"∀i∈ℕ. (*fn* a) i − (*fn* b) i ∈ HFinite".
```

In Section 5.5.2.1 we explained how this assumption is used. It is possible that this assumption could be proved from the others.

5.5.3.2 Structure of the proof

As McKinzie and Tuckey did, we used finite strong induction to show that $a_n \simeq b_n$ for all $n \in \mathbb{N}$ and this required us to prove our own induction rule within Isabelle for this purpose.

We separated out the main lemma as a separate lemma in Isabelle. This allowed us to be explicit about the assumptions required by it. Since it was far simpler to prove $a_0 \simeq b_0$ without using the main lemma, we only applied it once in our proof. Apart from this instance, we followed McKinzie and Tuckey's reasoning. However, in Section 5.5.2 we explained that there were several parts of their proof which were open to interpretation, and for these we gave additional reasoning which we replicated in our mechanisation.

5.5.3.3 Main lemma

For the main lemma, it is also sufficient to assume determinacy rather than Euler-convergence, although we do have to make the additional assumption that the coefficients are finite for finite indices. In Section 5.5.2.1 we discussed whether this assumption is necessary.

Lemma 13 (Main lemma for Hidden Lemma 3: version for formalisation)**.** *If any polynomial $p(x)$ given by $\sum_{k=0}^{L} c_k x^k$ is determinate for all finite x, and if $p(x) \simeq 0$ for all finite, noninfinitesimal x, and if c_k is finite for $k \in \mathbb{N}$, then $c_0 \simeq 0$.*

In the Isabelle version we assume that our series `hypsum (λn. ( *fn* c) n · x pow n) {a..< H}` is determinate from some arbitrary `a` $\in \mathbb{N}$ just to add some generality.

```
lemma hidden_lemma_3_main_lemma:
fixes c :: "nat ⇒ nat ⇒ real"
assumes
"∀x∈HFinite.
   determinate (λH. hypsum (λn. ( *fn* c) n · x pow n) {a..< H})"
and
"∀x∈(HFinite − Infinitesimal ). hypsum (λn. ( *fn* c) n · x pow n) {0..< L} ≈ 0"
```

```
and
"L ∈ HNatInfinite"
and
"a∈ ℕ"
and
"∀k∈ℕ. ( *fn* c) k ∈ HFinite"
shows "( *fn* c) 0 ≈ 0"
```

In our proof of the main lemma, to conclude that $p(1/m) \simeq 0$ for all finite, nonzero m, we first showed

```
"∀m∈(ℕ − {0}). hypsum (λn. ( *fn* c) n · ((1:: hypreal)/m∈ℕ) pow n) {0..< L} ≈ 0"
```

and then we derived

```
"∀m∈ℕ. hypsum (λn. ( *fn* c) n · ((1:: hypreal)/(m∈ℕ+1)) pow n) {0..< L} ≈ 0"
```

since to apply the Sequential Theorem, we needed a proposition quantified by $m \in \mathbb{N}$ not $m \in \mathbb{N} - \{0\}$. Another deviation from McKinzie and Tuckey's reasoning was that we explicitly performed a case-split on whether N, which was obtained from the Sequential Theorem, was less than L (see Section 5.5.1.3).

5.5.3.4 Induction step

Although we gave a pen-and-paper proof in Section 5.5.2.3, in Isabelle, we have only proved the determinacy of $\sum_{i=n}^{H}(a_i - b_i)x^i$ and we have not yet shown in Isabelle that this implies $\sum_{i=n}^{H}(a_i - b_i)x^{i-n}$ is also determinate. So we formalised this as a separate statement which is assumed in a locale given in Listing 5.13.

Listing 5.13: Gap in the Third Hidden Lemma

```
locale Hidden_Lemma3_Gap =
assumes hidden_lemma3_gap:
"⟦(x :: hypreal)∈HFinite; k∈ℕ;
 determinate (λM. hypsum (λi. ((*fn* a) (i + k) − (*fn* b) (i + k)) · x pow (i + k)
     ) {0..<M})⟧ ⟹
 determinate (λM. hypsum (λi. ((*fn* a) (i + k) − (*fn* b) (i + k)) · x pow i)
     {0..<M})"
```

We also found it necessary to use an explicit example of an infinite natural number. This is because we chose to interpret $f(x) \simeq g(x)$ as holding 'for all infinite degrees', rather than 'there exists an infinite degree for which $f(x) \simeq g(x)$'. Since an infinite natural number called whn has been defined in Isabelle, this did not present a challenge. It may have been a better choice to interpret $f(x) \simeq g(x)$ as holding for a single infinite-degree, since, although the two options are actually equivalent, proving that they are equal for a single degree would be easier, and would make the Third Hidden Lemma easier to reuse.

5.5.3.5 Challenges in the Isabelle proof

One additional difficulty of our Isabelle proof was that we explicitly proved the internality of all the functions, a step which McKinzie and Tuckey did not take. We also proved many new theorems about rewriting hyperpolynomial sums and extracting factors in order to replicate the mathematical reasoning.

Many of the issues we highlighted in Section 5.5.2 were discovered in the process of formalisation and all were understood more completely. This is the strength of formalisation: it forces us to have a complete understanding of the proof beyond simply convincing ourselves of its validity. This is especially true when manipulating infinitesimals or using the infinitely-close relation, since it is an easy mistake to treat the latter as equality and to ignore all its extra conditions.

We will discuss the Hidden Sublemma (i) and (ii) shortly: first we describe a result which the Hidden Sublemma (i) is based upon.

5.6 Hyperbolic sine is infinitely-close to 1 for values infinitely-close to 0

McKinzie and Tuckey prove

$$\left(1+\frac{x}{N}\right)^N \simeq 1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^N}{N!} \tag{5.10}$$

by taking the binomial expansion of the left hand side and applying the First Hidden Lemma, and from this they show a corollary. We quote this along with their proof.

Corollary. *If x is finite then $(1+x/N)^N$ is finite. If $x \simeq 0$ then $(1+x/N)^N \simeq 1$.*

Proof. The first result follows from the Euler-convergence of the series obtained from multiplying out the product. The second result follows from the First Hidden Lemma. [57, p. 46] □

The proof of the second part of the Corollary alone is 155 lines in Isabelle. To prove the corollary, we must first show (5.10) which we formalise as

lemma one_plus_x_over_N_pow_N_expansion:
assumes "N ∈HNatInfinite"
shows "(1+x/N$_{\in\mathbb{N}}$) pow N ≈hypsum (λn. x pow n/Hfact n) {0..< N +1}"

Proving the lemma above requires showing that the series $\sum_{n=0}^{M} \frac{x^n}{N!}$ and $\sum_{n=0}^{M} \binom{N}{n}\left(\frac{x}{N}\right)^n$ are determinate for infinite hypernatural N, and further that

$$\forall n \in \mathbb{N}.\ \frac{x^n}{N!} \simeq \binom{N}{n}\left(\frac{x}{N}\right)^n \tag{5.11}$$

Recall that in Section 5.2.2 we were able to give a pen-and-paper proof of the determinacy of the first series (which was not supplied by McKinzie and Tuckey). These proofs have not been formalised in Isabelle (see Section 5.2 for why), but we have assumed their statements and proven the lemma from them. Now we give our formalisation of the second part of the corollary as given by McKinzie and Tuckey. Its proof is complete in Isabelle.

lemma infml_imp_one_plus_x_over_N_pow_N_approx_one:
assumes "N ∈HNatInfinite" **and** "x ≈ 0"
shows "(1+x/N$_{\in\mathbb{N}}$) pow N ≈1"

Notice that McKinzie and Tuckey do not state the converse direction of the second part of the corollary and it is given no proof by them. However, they use the converse of the second part of the corollary in the their proof of Hidden Sublemma (i) so it is in fact necessary to prove it. The mechanised proof presents complications for us since it is only straightforward to prove in the case that x is nonnegative. It is still true for $x < 0$, but the alternating signs make the proof more involved, so we only give a complete Isabelle proof for $x \geq 0$.

lemma one_plus_x_over_N_pow_N_approx_one_imp_infml:
assumes "N ∈HNatInfinite" **and** "x≥0" **and**
"(1+x/N$_{\in\mathbb{N}}$) pow N ≈1"
shows "x ≈0"

The following lemma formalises the first part of the Corollary, which McKinzie and Tuckey state comes from the Euler-convergence of the series obtained by multiplying-out. Here when McKinzie and Tuckey say ‘multiplying-out’, they mean applying the Binomial Theorem. However, since it comes from the Euler-convergence of a series which we have not yet proven, we have not yet formalised the proof of this lemma.

```
lemma one_plus_x_over_N_pow_N_finite_x_finite_equivalence:
assumes "N ∈HNatInfinite" "x∈HFinite"
shows "(1+x/N∈ℕ) pow N ∈HFinite ≡x ∈ HFinite"
```

The Corollary is used in their proof of the Hidden Sublemma (i) which we discuss next.

5.7 Hidden Sublemma (i)

Hidden Sublemma (i) (see Section 4.4.8) allows McKinzie and Tuckey to rewrite Euler-convergence of sums as the Euler-convergence of products and vice-versa. It could also be useful to show that determinacy of products is equivalent to determinacy of sums, finite determinacy of products to finite determinacy of sums and Euler-determinacy of products is equivalent to Euler-determinacy of sums. Not all these combinations may be true however.

> **Determinacy**: McKinzie and Tuckey's proof of Hidden Sublemma (i) would not work for determinacy, since for products, Euler-determinacy is not equivalent to determinacy, and the proof relies on the Euler property (explained in Section 5.1.2).
>
> **Euler-determinacy** The finiteness property is also needed for the proof, so without modification, it would not hold for Euler-determinacy.
>
> **Finite determinacy** Euler-convergence of products is equivalent to finite determinacy, as long as the terms are non-infinitesimal. So given that condition, the Hidden Sublemma (i) works for finite determinacy.

We have not formalised the proof of Hidden Sublemma (i) as the infinitesimal-reasoning it contains is limited to an application of the Corollary. Additionally, the reasoning contains inequalities which, although uncontroversial, would be involved to prove in Isabelle. Therefore it is not of particular interest but would end up being technical to formalise. We discuss this in the wider context of the mechanisation in Section 6.1.2.2.

Our formalisation of the statement is as follows. We formalise it using finite determinacy, since that is the concept we use in our proof of the Basel problem.

Listing 5.14: Hidden Sublemma (i)

```
lemma hidden_sublemma_i:
assumes
  "∀n. (*fn* b) n ≥ 0"
shows
  "Fdeterminate (λM. hyprod (λn. (1::real star) + ( *fn* b) n) {0..< (M::hypnat)}) =
   Fdeterminate (λM. hypsum (λn. ( *fn* b) n) {0..< M})"
```

Hidden Sublemma (i) is written as ‘if and only if’. McKinzie and Tuckey use the left-to-right direction in the proof of Hidden Sublemma (ii) and the right-to-left direction directly in the proof of the Basel problem.

5.7.1 Partial proof

We give McKinzie and Tuckey’s proof of the Hidden Sublemma (i) before we discuss our partial mechanisation.

> Assume that $b_n \geqq 0$. for all n. If $x > 0$ then by the Binomial Theorem for natural exponents, the inequality $(1+x/n)^n > 1+x$ holds for all $n > 1$. Fix J, K, N, with $J \leqq K$. Let $B = \sum_{n=J}^{K} b_n$. By the Binomial Theorem and the Geometric Sum Theorem we get,
>
> $$1+B = 1+\sum_{n=J}^{K} b_n \leqq \prod_{n=J}^{K}(1+b_n) \leqq \prod_{n=J}^{K}\left(1+\frac{b_n}{N}\right)^N$$
>
> $$= \left[\prod_{n=J}^{K}\left(1+\frac{b_n}{N}\right)\right]^N \leqq \left[1+\frac{B}{N}+\left(\frac{B}{N}\right)^2+\ldots+\left(\frac{B}{N}\right)^{K-J+1}\right]^N$$
>
> $$= \left[1+\frac{B}{N}\underbrace{\left(\frac{1-(B/N)^{K-J+1}}{1-B/N}\right)}_{C}\right]^N = \left[1+\frac{BC}{N}\right]^N.$$
>
> In particular,
>
> $$1+\sum_{n=J}^{K} b_n \leqq \prod_{n=J}^{K}(1+b_n) \leqq \left[1+\frac{C\sum_{n=J}^{K} b_n}{N}\right]^N.$$
>
> Note that if B is finite and N is infinite then $C \simeq 1$. Recall that if N is infinite, then $(1+x/N)^N$ is finite iff x is finite, and $(1+x/N)^N \simeq 1$ iff $x \simeq 0$. It follows that

the sum $b_1 + b_2 + b_3 \ldots$ is Euler-convergent

$$\text{iff } \sum_{n=1}^{J} b_n \text{ is finite and } \sum_{n=J}^{K} b_n \simeq 0 \text{ for all infinite } J, K$$

$$\text{iff } \left(1 + \frac{\sum_{n=1}^{J} b_n}{N}\right)^N \text{ is finite}$$

$$\text{and } \left(1 + \frac{\sum_{n=J}^{K} b_n}{N}\right)^N \simeq 1 \text{ for all infinite } J, K$$

$$\text{iff } \left(1 + \frac{C\sum_{n=1}^{J} b_n}{N}\right)^N \text{ is finite}$$

$$\text{and } \left(1 + \frac{C\sum_{n=J}^{K} b_n}{N}\right)^N \simeq 1 \text{ for all infinite } J, K$$

$$\text{iff } \prod_{n=1}^{J} (1 + b_n). \text{ is finite and } \prod_{n=J}^{K} (1 + b_n) \simeq 1 \text{ for all infinite } J, K$$

iff the product $(1 + b_1)(1 + b_2)$ is Euler-convergent. [57, pp. 50-51]

We do not have a complete proof of Hidden Sublemma (i) although we have formalised some of the intermediate lemmas. The most significant of these is the Corollary given by McKinzie and Tuckey which we discuss in Section 5.6. McKinzie and Tuckey's proof consists of a chain of equivalences. These equivalences are supported by some inequalities which are obtained from the Binomial Theorem and the Geometric Sum Theorem. We have a formalisation of the Binomial Theorem (see Section 6.5) but we have not yet transferred the Geometric Sum Theorem. Since we formalised finite determinacy rather than Euler-convergence, we also have extra proof-steps. These are based on the lemmas in Section 5.1.3.

We define the same B and C used in McKinzie and Tuckey's proof

Listing 5.15: Definition of B and C in Hidden Sublemma (ii)

```
let ?B = "λJ K. hypsum ( *fn* b) {J..< K+1}"
let ?C = "λJ K. ((1 − (?B J K/N∈ℕ) pow (K − J + 1))/(1 − ?B J K/N∈ℕ))"
have "∀J∈HNatInfinite. ∀K∈HNatInfinite. hyprod (λn. 1 + ( *fn* b) n) {J..< K+1}
≤(1 + (?B J K/N∈ℕ) · ?C J K) pow N"
```

From our partial formalisation we do not identify any particular challenges to a complete mechanisation of the proof since we have already formalised the theorems and definitions that it relies on. The reasoning involved is mostly concerning standard manipulations of sums and apart from the one use of the Corollary does not involve nuanced manipulation of the infinitely-close relation. However, it does involve many inequalities which remain to be shown.

In the section Section 6.5 we briefly outline our formalisation of multiplying-out and how it ultimately lead us to find a representation in Isabelle for Hidden Sublemma (ii). Our description of the mechanisation of the statement of Hidden Sublemma (ii) depends on a concept that we introduce in that section so the reader may wish to refer to it in advance although we also give an informal explanation in context.

5.8 Hidden Sublemma (ii)

We do not have a complete proof of Hidden Sublemma (ii). In Section 6.9.1 we find that McKinzie and Tuckey are mistaken that it is necessary for the rehabilitation of the proof of the Basel Problem and thus Hidden Sublemma (ii) fits into the category of unformalised proofs which are not central to the reasoning of the proof of the Basel Problem (Section 1.3.1). We anticipate that a formal proof would present some challenges which we outline below. Our formalisation of the statement in Isabelle is

```
lemma hidden_sublemma_ii:
assumes
"Fdeterminate (λM. hyprod (λn. (1::real star) + ( *fn* b) n · x) {0..< M+1})"
and "∀n. (*fn* b) n ≥ 0"
shows
"Fdeterminate
    (λM. hypsum
        (λk. hypsum (λi.
                hyprod (*fn* b) (Iset i))
            (hyp_k_subset k (star_n (λn. {0..<unstarnum M n}))) · x pow k)
          {0..<M + 1})"
```

Every hypersum and hyperproduct needs to be indexed by something of type 'a star . However, one of these indexes i must be a set, therefore it would seem that needs to have the type 'b star set which would not be a valid indexing type. But internal sets can be defined as equivalence classes of sequences of sets and therefore can have type 'b set star. The Isabelle construct Iset takes an internal set of type 'b set star and translates it to type 'b star set so that we can apply further functions: in this case so that Iset i can be the indexing set for hyprod (*fn* b) (Iset i). We introduce hyp_k_subset in Section 6.2.2. The reader can refer to that section for a formal explanation, but it is enough to understand that this is a k-subset over the hypernaturals. Our definition of hyp_k_subset uses `Iset i`. This is so that it has type `nat star ⇒ 'a set star ⇒ 'a set star set` and thus we can use it as an indexing set for hypsum

5.8.1 Representing McKinzie and Tuckey's proof

As we have done for the other hidden lemmas and hidden sublemma, we shall quote McKinzie and Tuckey's proof for the reader to refer to.

> Fix x and assume that $(1+b_1x)(1+b_2x)\ldots$ is Euler-convergent. By (i), the sum $b_1x+b_2x+\ldots$ is Euler-convergent. Let H be infinite and let $B = \sum_{n=1}^{H} b_n$. Note that
>
> $$c_k = \sum_{1\leqq n_1<n_2<\ldots<n_k\leqq H} b_{n_1}b_{n_2}\ldots b_{n_k} \leqq B^k.$$
>
> Let J be infinite and less than H. Then
>
> $$\sum_{k=J}^{H} c_k|x|^k \leqq \sum_{k=J}^{H} B^k|x|^k = |Bx|^J\left(\frac{1-|Bx|^{H-J+1}}{1-|Bx|}\right).$$
>
> If it happens that $|Bx| \leqq 1/2$, then this last term is infinitesimal, as required. Otherwise, note that by Euler-convergence, $|x|\sum_{n=J}^{H} b_n$ is infinitesimal for all infinite J, and hence $|x|\sum_{n=J}^{H} b_n < 1/2$ for all infinite J. By the Overspill Theorem, there is a finite m such that $|x|\sum_{n=m}^{H} b_n < 1/2$. Now we may factor the original product as
>
> $$\prod_{n=1}^{H}(1+b_nx) = \prod_{n=1}^{m-1}(1+b_nx)\cdot\prod_{n=m}^{H}(1+b_nx)$$
>
> and then apply the reasoning in the previous case to conclude that multiplying out this last product yields an Euler-convergent polynomial. Its product with the polynomial of finite degree $m-1$ is Euler-convergent as well. [57, pp. 51-52]

McKinzie and Tuckey's proof of Hidden Sublemma (ii) involves expressing an n-fold sum of products

$$\sum_{1\leq n_1<n_2<\cdots<n_k\leq H} b_{n_1}b_{n_2}\ldots b_{n_k}$$

where H is an infinite hypernatural. We could use a variation on the concepts of folded sums of products defined in Section 6.2.2 to represent this in our mechanisation. McKinzie and Tuckey show several inequalities involving this sum. Formalising these would require proving and transferring several new theorems on folded sums of products. Their proof also uses the Overspill Theorem which we have formalised (see Section 3.4.1) so that part of the proof should not present too many difficulties.

5.8.2 Issues with showing internality

In Section 6.9.1 we explain why it is necessary to show that the sum from Hidden Sublemma (ii):

```
hypsum (λk.
  hypsum (λi.
    hyprod (*fn* b) (Iset i)) (hyp_k_subset k
  (star_n (λn. {0..<unstarnum M n}))) · x pow k)
{0..<M + 1})
```

has internal coefficients. Essentially, we must show that

```
λk. hypsum (λi. hyprod (*fn* b) (Iset i))
    (hyp_k_subset k (star_n (λn. {0..<unstarnum M n}))) · x pow k)
```

is an internal function of k. Following the rules of composition of internal functions, and since we know that λk. x pow k is internal, this requires showing that

```
hyprod (*fn* b) (Iset i)
```

is an internal function of i, and further that

```
λk. hypsum (λi. a i k) (hyp_k_subset k (star_n (λn. {0..<unstarnum M n})))
```

is internal when a i k is an internal function of both i and k. We have been able to show that

```
λk. hypsum (λi. *f2n* a i k) (*sn* S)
```

is internal. We use *f2n* because it is necessary that the coefficient is an internal function of i so that we can apply the characterising theorem of hypsum, and it is also a requirement that the coefficient is an internal function of k since we show that the overarching function of k is internal. We were able to show

```
lemma internal_hypsum_hyprod_Basel_k_variable:
"(λk. hypsum (λi. hyprod (*fn* b) (Iset i)) (hyp_k_subset k (N::'b set star))) =
(*fn* (λn k. sum (prod (b n)) {x. card x = k ∧ x ⊆ unstarnum N n}))"
```

and thus we know that the coefficients of multiplied-out polynomials are internal. We have also shown that the particular coefficient function to which we apply Hidden Sublemma (ii) is internal (see Section 6.9.3.1).

5.9 Summary

In this chapter we described our mechanisation of determinacy, Euler-convergence and general theorems on them, which we formulated and proved in Isabelle. We described the mechanisation in Isabelle of some of the specific determinate series which are involved in the proof of the Basel problem. We gave our formalisation and mechanised proofs of the three hidden lemmas.

- The First Hidden Lemma (Section 5.3) was straightforward to formalise. By contrast:
- The Second Hidden Lemma (Section 5.4) required an assumption beyond those that McKinzie and Tuckey explicitly stated. However, we give a pen-and-paper proof that this assumption is necessary.
- The Third Hidden Lemma (Section 5.5) had the most challenging proof. We discovered that there were two gaps in McKinzie and Tuckey's reasoning. We provided an additional justification for the gap in the main lemma, but have so far not been able to fill the other gap. It is quite likely that there is no way to 'fill' the gap and our solution, to give the lemma an additional assumption, is the correct approach. It is at least an unproblematic approach since the assumption can be discharged within the proof of the Basel Problem. We also found that the Third Hidden Lemma requires additional assumptions.

We do not have a complete mechanised proof of the Hidden Sublemma but we gave the formalisation of the statement of its two parts and we outlined some of the difficulties that would need to be overcome to produce a fully formalised proof. For each of the hidden lemmas and sublemmas we have worked out the minimum convergence condition: some can be formalised using the weaker condition of determinacy rather than Euler-convergence.

Chapter 6

Mechanising the Basel problem

The purpose of this chapter is to describe our mechanisation of the controversial parts of Euler's proof of the Basel problem and McKinzie and Tuckey's reconstruction. In the previous chapter we outlined our mechanisation of determinacy, Euler-convergence and the Hidden Lemmas: in this chapter we describe how they are used within our mechanisation of Euler's reasoning. We describe how we represent Euler's reasoning and what decisions were made. We also detail how our formalisation lead to the discovery of gaps in both Euler's and McKinzie and Tuckey's proof, how we filled these gaps and how our understanding of the proofs has been improved. Using the Hidden Lemmas identified by McKinzie and Tuckey allows us to construct deductive formal proofs which parallel parts of Euler's reasoning. However, there are limitations to how closely Euler's proof can be represented by deductive reasoning and formal logic-based mathematics and near the end of this chapter in Section 6.9.2 we explore an example of such a limitation. There are also parts of both Euler's and McKinzie and Tuckey's reasoning which are uncontroversial or not relevant to the main body of the proof: these have mostly been omitted from the formalisation. In the next section we describe the scope of the formalisation and which parts of Euler's proof have been included in the mechanisation.

6.1 The formalisation of Euler's reasoning

We would like to remind the reader that the parts of Euler's reasoning which have not been formalised by us fall into the following categories:

- They do not correspond to modern deductive reasoning

- They are uncontroversial, but long-winded in Isabelle.
- They are not relevant to the main body of the proof of the Basel Problem (and the truth of the statement is not under question)

In Figure 6.1 we gave an idea of the scope of the formalisation via illustrating how the formalised parts fit into the proof of the Basel problem. In Section 6.1.1 we explain why we chose to include certain parts in the formalisation: we also cover parts from the earlier chapters. In Table 6.2 and Table 6.1 we summarise the parts of the proof which are of interest and also the parts of the proof which have not been formalised. The tables contain some lemmas not mentioned in Figure 6.1 as that is concerned with only the top-level lemmas. In Section 6.1.2 we explain how they fit into the above categories. Combined, Figure 6.1, Table 6.2 and Table 6.1 also provide a way to navigate this chapter and Chapter 5 and section references have been provided in the table for this purpose.

6.1.1 The focus of the formalisation

In the formalisation, we found it was necessary to formalise the concepts and lemma statements as a starting point to understanding Euler's proof. This lead to the theory of polynomials in Isabelle (Section 3.2.5) including the theorem on equating coefficients (Section 4.5.1), the theory of determinacy and Euler-convergence in Isabelle (Section 5.1) and the theory of external sets and related theorems such as the Sequential and Overflow theorems in Isabelle (Section 3.4). The formalisation of concepts could not be limited to definitions since proving the consequences of the definitions (in particular, their characterising theorems) in Isabelle allowed us to verify that our formalisations of the definitions are correct. As for lemmas, we formalised at least the statement of every relevant lemma in the proof of the Basel problem which gave us a better understanding of how they fit into the structure of the proof and chain together.

We felt it was important to find the minimal assumptions for the hidden lemmas especially in terms of the level of determinacy/Euler-convergence required since this mathematically characterises the 'gaps' in Euler's reasoning. This was not explored in as much detail by McKinzie and Tuckey.

Several of the hidden lemmas contain subtle reasoning involving infinitesimals. In particular, Hidden Lemma 3 (see Section 5.5) contained some reasoning which puzzled us when we first read McKinzie and Tuckey's pen-and-paper proof. Formalisation allowed us to identify that there was indeed a gap in their reasoning at this point, and to

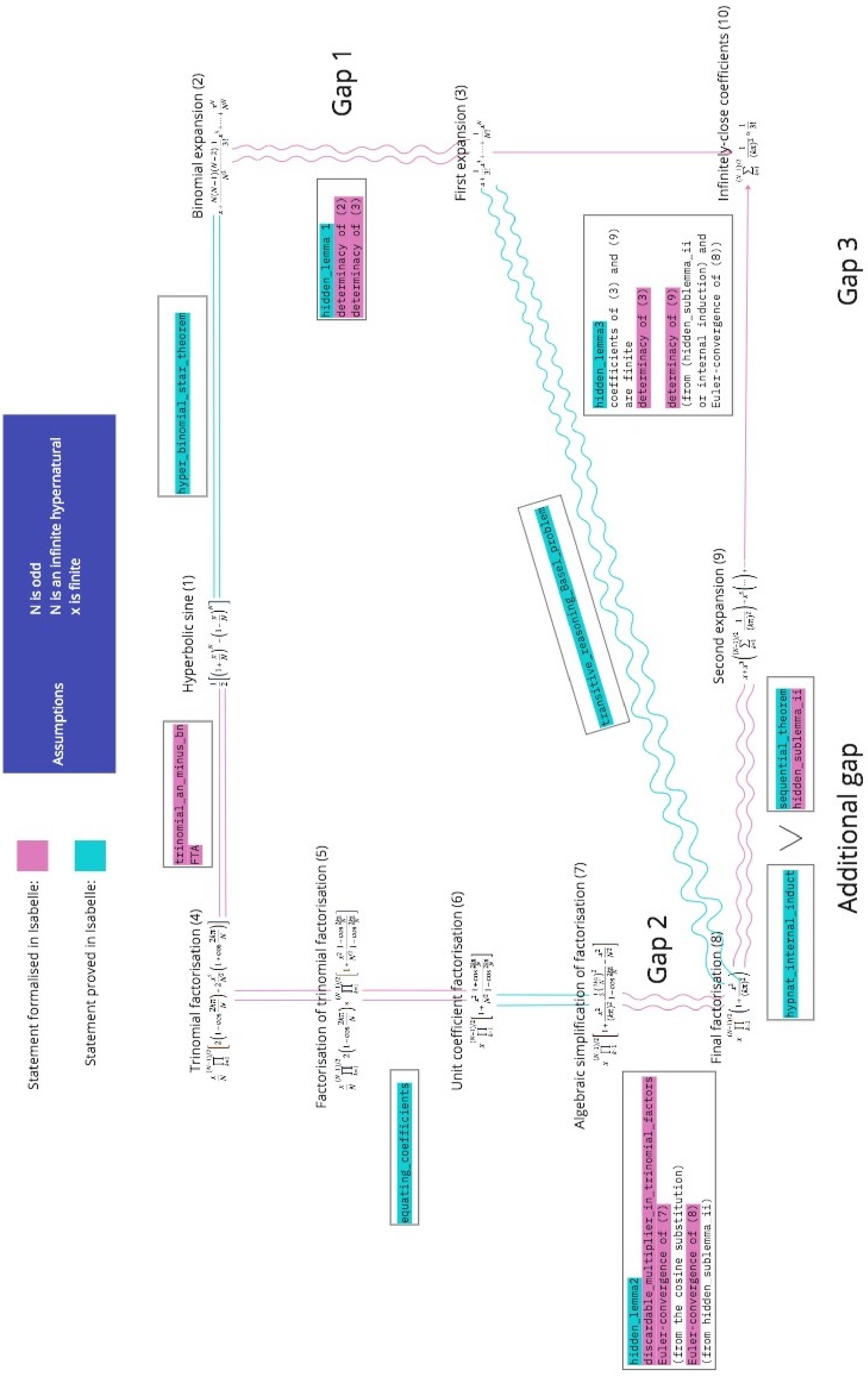

Figure 6.1: Overview of the structure and formalisation scope of the Isabelle mechanisation of Euler's proof

Table 6.1: Parts of the proof which are considered uncontroversial, not deductive, or outside the main body of reasoning

Lemma name	Category	Reference
trinomial_an_minus_bn	Not deductive reasoning Outside the main body of reasoning	Section 6.4.1
fundamental_theorem_of_algebra (FTA)	Outside the main body of reasoning	Section 3.3.3
discardable_multiplier_in_trinomial_factors (the cosine substitution)	Outside the main body of reasoning	Section 6.6
hyperbolic_sine_polynomial_and_trinomial_Hfactorisation	Uncontroversial but long-winded	Section 6.4.3
Determinacy and Euler-convergence lemmas	Uncontroversial but long-winded	Section 5.2
hidden_sublemmai	Uncontroversial but long-winded	Section 5.7
hidden_sublemmaii	Outside the main body of reasoning	Section 5.8
one_plus_x_over_N_pow_N_finite_x_finite_equivalence (the first direction of the Corollary)	Outside the main body of reasoning	Section 5.6

Table 6.2: Parts of the proof which are formalised

Lemma name	Point of interest	Reference
hidden_lemma1	Fills Gap 1 Uses infinitesimal reasoning	Section 5.3
hidden_lemma2	Fills Gap 2 Uses infinitesimal reasoning	Section 5.4
hidden_lemma3	Claimed to fill Gap 3 Has a gap in its own proof	Section 5.5
transitive_reasoning	Formalises the majority of the structure of Euler's reasoning	Section 6.8.8
hyper_binomial_star_theorem	Formalises the first expansion of sinh up to the gap.	Section 6.5
hyperbolic_sine_polynomial_and_binomial_star_expansion	Formalises the first expansion of sinh	Section 6.5
hypsum_prod_multiplying_out_binomial	Partially formalises Euler's §165 (Section 4.5) and influenced our representation of Hidden Lemma (ii)	Section 6.2.2
equating_coefficients	To confirm our representation of Euler's polynomials and to form part of the transitive reasoning of Euler's proof	Section 6.3
one_plus_x_over_N_pow_N_approx_one_imp_infml (the second direction of the Corollary)	Used in the proof of Hidden Lemma (i)	Section 5.6
hypnat_internal_induct	Fills additional gap (Not formalised by us but included in this table for completeness)	Section 6.9.2
sequential_theorem	Could be used to fill additional gap along with Hidden Sublemma ii Used to show Hidden Lemmas	Section 3.4.2

see how we could fix this gap. Only certain types of reasoning with infinitesimals and infinities are dangerous to the user: these include infinite sums of infinitesimals, multiplications of quantities which combine the infinitely-large and infinitely-small and the use of the infinitely-close relation. When the infinitely-close relation is involved, there are pitfalls (we demonstrate this with Theorem 11). This is because it resembles equality, and our intuitions sometimes falsely treat it as though it really is. Euler's reasoning was doubted by his contemporaries and later mathematicians partly because he did not disambiguate between the two (see Section 1.7.6).

We focus in this chapter particularly on the structure of the mechanised proof. To this end, we have formalised Euler's transitive reasoning which makes up the overarching structure. Some parts of this transitive reasoning do depend on lemmas left unproven in Isabelle: we discuss why those lemmas were not proven in Section 6.1.2. The other remaining part of the structure of the Basel problem was Gap 3. We formalised the structure of Gap 3 closely enough to discover that McKinzie and Tuckey's account of it was inaccurate: there is an additional gap that needs to be filled before Gap 3 completes the proof.

In the next section we explain and justify why some parts of the reasoning of the proof of the Basel problem were not included in the formalisation.

6.1.2 Categorisation of unformalised reasoning

We will now describe how each of the unformalised proofs falls into one of the three categories.

6.1.2.1 Proofs which do not correspond to modern deductive reasoning

We discover that the account of Gap 3 given by McKinzie and Tuckey is incomplete, and there is an additional gap in that part of the proof (see Section 6.9). Although we strictly find that Gap 3 (and the additional gap) cannot be completed using a modern deductive proof we do still discuss how the gap could be filled with a modern deductive proof which is as faithful as possible to Euler's original reasoning (see Section 6.9.2).

We argue in Section 4.6 that the trinomial lemma may also fall into this category. However, this category is in the minority. We generally agree with McKinzie and Tuckey's thesis that Euler's proof can be transformed into one of deductive reasoning with the addition of the hidden lemmas, apart from, as we have discovered, Gap 3, which the Third Hidden Lemma is not sufficient to fill.

6.1.2.2 Proofs which are uncontroversial, but long-winded in Isabelle.

Although the proof of Hidden Sublemma (i) (see Section 5.7) does make use of the infinitely-close relation, which could lead to pitfalls, in this case it is only used for a straightforward application of the Corollary (see Section 5.6). The proof of Hidden Sublemma (ii) (Section 5.8) does not use the infinitely-close relation at all. Therefore there was no part of the proofs which we felt required the more detailed examination which formalisation would bring. However, both parts of the Hidden Sublemma rely on inequalities which are uncontroversial, but difficult to prove in Isabelle. The automatic proof in Isabelle and in particular the simplifier is very useful for reducing calculational proofs to a human level of detail. However, nonstandard analysis in Isabelle requires explicit expressions of the internal functions and sets in terms of their underlying sequences. Theorems exist which show that e.g. the sum of two internal functions is well-defined and for pen-and-paper mathematics the distinction between ${}^*f(x) + {}^*g(x)$ and ${}^*(f(x)+g(x))$ can be ignored. In Isabelle, distinctions such as this cannot be ignored and thus calculational proofs involving such concepts have much greater complexity and so steps which in standard analysis could be handled by the automatic proof, in nonstandard analysis cannot always be. We revisit this in Section 7.2 and discuss it in relation to the transfer principle.

The determinacy lemmas also fall into this category (Section 5.2), although we have formalised some of the determinacy proofs as an example of what can be done. The first part of the Corollary (Section 5.6) comes directly (as a corollary in fact) from the Euler-convergence of one of these series.

6.1.2.3 Proofs which are not relevant to the main body of the proof

Every theorem which fits into this category also fits into the previous category: they are uncontroversial and well-established enough to not have their statements under doubt. However, the proofs or reasoning for them given by Euler, may still be controversial, although, we argue, not considered part of the reasoning of his proof of the Basel problem.

Not only is the Fundamental Theorem of Algebra (Section 3.3) now a well-established result, but Euler did not know of any proof of it. Therefore we did not think it was relevant to include any proof of it in our formalisation. The Trinomial Lemma (Section 4.6), which we have argued may not have a modern deductive proof (although perhaps this deserves further examination) is quite separate from Euler's main argument. Its statement is uncontroversial and its proof never been included in any criticisms of the proof, nor was it included in McKinzie and Tuckey's rehabilitation.

We argue in Section 6.9.1 that the Hidden Sublemma (ii) does not actually fill the gap in Euler's reasoning that McKinzie and Tuckey claim it does. We find internal induction to be closer representation of Euler's reasoning here. Hence we do not consider the Hidden Sublemma (ii) to be part of our rehabilitation of the reasoning of Euler's proof.

The cosine substitution (Section 6.6) is derived by McKinzie and Tuckey from geometric reasoning. Although geometric reasoning exists in Isabelle, it would be a large detour from the main focus of the proof. By Euler, the substitution is derived from the power series of cosine. The derivation of this in turn takes up a large part of the 'Introductio' and would again be a significant detour from the reasoning of the proof.

In the next few sections (Sections 6.2- 6.6) we describe our approach to formalising some of the controversial parts of Euler's reasoning and how these relate to the Hidden Lemmas provided by McKinzie and Tuckey. Finally, in (Sections 6.7- 6.10) we synthesise these parts into the overall mechanisation.

6.2 Approaches to formalising Euler's §165

In Section 4.5 we discussed a general statement made by Euler in §165 of his 'Introductio'. This statement allows him to multiply-out a potentially infinite product of binomials into a polynomial of determined coefficients. Euler explicitly states that his reasoning holds whether the polynomial in question is finite or infinite. The application of this piece of reasoning roughly corresponds to Gap 3 in Euler's proof of the Basel problem.

Our initial approach to formalising §165 was to represent 'multiplying-out' and separately represent 'gathering-terms'. Since the polynomials in question may be infinite, they may not in general be rearranged to 'gather terms'. However, as per the Riemann Rearrangement Theorem [2], if they are absolutely convergent they can be rearranged. The series we are interested in (the expansion of $\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right)$) is indeed absolutely convergent. We now believe that our initial approach was too general and thus presented some problems: rearrangement would require using hyperpermutations, our inductive definition would be difficult to prove internal and the generality means that it is unclear how closely it represents Euler's reasoning. We did define multiplying-out in general (see Section 6.5), and we proved some characterising properties of multiplying-out binomials. Our definition of multiplying-out ultimately influenced our formalisation of Hidden Sublemma (ii) (see Section 5.8). McKinzie and Tuckey's strategy is to represent §165 using Hidden Sublemma (ii) and Hidden Lemma 3. Their Hidden Sublemma (ii) represents the form of the multiplied-out polynomial. Their Hidden Lemma 3 gives the 'matching coefficients' part of §165. However, in Section 6.9.1 we observe that there is a missing piece in their reasoning and we give two options for how this missing piece can be filled. Out of these two interpretations of Euler's reasoning, we prefer McKinzie and Tuckey's approach as it is simpler and easier to relate to Euler's reasoning.

6.2.1 Multiplying-out

'Multiplying-out' is merely the distributivity property of series, and can be applied to any product of series, whether convergent or divergent. But 'gathering terms' involves rearrangement or commutativity and this property only holds for absolutely convergent or, in the case of our hypernatural polynomials, absolutely determinate products of series. When we combine 'multiplying-out' and 'gathering terms' for power series, this is known as the Cauchy-product theorem, a combination of distributivity and associativity. This is often expressed mathematically as

$$\left(\sum_{i=0}^{\infty} a_i\right) \cdot \left(\sum_{j=0}^{\infty} b_j\right) = \sum_{k=0}^{\infty} c_k \text{ where } c_k = \sum_{l=0}^{k} a_l b_{k-l}. \tag{6.1}$$

for infinite series $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ with complex terms, given that at least one those series is absolutely convergent (also known as Merten's theorem) [68, p. 74]. The standard Cauchy-product theorem is proven in Isabelle in the Series library as

⟦summable (λk. norm (a k)); summable (λk. norm (b k))⟧ $\Longrightarrow$
(λ. ∑≤k. a i · b (k − i)) sums ((∑k. a k) · (∑k. b k))

which expresses the fact that the limit of the partial sums of the terms $\sum_{l=0}^{k} a_l b_{k-l}$ indexed by k is $(\sum_{i=0}^{\infty} a_i) \cdot \left(\sum_{j=0}^{\infty} b_j\right)$ i.e. it is the same as (6.1) but with different notation. However, for our purpose, we wish to describe the theorem using sums of hypernatural length, rather than limits of sums of finite length. The distributivity property is formalised in Isabelle for finite sums as

sum f A · sum g B = (Σi∈A. Σj∈B. f i · g j)

The distributivity property can be transferred to hyperfinite sums, but this is quite a special-case of multiplying-out as it gives the product of only two sums. We would like to expand the product of a hypernatural number of hyperfinite sums. In Section 6.2.2, we define an n-fold sum which allows us to formally capture this notion generally. Although we only need the hyperfinite product of binomials in order to capture Euler's reasoning in §165, we give a more general mathematical treatment and mechanisation in Section 6.2.2. One reason we expand our formalisation is because our intention was to develop an expressive formalisation of hyperfinite polynomials that could be reused in other situations, e.g. for a nonstandard formalisation of Vieta's formulas, and to mechanise Euler's other derivations. Another is that the proof of Hidden Sublemma (ii) given by McKinzie and Tuckey relies on expressing an n-fold sum of products.

6.2.2 Defining hyperfinite sum of sums and the corresponding distributivity

In §165, Euler describes an infinite product of sums, which by the distributivity law, becomes an infinite sum of sums. Since the degree of the polynomial may be infinite, we represent it as a hypernatural number, and thus the sum is a hyperfinite sum. Therefore we are concerned with the concept of an n-fold sum, which we will attempt to define. We can define mathematical notation for the n-fold sum as follows.

$$\bigcirc_{l=1}^{n} \sum_{i_l=1}^{m_l} = \sum_{i_1=1}^{m_1} + \sum_{i_2=1}^{m_2} + \cdots + \sum_{i_n=1}^{m_n} \tag{6.2}$$

We have used $\bigcirc_{l=1}^{n}$, the fold operator, which can be defined separately, using an inductive definition, as follows. Inductive definitions can be used for functions of hypernatural numbers so long as the functions involved are internal, since there is an internal induction principle.

$$\bigcirc_{l=1}^{1} f(l,x) = f(1,x) \tag{6.3}$$

$$\bigcirc_{l=1}^{n+1} f(l,x) = f\left(n+1, \bigcirc_{l=1}^{n}\right) \tag{6.4}$$

The theorem encapsulating the notion of 'multiplying-out' can now be expressed mathematically as follows.

$$\prod_{l=1}^{k} \sum_{i_l=1}^{m_l} a_{l,i_l} = \bigcirc_{l=1}^{k} \sum_{i_l=1}^{m_l} \prod_{p=1}^{k} a_{p,i_p} \tag{6.5}$$

In Isabelle we do not represent the fold operator separately but use inductive definitions to define a k-fold sum and a sum of products. There is not currently any way of directly writing a definition by internal induction, so we must first define the concepts in standard analysis and then transfer them. The type variable 'a has the sorts comm_monoid_add (commutative monoid under addition) and times (a structure with multiplication) but we have removed the sorts for ease of reading. We write each definition first in mathematical notation and then give the representation in Isabelle.

Definition 7 (k-fold sum of products)**.** *The k-fold sum of products F is defined inductively by*

$$F(0,p,a,m) = \sum_{i=1}^{m_0} pa_{i,0} \tag{6.6}$$

$$F(k+1,p,a,m) = \sum_{i=1}^{m_{k+1}} F(k,pa_{i,k+1},a,m) \tag{6.7}$$

The type constructor of nat is Suc and here is equivalent with the function λn. n+1. We must use Suc in inductive definitions. We define the product of sums as follows.

```
fun k_fold_sum_prod ::
"nat ⇒ 'a ⇒ (nat ⇒ nat⇒ 'a) ⇒ (nat ⇒ nat) ⇒ 'a"
where
"k_fold_sum_prod 0 p a m = sum (λi. (p · a i 0)) {1 ..< m 0 + 1}" |
"k_fold_sum_prod (Suc k) p a m =
  sum (λi. k_fold_sum_prod k (p · a i (Suc k)) a m) {1 ..< m (Suc k) + 1}"
```

Definition 8 (product of sums)**.** *The product of sums G is defined inductively by*

$$G(0,a,m) = \sum_{i=1}^{m_0} a_{i,0} \tag{6.8}$$

$$G(k+1,a,m) = G(k,a,m) \sum_{i=1}^{m_{k+1}} a_{i,k+1} \tag{6.9}$$

```
fun prod_sum ::
"nat ⇒ (nat ⇒ nat ⇒ 'a) ⇒ (nat ⇒ nat) ⇒ 'a"
where
"prod_sum 0 a m = sum (λi. a i 0) {1 ..< m 0 + 1}" |
"prod_sum (Suc k) a m =
  sum (λi. a i (Suc k)){1 ..< m (Suc k) + 1} · prod_sum k a m"
```

We then define hyp_k_fold_sum_prod and hyp_prod_sum as internal functions using k_fold_sum_prod and prod_sum to define their underlying sequences of functions (see Section 2.3.2.2 on internal functions).

```
definition hyp_k_fold_sum_prod ::
"nat star ⇒ 'a star ⇒ (nat star ⇒ nat star ⇒ 'a star) ⇒ (nat star ⇒ nat star)
    ⇒ 'a star"
where
"hyp_k_fold_sum_prod k p a m =
 star_n (λn.
 k_fold_sum_prod (unstarnum k n) (unstarnum p n) (unstarfun2 a n) (unstarfun m n))"
```

```
definition hyp_prod_sum ::
"nat star ⇒ (nat star ⇒ nat star ⇒ 'a star) ⇒ (nat star ⇒ nat star) ⇒ 'a star"
where
"hyp_prod_sum k a m =
   star_n (λn. prod_sum (unstarnum k n) (unstarfun2 a n) (unstarfun m n))"
```

We prove a relation between hyp_k_fold_sum_prod and hyp_prod_sum.

```
theorem hypsum_prod_multiplying_out:
assumes A_def:"A = (*f2n* a)" and M_def:"M = (*fn* m)"
shows "hyp_prod_sum K A M = hyp_k_fold_sum_prod K (1::('a::field) star) A M"
```

For the proof of the Basel problem, it is enough to consider binomial products, and thus we have proven in Isabelle the following equivalence which formalises (6.5) for binomials.

```
lemma hypsum_prod_multiplying_out_binomial:
assumes A_def:"A = (*f2n* a)"
shows
  "hyp_prod_sum K A (λx. 2) = hyp_k_fold_sum_prod K (1::('a::field) star) A (λx. 2)"
```

So far we have expressed multiplying-out without gathered terms. We find that it is simpler to use k-subsets to express multiplying-out with gathering terms for Hidden Sublemma (ii). We define a hyper–k-subset as follows

```
definition hyp_k_subset where
"hyp_k_subset k N = {x. hypcard (Iset x) = k ∧ (Iset x) ⊆ (Iset N)}"
```

The term Iset indicates that the set Iset x is internal. It would seem more obvious to formalise this as

```
definition hyp_k_subset where
"hyp_k_subset k N = {x. hypcard x = k ∧ x ⊆ N"
```

In Section 5.8 we explained in more detail what Iset is, and why it is necessary to formalise this in the way that the first definition is expressed. We have shown that hyp_k_subset is internal. We used our formalisation of this concept to express Hidden Sublemma (ii).

6.3 Equating coefficients

In this section we present the mechanised proof of equating coefficients of hyperpolynomials. Gap 3 in Euler's proof (as identified by McKinzie and Tuckey) is not an application of equating coefficients and can instead be filled by the Third Hidden Lemma. However, equating coefficients is still a necessary result in the mechanised proof. First we give the mathematical representation and the Isabelle version of equating coefficients.

Theorem 14 (Equating Coefficients)**.** *If two polynomials A and B of degree N, and coefficients a_n and b_n respectively, are equal i.e. if*

$$\forall x. \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} b_n x^n \tag{6.10}$$

then

$$\forall n \in \{0 \ldots N\}.\, a_n = b_n \tag{6.11}$$

Our theorem is represented in Isabelle as

```
lemma equating_coefficients_hyperpolynomial:
assumes
  "∀x. hypsum (λn. ((*fn* a) n) · (x :: real star) pow n) {0..< (N::hypnat)} =
               hypsum (λn. ((*fn* b) n) · x pow n) {0..< N}"
shows "∀m ∈{0..< N}. (*fn* a) m = (*fn* b) m"
```

The usual mathematical proof of Equating Coefficients is by differentiating inductively, and this is how we proceed. We first prove the standard version of the theorem.

```
lemma equating_coefficients_real:
"∀a b. (∀(x::real). (Σn∈ {0..< (N::nat)}. ( ((a n))
 · (x ^ n))) = (Σn∈ {0..< (N)}. ( (b n) · (x ^ n)) ))
⟶(∀ m ∈{0..< N}. (a m) = (b m))"
```

We then transfer this theorem so that it can be applied to hyperpolynomials with hypernatural degree.

6.4 The trinomial expansion of hyperbolic sine

Euler expands hyperbolic sine $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ using his Trinomial Lemma. With a substitution of iy for x, this is similar to the infamous step where he expands sine as the product of its roots. However, since $\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]$ is already a polynomial (or hyperpolynomial when N is infinite), there is no real issue here with the representation in nonstandard analysis. In this section we describe our formalisation of his trinomial lemma, and the application of it to obtain the trinomial expansion.

6.4.1 Trinomial locale

We discussed Euler's Trinomial Lemma in Section 4.6. We have formalised the Trinomial Lemma in a locale (see Section 2.2.2).

```
locale Hyptrinomial = Fundamental_theorem_of_algebra +
assumes trinomial_an_minus_bn:
  "⟦0<2·k; 2·k<n; a = (λx. a1·x + a2); b = (λx. b1·x + b2)⟧ ⟹
   (λx. (((a x) pow 2) − 2·(a x)·(b x)·( *f* cos) (2·k∈ℕ·star_of pi / n∈ℕ)
   + ((b x) pow 2))) isHypfactorOf (λx. (((a x) pow n) − ((b x) pow n)))"
and trinomial_an_minus_bn_odd:
  "0<n ⟹
  (λx. ((a x)−(b x))) isHypfactorOf (λx. (((a x) pow n) − ((b x) pow n)))"
and trinomial_an_minus_bn_even:
  "0<2·k ⟹
  (λx. ((a x)+(b x))) isHypfactorOf (λx.(((a x) pow (2·k)) − ((b x) pow (2·k))))"
```

This locale only gives the factors of the polynomial $a^n - b^n$. It is also necessary to say that once the factors are found, their product is equal to the original polynomial. For that, we need to use the Fundamental Theorem of Algebra.

6.4.2 Using the Fundamental Theorem of Algebra together with the Trinomial Lemma

In Euler's proof of the Basel problem the Fundamental Theorem of Algebra is used to prove the Trinomial Lemma (see Section 4.6.1). Next we formulate a more specific version of the Fundamental Theorem of Algebra which allows us to put together the factors given by the Trinomial Lemma. First we give our mathematical representation.

Theorem 15 (Trinomial FTA)**.** *Let M and N be hypernaturals with $M \le 2N$ and $N \ge 1$. For each $n \in [0,N]$ let the trinomial $a_n x^2 + b_n x + 1$ be a factor of the polynomial with even degree $\sum_{n=M}^{2N} e_n x^n$. We assume that $e_{2N} \neq 0$ so that $2N$ truly is the degree of the polynomial. Finally, we assume that these are the only such trinomial factors of the polynomial, so if $cx^2 + dx + 1$ is a factor of the polynomial, with $c \neq 0$ then there is some $n \in [0,N]$ such that $c = a_n$ and $d = b_n$.*[1] *Then we may write our polynomial as a product of all the trinomials, multiplied by a constant which is the coefficient e_0.*

$$\sum_{n=M}^{2N} e_n x^n = e_0 \prod_{n=0}^{N} (a_n x^2 + b_n x + 1)$$

Our version in Isabelle is as follows.

```
locale Fundamental_theorem_of_algebra_trinomial_hypreal =
    Fundamental_theorem_of_algebra +
assumes Eulers_FTA_trinomial_hypreal:
"⟦∀n∈{0..<N+1}.
 (λx. a n · x pow 2 + b n · x + 1) isHypfactorOf
 (λx. hypsum (λn. (*fn* cf) n · x pow n) {M..<2 · N + 1});
 1 < 2 · N; M ≤2 · N; (*fn* cf) (2 · N) ≠ 0;
 ∀r. ∃c d. (r = (λx. c · x pow 2 + d · x + 1) ∧ c ≠ 0 ∧
 r isHypfactorOf (λx. hypsum (λn. (*fn* cf) n · x pow n) {M..<2 · N + 1})
 ⟶ (∃n∈{0..<N}. c = a n ∧ d = b n))⟧
 ⟹
 λx. hypsum (λn. (*fn* cf) n · x pow n) {M..<2 · N + 1}) =
 (λx. (*fn* cf) 0 · hyprod (λx. a n · x pow 2 + b n · x + 1) {0..<N})"
```

More work would be needed to prove the Trinomial FTA and Trinomial Lemma from the versions of the Fundamental Theorem of Algebra we have stated. Thus in

[1] A consequence of this is that $a_n \neq 0$ for all $n \in [0,N]$.

Section 6.4.1 we give a formalisation of the Trinomial Lemma in a locale. In Section 4.6 we argued that Euler's derivation of the Trinomial Lemma is not straightforward and has many points of interest, including his different concept of polynomials to the modern day and his use of meta-mathematical reasoning. McKinzie and Tuckey do not go into the proof of the Trinomial Lemma in their rehabilitation of Euler's proof of the Basel problem [57]. We leave the complete proof of the Trinomial Lemma for future work (see Section 7.4).

6.4.3 Trinomial factorisation

The following lemma is the trinomial factorisation of hyperbolic sine. Its proof is only partially complete in Isabelle. The proof is algebraic apart from the use of the Trinomial Lemma and does not contain any mathematically-controversial reasoning, however it does require some trigonometric reasoning and an expansion in order to compare coefficients both of which are routine but involved in Isabelle. Thus this fits into the second category of unformalised reasoning (see Section 1.3.1).

```
lemma hyperbolic_sine_polynomial_and_trinomial_Hfactorisation:
assumes "N = 2·k + 1" "N ∈HNatInfinite"
shows
"hyperbolic_sine N x ≈ x · hyprod (λn. 1+ (x pow 2)/((n∈ℕ · pi) pow 2)) {1..< k + 1}"
```

6.5 The binomial expansion of hyperbolic sine

As stated in Section 4.2, Euler's proof hinges on expressing the polynomial for hyperbolic sine in two different ways:

$$\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^{N}-\left(1-\frac{x}{N}\right)^{N}\right] \tag{6.12}$$

The Binomial Theorem gives one of these expansions, as discussed in Section 4.2.1. In order to obtain the second expansion, using the Fundamental Theorem of Algebra as a product of linear factors determined by its roots, we need to prove that it is a polynomial. This can be accomplished through the same expansion with the Binomial Theorem and then simplifying, to fit the definition of a polynomial given in our Isabelle theory. We have proven the nonstandard version of the Binomial Theorem from the standard version using properties of the ultrafilter (Section 2.3.1.3).

lemma hyper_binomial_star_theorem:
"(a+b) pow (N::nat star) =
hypsum (λn. (N · C n)$_{\in\mathbb{N}}$ · a pow (N − n) · b pow n) {0..< N +1}"

Using it to expand the formula is straightforward. However, we then have a difference between two hyperfinite sums: we want to combine them into a single sum, and we need to then simplify the terms of that sum. In order to manipulate the sums, we first need to prove that they are well-defined: in this case, we need to prove that the terms are internal functions of the index variable. Hence, using the theorems concerning composition, addition and subtraction of internal functions, we prove that the specific functions involved are internal. We first prove that the binomial function is internal:

lemma binomial_is_internal_function:
"⋀x. (λn. (N C n)$_{\in\mathbb{N}}$) = *fn* (λn m. of_nat (unstarnum N n choose m))"

The function k choose l is the binomial coefficient C_l^k. It is also necessary to prove that the functions (λn. (−x / N$_{\in\mathbb{N}}$) pow n) and are internal.

We express the binomial expansion and internality of hyperbolic sine as follows.

lemma hyperbolic_sine_polynomial_and_binomial_star_expansion:
"1 / 2 · ((1 + x/N$_{\in\mathbb{N}}$) pow N − (1 + (− x/N$_{\in\mathbb{N}}$)) pow N) =
hypsum (λn. if ∃k. n = 2 · k then 0 else (N · C n)$_{\in\mathbb{N}}$ · (x/N$_{\in\mathbb{N}}$) pow n) {0..< N +1}"

Our proof of this lemma is complete in Isabelle. Both $(1+x/N)^N$ and $(1-x/N)^N$ are expanded using the Binomial Theorem. The even terms cancel, and thus we use " if ... then ... else" to formalise this reasoning. We can express the binomial expansion of hyperbolic sine in two ways which we prove are equivalent. The first allows us to show that it satisfies the canonical representation of a hyperpolynomial. The second allows us to reason using sums of odd intervals, which we have shown some useful theorems about (see Section 3.2.5.1).

lemma equivalence_of_binomial_star_expansions_of_hyperbolic_sine:
assumes "N = 2·k +1"
shows "hypsum (λn. if ∃k. n = 2 · k then 0 else (N · C n)$_{\in\mathbb{N}}$ · (x / N$_{\in\mathbb{N}}$) pow n)
{0..< N +1} =
hypsum (λn. (N · C n)$_{\in\mathbb{N}}$ · (x / N$_{\in\mathbb{N}}$) pow n) {0..<odd k +1}"

6.6 The cosine substitution

McKinzie and Tuckey choose to substitute $\frac{1}{2}((\frac{2k\pi}{N})^2)/(1-\cos\frac{2k\pi}{N})$ for 1 at a different point in Euler's proof than Euler. These two quantities are only infinitely-close, not equal, and it is not always valid to substitute one for the other. Kanovei along with McKinzie and Tuckey argues that Euler simply made a mistake in where he chose to substitute [76, p. 75-80][57, p. 39]. We follow their method and make the substitution as part of proving the antecedent for the Second Hidden Lemma when it is applied to fill Gap 2 (see Section 6.8.7). We formalise the fact allowing the substitution as

```
locale discardable_multiplier_in_trinomial_factors =
assumes "⟦k∈HNatFinite; 0<2·k; 2·k<N⟧ ⟹
1/2 · (((2 · k∈ℕ · star_of pi)/N∈ℕ) pow 2)/(1−(*f* cos) ((2·k∈ℕ·star_of pi)/N∈ℕ)) ≈ 1"
```

We have given this fact as a locale and have not formalised its proof, since it falls into the category of reasoning which is external to the reasoning of the proof (see Section 1.3.1). McKinzie and Tuckey derive it from the geometric estimate $\sin\theta \le \theta \le \tan\theta$ for $0 \le \theta \le \pi/2$. Euler says

> ... note that the arc $\frac{2k\pi}{j}\pi$ is infinitely-small and according to §134 we have $\cos\frac{2k}{j}\pi = 1-2\frac{k^2}{j^2}\pi^2$. The other terms in the series are neglected since j is infinitely large.[29, p. 125]

In §134 he gave the power series for cosine. Thus McKinzie and Tuckey gave a different derivation from Euler.

6.7 Final structure of the formal proof

McKinzie and Tuckey's proof can be divided into the following parts:

1. A section of transitive reasoning where the polynomial representation of hyperbolic sine is rewritten in two different ways. This includes their Gap 1 and Gap 2, where infinitely-many infinitesimals are discarded, which they fill with the first two hidden lemmas along with Hidden Sublemma (i).

2. Matching coefficients of the two rewritten expressions. McKinzie and Tuckey do this only for the second term in order to obtain $\sum_{k=1}^{N} 1/k \simeq \pi^2/6$ but it could be extended to further coefficients as Euler does. This is covered by their Gap

3 which they fill using Hidden Sublemma (ii) and the Third Hidden Lemma. However, we will argue that they have a gap in their proof here, and we give two ways to fill this gap: by using Sequential Theorem or by using internal induction (see Section 6.9.1).

6.8 Transitive reasoning for the Basel problem

We formally represent seven lemmas in Isabelle that correspond to McKinzie and Tuckey's chain of transitive reasoning. We have mechanised the proofs of those which we find to be of interest for exploring Euler's reasoning and we state where we have done so. Some of the proofs would be technical in Isabelle but mathematically are straightforward and uninteresting, and we have left these unformalised. Here we give their mathematical representation as given in McKinzie and Tuckey's proof above the lemmas as formalised in Isabelle. They do not give the assumptions to each step explicitly throughout the proof, so we have added any necessary assumptions to the mathematical representation. As we stated in Section 5.2.2.2 it may be necessary to add the assumption $x \geq 0$ throughout since it would be required for our proof of the determinacy of the power series of e^x. It may be possible to extend our proof of the determinacy of e^x to this case also. However, we have omitted this assumption since even if this assumption was included through the transitive reasoning, it would not affect the overarching proof.

6.8.1 Gap 1

Lemma 16 (Gap 1)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$x + \frac{N(N-1)(N-2)}{N^3}\frac{1}{3!}x^3 + \cdots + \frac{x^N}{N^N} \simeq x + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N \tag{6.13}$$

```
lemma Basel_gap1:
assumes
"N = 2 · k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"hypsum (λn.
   (hyprod (λx. N − x) {0.. n})∈ℕ · (N∈ℕ pow n)·(x pow n)/(Hfact n∈ℕ))
 {1..<odd N+1} ≈
 hypsum (λn. (x pow n)/(Hfact n∈ℕ)) {1..<odd N+1}
```

We formalise $N(N-1)(N-2)\cdots$ as a product `hyprod (λx. N − x) {0.. n}`. There is a falling-factorial `pochhammer` defined in Isabelle which we could have used. However, we decided not to since defining the extension of that function would have only complicated the representation. Recall we use the subscript $_{\in\mathbb{N}}$ as an abbreviation for the coercion function from the hypernaturals to the hyperreals. The set `{1..<odd N+1}` is the interval of odd hypernaturals from 1 to N. Since N is odd in this case, it includes its endpoints. Gap 1 corresponds to the application of the First Hidden Lemma [57, p.47]. The proof has not been mechanised.

6.8.2 Binomial expansion of hyperbolic sine

Lemma 17 (Binomial expansion of hyperbolic sine)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]=x+\frac{N(N-1)(N-2)}{N^3}\frac{1}{3!}x^3+\cdots+\frac{x^N}{N^N} \tag{6.14}$$

```
lemma binomial_expansion_hyperbolic_sine:
assumes
"N = 2 · k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"(1/2 · ((1+x/N∈ℕ) pow N − (1−x / N∈ℕ) pow N)) =
hypsum (λn.
  (hyprod (λx. N − x) {0.. n})∈ℕ · (N∈ℕ pow n)·(x pow n)/(Hfact n∈ℕ))
{1..<odd N+1}"
```

This is the polynomial expression for hyperbolic sine expanded using the Binomial Theorem. The proof is complete in Isabelle.

6.8.3 Trinomial factorisation of hyperbolic sine

Lemma 18 (Trinomial factorisation of hyperbolic sine)**.** *Suppose x is a finitehyperreal and N is an infinitely-large odd hypernatural. Then*

$$\begin{gathered}\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]= \\ \frac{x}{N}\prod_{k=1}^{(N-1)/2}\left[2\left(1-\cos\frac{2k\pi}{N}\right)+2\frac{x^2}{N^2}\left(1+\cos\frac{2k\pi}{N}\right)\right]\end{gathered} \tag{6.15}$$

```
lemma trinomial_factorisation:
assumes "N = 2·k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"(1/2 · ((1+x/N∈ℕ) pow N − (1−x / N∈ℕ) pow N)) =
 x / N∈ℕ ·
 hyprod (λn.
   2 · (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) +
   2 · x pow 2 / N∈ℕ pow 2 · (1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)))
 {1..<k + 1}"
```

This is the statement of the trinomial expansion of hyperbolic sine before algebraic simplification. The proof has not been mechanised. Note it is no longer an expression over just odd indices. However, the product contains the same number of terms as the sum from the previous section since its upper-bound is now $(N-1)/2$. This upper-bound is only a hypernatural number for odd values of N. So in our representation we explicitly write N as a term of the form $2k+1$ which allows us to use k as an upper-bound without it being necessary to prove k is a hypernatural number.

6.8.4 Factorisation of trinomial factorisation

Lemma 19 (Factorisation of trinomial factorisation)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$\begin{gathered}\frac{x}{N}\prod_{k=1}^{(N-1)/2}\left[2\left(1-\cos\frac{2k\pi}{N}\right)+2\frac{x^2}{N^2}\left(1+\cos\frac{2k\pi}{N}\right)\right]= \\ \frac{x}{N}\prod_{k=1}^{(N-1)/2}2\left(1-\cos\frac{2k\pi}{N}\right)\times\prod_{k=1}^{(N-1)/2}\left[1+\frac{x^2}{N^2}\frac{1+\cos\frac{2k\pi}{N}}{1-\cos\frac{2k\pi}{N}}\right]\end{gathered} \tag{6.16}$$

```
lemma factorisation_of_trinomial_factorisation :
assumes "N = 2·k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"x / N∈ℕ ·
 hyprod (λn.
   2 · (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) +
   2 · x pow 2 / N∈ℕ pow 2 · (1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)))
 {1..<k + 1} =
 x / N∈ℕ ·
 hyprod (λn.
   2 · (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)))
 {1..<k + 1} ·
 hyprod (λn.
   1 + x pow 2 / N∈ℕ pow 2·
   ((1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) /
    (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))))
 {1..<k + 1}"
```

This factors out a constant term and a factor of x from the trinomial expansion of hyperbolic sine so that we have a product with a binomial of the form $(1+cx^2)$ where $c = \frac{1}{N^2}(1+\cos\frac{2k\pi}{N})/(1-\cos\frac{2k\pi}{N})$. The form of this binomial will allow McKinzie and Tuckey to apply their Second Hidden Lemma and Hidden Sublemmas. This also follows the pattern of Euler's reasoning. The proof has not been mechanised.

6.8.5 Trinomial factorisation with coefficient of unity

Lemma 20 (Trinomial factorisation with coefficient of unity)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$\begin{aligned}&\frac{x}{N}\prod_{k=1}^{(N-1)/2} 2\left(1-\cos\frac{2k\pi}{N}\right)\times\prod_{k=1}^{(N-1)/2}\left[1+\frac{x^2}{N^2}\frac{1+\cos\frac{2k\pi}{N}}{1-\cos\frac{2k\pi}{N}}\right]=\\&x\prod_{k=1}^{(N-1)/2}\left[1+\frac{x^2}{N^2}\frac{1+\cos\frac{2k\pi}{N}}{1-\cos\frac{2k\pi}{N}}\right]\end{aligned}\tag{6.17}$$

```
lemma unit_coefficient_trinomial_factorisation :
assumes "N=2·k+1" "N∈HNatInfinite" "x∈HFinite"
shows
"x/N∈ℕ ·
 hyprod(λn.
   2 · (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)))
 {1..<k+1} ·
 hyprod(λn.
   1 + x pow 2 / N∈ℕ pow 2·
   ((1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))/
    (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))))
 {1..<k+1} =
 x ·
 hyprod(λn.
   1 + x pow 2 / N∈ℕ pow 2·
   ((1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))/
    (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))))
 {1..<k+1}"
```

The proof has not been mechanised, however, to arrive at this from the previous step, it is necessary to compare coefficients. Since we are comparing coefficients between equal polynomials, we use the transferred version of equating-coefficients rather than the Third Hidden Lemma. We describe our formal proof of equating-coefficients in Section 6.3.

6.8.6 Algebraic simplification of trinomial factorisation

Lemma 21 (Algebraic simplification of trinomial factorisation)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$
\begin{aligned}
&x \prod_{k=1}^{(N-1)/2} \left[1 + \frac{x^2}{N^2} \frac{1+\cos\frac{2k\pi}{N}}{1-\cos\frac{2k\pi}{N}} \right] = \\
&x \prod_{k=1}^{(N-1)/2} \left[1 + \frac{x^2}{(k\pi)^2} \frac{\frac{1}{2}\left(\frac{2k\pi}{N}\right)^2}{1-\cos\frac{2k\pi}{N}} - \frac{x^2}{N^2} \right]
\end{aligned}
\tag{6.18}
$$

```
lemma simplification_of_trinomial_factorisation
assumes "N = 2·k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"x ·
 hyprod (λn.
   1 + x pow 2 / N∈ℕ pow 2 ·
   ((1 + (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) /
    (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ))))
 {1..<k + 1} =
 x ·
 hyprod (λn.
   1 + x pow 2 / (n∈ℕ · star_of pi) pow 2 · (1/2) ·
   (2 · n∈ℕ · star_of pi/N∈ℕ) pow 2 /
   (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) −
   x pow 2 / N∈ℕ pow 2)
 {1..<k + 1}"
```

This gives the trinomial expansion of hyperbolic sine after all algebraic simplification, but without Euler's substitution of $1 - \frac{1}{2}\left(\frac{2k\pi}{N}\right)^2$ for $\cos\frac{2k\pi}{N}$. This lemma has a complete proof in Isabelle which relies on the internality of the functions involved, properties of cosine and algebraic reasoning.

6.8.7 Gap 2

Lemma 22 (Gap 2)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$x \prod_{k=1}^{(N-1)/2}\left[1+\frac{x^2}{(k\pi)^2}\frac{\frac{1}{2}\left(\frac{2k\pi}{N}\right)^2}{1-\cos\frac{2k\pi}{N}}-\frac{x^2}{N^2}\right] \simeq x\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right) \tag{6.19}$$

```
lemma Basel_gap2:
assumes "N = 2·k + 1" "N∈HNatInfinite" "x ∈ HFinite"
shows
"x ·
 hyprod (λn.
   1 + x pow 2 / (n∈ℕ · star_of pi) pow 2 ·
   (1 / 2) · (2 · n∈ℕ · star_of pi/N∈ℕ) pow 2 /
   (1 − (*f* cos) (2 · n∈ℕ · hypreal_of_real pi / N∈ℕ)) − x pow 2 / N∈ℕ pow 2)
 {1..<k + 1} ≈
 x · hyprod (λn. 1 + x pow 2 / (n∈ℕ · star_of pi) pow 2) {1..<k + 1}"
```

McKinzie and Tuckey state that this follows from the Second Hidden Lemma, which requires that both products are Euler-convergent and that their respective finite factors differ infinitesimally. The structure of its proof is formalised in Isabelle, but depends on many unproven lemmas. Our Second Hidden Lemma has some assumptions requiring terms to be appreciable (Section 5.4.1) and not all of these assumptions have been discharged in our proof of Gap 2, although mathematically they are straightforward to show. In Isabelle this would require a proof in terms of bounding them above by some number known to be finite (or below by a non-infinitesimal), and since these proofs are not interesting or controversial and do not replicate any reasoning given by either Euler or McKinzie and Tuckey we did not formalise them.

6.8.8 Transitive reasoning

Finally we put all the previous lemmas together in a lemma which formalises the transitive reasoning in the Basel problem. In Isabelle, the proof of this lemma simply chains together the previously mentioned ones in Sections 6.8.1- 6.8.7.

Lemma 23 (Transitive reasoning)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$\frac{1}{2}\left[\left(1+\frac{x}{N}\right)^N-\left(1-\frac{x}{N}\right)^N\right]\simeq x\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right) \tag{6.20}$$

```
lemma transitive_reasoning_Basel_problem:
assumes "N ∈HNatInfinite" "N=2·k+1" "x ∈ HFinite"
shows
"hypsum (λn. (x pow n)/(Hfact n∈ℕ)) {1..<odd N+1} ≈
 x  ·  hyprod (λn. 1 + x pow 2 / (n∈ℕ  ·  star_of  pi)  pow 2) {1..<k + 1}"
```

6.9 Gap 3 and an additional gap

Our formalisation of Gap 3 is partial: in this section we detail our formalisation so far and analyse the remaining parts. First we describe an additional gap that we discovered in the proof, and how it may be filled. We further list our formalised lemmas which are used in the proof of Gap 3, and the difficulties we have identified with putting them together.

6.9.1 McKinzie and Tuckey's derivation for Gap 3 and a fourth gap

At the end of the transitive reasoning, McKinzie and Tuckey have obtained two polynomials which are infinitely-close

$$x+\frac{1}{3!}x^3+\cdots+\frac{1}{N!}x^N \simeq x\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right).$$

They expand the product by multiplying-out and gathering terms to

$$x+x^3\left(\sum_{k=1}^{(N-1)/2}\frac{1}{(k\pi)^2}\right)+x^5\left(\cdots\right)+\cdots. \tag{6.21}$$

They then wish to 'equate coefficients' as Euler did. However, since there is no true equality here, equating coefficients cannot be done directly. Thus they wish to apply the Third Hidden Lemma (see Section 5.5) to

$$x+\frac{1}{3!}x^3+\cdots+\frac{1}{N!}x^N \tag{6.22}$$

and to the expansion of the product (6.21) in order to conclude that their respective terms are infinitely-close. Explicitly they derive the Euler-convergence of both (6.22) and (6.21), and then they state that they apply the Third Hidden Lemma. However, in order to fulfill the assumption of the Third Hidden Lemma, they would need to additionally show that (6.21) is infinitely-close to the original product $\prod_{k=1}^{(N-1)/2}\left(1+\frac{x^2}{(k\pi)^2}\right)$ and thus that

$$x+\frac{1}{3!}x^3+\cdots+\frac{1}{N!}x^N \simeq x+x^3\left(\sum_{k=1}^{(N-1)/2}\frac{1}{(k\pi)^2}\right)+x^5\left(\cdots\right)+\cdots.$$

Certainly, for all finite degrees, we can prove that a product is equal to its rearrangement by finite induction. Furthermore, if the product and its rearrangement are internal, we could prove that they are equal for all hypernatural degrees by internal induction. It is also necessary for them to have internal coefficients for them to be Euler-convergent as McKinzie and Tuckey claim they are, and they do not prove that they are internal or 'elementary' in their terminology. However, although the multiplied-out product (6.21) is composed purely of internal functions, it consists of an infinite hypernatural number of internal functions. We have proven in Isabelle that hyperfinite sums and products are internal with respect to their upper and lower bounds. We have also been able to show that hyperfinite sums (and products) are internal with respect to an inner variable. In other words, that the function $\sum_{n=N}^{M} b_n(x)$ is internal when $b_n(x)$ is an internal function of n and x. Showing that all the coefficients of the multiplied-out product are internal is a nontrivial problem, and it was not initially obvious to us if it was possible. We discuss this further in Section 5.8.2 and Section 6.9.3.1. But we have been able to prove that the coefficients of (6.21) are internal functions of the exponent of x and furthermore that the polynomial itself is an internal function of x (see Section 6.9. Thus we have two options

1. We could prove by internal induction that they are equal for all hypernatural degrees.

2. We could instead use finite induction to show the equality for all natural numbers. Then we could extend this result with the Sequential Theorem to obtain some infinite N such that the polynomials are infinitely-close. The Hidden Sublemma (ii) would then be useful to show that both series are Euler-convergent and thus they are infinitely-close for all infinite degrees.[2]

[2] If we had chosen to understand the antecedent $f(x) \simeq g(x)$ of Hidden Lemma 3 differently: if we had interpreted it to mean that there exists a single infinite degree for which $f(x)$ and $g(x)$ are infinitely-close it would not actually be necessary to apply Hidden Sublemma (ii) to show $f(x) \simeq g(x)$. However, Hidden Sublemma (ii) would still be necessary to show the Euler-convergence of $f(x)$ and $g(x)$, thus it makes no material difference.

Option 1 is simpler but would make McKinzie and Tuckey's Hidden Sublemma (ii) redundant. Therefore we consider option 2 to represent their reasoning more closely. Let us evaluate which option would be a better representation of Euler's reasoning.

6.9.2 Filling the fourth gap with reasoning which could have been used by Euler

Both the Sequential Theorem and internal induction are results from nonstandard analysis and thus cannot directly be said to have been used or understood by Euler. McKinzie and Tuckey do not use internal induction at all in their reconstruction of Euler's proof. They manage to use finite induction wherever induction is necessary. Neither do they explicitly use the Sequential Theorem directly in their reconstruction of Euler's proof. They use the Sequential Theorem in the proofs of their hidden lemmas, but although they argue that Euler may have understood and used the hidden lemmas implicitly, they do not claim that he had proofs of them.

We argue that Euler did use a principle equivalent to internal induction. Euler's method of approaching induction was different from the modern conception of mathematical induction. He approached it in the style of empirical induction in the experimental sciences. McKinzie and Tuckey also believe that Euler used 'inductive arguments, in the style of the physical sciences, rather than as deductive proofs'[57, p. 32]. They base their reasoning for this on Pólya's analysis of Euler's first proof of the Basel problem [67]. Let us give some examples from Euler's 'Introductio'.

1. Euler obtains de Moivre's formula in the 'Introductio' by arguing that it is true for 1, 2, 3 etc. and thus concluding it is true for n[29, p.]. He will later go on to apply this formula for infinite n. There are other formulas he treats in this way, e.g. the Binomial Theorem which he applies in his proof of the Basel problem.

2. Another example is Euler's generalisation of the Basel problem which we discuss in Section 7.4.2. He does not give the nth term but expects us to be able to continue the pattern. This is typical of his presentation.

Euler's definition of a function would satisfy the requirements for an internal function (see Section 2.3.2.1). Thus we feel justified in saying that internal induction can parallel arguments made by Euler.

Informally, the Sequential Theorem allows transition from the finite to the infinite when the infinitely-close relation is involved. We can consider similarly whether the

Sequential Theorem could parallel reasoning used by Euler. We do not think that this is as likely since Euler did not make a strong distinction between terms differing by an infinitesimal and equality between terms. He represented both by '='. Although the hidden lemmas also use $\simeq$, they have analogous statements for equality. A second and perhaps stronger reason is that, on no occasion have we observed Euler state a fact only for infinite numbers up to a certain size (as would result from the Sequential Theorem). Whenever we have observed Euler make a statement involving infinite numbers, it is either for a specific infinite number or infinite numbers in general. We believe this is because he sees the infinite only as an extension of the finite, rather than true numbers with actual values. We do not rule out the possibility that the Sequential Theorem could have been intuitively used by Euler but we believe internal induction would be easier to justify.

6.9.3 Gap 3: Matching coefficients

Lemma 24 (Gap 3)**.** *Suppose x is a finite hyperreal and N is an infinitely-large odd hypernatural. Then*

$$\sum_{k=1}^{(N-1)/2} \frac{1}{(k\pi)^2} \simeq \frac{1}{3!}.$$

```
lemma gap3:
assumes "N ∈HNatInfinite" "N = 2·k + 1"
shows "hypsum (λn. 1/((n∈ℕ·star_of pi) pow 2)) {1..< k+1} ≈ 1/(Hfact 3)"
```

This lemma formalises Gap 3, which is the result of matching coefficients from the two expansions of hyperbolic sine. We have a partial formalisation of the proof of this lemma. We have broken down the proof into some lemmas which follow McKinzie and Tuckey's reasoning. We apply the Hidden Sublemma (ii) to

```
"∀m. hypsum (λn. (x pow n)/(Hfact n∈ℕ)) {1<.. m+1} ≈
   hypsum (λk.
     hypsum (λi.
       hyprod ( *fn* (λn x. 1 / (real x · unstarnum (hypreal_of_real pi) n)^2 ))
        (Iset i))
     (hyp_k_subset k (star_n (λn. {0..<unstarnum m n}))) · (x pow (2·k +1)))
   {1..<m+1}"
```

We intend to apply the Hidden Lemma 3 to the result of our application of Hidden Sublemma (ii). However, to apply it literally, as Isabelle requires, we will need to show that our polynomial is equal to one with a term of every power. Each of the even terms will be 0. We needed to use if ... then ... else to express this. The resulting function is unwieldy, and thus we defined an abbreviation for it.

```
abbreviation
"multiplied_out_term l  m ≡
 ( if  odd l  then
     hypsum (λi.
       hyprod
         (  *fn*  (λn x.  1 /  (real  x   ·   unstarnum (hypreal_of_real pi) n)²  ))
         ( lset  i ))
     (hyp_k_subset l (star_n  (λn.  {0..<unstarnum m n})))
   else  0)"
```

Similarly, we define an abbreviation for the coefficients of the other expansion of hyperbolic sine, since it is also unwieldy to read.

```
abbreviation
"hyp_sine_pow_series_term n ≡(if odd n then 1 /  Hfact n∈*ℕ else 0)"
```

Armed with these two abbreviations we are able to formalise a version of Hidden Lemma 3 applied to the particular case of Gap 3.

```
lemma hidden_lemma3_in_gap3:
assumes
"∀x∈HFinite.
   determinate (λM. hypsum (λn. hyp_sine_pow_series_term n ·x pow n) {0..<M})"
"∀x∈HFinite.
   determinate (λM. hypsum (λn. multiplied_out_term n m · x pow n) {0..<M})"
"∀x∈HFinite.
   ∀M∈HNatInfinite. hypsum (λn. hyp_sine_pow_series_term n ·x pow n) {0..<M} ≈
   hypsum (λn. multiplied_out_term n m ·  x pow n) {0..<M}"
"∀n∈ℕ.
   hyp_sine_pow_series_term n − multiplied_out_term n m ∈HFinite"
shows "∀n. hyp_sine_pow_series_term n ≈multiplied_out_term n m"
```

The proof of Gap 3 is reduced to showing the antecedents of this lemma.

We have identified some challenges and remaining pieces which will need to be overcome to complete the proof of Gap 3. We describe them here

6.9.3.1 Solution for the additional gap

In Section 6.9.1 we described our discovery of a potential gap in McKinzie and Tuckey's treatment of Gap 3. In short, the main antecedent of the Third Hidden Lemma needs to be shown: we must show that the two polynomials are infinitely-close for all infinite degrees. We outlined two solutions. Our preferred solution as we indicated in Section 6.9.2 is to prove the antecedent using internal induction. We have formalised the statement of this intermediate step. We have also shown that the polynomial resulting from multiplying-out the product is internal, a necessary step so that we may apply internal induction (see Section 5.8.2). Our Isabelle representation of this polynomial was complicated by the way that sums are indexed in Isabelle, and thus it was not trivial to show that it is internal. We thus have all the component results we need to complete the proof of the antecedent. However, although it is mathematically-straightforward, we still have some difficulty in connecting the representations: in particular, in translating between the results for series defined over only odd terms and series defined over all terms. We would need to show many further theorems to bridge this gap even though it is mathematically trivial.

6.9.3.2 Showing that the difference between coefficients is finite

Our current formalisation of the Third Hidden Lemma requires that the difference. between the coefficients is finite. When the lemma is applied to Gap 3, this becomes

```
"∀n∈ℕ. hyp_sine_pow_series_term n − multiplied_out_term n N ∈HFinite"
```

Note that the underlying goal which must be proven without the neat abbreviations is algebraically far more complex. This is the weakest condition the Third Hidden Lemma can have. However, it is unclear how to prove this on its own. It may be easier to prove a stronger assumption: i.e. that all the coefficients are finite. If we assume that all the coefficients are positive, which in the case of the Basel problem they are, and prove that the series are Euler-convergent or finite determinate, we can obtain the result that all the coefficients are finite.

6.9.3.3 Putting together the pieces

Our main challenge towards completing the proof of Gap 3 is in outlining the structure of the proof and putting together the lemmas which it consists of. As we outlined in the previous section, we defined abbreviations for the However, although all the pieces are formalised, there is the same difficulty in connecting the representations as we cited for the proof of the main antecendent to the Third Hidden Lemma: we would need to prove further theorems to unify series defined over only odd terms and series defined over all terms. Our formal proof thus remains incomplete although mathematically we have now characterised it in detail in terms of the Third Hidden Lemma and the additional gap we discovered.

6.10 Concluding the Basel problem

Below we give our formalisation of the statement of the Basel problem, and we also refer the reader back to the final formalisation of the structure of the case-split which we gave as an example of Isabelle proof in Section 2.2.

```
lemma Basel_problem:
assumes "N ∈HNatInfinite"
shows "hypsum (λn. 1/(n∈ℕ pow 2)) {1..< N+1} ≈(pi pow 2) / 6"
```

Our proof of this statement is by no means complete. We have focused on formalising those parts which formed a framework for understanding Euler's proof more precisely (e.g. the concepts of determinacy and Euler-convergence) and those points of reasoning where we found McKinzie and Tuckey's explanation lacking (e.g. the proof of Hidden Lemma 3).

The aim of our formalisation efforts has been to use the formalisation as a tool to achieve a greater level of detail in the description of the reasoning of Euler's proof. Some parts of Euler's (and McKinzie and Tuckey's) proof do not require a more detailed understanding as they are mathematically uninteresting and not considered controversial. These we have often omitted from our formalisation. In Section 6.1 we outlined the scope of the formalisation to formalise Euler's reasoning by categorising the unformalised parts.

However, we have formalised the majority of the pieces which need to be put together to achieve this result, and we have formalised the upper levels of the structure for putting these parts together into a whole.

A fully-formalised proof would still provide a neat counterpoint to Euler's proof, thus to aid future work and demarcate the scope of our proof we will now describe the formalisation state of those pieces which have not entirely made it into the formalisation, and outline some of the obstacles to integrating them into the formal proof. The proof assistant Isabelle and the current library of Nonstandard Analysis which we have used for the formalisation has a significant influence on what the difficulties in producing a complete formalisation would be. We discuss the challenges specific to Isabelle further in Section 7.2.

- Some pieces have proofs that remain to be understood. These are by far the minority. We have analysed nearly every part of the proof at the larger scale but some details remain to be explored. This includes the Trinomial Lemma (Section 6.4.1), Euler's Fundamental Theorem of Algebra (Section 3.3.3)

- Some pieces have been understood thoroughly but have not yet been formalised. For example, we have given pen-and-paper proofs of the determinacy of the power series of e^x (Section 5.2.2.1) and of the missing piece from the proof of Hidden Lemma 3 (Section 5.5.2.3). The determinacy and Euler-convergence of the related series (Section 5.2.2.3) would fit into this category. Also here would be the proof of Hidden Sublemma (i) (Section 5.7). It is still likely the case that our understanding would change or deepen if the formalisation is undertaken.

- Some pieces have been understood partially but still contain parts that would become clearer when formalised. For example, the proofs of Hidden Sublemmas (i) (Section 5.7) and (ii) (Section 5.8).

- Some have proofs which would be tedious to implement. For example, the converse of the second part of the Corollary for negative x (Section 5.6), the transitivity of hyperpolynomial factors (Section 3.2.5).

- Some of the pieces are formalised with proofs, but do not fit together well and require modification. For example, some of our lemmas assume that x (from the polynomial representation of hyperbolic sine) is nonnegative, including Hidden Sublemma (i) (Section 5.7) which depends on the Corollary (Section 5.6). We could rewrite our proof to accommodate this restriction. The Second Hidden Lemma also has a generalisation which we have mechanised and would fit more easily into the proof (Section 5.4). However, this generalisation has overlapping assumptions which could be winnowed down. Alternatively, the assumptions could simply be discharged when the lemma is applied in the proof of the Basel problem. This latter method is perhaps easier, but would be more ugly.

6.11 Summary

In this chapter we outlined our partial mechanisation of Euler's proof of the Basel problem. We illustrated the scope of our mechanisation with Figure 6.1, Table 6.2 and Table 6.1. We used these also to explain why we chose to focus our mechanisation of Euler's proof on certain controversial parts.

We described our formalisation of the central pieces of Euler's proof outside of the hidden lemmas (or 'gaps'). We explained how we put together the pieces which we have completed to form a partial reconstruction of Euler's proof of the Basel problem, and we outlined some of the pieces that are not mechanised and why. Most significantly, we raised the point that McKinzie and Tuckey's explanation of how Gap 3 may be filled with Hidden Lemma 3 and Hidden Sublemma (ii) is incomplete. We argue that this is a significant gap in their proof since it requires the use of either internal induction or the Sequential Theorem to complete it. Both of these are methods that they avoided using directly in their reconstruction of Euler's proof since they do not fit into the narrative of using hidden lemmas to rehabilitate the gaps in Euler's proof. We believe that this shows Euler's methods can be closely, but not identically, replicated in nonstandard analysis. Further, we believe it demonstrates the value of using a proof assistant to gain deeper insights into the mathematics behind Euler's reasoning.

Chapter 7

Conclusion

We conclude the thesis by discussing the decisions taken in the mechanisation process, how these influenced the current state of the formalisation and how our choices could be improved upon. We will revisit the aims we set out in the introduction and explain how this thesis has attained each of them. Finally, we will outline multiple avenues of future work.

7.1 Choices in our mechanisation

The act of formalisation can contribute to our understanding just as much as the end product of the formal proof. Throughout our journey, we made numerous decisions that informed the final formalisation. We can look back on these with hindsight and assess how we may have addressed them differently in the future. Although parts of our formalisation are unfinished, we have developed a fuller comprehension of effective methodology. This enables us to suggest approaches that would streamline the completion of the mechanisation as future work.

7.1.1 Top-down and bottom-up approaches

We employed a hybrid of top-down and bottom-up formalisation approaches in this project. Our bottom-up approaches consisted of mechanising many concepts and lemmas that we knew or believed to be useful. For example, theorems and definitions relating to determinacy (Section 5.2), hyperpolynomials (Section 3.2.5), even-and-odd intervals (Section 3.2.5.1). Our top-down techniques involved formally representing the lemmas necessary for a mechanised proof of the Basel problem, without proof themselves, at least initially. For example, those lemmas representing Euler's transitive reasoning (Section 6.8) and the lemma representing McKinzie and Tuckey's Gap 3 (Section 6.9.3). It also included outlining the proofs of central lemmas, for example, the hidden lemmas (Sections 5.3- 5.5), in terms of unproven intermediate statements.

From our experience, a combination of both approaches is ideal. The top-down approach was effective at highlighting the concepts and lemmas which were the most fundamental to the formalisation. The bottom-up approach enabled the creation of consistent definitions and theorems which were mutually compatible e.g. determinacy and hyperpolynomials, external sets (Section 3.4.3) and the Overspill Theorem (Section 3.4). In practice, we did not always make a systematic decision between the two approaches. Adhering entirely to a top-down approach would not have been possible or desirable. However, we believe it would have been better to give preference to the top-down approach, as this would have helped to focus on the priorities of the thesis. As an example, we spent significant effort in representing hyperpolynomials and the Fundamental Theorem of Algebra, and as part of this we mechanised a proof of equating-coefficients for complex polynomials. It was necessary to build a consistent theory of hyperpolynomials in order to formalise even the lemma statements in the Basel problem, and making sure of its consistency required proving theorems. However, ultimately, some parts of our representation of hyperpolynomials were not relevant to the main endeavour, including the proof of equating-coefficients for complex polynomials. This theorem is well-established and did not need to be a priority. There were omissions from the mechanisation which would have deserved formalisation instead of this e.g. the structure of the proof of Gap 3 which we discuss in Section 6.9.3.3.

7.1.2 The choice between Euler-convergence and determinacy

In Section 4.4.10 we outlined the concepts of Euler-convergence, determinacy, Euler-determinacy and finite determinacy. McKinzie and Tuckey conducted their rehabilitation of Euler's proof entirely using Euler-convergence. In a later paper they revisited Euler's Introductio, this time using the concept of determinacy. They rewrote some of their hidden lemmas using determinacy. In between writing these two papers, McKinzie conducted PhD research on the early history of series, and explored Euler's concept of convergence in further depth. As we stated in Section 4.4.1, McKinzie speculates that Euler never again mentioned his convergence principle because he became interested in series which may have infinite sums. This is a likely reason why McKinzie and Tuckey decided to use the notion of determinacy in their next paper, since even series with infinite sums may be determinate. Initially we preferred the notion of determinacy since it could be more simply and universally expressed. We attempted to use it to mechanise the proof of the Basel problem. However, we discovered that it was an inadequate replacement in some places, for example, it cannot be used to represent the Hidden Sublemma (ii) (see Section 5.8). We chose to discover the minimal assumptions for each hidden lemma and developed our categorisation of Euler-convergence, determinacy, finite determinacy and Euler-determinacy. We found some value in this endeavour since we were able to mathematically-characterise each 'gap' identified by McKinzie and Tuckey in the Basel problem. However, it would have been simpler and quicker to mechanise the entire proof using Euler-convergence since we could have followed McKinzie and Tuckey's proofs directly. The decision to use finite determinacy for Hidden Sublemma (i) rather than Euler-convergence was a more complex approach and contributes to the reason why the mechanisation of that proof is unfinished. This is since we need to additionally prove that Euler-convergence is equivalent to finite determinacy for sums and also for products in the proof. For products, this fact is only true under the assumption that coefficients are noninfinitesimal, which is in fact guaranteed for Hidden Sublemma (i) by its representation. [1]

7.1.3 Representing the internality of functions, sets and predicates

The internality of functions, sets and predicate can be specified in more than one way. For each lemma, we could

1. write the internality as an assumption. For example

```
lemma determinate_shift_argument:
  assumes "k∈ℕ" "i∈ℕ" "a ∈InternalFuns"
          "determinate (λM. hypsum a {i..<M})"
  shows  "determinate (λM. hypsum (λn. a (n+k)) {i..<M})"
```

   where InternalFuns is the set of internal functions.

[1] We also generalised other Hidden Lemmas including Hidden Lemma 2 (to a greater range of coefficients) and Hidden Lemma 3 (to other initial indices of the series) which did marginally increase the difficulty of the proofs. We also formulated a more general version of Hidden Lemma 2 which we did not complete a formal proof of.

2. Write the functions, sets and predicates involved in terms of their underlying sequences. For example

```
lemma determinate_shift_argument:
  assumes "k∈ℕ" "i∈ℕ"
          "determinate (λM. hypsum (λn. (*fn*  a) n) {i..<M})"
  shows  "determinate (λM. hypsum (λn. (*fn*  a) (n+k)) {i..<M})"
```

In most cases we made the latter choice. By taking a bottom-up approach to formalising internality, it was simpler to show that combinations of functions are internal using the latter representation (see Section 7.1.1). However, we now believe that the first choice would have been preferable. Statements in terms of underlying sequences are unwieldy and complex to read. For example, the function representing the coefficients of the binomial expansion of $(1+x/N)^N$ is expressed directly as

```
(λn. (N · C n)∈ℕ · ( x / N∈ℕ) pow n)
```

which is more easily read than the following expression in terms of the underlying sequence of real-valued functions.

```
*fn* (λn m. real (unstarnum N n choose m) ·unstarnum (x / N∈ℕ) n ^ m)
```

Statements in terms of underlying sequences must also be rewritten, potentially multiple times, to demonstrate that different functions and predicates that they contain are internal.

7.1.4 Declarative versus procedural proof

Connected to this is the choice between using declarative structured Isar proof (Section 2.2.1) and a procedural proof style consisting of successive rule applications. The structured Isar proof style is preferable in almost every way: it is more readable, less susceptible to small changes in the formalisation, more easily repaired when a change in the theory invalidates the proof and can more closely replicate pen-and-paper proofs. We used the structured style for all of our significant proofs. However, when proving that functions, sets and predicates were internal, we wrote some of our proofs using successive rule application as a quick way to obtain the result. These proofs were prone to breaking and difficult to read. In the previous section, we described our preferred approach of using assumptions to represent internality. If we had used this approach, all our proofs could have been written declaratively.

We now explore the extent to which this thesis has fulfilled its aims.

7.2 Criticisms of using Isabelle for formalising Euler's mathematics

Isabelle is not as flexible as pen-and-paper mathematics when it comes to formulating definitions. This has advantages and disadvantages. The advantage is that everything defined is well-defined (although it still may not be a definition of the intended concept). The disadvantages are that Isabelle is not as expressive, and that the definition must be carefully chosen, otherwise it may be overly complicated. Sometimes, there is not a simple and elegant way to express the concept. We see examples of this at several points in the thesis e.g. in Section 5.8.2 where we must express the internal function from Hidden Sublemma (i) using `Iset` which would be entirely omitted in pen-and-paper mathematics; in Section 5.1.1 where we must define Euler-convergence separately for sums and products rather than combining them into a single notion and in Section 3.2.5 where we cannot easily unify the algebraic representation of polynomials with the functional one. There are different reasons for this: often this is because there is not a one-to-one correspondence between the usual mathematical representation and the Isabelle representation. Then, since abstractions are built on abstractions in mathematics, the divergence between the notions in mathematics and the notions in Isabelle can grow. A choice that we needed to make at several points which had unintended consequences was whether to directly input the underlying sequence of an internal function as the argument of a definition or to instead input a function which we could not guarantee was internal. In other words, it was a choice between using the type `'a star ⇒ 'b star` and the type `nat ⇒ 'a ⇒ 'b` (equivalently `('a ⇒ 'b) star`). These are not entirely interchangeable. For some concepts, we tried both representations and propagated the effects through several layers of definitions before we could make an informed choice about which was simpler.

Another reason is insufficient integration between concrete instances of concepts and abstractions. Even when the correspondence has been proven, this does not always mean sufficient machinery has been developed to easily translate between the two. The transfer method is an example of how such machinery can work, however, this particular example could be extended further to apply to internal functions rather than simply extensions. It could also theoretically be integrated to allow definitions directly by hypernatural induction rather than requiring an intermediate concept defined by natural induction.

Yet another related reason why definitions in Isabelle may have an awkward representation is that often in mathematics two separate representations of one concept are used interchangeably. For example, embeddings of the naturals in the hypernaturals vs. naturals in the hyperreals. In Isabelle, these representations cannot be conflated although they can be proved to be equivalent. In Isabelle, machinery is required to translate from one to the other (e.g. automated proof methods such as simp) or to hide the clutter that using two representations causes (e.g. pretty-printing).

We did a significant amount of work formalising a background theory of hyper-polynomials for our formalisation of Euler's reasoning along with external sets (see Chapter 2). We prioritised this part of the background theory because the definitions from it allowed us to formalise the statements of all the lemmas required for our formalisation. Even this work was not a complete formalisation of all the mathematical background, however, and we would have had to do a great deal more further work to formalise proofs of, for example, the trinomial lemma (see Section 4.6) and Euler's cosine substitution (see Section 6.6), or to link up the Isabelle proof of the FTA with the version needed for the proof of the Basel problem (see Section 3.3). So although Isabelle has a good deal of mathematics already formalised in its libraries and in the Archive of Formal Proofs [3], the representations of mathematical concepts are specific and cannot be instantly unified with isomorphic representations.

7.3 The aims of the thesis

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. [29, Euler's 'Introductio', p. v]

Euler's mathematics in his proof of the Basel problem is not simply algebra. It draws together a nuanced understanding of revolutionary mathematical concepts, and combines them into a beautiful proof which he accomplished using methods different to the current mathematical standard. In the introduction to this thesis, we stated our aims which are each related to an aspect of understanding Euler's mathematics, and to analysing McKinzie and Tuckey's endeavour of rehabilitating Euler's proof using hidden lemmas. We will demonstrate how this thesis achieves these aims and provides a rounded understanding of Euler's proof along with McKinzie and Tuckey's reconstruction. In relation to each aim, we now summarise our accomplishments and conclusions.

7.3.1 Euler's concepts

Our aim has been to improve our understanding of the concepts used by Euler in the 'Introductio', in particular, 'infinitesimal', 'infinite number' and 'polynomial', by using the analogous concepts in nonstandard analysis and by following Euler's reasoning using formal proof in Isabelle

We analysed Euler's concepts alongside giving our representation of each in our formalisation. We aimed to discover how Euler conceived of his fundamental concepts including functions, polynomials, infinitesimals, equality, convergence and determinacy. This helped us to represent each of his concepts sympathetically, if not faithfully, in Isabelle, and use them appropriately. It also allowed us to demarcate the limitations of our representation by noting the differences between Euler's concepts and how we chose to formalise them.

Our formalisation by its nature could not be as expressive as Euler's mathematics. For example, we needed to make a choice between representing polynomials as functions or using their already existing representation as algebraic objects in Isabelle. The use of numbers of different types in the same expression required the use of explicit coercion functions. Euler was able to make transitions from the finite to the infinite by analogy or implicitly, whereas we made use of the internal induction principle and thus had to find the representations of the functions and sets involved in terms of the ultrafilter. However, the precision of formalising definitions and statements added clarity to our analysis and allowed us to characterise the gaps in Euler's and McKinzie and Tuckey's reasoning, especially with regards to the exact type of determinacy or Euler-convergence required in each case.

We believe that Euler regarded concepts such as infinite and infinitesimal numbers as tools, or stepping-stones to results about real numbers, as do many other scholars e.g. Petrie [66, p. 283], Ferraro [34, p. 45], Bair et al. [5, p. 205]. We observe that Euler did not obtain end-results in terms of infinite numbers or infinitesimals. This is similar to how Euler treated complex numbers. E.g. Euler's project of finding the trinomial factors of polynomials was founded on aiming to find the *real* linear and quadratic factors of polynomials. The trinomials were pairs of complex-conjugate roots multiplied-together to form factors with real-coefficients (see Section 4.6). Understanding the purpose of Euler's Trinomial Lemma helped us to represent it faithfully in Isabelle (see Section 6.4.1).

As McKinzie and Tuckey and Laugwitz observed, Euler understood that the transition from the finite to the infinite could not always be made, and placed conditions on when it could be allowed. For example, he stated that an infinite number N of increments of $1/N$ could be discarded. McKinzie and Tuckey and Laugwitz analysed Euler's notion of convergence and determinacy, and they used their interpretation of these concepts to rehabilitate and interpret Euler's proof. We found their interpretation useful and agree with their assessment that Euler's proof can be better understood through the lens of Euler-convergence and determinacy. We took these concepts further, and made distinctions between determinacy and Euler-determinacy, and between finite-determinate and Euler-convergence. This improved the understanding of what is occurring mathematically in each of the steps in Euler's proof that apparently rely on convergence notions.

We find that the expressivity of the concepts in nonstandard analysis is perfectly sufficient for representing Euler's reasoning. However, when it comes to proof methods, we believe that the modern rigorous proof methods that we, and McKinzie and Tuckey use, do not map perfectly onto Euler's. For example, in Section 6.9.2 we argue that Euler did not use mathematical induction in the same sense, and we demonstrated there are multiple methods in nonstandard analysis (either internal induction or finite induction plus the Sequential Theorem) which could reasonably represent his 'inductive' reasoning. This brings us to our second aim: to examine Euler's use of deductive reasoning.

7.3.2 Euler's use of deductive reasoning

Our aim has been to examine the validity of the logic of Euler's reasoning and the use of hidden lemmas (see Section 4.4.3) to explain the 'leaps of intuition' in Euler's proof

Laugwitz proposed the idea to use hidden lemmas to rehabilitate Euler's reasoning based on Lakatos' concept. McKinzie and Tuckey have already demonstrated that Euler's reasoning can be rehabilitated into a deductive proof by the use of hidden lemmas. By formalising the hidden lemmas, we obtained a better understanding of how they are used accomplish this, and the distinction between the deductive proof and Euler's original reasoning.

Excepting the First Hidden Lemma, we discovered that each of the hidden lemmas and sublemmas are more complex, both in statement and proof, than how they were expressed in McKinzie and Tuckey's paper. Hidden Lemma 2 and Hidden Lemma 3 required additional assumptions. The proofs, as given by McKinzie and Tuckey, have parts which either need additional clarification or actually contain gaps, for example, the gap in the proof of the Third Hidden Lemma. McKinzie and Tuckey gave precise, careful proofs which covered many cases which it would be easy to overlook, especially when concerned with the infinitely-close relation. One such case would be the conditions given on the main lemma in their proof of the Third Hidden Lemma. Several readings were required to fully understand and appreciate their reasons for stating these conditions. Using a proof assistant to verify proofs adds a level of detail which would be hard to achieve and to some extent would be undesirable in a pen-and-paper proof.

However, this level of detail is important for fully understanding the nuance of Euler's reasoning. We have discovered some limitations in their rehabilitation of Euler's proof. In particular, we have demonstrated that Gap 3 cannot be entirely filled with a single hidden lemma (Hidden Lemma 3) as they argue. We explored ways to complete the proof to modern standards of mathematical rigour, whilst also remaining as faithful as possible to Euler's methods of reasoning. We devised two alternatives: using internal induction or using finite induction along with the Sequential Theorem.

Verifying whether Euler's functions are 'elementary' (i.e. internal) is complex. The result of multiplying-out a product of binomials (Euler's §165) is a sum of sums of products, each of which may be infinitely-long. To represent this formally in Isabelle requires unusual constructs so that types agree (see Section 5.8.2). We were able to show that this folded sum of products is internal, but this function was sufficiently complex that prior to constructing the proof it was not obvious to us whether it truly was internal. Isabelle provides a useful tool to verify the nature of functions as internal without doubt. Mathematical intuition alone is not enough except in simple cases. At the same time, in Isabelle, the expression of internal functions in terms of their underlying sequences is unintuitive due to the constructs required for type-agreement and being overly-complex in notation (which could be fixed to some extent with better pretty-printing) thus reducing readability. Since the user must come up with these sequences constructively, and conduct the proofs manually, this is a significant obstacle to creating and reading the proofs.

There are some instances where McKinzie and Tuckey's reasoning requires additional justification, but predominantly this does not affect the understanding of Euler's proof. E.g. showing the determinacy of the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$. We argue that contrasting the work of modern mathematicians to formal proof is analogous to contrasting Euler's work with modern mathematics. We recognise that modern proofs, unlike

Euler's 18th Century proofs, are built on a well-defined mathematical theory and proof methods that are considered 'rigorous'. Similarly, formal proof, is built intrinsically on the axioms and logic we have chosen, losing some flexibility in comparison with McKinzie and Tuckey's pen-and-paper proofs. However, this analogy is not perfect. The flaws which we have discovered in McKinzie and Tuckey's proof are instances where the reasoning at first sight seems obvious, but due to some small oversight (e.g. not taking into account infinitesimal values or negative values) some additional justification is needed for a complete proof. Whereas, in Euler's proof, the gaps are not subtle, but exist in plain view and could arguably be intentional.

Euler does not fully articulate his thought process behind each step. For example, he gave his convergence principle explicitly in one single paper on the harmonic series, but he apparently used the principle implicitly elsewhere, as Laugwitz points out by formulating his hidden lemma [53]. As we stated in Section 4.4.1, McKinzie suggests that Euler did not use his convergence principle in later work because he became interested in series which take on infinite sums and did not wish to exclude them from consideration [56, pp. 116-117]. This may explain why he does not strictly adhere to this convergence principle but it does not explain why there is no mention of determinate series or why he did not formulate the criteria explicitly when he made a step which was valid only for series and products that satisfy those criteria. We can only hypothesise as to why Euler was not more explicit, but we can say with confidence that Euler does not have the modern expectation of a logically tight argument. He could not have, since the modern concept of foundations for mathematics in terms of axioms and logic had not been invented.[2] However, the reasoning Euler gives still feels persuasive: his aim was to convince the reader, and, at least in the 'Introductio', to help the reader to become familiar with the workings of the infinite.

Euler could make mistakes on his own terms: as Kanovei pointed out [76, p. 75-80], Euler made his substitution of $1-\frac{1}{2}(\frac{2k\pi}{N})^2$ for $\cos\frac{2k\pi}{N}$ prematurely.But many of the gaps that we wish to fill in as a modern reader have a different nature: Euler simply did not provide a complete account of all the steps in his reasoning, but there was an underlying consistency. We discovered that the '3rd gap' was more complicated mathematically than McKinzie and Tuckey considered, and consisted of multiple steps. Thus

[2] The reader may believe that Euclid is a counterexample to this. Eucliden geometry did give a standard of proof in geometry, but proofs in other disciplines of mathematics were not held to the same degree of rigour as they did not have the same groundwork. Non-Euclidean geometry had not been discovered yet and the consensus among mathematicians at the time was that Euclid's postulates were self-evident truths rather than axioms in the modern sense of the term. The modern notion of a formal system was introduced later by Hilbert [47, pp. 1026-1039].

the hidden lemmas do not always neatly bridge gaps to construct a perfect, rigorous modern proof. McKinzie and Tuckey were right to say that Euler was generally obeying consistent principles [55, p. 340]. We disagree, however, that '[Euler's arguments] with the adjunction of a few elementary lemmas, become a model of deductive reasoning' [57, p. 30]. This is accurate for some elements of Euler's proof, but Euler still incorporated empirically-inductive reasoning to some extent, as Pólya and McKinzie and Tuckey agree he did in his 1734 proof. We argue an example of this empirically-inductive reasoning is part of Gap 3. We do not believe this means that Euler is reckless or relies solely on intuition. He merely has a concept of proof that is not as narrow and specific as the modern one.

We reiterate McKinzie and Tuckey's claims. Euler did have an understanding of the difficulties inherent in the infinite. The hidden lemmas allow us to construct proofs in nonstandard analysis that parallel Euler's reasoning. However, we cannot translate Euler's methodology into nonstandard analysis or formal proof, since the kinds of arguments he made simply do not exist there. Our best option is to use methods of proof that share similarities with Euler's and to acknowledge the differences between the two.

We have observed that Euler had two main purposes when writing a proof.

- To give the reader the insight or understanding behind a fact, an explanation for how he arrived at it.
- To convince the reader, as he states in his 1736 paper on the Basel problem.

Modern mathematics is seen as a rule-based system. Only certain logical inferences may be made. Axioms are considered arbitrary starting points, only given significance because we believe they give rise to interesting or useful theories. Unlike Euler, we do not see 'self-evidence' as a reason for accepting a fact (we argued in Section 4.6.1 that this is how Euler viewed the Fundamental Theorem of Algebra).

Euler's 'magical intuition' comes from his experience with mathematics: thousands of examples that he had voraciously invented and explored during his lifetime, as one of the most prolific mathematicians ever. These examples and counterexamples thus gave him a nuanced understanding of when certain operations can and cannot be performed. Furthermore, he was able to choose examples that demonstrate exactly why a certain property holds. This is not to say that he never stated his principles or rules in the abstract. For example, his method of multiplying-out and equating coefficients in §165 or his statement of a principle under which a summation has a clear and definite value. However, he appears to have believed that such principles gain meaning in their application, and that others would best be able to learn or be convinced by examples. This is apparent in his iterative proofs of the Basel problem, where he demonstrates his *method* on several examples, in his application of the Trinomial Lemma and proposition from §165 to several polynomials or series, in his derivation of de Moivre's Formula from the 'Introductio' and, in fact, almost everywhere in his writings.

We argue that Euler's proofs are best understood with a layered approach. For a modern reader, the 'Introductio' is an exhilarating experience. We are dazzled by the ease with which beautiful and insightful results are produced, but also somewhat puzzled. To understand Euler's methods, we can try to put ourselves into 18th century shoes. This gives us sympathy and appreciation for the flexibility of his proof methods. But we are not in an 18th century mathematical landscape, and to relate his proofs to our own mathematical understanding, it is helpful to have a modern framework e.g. nonstandard analysis that contains concepts of infinitesimals and infinite polynomials that can mimic the behaviour of Euler's concepts. The third level of formal, computer verified proof provides two additional purposes. It brings us humility, as we realise that we are still to some extent relying on our intuition to write and understand our rigorous modern proofs. More importantly, it directs our attention to the nuances that are left implicit both in Euler's reasoning and our reconstructions of his reasoning.

At each point in this layered approach, it is important to note the aspects of Euler's reasoning that remain in our reconstruction, and those that can only be captured by the level above. We have argued, for example, that Euler's approach to inductive reasoning is not perfectly mirrored by either internal induction or finite induction together with the Sequential Theorem. Some level of his meaning has been lost. However, there is still value in choosing methods of proof that are as close as possible to Euler's original methodology. We chose to use internal induction rather than the Sequential Theorem in this example, since that better allowed us to represent Euler's use of the infinite as an extension of the finite. Thus, our understanding of the rehabilitated proof still provides insights into Euler's concepts.

7.3.3 Explanation of the proof

Our aim has been to provide insight into the details of the reasoning of Euler's proof of the Basel problem from the 'Introductio' and its relationship to the rehabilitation given by McKinzie and Tuckey [57].

Our formalisation comprises a concrete interpretation of Euler's proof and given a narrative to his reasoning which can be explored by the modern reader in detail. Since our interpretation is specific and exact, it can easily be distinguished from Euler's own proof and does not pretend to be something it is not. Yet we mark the similarities between the formalisation and Euler's original reasoning, and thus it should be clear to the reader which points are transferable.

McKinzie and Tuckey gave a simply-expressed reconstruction of Euler's proof in their paper. There are points where they gloss over detail in order to tell a succinct story. We were able to fill some additional details in and provide the specifics of what Euler did, as well as tracing the origin of results that he used. We have mechanised McKinzie and Tuckey's proofs of the three hidden lemmas and analysed the proofs of their hidden sublemmas. As we stated in the previous section, there are points in their proofs where we find that the reasoning is unclear. In these places, we have provided additional argument. Some of these we feel constitute gaps in their proofs: in particular, the Euler-convergence of $\sum_{k=n}^{H}(a_k - b_k)x^{k-n}$ for infinitesimal x in their Third Hidden Lemma (Section 5.5.2.3).

We specified the minimal assumptions to each of the hidden lemmas, allowing us to mathematically characterise the gaps in Euler's proof as presented by McKinzie and Tuckey. As a result of the formalisation we were able to build a coherent and thorough narrative of Euler's proof. It thus provides a resource to future work in multiple ways:

- Our formalisation provides a 'dictionary' to a reader of Euler's proof of the Basel problem from the 'Introductio'. We have given a formal interpretation to much of his reasoning and concepts and we have explained to what extent our interpretation is able to represent Euler's reasoning.

- Our formalisation of Euler's concepts and sections of Euler's reasoning can be used to explore his many related works and develop formalisations of those. We explore this possibility further in the next section.

7.4 Future work

Our mechanisation of Euler's proof of the Basel problem is still partial. A completion of this proof is the most obvious future work. We have analysed all the points that we believe are interesting and outlined an approach to formalising them, but a complete mechanised proof could still add valuable insights. Beyond this, there are several direct avenues for further research which we explain in the following sections.

7.4.1 The proof of the trinomial lemma

In Section 4.6.2 we gave a brief description of the kinds of reasoning used in the Trinomial Lemma. Euler does not use any infinitesimals or infinitely-large numbers in his derivation, although he will go on to apply his Trinomial Lemma to a polynomial with infinite degree. Nonetheless, there are interesting aspects of 18th century mathematics to explore in this derivation, in particular, Euler's use of the Fundamental Theorem of Algebra and trigonometry to represent complex numbers.

7.4.2 The generalisation of the Basel problem

In Section 4.2.5 we mentioned that Euler generalises his proof of the Basel problem to give the values of the following summations [29, §168, p. 139]

$$\begin{aligned}
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots &= \frac{2^2}{1\cdot2\cdot3\cdot4\cdot5}\frac{1}{3}\pi^4\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\cdots &= \frac{2^4}{1\cdot2\cdots7}\frac{1}{3}\pi^6\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\cdots &= \frac{2^6}{1\cdot2\cdot3\cdots9}\frac{3}{5}\pi^8\\
1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\cdots &= \frac{2^8}{1\cdot2\cdot3\cdots11}\frac{5}{3}\pi^{10}\\
1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\cdots &= \frac{2^{10}}{1\cdot2\cdot3\cdots13}\frac{691}{105}\pi^{12}
\end{aligned}$$

Euler provides more terms but we will not list them all here. These series are the Riemann zeta-function for even naturals $\zeta(2n)$. Euler's method for obtaining the values of these summations depends on formulas given in §166 of the 'Introductio' where he states 'The truth of these formulas is intuitively clear, but a rigorous proof will be given in the differential calculus'. Euler did in fact write a book on the differential calculus [23] and he may have given the derivation explicitly there, or perhaps elsewhere in his writings. In any case, an Eulerian proof of the generalisation of the Basel problem to all even powers could be reconstructed by deriving these formulas. One challenge in doing this this is apparent immediately: Euler expects us to find the general term of the sequence by ourselves. He states

> ... we have gone far enough to see a sequence which at first seems quite irregular, 1, 1/3, 1/3, 3/5, 5/3, 691/105, 35/1, ..., but it is of extraordinary usefulness in several places'.

These are in fact $(2n+1)B_{2n}$, where B_n are the Bernoulli numbers [63].

7.4.3 Extending our formalisation to other infinite series and products from the 'Introductio'

Two famous proofs which are often held as examples of Euler's intuitive genius and unrigorous methods, are his infinite factorisation of sine

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots \tag{7.1}$$

and his subsequent use of that result to prove Wallis' product

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot}\cdots. \tag{7.2}$$

These two proofs, although often considered separately from the Basel problem, are both from the 'Introductio' and their proofs either overlap with, or extend the proof of the Basel problem. So our existing formalisation could form a starting point. The formula for sine can be adapted from the formula for hyperbolic sine (which we have formalised) by replacing x with ix. This would require the extension of our formalisation to hypercomplex numbers. These proofs could be used as case-studies to show the application of the general methods and lemmas.

7.4.4 Related series and divergent series

A further, more ambitious project, which also draws on our existing formalisation is to examine, and partly formalise, Euler's paper on the relation between the alternating generalised Basel series, $1/1^n - 1/2^n + 1/3^n - 1/4^n + 1/5^n - 1/6^n + 1/7^n - 1/8^n + \cdots$, and the clearly-divergent alternating sum of the integers raised to powers, $1^n - 2^n + 3^n - 4^n + 5^n - 6^n + 7^n - 8^n + \cdots$ [79]. Because part of his investigation is based on his existing proof of the Basel problem, our mechanisation could be reused to formalise his argumentation. Euler's concept of attributing values to truly divergent series could also be explored: it would go beyond the concept of determinacy (which we discuss in Section 5.1). Kanovei states

> ... is it true that any series has a unique "natural" value of the sum, as Euler believed, and if so, how do we find that value? It would be of great interest to find an approach to this question from the standpoint of non-standard analysis. [76]

7.4.5 Extension to the transfer method in Isabelle

Currently the transfer method in Isabelle (see Section 2.3.2.5) only works for nonstandard extensions. A more general mechanisation of the transfer method to cover internal functions and sets would be useful for using nonstandard analysis in Isabelle. It would greatly reduce the tedious reasoning involved in reproving results that are already proved for standard analysis. A large part of our mechanisation effort was spent on proving that functions, sets and propositions are internal. McKinzie and Tuckey (and Euler) did not expend effort on proving that functions were 'elementary' (their corresponding concept to internal). As we state in Section 5.8.2, this could be an issue in their proof of Hidden Sublemma (ii). If the transfer method was extended, we would have the double advantage of ease of proof and absolute certainty.

However, we have come across functions which we did not immediately understand how to prove were internal. An example of this is our folded sum of products which we used to represent the multiplied-out polynomial from Euler's §165. Furthermore, in Isabelle, there are defined separate constructions to represent internal functions with different arity: e.g. `*fn*` and `*f2n*`. Thus, we cannot clearly say that the method for proving internality can be captured in a simple algorithm.

7.5 Concluding thoughts

Euler's proofs are inspiring to the modern reader because they demonstrate an entirely different approach to the one we use now: different concepts of function, polynomial, and infinities; different methods of proof and inductive reasoning; self-evident facts and convincing examples rather than axiomatic mathematical theories and logic. Euler's consistent and insightful mathematics reveals to us that the mathematics we practice now is not the only mathematics and, to some extent, it is arbitrary. We gravitated towards using limits in analysis, but we might have managed to stay with infinitesimals all along. In the future, mathematics and the standard for mathematical proof will be different yet again. We have already found new mathematical tools in the shape of theorem provers. This thesis at once unifies and distinguishes between three types of mathematical understanding and proof: there is, in the end, no reason to restrict ourselves to only one.

Bibliography

[1] W. S. Anglin. 'The Pythagoreans and Irrationality'. In: *Mathematics: A Concise History and Philosophy*. New York, NY: Springer New York, 1994, pp. 35–41. ISBN: 978-1-4612-0875-4. DOI: 10.1007/978-1-4612-0875-4_7. URL: https://doi.org/10.1007/978-1-4612-0875-4_7.

[2] T.M. Apostol. *Calculus, Volume I, 2nd Ed One-variable Calculus, with an Introduction to Linear Algebra*. Calculus v. 1. Wiley India Pvt. Limited, 2007. ISBN: 9788126515196. URL: https://books.google.co.uk/books?id=vTpbq0UPDaQC.

[3] *Archive of Formal Proofs*. URL: https://www.isa-afp.org/ (visited on 16/06/2022).

[4] Raymond Ayoub. 'Euler and the Zeta Function'. In: *The American Mathematical Monthly* 81.10 (1974), pp. 1067–1086. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2319041 (visited on 29/07/2022).

[5] Jacques Bair et al. 'Interpreting the Infinitesimal Mathematics of Leibniz and Euler'. In: *Journal for General Philosophy of Science* 48.2 (June 2017), pp. 195–238. ISSN: 1572-8587. DOI: 10.1007/s10838-016-9334-z. URL: https://doi.org/10.1007/s10838-016-9334-z.

[6] John L. Bell. *A Primer of Infinitesimal Analysis*. 2nd ed. Cambridge University Press, 2008. DOI: 10.1017/CBO9780511619625.

[7] David Benko. 'The Basel Problem as a Telescoping Series'. In: *The College Mathematics Journal* 43.3 (2012), pp. 244–250. DOI: 10.4169/college.math.j.43.3.244. eprint: https://doi.org/10.4169/college.math.j.43.3.244. URL: https://doi.org/10.4169/college.math.j.43.3.244.

[8] Yves Bertot and Pierre Castéran. *Interactive theorem proving and program development: Coq'Art: the calculus of inductive constructions*. Springer Science & Business Media, 2013.

[9] Enrico Bombieri. 'Problems of the Millennium : the Riemann Hypothesis'. In: 2000.

[10] C. B. Boyer. 'The Foremost Textbook of Modern Times'. In: *The American Mathematical Monthly* 58.4 (1951), pp. 223–226. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2306956.

[11] Florian Cajori. *A History of Mathematical Notations*. Vol. 2. The Open Court Publishing Company, 1929.

[12] Mario Carneiro. *Theorem basel*. 2014. URL: http://us.metamath.org/mpeuni/basel.html (visited on 15/08/2018).

[13] Daniel Daners. 'A Short Elementary Proof of $\sum 1/k^2 = \pi^2/6$'. In: *Mathematics Magazine* 85.5 (2012), pp. 361–364. DOI: 10.4169/math.mag.85.5.361. eprint: https://doi.org/10.4169/math.mag.85.5.361. URL: https://doi.org/10.4169/math.mag.85.5.361.

[14] Marcel Danesi. '93i: Imaginary numbers'. In: *Pythagoras' Legacy: Mathematics in Ten Great Ideas*. Oxford University Press, Feb. 2020. ISBN: 9780198852247. DOI: 10.1093/oso/9780198852247.003.0007. eprint: https://academic.oup.com/book/0/chapter/347973984/chapter-pdf/43242044/oso-9780198852247-chapter-7.pdf. URL: https://doi.org/10.1093/oso/9780198852247.003.0007.

[15] Martin Davis and Reuben Hersh. 'NONSTANDARD ANALYSIS'. In: *Scientific American* 226.6 (1972), pp. 78–89. ISSN: 00368733, 19467087. URL: http://www.jstor.org/stable/24927363 (visited on 22/09/2022).

[16] Lokenath Debnath. 'The legacy of Leonhard Euler - a tricentennial tribute'. In: *International Journal of Mathematical Education in Science and Technology* 40.3 (2009), pp. 353–388. DOI: 10.1080/00207390802642237. eprint: https://doi.org/10.1080/00207390802642237. URL: https://doi.org/10.1080/00207390802642237.

[17] William Dunham. 'Euler and the Fundamental Theorem of Algebra'. In: *The College Mathematics Journal* 22.4 (1991), pp. 282–293. DOI: 10.1080/07468342.1991.11973397. eprint: https://doi.org/10.1080/07468342.1991.11973397. URL: https://doi.org/10.1080/07468342.1991.11973397.

[18] Manuel Eberl. 'Symmetric Polynomials'. In: *Archive of Formal Proofs* (Sept. 2018). http://isa-afp.org/entries/Symmetric_Polynomials.html, Formal proof development. ISSN: 2150-914x.

[19] Manuel Eberl. *Theory Gamma_Function*. URL: https://isabelle.in.tum.de/library/HOL/HOL-Analysis/Gamma_Function.html (visited on 15/08/2018).

[20] W. F. Eberlein. 'The Circular Function(s)'. In: *Mathematics Magazine* 39.4 (1966), pp. 197–201.

[21] W.F Eberlein. 'On Euler's infinite product for the sine'. In: *Journal of Mathematical Analysis and Applications* 58.1 (1977), pp. 147–151. ISSN: 0022-247X. DOI: https://doi.org/10.1016/0022-247X(77)90234-7. URL: http://www.sciencedirect.com/science/article/pii/0022247X77902347.

[22] Paul Erdös and Underwood Dudley. 'Some Remarks and Problems in Number Theory Related to the Work of Euler'. In: *Mathematics Magazine* 56.5 (1983), pp. 292–298. DOI: 10.1080/0025570X.1983.11977060. eprint: https://doi.org/10.1080/0025570X.1983.11977060. URL: https://doi.org/10.1080/0025570X.1983.11977060.

[23] L. Euler. *Foundations of Differential Calculus*. Translation by J. D. Blanton, originally published 1755. Vol. 1. Berlin: Springer-Verlag, 2000, pp. xvi+194. ISBN: 0-387-98534-4.

[24] Leonhard Euler. 'De progressionibus harmonicis observationes [E43]'. In: *Opera Omnia*. 1st ser. 14 (1740), pp. 87–100. URL: http://eulerarchive.maa.org. Written 1734. No English translation is available. Original paper available online at the Euler Archive.

[25] Leonhard Euler. 'De summatione innumerabilium progressionum'. In: *Commentarii academiae scientiarum Petropolitanae* 5 (1738). Presented in 1731, pp. 91–105.

[26] Leonhard Euler. 'De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eaedem summationes ex fonte maxime diverso derivantur'. In: *Miscellanea Berolinensia* 7 (1743), pp. 172–192.

[27] Leonhard Euler. 'De summis serierum reciprocarum (On the sums of series of reciprocals)'. In: *arXiv* (Feb. 2008). Translated from the Latin by Jordan Bell. DOI: arXiv:math/0506415. URL: https://arxiv.org/abs/math/0506415.

[28] Leonhard Euler. 'Demonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + ...' In: *Journal litteraire d'Allemagne, de Suisse et du Nord* 2 (1743), pp. 115–127.

[29] Leonhard Euler. *Introductio in Analysin Infinitorum*. Lausanne: Tomus Primus, 1748. Reprinted in Opera Omnia, series 1 volume 8. Translated from the Latin by J. D. Blanton, *Introduction to Analysis of the Infinite*, Book 1, Springer-Verlag, 1988.

[30] Leonhard Euler. 'Methodus universalis serierum convergentium summas quam proxime inveniendi [E46]'. In: *Opera Omnia*. 1st ser. 14 (1741), pp. 101–107. URL: http://eulerarchive.maa.org. Written 1735. Translated into English by: Walter Jacob, IV and Thomas J. Osler as *A general method for finding approximations to the sums of convergent series*. Original paper available online along with the English translation at the Euler Archive.

[31] Leonhard Euler. *On the sums of series of reciprocals*. 2005. DOI: 10.48550/ARXIV.MATH/0506415. URL: https://arxiv.org/abs/math/0506415.

[32] Leonhard Euler. 'Vollstandige Anleitung zur Algebra [E387]'. In: *Opera Omnia*. 1st ser. 1 (1770). URL: http://eulerarchive.maa.org. Translated into English by: Hewlett, J., 1822. Elements of Algebra. Third edition. London. Original book available online along with an English translation by Hewlett, J., at the Euler Archive. All quotations are from the Hewlett translation.

[33] Leonhard Euler and A Aycock. *Another Dissertation on the sums of the series of reciprocals arising from the powers of the natural numbers, in which the same summations are derived from a completely different source*. 2020. URL: https://www.agtz.mathematik.uni-mainz.de/algebraische-geometrie/van-straten/euler-kreis-mainz/ (visited on 02/09/2020).

[34] Giovanni Ferraro. 'Differentials and differential coefficients in the Eulerian foundations of the calculus'. In: *Historia Mathematica* 31.1 (2004), pp. 34–61. ISSN: 0315-0860. DOI: https://doi.org/10.1016/S0315-0860(03)00030-2. URL: https://www.sciencedirect.com/science/article/pii/S0315086003000302.

[35] Giovanni Ferraro. 'Euler, infinitesimals and limits'. In: *HAL* (2012). URL: https://halshs.archives-ouvertes.fr/halshs-00657694v2.

[36] J. D. Fleuriot. *A Combination of Geometry Theorem Proving and Nonstandard Analysis with Application to Newton's Principia*. 1st ed. London: Springer, 2001.

[37] Jacques D Fleuriot and Lawrence C Paulson. 'A combination of nonstandard analysis and geometry theorem proving, with application to Newton's Principia'. In: *International Conference on Automated Deduction*. Springer. 1998, pp. 3–16.

[38] Ruben A Gamboa and Matt Kaufmann. 'Nonstandard Analysis in ACL2'. In: *Journal of Automated Reasoning* 27 (4 2001), pp. 323–351. DOI: 10.1023/A:1011908113514. URL: https://doi.org/10.1023/A:1011908113514.

[39] Carl Friedrich Gauss. 'New Proof of the Theorem That Every Algebraic Rational Integral Function In One Variable can be Resolved into Real Factors of the First or the Second Degree'. In: *Helmstedt, CG Fleckeisen* (1799). URL: https://www.quantresearch.org/Gauss_PhD_Dissertation.pdf (visited on 29/08/2022).

[40] R. Goldblatt. *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*. New York: Springer-Verlag, 1998.

[41] John Harrison. *HOL Light Source code: theorem EULER_HARMONIC_SUM*. 2014. URL: https://github.com/jrh13/hol-light/blob/master/100/piseries.ml (visited on 28/04/2023).

[42] D. Hilbert and E.J. Townsend. *The Foundations of Geometry,* Open Court Publishing Company, 1950.

[43] R. F. Hoskins. *Standard and nonstandard analysis: fundamental theory, techniques, and applications*. Ellis Horwood Ltd, 1990.

[44] Brian Huffman and Ondřej Kunčar. 'Lifting and Transfer: A Modular Design for Quotients in Isabelle/HOL'. In: *Certified Programs and Proofs*. Ed. by Georges Gonthier and Michael Norrish. Cham: Springer International Publishing, 2013, pp. 131–146. ISBN: 978-3-319-03545-1.

[45] Brian Huffman Jacques D. Fleuriot Lawrence C. Paulson. *Session HOL- Nonstandard_Analysis*. URL: https://isabelle.in.tum.de/library/HOL/HOL-Nonstandard_Analysis/index.html (visited on 17/08/2018).

[46] Israel Kleiner. 'Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)'. In: *The Mathematics Teacher* 81.7 (1988), pp. 583–592. DOI: 10.5951/MT.81.7.0583. URL: https://pubs.nctm.org/view/journals/mt/81/7/article-p583.xml.

[47] M. Kline. *Mathematics from Ancient to Modern Times*. Paperback. Oxford: Oxford University Press, 1990.

[48] Alexander Krauss et al. 'Termination of Isabelle Functions via Termination of Rewriting'. In: *Interactive Theorem Proving*. Ed. by Marko van Eekelen et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 152–167. ISBN: 978-3-642-22863-6.

[49] Ondřej Kunčar and Andrei Popescu. 'Comprehending Isabelle/HOL's Consistency'. In: *Programming Languages and Systems*. Ed. by Hongseok Yang. Berlin, Heidelberg: Springer Berlin Heidelberg, 2017, pp. 724–749. ISBN: 978-3-662-54434-1.

[50] Imre Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. Ed. by John Worrall and ElieEditors Zahar. Cambridge Philosophy Classics. Cambridge University Press, 2015. DOI: 10.1017/CBO9781316286425.

[51] D. Laugwitz. 'Hidden lemmas in the early history of infinite series.' In: *Aequationes mathematicae* 34 (1987), pp. 264–276. URL: http://eudml.org/doc/137244.

[52] Detlef Laugwitz. *Bernhard Riemann 1826–1866: Turning Points in the Conception of Mathematics*. Translated by Abe Schenitzer. Boston, MA: Birkhäuser Boston, 1999. ISBN: 978-0-8176-4777-3. DOI: 10.1007/978-0-8176-4777-3-4.

[53] Detlef Laugwitz. 'Definite Values of Infinite Sums: Aspects of the Foundations of Infinitesimal Analysis around 1820'. In: *Archive for History of Exact Sciences* 39.3 (1989), pp. 195–245. ISSN: 00039519, 14320657. URL: http://www.jstor.org/stable/41133853 (visited on 07/09/2022).

[54] Jean-Marie Madiot. *Coqtail.Reals.Rzeta2*. URL: https://coqtail.github.io/doc/Coqtail.Reals.Rzeta2.html (visited on 15/08/2018).

[55] M. McKinzie and C. Tuckey. 'Higher Trigonometry, Hyperreal Numbers, and Euler's Analysis of Infinities'. In: *Mathematics Magazine* 74.5 (2001), pp. 339–168.

[56] Mark B. McKinzie. 'The Halley -Euler method'. English. Copyright - Database copyright ProQuest LLC; ProQuest does not claim copyright in the individual underlying works; Last updated - 2021-10-26. PhD thesis. 2000, p. 167. ISBN: 978-0-599-63861-7. URL: https://www.proquest.com/dissertations-theses/halley-euler-method/docview/304637425/se-2.

[57] Mark McKinzie and Curtis Tuckey. 'Hidden lemmas in Euler's summation of the reciprocals of the squares'. In: *Archive for History of Exact Sciences* 51.1 (Mar. 1997), pp. 29–57. ISSN: 1432-0657. DOI: 10.1007/BF00376450. URL: https://doi.org/10.1007/BF00376450.

[58] L. Meikle and J. Fleuriot. 'Formalizing Hilbert's Grundlagen in Isabelle/Isar'. In: *Theorem Proving in Higher Order Logics*. Ed. by D. Basin and B. Wolff. Vol. 2758. LNCS. Heidelberg: Springer, 2003, pp. 319–334.

[59] I. I. Morris. 'An Axiomatic Formalisation of Trigonometric Functions in Isabelle'. 4th Year Project. University of Edinburgh, 2017.

[60] Leonardo Mendonça de Moura et al. 'The Lean Theorem Prover (System Description).' In: *CADE*. Ed. by Amy P. Felty and Aart Middeldorp. Vol. 9195. Lecture Notes in Computer Science. Springer, 2015, pp. 378–388. ISBN: 978-3-319-21400-9. URL: http://dblp.uni-trier.de/db/conf/cade/cade2015.html#MouraKADR15.

[61] Paul J Nahin. *Dr. Euler's fabulous formula: cures many mathematical ills*. Vol. 83. Princeton University Press, 2011.

[62] T. Nipkow. *A Tutorial Introduction to Structured Isar Proofs*. 2009. URL: http://isabelle.in.tum.de/website-Isabelle2009-1/dist/Isabelle/doc/isar-overview.pdf (visited on 23/02/2017).

[63] *Numerator of* $(2n+1)B_{2n}$*, where* B_n *are the Bernoulli numbers, Entry A002427 in The On-Line Encyclopedia of Integer Sequences*. 2022. URL: https://oeis.org/A002427 (visited on 29/08/2022).

[64] Karol Pąk and Artur Korniłowicz. 'Basel Problem'. In: 25 (July 2017).

[65] L. C. Paulson. *Isabelle: A Generic Theorem Prover*. 1st ed. Berlin: Springer-Verlag, 1994.

[66] Bruce J. Petrie. 'Leonhard Euler's use and understanding of mathematical transcendence'. In: *Historia Mathematica* 39.3 (2012), pp. 280–291. ISSN: 0315-0860. DOI: https://doi.org/10.1016/j.hm.2012.06.003. URL: https://www.sciencedirect.com/science/article/pii/S0315086012000353.

[67] G. Pólya. *Mathematics and plausible reasoning. I. Induction and analogy in mathematics. II. Patterns of plausible inference.* Princeton, NJ, US: Princeton University Press, 1954.

[68] Walter Rudin et al. *Principles of mathematical analysis*. Vol. 3. McGraw-hill New York, 1976.

[69] Ed Sandifer. 'How Euler Did It: Venn Diagrams'. In: *Mathematical Association of America Online* (Jan. 2004). URL: https://web.archive.org/web/20130126165922/http://maa.org/editorial/euler/How%20Euler%20Did%20It%2003%20Venn%20Diagrams.pdf.

[70] Edward Sandifer. 'Euler's Solution of the Basel Problem - The Longer Story'. In: 2003. URL: https://faculty.math.illinois.edu/~reznick/sandifer.pdf (visited on 29/07/2022).

[71] P. Scott. 'Mechanising Hilbert's Foundations of Geometry in Isabelle'. MSc. University of Edinburgh, 2008.

[72] Phil Scott. 'Ordered geometry in Hilbert's Grundlagen der Geometrie'. PhD thesis. The University of Edinburgh, 2015.

[73] Phil Scott and Jacques Fleuriot. 'An Investigation of Hilbert's Implicit Reasoning through Proof Discovery in Idle-Time'. In: *Automated Deduction in Geometry*. Ed. by Pascal Schreck, Julien Narboux and Jürgen Richter-Gebert. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 182–200. ISBN: 978-3-642-25070-5.

[74] Phil Scott and Jacques D Fleuriot. 'Where are the Natural Numbers in Hilbert's Foundations of Geometry?' In: *arXiv preprint arXiv:1911.07057* (2019).

[75] John Stillwell. *Mathematics and its history*. 3rd ed. Springer, 2010.

[76] V. G. Kanovei. 'The correctness of Euler's method for the factorization of the sine function into an infinite product'. In: *Russian Math. Surveys* 43 (4(262) 1988), pp. 57–81.

[77] V. S. Varadarajan. 'Euler and His Work on Infinite Series'. In: 2007.

[78] F. Viète. *Opera mathematica*. Ed. by F. van Schouten. 1st ed. Leiden, 1646.

[79] Lucas Willis and Thomas J. Osler. 'Remarks on a beautiful relation between direct as well as reciprocal power series'. In: *Euler Archive* (Aug. 2006). URL: http://eulerarchive.maa.org/docs/translations/E352.pdf.

[80] Richard Zach. 'Hilbert's Program'. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University, 2019.

Index of Named Theorems