The Stability of Hyperbolic PDEs in String Theory, Particle Physics and Cosmology

Zoe Wyatt

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Abstract

In this thesis we study hyperbolic PDEs arising from general relativity and the standard model of particle physics. In particular we prove the asymptotic stability of special solutions of these PDEs against small initial data perturbations. The study of stability elucidates our understanding of whether such PDEs can provide mathematically reasonable models for physical phenomena in our universe.

In Chapter 2, we prove the stability of the Kaluza-Klein spacetime to perturbations that depend only on the Minkowski coordinates. This chapter is based on the published work [Wya18] and uses the powerful direction-dependent decay and energy estimates established by Lindblad and Rodnianski for their proof of the stability of Minkowski spacetime.

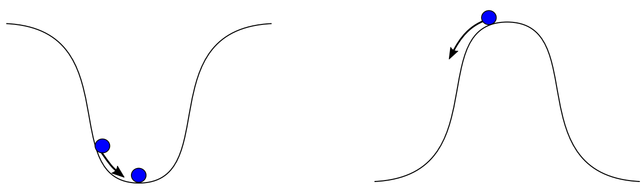
In Chapter 3, we present joint work with Shijie Dong and Philippe LeFloch [DLW19]. Using the hyperboloidal foliation method of LeFloch and Ma we establish the stability of the ground state of the $U(1)$ standard model of electroweak interactions. This amounts to establishing stability of the trivial solution of a coupled nonlinear wave–Klein-Gordon–Dirac PDE system. In Chapter 4, we study a different type of wave–Klein-Gordon PDE system which has quadratic wave-Klein-Gordon interactions in which there are no derivatives on the wave component. We show the stability of its trivial solution, and this chapter is based on a collaboration with Shijie Dong published in [SW20].

In Chapter 5, we show that the Milne spacetime is a stable solution to the Einstein Klein-Gordon equations, which is based on joint work with David Fajman appearing in [FW19]. The method utilises the CMCSH gauge and associated coercive energy estimates developed by Andersson and Moncrief for their proof of the stability of Milne as a solution to the vacuum Einstein equations.

Finally, in Chapter 6, we show the stability under the vacuum Einstein equations of the product of high-dimensional Minkowski space with a compact special holonomy manifold. This result provides a counterexample to an instability argument by Penrose and combines ideas from Chapters 2-5. Work presented in this final chapter is based on a collaboration with Lars Andersson, Pieter Blue and Shing-Tung Yau, and appears in [ABWY20].

Lay summary

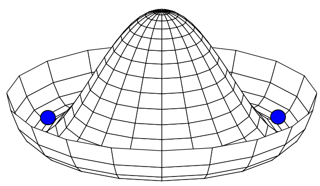
Consider two balls, one placed at the bottom of a valley and the other at the top of a hill. If no forces act then the balls will not move from their starting positions. If we now take the valley ball and instead move it up the side of the valley, then upon letting it go, it will naturally roll back down to the bottom stable point. By contrast, if the ball on the hill is moved slightly away from the hilltop, perhaps it has been pushed by a strong gust of Scottish wind, then the ball will rapidly roll off the hill and away from its unstable hilltop position.



Partial differential equations (PDEs) are one of the main tools used to model physical phenomena, from balls rolling off hilltops to the motion of the black hole thought to lie at the centre of our galaxy. Special 'positions', such as the bottom of the valley, arise as steady state solutions to the PDE. A steady state solution is asymptotically stable if all small initial perturbations of its starting position lead to solutions that, over time, asymptote back to the original steady state.

In this thesis we show the asymptotic stability of five steady state solutions to PDEs derived from two field theories: general relativity and the electroweak model of particle physics. The PDEs are closely tied to the wave equation, which models many kinds of waves, such as sound waves, electromagnetic waves, and, as recently discovered experimentally, gravitational waves.

To have any hope of proving asymptotic stability, the steady states we consider must be geometrically special, for example by having a high degree of symmetry, and the initial perturbations we consider must be small. The PDEs that we study are nonlinear; roughly speaking this allows for the ball to change shape as it moves around and down the valley, which makes proving stability more difficult. Other issues arise when we consider different types of balls, describing different fundamental particles, and even further interesting dynamics occur when we change the shape of the valley to the one below where there is a continuum of physically indistinguishable steady state solutions.



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Contents

Chapter 1

Introduction

In this thesis we study Einstein's theory of gravity. Special relativity is Einstein's framework for non-gravitational physics. Events are no longer labelled by absolute notions of space and time, but rather are represented as spacetime points in the Lorentzian manifold $(\mathbb{R}^{1+3}, \eta = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. This spacetime and any physics observed within it obey a symmetry, generated by Poincaré transformations, which modifies the Galilean symmetry of Newtonian mechanics in a way compatible with electromagnetism. Although observers can no longer agree on a preferred notion of time and space, they can at least agree on the speed of light c as a frame independent quantity.

General relativity is Einstein's geometric theory of the gravitational force. It extends the principles of relativity to non-inertial reference frames and gravitational physics. Events are now represented as points in a curved Lorentzian manifold $(\mathcal{M}, g)$. In general relativity a freely falling observer travels on a geodesic of this curved spacetime. Thus their motion due to gravity arises as the deviation of their path from otherwise inertial motion in the flat space $(\mathbb{R}^{1+3}, \eta)$.

The Einstein field equations determine the behaviour of the curved metric tensor g and, like Lorentz's formulation of electromagnetism, are independent of any reference frame. Instead of Poisson's equation for the scalar gravitational potential on $\mathbb{R}^3$ the Einstein equations are equations for tensor fields on the differentiable manifold $\mathcal{M}$. In natural units ($G = c = 1$) the field equations read

$$\mathrm{Ric}[g]_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathrm{R}[g] = 8\pi T_{\mu\nu}, \tag{1.0.1}$$

where the distribution of matter sourcing the gravitational field is represented by the energy-momentum tensor $T_{\mu\nu}$. The specific relationship (1.0.1) between curvature, expressed through the Ricci tensor $\mathrm{Ric}[g]_{\mu\nu}$ and Ricci scalar $\mathrm{R}[g]$, and matter was first introduced in [Ein16]. The left hand side of (1.0.1) is, up to an overall constant, fixed by the Bianchi identity and local conservation of energy. The overall constant is determined by comparing, in a weak curvature approximation, the tidal acceleration of two nearby particles in general relativity and in Newtonian gravity.

There are many known solutions to (1.0.1) which describe different phenomena within our universe, some of which have even been tested experimentally. Important solutions are Minkowski spacetime, the Schwarzschild and Kerr black holes and cosmological models such as the FLRW spacetimes. Even though the local form of these metrics depends on the coordinates used, the spacetime is only defined up to an equivalence class under diffeomorphisms.

The Einstein equations (1.0.1), together with the conservation law for the stress-energy tensor, form a system of PDEs for the metric components and matter terms. The overriding motivation of this thesis is to formulate the Cauchy problem of general relativity and to study steady state solutions to the associated PDEs and their behaviour under small perturbations of the fields. However, what does it mean to study a perturbation of a spacetime which the author perceives as being close to a particular spacetime as measured by some coordinates (t, x), since you the reader could have very easily used other coordinates (t', x')? How is it possible to measure the dynamical behaviour of a spacetime, when there is no natural choice of time parameter?

Thus, to study the question of spacetime stability we need to break the diffeomorphism freedom in (1.0.1). A key prediction of general relativity is that in a weak field approximation around a Minkowski background, equations (1.0.1) reduce to a system of wave equations for the perturbation of the metric $h_{\mu\nu}$ propagating at the speed of light. This is in stark contrast to the instantaneous action of the gravitational field in Newtonian theory and in its frame independent formulation, Newton-Cartan theory, which do not admit such wave-like solutions [HHO19]. Hyperbolic PDEs describe phenomena with a finite speed of propagation and so to study spacetime stability we break the diffeomorphism freedom in (1.0.1) in a way that uncovers this hyperbolic structure.

Structure of Chapter 1. Our goal in this chapter is to present the Cauchy problem in general relativity and summarise the main results of the thesis. In Section 1.1 we introduce the study of local solutions to the Cauchy formulation of (1.0.1). In Section 1.3 and 1.4 we introduce some of the key methods used to prove the existence of global solutions to wave and Klein-Gordon equations. In Section 1.5 we discuss properties of the linearised Lorentzian Einstein equations and special geometric properties of Riemannian Einstein spaces. In sections 1.6–1.9, we introduce the main results of this thesis which are presented in Chapters 2–6. Each section includes a brief introduction to the topic of interest, some pertinent references and the main ideas behind the proofs. Although the topics presented in Chapters 2–6 are somewhat diverse, we hope the presentation in this Chapter will emphasise their unifying features. Notation is, unless remarked upon, kept consistent throughout the thesis.

Notation 1.1 (Tensors, Indices)**.** Let $(\mathcal{M}, g)$ be a smooth $(1+m)$-dimensional Lorentzian manifold of signature $(-, +, \cdots, +)$. We sometimes write four-dimensional instead of $(1+3)$-dimensional. Let Greek letters denote spacetime indices $\mu, \nu \in \{0, \ldots, m\}$ and Roman letters denote spatial indices $a, b \in \{1, \ldots, m\}$.

For the metric $g_{\mu\nu}$ let $\nabla[g]$ denote its Levi-Civita connection, and let $\mathrm{Riem}[g]^{\gamma}{}_{\mu\rho\nu}$, $\mathrm{Ric}[g]_{\mu\nu}$ and $\mathrm{R}[g]$ denote its associated Riemann tensor, Ricci tensor and Ricci scalar respectively. With respect to some coordinate system $\{x^\mu\}$ let the Christoffel symbols be denoted $\Gamma[g]^{\gamma}_{\mu\nu}$ and the reduced wave operator be denoted $\widetilde{\Box}_g = g^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu}$. In Minkowski spacetime with Cartesian coordinates we write $\Box = -\partial_t^2 + \sum_{i=1}^n \partial_{x_i}^2$ and $\mathrm{d}^n x = \mathrm{d}x^1 \ldots \mathrm{d}x^n$ for the flat Euclidean volume form.

Let $\hat{e}$ be a fixed pseudo-Riemannian metric on $\mathcal{M}$ with similar definitions for $\nabla[\hat{e}]$ etc. Following [Bes87] we define the following contraction with respect to $\hat{e}$

$$(R[\hat{e}] \circ u)_{\mu\nu} = \mathrm{Riem}[\hat{e}]_{\mu\rho\nu\lambda}\hat{e}^{\rho\rho'}\hat{e}^{\lambda\lambda'}u_{\rho'\lambda'} = \mathrm{Riem}[\hat{e}]_{\mu}{}^{\rho}{}_{\nu}{}^{\lambda}u_{\rho\lambda}, \tag{1.0.2}$$

which acts on symmetric $(0,2)$-tensors $u_{\mu\nu}$. We occasionally use $\sim$ to denote a heuristic, or perhaps leading-order, relationship between variables.

Notation 1.2 (ADM decomposition, for full details see e.g. [Ren08])**.** Suppose $(\mathcal{M}, g)$ is a Lorentzian manifold and $\mathcal{M} = \mathbb{R} \times \Sigma_t$ is foliated by spacelike surfaces Σ_t defined as level sets of a time function $t : \mathcal{M} \to \mathbb{R}$. The gradient 1-form $(\mathrm{d}t)_\mu$ is associated, by metric duality, to a vector T^μ which defines the unique normal direction to Σ_t. We define $n^\mu = -NT^\mu$ to be the future-directed unit normal to Σ_t, where the lapse function $N = (-T^\mu(\mathrm{d}t)_\mu)^{-1/2} > 0$ is the normalisation factor. The induced Riemannian metric and second fundamental forms of Σ_t in $\mathcal{M}$ are given by $\bar{g}_{ab}$ and $\bar{K}_{ab}$. To compare these objects between each leaf Σ_t we introduce coordinates $\{x^a\}$ on Σ_t so that we can now use coordinates $\{t, x^a\}$ on $\mathcal{M}$. In general the coordinate vector ∂_t is not colinear to n^μ since the lines $x^a = const$ may not be orthogonal to the slices Σ_t. The shift vector X^a measures this difference as $(\partial_t)^a = Nn^a + X^a$. Finally, the definition of the second fundamental form implies the identity

$$\mathscr{L}_{\partial_t}\bar{g}_{ab} = -2N\bar{K}_{ab} + \mathscr{L}_X\bar{g}_{ab}, \tag{1.0.3}$$

and the spacetime line element can be expressed as

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \bar{g}_{ab}(\mathrm{d}x^a + X^a\mathrm{d}t)(\mathrm{d}x^b + X^b\mathrm{d}t). \tag{1.0.4}$$

The lapse and shift variables were first introduced in [FB56].

Given a stress-energy tensor $T_{\mu\nu}$ we follow the convention in [Ren08, (2.20)] and denote the energy density and matter current by

$$\rho = T^{\mu\nu}n_\mu n_\nu, \quad \jmath^\rho = -T^{\mu\nu}P^\rho{}_\nu n_\mu, \tag{1.0.5}$$

where $P^\rho{}_\nu = \delta^\rho_\nu + n^\rho n_\mu$ is the orthogonal projector onto Σ_t with respect to $g_{\mu\nu}$.

Notation 1.3 (Multi-indices)**.** Let $\mathcal{Z} = \{Z_1, \ldots, Z_q\}$ be a set consisting of q vector fields which do not necessarily commute. We use the multi-index notation $I = (\iota_1, \ldots, \iota_k)$ for $k \in \mathbb{Z}^+$ with each $\iota_j \in \{1, \ldots, q\}$ for $j \in \{1, \ldots, k\}$. In this case $|I| = k$ and $Z^I = Z_{\iota_1} \ldots Z_{\iota_k}$. When I is empty we denote $|I| = 0$ and $Z^I = id$.

Notation 1.4 (Inequality symbols)**.** Given two real functions f, g we write $f \lesssim g$ to imply that there exists a constant $C > 0$ such that $f \leq Cg$. This constant may depend on free parameters, but it will always be independent of the smallness parameter ε. We denote $f \simeq g$ when both $f \lesssim g$ and $g \lesssim f$.

1.1 The Cauchy problem in general relativity

Our goal over the next few pages is to understand the keys steps which are used to establish a local well-posedness theorem for the initial value formulation of the Einstein equations. We remark that although the following is standard, all of our later chapters rely on this theory (up to some modifications) and so it is worthwhile to review it here.

Definition 1.5 (Geometric vacuum initial data)**.** A vacuum initial data set is defined to be a triple $(\Sigma, \bar{g}_{ab}, \bar{K}_{ab})$ such that Σ is an m-dimensional manifold, $\bar{g}_{ab}$ is a Riemannian metric on Σ, $\bar{K}_{ab}$ is a symmetric 2-tensor on Σ and the following constraint equations are satisfied:

$$\begin{aligned} \nabla[\bar{g}]^b\bar{K}_{ab} - \nabla[\bar{g}]_a(\bar{g}^{bc}\bar{K}_{bc}) &= 0\,, \\ \mathrm{R}[\bar{g}] - \bar{K}_{ab}\bar{K}^{ab} + (\bar{g}^{bc}\bar{K}_{bc})^2 &= 0\,. \end{aligned} \tag{1.1.1}$$

The following theorem tells us that the above geometric initial data is sufficient to produce a globally hyperbolic Einstein spacetime. The result was shown by Choquet-Bruhat in [FB52] with later modifications concerning maximality added in [CBG69].

Theorem 1.6 (Local well-posedness of the vacuum Einstein equations)**.** *Let $(\Sigma, \bar{g}, \bar{K})$ be smooth geometric vacuum initial data. Then there exists a unique (up to diffeomorphism) smooth globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$, called the development, such that: $(\mathcal{M}, g)$ is Ricci flat, $(\Sigma, \bar{g})$ embeds as a Cauchy hypersurface into $(\mathcal{M}, g)$ with second fundamental form $\bar{K}$, the pull-back of the solution g to Σ is $\bar{g}$, and the development is maximal in the sense that any other smooth Ricci-flat spacetime with Σ a Cauchy hypersurface embeds isometrically into $\mathcal{M}$.*

The theorem also holds with appropriate matter fields and to initial data of lower regularity. For the first step towards theorem 1.6, we introduce a quite general gauge condition made with respect to $\hat{e}_{\mu\nu}$ [HE73], since this is used (in some form) in all but two of the later chapters.

Definition 1.7 ($\hat{e}$-wave gauge)**.** Define V^γ in local coordinates by

$$V^\gamma = g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta}[g] - \Gamma^\gamma_{\alpha\beta}[\hat{e}])\,. \tag{1.1.2}$$

Define also $V_\lambda = g_{\lambda\beta}V^\beta$. We say that g is in the $\hat{e}$-wave gauge if and only if

$$V^\gamma = 0\,. \tag{1.1.3}$$

The difference of two Christoffel symbols is a tensor, and so V^γ is a well-defined vector field on $\mathcal{M}$. Since V^γ is tensorial, and thus a coordinate-independent quantity, it is unclear how the condition (1.1.3) breaks the diffeomorphism covariance of the Einstein equations. However, we should really interpret (1.7) locally, with $\hat{e}$ being defined with respect to a fixed, local coordinate chart which we use to construct the $\hat{e}$-wave gauge coordinates.

To make this more precise, we recall that coordinate transformations and diffeomorphisms are two sides of the same coin. On the one hand, a chart $\varphi : \mathcal{U} \subseteq \mathcal{M} \to \mathbb{R}^{1+m}$ defines coordinate functions $\{x^\mu\}$ that label points $p \in \mathcal{U}$ on the manifold by a vector $\varphi(p)$ in $\mathbb{R}^{1+m}$. Another chart $\psi : \mathcal{U}' \subseteq \mathcal{M} \to\subseteq \mathbb{R}^{1+m}$ will define other coordinates and in the overlap region $\mathcal{U} \cap \mathcal{U}'$ we can move between the two coordinates by the transition map $\psi \circ \varphi^{-1}$. An alternative view of this process is to move the points on the manifold by a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ and evaluate the points $p \in \phi^{-1}(\mathcal{U})$ in terms of the original coordinates $\{x^\mu\}$. Thus the new coordinates are defined by the pullback maps $\{y^\mu = \phi_* x^\mu\}$, and the transition map is defined for points $p \in \phi^{-1}(\mathcal{U}) \cap \mathcal{U}$.

Keeping in mind the discussion in the previous paragraph, it turns out that condition (1.1.3) in fact identifies coordinates such that the identity diffeomorphism $\mathbf{i}$: $(\mathcal{M}, g_{\mu\nu}) \to (\mathcal{M}, \hat{e}_{AB})$ satisfies a particular property. To see this, we must first recall that a mapping between manifolds $f : (\mathcal{M}, g_{\mu\nu}) \to (\hat{\mathcal{M}}, \hat{e}_{AB})$, given in local coordinates by $\{x^\mu\} \mapsto \{y^A = f^A(x^\mu)\}$, is called a harmonic map if

$$g^{\mu\nu}(\partial_\mu\partial_\nu f^A - \Gamma[g]^\rho_{\mu\nu}\partial_\rho f^A + \Gamma[\hat{e}]^A_{BC}\partial_\mu f^B \partial_\nu f^C)|_{p\in\mathcal{M}} = 0. \tag{1.1.4}$$

By identifying $(\hat{\mathcal{M}}, \hat{e}_{AB}) = (\mathcal{M}, \hat{e}_{\mu\nu})$ and A with μ, a calculation then shows that condition (1.1.3) is equivalent to specifying coordinates $\{x^\mu\}$ such that the identity diffeomorphism $\mathbf{i} : (\mathcal{M}, g_{\mu\nu}) \to (\mathcal{M}, \hat{e}_{AB})$ satisfying $\partial_\mu \mathbf{i}^A = \delta^A_\mu$ is a harmonic map [CB09, VI§7.4]. Harmonic maps with a Lorentzian domain are often called wave maps, hence the gauge condition (1.1.3) is also often called a wave map gauge.

Having said all this, the easiest perspective of definition 1.7 comes when we have a foliation. In such a situation, we can choose lapse and shift functions in a way compatible with (1.1.3) and these variables clearly specify a preferred way to measure space and time. This is done later in (1.1.19).

If the spacetime $(\mathcal{M}, g)$ satisfies the vacuum Einstein equations and (1.1.3) is satisfied throughout the spacetime, then the following 'reduced' equations hold.

Definition 1.8 (Reduced Einstein equations)**.** A Lorentzian manifold $(\mathcal{M}, g)$ is defined to be a solution of the reduced Einstein equations if and only if

$$\begin{aligned} g^{\alpha\beta}\nabla[\hat{e}]_\alpha\nabla[\hat{e}]_\beta g_{\mu\nu} - g^{\gamma\delta}\Big(g_{\mu\lambda}\hat{e}^{\lambda\rho}\,\mathrm{Riem}[\hat{e}]_{\rho\gamma\nu\delta} + g_{\nu\lambda}\hat{e}^{\lambda\rho}\,\mathrm{Riem}[\hat{e}]_{\rho\gamma\mu\delta}\Big) \\ = \mathcal{F}_{\mu\nu}[g](\nabla[\hat{e}]g, \nabla[\hat{e}]g), \end{aligned} \tag{1.1.5a}$$

where the quadratic nonlinearity is defined as

$$\begin{aligned} \mathcal{F}_{\mu\nu}[g](\nabla[\hat{e}]g, \nabla[\hat{e}]g) = g^{\gamma\delta}g^{\alpha\beta}\Big(\nabla[\hat{e}]_\nu g_{\delta\beta}\nabla[\hat{e}]_\alpha g_{\mu\gamma} + \nabla[\hat{e}]_\mu g_{\gamma\alpha}\nabla[\hat{e}]_\beta g_{\nu\delta} \\ - \tfrac{1}{2}\nabla[\hat{e}]_\nu g_{\delta\beta}\nabla[\hat{e}]_\mu g_{\gamma\alpha} + \nabla[\hat{e}]_\gamma g_{\mu\alpha}\nabla[\hat{e}]_\rho g_{\mu\alpha} - \nabla[\hat{e}]_\gamma g_{\mu\alpha}\nabla[\hat{e}]_\beta g_{\nu\delta}\Big). \end{aligned} \tag{1.1.5b}$$

There are two key properties we should glean from (1.1.5). The first is that the equations are hyperbolic wave equations for the metric components $g_{\mu\nu}$ and so we should be able to make sense of an initial value problem. The second point is that the right hand side nonlinearity is quadratic in derivatives of the metric components $\nabla[\hat{e}]_\gamma g_{\mu\nu}$.

A very useful identity relating a solution of the reduced Einstein equations with the standard geometric Einstein equations is the following. For a solution $(\mathcal{M}, g)$ to the reduced Einstein equations (1.1.5) the following identity holds

$$\mathrm{Ric}[g]_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{R}[g] = -\nabla[\hat{e}]_{(\mu}V_{\nu)} + \frac{1}{2}g_{\mu\nu}\nabla[\hat{e}]_\gamma V^\gamma. \tag{1.1.6}$$

The proof follows by working at an arbitrary point in normal coordinates such that $\Gamma[\hat{e}] \equiv 0$. This identity is used (see (1.1.21)) to show that a solution to the reduced Einstein equations, under particular assumptions of the initial data, also satisfies the vacuum Einstein equations. Before we address the Cauchy problem for the reduced Einstein equations (1.1.5) we first comment on a special case of the $\hat{e}$-wave gauge.

Example 1.9 (Wave gauge)**.** If $\mathcal{M}$ admits a global coordinate system $\{y^\alpha\}$, then we can define $\hat{e}$ to be the Minkowski metric with respect to this coordinate system, i.e. $\hat{e} = \eta_{\alpha\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$. Consequently in these coordinates $\Gamma[\hat{e}] \equiv 0$ and (1.1.3) becomes

$$g^{\alpha\beta}\Gamma^{\gamma}_{\ \alpha\beta}[g] = 0\,. \tag{1.1.7}$$

By moving to the wave map point of view, it can be shown that (1.1.7) is equivalent to

$$g^{\alpha\beta}\nabla[g]_\alpha\nabla[g]_\beta y^\gamma = 0\,. \tag{1.1.8}$$

If the metric g, written with respect to particular coordinates, satisfies (1.1.7), either globally or locally, then we say it is in wave (also harmonic, de Donder) gauge and we call such coordinates wave (also harmonic, de Donder) coordinates. An example of a Lorentzian manifold admitting global coordinates is Minkowski spacetime $(\mathbb{R}^{1+n}, \eta)$.

Not all manifolds admit global coordinates. Nonetheless locally around any point $p \in \mathcal{M}$ there exists a neighbourhood and geodesic coordinates with respect to which the condition $\Gamma[\hat{e}] \equiv 0$ holds and (1.1.7) is satisfied. This local condition was used in the original proof of theorem 1.6. Furthermore if $(\mathcal{M}, g)$ satisfies the vacuum Einstein equations and is in wave gauge, then the reduced Einstein equations (1.1.5) simplify to

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = \mathcal{F}_{\mu\nu}[g](\partial g, \partial g). \tag{1.1.9}$$

where $\mathcal{F}_{\mu\nu}$ is given in (1.1.5b) (with $\nabla[\hat{e}]$ replaced by ∂ since $\Gamma[\hat{e}] \equiv 0$).

Local existence for quasilinear wave equations

The reduced Einstein equations (1.1.5) (also (1.1.9)) are quasilinear second-order hyperbolic equations for the metric components $g_{\mu\nu}$. The behaviour of a linear PDE is significantly dominated by its principal part. Quasilinear here means that the PDE is linear in its highest derivative terms, and consequently many results of linear systems apply at least locally in time. The following quite general result for local in time existence for quasilinear wave equations dates back to work of Leray where it was stated on general manifolds [Ler53]. For simplicity we state here the result on $\mathbb{R}^n$ given in [Rin09a, §9].

Proposition 1.10 (Local-in-time existence for geometric wave equations)**.** *Consider on $\mathbb{R}^{1+n}$ the Cauchy problem for the following quasilinear second-order hyperbolic PDE,*

$$\begin{aligned} G^{\mu\nu}(t,x,\psi)\partial_\mu\partial_\nu\psi &= F(t,x,\psi,\partial_\gamma\psi), \\ \psi(0,\cdot) &= \phi_0, \\ \partial_t\psi(0,\cdot) &= \phi_1. \end{aligned} \tag{1.1.10}$$

Here, for all $(t,x) \in \mathbb{R}^{1+n}$, G is a Lorentzian metric on $\mathbb{R}^{1+n}$. We assume F and all components of G are sufficiently regular with bounded derivatives (see [Rin09a, §9] for a precise statement). If $\mathbb{N} \ni N > n/2 + 1$ and $(\phi_0, \phi_1) \in H^{N+1}(\mathbb{R}^n) \times H^N(\mathbb{R}^n)$, then:

(i) Existence and uniqueness: there exists a $T > 0$, depending only on $\|\phi_0\|_{H^{N+1}}$ and $\|\phi_1\|_{H^N}$, such that there is a unique classical solution $u \in C^2([0,T] \times \mathbb{R}^n)$ to (1.1.10) *and such that the following maps are continuous*

$$t \in [0,T] \mapsto u(t,\cdot) \in H^{N+1}(\mathbb{R}^n), \tag{1.1.11}$$

$$t \in [0,T] \mapsto \partial_t u(t,\cdot) \in H^N(\mathbb{R}^n). \tag{1.1.12}$$

(ii) Domain of dependence: if $\Omega \subseteq \mathbb{R}^n$ and $(\phi_0, \phi_1)|_\Omega = 0$, then $\psi \equiv 0$ in the future domain of dependence $D^+(\Omega) \cap ([0,T] \times \mathbb{R}^n)$. Note that $D^+(\Omega)$ precisely consists of points connected to Ω by past timelike curves, and timelike here is with respect to the Lorentzian metric $G^{\alpha\beta}(x,\psi)$.

(iii) Continuation criterion: If we set T_N to be the supremum over all such times T such that there is a solution ψ defined on $[0,T] \times \mathbb{R}^n$ satisfying part (i)*, then either $T_N = +\infty$ or*

$$\lim_{t\to T_N^-}\big(\sup_{x\in\mathbb{R}^n}\sum_{|I|+|J|\le 2}|\partial_a^I\partial_t^J\psi(t,x)|\big) = \infty. \tag{1.1.13}$$

Furthermore T_N is independent of N.

(iv) Stability: suppose we have initial data $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^n)$ that are approximated by

a sequence of initial data $\varphi_0^k, \varphi_1^k \in C_0^\infty(\mathbb{R}^n)$ in the sense that

$$\lim_{k\to\infty} \|\phi_0 - \varphi_0^k\|_{H^{N+1}} + \lim_{k\to\infty} \|\phi_1 - \varphi_1^k\|_{H^N} = 0, \tag{1.1.14}$$

for some $\mathbb{N} \ni N > n/2 + 1$. Then the corresponding solution ψ is defined on some maximal time interval (T_-, T_+), and Ψ^k are defined on some maximal time intervals (T_-^k, T_+^k). For any fixed time $t \in (T_-, T_+)$ there exists a k_0 such that for all $k \geq k_0$ the approximate solution is defined there, i.e. $t \in (T_-^k, T_+^k)$, and the solution ψ is approximated by Ψ^k in the sense that

$$\lim_{k\to\infty} \|\psi(t,\cdot) - \Psi^k(t,\cdot)\|_{H^{N+1}} + \lim_{k\to\infty} \|\partial_t\psi(t,\cdot) - \partial_t\Psi^k(t,\cdot)\|_{H^N} = 0. \tag{1.1.15}$$

Let us highlight some key points of this proposition. Firstly part (i) tells us that for some small time away from the initial slice a solution exists and behaves just as smoothly as the initial data. Why should we care about initial data regularity, when, as argued in [HE73, §3.1], the 'order of differentiability of the metric is probably not physically significant'? A key motivation to understand at what point our theory breaks down, and perhaps when other physics kicks in. A prime example here is the study of low regularity solutions to the Einstein equations which are relevant to Penrose's strong cosmic censorship conjecture, see for example the discussion in [DL17, §1].

Part (ii) tells us that causality holds in $([0,T] \times \mathbb{R}^n, G_{\mu\nu})$. Part (iii) tells us that the solution either breaks down in a concrete way, or it exists forever, and as the data are made smoother the time of existence doesn't disappear to nothing. In particular if one makes the data more smooth then the solution inherits this regularity (also called 'persistence of regularity'). Finally part (iv) tells us that small perturbations of the initial data lead to small perturbations of the solution and those perturbed solutions exist for some slightly perturbed time interval. The key part is that the perturbation is measured using the natural regularity-norms of the data and solutions so that any divergent behaviour is detected and not somehow missed. A PDE satisfying parts (i) and (iv) is said to be well-posed in the sense of Hadamard.

Local existence for the reduced Einstein equations

The proof of theorem 1.6 is not just a simple application of proposition 1.10 to the reduced Einstein equations (1.1.5) with Cauchy data

$$\begin{aligned} g_{\mu\nu}|_\Sigma &= (g_0)_{\mu\nu}, \\ \partial_t g_{\mu\nu}|_\Sigma &= (g_1)_{\mu\nu}. \end{aligned} \tag{1.1.16}$$

This is obvious just by the fact that geometric initial data $(\Sigma, \bar{g}_{ab}, \bar{K}_{ab})$ specifies $m(m+1)$ functions while (1.1.16) requires $(m+1)(m+2)$ functions. The missing initial data are precisely what we can choose using the free degrees of freedom allowed by diffeomorphism covariance. One choice is to prescribe the lapse N and shift X^a on Σ and identify

$$(g_0)_{00} = -N^2, \quad (g_0)_{0a} = N^2 (g_0)_{ab} X^b\,. \tag{1.1.17}$$

We leave N an unspecified function but for simplicity set $X^a = 0$ so that ∂_t is colinear to n^μ. Following the ADM notation 1.2 we then set

$$(g_0)_{ab} = \bar{g}_{ab}, \quad (g_1)_{ab} = -2N\bar{K}_{ab}\,. \tag{1.1.18}$$

The initial data for $(\partial_t N, \partial_t X_a) = ((g_1)_{00}, (g_1)_{0a})$ is then chosen by satisfying $V^\gamma = 0$ on Σ, which amounts to satisfying the algebraic equations

$$\begin{aligned} (g_1)_{00} &= N^3 (\bar{g}^{-1})^{ab} \Gamma[\hat{e}]^0_{ab} - N^2 (\bar{g}^{-1})^{ab} \bar{K}_{ab}, \\ (g_1)_{0a} &= -N\partial_a N + N^2 (g_0)_{ad} (\bar{g}^{-1})^{bc} \Gamma^d_{bc}[\bar{g}] - N^2 (g_0)_{ac} (\bar{g}^{-1})^{de} \Gamma[\hat{e}]^c_{de} \,. \end{aligned} \tag{1.1.19}$$

We write $(\bar{g}^{-1})$ to emphasise that the indices are raised with respect to $\bar{g}$. We return to the identity (1.1.6). The Bianchi identities for g imply that the left-hand-side of (1.1.6) is divergence free with respect to $\nabla[g]^\mu$ (not $\nabla[\hat{e}]^\mu$). Thus the right-hand-side becomes

$$g^{\alpha\beta} \partial_\alpha \partial_\beta V^\gamma + A^{\mu\gamma}_\nu \partial_\gamma V^\nu + B^\gamma_\alpha V^\alpha = 0, \tag{1.1.20}$$

where $A^{\mu\gamma}_\nu, B^\gamma_\alpha$ are tensors depending on $g, \hat{e}$ and their derivatives. This is a second-order quasilinear system of wave equations for V^γ, and so proposition 1.10ii if $V^\gamma|_\Sigma = 0$ and $\partial_t V^\gamma|_\Sigma = 0$ then $V^\gamma \equiv 0$ in the Cauchy development of the initial data. However we have no more gauge freedom remaining to impose $\partial_t V^\gamma|_\Sigma = 0$. Thankfully though we can return again to (1.1.6) and contract the left-hand-side with $n^\mu Y^\nu$ on Σ where Y is orthogonal to n with respect to the metric (g_0). By the Gauss-Codazzi equations, the constraint equations (1.1.1) and the condition $V^\gamma = 0$ on Σ, the left hand side of (1.1.6) vanishes on Σ leaving us with

$$0 = \left(\mathrm{Ric}[g]_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R[g]\right) n^\mu Y^\nu|_\Sigma = -\tfrac{1}{2} n^\mu Y^a \nabla[\hat{e}]_\mu V_a|_\Sigma. \tag{1.1.21}$$

This shows $\partial_t V_a|_\Sigma = 0$. The same argument using $n^\mu n^\nu$ yields $\partial_t V_0|_\Sigma = 0$.

Multiple technical difficulties still remain however in order to transform the initial data (1.1.16) into something compatible with the $\mathbb{R}^n$ result of proposition 1.10. In particular, one must consider small patches of Σ on which the manifold locally looks like $\mathbb{R}^n$ and in local coordinates mollify the initial data (1.1.16) to satisfy the conditions of proposition 1.10. From each patch one obtains a Cauchy development, and the time of existence in this development needs to be related to both local (in space) and global Sobolev norms of the data. The Cauchy developments must then be carefully patched together, accounting for coordinate changes, in order to produce a spatially global Cauchy development. Finally the statement about continuous dependence on the initial data requires more care when the objects being perturbed are tensors (not scalars) and thus are also subject to diffeomorphism transformations. We gratefully refer to [CB09, §VI, §A.III] and [Rin09a, §14–15] for the details.

We have nevertheless broadly understood the steps behind theorem 1.6 and how the vacuum Einstein equations possess an inherent hyperbolic structure which can be exploited to produce a local spacetime solution from appropriately specified geometric data. Thus the arena is set for the study of the global stability of Einstein spacetimes.

1.2 The linear wave equation

We now aim to turn a local-in-time spacetime into a globally defined one, in the sense of future and/or past geodesic completeness. It is convenient to require that the geometric data is very close to the background spacetime, since a large perturbation of a spacetime can deform the geometry so much that a black hole forms [SY83].

The study of small-data global well-posedness for PDEs is a major and active area of reseach within mathematics beyond general relativity. We use the energy method applied to systems of quasilinear wave equations and, in Section 1.4, quasilinear Klein-Gordon equations. In order to put the main results of the thesis in context it is useful to study some of the key methods and issues that arise for these systems. To start us off we consider the following theorem, which can be found in [Sog08, Th. 1.1].

Theorem 1.11 (Decay of solutions to the linear wave equation on Minkowski spacetime)**.** *Let $n \geq 3$ and $\psi(t,x) \in C^2([0,\infty) \times \mathbb{R}^n)$ be the unique future global solution to the Cauchy problem*

$$\begin{aligned} \Box\psi &= 0, \\ \psi(0,x) &= \phi_0(x), \\ \partial_t\psi(0,x) &= \phi_1(x), \end{aligned} \tag{1.2.1}$$

where $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| < R\}$. Then the following properties hold.

(a) Finite speed of propagation: for any $t > 0$ the support of $\psi(t,\cdot)$ is contained in the ball $\{x \in \mathbb{R}^n : |x| \leq R + t\}$. Moreover if ϕ_0, ϕ_1 vanish then the solution vanishes inside the domain of dependence cone $\{(t,x) \in [0,R) \times \mathbb{R}^n : |x| \leq R - t\}$.

(b) Strong Huygen's principle: if n is odd and ϕ_0, ϕ_1 vanish on the sphere $S_t(x_0) = \{x \in \mathbb{R}^n : |x - x_0| = t\}$ of some radius $t > 0$, then $\psi(t, x_0) = 0$.

(c) Decay estimates: there exists a constant $C(n, \phi_0, \phi_1) = C$ such that for arbitrary rectangular spacetime derivatives $\partial_x^I \partial_t^J$ and $(t,x) \in [0,\infty) \times \mathbb{R}^n$

$$|\partial_x^I \partial_t^J \psi(t,x)| \leq \begin{cases} C(1+t)^{-\frac{n-1}{2}} & \text{if } n \text{ odd,} \\ C(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{n-1}{2}} & \text{if } n \text{ even.} \end{cases} \tag{1.2.2}$$

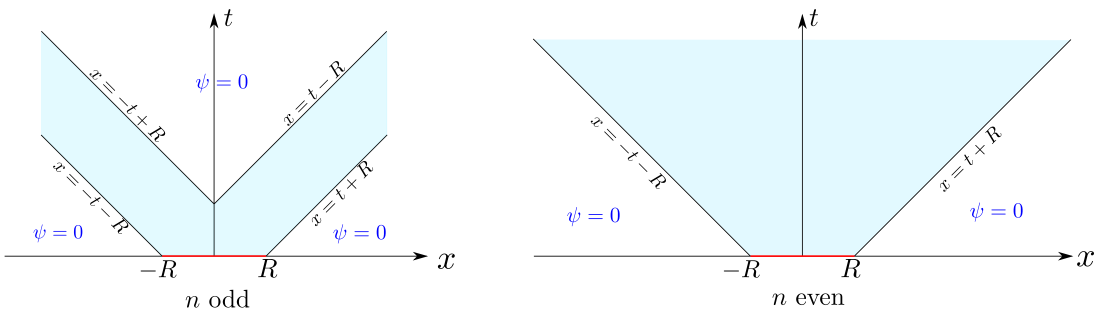


Figure 1.1: Strong and weak Huygen's principles. The blue regions indicate the support of the function emanating from data supported on the red line.

There are several important features in this theorem. The first is not stated, but using Kirchoff's representation formula the future global solution can be explicitly constructed in terms of the initial data. As expected the domain of influence of a point precisely coincides with a light cone in Minkowski centred at that point (part a). In odd dimensions a change in initial data ϕ_0, ϕ_1 at a point x_0 propagates entirely along the boundary of the light cone (part b). Without sourcing the wave on a fixed

Minkowski background has constant kinetic energy (KE_0) and so heuristically this energy is dispersed on larger and larger spheres:

$$KE_0 = \int_{\mathbb{R}^n} |\partial\psi|^2 \sim |\partial\psi|^2 \text{Area}(S_t(x_0)) \sim |\partial\psi|^2 t^{n-1}. \tag{1.2.3}$$

This agrees with (1.2.2). Although the behaviour concerning strict propagation along the light cone is lost in even dimensions the factor of $(1+|t-|x||)$ in (1.2.2) compensates by giving additional decay away from the wave zone $\{t \sim r\}$. Indeed the decay estimates of part c confirm the idea that if there are more directions (i.e. dimensions) for a wave to move into then its amplitude decays faster.

Comparing theorem 1.11 with proposition 1.10 illustrates some of the changes that occur when a nonlinearity is included. For one thing, the representation formula and strong Huygen's principle of solutions to the free wave equation are lost when new sources and nonlinearities are included. However the finite speed of propagation in theorem 1.11a (also called the weak Huygen's principle) also holds in proposition 1.10ii. The decay properties of theorem 1.11c, as the linear and hence predominant contributing factor to a quasilinear wave, play an essential role in establishing global results for quasilinear wave equations.

1.3 Nonlinear wave equations

We now state three important theorems concerning the existence (or non-existence!) of small data solutions to three nonlinear wave equations.

Theorem 1.12 (Global existence for semilinear wave equations). *Let $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^n)$ and consider on $[0,\infty)\times\mathbb{R}^n$ the Cauchy problem*

$$\begin{aligned} \Box\psi &= F(\partial\psi),\\ \psi(0,x) &= \varepsilon\phi_0(x),\\ \partial_t\psi(0,x) &= \varepsilon\phi_1(x), \end{aligned} \tag{1.3.1}$$

where $\partial\psi = (\partial_t\psi,\dots,\partial_n\psi)$ denotes the full spacetime gradient and $|F(\partial\psi)| \le C|\partial\psi|^p$ for $\mathbb{Z} \ni p \ge 2$. If $\varepsilon > 0$ is sufficiently small and either $n \ge 4$ or $(n,p) = (3,3)$ then (1.3.1) *has a unique global smooth solution that decays as*

$$\begin{aligned} |\partial\psi(t,x)| &\lesssim \varepsilon(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}},\\ |\psi(t,x)| &\lesssim \varepsilon(1+t)^{-\frac{n-1}{2}}, \quad n \text{ odd.} \end{aligned} \tag{1.3.2}$$

Remark 1.13. The assumption about smallness is necessary. The decay rate (1.3.2) is somewhat crude here, however the statement when $n = 3$ is relevant to later discussions.

Theorem 1.12 is known to fail when $n = 3$ and $p = 2$ due to the following example of John.

Theorem 1.14 (Semilinear wave equation with finite time blow-up [Joh81]). *Consider on $\mathbb{R}^{1+3}$ the Cauchy problem*

$$\begin{aligned} \Box\psi &= (\partial_t\psi)^2,\\ \psi(0,x) &= \phi_0(x),\\ \partial_t\psi(0,x) &= \phi_1(x). \end{aligned} \tag{1.3.3}$$

Then every non-trivial C^3 solution emanating from smooth data $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^3)$ blows up in finite time.

Clearly in three spatial dimensions additional structure is needed on quadratic derivative nonlinearities (i.e. $p = 2$). The following theorem was proved independently by Klainerman [Kla86] and Christodoulou [Chr86] and gives one important class of nonlinearities, called null forms, with good structure and hence stability of the trivial solution in the case $n = 3, p = 2$. Note that the reduced Einstein equations (1.1.9) about a Minkowski background $(\mathbb{R}^{1+3}, \eta)$ involve quadratic nonlinearities which do not have *a priori* good null structure, see the later discussion in Section 1.6.

Theorem 1.15 (Small-data global existence for a null form)**.** *Let $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^3)$ and consider on $\mathbb{R}^{1+3}$ the Cauchy problem*

$$\begin{aligned} \Box\psi &= (\partial_t\psi)^2 - \textstyle\sum_{a=1}^3 (\partial_a\psi)^2 + F(\psi, \partial\psi), \\ \psi(0,x) &= \varepsilon\phi_0(x), \\ \partial_t\psi(0,x) &= \varepsilon\phi_1(x), \end{aligned} \tag{1.3.4}$$

where $\partial\psi = (\partial_t\psi, \dots, \partial_n\psi)$ denotes the full spacetime gradient and $F(\psi, \partial\psi)$ denotes a cubic nonlinearity $\mathcal{O}(|\psi|^3 + |\partial\psi|^3)$. If $\varepsilon > 0$ is sufficiently small then (1.3.4) *has a unique global smooth solution.*

1.3.1 The vector-field method: Klainerman-Sobolev inequalities

We now outline a proof of theorem 1.12, in the present subsection 1.3.1 and in subsection 1.3.2. In doing so we illustrate the vector-field method, a powerful method used to show the stability of quasilinear wave equations which will be endemic throughout the thesis. The method uses L^2 energy estimates (themselves classical tools in PDE analysis) which are derived from the method of multipliers, together with L^∞ (dispersive) estimates derived from weighted and commuting vector fields. An influential presentation of these ideas appeared in [Kla86]. We follow the proof as given in [Sog08, §II].

Firstly we know from theorem 1.10iii that we can construct a smooth solution $\psi(t, x)$ on some maximal interval $[0, T_*) \times \mathbb{R}^n$ for $0 < T_*(\varepsilon) < \infty$. This is characterised, for a given ε, by the blow-up of the norm

$$\sum_{|I|+|J|\le 2} |\partial_a^I \partial_t^J \psi(t,x)| \notin L^\infty([0, T_*) \times \mathbb{R}^n). \tag{1.3.5}$$

In order to obtain the pointwise control that will allow us to extend the solution ψ to $[0, T_*] \times \mathbb{R}^n$, typically one uses a Sobolev inequality to convert the pointwise norms into L^2 norms which then obey certain conservation properties due to the PDE.

Lemma 1.16 (Sobolev Inequality)**.** *For every $\mathbb{N} \ni k > n/p$ there exists a constant $C = C(n, k) > 0$ such that for all $\phi \in W^{k,p}(\mathbb{R}^n)$*

$$\sup_{x\in\mathbb{R}^n} |\phi| = \|\phi\|_{L^\infty(\mathbb{R}^n)} \le C\|\phi\|_{W^{k,p}(\mathbb{R}^n)}. \tag{1.3.6}$$

Let us heuristically interpret lemma 1.16. For $k \in \mathbb{N}$ and $1 \le p < \infty$ the inhomogeneous Sobolev space $W^{k,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|\phi\|_{W^{k,p}} = \Big(\int_{\mathbb{R}^n} \sum_{0\le s\le k} |\partial^s\phi|^p \mathrm{d}^n x\Big)^{1/p}. \tag{1.3.7}$$

We refer to [Rin09a] for full definitions. Very roughly one can interpret a Sobolev space as measuring the amplitude A, support V and (single) frequency ω of a function ϕ as

$$\|\phi\|_{W^{k,p}} \sim (A^p\omega^{kp}V)^{1/p} = A\omega^k V^{1/p}. \tag{1.3.8}$$

The uncertainty principle states that a function of frequency ω must be spread out on a ball of radius ω^{-1}, i.e. support of volume $V \gtrsim \omega^{-n}$. Putting these together gives

$$A\omega^k V^{1/p} \gtrsim A\omega^{k-n/p} \gtrsim A \sim |\phi|, \tag{1.3.9}$$

which is as given in lemma 1.16 (note for inhomogeneous Sobolev spaces one essentially has $\omega \gtrsim 1$). Note also that (1.3.8) is dimensionful and so the constant in (1.3.6) is also.

In this thesis we primarily work with the Hilbert spaces $H^k = W^{k,2}$ and apply (1.3.6) to show that there exists a $C = C(n) > 0$ such that for all functions $\phi \in H^{\frac{n}{2}+1}(\mathbb{R}^n)$:

$$\sup_{x\in\mathbb{R}^n} |\phi(x)| \leq C \sum_{|I|\leq \frac{n}{2}+1} \|\partial_a^I \phi(\cdot)\|_{L^2(\mathbb{R}^n)}. \tag{1.3.10}$$

Unfortunately the estimate 1.3.10 does not encode any of the decay properties we see for the linear wave equation in theorem 1.11c. Such information is provided by the Klainerman-Sobolev inequality. The idea is to replace the rectangular spatial vector fields ∂_a used in (1.3.10) with the following larger collection of vector fields that are both weighted in t and x and preserve the equation $\Box\psi = 0$.

Definition 1.17 (Minkowski invariant vector fields)**.** Define respectively the generators of spacetime translations, spatial rotations on constant t level sets, and Lorentz boosts

$$\partial_t\,, \partial_a, \qquad 1 \leq a \leq n, \tag{1.3.11a}$$

$$\Omega_{ab} = \eta_{bc}x^c\partial_a - \eta_{ac}x^c\partial_b, \qquad 1 \leq a < b \leq n, \tag{1.3.11b}$$

$$\Omega_a = \eta_{ab}x^b\partial_0 + x^0\partial_a, \qquad 1 \leq a \leq n. \tag{1.3.11c}$$

These vector fields commute with the Minkowski wave operator $\Box$. Define the generator of spacetime dilations, also called the radial vector field since its integral curves are the rays through the origin, as

$$S = t\partial_t + \sum_{a=1}^n x^a\partial_a, \tag{1.3.11d}$$

for which $[\Box, S] = 2\Box$. The $(2n + 2 + \frac{1}{2}n(n-1))$-dimensional set of these vector fields is denoted

$$\mathbf{\Gamma} = \{\partial_\mu\,, \Omega_{ab}\,, \Omega_a\,, S\}\,. \tag{1.3.12}$$

Thus from solutions to $\Box\psi = 0$ we can generate new solutions $\Box\Gamma^I\psi = 0$. Note also that for arbitrary $\Gamma_i, \Gamma_j \in \Gamma$ we have $[\Gamma_i, \Gamma_j] = \sum_k c_{ijk}\Gamma_k$ for constants c_{ijk}.

Remark 1.18. We aim to keep notation consistent throughout this thesis. Sometimes Ω_a is written as Ω_{0a} (e.g. [LR10, Sog08]), as Z_{0a} (e.g. [Hör97]), or as L_a (e.g. [LM16a]). We prefer to reserve L for the vector field generating the forward Minkowski light cones, and thus introduce the notation Ω_a for the boosts.

The boosts and rotations form a representation of the Lie algebra of the Lorentz group, while the full set (1.3.11) forms a representation of the Lie algebra of the Poincaré group, the group of Minkowski spacetime isometries. If one were to consider the wave equation on a spacetime without a maximal symmetry group then such a large collection would not exist. Using the $\mathbf{\Gamma}$ vector fields instead of merely ∂ we obtain the following Sobolev-type inequality [Kla84, Kla85b].

Theorem 1.19 (Klainerman-Sobolev Inequality)**.** *There exists a constant $C = C(n) > 0$ such that for all functions $\phi(t,\cdot) \in H^{\frac{n}{2}+1}(\mathbb{R}^n)$ and for all $t \geq 0$:*

$$\sup_{x\in\mathbb{R}^n} (1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|\phi(t,x)| \leq C \sum_{|I|\leq\frac{n}{2}+1, \Gamma\in\mathbf{\Gamma}} \|\Gamma^I\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)}. \quad (1.3.13)$$

Recall our goal is to show that, for sufficiently small ε, $T_* = +\infty$. By applying (1.3.13) and remark 1.22 to (1.3.5), we see that it will suffice to prove

$$\sup_{0\leq t<T_*} \sum_{|I|\leq\frac{n}{2}+3} \|\partial\Gamma^I\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty. \quad (1.3.14)$$

By multiplying the PDE (1.3.1) by $\partial_t\psi$ and integrating (by parts) the resulting expression over $[0,t]\times\mathbb{R}^n$ we obtain the following energy identity for all $0 \leq t < T_*$

$$\|\partial\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|\partial\psi(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^t \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau. \quad (1.3.15)$$

Note the squared norm $\|\partial\psi\|^2_{L^2}$ measures the wave's kinetic energy

$$\|\partial\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = \int_{\{t\}\times\mathbb{R}^n} \Big(|\partial_t\psi|^2 + \sum_{a=1}^n |\partial_a\psi|^2\Big)\mathrm{d}^n x. \quad (1.3.16)$$

If there is no sourcing, i.e. $F = 0$, then (1.3.15) would imply that the wave has constant kinetic energy. Although equation (1.3.16) does not control the wave amplitude $\|\psi\|_{L^2}$, it is (up to squaring) equal to the $|I| = 0$ term of (1.3.14). Thus we define the following higher-order energy

$$\mathcal{E}_N(t) = \sum_{|I|\leq N} \|\partial\Gamma^I\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}. \quad (1.3.17)$$

By commuting the vector fields Γ^I through the PDE, where the total number N of these vector fields is yet to be fixed, the energy identity (1.3.15) yields the following for all $0 \leq t < T_*$

$$\mathcal{E}_N(t)^{1/2} \lesssim \mathcal{E}_N(0)^{1/2} + \sum_{|I|\leq N}\int_0^t \|\Gamma^I F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau. \quad (1.3.18)$$

1.3.2 The vector-field method: the bootstrap argument

In this section we complete the proof of Theorem 1.12. We begin by stating two useful lemmas. The first is an important estimate called Grönwall's inequality. It is a key tool used throughout the study of PDEs. We refer to [Sog08] for a proof.

Lemma 1.20 (Grönwall's inequality)**.** *Let $F(t), A(t), B(t) : [0,T] \to [0,\infty)$.*

1. Differential form: assume $F(t)$ is differentiable and satisfies

$$F'(t) \leq A(t) \cdot F(t), \quad t \in [0,T]. \tag{1.3.19}$$

Then, F also satisfies

$$F(t) \leq F(0) \cdot \exp\Big(\int_0^t A(s)\,\mathrm{d}s\Big). \tag{1.3.20}$$

2. Integral form: assume $F(t)$ is continuous and satisfies

$$F(t) \leq B(t) + \int_0^t A(s)F(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{1.3.21}$$

and $B(t)$ is non-decreasing on $[0,T]$. Then, F also satisfies

$$F(t) \leq B(t) \cdot \exp\Big(\int_0^t A(s)\,\mathrm{d}s\Big). \tag{1.3.22}$$

The final step to complete the proof of Theorem 1.12 is a bootstrap, or continuity, argument. This involves first assuming the required estimate and then proving a strictly better estimate. The following lemma justifies this approach.

Lemma 1.21. *Fix a constant $A > 0$ and let $F(t) : [0,T_*) \to [0,\infty)$ be continuous, where $0 < T_* \leq \infty$. Suppose $F(0) \leq A$, and assume the following holds for all $0 \leq T < T_*$*

1. If $F(t) \leq 4A$ for each $t \in [0,T]$, then in fact $F(t) \leq 2A$.

Then $F(t) \leq 4A$ (and hence $F(t) \leq 2A$) for all $t \in [0,T_*)$.

Proof. Define the set $S = \{T \in [0,T_*) : F(t) \leq 4A \text{ for all } 0 \leq t \leq T\}$. The set S is nonempty since $0 \in S$, and it is relatively closed in $[0,T_*)$ since F is continuous. For a given $T \in S$, assumption (1) implies that $F(t) \leq 2A$ for all $0 \leq t \leq T$. Since $F(t)$ is continuous it follows that $T+\delta \in S$ for small enough $|\delta|$. Thus S is a nonempty, closed and open subset of the connected set $[0,T_*)$ and hence $S = [0,T_*)$. □

We return now to the proof of theorem 1.12. For some constant $C_0 > 0$ we have

$$\mathcal{E}_N(0)^{1/2} \leq C_0\varepsilon. \tag{1.3.23}$$

Since $\mathcal{E}_N(0)$ depends only the initial data ϕ_0, ϕ_1 we can choose C_0 independent of ε. For $C_1 \gg C_0$ another large constant, assume that

$$\sup_{0\leq t\leq T} \mathcal{E}_N(t)^{1/2} \leq C_1\varepsilon, \tag{1.3.24}$$

for some fixed $0 < T < T_*$. We now return to the higher-order energy inequality (1.3.18). By distributing and commuting derivatives (note $[\partial,\Gamma] \propto \partial$ and $\mathbb{Z} \ni p \geq 2$) we find

$$\begin{aligned} &\sum_{|I|\leq N} \int_0^t \|\Gamma^I F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau \\ &\quad \leq C\int_0^t \sum_{|J|\leq \frac{N}{2}+1} \|\partial\Gamma^J\psi(\tau,\cdot)\|^{p-1}_{L^\infty(\mathbb{R}^n)} \cdot \sum_{|I|\leq N} \|\partial\Gamma^I\psi(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau. \end{aligned} \tag{1.3.25}$$

We apply the Klainerman-Sobolev inequality to the L^∞ terms above, provided that $N \in \mathbb{N}$ is large enough to absorb the right-hand-side of the Klainerman-Sobolev inequality into $\mathcal{E}_N$ (i.e. provided $\frac{N}{2}+1+\frac{n}{2}+1 \leq N$). By considering now the higher-order energy inequality (1.3.18), together with our assumption (1.3.24), we have for any $0 \leq t \leq T$,

$$\mathcal{E}_N(t)^{1/2} \leq C\mathcal{E}_N(0)^{1/2} + C(C_1\varepsilon)^{p-1} \int_0^t \frac{\mathcal{E}(\tau)^{1/2}}{(1+\tau)^{\frac{(n-1)(p-1)}{2}}} \mathrm{d}\tau. \tag{1.3.26}$$

Since $\mathcal{E}_N(t)^{1/2} \in C^0([0,T_*))$, we can apply the integral form of Grönwall's inequality from lemma 1.20. In particular we have $\mathcal{E}_N(t)^{1/2} \leq C(C_0\varepsilon)$ for all $0 \leq t \leq T$ if

$$C(C_1\varepsilon)^{p-1} \int_0^t (1+\tau)^{-\frac{(n-1)(p-1)}{2}} \mathrm{d}\tau \leq \ln 2. \tag{1.3.27}$$

We can choose ε sufficiently small to enforce (1.3.27), provided that n and p satisfy

$$\frac{(n-1)(p-1)}{2} > 1. \tag{1.3.28}$$

Thus, after possibly increasing C_1, we have shown that for sufficiently small ε

$$\sup_{0\leq t \leq T} \mathcal{E}_N(t)^{1/2} \leq C(C_0\varepsilon) \leq \frac{1}{2}C_1\varepsilon. \tag{1.3.29}$$

To apply lemma 1.21, we now let $F(t) = \mathcal{E}_N(t)^{1/2}$, $4A = C_1\varepsilon$ and T_* be the maximal time of existence given by the local existence theorem. Thus the bootstrap argument has allowed us to show

$$\sup_{0\leq t<T_*} \sum_{|I|\leq \frac{n}{2}+3} \|\partial\Gamma^I\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty. \tag{1.3.30}$$

By a Sobolev estimate, (1.3.10) or (1.3.13), and the small point made in the following remark, this proves that $\sum_{|I|+|J|\leq 2} |\partial_a^I \partial_t^J \psi|$ is uniformly bounded on $[0,T_*) \times \mathbb{R}^n$. By the continuation criterion of theorem 1.10iii, we may extend the solution to $[0, T_*+\delta)$ for a small $\delta > 0$. This contradicts the maximality of T_*, and so we must have $T_* = +\infty$. The decay estimates stated in theorem 1.12 can be shown using the Klainerman-Sobolev inequality and the following remark.

Remark 1.22. Some care is in fact required to ensure that (1.3.30) implies

$$\sup_{(t,x)\in[0,T_*)\times\mathbb{R}^n} \sum_{|I|+|J|\leq 2} |\partial_a^I \partial_t^J \psi(t,x)| < \infty. \tag{1.3.31}$$

This is because a naive application of either (1.3.10) or (1.3.13) would indicate that we need to control terms such as $\|\Gamma^J\psi(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$ which are not included in the natural energy $\mathcal{E}_N(t)$. Since $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^n)$ we can choose $R > 0$ such that $\operatorname{supp}(\phi_0) \cup \operatorname{supp}(\phi_1) \subset \{|x| \leq R\}$. By the finite speed of propagation, $\psi(t,x) = 0$ if $0 \leq t < T_*$ and $|x| \geq R+t$. Thus for $|J| \leq 2$ the estimate

$$|\partial\Gamma^J\psi(t,x)| \leq C(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}} \sum_{|I|\leq\frac{n}{2}+3} \|\partial\Gamma^I\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}, \tag{1.3.32}$$

can be integrated up in the direction $u = t - r$ to give, for all $(t, x) \in [0, T_*) \times \mathbb{R}^n$,

$$|\Gamma^J \psi(t,x)| \leq C(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}} \sum_{|I| \leq \frac{n}{2}+3} \|\partial \Gamma^I \psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty. \quad (1.3.33)$$

The finite speed of propagation, essentially proposition 1.10ii, is a major feature for hyperbolic PDEs. Indeed theorem 1.12 still holds if we considered a perturbed principal operator $\Box + h^{\mu\nu}(\partial\psi)\partial_\mu\partial_\nu$ for some appropriate conditions on $h^{\mu\nu}$. Although the light cones and domain of dependence for the quasilinear equation are then slightly shifted from the background Minkoski geometry, the hyperbolic property of finite propagation is still present and plays a key role.

1.3.3 Alternative vector fields

To derive (1.3.15) we multiply the PDE by $X\psi = \partial_t \psi$ and integrate by parts to obtain the identity

$$\int_0^t \int_{\mathbb{R}^3} X\psi \cdot \Box\psi = \mathcal{E}_0(t) - \mathcal{E}_0(0). \quad (1.3.34)$$

Typically one calls X the multiplier. If X is a vector field, then we have the more natural interpretation via a conserved current. Let $T[\psi]_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\psi\partial_\rho\psi$. If X is Killing with respect to η and ψ is a solution to the linear wave equation then, by Stokes' theorem, we have the identity

$$\begin{aligned} 0 &= \int_0^t \int_{\mathbb{R}^3} \Big(X\psi \cdot \Box\psi + \tfrac{1}{2} T[\psi]^{\mu\nu} (\mathscr{L}_X \eta)_{\mu\nu} \Big) \mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t \int_{\mathbb{R}^3} \partial_\mu (T[\psi]^\mu{}_\nu X^\nu) \mathrm{d}x\mathrm{d}\tau \\ &= \int_{\{t\}\times\mathbb{R}^3} T[\psi]^\mu{}_\nu X^\nu (\mathrm{d}t)_\mu - \int_{\{0\}\times\mathbb{R}^3} T[\psi]^\mu{}_\nu X^\nu (\mathrm{d}t)_\mu. \end{aligned} \quad (1.3.35)$$

Thus the energy identity arises from the conserved current $T[\psi]^\mu{}_\nu X^\nu$. Even in the inhomogeneous case the above construction can still lead to useful estimates, as in (1.3.15). There is a beautiful interplay between a good choice of X and the geometry of the region of integration, see for example [DR13]. The use of vector fields as *multipliers* dates back at least to work of Morawetz who used each of the following vector fields

$$\begin{aligned} S &= t\partial_t + r\partial_r, \\ K_0 &= (r^2 + t^2)\partial_t + 2rt\partial_r, \\ M &= \partial_r, \end{aligned} \quad (1.3.36)$$

to derive decay rates for different wave and Klein-Gordon equations in [Mor61, Mor62] and [Mor68] respectively. Note, by contrast, that the use of vector fields as *commutators* dates back at least to work of Klainerman [Kla84, Kla85b].

K_0 is often called the conformal Morawetz vector field[1] and satisfies $\mathscr{L}_{K_0}\eta = 4t\eta$. Even though the wave equation is only conformally invariant in one spatial dimension, the vector field K_0 can still produce useful conservation laws for the wave equation.

[1] K_0 is one of the $1+n$ vector fields $K_\mu = m_{\rho\nu}x^\rho x^\nu \partial_\mu - 2m_{\mu\nu}x^\nu x^\rho \partial_\rho$ that generate the special conformal transformations of Minkowski. Together with $\mathbf{\Gamma}$ these vector fields generate the full set conformal Killing vector fields of Minkowski spacetime.

Indeed using the multiplier $(K_0+(n-1)t)\psi$ one eventually arrives at the energy estimate

$$\mathbb{E}_{\text{con}}[\psi](t)^{1/2} \leq C\mathbb{E}_{\text{con}}[\psi](0)^{1/2} + C\int_0^t \|(r^2+\tau^2)^{1/2}F(x,\tau)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau, \tag{1.3.37}$$

for the conformal energy

$$\mathbb{E}_{\text{con}}[\psi](t) = \int_{\mathbb{R}^3}\Big(|S\psi|^2 + \sum_a |\Omega_a\psi|^2 + \sum_{a\neq b}|\Omega_{ab}\psi|^2 + |\psi|^2\Big)\mathrm{d}x. \tag{1.3.38}$$

A key feature of this energy is that it includes ψ in L^2. Using (1.3.37) the decay estimates (1.3.2), for simplicity in the homogeneous case, can be improved to

$$\begin{aligned} |\partial\psi(t,x)| &\lesssim \varepsilon(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{3}{2}}, \\ |\psi(t,x)| &\lesssim \varepsilon(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}}. \end{aligned} \tag{1.3.39}$$

In the inhomogeneous case see for example [Ali09, §6.7]. The work [MH17] establishes a parallel conformal energy using a different foliation of Minkowski spacetime which is relevant to our work in Chapter 4.

1.4 The Klein-Gordon equation

The Klein-Gordon equation reads

$$\Box\psi - m^2\psi = 0, \tag{1.4.1}$$

where $m > 0$ is a constant mass parameter. If we multiply this equation by $\partial_t\psi$ and integrate over $[0,t]\times\mathbb{R}^3$, assuming some large x decay in ψ, the naturally controlled energy functional on constant t slices is

$$\mathbb{E}_m[\psi](t) = \int_{\{t\}\times\mathbb{R}^n}\big(|\partial_t\psi|^2 + \sum_{a=1}^n|\partial_a\psi|^2 + m^2|\psi|^2\big)\mathrm{d}^nx. \tag{1.4.2}$$

Note $\mathbb{E}_m[\psi](t)$ controls the inhomogeneous Sobolev norm $H^1(\mathbb{R}^n)$

$$\mathbb{E}_m[\psi](t) \simeq \|\psi(t,\cdot)\|^2_{H^1(\mathbb{R}^n)} = \sum_{|I|\leq 1}\|\partial^I\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}. \tag{1.4.3}$$

This is in stark comparison to the energy for a wave equation which, as we have seen, only controls the homogeneous Sobolev norm $\|\psi(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} = \|\partial\psi(t,\cdot)\|_{L^2}$. Indeed handling the L^2 norm $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ is not natural for waves. However at the expense of r-weights we can obtain L^2-control on a wave in terms of $\mathbb{E}[\psi]^{1/2}$ using the following Hardy inequality. We make use of Hardy-type inequalities in Chapters 2, 4 and 6.

Lemma 1.23 (Hardy inequality)**.** *There exists a constant $C = C(n) > 0$ such that for all functions $\psi \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$ we have*

$$\big\|r^{-1}\psi\big\|_{L^2(\mathbb{R}^n)} \leq C\sum_{a=1}^n\|\partial_a\psi\|_{L^2(\mathbb{R}^n)}. \tag{1.4.4}$$

1.4.1 Nonlinear Klein-Gordon equations

The following theorem dates back to [Kla85a, Sha85]. It indicates that the special structure required in theorem 1.15 is not required for Klein-Gordon fields.

Theorem 1.24 (Global existence for semilinear Klein-Gordon equations)**.** *Let $n \geq 3$, $p \geq 2$ and consider on $[0, \infty) \times \mathbb{R}^n$ the Cauchy problem*

$$\begin{aligned} \Box\psi - m^2\psi &= F(\partial\psi), \\ \psi(0, x) &= \varepsilon\phi_0(x), \\ \partial_t\psi(0, x) &= \varepsilon\phi_1(x), \end{aligned} \tag{1.4.5}$$

where $|F(\partial\psi)| \leq C|\partial\psi|^p$, $m > 0$ and $\phi_0, \phi_1 \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| < R\}$. If $\varepsilon > 0$ is sufficiently small, then (1.4.5) *has a unique global smooth solution obeying, for $t \geq 2R$,*

$$|\psi(t, x)| \lesssim \varepsilon m^{-1} t^{-\frac{n}{2}}. \tag{1.4.6}$$

Remark 1.25. Compare the fast decay rate (1.4.6) with that of (1.3.2).

A problem occurs if we try to study equation (1.4.5) using the vector-field method. The dilation vector does not produce new solutions to the Klein-Gordon equation, and hence the Klainerman-Sobolev estimate is not useful to us. However we can convert L^∞ estimates into L^2 estimates using vector fields in the smaller set

$$\mathcal{Z} = \{\partial_\mu, \Omega_{ab}, \Omega_a\} = \Gamma \backslash \{S\}, \tag{1.4.7}$$

provided we evaluate the L^2 norms on hyperboloids. This idea to use hyperboloids dates back to work of Klainerman [Kla85a] and Hörmander [Hör97].

Definition 1.26 (Hyperboloidal foliation)**.** Define $|x|^2 = \sum_{a=1}^n (x^a)^2$ and define, in the region $t \geq |x|$, the hyperboloidal coordinates to be

$$\begin{aligned} s &= (t^2 - |x|^2)^{1/2}, \\ y &= x. \end{aligned} \tag{1.4.8}$$

Define, for $a \in \{1, \ldots, n\}$, the vector fields Y_a so that, in the hyperboloidal coordinates, they are given by

$$Y_a = \partial_{y^a}. \tag{1.4.9}$$

For $s \geq 0$, define the spacelike hyperboloidal hypersurface

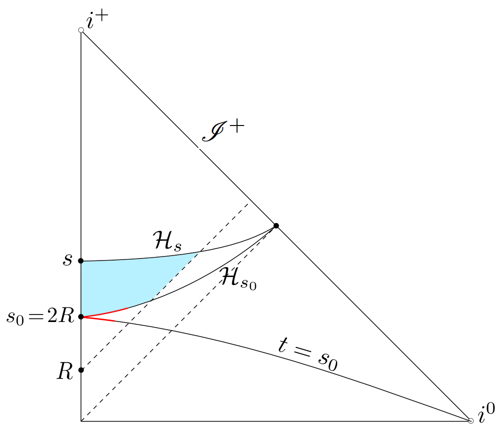
$$\mathcal{H}_s = \{(t, x) \in (0, \infty) \times \mathbb{R}^n : t^2 - x^2 = s^2\}. \tag{1.4.10}$$

The hyperboloidal radius $s > 0$ is used to label the hyperboloidal slices of the light cone on which we define energies. We then have the following Sobolev inequality, see [Hör97, §7.6].

Theorem 1.27 (Hyperboloidal Sobolev Inequality)**.** *Let $\phi \in C^\infty(\mathbb{R}^{1+n})$ be supported in $|x| < t - R$ and $s \geq 2R$. Then there exists a constant $C > 0$ such that*

$$\sup_{(t,x)\in\mathcal{H}_s} t^{\frac{n}{2}} |\phi(t, x)| \leq C \sum_{|I| \leq \frac{n}{2}+1, Z \in \mathcal{Z}} \|Z^I \phi(t, x)\|_{L^2(\mathcal{H}_s)}. \tag{1.4.11}$$

We consider the Cauchy problem of (1.4.5) at $t_0 = s_0 = 2R$. Initial data supported in $\{x \in \mathbb{R}^n : |x| < R\}$, by a finite speed of propagation argument, can be shown to lead to solutions supported in the light cone $\{(t,x) \in \mathbb{R}^{1+n} : |x| \leq t - R\}$, see also the figure.



The hyperboloidal foliation.

In a similar way to the derivation of (1.3.15) we can apply the multiplier $\partial_t\psi$ to the PDE and integrate over the region $\cup_{2R\leq s'\leq s}\mathcal{H}_{s'}$ to obtain the basic energy inequality

$$\mathcal{E}_m[\psi](s)^{1/2} \leq \mathcal{E}_m[\psi](s_0)^{1/2} + C\int_{s_0}^{s} \|Z^I F\|_{L^2(\mathcal{H}_\tau)}\mathrm{d}\tau \tag{1.4.12}$$

where the hyperboloidal energy functional takes the form

$$\mathcal{E}_m[\psi](s) = \int_{\mathcal{H}_s}\Big(|(s/t)\partial_t\psi|^2 + \sum_{a=1}^{n}|Y_a\psi|^2 + m^2|\psi|^2\Big)\mathrm{d}^n x \tag{1.4.13}$$

$$= \int_{\mathcal{H}_s}\Big(|\partial_t\psi|^2 + \sum_{a=1}^{n}|\partial_a\phi|^2 + 2\sum_a \frac{x^a}{t}\partial_t\psi\partial_a\psi + m^2|\psi|^2\Big)\mathrm{d}^n x. \tag{1.4.14}$$

Theorem 1.24 is proven in a similar way to theorem 1.12, with a commutator-boosted energy inequality on hyperboloids together with the hyperboloidal Sobolev inequality (1.4.11). In particular the bootstrap argument is very similar to that shown in section 1.3.2 except the application of Grönwall's inequality only requires

$$\frac{n(p-1)}{2} > 1, \tag{1.4.15}$$

instead of (1.3.28). Clearly (1.4.15) is satisfied in the critical case $n = 3, p = 2$. Although Klein-Gordon fields decay faster (1.4.6) than a wave field, their derivatives do not decay faster or enjoy direction dependent decay properties. The different decay rates of massive and massless fields reflects the different conformal structure of such particles.

1.5 Symmetries and Riemannian geometry

Having spent some time talking about wave and Klein-Gordon equations, we briefly give a short discussion here on Lorentzian and Riemannian Einstein spacetimes. This will leave us well-placed to discuss the main results of the thesis in the subsequent section.

1.5.1 Riemannian Einstein spaces

The ideas and results in this section can be found in [Bes87], see also the clear presentation in [Krö14]. It is relevant to Chapters 5 and 6.

Notation 1.28 (Compact spaces)**.** Let $(\mathcal{K}, \gamma)$ be a compact, connected, orientable smooth Riemannian manifold without boundary of dimension $d \geq 3$ with the obvious definitions for $\nabla[\gamma], \Gamma[\gamma]$ etc. Let capital Roman letters denote abstract indices $A, B \in \{1, \dots, d\}$. Define the pointwise and L^2 inner products

$$\langle u, v\rangle_\gamma = \gamma^{AC}\gamma^{BD}u_{AB}v_{CD}, \tag{1.5.1}$$

$$(u, v)_{L^2(\mathcal{K})} = \int_{\mathcal{K}} \langle u, v\rangle_\gamma \mathrm{d}\mu_\gamma, \tag{1.5.2}$$

where $u, v \in \Gamma(S^2(\mathcal{K}))$ are arbitrary symmetric two-tensors. Let $\mathscr{M}$ denote the space of all Riemannian metrics on $\mathcal{K}$ and

$$\mathscr{M}_1 = \Big\{\gamma \in \mathscr{M} : \mathrm{vol}_\gamma(\mathcal{K}) = \int_{\mathcal{K}} \mathrm{d}\mu_\gamma = 1\Big\}. \tag{1.5.3}$$

Define the total scalar curvature functional as

$$S(\gamma) = \int_{\mathcal{K}} \mathrm{R}[\gamma]\mathrm{d}\mu_\gamma, \tag{1.5.4}$$

where $\mathrm{R}[\gamma]$ is the scalar curvature, also called the Ricci scalar.

The functional $S(\gamma) : \mathscr{M} \to \mathbb{R}$ is diffeomorphism invariant. To also make it scale invariant typically it is restricted to either $\mathscr{M}_1$ or to $S(\gamma)/(\mathrm{vol}_\gamma(\mathcal{K})^{(d-2)/d})$. Under such a restriction, the critical points of $S(\gamma)$ are given by the following definition.

Definition 1.29 (Riemannian Einstein metrics)**.** A Riemannian metric $\gamma \in \mathscr{M}$ on $\mathcal{K}$ is said to be Einstein with Einstein constant $k \in \mathbb{R}$ if

$$\mathrm{Ric}[\gamma]_{AB} = k\gamma_{AB}. \tag{1.5.5}$$

If $k < 0$ then γ is called a negative Einstein metric.

A result of Hilbert then states that $\gamma \in \mathscr{M}_1$ is Einstein if and only if it is a critical point of $S(\gamma)|_{\mathscr{M}_1}$ [Hil24]. The second variation of $S(\gamma)$ requires the following definition.

Definition 1.30 (Lichnerowicz Laplacian)**.** Define the operators $\Delta_L, \mathcal{L}_\gamma$ acting on symmetric two-tensors u_{AB} by

$$(\Delta_L u)_{AB} = -\Delta_\gamma u_{AB} - 2(R[\gamma] \circ u)_{AB} + \mathrm{Ric}[\gamma]_{AC}u^C{}_B + \mathrm{Ric}[\gamma]^C{}_B u_{AC}, \tag{1.5.6}$$

$$(\mathcal{L}_\gamma u)_{AB} = -\Delta_\gamma u_{AB} - 2(R[\gamma] \circ u)_{AB}. \tag{1.5.7}$$

These operators are elliptic and self-adjoint with respect to (1.5.2). Note Δ_L is called the Lichnerowicz operator and $\mathcal{L}_\gamma$ is often denoted Δ_E and called the Einstein operator. Here $\Delta_\gamma = \gamma^{AB}\nabla[\gamma]_A\nabla[\gamma]_B$ is the standard Laplacian.

When $(\mathcal{K},\gamma)$ is Einstein note the relationship

$$\Delta_L = \mathcal{L}_\gamma + 2k \cdot \mathrm{id}. \tag{1.5.8}$$

For simplicity fix some Einstein metric $\gamma \in \mathscr{M}_1$ with Einstein constant $k \in \mathbb{R}$ which is not the standard sphere. We consider now the second variation of $S(g)|_{\mathscr{M}_1}$ at $\gamma \in \mathscr{M}_1$ in the direction of $h \in T_\gamma(\mathscr{M}_1)$. First, recall that an arbitrary two-tensor h_{AB} can be decomposed into antisymmetric, symmetric traceless and trace parts. This decomposition is orthogonal with respect to the induced metric acting on two-tensors. In an analogous way, a result of Koiso [Koi79] implies that a symmetric two-tensor $h \in T_\gamma(\mathscr{M}_1)$ can be decomposed as

$$h_{AB} = h^1_{AB} + h^2_{AB} + \bar{h}_{AB}, \tag{1.5.9}$$

where $h^1 = \frac{1}{2}\mathscr{L}_X\gamma$ for some vector field X on $\mathcal{K}$, $h^2 = f\gamma$ for some function $f \in C^\infty(\mathcal{K})$ with $\int_{\mathcal{K}} f \mathrm{d}\mu_\gamma = 0$, and $\bar{h}$ is traceless $\mathrm{tr}_\gamma \bar{h} = 0$ and transverse $\nabla[\gamma]^A \bar{h}_{AB} = 0$. Note $\bar{h}$ is called a TT-tensor. This decomposition is orthogonal with respect to the bilinear form induced by $S''_\gamma(h,h)$ and so we can consider the second variation evaluated on each component of (1.5.9) separately.

Since h^1 is a variation coming from the group of diffeomorphisms of $\mathcal{K}$ acting on γ, and $S(\gamma)$ is diffeomorphism invariant, it quite quickly follows that $S''_\gamma(h^1,h^1) = 0$. Next one can show that $S''_\gamma(h^2,h^2) \geq 0$ using Lichnerowicz's eigenvalue estimate of the smallest nonzero eigenvalue of the Laplacian [Lic58]. Thus an Einstein metric γ is a local minimum of $S(\gamma)$ when restricted to metrics of the same volume in its conformal class. Finally a calculation shows that

$$S''_\gamma(\bar{h},\bar{h}) = -\frac{1}{2}\int_{\mathcal{K}} \langle \mathcal{L}_\gamma \bar{h}, \bar{h}\rangle_\gamma \mathrm{d}\mu_\gamma. \tag{1.5.10}$$

The operator $\mathcal{L}_\gamma$ is a self-adjoint elliptic operator with respect to the above L^2 inner product and thus has a discrete spectrum and finite dimensional kernel. The unknown sign of these eigenvalues and thus the ambiguous sign in (1.5.10) leads us to the following definition.

Definition 1.31 (Riemannian linear stability, λ_0, infinitesmal Einstein deformations)**.** Let $(\mathcal{K},\gamma)$ be an Einstein manifold. We say $(\mathcal{K},\gamma)$ is Riemannian linearly stable if for all symmetric tensors $\bar{h}_{AB}$ such that $\mathrm{tr}_\gamma \bar{h} = 0$ and $\nabla[\gamma]^A \bar{h}_{AB} = 0$ one has

$$(\mathcal{L}_\gamma \bar{h}, \bar{h})_{L^2(\mathcal{K})} = \int_{\mathcal{K}} \langle \mathcal{L}_\gamma \bar{h}, \bar{h}\rangle_\gamma \mathrm{d}\mu_\gamma \geq 0, \tag{1.5.11}$$

and otherwise $(\mathcal{K},\gamma)$ is Riemannian linearly unstable. If instead of (1.5.11) we have $(\mathcal{L}_\gamma \bar{h}, \bar{h})_{L^2(\mathcal{K})} > c(\bar{h},\bar{h})_{L^2(\mathcal{K})}$ for some constant $c > 0$ then we say the manifold is Riemannian strictly stable. We define λ_0 to be the smallest eigenvalue of $\mathcal{L}_\gamma|_{TT}$ and we call $\ker(\mathcal{L}_\gamma|_{TT})$ the space of infinitesmal Einstein deformations.

The sign of λ_0 and properties of $\ker \mathcal{L}_\gamma$ play a central role when studying the evolution of γ under Ricci flow (see for example [CCG+15]) and under the Einstein equations, which is our focus. Although definition 1.31 and the sign appearing in (1.5.10) may seem contradictory, this is in fact correct when one considers Ricci flow or the evolution of a Lorentzian spacetime constructed from $(\mathcal{K},\gamma)$. More generally, we see that an Einstein metric is neither a local minimum nor maximum of $S(\gamma)$.

We now give a brief discussion of the properties of λ_0 and $\ker \mathcal{L}_\gamma$, referrring to [Bes87, §12] for complete and rigorous details. We say metrics $\gamma_1, \gamma_2 \in \mathscr{M}$ are equivalent, denoted $\gamma_1 \sim \gamma_2$, if there exists a positive constant $c > 0$ and a diffeomorphism ϕ of $\mathcal{K}$ such that $\gamma_1 = c \cdot \phi^* \gamma_2$. The set of Einstein metrics under this equivalence relation is called the moduli space of Einstein structures and denoted $\mathscr{E}(\mathcal{K})$. Note that $\mathscr{E}(\mathcal{K})$ is not a manifold and can be disconnected, e.g. Hopf fibrations can give different Einstein structures on $\mathbb{S}^{4n+3}$ with different Einstein constants [Bes87, §12.53].

Consider now a fixed Einstein metric $\gamma \in \mathscr{M}_1$. We would like to understand the conditions on a symmetric 2-tensor h that ensure, at least infinitesimally, that $\gamma + h$ is also an Einstein metric. Since $\mathscr{E}(\mathcal{K})$ is not a manifold, and since we would like to rule out $(\gamma + h) \sim \gamma$, some technical set-up is required. Ebin's slice theorem posits the existence of $\mathfrak{S}_\gamma$, a submanifold in $\mathscr{M}_1$ containing γ which is a slice of the action of the diffeomorphism group[2]. We next consider a curve of Einstein metrics in this submanifold, i.e. $\gamma(t) \in \mathfrak{S}_\gamma$ with $\gamma(0) = \gamma$. Since $\gamma(t)$ are all critical points of $S(\gamma)|_{\mathscr{M}_1}$ we have that $t \mapsto \mathrm{vol}_{\gamma(t)}(\mathcal{K}) \cdot \mathrm{R}[\gamma(t)]$ is a constant function. The first derivative $h = \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}|_{t=0}$ satisfies the system

$$\nabla[\gamma(t)]^A h_{AB} = 0, \quad \int_{\mathcal{K}} \mathrm{tr}_{\gamma(t)} h \, \mathrm{d}\mu_{\gamma(t)} = 0, \tag{1.5.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Ric}[\gamma(t)] - \frac{\mathrm{R}[\gamma(t)]}{d} \cdot \gamma(t) \Big)\Big|_{t=0} = 0. \tag{1.5.12b}$$

Note that (1.5.12b) is the linearised equations of motion. Two key results now state that solutions to the system (1.5.12) precisely coincide with $\ker(\mathcal{L}_\gamma|_{TT})$ [Bes87, Th. 12.30] and, moreover, $\ker(\mathcal{L}_\gamma|_{TT})$ is equal to the tangent space at γ of a finite dimensional submanifold $Z \subset \mathfrak{S}_\gamma$ [Bes87, Th. 12.30].

If $\ker(\mathcal{L}_\gamma|_{TT}) = \{0\}$ then $Z \cap \mathscr{E}(\mathcal{K}) = \{\gamma\}$ and so up to the equivalence relation $\sim$ there are no other Einstein metrics with the same Einstein constant close to γ in $\mathscr{E}(\mathcal{K})$. Such Einstein metrics are said to be rigid, and so under evolution by the nonlinear Einstein equations, we can expect to converge back to γ. Clearly a Riemannian strictly linearly stable manifold is rigid. Unfortunately the other case is not so clean. If there exists a nonzero $h \in \ker(\mathcal{L}_\gamma|_{TT})$ then Z is a submanifold of positive dimension. However because the Einstein equations are nonlinear, there may not exist a curve of Einstein metrics tangent to h. If such a curve does exist however, then h is said to be integrable, and one could expect to converge back to a member of this curve (not necessarily $\gamma(0)$) under Einstein evolution.

We now give two important examples of Riemannian linearly stable and strictly stable manifolds that are studied in Chapters 5 and 6.

Example I - Hyperbolic manifolds

We first consider a negative Riemannian Einstein metric with Einstein constant $k < 0$, given in definition 1.29. In the case $d = 3$ such a space necessarily has constant sectional curvature (and thus is hyperbolic) [Bes87]. A compact manifold $\mathcal{K}$ admits either no hyperbolic metric or precisely one [Mos68]. Thus if $\mathcal{K}$ has a negative Einstein metric it is isolated in the moduli space of Einstein structures. This can also be proved using eigenvalue estimates [Koi79, Krö15], leading to the following result.

[2] i.e. there exists a neighbourhood $\mathcal{U} \subset \mathscr{M}_1$ of γ such that any metric in $\mathcal{U}$ can be related to an element of $\mathfrak{S}_\gamma$ by a diffeomorphism. Roughly speaking this means we can fix the diffeomorphism freedom.

Proposition 1.32. *Let $(\mathcal{K}, \gamma)$ be a negative Einstein three-manifold with Einstein constant $k = -2/9$. Then $(\mathcal{K}, \gamma)$ is strictly stable, with $\lambda_0 \geq 1/9$ and $\ker(\mathcal{L}_\gamma) = \{0\}$.*

In the case $d > 3$ hyperbolic Einstein spaces are still strictly stable and hence isolated. However there are also examples of families of non-hyperbolic, negative Einstein metrics. Thus $\ker(\mathcal{L}_\gamma|_{TT}) \neq \{0\}$ for such metrics. These non-hyperbolic, negative Einstein metrics are still Riemannian linear stable and integrable however they are not isolated in the moduli space of Einstein structures. Thus under the Einstein flow it is possible to evolve from one member of the family to another. This is relevant in [AM11] however does not occur in our Chapter 5 where we restrict to $d = 3$. For further discussion on negative Riemannian Einstein metrics see [Bes87, Ch.12].

Example II - Special Holonomy manifolds

Our next example concerns manifolds $(\mathcal{K}, \gamma)$ with a spin structure. We refer to [Bes87, Ch.6§F] for rigorous definitions of the concepts. Heuristically speaking, these are manifolds where one can write down a well-defined Clifford algebra and globally defined spinor ψ that transforms under the double cover of the Lorentz group. Just as it is not possible to write down a nowhere vanishing vector field on even-dimensional spheres $\mathbb{S}^{2n}$, the existence of a nowhere vanishing spinor also involves topological restrictions.

Definition 1.33 (Killing spinor)**.** Suppose $(\mathcal{K}, \gamma)$ has a spin structure and admits a nonzero spinor ψ. We say the spinor is Killing if

$$\nabla^S[\gamma]_X \psi = cX \cdot \psi, \tag{1.5.13}$$

where $\cdot$ denotes Clifford multiplication, $\nabla^S[\gamma]$ is the spinor covariant derivative and $c \in \mathbb{C}$ is a constant. If $c = 0$ then ψ is said to be parallel.

A manifold $(\mathcal{K}, \gamma)$ admitting such a nonzero spinor is necessarily Einstein with Einstein constant $k = 4c^2(d-1)$. Thus if ψ is parallel then $(\mathcal{K}, \gamma)$ is Ricci-flat and, moreover, such manifolds are linearly stable due to the following result.

Theorem 1.34 ([DWW05, Theorem 1.1])**.** *If a compact Riemannian manifold $(\mathcal{K}, \gamma)$ has a cover which is spin and admits a nonzero parallel spinor then, for all symmetric two-tensors u,*

$$\int_{\mathcal{K}} \langle \mathcal{L}_\gamma u, u \rangle_\gamma \mathrm{d}\mu_\gamma \geq 0. \tag{1.5.14}$$

Thus the Calabi-Yau 3-fold, G_2, Spin(7) and hyperkähler manifolds are Riemannian linearly stable.

The proof works by arguing that in the presence of a nonzero parallel spinor, it is possible to relate $\mathcal{L}_\gamma$ to the square of the Dirac operator. In fact this idea dates to earlier work of [Wan91] on the deformation theory of Killing spinors. All known examples of compact Ricci-flat manifolds admit a spin cover with nonzero parallel spinors and thus are Riemannian linearly stable. A major open question is whether all compact Einstein manifolds with nonpositive scalar curvature are stable [KW75]. This fails in the noncompact case by the Riemannian Schwarzschild metric [GPY82]. For completeness we mention that a spin manifold $(\mathcal{K}, \gamma)$ admitting a parallel spinor can be classified according to its holonomy group, see for example [Wan89].

In Chapters 2 and 6 we consider the evolution under the Einstein equations of a spacetime constructed from Minkowski with a compact Riemannian space covered by theorem 1.34.

1.5.2 Linearised Lorentzian equations

The global solutions produced in theorem 1.12 were very close to the trivial zero solution. Motivated by this, we now rewrite the reduced Einstein equations (1.1.5) in terms of a small perturbation away from a fixed spacetime $(\mathcal{M}, \hat{g})$ which satisfies the vacuum Einstein equations.

Definition 1.35 (Metric perturbation)**.** A spacetime g can be written in terms of a perturbation and inverse perturbation away from $\hat{g}$ as

$$h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}, \tag{1.5.15}$$
$$H^{\mu\nu} = g^{\mu\nu} - \hat{g}^{\mu\nu}. \tag{1.5.16}$$

Note care is required to raise indices on the perturbation $h_{\mu\nu}$ since it does not necessarily agree with the inverse $H^{\mu\nu}$. In particular

$$H^{\mu\nu} = -\hat{g}^{\mu\beta}\hat{g}^{\nu\alpha}h_{\beta\alpha} - h_{\alpha\beta}H^{\beta\nu}\hat{g}^{\alpha\mu} = -h^{\mu\nu} + \mathcal{O}(h^2). \tag{1.5.17}$$

If we choose $\hat{e} = \hat{g}$ then the reduced Einstein equations (1.1.5) become

$$(\hat{g}^{\alpha\beta} + H^{\alpha\beta})\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta h_{\mu\nu} + 2(R[\hat{g}] \circ h)_{\mu\nu} = \mathcal{F}_{\mu\nu}[g](\nabla[\hat{g}]h, \nabla[\hat{g}]h) + F_{\mu\nu}(H, h), \tag{1.5.18a}$$

where $\mathcal{F}_{\mu\nu}$ was defined in (1.1.5b) and $F_{\mu\nu}$ is given by

$$\begin{aligned} F_{\mu\nu}(H, h) = {} & H^{\alpha\beta}\left(h_{\alpha\delta}\operatorname{Riem}[\hat{g}]^{\delta}{}_{\mu\nu\beta} + h_{\alpha\delta}\operatorname{Riem}[\hat{g}]^{\delta}{}_{\nu\mu\beta}\right) \\ & + H^{\alpha\beta}\left(h_{\mu\delta}\operatorname{Riem}[\hat{g}]^{\delta}{}_{\alpha\nu\beta} + h_{\nu\delta}\operatorname{Riem}[\hat{g}]^{\delta}{}_{\alpha\mu\beta}\right). \end{aligned} \tag{1.5.18b}$$

Note in this calculation we used that $\operatorname{Ric}[\hat{g}] = 0$ so that

$$g^{\gamma\delta}g_{\mu\lambda}\operatorname{Riem}[\hat{g}]^{\lambda}{}_{\gamma\nu\delta} = (\hat{g}^{\gamma\delta} + H^{\gamma\delta})(\hat{g}_{\mu\lambda} + h_{\mu\lambda})\operatorname{Riem}[\hat{g}]^{\lambda}{}_{\gamma\nu\delta} \tag{1.5.19}$$
$$= H^{\gamma\delta}\hat{g}_{\mu\lambda}\operatorname{Riem}[\hat{g}]^{\lambda}{}_{\gamma\nu\delta} + H^{\gamma\delta}h_{\mu\lambda}\operatorname{Riem}[\hat{g}]^{\lambda}{}_{\gamma\nu\delta}. \tag{1.5.20}$$

The linearisation of equations (1.5.18a) reads

$$\nabla[\hat{g}]^\alpha\nabla[\hat{g}]_\alpha h_{\mu\nu} + 2(R[\hat{g}] \circ h)_{\mu\nu} = 0, \tag{1.5.21}$$

where we used definition (1.0.2). The differential operator in (1.5.21) is, up to an overall minus sign, the Lorentzian version of the operator $\mathcal{L}_\gamma$ from definition 1.30.

Remark 1.36. It is enjoyable to write out the notation $R[\hat{g}]\circ$ and enjoy the similarity between the linearised Einstein equations and the Maxwell equations in Lorenz gauge

$$\nabla[\hat{g}]^\alpha\nabla[\hat{g}]_\alpha h_{\mu\nu} - 2\operatorname{Riem}[\hat{g}]^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = 0, \tag{1.5.22}$$
$$\nabla[\hat{g}]^\alpha\nabla[\hat{g}]_\alpha A_\mu - 2\operatorname{Ric}[\hat{g}]^{\alpha}{}_{\mu}A_\alpha = 0. \tag{1.5.23}$$

Moreover there are close relationships between solutions of these equations, see for example recent work on a Schwarzschild background [Joh20] and in supergravity [BCC+19].

In Chapters 2 and 6 we study equations (1.5.18a) when $(\mathcal{M}, \hat{g})$ is a product spacetime.

Definition 1.37 (Product spacetimes)**.** Given a $(1+n+d)$-dimensional manifold $\mathcal{M} = M \times \mathcal{K}$ the product spacetime metric takes the form

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = g_{ij}\mathrm{d}x^{i}\mathrm{d}x^{j} + \gamma_{AB}\mathrm{d}x^{A}\mathrm{d}x^{B}. \tag{1.5.24}$$

We call (M, g) the external manifold and $(\mathcal{K}, \gamma)$ the internal manifold. We use $\mu, \nu \in \{0, \dots, 1+n+d\}$ for spacetime indices, $a, b \in \{1, \dots, 1+n+d\}$ for spatial indices, $i, j \in \{0, \dots, n\}$ for external indices and $A, B \in \{1+n+1, \dots, 1+n+d\}$ for internal indices.

The presence of the internal space breaks the symmetries of the operator in (1.5.21). In Chapter 2 we consider a product spacetime with initial data with unbroken internal symmetry. In Chapter 6 we do not do this and the compact space produces 'effective' metric masses in the operator (1.5.21). These 'effective masses' will at least have the right sign due to the condition of Riemannian linear stability. We compensate for the broken symmetries in this case by taking the Minkowski dimension sufficiently high.

Main results of the thesis and their context

1.6 Kaluza-Klein theory

Just as electricity and magnetism were unified by Maxwell's theory of electromagnetism, and spacetime and gravity were unified through Einstein's theory of general relativity, string theory is a candidate theory unifying quantum theory with gravity. The objects of interest in string theory are higher-dimensional spacetimes. The simplest higher-dimensional spacetime, the Kaluza-Klein spacetime, attempts to produce a unified system of general relativity and electromagnetism. It has been known since the 1920s [Kal21, Kle26]. The spacetime manifold takes the form $\mathbb{R}^{1+3} \times \mathbb{S}^1$ with metric

$$g_{KK} = -\mathrm{d}t^2 + \sum_{a=1}^{3} (\mathrm{d}x^a)^2 + (\mathrm{d}x^A)^2, \tag{1.6.1}$$

and $(t, x^a, x^A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^1$. In Chapter 2 we prove the following stability result.

Theorem 1.38 (§2 Th. 2.9, see also [Wya18])**.** *The Kaluza-Klein spacetime $(\mathbb{R}^{1+3} \times \mathbb{S}^1, g_{KK})$ is stable as a solution to the classical vacuum Einstein equations provided the internal $\mathbb{S}^1$ symmetry is not broken. That is, the spacetime is stable against perturbations that depend only on the Minkowski coordinates $(t, x) \in \mathbb{R}^{1+3}$.*

The initial data restriction in theorem 1.38 appears in string theory, where one considers the limit of a small circle radius such that at low energies the theory should be independent of the internal space. The equations of motion in five dimensions reduce to a nontrivially coupled Einstein-matter system in four-dimensions. To see this reduction consider a five-dimensional vacuum Einstein spacetime $G_{\mu\nu}$ which does not depend on the internal coordinate x^A. That is

$$G_{\mu\nu}(x^i, x^A) = G_{\mu\nu}(x^i), \tag{1.6.2}$$

with indices given in definition 1.37. For fixed constants α, β we can reparametrise the metric as

$$G_{\mu\nu} = \begin{pmatrix} e^{2\alpha\phi} g_{ij} + e^{2\beta\phi} A_i A_j & e^{2\beta\phi} A_i \\ e^{2\beta\phi} A_j & e^{2\beta\phi} \end{pmatrix}. \tag{1.6.3}$$

Indeed provided $\beta \neq 0$ this choice fully parametrises the higher-dimensional metric, see for example [Pop]. This ansatz is chosen so that g_{ij} transforms as a 2-tensor, A_i as a vector and ϕ a dilaton (scalar field). The five-dimensional vacuum Einstein equations for $G_{\mu\nu}$ are equivalent (i.e. we have a consistent truncation) to the following

$$\begin{aligned}
\mathrm{Ric}[g]_{ij} &= \tfrac{1}{2}\partial_i\phi\partial_j\phi + \tfrac{1}{2}e^{-6\alpha\phi}\Big(F_{ik}F_j{}^k - \tfrac{1}{4}F_{kl}F^{kl}g_{ij}\Big)\,, && (1.6.4a)\\
\nabla[g]_i\big(e^{-6\alpha\phi}F^{ij}\big) &= 0\,, && (1.6.4b)\\
g^{ij}\nabla[g]_i\nabla[g]_j\phi &= -\tfrac{3}{2}\alpha e^{-6\alpha\phi}F_{kl}F^{kl}\,, && (1.6.4c)
\end{aligned}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ and $\alpha = -\frac{1}{2}\beta = 1/\sqrt{12}$. According to theorem 1.38, the trivial Minkowskian solution of equations (1.6.4) is stable. It is very important to note the *nontrivial coupling* between the vector potential and scalar field in equations (1.6.4) which prevents us from setting $\phi \equiv 0$ and thus obtaining a pure minimally coupled Einstein-Maxwell theory, whose trivial Minkowskian solution was already known to be stable [Loi09]. A similar reduction also occurs if one considers the flat d-dimensional torus $\mathbb{T}^d$, and indeed theorem 1.38 applies to this case also. Theorem 1.38 was also followed by an analogous result for cosmological Kaluza-Klein spacetimes where the Milne spacetime replaces the Minkowski manifold [BFK19].

1.6.1 Witten's 'bubble of nothing'

In a remarkable and highly influential work by Witten, it was shown that the Kaluza-Klein vacuum $(\mathbb{R}^{1+3}\times\mathbb{S}^1, g_{KK})$ is unstable at the semi-classical level [Wit82]. We now briefly discuss part of Witten's result in relation to theorem 1.38.

By starting with the five-dimensional Schwarzschild black hole, Witten performed two appropriate Wick transformations (which are not so relevant to our discussion) leading to the following Lorentzian metric

$$\mathrm{d}s^2 = -\rho^2\mathrm{d}T^2 + \rho^2\cosh(T)^2\mathrm{d}\Omega^2 + \frac{\mathrm{d}\rho^2}{1-(R/\rho)^2} + (1-(R/\rho)^2)\mathrm{d}\phi^2. \tag{1.6.5}$$

Here $\mathrm{d}\Omega^2$ is the line element of the unit sphere $\mathbb{S}^2$, $R>0$ is a constant and the variables range over $T\in\mathbb{R}$, $\rho\in[R,\infty)$ and $\phi\in[0,2\pi R)$ is a periodic variable. The metric (1.6.5) is a solution to the five-dimensional vacuum Einstein equations. The spacetime does not exist in the region $\rho<R$. The radius of the fifth dimension, given by $R\sqrt{1-(R/\rho)^2}$, shrinks to zero as $\rho\to R$ in a way that implies (1.6.5) describes a nonsingular and geodesically complete spacetime.

Witten argued that, under a semi-classical process which should be thought of as the field-theory equivalent of quantum tunnelling through a finite barrier, the Kaluza-Klein vacuum $(\mathbb{R}^{1+3}\times\mathbb{S}^1, g_{KK})$ decays into the solution (1.6.5). Thus Witten argues that the Kaluza-Klein vacuum is semi-classically unstable.

However the process Witten considers is different to the classical perturbative approach studied in theorem 1.38. A useful perspective illustrating how these approaches are disjoint comes by looking at the initial value surface. The $t=0$ initial value hypersurface of the Kaluza-Klein vacuum $(\mathbb{R}^{1+3}\times\mathbb{S}^1, g_{KK})$ has topology $\mathbb{R}^3\times\mathbb{S}^1$. In theorem 1.38 we consider perturbations of the vacuum which have the same topology as this slice. By contrast, we can look at the $T=0$ hypersurface of Witten's solution, where the metric induced from (1.6.5) is

$$\mathrm{d}s^2 = \rho^2\mathrm{d}\Omega^2 + \frac{\mathrm{d}\rho^2}{1-(R/\rho)^2} + (1-(R/\rho)^2)\mathrm{d}\phi^2. \tag{1.6.6}$$

It is crucial to note that $\rho\geq R>0$. This means that (1.6.6) does not describe a conical geometry, but is a metric on the topological space $\mathbb{R}^2 \times \mathbb{S}^2$ with ρ and ϕ the polar coordinates on $\mathbb{R}^2$. Thus Witten considers excitations of the Kaluza-Klein vacuum with a different, $\mathbb{R}^2 \times \mathbb{S}^2$, topology. By definition, there is no continuous perturbation of any size, let alone of the small size we consider, that could lead to a change in our initial topology $\mathbb{R}^3 \times \mathbb{S}^1$. Thus the two results do not disagree. For completeness we remark that at $\rho = +\infty$, the spacetime (1.6.5) does actually have the same topology as the Kaluza-Klein vacuum, and Witten argues that this makes such excitations reasonable from a semi-classical point of view.

Finally, in terms of the usual notion of ADM energy, (1.6.6) has zero energy and Witten claims that, using methods of Brill and Deser [BD68], it is even possible to have negative energy solutions. This is a crucial part of Witten's argument, since the semi-classical process cannot decay into a spacetime with more energy than it started with. By comparison, the $t = 0$ slice of Minkowski spacetime has zero energy and, by the positive energy theorem [Wit81, SY81], every Riemannian manifold that is asymptotically Euclidean with non-negative scalar curvature has strictly positive energy. Thus Minkowski spacetime is semiclassically stable since no nontrivial states with zero energy exist. Although (1.6.6) shows that a Kaluza-Klein-type positive mass theorem cannot in general be true, Witten makes the point that if we only consider perturbations of the Kaluza-Klein vacuum in which the topology is not changed, then in fact the methods of [Wit81] still apply.

1.6.2 Null conditions

The proof of theorem 1.38 is shown by studying a class of quasilinear wave equations with particular structure in the nonlinearities quadratic in ∂g. To explain this structure let us introduce the following notation.

Definition 1.39 (Null variables)**.** Introduce the null variables $s = t + r$, $q = t - r$ and $\omega^a := x^a/|x|$ the radial unit vector so that $\partial_r = \omega^a \partial_a$. Define the Minkowski null vectors $2\partial_s = \partial_t + \partial_r$ and $2\partial_q = \partial_t - \partial_r$ and the angular derivatives $\not{\partial}_a := \partial_a - \omega_a \omega^b \partial_b$. We define 'good' $\bar{\partial}$ derivatives by

$$|\bar{\partial}\psi|^2 = |\partial_s \psi|^2 + \sum_{a=1}^{3} |\not{\partial}_a \psi|^2. \tag{1.6.7}$$

Note that the set $\{\partial_s, \not{\partial}_a\}$ spans the tangent space of the outgoing light cone $|x| - t =$ constant. See also Section 2.2. By direct calculation (e.g. [LR10, Lemma 5.1]) we have the very useful estimate

$$(1 + |t - r|)|\partial\psi| + (1 + t + r)|\bar{\partial}\psi| \leq C \sum_{|I|=1} |\Gamma^I \psi|. \tag{1.6.8}$$

Combining (1.6.8) with (1.3.33) we can in fact return to theorem 1.12 and state the following refined estimates

$$|\bar{\partial}\psi(t, x)| \lesssim \varepsilon(1 + t)^{-\frac{n+1}{2}}(1 + |t - |x||)^{\frac{1}{2}}, \tag{1.6.9a}$$

$$|\partial_q \psi(t, x)| \lesssim \varepsilon(1 + t)^{-\frac{n-1}{2}}(1 + |t - |x||)^{-\frac{1}{2}}. \tag{1.6.9b}$$

These estimates, part of a broader class of peeling estimates, show the preference for the wave to move along directions tangential to the light cone (1.6.9a), and not in directions transversal to it (1.6.9b). Note since $t - |x| = \mathcal{O}(t)$ in the support of ψ, the decay rate of (1.6.9a) is integrable in dimension $n = 3$.

The following definition introduces a large class of nonlinearities which have the property that at least one of the derivatives is a good derivative $\bar{\partial}$.

Definition 1.40 (Null condition)**.** Consider a system of unknowns $\psi = (\psi_1, \dots, \psi_m)$ satisfying for $K \in \{1, \dots, m\}$ the PDEs

$$\Box \psi_K = F_K(\partial \psi). \tag{1.6.10}$$

The nonlinearity $F(\partial\psi) = (F_1(\partial\psi), \dots, F_m(\partial\psi))$ is said to satisfy the null condition if at lowest order it takes the form

$$F_K(\partial\psi) = \sum_{L,M=1}^{m} q^{\mu\nu}_{KLM} \partial_\mu \psi_L \partial_\nu \psi_M, \tag{1.6.11}$$

and the constants $q^{\mu\nu}_{KLM}$ satisfy

$$q^{\mu\nu}_{KLM} \xi_\mu \xi_\nu = 0 \text{ whenever } \eta^{\mu\nu}\xi_\mu\xi_\nu = 0. \tag{1.6.12}$$

In particular this implies that F is a linear combination of the following bilinear forms, called (classical) null forms,

$$Q_0(\psi, \varphi) = \eta^{\mu\nu} \partial_\mu \psi \partial_\nu \varphi, \tag{1.6.13a}$$
$$Q_{\mu a}(\psi, \varphi) = \partial_\mu \psi \partial_a \varphi - \partial_\mu \varphi \partial_a \psi. \tag{1.6.13b}$$

The null condition was first introduced in [Kla80] where it also covered more general nonlinearities of the type $F(\partial\psi, \partial^2\psi)$.

Remark 1.41 (Strong null forms)**.** The null forms $Q_{\mu a}$ are sometimes called strong null forms. If we write the null forms in terms of the Γ vector fields we find

$$Q_0(\psi, \varphi) = t^{-1}\big(S\psi \cdot \partial_t \varphi - \textstyle\sum_{a=1}^{n} \partial_a \psi \cdot \Omega_{0a}\varphi\big), \tag{1.6.14a}$$
$$Q_{ab}(\psi, \varphi) = t^{-1}\left(\partial_a \psi \cdot \Omega_{0b}\varphi - \partial_b \psi \cdot \Omega_{0a}\varphi + \Omega_{ab}\psi \cdot \partial_t \varphi\right), \tag{1.6.14b}$$
$$Q_{0a}(\psi, \varphi) = t^{-1}\left(\partial_t \psi \cdot \Omega_{0a}\varphi - \Omega_{0a}\psi \cdot \partial_t \varphi\right). \tag{1.6.14c}$$

From this decomposition we can see that the strong null forms do not involve the scaling vector field S and so are compatible with both massive and massless wave equations. That is, it is possible to study a Klein-Gordon equation with a nonlinearity only involving $Q_{\mu a}$ using the energies (1.4.2) on constant t-slices provided an adapted Sobolev estimate (which does not require S commutation) is also proved, see [Geo90].

In a similar spirit to (1.6.14), albeit using the null frame $\{\partial_s, \partial_q, \not{\partial}_a\}$ instead, one can show the following key estimates for null forms:

$$|Q(\psi, \varphi)| \le C\left(|\bar{\partial}\psi||\partial\varphi| + |\partial\psi||\bar{\partial}\varphi|\right), \tag{1.6.15}$$
$$\Gamma(Q(\psi, \varphi)) = Q(\Gamma\psi, \varphi) + Q(\psi, \Gamma\varphi) + \tilde{Q}(\psi, \varphi), \tag{1.6.16}$$

where $\tilde{Q}$ is a linear combination of classical null forms (for example if $\Gamma = S$ then $\tilde{Q} = 2Q_0$). The first estimate shows that effectively null forms always involve one good derivative. The second estimate tells us that this good structure is preserved when we apply the vector fields Γ.

Returning now to theorem 1.15 we see the nonlinearity is in fact $Q_0(\psi,\psi)$. To prove this theorem along the lines of distributing derivatives as in (1.3.25), we need to estimate $|\bar{\partial}\psi|$ in both L^2 with a high number of Γ derivatives, and L^∞ with a low number of Γ derivatives. Can we obtain improved estimates for $\|\bar{\partial}\psi\|_{L^2}$, better than those for $\|\partial\psi\|_{L^2}$ in the energy inequality (1.3.15)? Emulating the heuristic idea of (1.2.3), we can use (1.6.9a) to see that we should expect the following rough behaviour

$$\int_0^t \int_{\mathbb{R}^n} \frac{|\bar{\partial}\psi|^2}{(1+|\tau-|x||)}\mathrm{d}x\mathrm{d}\tau \lesssim \int_0^t \frac{KE_0}{\tau^{n+1}}\mathrm{Area}(S_t(x_0))\mathrm{d}\tau \sim \int_0^t \frac{KE_0}{\tau^2}\mathrm{d}\tau \sim KE_0. \tag{1.6.17}$$

The following lemma, dating to work in [Ali04, LR05], formalises this idea. See also [Ali09] for a proof.

Lemma 1.42. *Consider the equation $\Box\psi = F$. For every $\delta > 0$ there exists a constant $C = C(\delta)$ such that*

$$\left(\int_0^t \int_{\mathbb{R}^3} \frac{|\bar{\partial}\psi|^2}{(1+|\tau-|x||)^{1+\delta}}\mathrm{d}x\mathrm{d}\tau\right)^{1/2} \leq C\|\partial\psi(0,\cdot)\|_{L^2(\mathbb{R}^3)} + C\int_0^t \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}\mathrm{d}\tau. \tag{1.6.18}$$

Away from the light cone we have $(1+|r-t|)^{-1-\delta} \leq (1+t)^{-1-\delta}$. Since $(1+t)^{-1-\delta}$ is integrable, the above lemma is really improving our estimates of the good derivatives near the light cone. As first noted in [LR10], it is possible to now prove small data global existence for theorem 1.15, or more generally for systems of wave equations satisfying the null condition of Definition 1.40, by repeating the argument as in theorem 1.12 combined with (1.6.8), (1.6.15), (1.6.16) and lemma 1.42.

1.6.3 The weak null condition and Kaluza-Klein spacetimes

The vacuum Einstein equations written in terms of the metric components do not satisfy the null condition, but rather a 'weak null' condition first identified by Lindblad and Rodnianski [LR03]. The weak null condition essentially measures how much residual energy of a quasilinear wave remains near null infinity $\mathscr{I}^+$. The amount of residual energy corresponds to the existence of solutions for some asymptotic PDE system at $\mathscr{I}^+$. If there is too much residual energy, i.e. if the asymptotic system does not have solutions, then one expects that problems would occur for the original PDE. The use of asymptotic systems has its origins in work by Hörmander [Hör87]. The following is a simple example of a weak null system discussed in [LR03], see also the presentation in [Luk16].

Theorem 1.43 (Global existence for a weak null system)**.** *Let $\phi_0, \phi_1, \Psi_0, \Psi_1 \in C_0^\infty(\mathbb{R}^3)$ and consider on $\mathbb{R}^{1+3}$ the Cauchy problem*

$$\Box\psi = (\partial_t\varphi)^2, \tag{1.6.19a}$$
$$\Box\varphi = Q_0(\psi,\psi), \tag{1.6.19b}$$

with initial data

$$\begin{aligned}\psi(0,x) &= \varepsilon\Psi_0(x), \quad \partial_t\psi(0,x) = \varepsilon\Psi_1(x),\\ \varphi(0,x) &= \varepsilon\phi_0(x), \quad \partial_t\varphi(0,x) = \varepsilon\phi_1(x).\end{aligned} \tag{1.6.20}$$

Then for $\varepsilon > 0$ sufficiently small there exists a global smooth solution which satisfies

$$|\partial\varphi| \lesssim \varepsilon(1+t)^{-1}, \quad |\partial\psi| \lesssim \varepsilon\frac{\log(2+t)}{1+t}. \tag{1.6.21}$$

There is a kind of feedback loop occurring in (1.6.19). Note that the bad nonlinearity seen in theorem 1.14 sources the field ψ. This leads to the slower decay rate for the field ψ in (1.6.21), which one can see as an indication of the residual energy of ψ. However when ψ sources the field φ it does so as a null form Q_0 which has better properties as seen in theorem 1.15. Thus global existence can still be shown for the system (1.6.19) using a combination of lemma 1.42, properties of null forms and energy estimates discussed before.

The feedback loop above was actually believed to lead to a break down in global solutions to the Einstein equations when written in harmonic gauge [CB73]. Nonetheless the powerful direction-dependent decay and energy integral method of [LR05, LR10] established that, even when using harmonic gauge, Minkowski is a stable solution to both the vacuum Einstein equations and the Einstein equations coupled to a massless scalar field. Of course the Einstein equations are significantly more complicated than (1.6.19) and so the proof in [LR05, LR10] required several other key insights. For our purposes, if the internal symmetry of the product spacetime on $\mathbb{R}^{1+3}\times\mathbb{S}^1$ is not broken, then in a higher-dimensional $\hat{e}$-wave gauge, i.e. (1.1.3) with $\hat{e} = g_{KK}$, then in terms of the perturbation $h = g - g_{KK}$ the vacuum Einstein equations reduce to the following non-linear wave system

$$\begin{aligned} g^{kl}\partial_k\partial_l h_{ij} &= P(\partial_i h, \partial_j h) + Q_{ij}(\partial h, \partial h) + G_{ij}(h)(\partial h, \partial h)\,,\\ g^{kl}\partial_k\partial_l h_{iA} &= Q_{iA}(\partial h, \partial h) + G_{iA}(h)(\partial h, \partial h)\,,\\ g^{kl}\partial_k\partial_l h_{AA} &= Q_{AA}(\partial h, \partial h) + G_{AA}(h)(\partial h, \partial h). \end{aligned} \tag{1.6.22a}$$

The quadratic non-null terms are

$$\begin{aligned} P(\partial_i h, \partial_j h) = {}& \eta^{kl}\eta^{cd}\left(\frac{1}{4}\partial_i h_{kl}\partial_j h_{cd} - \frac{1}{2}\partial_i h_{kc}\partial_j h_{ld}\right)\\ &+ \delta^{AB}\delta^{CD}\left(\frac{1}{4}\partial_i h_{AB}\partial_j h_{CD} - \frac{1}{2}\partial_i h_{AC}\partial_j h_{BD}\right)\\ &+ \delta^{AB}\eta^{kl}\left(\frac{1}{4}\partial_i h_{AB}\partial_j h_{kl} + \frac{1}{4}\partial_j h_{AB}\partial_i h_{kl} - \partial_i h_{Ak}\partial_j h_{Bl}\right), \end{aligned} \tag{1.6.22b}$$

and the other terms are standard null (Q) or cubic terms (G) given in proposition 2.52. Since we have such a large group of symmetries there are a few ways the proof of theorem 1.38 could proceed. Under the $\mathbb{S}^1$-symmetry assumption the five-dimensional vacuum Einstein equations reduce to a non-minimally coupled four-dimensional Einstein-Maxwell-scalar field system (1.6.4) whose stability could be studied. This was a key part of the approach in [BFK19]. Another alternative approach is to study the five-dimensional quasilinear wave equations (1.6.22) by adapting the method of [LR05, LR10] to include the internal space.

The approach we take is to treat the addition fields $\{h_{iA}, h_{AB}\}$ appearing in (1.6.22) as just a collection of fields obeying four-dimensional quasilinear wave equations. Thus the proof of theorem 1.38 essentially extends the result of [LR05, LR10] to the Einstein equations coupled to a system of nontrivially coupled wave equations which are not covered by the Einstein scalar field system but include interactions expressed in (1.6.22b).

Note that, in a similar way to (1.6.21), we obtain the following decay estimates

$$|\partial h_{iA}| + |\partial h_{AA}| + |\partial h_{ij}| \lesssim \varepsilon(1+t)^{-1}\,, \tag{1.6.23a}$$
$$|\partial h_{\underline{LL}}| \lesssim \varepsilon(1+t)^{-1}\log(2+t)\,. \tag{1.6.23b}$$

We omit the full definition of $h_{\underline{LL}}$ for now, but remark that the main point to take away is that most metric components behave like φ (1.6.23a) while one component of the metric $g_{\mu\nu}$ behaves like ψ (1.6.23b).

As established in [LR03, LR05, LR10], the weak null condition is an important criterion for global existence of quasilinear wave equations. We note the recent work [Kei18] which also studied the weak null condition using different vector fields as multipliers.

1.7 The Higgs Mechanism

The four fundamental interactions known to exist in nature are gravitational, electromagnetic, strong and weak interactions. These interactions are each described by a field theory. So far we have talked about Einstein's classical field theory of gravity, general relativity. The other major field theory of 20th century physics is that of the standard model of particle physics which brings together the strong, weak and electromagnetic interactions. An essential ingredient of this theory is the Higgs mechanism which leads to a new type of symmetry breaking. In Chapter 3 we study a classical gauge theory coupled to a Higgs field on a fixed Minkowski background, as a toy model for the electroweak sector of the standard model. In particular we prove the following result.

Theorem 1.44 (§3 Th. 3.2, see also [DLW19])**.** *The ground state of the Higgs mechanism applied to an abelian gauge field and Yukawa-coupled spinor field is classically stable to compact perturbations.*

1.7.1 An abelian gauge theory and the Dirac-Proca equations

Given the theory considered above is quite different to the gravitational theory we have discussed so far, let us briefly step back to Maxwell's theory of electromagnetism. For full details of the theory discussed in this section, see for example [AH12]. Consider on a fixed Minkowski background $(\mathbb{R}^{1+3}, \eta)$ the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.7.1}$$

We require the electromagnetic tensor to satisfy the Bianchi property

$$\partial_{[\mu}F_{\nu\rho]} = 0. \tag{1.7.2a}$$

This identity combined with the Poincaré lemma implies that we can write the potential as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where A_μ is a four-potential. The Euler-Lagrange equations of (1.7.1) for the potential A^μ are

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0. \tag{1.7.2b}$$

The system (1.7.2) precisely encodes Maxwell's (source-free) field equations and are invariant under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu\alpha$ for some function α. In order to obtain a well-posed PDE for A^ν we need to choose a gauge. The analogue of the harmonic gauge discussed in Example 1.9 is the Lorenz gauge

$$\partial_\mu A^\mu = 0. \tag{1.7.3}$$

Note it is named after Ludvig Lorenz and not Hendrik Loren*tz*. Under the Lorenz gauge condition the equations of motion (1.7.2b) (c.f. the geometric Einstein equations (1.0.1)) reduce to uncoupled linear wave equations for A^ν (c.f. the reduced Einstein equations (1.1.5)). It is a standard calculation to show that initial data $(A_\mu(0,x), \partial_t A_\mu(0,x))$ satisfying (1.7.3) is propagated, in the sense that solutions to these linear wave equations evolving from such data will also satisfy (1.7.3). The photon is described by such a massless vector boson.

Remark 1.45. Note that (1.7.3) still enjoys some residual gauge freedom, namely $A_\mu \mapsto A_\mu + \partial_\mu \alpha$ for some function α satisfying the linear wave equation. Such residual freedom can be removed by imposing further conditions, such as some restrictions on one of the initial data components $(A_\mu(0,x), \partial_t A_\mu(0,x))$ or some other asymptotic condition on the initial data. A similar feature arises in general relativity where there also exist residual gauge degrees of freedom. For example for linearised gravity about Minkowski one can easily write down additional gauge transformations compatible with the harmonic gauge.

The massive Z^0 and $W^\pm$ bosons are fundamental particles mediating the weak interaction. In order to describe a massive vector boson we could artificially include a mass term to (1.7.1) in the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{1.7.4}$$

The action arising from (1.7.4) is often called the Proca action. Note that the Lorenz gauge in (1.7.3) is implied by the Euler-Lagrange equations for A^ν and cannot be fixed by a gauge choice.

The Dirac-Proca equations were an early model for electroweak interactions and describe the interactions between a massive vector boson and a massive spin 1/2 field of mass $M \geq 0$. In the Lorenz gauge the equations of motion for this model read

$$\begin{aligned} \Box A^\nu - m^2 A^\nu &= -\frac{1}{2} \psi^* \gamma^0 \gamma^\nu (\mathbf{I}_4 - \gamma^5) \psi, \\ -i\gamma^\mu \partial_\mu \psi + M\psi &= -\frac{1}{2} \gamma^\mu A_\mu (\mathbf{I}_4 - \gamma^5) \psi. \end{aligned} \tag{1.7.5}$$

The Dirac matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ are 4×4 matrices satisfying the identities

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu = -2\eta^{\mu\nu} \mathbf{1}_4, \tag{1.7.6}$$

and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Note the unusual sign in (1.7.6) comes from our mostly plus sign convention.

1.7.2 The Higgs mechanism in an abelian gauge theory

The Lagrangian density (1.7.4) is no longer gauge invariant under the transformation $A_\mu \mapsto A_\mu + \partial_\mu \alpha$. The Higgs mechanism allows us to maintain symmetry in the Lagrangian while still producing an effective mass to the vector bosons. Theorem 1.44

treats just one abelian vector boson and one fermion field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* D^\mu\phi - V(\phi^*\phi) - i\psi^*\gamma^0\gamma^\mu D_\mu\psi + g\phi^*\phi\psi^*\gamma^0\psi, \tag{1.7.7}$$

where $\lambda > 0, g, q, \mu$ are real constants and we use the following definitions for the Higgs potential, gauge curvature and gauge covariant derivatives:

$$\begin{aligned} V(\phi^*\phi) &= -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, & D_\mu\phi &= (\partial_\mu - iqA_\mu)\phi, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, & D_\mu\psi &= (\partial_\mu - iqA_\mu)\psi. \end{aligned} \tag{1.7.8}$$

The unknowns are a gauge vector $A = (A^\mu) : \mathbb{R}^{1+3} \to \mathbb{R}^4$ with gauge group $U(1)$, a Dirac fermion $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ and a complex scalar field $\phi : \mathbb{R}^{1+3} \to \mathbb{C}$ representing the Higgs field with $U(1)$ charge q that couples to itself, the gauge vector and the fermion.

The Lagrangian (1.7.7) is invariant under the following transformations

$$A_\mu \mapsto A'_\mu := A_\mu + \partial_\mu\alpha, \quad \phi \mapsto \phi' := e^{iq\alpha}\phi, \quad \psi \mapsto \psi' := e^{iq\alpha}\psi, \tag{1.7.9}$$

where $\alpha = \alpha(t, x^a)$ is some arbitrary function of spacetime. The Euler-Lagrange equations for A^ν, ϕ and ψ derived from (1.7.7) are the following

$$\begin{aligned} \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) &= iq\phi^*(D^\nu\phi) - iq(D^\nu\phi)^*\phi + q\psi^*\gamma^0\gamma^\nu\psi, \\ D^\mu D_\mu\phi &= V'(\phi^*\phi)\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu D_\mu\psi &= g\phi^*\phi\psi, \end{aligned} \tag{1.7.10}$$

where $V' = \frac{\partial V}{\partial(\phi^*\phi)}$. If $\mu^2 < 0$ then the trivial solution of (1.7.10) is

$$(A_0^\mu, \phi_0, \psi_0) = (0, 0, 0), \tag{1.7.11}$$

and this is invariant under the symmetries (1.7.9) of the Lagrangian. By contrast if $\mu^2 > 0$ the scalar field potential V now has a nonzero minimum defined by

$$\frac{\partial V}{\partial\phi}\Big|_{\langle\phi_0\rangle} = 0, \qquad \text{where } \langle\phi_0\rangle = \sqrt{\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}. \tag{1.7.12}$$

Consequently the field ϕ has a non-trivial minimum $\langle\phi_0\rangle$. The system (1.7.10) now has a continuous nontrivial set of solutions, often called ground states, labelled by a parameter $\theta \in [0, 2\pi)$:

$$(A_0^\mu, \phi_0, \psi_0) = (0, |v|e^{i\theta}, 0). \tag{1.7.13}$$

Symmetry breaking describes the situation where a Lagrangian (or its equations of motion) obey a symmetry that the ground state of the theory does not inherit. Although the *set* of ground states in (1.7.13) is $U(1)$ invariant (note $\phi_0^*\phi_0 = v^2$), there is no nontrivial α such that $\phi'_0 = \phi_0$. This implies that the $U(1)$ symmetry is broken since any 'spontaneous' choice of a ground state, that is any choice of θ, is not invariant under the symmetry. In this situation Goldstone's theorem implies that the spontaneous breaking of the continuous $U(1)$ symmetry generates a massless scalar boson [Gol61]. By choosing $\phi_0 = ve^{i\theta_0}$ for some constant θ_0 we can parametrise ϕ as

$$\phi = (ve^{i\theta_0} + h)e^{i\Phi}. \tag{1.7.14}$$

One can show using (1.7.10) that the field h is a massive boson, called the Higgs field, with Φ being the new massless Goldstone boson, which can be removed by gauge fixing.

Remark 1.46. By contrast to the above, note that the Einstein-Hilbert action of general relativity and solutions to the Einstein equations are respectively diffeomorphism in/co-variant.

1.7.3 Global evolution of the U(1) Higgs Boson: nonlinear stability and uniform energy bounds

All ground states of (1.7.13) are physically indistinguishable. We consider constant ground states satisfying $\partial_\mu \phi_0 = 0$ and express the field ϕ as a perturbation $\chi = \phi - \phi_0$. The equations of motion (1.7.10) now become

$$\Box A^\nu - \partial^\nu(\mathrm{div} A) - 2q^2 A^\nu \phi^* \phi = iq\big(\phi^*(\partial^\nu \phi) - (\partial^\nu \phi^*)\phi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \tag{1.7.15a}$$

$$\Box\phi - V'(\phi^*\phi)\phi - iq\phi\partial_\mu A^\mu = 2iqA_\mu\partial^\mu\phi + q^2 A^\mu A_\mu \phi - g\phi\psi^*\gamma^0\psi, \tag{1.7.15b}$$

$$i\gamma^\mu\partial_\mu\psi - g\phi^*\phi\psi = -q\gamma^\mu A_\mu \psi. \tag{1.7.15c}$$

As in the discussion of the Lorenz gauge (1.7.3) we need to fix a gauge condition in order to obtain PDEs of a definite type. A key aspect of our proof is to choose the following modified Lorenz gauge

$$\partial_\mu A^\mu + iq\big(\phi_0^*\chi - \chi^*\phi_0\big) = 0. \tag{1.7.16}$$

This choice removes a first order term from the nonlinearities in (1.7.15a), however it also feeds back into equation (1.7.15b). We obtain the following system of four coupled equations

$$\big(\Box - m_q^2\big)A^\nu = q\psi^*\gamma^0\gamma^\nu\psi + F_{A^\nu}, \tag{1.7.17a}$$

$$\Box\chi - m_q^2\,\frac{\phi_0}{2v^2}\big(\phi_0^*\chi - \chi^*\phi_0\big) - m_\lambda^2\,\frac{\phi_0}{2v^2}\big(\phi_0^*\chi - \chi^*\phi_0\big) = -g\phi_0\psi^*\gamma^0\psi + F_\chi, \tag{1.7.17b}$$

$$i\gamma^\mu\partial_\mu\psi - m_g\psi = F_\psi. \tag{1.7.17c}$$

The nonlinearities are defined in (3.4.14) and the masses are

$$m_q^2 = 2q^2v^2 > 0, \qquad m_\lambda^2 = 4\lambda v^2 > 0, \qquad m_g = gv^2, \tag{1.7.18}$$

where $\lambda > 0, g, v, q$ are constants. Using (1.7.17b) we derive two Klein-Gordon equations for the real functions $\chi_\pm = \phi_0^*\chi \pm \chi^*\phi_0$ (see Section 3.5.3). As is well known, the Dirac equation can be squared to obtain a Klein-Gordon equation with mass m_g^2 (see (3.3.18)). We can now state a slightly more precise version of theorem 1.44.

Theorem 1.47 (§3 Th. 3.2, see also [DLW19])**.** *Consider the system* (1.7.17)*. There exists $\varepsilon > 0$, which is independent of m_g, such that for all compactly supported, Lorenz compatible initial data, with Sobolev norm (see* (3.1.9)*) sufficiently smaller than ε, there exists a global-in-time solution (A, χ, ψ) with*

$$|A| \lesssim \varepsilon t^{-3/2}, \qquad |\chi| \lesssim \varepsilon t^{-3/2}, \tag{1.7.19a}$$

$$|\psi| \lesssim \varepsilon \min\big(t^{-1}, |m_g|^{-1}t^{-3/2}\big). \tag{1.7.19b}$$

Since we are dealing with Klein-Gordon equations, our proof uses the hyperboloidal foliation discussed in Section 1.4. For simplicity, we study $m_g \geq 0$ since the case $m_g < 0$

follows in the same way. The result of theorem 1.47 holds in a fairly straightforward way for the cases $m_g = 0$ or $m_g \simeq \min(m_q, m_\lambda)$. However much more is required to obtain a result uniform in terms of the mass parameter $m_g \in [0, \min(m_q, m_\lambda)]$. To achieve this, we define an energy for the first order Dirac equation on hyperboloidal slices which gives a weighted L^2 estimate

$$\int_{\mathcal{H}_s} \Big(\psi^* \psi - \frac{x_i}{t} \psi^* \gamma^0 \gamma^i \psi \Big) \, dx \geq \frac{1}{2} \|(s/t)\psi\|_{L^2(\mathcal{H}_s)}. \tag{1.7.20}$$

The weighted L^2 estimate for the Dirac equation is independent of its mass, and thus in stark contrast to what one normally obtains for Klein-Gordon equations (as discussed at the start of Section 1.4). Since the Dirac equation can be used to obtain a Klein-Gordon equation, we also have available standard hyperboloidal energies like (1.4.13). The combination of estimates satisfied by these energy integral quantities allows us to obtain the interpolated decay estimate (1.7.19b).

Indeed there is typically a dichotomy between the decay of a massive scalar (1.4.6) and the decay of a massless scalar (1.2.2), in the sense that taking the zero-mass limit in the decay rate of the former does not reduce to the decay rate of the latter. However in the case of a Dirac field, which of course is not technically a scalar, the decay estimate (1.7.19b) indicates that this limit is possible.

Finally note that our proof also relies on an additional transformation on the variables A^ν and $\chi_\pm$. These are heuristically of the form

$$A^\nu = \widetilde{A}^\nu + \mathcal{O}(|\psi|^2), \qquad \chi_+ = \widetilde{\chi}_+ + \mathcal{O}(|\psi|^2). \tag{1.7.21}$$

The transformation follows a similar idea used in [Tsu03a] and allows us to remove the slowly decaying nonlinearities (the non-F terms) given in (1.7.17). By theorem 1.24 the F nonlinearities in (1.7.17)) are easy to control. Finally, we note that our method can also be applied to prove the stability of the ground state $(A^\mu_0, \psi_0) = (0, 0)$ of the Dirac-Proca equations (1.7.5) for an appropriate class of gauge-fixed initial data.

Theorem 1.48 ([DLW19])**.** *The ground state of the Dirac-Proca equations* (1.7.5) *is classically stable to compactly supported perturbations.*

1.7.4 Stability of a coupled wave–Klein-Gordon system with quadratic nonlinearities

We view theorem 1.47 as a stepping-stone towards the Glashow-Weinberg-Salam (GSW) model of the electroweak, Higgs and Yukawa sectors of the Standard Model. Maxwell's theory of electromagnetism describes a gauge theory with an abelian group $U(1)$ [Max73]. Yang and Mills [YM54] extended this concept to nonabelian groups, in particular $SU(3)$, which was then used by Glashow [Gla61] to describe weak and electromagnetic interactions using the gauge group $SU(2) \times U(1)$. By work of Weinberg [Wei67] and Salam [Sal68], Glashow's theory was combined with the Higgs mechanism [Hig64] as a way to explain the origin of the Z^0 and $W^\pm$ boson masses. The $Z^0, W^\pm$ bosons and the Higgs boson were experimentally detected in 1983 and 2012 respectively. The analogous result of theorem 1.44 for the GSW model remains open:

Conjecture 1.49. *The ground state of the Glashow-Salam-Weinberg model is classically stable.*

Substantial progress towards addressing this conjecture has been made by Shijie Dong, Philippe LeFloch and the author. The main distinction between conjecture 1.49

and theorem 1.44 is going from an abelian gauge group to a rather special nonabelian gauge group which, from the point of view of the PDEs, leads to nontrivial (quadratic yet no derivatives) wave–Klein-Gordon interactions. Note that previous PDE work studying Higgs fields assume that the Higgs potential does not have a non-zero constant term, see [EM82, eq. 2.49]) and [CBC81, eq. 2.12], which prevents these important nontrivial wave–Klein-Gordon interactions from occurring.

In Chapter 4 we present some of the preliminary work addressing conjecture 1.49, in particular the following theorem.

Theorem 1.50 (§4 Thm. 4.1, see also [SW20]). *Consider the Cauchy problem for the coupled wave–Klein-Gordon equations*

$$\Box u = uv + u\partial_t v, \tag{1.7.22a}$$

$$\Box v - m^2 v = uv, \tag{1.7.22b}$$

For all compactly supported initial data smaller (in the sense of (4.1.3)*) than ε there exists a global-in-time solution (u, v) with*

$$|u(t,x)| \lesssim \varepsilon t^{-1}, \quad |v(t,x)| \lesssim \varepsilon m^{-1} t^{-3/2}. \tag{1.7.23}$$

The overall sign in (1.7.22) is irrelevant to that given later in theorem 4.1. The first key idea in the proof of theorem 1.50 is to treat the nonlinearity in equation (1.7.22b) on the left-hand-side as a perturbed mass $\sqrt{m^2+u}$. We then adapt robust pointwise decay estimates for Klein-Gordon equations given in [LM16a] and energy estimates given in [LM14]. To prove these modified estimates we require improved L^2-type and L^∞ norms for the wave component u. We obtain the former from a conformal-type energy estimate, first introduced on hyperboloids by Huang and Ma [MH17], and the latter from robust pointwise decay estimates for linear wave equations given in [LM16a].

The second key insight involves a combination of two transformations introduced in [Tsu03a, Kat12]. To treat the nonlinear term $u\partial_t v$ in (1.7.22a) we first note the identity $u\partial_t v = \partial_t(uv) - v\partial_t u$. Then we split the wave as $u = U_1 + \partial_t U_2$. This leads to two new wave equations for U_1 and U_2. Significantly the first of these equations now contains the nonlinearity $v\partial_t u$ which has much better properties than $u\partial_t v$. However both equations still retain uv nonlinear terms. To treat these we transform the variables again. This transformation heuristically takes the form

$$U_p = \widetilde{U}_p + \mathcal{O}(uv), \tag{1.7.24}$$

for $p \in \{1, 2\}$. This transformation is similar to that used in (1.7.21) for theorem 1.47. We then obtain two new wave equations for $\tilde{U}_1$ and $\tilde{U}_2$ whose nonlinearities now contain additional null forms which, as seen in theorem 1.15, can be controlled.

Although theorem 1.50 unfortunately does not deal with all of the problematic terms required to prove conjecture 1.49, it does allow us to show the stability of the trivial solution of the Klein-Gordon-Zakharov equations. These equations read

$$\begin{aligned} \Box u &= -\sum_{K=1}^{3} \Delta_{\mathbb{R}^3} |v_K|^2, \\ \Box v_K - v_K &= -u v_K, \end{aligned} \tag{1.7.25}$$

where $u : \mathbb{R}^{1+3} \to \mathbb{R}$ and $v_K : \mathbb{R}^{1+3} \to \mathbb{C}$ for $K = 1, 2, 3$. The system describes the turbulence of Langmuir waves in high-frequency plasmas, see for example [TtH78].

Theorem 1.47 and theorem 1.48 are known for non-compactly supported data, although *without* the uniform result in the fermion mass, in [Tsu03b] and [Tsu03a] respectively. The method in [Tsu03b, Tsu03a] relies on constant time slices, and thus requires some very subtle decomposition of the nonlinearities into strong null forms (see Remark 1.41). Our theorems 1.47 and 1.48 do not require such subtle decompositions, which is important since the decomposition does not easily extend to the GSW model. The Klein-Gordon-Zakharov equations have also been studied before using constant time slices or phase-space methods in [OTT95, Kat12, Tsu96]. The work in Chapters 3 and 4 nonetheless gives a new application of the hyperboloidal foliation method, in particular following the recent series of works by LeFloch and Ma [LM14, LM16b, LM16a, LM17a, LM18], which hopefully will allow us to resolve Conjecture 1.49.

1.8 Slowly expanding cosmological spacetimes

In Chapter 5 we return to general relativity and study slowly expanding cosmological spacetimes. Typically in cosmological models the decay mechanism comes from the cosmological expansion, and not dispersion. Milne is the FLRW spacetime with the fastest cosmological expansion, and thus volume growth, amongst $\Lambda = 0$ spacetimes (see e.g. [And14]) and so it is the most natural candidate for a future-stable vacuum cosmological spacetime with simple asymptotics and no cosmological constant. We show the following result for the Milne spacetime.

Theorem 1.51 (§5 Th. 5.12, also [FW19])**.** *The four-dimensional generalised Milne spacetime is a stable solution to the Einstein Klein-Gordon equations in the direction of cosmological expansion.*

1.8.1 Friedman-Lemaître-Robertson-Walker spacetimes

We first motivate the study of the Milne spacetime. General relativity has been frequently used in the study of cosmology, from providing a robust explanation for the anomalous precession observed in the perihelion of Mercury [Le 59], to predicting new phenomena, such as the deflection of light in the strong gravitational field around the Sun [DED20]. Standard comological models are based on the cosmological principle which comprises two postulates. The first postulate is that there exists a family of fundamental observers which follow timelike geodesics spanning the spacetime manifold. Their proper time is called cosmic time, which we denote t_c. The second postulate states that the universe is spatially homogeneous and isotropic.

Under the conditions of the cosmological principle, Friedman proposed in [Fri22] a model of an expanding universe $\mathcal{M} = \mathbb{R} \times \Sigma$ with the following line element

$$g = -\mathrm{d}t_c^2 + a(t_c)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\mathbb{S}^2 \right). \tag{1.8.1}$$

Here $a(t_c)$ is the scale factor (expanding if $\dot{a}(t_c) > 0$) and Σ is a Riemannian three-manifold of constant sectional curvature $k \in \{-1, 0, 1\}$. We refer to line elements of the form (1.8.1) as Friedman-Lemaître-Robertson-Walker (FLRW) metrics. The FLRW metric can be supported on the simply-connected Riemmanian manifolds $\mathbb{R}^3, \mathbb{S}^3$ and $\mathbb{H}^3$ which are globally maximally symmetric (i.e. they admit a full set of globally defined independent Killing vector fields). It is also possible to allow a cosmological spacetime to be isotropic but only locally homogeneous, since although the former is rather well confirmed it is difficult, from our position on Earth, to confirm the latter. Thus it is reasonable to allow quotients of the spaces Σ by a subgroup of their isometry group and to consider FLRW metrics with spatial slices with nontrivial topologies that are only locally isometric to $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$ (e.g. $k = 0$ and $\Sigma = \mathbb{R}^3\backslash\mathbb{Z}^3 = \mathbb{T}^3$, or $k = -1$ discussed below). Furthermore from a PDE point of view, when the spacetime undergoes accelerated expansion ($\ddot{a}(t_c) > 0$) the topology of Σ is broadly speaking irrelevant to the analysis, see the discussion in [Rin08, §1.2].

How does a free wave behave on a spacetime of the form (1.8.1)? Heuristically one could argue that the wave length should scale with the expansion factor like $\lambda \sim a$, while the wave's energy is inversely proportional to the wave length. The energy density is then the energy of the wave divided by the volume over which it is distributed. Thus (1.2.3) is replaced instead with:

$$|\partial\psi|^2 \sim \rho \sim \frac{a^{-1}}{a^3} \Rightarrow |\partial\psi| \sim a(t_c)^{-2}. \tag{1.8.2}$$

This argument however is too simplistic, since the waves are sourced by the background curvature and hence scatter as they propagate through the spacetime. Indeed in the flat ($k = 0$) Einstein-de Sitter universe one can show that linear waves decay as $|\partial\psi| \lesssim t_c^{-1}$ in agreement with the standard Minkowski decay rate, while in $k = -1$ the rate becomes faster $|\partial\psi| \lesssim t_c^{-2}$ [AC14]. More significantly, in these cases the strong Huygen's principle (theorem 1.11b) no longer holds, see for example [AC14, CNO19] and references cited within. This indicates that the stability mechanism (i.e. decay) comes from the cosmological expansion and thus depends on properties of the scale factor $a(t_c)$.

We return to the FLRW ansatz (1.8.1). The next step in producing a cosmological model is to describe the matter content that interacts with the spacetime. The strong symmetry of the spacetime forces the stress-energy tensor to take the form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}. \tag{1.8.3}$$

This is the stress-energy for a perfect fluid with (average) density ρ, pressure P and four-velocity u^μ. The governing PDEs for this gravity-fluid system are the Einstein relativistic-Euler equations which, due to the strong symmetry, reduce to the following ODEs

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}, \tag{1.8.4a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P), \tag{1.8.4b}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P). \tag{1.8.4c}$$

To close the system we impose the baryonic equation of state $P = c_s^2\rho$ relating the pressure to the density with $|c_s| \leq 1$. Three important solutions to these equations are the Milne, de Sitter and Einstein de-Sitter spacetimes, which undergo linear, accelerated and decelerated expansion respectively.

$$g_{\text{Milne}} = -\mathrm{d}t_c^2 + t_c^2 g_{\mathbb{H}^3}, \qquad (\rho, P) = (0, 0), \tag{1.8.5a}$$

$$g_{\text{dS}} = -\mathrm{d}t^2 + e^{2t_c/\ell} g_{\mathbb{R}^3}, \qquad (\rho, P) = (\Lambda = 3/\ell^2, -\Lambda). \tag{1.8.5b}$$

$$g_{\text{EdS}} = -\mathrm{d}t_c^2 + t_c^{4/3}\mathrm{d}\mathbb{S}^3, \qquad (\rho, P) = (\tfrac{4}{3}t_c^{-2}, 0), \tag{1.8.5c}$$

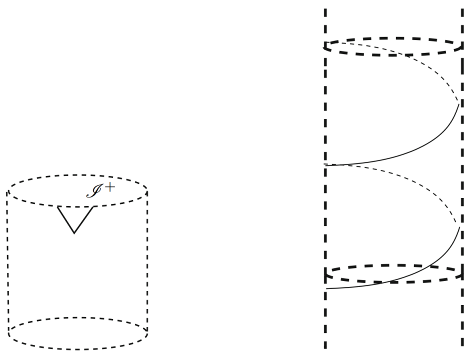


Figure 1.2: Conformal diagrams of de Sitter (left) and Milne spacetimes. Reproduced with permission from [And14].

Firstly we see that the Milne spacetime has linear expansion rate $a(t_c) \sim t_c$ which a priori seems very slow compared to the accelerated expansion rate of de Sitter $a(t_c) \sim e^{t_c}$. However the system (1.8.4) can be studied from a dynamical systems perspective with the solutions (1.8.5) arising as equilibrium points. In the case $\Lambda = 0$ there are two equilibrium points. The first is a source, corresponding to the unstable Einstein de-Sitter universe (1.8.5c) which has slow volume growth, while the second is a sink, corresponding to the stable Milne universe (1.8.5a) which has maximal volume growth amongst $\Lambda = 0$ models. This why Milne is sometimes referred to as an 'attractor' amongst $\Lambda = 0$ FLRW models. A similar analysis when $\Lambda \geq 0$ shows that de Sitter arises as an asymptotic attractor state. For further details see [And14].

From a PDE perspective the Milne spacetime behaves differently to spacetimes undergoing accelerated expansion. Note first that de Sitter spacetime is conformal to part of the Einstein cylinder with a spacelike conformal boundary $\mathscr{I}^+$, see Figure 1.2. For spacetimes undergoing accelerated expansion one can typically 'localise' the PDE arguments so that stability only needs to be shown for initial data that is *local* in space. For example in [Rin08] initial data for (1.8.5b) is considered at a point $p \in \Sigma$ on the initial hypersurface. Due to the rapid volume expansion:

$$J^+(B_\ell(p)) \subseteq D^+(B_{3\ell}(p)). \tag{1.8.6}$$

That is, the solution in the causal future of a ball $B_\ell(p)$ of radius ℓ around a point p is fully determined by data in the set $B_{3\ell}(p)$. Contrast this to the wave equation in Minkowski (theorem 1.11a) where $J^+(B_R(p)) \subseteq D^+(\mathbb{R}^3)$. This localisation analysis also holds for spacetimes undergoing power-law inflation, where the metric takes the form

$$g_{\mathrm{PL}} = -\mathrm{d}t^2 + t^{2p} g_{\mathbb{T}^3}, \tag{1.8.7}$$

and stability of such spacetimes can be shown in the accelerating regime $p > 1$ [Rin09b]. By contrast the Milne spacetime is conformal to an infinite cyclinder, see Figure 1.2, and so topology is relevant since an observer is able to see the whole past of their spacetime. Furthermore there are no particle or event horizons in Milne and so it sits precisely at the threshold where the localisation methods fails.

1.8.2 Attractors of the Einstein–Klein-Gordon system

In Chapter 5 we investigate the stability of the following generalised Milne spacetime as a solution to the Einstein–Klein-Gordon equations.

Definition 1.52 (Generalised Milne spacetime)**.** Let $(\mathcal{K}, \gamma)$ be a Riemannian negative Einstein three-manifold with Einstein constant $k = -2/9$ (see definition 1.29). The generalised Milne spacetime is the Lorentz cone spacetime $\mathcal{M} = (0, \infty) \times \mathcal{K}$ with metric

$$g_{\text{Milne}} = -\mathrm{d}t_c^2 + \frac{t_c^2}{9}\gamma. \tag{1.8.8}$$

The generalised Milne spacetime is globally hyperbolic and a solution to the four-dimensional vacuum Einstein equations. By taking an appropriate quotient of $\mathbb{H}^3$ the definition includes the spatially compact $k = -1$ case of (1.8.1).

Given the scaling apparent in cosmological spacetimes of the form (1.8.8) it is natural to put a $\tilde{}$ or overline on all 'physical' quantities from now on which we then rescale. For simplicity here however, we refrain from introducing the rescaled variables until Chapter 5. The dynamical metric $\overline{g}_{\mu\nu}$ evolves as a small perturbation of (1.8.8). It is governed by the Einstein equations

$$\mathrm{Ric}[\overline{g}]_{\mu\nu} - \frac{1}{2}\mathrm{R}[\overline{g}]\overline{g}_{\mu\nu} = 8\pi T_{\mu\nu}[\tilde{\phi}], \tag{1.8.9a}$$

$$\overline{g}^{\mu\nu}\nabla[\overline{g}]_\mu\nabla[\overline{g}]_\nu\tilde{\phi} = m^2\tilde{\phi}, \tag{1.8.9b}$$

with the Klein-Gordon stress-energy tensor

$$\tilde{T}_{\mu\nu}[\tilde{\phi}] = \nabla[\overline{g}]_\mu\tilde{\phi}\nabla[\overline{g}]_\nu\tilde{\phi} - \tfrac{1}{2}\overline{g}_{\mu\nu}\left(\overline{g}^{\rho\sigma}\nabla[\overline{g}]_\rho\tilde{\phi}\nabla[\overline{g}]_\sigma\tilde{\phi} + m^2\tilde{\phi}^2\right). \tag{1.8.9c}$$

We express $\overline{g}_{\mu\nu}$ in terms of the ADM variables (see Notation 1.2)

$$\overline{g} = -\tilde{N}^2dt^2 + \tilde{g}_{ab}(dx^a + \tilde{X}^a dt)(dx^b + \tilde{X}^b dt). \tag{1.8.10}$$

For $t = t_c$ the generalised Milne spacetime (1.8.8) corresponds to

$$(\tilde{g}_{ab}, \tilde{k}_{ab}, \tilde{N}, \tilde{X}^a)|_{\text{Milne}} = \left(\frac{t_c^2}{9}\gamma_{ab}, -\frac{t_c}{9}\gamma_{ab}, 1, 0\right). \tag{1.8.11}$$

Note that the mean curvature $\tau = \tilde{g}^{ab}\tilde{k}_{ab}$ for Milne is given by $\tau|_{\text{Milne}} = -3t_c^{-1}$. Thus hypersurfaces of constant t_c are also surfaces of constant mean curvature (CMC) and so we could instead use $\tau \in (-\infty, 0]$ as a time function where $\tau \to 0$ corresponds to the direction of cosmological expansion.

This motivates the gauge choice for equations (1.8.9a). The future ($t_c \to \infty$) stability of the generalised Milne spacetime as a solution to the vacuum Einstein equations is known due to a series of works by Andersson and Moncrief [AM03, AM11]. They use the constant mean curvature, spatially harmonic gauge (CMCSH):

$$t = \tau, \tag{1.8.12a}$$

$$V^a = \tilde{g}^{bc}(\Gamma^a_{bc}[\tilde{g}] - \Gamma^a_{bc}[\gamma]) = 0. \tag{1.8.12b}$$

The first condition is in fact a geometric restriction, since not all spacetimes admit a time function whose leaves Σ_t are CMC [CIP04]. For our purposes however, the Milne spacetime, and small perturbations, do admit CMC slicing. The second condition is the spatial version of the $\hat{e}$-wave gauge discussed in Definition 1.7.

With respect to (1.8.12) the Einstein equations reduce to hyperbolic equations for the first and second fundamental forms $\tilde{g}_{ab}$ and $\tilde{k}_{ab}$ and elliptic equations for the lapse $\tilde{N}$ and shift $\tilde{X}$. The analogous version of theorem 1.6 for this mixed hyperbolic-elliptic system was given in [AM03]. As an example two of these equations read

$$\begin{aligned}\partial_t\tilde{k}_{ab} &= -\nabla[\tilde{g}]_a\nabla[\tilde{g}]_b\tilde{N} + \tilde{N}\left(\mathrm{Ric}[\tilde{g}]_{ab} + \tau\tilde{k}_{ab} - 2\tilde{k}_{ac}\tilde{k}^c{}_b\right) + \mathscr{L}_{\tilde{X}}\tilde{k}_{ab} \\ &\quad - 8\pi\tilde{N}\tilde{S}_{ab} - 4\pi\tilde{\rho}\tilde{g}_{ab}, \qquad (1.8.13)\\ -\Delta\tilde{N} &= 1 - \left(\tilde{k}_{ab}\tilde{k}^{ab} + 4\pi(\tilde{\rho} + \mathrm{tr}_{\tilde{g}}\tilde{T})\right). \qquad (1.8.14)\end{aligned}$$

Condition (1.8.12b) implies that the Ricci tensor appearing in (1.8.13) becomes an elliptic operator involving the Lichnerowicz Laplacian of definition 1.30

$$\mathrm{Ric}[\tilde{g}]_{ab} = -\frac{1}{2}\mathcal{L}_\gamma\tilde{g}_{ab} + S_{ab}, \qquad (1.8.15)$$

and where S_{ab} denotes some higher-order error terms (essentially all quasilinear and nonlinear terms of (1.5.18a)). A major insight of [AM11] was to use the lowest eigenvalue λ_0 of $\mathcal{L}_\gamma$, see Section 1.5 and in particular proposition 1.32, to define L^2-energy norms based on the operator $\mathcal{L}_\gamma$ with small correction terms related to λ_0. These corrected energy norms yield strong energy estimates that yield decay of the geometric perturbations at a rate that implies future geodesic completeness of the spacetime.

For our purposes the key difficulty in addressing theorem 1.51 is the matter field. A crucial difficulty for massive matter models coupled to the Einstein equations for data close to the Milne model results from the slow decay of the lapse. When the lapse decay obtained from (1.8.14) is bootstrapped into the equation of motion (1.8.9b) for the matter, the resulting loss of decay for the matter field is too strong to close the argument. To address this issue we derive an independent estimate using the continuity equation

$$\partial_t\tilde{\rho} = \tilde{X}^a\nabla[\tilde{g}]_a\tilde{\rho} + \tilde{N}(\mathrm{tr}\tilde{k})\tilde{\rho} - \tilde{N}^{-1}\nabla[\tilde{g}]_a(\tilde{N}^2\tilde{\jmath}^a) + \tilde{N}\tilde{k}_{ab}\tilde{T}^{ab}, \qquad (1.8.16)$$

which is a first order evolution equation for the energy density $\tilde{\rho}$. This follows a similar idea used in [AF20] for the massive Einstein-Vlasov system. However unlike that work, we must correct the energy density with a small indefinite term

$$\tilde{\rho} - \frac{1}{2}\tau\tilde{\phi}n^\mu\partial_\mu\tilde{\phi}. \qquad (1.8.17)$$

This corrected energy density fulfils an evolution equation, given in (5.5.10), with only time-integrable terms on the right-hand side. Consequently we obtain improved pointwise bounds on the energy density and, in turn, for the Klein-Gordon field. This allows us to close the bootstrap argument for the full system.

The second feature in the proof of theorem 1.51 is that we use the Laplacian, and its higher powers, in the energy norms of the Klein-Gordon field (e.g. equation (5.3.1)). These norms are equivalent to the standard Sobolev norms $\|\phi\|_{H^k(\mathcal{K})}$ by use of elliptic regularity results on the compact manifold $\mathcal{K}$. In particular for $k \in \mathbb{N}$ we can use the following equivalence

$$\|\phi\|_{H^{k+2}(\mathcal{K})} \cong \|\Delta_\gamma\phi\|_{H^k(\mathcal{K})} + \|\phi\|_{L^2(\mathcal{K})}, \qquad (1.8.18)$$

since our Klein-Gordon energy controls the term $\|m\phi\|_{L^2(\mathcal{K})}$. Note this crucially relies on our spatial slices being compact, which is in stark difference to the Klein-Gordon methods on $\mathbb{R}^n$ discussed in Section 1.4. Using Laplacians in the energy norms, the modified continuity equation, and the correct energy norms of [AM11] for the geometric perturbation makes our proof significantly shorter than another in the literature [Wan19].

1.9 String theory compactifications

In Chapter 6 we consider the stability of product spacetimes where the internal manifold admits a parallel spinor. These spacetimes generalise the Kaluza-Klein spacetimes of Chapter 2 and play an essential role in supergravity and string theory, see for example [Pol07, FVP12, CHSW85]. In particular unlike the toroidal-reduction seen in (1.6.4), the goal in these more complicated product spacetimes is to produce, in the low-energy limit, gravity coupled to a non-abelian Yang Mills gauge theory and chiral fermions. Although the torus admits parallel spinors, to obtain more complicated chiral matter fields typically one considers a non-trivial special holonomy manifold such as a Calabi-Yau 3-fold [CHSW85]. We prove the following result in Chapter 6.

Theorem 1.53 (§6 Th. 6.1, see also [ABWY20])**.** *Let $n \geq 9$ and let $(\mathcal{K}, \gamma)$ be a compact Riemannian manifold which has a spin cover and admits a parallel spinor (e.g. $\mathbb{T}^d$, Calabi-Yau 3-fold, G_2 or Spin(7)). The product spacetime $(\mathbb{R}^{1+n} \times \mathcal{K}, \eta_{\mathbb{R}^{1+n}} + \gamma)$ is a stable solution to the vacuum Einstein equations for sufficiently small initial data that is exactly the product of Schwarzschild with $(\mathcal{K}, \gamma)$ outside of a compact set.*

The internal manifolds considered in theorem 1.53 satisfy the assumptions of theorem 1.34. We refer to the product spacetime $\mathbb{R}^{1+n} \times \mathcal{K}$ with metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + \gamma \tag{1.9.1}$$

as a spacetime with a supersymmetric compactification[3]. Note that $(\mathbb{R}^{1+n} \times \mathcal{K}, \hat{g})$ is globally hyperbolic and a solution to the $(1+n+d)$-dimensional vacuum Einstein equations. The simplest spacetime with a supersymmetric compactification is the Kaluza-Klein spacetime $(\mathbb{R}^{1+3} \times \mathbb{T}^d, \eta + \delta)$. It is fairly straightforward to believe that the analogue of Theorem 1.38 holds when the Minkowski dimension is large, i.e. when $n \geq 9$, and for toroidal-independent data that is exactly the product of Schwarzschild with $(\mathbb{T}^d, \delta)$ outside of a compact set. In this very specialised setting, the result of Theorem 1.53 allows us to remove the restriction to toroidal-independent initial data.

1.9.1 On the instability of extra space dimensions

Penrose has sketched an argument intended to show that spacetimes with supersymmetric compactifications are generically classically unstable, for every dimension n and all internal manifolds, except possibly when the internal manifold is a flat d-dimensional torus [Pen05, Pen03]. We briefly discuss some aspects of his argument and their relation to our theorem 1.53. In a similar way to the discussion of permissible Kaluza-Klein topologies in section 1.6.1, the main issue seems to be about what kind of perturbations are allowed and deemed reasonable.

[3]This a slight abuse of terminology since for supersymmetry to hold and not produce unobserved high-spin particles there are somewhat stringent conditions on the maximal dimension $1+d+n$ [Nah78].

Penrose's main claim is that initial data perturbations which depend only on $\mathcal{K}$, and thus do not 'leak out' into the external spatial geometry $\mathbb{R}^3$, must contain an appropriately trapped submanifold in $\mathcal{K}$ which, by an appropriate extension of Penrose and Hawking's singularity theorem [HP70], would imply that the full spacetime is geodesically incomplete. There are theorems motivated by these considerations that generalize the classical singularity theorems to trapped surfaces of arbitrary co-dimension [GS10] and have been applied to certain product spacetimes [CS19], thus substantiating his claim.

However the initial data considered by Penrose, and the later work [CS19], should be thought of as 'globally finite' and not compatible with theorem 1.53 where only 'globally decaying' data is allowed. To explain these terms more precisely, in theorem 1.53 we require appropriate asymptotic decay of the initial perturbation in the sense that the derivatives of the metric perturbation and second fundamental form must have small $L^2(\mathbb{R}^n \times \mathcal{K})$ norm, see (6.1.5). By contrast a 'globally finite' perturbation, say of uniform size ε on $\mathcal{K}$ and independent of $\mathbb{R}^n$, will, roughly speaking, lead to an integral of $|\varepsilon|\mathrm{vol}(\mathcal{K})$ over $\mathbb{R}^n$, and thus have an infinite $L^2(\mathbb{R}^n)$ norm.

The work [CS19] looks at spacetimes $\mathbb{R}^{1+3} \times \mathcal{K}$ with a warped metric $g = \eta_{\mathbb{R}^{1+n}} + f^2(t, x^a)\gamma$ such that the warping function $f(t, x^a)$ depends only on the Minkowski coordinates. Such spacetimes can be interpreted as zero-mode scalar perturbations of the background spacetime with $f \equiv 1$. The authors derive conditions on the warping function which lead to geodesic incompleteness. However these conditions are clearly very strong, since they also find that for warping functions such that $|f - 1| \leq 1/2$ for all $(t, x) \in \mathbb{R}^{1+3}$, the spacetime is geodesically complete [CS19, Theorem 3.3]. A spacetime evolving from 'globally decaying' initial data would certainly be covered by this situation.

Penrose also claims that perturbations that also depend on the $\mathbb{R}^3$ geometry will result in spacetime singularities forming, due to the large Planck-scale curvatures in $\mathcal{K}$ (which is postulated to be very small) 'spilling over' into $\mathbb{R}^3$. In this situation Theorem 1.53 does apply and shows that this isn't the case, at least when $n = 9$. Penrose's argument is made for Minkowski spatial dimension $n = 3$ however it does not depend on this fact. In particular, Penrose's argument does not take into account properties of dispersion, such as how waves decay faster with increasing n, seen in (1.2.2), or how massive fields decay at faster rates to massless ones, seen in (1.4.6). We expect that the assumptions that $n \geq 9$, and that the Cauchy data is Schwarzschild near infinity can be relaxed. In fact we make the following loose conjecture.

Conjecture 1.54. *Theorem 1.53 holds when $n = 3$ and for all small data that is asymptotically flat in the external directions.*

1.9.2 Global stability of spacetimes with supersymmetric compactifications

The proof of theorem 1.53 should be seen as a kind of unification of the ideas discussed so far: the motivation for product spacetimes comes from Chapter 2, the hyperboloidal method used to study Klein-Gordon equations in Chapter 3 and the importance of spectral properties of the Lichnerowicz Laplacian from Chapter 5.

We use the $\hat{e}$-wave gauge from definition 1.7 with $\hat{e} = \hat{g}$. The associated reduced Einstein equations written in terms of the perturbation and inverse perturbation are given in (1.5.18). The linearisation of (1.5.18) is

$$(\Box_\eta + \Delta_\gamma + 2R[\hat{g}]\circ)h_{\mu\nu} = 0. \tag{1.9.2}$$

From these equations we can see that the non-negativity of the spectrum of the Lichnerowicz Laplacian on symmetric 2-tensors, which holds for the internal spaces by theorem 1.34, plays a crucial role. By using definition 1.30 note that

$$\begin{aligned}\mathcal{L}_\gamma h_{i\mu} &= -(\Delta_\gamma + 2R[\hat{g}]\circ)h_{i\mu} = -\Delta_\gamma h_{i\mu},\\ \mathcal{L}_\gamma h_{AB} &= -(\Delta_\gamma + 2R[\hat{g}]\circ)h_{AB} = -(\Delta_\gamma + 2R[\gamma]\circ)h_{AB}.\end{aligned} \tag{1.9.3}$$

Since $\mathcal{K}$ is compact these self-adjoint operators each have a discrete spectrum of eigenvalues. We can consider decomposing solutions of the linearised equations (1.9.2) in terms of eigenfunctions of the operators $-\Delta_\gamma$ and $\mathcal{L}_\gamma$ appearing in (1.9.3). On a compact manifold it is standard to show that the eigenvalues of $-\Delta_\gamma$ are nonnegative. Theorem 1.34 implies that the eigenvalues of $\mathcal{L}_\gamma$ are nonnegative. Thus equations (1.9.2) lead to an infinite collection of fields $h^{(n)}_{\mu\nu}$ (sometimes called a Kaluza-Klein tower of modes) each obeying Klein-Gordon equations

$$(\Box_\eta - \lambda_n)h^{(n)}_{\mu\nu} = 0, \tag{1.9.4}$$

where the mass squared is given by the eigenvalues $0 \leq \lambda_n \to \infty$ of the appropriate operator, i.e $-\Delta_\gamma$ or $\mathcal{L}_\gamma$. A decomposition of this type for the operator Δ_γ has previously been used in the analysis of wave guides, where $\mathcal{K}$ is replaced by a compact subset of $\mathbb{R}^d$ with Neumann boundary conditions, see e.g. [MSS05, MS08], and in the study of a semilinear wave equation describing a 3-form field on a fixed product spacetime [Ett15].

To study conservation properties of solutions of the linearised equations (1.9.2) we consider the divergence of a novel stress-energy tensor

$$T^\mu{}_\nu = \hat{g}^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha h, \nabla[\hat{g}]_\nu h\rangle_E - \frac{1}{2}\hat{g}^{\alpha\beta}\langle\nabla[\hat{g}]_\beta h, \nabla[\hat{g}]_\alpha h\rangle_E\delta^\mu_\nu + \langle R[\hat{g}]\circ h, h\rangle_E\delta^\mu_\nu, \tag{1.9.5}$$

with $\langle\cdot,\cdot\rangle_E$ given in (6.1.4). This stress-energy tensor is specifically adapted to the tensorial operator appearing in (1.9.2). Indeed using (1.9.5) it is fairly straightforward to show that a spacetime with a supersymmetric compactification is a stable solution to the *linearised* vacuum Einstein equations for $n \geq 3$.

For the quasilinear reduced Einstein equations (1.5.18) however, we refrain from performing a spectral decomposition and also require a perturbed version of (1.9.5). Nonetheless one should think that effectively our system (1.5.18) contains terms with zero eigenvalue, corresponding to a wave equation, as well as terms with the positive eigenvalues, corresponding to effective Klein-Gordon equations.

Consequently our proof uses a relatively simple vector-field argument which lies at the intersection of the wave *and* Klein-Gordon methods discussed in Sections 1.3 and 1.4. In particular, we obtain a decay rate of $|h| \lesssim t^{-\delta(n)}$ with $\delta(n) = (n-2)/4$. This is far worse than the rate $t^{-(n-1)/2}$ seen in (1.3.2) and (1.6.23) for quasilinear wave equations, and $t^{-n/2}$ seen in (1.4.6) for Klein-Gordon equations.

In light of this, it seems likely that some novel refinement should allow for a significantly better decay rate. Our proof already contains two types of refinement. First, the decay rate is shown to be $s^{-2\delta(n)}$ where $s^2 = t^2 - x^2$ inside light cones. The exponent $2\delta(n) = (n-2)/2$ is much closer to the decay rate for the wave and Klein-Gordon equation. Second, the same decay rates are proved for $\Gamma^I h$ as for h, but, since the Γ contain t- and x-dependent weights, with respect to a translation invariant basis in Minkowski space, derivatives decay faster than the field h itself.

Having obtained a linear estimate that improves with increasing n, we take n sufficiently large so that the nonlinear terms decay sufficiently fast for the linear estimates to remain valid. This is obviously in stark contrast to the analysis of null and weak null structures discussed in Section 1.6 and Chapter 2.

A solution of the $(1+n)$-dimensional vacuum Einstein equations for $n \leq 11$ can be considered as a particular solution of the supergravity equations. These are roughly speaking the Einstein equations coupled to specific matter, including p-form gauge fields, spinors and massless scalars. See for example [FVP12]. Local-in-time existence results are known for the supergravity equations [CB85]. The field equation studied in [Ett15] describes a particular 3-form field arising from the supergravity equations with the gravitational interaction turned off. In future work, and in the hope of resolving conjecture 1.54, we intend to study the stability of spacetimes with supersymmetric compactifications under the supergravity field equations.

Chapter 2

The Zero-Mode Stability of Kaluza-Klein Spacetimes

2.1 Introduction

In this chapter, we study the classical stability of the Kaluza-Klein spacetime $\mathbb{R}^{1+3} \times \mathbb{T}^d$ with metric

$$\hat{G} = \eta_{\mathbb{R}^{1+3}} + \delta_{\mathbb{T}^d}, \tag{2.1.1}$$

and prove the following result.

Theorem 2.1. *The Minkowski vacuum of the Einstein-Maxwell-Scalar system arising from the zero modes of* $(3 + d + 1)-$*dimensional pure Einstein theory compactified on a flat* $\mathbb{T}^d$ *is nonlinearly stable. Furthermore, the radii of the* $\mathbb{T}^d$ *are nonlinearly stable to perturbations of the zero modes.*

The nonlinear stability we consider is subject to the symmetry assumption that the perturbations only depend on the non-compact directions. This symmetry assumption, also called the zero-mode truncation, is, in the physics literature, called consistent since it yields solutions of the full equations of motion of the higher dimensional theory. Indeed initial data obeying this symmetry will yield a solution similarly invariant in the compact directions.

As frequently done from an effective theory point of view, one can further make a heuristic physical argument that for sufficiently small initial compact radii, it is in fact sufficient to only consider zero-mode perturbations [Pop]. Our result shows that the radii are nonlinearly stable to zero-mode perturbations. Of course from the nonlinear PDE point of view, the dynamics from the non-zero-modes are still relevant, see Chapter 6. Nonetheless, stability of the zero-modes is a necessary first step.

We study the perturbed spacetime Kaluza-Klein spacetime using the $\hat{e}$-gauge condition of definition 1.7 and choosing $\hat{e} = \hat{G}$. Clearly on Minkowski indices this reduces to the standard wave gauge condition introduced in definition 1.9. We now introduce the PDE, and defer the application to Kaluza-Klein and theorem 2.1 until section 2.9.

Remark 2.2. The indices in this chapter, except for subsection 2.9, are the four-dimensional ones of definition 1.1.

Our proof very closely follows the seminal stability method of [LR05, LR10] which showed the stability of Minkowski spacetime using wave gauge. We now briefly restate their set-up. The four-dimensional Einstein equations in wave coordinates coupled to a scalar field, written in terms of the perturbation $h_{\mu\nu} = g_{\mu\nu} - (\eta_{\mathbb{R}^{1+3}})_{\mu\nu}$ away from Minkowski, take the form

$$\begin{aligned} \widetilde{\Box}_g h_{\mu\nu} &= F_{\mu\nu}(h)(\partial h, \partial h) + 2\partial_\mu \psi \partial_\nu \psi \,, \\ \widetilde{\Box}_g \psi &= 0 \,, \end{aligned} \tag{2.1.2a}$$

where $\widetilde{\Box}_g g_{\mu\nu} = g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu}$ is the reduced wave operator written in terms of the wave coordinates and the inhomogeneity takes the form

$$\begin{aligned} F_{\mu\nu}(h)(\partial h, \partial h) &= P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h) \,, \\ P(\partial_\mu h, \partial_\nu h) &= \frac{1}{4}\partial_\mu(\mathrm{tr}_\eta h)\partial_\nu(\mathrm{tr}_\eta h) - \frac{1}{2}\partial_\mu(\eta^{\rho\lambda}\eta^{\sigma\alpha}h_{\lambda\alpha})\partial_\nu h_{\rho\sigma}. \end{aligned} \tag{2.1.2b}$$

Note the term involving ψ in (2.1.2a) comes from adding a stress energy tensor

$$T^{\psi}_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \tfrac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\psi\partial_\sigma\psi), \tag{2.1.3}$$

to the Einstein equations. Here $Q_{\mu\nu}$ is a linear combination of the classic null forms (1.6.13) and $G_{\mu\nu}(h)(\partial h, \partial h)$ represents terms quadratic in ∂h with coefficients that smoothly depend on h and vanishing for $h = 0$: i.e. $G(0)(\partial h, \partial h) = 0$. Note for simplicity we use here the notation $F_{\mu\nu}$ instead of $\mathcal{F}_{\mu\nu}$ as in (1.1.5).

Definition 2.3 (Generalised PDE system)**.** In this chapter we consider the unknowns

$$W = \{h_{\mu\nu}\}_{\mu,\nu\in\{0,1,2,3\}} \cup \{\psi_K\}_{K\in\{1,\dots,m\}} \,, \tag{2.1.4}$$

for some $m \in \mathbb{N}$. The index on ψ_K is a label and not a covariant index. These unknowns satisfy the following generalised PDE system in *four* spacetime dimensions:

$$\begin{aligned} \widetilde{\Box}_g h_{\mu\nu} &= F_{\mu\nu}(W)(\partial W, \partial W) \,, \\ \widetilde{\Box}_g \psi_K &= F_K(W)(\partial W, \partial W) \,, \end{aligned} \tag{2.1.5a}$$

where we define the nonlinearities by

$$\begin{aligned} F_{\mu\nu}(W)(\partial W, \partial W) &= P(\partial_\mu W, \partial_\nu W) + Q_{\mu\nu}(\partial W, \partial W) + G_{\mu\nu}(W)(\partial W, \partial W), \\ F_K(W)(\partial W, \partial W) &= Q_K(\partial W, \partial W) + G_K(W)(\partial W, \partial W), \end{aligned} \tag{2.1.5b}$$

together with the *four*-dimensional wave-coordinate condition

$$\partial_\rho\left(g^{\rho\mu}\sqrt{|\det g|}\right) = 0, \tag{2.1.5c}$$

where we have the relationship

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.1.5d}$$

The quadratic terms $Q_{\mu\nu}, Q_K$ are unspecified linear combinations of the null forms (1.6.13) in terms of ∂W variables, contracting the arguments of the null forms with $\eta_{\mu\nu}$ and/or arbitrary $N^K \in \mathbb{R}^m$ as appropriate. The remaining non-null $\mathcal{O}((\partial W)^2)$ terms are defined by

$$P(\partial_\mu W, \partial_\nu W) = \frac{1}{4}\partial_\mu(\mathrm{tr}_\eta h)\partial_\nu(\mathrm{tr}_\eta h) - \frac{1}{2}\partial_\mu(\eta^{\rho\lambda}\eta^{\sigma\alpha}h_{\lambda\alpha})\partial_\nu h_{\rho\sigma} \\ + N^K\eta^{\rho\sigma}\left(\partial_\mu h_{\rho\sigma}\partial_\nu\psi_K + \partial_\nu h_{\rho\sigma}\partial_\mu\psi_K\right) \\ + N^{KK'}\left(\partial_\mu\psi_K\partial_\nu\psi_{K'} + \partial_\nu\psi_K\partial_\mu\psi_{K'}\right). \tag{2.1.5e}$$

Here $N^K \in \mathbb{R}^m, N^{KK'} \in \mathbb{R}^{m^2}$ are some arbitrary constant coefficients and $G_{\mu\nu}, G_K$ are terms quadratic in ∂W with coefficients smoothly dependent on W and vanishing for $G(0)(\partial W, \partial W) = 0$. Note we use compact notation in (2.1.5), writing for example $Q(\partial W, \partial W)$ to represent arbitrary combinations of $Q(\partial h, \partial h), Q(\partial h, \partial\psi), Q(\partial\psi, \partial\psi)$.

Although we have added additional nonlinearities to both the $h_{\mu\nu}$ and ψ_K terms, we have specifically only added $\mathcal{O}((\partial W)^2)$ terms which are null forms to F_K. This choice is so that the variables ψ_K obey the same estimates as the 'best' components of $h_{\mu\nu}$.

Definition 2.4. Define the standard spacetime coordinates $\{x^\mu\} = (t, x)$ with $x = (x_1, x_2, x_3)$ and $r = |x|$. Define the collection of vector fields

$$\mathbf{\Gamma} = \{\partial_\mu\,, \Omega_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu\,, S = x^\mu\partial_\mu\}\ .$$

Denote the above vector fields by Γ^ι where $\iota = (0, .., 1, .., 0)$ is an 11-dimensional integer index. Let $I = (\iota_1, \dots, \iota_k)$, where $|\iota_i| = 1$, be a multi-index of length $|I| = k$ and let $\Gamma^I = \Gamma^{\iota_1}\cdots\Gamma^{\iota_k}$ denote a product of k vector fields from the family $\mathbf{\Gamma}$. A sum $I_1 + I_2 = I$ denotes a sum over all possible order preserving partitions of the multi-index I into two multi-indices I_1 and I_2. Let ∇ denote spatial derivatives $\{\partial_1, \partial_2, \partial_3\}$, so that for a multi-index I of length $|I| = k$, ∇^I denotes a product of k spatial derivatives.

Let $\mathcal{U}$ denote the full null frame (see section 2.2 for further details). For any two families $\mathcal{V}$ and $\mathcal{W}$ of vector fields and an arbitary 2-tensor π, define the following pointwise seminorms

$$|\pi|_{\mathcal{V}\mathcal{W}} = \sum_{V\in\mathcal{V}, W\in\mathcal{W}} |\pi_{\mu\nu}V^\mu W^\nu|, \quad |\partial\pi|_{\mathcal{V}\mathcal{W}} = \sum_{\substack{U\in\mathcal{U}\\ V\in\mathcal{V}, W\in\mathcal{W}}} |(\partial_\rho\pi_{\mu\nu})U^\rho V^\mu W^\nu|. \tag{2.1.6}$$

For a collection of scalar fields $\{\Psi_K\}_{K\in\mathcal{K}}$ where $\mathcal{K} = \{1, \dots, m\}$, let

$$|\Psi|_{\mathcal{K}} = \sum_{K=1}^m |\Psi_K|. \tag{2.1.7}$$

Definition 2.5 (Initial data for the reduced system (2.1.5))**.** The initial data consists of the set $(\Sigma_0, \bar{g}_{ab}, \bar{K}_{ab}, \phi_0^K, \phi_1^K)$ where $(\Sigma_0, \bar{g}_{ab})$ is a three-dimensional Riemannian manifold, Σ_0 is diffeomorphic to $\mathbb{R}^3$, $\bar{K}_{ab}$ a symmetric two-tensor on Σ_0, (ϕ_0^K, ϕ_1^K) are smooth functions on Σ_0.

Definition 2.6 (Initial data asymptotics)**.** Let

$$\bar{g}^1_{ab} = \bar{g}_{ab} - \left(1 + \chi(r)\frac{M}{r}\right)\delta_{ab} \tag{2.1.8}$$

where M is the ADM mass parameter for $\bar{g}$ and $\chi(s) \in C^\infty$ is 1 when $s \geq 3/4$ and 0 when $s \leq 1/2$. We assume that as $r \to \infty$ for $\alpha > 0$ the initial data is asymptotically flat in the sense that

$$\begin{aligned} \bar{g}^1_{ab} &= o(r^{-1-\alpha})\,, & \bar{K}_{ab} &= o(r^{-2-\alpha})\,, \\ \phi^K_0 &= o(r^{-1-\alpha})\,, & \phi^K_1 &= o(r^{-2-\alpha}). \end{aligned} \tag{2.1.9}$$

Definition 2.7 (Initial energy)**.** For some $N \in \mathbb{N}, N \geq 6$ and constant $0 < \gamma < 1/2$, we define an initial weighted energy by

$$\begin{aligned} E_N(0) = \sum_{|I|\leq N} \Big(&\|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \bar{g}^1\|^2_{L^2(\Sigma_0)} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I \bar{K}\|^2_{L^2(\Sigma_0)} \\ &+ \|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \phi^K_0\|^2_{L^2(\Sigma_0)} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I \phi^K_1\|^2_{L^2(\Sigma_0)}\Big). \end{aligned} \tag{2.1.10}$$

The choice of $\gamma > 0$ will, amongst other things, allow us to use the modified Hardy estimate given in corollary 2.48.

Definition 2.8 (Full energy, weight function and $h^1_{\mu\nu}$)**.** Define a weight function w and variable q by

$$w := w(q) = \begin{cases} 1+(1+|q|)^{1+2\gamma}\,, & q \geq 0 \\ 1+(1+|q|)^{-2\mu}\,, & q < 0 \end{cases}\,, \quad q = r - t. \tag{2.1.11}$$

Here $\mu > 0$ is a constant to be fixed later. Define the 2-tensor $h^1_{\mu\nu}$ by

$$h_{\mu\nu} = \chi(r)\chi(r/t)\frac{M}{r}\delta_{\mu\nu} + h^1_{\mu\nu}, \tag{2.1.12}$$

where $\chi(s) \in C^\infty$ is 1 when $s \geq 3/4$ and 0 when $s \leq 1/2$. This is also depicted in Figure 2.1 on page 60. Denote the unknown dynamical variables by

$$W^1 = \{h^1_{\mu\nu}\}_{\mu,\nu\in\{0,1,2,3\}} \cup \{\psi_K\}_{K\in\{1,\ldots,m\}}. \tag{2.1.13}$$

Define the weighted energy

$$\mathcal{E}_N[W^1](t) = \sup_{0\leq\tau\leq t} \sum_{|I|\leq N,\,\Gamma\in\mathbf{\Gamma}} \|(w^{1/2})|\partial\Gamma^I h^1(\tau,\cdot)|_{\mathcal{U}\mathcal{U}}\|^2_{L^2} + \|(w^{1/2})|\partial\Gamma^I\psi(\tau,\cdot)|_{\mathcal{K}}\|^2_{L^2}. \tag{2.1.14}$$

Our main result is now:

Theorem 2.9. *Let $(\Sigma_0, \bar{g}_{ab}, \bar{K}_{ab}, \phi^K_0, \phi^K_1)$ be smooth initial data for the PDE system (2.1.5) satisfying definitions 2.5 and 2.6. There exists a constant $\varepsilon > 0$ such that for all initial data satisfying*

$$E_N(0)^{1/2} + M \leq \varepsilon\,, \tag{2.1.15}$$

for some $\gamma > 0$, there exists a global in time solution $(\eta_{\mu\nu}+h_{\mu\nu}(t)\,, \psi_K(t))$ to the system (2.1.5). Moreover the solution satisfies

$$\mathcal{E}_N[W^1](t)^{1/2} \leq C_N\varepsilon(1+t)^{c\varepsilon}, \tag{2.1.16}$$

where C_N is a large constant depending only on N and $c > 0$ is independent of ε.

Furthermore the perturbation decays to zero as

$$|\partial\Gamma^I W^1| \leq \begin{cases} C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-1-\gamma}, & r>t, \\ C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-1/2}, & r\leq t, \end{cases} \quad |I|\leq N-2 \tag{2.1.17}$$

$$|\Gamma^I W^1| \leq \begin{cases} C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-\gamma}, & r>t, \quad |I|\leq N-2, \\ C_N\varepsilon(1+t+|q|)^{-1+2C_N\varepsilon}, & r\leq t, \quad |I|\leq N-2. \end{cases} \tag{2.1.18}$$

where we have used the notation $|W^1| = |h^1|_{\mathcal{UU}} + |\psi|_{\mathcal{K}}$ *and* $|\partial W^1| = |\partial h^1|_{\mathcal{UU}} + |\partial\psi|_{\mathcal{K}}$.

Remark 2.10. We assume that the initial data considered in theorem 2.9 exists. Note that since the data is evolved according to the reduced system of equations (2.1.5), which includes the gauge condition (2.1.5c), it does not necessarily need to satisfy the constraint equations. Only when we wish to relate the solution to a spacetime satisfying the Einstein equations will the constraint equations play an important role.

The bootstrap argument. The proof of theorem 2.9 relies on a continuous induction argument, see in particular the discussion in section 1.3.2. By standard theory of nonlinear wave equations, i.e. proposition 1.10, we can obtain a local-in-time smooth solution $(g_{\mu\nu}(t)\,, \psi_K(t))$ of our PDE obeying the wave-gauge condition in a maximal time of existence $[0, T_*)$. The maximal time of existence T_* is defined by a continuation criterion, namely blow-up of the energy: $\mathcal{E}_N[W^1](t) \to \infty$ as $t \to T_*^-$.

Next, let $0 < \delta < 1/4$ be a fixed number with $\delta < \gamma$. For C_N a very large constant we define

$$T = \sup\{t \geq 0 : \mathcal{E}_N[W^1](t')^{1/2} \leq 2C_N\varepsilon(1+t')^{\delta} \quad \forall\, 0 \leq t' \leq t\} < T_*. \tag{2.1.19}$$

We have, given the smallness assumptions of the theorem, $T > 0$. Note, in contrast to (1.3.24), that we allow here for a small growth in t. From our assumption that

$$\mathcal{E}_N[W^1](t)^{1/2} \leq 2C_N\varepsilon(1+t)^{\delta}\,, \quad 0 \leq t \leq T\,, \tag{2.1.20}$$

we will show that, if in fact ε is chosen sufficiently small, then inequality (2.1.20) implies

$$\mathcal{E}_N[W^1](t)^{1/2} \leq C_N\varepsilon(1+t)^{c\varepsilon}\,, \quad 0 \leq t \leq T\,. \tag{2.1.21}$$

Since $\mathcal{E}_N[W^1](t)$ is continuous this will contradict the maximality of T and thus the estimate

$$\mathcal{E}_N[W^1](t)^{1/2} \leq C_N\varepsilon(1+t)^{c\varepsilon} \tag{2.1.22}$$

will hold for all $T \leq T_*$. Consquently we have shown $\mathcal{E}_N[W^1](t) \not\to \infty$ as $t \to T_*^-$. Since the energy is now finite at $t = T_*$ we can extend the solution beyond this time thus contradicting the maximality of T_* and showing $T_* = \infty$. Thus the aim of the rest of this chapter is to show (2.1.21).

Outline of this chapter. In section 2.2 we set up the null frame. In section 2.3 we discuss the generalised PDE system (2.1.5) and its form when written with respect to this null frame. Section 2.4 is where we derive estimates coming from the wave coordinates and then apply these to the inhomogeneity. In sections 2.5 and 2.6 we derive the main decay estimates which are then used to derive an integrated energy inequality in section 2.7 which concludes our proof of (2.1.21). In section 2.8 we state some useful identities from [LR10]. Finally we conclude in section 2.9 by relating the main theorem to the Kaluza-Klein spacetime.

2.2 The null frame

We first introduce the following notation from [LR10].

Definition 2.11 (The null frame)**.** Define the local pair of null vectors $L, \underline{L}$

$$L^0 = 1, \quad L^a = x^a/|x|, \qquad \underline{L}^0 = 1, \quad \underline{L}^a = -x^a/|x|, \tag{2.2.1}$$

where $a = 1, 2, 3$. Note that L is tangent to the outgoing Minkowski null cones $\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : |x| - t = q\}$ and $\underline{L}$ is tangent to the ingoing cones $\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : |x| + t = s\}$. Furthermore

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r. \tag{2.2.2}$$

Let S_1, S_2 be orthonormal smooth vector fields spanning the tangent space of the sphere $\mathbb{S}^2$. The set $\mathcal{U} = (L, \underline{L}, S_1, S_2)$ forms a null frame, although it is not globally defined. Define the outgoing and ingoing null derivatives by

$$\partial_s = \frac{1}{2}(\partial_r + \partial_r), \partial_q = \frac{1}{2}(\partial_r - \partial_t). \tag{2.2.3}$$

Relative to the null frame $\mathcal{U}$ the Minkowski metric $\eta_{\mu\nu}$ takes the form

$$\eta_{LL} = \eta_{\underline{L}\underline{L}} = \eta_{Lp} = \eta_{\underline{L}p} = 0, \quad \eta_{L\underline{L}} = \eta_{\underline{L}L} = -2, \quad \eta_{pq} = \delta_{pq}, \tag{2.2.4}$$

where p, q denote any of the vectors S_1 and S_2. Since $\mathbb{S}^2$ does not admit a global orthonormal frame, we consider the following projections of S_1 and S_2 onto the sphere

$$\not{\partial}_a = \partial_a - \omega_a \omega^b \partial_b, \quad \omega^a = x^a/|x|, \quad a = 1, 2, 3. \tag{2.2.5}$$

Note these are angular derivatives since $\omega^a \not{\partial}_a = 0$. If we denote $\bar{\partial}_a = \not{\partial}_a$ then the *five-dimensional* set $\{L, \underline{L}, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3\}$ is globally defined, and at any point we can choose four, thus giving a global frame. Furthermore if we define $\bar{\partial}_0 = L^\mu \partial_\mu$ then the set

$$\bar{\partial} = \{\bar{\partial}_0, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3\} \tag{2.2.6}$$

spans the tangent space of the light cones $t - r =$constant.

Definition 2.12 (Collections of frame vector fields, seminorms)**.** Let

$$\mathcal{U} = \{L, \underline{L}, S_1, S_2\}, \quad \mathcal{T} = \{L, S_1, S_2\}, \quad \mathcal{L} = \{L\}, \tag{2.2.7}$$

where $\mathcal{T}$ is the set of null frame vector fields tangent to the outgoing cones. For any two families $\mathcal{V}$ and $\mathcal{W}$ of vector fields and an arbitary 2-tensor π, define

$$|\bar{\partial}\pi|_{\mathcal{V}\mathcal{W}} = \sum_{T \in \mathcal{T}, V \in \mathcal{V}, W \in \mathcal{W}} |(\partial_\rho \pi_{\mu\nu}) T^\rho V^\mu W^\nu|. \tag{2.2.8}$$

Recall $|\pi|_{\mathcal{V}\mathcal{W}}$ and $|\partial\pi|_{\mathcal{V}\mathcal{W}}$ are given in definition 2.4.

2.3 The (extended) Einstein Equations and Wave Coordinates

In this section we look at the structure of the nonlinearity of the PDE (2.1.5) with respect to the null frame.

Definition 2.13 (Quadratic non-null terms)**.** The quadratic non-null nonlinearities can be broken up as

$$\begin{aligned} P(\partial_\mu W, \partial_\nu W) &= P^1(\partial h, \partial h)_{\mu\nu} + P^2(\partial h, \partial \psi)_{\mu\nu} + P^3(\partial \psi, \partial \psi)_{\mu\nu}, \\ P^1(\partial h, \partial h)_{\mu\nu} &= m^{\rho\sigma} m^{\lambda\tau} \left(\frac{1}{4} \partial_\mu h_{\rho\sigma} \partial_\nu h_{\lambda\tau} - \frac{1}{2} \partial_\mu h_{\rho\lambda} \partial_\nu h_{\sigma\tau} \right), \\ P^2(\partial h, \partial \psi)_{\mu\nu} &= N^K m^{\rho\sigma} \left(\partial_\mu h_{\rho\sigma} \partial_\nu \psi_K + \partial_\nu h_{\rho\sigma} \partial_\mu \psi_K \right), \\ P^3(\partial \psi, \partial \psi)_{\mu\nu} &= N^{KK'} \left(\partial_\mu \psi_K \partial_\nu \psi_{K'} + \partial_\nu \psi_K \partial_\mu \psi_{K'} \right). \end{aligned} \tag{2.3.1}$$

We now study the structure of the nonlinearity of the PDE (2.1.5) with respect to the null frame. The following notation expresses the nonlinearity in terms of general tensors and/or scalars. Note that for the null terms, it is irrelevant which components of the unknowns are being considered, but it is crucial which derivatives appear.

Notation 2.14 (Quadratic forms)**.** For symmetric 2-tensors p, k we define

$$P^1(p, k) = \eta^{\rho\sigma} \eta^{\lambda\tau} \left(\frac{1}{4} p_{\rho\sigma} k_{\lambda\tau} - \frac{1}{2} p_{\rho\lambda} k_{\sigma\tau} \right). \tag{2.3.2}$$

Note there are no free indices above. Similar definitions holds for P^2, P^3 with Φ, Ψ some arbitrary functions. All together, let

$$P(p, k, \Psi, \Phi) = P^1(p, k) + P^2(p, \Psi) + P^3(\Psi, \Phi).$$

Similarly for some Π, Θ, which are either 2-tensors or scalars as required, define

$$|Q(\partial \Pi, \partial \Theta)| = |Q_{\mu\nu}(\partial \Pi, \partial \Theta)|_{\mathcal{U}\mathcal{U}} + |Q_K(\partial \Pi, \partial \Theta)|_{\mathcal{K}}. \tag{2.3.3}$$

Furthermore to keep track of the derivatives as well as the components, introduce the following notation for arbitrary 2-tensors π, θ and functions Ψ, Φ

$$P(\partial \pi, \partial \theta, \partial \Psi, \partial \Phi)_{\mu\nu} = P^1(\partial \pi, \partial \theta)_{\mu\nu} + P^2(\partial \pi, \partial \Psi)_{\mu\nu} + P^3(\partial \Psi, \partial \Phi)_{\mu\nu},$$

where for example

$$P^1(\partial \pi, \partial \theta)_{\mu\nu} = m^{\rho\sigma} m^{\lambda\tau} \left(\frac{1}{4} \partial_\mu \pi_{\rho\sigma} \partial_\nu \theta_{\lambda\tau} - \frac{1}{2} \partial_\mu \pi_{\rho\lambda} \partial_\nu \theta_{\sigma\tau} \right),$$

and similarly for P^2 and P^3.

The reason for using this notation is that eventually we will want to calculate $\Gamma^I F_{\mu\nu}$ where $\Gamma \in \mathbf{\Gamma}$. This notation allows us to derive estimates which still hold even when we have distributed the Γ^I derivatives across the terms in the nonlinearity. See for example corollary 2.21.

Lemma 2.15 (Modified Lemma 4.2 from [LR10])**.** *Let π, θ be arbitrary 2-tensors, Φ, Ψ arbitrary functions and Π, Θ 2-tensors or scalars as required. For the $\mathcal{O}((\partial W)^2)$*

nonlinearities given in (2.3.1) *we obtain*

$$\begin{aligned}|P(p,k,\Psi,\Phi)| &\lesssim |p|_{\mathcal{T}\mathcal{U}}|k|_{\mathcal{T}\mathcal{U}} + |p|_{\mathcal{L}\mathcal{L}}|k|_{\mathcal{U}\mathcal{U}} + |p|_{\mathcal{U}\mathcal{U}}|k|_{\mathcal{L}\mathcal{L}} + |p|_{\mathcal{T}\mathcal{U}}|\Psi|_{\mathcal{K}} + |\Psi|_{\mathcal{K}}|\Phi|_{\mathcal{K}},\\ |Q(\partial\Pi,\partial\Theta)| &\lesssim |\overline{\partial}\Pi||\partial\Theta| + |\partial\Pi||\overline{\partial}\Theta|.\end{aligned}$$

Proof. Expanding with respect to the null frame we find

$$\begin{aligned}|P^1(p,k)| &\lesssim |p|_{\mathcal{T}\mathcal{U}}|k|_{\mathcal{T}\mathcal{U}} + |p|_{\mathcal{L}\mathcal{L}}|k|_{\mathcal{U}\mathcal{U}} + |p|_{\mathcal{U}\mathcal{U}}|k|_{\mathcal{L}\mathcal{L}},\\ |P^2(p,\Psi)| &\lesssim |p|_{\mathcal{T}\mathcal{U}}|\Psi|_{\mathcal{K}},\\ |P^3(\Psi,\Phi)| &\lesssim |\Psi|_{\mathcal{K}}|\Phi|_{\mathcal{K}}.\end{aligned}$$

Note that $P^2(\partial h, \partial\psi)_{\mu\nu}$ involves $\eta^{\rho\sigma}h_{\rho\sigma} = \mathrm{tr}_\eta h$. By considering possible indices, we see another possible term we could have tried to include is of the form

$$N^K m^{\rho\sigma}\left(\partial_\rho h_{\mu\sigma}\partial_\nu\psi_K + \partial_\rho h_{\nu\sigma}\partial_\mu\psi_K\right).$$

However this would lead to terms of the form $|p|_{\mathcal{U}\mathcal{U}}|\Psi|_{\mathcal{K}}$ which cannot be controlled. The estimate for $|Q|$ comes from the estimate (1.6.15) for null forms. □

An important point to take away from lemma 2.15 is that, thanks to the null frame decomposition, a 'bad component' $|k|_{\mathcal{U}\mathcal{U}}$ always appears with a 'good component' $|p|_{\mathcal{L}\mathcal{L}}$ which obeys better estimates thanks to the wave-coordinate condition, see the next section.

Remark 2.16. The above lemma is the key estimate that allows us to very closely follow the proof of [LR10]. To be self-contained, this chapter repeats several results directly from this paper. Where a result here is given as 'modified' from a result in [LR10], it generally follows from their proofs with a very simple calculation difference.

2.4 Wave Gauge and Estimates of the Inhomogeneity

We will now use the wave-coordinate condition

$$\partial_\rho\left(g^{\rho\nu}\sqrt{|\det g|}\right) = 0. \tag{2.4.1}$$

to exchange a full derivative ∂ on certain 'good' components of the metric, with a good derivative $\overline{\partial}$ on all components of the metric, plus some higher order terms.

Lemma 2.17 (Exchanging good metric components for good derivatives, [LR10, Lemma 8.1])**.** *Assume g satisfies the wave-coordinate condition* (2.4.1) *and that the inverse perturbation satisfies $|H|_{\mathcal{U}\mathcal{U}} \le \frac{1}{4}$, then*

$$|\partial H|_{\mathcal{L}\mathcal{T}} \le |\overline{\partial} H|_{\mathcal{U}\mathcal{U}} + |H|_{\mathcal{U}\mathcal{U}}|\partial H|_{\mathcal{U}\mathcal{U}}. \tag{2.4.2}$$

One can also commute through vector fields $\Gamma \in \mathbf{\Gamma}$ in (2.4.2) to estimate 'good' components $|\partial\Gamma^I H|_{\mathcal{T}\mathcal{U}}$ in terms of good derivatives and error terms.

Proposition 2.18 (Preservation of lemma 2.17 under commutation, [LR10, Proposition 8.2])**.** *Suppose that g satisfies the wave-coordinate condition* (2.4.1) *and that for some $|I|$ we have $|\Gamma^J H| \le C$ for all $|J| \le |I|/2$ and for all $\Gamma \in \mathbf{\Gamma}$. Then*

$$|\partial\Gamma^I H|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|J|\le|I|}|\overline{\partial}\Gamma^J H|_{\mathcal{U}\mathcal{U}} + \sum_{|J|\le|I|-1}|\partial\Gamma^J H|_{\mathcal{U}\mathcal{U}} + \sum_{|J_1|+|J_2|\le|I|}|\Gamma^{J_1}H|_{\mathcal{U}\mathcal{U}}|\partial\Gamma^{J_2}H|_{\mathcal{U}\mathcal{U}},$$

$$|\partial \Gamma^I H|_{\mathcal{LL}} \lesssim \sum_{|J|\le|I|} |\overline{\partial}\Gamma^J H|_{\mathcal{UU}} + \sum_{|J|\le|I|-2} |\partial\Gamma^J H|_{\mathcal{UU}} + \sum_{|J_1|+|J_2|\le|I|} |\Gamma^{J_1} H|_{\mathcal{UU}} |\partial\Gamma^{J_2} H|_{\mathcal{UU}}.$$

Using lemma 2.17 and proposition 2.18 we now aim to estimate the quadratic non-null nonlinearities in (2.1.5b) and analyse where the good metric components and good derivatives appear.

Lemma 2.19 (Modified from [LR10, Lemma 9.6]). *The quadratic form $P_{\mu\nu}$ defined in* (2.3.1) *satisfies the following estimates*

$$\begin{aligned}
|P(\partial\pi,\partial\theta,\partial\Psi,\partial\Phi)|_{\mathcal{TU}} &\lesssim |\overline{\partial}\pi|_{\mathcal{UU}}|\partial\theta|_{\mathcal{UU}} + |\partial\pi|_{\mathcal{UU}}|\overline{\partial}\theta|_{\mathcal{UU}} + |\overline{\partial}\pi|_{\mathcal{TU}}|\partial\Psi|_{\mathcal{K}} \\
&\quad + |\partial\pi|_{\mathcal{TU}}|\overline{\partial}\Psi|_{\mathcal{K}} + |\overline{\partial}\Psi|_{\mathcal{K}}|\partial\Phi|_{\mathcal{K}} + |\partial\Psi|_{\mathcal{K}}|\overline{\partial}\Phi|_{\mathcal{K}}, \\
|P(\partial\pi,\partial\theta,\partial\Psi,\partial\Phi)|_{\mathcal{UU}} &\lesssim |\partial\pi|_{\mathcal{TU}}|\partial\theta|_{\mathcal{TU}} + |\partial\pi|_{\mathcal{LL}}|\partial\theta|_{\mathcal{UU}} + |\partial\pi|_{\mathcal{UU}}|\partial\theta|_{\mathcal{LL}} \\
&\quad + |\partial\pi|_{\mathcal{TU}}|\partial\Psi|_{\mathcal{K}} + |\partial\Psi|_{\mathcal{K}}|\partial\Phi|_{\mathcal{K}}.
\end{aligned}$$

Proof. The first estimate follows by contracting (2.3.1) with $T^\mu S^\nu$ for $T\in\mathcal{T}, S\in\mathcal{U}$. For the second estimate, replace p,Ψ,k,Φ from lemma 2.15 with $U^\mu\partial_\mu p, U^\mu\partial_\mu\Psi, S^\nu\partial_\nu k, S^\nu\partial_\nu\Phi$ respectively. For $S,U\in\mathcal{U}$ one obtains

$$\begin{aligned}
|S^\mu U^\nu P(\partial\pi,\partial\theta,\partial\Psi,\partial\Phi)_{\mu\nu}| &\lesssim |U^\mu\partial_\mu\pi|_{\mathcal{TU}}|S^\nu\partial_\nu\theta|_{\mathcal{TU}} \\
&\quad + |U^\mu\partial_\mu\pi|_{\mathcal{LL}}|S^\nu\partial_\nu\theta|_{\mathcal{UU}} + |U^\mu\partial_\mu\pi|_{\mathcal{UU}}|S^\nu\partial_\nu\theta|_{\mathcal{LL}} \\
&\quad + |U^\mu\partial_\mu\pi|_{\mathcal{TU}}|S^\nu\partial_\nu\Psi|_{\mathcal{K}} + |U^\mu\partial_\mu\Psi|_{\mathcal{K}}|S^\nu\partial_\nu\Phi|_{\mathcal{K}}.
\end{aligned}$$

Lastly note that $|\overline{\partial}\phi| = \sum_{i=0}^3 |\overline{\partial}_i\phi|$ is equivalent to $\sum_{T\in\mathcal{T}}|T^\mu\partial_\mu\phi|$, similarly for $|\partial\phi|$ and $\sum_{U\in\mathcal{U}}|U^\mu\partial_\mu\phi|$. □

It turns out the wave-coordinate condition will imply a hierarchy of components, with certain 'good' variables W_G having better decay rates than the full set W. The following notation is meant to simplify the notation and distinguish between components with different decay rates. Note for comparison theorem 1.43.

Definition 2.20 (Good metric components and derivatives). Let

$$\begin{aligned}
W &= \{U^\mu V^\nu h_{\mu\nu}, \psi_K : U,V\in\mathcal{U}, K\in\mathcal{K}\}, \\
W_G &= \{U^\mu T^\nu h_{\mu\nu}, \psi_K : U\in\mathcal{U}, T\in\mathcal{T}, K\in\mathcal{K}\}.
\end{aligned} \tag{2.4.3}$$

In particular we define

$$|W| = |h|_{\mathcal{UU}} + |\psi|_{\mathcal{K}}, \qquad |W|_G = |h|_{\mathcal{TU}} + |\psi|_{\mathcal{K}}, \tag{2.4.4}$$

$$|\partial W| = |\partial h|_{\mathcal{UU}} + |\partial\psi|_{\mathcal{K}}, \qquad |\overline{\partial} W| = |\overline{\partial} h|_{\mathcal{UU}} + |\overline{\partial}\psi|_{\mathcal{K}}, \tag{2.4.5}$$

and analogous definitions for $|\partial W|_G$ and $|\overline{\partial}W|_G$.

The norms without subscripts, $|\cdot|$ and (2.3.3), indicate a sum over all the possible components of the $\mathcal{UU}$ and $\mathcal{K}$ terms. The subscript $|\cdot|_G$ indicates only the 'good' components are being considered. This means $|W_G| = |W|_G$, however we generally use the latter throughout.

Eventually the inhomogeneity to be studied will come from commuting Γ^I through the PDE (2.1.5). Thus the next two results allow us to estimate the nonlinearities $F_{\mu\nu}, F_K$ as well as $\Gamma^I F_{\mu\nu}, \Gamma^I F_K$.

Corollary 2.21 (Modified from [LR10, Corollary 9.7]). *Assume g satisfies the wave-coordinate condition* (2.4.1) *and W, W_G are defined as in* (2.4.3)*, then*

$$\begin{aligned} &|P(\partial h, \partial h, \partial\psi, \partial\psi)|_{\mathcal{TU}} \lesssim |\overline{\partial} W||\partial W|, \\ &|P(\partial h, \partial h, \partial\psi, \partial\psi)|_{\mathcal{UU}} \lesssim |\partial W|_G^2 + |\overline{\partial} W||\partial W| + |W||\partial W|^2. \end{aligned} \tag{2.4.6}$$

Furthermore, assuming that $|\Gamma^J W| \leq C$ for all $|J| \leq |I|$ and for all $\Gamma \in \mathbf{\Gamma}$, we have

$$\begin{aligned} |\Gamma^I P(\partial h, \partial h)|_{\mathcal{UU}} \lesssim &\sum_{|J|+|K|\leq |I|} |\partial \Gamma^J W|_G |\partial \Gamma^K W|_G + |\overline{\partial}\Gamma^J W||\partial \Gamma^K W| \\ &+ \sum_{|J|+|K|\leq |I|} |\partial \Gamma^K h|_{\mathcal{UU}} \left[\sum_{|J'|\leq |J|-1} |\partial \Gamma^{J'} h|_{\mathcal{LT}} + \sum_{|J''|\leq |J|-2} |\partial \Gamma^{J''} h| + \sum_{|J_1|+|J_2|\leq |J|} |\Gamma^{J_2} h||\partial \Gamma^{J_1} h| \right]. \end{aligned}$$

It also holds that

$$|Q(\partial W, \partial W)| \lesssim |\overline{\partial} W||\partial W|.$$

Proof. The first identities (2.4.6) follow from lemma 2.19 and lemma 2.17. For the estimate of $\Gamma^I P$ we use lemma 2.19, proposition 2.18 and note that

$$|\Gamma^I P(\partial h, \partial h, \partial\psi, \partial\psi)|_{\mathcal{UU}} \lesssim \sum_{|J_1|+\cdots+|J_4|\leq |I|} |P(\partial\Gamma^{J_1} h, \partial\Gamma^{J_2} h, \partial\Gamma^{J_3}\psi, \partial\Gamma^{J_4}\psi)|.$$

Lastly lemma 2.15 implies the estimate for $|Q|$. □

We can now put these results together in proposition 2.22 to estimate the inhomogeneous terms $F_{\mu\nu}$ and F_K.

Proposition 2.22 (Modified from [LR10, Proposition 9.8]). *Assume g satisfies the wave-coordinate condition* (2.4.1)*, W and W_G are defined as in* (2.4.3) *and $F_{\mu\nu}$ and F_K be as defined in* (2.1.5b)*. Then*

$$\begin{aligned} |F|_{\mathcal{K}} + |F|_{\mathcal{TU}} &\lesssim |\overline{\partial} W||\partial W| + |W||\partial W|^2, \\ |F|_{\mathcal{UU}} &\lesssim |\partial W|_G^2 + |\overline{\partial} W||\partial W| + |W||\partial W|^2. \end{aligned} \tag{2.4.7}$$

Furthermore if $|\Gamma^J W| \leq C$ for all $|J| \leq |I|$ and for all $\Gamma \in \mathbf{\Gamma}$, then

$$\begin{aligned} |\Gamma^I F|_{\mathcal{UU}} \lesssim &\sum_{|J|+|K|\leq |I|} \Big[|\partial\Gamma^J W|_G |\partial\Gamma^K W|_G + |\overline{\partial}\Gamma^J W||\partial\Gamma^K W| \Big] \\ &+ \sum_{|J|+|K|\leq |I|-2} |\partial\Gamma^J h|_{\mathcal{UU}} |\partial\Gamma^K h|_{\mathcal{UU}} + \sum_{|J_1|+|J_2|+|J_3|\leq |I|} |\Gamma^{J_3} W||\partial\Gamma^{J_2} W||\partial\Gamma^{J_1} W|, \end{aligned} \tag{2.4.8}$$

$$|\Gamma^I F|_{\mathcal{K}} \lesssim \sum_{|J|+|K|\leq |I|} |\overline{\partial}\Gamma^J W||\partial\Gamma^K W| + \sum_{|J_1|+|J_2|+|J_3|\leq |I|} |\Gamma^{J_3} W||\partial\Gamma^{J_2} W||\partial\Gamma^{J_1} W|. \tag{2.4.9}$$

Proof. The required estimates come from lemma 2.19, corollary 2.21 and noting that

$$|\Gamma^I Q(\partial W, \partial W)| \lesssim \sum_{|J_1|+|J_2|\leq |I|} |Q(\partial\Gamma^{J_1} W, \partial\Gamma^{J_2} W)|$$

and

$$|\Gamma^I G(W)(\partial W, \partial W)| \lesssim \sum_{|J_1|+\cdots|J_3|\leq|I|} |\Gamma^{J_1} W||\partial\Gamma^{J_2} W||\partial\Gamma^{J_3} W|.$$

□

It is important to note in proposition 2.22 that the terms $|\partial\Gamma^J h|_{\mathcal{UU}}|\partial\Gamma^K h|_{\mathcal{UU}}$ in (2.4.8), which involve both bad derivatives and bad components, only occur at two orders of derivatives lower, i.e. $|I|-2$.

2.5 Decay Estimates – Part I

The estimates in this section fall into two types. The first type, discussed in sections 2.5.1 and 2.5.2, involve sharp decay estimates for solutions ϕ of $\widetilde{\Box}_g\phi = F$ where F will eventually be of the form in (2.1.5b). The second type, discussed in section 2.5.3, involves 'weak decay' estimates coming from a weighted Klainerman-Sobolev inequality and a bootstrap assumption on our energy.

2.5.1 Motivating example for weighted L^∞ pointwise estimate

One of the key ingredients in [LR10] was the use of an additional 'independent' estimate designed to boost the weak decay estimates derived via the Klainerman-Sobolev inequality. These sharp decay estimates do not rely on estimates of the fundamental solution but rather use integration along characteristics, and are derived from the earlier works [Lin90, Lin92]. We start with a simple scalar wave equation to explain this independent estimate.

Definition 2.23 (Regions near and away from light cone)**.** For $\delta > 0$ let

$$S_{t,\delta} = \{(t,x) \in \mathbb{R}^{1+3} : |x| < (1-\delta)t\} \cup \{|x| > (1+\delta)t\}, \tag{2.5.1}$$
$$D_{t,\delta} = \{(t,x) \in \mathbb{R}^{1+3} : (1-\delta)t \leq |x| \leq (1+\delta)t\}. \tag{2.5.2}$$

We mainly consider, for simplicity, $\delta = 1/2$ and denote $S_{t,1/2} = S_t$ and $D_{t,1/2} = D_t$.

We have the identity

$$(1+|t-r|)|\partial\phi| + (1+t+r)|\bar{\partial}\phi| \leq C \sum_{|I|=1} |\Gamma^I\phi|. \tag{2.5.3}$$

In the region S_t we can convert $(1+|t-r|)^{-1} \leq (1+t)^{-1}$ which allows us to to control the transversal derivative $\partial_q\phi$. This leads to the following lemma.

Lemma 2.24 (Decay away from the light cone)**.** *Let ϕ be a solution to $\widetilde{\Box}_g\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi = (\eta+H)^{\mu\nu}\partial_\mu\partial_\nu\phi = F$ with $C_c^\infty(\mathbb{R}^3)$ data. Suppose also that the coefficients $H^{\mu\nu}$ satisfy*

$$|H|_{\mathcal{UU}} \leq \varepsilon', \quad |H|_{\mathcal{LT}} \leq \varepsilon' \frac{|q|+1}{1+t+|x|}, \tag{2.5.4}$$

for some $0 < \varepsilon' < \frac{1}{10}$ and $t \geq 0$. Then for all $\delta > 0$ there exists a constant $C = C(\delta) > 0$ such that for $t \geq 0$ we have

$$\sup_{x\in S_{t,\delta}} |\partial\phi| \leq C(1+t)^{-\frac{3}{2}} \sum_{|I|\leq 2} \|\partial\Gamma^I\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)}. \tag{2.5.5}$$

See [Lin08, Lemma 4.1] for a proof. Lemma (2.24) reinforces the idea that the region of slowest pointwise decay occurs on the light cone. From (2.5.3) we see the main estimate we are missing is control of the derivative perpendicular to the light cones, $|\partial_q \phi|$, in the region D_t. Indeed because of the t^δ growth in the bootstrap assumption a standard application of the Klainerman-Sobolev inequality would lead to small loss in decay as $|\partial_q \phi| \lesssim (1+t)^{-1+\delta}$. The following estimate, when combined with weak-decay estimates from the Klainerman-Sobolev inequality, allows us to avoid this loss (as done in section 2.6).

Proposition 2.25 (Decay near the light cone)**.** *Under the same conditions as lemma 2.24, we have*

$$\begin{aligned} |(1+t+r)\partial\phi(t,x)| \leq C \sup_{\tau\in[0,t]} \sum_{|I|\leq 1} \|\Gamma^I \phi(\tau)\|_{L^\infty(\mathbb{R}^3)} \\ + C\int_0^t \Big((1+\tau)\|F\|_{L^\infty(D_t)} + \sum_{|I|\leq 2} (1+\tau)^{-1} \|\Gamma^I\phi\|_{L^\infty(D_t)} \Big) \mathrm{d}\tau. \end{aligned} \tag{2.5.6}$$

This proposition was first derived for $H \equiv 0$ in [Lin90, Proposition 1.8.], with later adaption to $H \not\equiv 0$ and tensorial systems in [LR05, Lin08, LR10]. The very rough idea of the proof is to write the equation as

$$\Box\phi + H^{\mu\nu}\partial_\mu\partial_\nu\phi = \frac{4}{r}\partial_s\partial_q(r\phi) + \Delta_\omega\phi + H^{\mu\nu}\partial_\mu\partial_\nu\phi = F, \tag{2.5.7}$$

where Δ_ω is the spherical Laplacian. Expressing $H^{\mu\nu}$ with respect to the null frame leads to the inequality

$$|H^{\mu\nu}\partial_\mu\partial_\nu\phi| \lesssim |H_{LL}\partial_q^2\phi| + |H|_{\mathcal{U}\mathcal{U}}|\overline{\partial}\overline{\partial}\phi| + |H|_{\mathcal{T}\mathcal{U}}|\overline{\partial}\partial\phi|. \tag{2.5.8}$$

So by keeping the $H_{LL}\partial_q^2\phi$ term on the left hand side in (2.5.7), one eventually arrives at the estimate

$$\begin{aligned} |(4\partial_s + H_{LL}\partial_q)\partial_q(r\phi)| &\lesssim r|F| + r|\Delta_\omega\phi| + |H|_{\mathcal{L}\mathcal{L}}|\partial\phi| + r|H|_{\mathcal{T}\mathcal{U}}|\overline{\partial}\partial\phi| + r(1+|H|_{\mathcal{U}\mathcal{U}})|\overline{\partial}\overline{\partial}\phi| \\ &\lesssim \sum_{|I|\leq 2}(1+t)^{-1}|\Gamma^I\phi| + (1+t)|F|. \end{aligned} \tag{2.5.9}$$

The estimate (2.5.6) is then obtained by integrating the above along the integral curves of $4\partial_s + H_{LL}\partial_q$ in the region D_t. We refer to [Lin08] for the many omitted details.

2.5.2 Weighted L^∞ pointwise estimate

To treat the quasilinear Einstein equations with non-compact initial data we in fact require a weighted version of proposition 2.25.

Definition 2.26 (Decay weight function)**.** We define the *decay weight function* as

$$\varpi = \varpi(q) = \begin{cases} (1+|q|)^{1+\gamma'}, & q \geq 0, \\ (1+|q|)^{1/2-\mu'}, & q < 0, \end{cases} \tag{2.5.10}$$

where $\gamma' \geq -1$ and $\mu' \leq 1/2$ are some constants and $q = r - t$.

Corollary 2.27 ([LR10, Corollary 7.2])**.** *Let $\phi_{\mu\nu}$ be a solution of the system*

$$\widetilde{\Box}_g \phi_{\mu\nu} = F_{\mu\nu},$$

where $F_{\mu\nu}$ is some as yet unspecified nonlinearity. Assume that $H^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ satisfies

$$|H|_{\mathcal{U}\mathcal{U}} \le \varepsilon', \quad \int_0^\infty \| |H(t,\cdot)|_{\mathcal{U}\mathcal{U}} \|_{L^\infty(D_t)} \frac{\mathrm{d}t}{1+t} \le \frac{1}{4}, \quad |H|_{\mathcal{L}\mathcal{T}} \le \varepsilon' \frac{|q|+1}{1+t+|x|}, \tag{2.5.11}$$

in the region D_t for some $\varepsilon' > 0$. Then for any $U, V \in \mathcal{U}, \mathcal{L}$ or $\mathcal{T}$ and $(x,t) \in [0,T) \times \mathbb{R}^3$

$$\begin{aligned}(1+t+|q|)\varpi(q)|\partial\phi(t,x)|_{UV} \lesssim &\sup_{0\le\tau\le t} \sum_{|I|\le 1} \| |\varpi(q)\Gamma^I \phi(\tau,\cdot)|_{UV} \|_{L^\infty} \\ &+ \int_0^t \Big\{ \varepsilon'\alpha \| |\varpi(q)\partial\phi(\tau,\cdot)|_{UV} \|_{L^\infty} + (1+\tau) \| |\varpi(q) F(\tau,\cdot)|_{UV} \|_{L^\infty(D_\tau)} \\ &\qquad + \sum_{|I|\le 2} (1+\tau)^{-1} \| |\varpi(q)\Gamma^I\phi(\tau,\cdot)|_{UV} \|_{L^\infty(D_\tau)} \Big\} \mathrm{d}\tau,\end{aligned}$$

where $\alpha = \max(1+\gamma', 1/2-\mu') \ge 0$. Note here $|H|_{\mathcal{U}\mathcal{U}} = \sum_{U,V\in\mathcal{U}} |U^\mu V^\nu \eta_{\mu\rho} m_{\nu\sigma} H^{\rho\sigma}|$.

The above estimate is obtained for $\phi_{\mu\nu}$ and then we contract with any $U, V \in \mathcal{U}$ since ∂_s and ∂_q, defined in section 2.2, commute with $\mathcal{U}$. Similarly we could have considered a system $\widetilde{\Box}_g \phi_K = F_K$ and contracted the estimate with arbitrary coefficients $N^K \in \mathbb{R}^m$.

2.5.3 Weak decay estimates

We next turn to a generalised version of the Klainerman-Sobolev inequality. Recall from (2.1.11) the *energy weight* function w

$$w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & q \ge 0, \\ 1 + (1+|q|)^{-2\mu}, & q < 0, \end{cases} \tag{2.5.12}$$

where $\mu > 0$ and $\gamma \in (0, 1/2)$ are constants and $q = r - t$. The weight w is used to provide additional decay in the region $q \gg 0$, see corollary (2.31) below. Furthermore the conditions on w imply $w'(q) \ge 0$, which allows for an additional spacetime integral of $\overline{\partial}\phi$ to appear in the energy estimate, see theorem 2.43 and c.f. lemma 1.42.

We have the following weighted version of the Klainerman-Sobolev inequality.

Proposition 2.28 ([LR10, Proposition 14.1])**.** *For any function $\phi \in C_0^\infty(\mathbb{R}^3_x)$, at an arbitrary point (t,x) we have*

$$|\phi(t,x)|(1+t+|q|)\left((1+|q|)w(q)\right)^{1/2} \le C \sum_{|I|\le 3} \|w^{1/2}\Gamma^I \phi(t,\cdot)\|_{L^2_x}.$$

We now combine the bootstrap assumption with proposition 2.28 which gives the weak decay estimates. First we recall one part of definition 2.8.

Definition 2.29 (Schwarzschildean variable)**.** Define

$$\begin{aligned} h^0_{\mu\nu} &= \chi(r)\chi(r/t)\frac{M}{r}\delta_{\mu\nu}, \\ h^1_{\mu\nu} &= h_{\mu\nu} - h^0_{\mu\nu}, \end{aligned} \tag{2.5.13}$$

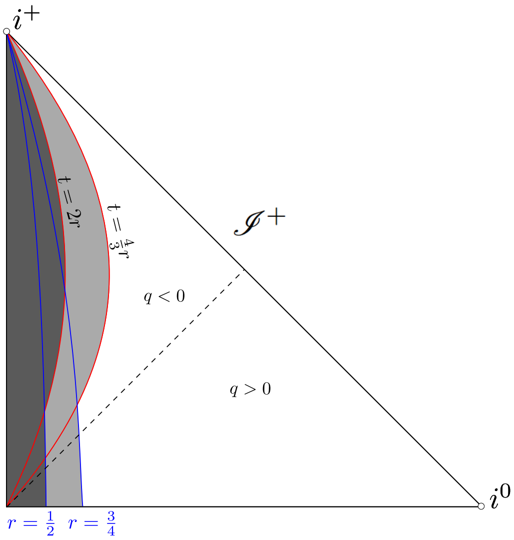


Figure 2.1: Regions where $\chi(r)\chi(r/t)$ is vanishing (dark grey), between 0 and 1 (light grey) and identically 1 (white). In particular, h^0 is identically 1 towards i^0 and along $\mathscr{I}^+$, and vanishes when $r = 0$.

where $\chi(s) \in C^\infty$ is 1 when $s \geq 3/4$ and 0 when $s \leq 1/2$. See also Figure 2.1.

Since h^0 is a known quantity, we introduce some notation to separate between the decay for terms involving h^0 and the decay for h^1.

Definition 2.30 (W^p variables)**.** For $p = 0, 1$ we define

$$\begin{aligned} W^p &= \left\{ U^\mu V^\nu h^p_{\mu\nu}, \psi_K : U, V \in \mathcal{U}, K \in \mathcal{K} \right\}, \\ W^p_G &= \left\{ T^\mu U^\nu h^p_{\mu\nu}, \psi_K : U \in \mathcal{U}, T \in \mathcal{T}, K \in \mathcal{K} \right\}. \end{aligned} \tag{2.5.14}$$

Similar to definition 2.20, for $p = 0, 1$ we define

$$|W^p| = |h^p|_{\mathcal{U}\mathcal{U}} + |\psi|_{\mathcal{K}}, \qquad |W^p|_G = |h^p|_{\mathcal{T}\mathcal{U}} + |\psi|_{\mathcal{K}}, \tag{2.5.15}$$

$$|\partial W^p| = |\partial h|_{\mathcal{U}\mathcal{U}} + |\partial \psi|_{\mathcal{K}}, \qquad |\overline{\partial} W^p| = |\overline{\partial} h^p|_{\mathcal{U}\mathcal{U}} + |\overline{\partial} \psi|_{\mathcal{K}}. \tag{2.5.16}$$

As discussed after theorem 2.9, we have the bootstrap assumption (2.1.20) and wave gauge (2.1.5c). Using the weighted Klainerman-Sobolev inequality and the bootstrap assumption one can now derive the following.

Corollary 2.31 (Weak decay estimates, modified from [LR10, Corollary 9.4])**.** *Assume* (2.1.20) *holds and W^p is defined as in* (2.5.14)*, then for $p = 0, 1$ and $|I| \leq N - 2$ we have*

$$|\Gamma^I W^p(t, x)| \leq \begin{cases} C\varepsilon(1 + t + |q|)^{-1+\delta}(1 + |q|)^{-\delta'}, & q > 0, \\ C\varepsilon(1 + t + |q|)^{-1+\delta}(1 + |q|)^{1/2}, & q < 0, \end{cases} \tag{2.5.17}$$

$$|\partial\Gamma^I W^p(t, x)| \leq \begin{cases} C\varepsilon(1 + t + |q|)^{-1+\delta}(1 + |q|)^{-1-\delta'}, & q > 0, \\ C\varepsilon(1 + t + |q|)^{-1+\delta}(1 + |q|)^{-1/2}, & q < 0, \end{cases} \tag{2.5.18}$$

where $\delta' = \delta$ if $p = 0$ and $\delta' = \gamma > \delta$ if $p = 1$. Furthermore if $|I| \leq N - 3$ then

$$|\overline{\partial}\Gamma^I W^p(t, x)| \leq \begin{cases} C\varepsilon(1 + t + |q|)^{-2+\delta}(1 + |q|)^{-\delta'}, & q > 0, \\ C\varepsilon(1 + t + |q|)^{-2+\delta}(1 + |q|)^{1/2}, & q < 0. \end{cases} \tag{2.5.19}$$

Proof. The proof of (2.5.18) follows from the weighted Klainerman-Sobolev estimate of proposition 2.28 and the energy assumption (2.1.20). Note the additional decay in q comes from the fact that we use a *weighted* Klainerman-Sobolev inequality. We obtain (2.5.17) for $r > t$ by integrating (2.5.18) from the hypersurface $t = 0$ along lines with $t + r$ and $w = x/|x|$ fixed. One also needs the initial condition from the assumptions of our main theorem 2.9, namely (2.1.9) implies

$$\liminf_{|x|\to\infty} \left(|h^1(0,x)| + |\psi_K(0,x)|\right) = 0. \tag{2.5.20}$$

The final estimates for $|\bar{\partial}\Gamma^I W^p(t,x)|$ come from using (2.5.3). □

Note that by introducing h^0, we can obtain better decay for h^1 in the region $q > 0$, i.e. near i^0. These weak decay estimates give the following decay on the inhomogeneities.

Lemma 2.32 (Inhomogeneity decay estimates, modified from [LR10, Lemma 10.5])**.** *Assume* (2.1.20) *and* (2.1.5c) *hold and let F_K be as given in* (2.1.5b)*. Then*

$$\begin{aligned} |F|_{\mathcal{TU}} + |F|_{\mathcal{K}} &\lesssim \varepsilon t^{-3/2+\delta}|\partial W|, \\ |F|_{\mathcal{UU}} &\lesssim \varepsilon t^{-3/2+\delta}|\partial W| + |\partial W|^2_G. \end{aligned}$$

Proof. This follows from proposition 2.22 and the weak decay estimates of corollary 2.31. □

We end this section with a stand-alone lemma which involves estimating the second-order derivatives landing on the Schwarzschild-like term $h^0_{\mu\nu}$. Since the form of $h^0_{\mu\nu}$ has been chosen in (2.5.13), this result comes from commuting through copies of $\Gamma \in \mathbf{\Gamma}$ and applying the results from corollary 2.31. Note that $\Box(M/r) = 0$ away from $r = 0$ so the important contribution of $\Box h^0$ is in the support of the derivatives of $\chi(r/t)$, which is in the interior (i.e. $q < 0$) region $\frac{1}{2}t \leq r \leq \frac{3}{4}t$.

Lemma 2.33 (Schwarzschildean decay estimates, [LR10, Lemma 9.9])**.** *Let* $F^0_{\mu\nu} = \widetilde{\Box}_g h^0_{\mu\nu}$ *where h^0 is as defined in* (2.5.13)*. Then for low orders* $|I| \leq N-2$

$$|\Gamma^I F^0|_{\mathcal{UU}} \leq \begin{cases} C\varepsilon^2(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta}, & q > 0, \\ C\varepsilon(1+t+|q|)^{-3}, & q < 0, \end{cases}$$

and for high orders $|I| = N-1, |I| = N$

$$|\Gamma^I F^0|_{\mathcal{UU}} \lesssim \begin{cases} C\varepsilon^2(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta}, & q > 0, \\ C\varepsilon(1+t+|q|)^{-3}, & q < 0, \end{cases} + \frac{C\varepsilon}{(1+t+|q|)^3}\sum_{|J|\leq|I|}|\Gamma^J h^1|_{\mathcal{UU}}.$$

2.6 Decay Estimates – Part II

In this section we prove upgraded decay estimates for $\partial\Gamma^I W$ by combining together the results from section 2.5 . These decay estimates are valid only for a smaller number of $\mathbf{\Gamma}$ vector fields than that given in the weak decay estimates, namely for $|I| \leq N/2 + 2$. We first obtain estimates for $|\partial W|$ and $|\partial W|_G$ given in proposition 2.36, followed by estimates of $\partial\Gamma^I h$ for $|I| \leq 1$ in proposition 2.38.

Lemma 2.34 (Modified from [LR10, Lemma 10.6]). *Let $(h_{\mu\nu},\psi_K)$ be a local-in-time solution of* (2.1.5) *such that* (2.1.20) *and* (2.1.5c) *hold. Let $F_{\mu\nu}$ and F_K be as defined in* (2.1.5b). *Then*

$$(1+t)\|\partial W_G(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon\int_0^t (1+\tau)^{\delta-1/2}\|\partial W(\tau,\cdot)\|_{L^\infty}\mathrm{d}\tau,$$

$$(1+t)\|\partial W(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\int_0^t \Big\{\varepsilon(1+\tau)^{\delta-1/2}\|\partial W(\tau,\cdot)\|_{L^\infty} + (1+\tau)\|\partial W_G(\tau,\cdot)\|^2_{L^\infty}\Big\}\mathrm{d}\tau.$$

Proof. Apply corollary 2.27 with $\varpi(q)=1$, ie, $\alpha=0$, and estimate $|F|_{\mathcal{TU}},|F|_{\mathcal{K}}$ and $|F|_{\mathcal{UU}}$ with lemma 2.32 and the weak decay estimates of corollary 2.31. Note $\delta < 1/4$. $\square$

The following algebraic lemma is needed for proposition 2.36.

Lemma 2.35 ([LR10, Lemma 10.7]). *If $b(t)\ge 0$ and $c(t)\ge 0$ satisfy*

$$\begin{aligned} b(t) &\le C\varepsilon\left(\int_0^t (1+s)^{-1-a}c(s)ds+1\right),\\ c(t) &\le C\varepsilon\left(\int_0^t (1+s)^{-1-a}c(s)ds+1\right) + C\int_0^t (1+s)^{-1}b^2(s)ds, \end{aligned}$$

for some positive constants such that $a\ge C^2\varepsilon$ and $a\ge 4C\varepsilon/(1-2C\varepsilon)$, then

$$b(t)\le 2C\varepsilon, \qquad \textit{and} \qquad c(t)\le 2C\varepsilon(1+a\ln(1+t)).$$

Proposition 2.36 (Modified from [LR10, Proposition 10.1]). *Let $(h_{\mu\nu},\psi_K)$ be a local-in-time solution of* (2.1.5) *such that* (2.1.20) *and* (2.1.5c) *hold. Then*

$$\begin{aligned} |\partial W|_G &\le C\varepsilon(1+t+|q|)^{-1},\\ |\partial W| &\le C\varepsilon(1+t)^{-1}\ln(1+t). \end{aligned}$$

Proof. Take $a=1/2-\delta$, $b(t)=(1+t)\|\partial W_G(\tau,\cdot)\|_{L^\infty}$ and $c(t)=(1+t)\|\partial W(\tau,\cdot)\|_{L^\infty}$ in lemma 2.35. $\square$

In the works [LR05, LR10], it was shown that all but one component of $\partial h_{\mu\nu}$ obeys a $(1+t+|q|)^{-1}$ decay rate. The 'worst' component $\partial h_{\underline{LL}}$ has the slower $t^{-1}\ln t$ decay rate. Our choice of additional nonlinearity has meant that the new variables ψ_K pick up the better of these two decay rates (c.f. theorem 1.43 also).

We now turn to estimates of even better components of $\partial\Gamma^I h$ for $|I|\le 1$, given in proposition 2.38. These 'best' components have much better decay rates than the full set given in proposition 2.36. Note however that since the control is obtained from the wave-coordinate condition condition, we only obtain estimates on $h_{\mu\nu}$ and not ψ_K. Thus lemma 2.37 and proposition 2.38 are completely unchanged from [LR10].

Lemma 2.37 ([LR10, Lemma 10.4]). *Let $(h_{\mu\nu},\psi_K)$ be a local-in-time solution of* (2.1.5) *such that* (2.1.20) *and* (2.1.5c) *hold. Then*

$$\sum_{|I|\le k}|\partial\Gamma^I h|_{\mathcal{LL}} + \sum_{|I|\le k-1}|\partial\Gamma^I h|_{\mathcal{LT}} \lesssim \sum_{|I|\le k-2}|\partial\Gamma^I h|_{\mathcal{UU}} + \begin{cases} \varepsilon(1+t+|q|)^{-2-2\delta}(1+|q|)^{-2\delta}, & q>0,\\ \varepsilon(1+t+|q|)^{-2-2\delta}(1+|q|)^{-1/2-\delta}, & q<0, \end{cases}$$

$$\sum_{|I|\le k} |\Gamma^I h|_{\mathcal{LL}} + \sum_{|I|\le k-1} |\Gamma^I h|_{\mathcal{LT}} \lesssim \sum_{|I|\le k-2} \int_{s,\omega=const} |\Gamma^I h|_{\mathcal{UU}}$$
$$+ \begin{cases} \varepsilon(1+t+|q|)^{-1}, & q>0, \\ \varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta}, & q<0, \end{cases}$$

where here the sums over $k-2$ (resp. $k\le 1$) are absent if $k\le 1$ (resp. $k=0$).

The proof of the above lemma follows by combining proposition 2.18 with the weak decay estimates of corollary 2.31.

Proposition 2.38 ([LR10, Proposition 10.1])**.** *Let $(h_{\mu\nu},\psi_K)$ be a local-in-time solution of* (2.1.5) *such that* (2.1.20) *and* (2.1.5c) *hold. Then*

$$|\partial h|_{\mathcal{LT}} + |\partial\Gamma h|_{\mathcal{LL}} \le \begin{cases} \varepsilon(1+t+|q|)^{-2+2\delta}(1+q)^{-\delta}, & q>0, \\ \varepsilon(1+t+|q|)^{-2+2\delta}(1+q)^{1/2}, & q<0, \end{cases}$$
$$|h|_{\mathcal{LT}} + |\Gamma h|_{\mathcal{LL}} \le \begin{cases} \varepsilon(1+t+|q|)^{-1}, & q>0, \\ \varepsilon(1+t+|q|)^{-1}(1+q)^{1/2+\delta}, & q<0. \end{cases}$$

We now continue to the crux of this section, and derive in proposition 2.41 'stronger' estimates of all components of $\partial\Gamma^I W$ for the restricted range $|I|\le N/2+2$. The proof relies on using corollary 2.27 again, this time with a non-trivial weight ϖ. Recall that in corollary 2.27 there was a spacetime integral of $|\varpi F|$ where F is the particular inhomogeneity being considered. The inhomogeneity we consider comes from commuting Γ^I through the PDE (2.1.5). Note the recent work [LT20] used commutation properties of the Lie derivative which simplifies the analysis. Nonetheless we follow the original approach of [LR10].

Definition 2.39 (Commuted PDE system)**.** Let $\hat\Gamma = \Gamma + c_\Gamma$ where c_Γ is a constant defined by $\partial_\mu\Gamma_\nu + \partial_\nu\Gamma_\mu = c_\Gamma\eta_{\mu\nu}$. In particular, $c_\Gamma = 0$ except when $\Gamma = S$ in which case $c_S = 2$. Let D^I be defined by $D^I = \widetilde\Box_g\Gamma - \hat\Gamma\widetilde\Box_g$. Then,

$$\begin{aligned} \widetilde\Box_g\Gamma^I h^1_{\mu\nu} &= D^I h^1_{\mu\nu} + \hat\Gamma^I F_{\mu\nu} - \hat\Gamma^I F^0_{\mu\nu}, \\ \widetilde\Box_g\Gamma^I\psi_K &= D^I\psi_K + \hat\Gamma^I F_K. \end{aligned} \tag{2.6.1}$$

The estimates of $D^I h^1_{\mu\nu}$ and $D^1\psi_K$ can be found in proposition 2.50. On the right hand side of (2.6.1) are the terms $\hat\Gamma^I F_{\mu\nu}$ and $\hat\Gamma^I F_K$ which are estimated in the following lemma.

Lemma 2.40 (Modified from [LR10, Lemma 10.8])**.** *Let $(h_{\mu\nu},\psi_K)$ be a local-in-time solution of* (2.1.5) *such that* (2.1.20) *and* (2.1.5c) *hold. Let $F_{\mu\nu}$ and F_K be as given in* (2.1.5b)*. Then for $|I|\le N-2$ we have*

$$|\Gamma^I F|_{\mathcal{K}} + |\Gamma^I F|_{\mathcal{UU}} \lesssim \varepsilon(1+t)^{-1}\sum_{|J|\le|I|}|\partial\Gamma^J W| + \sum_{|J|+|K|\le|I|,|K|<|I|} |\partial\Gamma^J W||\partial\Gamma^K W|,$$

where it is understood that the term with $|K|<|I|$ vanishes if $|I|=0$.

Proof. The result follows from proposition 2.22, corollary 2.31 and the first estimate in proposition 2.36. □

Proposition 2.41 (Modified from [LR10, Proposition 10.2])**.** *Let* $(h_{\mu\nu}, \psi_K)$ *be a solution of the generalised PDE* (2.1.5)*. Assume* (2.1.20) *holds,* W^p *is defined as in* (2.5.14) *and* $F_{\mu\nu}$ *and* F_K *are defined as in* (2.1.5b)*. Let* $\gamma' < \gamma - \delta$ *and* $\mu' > \delta > 0$ *be fixed. Then there exist constants* M_k, C_k *and* ε *depending on* γ', μ' *and* δ *such that for all* $|I| = k \leq N/2 + 2$

$$|\partial \Gamma^I W^p| \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1-\delta'}, & q > 0, \\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1/2-\mu'}, & q < 0, \end{cases} \tag{2.6.2}$$

$$|\Gamma^I W^p| \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-\delta'}, & q > 0, \\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{1/2-\mu'}, & q < 0, \end{cases} \tag{2.6.3}$$

where $\delta' = \gamma'$ *if* $i = 1$ *or* $\delta' = M_k\varepsilon$ *if* $i = 0$.

Proof. The proof follows by induction. The base case $k = 0$ follows in a simpler but similar way to the main case. One assumes the step for $|I| \leq k$ and then considers the case of $|I| = k+1$. From (2.6.1) the following holds

$$\begin{aligned} |\widetilde{\square}_g \Gamma^I h^1|_{\mathcal{UU}} &\leq |D^I h^1|_{\mathcal{UU}} + |\Gamma^I F|_{\mathcal{UU}} + |\Gamma^I F^0|_{\mathcal{UU}}, \\ |\widetilde{\square}_g \Gamma^I \psi|_{\mathcal{K}} &\leq |D^I \psi|_{\mathcal{K}} + |\Gamma^I F|_{\mathcal{K}}. \end{aligned}$$

From here we use proposition 2.22 and lemma 2.40 to estimate $|\Gamma^I F|_{\mathcal{UU}}$ and $|\Gamma^I F|_{\mathcal{K}}$. Lemma 2.33 gives an estimate for $|\Gamma^I F^0|_{\mathcal{UU}}$. Lastly proposition 2.50 gives an estimate on $|D^I h^1|_{\mathcal{UU}}$ and $|D^I \psi|_{\mathcal{K}}$. One then collects together terms according to whether $|K| = |I|$ or $|K| < |I|$. When $|K| = |I|$ we require strong estimates on the terms with a low number of Γ's acting on H, ie, of the form

$$\sum_{|J|\leq 1} \left(|\Gamma^J H|_{\mathcal{LL}} + |H|_{\mathcal{LT}} \right).$$

Since there are no ψ_K terms appearing here, we may use proposition 2.38 which gives the necessary decay. When $|K| < |I|$ one must use the induction hypothesis.

Putting this altogether gives an estimate on $|\widetilde{\square}_g \Gamma^I W^1|$. Corollary 2.31 implies an estimate on $\varpi(q)|\Gamma^I W^1(t,x)|$. Then inserting all this in corollary 2.27 yields an integral inequality and the result then follows by Grönwall's inequality from lemma 1.20. $\square$

Note the important fact that in proposition 2.41 we have gained more $|q|$ decay and replaced the δ-loss in $(1+t)^{-1+\delta}$ with a term we can control better, namely $(1+t)^{-1+\varepsilon}$.

2.7 Energy Estimates

In this final section we obtain an energy inequality with an additional spacetime integral involving $\overline{\partial}$ derivatives. Using this energy estimate we can apply a Grönwall type argument to deduce the improved inequality (2.1.21) for the energy.

Proposition 2.42 (Energy inequality, [LR10, Proposition 6.2])**.** *Let* ϕ *be a solution to* $\widetilde{\square}_g \phi = F$ *such that for* $H^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ *we have*

$$\begin{aligned} (1+|q|)^{-1}|H|_{\mathcal{LL}} + |\partial H|_{\mathcal{LL}} + |\overline{\partial} H|_{\mathcal{UU}} &\leq C\varepsilon'(1+t)^{-1}, \\ (1+|q|)^{-1}|H|_{\mathcal{UU}} + |\partial H|_{\mathcal{UU}} &\leq C\varepsilon'(1+t)^{-1/2}(1+|q|)^{1/2}(1+q_-)^{-\mu}. \end{aligned} \tag{2.7.1}$$

where $q_- = |q|$ *when* $q < 0$ *and* $q_- = 0$ *when* $q > 0$. *Then for any* $0 < \gamma \leq 1$ *and* $0 < \varepsilon' \leq \gamma/C$ *we have*

$$\begin{aligned}\int_{\Sigma_t} |\partial\phi|^2 w \mathrm{d}^3x + \int_0^t \int_{\Sigma_\tau} |\overline{\partial}\phi|^2 w' \mathrm{d}^3x \mathrm{d}\tau \leq 8 \int_{\Sigma_0} |\partial\phi|^2 w \mathrm{d}^3x \\ + 16 \int_0^t \int_{\Sigma_\tau} \varepsilon \left(\frac{|\partial\phi|^2}{1+t} w + |F||\partial\phi| w \right) \mathrm{d}^3x \mathrm{d}\tau.\end{aligned} \tag{2.7.2}$$

Note the additional positive space-time integral for $\overline{\partial}$ derivatives on the left-hand-side of (2.7.2). The conditions on the weight w imply

$$w' \leq w(1+|q|)^{-1} \leq 16\gamma^{-1} w'(1+q_-)^{2\mu}. \tag{2.7.3}$$

Using this identity, we can see the close comparison with the spacetime integral term and lemma 1.42 of Chapter 1.

Applying Young's inequality and proposition 2.42 on the PDE system (2.6.1) yields

$$\begin{aligned}&\int_{\Sigma_t} \left(|\partial\Gamma^I h^1|^2_{\mathcal{UU}} + |\partial\Gamma^I \psi|^2_{\mathcal{K}} \right) w \mathrm{d}^3x + \int_0^t \int_{\Sigma_\tau} \left(|\overline{\partial}\Gamma^I h^1|^2_{\mathcal{UU}} + |\overline{\partial}\Gamma^I \psi|^2_{\mathcal{K}} \right) w' \mathrm{d}^3x \mathrm{d}\tau \\ &\leq 8 \int_{\Sigma_0} \left(|\partial\Gamma^I h^1|^2_{\mathcal{UU}} + |\partial\Gamma^I \psi|^2_{\mathcal{K}} \right) w \mathrm{d}x + 16 \int_0^t \int_{\Sigma_\tau} \frac{\varepsilon}{1+t} \left(|\partial\Gamma^I h^1|^2_{\mathcal{UU}} + |\partial\Gamma^I \psi|^2_{\mathcal{K}} \right) w \mathrm{d}^3x \mathrm{d}\tau \\ &+ \int_0^t \int_{\Sigma_\tau} \varepsilon^{-1} \left(|\hat{\Gamma}^I F|^2_{\mathcal{UU}} + |\hat{\Gamma}^I F|^2_{\mathcal{K}} + |D^I h^1|^2_{\mathcal{UU}} + |D^I \psi|^2_{\mathcal{K}} \right) (1+t) w \mathrm{d}^3x \mathrm{d}\tau \\ &+ \int_0^t \int_{\Sigma_\tau} |\Gamma^I F^0|_{\mathcal{UU}} |\partial\Gamma^I h^1|_{\mathcal{UU}} w \mathrm{d}^3x \mathrm{d}\tau.\end{aligned} \tag{2.7.4}$$

The terms not 'roughly' in the form of the energy $\mathcal{E}_N[W^1](t)$, defined in (2.1.14), are

$$\varepsilon^{-1}(|\hat{\Gamma}^I F|^2_{\mathcal{UU}} + |\hat{\Gamma}^I F|^2_{\mathcal{K}} + |D^I h^1|^2_{\mathcal{UU}} + |D^I \psi|^2_{\mathcal{K}})(1+t)w + |\Gamma^I F^0|_{\mathcal{UU}} |\partial\Gamma^I h^1|_{\mathcal{UU}} w, \tag{2.7.5}$$

and so these are estimated in the following lemmas 2.44–2.46. First we state the main theorem 2.43, as its assumptions will be the same for lemmas 2.44–2.46, but leave the proof of the theorem until after the lemmas.

Theorem 2.43 (Main theorem, modified from [LR10, Theorem 11.1])**.** *Let* $(h_{\mu\nu}, \psi_K)$ *be a local in time solution to* (2.1.5) *satisfying the wave gauge condition* (2.4.1) *on some maximal interval* $[0, T)$. *Suppose for fixed* $\mu' \in (0, 1/2)$ *and* $\gamma \in (0, 1/2)$, *and* μ *satisfying* $0 < \mu < 1/2 - \mu'$, *that for all* $0 \leq t \leq T$ *and all* $|I| \leq N/2 + 2$ *we have*

$$|\partial W|_G + (1+|q|)^{-1}|H|_{\mathcal{TL}} + (1+|q|)^{-1}|\Gamma H|_{\mathcal{LL}} \leq C\varepsilon(1+t)^{-1}, \tag{2.7.6a}$$

$$|\partial\Gamma^I W| + \frac{|\Gamma^I W|}{1+|q|} + \frac{1+t+|q|}{1+|q|}|\overline{\partial}\Gamma^I W| \leq \begin{cases} C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1-C\varepsilon}, \\ C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} \end{cases} \tag{2.7.6b}$$

$$E_N[W^1](0) + M \leq \varepsilon. \tag{2.7.6c}$$

Then there exist constants C_N *and* C, *independent of* T, *such that for* $\varepsilon > 0$ *sufficiently small the following holds for all* $0 \leq t \leq T$

$$\mathcal{E}_N[W^1](t) \leq C_N \varepsilon^2 (1+t)^{C\varepsilon}. \tag{2.7.7}$$

We begin by estimating the first pair of terms of (2.7.5) in lemmas 2.44 and 2.45.

Lemma 2.44 (Modified from [LR10, Lemma 11.2])**.** *Assuming the conditions of theorem 2.43 then*

$$
\begin{aligned}
|\Gamma^I F|_{\mathcal{U}\mathcal{U}} \lesssim & \sum_{|J|\le|I|} \Big(\frac{\varepsilon|\partial\Gamma^J W^1|}{1+t} + \frac{\varepsilon(1+|q|)^{\mu'-1/2}}{(1+t+|q|)^{1-C\varepsilon}}|\bar\partial\Gamma^J W^1| + \frac{\varepsilon^2}{1+t+|q|}\frac{|\Gamma^J W^1|}{1+|q|}\Big) \\
& + \sum_{|J|\le|I|-1} \frac{\varepsilon}{(1+t)^{1-C\varepsilon}}|\partial\Gamma^J h^1|_{\mathcal{U}\mathcal{U}} + \frac{\varepsilon^2}{(1+t+|q|)^4}, \\
|\Gamma^I F|_{\mathcal{K}} \lesssim & \sum_{|J|\le|I|} \Big(\frac{\varepsilon|\partial\Gamma^J W^1|}{1+t} + \frac{\varepsilon(1+|q|)^{\mu'-1/2}}{(1+t+|q|)^{1-C\varepsilon}}|\bar\partial\Gamma^J W^1| + \frac{\varepsilon^2}{1+t+|q|}\frac{|\Gamma^J W^1|}{1+|q|}\Big) \\
& + \frac{\varepsilon^2}{(1+t+|q|)^4}.
\end{aligned}
$$

Proof. The proof follows from proposition 2.22 and by estimates coming from the assumptions (2.7.6a), (2.7.6b) and from calculating $\partial\Gamma^I h^0, \Gamma^I h^0$. □

Lemma 2.45 (Modified from [LR10, Lemma 11.3])**.** *Assuming the conditions of theorem 2.43 then*

$$
\begin{aligned}
\varepsilon^{-1}\int_0^t\int_{\Sigma_\tau} & \left(|\Gamma^I F|^2_{\mathcal{U}\mathcal{U}} + |\Gamma^I F|^2_{\mathcal{K}}\right)(1+\tau)w\,\mathrm{d}\tau\mathrm{d}^3x \\
& \lesssim \sum_{|J|\le|I|}\int_0^t\int_{\Sigma_\tau}\varepsilon\Big(\frac{|\partial\Gamma^J W^1|^2}{1+\tau}w + |\bar\partial\Gamma^J W^1|^2 w'\Big)\mathrm{d}\tau\mathrm{d}^3x \\
& + \sum_{|J|\le|I|-1}\int_0^t\int_{\Sigma_\tau}\varepsilon\frac{|\partial\Gamma^J h^1|^2_{\mathcal{U}\mathcal{U}}}{(1+\tau)^{1-2C\varepsilon}}w\,\mathrm{d}\tau\mathrm{d}^3x + \varepsilon^3.
\end{aligned}
$$

Proof. Insert the estimate from lemma 2.44 and then apply the variant of Hardy's inequality given in corollary 2.48 on the $|\Gamma^J W^1|$ term. □

Next we estimate the second pair of terms from (2.7.5).

Lemma 2.46 (Modified Lemma 11.5 from [LR10])**.** *Assuming the conditions of theorem 2.43 then*

$$
\begin{aligned}
\varepsilon^{-1}\int_0^t\int_{\Sigma_\tau} & \left(|D^I h^1|^2_{\mathcal{U}\mathcal{U}} + |D^I\psi|^2_{\mathcal{K}}\right)(1+\tau)w\,\mathrm{d}x\mathrm{d}t \\
& \lesssim \varepsilon\sum_{|J|\le|I|}\int_0^t\int_{\Sigma_\tau}\Big(\frac{w}{1+t}|\partial\Gamma^J W^1|^2 + |\bar\partial\Gamma^J W^1|^2 w'\Big)\mathrm{d}x\mathrm{d}t \\
& + \varepsilon\sum_{|J|\le|I|-1}\int_0^t\int_{\Sigma_\tau}\frac{w}{(1+t)^{1-2C\varepsilon}}|\partial\Gamma^J W^1|^2\mathrm{d}\tau\mathrm{d}^3x + \varepsilon^3.
\end{aligned}
$$

Proof. The proof follows as in [LR10], by using the very technical proposition 2.50 (given in the next section) together with the decay estimates for $|\partial\Gamma^I h|_{\mathcal{U}\mathcal{U}}$ and $|\partial\Gamma^I\psi|_{\mathcal{K}}$ given in (2.7.6b). The stronger estimates required on $|\Gamma^J H|_{\mathcal{L}\mathcal{L}}$ and $|H|_{\mathcal{L}\mathcal{T}}$, are unchanged and given in (2.7.6a). □

For the last term in (2.7.5), recall that $h^0_{\mu\nu}$, and thus $\Gamma^I \widetilde{\Box}_g h^0_{\mu\nu}$, is determined by calculating directly from the definition in (2.5.13). Thus the next lemma follows identically from [LR10], with the proof using lemma 2.33 and corollary 2.48 on the $|\Gamma^I h^1|$ term.

Lemma 2.47 ([LR10, Lemma 11.4]). *Assuming the conditions of theorem 2.43 then*

$$\int_0^t \int_{\Sigma_\tau} |\Gamma^I F^0|_{\mathcal{U}\mathcal{U}} |\partial \Gamma^I h^1|_{\mathcal{U}\mathcal{U}} w \mathrm{d}\tau \mathrm{d}^3 x$$
$$\lesssim \varepsilon \sum_{|J|\leq|I|} \Big(\int_0^t \int_{\Sigma_\tau} \frac{|\partial \Gamma^J h^1|^2_{\mathcal{U}\mathcal{U}}}{(1+\tau)^2} w \mathrm{d}\tau \mathrm{d}^3 x + \int_0^t \Big(\int_{\Sigma_\tau} |\partial \Gamma^J h^1|^2_{\mathcal{U}\mathcal{U}} w \mathrm{d}^3 x \Big)^{1/2} \frac{\mathrm{d}\tau}{(1+\tau)^{3/2}} \Big).$$

We can now complete the proof of the main theorem.

Proof of Theorem 2.43. Precisely as in [LR10], lemmas 2.45, 2.47, 2.46 and the integrated energy inequality (2.7.4) imply

$$\begin{aligned}
&\int_{\Sigma_t} \big(|\partial \Gamma^I h^1|^2_{\mathcal{U}\mathcal{U}} + |\partial \Gamma^I \psi|^2_{\mathcal{K}} \big) w \mathrm{d}^3 x + \int_0^t \int_{\Sigma_\tau} \big(|\bar{\partial} \Gamma^I h^1|^2_{\mathcal{U}\mathcal{U}} + |\bar{\partial} \Gamma^I \psi|^2_{\mathcal{K}} \big) w' \mathrm{d}^3 x \mathrm{d}\tau \\
&\leq 8 \int_{\Sigma_0} \big(|\partial \Gamma^I h^1|^2_{\mathcal{U}\mathcal{U}} + |\partial \Gamma^I \psi|^2_{\mathcal{K}} \big) w \mathrm{d}x + C\varepsilon \sum_{|J|\leq|I|} \int_0^t \int_{\Sigma_\tau} \frac{|\partial \Gamma^J W^1|^2}{1+\tau} w \mathrm{d}^3 x \mathrm{d}\tau \\
&\quad + C\varepsilon \sum_{|J|\leq|I|} \int_0^t \int_{\Sigma_\tau} |\bar{\partial} \Gamma^J W^1|^2 w' \mathrm{d}^3 x \mathrm{d}\tau + C\varepsilon \sum_{|J|\leq|I|-1} \int_0^t \int_{\Sigma_\tau} \frac{|\partial \Gamma^J W^1|^2}{(1+\tau)^{1-2C\varepsilon}} w \mathrm{d}^3 x \mathrm{d}\tau \\
&\quad + C_N \varepsilon \sum_{|J|\leq|I|} \int_0^t \Big(\int_{\Sigma_\tau} |\partial \Gamma^J h^1|^2_{\mathcal{U}\mathcal{U}} w \mathrm{d}^3 x \Big)^{1/2} \frac{\mathrm{d}\tau}{(1+\tau)^{3/2}} + C\varepsilon^3.
\end{aligned} \tag{2.7.8}$$

Define the following notation for the spacetime integral of the 'good' derivative

$$S_m(t) = \sum_{|I|\leq m,\, \Gamma\in\mathbf{\Gamma}} \int_0^t \int_{\Sigma_\tau} \big(|\bar{\partial} \Gamma^I h^1|^2_{\mathcal{U}\mathcal{U}} + |\bar{\partial} \Gamma^I \psi|^2_{\mathcal{K}} \big) w' \mathrm{d}^3 x. \tag{2.7.9}$$

Then for $m \leq N$ we obtain the inequality

$$\begin{aligned}
\mathcal{E}_m(t) + S_m(t) &\leq 8\mathcal{E}_m(0) + C\varepsilon S_m(t) + C\varepsilon^3 \\
&\quad + C_N \varepsilon \int_0^t \frac{\mathcal{E}_m(\tau)^{1/2}}{(1+\tau)^{3/2}} \mathrm{d}\tau + C\varepsilon \int_0^t \frac{\mathcal{E}_m(\tau)}{1+\tau} \mathrm{d}\tau + C\varepsilon \int_0^t \frac{\mathcal{E}_{m-1}(\tau)}{(1+\tau)^{1-C\varepsilon}} \mathrm{d}\tau.
\end{aligned} \tag{2.7.10}$$

Note that the smallness assumption of the main theorem, (2.1.15), implies

$$\mathcal{E}_N(0)^{1/2} + M \leq \varepsilon. \tag{2.7.11}$$

We now use Grönwall's inequality from lemma 1.20 and induction. First by choosing $C\varepsilon \leq 1/2$ we can absorb the term $S_m(t)$ on the right-hand-side of (2.7.10) into the left-hand-side at the expense of doubling the constants on the right-hand-side. Next, by applying Young's inequality as $C_N \varepsilon \mathcal{E}_m(\tau)^{1/2} \leq \mathcal{E}_m(\tau) + C_N^2 \varepsilon^2$, and noting that $\mathcal{E}_m(\tau)$

is increasing with τ, we have

$$16C_N\varepsilon\int_0^t \frac{(\mathcal{E}_m(\tau)/4)^{1/2}}{(1+\tau)^{3/2}}\mathrm{d}\tau \le \int_0^t \frac{(\mathcal{E}_m(\tau)/4)}{(1+\tau)^{3/2}}+\frac{(16C_N\varepsilon)^2}{(1+\tau)^{3/2}}\mathrm{d}\tau \le \frac{1}{2}\mathcal{E}_m(t)+CC_N^2\varepsilon^2. \tag{2.7.12}$$

Substituting this information into (2.7.10) we find

$$\mathcal{E}_m(t)+S_m(t)\le 32\mathcal{E}_m(0)+C\varepsilon\int_0^t \frac{\mathcal{E}_m(\tau)}{1+\tau}\mathrm{d}\tau + C\varepsilon\int_0^t \frac{\mathcal{E}_{m-1}(\tau)}{(1+\tau)^{1-C\varepsilon}}\mathrm{d}\tau. \tag{2.7.13}$$

For $k=0$ this gives

$$\mathcal{E}_0(t)\le 32\varepsilon + C\varepsilon\int_0^t \frac{\mathcal{E}_0(\tau)}{1+\tau}\mathrm{d}\tau, \tag{2.7.14}$$

and an application of Grönwall's inequality from lemma 1.20 implies

$$\mathcal{E}_0(t)\le 32\varepsilon^2(1+t)^{2c\varepsilon}. \tag{2.7.15}$$

By possibly increasing C_N, we have for sufficiently small ε,

$$\mathcal{E}_0(t)^{1/2}\le C_N\varepsilon(1+t)^{c\varepsilon}. \tag{2.7.16}$$

Assuming (2.7.7) for $m-1$ we then have from (2.7.13)

$$\mathcal{E}_m(t)\le 32\mathcal{E}_m(0)+C\varepsilon\int_0^t \frac{\mathcal{E}_m(\tau)}{1+\tau}\mathrm{d}\tau + \int_0^t \frac{C\varepsilon^3}{(1+\tau)^{1-C\varepsilon}}\mathrm{d}\tau. \tag{2.7.17}$$

Again by Grönwall's inequality this implies

$$\mathcal{E}_m(t)\le \left(32\mathcal{E}_m(0)+C\varepsilon^2(1+t)^{2C\varepsilon}\right)(1+t)^{2c\varepsilon}. \tag{2.7.18}$$

Clearly for ε sufficiently small we have for all $0\le t\le T$

$$\mathcal{E}_N(t)\le C_N\varepsilon^2(1+t)^{C\varepsilon}. \tag{2.7.19}$$

□

Proof of Theorem 2.9. To conclude the proof of theorem 2.9 we remark that (2.1.20) then allows us to use propositions 2.36 and 2.38 to derive (2.7.6a) and proposition 2.41 to derive (2.7.6b). □

2.8 Additional results

This section states without proof some necessary results from [LR10]. We first state a Hardy-type estimate. Note in comparison to the Hardy estimate given in lemma 1.23 the weight depends on the distance $|q|$ to the light cone, rather than the distance r to the origin.

Corollary 2.48 ([LR10, Corollary 13.3]). *If $\gamma>0$ and $\mu>0$, then for any $a\in[-1,1]$ and any $\phi\in C_0^\infty(\mathbb{R}^3)$ we have*

$$\int \frac{|\phi|^2}{(1+|q|)^2}\frac{w\mathrm{d}x}{(1+t+|q|)^{1-a}} \lesssim \int |\partial\phi|^2\frac{w\mathrm{d}x}{(1+t+|q|)^{1-a}}.$$

If additionally $a < 2\min(\gamma,\mu)$ then

$$\int \frac{|\phi|^2}{(1+|q|)^2}\frac{(1+|q|)^{-a}}{(1+t+|q|)^{1-a}}\frac{w\mathrm{d}x}{(1+q_-)^{2\mu}} \lesssim \int |\partial\phi|^2 \min\left(w', \frac{w}{(1+t+|q|)^{1-a}}\right)\mathrm{d}x,$$

where $q_- = |q|$ when $q<0$ and $q_- = 0$ when $q>0$.

We next briefly state some other key results from [LR10], the first two making use of the null frame. The final proposition 2.50 gives control on the commutator $\widetilde{\Box}_g\Gamma^I\phi - \hat{\Gamma}^I\widetilde{\Box}_g\phi$.

Lemma 2.49 ([LR10, Lemma 5.1]). *If ϕ is a scalar and π a symmetric 2-tensor then*

$$(1+t+|q|)|\overline{\partial}\phi| + (1+|q|)|\partial\phi| \le C\sum_{|I|=1}|\Gamma^I\phi|, \tag{2.8.1}$$

$$|\overline{\partial}^2\phi| + r^{-1}|\overline{\partial}\phi| \le \frac{C}{r}\sum_{|I|\le 2}\frac{|\Gamma^I\phi|}{1+t+|q|}, \tag{2.8.2}$$

$$|\pi^{\mu\nu}\partial_\mu\partial_\nu\phi| \le C\left(\frac{|\pi|_{\mathcal{U}\mathcal{U}}}{1+t+|q|} + \frac{|\pi|_{\mathcal{L}\mathcal{L}}}{1+|q|}\right)\sum_{|I|\le 1}|\partial\Gamma^I\phi|. \tag{2.8.3}$$

Proposition 2.50 ([LR10, Proposition 5.3]). *Recall definition 2.39. We have*

$$|\widetilde{\Box}_g\Gamma\phi - \hat{\Gamma}\widetilde{\Box}_g\phi| \lesssim \Big(\frac{|\Gamma H|_{\mathcal{U}\mathcal{U}} + |H|_{\mathcal{U}\mathcal{U}}}{1+t+|q|} + \frac{|\Gamma H|_{\mathcal{L}\mathcal{L}} + |H|_{\mathcal{L}\mathcal{T}}}{1+|q|}\Big)\sum_{|I|\le 1}|\partial\Gamma^I\phi|.$$

Furthermore for any $\Gamma \in \mathbf{\Gamma}$

$$\begin{aligned}
|\widetilde{\Box}_g\Gamma^I\phi - \hat{\Gamma}^I\widetilde{\Box}_g\phi| \lesssim{}& \frac{1}{1+t+|q|}\sum_{|K|\le|I|}\ \sum_{|J|+(|K|-1)_+\le|I|}|\Gamma^J H|_{\mathcal{U}\mathcal{U}}|\partial\Gamma^K\phi| \\
&+ \frac{1}{1+|q|}\sum_{|K|\le|I|}\Big(\sum_{|J|+(|K|-1)_+\le|I|}|\Gamma^J H|_{\mathcal{L}\mathcal{L}} + \sum_{|J|+(|K|-1)_+\le|I|-1}|\Gamma^J H|_{\mathcal{L}\mathcal{T}} \\
&\qquad + \sum_{|J|+(|K|-1)_+\le|I|-2}|\Gamma^J H|_{\mathcal{U}\mathcal{U}}\Big)|\partial\Gamma^K\phi|,
\end{aligned}$$

where $(|K|-1)_+ = |K|-1$ if $|K|\ge 1$ and $(|K|-1)_+ = 0$ if $|K| = 0$.

A key feature of proposition 2.50 is that the factors involving the poorly decaying term $(1+|q|)^{-1}$ involves the very good components $|\Gamma H|_{\mathcal{L}\mathcal{L}}$ and $|H|_{\mathcal{L}\mathcal{T}}$ whose estimates are improved using the wave gauge condition (e.g. by proposition 2.38).

2.9 Zero-mode perturbations of Kaluza Klein spacetimes

The motivation for considering the system (2.1.5) is to prove theorem 2.1, and we now discuss how this is achieved.

Remark 2.51. The indices in the remainder of this Chapter are those of definition 1.37.

The Kaluza-Klein spacetime $\mathbb{R}^{1+3}\times\mathbb{T}^d$ with metric

$$\hat{G}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = (\eta_{\mathbb{R}^{1+3}})_{ij}\mathrm{d}x^i\mathrm{d}x^j + \delta_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{2.9.1}$$

is Ricci flat. Consider another spacetime $G_{\mu\nu}$ defined on $\mathbb{R}^{1+3}\times\mathbb{T}^d$ which is 'close' to $\hat{G}$ and is Ricci flat. As shown earlier in (1.1.9) with respect to the *five*-dimensional $\hat{e}$-wave gauge, when $\hat{e}=\hat{G}$ the field equations for $G_{\mu\nu}$ becomes a system of nonlinear wave equations

$$G^{\rho\sigma}\partial_\rho\partial_\sigma G_{\mu\nu}=\mathcal{F}_{\mu\nu}(G)(\partial G,\partial G). \tag{2.9.2}$$

For simplicity, we briefly restrict to the case of the one-dimensional torus $\mathbb{S}^1$. Since $\mathbb{S}^1$ is compact, say of radius R, we can Fourier expand the metric as

$$G_{\mu\nu}(x^i,x^A)=\sum_{n\in\mathbb{Z}}\exp\left(\frac{inx^A}{R}\right)G^{(n)}_{\mu\nu}(x^i). \tag{2.9.3}$$

If we substitute the expansion (2.9.3) into the left-hand-side of (2.9.2) and look at just the terms coming from the flat background $\hat{G}_{\mu\nu}$ we obtain

$$\hat{G}^{\rho\sigma}\partial_\rho\partial_\sigma G_{\mu\nu}=\sum_{n\in\mathbb{Z}}\left((\eta_{\mathbb{R}^{1+3}})^{ij}\partial_i\partial_j-\left(\frac{n}{R}\right)^2\right)G^{(n)}_{\mu\nu}.$$

Heuristically one can see that the modes $G^{(n)}_{\mu\nu}$ with $n\neq 0$, will satisfy nonlinear Klein-Gordon equations with mass $|n|/R$. At this point, the standard physical argument is to ignore these $n\neq 0$ modes by taking R sufficiently small that the mass $|n|/R$ is larger than any probable energy [Pop].

Hence we consider only the $n=0$ modes by setting $G^{(n)}_{\mu\nu}=0$ for all $n\neq 0$. This implies the higher-dimensional metric $G_{\mu\nu}$ depends only on the internal coordinates

$$G_{\mu\nu}(x^i,x^A)=G_{\mu\nu}(x^i). \tag{2.9.4}$$

If the initial data satisfies (2.9.4), and the PDE is invariant in the internal directions, then the solution will be also. The perturbation and its inverse are defined by

$$h_{\mu\nu}=G_{\mu\nu}-\hat{G}_{\mu\nu}, \tag{2.9.5}$$
$$H^{\mu\nu}=G^{\mu\nu}-\hat{G}^{\mu\nu}, \tag{2.9.6}$$

where $G_{\mu\rho}G^{\rho\nu}=\delta^\nu_\mu$. Condition (2.9.4) can be rewritten as

$$\partial_A h_{\mu\nu}=0. \tag{2.9.7}$$

Differentiating this identity and using (2.9.7) implies $\partial_A H^{\mu\nu}=0$.

Proposition 2.52. *The zero-mode perturbations of the Kaluza Klein spacetime on* $\mathbb{R}^{1+3}\times\mathbb{T}^d$ *about the background spacetime* (2.9.1) *are covered by the generalised PDE system* (2.1.5).

Proof. Under (2.9.7) and the $\hat{G}$-wave gauge condition

$$\partial_\rho\left(G^{\rho\mu}\sqrt{|\det G|}\right)=0\,, \tag{2.9.8}$$

the vacuum Einstein equations in $\mathbb{R}^{1+3}\times\mathbb{T}^d$ reduce to the system

$$\widetilde{\Box}_g h_{ij}=P(\partial_i h,\partial_j h)+Q_{ij}(\partial h,\partial h)+G_{ij}(h)(\partial h,\partial h), \tag{2.9.9a}$$
$$\widetilde{\Box}_g h_{iB}=Q_{iB}(\partial h,\partial h)+G_{iB}(h)(\partial h,\partial h), \tag{2.9.9b}$$
$$\widetilde{\Box}_g h_{AB}=Q_{AB}(\partial h,\partial h)+G_{AB}(h)(\partial h,\partial h). \tag{2.9.9c}$$

The $\mathcal{O}((\partial h)^2)$ non-null nonlinearities are given by

$$P(\partial_i h, \partial_j h) = P^1(\partial_i h, \partial_j h) + P^2(\partial_i h, \partial_j h) + P^3(\partial_i h, \partial_j h),$$

where we have

$$\begin{aligned}
P^1(\partial_i h, \partial_j h) &= \eta^{kl}\eta^{pq}\left(\frac{1}{4}\partial_i h_{kl}\partial_\nu h_{pq} - \frac{1}{2}\partial_i h_{kp}\partial_j h_{lq}\right),\\
P^2(\partial_i h, \partial_j h) &= \delta^{AB}\eta^{kl}\left(\frac{1}{4}\partial_i h_{AB}\partial_j h_{kl} + \frac{1}{4}\partial_j h_{AB}\partial_i h_{kl}\right),\\
P^3(\partial_i h, \partial_j h) &= -\delta^{AB}\eta^{kl}\left(\partial_i h_{Ak}\partial_j h_{Bl}\right) + \delta^{AB}\delta^{CD}\left(\frac{1}{4}\partial_i h_{AB}\partial_j h_{CD} - \frac{1}{2}\partial_i h_{AC}\partial_j h_{BD}\right),
\end{aligned}$$

and the null forms are given by

$$\begin{aligned}
Q_{ij}(\partial h, \partial h) &= \eta^{kl}Q_0\left(h_{ki}, h_{lj}\right) + \delta^{AB}Q_0\left(h_{Ai}, h_{Bj}\right)\\
&\quad + \eta^{pq}\left(\eta^{kl}Q_{pl}(h_{ki}, h_{qj}) + \delta^{AB}Q_{ip}(h_{qB}, h_{Aj})\right) + (i \leftrightarrow j)\\
&\quad + \frac{1}{2}\eta^{kl}\left(\eta^{pq}Q_{li}(h_{pq}, h_{kj}) + \delta^{AB}Q_{li}(h_{AB}, h_{kj})\right) + (i \leftrightarrow j),\\
Q_{iB}(\partial h, \partial h) &= \delta^{CD}Q_0(h_{Ci}, h_{DB}) + \eta^{kj}\delta^{CD}\left(Q_{ik}(h_{jD}, h_{CB}) + \frac{1}{2}Q_{ji}(h_{CD}, h_{kB})\right),\\
Q_{AB}(\partial h, \partial h) &= \delta^{CD}Q_0(h_{CA}, h_{DB}).
\end{aligned}$$

Here $(i \leftrightarrow j)$ indicates the previous bracketed term are repeated with i and j swapped. If we consider the set $\{\psi_K\} = \{h_{aB}, h_{AB}\}$ then we see the PDE (2.9.9) falls into the system described in (2.1.5) □

Using theorem 2.9 we have the following.

Theorem 2.53. *Let* $(\Sigma_0, \bar{g}_{ab}, \bar{K}_{ab})$ *be smooth initial data for the equations of motion* (2.9.9) *arising from the* $(3+d+1)-$*dimensional vacuum Einstein equations under the condition* (2.9.7)*. Define, for* $i', j' \in \{1,2,3\}$, $r = ((\eta_{\mathbb{R}^{1+3}})_{i'j'}x^{i'}x^{j'})^{1/2} \to \infty$ *and the 2-tensor* $\bar{g}^1_{ab}$ *by*

$$\bar{g}_{ab} = \begin{pmatrix} \left(1 + \chi(r)\frac{M}{r}\right)\delta_{i'j'} & 0 \\ 0 & \delta_{AB} \end{pmatrix} + \bar{g}^1_{ab},$$

with $\chi(r)$ *defined in definition 2.6 and* $M \in (0, \infty)$*. Suppose also that* Σ_0 *is diffeomorphic to* $\mathbb{R}^3 \times \mathbb{T}^d$ *and the initial data satisfies the constraint equations* (1.1.1) *and is asymptotically Kaluza-Klein in the sense that*

$$\bar{g}^1_{ab} = o(r^{-1-\alpha}), \qquad \bar{K}_{ab} = o(r^{-2-\alpha}), \qquad \alpha > 0. \tag{2.9.10}$$

Furthermore for $N \in \mathbb{Z}$ *large enough and* $\gamma \in (0, 1/2)$*, let* $\nabla = \{\partial_1, \partial_2, \partial_3\}$ *denote spatial derivatives and define*

$$E_N(0) = \sum_{|I|\le N}\int_{\Sigma_0}(1+r)^{1+2\gamma+2|I|}\left(|\nabla\nabla^I\bar{g}^1|^2 + |\nabla^I K|^2\right)d^{3+d}x.$$

Then there exists a constant $\varepsilon_0 > 0$ *such that for all* $\varepsilon \le \varepsilon_0$ *and initial data with*

$E_N(0) + M \leq \varepsilon$, the solution

$$G_{\mu\nu}(t) = \hat{G}_{\mu\nu} + \begin{pmatrix} \left(1 + \chi(r/t)\chi(r)\frac{M}{r}\right)\delta_{i'j'} & 0 \\ 0 & 0 \end{pmatrix} + h^1_{\mu\nu}(t)$$

to (2.9.9) *exists for all times and obeys the estimate*

$$\mathcal{E}_N(t) \leq C_N \varepsilon^2 (1+t)^{C\varepsilon}. \tag{2.9.11}$$

Moreover,

$$|\partial\Gamma^I h^1| \leq \begin{cases} C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-1-\gamma}, & r > t, \\ C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-1/2}, & r \leq t, \end{cases} \quad |I| \leq N-2 \tag{2.9.12}$$

$$|\Gamma^I h^1| \leq \begin{cases} C_N\varepsilon(1+t+|q|)^{-1+C_N\varepsilon}(1+|q|)^{-\gamma}, & r > t \quad |I| \leq N-2, \\ C_N\varepsilon(1+t+|q|)^{-1+2C_N\varepsilon}, & r \leq t \quad |I| \leq N-2. \end{cases} \tag{2.9.13}$$

Hence $G_{\mu\nu}(t) \to \hat{G}_{\mu\nu}$ as $t \to \infty$ and so the perturbed $\mathbb{T}^d$ radii decay to the radii of the background geometry.

Note the radii of the $\mathbb{T}^d$ in the background geometry do not necessarily have to be taken small, merely non-zero. In the earlier work [LR05, §16] it was shown that the perturbed solution defines a future geodesically complete manifold. Since our variables obey the same decay rates, it follows in a similar way that the perturbed solution $G_{\mu\nu}(t)$ yields a future causally geodesically complete solution asymptotically converging to $\mathbb{R}^{1+3} \times \mathbb{T}^d$.

We assume the initial data required by theorem 2.53 exists. Note that the case of $M = 0$ is excluded since, by the positive mass theorem, such initial data would be identically Euclidean. The constraint equations have a perhaps more natural interpretation from the perspective of the $(3+1)-$dimensional non-minimally coupled Einstein-Maxwell-Scalar system. This was derived in [Wya18, Appendix A] however we refer to the clearer derivation in [BFK19, §3].

Comparison with other results. Although our method of proof follows [LR10], the terms h_{AB} and h_{aB} cannot be treated directly as ψ variables in the original system (2.1.2). For one, these variables have non-trivial inhomogeneities which is unlike those presented in (2.1.2). In the Kaluza-Klein example there is also a physical interpretation. Equation (2.9.9a) includes a term of the form

$$\delta^{AB} m^{kl}\left(\frac{1}{4}\partial_i h_{AB}\partial_j h_{kl} + \frac{1}{4}\partial_j h_{AB}\partial_i h_{kl}\right).$$

Up to redefining variables, as done in (1.6.3), this describes interactions between the $(3+1)-$dimensional metric and the scalar fields g_{AB}. Similar terms exist in $Q_{\mu\nu}$ and $G_{\mu\nu}$. This non-trivial coupling cannot come from the stress-energy tensor of a massless scalar field, and so the more general PDE system (2.1.5) is required.

In fact, we can return to the system (1.6.4) when $\alpha \neq 1/\sqrt{12}$. We recall that these equations read

$$\mathrm{Ric}[g]_{ij} = \tfrac{1}{2}\partial_i\phi\partial_j\phi + \tfrac{1}{2}e^{-6\alpha\phi}\Big(F_{ik}F_j{}^k - \tfrac{1}{4}F_{kl}F^{kl}g_{ij}\Big)\,, \tag{2.9.14a}$$

$$\nabla[g]_i\big(e^{-6\alpha\phi}F^{ij}\big) = 0\,, \tag{2.9.14b}$$

$$g^{ij}\nabla[g]_i\nabla[g]_j\phi = -\tfrac{3}{2}\alpha e^{-6\alpha\phi}F_{kl}F^{kl}\,, \tag{2.9.14c}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. Even when $\alpha \neq 1/\sqrt{12}$ such a PDE system is still relevant in various four-dimensional theories although it does not come from the higher-dimensional Einstein equations compactified on a torus. With respect to the harmonic and Lorenz gauges

$$\partial_\rho\left(g^{\rho\mu}\sqrt{|\det g|}\right) = 0, \quad \partial_\rho\left(\sqrt{|\det g|}A^\rho\right) = 0\,, \tag{2.9.15}$$

the system (2.9.14) for general $\alpha \in \mathbb{R}$ reduces to

$$\begin{aligned}
\widetilde{\Box}_g h_{ij} =& F_{ij}(h)(\partial h,\partial h) - \partial_i\phi\partial_j\phi - e^{-6\alpha\phi}\Big(\eta^{kl}(\partial_i A_k - \partial_k A_i)(\partial_j A_l - \partial_l A_j) \\
&- \tfrac{1}{4}\eta_{ij}\eta^{mk}\eta^{nl}(\partial_k A_l - \partial_l A_k)(\partial_m A_n - \partial_n A_m)\Big) + O(|h||\partial A|^2)\,, \qquad (2.9.16a)\\
\widetilde{\Box}_g A^i =& \eta^{jk}\eta^{lm}\partial_l h_{ij}(\partial_m A_k - \partial_k A_m) + \eta^{jk}\eta^{lm}\partial_i h_{jl}\partial_k A_m \\
&- 6\alpha\eta^{ik}(\partial_k A_j - \partial_j A_k)\partial_i\phi + O(|h||\partial h||\partial A|) + \alpha O(|h||\partial h||\partial\phi|)\,, \qquad (2.9.16b)\\
\widetilde{\Box}_g\phi =& -\tfrac{3}{2}\alpha e^{-6\alpha\phi}\eta^{ik}\eta^{jl}(\partial_k A_l - \partial_l A_k)(\partial_i A_j - \partial_j A_i)\,. \qquad (2.9.16c)
\end{aligned}$$

Analagous to (2.4.2) we obtain good control on one component of A^ν by expanding out the Lorenz gauge condition (2.9.15) to find

$$|\partial A|_{\mathcal{L}} \leq C|\overline{\partial} A|_{\mathcal{U}} + C|h|_{\mathcal{U}\mathcal{U}}|\partial A|_{\mathcal{U}}. \tag{2.9.17}$$

Thus using the same idea as in section 2.3, we can use (2.4.2), (2.9.17) and the trivial estimate $|e^{-6\alpha\phi}| \leq C$ for $|\phi| \leq \frac{1}{10}$, to estimate the nonlinearities in (2.9.16) as

$$\begin{aligned}
|\widetilde{\Box}_g h_{ij}|_{\mathcal{U}\mathcal{U}} &\leq |\partial h|^2_{\mathcal{T}\mathcal{U}} + |\overline{\partial} h|_{\mathcal{U}\mathcal{U}}|\partial h|_{\mathcal{U}\mathcal{U}} + |\partial\phi|^2 + |\partial A|^2_{\mathcal{U}} + O(\text{cubic}),\\
|\widetilde{\Box}_g h_{ij}|_{\mathcal{T}\mathcal{U}} &\leq |\overline{\partial} h|_{\mathcal{U}\mathcal{U}}|\partial h|_{\mathcal{U}\mathcal{U}} + |\overline{\partial}\phi||\partial\phi| + |\overline{\partial} A|_{\mathcal{U}}|\partial A|_{\mathcal{U}} + O(\text{cubic})\\
|\widetilde{\Box}_g A|_{\mathcal{U}} &\lesssim |\overline{\partial} A|_{\mathcal{U}}|\partial\phi| + |\overline{\partial}\phi||\partial A|_{\mathcal{U}} + |\overline{\partial} h|_{\mathcal{U}\mathcal{U}}|\partial A|_{\mathcal{U}} + |\partial h|_{\mathcal{U}\mathcal{U}}|\overline{\partial} A|_{\mathcal{U}} + O(\text{cubic})\\
|\widetilde{\Box}_g\phi| &\lesssim |\overline{\partial} A|_{\mathcal{U}}|\partial A|_{\mathcal{U}} + O(\text{cubic})
\end{aligned} \tag{2.9.18}$$

In the above the implicit constant in $\lesssim$ depends on α, i.e. $C = C(|\alpha|)$. Also we write $O(\text{cubic}) = O(|h|; |\partial\phi|, |\partial h|, |\partial A|; |\partial\phi|, |\partial h|, |\partial A|)$ to denote a cubic nonlinearity. Similar to definition 2.20, if we make the identification

$$\begin{aligned}
W &= \{U^\mu V^\nu h_{\mu\nu}, U^\mu A_\mu, \phi : U, V \in \mathcal{U}\},\\
W_G &= \{U^\mu T^\nu h_{\mu\nu}, U^\mu A_\mu, \phi : U \in \mathcal{U}, T \in \mathcal{T}\},
\end{aligned} \tag{2.9.19}$$

then the estimates (2.9.18) become

$$\begin{aligned}
|\widetilde{\Box}_g h_{ij}|_{\mathcal{T}\mathcal{U}} + |\widetilde{\Box}_g A|_{\mathcal{U}} + |\widetilde{\Box}_g\phi| &\lesssim |\overline{\partial} W||\partial W| + |W||\partial W|^2,\\
|\widetilde{\Box}_g h_{ij}|_{\mathcal{U}\mathcal{U}} &\lesssim |\partial W|^2_G + |\overline{\partial} W||\partial W| + |W||\partial W|^2.
\end{aligned} \tag{2.9.20}$$

This establishes the first estimate in proposition 2.22, which is the key estimate that we used to extend the result of [LR10] to prove theorem 2.9. Thus, up to precisely writing down the full details, the proof presented for theorem 2.9 will clearly also extend to (2.9.14) with $\alpha \neq 1/\sqrt{12}$.

One example of where this extension is useful occurs when $\alpha = 0$. The system

(2.9.14) becomes

$$\begin{aligned} \mathrm{Ric}[g]_{ij} &= \tfrac{1}{2}\partial_i\phi\partial_j\phi + \tfrac{1}{2}\Big(\mathcal{F}_{ik}\mathcal{F}_j{}^k - \frac{1}{4}g_{ij}\mathcal{F}^{kl}\mathcal{F}_{kl}\Big), \\ \nabla[g]_i\mathcal{F}^{ij} &= 0, \\ \widetilde{\Box}_g\phi &= 0. \end{aligned} \tag{2.9.21}$$

In this situation it is consistent to set $\phi = 0$ which gives the bosonic part of $\mathcal{N} = 2$ supergravity in four spacetime dimensions, or, more simply, Einstein-Maxwell theory. Thus establishing estimates of the form (2.9.20), allows one to prove that Minkowski is a stable solution to the Einstein-Maxwell equations. This is precisely the approach taken in [Loi09].

The PDE system (2.1.5) is certainly not the most general possible and it is unclear what generalisation of (2.1.5) would be the most fruitful at capturing physically interesting scenarios. The recent expansive work [Kei18] has made progress in this direction, by considering a system of quasilinear wave equations ordered into a hierarchy, so that the worst term (i.e. ψ in theorem 1.43, $h_{\underline{LL}}$ here), never sources itself quadratically as $\partial_t\psi\partial_t\psi$ and when it does source better behaved terms (which are lower down in the hierarchy) it does so as a null form.

However not all quasilinear wave equations derived from general relativity are included in our class (2.1.5) nor indeed in [Kei18]. For example, nonlinearities of the type $A^\mu\partial_\mu\phi$ are not covered, yet these arise when considering the Einstein-Maxwell-scalar field system where the scalar field is charged under the $U(1)$. Such a system has been studied on a fixed asymptotically-Minkowskian background in [Kau18]. Another example is the Einstein equations coupled to nonlinear electromagnetic fields, such as the Born-Infeld system, considered in [Spe14].

Chapter 3

The Ground State Stability of $U(1)$-Higgs Model

3.1 Introduction

Main objective. In this chapter, we[1] study the equations of motion arising from the Higgs mechanism applied to an abelian $U(1)$ gauge theory on a fixed Minkowski spacetime after spontaneous symmetry breaking. We view this model as a stepping-stone towards the full non-abelian Glashow-Weinberg-Salam theory (GSW), also known as the electroweak Standard Model. For background physics information on the models treated in this chapter, see for example [AH12].

In short, we study in this chapter a class of semilinear hyperbolic equations which involve the first-order Dirac equation coupled to second-order wave or Klein-Gordon equations. We are interested in the initial value problem for such systems, when the initial data have sufficiently small Sobolev-type norm. In particular we provide a new application of the hyperboloidal foliation method which is well suited to study such coupled systems involving massive fields.

The hyperboloidal foliation method takes its root in pioneering work by Klainerman [Kla85a] and Hörmander [Hör97] on the (uncoupled) Klein-Gordon equation. Recently it has been extensively developed by LeFloch and Ma [LM14] to establish global-in-time existence results for nonlinear systems of coupled wave and Klein-Gordon equations. More generally there has been much recent study on nonlinear interaction terms between coupled wave and Klein-Gordon equations [Kat12, LM14, Smu16, IP19a] with results even extending to the Einstein Klein-Gordon equations [HW15, LM16b, FJS17, LM16a, LM17a, LM18, IP19b, Wan20].

The model of interest. In the abelian $U(1)$ gauge model, the set of unknowns consists of a Dirac field $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ representing a fermion of mass m_g with spin 1/2, a vector field $A = (A^\mu)$ representing a massive boson of mass m_q with spin 1, and a complex scalar field ϕ representing the Higgs field. At the non-zero points $\phi_0 = ve^{i\theta}$ where $\theta : \mathbb{R}^{1+3} \to \mathbb{R}$ is arbitrary, the Higgs potential vanishes

$$V(\phi_0^*\phi_0) = 0. \tag{3.1.1}$$

[1]The results in this chapter were obtained in collaboration with Philippe LeFloch and Shijie Dong at Sorbonne University (Pierre and Marie Curie Campus). This research was also supported in part by the Innovative Training Networks (ITN) grant 642768 'ModCompShock'.

As in [Tsu03b], we consider only constant ground states satisfying $\partial_\mu \phi_0 = 0$. Thus we consider perturbations of the form

$$\chi = \phi - \phi_0, \tag{3.1.2}$$

where ϕ_0 is *constant* in space and time and has magnitude $|\phi_0| = v$. Given such a constant ground state ϕ_0 and the three physical parameters m_q, m_λ, m_g, the equations of motion in a modified Lorenz gauge consist of three evolution equations

$$(\Box - m_q^2)A^\nu = Q_{A^\nu}, \tag{3.1.3a}$$

$$\Box\chi - m_q^2 \frac{\phi_0}{2v^2}(\phi_0^*\chi - \chi^*\phi_0) - m_\lambda^2 \frac{\phi_0}{2v^2}(\phi_0^*\chi - \chi^*\phi_0) = Q_\chi, \tag{3.1.3b}$$

$$i\gamma^\mu \partial_\mu \psi - m_g \psi = Q_\psi, \tag{3.1.3c}$$

and a constraint equation

$$\mathrm{div}_\eta A + i\frac{m_q}{v\sqrt{2}}(\phi_0^*\chi - \chi^*\phi_0) = 0. \tag{3.1.4}$$

The quadratic nonlinearities Q_{A^ν}, Q_χ, Q_ψ and Dirac matrices γ^μ and γ_5 are defined in (3.4.14) and section 3.2.1). The mass coefficients are

$$m_q^2 = 2q^2v^2 > 0, \qquad m_\lambda^2 = 4\lambda v^2 > 0, \qquad m_g = gv^2. \tag{3.1.5}$$

Without loss of generality we can restrict to $m_g \geq 0$. Our main result concerns the system (3.1.3) and is a proof of the global-in-time existence of solutions for sufficiently small perturbations away from the constant vacuum state, defined by the conditions

$$A^\mu \equiv 0, \quad \phi \equiv \phi_0 \text{ and } \partial_\mu \phi_0 = 0, \quad \psi \equiv 0. \tag{3.1.6}$$

It is convenient to work with the perturbed Higgs field $\chi = \phi - \phi_0$ as our main unknown.

Definition 3.1 (Initial data for theorem 3.2)**.** The initial data set consists of functions $A_0^\nu, A_1^\nu : \mathbb{R}^3 \to \mathbb{R}^4$, $\chi_0, \chi_1 : \mathbb{R}^3 \to \mathbb{C}$ and $\psi_0 : \mathbb{R}^3 \to \mathbb{C}^4$ compactly supported in the ball $\{|x| \leq R\}$ of radius $R > 0$. We consider the initial value problem at $t_0 = R + 1$ with

$$(A^\nu, \chi, \psi)(t_0, \cdot) = (A_0^\nu, \chi_0, \psi_0), \qquad (\partial_t A^\nu, \partial_t \chi)(t_0, \cdot) = (A_1^\nu, \chi_1), \tag{3.1.7}$$

and $A_0, \chi_0, \psi_0 \in H^{N_0+1}(\mathbb{R}^3)$ and $A_1, \chi_1 \in H^{N_0}(\mathbb{R}^3)$ for N_0 a large integer. These data are 'Lorenz compatible' in the sense of satisfying the following equations

$$\partial_a A_0^a = -A_1^0 + 2q\mathrm{Im}(\phi_0^*\chi_0), \tag{3.1.8a}$$

$$\begin{aligned}\Delta A_0^0 - m_q^2 A_0^0 = {} & -\partial_a A_1^a + 2q\mathrm{Im}(\phi_0^*\chi_1) - 2q\mathrm{Im}(\chi_0^*\chi_1) \\ & + 2q^2 A_0^0\big(2\mathrm{Re}(\phi_0^*\chi_0) + \chi_0^*\chi_0\big) + q\psi_0^*\psi_0.\end{aligned} \tag{3.1.8b}$$

The elliptic-type system (3.1.8) consists of two equations for 16 unknowns. Due to our restriction to compactly supported data, we are *not* able to study all solutions to (3.1.8). Nevertheless nontrivial data can be found. For example, we can freely pick $A_0^\mu, A_1^\mu, \mathrm{Re}(\chi_0), \mathrm{Re}(\chi_1), \psi_0 \in C_c^\infty(\mathbb{R}^3)$, then solve for $\mathrm{Im}(\chi_0)$ using (3.1.8a) and finally solve for $\mathrm{Im}(\chi_1)$ using (3.1.8b).

In the following statement, we have $m_g \geq 0$, and the coefficient m_g^{-1} is interpreted as $+\infty$ when $m_g = 0$.

Theorem 3.2 (Nonlinear stability of the ground state for the Higgs boson)**.** *Consider the system* (3.1.3) *with parameters* $m_q, m_\lambda > 0$, $m_g \in [0, \min(m_q, m_\lambda)]$ *and let* N_0 *be a sufficiently large integer. There exists* $\varepsilon_0 > 0$, *which is* <u>*independent*</u> *of* m_g, *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and all initial data in the sense of definition 3.1 satisfying*

$$\|A_0, \chi_0, \psi_0\|_{H^{N_0+1}(\mathbb{R}^3)} + \|A_1, \chi_1\|_{H^{N_0}(\mathbb{R}^3)} \leq \varepsilon, \tag{3.1.9}$$

the initial value problem of (3.1.3) *admits a global-in-time solution* (A, χ, ψ). *Furthermore there exists a constant* C_1 *such that the solution satisfies the following uniform energy estimates at highest order*

$$\big\|(s/t)\partial^I\Omega^J(\partial_\mu A_\nu, \partial_\mu\chi)\big\|_{L^2(\mathcal{H}_s)} \lesssim C_1\varepsilon, \quad |I| + |J| = N_0, \tag{3.1.10}$$

and logarithmic energy growth for the Dirac field

$$\big\|(s/t)\partial^I\Omega^J\partial_\mu\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim C_1\varepsilon\log s, \quad |I| + |J| = N_0. \tag{3.1.11}$$

together with the decay estimates

$$|A| \lesssim \varepsilon t^{-3/2}, \qquad |\chi| \lesssim \varepsilon t^{-3/2}, \qquad |\psi| \lesssim \varepsilon \min\big(t^{-1}, m_g^{-1}t^{-3/2}\big). \tag{3.1.12}$$

The result in theorem 3.2 holds in a fairly straightforward way for the cases $m_g = 0$ or $m_g \simeq \min(m_q, m_\lambda)$. However much more is required to obtain a result uniform in terms of the mass parameter $m_g \in [0, \min(m_q, m_\lambda)]$. Indeed the assumption $m_g \in [0, \min(m_q, m_\lambda)]$ in theorem 3.2 can be understood both mathematically and physically. Consider the Klein-Gordon equation $\Box w - m_g^2 w = F$. In order to obtain an estimate of the energy on a constant-time slice t, we use the standard energy estimate (see also section 1.4.1)

$$\begin{aligned}\int_{\mathbb{R}^3} \big|\partial w(t)\big|^2 + m_g^2|w(t)|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^3} \big|\partial w(t_0)\big|^2 + m_g^2|w(t_0)|^2\,\mathrm{d}x \\ + \int_{t_0}^t\int_{\mathbb{R}^3} |F(\tau)\partial_t w(\tau)|\,\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{3.1.13}$$

Any energy estimate for $t \geq t_0$ clearly requires control over the initial term $m_g^2\|w(t_0)\|^2_{L^2(\mathbb{R}^3)}$. Since this term depends on the mass parameter m_g, we clearly require m_g to have an upper bound in order for any energy argument to be made independent of m_g. What the upper bound on m_g should be is motivated from physics. Fermion masses are extremely small compared to both m_q, m_λ, and so it is particularly relevant to study the system in the entire range $m_g \in [0, \min(m_q, m_\lambda)]$.

A simpler model: the Dirac-Proca equations. It is possible to also understand a simpler set of PDEs, called the Dirac-Proca equations. In the Lorenz gauge the equations of motion for this model read

$$\begin{aligned}\Box A^\nu - m^2 A^\nu &= -\psi^*\gamma^0\gamma^\nu(P_L\psi), \\ -i\gamma^\mu\partial_\mu\psi + M\psi &= -\gamma^\mu A_\mu(P_L\psi),\end{aligned} \tag{3.1.14}$$

where $P_L = \frac{1}{2}(\mathbf{I_4} - \gamma^5)$. This system describes a spinor field $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ representing a fermion of mass M with spin 1/2 and a vector field $A = (A^\mu)$ representing a massive boson of mass $m > 0$ with spin 1. Using our methods we can study both the case

$M = 0$ (i.e. the standard Dirac-Proca equations) as well as in the case $M > 0$ (massive field). Observe that the Dirac-Proca system (3.1.14) contains no Higgs field and so the masses M, m are introduced "artificially" in the model, instead of arising from the Higgs mechanism.

Strategy of proof. The small data global existence problem for (3.1.14) with $M \equiv 0$ and for (3.1.3) with $\psi \equiv 0$ was solved by Tsutsumi in [Tsu03a, Tsu03b] respectively without a restriction to compactly supported initial data. Nonetheless it is still interesting to revisit this system via the hyperboloidal foliation method which is well suited to the study of massive fields. We thus introduce a hyperboloidal foliation which covers the interior of a light cone in Minkowski spacetime, and we then construct the solutions of interest in the future of an initial hyperboloid. For compactly-supported initial data this is equivalent to solving the initial value problem for a standard t=constant initial hypersurface. Festricting to compactly supported initial data is a strong assumption which limits the set of initial data we can treat compared to the works [Tsu03a, Tsu03b]. In future work we hope to remove the restriction to compact data by using other work which has achieved this for hyperboloidal foliations [LM16b, LM17b, KWY20].

A key new insight in this chapter arises from using L^2 norms defined on hyperboloids. In particular we are able to derive the following functional on hyperboloids

$$E^{\mathcal{H}}(s, \psi) = \int_{\mathcal{H}_s} \Big(\psi^*\psi - \frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi\Big)\, \mathrm{d}x. \tag{3.1.15}$$

We can show that this energy is positive definite and controls the norm $\|(s/t)\psi\|_{L^2}$ (see proposition 3.9) for spinors ψ supported in the region $|x| < t - 1$. This functional also satisfies an energy conservation property between hyperboloids (see proposition 3.12) which we use in combination with Sobolev-type estimates in the case $0 \leq M \ll m$.

There are several difficulties in dealing with the system of coupled wave–Klein–Gordon equations (3.1.3). The first and most well-known one is that the standard Klein-Gordon equation does not commute with the scaling Killing field $S = x^\mu\partial_\mu$ of Minkowski spacetime, which prevents us from applying the standard vector-field method in a direct manner.

Second, the nonlinearities in the Klein–Gordon equation (or the Proca equation) include a term describing ψ–ψ interactions, which can be regarded as a wave–wave interaction term and does not have good decay. Tsutsumi [Tsu03a] was able to overcome this difficulty by defining a new variable whose quadratic nonlinearity consists of linear combinations of 'strong null forms' $Q_{\mu j}$ (see Remark 1.41) which are compatible with the scaling vector field. A similar, but complicated new variable, was also defined in [Tsu03b] to overcome the same issue. Due to our use of the hyperboloidal foliation method we do not need to find new variables leading solely to strong null forms, which is an important simplification useful in our future work studying conjecture 1.49. Note also that these ψ–ψ interactions essentially arise from the Yukawa coupling between the spinor and the Higgs field. The stability of a massive Dirac equation with Yukawa coupling to a massive Klein-Gordon equation (emulating a Higgs field) was studied in [Bac88].

Next, our global-in-time existence result is established under a low regularity assumption on the initial data. In the main statement in [Tsu03a, Tsu03b], a slow growth of the energy of all high-order derivatives occurs. By contrast, with our method of proof we do not have any growth factor in the L^2 norm (3.1.10) and the L^∞ norms of the Higgs scalar field and vector boson field. See section 3.6 for a further discussion of these improved estimates.

Furthermore, it is challenging to establish a global stability result uniformly for all $m_g \geq 0$, while it is relatively easy to complete the proof for either $m_g = 0$ or m_g a large constant. In the case where the mass is small, say $m_g = \varepsilon^2$, if we treat ψ as a Klein-Gordon field then the field decays like

$$|\psi| \lesssim C_1 \varepsilon^{-1} t^{-3/2}. \tag{3.1.16}$$

However with this decay we cannot arrive at the improved estimate $(1/2)C_1\varepsilon$ if we start from the *a priori* estimate $C_1\varepsilon$, where C_1 is some large constant introduced in the bootstrap method. This is because the "improved" estimates we find are $\varepsilon + C_1^2$ instead of $\varepsilon + (C_1\varepsilon)^2$. Hence ψ behaves more like a wave component when $m_g \ll 1$, but since the mass m_g may be very small but non-zero, we cannot apply the standard techniques for wave equations (i.e. commute with S). We find it possible to overcome these difficulties by analysing the first–order Dirac equation, which admits the positive energy functional (3.1.15), and this energy plays a key role in our analysis. Furthermore although we obtain logarithmic growth for the L^2 norm of our Dirac field, we obtain interesting L^∞ estimates interpolating between wave and Klein-Gordon decay.

Outline of this chapter. In section 3.2 we give some preliminaries, including basic definitions for the Dirac equation in section 3.2.1 and for the hyperboloidal foliation in section 3.2.2. In section 3.3 we study the Dirac field: in section 3.3.1 we define the energy for the Dirac field on hyperboloids and give an energy estimate in proposition 3.12, in section 3.3.2 we convert the Dirac equation into a second order wave/Klein-Gordon equation and define the appropriate energy functional, and in section 3.3.3 we state Sobolev estimate for the Dirac field. We then revise the abelian model in section 3.4. In section 3.5 we discuss the system of equations (3.1.3) and introduce transformations that allow us to control the nonlinearities. In section 3.6 we use a bootstrap argument to prove the desired the stability result. Finally in section 3.7 we provide complementing views on the Dirac energy $E^{\mathcal{H}}$ given in section 3.3.1. The first comes from using a Cholesky decomposition for the energy integrand in section 3.7.1, and the second from using a Weyl decomposition for the Dirac spinor in section 3.7.2.

3.2 Preliminaries

3.2.1 Dirac spinors and matrices

This section is devoted to analyzing energy functionals for the Dirac equation with respect to a hyperboloid foliation of Minkowski spacetime. The Dirac equation for the unknown $\psi \in \mathbb{C}^4$ is

$$-i\gamma^\mu \partial_\mu \psi + M\psi = F, \tag{3.2.1}$$

with prescribed right-hand side $F \in \mathbb{C}^4$ and mass $M \in \mathbb{R}$. To make sense of this equation we need to define various complex vectors and matrices.

Notation 3.3 (Complex objects)**.** For a complex vector $z = (z_0, z_1, z_2, z_3)^T \in \mathbb{C}^4$ let $\bar{z}$ denote the conjugate, and $z^* = (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ denotes the conjugate transpose. If also $w \in \mathbb{C}^4$ then the conjugate inner product is defined by

$$\langle z, w\rangle = z^* w = \sum_{\alpha=1}^{4} \bar{z}_\alpha w_\alpha. \tag{3.2.2}$$

The Hermitian conjugate[2] of a matrix A is denoted by A^*, meaning

$$(A^*)_{\alpha\beta} = (\bar{A})_{\beta\alpha}. \tag{3.2.3}$$

Definition 3.4 (Dirac matrices and their representations)**.** The Dirac matrices γ^μ for $\mu = 0, 1, 2, 3$ are 4×4 matrices satisfying the identities

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu = -2\eta^{\mu\nu}\mathbf{1}_4, \\ &(\gamma^\mu)^* = -\eta_{\mu\nu}\gamma^\nu, \end{aligned} \tag{3.2.4}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. In the Dirac representation the Dirac matrices take the following form

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.2.5}$$

where σ^i's are the standard Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2.6}$$

We often use the following product of the Gamma matrices

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3.2.7}$$

The γ_5 matrix squares to $\mathbf{I_4}$ and so we also define the following projection operators

$$P_L = \frac{1}{2}\left(\mathbf{1}_4 - \gamma_5\right), \qquad P_R = \frac{1}{2}\left(\mathbf{1}_4 + \gamma_5\right), \tag{3.2.8}$$

to extract the 'left-handed' and 'right-handed' parts of a spinor, that is, to extract its chiral parts.

We note the following useful identities

$$(\gamma^\mu)^* = \gamma^0\gamma^\mu\gamma^0, \quad (\gamma^0\gamma^\mu)^* = \gamma^0\gamma^\mu, \quad \{\gamma_5, \gamma^\mu\} = 0. \tag{3.2.9}$$

For further details on Dirac matrices and the representation considered, see [AH12].

3.2.2 Hyperboloidal foliation of Minkowski spacetime

In order to introduce the energy formula of the Dirac component ψ on hyperboloids, we first need to use some notation concerning the hyperboloidal foliation method. This is primarily taken from [LM14] however we make some modifications to keep notation consistent with other chapters (eg section 1.4.1).

Definition 3.5 (Hyperboloidal foliation)**.** Given a point $(t, x) = (x^0, x^1, x^2, x^3)$ in Cartesian coordinates we denote its spatial radius as $r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We write $\partial_\alpha = \partial_{x^\alpha}$ (for $\alpha = 0, 1, 2, 3$) for partial derivatives and

$$\Omega_a = x^a\partial_t + t\partial_a, \qquad a = 1, 2, 3, \tag{3.2.10}$$

for the Lorentz boosts. Note $[\partial_0, \Omega_a] = \partial_a$ and $[\partial_b, \Omega_a] = \delta^a_b\partial_0$.

[2] In mathematical physics literature, A^* is often denoted by $A^\dagger$ and the notation $\bar{\psi} = \psi^\dagger\gamma^0$ is often used, however we keep here the notation of [DLW19].

We introduce hyperboloidal hypersurfaces $\mathcal{H}_s = \{(t,x) : t^2 - r^2 = s^2\}$ in the interior of the future light cone $\mathcal{C} = \{(t,x) : r < t-1\}$. The truncated region

$$\mathcal{C}_{[s_0,s_1]} = \{(t,x) : s_0^2 \le t^2 - r^2 \le s_1^2\} \cap \mathcal{C}, \tag{3.2.11}$$

is used to denote subsets of $\mathcal{C}$ and is naturally foliated by hyperboloids $\mathcal{H}_s$. The semi-hyperboloidal frame is defined by

$$Y_0 = \partial_t, \qquad Y_a = \frac{\Omega_a}{t} = \frac{x^a}{t}\partial_t + \partial_a. \tag{3.2.12}$$

Observe that the vectors Y_a generate the tangent space to the hyperboloids. We also introduce the vector field $Y^\perp = \partial_t + \frac{x^a}{t}\partial_a$ which is orthogonal to the hyperboloids.

The dual frame of $\{Y_0, Y_a\}$ is given by $\underline{\theta}^0 = \mathrm{d}t - \frac{x^a}{t}\mathrm{d}x^a$, $\underline{\theta}^a = \mathrm{d}x^a$. The (dual) semi-hyperboloidal frame and the (dual) natural Cartesian frame are connected by the relation

$$Y_\alpha = \Phi_\alpha^{\alpha'}\partial_{\alpha'}, \quad \partial_\alpha = \Psi_\alpha^{\alpha'} Y_{\alpha'}, \quad \underline{\theta}^\alpha = \Psi^\alpha_{\alpha'}\mathrm{d}x^{\alpha'}, \quad \mathrm{d}x^\alpha = \Phi^\alpha_{\alpha'}\underline{\theta}^{\alpha'}, \tag{3.2.13}$$

where the transition matrix (Φ_α^β) and its inverse (Ψ_α^β) are given by

$$(\Phi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix}, \qquad (\Psi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{3.2.14}$$

In this chapter we use roman font E to denote energies coming from a first-order PDE (see below) and, calligraphic font $\mathcal{E}$ to denote energies coming from a second-order PDE.

Definition 3.6 (Hyperboloidal energies for wave or Klein-Gordon equations)**.** We introduce the following energy functional defined on a hyperboloid $\mathcal{H}_s$ for scalar-valued or vector-valued maps ϕ

$$\mathcal{E}_m(s,\phi) = \int_{\mathcal{H}_s} \Big(|\partial_t\phi|^2 + \sum_a |\partial_a\phi|^2 + (x^a/t)(\partial_t\phi^*\partial_a\phi + \partial_t\phi\partial_a\phi^*) + m^2|\phi|^2 \Big)\,\mathrm{d}x. \tag{3.2.15}$$

We write $\mathcal{E}(s,\phi) = \mathcal{E}_0(s,\phi)$ for simplicity. All of our integrals in L^1, L^2, etc. are defined from the standard (flat) metric in $\mathbb{R}^3$, so

$$\|\phi\|_{L^p_f(\mathcal{H}_s)} = \Big(\int_{\mathcal{H}_s} |\phi(t,x)|^p\,\mathrm{d}x\Big)^{1/p} = \Big(\int_{\mathbb{R}^3} \big|\phi(\sqrt{s^2+r^2},x)\big|^p\,\mathrm{d}x\Big)^{1/p}. \tag{3.2.16}$$

Note the identity

$$\begin{aligned}\mathcal{E}_m(s,\phi) &= \int_{\mathcal{H}_s} \Big(|(s/t)\partial_t\phi|^2 + \sum_a |Y_a\phi|^2 + m^2|\phi|^2 \Big)\,\mathrm{d}x \\ &= \int_{\mathcal{H}_s} \Big(|Y^\perp\phi|^2 + \sum_a |(s/t)\partial_a\phi|^2 + \sum_{a<b} |t^{-1}\Omega_{ab}\phi|^2 + m^2|\phi|^2 \Big)\,\mathrm{d}x.\end{aligned} \tag{3.2.17}$$

The following lemma is discussed in Chapter 1 in (1.4.12).

Lemma 3.7 (Properties of the Klein-Gordon energy)**.** *Sufficiently regular solutions ϕ to $\Box\phi - m^2\phi = F$, satisfy*

$$\mathcal{E}_m(s,\psi)^{1/2} \leq \mathcal{E}_m(s_0,\psi)^{1/2} + \int_{s_0}^{s} \|F\|_{L^2(\mathcal{H}_\tau)} d\tau. \tag{3.2.18}$$

3.3 Properties of Dirac spinors on hyperboloids

3.3.1 Hyperboloidal energy of the Dirac equation

We now derive a hyperboloidal energy for the Dirac equation (3.2.1). Premultiplying the PDE (3.2.1) by $\psi^*\gamma^0$ gives

$$\psi^*\partial_0\psi + \psi^*\gamma^0\gamma^a\partial_a\psi + iM\psi^*\gamma^0\psi = i\psi^*\gamma^0 F\,. \tag{3.3.1}$$

The conjugate of (3.2.1) is

$$(\partial_\mu\psi^*)(\gamma^\mu)^* - i\psi^* M = -iF^*.$$

Multiplying this equation by ψ gives

$$(\partial_0\psi^*)\psi + (\partial_a\psi^*)\gamma^0\gamma^a\psi - iM\psi^*\gamma^0\psi = -iF^*\gamma^0\psi. \tag{3.3.2}$$

Adding (3.3.1) and (3.3.2) together yields

$$\partial_0(\psi^*\psi) + \partial_a(\psi^*\gamma^0\gamma^a\psi) = i\psi^*\gamma^0 F - iF^*\gamma^0\psi. \tag{3.3.3}$$

Note the mass term does not appear in (3.3.3). Moreover $2\mathrm{Re}[z] = z + \bar{z}$ for some $z \in \mathbb{C}$ and so we see that (3.3.3) is the real part of (3.3.1). It would appear however if we subtracted (3.3.1) from (3.3.2), that is the imaginary part of (3.3.1), then we find

$$\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi + \psi^*\gamma^0\gamma^j\partial_j\psi - \partial_a\psi^*\cdot\gamma^0\gamma^a\psi + 2iM\psi^*\gamma^0\psi = i\psi^*\gamma^0 F + iF^*\gamma^0\psi.$$

However such an expression does not appear to be useful. We return to (3.3.3) and integrate in spacetime to obtain energy inequalities. These give the following two definitions of first-order energy functionals.

Definition 3.8 (Energies for the Dirac equation)**.** On $t = const$ slices, define the energy functional

$$E^{\mathrm{t}}(t,\psi) = \int_{\mathbb{R}^3} (\psi^*\psi)(t,x)\mathrm{d}x. \tag{3.3.4}$$

On hyperboloidal slices $\mathcal{H}_s$ define the energy functionals

$$E^{\mathcal{H}}(s,\psi) = \int_{\mathcal{H}_s} \Big(\psi^*\psi - \frac{x_a}{t}\psi^*\gamma^0\gamma^a\psi\Big)\,\mathrm{d}x. \tag{3.3.5}$$

$$E^{+}(s,\psi) = \int_{\mathcal{H}_s} \Big(\psi - \frac{x_a}{t}\gamma^0\gamma^a\psi\Big)^* \Big(\psi - \frac{x_b}{t}\gamma^0\gamma^b\psi\Big)\,\mathrm{d}x. \tag{3.3.6}$$

We now show that the energy $E^{\mathcal{H}}(s,\psi)$ is indeed positive definite.

Proposition 3.9 (Properties of the hyperboloidal energy for the Dirac equation)**.** *For* *a sufficiently regular Dirac spinor ψ we have the identity*

$$E^{\mathcal{H}}(s,\psi) = \frac{1}{2}E^{+}(s,\psi) + \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2}\psi^*\psi \,\mathrm{d}x, \tag{3.3.7}$$

which in particular implies the positivity of the energy

$$E^{\mathcal{H}}(s,\psi) \geq \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2}\psi^*\psi \,\mathrm{d}x \geq 0. \tag{3.3.8}$$

Proof. Expanding out the bracket in $E^{+}(s,\psi)$ gives

$$\begin{aligned}
E^{+}(s,\psi) &= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_a}{t}\psi^*\gamma^0\gamma^a i\psi + \psi^*\frac{x^a x^k}{t^2}\gamma^0\gamma^b\gamma^0\gamma^l\psi\delta_{ab}\delta_{kl}\Big)\,\mathrm{d}x \\
&= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_a}{t}\psi^*\gamma^0\gamma^a\psi - \psi^*\frac{x_a x_b}{t^2}\gamma^{(a}\gamma^{b)}\psi\Big)\,\mathrm{d}x \\
&= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_a}{t}\psi^*\gamma^0\gamma^a\psi + \psi^*\psi\frac{x^a x^b}{t^2}\delta_{ab}\Big)\,\mathrm{d}x \\
&= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_a}{t}\psi^*\gamma^0\gamma^a\psi + \psi^*\psi\frac{t^2-s^2}{t^2}\Big)\,\mathrm{d}x \\
&= 2E^{\mathcal{H}}(s,\psi) - \int_{\mathcal{H}_s}\frac{s^2}{t^2}\psi^*\psi\,\mathrm{d}x,
\end{aligned} \tag{3.3.9}$$

and re-arranging gives (3.3.7). □

Lemma 3.10 (Energy conservation on constant time slices for the Dirac equation)**.** *Sufficiently regular solutions to* (3.3.3) *satisfy*

$$E^t(t,\psi) = E^t(t_0,\psi) + \int_{t_0}^{t}\int_{\mathbb{R}^3}\big(i\psi^*\gamma^0 F - iF^*\gamma^0\psi\big)\,\mathrm{d}x\mathrm{d}t. \tag{3.3.10}$$

Such functionals on a constant-time foliation have been considered frequently in the literature, see for instance [DFS07, Bou99]. The following lemma gives a new perspective using a hyperboloidal foliation.

Lemma 3.11 (Energy conservation on hyperboloids for the Dirac equation)**.** *Sufficiently regular solutions to* (3.3.3) *satisfy*

$$E^{\mathcal{H}}(s,\psi) = E^{\mathcal{H}}(s_0,\psi) + \int_{s_0}^{s}\int_{\mathcal{H}_\tau}(\tau/t)(i\psi^*\gamma^0 F - iF^*\gamma^0\psi)\,\mathrm{d}x\mathrm{d}\tau. \tag{3.3.11}$$

Proof. Integrating (3.3.3) over $\mathcal{C}_{[s_0,s]}$ gives

$$\begin{aligned}
\int_{\mathcal{H}_s}\big(\psi^*\psi, \psi^*\gamma^0\gamma^j\psi\big)\cdot n\,d\sigma - \int_{\mathcal{H}_{s_0}}\big(\psi^*\psi, \psi^*\gamma^0\gamma^j\psi\big)\cdot n\,d\sigma \\
= \int_{\mathcal{C}_{[s_0,s]}}(i\psi^*\gamma^0 F - iF^*\gamma^0\psi)\mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{3.3.12}$$

Here n and $d\sigma$ are the unit normal and induced Lebesgue measure on the hyperboloids respectively

$$n = (t^2+r^2)^{-1/2}(t,-x^i), \quad d\sigma = t^{-1}(t^2+r^2)^{1/2}\mathrm{d}x. \tag{3.3.13}$$

Using this explicit form of n and $d\sigma$ gives the result. $\square$

Using the positivity property of proposition 3.9 we can now establish the following energy inequality.

Proposition 3.12 (Properties of the hyperboloidal energy for the Dirac equation)**.** *For the massive Dirac spinor described by* (3.2.1) *the following estimate holds*

$$E^{\mathcal{H}}(s,\psi)^{1/2} \leq E^{\mathcal{H}}(s_0,\psi)^{1/2} + \int_{s_0}^{s} \|F\|_{L^2(\mathcal{H}_\tau)}\,\mathrm{d}\tau. \tag{3.3.14}$$

Proof. Differentiating (3.3.11) with respect to s yields

$$\begin{aligned} E^{\mathcal{H}}(\tau,\psi)^{1/2}\frac{d}{\mathrm{d}\tau}E^{\mathcal{H}}(\tau,\psi)^{1/2} &\leq \frac{1}{2}\int_{\mathcal{H}_\tau}(\tau/t)\big(|F^*\gamma^0\psi| + |\psi^*\gamma^0 F|\big)\,\mathrm{d}x \\ &\leq \|(\tau/t)\psi\|_{L^2(\mathcal{H}_\tau)}\|F\|_{L^2(\mathcal{H}_\tau)}. \end{aligned} \tag{3.3.15}$$

From proposition 3.9, we have $\|(\tau/t)\psi\|_{L^2(\mathcal{H}_\tau)} \leq E^{\mathcal{H}}(\tau,\psi)^{1/2}$. Thus we have

$$\frac{d}{\mathrm{d}\tau}E^{\mathcal{H}}(\tau,\psi)^{1/2} \leq \|F\|_{L^2(\mathcal{H}_\tau)}, \tag{3.3.16}$$

and the conclusion follows by integrating over $[s_0, s]$. $\square$

3.3.2 Hyperboloidal energy based on the second-order formulation of the Dirac equation

In this section we convert the Dirac equation into a second-order PDE and define associated energy functionals. Apply the first-order Dirac operator $-i\gamma^\nu\partial_\nu$ to the Dirac equation (3.2.1) and use the identity (3.2.4) to obtain

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\psi + M(-i\gamma^\nu\partial_\nu\psi) = -i\gamma^\nu\partial_\nu F. \tag{3.3.17}$$

Substituting the PDE into the bracketed term gives the following second-order PDE which we call the squared Dirac equation.

Definition 3.13 (The squared Dirac equation)**.**

$$\begin{aligned} \Box\psi - M^2\psi &= -MF - i\gamma^\nu\partial_\nu F, \\ (\psi,\partial_t\psi)(0,x) &= (\psi_0(x),\psi_1(x)) = (\psi_0(x), -(\gamma^0)^*\gamma^a\partial_a\psi_0(x)). \end{aligned} \tag{3.3.18}$$

We denote the right hand side of (3.3.18) by $G = G_\psi = -MF - i\gamma^\nu\partial_\nu F$. Note that $\psi_1(x)$ is consistent with the regularity assumed in (3.1.9).

This provides us with another approach for deriving an energy estimate for the Dirac equation. We now check the hyperboloidal energy coming from (3.3.18).

Lemma 3.14 (Properties of the hyperboloidal energy for the squared Dirac equation)**.** *For a solution $\psi \in \mathbb{C}^4$ of*

$$\Box\psi - M^2\psi = G, \tag{3.3.19a}$$

we have

$$\mathcal{E}_M(s,\psi)^{1/2} \leq \mathcal{E}_M(s_0,\psi)^{1/2} + \int_{s_0}^{s}\|G\|_{L^2(\mathcal{H}_\tau)}d\tau, \tag{3.3.19b}$$

where $\mathcal{E}_M(s,\psi)$ is given in (3.2.15).

Proof. The conjugate of (3.3.19a) reads $\Box\psi^* - M^2\psi^* = G^*$. Using $-\partial_t\psi^*$ and $-\partial_t\psi$ as multipliers on (3.3.19a) and its conjugate respectively we obtain

$$\begin{aligned}
&\partial_t\psi^*\partial_t^2\psi - \partial_t\psi^*\sum_a \partial_a^2\psi + M^2\partial_t\psi^*\cdot\psi = -\partial_t\psi^* G,\\
&\partial_t^2\psi^*\partial_t\psi - \sum_a \partial_a^2\psi^*\partial_t\psi + M^2\partial_t\psi\cdot\psi^* = -G^*\partial_t\psi.
\end{aligned} \tag{3.3.20}$$

Adding these two equations together gives

$$\begin{aligned}
&\partial_t\big(\partial_t\psi^*\partial_t\psi + \partial^a\psi^*\partial_a\psi + M^2\psi^*\psi\big) - \sum_a \partial_a\big(\partial_t\psi^*\partial_i\psi + \partial_t\psi\partial_i\psi^*\big)\\
&= -\big(\partial_t\psi^* G + G^*\partial_t\psi\big) = -2\mathrm{Re}\big[G^*\partial_t\psi\big].
\end{aligned} \tag{3.3.21}$$

Integrating this equation in the region $\mathcal{C}_{[s_0,s]}$, we have

$$\begin{aligned}
&\int_{\mathcal{H}_s}\Big(|\partial_t\psi|^2 + \sum_a|\partial_a\psi|^2 + M^2|\psi|^2, -(\partial_t\psi^*\partial_b\psi + \partial_b\psi^*\partial_t\psi)\Big)\cdot n\,d\sigma\\
&-\int_{\mathcal{H}_{s_0}}\Big(|\partial_t\psi|^2 + \sum_a|\partial_a\psi|^2 + M^2|\psi|^2, -(\partial_t\psi^*\partial_b\psi + \partial_b\psi^*\partial_t\psi)\Big)\cdot n\,d\sigma\\
&= -2\int_{\mathcal{C}_{[s_0,s]}}\mathrm{Re}\big[G^*\partial_t\psi\big]\mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{3.3.22}$$

Using the explicit form of n and $d\sigma$ given in (3.3.13) and noting that $2\mathrm{Re}[\partial_a\psi^*\partial_t\psi] = \partial_t\psi^*\partial_a\psi + \partial_a\psi^*\partial_t\psi$ we find

$$\mathcal{E}_M(s,\psi) - \mathcal{E}_M(s_0,\psi) = -2\int_{s_0}^s\int_{\mathcal{H}_t}\mathrm{Re}\big[G^*\partial_t\psi\big]\mathrm{d}t\mathrm{d}x. \tag{3.3.23}$$

We next use the change of variable formula $\partial_a = Y_a - (x^a/t)\partial_t$ to rewrite the energy term:

$$\begin{aligned}
&\int_{\mathcal{H}_s}\Big(|\partial_t\psi|^2 + \sum_a|\partial_a\psi|^2 + M^2|\psi|^2 + \frac{x^a}{t}\big(\partial_t\psi^*\partial_a\psi + \partial_a\psi^*\partial_t\psi\big)\Big)\mathrm{d}x\\
&= \int_{\mathcal{H}_s}\Big(|(s/t)\partial_t\psi|^2 + \sum_a|Y_a\psi|^2 + M^2|\psi|^2\Big)\mathrm{d}x.
\end{aligned} \tag{3.3.24}$$

We can now estimate the nonlinearity on the RHS using the change of variables $\tau = (t^2-r^2)^{1/2}$ and $\mathrm{d}t\mathrm{d}x = (\tau/t)\mathrm{d}\tau\mathrm{d}x$. In particular we have

$$-2\int_{\mathcal{H}_\tau}\mathrm{Re}\big[G^*\partial_t\psi\big](\tau/t)\,\mathrm{d}x \le 2\|(\tau/t)\partial_t\psi\|_{L^2(\mathcal{H}_\tau)}\|G\|_{L^2(\mathcal{H}_\tau)} \le 2\mathcal{E}_M(\tau,\psi)^{1/2}\|G\|_{L^2(\mathcal{H}_\tau)}. \tag{3.3.25}$$

Thus by differentiating (3.3.23) and using the above we have

$$\frac{d}{d\tau}\mathcal{E}_M(\tau,\psi)^{1/2} \le \|G\|_{L^2(\mathcal{H}_\tau)}. \tag{3.3.26}$$

Integrating this expression over $[s_0,s]$ gives the desired result. $\square$

Note that the mass did not appear in the hyperboloidal energy $E^{\mathcal{H}}$ defined in (3.3.5). This implies that spinors with equal masses but opposite signs ($\pm M$) still obey the same energy estimates of proposition 3.12. This is consistent with the second-order equation (3.3.18) for the Dirac field where the mass M^2 appears. In this equation the mass appears squared; so spinors with equal masses, but of opposite signs, obey the *same* second order equation.

3.3.3 Sobolev-type estimates for a Dirac spinor

In this section we obtain novel decay estimates for the Dirac spinor. First we use the following Sobolev estimate on hyperboloids for functions compactly supported in $|x| < t - 1$ [LM14, Proposition 5.1.1].

Proposition 3.15 (Sobolev-type inequality on hyperboloids)**.** *For all sufficiently smooth functions φ supported in the region $\mathcal{C}$, then for $s \geq 2$*

$$\sup_{(t,x)\in\mathcal{H}_s} \left|t^{3/2}\varphi(t,x)\right| \leq C \sum_{|J|\leq 2} \left\|\Omega^J \varphi\right\|_{L^2(\mathcal{H}_s)}, \tag{3.3.27}$$

where the summation is over Lorentz boosts Ω. The constant $C > 0$ is uniform in s, and we note that $t = \sqrt{s^2 + |x|^2}$ on $\mathcal{H}_s$.

This Sobolev-type inequality is easily adapted to include boosts which commute with the Dirac operator $i\gamma^\nu\partial_\nu$.

Definition 3.16 (Modified Lorentz boosts)**.** Following [Bac88] (albeit in a different sign convention) define modified boosts $\widehat{\Omega}_a$ that differ from Ω_a by a constant matrix

$$\widehat{\Omega}_a = \Omega_a - \frac{1}{2}\gamma^0\gamma^a. \tag{3.3.28}$$

It then holds that $[\widehat{\Omega}_a, i\gamma^\nu\partial_\nu] = 0$.

The following result is a simple extension of proposition 3.15.

Corollary 3.17 (Sobolev-type inequality for spinors on hyperboloids)**.** *Suppose ψ is a sufficiently smooth spinor field supported in $\mathcal{C}$, then it holds for $s \geq 2$*

$$\sup_{\mathcal{H}_s} \left|t^{3/2}\psi(t,x)\right| \lesssim \sum_{|J|\leq 2} \left\|\widehat{\Omega}^J \psi\right\|_{L^2(\mathcal{H}_s)}, \tag{3.3.29}$$

where $\widehat{\Omega}$ denotes a modified Lorentz boost.

We can combine this Sobolev estimate (3.3.29) with an appropriate energy bound in a standard way to obtain the following 'weak' estimate for the Dirac field.

Corollary 3.18 (Weak decay estimate for the Dirac equation)**.** *Suppose ψ is a sufficiently smooth Dirac field supported in $\mathcal{C}$, then for $s \geq 2$*

$$\sup_{\mathcal{H}_s} \left|st^{1/2}\psi\right| \lesssim \sum_{|J|\leq 2} E^{\mathcal{H}}(s, \widehat{\Omega}^J\psi)^{1/2}. \tag{3.3.30}$$

Proof. By the Sobolev inequality (3.3.29) on a hyperboloid $\mathcal{H}_s$ and proposition 3.9 we have

$$\left|st^{1/2}\psi\right| = \left|t^{3/2}(s/t)\psi\right| \lesssim \sum_{|J|\leq 2} \left\|\widehat{\Omega}^J\big((s/t)\psi\big)\right\|_{L^2(\mathcal{H}_s)} \lesssim \sum_{|J|\leq 2} E^{\mathcal{H}}(s, \widehat{\Omega}^J\psi)^{1/2}. \tag{3.3.31}$$

$\square$

Since $s \le t \le Cs^2$ on $\mathcal{H}_s \cap \mathcal{C}$, the decay rate (3.3.30) is not as good as that of standard Klein-Gordon fields. However we know that the Dirac equation, when squared, leads to a Klein-Gordon equation. This leads us to seek help from the other component of the functional $E^{\mathcal{H}}(s,\psi)$. In proposition 3.9, we have shown the decomposition identity (3.3.7) for the hyperboloidal energy functional $E^{\mathcal{H}}(s,\psi)$:

$$E^{\mathcal{H}}(s,\psi) = \frac{1}{2}E^{+}(s,\psi) + \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2}\psi^*\psi \,\mathrm{d}x, \tag{3.3.32}$$

where we can rewrite E^+ as

$$\begin{aligned} E^{+}(s,\psi) &= \int_{\mathcal{H}_s} \Big(\psi - \frac{x_a}{t}\gamma^0\gamma^a\psi\Big)^* (\gamma^0)^*\gamma^0 \Big(\psi - \frac{x_b}{t}\gamma^0\gamma^b\psi\Big)\,\mathrm{d}x \\ &= \int_{\mathcal{H}_s} \Big((\gamma^0 - \frac{x_a}{t}\gamma^a)\psi\Big)^* \Big((\gamma^0 - \frac{x_b}{t}\gamma^b)\psi\Big)\,\mathrm{d}x. \end{aligned} \tag{3.3.33}$$

Although there is no information about the mass M in the Dirac energy $E^{\mathcal{H}}$, by closely studying each component of the functional $E^{\mathcal{H}}(s,\psi)$, we can in fact recover the Klein-Gordon-like decay for $M \neq 0$.

Proposition 3.19 (Strong decay estimate for the Dirac equation)**.** *Consider a sufficiently regular solution to the Dirac equation* (3.2.1) *supported in* $\mathcal{C}$*, then for* $s \ge 2$

$$\sup_{\mathcal{H}_s} |Mt^{3/2}\psi| \lesssim \sup_{\mathcal{H}_s} |t^{3/2}F| + \sum_{|I|+|J|\le 3} E^{\mathcal{H}}(s, \partial^I \widehat{\Omega}^J\psi) + \sum_{|J|\le 2} \big\|Y_a \Omega^J \psi\big\|_{L^2(\mathcal{H}_s)}. \tag{3.3.34}$$

Proof. First note that by proposition 3.9, $\|(s/t)\partial^I\widehat{\Omega}^J\psi\|_{L^2(\mathcal{H}_s)} \le E^{\mathcal{H}}(s,\partial^I\widehat{\Omega}^J\psi)$ for $|I|+|J| \le 3$. We express the Dirac equation (3.2.1) in the semi-hyperboloidal frame

$$-i\big(\gamma^0 - \gamma^a\frac{x_a}{t}\big)\partial_t\psi - i\gamma^a Y_a\psi + M\psi = F, \tag{3.3.35}$$

which can be rewritten as

$$M\psi = F + i\big(\gamma^0 - \gamma^a\frac{x_a}{t}\big)\partial_t\psi + i\gamma^a Y_a\psi. \tag{3.3.36}$$

For the third term, note that by corollary 3.17 we have

$$|t^{3/2}Y_a\psi| \lesssim \sum_{|J|\le 2} \big\|\widehat{\Omega}^J Y_a\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim \sum_{|J|\le 2} \big\|Y_a\Omega^J\psi\big\|_{L^2(\mathcal{H}_s)}. \tag{3.3.37}$$

Again by corollary 3.17 we have

$$\big|\big(\gamma^0 - \gamma^a\frac{x_a}{t}\big)\partial_t\psi\big| \lesssim \sum_{|J|\le 2} \big\|\widehat{\Omega}^J\big((\gamma^0 - \gamma^a(x_a/t))\partial_t\psi\big)\big\|_{L^2(\mathcal{H}_s)}. \tag{3.3.38}$$

We note that the decomposition 3.3.33 implies

$$\big\|\big(\gamma^0 - \gamma^a(x_a/t)\big)\widehat{\Omega}^J\partial_t\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim E^{\mathcal{H}}(s,\partial\widehat{\Omega}^J\psi)^{1/2}, \tag{3.3.39}$$

which suggests we study the commutator $\big[\widehat{\Omega}^J, \gamma^0 - \gamma^b(x_b/t)\big]$. A computation gives

$$\big[\widehat{\Omega}_a, \gamma^0 - \gamma^b(x_b/t)\big] = -\frac{x_a}{t}\big(\gamma^0 - \gamma^b(x_b/t)\big), \tag{3.3.40}$$

which we can control since $r \le t$ in the cone. An induction shows $\big[\widehat{\Omega}^J, \gamma^0 - \gamma^b(x_b/t)\big]$ with $|J| \le 2$ also only contains good lower-order terms. Hence the proof is complete. □

3.4 Nonlinear stability of the ground state for the $U(1)$ model

The main result of this chapter, theorem 3.2 was stated for a system of PDEs (3.1.3). We now derive these PDEs.

3.4.1 The abelian action and $U(1)$ invariance

The Lagrangian we consider is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* D^\mu\phi - V(\phi^*\phi) - i\psi^*\gamma^0\gamma^\mu D_\mu\psi + g\phi^*\phi\psi^*\gamma^0\psi, \tag{3.4.1}$$

where we use the following definitions for the Higgs potential, gauge curvature and gauge covariant derivatives:

$$\begin{aligned} V(\phi^*\phi) &= -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \qquad & D_\mu\phi &= (\partial_\mu - iqA_\mu)\phi, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, & D_\mu\psi &= (\partial_\mu - iqA_\mu)\psi. \end{aligned} \tag{3.4.2}$$

Furthermore $\lambda > 0, g, q, \mu$ are real constants. A calculation shows that the Euler-Lagrange equations for (3.4.1) are the following

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= iq\phi^*(D^\nu\phi) - iq(D^\nu\phi)^*\phi + q\psi^*\gamma^0\gamma^\nu\psi, \\ D^\mu D_\mu\phi &= V'(\phi^*\phi)\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu D_\mu\psi &= g\phi^*\phi\psi. \end{aligned} \tag{3.4.3}$$

where we denote $V'(\phi^*\phi) = \frac{\mathrm{d}V}{\mathrm{d}(\phi^*\phi)}$. These PDEs can also be expressed as

$$\Box A^\nu - \partial^\nu(\mathrm{div}A) - 2q^2A^\nu\phi^*\phi = iq\big(\phi^*(\partial^\nu\phi) - (\partial^\nu\phi^*)\phi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \tag{3.4.4a}$$

$$\Box\phi - V'(\phi^*\phi)\phi - iq\phi\partial_\mu A^\mu = 2iqA_\mu\partial^\mu\phi + q^2A^\mu A_\mu\phi - g\phi\psi^*\gamma^0\psi, \tag{3.4.4b}$$

$$i\gamma^\mu\partial_\mu\psi - g\phi^*\phi\psi = -q\gamma^\mu A_\mu\psi. \tag{3.4.4c}$$

Remark 3.20. If we were to consider perturbations of ground states ϕ_0 which are not constant in spacetime, then we would pick up, for example, a term of the form

$$iq\big(\phi_0^*(\partial^\nu\phi_0) - (\partial^\nu\phi_0^*)\phi_0\big) \sim \mathcal{O}(1) \tag{3.4.5}$$

in (3.4.4a) which is certainly not small. This is why we impose the restriction that $\partial_\mu\phi_0 = 0$. However even with this choice, we could still pick up linear terms

$$iq(\phi_0^*\partial^\nu\chi - (\partial^\nu\chi)^*\phi_0) \sim \mathcal{O}(\varepsilon) \tag{3.4.6}$$

in (3.4.4a). We use the gauge freedom to remove such a term.

Lemma 3.21 (Gauge symmetry)**.** *$\mathcal{L}$ has a $U(1)$-gauge symmetry under the transformations*

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \alpha, \quad \phi \mapsto \phi' = e^{iq\alpha}\phi, \quad \psi \mapsto \psi' = e^{iq\alpha}\psi, \tag{3.4.7}$$

where $\alpha = \alpha(t,x)$ is some arbitrary function of space and time.

Proof. Using (3.4.7) and (3.4.2) we find

$$\begin{aligned} &D_\mu\phi \mapsto D'_\mu\phi' = e^{iq\alpha}D_\mu\phi, \\ &D_\mu\psi \mapsto D'_\mu\psi' = \partial_\mu(e^{iq\alpha}\psi) - iq(A_\mu + \partial_\mu\alpha)e^{iq\alpha}\psi = e^{iq\alpha}D_\mu\psi. \end{aligned} \tag{3.4.8}$$

By the commutation of partial derivatives we immediately see that $F_{\mu\nu}$ is invariant under $U(1)$ gauge transformations. Inserting these transformations into the Lagrangian one obtains

$$\begin{aligned} \mathcal{L}' &= -i\psi'^*\gamma^0\gamma^\mu D'_\mu\psi' - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - (D'_\mu\phi')^* D'^\mu\phi' - V((\phi')^*\phi') + \phi'^*\phi' g\psi'^*\gamma^0\psi' \\ &= -ie^{-iq\alpha}\psi^*\gamma^0\gamma^\mu\partial_\mu(e^{iq\alpha}\psi) - i\big(e^{iq\alpha}\psi, -iq\gamma^\mu(A_\mu + \partial_\mu\alpha)e^{iq\alpha}\psi)\big) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad - (D_\mu\phi)^* e^{-iq\alpha}e^{iq\alpha}D_\mu\phi - V(\phi^* e^{-iq\alpha}e^{iq\alpha}\phi) + g\phi^*\phi e^{-iq\alpha}\psi^*\gamma^0 e^{iq\alpha}\psi \\ &= -i\psi^*\gamma^0\gamma^\mu D_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* D^\mu\phi - V(\phi^*\phi) + g\phi^*\phi\psi^*\gamma^0\psi. \end{aligned} \tag{3.4.9}$$

Thus $\mathcal{L}$ is invariant under the transformation (3.4.7). □

3.4.2 Propagation of an inhomogeneous Lorenz gauge

Next, with a similar aim to that of section 1.7.1, albeit keeping in mind remark 3.20, we turn (3.4.4) into a PDE of definite type by specifying a particular gauge for A^μ.

Lemma 3.22 (Propagation of an inhomogeneous Lorenz gauge)**.** *Let X be a suitably regular scalar field. Consider the modified system*

$$\begin{aligned} \Box A^\nu - 2q^2 A^\nu\phi^*\phi &= \partial^\nu X + iq\big(\phi^*(\partial^\nu\phi) - (\partial^\nu\phi^*)\phi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \\ \Box\phi - V'(\phi^*\phi)\phi - iq\phi X &= 2iqA_\mu\partial^\mu\phi + q^2A^\mu A_\mu\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu\partial_\mu\psi - g\phi^*\phi\psi &= -q\gamma^\mu A_\mu\psi. \end{aligned} \tag{3.4.10}$$

Suppose the initial data for (3.4.10) *satisfy*

$$\begin{aligned} \big(divA - X\big)(t_0,\cdot) &= 0, \\ \partial_t\big(divA - X\big)(t_0,\cdot) &= 0. \end{aligned} \tag{3.4.11}$$

Then as long as a sufficiently regular solution to (3.4.10) *exists, it satisfies $divA = X$.*

Proof. To propagate the gauge choice $\mathrm{div}A = X$ imposed on the initial data we take the divergence of the first equation in (3.4.10) and use the second evolution equation

(3.4.10):

$$
\begin{aligned}
\Box(\mathrm{div}A - X) &= iq(\phi^*\Box\phi - \Box\phi^*\phi) + 2q^2\partial_\nu(A^\nu\phi^*\phi) + q\partial_\nu(\psi^*\gamma^0\gamma^\nu\psi) \\
&= iq\phi^*\Big(2iqA_\mu\partial^\mu\phi + iq\phi\partial_\mu A^\mu + q^2A_\mu A^\mu\phi + V'\phi - g\phi\psi^*\gamma^0\psi\Big) \\
&\quad - iq\Big(- 2iqA_\mu\partial^\mu\phi^* - iq\phi^*\partial_\mu A^\mu + q^2A_\mu A^\mu\phi^* + (V'\phi)^* - g\phi^*\psi^*\gamma^0\psi\Big)\phi \\
&\quad + 2q^2\partial_\nu(A^\nu\phi^*\phi) + q\psi^*\gamma^0\gamma^\mu\partial_\mu\psi + q\partial_\mu\psi^*\gamma^0\gamma^\mu\psi \\
&= iq\big(\phi^*V'\phi - (V'\phi)^*\phi\big) = 0.
\end{aligned}
$$

In the last line we used the fact that $V' = (V')^*$ which is a consequence of V being chosen real since it appears in the (real-valued) action. □

3.4.3 Gauge choice for the abelian model

As discussed in (3.1.2) and remark 3.20 we consider perturbations $\chi = \phi - \phi_0$ where ϕ_0 is *constant* in space and time and has magnitude $|\phi_0| = v$. The following result is a consequence of lemma 3.22 by choosing $X = -iq(\phi_0^*\chi - \chi^*\phi_0)$.

Corollary 3.23 (Propagation of an inhomogeneous Lorenz gauge)**.** *Suppose the initial data satisfy the following gauge condition*

$$
\begin{aligned}
\big(divA + iq(\phi_0^*\chi - \chi^*\phi_0)\big)(t_0,\cdot) &= 0, \\
\partial_t\big(divA + iq(\phi_0^*\chi - \chi^*\phi_0)\big)(t_0,\cdot) &= 0.
\end{aligned}
\tag{3.4.12}
$$

Then the Euler-Lagrange equations for (3.4.1) *are equivalent to those written in* (3.4.13).

3.4.4 The U(1) model as a PDE system

We have now derived the equations originally given in (3.1.3). Thus in the Lorenz gauge of corollary 3.23 the field equations take the form

$$
\begin{aligned}
(\Box - m_q^2)A^\nu &= Q_{A^\nu}, \\
\Box\chi - m_q^2\frac{\phi_0}{2v^2}\big(\phi_0^*\chi - \chi^*\phi_0\big) - m_\lambda^2\frac{\phi_0}{2v^2}\big(\phi_0^*\chi + \chi^*\phi_0\big) &= Q_\chi, \\
i\gamma^\mu\partial_\mu\psi - m_g\psi &= Q_\psi,
\end{aligned}
\tag{3.4.13}
$$

with quadratic or higher order terms given by

$$
Q_{A^\nu} = iq\big(\chi^*(\partial^\nu\chi) - (\partial^\nu\chi^*)\chi\big) + 2q^2A^\nu\big(\chi^*\phi_0 + \phi_0^*\chi + \chi^*\chi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \tag{3.4.14a}
$$

$$
\begin{aligned}
Q_\chi &= 2iqA_\mu\partial^\mu\chi + q^2\chi\big(\phi_0^*\chi - \chi^*\phi_0\big) + q^2A^\mu A_\mu\big(\phi_0 + \chi\big) \\
&\quad + 2\lambda\chi^*\chi\phi_0 + 2\lambda\chi\big(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi\big) - g\big(\phi_0 + \chi\big)\psi^*\gamma^0\psi,
\end{aligned}
\tag{3.4.14b}
$$

$$
Q_\psi = g\big(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi\big)\psi - q\gamma^\mu A_\mu\psi. \tag{3.4.14c}
$$

The mass coefficients

$$
m_q^2 = 2q^2v^2, \quad m_\lambda^2 = 4\lambda v^2, \quad m_g = gv^2, \tag{3.4.15}
$$

depend themselves on given coupling constants denoted by q, g, λ, as well as the vacuum expectation value v of the Higgs field.

The initial data set are denoted by

$$\big(A^\nu, \chi, \psi\big)(t_0, \cdot) = \big(A_0^\nu, \chi_0, \psi_0\big), \qquad \big(\partial_t A^\nu, \partial_t \chi\big)(t_0, \cdot) = \big(A_1^\nu, \chi_1\big), \tag{3.4.16}$$

and are said to be Lorenz compatible if they satisfy

$$\begin{aligned} \partial_a A_0^a &= -A_1^0 - iq\big(\phi_0^* \chi_0 - \chi_0^* \phi_0\big), \\ \Delta A_0^0 - m_q^2 A_0^0 &= -\partial_a A_1^a - iq\big(\phi_0^* \chi_1 - \chi_1^* \phi_0\big) + iq\big(\chi_0^* \chi_1 - \chi_1^* \chi_0\big) \\ &\quad + 2q^2 A_0^0 \big(\phi_0^* \chi_0 + \chi_0^* \phi_0 + \chi_0^* \chi_0\big) + q\psi_0^* \psi_0. \end{aligned} \tag{3.4.17}$$

The derivation of (3.4.17) follows from results of the following section 3.4.1, in particular the gauge choice condition given in corollary 3.23.

3.5 Structure and nonlinearity of the models

3.5.1 Aim of this section

In this section we transform the variables in (3.4.13) in order to treat some difficult terms in the nonlinearities (3.4.14) of the following two types

$$q\psi^* \gamma^0 \gamma^\nu \psi, \qquad -g\phi_0 \psi^* \gamma^0 \psi. \tag{3.5.1}$$

These appear in (3.4.14a) and (3.4.14b) respectively. If the mass of the Dirac spinor is small or zero, that is $0 \leq m_g \ll \min(m_q, m_\lambda)$, then the nonlinearities in (3.5.1) have insufficiently fast decay for the bootstrap argument to close. This is because in the case $0 \leq m_g \ll \min(m_q, m_\lambda)$, the Dirac field behaves more like a nonlinear wave equation and we do not have good bounds for either the L^2 or L^∞ norm of ψ.

To address these issues, we employ a transformation used in [Tsu03a, (3.1)] for the Dirac-Proca equations. The transformation (see (3.5.2)) will introduce an additional factor of m_g^2 in front of the ψ–ψ interactions (3.5.1) which we can then control using Proposition 3.19. However the transformation will also produce new ψ–ψ interactions in the form of null forms $Q(\psi, \gamma^0\gamma^\nu, \psi)$. This was problematic in [Tsu03a] since a constant t-foliation is used. Thus an essential part of their proof relies on showing that these new null forms are only strong null forms, see remark 1.41, compatible with Klein-Gordon equations [Tsu03a, Lemma 2.3]. This is also a significant part of the analysis in [Tsu03b, §2]. Thus an added benefit of using the hyperboloidal foliation is that that we do not need to reduce our nonlinearities to strong null forms.

In the case $m_g \sim \min(m_q, m_\lambda)$, better Klein-Gordon bounds are available and we do not require the above transformation anymore. Finally although the equation satisfied by χ is not obviously a semilinear Klein-Gordon equation, we find that χ can be decomposed into two components, with each component satisfying a Klein-Gordon equation.

3.5.2 Hidden null structure from Tsutsumi

We follow a transformation given in Tsutsumi [Tsu03a, (3.1)]. Define a new variable

$$\widetilde{A}^\nu = A^\nu + \frac{q}{m_q^2} \psi^* \gamma^0 \gamma^\nu \psi, \tag{3.5.2}$$

which satisfies the non-linear Klein-Gordon equation

$$\left(\Box - m_q^2\right)\widetilde{A}^\nu = Q_{\widetilde{A}^\nu}, \tag{3.5.3}$$

in which the nonlinearities are

$$Q_{\widetilde{A}^\nu} = \frac{2q}{m_q^2}Q_0(\psi, \gamma^0\gamma^\nu\psi) + \frac{q}{m_q^2}G_\psi^*\gamma^0\gamma^\nu\psi + \frac{q}{m_q^2}\psi^*\gamma^0\gamma^\nu G_\psi + \frac{2q}{m_q^2}m_g^2\psi^*\gamma^0\gamma^\nu\psi, \tag{3.5.4}$$

and G_ψ is computed using (3.3.18) and (3.4.14)

$$\begin{aligned} G_\psi &= -m_g Q_\psi - i\gamma^\nu\partial_\nu Q_\psi \\ &= -m_g\big(g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)\psi - q\gamma^\mu A_\mu\psi\big) - ig\gamma^\nu\partial_\nu(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)\psi \\ &\quad + iq\gamma^\nu\gamma^\mu\partial_\nu A_\mu\psi - g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)(m_g\psi + Q_\psi) \\ &\quad - 2igA^\mu\partial_\mu\psi - gA_\mu\gamma^\mu(m_g\psi + Q_\psi). \end{aligned} \tag{3.5.5}$$

Note the nonlinearity $\psi^*\gamma^0\gamma^\nu\psi$ in (3.5.4) now appears with a good factor of m_g^2. We can control this term using Proposition 3.19. Note that the null forms appearing in (3.5.4) extend in the obvious way from definition 1.40 to complex-valued functions $\Phi(t,x), \Psi(t,x) : \mathbb{R}^{3+1} \to \mathbb{C}^n$, for example $Q_0(\Phi, \Psi) = -\eta^{\mu\nu}\langle\partial_\mu\Phi, \partial_\nu\Psi\rangle$.

3.5.3 Decomposition of χ

In order to study the behaviour of χ, it is more convenient to consider equations for the following two variables

$$\chi_\pm = \phi_0^*\chi \pm \chi^*\phi_0. \tag{3.5.6}$$

Since the following identity holds

$$|\chi_+|^2 + |\chi_-|^2 = 2v^2|\chi|^2, \tag{3.5.7}$$

it is equivalent to estimate either χ or $\chi_\pm$. These new variables satisfy the Klein-Gordon equations

$$\Box\chi_+ - m_\lambda^2\chi_+ = Q_{\chi_+}, \tag{3.5.8}$$

$$\Box\chi_- - m_q^2\chi_- = Q_{\chi_-}, \tag{3.5.9}$$

with a computation showing the nonlinearities are

$$\begin{aligned} Q_{\chi_+} &= 2iqA_\mu\partial^\mu\chi_- + q^2\chi_-^2 + q^2A^\mu A_\mu\big(2v^2 + \chi_+\big) \\ &\quad + 4\lambda v^2\chi^*\chi + 2\lambda\chi_+^2 + 2\lambda\chi_+\chi^*\chi - g\big(2v^2 + \chi_+\big)\psi^*\gamma^0\psi, \end{aligned} \tag{3.5.10}$$

$$\begin{aligned} Q_{\chi_-} &= 2iqA_\mu\partial^\mu\chi_+ + q^2\chi_-\chi_+ + q^2A_\mu A^\mu\chi_- \\ &\quad + 2\lambda\chi_-\chi_+ + 2\lambda\chi_-\chi^*\chi - g\chi_-\psi^*\gamma^0\psi. \end{aligned} \tag{3.5.11}$$

Following Tsutsumi again [Tsu03a], we define the new variable

$$\widetilde{\chi}_+ = \chi_+ - \frac{2m_g}{m_\lambda^2}\psi^*\gamma^0\psi, \tag{3.5.12}$$

which satisfies the following Klein-Gordon equation

$$\Box\widetilde{\chi}_+ - m_\lambda^2\widetilde{\chi}_+ = Q_{\widetilde{\chi}_+}, \tag{3.5.13}$$

with the nonlinearity

$$
\begin{aligned}
Q_{\widetilde{\chi}_+} &= -4\frac{m_g}{m_\lambda^2}Q_0(\psi,\gamma^0\psi) + 2\frac{m_g}{m_\lambda^2}G_\psi^*\gamma^0\psi + 2\frac{m_g}{m_\lambda^2}\psi^*\gamma^0 G_\psi - 4\frac{m_g^3}{m_\lambda^2}\psi^*\gamma^0\psi \\
&\quad + Q_{\chi_+} + 2m_g\psi^*\gamma^0\psi \\
&= -4\frac{m_g}{m_\lambda^2}Q_0(\psi,\gamma^0\psi) + 2\frac{m_g}{m_\lambda^2}G_\psi^*\gamma^0\psi + 2\frac{m_g}{m_\lambda^2}\psi^*\gamma^0 G_\psi - 4\frac{m_g^3}{m_\lambda^2}\psi^*\gamma^0\psi \\
&\quad + 2iqA_\mu\partial^\mu\chi_- + q^2\chi_-^2 + q^2A^\mu A_\mu\big(2v^2+\chi_+\big) + 4\lambda v^2\chi^*\chi + 2\lambda\chi_+^2 \\
&\quad + 2\lambda\chi_+\chi^*\chi - g\chi_+\psi^*\gamma^0\psi. \qquad (3.5.14)
\end{aligned}
$$

The term $2m_g\psi^*\gamma^0\psi$ has cancelled with the problematic term in Q_{χ_+} and all other nonlinearities of the form $\psi^*\gamma^0\psi$ now appear with a good factor of m_g^3 in front.

3.6 Bootstrap argument

This section is devoted to using a bootstrap argument to prove theorem 3.2. To briefly summarise from section 3.5, we now deal with the unknowns

$$
\begin{aligned}
\widetilde{A}^\nu &= A^\nu + \frac{q}{m_q^2}\psi^*\gamma^0\gamma^\nu\psi, && (3.6.1a)\\
\widetilde{\chi}_+ &= \chi_+ - \frac{2m_g}{m_\lambda^2}\psi^*\gamma^0\psi, && (3.6.1b)\\
\chi_- &= \phi_0^*\chi - \chi^*\phi_0, && (3.6.1c)\\
&\psi, && (3.6.1d)
\end{aligned}
$$

which satisfy the equations

$$
\begin{aligned}
\Box\widetilde{A}^\nu - m_q^2\widetilde{A}^\nu &= Q_{\widetilde{A}^\nu}, && (3.6.2a)\\
\Box\widetilde{\chi}_+ - m_\lambda^2\widetilde{\chi}_+ &= Q_{\widetilde{\chi}_+}, && (3.6.2b)\\
\Box\chi_- - m_q^2\chi_- &= Q_{\chi_-}, && (3.6.2c)\\
i\gamma^\mu\partial_\mu\psi - m_g\psi &= Q_\psi. && (3.6.2d)
\end{aligned}
$$

The mass parameters are defined in (3.1.5) and the nonlinearities are defined in (3.5.4), (3.5.14), (3.5.11) and (3.4.14c) respectively.

The proof of theorem 3.2 is based on a bootstrap argument. In section 3.6.1 we state standard estimates for null terms, various commutators and Sobolev-type estimates on hyperboloids. The bootstrap assumptions are made in (3.6.15). These bootstraps, combined with some standard commutator estimates and Sobolev-type inequalities, lead to certain weak estimates in (3.6.18) and (3.6.19). In section 3.6.3 we use our first-order hyperboloidal energy to upgrade our estimates for the Dirac component in proposition 3.27 and corollary 3.28. In section 3.6.4 we obtain estimates for the transformed variables $\tilde{A}^\nu$ and $\tilde{\chi}_+$ defined above. Putting all of this together we finally are able to close our bootstrap assumptions.

3.6.1 Standard Estimates: null forms, commutators and Sobolev estimates

We first illustrate estimates for the quadratic null terms, referring to [LM14, Prop. 4.1.2] for the proof.

Lemma 3.24 (Estimate of null forms)**.** *Let $Q(\phi,\varphi)$ be one of the null forms given in definition 1.40 and let ϕ,φ be some sufficiently regular functions defined in the region $\mathcal{C}$. Then for any $|I|+|J|\in\mathbb{N}$*

$$\begin{aligned}\big|\partial^I\Omega^J Q(\phi,\varphi)\big| \lesssim (s/t)^2 &\sum_{\substack{|I_1|+|I_2|\le|I|\\|J_1|+|J_2|\le|J|}} \big|\partial^{I_1}\Omega^{J_1}\partial_t\phi\big|\big|\partial^{I_2}\Omega^{J_2}\partial_t\varphi\big|.\\ &+\sum_{\substack{|I_1|+|I_2|\le|I|\\|J_1|+|J_2|\le|J|}}\sum_{a,\beta}\Big(\big|\partial^{I_1}\Omega^{J_1}Y_a\phi\big|\big|\partial^{I_2}\Omega^{J_2}Y_\beta\varphi\big|+\big|\partial^{I_1}\Omega^{J_1}Y_\beta\phi\big|\big|\partial^{I_2}\Omega^{J_2}Y_a\varphi\big|\Big).\end{aligned} \tag{3.6.3}$$

The definition of the semi-hyperboloidal frame $\{Y_0,Y_a\}$ is given in (3.2.12). The key outcome from lemma 3.24 is the factor of (s/t) in front of the component $\partial^{I_1}\Omega^{J_1}\partial_t\phi\cdot\partial^{I_2}\Omega^{J_2}\partial_t\varphi$ since this allows such a term to be absorbed by the energy using (3.2.17). This is important since the derivatives Y_a, like $\bar\partial$ in Chapter 1, enjoy better L^∞ and L^2 properties. The derivatives ∂_t require the factor of (s/t) in order to also have good L^∞ and L^2 estimates.

We next state an estimate for commutators, referring to [LM14, Lemmas 3.3.1-3.3.3] for the proof. Roughly speaking, the lemma allows us to commute $\partial^I\Omega^J$ with derivatives in the set $\{\partial_\alpha, Y_\alpha, \partial_\alpha\partial_\beta, (s/t)\partial_\alpha\}$ at the expense of generating lower-order terms.

Lemma 3.25 (Estimates of commutators)**.** *Let φ be a sufficiently regular function defined in the region $\mathcal{C}$. Then for any $|I|+|J|\in\mathbb{N}$ there exists a constant $C=C(|I|,|J|)$ such that*

$$\big|[\partial^I\Omega^J,\partial_\alpha]\varphi\big|+\big|[\partial^I\Omega^J,Y_\alpha]\varphi\big|\le C\sum_{|J_1|<|J|}\sum_\beta\big|\partial_\beta\partial^I\Omega^{J_1}\varphi\big|, \tag{3.6.4}$$

$$\big|[\partial^I\Omega^J,Y_a]\varphi\big|\le C\Big(\sum_{\substack{|I_1|<|I|\\|J_1|<|J|}}\sum_b|Y_b\partial^{I_1}\Omega^{J_1}\varphi|+t^{-1}\sum_{\substack{|I_1|<|I|\\|J_1|<|J|}}\sum_\beta|\partial_\beta\partial^{I_1}\Omega^{J_1}\varphi|\Big), \tag{3.6.5}$$

$$\big|[\partial^I\Omega^J,\partial_\alpha\partial_\beta]\varphi\big|\le C\sum_{\substack{|I_1|<|I|\\|J_1|<|J|}}\sum_{\alpha,\beta}\big|\partial_\alpha\partial_\beta\partial^{I_1}\Omega^{J_1}\varphi\big|, \tag{3.6.6}$$

$$\big|\partial^I\Omega^J((s/t)\partial_\alpha\varphi)\big|\le\big|(s/t)\partial_\alpha\partial^I\Omega^J\varphi\big|+C\sum_{\substack{|I_1|<|I|\\|J_1|<|J|}}\sum_\beta\big|(s/t)\partial_\beta\partial^{I_1}\Omega^{J_1}\varphi\big|. \tag{3.6.7}$$

Note the conventions here: $\partial^I\Omega^J=0$ if $|I|+|J|<0$, $b\in\{1,2,3\}$ and $\alpha,\beta\in\{0,1,2,3\}$.

We frequently make use of the following identity which follows under the assumptions of proposition 3.15 and by using lemma 3.25:

$$\sup_{\mathcal{H}_s}\big|st^{1/2}u(t,x)\big|\lesssim\sum_{|J|\le2}\|(s/t)\Omega^J u\|_{L^2(\mathcal{H}_s)}. \tag{3.6.8}$$

3.6.2 Bootstrap assumptions and basic estimates

Recall the assumption on initial data in theorem 3.2. This initial data is supported in $\{|x| \leq R\}$ and is posed on the hypersurface of constant time $t = t_0$. The following proposition allows us to consider the Cauchy problem posed on the hyperboloid $s_0 = R+1$ and to construct a local solution from this data. The proof follows using standard arguments given in [LM14, §11].

Proposition 3.26 (Existence of data and local existence in hyperboloidal time).

1. *Let $\mathbb{Z} \ni N_0 \geq 5$. For a sufficiently large constant $A > 0$ there exists $\varepsilon_0' > 0$ depending only on R and A such that under the smallness assumption*

$$\|A_0, \chi_0, \psi_0\|_{H^{N_0+1}(\mathbb{R}^3)} + \|A_1, \chi_1\|_{H^{N_0}(\mathbb{R}^3)} \leq \varepsilon, \tag{3.6.9}$$

for $\varepsilon \leq \varepsilon_0'$ the local in t-time solution to (3.1.3) *defines compactly supported data on the hyperboloid $\mathcal{H}_{s_0}$, $s_0 = R+1$, satisfying*

$$\begin{aligned}\sum_{|I|\leq N_0+1} &\Big(\|\partial^I A_0\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi_0\|_{L^2(\mathcal{H}_s)} + \|\partial^I \psi_0\|_{L^2(\mathcal{H}_s)}\Big) \\ &+ \sum_{|I|\leq N_0} \Big(\|\partial^I A_1\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi_1\|_{L^2(\mathcal{H}_s)}\Big) \leq A\varepsilon,\end{aligned} \tag{3.6.10}$$

for all $\varepsilon \leq \varepsilon_0'$.

2. *Let $\mathbb{Z} \ni N_0 \geq 5$. Consider the Cauchy problem for* (3.1.3) *with initial data*

$$\big(A^\nu, \chi, \psi\big)(s_0, \cdot) = \big(A_0^\nu, \chi_0, \psi_0\big), \qquad \big(\partial_t A^\nu, \partial_t \chi\big)(s_0, \cdot) = \big(A_1^\nu, \chi_1\big), \tag{3.6.11}$$

posed on the hyperboloid $\mathcal{H}_{s_0}$. There exists a sufficiently small $\varepsilon_0 > 0$ such that for all data satisfying

$$\begin{aligned}\sum_{|I|\leq N_0+1} &\Big(\|\partial^I A_0\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi_0\|_{L^2(\mathcal{H}_s)} + \|\partial^I \psi_0\|_{L^2(\mathcal{H}_s)}\Big) \\ &+ \sum_{|I|\leq N_0} \Big(\|\partial^I A_1\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi_1\|_{L^2(\mathcal{H}_s)}\Big) \leq \varepsilon,\end{aligned} \tag{3.6.12}$$

where $\varepsilon \leq \varepsilon_0$, there exists a unique local in s-time solution on $[s_0, s_1]$ with $s_1 > s_0$ which satisfies

$$\sum_{|I|\leq N_0+1} \Big(\|\partial^I A^\nu\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi\|_{L^2(\mathcal{H}_s)} + \|\partial^I \psi\|_{L^2(\mathcal{H}_s)}\Big) \leq B\varepsilon, \tag{3.6.13}$$

for some constant $B > 0$ and all $s \in [s_0, s_1]$. Furthermore if T_* denotes the supremum of all such times (for a fixed ε) then either $T_* = +\infty$ or

$$\lim_{s\to T_*^-} \sum_{|I|\leq N_0+1} \Big(\|\partial^I A^\nu\|_{L^2(\mathcal{H}_s)} + \|\partial^I \chi\|_{L^2(\mathcal{H}_s)} + \|\partial^I \psi\|_{L^2(\mathcal{H}_s)}\Big) = \infty. \tag{3.6.14}$$

We proceed with a bootstrap argument, referring to section 1.3.2 for a more detailed discussion of this approach. For some large constant C_1 we assume that, in a time interval $[s_0, s_1]$ where $s_1 > s_0$, the local solution satisfies for all $s \in [s_0, s_1]$ the bounds

$$
\begin{aligned}
\mathcal{E}_{m_q}\big(s, \partial^I \Omega^J A_\nu\big)^{1/2} + \mathcal{E}_{m_\lambda}\big(s, \partial^I \Omega^J \chi\big)^{1/2} &\le C_1 \varepsilon, && |I| + |J| \le N_0,\\
\mathcal{E}_{m_g}(s, \partial^I \Omega^J \psi)^{1/2} &\le C_1 \varepsilon, && |I| + |J| \le N_0 - 1,\\
\mathcal{E}_{m_g}(s, \partial^I \Omega^J \psi)^{1/2} &\le C_1 \varepsilon \log s, && |I| + |J| = N_0.
\end{aligned}
\tag{3.6.15}
$$

We suppose

$$
s_* = \sup\{s_1 : (3.6.15) \text{ holds for all } s \in [s_0, s_1]\} < \infty. \tag{3.6.16}
$$

Our aim is to prove the refined estimates

$$
\begin{aligned}
\mathcal{E}_{m_q}\big(s, \partial^I \Omega^J A_\nu\big)^{1/2} + \mathcal{E}_{m_\lambda}\big(s, \partial^I \Omega^J \chi\big)^{1/2} &\le \frac{1}{2} C_1 \varepsilon, && |I| + |J| \le N_0,\\
\mathcal{E}_{m_g}(s, \partial^I \Omega^J \psi)^{1/2} &\le \frac{1}{2} C_1 \varepsilon, && |I| + |J| \le N_0 - 1,\\
\mathcal{E}_{m_g}(s, \partial^I \Omega^J \psi)^{1/2} &\le \frac{1}{2} C_1 \varepsilon \log s, && |I| + |J| = N_0,
\end{aligned}
\tag{3.6.17}
$$

for all $s \in [s_0, s_1]$.

Combining the bootstrap assumptions (3.6.15) with the estimates for commutators in lemma 3.25, the following set of estimates is obtained, for $|I| + |J| \le N_0$,

$$
\begin{aligned}
\big\|(s/t)\partial^I \Omega^J \partial_\mu (A_\nu, \chi)\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I \Omega^J (A_\nu, \chi)\big\|_{L^2(\mathcal{H}_s)} &\le C_1 \varepsilon,\\
\big\|\partial^I \Omega^J (A_\nu, \chi)\big\|_{L^2(\mathcal{H}_s)} &\le C_1 \varepsilon,\\
|m_g| \big\|\partial^I \Omega^J \psi\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I \Omega^J \psi\big\|_{L^2(\mathcal{H}_s)} &\le C_1 \varepsilon \log s,
\end{aligned}
\tag{3.6.18a}
$$

and, for $|I| + |J| \le N_0 - 1$,

$$
|m_g| \big\|\partial^I \Omega^J \psi\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I \Omega^J \psi\big\|_{L^2(\mathcal{H}_s)} \le C_1 \varepsilon. \tag{3.6.18b}
$$

Combining (3.6.18) with proposition 3.15 and corollary 3.3.28 the following hold, for $|I| + |J| \le N_0 - 2$

$$
\begin{aligned}
\sup_{(t,x) \in \mathcal{H}_s} \big(t^{1/2} s \big|\partial_\alpha \partial^I \Omega^J (A_\nu, \chi)\big| + t^{1/2} s \big|\partial^I \Omega^J \partial_\alpha (A_\nu, \chi)\big|\big) &\lesssim C_1 \varepsilon,\\
\sup_{(t,x) \in \mathcal{H}_s} \big(t^{3/2} \big|\partial^I \Omega^J A_\nu, \partial^I \Omega^J \chi\big|\big) &\lesssim C_1 \varepsilon,\\
\sup_{(t,x) \in \mathcal{H}_s} \Big(t^{3/2} \big|m_g \partial^I \Omega^J \psi\big| + t^{1/2} s \big|\partial_\alpha \partial^I \Omega^J \psi, \partial^I \Omega^J \partial_\alpha \psi\big|\Big) &\lesssim C_1 \varepsilon \log s,
\end{aligned}
\tag{3.6.19a}
$$

and, for $|I| + |J| \le N_0 - 3$

$$
\sup_{(t,x) \in \mathcal{H}_s} \Big(t^{3/2} \big|m_g \partial^I \Omega^J \psi\big| + t^{1/2} s \big|\partial_\alpha \partial^I \Omega^J \psi, \partial^I \Omega^J \partial_\alpha \psi\big|\Big) \lesssim C_1 \varepsilon. \tag{3.6.19b}
$$

Note that we could in principle prove the two estimates for $|m_g \partial^I \Omega^J \psi|$ using proposition 3.19 however that would involve making a bootstrap assumption $E^{\mathcal{H}}(s, \partial^I \Omega^J \psi)$ which we avoid doing. The main thing to be aware of is that the estimate for $|m_g \partial^I \Omega^J \psi|$ breaks down as $m_g \to 0$.

3.6.3 First-order energy estimate for the Dirac field

To obtain decay estimates for the Dirac component ψ, a standard method is to analyse the second-order form of the Dirac equation (3.3.18). This is then a semilinear Klein-Gordon equation with mass m_g^2 and so there are standard techniques to estimate the nonlinearity, for example [Ali06] and [LM16a].

However, the right-hand side term appearing in our wave equation (3.3.18) does not decay sufficiently fast for this argument to close, which is due to the possibly vanishing mass $m_g^2 \geq 0$. Thus at this point we use proposition 3.9 and the lower bound (3.3.8) for the energy $E^{\mathcal{H}}$. This motivates us to analyse the first-order form of the Dirac equation in the following theorem to obtain certain improved L^2 and L^∞ estimates for ψ that are uniform in m_g.

Proposition 3.27. *Under the bootstrap assumption* (3.6.15) *the Dirac field* ψ *satisfies*

$$\big\|(s/t)\widehat{\Omega}^J\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^2, \quad |J| \leq N_0, \tag{3.6.20}$$

$$\sup_{\mathcal{H}_s}\big|t^{1/2}s\widehat{\Omega}^J\psi\big| \lesssim \varepsilon + (C_1\varepsilon)^2, \quad |J| \leq N_0 - 2, \tag{3.6.21}$$

as well as the following sup-norm estimate for ψ

$$\sup_{\mathcal{H}_s}\big|t\partial^I\widehat{\Omega}^J\psi\big| \lesssim \varepsilon + (C_1\varepsilon)^2, \qquad |I| + |J| \leq N_0 - 2. \tag{3.6.22}$$

Corollary 3.28. *Under the bootstrap assumption* (3.6.15) *the Dirac field* ψ *satisfies*

$$\sup_{\mathcal{H}_s}\big|t^{1/2}s\partial^I\Omega^J\psi\big| \lesssim \varepsilon + (C_1\varepsilon)^2, \qquad |I| + |J| \leq N_0 - 2. \tag{3.6.23}$$

Note that, in contrast to the estimate in (3.6.19a), corollary 3.28 controls $|\psi|$ instead of merely $|\partial\psi|$.

Proof of proposition 3.27. We write the Dirac equation (3.4.13) as

$$i\gamma^\mu\partial_\mu\psi - m_g\psi = H\psi, \tag{3.6.24}$$

where $H = g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi) - g\gamma^\mu A_\mu$. Since $i\psi^*\gamma^0 H - iH^*\gamma^0\psi = 0$ from lemma 3.11 we have the following conserved energy

$$E^{\mathcal{H}}(s,\psi) = E^{\mathcal{H}}(s_0,\psi). \tag{3.6.25}$$

Thus using inequality (3.3.8) from proposition 3.9, we have

$$\|(s/t)\psi\|_{L^2(\mathcal{H}_s)} \lesssim E^{\mathcal{H}}(s_0,\psi) \lesssim \varepsilon. \tag{3.6.26}$$

This will be our first induction step.

Next we assume that for $|J| = k-1$, where $0 \leq |J| \leq k-1 \leq N_0 - 3$, the following holds

$$\big\|(s/t)\widehat{\Omega}^J\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^2. \tag{3.6.27}$$

Consider now the case $|J| = k$ where $1 \leq k \leq N_0 - 2$. Applying $\widehat{\Omega}^J$ to the Dirac equation above we find

$$\gamma^\mu\partial_\mu(\widehat{\Omega}^J\psi) + im_g\widehat{\Omega}^J\psi = -iH(\widehat{\Omega}^J\psi) - iR^J\psi, \tag{3.6.28}$$

with $R^J = [\widehat{\Omega}^J, H]$ and H as before. Observe that $R^J\psi$ contains only terms, up to some constant matrices, of type

$$\widehat{\Omega}^{J_1} H \cdot \widehat{\Omega}^{J_2}\psi, \quad |J_1| + |J_2| \leq |J|, \quad |J_2| \leq |J| - 1. \tag{3.6.29}$$

Using the induction assumption (3.6.27) and sup-norm estimate from (3.6.19a), we have

$$\begin{aligned}
\|R^J\psi\|_{L^2(\mathcal{H}_s)} &\lesssim \sum_{1\leq|J_1|\leq|J|} \big\|(t/s)\widehat{\Omega}^{J_1}H\big\|_{L^\infty(\mathcal{H}_\tau)} \sum_{|J_2|\leq|J|-1} \big\|(s/t)\widehat{\Omega}^{J_2}\psi\big\|_{L^2(\mathcal{H}_\tau)} \\
&\lesssim \sum_{1\leq|J_1|\leq N_0-2} \big\|(t/s)\widehat{\Omega}^{J_1}H\big\|_{L^\infty(\mathcal{H}_\tau)} \sum_{|J_2|\leq k-1} \big\|(s/t)\widehat{\Omega}^{J_2}\psi\big\|_{L^2(\mathcal{H}_\tau)} \\
&\lesssim \varepsilon^2 s^{-3/2}.
\end{aligned} \tag{3.6.30}$$

Thus using proposition 3.12 and the initial data condition (3.6.15) we have, for $|J| = k \leq N_0 - 2$,

$$\begin{aligned}
E^{\mathcal{H}}(s, \widehat{\Omega}^J\psi)^{1/2} &\leq E^{\mathcal{H}}(s_0, \widehat{\Omega}^J\psi)^{1/2} + \int_{s_0}^{s} \|R^J\psi\|_{L^2(\mathcal{H}_\tau)} \mathrm{d}\tau \\
&\lesssim \varepsilon + (C_1\varepsilon)^2 \int_{s_0}^{s} \tau^{-3/2}\, \mathrm{d}\tau, \\
&\lesssim \varepsilon + (C_1\varepsilon)^2.
\end{aligned} \tag{3.6.31}$$

From proposition 3.9, we have, for $|J| = k \leq N_0 - 2$,

$$\big\|(s/t)\widehat{\Omega}^J\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim E^{\mathcal{H}}(s, \widehat{\Omega}^J\psi) \lesssim \varepsilon + (C\varepsilon)^2. \tag{3.6.32}$$

We have thus shown the induction for $|J| \leq N_0 - 2$. By corollary 3.17 this implies for $|J| \leq N_0 - 4$

$$\sup_{\mathcal{H}_s} \big|t^{1/2} s \widehat{\Omega}^J\psi\big| \lesssim \sum_{|J_1|\leq|J|+2} \big\|(s/t)\widehat{\Omega}^J\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^2. \tag{3.6.33}$$

We now consider the case $|J| = N_0 - 1$, choosing N_0 large enough that $\frac{N_0-1}{2} + 1 \leq N_0 - 4$ ($N_0 \geq 8$ suffices). We distribute derivatives according to (3.6.29) and apply the estimates from (3.6.18a), (3.6.19a), (3.6.32) and (3.6.33) to find

$$\begin{aligned}
\|R^J\psi\|_{L^2(\mathcal{H}_s)} \lesssim &\sum_{1\leq|J_1|\leq N_0-1} \big\|\widehat{\Omega}^{J_1}H\big\|_{L^2(\mathcal{H}_\tau)} \sum_{|J_2|\leq N_0-4} \big\|\widehat{\Omega}^{J_2}\psi\big\|_{L^\infty(\mathcal{H}_\tau)} \\
&+ \sum_{1\leq|J_1|\leq N_0-4} \big\|(t/s)\widehat{\Omega}^{J_1}H\big\|_{L^\infty(\mathcal{H}_\tau)} \sum_{|J_2|\leq N_0-2} \big\|(s/t)\widehat{\Omega}^{J_2}\psi\big\|_{L^2(\mathcal{H}_\tau)} \\
\lesssim &\, \varepsilon^2 s^{-3/2}.
\end{aligned} \tag{3.6.34}$$

As before, using proposition 3.12, the initial data condition (3.6.15) and proposition 3.9, we have, for $|J| = N_0 - 1$,

$$\big\|(s/t)\widehat{\Omega}^J\psi\big\|_{L^2(\mathcal{H}_s)} \lesssim E^{\mathcal{H}}(s, \widehat{\Omega}^J\psi) \lesssim \varepsilon + (C_1\varepsilon)^2. \tag{3.6.35}$$

We can now state the sup-estimate (3.6.33) for $|J| \leq N_0 - 3$.

The same analysis applies for $|J| = N_0 - 1$, and also for the case $|J| = N_0$. This then allows us to state the sup-estimate (3.6.33) for $|J| \leq N_0 - 2$. To conclude the proof, we can now use $t \lesssim s^2$ in $\mathcal{C}$ to prove (3.6.22). $\square$

3.6.4 Refined estimates for A^ν and χ

In this final subsection we close our bootstrap argument. For this to work we move to the transformed vector field $\widetilde{A}^\nu$ defined in (3.5.2) and the transformed scalar field $\widetilde{\chi}_+$ defined in (3.5.12), which are heuristically of the form

$$A^\nu = \widetilde{A}^\nu + \mathcal{O}(|\psi|^2), \qquad \chi_+ = \widetilde{\chi}_+ + \mathcal{O}(|\psi|^2). \tag{3.6.36}$$

Conclusion of proof of Theorem 3.2. Recall that our goal is to prove (3.6.17). Using the estimates for A^ν and χ_+ coming from (3.6.18) and (3.6.19), together with the previous energy and sup-norm estimates for ψ from proposition 3.27, the following estimates for $|I| + |J| \le N_0$ hold

$$\big\|(s/t)\partial^I\Omega^J\partial_\mu(\widetilde{A}_\nu, \widetilde{\chi}_+)\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu\partial^I\Omega^J(\widetilde{A}_\nu, \widetilde{\chi}_+)\big\|_{L^2(\mathcal{H}_s)} \lesssim C_1\varepsilon, \tag{3.6.37a}$$

$$\big\|\partial^I\Omega^J(\widetilde{A}_\nu, \widetilde{\chi}_+)\big\|_{L^2(\mathcal{H}_s)} \lesssim C_1\varepsilon, \tag{3.6.37b}$$

while for $|I| + |J| \le N_0 - 2$ we have

$$\sup_{\mathcal{H}_s}\Big(t^{1/2}s\big|\partial_\alpha\partial^I\Omega^J(\widetilde{A}_\nu, \widetilde{\chi}_+)\big| + t^{1/2}s\big|\partial^I\Omega^J\partial_\alpha(\widetilde{A}_\nu, \widetilde{\chi}_+)\big|\Big) \lesssim C_1\varepsilon, \tag{3.6.37c}$$

$$\sup_{\mathcal{H}_s}\Big(t^{3/2}\big|\partial^I\Omega^J(\widetilde{A}_\nu, \widetilde{\chi}_+)\big|\Big) \lesssim C_1\varepsilon. \tag{3.6.37d}$$

We next look at the energy for ψ in the case $|I| + |J| = N_0$. From lemma 3.14 and (3.6.24) we have

$$\begin{aligned}\mathcal{E}_{m_g}(s, \partial^I\Omega^J\psi)^{1/2} \le \varepsilon + C\int_{s_0}^{s}\Big(\big\|\partial^I\Omega^J(H\psi)\big\|_{L^2(\mathcal{H}_\tau)} + \sum_\mu\big\|\partial^I\Omega^J((\partial_\mu H)\psi)\big\|_{L^2(\mathcal{H}_\tau)}\\ + \sum_\mu\big\|\partial^I\Omega^J(H\partial_\mu\psi)\big\|_{L^2(\mathcal{H}_\tau)}\Big)\,\mathrm{d}\tau,\end{aligned} \tag{3.6.38}$$

where H is defined after (3.6.24). By distributing derivatives we see the worst term to estimate is the following.

$$\begin{aligned}&\|\partial^I\Omega^J((\partial_\mu H)\psi)\|_{L^2(\mathcal{H}_\tau)}\\ &\le \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\le N_0-2}}\sum_\mu\big\|(t/s)\partial^{I_1}\Omega^{J_1}\partial_\mu H\big\|_{L^\infty(\mathcal{H}_\tau)}\big\|(s/t)\partial^{I_2}\Omega^{J_2}\psi\big\|_{L^2(\mathcal{H}_\tau)}\\ &\quad + \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\ge N_0-1}}\sum_\mu\big\|(s/t)\partial^{I_1}\Omega^{J_1}\partial_\mu H\big\|_{L^2(\mathcal{H}_\tau)}\big\|(t/s)\partial^{I_2}\Omega^{J_2}\psi\big\|_{L^\infty(\mathcal{H}_\tau)}\\ &\lesssim (C_1\varepsilon)^2\tau^{-3/2}\log\tau + (C_1\varepsilon)^2\tau^{-1}\\ &\lesssim (C_1\varepsilon)^2\tau^{-1}.\end{aligned} \tag{3.6.39}$$

Note this term leads to the $\log s$ growth in the third bootstrap assumption in (3.6.15). Inserting this information into (3.6.38) we obtain

$$\mathcal{E}_{m_g}(s, \partial^I\Omega^J\psi)^{1/2} \le \varepsilon + C(C_1\varepsilon)^2\log s. \tag{3.6.40}$$

A similar argument, however this time only for $|I|+|J| \le N_0 - 1$, allows us to find

$$\mathcal{E}_{m_g}(s, \partial^I \Omega^J \psi)^{1/2} \le \varepsilon + C(C_1\varepsilon)^2. \tag{3.6.41}$$

To obtain estimates for A^ν, we first bound the energy for $\widetilde{A}^\nu$ using lemma 3.7. For all $|I|+|J| \le N_0$ we have

$$\begin{aligned}
\mathcal{E}_m(s, \partial^I \Omega^J \tilde{A}^\nu)^{1/2} &\le \varepsilon + \int_{s_0}^{s} \|Q_{\tilde{A}^\nu}\|_{L^2(\mathcal{H}_\tau)}\, \mathrm{d}\tau \\
&\le \varepsilon + \int_{s_0}^{s} \Big(\big\|\partial^I \Omega^J Q_0(\psi, \gamma^0\gamma^\nu\psi)\big\|_{L^2(\mathcal{H}_\tau)} + m_g^2 \big\|\partial^I \Omega^J (\psi^* \gamma^0 \gamma^\nu \psi)\big\|_{L^2(\mathcal{H}_\tau)} \\
&\qquad + \big\|\partial^I \Omega^J (\psi^2 \partial A + \psi A \partial \psi + \psi^3 \partial \psi)\big\|_{L^2(\mathcal{H}_\tau)}\Big)\, \mathrm{d}\tau \\
&\le \varepsilon + C(C_1\varepsilon)^2.
\end{aligned} \tag{3.6.42}$$

Note we used the definition of $Q_{\tilde{A}^\nu}$ given in (3.5.4) and the null form estimate of lemma 3.24. Now using (3.6.42) and Young's inequality we can bound the energy for A^ν. For all $|I|+|J| \le N_0$ we have

$$\begin{aligned}
\mathcal{E}_{m_q}(s, \partial^I \Omega^J A^\nu)^{1/2} &\le \mathcal{E}_{m_q}(s, \partial^I \Omega^J \widetilde{A}^\nu)^{1/2} + \mathcal{E}_{m_q}(s, \partial^I \Omega^J (\psi)^2)^{1/2} \\
&\le (3/2)\varepsilon + C \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ I_1+J_1 \le N_0-2}} \big\|(t/s)\partial^{I_1}\Omega^{J_1}\psi\big\|_{L^\infty(\mathcal{H}_s)} \times \\
&\qquad \Big(\big\|(s/t)\partial_\mu \partial^{I_2}\Omega^{J_2}\psi\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial^{I_2}\Omega^{J_2}\psi\big\|_{L^2(\mathcal{H}_s)}\Big) \\
&\lesssim \varepsilon + (C_1\varepsilon)^2.
\end{aligned} \tag{3.6.43}$$

Note in this we used that m_q is some finite, non-zero number.

A similar procedure as just argued for A^ν also gives the refined estimates for χ_+

$$\mathcal{E}_{m_\lambda}(s, \partial^I \Omega^J \chi_+)^{1/2} \le (3/2)\varepsilon + C(C_1\varepsilon)^2, \tag{3.6.44}$$

for all $|I|+|J| \le N_0$. The refined estimates for χ_-,

$$\mathcal{E}_{m_q}(s, \partial^I \Omega^J \chi_-)^{1/2} \le \varepsilon + C(C_1\varepsilon)^2, \tag{3.6.45}$$

can be obtained directly using the bootstraps and lemma 3.7 since the nonlinearity Q_{χ_-} contains no problematic quadratic terms. A combination of (3.6.44), (3.6.45) and (3.5.7) gives the refined estimates for χ

$$\mathcal{E}_{m_\lambda}(s, \partial^I \Omega^J \chi)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^2. \tag{3.6.46}$$

By choosing C_1 sufficiently large and ε sufficiently small, we collect together (3.6.40), (3.6.41), (3.6.43) and (3.6.46) to arrive at the refined bounds (3.6.17). This establishes the refined bootstrap estimates. Repeating the discussion of the bootstrap argument in section 1.3.2 we have shown global existence and thus completed the proof the main theorem. Furthermore using the relation $t < s^2$ within the cone $\mathcal{C}$, then (3.6.19) and (3.6.23) verify (3.1.12). $\square$

3.7 Additional properties of Dirac spinors on hyperboloids

We end this chapter with a short alternative presentation of the Dirac energy on hyperboloids $E^{\mathcal{H}}$. Although we don't use this in proving the theorem, it's quite fun to derive. The flagging reader may prefer to jump ahead to Chapter 4.

3.7.1 Hyperboloidal energy based on a Cholesky decomposition

Our first task is to obtain a hyperboloidal energy for the Dirac field ψ expressed in terms of a product of complex vectors $z(\psi)^* z(\psi)$. Such an expression is then easily seen to be positive semi-definite which, clearly, is in contrast to the form given in (3.3.5) and proposition 3.9.

Recall the standard Cholesky decomposition: any Hermitian, positive-definite matrix A can be decomposed uniquely as

$$A = P^* P, \tag{3.7.1}$$

where P is a lower triangular matrix with real and positive diagonal entries. In particular if A is positive semi-definite then the decomposition exists however one loses uniqueness and the diagonal entries of P may be zero.

We now prove the following result.

Proposition 3.29 (Properties of the hyperboloidal energy for the Dirac equation)**.** *There exists a lower triangular matrix P with real and positive diagonal entries such that*

$$E^{\mathcal{H}}(s, \psi) = \int_{\mathcal{H}_s} (P\psi)^* (P\psi) \mathrm{d}x, \tag{3.7.2}$$

and specifically

$$P = \begin{pmatrix} s/t & 0 & 0 & 0 \\ 0 & s/t & 0 & 0 \\ x^3/t & x^1/t - ix^2/t & 1 & 0 \\ x^1/t + ix^2/t & -x^3/t & 0 & 1 \end{pmatrix}, \tag{3.7.3}$$

which can also be expressed as

$$P = \frac{(s/t) + 1}{2} \mathbf{I}_4 + \frac{(s/t) - 1}{2} \gamma^0 + \delta_{ab} \frac{x^a}{t} \gamma^0 \gamma^b. \tag{3.7.4}$$

The above expression is quite natural and resembles what is known for the wave equation: the factor x^a/t comes from Stokes' theorem applied to hyperboloids and we cannot expect to fully control the standard L^2 norm, namely $\int_{\mathcal{H}_s} \psi^* \psi \, \mathrm{d}x$.

Proof. Step 1. Existence of the decomposition. Before we proceed with the derivation of the identity, we present an argument showing that such a decomposition exists by proving positive semi-definiteness. For simplicity of notation, let $N_a = x^a/t$. The integrand of $E^{\mathcal{H}}(s, \psi)$ can be written as $\psi^* A \psi$ where $A = \mathbf{I_4} + N_a \gamma^0 \gamma^a$. Here the spatial indices are contracted with δ_{ab}, so that $N_a \gamma^a = \delta_{ab} N^a \gamma^b$. Note A is hermitian since $A^* = \mathbf{I_4} + N_a (\gamma^0 \gamma^a)^* = A$. Also

$$(N_a \gamma^0 \gamma^a)(N_b \gamma^0 \gamma^b) = -N_a N_b \gamma^0 \gamma^0 \gamma^a \gamma^b = -N_a N_b \gamma^{(a} \gamma^{b)} = N_a N^a \mathbf{I_4}. \tag{3.7.5}$$

Then for all $z \in \mathbb{C}^4$ we have

$$\begin{aligned} 0 \le \big(Az\big)^*\big(Az\big) &= (1 + N_a N^a) z^* \mathbf{I_4} z + 2 z^* N_a \gamma^0 \gamma^a z \\ &\le 2\left(z^* \mathbf{I_4} z + z^* N_a \gamma^0 \gamma^a z\right) \\ &= 2 z^* A z. \end{aligned} \tag{3.7.6}$$

We used that $N_a N^a = (r/t)^2 \le 1$ which holds in the light-cone $\mathcal{C}$. Thus A is positive semi-definite.

Step 2. Computing the matrix P. With respect to the Dirac representation (3.2.5) we have

$$\begin{aligned} A &= \mathbf{I_4} + N_a \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix} \begin{pmatrix} 0_2 & \sigma^a \\ -\sigma^a & 0_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} + N_a \begin{pmatrix} 0_2 & \sigma^a \\ \sigma^a & 0_2 \end{pmatrix}. \end{aligned} \tag{3.7.7}$$

Here I_2 and 0_2 represent the 2×2 identity and zero matrices respectively. Calculate the second term above using the Pauli matrices:

$$N_a \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} = N_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + N_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + N_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} N_3 & N_1 - iN_2 \\ N_1 + iN_2 & -N_3 \end{pmatrix}. \tag{3.7.8}$$

Define $\omega = N_1 + iN_2$ and note $N_i \in \mathbb{R}$. Thus we have

$$A = \begin{pmatrix} \begin{matrix} & \\ & I_2 \\ & \end{matrix} & \begin{matrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{matrix} \\ \begin{matrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{matrix} & I_2 \end{pmatrix}. \tag{3.7.9}$$

Consider now 2×2 complex matrices B, C, D such that

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}^* \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} I_2 & \begin{matrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{matrix} \\ \begin{matrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{matrix} & I_2 \end{pmatrix}. \tag{3.7.10}$$

This implies the following identities

$$\begin{aligned} D^* D &= I_2, \\ C^* D = D^* C &= \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}, \\ B^* B + C^* C &= I_2. \end{aligned} \tag{3.7.11}$$

If we let $D = I_2$ and $C = \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}$ then we must solve

$$B^* B = I_2 - \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}^* \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.7.12}$$

where $\lambda = 1 - (N_3^2 + \bar{\omega}\omega) = 1 - (N_1^2 + N_2^2 + N_3^2)$. Indeed $\lambda = 1 - (r/t)^2 = (s/t)^2 \ge 0$ so we can take $B = \sqrt{\lambda} I_2 = (s/t) I_2$. $\square$

3.7.2 Hyperboloidal energy based on the Weyl spinor representation of the Dirac equation

Yet one more approach in deriving energy estimates is obtained by expressing the Dirac spinors in terms of Weyl spinors and then studying the energy of Weyl spinors (3.7.15) instead. This provides another convenient way to study Dirac equations. Decompose the spinor ψ and source term F as

$$\psi = \begin{pmatrix} u+v \\ u-v \end{pmatrix}, \quad F = \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}, \tag{3.7.13}$$

where $u, v : \mathbb{R}^{1+3} \to \mathbb{C}^2$ are Weyl spinors and $F_\pm \in \mathbb{C}^2$. Defining $\partial_\pm = \partial_0 \pm \sigma^i \partial_i$ the PDE (3.2.1) can be shown to be equivalent to

$$\begin{aligned} \partial_- v + iMu &= iF_+, \\ \partial_+ u + iMv &= iF_-. \end{aligned} \tag{3.7.14}$$

A Dirac-Klein-Gordon system with respect to such a Weyl spinor decomposition has been studied in the low-regularity setting in [Bou99]. Following a similar approach to section 3.3.1 we find an analogous hyperboloidal Weyl spinor energy:

$$E^\sigma_\pm(s,u) = \int_{\mathcal{H}_s} \big(u^* u \pm \frac{x_a}{t} u^* \sigma^a u \big) \, \mathrm{d}x. \tag{3.7.15}$$

Similar to propositions 3.9 and 3.12 we can prove positivity and an energy estimate for $E^\sigma_\pm$.

Proposition 3.30 (Properties of the hyperboloidal energy for the Weyl equation)**.** *For a $\mathbb{C}^2$–valued function w the following holds:*

$$E^\sigma_\pm(s,w) \geq \frac{1}{2} \int_{\mathcal{H}_s} \frac{s^2}{t^2} w^* w \, \mathrm{d}x \geq 0. \tag{3.7.16}$$

Furthermore for solutions u, v to (3.7.14) *we have*

$$\big(E^\sigma_+(s,u) + E^\sigma_-(s,v)\big)^{1/2} \leq \big(E^\sigma_+(s_0,u) + E^\sigma_-(s_0,v)\big)^{1/2} + \int_{s_0}^s \|F_+\|_{L^2(\mathcal{H}_\tau)} + \|F_-\|_{L^2(\mathcal{H}_\tau)} \mathrm{d}\tau. \tag{3.7.17}$$

Proof. Step 1. Using the Dirac representation (3.2.5) and the decomposition (3.7.13), the PDE (3.2.1) becomes

$$\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \partial_0 \begin{pmatrix} u+v \\ u-v \end{pmatrix} + \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \partial_a \begin{pmatrix} u+v \\ u-v \end{pmatrix} + iM \begin{pmatrix} u+v \\ u-v \end{pmatrix} = i \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}. \tag{3.7.18}$$

Defining $\partial_\pm = \partial_0 \pm \sigma^i \partial_i$ this becomes

$$\begin{pmatrix} \partial_+ u + \partial_- v \\ \partial_- v - \partial_+ u \end{pmatrix} + iM \begin{pmatrix} u+v \\ u-v \end{pmatrix} = i \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}. \tag{3.7.19}$$

Adding and subtracting the two rows above gives the following

$$\begin{aligned} \partial_- v + iMu &= iF_+, \\ \partial_+ u + iMv &= iF_-. \end{aligned} \tag{3.7.20}$$

Following a similar approach to deriving (3.3.3), we multiply the first and second equation by v^* and u^* respectively.

$$\begin{aligned} u^*\partial_0 u + u^*\sigma^a\partial_a u + iMu^*v &= iu^*F_-, \\ v^*\partial_0 v - v^*\sigma^a\partial_a v + iMu^*v &= -iv^*F_+. \end{aligned} \tag{3.7.21}$$

One then adds these equations to their conjugate to obtain the following:

$$\begin{aligned} \partial_0(u^*u) + \partial_a(u^*\sigma^a u) + iM(u^*v - v^*u) &= iu^*F_- - iF_-^*u, \\ \partial_0(v^*v) - \partial_a(v^*\sigma^a v) + iM(v^*u - u^*v) &= iv^*F_+ - iF_+^*v. \end{aligned} \tag{3.7.22}$$

Note the mass terms appear above. However if add these equations together we obtain

$$\partial_0(u^*u + v^*v) + \partial_a\big(u^*\sigma^a u - v^*\sigma^a v\big) = iu^*F_- - iF_-^*u + iv^*F_+ - iF_+^*v, \tag{3.7.23}$$

which does not contain a term involving M. This equation is the analogous Weyl spinor version of (3.3.3). Clearly integrating (3.7.23) over $\mathcal{C}_{[s_0,s]}$ gives the energy functional $E^\sigma_\pm(s,u)$ defined in (3.7.15).

Step 2. Next we establish that for a $\mathbb{C}^2$–valued function w the following holds:

$$E^\sigma_\pm(s,w) \geq \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2} w^*w\,\mathrm{d}x \geq 0. \tag{3.7.24}$$

The idea is in the spirit of proposition 3.9. Observe that the sigma matrices are Hermitian and satisfy the following anti-commutator relation: $\{\sigma^a,\sigma^b\} = 2\delta^{ab}I_2$. Then we have

$$\begin{aligned} \int_{\mathcal{H}_s}\Big(w \pm \frac{x_a}{t}\sigma^a w\Big)^*\Big(w \pm \frac{x_b}{t}\sigma^b w\Big)\,\mathrm{d}x &= \int_{\mathcal{H}_s}\Big(w^*w \pm 2\frac{x_a}{t}w^*\sigma^a w + \frac{x_a x_b}{t^2}w^*\sigma^{(a}\sigma^{b)}w\Big)\,\mathrm{d}x \\ &= \int_{\mathcal{H}_s}\Big(w^*w(1+(r/t)^2) \pm 2\frac{x_a}{t}w^*\sigma^a w\Big)\,\mathrm{d}x \\ &= 2E^\sigma_\pm(s,w) - \int_{\mathcal{H}_s}\frac{s^2}{t^2}w^*w\,\mathrm{d}x. \end{aligned} \tag{3.7.25}$$

Thus we have

$$E^\sigma_\pm(s,w) = \frac{1}{2}\int_{\mathcal{H}_s}\Big(w \pm \frac{x_a}{t}\sigma^a w\Big)^*\Big(w \pm \frac{x_a}{t}\sigma^a w\Big)\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{H}_s}\frac{s^2}{t^2}w^*w\,\mathrm{d}x \geq 0. \tag{3.7.26}$$

Step 3. Next, let us show that the following hyperboloidal energy inequality holds for the Weyl spinor equation (3.7.14)

$$\big(E^\sigma_+(s,u) + E^\sigma_-(s,v)\big)^{1/2} \leq \big(E^\sigma_+(s_0,u) + E^\sigma_-(s_0,v)\big)^{1/2} + \int_{s_0}^s \|F_+\|_{L^2(\mathcal{H}_\tau)} + \|F_-\|_{L^2(\mathcal{H}_\tau)}\mathrm{d}\tau. \tag{3.7.27}$$

Namely, integrating (3.7.23) over $\mathcal{C}_{[s_0,s]}$ we obtain

$$\begin{aligned} &E^\sigma_+(s,u) + E^\sigma_-(s,v) \\ &= E^\sigma_+(s_0,u) + E^\sigma_-(s_0,v) + \int_{s_0}^s \mathrm{d}\tau\int_{\mathcal{H}_\tau}(\tau/t)(iu^*F_- - iF_-^*u + iv^*F_+ - iF_+^*v)\,\mathrm{d}x. \end{aligned} \tag{3.7.28}$$

Differentiating in s and noting that $\|(s/t)u, (s/t)v\|_{L^2(\mathcal{H}_s)} \le \|u, v\|_{L^2(\mathcal{H}_s)} \le (E^\sigma_+(s,u) + E^\sigma_-(s,v))^{1/2}$ gives

$$\frac{d}{\mathrm{d}s}\Big(E^\sigma_+(s,u) + E^\sigma_-(s,v)\Big)^{1/2} \le \|F_+\|_{L^2(\mathcal{H}_\tau)} + \|F_-\|_{L^2(\mathcal{H}_\tau)}. \tag{3.7.29}$$

$\square$

Chapter 4

Stability of a coupled wave–Klein-Gordon system with quadratic nonlinearities

4.1 Introduction

In this chapter we[1] study a semilinear coupled wave–Klein-Gordon system using the hyperboloidal foliation method of LeFloch and Ma [LM14]. Consider in $\mathbb{R}^{1+3}$ the PDEs

$$\Box u = F_u = -uv - u\partial_t v, \tag{4.1.1a}$$
$$\Box v - v = F_v = -uv, \tag{4.1.1b}$$

with initial data prescribed on the time slice $t = 2$

$$\begin{aligned} u(2,x) &= \varepsilon\Psi_0(x), \quad \partial_t u(2,x) = \varepsilon\Psi_1(x), \\ v(2,x) &= \varepsilon\phi_0(x), \quad \partial_t v(2,x) = \varepsilon\phi_1(x). \end{aligned} \tag{4.1.1c}$$

We remind the reader that the motivation for studying this system is to resolve conjecture 1.49 where coupled wave–Klein-Gordon nonlinearities, closely related to those in (4.1.1), are not yet fully understood.

Our aim is to prove that initial data, sufficiently small in some norm, yield global-in-time solutions to (4.1.1) that decay back to the trivial solution $(u, v) = (0, 0)$. The main difficulty is that there are no derivatives on the wave component u on the right-hand-side terms F_u and F_v of equation (4.1.1), and thus the nonlinearities appear to decay insufficiently fast. To be more precise, the best we can expect is that

$$\|F_u\|_{L^2} = \|uv + u\partial_t v\|_{L^2} \lesssim t^{-1}, \qquad \|F_v\|_{L^2} = \|uv\|_{L^2} \lesssim t^{-1}, \tag{4.1.2}$$

both of which are not integrable.

We now state the main theorem for this chapter.

Theorem 4.1 (Nonlinear stability of a wave–Klein-Gordon model)**.** *Consider the system (4.1.1) and let $N_0 \geq 8$ be an integer. Then there exists $\varepsilon_0 > 0$ such that for all*

[1]The results in this chapter were obtained in collaboration with Shijie Dong at Sorbonne University (Pierre and Marie Curie Campus). This research was supported in part by the Innovative Training Networks (ITN) grant 642768 'ModCompShock'.

$\varepsilon \in (0, \varepsilon_0)$ and all compactly supported initial data $(\phi_0, \phi_1, \Psi_0, \Psi_1)$ satisfying for all $s \geq 2$ the smallness condition

$$\|\phi_0, \Psi_0\|_{H^{N_0+1}(\mathbb{R}^3)} + \|\phi_1, \Psi_1\|_{H^{N_0}(\mathbb{R}^3)} \leq \varepsilon, \tag{4.1.3}$$

the initial value problem (4.1.1) *admits a global-in-time solution* (u, v) *satisfying*

$$\|(s/t)\partial^I \partial_\mu u\|_{L^2(\mathcal{H}_s)} + \|(s/t)\partial^I \partial_\mu v\|_{L^2(\mathcal{H}_s)} + \|\partial^I v\|_{L^2(\mathcal{H}_s)} \leq C\varepsilon s^\delta, \quad |I| = N_0, \tag{4.1.4}$$

where $C > 0$ is a large constant and $\delta \ll 1$, and which satisfies the pointwise estimates

$$|u(t,x)| \lesssim \varepsilon t^{-1}, \quad |v(t,x)| \lesssim \varepsilon t^{-3/2}. \tag{4.1.5}$$

Remark 4.2 (Spatially compactly supported functions in $\mathcal{C}$)**.** In the proof of theorem 4.1, we assume the initial data are prescribed on the slice $t_0 = 2$, and that the initial data are supported in the ball $\{x : |x| \leq R\}$ for $R < 1$. The evolution of such data will lead to functions that are spatially compactly supported in $\mathcal{C} = \{r < t - 1\}$, i.e. the functions are supported in the region $\{|x| \leq 1\}$ and vanish identically in a neighborhood of the light cone $\{t = r + 1\}$. We remind the reader that on a hyperboloid $\mathcal{H}_s \cap \mathcal{C}$ we have the estimate $s \leq t \leq Cs^2$.

For the proof of the main theorem, we use the hyperboloidal foliation of the strategy introduced by LeFloch and Ma [LM14]. In particular, we consider (4.1.1b) as a Klein-Gordon equation with a varying mass $m = \sqrt{1-u}$ and adapt energy estimates given in [LM14] and robust pointwise decay estimates for Klein-Gordon equations given in [LM16a]. Our approach relies on improved L^2-type norms for the wave component u which we obtain from a conformal-type energy estimate, first introduced on hyperboloids by Huang and Ma [MH17], and on improved pointwise decay estimates for the the wave component u given in [LM16a]. We also perform two transformations, given in 4.20 and 4.21, on the wave component u, inspired by work in [Kat12, Tsu03a]. This produces two transformed variables, $\tilde{U}_p$, $p \in \{1, 2\}$, whose nonlinearities are easier to study.

Our main theorem allows us to treat the Klein-Gordon-Zakharov equations. These equations have been studied before using constant time slices or Fourier methods in [OTT95, Kat12, Tsu96]. In Chapter 1 we introduced the Klein-Gordon-Zakharov equations

$$\begin{aligned} \Box u &= -\sum_K \Delta |v_K|^2, \\ \Box v_K - v_K &= -u v_K, \end{aligned} \tag{4.1.6}$$

where the unknown u is real valued and v_K are complex valued for $K = 1, 2, 3$. The initial data are denoted by

$$\big(u, \partial_t u\big)(t=2, \cdot) = \big(u^{(0)}, u^{(1)}\big), \qquad \big(v_K, \partial_t v_K\big)(t=2, \cdot) = \big(v_K^{(0)}, v_K^{(1)}\big). \tag{4.1.7}$$

In order to apply the strategy of theorem 4.1, we rewrite the equations (4.1.6) in terms of the variables $x_K = \mathrm{Re}(v_K)$ and $y_K = \mathrm{Im}(v_K)$ to obtain

$$\begin{aligned} &\Box u = -\textstyle\sum_{a=1}^3 \partial_a \partial^a \big(x_K^2 + y_K^2\big), \\ &\Box x_K - x_K = -u x_K, \\ &\Box y_K - y_K = -u y_K. \end{aligned} \tag{4.1.8}$$

We note that due to the second order derivative appearing in the nonlinearity for u in (4.1.8) the regularity of u is one order less than that of v_K. This is reflected in the initial data

$$\|u^{(0)}\|_{H^{N_0}}, \quad \|u^{(1)}\|_{H^{N_0-1}}, \quad \|v_K^{(0)}\|_{H^{N_0+1}}, \quad \|v_K^{(1)}\|_{H^{N_0}},$$

with N_0 some large integer. Note that the wave nonlinearity in (4.1.8) is of divergence form, and thus easier to handle than that considered in theorem 4.1. Thus our method of proof applies to this system in a very similar way and for this reason we omit the details.

In principle one can now use our methods to extend theorem 4.1 to the following more general system

$$\begin{aligned} \Box u &= Q(u, v, \partial v; v, \partial v), \\ \Box v - v &= Q(u; u, v) + Q(\partial u, v, \partial v; v, \partial v), \end{aligned} \tag{4.1.9}$$

where we use the short-hand notation $Q(\cdots;\cdots)$ to denote quadratic nonlinearities involving interactions between one term from each side of the semicolon. Note further that, compared to the work of [OTT95] and [Kat12], we can treat a wider class of nonlinearities. In [Kat12, (2.14)] any nonlinearity for the wave equation involving at most one derivative, needed to be of divergence form. This is not needed in our setting. We remind one that the u–u interaction terms were treated by Tsutsumi in [Tsu03a]. Finally, it was speculated in [LM14] that nonlinear interaction terms of the form $Q(u; v, \partial v)$ may lead to finite time blow-up. Thus this chapter partially answers their question by showing that certain terms of this form do not lead to finite time blow-up.

Outline. The rest of this chapter is organised as follows. In section 4.2 and 4.3 we establish hyperboloidal L^2 and L^∞ estimates respectively for both wave and Klein-Gordon components. In section 4.4, we state the bootstrap assumptions and derive some basic estimates. In section 4.5 and section 4.6 we derive refined estimates for Klein-Gordon and wave components respectively. In section 4.7 we give the proof of the main theorem.

4.2 Energy estimates

First, we state energy estimates for the hyperboloidal energy functionals $\mathcal{E}$ and $\mathcal{E}_1$ given in definition 3.6.

Proposition 4.3 (Energy estimate for wave equation)**.** *Let u be a sufficiently regular solution to* (4.1.1a) *supported in the region $\mathcal{C}$ then*

$$\mathcal{E}(s, u)^{1/2} \leq \mathcal{E}(2, u)^{1/2} + \int_2^s \|F_u\|_{L^2(\mathcal{H}_\tau)} \, \mathrm{d}\tau. \tag{4.2.1}$$

For the proof we refer to [LM14, Lemma 6.3.1].

Proposition 4.4 (Energy estimate for Klein-Gordon equation with varying mass)**.** *Let u be a sufficiently regular function supported in the region $\mathcal{C}$ and satisfying*

$$|u| \leq \frac{1}{10}. \tag{4.2.2}$$

Let v be a sufficiently regular function supported in the region $\mathcal{C}$ and governed by the following Klein-Gordon equation with squared mass $1-u$:

$$\Box v - (1-u)v = f. \tag{4.2.3}$$

Then the energy functional for v on the hyperboloid $\mathcal{H}_s$ can be controlled by both

$$\mathcal{E}_1(s,v)^{1/2} \leq \mathcal{E}_1(2,v)^{1/2} + \int_2^s \Big(\|uv\|_{L^2(\mathcal{H}_\tau)} + \|f\|_{L^2(\mathcal{H}_\tau)} \Big) \mathrm{d}\tau, \tag{4.2.4}$$

and

$$\mathcal{E}_1(s,v)^{1/2} \leq 2\mathcal{E}_1(2,v)^{1/2} + 2\int_2^s \Big(\|(\tau/t) v \partial_t u\|_{L^2(\mathcal{H}_\tau)} + \|f\|_{L^2(\mathcal{H}_\tau)} \Big) \mathrm{d}\tau. \tag{4.2.5}$$

The energy estimate (4.2.5) is better than (4.2.4) in the cases where $\partial_t u$ decays faster than u, which is the case when u is a solution to a wave equation.

Proof. The energy estimate (4.2.4) is the standard application of lemma 3.7 to the system written in the form (4.1.1b). In order to prove the energy estimate (4.2.5), we first multiply the equation (4.2.3) by the multiplier $\partial_t v$ and write the resulting equation in the following favourable form

$$\frac{1}{2}\partial_t\big((\partial_t v)^2 + \sum_a (\partial_a v)^2 + (1-u)v^2\big) + \sum_a \partial_a\big(-\partial_t v \partial_a v\big) = -\frac{1}{2}v^2\partial_t u + \partial_t v f. \tag{4.2.6}$$

We then integrate the identity (4.2.6) over the region $\mathcal{C}_{[2,s]}$ and integrate by parts to arrive at

$$\begin{aligned}
\mathcal{E}_{\sqrt{1-u}}(s,v)^{1/2} \frac{d}{ds}\mathcal{E}_{\sqrt{1-u}}(s,v)^{1/2} &= \int_{\mathcal{H}_s} (s/t)\big(-\frac{1}{2}v^2\partial_t u + \partial_t v f\big)\,\mathrm{d}x \\
&\leq \|(s/t)\partial_t u v\|_{L^2(\mathcal{H}_s)} \|v\|_{L^2(\mathcal{H}_s)} \\
&\quad + \|f\|_{L^2(\mathcal{H}_s)} \|(s/t)\partial_t v\|_{L^2(\mathcal{H}_s)}.
\end{aligned} \tag{4.2.7}$$

Next by using the assumption that $|u| \leq 1/10$, we have

$$\frac{9}{10}\mathcal{E}_1(s,v)^{1/2} \leq \mathcal{E}_{\sqrt{1-u}}(s,v)^{1/2} \leq \frac{11}{10}\mathcal{E}_1(s,v)^{1/2},$$

which together with (4.2.7) leads to

$$\mathcal{E}_{\sqrt{1-u}}(s,v)^{1/2} \leq \mathcal{E}_{\sqrt{1-u}}(2,v)^{1/2} + \frac{10}{9}\int_2^s \Big(\|v\partial_t u\|_{L^2(\mathcal{H}_\tau)} + \|f\|_{L^2(\mathcal{H}_\tau)} \Big) \mathrm{d}\tau,$$

and finally (4.2.5). □

4.2.1 Conformal-type energy estimates on hyperboloids

We now introduce a conformal-type energy, see the discussion near (1.3.38), which was adapted to the hyperboloidal foliation setting in [MH17]. This energy is key to obtaining a robust estimate of the L^2-type norm for the wave component u.

First we recall the following Hardy-type inequality.

Lemma 4.5 (Hardy inequality on the hyperboloid)**.** *Let u be a sufficiently regular function supported in the region $\mathcal{C}$ then for all $s \geq 2$*

$$\|r^{-1}u\|_{L^2(\mathcal{H}_s)} \lesssim \sum_a \|Y_a u\|_{L^2(\mathcal{H}_s)}. \tag{4.2.8}$$

The proof follows by adapting lemma 1.23 to the hyperboloidal foliation, see for example [LM14, §5.3].

Definition 4.6 (Conformal-type energy)**.** Define the following conformal energy

$$\mathcal{E}_{\text{con}}(u,s) = \int_{\mathcal{H}_s} \big(\sum_a \big(sY_a u\big)^2 + \big(Ku + 2u\big)^2\big)\,\mathrm{d}x, \tag{4.2.9}$$

where we define $Ku = \big(s(s/t)\partial_t + 2x^a Y_a\big)u$.

We have the bound

$$\int_{\mathcal{H}_s} \big(|s(s/t)\partial_t u|^2 + \sum_a |(s/t)\Omega_a u|^2 + |(s/t)u|^2\big)\,\mathrm{d}x \leq \mathcal{E}_{\text{con}}(u,s). \tag{4.2.10}$$

By using definitions 3.5 and 1.40, we see that $K = t^{-1}K_0$, where K_0 was discussed in remark 1.3.3. Indeed up to hyperboloidal factors of (s/t) one can see the close connections between the structure of the left hand side of (4.2.10) with the conformal energy on constant t-slices given in remark 1.3.3 in equation (1.3.38).

Lemma 4.7 (Conformal energy estimate)**.** *Let u be a sufficiently regular spatially compactly supported solution to* (4.1.1a) *in $\mathcal{C}$, then*

$$\mathcal{E}_{con}(u,s)^{1/2} \leq \mathcal{E}_{con}(u,s_0)^{1/2} + 2\int_{s_0}^{s} \tau \|F_u\|_{L^2(\mathcal{H}_\tau)}\,\mathrm{d}\tau, \tag{4.2.11}$$

with moreover

$$\big\|(s/r)u\big\|_{L^2(\mathcal{H}_s)} \leq 2\mathcal{E}_{con}(u,s)^{1/2}. \tag{4.2.12}$$

The estimate (4.2.11) follows by using $s(K+2)$ as a multiplier on $\Box u$, see [MH17, Proposition 2.1]. The estimate (4.2.12) can be shown by combining (4.2.11) with lemma 4.5, see also [MH17, Proposition 2.2].

4.3 Pointwise estimates

4.3.1 Sup-norm estimates for wave components

We state the following lemma which is essential in proving the sup-norm bound for wave components. A proof may be found in [LM16a, Proposition 3.1] or [Ali06].

Lemma 4.8 (Pointwise estimates for wave components)**.** *Suppose u, f are sufficiently regular spatially compactly supported functions in $\mathcal{C}$. Suppose u is a solution to the wave equation*

$$\begin{aligned} \Box u &= f, \\ u(t_0,x) &= \partial_t u(t_0,x) = 0, \end{aligned} \tag{4.3.1}$$

where the source f satisfies

$$|f| \leq C_f t^{-2-\nu}(t-r)^{-1+\mu}, \tag{4.3.2}$$

for some constants $C_f > 0$, $0 < \mu \le 1/2$ and $0 < \nu \le 1/2$. Then we have

$$|u(t,x)| \lesssim \frac{C_f}{\nu\mu}(t-r)^{\mu-\nu}t^{-1}. \tag{4.3.3}$$

Note that this estimate plays an essential role in controlling the quasi-linear term $h^{00}Y_0Y_0\phi$ in the Einstein Klein-Gordon equations studied in [LM16a].

4.3.2 Sup-norm estimates for Klein-Gordon components

First we state a slight adaption of Grönwall's inequality given in lemma 1.20.

Lemma 4.9. *Assume $u(t) : [0,T] \to [0,\infty)$ is continuous and satisfies*

$$u(t) \le C + \int_0^t \Big(a(s)u(s) + b(s)\Big)\,\mathrm{d}s, \quad t \in [0,T], \tag{4.3.4}$$

where $a(t), b(t) : [0,T] \to [0,\infty)$ are integrable functions and C is a nonnegative constant. Then u satisfies

$$u(t) \le \Big(C + \int_0^t b(s)\,\mathrm{d}s\Big)\exp\Big(\int_0^t a(s)\,\mathrm{d}s\Big), \quad t \in [0,T]. \tag{4.3.5}$$

The following pointwise estimate for the Klein-Gordon equation dates back to work of Klainerman [Kla85a]. We state the result for the hyperboloidal foliation setting as given in [LM16a, Proposition 3.3] and adapt it to the PDE (4.2.3) where the mass of the Klein-Gordon field varies.

Proposition 4.10 (Pointwise estimates for Klein-Gordon components with varying mass)**.** *Suppose v, f are sufficiently regular spatially compactly supported functions in $\mathcal{C}$. Suppose v satisfies the Klein-Gordon equation*

$$\begin{aligned} &\Box v - (1-u)v = f, \\ &v|_{\mathcal{H}_2} = \Psi_0, \quad \partial_t v|_{\mathcal{H}_2} = \Psi_1, \end{aligned} \tag{4.3.6}$$

with the assumption $|u| \le 1/10$. Then we have

$$s^{3/2}\big|v(t,x)\big| + (s/t)^{-1}s^{3/2}\big|Y^{\perp}v(t,x)\big| \lesssim V(t,x), \tag{4.3.7}$$

where we have defined

$$s_0 = \begin{cases} 2, & 0 \le r/t \le 3/5, \\ \sqrt{\frac{t+r}{t-r}}, & 3/5 \le r/t < 1, \end{cases} \tag{4.3.8}$$

$$V(t,x) = \begin{cases} e^{\int_{s_0}^s |\frac{d}{d\lambda}u(\lambda t/s, \lambda x/s)|\,d\lambda}\Big(\|\Psi_0\|_{L^\infty(\mathcal{H}_2)} + \|\Psi_1\|_{L^\infty(\mathcal{H}_2)} + F(s)\Big), & 0 \le r/t \le 3/5, \\ e^{\int_{s_0}^s |\frac{d}{d\lambda}u(\lambda t/s, \lambda x/s)|\,d\lambda}F(s), & 3/5 \le r/t < 1, \end{cases}$$

as well as

$$R[v] = s^{3/2}\sum_a Y_aY_av + \frac{x^ax^b}{s^{1/2}}Y_aY_bv + \frac{3}{4s^{1/2}}v + \frac{3x^a}{s^{1/2}}Y_av. \tag{4.3.9}$$

$$F(s) = \int_{s_0}^s \Big|R[v](\lambda t/s, \lambda x/s) + \lambda^{3/2}f(\lambda t/s, \lambda x/s)\Big|\,d\lambda. \tag{4.3.10}$$

The proof of proposition 4.10 is based on a decomposition result in lemma 4.11 and an ODE estimate in lemma 4.12. We refer to [LM16a] for the detailed proofs, but give a simpler proof of lemma 4.12 below, which provides a neater expression of the estimate for the ODE.

Lemma 4.11 (A decomposition identity, [LM16a, Lemma 3.4])**.** *Assume v is a sufficiently regular solution to the Klein-Gordon equation* (4.3.6)*, and let*

$$w_{t,x}(\lambda) = \lambda^{3/2} v(\lambda t/s, \lambda x/s), \quad (t,x) \in \mathcal{C},$$

then the following second-order ODE with respect to $\lambda > 0$ holds

$$\frac{d^2}{d\lambda^2} w_{t,x}(\lambda) + \big(1 - u(\lambda t/s, \lambda x/s)\big) w_{t,x}(\lambda) = \big(R[v] + s^{3/2} f\big)(\lambda t/s, \lambda x/s). \tag{4.3.11}$$

Lemma 4.12 (Technical ODE estimate, modified from [LM16a, Lemma 3.5])**.** *Let G be a function defined on an interval $[s_0, s_1]$ and satisfying* $\sup |G(\lambda)| \leq 1/10$ *and let k be an integrable function defined on $[s_0, s_1]$. Then the solution z to the following ODE*

$$\begin{aligned} &z''(\lambda) + \big(1 - G(\lambda)\big) z(\lambda) = k(\lambda), \\ &z(s_0) = z_0, \quad z'(s_0) = z_1, \end{aligned} \tag{4.3.12}$$

satisfies the uniform bound

$$\big((z')^2(s) + (1 - G(s)) z^2(s)\big)^{1/2} \lesssim \Big(\big((z')^2(s_0) + z^2(s_0)\big)^{1/2} + \int_{s_0}^{s} |k(\lambda)|\, d\lambda\Big) \exp\Big(\int_{s_0}^{s} |G'(\lambda)|\, d\lambda\Big). \tag{4.3.13}$$

Proof. We set $Y(\lambda) = \big((z')^2(\lambda) + (1 - G(\lambda)) z^2(\lambda)\big)^{1/2}$, and then by multiplying $z'(\lambda)$ in (4.3.12), we get

$$\begin{aligned} \frac{d}{d\lambda} Y^2(\lambda) &= z'(\lambda) k(\lambda) - G'(\lambda) z^2(\lambda) \\ &\leq C Y(\lambda) \big(|k(\lambda)| + Y(\lambda) |G'|\big). \end{aligned} \tag{4.3.14}$$

In order to proceed, we divide $Y(\lambda)$ in the above inequality and, integrate to get

$$Y(s) \leq Y(s_0) + C \int_{s_0}^{s} \big(|k(\lambda)| + Y(\lambda) |G'|\big)\, d\lambda. \tag{4.3.15}$$

Finally, we apply lemma 1.20 to end the proof. □

4.4 Bootstrap method

In a similar manner to the argument given in proposition 3.26, for initial data sufficiently small in the sense of (4.1.3) there exists a local-in-hyperboloidal time solution to (4.1.1). We then assuming the following bootstrap assumptions hold in the interval $[2, s)$

$$\begin{aligned} \mathcal{E}(s, \partial^I \Omega^J u)^{1/2} &\leq C_1 \varepsilon, & |I| + |J| &\leq N_0 - 1, & \text{(4.4.1a)} \\ \mathcal{E}(s, \partial^I u)^{1/2} &\leq C_1 \varepsilon s^{\delta}, & |I| &= N_0, & \text{(4.4.1b)} \\ \mathcal{E}(s, \partial^I \Omega^J u)^{1/2} &\leq C_1 \varepsilon s^{|J|\delta}, & |I| + |J| &\leq N_0, |J| \geq 1 & \text{(4.4.1c)} \\ \mathcal{E}_1(s, \partial^I \Omega^J v)^{1/2} &\leq C_1 \varepsilon s^{|J|\delta}, & |I| + |J| &\leq N_0, & \text{(4.4.1d)} \\ \|(s/t) \partial^I \Omega^J u\|_{L^2(\mathcal{H}_s)} &\leq C_1 \varepsilon s^{1/2 + |J|\delta}, & |I| + |J| &\leq N_0, & \text{(4.4.1e)} \end{aligned}$$

$$|\partial^I \Omega^J u| \le C_1 \varepsilon t^{-1} s^{|J|\delta}, \qquad |I| + |J| \le N_0 - 4, \tag{4.4.1f}$$
$$|\partial^I \Omega^J v| \le (C_1 \varepsilon)^{1/2} t^{-3/2} s^{|J|\delta}, \qquad |I| + |J| \le N_0 - 4. \tag{4.4.1g}$$

In the above C_1 is some large constant which is fixed and chosen to satisfy $C_1\varepsilon \ll 1$, δ is some fixed small constant, such that $0 < \delta \ll 1/N_0$. We will let $s_* = \sup\{s :$ (4.4.1) *hold*$\}$ and note that C_1 and δ are independent of s_*. In order to prove the stability result stated in theorem 4.1, it suffices (for more details see the proof in section 1.3.2) to replace C_1 in (4.4.1) with $\frac{1}{2}C_1$ since this will indicate that s_* cannot be of finite value, which thus completes the proof of a global-in-time solution stated in the main theorem 4.1.

Direct consequences of (4.4.1a) and (4.4.1d) are the following:

$$\begin{aligned} |\partial^I \Omega^J \partial u| + |\partial \partial^I \Omega^J u| &\lesssim C_1 \varepsilon t^{-1/2} s^{-1}, && |I| + |J| \le N_0 - 3, \\ |\partial^I \Omega^J v| &\lesssim C_1 \varepsilon t^{-3/2} s^{(|J|+2)\delta}, && |I| + |J| \le N_0 - 2. \end{aligned} \tag{4.4.2}$$

These follow from the Sobolev–type inequality of lemma 3.15 and estimates for commutators in lemma 3.25.

Note the bootstrap assumption (4.4.1g) allows us to avoid the $s^{2\delta}$ loss appearing in the second estimate of (4.4.2) for $|\partial^I \Omega^J v|$. This is however at the expense of using $(C_1\varepsilon)^{1/2}$ instead of $(C_1\varepsilon)$. In proposition 4.18 we will improve (4.4.1g) using estimates on the fundamental solution taken from proposition 4.10. In this process we obtain first-order factors of $CC_1\varepsilon$, not $C(C_1\varepsilon)^2$, which is why we can only assume $(C_1\varepsilon)^{1/2}$ to begin with in (4.4.1g).

Assumptions (4.4.1a)–(4.4.1c) also imply the following L^2 estimates

$$\begin{aligned} \|(s/t)\partial^I \Omega^J \partial u\|_{L^2(\mathcal{H}_s)} + \|(s/t)\partial \partial^I \Omega^J u\|_{L^2(\mathcal{H}_s)} &\lesssim C_1 \varepsilon, && |I| + |J| \le N_0 - 1, \\ \|(s/t)\partial^I \partial u\|_{L^2(\mathcal{H}_s)} + \|(s/t)\partial \partial^I u\|_{L^2(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{\delta}, && |I| = N_0, \\ \|(s/t)\partial^I \Omega^J \partial u\|_{L^2(\mathcal{H}_s)} + \|(s/t)\partial \partial^I \Omega^J u\|_{L^2(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{|J|\delta}, && |I| + |J| = N_0, |J| \ge 1. \end{aligned} \tag{4.4.3}$$

4.5 Refined estimates for the Klein-Gordon component

4.5.1 Refined energy estimates for v

We show here the refined estimates for the Klein-Gordon equation (4.1.1b), and we see that the most difficult part is to get a refined estimate for $|\partial^I u|$. The difficulty comes because the integral of

$$\int_2^s \tau^{-1}\, \mathrm{d}\tau$$

is not uniformly bounded in s. We can circumvent it by moving the nonlinear term uv in the Klein-Gordon equation in (4.1.1) to the left hand side, however this requires improved L^2 and L^∞ bounds on the wave component u.

Lemma 4.13 (Commutator estimates with $1-u$ mass)**.** *Under the assumptions* (4.4.1) *we have*

$$\big\|[1-u, \partial^I \Omega^J]v\big\|_{L^2(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^{3/2} s^{-1+|J|\delta}, \quad |I| + |J| \le N_0, \tag{4.5.1}$$
$$\big\|[1-u, \partial^I]v\big\|_{L^2(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^{3/2} s^{-3/2}, \quad |I| \le N_0. \tag{4.5.2}$$

Proof. We first prove (4.5.1) and note the expansion of the commutator

$$[1-u, \partial^I\Omega^J]v = \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\geq 1}} \partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}v. \tag{4.5.3}$$

For the case of $|J|\geq 1$, we have the following

$$\begin{aligned}
&\big\|[1-u, \partial^I\Omega^J]v\big\|_{L^2(\mathcal{H}_s)}\\
\lesssim &\sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\geq |I_2|+|J_2|}} \|(s/t)\partial^{I_1}\Omega^{J_1}u\|_{L^2(\mathcal{H}_s)}\|(t/s)\partial^{I_2}\Omega^{J_2}v\|_{L^\infty(\mathcal{H}_s)}\\
+ &\sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ 1\leq |I_1|+|J_1|\leq |I_2|+|J_2|}} \|\partial^{I_1}\Omega^{J_1}u\|_{L^\infty(\mathcal{H}_s)}\|\partial^{I_2}\Omega^{J_2}v\|_{L^2(\mathcal{H}_s)},
\end{aligned} \tag{4.5.4}$$

and the L^2–type estimates for u in (4.4.1) verify

$$\begin{aligned}
\big\|[1-u, \partial^I\Omega^J]v\big\|_{L^2(\mathcal{H}_s)} \lesssim \sum_{J_1+J_2=J} &C_1\varepsilon s^{1/2+|J_1|\delta}(C_1\varepsilon)^{1/2}t^{-1/2}s^{-1+|J_2|\delta}\\
&+ C_1\varepsilon t^{-1}s^{|J_1|\delta}C_1\varepsilon s^{|J_2|\delta},
\end{aligned} \tag{4.5.5}$$

which leads to (4.5.1). For the proof of (4.5.2) with $|I|\geq 1$, we proceed in the same way

$$[1-u, \partial^I]v = \sum_{\substack{I_1+I_2=I\\ |I_1|\geq 1}} \partial^{I_1}u\partial^{I_2}v. \tag{4.5.6}$$

We note that there exists at least one derivative hitting the wave component u, and use the fact that

$$\begin{aligned}
\big\|(s/t)\partial^{I_1}u\big\|_{L^2(\mathcal{H}_s)} &\lesssim C_1\varepsilon, \qquad & 1\leq |I_1|\leq N_0,\\
\big\|(s/t)\partial^{I_1}u\big\|_{L^\infty(\mathcal{H}_s)} &\lesssim C_1\varepsilon t^{-3/2}, \qquad & 1\leq |I_1|\leq N_0-4.
\end{aligned} \tag{4.5.7}$$

Then we have

$$\begin{aligned}
\big\|[1-u, \partial^I]v\big\|_{L^2(\mathcal{H}_s)} \lesssim &\sum_{\substack{I_1+I_2=I\\ |I_1|\geq 1, |I_1|\geq |I_2|}} \|(s/t)\partial^{I_1}u\|_{L^2(\mathcal{H}_s)}\|(t/s)\partial^{I_2}v\|_{L^\infty(\mathcal{H}_s)}\\
&+ \sum_{\substack{I_1+I_2=I\\ 1\leq |I_1|\leq |I_2|}} \|\partial^{I_1}u\|_{L^\infty(\mathcal{H}_s)}\|\partial^{I_2}v\|_{L^2(\mathcal{H}_s)}\\
&\lesssim (C_1\varepsilon)^{3/2}s^{-1}t^{-1/2}\\
&\lesssim (C_1\varepsilon)^{3/2}s^{-3/2}.
\end{aligned} \tag{4.5.8}$$

Finally, since $[1-u, \partial^I\Omega^J] = [1-u, id] = 0$ when $|I| = |J| = 0$, the proof is complete. ☐

Proposition 4.14 (Refined energy estimates for v)**.** *Under the bootstrap assumptions*

(4.4.1) *we have*

$$\mathcal{E}_1(s, \partial^I \Omega^J v)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2} s^{|J|\delta}, \quad |I| + |J| \le N_0. \tag{4.5.9}$$

Proof. We first act $\partial^I \Omega^J$ on the Klein-Gordon equation in (4.1.1b) to get

$$\Box \partial^I \Omega^J v - (1-u)\partial^I \Omega^J v = - \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\ge 1}} \partial^{I_1}\Omega^{J_1} u \partial^{I_2}\Omega^{J_2} v. \tag{4.5.10}$$

We then apply the energy estimate (4.2.5) for Klein-Gordon equations with varying masses and use lemma 4.13 to show

$$\begin{aligned}
&\mathcal{E}_1(s, \partial^I \Omega^J v)^{1/2} \\
&\le 2\mathcal{E}(2, \partial^I \Omega^J v)^{1/2} + 2\int_2^s \|(\tau/t)\partial_t u \partial^I \Omega^J v\|_{L^2(\mathcal{H}_\tau)} + \|[1-u, \partial^I \Omega^J] v\|_{L^2(\mathcal{H}_\tau)} \mathrm{d}\tau \\
&\lesssim \varepsilon + \int_2^s \Big(\|\partial_t u\|_{L^\infty(\mathcal{H}_\tau)} \|\partial^I \Omega^J v\|_{L^2(\mathcal{H}_\tau)} + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\ge 1}} \|\partial^{I_1}\Omega^{J_1} u \partial^{I_2}\Omega^{J_2} v\|_{L^2(\mathcal{H}_\tau)} \Big) \,\mathrm{d}\tau.
\end{aligned} \tag{4.5.11}$$

Successively, in the case of $|J| \ge 1$, it is true that

$$\mathcal{E}_1(s, \partial^I \Omega^J v)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2} \int_2^s \tau^{-1+|J|\delta} \,\mathrm{d}\tau \lesssim \varepsilon + (C_1\varepsilon)^{3/2} s^{|J|\delta}, \tag{4.5.12}$$

while in the case of $|J| = 0$, better estimates on $\partial^{I_1} u$ with $|I_1| \ge 1$ enable us to obtain

$$\mathcal{E}_1(s, \partial^I \Omega^J v)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2} \int_2^s \tau^{-3/2+\delta} \,\mathrm{d}\tau \lesssim \varepsilon + (C_1\varepsilon)^{3/2}, \tag{4.5.13}$$

which finishes the proof. □

4.5.2 Refined pointwise estimates for v

We now prove refined sup-norm bounds for the Klein-Gordon field v which, due to proposition 4.10, requires some preliminary lemmas.

Lemma 4.15. *Under the bootstrap assumptions* (4.4.1) *the solution u to our wave equation satisfies*

$$e^{\int_{s_0}^s |\frac{d}{d\lambda} u(\lambda t/s, \lambda x/s)| \, d\lambda} \lesssim 1. \tag{4.5.14}$$

Proof. We observe that

$$\frac{d}{d\lambda} u(\lambda t/s, \lambda x/s) = (t/s) Y^\perp u(\lambda t/s, \lambda x/s), \tag{4.5.15}$$

where we recall $Y^\perp = \partial_t + \frac{x^a}{t}\partial_a$ from definition 3.5. Note also the identity

$$Y^\perp u(t,x) = \frac{s^2}{t^2}\partial_t u(t,x) + \frac{x^a}{t^2}\Omega_a u(t,x). \tag{4.5.16}$$

Hence by using the pointwise bootstrap (4.4.1f) of u that

$$|\Omega_a u(t,x)| \le C_1 \varepsilon t^{-1} s^\delta, \tag{4.5.17}$$

we find

$$\big|(t/s)Y^{\perp}u(t,x)\big| \lesssim C_1\varepsilon s^{-3/2}. \tag{4.5.18}$$

This implies that

$$\Big|\frac{d}{d\lambda}u(\lambda t/s, \lambda x/s)\Big| \lesssim C_1\varepsilon\lambda^{-3/2}, \tag{4.5.19}$$

and the result now follows easily. $\square$

Lemma 4.16 ([LM16a, Lemma 7.3])**.** *Under the bootstrap assumptions* (4.4.1) *we have the following estimate for $R[\partial^I\Omega^J v]$ in the region $\mathcal{C}_{[2,s_1)}$*

$$\big|R[\partial^I\Omega^J v](\lambda t/s, \lambda x/s)\big| \lesssim C_1\varepsilon(s/t)^{3/2}\lambda^{-3/2+N_0\delta}, \quad |I|+|J| \le N_0-4. \tag{4.5.20}$$

One last ingredient is the commutator estimate stated below.

Lemma 4.17 (Pointwise commutator estimate with $1-u$ mass)**.** *Under the bootstrap assumptions* (4.4.1) *the following estimates hold*

$$\big|\big([1-u, \partial^I\Omega^J]v\big)(\lambda t/s, \lambda x/s)\big| \lesssim (C_1\varepsilon)^{3/2}(s/t)^{5/2}\lambda^{-5/2+|J|\delta}, \quad |I|+|J| \le N_0-4. \tag{4.5.21}$$

Moreover, in the case of $|J|=0$, we have

$$\big|\big([1-u, \partial^I]v\big)(\lambda t/s, \lambda x/s)\big| \lesssim (C_1\varepsilon)^{3/2}(s/t)^2\lambda^{-3}, \quad |I| \le N_0-4. \tag{4.5.22}$$

Proof. In order to show (4.5.21), first use the expansion of the commutator

$$[1-u, \partial^I\Omega^J]v = \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\ge 1}} \partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}v. \tag{4.5.23}$$

Next we use the pointwise estimates in (4.4.1) to find

$$\begin{aligned} |\big([1-u, \partial^I\Omega^J]v\big)(t,x)| &\lesssim \sum_{J_1+J_2=J} C_1\varepsilon t^{-1}s^{|J_1|\delta}(C_1\varepsilon)^{1/2}t^{-3/2}s^{|J_2|\delta} \\ &\lesssim (C_1\varepsilon)^{3/2}t^{-5/2}s^{|J|\delta} \\ &= (C_1\varepsilon)^{3/2}(s/t)^{5/2}s^{-5/2+|J|\delta}, \end{aligned} \tag{4.5.24}$$

which finishes the proof of (4.5.21).

For the proof of (4.5.22), we have

$$[1-u, \partial^I\Omega^J]v = \sum_{\substack{I_1+I_2=I\\ |I_1|\ge 1}} \partial^{I_1}u\partial^{I_2}v, \tag{4.5.25}$$

and note that there is at least one partial derivative hitting the wave component u, which is good. We proceed in the same way as before but use instead the estimate (4.4.2) to find

$$|\partial^{I_1}u| \lesssim C_1\varepsilon t^{-1/2}s^{-1}, \quad 1 \le |I_1| \le N_0-4. \tag{4.5.26}$$

This allows us to conclude that

$$|\big([1-u, \partial^I]v\big)(t,x)| \lesssim (C_1\varepsilon)^{3/2}t^{-2}s^{-1} = (C_1\varepsilon)^{3/2}(s/t)^2s^{-3}. \tag{4.5.27}$$

□

We are now in a position to give the proof of the refined sup-norm bounds for the Klein-Gordon field.

Proposition 4.18 (Refined pointwise estimates for v)**.** *Under the bootstrap assumptions* (4.4.1) *the following estimates hold*

$$\big|\partial^I\Omega^J v\big| + \big|(t/s)Y^\perp\partial^I\Omega^J v\big| \lesssim C_1\varepsilon t^{-3/2}s^{|J|\delta}, \quad |I|+|J|\le N_0-4. \tag{4.5.28}$$

Proof. We act $\partial^I\Omega^J$ on the Klein-Gordon equation in (4.1.1) to get

$$\Box\partial^I\Omega^J v - (1-u)\partial^I\Omega^J v = -[1-u,\partial^I\Omega^J]v. \tag{4.5.29}$$

In order to bound the quantity

$$\big|\partial^I\Omega^J v\big| + \big|(t/s)Y^\perp\partial^I\Omega^J v\big|, \tag{4.5.30}$$

we now apply the pointwise estimates from proposition 4.10. We need to bound the term $V(t,x)$ in (4.3.7). According to the definition of $V(t,x)$, we have to bound

$$F(s) \le \int_{s_0}^{s}\Big(\big|R[\partial^I\Omega^J v](\lambda t/s,\lambda x/s)\big| + \lambda^{3/2}\big|[1-u,\partial^I\Omega^J]v\big|(\lambda t/s,\lambda x/s)\Big)\,d\lambda, \tag{4.5.31}$$

in which $F(s)$ was defined in (4.3.10) in proposition 4.10, and we have estimated

$$e^{\int_{s_0}^{s}|\frac{d}{d\lambda}u(\lambda t/s,\lambda x/s)|\,d\lambda} \lesssim 1, \tag{4.5.32}$$

using lemma 4.15. Then by using the estimate (4.5.20) and the commutator estimates (4.5.22) from lemma 4.16 and lemma 4.17, we have

$$|F(s)| \lesssim C_1\varepsilon(s/t)^{3/2}s^{|J|\delta}. \tag{4.5.33}$$

Using proposition 4.10 we are led to the desired result

$$\big|\partial^I\Omega^J v\big| + \big|(t/s)Y^\perp\partial^I\Omega^J v\big| \lesssim s^{-3/2}V(t,x) \lesssim s^{-3/2}|F| \lesssim C_1\varepsilon t^{-3/2}s^{|J|\delta}. \tag{4.5.34}$$

□

Corollary 4.19 (Pointwise estimate on ∂v)**.** *Under the bootstrap assumptions* (4.4.1) *the following estimates hold*

$$|\partial\partial^I\Omega^J v| \lesssim C_1\varepsilon t^{-1/2}s^{-1+|J|\delta}, \quad |I|+|J|\le N_0-4, \tag{4.5.35}$$

Proof. The proof follows by using proposition 4.18 together with the following identities

$$\partial_t = \frac{t^2}{s^2}\big(Y^\perp - (x^a/t)Y_a\big), \qquad \partial_a = -\frac{tx^a}{s^2}Y^\perp + \frac{x^a x^b}{t^2}Y_b + Y_a. \tag{4.5.36}$$

□

4.6 Refined estimates for the wave component

4.6.1 Transformation of u

If we deal directly with the nonlinearity uv for the wave equation in (4.1.1a), it is not possible to close our bootstrap assumptions. Due to this difficulty, we perform two transformations of our variable u. In our first transformation we 'split' the equation (4.1.1a) into two wave equations (4.6.1). This is inspired by a method introduced by Katayama [Kat12]. In our second transformation we seek a new unknown which satisfies a wave equation with good nonlinearities and which is 'close' to the original wave component u up higher order terms. This is inspired by a method used to study *wave–wave* interactions introduced by Tsutsumi [Tsu03a], and is used also in Chapter 3.

Definition 4.20 (Transformation I)**.** Let (u, v) be a solution to the model problem (4.1.1) then we can split u into the following form

$$u = U_1 + \partial_t U_2, \tag{4.6.1}$$

in which U_1 and U_2 are solutions to the two wave equations below:

$$\begin{aligned} \Box U_1 &= -uv + v\partial_t u, \\ \big(U_1, \partial_t U_1\big)(2, \cdot) &= \big(\phi_0, \phi_1 + \phi_0 \Psi_0\big), \end{aligned} \tag{4.6.2}$$

and

$$\begin{aligned} \Box U_2 &= -uv, \\ \big(U_2, \partial_t U_2\big)(2, \cdot) &= \big(0, 0\big). \end{aligned} \tag{4.6.3}$$

Next, we do a transformation to make the nonlinearities in the U_1 and U_2 equations easier to deal with.

Definition 4.21 (Transformation II)**.** Consider the wave equations of U_1 and U_2 in definition 4.20, and set

$$\widetilde{U}_1 = U_1 + uv, \qquad \widetilde{U}_2 = U_2 + uv.$$

Then the new unknowns $\widetilde{U}_1$ and $\widetilde{U}_2$ satisfy wave equations with nonlinearities $F_{\widetilde{U}_1}, F_{\widetilde{U}_2}$, which are easy to handle.

$$\Box \widetilde{U}_1 = F_{\widetilde{U}_1} = \partial^\alpha u \partial_\alpha v + v\partial_t u - u^2 v - uv^2 - uv\partial_t v, \tag{4.6.4a}$$

$$\Box \widetilde{U}_2 = F_{\widetilde{U}_2} = \partial^\alpha u \partial_\alpha v - u^2 v - uv^2 - uv\partial_t v. \tag{4.6.4b}$$

The derivation of (4.6.4) follows by simple calculations. We only do it for $\widetilde{U}_2$

$$\Box \widetilde{U}_2 = \Box(U_2 + uv) = \Box U_2 + \partial^\alpha u \partial_\alpha v + (\Box u)v + u(\Box v + v) + uv, \tag{4.6.5}$$

then by utilising the equations in (4.6.3), we finally arrive at (4.6.4b).

The following lemma allows us to measure L^2-norms for U_p using energies for $\widetilde{U}_p$.

Proposition 4.22 (Energy equivalence)**.** *Let the bootstrap assumptions in* (4.4.1) *hold. Then for $p = 1, 2$ and $|I| + |J| \le N_0$, we have, for ε sufficiently small,*

$$\mathcal{E}(s, \partial^I \Omega^J \widetilde{U}_p)^{1/2} \simeq \mathcal{E}(s, \partial^I \Omega^J U_p)^{1/2}, \tag{4.6.6a}$$

and

$$
\begin{aligned}
\frac{1}{2}\mathcal{E}_{con}(s, \partial^I\Omega^J U_p)^{1/2} - (C_1\varepsilon)^{3/2}s^{1/2} &\leq \mathcal{E}_{con}(s, \partial^I\Omega^J \widetilde{U}_p)^{1/2} \\
&\leq 2\mathcal{E}_{con}(s, \partial^I\Omega^J U_p)^{1/2} + 2(C_1\varepsilon)^{3/2}s^{1/2}. \quad (4.6.6b)
\end{aligned}
$$

For $|I| + |J| \leq N_0 - 4$, we have

$$
|\partial^I\Omega^J(U_p - \widetilde{U}_p)| \leq (C_1\varepsilon)^{3/2}t^{-3/2}. \tag{4.6.7}
$$

Proof. The proof follows by the fact that the difference between U_p and $\widetilde{U}_p$ is a quadratic term uv, which has good decay properties. Using Young's inequality we obtain

$$
\begin{aligned}
\mathcal{E}(s, \partial^I\Omega^J\widetilde{U}_1) &= \big\|(s/t)\partial_t\partial^I\Omega^J\widetilde{U}_1\big\|^2_{L^2(\mathcal{H}_s)} + \sum_a \big\|Y_a\partial^I\Omega^J\widetilde{U}_1\big\|^2_{L^2(\mathcal{H}_s)} \\
&\leq 2\big\|(s/t)\partial_t\partial^I\Omega^J U_1\big\|^2_{L^2(\mathcal{H}_s)} + 2\sum_a \big\|Y_a\partial^I\Omega^J U_1\big\|^2_{L^2(\mathcal{H}_s)} \\
&\quad + 2\big\|(s/t)\partial_t\partial^I\Omega^J (uv)\big\|^2_{L^2(\mathcal{H}_s)} + 2\sum_a \big\|Y_a\partial^I\Omega^J (uv)\big\|^2_{L^2(\mathcal{H}_s)} \\
&\leq 2\mathcal{E}(s, \partial^I\Omega^J U_1) + 2(C_1\varepsilon)^3.
\end{aligned} \tag{4.6.8}
$$

Similarly

$$
\mathcal{E}_{\text{con}}(s, \partial^I\Omega^J\tilde{U}_p)^{1/2} \leq 2\mathcal{E}_{\text{con}}(s, \partial^I\Omega^J U_p)^{1/2} + \mathcal{E}_{\text{con}}(s, \partial^I\Omega^J(uv))^{1/2}. \tag{4.6.9}
$$

One of the most problematic term in $\mathcal{E}_{\text{con}}(s, \partial^I\Omega^J(uv))^{1/2}$ is estimated, using $|x^a| \leq r \leq t$, as

$$
\begin{aligned}
\sum_{\substack{I_i+J_i=I+J \\ |I_2|+|J_2|\geq 1}} \|\partial^{I_1}\Omega^{J_1}u\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}\Omega^{J_2}x^aY_av\big\|_{L^2(\mathcal{H}_s)} &\lesssim \sum_{\substack{I_i+J_i=I+J \\ |I_2|+|J_2|\geq 1}} \varepsilon t^{-1}s^{\delta|J_1|}t\big\|\partial^{I_2}\Omega^{J_2}Y_av\big\|_{L^2(\mathcal{H}_s)} \\
&\lesssim \varepsilon^2 s^{2\delta N_0}.
\end{aligned}
$$

□

4.6.2 Estimates of U_1

We now derive various estimates for U_1 based on the analysis of the unknown $\widetilde{U}_1$ and its nonlinearities $F_{\widetilde{U}_1}$.

Lemma 4.23 (Estimate for $v\partial_t u$ nonlinearity)**.** *Under the bootstrap assumptions* (4.4.1)

$$
\big\|\partial^I\Omega^J(v\partial_t u)\big\|_{L^2(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^{3/2}s^{-3/2+|J|\delta}, \quad |I| + |J| \leq N_0, \tag{4.6.10}
$$

$$
\big|\partial^I\Omega^J(v\partial_t u)\big| \lesssim (C_1\varepsilon)^{3/2}t^{-2}s^{-1+|J|\delta}, \quad |I| + |J| \leq N_0 - 4. \tag{4.6.11}
$$

Proof. We directly do the estimates

$$
\begin{aligned}
\big\|\partial^I\Omega^J(v\partial_t u)\big\|_{L^2(\mathcal{H}_s)} &\leq \sum_{\substack{I_1+I_2=I \\ J_1+J_2=J}} \big\|\partial^{I_1}\Omega^{J_1}\partial_t u\partial^{I_2}\Omega^{J_2}v\big\|_{L^2(\mathcal{H}_s)} \\
&\leq \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\leq|I_2|+|J_2|}} \|\partial^{I_1}\Omega^{J_1}\partial_t u\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}\Omega^{J_2}v\big\|_{L^2(\mathcal{H}_s)}
\end{aligned}
$$

$$+ \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\geq|I_2|+|J_2|}} \big\|(s/t)\partial^{I_1}\Omega^{J_1}\partial_t u\|_{L^2(\mathcal{H}_s)}\big\|(t/s)\partial^{I_2}\Omega^{J_2}v\big\|_{L^\infty(\mathcal{H}_s)},$$

and use (4.4.2) and (4.4.3). For the sup-norm bound, we note that

$$\big|\partial^I\Omega^J(v\partial_t u)\big| \leq \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big|\partial^{I_1}\Omega^{J_1}\partial_t u\partial^{I_2}\Omega^{J_2}v\big|, \tag{4.6.12}$$

and then use the bootstrap assumptions (4.4.1) as well as the pointwise estimates (4.4.2) for $\partial^{I_1}\Omega^{J_1}\partial_t u$. ☐

Lemma 4.24 (L^2 estimate for $\tilde{U}_1$ nonlinearity)**.** *Under the bootstrap assumptions* (4.4.1) *we have, for* $|I|+|J|\leq N_0$*,*

$$\big\|\partial^I\Omega^J F_{\widetilde{U}_1}\big\|_{L^2(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^{3/2}s^{-3/2+|J|\delta}, \tag{4.6.13}$$

as well as, for $|I|+|J|\leq N_0-4$*,*

$$\big|\partial^I\Omega^J F_{\widetilde{U}_1}\big| \lesssim (C_1\varepsilon)^{3/2}t^{-2}s^{-1+|J|\delta}. \tag{4.6.14}$$

Proof. From (4.6.4a) we see that the terms to be estimated are either null forms, terms of the form $\partial^I\Omega^J(\partial_t uv)$ or cubic terms. The term $\partial^I\Omega^J(\partial_t uv)$ is treated in lemma 4.23. Cubic terms behave very nicely, see Theorem 1.24, and we use lemma 3.24 and corollary 4.19 to treat the null forms. ☐

Proposition 4.25 (Energy estimates for U_1)**.** *Consider the wave equation in* (4.6.2) *and assume the bounds in* (4.4.1) *hold, then we have the following energy estimates for* U_1

$$\mathcal{E}(s,\partial^I\Omega^J U_1)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}, \quad |I|+|J|\leq N_0. \tag{4.6.15}$$

Proof. Firstly, by (4.6.6), we know

$$\mathcal{E}(2,\partial^I\Omega^J\widetilde{U}_1)^{1/2} \leq 2\varepsilon.$$

Then the estimate follows by combining the energy estimates (4.2.1) for wave equations with (4.6.13). By using the equivalence relation (4.6.6) between U_1 and $\widetilde{U}_1$ we complete the proof. ☐

The ideas of the proofs for the two propositions below are very similar to the one above, i.e. we can get good estimates for the auxiliary unknown $\widetilde{U}_1$ easily, and then an application of the equivalence relation (4.6.6) in turn gives us good estimates of the unknown U_1. Given this close similarity, we omit the proofs.

Lemma 4.26 (Conformal-type energy estimates for U_1)**.** *Under the bootstrap assumptions* (4.4.1) *the conformal-type energy introduced in Subsection 4.2.1 satisfies for* $|I|+|J|\leq N_0$

$$\mathcal{E}_{con}(s,\partial^I\Omega^J U_1)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{1/2+|J|\delta}. \tag{4.6.16}$$

Thus for $|I|+|J|\leq N_0$ *we have*

$$\big\|(s/r)\partial^I\Omega^J U_1\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{1/2+|J|\delta}. \tag{4.6.17}$$

Lemma 4.27 (Pointwise estimates for U_1)**.** *Under the bootstrap assumptions* (4.4.1) *we have*

$$|\partial^I \Omega^J U_1| \lesssim \big(\varepsilon + (C_1\varepsilon)^{3/2}\big) t^{-1} s^{|J|\delta}, \quad |I| + |J| \leq N_0 - 4. \tag{4.6.18}$$

The proof of this lemma clearly follows from lemma 4.8 and the sup-estimate obtained in (4.6.14).

4.6.3 Estimates of U_2

We state the following propositions about estimates of U_2, but we do not provide proofs as they are either the same as or easier than those of U_1.

Lemma 4.28 (Energy estimates for U_2)**.** *Under the bootstrap assumptions* (4.4.1) *we have*

$$\mathcal{E}(s, \partial^I \Omega^J U_2)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}, \quad |I| + |J| \leq N_0. \tag{4.6.19}$$

As a consequence, this lemma gives for $|I| + |J| \leq N_0$,

$$\big\|(s/t)\partial_t \partial^I \Omega^J U_2\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial^I \Omega^J \partial_t U_2\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}. \tag{4.6.20}$$

Lemma 4.29 (Pointwise estimates for U_2)**.** *Under the bootstrap assumptions* (4.4.1) *we have*

$$|\partial_t \partial^I \Omega^J U_2| + |\partial^I \Omega^J \partial_t U_2| \lesssim \big(\varepsilon + (C_1\varepsilon)^{3/2}\big) t^{-1/2} s^{-1}, \quad |I| + |J| \leq N_0 - 4. \tag{4.6.21}$$

The proof of this lemma clearly follows from lemma 4.8 and the Sobolev embedding of lemma 3.15.

4.6.4 Refined estimates for u

We are ready to derive the refined estimates for u, which are based on the analysis of the unknowns U_p. To clarify the role played by the U_p and $\widetilde{U}_p$ $(p = 1, 2)$ unknowns, we revisit the difficulties in estimating directly the original wave component u. The nonlinearities in the u equation are $F_u = uv + u\partial_t v$, and the energy does not decay sufficiently fast, i.e.

$$\begin{aligned}\|F_u\|_{L^2(\mathcal{H}_s)} &\lesssim \min\big\{\|(s/t)u\|_{L^2(\mathcal{H}_s)}\|(t/s)v, (t/s)\partial_t v\|_{L^\infty(\mathcal{H}_s)}, \|u\|_{L^\infty(\mathcal{H}_s)}\|v, \partial_t v\|_{L^2(\mathcal{H}_s)}\big\} \\ &\lesssim s^{-1}.\end{aligned}$$

This tells us that closing the bootstrap assumptions on the natural wave energy $\mathcal{E}(s, u)^{1/2}$ is critical, and even worse, closing the bootstrap assumptions on $\|(s/t)u\|_{L^2(\mathcal{H}_s)}$ using the energy estimate (4.2.11) is far from possible.

So how do the nonlinear transformations help? It is easier to explain if we look the procedure backward. First, we find the wave equations for $\widetilde{U}_p$ have good nonlinearities ((4.6.4a) and (4.6.4b)) which are possible to control. Next, the difference between the unknowns U_p and $\widetilde{U}_p$ is a simple quadratic term uv, which indicates that all of the estimates valid for $\widetilde{U}_p$ are also true for U_p (more precisely $U_p/2$), and we can bound U_p using $\widetilde{U}_p$. Finally, we can control the original wave component u by the relation $u = U_1 + \partial_t U_2$.

Proposition 4.30 (Refined energy estimates for u)**.** *Consider the wave equation in* (4.1.1) *and assume the bounds in* (4.4.1) *hold, then we have the following refined ones*

$$
\begin{aligned}
\mathcal{E}(s,\partial^I\Omega^J u)^{1/2} &\lesssim \varepsilon + (C_1\varepsilon)^{3/2}, && |I|+|J|\le N_0-1,\\
\mathcal{E}(s,\partial^I u)^{1/2} &\lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{\delta}, && |I| = N_0,\\
\mathcal{E}(s,\partial^I\Omega^J u)^{1/2} &\lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{|J|\delta}, && |I|+|J|\le N_0, |J|\ge 1.
\end{aligned}
\tag{4.6.22}
$$

Proof. For $|I|+|J|\le N_0-1$, we have

$$
\mathcal{E}(s,\partial^I\Omega^J u)^{1/2} \lesssim \mathcal{E}(s,\partial^I\Omega^J U_1)^{1/2} + \mathcal{E}(s,\partial^I\Omega^J\partial_t U_2)^{1/2}, \tag{4.6.23}
$$

then the energy estimates of U_1 and U_2 and the commutators give the desired result.

Next, for the case of $|I|+|J| = N_0$ with $|J|\ge 1$, we use the original equation in (4.1.1) and have

$$
\Box\partial^I\Omega^J u = \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \Big(\partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}v + \partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}\partial_t v\Big). \tag{4.6.24}
$$

Then by the energy estimates for wave components (4.2.1), it is true that

$$
\begin{aligned}
&\mathcal{E}(s,\partial^I\Omega^J u)^{1/2} && (4.6.25)\\
&\le \mathcal{E}(2,\partial^I\Omega^J u)^{1/2} + \int \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big\|\partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}v + \partial^{I_1}\Omega^{J_1}u\partial^{I_2}\Omega^{J_2}\partial_t v\big\|_{L^2(\mathcal{H}_\tau)}\,\mathrm{d}\tau. && (4.6.26)
\end{aligned}
$$

Successively, we arrive at

$$
\mathcal{E}(s,\partial^I\Omega^J u)^{1/2} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{|J|\delta}, \tag{4.6.27}
$$

which is based on the estimates we already have obtained. The case of $|I| = N_0$ can be treated in a similar way, and hence the proof is done. □

Proposition 4.31 (Refined L^2-type energy estimates for u)**.** *Under the bootstrap assumptions* (4.4.1) *we have the following*

$$
\big\|(s/t)\partial^I\Omega^J u\big\|_{L^2(\mathcal{H}_s)} \lesssim \varepsilon + (C_1\varepsilon)^{3/2}s^{1/2+|J|\delta}, \quad |I|+|J|\le N_0. \tag{4.6.28}
$$

Proof. We obtain

$$
\|(s/t)\partial^I\Omega^J u\|_{L^2(\mathcal{H}_s)} \lesssim \big\|(s/r)\partial^I\Omega^J U_1\big\|_{L^2(\mathcal{H}_s)} + \big\|(s/t)\partial^I\Omega^J\partial_t U_2\big\|_{L^2(\mathcal{H}_s)}, \tag{4.6.29}
$$

and finish the proof by using the estimates (4.6.17) and (4.6.20). □

Proposition 4.32 (Refined pointwise estimates for u)**.** *Under the bootstrap assumptions* (4.4.1) *we have the following pointwise estimates*

$$
|\partial^I\Omega^J u| \lesssim \big(\varepsilon + (C_1\varepsilon)^{3/2}\big)t^{-1}s^{|J|\delta}, \quad |I|+|J|\le N_0-4. \tag{4.6.30}
$$

Proof. It is true that

$$
|\partial^I\Omega^J u| \le |\partial^I\Omega^J U_1| + |\partial^I\Omega^J\partial_t U_2|, \tag{4.6.31}
$$

and the proof is done by the use of (4.6.18) and (4.6.21). □

4.7 Proof of the stability result and further remarks

Proof of theorem 4.1. By collecting together the refined estimates for wave and Klein-Gordon components, namely equations (4.5.9), (4.5.28), (4.6.22), (4.6.28) and (4.6.30), we choose large $C_1 \gg 1$ and sufficiently small $\varepsilon \ll 1$ such that $C_1\varepsilon \ll 1$. From this we have closed the bootstrap assumptions (4.4.1). $\square$

Chapter 5

Attractors of the Einstein–Klein-Gordon system

5.1 Introduction

In this chapter, we[1] are interested in the stability of cosmological spacetimes. If the cosmological constant is non-vanishing a large class of de Sitter type universes, including the Kerr de-Sitter black hole, are known to be stable [Fri86, Rin08, HV18] and also towards the singularity in certain cases [RS18].

In the case of vanishing cosmological constant and restricting to stability results towards the complete direction of spacetime, we are interested in the generalised Milne spacetime given in definition 1.52. The spacetime takes the form $((0,\infty)\times\mathcal{K},\overline{g})$ with metric

$$\overline{g} = -dt_c^2 + \frac{t_c^2}{9}\gamma_{ab}dx^a dx^b, \tag{5.1.1}$$

Here $\mathcal{K}$ is a closed Riemannian three-manifold admitting an Einstein metric γ with Einstein constant $k = -\frac{2}{9}$, see section 1.5.1. Note that even though $\mathcal{K}$ is compact we use indices $a,b \in \{1,2,3\}$ (instead of say $A,B \in \{1,2,3\}$) since this follow the convention established in [AM11]. Note also that in this chapter we use $\overline{g}$ to denote the physical metric which we will rescale shortly.

The Milne model is known to be a stable solution solution to the Einstein vacuum equations [AM11], the Einstein massive-Vlasov equations [AF20], the coupled Einstein-Maxwell-scalar field system arising from a Kaluza-Klein reduction [BFK19], and the Einstein Klein-Gordon equations [Wan19, FW19]. This chapter is devoted to presenting the work [FW19].

It is an open question whether the Milne spacetime is a stable solution to the Einstein-relativistic Euler equations. The author has shown the following result which proves the stability of a specific fluid on a fixed Milne-like cosmological spacetime. It is the first fluid stabilization result known when the spacetime expansion rate is non-accelerated.

Theorem 5.1 ([FOW20]). *The trivial solution of the irrotational relativistic Euler equations with equation of state $P = c_s^2\rho$, where $0 < c_s^2 < 1/3$, for small initial data in the expanding direction of FLRW spacetimes of the form $(\mathbb{R}\times\mathbb{T}^3, -\mathrm{d}t_c^2 + t_c^2\delta_{ab}dx^a dx^b)$ is stable.*

[1]The results in this chapter were obtained in collaboration with David Fajman at the University of Vienna. This research was supported in part by the Austrian Science Fund (FWF) project P29900-N27 'Geometric Transport equations and the non-vacuum Einstein-flow'.

Theorem 5.1 is known to fail in the end-point radiation case $c_s^2 = 1/3$ [Spe13] and on the Minkowski background [Chr07]. The method used to proved this result is somewhat complementary to what we have presented so far, hence the omission of the proof. Nevertheless one can view results in the present chapter on the EKGS as potentially useful for future study extending theorem 5.1 to the full Einstein-relativistic Euler equations about a Milne background.

A crucial difficulty that arises for massive matter models coupled to the Einstein equations for data close to the Milne model results from the slow decay of the lapse gradient. This is due to the matter quantity appearing in the elliptic equation for the lapse function ($\tau\eta$ in (5.2.15a)). Roughly speaking this implies that the decay of the gradient of the lapse, after rescaling, takes the form $\nabla N \approx \varepsilon e^{-T}$ where ε denotes the size of the initial perturbation. Then in the evolution of the L^2 energy of the Klein-Gordon field, the critical term at lowest order (see (5.6.10) for higher orders) reads

$$m^2 \nabla N \nabla \phi, \tag{5.1.2}$$

when written in rescaled variables. Given the coupling to the lapse gradient, this leads to a small growth of $e^{\varepsilon T}$ in the L^2 energy of the Klein-Gordon field. When the matter field couples back into the lapse equation (via $\tau\eta$) it reduces the decay of the gradient of the lapse to $\varepsilon e^{(-1+\varepsilon)T}$ and consequently one cannot close the straight-forward bootstrap argument.

The above issue was first observed for the Einstein-Vlasov system in [AF20] where the problem was overcome by using the continuity equation (see (5.5.2)), which gives a first order evolution equation for the energy density. Given the similarity between massive matter models we pursue this idea in the present chapter. In particular, we consider the rescaled energy density ρ, defined in (5.2.10), and correct it with a small indefinite term to obtain the corrected energy density

$$\hat{\rho} = \rho - \frac{1}{2}\tau^2 \phi \left(\frac{3}{2} N^{-1} \phi - \phi' \right), \tag{5.1.3}$$

where τ, ϕ and ϕ' are the mean-curvature, Klein-Gordon field and its time derivative (defined in Section 5.2.2). The corrected energy density fulfils an evolution equation, given in (5.5.10), with only time-integrable terms on the right-hand side. This equation then yields uniform pointwise bounds on the energy density and, in turn, on the Klein-Gordon field. This approach is sufficiently strong to close a bootstrap argument for the full system.

We remark that the main theorem of this chapter is also shown by work of Wang [Wan19], who uses the CMC-vanishing-shift gauge and Bel-Robinson-type energies (cf. [AM03]) to control the geometric perturbations. The issue of the slow decay of the lapse gradient is resolved therein by a hierarchy which is initiated at lowest order of regularity using an estimate for the Klein-Gordon field that has been proven in the work on Minkowski stability by LeFloch and Ma [LM18] but surprisingly applies in the cosmological setting as well.

By contrast to [Wan19] we work in CMC and spatially harmonic gauge. Consequently we have access to the energy-method of [AM11, AF20] based on the modified Einstein operator, see section 5.4.2, to control the perturbation of the geometry, which is significantly more concise than the one based on Bel-Robinson energies used in [AM03, Wan19]. We have access to this approach since we do not require the shift vector to vanish as our auxiliary estimate for the energy density, based on the continuity equation, is sufficiently robust to handle a non-vanishing shift vector field. In consequence, we obtain a significantly shorter proof of the nonlinear stability problem which avoids many of the technical details used in [Wan19].

Finally we remark that issue between the lapse and matter field decay does not occur for massless matter models. For example the spacetime $(0,\infty)\times\mathbb{S}^1\times\Sigma$ with metric $g_U = -\mathrm{d}t^2 + \mathrm{d}\theta^2 + t^2 g_\Sigma$ where (Σ, g_Σ) is an closed hyperbolic two-manifold is known to be future stable to $U(1)$-symmetric perturbations [CBM01, CB04]. The $U(1)$ symmetry leads, after a Kaluza-Klein reduction, to three-dimensional gravity with (massless) wave maps matter. Similarly the issue does not occur for the massless coupled Einstein-Maxwell-scalar field system arising from a Kaluza-Klein reduction studied about a Milne background in [BFK19].

Overview on the chapter. This chapter is organized as follows. Section 5.2 introduces notations and the fundamental equations which allow us to state the main theorem 5.12. Section 5.3 discusses the L^2-energies for the Klein-Gordon field. Section 5.4 states the bootstrap assumptions and introduces the L^2-energies for the perturbation of the geometry. Section 5.5 discusses the continuity equation and its modification for the Klein-Gordon field. This is the crucial part of our argument. In Section 5.6 the energy estimates for the Klein-Gordon field are derived. Section 5.7 introduces the elliptic estimates for lapse and shift. Section 5.8 discusses the hierarchy of decay for the lapse function and the Klein-Gordon field. Finally, Section 5.9 closes estimates for the shift and the perturbation of the geometry and ends the proof of Theorem 5.12.

5.2 Preliminaries

5.2.1 The Einstein–Klein-Gordon system

We consider a four-dimensional spacetime $((0,\infty)\times\mathcal{K},\bar{g})$ governed by the Einstein–Klein-Gordon system (EKGS)

$$\mathrm{Ric}[\bar{g}]_{\mu\nu} - \frac{1}{2}\mathrm{R}[\bar{g}]\bar{g}_{\mu\nu} = 2T_{\mu\nu}[\tilde{\phi}], \tag{5.2.1a}$$

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\tilde{\phi} = m^2\tilde{\phi}. \tag{5.2.1b}$$

where we have set $c = 1, 4\pi G = 1$ and the stress-energy tensor is given by

$$\tilde{T}_{\mu\nu}(\tilde{\phi}) = \bar{\nabla}_\mu\tilde{\phi}\bar{\nabla}_\nu\tilde{\phi} - \tfrac{1}{2}\bar{g}_{\mu\nu}\left(\bar{g}^{\rho\sigma}\bar{\nabla}_\rho\tilde{\phi}\bar{\nabla}_\sigma\tilde{\phi} + m^2\tilde{\phi}^2\right). \tag{5.2.2}$$

Note in this chapter we denote $\bar{\nabla}$ to be the Levi-Civita connection $\nabla[\bar{g}]$ with respect to $\bar{g}$. The constant $m > 0$ is the Klein-Gordon mass parameter.

5.2.2 Negative Einstein metrics, gauge choice and rescaled variables

The following setup is similar to earlier papers on the vacuum case or different matter models [AM11, AF20]. We state it briefly for the sake of completeness. Throughout the chapter let γ denote a fixed negative Einstein metric. We choose the constant $k = -2/9$ for convenience, but any negative Einstein metric can be treated in the same way. To model the dynamical spacetime $\bar{g}$ we use the ADM decomposition

$$\bar{g} = -\tilde{N}^2 dt^2 + \tilde{g}_{ab}(dx^a + \tilde{X}^a dt)(dx^b + \tilde{X}^b dt), \tag{5.2.3}$$

as given in definition 1.2. Note that $\bar{g}_{ab} = \tilde{g}_{ab}$ but in general $\bar{g}^{ab} \neq \tilde{g}^{ab}$.

Definition 5.2 (τ, Σ)**.** We let τ be the trace of the second fundamental form $\tilde{k}$ with respect to $\tilde{g}$ and define Σ to be the trace-free part of $\tilde{k}$. That is:

$$\tau = \mathrm{tr}_{\tilde{g}}\tilde{k}, \tag{5.2.4}$$
$$\tilde{k} = \tilde{\Sigma} + \tfrac{1}{3}\tau\tilde{g}. \tag{5.2.5}$$

We impose the CMCSH gauge, see also (1.8.12),

$$t = \tau, \qquad V^a = \tilde{g}^{cb}(\Gamma[\tilde{g}]^a_{cb} - \Gamma[\gamma]^a_{cb}) = 0. \tag{5.2.6}$$

We rescale the variables $(\tilde{g}, \tilde{\Sigma}, \tilde{N}, \tilde{X}, \tilde{\phi})$ with respect to mean curvature time $t = \tau$, calling the rescaled variables (g, Σ, N, X, ϕ). This coincides with earlier works, except for the Klein-Gordon field, which is rescaled here as follows. The rescaling is found using dimensional arguments, see for example [AM11, §1.2].

Definition 5.3 (Rescaled variables and new time coordinate)**.** Define the rescaled variables

$$\begin{aligned} g_{ab} &= \tau^2\tilde{g}_{ab}, & N &= \tau^2\tilde{N}, \\ g^{ab} &= (\tau^2)^{-1}\tilde{g}^{ab}, & \Sigma_{ab} &= \tau\tilde{\Sigma}_{ij}, \\ \phi &= -(-\tau)^{-3/2}\tilde{\phi}, & X^a &= \tau\tilde{X}^a. \end{aligned} \tag{5.2.7}$$

Define also the logarithmic time

$$T = -\ln(\tau/(e\tau_0)), \tag{5.2.8}$$

which satisfies $\partial_T = -\tau\partial_\tau$. Define also $\widehat{N} = \frac{N}{3} - 1$.

Note in vacuum the new time variable removes all explicit dependence on τ from the equations of motion, see [AM11, §4.4]. We have the following ranges $\tau_0 \leq \tau \nearrow 0$ and $1 \leq T \nearrow \infty$ where $\tau \nearrow 0$ corresponds to the direction of cosmological expansion.

Definition 5.4 (Matter variables and rescaling)**.** The energy density and energy current are defined by

$$\tilde{\rho} = \tilde{N}^2\tilde{T}^{00}\,,\; \tilde{\jmath}_a = \tilde{N}\tilde{T}^0{}_a, \tag{5.2.9}$$

respectively (note this is consistent with (1.0.5)). Furthermore $\tilde{\jmath}^a$ is defined by raising the index using $\tilde{g}^{ab}$. Following [AF20, (2.22)] we define the rescaled matter quantities:

$$\begin{aligned} \rho &= \tilde{\rho}(-\tau)^{-3}, & \eta &= (\tilde{\rho} + \tilde{g}^{ab}\tilde{T}_{ab})(-\tau)^{-3}, \\ \jmath^a &= \tilde{\jmath}^a(-\tau)^{-5}, & S_{ab} &= (\tilde{T}_{ab} - \tfrac{1}{2}\tilde{g}_{ab}\tilde{T})(-\tau)^{-1}, \\ T^{ab} &= \tilde{T}^{ab}(-\tau)^{-7}. \end{aligned} \tag{5.2.10}$$

Remark 5.5. Note here and throughout this chapter we let $\nabla = \nabla[g]$ denote the Levi-Civita connection for the rescaled 3-metric g, μ_g the volume form with respect to g, and $\Delta = g^{ab}\nabla_a\nabla_b$ the Laplacian with respect to this metric. Furthermore spatial indices for rescaled quantities are raised and lowered using the rescaled metric g. Thus $\jmath_a$ is defined using the rescaled metric g_{ab}, and moreover $\jmath_a$ scales as $\tilde{\jmath}_a(-\tau)^{-3}$.

For the Klein-Gordon field the matter quantities (5.2.10) are evaluated as

$$\rho = \tfrac{1}{2}m^2\phi^2 + \tfrac{\tau^2}{2}\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)^2 + \tfrac{1}{2}\tau^2 g^{ab}\nabla_a\phi\nabla_b\phi, \tag{5.2.11a}$$
$$\jmath^a = -\tau\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)g^{ab}\nabla_b\phi, \tag{5.2.11b}$$
$$\eta = -\tfrac{1}{2}m^2\phi^2 + 2\tau^2\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)^2, \tag{5.2.11c}$$

$$S_{ab} = \tfrac{1}{2}m^2\phi^2 g_{ab} + \tau^2\nabla_a\phi\nabla_b\phi, \tag{5.2.11d}$$

$$g_{ab}T^{ab} = -\tfrac{3}{2}\tau^{-2}m^2\phi^2 + \tfrac{3}{2}\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)^2 - \tfrac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi, \tag{5.2.11e}$$

where we have used the following notation (adapted from [CBM01])

$$\hat{\partial}_0 = \partial_T + \mathcal{L}_X\,, \quad \phi' = N^{-1}\hat{\partial}_0\phi. \tag{5.2.12}$$

Note that the unit normal vector acting on $\tilde{\phi}$ becomes, in the rescaled variables,

$$\tilde{N}^{-1}(\partial_\tau - \tilde{X}^a\partial_a)\tilde{\phi} = \tau^2 N^{-1}\cdot\tau^{-1}\hat{\partial}_0\big((-\tau)^{3/2}\phi\big) = (-\tau)^{5/2}\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big). \tag{5.2.13}$$

This is partly why we so often see the combination $\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)$ in our calculations.

Definition 5.6 (Rescaled EKGS in CMCSH gauge)**.** In the CMCSH gauge the constraint equations take the following form

$$\mathrm{R}(g) - |\Sigma|_g^2 + \tfrac{2}{3} = 4\tau\rho, \tag{5.2.14a}$$

$$\nabla^a\Sigma_{ab} = 2\tau^2\jmath_b, \tag{5.2.14b}$$

while the reduced and rescaled Einstein equations become

$$(\Delta - \tfrac{1}{3})N = N\left(|\Sigma|_g^2 - \tau\eta\right) - 1, \tag{5.2.15a}$$

$$\begin{aligned}\Delta X^a + \mathrm{Ric}[g]^a{}_bX^b &= 2\nabla_b N\Sigma^{ba} - \nabla^a\widehat{N} + 2N\tau^2\jmath^a \\ &\quad - (2N\Sigma^{bc} - \nabla^b X^c)(\Gamma[g]^a_{bc} - \Gamma[\gamma]^a_{bc}),\end{aligned} \tag{5.2.15b}$$

$$\partial_T g_{ab} = 2N\Sigma_{ab} + 2\widehat{N}g_{ab} - \mathscr{L}_X g_{ab}, \tag{5.2.15c}$$

$$\begin{aligned}\partial_T\Sigma_{ab} &= -2\Sigma_{ab} - N(\mathrm{Ric}[g]_{ab} + \tfrac{2}{9}g_{ab}) + \nabla_a\nabla_b N + 2N\Sigma_{ac}\Sigma^c_b \\ &\quad - \tfrac{1}{3}\widehat{N}g_{ab} - \widehat{N}\Sigma_{ab} - \mathscr{L}_X\Sigma_{ab} + N\tau S_{ab}.\end{aligned} \tag{5.2.15d}$$

Finally, the rescaled Klein-Gordon equation takes the form

$$\hat{\partial}_0\phi' = \nabla^a(N\nabla_a\phi) + (4-N)\phi' + \tfrac{3}{2}\phi - \tfrac{15}{4}N^{-1}\phi - \tfrac{3}{2}N^{-2}\phi\hat{\partial}_0 N - \tau^{-2}m^2N\phi. \tag{5.2.15e}$$

To derive (5.2.15e) it was convenient to move to a Cauchy adapted frame, see for example [CB09, VI§3]. To derive the other equations see [AF20]. The propagation of the gauge constraints (5.2.6) under (5.2.14)-(5.2.15) is known from [AM03]. This ends the setup of the EKGS in the CMCSH gauge with appropriate rescaling. In the following we work solely with these equations.

Remark 5.7 (Background solution)**.** With respect to these rescaled variables the background Milne solution to (5.2.15) becomes

$$(g_{ab}, \Sigma_{ab}, \widehat{N}, X_a, \phi) = (\gamma, 0, 0, 0, 0). \tag{5.2.16}$$

5.2.3 Local well-posedness and main theorem

Local existence theory is a prerequisite for addressing the global existence and stability problem for any Einstein-matter system. The local existence problem for the vacuum Einstein equations in CMCSH gauge was proven in [AM03]. We provide the corresponding result for the EKGS below. As it differs from the vacuum system only by coupling an additional nonlinear hyperbolic equation to the elliptic-hyperbolic system there is essentially no difference in the proof. One issue is however important to remark, which concerns the elliptic system. To preserve the crucial feature that the elliptic operators are isomorphisms we need to impose a smallness condition on the matter variables. This has been observed already in [Faj16] for collisionless matter and, for simplicity, turned into a smallness assumption for the full perturbation. Following the strategy of proof in [AM03] and making an additional smallness assumption analogously to that in [Faj16, Theorem 6.2] yields the following local-existence theorem for the EKGS.

Definition 5.8 (CMC initial data)**.** A CMC initial data set for the system (5.2.15) consists of the set $(\mathcal{K}, (g_0)_{ab}, (k_0)_{ab}, \phi_0, \phi_1)$ such that the following hold. $(\mathcal{K}, (g_0)_{ab})$ is a three-dimensional Riemannian manifold and $(k_0)_{ab}$ is a symmetric two-tensor field on $\mathcal{K}$ such that $\tau_0 = \mathrm{tr}_{g_0} k_0$ is constant. Furthermore ϕ_0, ϕ_1 are functions on $\mathcal{K}$ and, with the identification $\partial_T \phi = \phi_1$, the constraint equations (5.2.14) hold.

Lemma 5.9 (Local-existence theorem and continuation criterion in CMC time)**.** *There exists a $\delta > 0$ such that for any CMC initial data set $(\mathcal{K}, (g_0)_{ab}, (k_0)_{ab}, \phi_0, \phi_1)$ satisfying the smallness conditions*

$$\|g_0 - \gamma\|_{H^5} + \|\Sigma_0\|_{H^4} + \sqrt{|\tau_0|}(\|\phi_0\|_{H^5} + \|\phi_1\|_{H^4}) \leq \delta, \tag{5.2.17}$$

there exists a local-in-CMC-time solution on $[T_0, T)$, where $T > T_0$, to the reduced equations (5.2.15) *which agrees with the initial data at CMC time $\tau = \tau_0$ ($T_0 = 1$).*

Furthermore, let T_* be the supremum of all $T > T_0$ such that the corresponding solution exists up until T. Then either $T_* = +\infty$ or

$$\limsup_{T \to T_*} \|g - \gamma\|_{H^5} + \|\Sigma\|_{H^4} + \sqrt{|\tau_0|}(\|\phi_0\|_{H^5} + \|\phi_1\|_{H^4}) = \infty. \tag{5.2.18}$$

Remark 5.10. The mean-curvature factor in front of the Klein-Gordon field terms results from the lapse equation (5.2.15a), where this condition assures smallness of the corresponding matter term $-\tau\eta$. As η is quadratic in the rescaled Klein-Gordon field, each terms obtains a factor of $\sqrt{|\tau|}$.

Definition 5.11 (Function spaces)**.** For functions and symmetric tensor fields on $\mathcal{K}$ we denote the standard Sobolev norm with respect to the fixed metric γ of order $k \geq 0$ by $\|\cdot\|_{H^k}$. The corresponding function spaces are denoted by $H^k = H^k(\mathcal{K})$. We let $\mathscr{B}_\varepsilon^{j,k,l,m}(\gamma, \frac{1}{3}\gamma, 0, 0)$ denote the ball of radius ε in the space $H^j \times H^k \times H^l \times H^m$ centered at $(\gamma, \frac{1}{3}\gamma, 0, 0)$.

We can now formulate the main theorem.

Theorem 5.12. *Let $(\mathcal{K}, \gamma)$ be a negative, closed 3-dimensional Einstein manifold with Einstein constant $\mu = -2/9$. Let $\varepsilon > 0$ and $(g_0, k_0, \phi_0, \phi_1)$ be rescaled initial data at $T_0 = 1$ satisfying* (5.2.14) *and such that*

$$(g_0, k_0, \phi_0, \phi_1) \in \mathscr{B}_\varepsilon^{5,4,5,4}\left(\gamma, \frac{1}{3}\gamma, 0, 0\right). \tag{5.2.19}$$

Then, for ε sufficiently small the corresponding future development under the Einstein–Klein-Gordon system is future complete and the variables converge as

$$(g, k, \phi) \to \left(\gamma, \frac{1}{3}\gamma, 0\right) \text{ as } T \nearrow \infty, \tag{5.2.20}$$

with decay rates as given in (5.8.4) *and* (5.9.5)*. In particular, any four-dimensional*

Milne model is future asymptotically stable for the Einstein–Klein-Gordon equations in the class of initial data given above.

To prove theorem 5.12 one can show, using work developed in [FK], that the local development of non-CMC data sufficiently near a Milne background contains a CMC hypersurface. This in turn means that we can apply Lemma 5.9. Using certain bootstrap assumptions, given in (5.4.1), we can then establish global existence of the perturbed solution by proving decay estimates that assure, in particular, that the continuation criterion (5.2.18) is satisfied.

5.3 Energy functionals for the Klein-Gordon field

In this section we define the L^2-energy of the Klein-Gordon field in two steps. First, we define the natural L^2-norm of a massive scalar field. In the second step we modify this energy with two non-definite terms to obtain the corrected energy, which turns out to fulfil the desired energy estimate, which is derived later.

5.3.1 Natural energy

The following energy is the natural L^2-energy expressed in the rescaled variables.

Definition 5.13.

$$\begin{aligned} E_k(\phi) &= \int_{\mathcal{K}} \tau^2(-1)^k \left(\phi' \Delta^k \phi' - \phi \Delta^{k+1}\phi\right) \mathrm{d}\mu_g + \int_{\mathcal{K}} m^2 (-1)^k \phi \Delta^k \phi \mathrm{d}\mu_g, \\ \mathcal{E}_\ell(\phi) &= \sum_{k=0}^{\ell} E_k(\phi). \end{aligned} \tag{5.3.1}$$

Note that when $k = 2\ell + 1$ for $\ell \in \mathbb{Z}$, i.e. when k is odd, we can integrate by parts to obtain

$$\begin{aligned} E_{2\ell+1}(\phi) &= \int_{\mathcal{K}} \tau^2(-1) \left(\Delta^\ell \phi' \Delta^{\ell+1} \phi' - \phi \Delta^{k+1}\phi\right) \mathrm{d}\mu_g + \int_{\mathcal{K}} m^2 (-1) \Delta^\ell \phi \Delta^{\ell+1} \phi \, \mathrm{d}\mu_g, \\ &= \int_{\mathcal{K}} \tau^2 \left(g^{ab} \nabla[g]_a \Delta^\ell \phi' \cdot \nabla[g]_b \Delta^\ell \phi' + \Delta^{\ell+1}\phi \cdot \Delta^{\ell+1}\phi\right) \mathrm{d}\mu_g \\ &\quad + \int_{\mathcal{K}} m^2 g^{ab} \nabla[g]_a \Delta^\ell \phi \cdot \nabla[g]_b \Delta^\ell \phi \, \mathrm{d}\mu_g. \end{aligned} \tag{5.3.2}$$

The case when k is even is similar, and since g is Riemannian we have $E_k(\phi) \geq 0$.

We need the following lemma further below.

Lemma 5.14. *The following equivalence (denoted $\simeq$) holds*

$$\|\phi\|_{H^{k+2}} \simeq \|\Delta\phi\|_{H^k} + \|\phi\|_{L^2}, \tag{5.3.3}$$

for a sufficiently regular function ϕ. This implies

$$\|\phi\|_{H^k} \simeq \|\Delta^{\lfloor k/2 \rfloor}\phi\|_{L^2} + \|\nabla^{\mathring{k}} \Delta^{\lfloor k/2 \rfloor}\phi\|_{L^2} + \|\phi\|_{L^2}, \tag{5.3.4}$$

where $\mathring{k} = \lceil k/2 \rceil - \lfloor k/2 \rfloor$. Thus

$$\mathcal{E}_k(\phi)^{1/2} \simeq \|\tau\phi'\|_{H^k} + \|\tau\phi\|_{H^{k+1}} + \|m\phi\|_{H^k}. \tag{5.3.5}$$

Proof. The proof follows from [Bes87, App. H, Thm. 27], while the last part explicitly for $k = 2\ell$ even and $k = 2\ell + 1$ odd:

$$\begin{aligned} \|\phi\|_{H^{2l}} &\simeq \|\Delta^l \phi\|_{L^2} + \|\phi\|_{L^2}, \\ \|\phi\|_{H^{2l+1}} &\simeq \|\Delta^l \phi\|_{H^1} + \|\phi\|_{L^2} = \|\Delta^l \phi\|_{L^2} + \|\nabla \Delta^l \phi\|_{L^2} + \|\phi\|_{L^2}. \end{aligned} \tag{5.3.6}$$

□

It is important to note the appearance of $|\tau|$ weights in (5.3.5).

5.3.2 Modified energy

We now introduce the modified energy, which contains two indefinite terms. This modified energy is equivalent, up to an overall scaling of τ^2, with the modified energy considered in [Wan19, (3.27)]. Since the modified energy arises by using the unit normal vector field (instead of merely ∂_τ) as a multiplier vector field it yields a better energy estimate, see Proposition 5.21 and (5.6.7), than the standard energy. Note furthermore that this procedure of adding additional off-diagonal terms to produce a modified energy with improved estimates is very similar in spirit to the corrected geometric energies given in Definition 5.18, see in particular (5.4.10). Its equivalence to the standard energy is then shown.

Definition 5.15.

$$\begin{aligned} \tilde{E}_k(\phi) = \int_{\mathcal{K}} \tau^2 (-1)^k \Big[& \phi' \Delta^k \phi' - \phi \Delta^{k+1} \phi + 3\phi' \Delta^k \big(N^{-1} \phi \widehat{N}\big) \\ & + \tfrac{3}{2} N^{-1} \phi \Delta^k \left((\tfrac{3}{2} - N) N^{-1} \phi \right) \Big] \mathrm{d}\mu_g + \int_{\mathcal{K}} m^2 (-1)^k \phi \Delta^k \phi \mathrm{d}\mu_g, \end{aligned} \tag{5.3.7}$$

$$\tilde{\mathcal{E}}_\ell(\phi) = \sum_{k=0}^{\ell} \tilde{E}_k(\phi). \tag{5.3.8}$$

We have $\tilde{E}_k(\phi) \geq 0$ since under the integral

$$\phi' \Delta^k \phi' - 3\Delta^k (N^{-1}\phi)\phi' + \tfrac{9}{4} N^{-1} \phi \Delta^k (N^{-1} \phi) = \left(\phi' - \tfrac{3}{2} N^{-1} \phi\right) \Delta^k \left(\phi' - \tfrac{3}{2} N^{-1} \phi\right). \tag{5.3.9}$$

Lemma 5.16. *Let $\mathbb{N} \ni N_0 \geq 4$. Assume that there exists a constant $C > 0$ such that $\|N\|_{L^\infty} + \|N^{-1}\|_{L^\infty} + \|g - \gamma\|_{H^{N_0}} < C$ and $\|\widehat{N}\|_{H^{N_0+1}} \leq Ce^{-\frac{1}{2}T}$. Then there exists a τ_0 such that for all $\tau > \tau_0$ the following equivalence holds.*

$$\tilde{\mathcal{E}}_\ell(\phi) \simeq \mathcal{E}_\ell(\phi), \quad \ell \leq N_0. \tag{5.3.10}$$

Proof. We first write the difference between the two energies (note without summing in k).

$$\begin{aligned} E_k(\phi) - \tilde{E}_k(\phi) = & \int_{\mathcal{K}} (-1)^k \tau^2 \Big(3\phi' \Delta^k (N^{-1}\phi) - \tfrac{9}{4} N^{-1} \phi \Delta^k (N^{-1}\phi) \Big) \mathrm{d}\mu_g \\ & + \int_{\mathcal{K}} (-1)^k \tau^2 \Big(\tfrac{3}{2} N^{-1} \phi \Delta^k \phi - \phi' \Delta^k \phi \Big) \mathrm{d}\mu_g. \end{aligned} \tag{5.3.11}$$

Examine the first term on the right hand side of (5.3.11). The claim for $\ell = 0$ is easily seen from:

$$\left|\int_{\mathcal{K}} \tau^2 \left(-3(N^{-1}\phi)\phi'\right)\, \mathrm{d}\mu_g\right| \le C\delta \int_{\mathcal{K}} \tau^2(\phi')^2\, \mathrm{d}\mu_g + \frac{C\tau_0^2}{\delta m^2}\int_{\mathcal{K}} m^2\phi^2\, \mathrm{d}\mu_g, \tag{5.3.12}$$

where we used $\|N^{-1}\|_{L^\infty} \le C$ and δ is a constant we are free to choose. For sufficiently small δ and $\tau_0 = \mathrm{tr}\widetilde{k}_0$ one can ensure all coefficients are strictly less than 1. Note that $0 > \tau > \tau_0$ implies $|\tau| < |\tau_0|$. Thus this term can be absorbed by the clearly positive terms of $E_0(\phi)$.

A similar argument holds for $\ell \ge 1$. In particular for some smooth functions v, w we have

$$\Delta^i(vw) = (\Delta^i v)w + \sum_{|I|+|J|+1=2i} c_{IJ}\nabla^I v \nabla^{J+1} w, \tag{5.3.13}$$

for some coefficients c_{IJ} depending on g. So for a fixed value of $k \ge 1$, integration by parts (where there are $2\lfloor k/2\rfloor + \mathring{k} = k$ derivatives distributed) gives

$$\begin{aligned}
&\left|\int_{\mathcal{K}} \tau^2(-1)^k \left(3\Delta^k(N^{-1}\phi)\phi'\right)\, \mathrm{d}\mu_g\right| \\
&\quad = 3\left|\int_{\mathcal{K}} \tau^2 \left((\nabla_a)^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}(N^{-1}\phi)\right)\left((\nabla^a)^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi'\right)\, \mathrm{d}\mu_g\right| \\
&\quad \le C\int \tau^2 N^{-1}\left|\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi'\right|\, \mathrm{d}\mu_g \\
&\qquad + C\sum_{|I|+|J|+1=k}\int \tau^2\left|\nabla^I\phi\nabla^{J+1}N^{-1}\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\right|\, \mathrm{d}\mu_g \\
&\quad \le C\|N^{-1}\|_{L^\infty}\|\tau\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi\|_{L^2}\|\tau\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi'\|_{L^2} \\
&\qquad + C\sum_{|I|+|J|+1=k}\|\tau\nabla^I\phi\|_{L^4}\|\nabla^{J+1}N^{-1}\|_{L^4}\|\tau\nabla^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}\phi'\|_{L^2}^2 \\
&\quad \le C\|\tau\phi\|_{H^k}\|\tau\phi'\|_{H^k} + C\sum_{|I|\le k-1}\|\tau\nabla^I\phi\|_{H^1}\cdot\sum_{1\le|J|\le k-1}\|\nabla^J N^{-1}\|_{H^1}\cdot\|\tau\phi'\|_{H^k} \qquad (5.3.14)\\
&\quad \le C\|\tau\phi\|_{H^k}\|\tau\phi'\|_{H^k} + C\|\tau\phi\|_{H^k}\cdot\|\widehat{N}\|_{H^{k+1}}\|\tau\phi'\|_{H^k} \\
&\quad \le C\delta\|\tau\phi'\|_{H^k}^2 + \frac{C\tau_0^2}{\delta m^2}\|m\phi\|_{H^k}^2. \qquad (5.3.15)
\end{aligned}$$

To get to the line (5.3.14) we used the following estimate

$$\begin{aligned}
\sum_{1\le|J|\le k}\|\nabla^J N^{-1}\|_{L^2} &\le \|N^{-2}\|_{L^\infty}\|\widehat{N}\|_{H^k} + C(\|N^{-1}\|_{L^\infty})\sum_{|I|\le\lfloor k/2\rfloor}\|\nabla^I\widehat{N}\|_{L^\infty}\|\widehat{N}\|_{H^k} \\
&\le C\|\widehat{N}\|_{H^k} + C\|\widehat{N}\|_{H^{k+1}}\|\widehat{N}\|_{H^k} \le C\|\widehat{N}\|_{H^k}.
\end{aligned} \tag{5.3.16}$$

In this estimate we used $\lfloor k/2\rfloor + 2 \le k+1$ for $k \ge 1$ in order to take the terms with low derivatives on $\widehat{N}$ out in L^∞ and embed using Sobolev, recalling also that N is controlled at one order of regularity higher than ϕ. Note we also used the three-dimensional Sobolev embedding $H^1 \hookrightarrow L^4$.

In a similar way we can show

$$\left|\int_{\mathcal{K}}\tau^2 N^{-1}\phi\Delta^k(N^{-1}\phi)\, \mathrm{d}\mu_g\right| + \left|\int_{\mathcal{K}}\tau^2\left(\phi'\Delta^k\phi - \tfrac{3}{2}N^{-1}\phi\Delta^k\phi\right)\, \mathrm{d}\mu_g\right|$$

$$
\begin{aligned}
&\leq \int_{\mathcal{K}} \left|\tau \mathring{\nabla}^k \Delta^{\lfloor k/2 \rfloor}(N^{-1}\phi)\right|^2 \mathrm{d}\mu_g + C\int_{\mathcal{K}} \tau^2 \left|\mathring{\nabla}^k \Delta^{\lfloor k/2 \rfloor}(N^{-1}\phi)\mathring{\nabla}^k \Delta^{\lfloor k/2 \rfloor}\phi\right| \mathrm{d}\mu_g \\
&\quad + C\delta \int_{\mathcal{K}} \tau^2 |\mathring{\nabla}^k \Delta^{\lfloor k/2 \rfloor}\phi'|^2 \mathrm{d}\mu_g + \frac{C\tau_0^2}{\delta m^2}\int_{\mathcal{K}} m^2 |\mathring{\nabla}^k \Delta^{\lfloor k/2 \rfloor}\phi|^2 \mathrm{d}\mu_g \\
&\leq \frac{C\tau_0^2}{m^2}\|m\phi\|_{H^k}^2 + \|\widehat{N}\|_{H^k}^2 \|\tau\phi\|_{H^{k+1}}^2 + C\delta\|\tau\phi'\|_{H^k}^2 + \frac{C\tau_0^2}{\delta m^2}\|m\phi\|_{H^k}^2 \\
&\leq \frac{C\tau_0^2}{\delta m^2}\mathcal{E}_k(\phi) + |\tau_0|\mathcal{E}_k(\phi) + C\delta\mathcal{E}_k(\phi) + \frac{C\tau_0^2}{\delta m^2}\mathcal{E}_k(\phi).
\end{aligned} \tag{5.3.17}
$$

Thus the claim holds by summing in k from 0 to ℓ and reducing δ and $|\tau_0|$ to be sufficiently small so that all coefficients can be made to be strictly smaller than 1. □

5.4 Energy norms and smallness assumptions

In this section we state the global bootstrap assumptions to provide for a simpler notation in all the estimates to follow. Also, we introduce the definition of the corrected L^2-energy to control the perturbation of the geometry. This was given in the earlier works [AM11, AF20].

5.4.1 Bootstrap assumptions

Fix the regularity $\mathbb{N} \ni N_0 \geq 4$ and some constant $0 < \kappa \ll 1$. We employ a bootstrap argument, referring to section 1.3.2 for a more detailed discussion of this approach. Let $C_I > 0$ be a large constant. We assume that for all $T_0 \leq T' \leq T$ the following estimates hold

$$\|g-\gamma\|_{H^{N_0+1}} + \|\Sigma\|_{H^{N_0}} \leq C_I \varepsilon e^{-\frac{3}{4}T'}, \tag{5.4.1a}$$

$$\|\widehat{N}\|_{H^{N_0+1}} \leq C_I \varepsilon e^{(-1+\kappa)T'}, \tag{5.4.1b}$$

$$\|X\|_{H^{N_0+1}} \leq C_I \varepsilon e^{(-1+\kappa)T'}, \tag{5.4.1c}$$

$$\mathcal{E}_{N_0}(\phi)^{1/2} \leq C_I \varepsilon e^{\kappa T'}, \tag{5.4.1d}$$

where $T < T_*$ is fixed. A simple check shows the assumptions of lemma 5.16 are consistent with (5.4.1).

5.4.2 Energy for the perturbation of the geometry

Following [AM11, AF20], we now define an energy for the geometric perturbation of the first and second fundamental forms.

Definition 5.17 (Lichnerowicz Laplacian in Milne)**.** Following the notation of [AM11] we introduce the following operators on symmetric two-tensors u_{cd}

$$\hat{\Delta}_{g,\gamma} u_{cd} = \mu_g^{-1}\nabla[\gamma]_a (g^{ab}\mu_g \nabla[\gamma]_b u_{cd}), \tag{5.4.2}$$

$$\mathcal{L}_{g,\gamma} u_{cd} = -\Delta_{g,\gamma} u_{cd} - 2(R[\gamma]\circ u)_{cd}, \tag{5.4.3}$$

where $\mu_g = \sqrt{\det g}$ is the volume element on $\mathcal{K}$.

We note that the operator (5.4.2) can be rewritten as

$$\hat{\Delta}_{g,\gamma} u_{cd} = g^{ab}\nabla[\gamma]_a \nabla[\gamma]_b u_{cd} - V^a \nabla[\gamma]_a u_{cd}. \tag{5.4.4}$$

Thus in the spatially harmonic gauge of (5.2.6) the operator (5.4.3) agrees with the Lichnerowicz operator given in (1.5.7). However, we follow [AM11, AF20] and use the notation $\mathcal{L}_{g,\gamma}$ in this chapter. As alluded to in (1.8.15), in the spatially harmonic gauge we have

$$\mathrm{Ric}[g]_{ab} + \frac{2}{9}g_{ab} = \frac{1}{2}\mathcal{L}_{g,\gamma}(g-\gamma)_{ab} + J_{ab}, \tag{5.4.5}$$

where J_{ab} are higher-order terms satisfying for $k \geq 1$

$$\|J\|_{H^{k-1}} \leq C\|g-\gamma\|_{H^k}. \tag{5.4.6}$$

We now construct higher-order energies out of $\mathcal{L}_{g,\gamma}$, following for example [AF20, §8.2]

Definition 5.18 (Geometric energy)**.** Recall that λ_0 is the lowest eigenvalue of the operator $\mathcal{L}_{g,\gamma}$, with lower bounds given in proposition 1.32. We define the correction parameter $\alpha = \alpha(\lambda_0, \delta_\alpha)$ by

$$\alpha = \begin{cases} 1 & \lambda_0 > 1/9 \\ 1-\delta_\alpha & \lambda_0 = 1/9, \end{cases} \tag{5.4.7}$$

where $\delta_\alpha = \sqrt{1-9(\lambda_0-\varepsilon')}$ with $1 \gg \varepsilon' > 0$ remains a variable to be determined in the course of the argument to follow. By fixing ε' once and for all, δ_α can be made suitably small when necessary. The corresponding correction constant, relevant for defining the corrected energies, is defined by

$$c_E = \begin{cases} 1 & \lambda_0 > 1/9 \\ 9(\lambda_0-\varepsilon') & \lambda_0 = 1/9. \end{cases} \tag{5.4.8}$$

We are now ready to define the energy for the geometric perturbation. For $\mathbb{Z} \ni m \geq 1$ let

$$\mathcal{E}_{(m)} = \frac{1}{2}\int_{\mathcal{K}} \langle 6\Sigma, \mathcal{L}_{g,\gamma}^{m-1} 6\Sigma\rangle \mu_g + \frac{9}{2}\int_{\mathcal{K}} \langle (g-\gamma), \mathcal{L}_{g,\gamma}^{m}(g-\gamma)\rangle \mu_g, \tag{5.4.9}$$

$$\Gamma_{(m)} = \int_{\mathcal{K}} \langle 6\Sigma, \mathcal{L}_{g,\gamma}^{m-1}(g-\gamma)\rangle \mu_g. \tag{5.4.10}$$

The energy measuring the geometric perturbation is then defined by

$$E_k(g,\Sigma) = \sum_{1\leq m\leq k} \big(\mathcal{E}_{(m)} + c_E\Gamma_{(m)}\big). \tag{5.4.11}$$

The following lemma states that the corrected geometric energy $E_k(g,\Sigma)$ is in fact coercive over the standard Sobolev norms of the geometric variables g_{ab} and Σ_{ab}.

Lemma 5.19. *Under bootstrap assumptions* (5.4.1) *there exists a constant* $C > 0$ *such that*

$$\|g-\gamma\|^2_{H^{N_0+1}} + \|\Sigma\|^2_{H^{N_0}} \leq C E^g{}_{N_0}(g,\Sigma). \tag{5.4.12}$$

For a proof see [AF20, Lemma 19] or [AM11, Lemma 7.2].

5.5 Modified continuity equation

In this section we cast the rescaled Klein-Gordon equation in the particular form (5.5.1) and derive a modified continuity equation (5.5.10). This is essential to prove an improved bound for the pointwise norm of the energy-density, which in turn allow for an initialization of the hierarchy that improves bounds on the Klein-Gordon field and the lapse function.

The rescaled Klein-Gordon equation (5.2.15e) when written in terms of the small quantity $\widehat{N}$ reads

$$\hat{\partial}_0\phi' = \nabla^a(N\nabla_a\phi) - 3\widehat{N}\phi' + \phi' + \tfrac{9}{2}N^{-1}\phi\widehat{N} + \tfrac{3}{4}N^{-1}\phi + \tfrac{3}{2}\phi\hat{\partial}_0 N^{-1} - \tau^{-2}m^2N\phi. \tag{5.5.1}$$

The rescaled continuity equation, see for example [AF20, (10.16)], is

$$\begin{aligned}\partial_T\rho &= (3-N)\rho - X^a\nabla_a\rho + \tau N^{-1}\nabla_a(N^2 j^a) - \tau^2\tfrac{N}{3}g_{ab}T^{ab} - \tau^2 N\Sigma_{ab}T^{ab} \qquad (5.5.2)\\ &= -3\widehat{N}\rho - X^a\nabla_a\rho - \tau^2N^{-1}\nabla_a\Big(N\nabla^a\phi(\tfrac{3}{2}\phi - \hat{\partial}_0\phi)\Big) + \tfrac{1}{2}Nm^2\phi^2\\ &\quad + \tau^2\tfrac{N}{6}\nabla^a\phi\nabla_a\phi - \tau^2\tfrac{N-1}{2}\Big(\tfrac{3}{2}\phi - \hat{\partial}_0\phi\Big)^2 + \tau^2\tfrac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi. \qquad (5.5.3)\end{aligned}$$

The problematic terms in this expression are $Nm^2\phi^2$ and $N(\tau\phi')^2$. This is because naively estimating such terms using the standard Sobolev embedding $L^\infty \hookrightarrow H^2$ leads to a problematic $e^{\kappa T}$ growth for ρ. Nonetheless, motivated by the notion of a 'modified' energy, we consider a modified energy density. Consider the quantity

$$P = \phi\Big(\tfrac{3}{2}N^{-1}\phi - \phi'\Big). \tag{5.5.4}$$

Up to a factor of τ^2, this is similar to one of the terms subtracted from $E_0(\phi)$ to obtain $\tilde{E}_0(\phi)$ in (5.3.7). Using the Klein-Gordon equation (5.2.15e) its evolution equation is the following.

$$\begin{aligned}\partial_T P &= -\mathcal{L}_X P + \hat{\partial}_0 P\\ &= -\mathcal{L}_X P + 3\phi\phi' + \tfrac{3}{2}\phi^2\hat{\partial}_0 N^{-1} - N(\phi')^2 - \phi\hat{\partial}_0(\phi')\\ &= -\mathcal{L}_X P + 3\phi\phi' + \tfrac{3}{2}\phi^2\hat{\partial}_0 N^{-1} - N(\phi')^2\\ &\quad - \phi\Big(\nabla^a(N\nabla_a\phi) - N\phi' + \tfrac{3}{2}\phi + 4\phi' - \tfrac{3}{2}N^{-2}\phi\hat{\partial}_0 N - \tfrac{15}{4}N^{-1}\phi - \tau^{-2}m^2Nu\Big)\\ &= -\mathcal{L}_X P + 3\phi\phi' - \phi\nabla^a(N\nabla_a\phi) + N\phi'\phi - \tfrac{3}{2}\phi^2 - 4\phi'\phi + \tfrac{15}{4}N^{-1}\phi^2\\ &\quad - N(\phi')^2 + \tau^{-2}m^2N\phi^2. \qquad (5.5.5)\end{aligned}$$

For some constant λ define

$$\hat{\rho} = \rho + \lambda\tau^2 P. \tag{5.5.6}$$

Proposition 5.20. *Assume the bootstrap assumptions* (5.4.1) *hold. If* $\lambda = -1/2$ *then there exists sufficiently small* τ_0 *such that for all* $\tau > \tau_0$ *the following holds*

$$\tfrac{1}{2}\rho \le \hat{\rho} \le 2\rho, \tag{5.5.7}$$

and also the estimates

$$|\partial_T\hat{\rho}| \lesssim \big(\|\widehat{N}\|_{H^3} + \|X\|_{H^2} + \|\Sigma\|_{H^2} + |\tau|\big)\mathcal{E}_4(\phi). \tag{5.5.8}$$

Consequently we have

$$\rho|_T \lesssim \rho|_{T_0} + C\varepsilon^3 \int_{T_0}^{T} e^{(-1+2\kappa)s}\mathrm{d}s \le C\varepsilon^2. \tag{5.5.9}$$

Proof. We combine (5.5.3) and (5.5.5).

$$\begin{aligned}
\partial_T\hat\rho &= \partial_T\rho + \lambda\tau^2\partial_T P - 2\lambda\tau^2 P\\
&= -3\widehat{N}\rho - X^a\nabla_a\hat\rho + Nm^2\phi^2(\tfrac12+\lambda) - \tau^2N(\phi')^2(\tfrac12+\lambda)\\
&\quad - \tau^2N^{-1}\nabla_aN\nabla^a\phi\Big(\tfrac32\phi - \hat\partial_0\phi\Big) - \tau^2\Delta\phi\Big(\tfrac32\phi - \hat\partial_0\phi\Big) - \tau^2\nabla^a\phi\nabla_a\Big(\tfrac32\phi - \hat\partial_0\phi\Big)\\
&\quad + \tau^2\tfrac{N}{6}\nabla_a\phi\nabla^a\phi - \tau^2\tfrac98 N^{-1}\phi^2 + \tau^2\tfrac32\phi\phi' + \tau^2\tfrac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi\\
&\quad + \lambda\tau^2\big(-\phi\phi' - \phi\nabla^a(N\nabla_a\phi) + N\phi\phi' - \tfrac32\phi^2 + \tfrac{15}{4}N^{-1}\phi^2 - 3N^{-1}\phi^2 + 2\phi\phi'\big).
\end{aligned}$$

Choosing $\lambda = -1/2$ we can remove the problematic terms. Indeed the evolution equation is now

$$\begin{aligned}
\partial_T\hat\rho &= -3\widehat{N}\rho - X^a\nabla_a\hat\rho + \tau^2\tfrac12(1+3/N)\phi\nabla_aN\nabla^a\phi - \tau^2\tfrac12(3+N)\phi\Delta\phi\\
&\quad + \tau^2N\phi'\Delta\phi + \tau^2\tfrac32(-1+N/9)\nabla_a\phi\nabla^a\phi + \tau^2N\nabla^a\phi\nabla_a\phi' + 2\tau^2\phi'\nabla^a\phi\nabla_aN\\
&\quad + \tau^2\tfrac34(1-2/N)\phi^2 + \tau^2\tfrac12(1-N/2)\phi\phi' + \tau^2\tfrac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi.
\end{aligned}\tag{5.5.10}$$

To show the equivalence, note

$$|\rho - \hat\rho| = \frac{\tau^2}{2}\Big|\phi\Big(\tfrac32N^{-1}\phi - \phi'\Big)\Big| \le \frac{\tau_0^2}{\delta m^2}m^2\phi^2 + \delta\tau^2\Big(\tfrac32N^{-1}\phi - \phi'\Big)^2. \tag{5.5.11}$$

Thus for sufficiently small δ and $\tau \ge \tau_0$ equivalency holds. Finally, using (5.3.5) and the standard Sobolev embedding $H^2 \hookrightarrow L^\infty$, we have

$$\begin{aligned}
|\partial_T\hat\rho| &\lesssim \|\widehat{N}\|_{L^\infty}\big(\|m\phi\|_{L^\infty}^2 + \|\tau N^{-1}\phi\|_{L^\infty}^2 + \|\tau\phi'\|_{L^\infty}^2\big) + \|X\|_{L^\infty}\|m^2\phi\nabla\phi\|_{L^\infty}\\
&\quad + \|X\|_{L^\infty}(\|\tau N^{-1}\phi\|_{L^\infty} + \|\tau\phi'\|_{L^\infty})(\|\tau\nabla(N^{-1}\phi)\|_{L^\infty} + \|\tau\nabla\phi'\|_{L^\infty})\\
&\quad + \|\nabla\widehat{N}\|_{L^\infty}\|\tau^2\phi\nabla\phi\|_{L^\infty} + |\tau|\|\tau\phi'\|_{L^\infty}\|\Delta\phi\|_{L^\infty} + \tau^2\|\nabla\phi\|_{L^\infty}^2\\
&\quad + |\tau|\|\nabla\phi\|_{L^\infty}\|\tau\nabla\phi'\|_{L^\infty} + \|\tau\phi'\|_{L^\infty}\|\tau\nabla\phi\|_{L^\infty}\|\nabla\widehat{N}\|_{L^\infty} + \tau^2\|\phi\|_{L^\infty}^2\\
&\quad + |\tau|\|\tau\phi'\|_{L^\infty}\|\phi\|_{L^\infty} + \|\Sigma\|_{L^\infty}\|\tau\nabla\phi\|_{L^\infty}^2\\
&\lesssim \|\widehat{N}\|_{H^3}\mathcal{E}_3(\phi) + \|X\|_{H^2}\mathcal{E}_3(\phi) + |\tau|\mathcal{E}_4(\phi) + \|\Sigma\|_{H^2}\mathcal{E}_2(\phi)\,.
\end{aligned}\tag{5.5.12}$$

We can also estimate the initial value of ρ

$$\rho\big|_{T_0} \lesssim \big(\|m\phi\|_{L^\infty}^2 + \|\tau N^{-1}\phi\|_{L^\infty}^2 + \|\tau\phi'\|_{L^\infty}^2\big)\big|_{T_0} \lesssim \mathcal{E}_2(\phi)\big|_{T_0} \lesssim \varepsilon^2 e^{4\kappa T_0}\,. \tag{5.5.13}$$

□

5.6 Energy inequalities

In this section we derive decay inequalities for the time derivative ∂_T of the modified L^2-energy norm of the Klein-Gordon field defined in Section 5.3.2. The main results in this section, propositions 5.21 and 5.22, are then combined with estimates for the lapse in lemma 5.30 in order to close the bootstrap argument.

5.6.1 Lowest-order Klein-Gordon energy

We restate the definition

$$\tilde{\mathcal{E}}_0(\phi) = \int_{\mathcal{K}} \tau^2 \left[(\phi')^2 + g^{ab}\nabla_a\phi\nabla_b\phi + 3N^{-1}\phi\phi'\widehat{N} + \tfrac{3}{2}N^{-2}\phi^2\left(\tfrac{3}{2} - N\right)\right] \mathrm{d}\mu_g + \int_{\mathcal{K}} m^2\phi^2 \,\mathrm{d}\mu_g.$$

Proposition 5.21. *Assume the bootstrap assumptions* (5.4.1) *hold. Then the lowest order energy* $\tilde{\mathcal{E}}_0(\phi)$ *obeys*

$$\partial_T\tilde{\mathcal{E}}_0(\phi) \lesssim \big(\|\Sigma\|_{H^2} + \|\widehat{N}\|_{H^3} + |\tau|\big)\tilde{\mathcal{E}}_0(\phi), \tag{5.6.1}$$

and thus

$$\tilde{\mathcal{E}}_0(\phi)\big|_T \lesssim \tilde{\mathcal{E}}_0(\phi)\big|_{T_0} \cdot \exp\Big(C\int_{T_0}^{T} e^{(-1+\kappa)s}\mathrm{d}s\Big). \tag{5.6.2}$$

Proof. The modified energy takes the form

$$\tilde{\mathcal{E}}_0(\phi) = \int_{\mathcal{K}} (\tau^2 f_0(\phi) + m^2\phi^2)\,\mathrm{d}\mu_g, \tag{5.6.3}$$

where $f_0(\phi)$ is the expression between the square brackets above. An identity taken from [AF20, (6.5)], valid for some function u on Σ, is the following

$$\partial_T \int_{\mathcal{K}} u\mu_g = 3\int_{\mathcal{K}} \widehat{N}u\mu_g + \int_{\mathcal{K}} \hat{\partial}_0(u)\mu_g\,. \tag{5.6.4}$$

Thus we find

$$\begin{aligned}
\partial_T\tilde{\mathcal{E}}_0(\phi) &= -\int_{\mathcal{K}} (3-N)(\tau^2 f_0(\phi) + m^2\phi^2)\,\mathrm{d}\mu_g + \int_{\mathcal{K}} \Big(\tau^2\hat{\partial}_0(f_0) - 2\tau^2 f_0 + m^2\hat{\partial}_0(\phi^2)\Big)\,\mathrm{d}\mu_g \\
&\le \|\widehat{N}\|_{L^\infty}\tilde{\mathcal{E}}_0(\phi) + \Big|\int_{\mathcal{K}} \tau^2\hat{\partial}_0(f) - 2\tau^2 f + m^2\hat{\partial}_0(\phi^2)\,\mathrm{d}\mu_g\Big|\,.
\end{aligned} \tag{5.6.5}$$

Another identity taken from [AF20, (6.4)] and relevant for this and later calculations is

$$\hat{\partial}_0 g^{ab} = -2N\Sigma^{ab} - 2\widehat{N}g^{ab}\,. \tag{5.6.6}$$

Using these and the Klein-Gordon equation in the form (5.5.1) we find

$$\begin{aligned}
&\tau^2\hat{\partial}_0(f_0) - 2\tau^2 f_0 + m^2\hat{\partial}_0(\phi^2) \\
&= \tau^2\hat{\partial}_0\phi'\big(2\phi' + 3N^{-1}\phi\widehat{N}\big) + 2\tau^2\nabla^a\hat{\partial}_0\phi\nabla_a\phi - 3\tau^2(\phi')^2\widehat{N} + 3\tau^2N^{-1}\phi\phi'(\tfrac{3}{2} - N) \\
&\quad + \tau^2\big(-3\phi\phi' + \tfrac{9}{2}\phi^2N^{-1} - \tfrac{3}{2}\phi^2\big)\hat{\partial}_0N^{-1} + \tau^2(-2N\Sigma^{ab} - 2\widehat{N}g^{ab})\nabla_a\phi\nabla_b\phi \\
&\quad - 2\tau^2 f_0 + 2m^2\phi\hat{\partial}_0\phi \\
&= 2\tau^2\nabla^a(\hat{\partial}_0\phi\nabla_a\phi) + 3\tau^2\nabla^a(\phi\widehat{N}\nabla_a\phi) + \tau^2\nabla_aN\big(3\widehat{N}N^{-1}\phi\nabla_a\phi + 2\phi'\nabla_a\phi - \phi\nabla_a\phi\big) \\
&\quad + \tau^2\widehat{N}\Big(-3(\phi')^2 - 5(\nabla\phi)^2 + 6N^{-1}\phi\phi' - 9\widehat{N}(N^{-1}\phi)\phi' + \tfrac{9}{2}(N - \tfrac{3}{2})(N^{-1}\phi)^2\Big) \\
&\quad - \tau^2\Big(3N^{-2}\phi^2(\tfrac{3}{2} - N) + 2(\nabla\phi)^2\Big) + 3\tau^2N^{-1}\phi\phi'(2 - N) \\
&\quad + \tau^2\hat{\partial}_0N^{-1}\Big(3\phi\phi' + \tfrac{9}{2}N^{-1}\phi^2\widehat{N} - 3\phi\phi' + \tfrac{9}{2}\phi^2N^{-1} - \tfrac{3}{2}\phi^2\Big) \\
&\quad - 2N\tau^2\Sigma^{ab}\nabla_a\phi\nabla_b\phi - 3m^2\widehat{N}\phi^2.
\end{aligned} \tag{5.6.7}$$

Note the terms involving $\hat{\partial}_0 N$ cancel. Combining this with (5.6.5) we find

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_0(\phi) &\lesssim \big(\|\Sigma\|_{H^2} + \|\widehat{N}\|_{H^3}\big)\tilde{\mathcal{E}}_0(\phi) + \tau^2 \int \phi^2 \,\mathrm{d}\mu_g + \int ||\tau|^{1/2}\phi|\tau|^{3/2}\phi'| \,\mathrm{d}\mu_g \\ &\lesssim \big(\|\Sigma\|_{H^2} + \|\widehat{N}\|_{H^3} + |\tau|\big)\tilde{\mathcal{E}}_0(\phi). \end{aligned} \tag{5.6.8}$$

Applying Grönwall's inequality from lemma 1.20 yields the result. □

5.6.2 Higher order modified Klein-Gordon energies

Now we calculate the time derivatives of higher-order energy norms for the Klein-Gordon field. Recall definition (5.3.7) for the modified L^2-energy. The main energy estimate for the modified higher order energies is given in the following proposition.

Proposition 5.22. *Assume the bootstrap assumptions* (5.4.1) *hold, then the higher order energies for* $1 \leq \ell \leq N_0$ *satisfy*

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_\ell(\phi) &\lesssim \Big(\|\widehat{N}\|_{H^3} + \|\Sigma\|_{H^3} + \|\Sigma\|_{H^\ell} + \|\widehat{N}\|_{H^{\ell+1}} + |\tau|\Big)\tilde{\mathcal{E}}_\ell(\phi) \\ &\quad + \Big(\|\widehat{N}\|_{H^{\ell+1}} + \|\Sigma\|_{H^\ell}\Big)\tilde{\mathcal{E}}_3(\phi) + |\tau|\varepsilon^4 + \sum_{k=1}^{\ell}\Big|\int_{\mathcal{K}} B_k \,\mathrm{d}\mu_g\Big|, \end{aligned} \tag{5.6.9}$$

where B_k *denote the border-line terms, which for* $k \geq 1$ *are defined by*

$$B_k = m^2 \phi' [N, \Delta^k]\phi. \tag{5.6.10}$$

Proof. In a similar way to proposition 5.21, we have

$$\partial_T \tilde{E}_k(\phi) \leq \|\widehat{N}\|_{L^\infty} \tilde{E}_k(\phi) + \Big|\int_{\mathcal{K}} (-1)^k \big(\tau^2 \hat{\partial}_0(f_k) - 2\tau^2 f_k + m^2 \hat{\partial}_0(\phi\Delta^k\phi)\big)\,\mathrm{d}\mu_g\Big|, \tag{5.6.11}$$

where f_k is the integrand inside the square brackets of $\tilde{E}_k(\phi)$ above. Hence we calculate the final term above and use the Klein-Gordon equation (5.5.1) to simplify. For some function u we have, by repeated applications of (5.6.6),

$$\begin{aligned}\hat{\partial}_0(\Delta^k u) &= \hat{\partial}_0\big(g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u\big) \\ &= (\hat{\partial}_0 g^{a_1a_2})\nabla_{a_1}\nabla_{a_2}\Delta\cdots\Delta u + \ldots (\hat{\partial}_0 g^{a_{2k-1}a_{2k}})\Delta\cdots\Delta\nabla_{a_{2k-1}}\nabla_{a_{2k}}u \\ &\quad + g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}\hat{\partial}_0\big(\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u\big) \\ &= \Delta^k(\hat{\partial}_0 u) - 2\widehat{N}|k|\Delta^k u + g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}[\hat{\partial}_0, \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}]u \\ &\quad - 2N\sum_{i=1}^{k} g^{a_1a_2}\cdots\Sigma^{a_{2i-1}a_{2i}}\cdots g^{a_{2k}a_{2k-1}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2i}}\nabla_{a_{2i-1}}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u. \end{aligned}$$

We introduce the following compact notation for the second and last terms.

$$\begin{aligned}\Sigma^I \Delta_I^k u &= \sum_{i=1}^{k} g^{a_1a_2}\cdots\Sigma^{a_{2i-1}a_{2i}}\cdots g^{a_{2k}a_{2k-1}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2i}}\nabla_{a_{2i-1}}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u, \\ [\hat{\partial}_0, \Delta^k]u &= g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}[\hat{\partial}_0, \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}]u\,. \end{aligned} \tag{5.6.12}$$

Thus

$$\Big|\int_{\mathcal{K}}(-1)^k\big(\tau^2\hat{\partial}_0(f_k)-2\tau^2 f_k+m^2\hat{\partial}_0(\phi\Delta^k\phi)\big)\,\mathrm{d}\mu_g\Big| \lesssim \sum_{i=1}^{3}|I_k^i|+|C_k^1|+|C_k^2| + \Big|\int_{\mathcal{K}}B_k\,\mathrm{d}\mu_g\Big|, \tag{5.6.13}$$

where we define the lower-order integrals by

$$\begin{aligned} I_k^1 &= \int_{\mathcal{K}}\tau^2\widehat{N}\big(|k|f_k-\phi\Delta^{k+1}\phi\big)\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2N\phi'\Sigma^I\Delta_I^k\phi'\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2N\phi\Sigma^I\Delta_I^{k+1}\phi\,\mathrm{d}\mu_g \\ &\quad+\int_{\mathcal{K}}\tau^2N\phi'\Sigma^I\Delta_I^k(N^{-1}\phi\widehat{N})\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2\phi\Sigma^I\Delta_I^k\big((\tfrac{3}{2}-N)N^{-1}\phi\big)\,\mathrm{d}\mu_g \\ &\quad+\int_{\mathcal{K}}\widehat{N}m^2\phi\Delta^k\phi\,\mathrm{d}\mu_g+\int_{\mathcal{K}}m^2\phi\Sigma^I\Delta_I^k\phi\,\mathrm{d}\mu_g, \end{aligned} \tag{5.6.14a}$$

$$\begin{aligned} I_k^2 &= \int_{\mathcal{K}}\tau^2N\Delta\phi\Delta^k(N^{-1}\phi\widehat{N})\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2\nabla^aN\nabla_a\phi\Delta^k\phi'\,\mathrm{d}\mu_g \\ &\quad+\int_{\mathcal{K}}\tau^2\nabla^aN\nabla_a\phi\Delta^k(N^{-1}\phi\widehat{N})\,\mathrm{d}\mu_g, \end{aligned} \tag{5.6.14b}$$

$$\begin{aligned} I_k^3 &= \int_{\mathcal{K}}\tau^2\Big(-3\widehat{N}\phi'\Delta^k\phi'+(15-3N)\phi'\Delta^k(\widehat{N}N^{-1}\phi)+\tfrac{9}{2}(N-\tfrac{3}{2})N^{-1}\phi\Delta^k(N^{-1}\phi\widehat{N})\Big)\,\mathrm{d}\mu_g \\ &\quad+\int_{\mathcal{K}}\tau^2(2-N)N^{-1}\phi\Delta^k\phi'\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2(\tfrac{3}{2}-N)\phi\Delta^k(N^{-1}\phi)\,\mathrm{d}\mu_g, \end{aligned} \tag{5.6.14c}$$

and the integrals involving commutators by

$$\begin{aligned} C_k^1 &= \int_{\mathcal{K}}m^2\phi[\hat{\partial}_0,\Delta^k]\phi\,\mathrm{d}\mu_g+\int_{\mathcal{K}}\tau^2\Big(\phi'[\hat{\partial}_0,\Delta^k]\phi'-\phi[\hat{\partial}_0,\Delta^{k+1}]\phi+3\phi'[\hat{\partial}_0,\Delta^k]\big(N^{-1}\phi\widehat{N}\big) \\ &\quad+\tfrac{3}{2}(N^{-1}\phi)[\hat{\partial}_0,\Delta^k]\big((\tfrac{3}{2}-N)N^{-1}\phi\big)\Big)\,\mathrm{d}\mu_g, \end{aligned} \tag{5.6.15a}$$

$$C_k^2 = \int_{\mathcal{K}}\tau^2\Big(-2\Delta\phi[\Delta^k,N]\phi'-\tfrac{3}{2}N^{-1}\phi[\Delta^k,N]\phi'\Big)\,\mathrm{d}\mu_g. \tag{5.6.15b}$$

Finally the terms without decaying factors (for example, without factors of $|\tau|$ or $\|\widehat{N}\|_{L^\infty}$), and which require additional care to control, are the following.

$$B_k = m^2\phi'[N,\Delta^k]\phi.$$

Note the terms involving $\hat{\partial}_0 N$ have cancelled with each other. The terms I_k are controlled using lemmas 5.24 and 5.25. The commutator terms C_k are controlled using lemmas 5.26 and 5.28 below. Summing these estimates for $k=0$ to $k=\ell$, noting that $k=0$ is covered using proposition 5.21, yields the claim. □

5.6.3 Auxiliary lemmas

As mentioned in the foregoing proof we require a series of lemmas that are used in the proof of the main energy estimate above. We list and prove those in the following. The main strategy throughout this section is to integrate by parts on each term and distribute $k\geq 1$ derivatives while also making use of the Sobolev embeddings $H^2\hookrightarrow L^\infty$ and $H^1\hookrightarrow L^4$.

Lemma 5.23. *For $k \geq 1$ and general functions $v, u,$ and w we have*

$$\Big|\int_{\mathcal{K}} vu\Delta^k w \,\mathrm{d}\mu_g\Big| \lesssim \|v\|_{L^\infty}\|u\|_{H^k}\|w\|_{H^k} + \|u|_{L^\infty}\|v\|_{H^k}\|w\|_{H^k} + \|v\|_{H^k}\|u\|_{H^k}\|w\|_{H^k}.$$

The sums involving $|I|, |J| \leq k-1$ do not appear if $k = 1$. Also assuming the bootstrap assumptions (5.4.1) *hold, then for $k \leq N_0$*

$$\|\tau N^{-1}\phi\|_{H^k} \lesssim \big(|\tau| + \|\widehat{N}\|_{H^{k+1}}\big)\tilde{\mathcal{E}}_k(\phi)^{1/2}, \tag{5.6.16}$$

$$\|\tau(3-N)N^{-1}\phi\|_{H^k} \lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|\varepsilon^2, \tag{5.6.17}$$

$$\|\tau(\alpha-N)N^{-1}\phi\|_{H^k} \lesssim \big(|\tau| + \|\widehat{N}\|_{H^{k+1}}\big)\tilde{\mathcal{E}}_k(\phi)^{1/2} + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_1(\phi)^{1/2}, \tag{5.6.18}$$

where $\alpha \neq 3$.

Proof. Using the Sobolev embeddings $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^4$ and integration by parts on general functions $v, u,$ and w gives, for $k \geq 1$,

$$\begin{aligned}
\Big|\int_{\mathcal{K}} vu\Delta^k w \,\mathrm{d}\mu_g\Big| &\lesssim \int_{\mathcal{K}} \Big|(\nabla)^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}(vu)(\nabla)^{\mathring{k}}\Delta^{\lfloor k/2\rfloor}(w)\Big| \,\mathrm{d}\mu_g \\
&\lesssim \|v\|_{L^\infty}\|u\|_{H^k}\|w\|_{H^k} + \|u\|_{L^\infty}\|v\|_{H^k}\|w\|_{H^k} \\
&\quad + \sum_{1\leq|I|\leq k-1}\|\nabla^I v\|_{L^4}\sum_{1\leq|J|\leq k-1}\|\nabla^J\phi\|_{L^4}\|w\|_{H^k} \\
&\lesssim \|v\|_{L^\infty}\|u\|_{H^k}\|w\|_{H^k} + \|u|_{L^\infty}\|v\|_{H^k}\|w\|_{H^k} + \|v\|_{H^k}\|u\|_{H^k}\|w\|_{H^k}.
\end{aligned}$$

In general sums involving $|I|, |J| \leq k-1$ do not appear if $k = 1$. We also have

$$\begin{aligned}
\|\tau N^{-1}\phi\widehat{N}\|_{H^k} &\lesssim \|\tau\phi\|_{H^k} + \|\tau N^{-1}\phi\|_{H^k} \\
&\lesssim \frac{|\tau|}{m^2}\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|\|N^{-1}\|_{L^\infty}\tilde{\mathcal{E}}_k(\phi)^{1/2} + \|\tau\phi\|_{L^\infty}\|\widehat{N}\|_{H^k} \\
&\quad + |\tau|\|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_k(\phi)^{1/2} \\
&\lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|\varepsilon^2,
\end{aligned} \tag{5.6.19}$$

and also

$$\begin{aligned}
\|\tau N^{-1}\phi\|_{H^k} &= |\tau|\|N^{-1}\|_{L^\infty}\|\phi\|_{H^k} + \Big(\sum_{|I|+|J|+1\leq k}\int_{\mathcal{K}}\tau^2|\nabla^{I+1}N^{-1}|^2|\nabla^J\phi|^2\,\mathrm{d}\mu_g\Big)^{1/2} \\
&\lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + \sum_{1\leq|I|\leq k}\|\nabla^I N^{-1}\|_{L^4}\sum_{|J|\leq k-1}\|\tau\nabla^J\phi\|_{L^4} \\
&\lesssim \big(|\tau| + \|\widehat{N}\|_{H^{k+1}}\big)\tilde{\mathcal{E}}_k(\phi)^{1/2}.
\end{aligned} \tag{5.6.20}$$

Finally although $\widehat{N}$ is small, there are several terms involving $N-\alpha$ where $\alpha \neq 3$. In this case we take care to extract $N-\alpha$ in L^∞ when no derivatives hit it, but when derivatives do hit this term we note that $\nabla(N-\alpha) = \nabla\widehat{N}$ and this is small.

$$\begin{aligned}
\|\tau(\alpha-N)N^{-1}\phi\|_{H^k} &\lesssim \|\alpha-N\|_{L^\infty}\|\tau N^{-1}\phi\|_{H^k} + \|\tau N^{-1}\phi\|_{L^\infty}\|\widehat{N}\|_{H^k} \\
&\quad + \|\tau N^{-1}\phi\|_{H^k}\|\widehat{N}\|_{H^k} \\
&\lesssim \|\tau N^{-1}\phi\|_{H^k} + \|N^{-1}\|_{L^\infty}\|\tau\phi\|_{H^2}\|\widehat{N}\|_{H^k} \\
&\lesssim \big(|\tau| + \|\widehat{N}\|_{H^{k+1}}\big)\tilde{\mathcal{E}}_k(\phi)^{1/2} + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_1(\phi)^{1/2}.
\end{aligned} \tag{5.6.21}$$

$\square$

The first set of lower-order integrals I_k^1 are controlled in the following lemma.

Lemma 5.24. *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $k \leq N_0$ *the following estimate holds.*

$$|I_k^1| \lesssim \big(\|\widehat{N}\|_{H^2} + \|\Sigma\|_{H^2}\big)\tilde{\mathcal{E}}_k(\phi) + \big(\|\widehat{N}\|_{H^k} + \|\Sigma\|_{H^k}\big)\tilde{\mathcal{E}}_2(\phi) + \big(\|\widehat{N}\|_{H^k} + \|\Sigma\|_{H^k}\big)\tilde{\mathcal{E}}_k(\phi). \tag{5.6.22}$$

Proof. The results of lemma 5.23 allow us to easily obtain

$$\begin{aligned} \Big|\int_{\mathcal{K}} \widehat{N} m^2 \phi \Delta^k \phi \, \mathrm{d}\mu_g\Big| + \Big|\int_{\mathcal{K}} \tau^2 \widehat{N}\big(|k| f_k - \phi \Delta^{k+1}\phi\big) \, \mathrm{d}\mu_g\Big| \\ \lesssim \|\widehat{N}\|_{H^2}\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_2(\phi) + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_k(\phi). \end{aligned} \tag{5.6.23}$$

The remaining terms involving contractions with Σ^{ab} can similarly be estimated. For example

$$\begin{aligned} \Big|\int_{\mathcal{K}} m^2 \phi \Sigma^I \Delta_I^k \phi \, \mathrm{d}\mu_g\Big| &\lesssim \Big(\|\Sigma\|_{H^2}\|m\phi\|_{H^k} + \|m\phi\|_{L^\infty}\|\Sigma\|_{H^k} + \|m\phi\|_{H^k}\|\Sigma\|_{H^k}\Big)\|m\phi\|_{H^k} \\ &\lesssim \|\Sigma\|_{H^2}\tilde{\mathcal{E}}_k(\phi) + \|\Sigma\|_{H^k}\tilde{\mathcal{E}}_2(\phi) + \|\Sigma\|_{H^k}\tilde{\mathcal{E}}_k(\phi). \end{aligned} \tag{5.6.24}$$

$\square$

Lemma 5.25. *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $k \leq N_0$ *the following estimate holds.*

$$|I_k^2| + |I_k^3| \lesssim \big(\|\widehat{N}\|_{H^{k+1}} + \|\widehat{N}\|_{H^3}\big)\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + |\tau|\varepsilon^4. \tag{5.6.25}$$

Proof. We make frequent use of the identities from lemma 5.23 and also the identity (5.3.16) for derivatives of N^{-1}. For the first term in L_k^2, we integrate by parts only $k-1$ times so that we avoid terms such as $\|\tau\phi\|_{H^{k+2}}$ since this is only controlled by $\tilde{\mathcal{E}}_{k+1}(\phi)$.

$$\begin{aligned} &\Big|\int_{\mathcal{K}} (-1)^k \tau^2 N \Delta\phi \Delta^k (N^{-1}\phi\widehat{N}) \, \mathrm{d}\mu_g\Big| \lesssim \|\tau N \Delta\phi\|_{H^{k-1}} \|\tau(N^{-1}\widehat{N})\phi\|_{H^{k+1}} \\ &\lesssim \Big(\|N\|_{L^\infty}\|\tau\phi\|_{H^{k+1}} + \|\tau\Delta\phi\|_{L^\infty} \sum_{1\leq|I|\leq k-1} \|\nabla^I N\|_{L^2} \\ &\qquad + \sum_{1\leq|I|\leq k-2} \|\nabla^I N\|_{L^4} \sum_{1\leq|J|\leq k-2} \|\tau\nabla^J \Delta\phi\|_{L^4}\Big) \\ &\quad \times \Big(\|N^{-1}\widehat{N}\|_{L^\infty}\|\tau\phi\|_{H^{k+1}} + |\tau|\|\phi\|_{L^\infty} \sum_{1\leq|I|\leq k+1} \|\nabla^I N^{-1}\|_{L^2} \\ &\qquad + \sum_{1\leq|I|\leq k} \|\nabla^I N^{-1}\|_{L^4} \sum_{1\leq|J|\leq k} \|\tau\nabla^J\phi\|_{L^4}\Big) \end{aligned} \tag{5.6.26}$$

$$\begin{aligned} &\lesssim \Big(\|\tau\phi\|_{H^{k+1}} + \|\tau\phi\|_{H^4}\|\widehat{N}\|_{H^{k-1}}\Big)\Big(|\tau|\|\phi\|_{H^2}\|\widehat{N}\|_{H^{k+1}} + \|\widehat{N}\|_{H^{k+1}}\|\tau\phi\|_{H^{k+1}}\Big) \\ &\lesssim \|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + \|\widehat{N}\|_{H^{k+1}}\mathcal{E}_k(\phi). \end{aligned} \tag{5.6.27}$$

Sums involving $|I| \leq k-1$ or $|I| \leq k-2$ do not exist for $k = 1$ and $k = 2$ respectively.

The key point above is the estimate

$$\|\tau N^{-1}\widehat{N}\phi\|_{H^{k+1}} \lesssim \|\tau\phi\|_{H^{k+1}}\|\widehat{N}\|_{H^{k+1}} + |\tau|\varepsilon^2 \lesssim \tilde{\mathcal{E}}_k(\phi)^{1/2}\|\widehat{N}\|_{H^{k+1}} + |\tau|\varepsilon^2. \tag{5.6.28}$$

Note the first term $\tilde{\mathcal{E}}_k(\phi)^{1/2}\|\widehat{N}\|_{H^{k+1}}$ above is worse than the term $\tilde{\mathcal{E}}_k(\phi)^{1/2}|\tau|$ from (5.6.17). This is because have more derivatives to distribute and so we must allow for a term with both high derivatives in $\widehat{N}$ and ϕ.

For the remaining terms of L^1_k we integrate by parts k times to obtain

$$\begin{aligned}
&\Big|\int_{\mathcal{K}} \tau^2\nabla^a N\nabla_a\phi\Delta^k\phi'\Big| + \Big|\int_{\mathcal{K}} \tau^2\nabla^a N\nabla_a\phi\Delta^k(N^{-1}\phi\widehat{N})\Big| \\
&\lesssim \Big(\|\nabla N\|_{L^\infty}\|\tau\nabla\phi\|_{H^k} + \|\tau\nabla\phi\|_{L^\infty}\|\nabla N\|_{H^k} + \|\nabla N\|_{H^k}\|\tau\nabla\phi\|_{H^k}\Big) \\
&\qquad \times \big(\|\tau\phi'\|_{H^k} + \|\tau N^{-1}\phi\widehat{N}\|_{H^k}\big) \\
&\lesssim \big(\|\widehat{N}\|_{H^3} + \|\widehat{N}\|_{H^{k+1}}\big)\mathcal{E}_k(\phi) + |\tau|\|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + \varepsilon^4|\tau|\|\widehat{N}\|_{H^2}.
\end{aligned} \tag{5.6.29}$$

Thus

$$|I^2_k| \lesssim \big(\|\widehat{N}\|_{H^{k+1}} + \|\widehat{N}\|_{H^3}\big)\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + |\tau|\varepsilon^4. \tag{5.6.30}$$

Turning to I^3_k, we see that the last two terms contain no factors of $\widehat{N}$. We estimate these using (5.6.18) to obtain

$$\begin{aligned}
\Big|\int_{\mathcal{K}} \tau^2(2-N)N^{-1}\phi\Delta^k\phi'\,\mathrm{d}\mu_g\Big| &\lesssim \|\tau(2-N)N^{-1}\phi\|_{H^k}\|\tau\phi'\|_{H^k} \\
&\lesssim \big(|\tau| + \|\widehat{N}\|_{H^{k+1}}\big)\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_2(\phi).
\end{aligned} \tag{5.6.31}$$

The other terms of I^3_k are estimated in a similar manner, using instead (5.6.17).

$$\begin{aligned}
&\Big|\int_{\mathcal{K}} \tau^2\Big(-3\widehat{N}\phi'\Delta^k\phi' + (15-3N)\phi'\Delta^k(\widehat{N}N^{-1}\phi) + \tfrac{9}{2}(N-\tfrac{3}{2})N^{-1}\phi\Delta^k(N^{-1}\phi\widehat{N})\Big)\,\mathrm{d}\mu_g\Big| \\
&\lesssim \big(\|\widehat{N}\|_{H^{k+1}} + |\tau| + \|\widehat{N}\|_{H^2}\big)\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_2(\phi) + |\tau|\varepsilon^4.
\end{aligned}$$

Thus

$$|I^3_k| \lesssim \big(\|\widehat{N}\|_{H^{k+1}} + |\tau| + \|\widehat{N}\|_{H^2}\big)\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^k}\tilde{\mathcal{E}}_2(\phi) + |\tau|\varepsilon^4. \tag{5.6.32}$$

□

We now estimate the commutator terms C_k, divided into those commutators of the form $[\hat{\partial}_0, \Delta^k]$, see lemma 5.26, or those of the form $[\Delta^k, N]$, see lemma 5.28. We start with the following identity, adapted from [AF20, (6.6)] and [CBC02].

Lemma 5.26 (Commutator identity)**.** *For some appropriately smooth functions v, w and $k \geq 1$ we have*

$$\begin{aligned}
&\Big|\int_{\mathcal{K}} v[\hat{\partial}_0, \Delta^k]w\,\mathrm{d}\mu_g\Big| \\
&\lesssim \big(\|v\|_{H^k}\|w\|_{H^{k-1}} + \|w\|_{H^k}\|v\|_{H^{k-1}}\big)\big(\|\Sigma\|_{H^3} + \|\widehat{N}\|_{H^3} + \|\Sigma\|_{H^k} + \|\widehat{N}\|_{H^k}\big).
\end{aligned} \tag{5.6.33}$$

Proof. From [AF20] we have

$$[\hat\partial_0, \Delta^k]w = g^{a_1a_2}\cdots g^{a_{2k-1}}g^{a_{2k}}[\hat\partial_0, \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}]w$$
$$= g^{a_1a_2}\cdots g^{a_{2k-1}}g^{a_{2k}}\cdot$$
$$\Big[\sum_{i\le 2k-1}\sum_{i+1\le j\le 2k}\nabla_{a_1}\cdots\nabla_{a_{i-1}}\Big(\nabla_{a_{i+1}}\cdots\nabla_{a_{j-1}}\nabla_c\nabla_{a_{j+1}}\cdots\nabla_{a_{2k}}(w)\cdot K^c_{a_ja_i}\Big)\Big],$$

where

$$K^a_{bc} = \nabla_b(Nk^a_c) + \nabla_c(Nk^a_b) - \nabla^a(k_{cb}). \tag{5.6.34}$$

Thus after integration by parts, we find

$$\Big|\int_{\mathcal{K}} v[\hat\partial_0, \Delta^k]w\,\mathrm{d}\mu_g\Big|$$
$$= \sum_{2\le i\le 2k-1}\sum_{i+1\le j\le 2k}\int_{\mathcal{K}} vg^{a_1a_2}\cdots g^{a_{2k-1}}g^{a_{2k}}\nabla_{a_1}\cdots\nabla_{a_{i-1}}\cdot$$
$$\Big(\nabla_{a_{i+1}}\cdots\nabla_{a_{j-1}}\nabla_c\nabla_{a_{j+1}}\cdots\nabla_{a_{2k}}(w)\cdot K^c_{a_ja_i}\Big)\,\mathrm{d}\mu_g \tag{5.6.35}$$
$$= \Big|\sum_{\substack{|I|+|J|=2k-1\\|J|\ge 1}}\int_{\mathcal{K}} c_{IJ}v\nabla^I(\nabla^J(w)K)\,\mathrm{d}\mu_g\Big|$$
$$\lesssim \Big(\sum_{\substack{|I|+|J|=2k-1\\1\le|J|\le k}} + \sum_{\substack{|I|+|J|=2k-1\\|I|\le k}}\Big)c_{IJ}\int_{\mathcal{K}} v\nabla^I(\nabla^J(w)K)\,\mathrm{d}\mu_g$$
$$\lesssim \sum_{\substack{|I''|+|J|=k-1\\|I'|=k\\|J|\ge 1}}\int_{\mathcal{K}}\big|\nabla^{I'}v\nabla^{I''}\left(\nabla^J w\cdot K\right)\big|\,\mathrm{d}\mu_g + \sum_{\substack{|I|+|J''|=k-1\\|J'|=k\\|J''|\ge 1}}\int_{\mathcal{K}}\big|\nabla^{J'}w\nabla^{J''}\left(\nabla^I v\cdot K\right)\big|\,\mathrm{d}\mu_g$$
$$\lesssim \|v\|_{H^k}\left(\|K\|_{L^\infty}\|w\|_{H^{k-1}} + \|w\|_{H^{k-1}}\|K\|_{H^{k-1}}\right)$$
$$+ \|w\|_{H^k}\left(\|K\|_{L^\infty}\|v\|_{H^{k-1}} + \|v\|_{H^{k-1}}\|K\|_{H^{k-1}}\right)$$
$$\lesssim \left(\|v\|_{H^k}\|w\|_{H^{k-1}} + \|w\|_{H^k}\|v\|_{H^{k-1}}\right)\left(\|K\|_{L^\infty} + \|K\|_{H^{k-1}}\right). \tag{5.6.36}$$

Finally $k = \Sigma + g/3$ so that $K = \nabla(Nk) = \nabla(N\Sigma) + \frac{g}{3}\nabla N$. Thus

$$\|K\|_{L^\infty} + \|K\|_{H^{k-1}} \lesssim \|N\|_{L^\infty}\|\nabla\Sigma\|_{L^\infty} + \|\Sigma\|_{L^\infty}\|\nabla N\|_{L^\infty} + \|\nabla N\|_{L^\infty} + \|N\nabla\Sigma\|_{H^{k-1}}$$
$$+ \|\Sigma\nabla N\|_{H^{k-1}} + \|\nabla N\|_{H^{k-1}}$$
$$\lesssim \|\Sigma\|_{H^3} + \|\widehat{N}\|_{H^3} + \|\Sigma\|_{H^k} + \|\widehat{N}\|_{H^k}. \tag{5.6.37}$$

□

Lemma 5.27. *Assume the bootstrap assumptions* (5.4.1) *hold, then for $k \le N_0$*

$$|C^1_k| \lesssim \big(\tilde{\mathcal{E}}_k(\phi) + |\tau|\varepsilon^4\big)\big(\|K\|_{L^\infty} + \|K\|_{H^{k-1}}\big). \tag{5.6.38}$$

Proof. Using lemma 5.26 the 'symmetric' terms are controlled by

$$\Big|\int_{\mathcal{K}}\big(\tau^2\phi'[\hat\partial_0, \Delta^k]\phi' + m^2\phi[\hat\partial_0, \Delta^k]\phi\big)\,\mathrm{d}\mu_g\Big|$$
$$\lesssim \|K\|_{L^\infty}\mathcal{E}_k(\phi)^{1/2}\mathcal{E}_{k-1}(\phi)^{1/2} + \mathcal{E}_k(\phi)^{1/2}\mathcal{E}_{k-1}(\phi)^{1/2}\|K\|_{H^{k-1}}$$

$$\lesssim (\|K\|_{L^\infty} + \|K\|_{H^{k-1}})\,\tilde{\mathcal{E}}_k(\phi). \tag{5.6.39}$$

The remaining terms are

$$\begin{aligned}
&\Big|\int_{\mathcal{K}} \tau^2\phi[\hat{\partial}_0, \Delta^{k+1}]\phi\,\mathrm{d}\mu_g\Big| + \Big|\int_{\mathcal{K}} \tau^2\phi'[\hat{\partial}_0, \Delta^{k}]\big(N^{-1}\phi\widehat{N}\big)\,\mathrm{d}\mu_g\Big| \\
&\quad + \Big|\int_{\mathcal{K}} \tau^2(N^{-1}\phi)[\hat{\partial}_0, \Delta^{k}]\big((\tfrac{3}{2}-N)N^{-1}\phi\big)\,\mathrm{d}\mu_g\Big| \\
&\quad \lesssim \Big(|\tau|\|\tau\phi\|_{H^{k+1}}\|\phi\|_{H^k} + \|\tau\phi'\|_{H^k}\|\tau N^{-1}\phi\widehat{N}\|_{H^k} + \|\tau N^{-1}\phi\|_{H^k}\|\tau\phi(1-3/2N)\|_{H^k}\Big) \\
&\qquad \times \big(\|K\|_{L^\infty} + \|K\|_{H^{k-1}}\big) \\
&\quad \lesssim \Big(|\tau|\mathcal{E}_k(\phi) + |\tau|\varepsilon^4 + \|\widehat{N}\|_{H^{k+1}}\mathcal{E}_k(\phi)\Big)\big(\|K\|_{L^\infty} + \|K\|_{H^{k-1}}\big).
\end{aligned} \tag{5.6.40}$$

In the final line we used lemma 5.23.

□

The final result in this section controls commutators involving N.

Lemma 5.28. *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $k \le N_0$

$$|C_k^2| \lesssim \|\widehat{N}\|_{H^3}\tilde{\mathcal{E}}_k(\phi) + \|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + \|\widehat{N}\|_{H^{k+1}}\tilde{\mathcal{E}}_k(\phi). \tag{5.6.41}$$

Proof. First note the expansion

$$[\Delta^k, N]w = \sum_{|I|+|J|=2k-1} c_{IJ}\nabla^{I+1}N\nabla^J w, \tag{5.6.42}$$

where the constants c_{IJ} are functions of g and so are bounded below by some large overall constant. For general functions w, v and $k \ge 1$ we have the following

$$\begin{aligned}
&\int_{\mathcal{K}} v[\Delta^k, N]w\,\mathrm{d}\mu_g \\
&\lesssim \sum_{\substack{|I|+|J'|=k-1\\|J''|=k}} \|\nabla^{J'}v\nabla^{I+1}N\nabla^{J''}w\|_{L^1} + \sum_{\substack{|I'|+|J|=k-1\\|I''|=k+1}} \|\nabla^{I'}v\nabla^{J}w\nabla^{I''}N\|_{L^1} \\
&\lesssim \Big(\|v\|_{L^\infty}\|\widehat{N}\|_{H^k} + \|\nabla N\|_{L^\infty}\|v\|_{H^{k-1}} + \sum_{|\alpha|\le k-2}\|\nabla^\alpha v\|_{L^4}\sum_{|\beta|\le k-1}\|\widehat{N}\|_{L^4}\Big)\|\tau w\|_{H^k} \\
&\quad + \Big(\|w\|_{L^\infty}\|v\|_{H^{k-1}} + \|v\|_{L^\infty}\|w\|_{H^{k-1}} + \sum_{|\alpha|\le k-2}\|\nabla^\alpha v\|_{L^4}\sum_{|\beta|\le k-2}\|w\|_{L^4}\Big)\|\widehat{N}\|_{H^{k+1}} \\
&\lesssim \Big(\|v\|_{H^2} + \|v\|_{H^{k-1}}\Big)\|\widehat{N}\|_{H^{k+1}}\|w\|_{H^k} \\
&\quad + \Big(\|w\|_{H^k}\|\widehat{N}\|_{H^3} + \|w\|_{H^2}\|\widehat{N}\|_{H^{k+1}}\Big)\|\widehat{N}\|_{H^{k-1}}\|w\|_{H^k}.
\end{aligned} \tag{5.6.43}$$

The claim follows, using for example the estimate (5.6.16).

□

5.7 Lapse and shift estimates

We first state the following elliptic estimates from [AF20, Proposition 17] for the lapse and shift.

Proposition 5.29. *Under appropriate smallness conditions we have the pointwise estimate $N \in (0,3]$ and the following estimates for $\ell \leq N_0 + 1$*

$$
\begin{aligned}
&\|\widehat{N}\|_{H^\ell} \leq C\big(\|\Sigma\|^2_{H^{\ell-2}} + |\tau|\|\eta\|_{H^{\ell-2}}\big), \\
&\|X\|_{H^\ell} \leq C\big(\|\Sigma\|^2_{H^{\ell-2}} + \|g-\gamma\|^2_{H^{\ell-1}} + |\tau|\|\eta\|_{H^{\ell-3}} + \tau^2\|Nj\|_{H^{\ell-2}}\big).
\end{aligned}
\tag{5.7.1}
$$

Applied to the present case this yields the following estimate for the lapse function.

Lemma 5.30 (Lapse estimate)**.** *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $2 \leq \ell \leq N_0 + 1$ *we have*

$$
\|\widehat{N}\|_{H^\ell} \lesssim \|\Sigma\|^2_{H^{\ell-2}} + |\tau|\|\rho\|_{L^\infty} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi), \tag{5.7.2}
$$

and furthermore

$$
\|\widehat{N}\|_{H^\ell} \lesssim \varepsilon^2|\tau| + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi). \tag{5.7.3}
$$

Proof. For $k \geq 0$ and some function w we have

$$
\begin{aligned}
\|w^2\|^2_{H^k} &\lesssim \|w\|^2_{L^\infty}\sum_{|I|\leq k}\int_{\mathcal{K}}|\nabla^I w|^2\,\mathrm{d}\mu_g + \sum_{|I|+|J|+2\leq k}\int_{\mathcal{K}}\big(|\nabla^{I+1}w|^4 + |\nabla^{J+1}w|^4\big)\,\mathrm{d}\mu_g \\
&\lesssim \|w\|^2_{L^\infty}\|w\|^2_{H^\ell} + \sum_{|I|+1\leq k-1}\|\nabla^{I+1}w\|^4_{L^4} \\
&\lesssim \|w\|^2_{L^\infty}\|w\|^2_{H^k} + \|w\|^4_{H^k}.
\end{aligned}
\tag{5.7.4}
$$

Using the definition of $\eta = -\frac{1}{2}(mu)^2 + 2\left(\tau\left(\frac{3}{2}N^{-1}\phi - \phi'\right)\right)^2$ from (5.2.11c) and the estimate from proposition 5.29 we see

$$
\begin{aligned}
\|\widehat{N}\|_{H^\ell} &\lesssim \|\Sigma\|^2_{H^{\ell-2}} + |\tau|\|\eta\|_{H^{\ell-2}} \\
&\lesssim \|\Sigma\|^2_{H^{\ell-2}} + |\tau|\big(\|m\phi\|_{L^\infty} + \|\tau N^{-1}\phi\|_{L^\infty} + \|\tau\phi'\|_{L^\infty}\big)\tilde{\mathcal{E}}_{\ell-2}(\phi)^{1/2} + |\tau|\tilde{\mathcal{E}}_{k-2}(\phi) \\
&\lesssim \|\Sigma\|^2_{H^{\ell-2}} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi) + |\tau|\|\rho\|_{L^\infty}.
\end{aligned}
\tag{5.7.5}
$$

□

Remark 5.31. It is crucial in the final line of the previous proof (specifically for $\ell = 2$) to use the pointwise estimate from proposition 5.20 instead of the standard Sobolev estimate invoking $\tilde{\mathcal{E}}_2(\phi)$, since the latter would have created a $e^{\kappa T}$ growth preventing the envisioned bootstrap argument.

Lemma 5.32 (Shift estimate)**.** *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $3 \leq \ell \leq N_0 + 1$

$$
\|X\|_{H^\ell} \lesssim \|\Sigma\|^2_{H^{\ell-2}} + \|g-\gamma\|^2_{H^{\ell-1}} + |\tau|\|\rho\|_{L^\infty} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi), \tag{5.7.6}
$$

and furthermore

$$
\|X\|_{H^\ell} \lesssim \varepsilon^2|\tau| + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi). \tag{5.7.7}
$$

Proof. We use the estimate from proposition 5.29. We may use the estimate for $\|\eta\|_{H^k}$ derived in lemma 5.30, and also need an estimate for the rescaled matter current

$$
\jmath^a = -\tau\big(\tfrac{3}{2}N^{-1}\phi - \phi'\big)\nabla^a\phi. \tag{5.7.8}
$$

For $k \geq 0$ using the standard Sobolev embeddings, we have

$$\begin{aligned}\tau\|j\|_{H^k} &\lesssim \|\tau\nabla\phi\|_{L^\infty}\tilde{\mathcal{E}}_k(\phi)^{1/2} + \big(\|\tau N^{-1}\phi\|_{L^\infty} + \|\tau\phi'\|_{L^\infty}\big)\tilde{\mathcal{E}}_k(\phi)^{1/2} + \tilde{\mathcal{E}}_k(\phi) \\ &\lesssim \|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_k(\phi).\end{aligned} \tag{5.7.9}$$

So for $3 \leq \ell \leq N+1$ we have

$$\begin{aligned}\|X\|_{H^\ell} &\lesssim \|\Sigma\|^2_{H^{\ell-2}} + \|g-\gamma\|^2_{H^{\ell-1}} + |\tau|\|\eta\|_{H^{\ell-3}} + \tau^2\|Nj\|_{H^{\ell-2}} \\ &\lesssim \|\Sigma\|^2_{H^{\ell-2}} + \|g-\gamma\|^2_{H^{\ell-1}} + |\tau|\tilde{\mathcal{E}}_{\ell-3}(\phi) + |\tau|\|\rho\|_{L^\infty} \\ &\quad + |\tau|\Big(\|N\|_{L^\infty}\|\tau j\|_{H^{\ell-2}} + \sum_{|I|\leq\ell-2}\|\nabla^I\widehat{N}\|_{L^\infty}\|\tau j\|_{L^2} + \|\widehat{N}\|_{H^{\ell-2}}\|\tau j\|_{H^{\ell-2}}\Big) \\ &\lesssim \|\Sigma\|^2_{H^{\ell-2}} + \|g-\gamma\|^2_{H^{\ell-1}} + |\tau|\tilde{\mathcal{E}}_{\ell-3}(\phi) + |\tau|\big(\|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_{\ell-2}(\phi) + \|\widehat{N}\|_{H^\ell}\tilde{\mathcal{E}}_2(\phi)\big).\end{aligned}$$

□

5.8 Hierarchy between lapse and Klein-Gordon field

In the following lemma we estimate the borderline terms for the Klein-Gordon energy.

Lemma 5.33 (Borderline terms). *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $1 \leq \ell \leq N_0$ *we have*

$$\begin{aligned}\sum_{k=1}^{\ell}\Big|\int_{\mathcal{K}} B_k\,\mathrm{d}\mu_g\Big| &\lesssim |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_\ell(\phi)^{1/2}\|\widehat{N}\|_{H^\ell} + |\tau|^{-1}\tilde{\mathcal{E}}_\ell(\phi)\|\widehat{N}\|_{H^2} \\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_\ell(\phi)^{1/2}\tilde{\mathcal{E}}_{\ell-1}(\phi)^{1/2}\|\widehat{N}\|_{H^{\ell+1}}.\end{aligned} \tag{5.8.1}$$

Proof.

$$\begin{aligned}&\Big|\int_{\mathcal{K}} B_k\,\mathrm{d}\mu_g\Big| = \Big|m^2\int_{\mathcal{K}}\phi'\left(N\Delta^k\phi - \Delta^k(N\phi)\right)\,\mathrm{d}\mu_g\Big| \\ &\lesssim \int_{\mathcal{K}}\sum_{|I|+1+|J|=k}|m^2\nabla^{I+1}N\nabla^J\phi||\sum_{|I'|\leq k}\nabla^{I'}\phi'|\,\mathrm{d}\mu_g \\ &\quad + \int_{\mathcal{K}}\sum_{|I|+1+|J|=k}|\nabla^{I+1}N\nabla^J\phi'|\sum_{|I'|\leq k}|m^2\nabla^{I'}\phi|\,\mathrm{d}\mu_g \\ &\lesssim \|m\phi\|_{L^\infty}|\tau|^{-1}\|\tau\phi'\|_{H^k}\|\widehat{N}\|_{H^k} + |\tau|^{-1}\|\tau\phi'\|_{H^k}\sum_{|I|+1+|J|=k}\|\nabla^{I+1}N\|_{H^1}\|m\nabla^J\phi\|_{H^1} \\ &\quad + |\tau|^{-1}\|\tau\phi'\|_{L^\infty}\|m\phi\|_{H^k}\|\widehat{N}\|_{H^k} + |\tau|^{-1}\|m\phi\|_{H^k}\sum_{|I|+1+|J|=k}\|\nabla^{I+1}N\|_{H^1}\|\tau\nabla^J\phi'\|_{H^1} \\ &\lesssim |\tau|^{-1}\|\rho\|^{1/2}_{L^\infty}\mathcal{E}_k(\phi)^{1/2}\|\widehat{N}\|_{H^k} \\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}\Big(\|\nabla N\|_{H^1}\|\phi\|_{H^k} + \|\phi\|_{H^{k-1}}\|\widehat{N}\|_{H^{k+1}}\Big) \\ &\quad + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_k(\phi)^{1/2}\|\widehat{N}\|_{H^k} \\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}\Big(\|\nabla N\|_{H^1}\|\tau\phi'\|_{H^k} + \|\widehat{N}\|_{H^{k+1}}\|\tau\phi'\|_{H^{k-1}}\Big) \\ &\lesssim |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_k(\phi)^{1/2}\|\widehat{N}\|_{H^k} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)\|\widehat{N}\|_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}\tilde{\mathcal{E}}_{k-1}(\phi)^{1/2}\|\widehat{N}\|_{H^{k+1}}.\end{aligned}$$

Summing from $k=1$ to ℓ gives the required result. □

Remark 5.34. We now outline the key ideas behind closing the lapse and Klein-Gordon bootstrap assumptions, as proved below in proposition 5.35. The estimates for $\tilde{\mathcal{E}}_0(\phi)$ and $\|\widehat{N}\|_{H^2}$ are readily improved. Then, starting from $\ell = 1$, the most problematic terms needed for the $\tilde{\mathcal{E}}_\ell(\phi)$ estimate are contained in lemma 5.33. Nonetheless, lemma 5.33 tells us that we need information about $\|\widehat{N}\|_{H^2}$, $\|\widehat{N}\|_{H^{\ell+1}}$ and $\tilde{\mathcal{E}}_{\ell-1}(\phi)$, all of which have been upgraded from the previous steps. The upgraded estimate for $\tilde{\mathcal{E}}_\ell(\phi)$ is then used, via lemma 5.30, to close the bootstrap estimate for $\|\widehat{N}\|_{H^{\ell+2}}$. One then moves onto improving the estimate for $\tilde{\mathcal{E}}_{\ell+1}(\phi)$ and continues until $\ell = N_0 - 1$.

Proposition 5.35 (Upgraded lapse and Klein-Gordon estimates)**.** *Assume the bootstrap assumptions* (5.4.1) *hold, then*

$$\|\widehat{N}\|_{H^2} \lesssim \varepsilon^2 e^{-T}, \tag{5.8.2}$$
$$\tilde{\mathcal{E}}_0(\phi) \lesssim \varepsilon^2, \tag{5.8.3}$$

and for higher orders $1 \le \ell \le N_0$

$$\|\widehat{N}\|_{H^{\ell+1}} \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}, \tag{5.8.4a}$$
$$\tilde{\mathcal{E}}_\ell(\phi) \lesssim \varepsilon^2 e^{C\varepsilon T}. \tag{5.8.4b}$$

Proof. From proposition 5.21

$$\tilde{E}_0(\phi)\big|_T \lesssim \tilde{E}_0(\phi)\big|_{T_0}. \tag{5.8.5}$$

The lapse estimate lemma 5.30 with $\ell = 2$ then implies

$$\|\widehat{N}\|_{H^2} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_0(\phi) \lesssim \varepsilon^2 e^{-T}. \tag{5.8.6}$$

The Klein-Gordon estimate of proposition 5.22 combined with the borderline estimate of lemma 5.33 for $\ell = 2 - 1 = 1$ together imply

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_1(\phi) \lesssim{}& \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_1(\phi) + |\tau|\tilde{\mathcal{E}}_1(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4|\tau| + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_1(\phi)^{1/2}\|\widehat{N}\|_{H^2} \\ &+ |\tau|^{-1}\tilde{\mathcal{E}}_1(\phi)\|\widehat{N}\|_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_1(\phi)^{1/2}\tilde{\mathcal{E}}_0(\phi)^{1/2}\|\widehat{N}\|_{H^2} \\ \lesssim{}& \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4 + \big(\varepsilon e^{(-1+\kappa)T} + |\tau| + \varepsilon^2\big)\tilde{\mathcal{E}}_1(\phi).\end{aligned} \tag{5.8.7}$$

Thus using Grönwall's inequality from lemma 1.20 implies

$$\begin{aligned}\tilde{\mathcal{E}}_1(\phi)\big|_T \le{}& \Big(\tilde{\mathcal{E}}_1(\phi)\big|_{T_0} + C\int_{T_0}^T \big(\varepsilon^3 e^{(-1+\kappa)s} + \varepsilon^4\big)\mathrm{d}s\Big)\exp\Big(C\int_{T_0}^T \big(e^{-s} + \varepsilon e^{(-1+\kappa)s} + \varepsilon^2\big)\mathrm{d}s\Big) \\ \lesssim{}& \big(\tilde{\mathcal{E}}_1(\phi)\big|_{T_0} + \varepsilon^3 e^{C\varepsilon T}\big)\exp(C\varepsilon T).\end{aligned} \tag{5.8.8}$$

Note we used the identity: $x \le 1 + x \le e^x$ for $x \ge 0$. Returning to the lapse estimate from lemma 5.30 with $\ell = 3$ now implies

$$\|\widehat{N}\|_{H^3} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_1(\phi) \lesssim \varepsilon^2 e^{(-1+2C\varepsilon)T}. \tag{5.8.9}$$

Now we use this result to improve the $\ell = 2$ estimate for the Klein-Gordon field. From lemma 5.33, proposition 5.22 and the upgraded estimates obtained so far, we have

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_2(\phi) \lesssim{}& \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_2(\phi) + |\tau|\tilde{\mathcal{E}}_2(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4|\tau| + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_2(\phi)^{1/2}\|\widehat{N}\|_{H^2} \\ &+ |\tau|^{-1}\tilde{\mathcal{E}}_2(\phi)\|\widehat{N}\|_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_2(\phi)^{1/2}\tilde{\mathcal{E}}_1(\phi)^{1/2}\|\widehat{N}\|_{H^3}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_2(\phi) + |\tau|\tilde{\mathcal{E}}_2(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^3\tilde{\mathcal{E}}_2(\phi)^{1/2} + \varepsilon^2\tilde{\mathcal{E}}_2(\phi) \\
&\quad + \varepsilon^3 e^{C\varepsilon T}\tilde{\mathcal{E}}_2(\phi)^{1/2} \\
&\lesssim \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4 e^{2C\varepsilon T} + \big(\varepsilon e^{(-1+\kappa)T} + |\tau| + \varepsilon^2\big)\tilde{\mathcal{E}}_2(\phi). \qquad (5.8.10)
\end{aligned}
$$

Thus by Grönwall's inequality from lemma 1.20:

$$
\begin{aligned}
\tilde{\mathcal{E}}_2(\phi)\big|_T &\lesssim \Big(\tilde{\mathcal{E}}_2(\phi)\big|_{T_0} + \int_{T_0}^T \big(\varepsilon^3 e^{(-1+\kappa)s} + \varepsilon^4 e^{C\varepsilon s}\mathrm{d}s\big)\Big)\exp\Big(C\int_{T_0}^T (e^{-s}+\varepsilon)\mathrm{d}s\Big) \\
&\lesssim \big(\tilde{\mathcal{E}}_2(\phi)\big|_{T_0} + \varepsilon^3 e^{C\varepsilon T}\big)e^{C\varepsilon T}. \\
&< \varepsilon^2 e^{C\varepsilon T}. \qquad (5.8.11)
\end{aligned}
$$

Continuing in this way we stop at $\ell = N+1$ for the lapse estimate.

$$
\|\widehat{N}\|_{H^{N+1}} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_{N-1}(\phi) \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}. \qquad (5.8.12)
$$

Using this to estimate the final $\ell = N$ energy (note $N \geq 4$) for the Klein-Gordon field gives

$$
\begin{aligned}
\partial_T\tilde{\mathcal{E}}_N(\phi) &\lesssim \varepsilon e^{(-1+C\varepsilon)T}\tilde{\mathcal{E}}_N(\phi) + |\tau|\tilde{\mathcal{E}}_N(\phi) + \varepsilon^3 e^{(-1+C\varepsilon)T} + |\tau|\varepsilon^4 + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_N(\phi)^{1/2}\|\widehat{N}\|_{H^N} \\
&\quad + |\tau|^{-1}\tilde{\mathcal{E}}_N(\phi)\|\widehat{N}\|_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_N(\phi)^{1/2}\tilde{\mathcal{E}}_{N-1}(\phi)^{1/2}\|\widehat{N}\|_{H^{N+1}} \\
&\lesssim \varepsilon^3 e^{(-1+C\varepsilon)T} + |\tau|\tilde{\mathcal{E}}_N(\phi) + \varepsilon^3 e^{C\varepsilon T}\tilde{\mathcal{E}}_N(\phi)^{1/2} + \varepsilon^2\tilde{\mathcal{E}}_N(\phi) \\
&\lesssim \varepsilon^3 e^{(-1+C\varepsilon)T} + \varepsilon^4 e^{2C\varepsilon T} + \big(\varepsilon e^{(-1+C\varepsilon)T} + |\tau| + \varepsilon^2\big)\tilde{\mathcal{E}}_N(\phi). \qquad (5.8.13)
\end{aligned}
$$

Thus by lemma 1.20 we have

$$
\begin{aligned}
\tilde{\mathcal{E}}_N(\phi)\big|_T &\lesssim \Big(\tilde{\mathcal{E}}_N(\phi)\big|_{T_0} + \int_{T_0}^T \big(\varepsilon^2 e^{(-1+C\varepsilon)s} + \varepsilon^4 e^{C\varepsilon s}\big)\mathrm{d}s\Big)\exp\Big(C\int_{T_0}^T (\varepsilon+e^{-s})\mathrm{d}s\Big) \\
&\lesssim \big(\tilde{\mathcal{E}}_N(\phi)\big|_{T_0} + \varepsilon^3 e^{C\varepsilon T}\big)e^{C\varepsilon T}. \\
&< \varepsilon^2 e^{C\varepsilon T}. \qquad (5.8.14)
\end{aligned}
$$

□

5.9 Energy estimate - Geometry

In this final section we obtain improved estimates for the shift vector field and the second fundamental form and metric perturbation.

Corollary 5.36 (Improved shift estimate)**.** *Assume the bootstrap assumptions* (5.4.1) *hold, then for* $3 \leq \ell \leq N_0 + 1$

$$
\|X\|_{H^\ell} \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}. \qquad (5.9.1)
$$

Proof. This follows clearly from lemma 5.32 and proposition 5.35. □

Using the energy estimate given in [AF20, Lemma 20], itself adapted from [AM11] we have the following estimate for the second fundamental form and metric perturbation.

Corollary 5.37 (Improved geometry estimate). *Assume the bootstrap assumptions (5.4.1) hold, then for $1 \leq \ell \leq N_0+1$*

$$\partial_T E^g{}_\ell \leq -2\alpha E^g{}_\ell + 6 E^g{}_\ell^{1/2} |\tau| \|NS\|_{H^{\ell-1}} + C E^g{}_\ell^{3/2} + C E^g{}_\ell^{1/2} \big(|\tau| \|\eta\|_{H^{\ell-1}} + \tau^2 \|Nj\|_{H^{\ell-2}}\big). \tag{5.9.2}$$

Furthermore

$$\partial_T E^g{}_\ell \leq -2\alpha E^g{}_\ell + C E^g{}_\ell^{1/2} \varepsilon^2 e^{(-1+C\varepsilon)T} + C E^g{}_\ell^{3/2}. \tag{5.9.3}$$

Finally

$$E^g{}_\ell\big|_T \leq C\varepsilon^2 e^{-2\alpha\zeta T}, \tag{5.9.4}$$

and hence

$$\|g-\gamma\|^2_{H^{\ell+1}} + \|\Sigma\|^2_{H^\ell} \leq C\varepsilon^2 e^{-2\alpha\zeta T}, \tag{5.9.5}$$

where for sufficiently small ε we may choose ζ arbitrarily close to 1, in particular $\zeta \leq 1 - \frac{C\varepsilon}{\alpha}$.

Proof. The estimate (5.9.2) comes from [AF20, Lemma 20]. S_{ij} was given in (5.2.11d). So for $2 \leq \ell \leq N_0+1$ we find

$$\begin{aligned}
\|NS\|_{H^{\ell-1}} &\lesssim \|N\|_{L^\infty} \Big(\|\phi\|_{L^\infty} \|\phi\|_{H^{\ell-1}} + \|\phi\|^2_{H^{\ell-1}} + \|\tau\nabla\phi\|_{L^\infty} \|\tau\nabla\phi\|_{H^{\ell-1}} + \|\tau\nabla\phi\|^2_{H^{\ell-1}} \Big) \\
&\quad + \|\widehat{N}\|_{H^{\ell-1}} \Big(\|\phi\|^2_{L^\infty} + \|\phi\|^2_{L^\infty} \|\phi\|_{H^{\ell-1}} + \|\phi\|^2_{H^{\ell-1}} + \|\tau\nabla\phi\|^2_{L^\infty} \\
&\qquad + \|\tau\nabla\phi\|_{L^\infty} \|\tau\nabla\phi\|_{H^{\ell-1}} + \|\tau\nabla\phi\|^2_{H^{\ell-1}} \Big) \\
&\lesssim \|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi) + \|\widehat{N}\|_{H^{\ell-1}} \big(\|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi) \big),
\end{aligned} \tag{5.9.6}$$

where we took note of the product identity (5.7.4). Also following the method of lemmas 5.30 and 5.32 we have

$$\begin{aligned}
|\tau| \|\eta\|_{H^{\ell-1}} + \tau^2 \|Nj\|_{H^{\ell-2}} &\lesssim |\tau| \tilde{\mathcal{E}}_{\ell-1}(\phi) + |\tau| \|\rho\|_{L^\infty} \\
&\quad + |\tau| \big(\|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_{\ell-2}(\phi) \big) \big(\|N\|_{L^\infty} + \|\widehat{N}\|_{H^\ell} \big).
\end{aligned} \tag{5.9.7}$$

Putting these together gives

$$\begin{aligned}
\partial_T E^g{}_\ell &\leq -2\alpha E^g{}_\ell + 6 E^g{}_\ell^{1/2} \tau \|NS\|_{H^{\ell-1}} + C E^g{}_\ell^{3/2} + C E^g{}_\ell^{1/2} \big(|\tau| \|\eta\|_{H^{\ell-1}} + \tau^2 \|Nj\|_{H^{\ell-2}} \big) \\
&\leq -2\alpha E^g{}_\ell + 6 E^g{}_\ell^{1/2} |\tau| \big(\|\rho\|_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi) \big) + C E^g{}_\ell^{3/2} \\
&\quad + C E^g{}_\ell^{1/2} \Big(|\tau| \tilde{\mathcal{E}}_{\ell-1}(\phi) + |\tau| \|\rho\|_{L^\infty} + |\tau| \|\rho\|_{L^\infty} \|\widehat{N}\|_{H^\ell} + |\tau| \tilde{\mathcal{E}}_{\ell-2}(\phi) \|\widehat{N}\|_{H^\ell} \Big).
\end{aligned} \tag{5.9.8}$$

Thus

$$\partial_T \big(E^g{}_\ell^{1/2} \big) \leq -\alpha E^g{}_\ell^{1/2} + C\varepsilon^2 e^{-T} + C E^g{}_\ell. \tag{5.9.9}$$

Now $\alpha \in [1-\delta_\alpha, 1]$ where δ_α can be made suitably small. Given α, pick ζ such that $\frac{3}{4} < \zeta < 1$ and $-(\alpha\zeta - \frac{3}{4}) < 0$ (ie, $\alpha\zeta > \frac{3}{4}$). Indeed we can guarantee $\alpha\zeta > \frac{3}{4}$ holds by choosing ε sufficiently small such that $1 - \delta_\alpha(\varepsilon) > \frac{3}{4\zeta}$. Then we have

$$\partial_T \big(e^{\frac{3}{4}T} E^g{}_\ell^{1/2} \big) \leq -(\alpha\zeta - \tfrac{3}{4}) e^{\frac{3}{4}T} E^g{}_\ell^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T} - e^{\frac{3}{4}T} E^g{}_\ell^{1/2} \Big(\alpha(1-\zeta) - C E^g{}_\ell^{1/2} \Big)$$

$$\leq -(\alpha\zeta - \tfrac{3}{4})e^{\frac{3}{4}T} E^{g}{}_{\ell}^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T} - e^{\frac{3}{4}T} E^{g}{}_{\ell}^{1/2}\Big(\alpha(1-\zeta) - C\varepsilon\Big)$$
$$\leq -(\alpha\zeta - \tfrac{3}{4})e^{\frac{3}{4}T} E^{g}{}_{\ell}^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T}, \tag{5.9.10}$$

where we dropped the final term by picking ε small enough so that $\alpha(1-\zeta) - C\varepsilon \geq 0$. Then by Grönwall's inequality, from lemma 1.20, we have

$$e^{\delta T} E^{g}{}_{\ell}^{1/2}\big|_T \leq \Big(e^{\frac{3}{4}T_0} E^{g}{}_{\ell}^{1/2}\big|_{T_0} + C\varepsilon^2 \int_{T_0}^{T} e^{(-1+\frac{3}{4})s} \mathrm{d}s\Big)\Big) \exp\Big(-\int_{T_0}^{T} (\alpha\zeta - \tfrac{3}{4})\mathrm{d}s\Big). \tag{5.9.11}$$

This implies

$$E^{g}{}_{\ell}^{1/2}\big|_T \leq \big(e^{\frac{3}{4}T_0} E^{g}{}_{\ell}^{1/2}\big|_{T_0} + C\varepsilon^2\big) e^{-\frac{3}{4}T} e^{-(\alpha\zeta - \frac{3}{4})(T-T_0)}, \tag{5.9.12}$$

and thus

$$E^{g}{}_{\ell}\big|_T \leq (E^{g}{}_{\ell}|_{T_0} + C\varepsilon^4) e^{-2\alpha\zeta T} < C\varepsilon^2 e^{-\frac{3}{2}T}. \tag{5.9.13}$$

□

Proof of theorem 5.12. We now bring together the previous results and prove the main theorem. Consider initial data at time T_0 which is close to the induced data of the Milne model in the sense of the smallness assumption (5.2.19). This data is not necessarily CMC. The maximal globally hyperbolic development of this data under the EKGS is, locally in time, sufficiently close to the background geometry. The existence of a CMC surface in this local solution then can be shown by following the corresponding argument in the vacuum case presented for instance in [FK].

We can then consider the initial data induced on this CMC surface. By choosing (5.2.19) sufficiently small, the new initial data satisfies

$$E_{tot}(T_0) \leq C_0\varepsilon, \tag{5.9.14}$$

where $C_0 > 0$ is a constant and we have defined

$$E_{tot}(T) = (\|g-\gamma\|_{H^{N_0+1}} + \|\Sigma\|_{H^{N_0}})\, e^{\frac{3}{4}T} + \Big(\|\widehat{N}\|_{H^{N_0+1}} + \|X\|_{H^{N_0+1}}\Big) e^{(1-\kappa)T}$$
$$+ \mathcal{E}_{N_0}(\phi)^{1/2} e^{-\kappa T}. \tag{5.9.15}$$

Recall also that $T_0 = 1$.

We next evolve this CMC inital data by the rescaled EKGS system (5.2.15). By lemma 5.9 there exists a solution to (5.2.15) defined up to a maximal CMC time of existence denoted T_* such that either $T_* = +\infty$ or

$$\limsup_{T\to T_*} \|g-\gamma\|_{H^5} + \|\Sigma\|_{H^4} + \sqrt{|\tau_0|}(\|\phi\|_{H^5} + \|\phi_1\|_{H^4}) = \infty. \tag{5.9.16}$$

For $C_I > 0$ a sufficiently large constant, we then define

$$T_+ = \sup\{T > T_0 : E_{tot}(T') \leq C_I\varepsilon \text{ hold for all } T' \in [T_0, T)\}. \tag{5.9.17}$$

By lemma 5.9, together with the smallness assumptions (5.2.19), T_+ exists. The results of proposition 5.9, corollary 5.36 and corollary 5.9 then imply that for sufficiently small

ε, we have

$$E_{tot}(T') \leq \tfrac{1}{2} C_I \varepsilon \tag{5.9.18}$$

for all $T_0 \leq T' \leq T$. Note that in the proofs of these propositions and corollaries, all the estimates were independent of the smallness of the initial data once ε is chosen sufficiently small, and so we can decrease ε in the course of the argument. The same holds for decreasing $|\tau_0| = |\mathrm{tr}_{g_0} k_0|$.

By the continuous dependence of E_{tot} on T, (5.9.18) then contradicts the maximality of T_+ and implies that $E_{tot}(T) \leq C_I \varepsilon$ must hold for all $T \leq T_*$. Thus by the continuity criterion (5.9.16) we can extend the solution to the interval $[T_0, T_* + \varepsilon')$ for a small $\varepsilon' > 0$, where $E_{tot}(T) \leq C_I \varepsilon$, or equivalently (5.4.1), hold on this extended interval. A standard continuity argument, see for example section 1.3.2, implies that $T_* = +\infty$.

Finally, one can use the rate of decay of the perturbation of the geometric variables to show that the solution we have constructed is future complete. Since we have the same decay rate as the work [AF20] on the Einstein-Vlasov equations, this follows exactly as in [AF20, §10.6]. Note that the proof in this work is based on a completeness criterion given by Choquet-Bruhat and Cotsakis in [CBC02]. $\square$

Chapter 6

Stability of spacetimes with supersymmetric compactifications

In this chapter, we[1] study the stability, with respect to the evolution determined by the vacuum Einstein equations, of the Cartesian product of high-dimensional Minkowski space with a compact, Ricci-flat Riemannian manifold that admits a spin structure and a nonzero parallel spinor. Such a product includes the example of Calabi-Yau and other special holonomy compactifications, which play a central role in supergravity and string theory [CHSW85].

6.1 Introduction

Let $(\mathbb{R}^{1+n}, \eta_{\mathbb{R}^{1+n}})$ be the $(1+n)$-dimensional Minkowski spacetime, and let $(\mathcal{K}, \gamma)$ be a compact, d-dimensional Ricci-flat Riemannian manifold that has a cover that admits a spin structure and a nonzero parallel spinor. The spacetime $\mathcal{M} = \mathbb{R}^{1+n} \times \mathcal{K}$ with metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + \gamma \tag{6.1.1}$$

is globally hyperbolic and Ricci flat, i.e, it is a solution to the $(1+n+d)$-dimensional vacuum Einstein equations. In this chapter we refer to $(\mathcal{M}, \hat{g})$ as a spacetime with a supersymmetric compactification and $(\mathcal{K}, \gamma)$ as the internal manifold.

We first fix some notation. The indices are the following, see also definition 1.37:

$$\begin{aligned}
\text{spacetime indices } \mu, \nu &\in \{0, \ldots, 1+n+d\}, \\
\text{spatial indices } a, b &\in \{1, \ldots, 1+n+d\}, \\
\text{external indices } i, j &\in \{0, \ldots, n\}, \\
\text{internal indices } A, B &\in \{1+n+1, \ldots, 1+n+d\}.
\end{aligned}$$

We will sometimes use i, j to denote only external spatial indices, that is $i, j \in \{1, \ldots, n\}$.

[1]The results in this chapter were obtained in collaboration with Lars Andersson (Albert Einstein Institute Potsdam), Pieter Blue (University of Edinburgh) and Shing-Tung Yau (Harvard University). This research was supported in part by the Swedish Research Council under grant no. 2016-06596 as part of the program 'General Relativity, Geometry and Analysis: beyond the first 100 years after Einstein'.

This will be made clear in context.

Given the supersymmetric spacetime $(\mathbb{R}^{1+n}\times\mathcal{K},\hat{g})$ consider the Euclidean metric

$$(g_E)_{\mu\nu} = \hat{g}_{\mu\nu} + 2(\mathrm{d}t)_\mu(\mathrm{d}t)_\nu, \tag{6.1.2}$$

where $\mathrm{d}t$ is with respect to the standard Cartesian coordinates on $\mathbb{R}^{1+n}$. On $\mathcal{K}$ and $\mathbb{R}^{1+n}\times\mathcal{K}$ respectively, define the following inner products on $(0,2)$ tensors

$$\langle u,v\rangle_\gamma = \gamma^{AC}\gamma^{BD}u_{AB}v_{CD}, \tag{6.1.3}$$
$$\langle u,v\rangle_E = g_E^{\mu\nu}g_E^{\rho\sigma}u_{\mu\rho}v_{\nu\sigma}. \tag{6.1.4}$$

Define $|u|_\gamma = (\langle u,u\rangle_\gamma)^{1/2}$, and similarly for $|u|_E$.

The following is our main result. The details of some of the concepts appearing in the statement of the theorem appear in definitions 1.5, 1.7, 6.12 and theorem 6.13.

Theorem 6.1. *Let $n,d\in\mathbb{Z}^+$ be such that $n\geq 9$, and let $N_0\in\mathbb{Z}^+$ be sufficiently large. Let $(\mathbb{R}^{1+n}\times\mathcal{K},\hat{g}=\eta_{\mathbb{R}^{1+n}}+\gamma)$ be a spacetime with a supersymmetric compactification. Let g_S denote the Schwarzschild metric in the $\eta_{\mathbb{R}^{1+n}}$-wave gauge with mass parameter $C_S>0$.*

There is an $\varepsilon>0$ such that if $(\mathbb{R}^n\times\mathcal{K},\bar{g}_{ab},\bar{K}_{ab})$ is an initial data set satisfying $\bar{g}=g_S+\gamma$ and $\bar{K}=0$ where $|x|\geq 1$ and satisfying

$$\sum_{|I|\leq N_0}\|\nabla[\bar{g}]^I(\bar{g}-\hat{g}|_{t=0})\|^2_{L^2(\mathbb{R}^n\times\mathcal{K})} + \sum_{|I|\leq N_0-1}\|\nabla[\bar{g}]^I\bar{K}\|^2_{L^2(\mathbb{R}^n\times\mathcal{K})} + C_S^2 \leq \varepsilon, \tag{6.1.5}$$

then there is a solution g of the vacuum Einstein equations on $\mathbb{R}^{1+n}\times\mathcal{K}$ with initial data $(\mathbb{R}^n\times\mathcal{K},\bar{g}_{ab},\bar{K}_{ab})$ and satisfying the $\hat{g}$-wave gauge. There is the bound

$$\sup_{(t,x^i,\omega)\in\mathcal{H}_s\times\mathcal{K}} t^{2\delta(n)}|g(t,x^i,\omega)-\hat{g}(t,x^i,\omega)|^2_E \lesssim \varepsilon, \tag{6.1.6}$$

where the decay rate is given by

$$\delta(n) = \frac{n-2}{4}. \tag{6.1.7}$$

Finally $(\mathbb{R}^{1+n}\times\mathcal{K},g)$ is globally hyperbolic and causally geodesically complete.

Remark 6.2. A positive energy theorem holds for spacetimes with a supersymmetric spacetime [Dai04], in the sense that a Riemannian manifold with the same initial topology $\mathbb{R}^n\times\mathcal{K}$ as (6.1.1) and which asymptotically approaches in the non-compact directions the metric $\delta_{\mathbb{R}^n}+\gamma$ must have non-negative mass and this mass is zero if and only if the space is precisely $\delta_{\mathbb{R}^n}+\gamma$. The restriction in initial data, which is then 'compactly supported' as much as this positive mass theorem allows, mirrors the proof of the stability of Minkowski in four-dimensions as a solution to the vacuum Einstein equations given in [LR05]. That said, the existence of nontrivial initial data for the above theorem is not yet known, and may require significant modifications of gluing theorems used to construct nontrivial data for Minkowski spacetime [Cor00, CD02].

Remark 6.3. In section 1.5 we remarked that $(\mathcal{K},\gamma)$ is Riemannian linear stable but not necessarily isolated in the moduli space of Einstein structures. The above theorem shows that we do not evolve to another nearby Einstein metric in the moduli space. This is not *a priori* obvious and relies on the external space having infinite volume.

Theorem 6.1 can be compared with the result of [BFK19]. In this work, the authors prove the stability to zero-mode perturbations of cosmological Kaluza-Klein spacetimes constructed by replacing the Minkowski geometry with the 4-dimensional Milne model considered in Chapter 5. In particular, they show that the spacetime radii only converge to values *close* to the background radii values. In some sense, the compact spatial slices of the Milne can only absorb so much energy, while the rest is transferred into the moduli space of the torii.

The proof of theorem 6.1 uses a relatively simple vector-field argument, while, for example, the proof of global stability for the coupled Einstein–Klein-Gordon system in $(1+3)$-dimensions [LM16a] required combining vector-field arguments with estimates arising from control on the fundamental solution for the wave and Klein-Gordon equation. Such detailed analysis is beyond the scope of this chapter, but we intend to explore this in future work. Note that the method in the present chapter can be easily used to show linear stability as far as $n = 3$.

The decay rate of $|h| \lesssim t^{-\delta(n)}$, given in lemma 6.15, arises essentially as a linear estimate. The linearisation of the Einstein equation is

$$(\Box_\eta + \Delta_\gamma + 2R[\hat{g}]\circ)h_{\mu\nu} = 0. \tag{6.1.8}$$

To study conservation properties of the linear equations we introduce a new stress-energy tensor

$$T[h]^\mu{}_\nu = \hat{g}^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha h, \nabla[\hat{g}]_\nu h\rangle_E - \frac{1}{2}\hat{g}^{\alpha\beta}\langle\nabla[\hat{g}]_\beta h, \nabla[\hat{g}]_\alpha h\rangle_E \delta^\mu_\nu + \langle R[\hat{g}]\circ h, h\rangle_E \delta^\mu_\nu. \tag{6.1.9}$$

The conditions on $(\mathcal{K}, \gamma)$ imply that the energy integral derived from (6.1.9) is nonnegative. Indeed the conditions on $(\mathcal{K}, \gamma)$ imply that the operator $-(\Delta_\gamma + 2R\circ)$ has a nonnegative discrete spectrum, and so a spectral decomposition can be applied to solutions h of the linearised Einstein equation (6.1.8). The spectral component corresponding to the zero eigenvalue satisfies an effective wave equation, $\Box_\eta(h^0)_{\mu\nu} = 0$; the components corresponding to positive eigenvalues λ satisfy effective Klein-Gordon equations $(\Box_\eta - \lambda)(h^\lambda)_{\mu\nu} = 0$.

For the quasilinear Einstein equation, we refrain from performing a spectral decomposition into wave and Klein-Gordon components. Thus, we use the intersection of wave and Klein-Gordon methods together with an energy integral derived from a perturbed version of (6.1.9), see definition 6.16. This follows especially the treatment of quasilinear Klein-Gordon equations in [Hör97]. This approach leaves us with a decay rate, as given in (6.1.7), that is far from the sharp decay rates of the wave and Klein-Gordon equations.

Having obtained a linear estimate that improves with increasing n, we take n sufficiently large so that we can ignore all nonlinear structure in the Einstein equation. Note that the dimension of the compact manifold only appears in the required regularity of the initial data, which is given explicitly in theorem 6.23.

Outline of chapter. In section 6.2 we introduce: the Lichnerowicz Laplacian, the foliation by hyperboloids, the gauge condition and the higher dimensional Schwarzschild -product spacetime. In section 6.3 we prove a Sobolev estimate on hyperboloids with respect to wave-like energies. In section 6.4 we define an energy functional adapted to the internal manifold and to hyperboloids. Finally in section 6.5 we prove the main theorem.

6.2 Preliminaries

6.2.1 Parallel Spinors and the Lichnerowicz Laplacian

In this subsection we briefly detail how the condition on the internal manifold in the main theorem relates to a linear stability condition involving the eigenvalues of an operator closely related to the Lichnerowicz Laplacian. We refer to the expansive discussion in section 1.5.1, in particular definitions 1.30 and 1.31. However for convenience we restate theorem 6.4 here.

Theorem 6.4 ([DWW05]). *If a compact Riemannian manifold $(\mathcal{K}, \gamma)$ has a cover which is spin and admits a nonzero parallel spinor then it is Riemannian linearly stable. That is, for all symmetric 2-tensors u_{AB} we have*

$$\int_{\mathcal{K}} \langle \mathcal{L}_\gamma u, u\rangle_\gamma \mathrm{d}\mu_\gamma \geq 0, \tag{6.2.1}$$

where $(\mathcal{L}_\gamma u)_{AB} = -\Delta_\gamma u_{AB} - 2(R[\gamma]\circ u)_{AB}$.

The operator $\mathcal{L}_\gamma$ is self-adjoint and elliptic, and consequently by the compactness of $\mathcal{K}$ and spectral theory, it has a discrete set of eigenvalues of finite multiplicity. Consequently theorem 6.4 implies a condition $\lambda_{\min} \geq 0$ on the lowest eigenvalue $\lambda_{\min}$ of $\mathcal{L}_\gamma$.

Our main theorem 6.1 in fact applies more generally to internal manifolds which are Riemannian linearly stable. However all known examples of compact Ricci-flat manifolds admit a spin cover with nonzero parallel spinors. In particular, all known examples of compact Ricci-flat manifolds are Riemannian linearly stable manifolds.

6.2.2 Cartesian, hyperbolic, and hyperbolic polar coordinates

We restate some concepts given in definitions 1.17 1.26 and 3.5, as well as notation from (1.4.7).

Definition 6.5 (Minkowski space). Let $n \geq 1$ be an integer. Define Cartesian coordinates to be $(x^0, x^1, \ldots, x^n) = (t, x^1, \ldots, x^n) = (t, \vec{x})$ parameterising $\mathbb{R}^{1+n}$, and define

$$\eta_{\mathbb{R}^{1+n}} = -\,\mathrm{d}t^2 + \sum_{i=1}^{n} (\mathrm{d}x^i)^2. \tag{6.2.2}$$

Define, for $i \in \{1, \ldots, n\}$, the translation vector fields T and X_i so that, in the Cartesian coordinates, they are given by

$$\begin{aligned} X_i &= \partial_{x^i}, \\ T &= X_0 = \partial_t. \end{aligned} \tag{6.2.3}$$

Define the vector fields $\{\Omega_i\}_{i\in\{1,\ldots,n\}}$ and $\{\Omega_{ij}\}_{1\leq i<j\leq n}$ so that, in the Cartesian coordinates, they are given by

$$\Omega_i = t\partial_{x^i} + x_i\partial_t, \quad 1 \leq i \leq n, \tag{6.2.4}$$

$$\Omega_{ij} = (\eta_{\mathbb{R}^{1+n}})_{jk} x^k \partial_i - (\eta_{\mathbb{R}^{1+n}})_{ik} x^k \partial_j, \quad 1 \leq i < j \leq n. \tag{6.2.5}$$

Define the collection of Lorentz generators by

$$Z = \{\Omega_i, \Omega_{ij}, T, X_i\}. \tag{6.2.6}$$

Define $|x|^2 = \sum_{i=1}^n (x^i)^2$ and define, in the region $t \geq |x|$, the hyperboloidal coordinates to be

$$\begin{aligned} s &= (t^2 - |x|^2)^{1/2}, \\ y &= x. \end{aligned} \tag{6.2.7}$$

Define, for $i \in \{1, \dots, n\}$, the vector fields Y_i so that, in the hyperboloidal coordinates, they are given by

$$Y_i = \partial_{y^i}. \tag{6.2.8}$$

For $s_0 \geq 0$, define the spacelike hyperboloidal hypersurface

$$\Sigma_{s_0} = \{(t, x) \in \mathbb{R}^{1+n} : t > 0, s = s_0\}. \tag{6.2.9}$$

In terms of the previous notation from definition (1.4.7), the set Z is the same as the set $\mathcal{Z}$. Furthermore the collection Z is closed under commutation and forms a basis for the Poincaré Lie algebra $\mathfrak{iso}(n, 1)$.

Definition 6.6 (Spacetimes with a supersymmetric compactification)**.** On $\mathbb{R}^{1+n} \times \mathcal{K}$, define, for $i, j \in \{0, \dots, n\}$, X_i, Y_i, and Ω_i, Ω_{ij} to be as in $\mathbb{R}^{1+n}$. Define the following collection of differential operators

$$\Gamma = Z \cup \{\Delta_\gamma\}. \tag{6.2.10}$$

Note $[Z, \Delta_\gamma] = 0$. Define $\mathbb{N} = \{0, 1, 2 \dots\}$. Define $\{Z_i\}_{i=1}^{(n+1)(n+2)/2}$ to be a reindexing of $\{X_i\}_{i \in \{1,\dots,n\}} \cup \{\Omega_i\}_{i \in \{1,\dots,n\}} \cup \{\Omega_{ij}\}_{0 \leq i < j \leq n}$, define a multi-index to be an ordered list of arbitrary length of elements from $\{1, \dots, (n+1)(n+2)/2\}$, and for a multi-index $I = (i_1, \dots, i_k)$ define the length $|I| = k$ and the differential operator $Z^I = Z_{i_k} \circ \dots \circ Z_{i_1}$. For $I \in \mathbb{N}$ and $u_{\mu\nu}$ a tensor defined on $\mathbb{R}^{1+n} \times \mathcal{K}$, define the following generalised multi-index notation

$$|\Gamma^I u|_E^2 = \sum_{I_1 : |I_1| + 2\ell = |I|} |Z^{I_1} \Delta_\gamma^j u|_E^2,$$

where the sum is taken over all multi-indices I_1 of length $|I_1| = k$ and integers ℓ such that $k + 2\ell = |I|$.

Definition 6.7 (Sobolev norms)**.** Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n} \times \mathcal{K}$ and $\ell \in \mathbb{N}$. Define

$$|\nabla[\gamma]^\ell u|_E^2 = \gamma^{A_1 B_1} \dots \gamma^{A_\ell B_\ell} g_E^{\mu\nu} g_E^{\rho\sigma} (\nabla[\gamma]_{A_\ell} \dots \nabla[\gamma]_{A_1} u_{\mu\rho})(\nabla[\gamma]_{B_\ell} \dots \nabla[\gamma]_{B_1} u_{\nu\sigma}). \tag{6.2.11}$$

Define the norms

$$\|u(\cdot, \cdot, \omega)\|_{H^\ell(\mathcal{K})} = \left(\int_{\mathcal{K}} \sum_{0 \leq j \leq \ell} |\nabla[\gamma]^j u(\cdot, \cdot, \omega)|_E^2 \mathrm{d}\mu_\gamma \right)^{1/2}, \tag{6.2.12}$$

$$\|u(t,x,\omega)\|_{L^2(\mathcal{H}_s\times\mathcal{K})} = \left(\int_{\mathcal{H}_s\times\mathcal{K}} |u(t,x,\omega)|_E^2 \mathrm{d}x\mathrm{d}\mu_\gamma\right)^{1/2}, \tag{6.2.13}$$

where $\mathrm{d}x = \mathrm{d}x^1 \dots \mathrm{d}x^n$ is defined to be the flat Euclidean volume form.

Lemma 6.8.

$$Y_i = X_i + \frac{x_i}{t}T, \tag{6.2.14}$$

$$\Omega_i = tY_i, \tag{6.2.15}$$

$$\Omega_{ij} = y_iY_j - y_jY_i. \tag{6.2.16}$$

Proof. Since $t = \sqrt{s^2+y^2}$ on $\mathcal{H}_s$, by the chain rule, for $j \in \{1,\dots,n\}$, $\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j} + \frac{\partial t}{\partial y^j}\frac{\partial}{\partial t} = \frac{\partial}{\partial x^j} + \frac{y_j}{t}\frac{\partial}{\partial t}$, which gives the first result. The second follows from multiplying both sides of the first by t. The third follows from $\Omega_{ij} = x_iX_j - x_jX_i = x_i(X_j + x_jt^{-1}T) - x_j(X_i + x_it^{-1}T)$. □

The following two lemmas relate the t coordinate to the s coordinate.

Lemma 6.9. *Let $s \geq 1$. Suppose $(t_0, x_0) \in \mathcal{H}_s$ and $(t,x) \in \mathcal{H}_s$ with $|x - x_0| \leq t_0/2$. In this case, $t_0/2 \leq t \leq 2t_0$.*

Proof. For the graph $t = \sqrt{s^2+|x|^2}$, the gradient $|\frac{\partial t}{\partial x}| = |\frac{x}{\sqrt{s^2+|x|^2}}| \leq 1$, so the change from t to t_0 is less than the change in $|x|$ to $|x_0|$. □

Lemma 6.10. *For all $s > 1$, in the portion of $\mathcal{H}_s$ where $|x| \leq t-1$, one has $2t-1 \leq s^2 \leq t^2$.*

Proof. First, observe that $t^2 = s^2 + |x|^2 \geq s^2$. Second, since $|x|^2 \leq t^2 - 2t + 1$, one has $s^2 = t^2 - |x|^2 \geq 2t - 1$. □

The following are standard elliptic estimates, see for example [Bes87, §Appx. H].

Lemma 6.11 (Elliptic estimates on $(\mathcal{K},\gamma)$)**.** *For $\ell \in \mathbb{N}$ and $u_{\mu\nu}$ a sufficiently regular tensor defined on $\mathbb{R}^{1+n}\times\mathcal{K}$ there exist constants $c_1, c_2, c_3 > 0$ such that*

$$\|u\|_{H^{2\ell}(\mathcal{K})} \leq c_1\|(\Delta_\gamma)^\ell u\|_{L^2(\mathcal{K})} + c_2\|u\|_{L^2(\mathcal{K})} \leq c_3\|u\|_{H^{2\ell}(\mathcal{K})}. \tag{6.2.17}$$

6.2.3 The higher-dimensional Schwarzschild spacetime

In this subsection, the higher-dimensional Schwarzschild solution is considered and its relationship to the initial data for the Einstein equations and the reduced Einstein equations (1.1.5) is discussed. The form of the metric is presented in the following definition.

Definition 6.12. Let $n \in \mathbb{Z}$ be such that $n \geq 5$ and $C_S \in [0,\infty)$. The Schwarzschild metric (in Schwarzschild coordinates) is defined for $(t,\bar{r},\omega) \in \mathbb{R}\times(C_S^{1/(n-2)},\infty)\times S^{n-1}$ to be

$$g_S = -\left(1 - \frac{C_S}{\bar{r}^{n-2}}\right)\mathrm{d}t^2 + \left(1 - \frac{C_S}{\bar{r}^{n-2}}\right)^{-1}\mathrm{d}\bar{r}^2 + \bar{r}^2\sigma_{S^{n-1}}. \tag{6.2.18}$$

Note $\sigma_{S^{n-1}}$ is the metric on the unit round sphere.

The above metric can also be written in the wave gauge. For $n = 3$, it is sufficient to replace $(t, \bar{r}, \omega) \in \mathbb{R} \times (C_S^{1/(n-2)}, \infty) \times S^{n-1}$ by $(t, x) = (t, r\omega)$ with $r = \bar{r} - M$; the resulting explicit metric can be found in [LR05, LM16a]. Although the case $n = 4$ leads to complicated terms involving logarithms, for $n \geq 5$, there is the following theorem.

Theorem 6.13 ([CBCL06, Section 5.2])**.** *Let $n \in \mathbb{Z}$ be such that $n \geq 5$ and $C_S \in [0, \infty)$. There are coordinates (t, x) related to those in definition 6.12 by $(x^i)_{i=0}^n = (t, r(\bar{r})\omega)$ with*

$$r(\bar{r}) = \bar{r} - \frac{C_S}{2\bar{r}^{n-3}} + O(\bar{r}^{5-2n}),$$

such that $(x^i)_{i=0}^n$ satisfy the harmonic gauge, that is, the $\eta_{\mathbb{R}^{1+n}}$-wave gauge. Furthermore, there exist functions $h_{00}(R)$, $h(R)$, and $\hat{h}(R)$, defined on an interval around $R = 0$, that are analytic and bounded by a multiple of C_S near $R = 0$, and such that

$$g_S = -\left(1 - \frac{h_{00}(r^{-1})}{r^{n-2}}\right)(\mathrm{d}x^0)^2 + \sum_{i,j=1}^{n}\left[\left(1 + \frac{h(r^{-1})}{r^{n-2}}\right)\delta^{ij} + \frac{\hat{h}(r^{-1})}{r^{n-2}}\frac{x^i x^j}{r^2}\right]\mathrm{d}x^i \mathrm{d}x^j. \tag{6.2.19}$$

In particular, the difference between the components of g_S with respect to the harmonic coordinates and the corresponding components of the Minkowski metric are such that any ∂^I derivative decays at least as fast as $C_S r^{-(n-2)-|I|}$.

6.3 Sobolev estimates on hyperboloids

We begin in lemma 6.14 by recalling Hörmander's proof of a Sobolev estimate on hyperboloids. This allows us to introduce some of the key ideas that appear in our proof of the main result of this section, lemma 6.15.

Lemma 6.14 (Sobolev estimate for compactly supported functions on hyperboloids in Minkowski space [Hör97, Lemma 7.6.1])**.** *Let ν be the smallest integer greater than $n/2$ and $v \in C^\nu(\mathbb{R}^{1+n})$ have support in $|x| < t - 1$. There is a constant C such that*

$$\sup_{\mathcal{H}_s} t^n |v(t,x)|^2 \leq C \sum_{|I| \leq \nu} \int_{\mathcal{H}_s} |Z^I v|^2 \mathrm{d}x. \tag{6.3.1}$$

Proof. Consider a point $(t_0, x_0) \in \mathcal{H}_s$ with $|x_0|^2 \leq t_0^2 - 1$. Set $r_0 = t_0/2$ and $y_0 = x_0$. Set Σ to be the portion of $\mathcal{H}_s$ on which $|x - x_0| \leq r_0$. Let $(t, x) \in \Sigma$. This implies $|t - t_0| \leq r_0$, which implies $t/2 \leq t_0 \leq 2t$. Thus,

$$\sum_{|I| \leq \nu} \int_{\mathcal{H}_s} |Z^I v(t,x)|^2 \mathrm{d}x \geq C \sum_{|I| \leq \nu} \int_{\mathcal{H}_s} |t_0^{|I|} Y^I v(t,y)|^2 \mathrm{d}y.$$

The right can be rewritten, by introducing rescaled coordinates $\tilde{y} = 2t_0^{-1}(y - y_0)$ and $\tilde{v}(\tilde{y}) = v(t, y)$. Let $\chi(\tilde{y})$ be a smooth cut-off such that χ is 1 on a neighbourhood of 0 and is 0 for $|\tilde{y}| \geq 1/2$. A Sobolev estimate can be applied on $\chi \cdot \tilde{v}$ to give a lower bound on v. The error terms arising from derivatives hitting the χ factor can be absorbed into the L^2 norm on $\tilde{v}$. We obtain

$$\sum_{|I| \leq \nu} \int_{\mathcal{H}_s} |t_0^{|I|} Y^I v(t,x)|^2 \mathrm{d}y = \sum_{|I| \leq \nu} \int_{|\tilde{y}| \leq 1} |\partial_{\tilde{y}}^I \tilde{v}(\tilde{y})|^2 t_0^n \mathrm{d}\tilde{y}$$

$$\begin{aligned}&\geq Ct_0^n \sum_{|I|\leq\nu} \int_{|\tilde y|\leq 1} \left|\partial_{\tilde y}^I\left((\chi(\tilde y)\tilde v(\tilde y)\right)\right|^2 \mathrm{d}\tilde y\\ &\geq Ct_0^n|\tilde v(0)|^2\\ &= Ct_0^n|v(t_0,x_0)|^2,\end{aligned}$$

which completes the proof. □

In the following lemma we obtain a Sobolev estimate for functions supported on product spacetimes with specified properties outside a compact set. In particular we obtain a pointwise estimate (6.3.3) in terms of the hyperboloidal time s, as well as a t-weighted pointwise estimate on a fixed hyperboloid (6.3.4).

Lemma 6.15 (Sobolev estimate for eventually prescribed functions on hyperboloids foliating product spacetimes)**.** *Let $n\geq 4$, let $\tilde d$ be the smallest even integer larger than $d/2$ and let $\tilde\nu$ be the smallest integer greater than $n/2+\tilde d$. Let $u_{\mu\nu}, f_{\mu\nu}$ be tensors on $\mathbb{R}^{1+n}\times\mathcal{K}$ with f depending only the Minkowski coordinates x^i. Let $u\in C^{\tilde\nu}(\mathbb{R}^{1+n}\times\mathcal{K})$ satisfy $u=f$ for $|x|\geq t-1$. Let $f\in C^\infty(\mathbb{R}^{1+n}\times\mathcal{K})$ be smooth and such that for all $I\in\mathbb{N}$, there is a C_I such that*[2]

$$|\nabla[\hat g]^I f|_E \leq C_{|I|}|x|^{-(n-1)/2-|I|}. \tag{6.3.2}$$

Let $\delta(n)=\frac{n-2}{4}$. There is a constant C such that,

$$\begin{aligned}\sup_{(t,x^i,\omega)\in\mathcal{H}_s\times\mathcal{K}} s^{4\delta(n)}|u(t,x^i,\omega)|_E^2 \leq{}& C\sum_{|I|\leq\tilde\nu}\sum_{i=1}^n \int_{\substack{\mathcal{H}_s\times\mathcal{K}\\|x|\leq t-1}} |Y_iZ^Iu|_E^2\mathrm{d}x\mathrm{d}\mu_\gamma\\ &+C\sum_{|I|\leq\tilde\nu-1} C_I^2.\end{aligned} \tag{6.3.3}$$

Furthermore there is a constant C such that,

$$\begin{aligned}\sup_{(t,x^i,\omega)\in\mathcal{H}_s\times\mathcal{K}} t^{2\delta(n)}|u(t,x^i,\omega)|_E^2 \leq{}& C\sum_{|I|\leq\tilde\nu}\sum_{i=1}^n \int_{\substack{\mathcal{H}_s\times\mathcal{K}\\|x|\leq t-1}} |Y_iZ^Iu|_E^2\mathrm{d}x\mathrm{d}\mu_\gamma\\ &+C\sum_{|I|\leq\tilde\nu-1} C_I^2.\end{aligned} \tag{6.3.4}$$

Proof. Lemma 6.11 and the standard Sobolev estimate imply

$$\sup_{\omega\in\mathcal{K}}|u(\cdot,\cdot,\omega)|_E \leq \|u\|_{H^{\tilde d}(\mathcal{K})} \leq \|(\Delta_\gamma)^{\tilde d/2}u\|_{L^2(\mathcal{K})}+\|u\|_{L^2(\mathcal{K})}, \tag{6.3.5}$$

for $\tilde d$ the smallest even integer greater than $d/2$. This choice of $\tilde d$ being even is simply to make the elliptic estimate cleaner. Note the trivial estimate

$$\sum_{|I|\leq\tilde\nu-\tilde d}\left(|Y_iZ^I(\Delta_\gamma)^{\tilde d/2}u|_E^2+|Y_iZ^Iu|_E^2\right)\leq \sum_{|I|+2j\leq\tilde\nu}|Y_iZ^I(\Delta_\gamma)^ju|_E^2\,.$$

[2]The exponent on f is set to match that corresponding to the exponent arising from the pointwise estimate (6.3.3) on u in the region $|t-r|\leq C$. The limiting factor on the exponent in (6.3.3) arises from estimates on the hyperboloid, not from the decay of the prescribed function f. If a faster decay rate $t^{-\beta}$ could be proved (using similar methods) on hyperboloids for compact data, then a similar $t^{-\beta}$ decay could be proved for prescribed functions satisfying $f\leq r^{-\beta}$.

It is thus sufficient to prove in Minkowski space that

$$\sup_{\mathcal{H}_s} s^{n-2}|u(t,x)|_E^2 \le C \sum_{|I|\le\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \int_{\mathcal{H}_s} |Y_i Z^I u|_E^2 \mathrm{d}x + C \sum_{|I|\le\tilde{\nu}-1} C_I^2, \tag{6.3.6}$$

since this would then imply

$$\begin{aligned}\sup_{\mathcal{H}_s\times\mathcal{K}} s^{n-2}|u(t,x^i,\omega)|_E^2 &\lesssim \sum_{|I|\le\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \| \sup_{\mathcal{K}}(Y_i Z^I u)\|_{L_x^2}^2 + C \sum_{|I|\le\tilde{\nu}-1} C_I^2 \\ &\lesssim \sum_{|I|+2j\le\tilde{\nu}} \sum_{i=1}^{n} \|Y_i Z^I (\Delta_\gamma)^j u\|_{L_x^2 L_{\mathcal{K}}^2}^2 + C \sum_{|I|\le\tilde{\nu}-1} C_I^2.\end{aligned}$$

For $|x| \ge t-1$ and $(t,x)\in\mathcal{H}_s$, one has $t\sim|x|$, and so

$$s^{n-2}|u(t,x)|_E^2 \le t^{n-2}|u(t,x)|_E^2 \le C|x|^{n-2}|u(t,x)|_E^2 \le C|x|^{n-2}|f(t,x)|_E^2 \le CC_0^2.$$

Thus, it remains to prove (6.3.6) for $|x|\le t-1$.

Consider the region $|x|\le t-1$. Set $t_{\max} = (s^2+1)/2$, which is the value of t at which $\mathcal{H}_s$ intersects $|x| = t-1$ and which satisfies $t\le t_{\max}\le (t^2+1)/2$ on the portion of $\mathcal{H}_s$ where $|x|\le t-1$ by lemma 6.10. Let $\chi:\mathbb{R}\to[0,1]$ be a smooth cut-off function such that $\chi(\alpha)=1$ for $\alpha<1$ and $\chi(\alpha)=0$ for $\alpha>2$, and define the $(0,2)$ tensor $v_{\mu\nu}(t,x) = \chi(|x|/t_{\max})u_{\mu\nu}(t,x)$. Observe that $u_{\mu\nu}=v_{\mu\nu}$ in the region $|x|\le t-1$.

Hormander's proof of lemma 6.14 relies on a carefully chosen rescaling of a portion of the hyperboloid, and the rest of this lemma follows the same idea, although the scaling is chosen differently. Recall both the Cartesian (t,x) and hyperboloidal (s,y) in Minkowski space, which are related via $(s,y) = (\sqrt{t^2-|x|^2},x)$. Given a choice of s, define $\tilde{y} = s^{-1}y$ and $\tilde{v}(\tilde{y})$ to be the value of v at hyperboloidal coordinates $(s,s\tilde{y})$. With this $\mathrm{d}^n\tilde{y} = s^{-n}\mathrm{d}y$, $\partial_{\tilde{y}^i} = s\partial_{y^i} = sY_i$. Recall $Z_i = tY_i$. Thus, by a Sobolev estimate that exploits the fact that $1<n/2<n/2+1$,

$$\begin{aligned}\sup_{\mathcal{H}_s}|v(t,x)|_E^2 &= \sup|\tilde{v}(\tilde{y})|_E^2 \\ &\lesssim \sum_{1\le|J|\le\frac{n}{2}+1} \int |\partial_{\tilde{y}}^J \tilde{v}|_E^2 \mathrm{d}^n\tilde{y}.\end{aligned}$$

From rescaling and the facts that $s\le t$ and that $\Omega_i = tY_i$, it follows that

$$\begin{aligned}\sup_{\mathcal{H}_s}|v(t,x)|_E^2 &\lesssim s^{-n} \sum_{1\le|J|\le\frac{n}{2}+1} \int_{\mathcal{H}_s} |(sY)^J v|_E^2 \mathrm{d}^n y \\ &\lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_{i=1}^{n} \int_{\mathcal{H}_s} s^{2|J|}|Y^J Y_i v|_E^2 \mathrm{d}^n y \\ &\lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_{i=1}^{n} \int_{\mathcal{H}_s} t^{2|J|}|Y^J Y_i v|_E^2 \mathrm{d}^n y \\ &\lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_{i=1}^{n} \int_{\mathcal{H}_s} |Y_i Z^J v|_E^2 \mathrm{d}^n y.\end{aligned}$$

The integral on the right can now be decomposed into the parts where $|x| \leq t-1$ and $|x| > t-1$. Where $|x| \leq t-1$, the integral can be bounded by the integral term on the right-hand-side of (6.3.6) since $\tilde{\nu} - \tilde{d} > n/2$. Now consider the region $|x| > t-1$. Because of the support of χ, it is sufficient to consider the region $t_{\max} - 1 \leq |x| \leq 2t_{\max}$. In this region, $v = \chi f$. When a derivative is applied to v, it is applied to either χ or to f, in which case one obtains an additional factor of $t_{\max}^{-1}$ or $|x|^{-1}$, from the properties of χ and f respectively. Since $|x|/t_{\max} \in [1,2]$ in the support of $\partial\chi$, effectively, one obtains an extra factor of $|x|^{-1}$ in all cases, so $|Y_i Z^J v|_E \leq CC_{|J|+1}|x|^{-(n-1)/2-1}$, and

$$\begin{aligned}
\int_{|x|\geq t_{\max}-1} |Y_i Z^J u|_E^2 \mathrm{d}x &\leq CC_{|J|+1}^2 \int_{\mathbb{S}^{n-1}} \int_{t_{\max}-1}^{2t_{\max}} (|r|^{-(n-1)/2-1})^2 |r|^{n-1} \mathrm{d}r \mathrm{d}^{n-1}\omega_{\mathbb{S}^{n-1}} \\
&\leq CC_{|J|+1}^2.
\end{aligned}$$

Observing that $s \geq Ct^{1/2}$ in the region $|x| \leq t-1$ allows us to obtain

$$\begin{aligned}
\sup_{\mathcal{H}_s\times\mathcal{K}} t^{2\delta(n)}|u|_E^2 &\leq \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|\leq t-1\}} t^{2\delta(n)}|u|_E^2 + \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\}} t^{2\delta(n)}|u|_E^2 \\
&\lesssim \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|\leq t-1\}} s^{4\delta(n)}|u|_E^2 + \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\}} r^{2\delta(n)}|f|_E^2. \\
&\lesssim \sum_{|I|\leq\tilde{\nu}} \sum_{i=1}^{n} \int_{\substack{\mathcal{H}_s\times\mathcal{K}\\ |x|\leq t-1}} |Y_i Z^I u|_E^2 \mathrm{d}x \mathrm{d}\mu_\gamma + \sum_{|I|\leq\tilde{\nu}-1} C_I^2 \\
&\quad + C_0 \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\}} r^{\frac{n-2}{2}} r^{-\frac{n-1}{2}}.
\end{aligned}$$

In the final line we applied estimate (6.3.3) to the first term and assumption (6.3.2) to the second term. □

6.4 Energy integrals and inequalities

6.4.1 Basic properties of the energy

The energy introduced in the following definition is related to the standard energy used to study quasilinear hyperbolic PDEs, albeit with additional terms included in order to be compatible with the linearised equations (6.1.8).

Definition 6.16 (Lichnerowicz-type energy on hyperboloids)**.** Let $n \in \mathbb{Z}^+$ and let $v^{\mu\nu}, u_{\mu\nu}$ be tensors defined on $\mathbb{R}^{1+n}\times\mathcal{K}$. For $u, v \in C^1(\mathbb{R}^{1+n}\times\mathcal{K})$ and $s \geq 2$ define

$$\begin{aligned}
\mathcal{E}[v;u;s] = \int_{\mathcal{H}_s\times\mathcal{K}} \Big(&(s/t)^2|\partial_t u|_E^2 + \sum_{i=1}^{n} |Y_i u|_E^2 + \langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E - 2\langle R[\hat{g}]\circ u, u\rangle_E \\
&- 2v^{\alpha\beta}\langle \nabla[\hat{g}]_\beta u, \partial_t u\rangle_E n_\alpha + v^{\alpha\beta}\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E \Big) \mathrm{d}x\mathrm{d}\mu_\gamma, \qquad (6.4.1)
\end{aligned}$$

where $n_0 = 1, n_i = -x_i/t$ for $i \in \{1,\ldots,n\}$ and $n_A = 0$, and $\mathrm{d}x$ is the flat Euclidean volume form.

Note that, following [Hör97, LM16a], we have endowed $\mathcal{H}_s$ with the flat Euclidean volume form $\mathrm{d}x$, instead of the induced Riemannian volume form $(s/t)\mathrm{d}x$.

The following lemma provides us with an energy functional which allows us to measure the perturbation of the spacetime. Note that in (6.4.2) we require some weighted $t-$decay on hyperboloids which we recover from (6.3.4) in lemma 6.15.

Lemma 6.17 (Basic properties of the energy)**.** *Take the conditions of definition 6.16.*

1. *There is an* $\varepsilon_n > 0$*, such that if*

$$\sup_{\mathcal{H}_s \times \mathcal{K}} t|v|_E \leq C\varepsilon_n, \tag{6.4.2}$$

then for $s \geq 2$*,*

$$\frac{1}{2}\mathcal{E}[v;u;s] \leq \mathcal{E}[0;u;s] \leq 2\mathcal{E}[v;u;s]. \tag{6.4.3}$$

2. *If* $u_{\mu\nu}$ *is a solution of*

$$(\hat{g}+v)^{\alpha\beta}\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u_{\mu\nu} + 2(R[\hat{g}]\circ u)_{\mu\nu} = F_{\mu\nu}, \tag{6.4.4}$$

then

$$\begin{aligned}\mathcal{E}[v;u;s_1] &= \mathcal{E}[v;u;s_2] + \int_{s_1}^{s_2}\int_{\mathcal{H}_s\times\mathcal{K}} \langle F, \partial_t u\rangle_E \frac{s}{t}\mathrm{d}y\mathrm{d}\mu_\gamma\mathrm{d}s \qquad (6.4.5)\\ &+ \int_{s_1}^{s_2}\int_{\mathcal{H}_s\times\mathcal{K}}\Big(- 2(\nabla[\hat{g}]_\alpha v^{\alpha\beta})\langle\nabla[\hat{g}]_\beta u, \partial_t u\rangle_E + (\partial_t v^{\alpha\beta})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E\Big)\frac{s}{t}\mathrm{d}y\mathrm{d}\mu_\gamma\mathrm{d}s.\end{aligned}$$

Proof. We first derive the energy $\mathcal{E}[v;u;s]$ by considering the following nonlinear version of the stress energy tensor (6.1.9)

$$\begin{aligned}T[v;u]^\mu{}_\nu = (\hat{g}+v)^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E - \frac{1}{2}(\hat{g}+v)^{\alpha\beta}\langle\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\alpha u\rangle_E\delta^\mu_\nu \\ + \langle R[\hat{g}]\circ u, u\rangle_E\delta^\mu_\nu.\end{aligned} \tag{6.4.6}$$

We calculate

$$\begin{aligned}&\nabla[\hat{g}]_\mu T[v;u]^\mu{}_\nu \\ &= \langle(\hat{g}+v)^{\alpha\beta}\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\nu u\rangle_E + (\hat{g}+v)^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\mu\nabla[\hat{g}]_\nu u\rangle_E \\ &- (\hat{g}+v)^{\alpha\beta}\langle\nabla[\hat{g}]_\nu\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\alpha u\rangle_E + \nabla[\hat{g}]_\nu\langle R[\hat{g}]\circ u, u\rangle_E \\ &+ (\nabla[\hat{g}]_\mu v^{\mu\alpha})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E - \frac{1}{2}(\nabla[\hat{g}]_\nu v^{\alpha\beta})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E.\end{aligned} \tag{6.4.7}$$

Let X^μ be a vector field on $\mathbb{R}^{1+n}\times\mathcal{K}$ tangent to $\mathbb{R}^{1+n}$. We have

$$\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u_{\gamma\delta} = \nabla[\hat{g}]_\beta\nabla[\hat{g}]_\alpha u_{\gamma\delta} + \mathrm{Riem}[\hat{g}]_{\alpha\beta\gamma}{}^\rho u_{\rho\delta} + \mathrm{Riem}[\hat{g}]_{\alpha\beta\delta}{}^\rho u_{\rho\gamma}.$$

However since $\mathrm{Riem}[\eta_{\mathbb{R}^{1+n}}] \equiv 0$ we have

$$\mathrm{Riem}[\hat{g}]_{\alpha\beta\gamma\delta}X^\delta = 0. \tag{6.4.8}$$

Consequently

$$\langle\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\nu u\rangle_E X^\alpha = \langle\nabla[\hat{g}]_\beta\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E X^\alpha. =, \tag{6.4.9}$$

and also

$$\nabla[\hat{g}]_\nu\langle R[\hat{g}]\circ u, u\rangle_E X^\nu = 2\langle R[\hat{g}]\circ u, X^\nu\nabla[\hat{g}]_\nu u\rangle_E. \tag{6.4.10}$$

This allows us to calculate

$$\begin{aligned}
&\nabla[\hat{g}]_\mu(T[v;u]^\mu{}_\nu X^\nu) \\
&= T^\mu{}_\nu[v]\nabla[\hat{g}]_\mu X^\nu + \langle F, X^\nu \nabla[\hat{g}]_\nu u\rangle_E \\
&+ (\nabla[\hat{g}]_\mu v^{\mu\alpha})\langle \nabla[\hat{g}]_\alpha u, X^\nu \nabla[\hat{g}]_\nu u\rangle_E - \frac{1}{2}(X^\nu \nabla[\hat{g}]_\nu v^{\alpha\beta})\langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E. \qquad (6.4.11)
\end{aligned}$$

Consider the hyperboloidal energy

$$\begin{aligned}
\mathcal{E}[v;u;s] &= \int_{\mathcal{H}_s\times\mathcal{K}} -2T[v;u]^\mu{}_\nu(\partial_t)^\nu n_\mu \mathrm{d}x\mathrm{d}\mu_\gamma \\
&= \int_{\mathcal{H}_s\times\mathcal{K}} \Big(|\partial_t u|_E^2 + \sum_{i=1}^n |\partial_i u|_E^2 + \sum_{i=1}^n 2\frac{x^i}{t}\langle \partial_t u, \partial_i u\rangle_E + \gamma^{AB}\langle \nabla[\hat{g}]_A u, \nabla[\hat{g}]_B u\rangle_E \\
&\quad - 2\langle R[\hat{g}]\circ u, u\rangle_E - 2v^{\mu\rho}\langle \nabla[\hat{g}]_\rho u, \partial_t u\rangle_E n_\mu + v^{\rho\lambda}\langle \nabla[\hat{g}]_\rho u, \nabla[\hat{g}]_\lambda u\rangle_E\Big)\mathrm{d}x\mathrm{d}\mu_\gamma,
\end{aligned}$$

where $n_0 = 1, n_i = -\eta_{ij}x^j/t$ for $i \in \{1,\dots,n\}$ and $n_A = 0$. Note that

$$\begin{aligned}
\mathcal{E}[0;u;s] = \int_{\mathcal{H}_s\times\mathcal{K}} \Big(&|\partial_t u|_E^2 + \sum_{i=1}^n |\partial_i u|_E^2 + 2\frac{x^i}{t}\langle \partial_t u, \partial_i u\rangle_E \\
&+ \langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E - 2\langle R[\hat{g}]\circ u, u\rangle_E\Big)\mathrm{d}x\mathrm{d}\mu_\gamma,
\end{aligned} \qquad (6.4.12)$$

which alternatively can be written in hyperboloidal coordinates as

$$\begin{aligned}
\mathcal{E}[0;u;s] = \int_{\mathcal{H}_s\times\mathcal{K}} \Big(&(s/t)^2\,|\partial_t u|_E^2 + \sum_{i=1}^n |Y_i u|_E^2 + \langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E \\
&- 2\langle R[\hat{g}]\circ u, u\rangle_E\Big)\mathrm{d}x\mathrm{d}\mu_\gamma.
\end{aligned} \qquad (6.4.13)$$

Since the contraction of $R[\hat{g}]$ with any direction tangent to $\mathbb{R}^{1+n}$ vanishes, and since $|w|_E \geq |w|_\gamma$ for any tensor field w, it follows from the definition of $\mathcal{L}_\gamma$ that

$$\begin{aligned}
&\int_{\mathcal{K}} \Big(\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E - 2\langle R[\hat{g}]\circ u, u\rangle_E\Big)\mathrm{d}\mu_\gamma \\
&\geq \int_{\mathcal{K}} \Big(\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_\gamma - 2\langle R[\hat{g}]\circ u, u\rangle_\gamma\Big)\mathrm{d}\mu_\gamma = \int_{\mathcal{K}} \langle \mathcal{L}_\gamma u, u\rangle_\gamma \mathrm{d}\mu_\gamma. \qquad (6.4.14)
\end{aligned}$$

Thus, from theorem 6.4 and the condition of Riemannian linear stability (6.2.1), it follows that

$$\int_{\mathcal{K}} \Big(\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E - 2\langle R[\hat{g}]\circ u, u\rangle_E\Big)\mathrm{d}\mu_\gamma \geq 0. \qquad (6.4.15)$$

This implies $\mathcal{E}[0,u,s] \geq 0$. Using our previously calculated expression for the divergence of $T[v;u]^\mu{}_\nu X^\nu$ we obtain via Stoke's theorem

$$\begin{aligned}
&\mathcal{E}[v;u;s_1] = \mathcal{E}[v;u;s_2] + \int_{s_1}^{s_2}\int_{\mathcal{H}_s\times\mathcal{K}} \langle -2F, \partial_t u\rangle_E \frac{s}{t}\mathrm{d}y\mathrm{d}\mu_\gamma\mathrm{d}s \qquad (6.4.16) \\
&+ \int_{s_1}^{s_2}\int_{\mathcal{H}_s\times\mathcal{K}} \Big(-2(\nabla[\hat{g}]_\alpha v^{\alpha\beta})\langle \nabla[\hat{g}]_\beta u, \partial_t u\rangle_E + (\partial_t v^{\alpha\beta})\langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E\Big)\frac{s}{t}\mathrm{d}y\mathrm{d}\mu_\gamma\mathrm{d}s.
\end{aligned}$$

This proves equality (6.4.5).

Condition (6.4.2) combined with $s \geq Ct^{1/2}$ implies $\sup_{\mathcal{H}_s \times \mathcal{K}} |v|_E (t/s)^2 \leq C\varepsilon_n$. For simplicity denote $\gamma^{AB}\langle \nabla[\hat{g}]_A u, \nabla[\hat{g}]_B u\rangle_E$ by $|\partial_A u|_E^2$, then

$$\frac{s^2}{2t^2}(|\partial_t u|_E^2 \sum_i + |\partial_i u|_E^2 + |\nabla[\gamma]u|_E^2) \leq (|\partial_t u|^2 + \sum_i |\partial_i u|^2 + |\nabla[\gamma]u|_E^2)(1 - |x|/t)$$
$$\leq |\partial_t u|_E^2 + |\partial_i u|_E^2 + 2\frac{x^i}{t}\langle \partial_t u, \partial_i u\rangle_E + |\nabla[\gamma]u|_E^2.$$

Using this and Young's inequality we find

$$\begin{aligned}
&|\mathcal{E}[v; u; s] - \mathcal{E}[0; u; s]| \\
&= \left| \int_{\mathcal{H}_s \times \mathcal{K}} \left(2v^{\alpha\beta}\langle \nabla[\hat{g}]_\alpha u, \partial_t u\rangle_E n_\beta - v^{\alpha\beta}\langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E \right) \mathrm{d}x\mathrm{d}\mu_\gamma \right| \\
&\leq C\varepsilon_n \mathcal{E}[0; u; s],
\end{aligned}$$

and thus the energies are equivalent for sufficiently small ε_n. This proves estimate (6.4.3), completing the proof of the lemma. □

Having defined the energy which involves first-order derivatives, we now introduce higher-order energies.

Definition 6.18 (Symmetry boosted energy)**.** Let $(\mathbb{R}^{1+n} \times \mathcal{K}, \hat{g})$ be a spacetime with a supersymmetric compactification and $N_0 \in \mathbb{N}$. For $k \leq N_0$, define the energy of a symmetric tensor field g to be

$$\mathcal{E}_{k+1}(s) = \sum_{|I| \leq k} \mathcal{E}[g^{-1} - \hat{g}^{-1}; v^I g; s]. \tag{6.4.17}$$

We end this section with the following Hardy estimate on hyperboloids. The proof is standard, see for example [LM16a, Lemma 2.4].

Lemma 6.19 (Hardy estimate on hyperboloids)**.** *Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n}$, then one has*

$$\|r^{-1}u\|_{L^2(\mathcal{H}_s)} \lesssim \sum_{i=1}^{n} \|Y_i u\|_{L^2(\mathcal{H}_s)}. \tag{6.4.18}$$

6.4.2 Preliminary L^2 and L^∞-estimates

In our nonlinear estimates we will need to estimate terms of the form

$$Z^I (\Delta_\gamma)^\ell (uv) = \sum_{\substack{|I_1|+|I_2|=|I| \\ |J_1|+|J_2|=2\ell}} Z^{I_1}\nabla[\gamma]^{J_1} u \cdot Z^{I_2}\nabla[\gamma]^{J_2} v. \tag{6.4.19}$$

In the following lemma we estimate terms which appear as factors in the right hand side of (6.4.19) in L^2 by using the elliptic estimates of lemma 6.11 and the Hardy estimate of lemma 6.19. Note the use of elliptic estimates allows us to avoid commuting derivatives, such as $[\nabla[\gamma], \Delta_\gamma]$, which makes the argument shorter.

Lemma 6.20 (L^2 estimate for distributed derivatives)**.** *Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n} \times \mathcal{K}$. Suppose N_0 is even, $\ell \in \mathbb{N}$ and $\ell \le N_0 + 1$, then*

$$\sum_{|I|+|J| \le \ell} \|t^{-1} Z^I \nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \lesssim \mathcal{E}_{N_0+1}(s)^{1/2}. \tag{6.4.20}$$

Proof. We prove the estimate by considering separately the cases of $|I| = 0$ and $|I| \neq 0$. Firstly take $|I| \ge 1$, suppose $|J| = 2m$ where $m \in \mathbb{N}$ and consider $|I| + |J| = \ell \le N_0 + 1$. Using the elliptic estimates of lemma 6.11 we find

$$\begin{aligned}
\|t^{-1} Z^I \nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} &\lesssim \| \|t^{-1} Z^I u\|_{H^{2m}(\mathcal{K})} \|_{L^2(\mathcal{H}_s)} \\
&\lesssim \|t^{-1} Z^I (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \|t^{-1} Z^I u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \\
&\lesssim \sum_{i=1}^n \|Y_i Z^{I-1} (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \sum_{i=1}^n \|Y_i Z^{I-1} u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \\
&\lesssim \mathcal{E}[0; Z^{I-1} (\Delta_\gamma)^m u; s]^{1/2} + \mathcal{E}[0; Z^{I-1} u; s]^{1/2} \\
&\lesssim \mathcal{E}_\ell(s)^{1/2}.
\end{aligned} \tag{6.4.21}$$

Next take $|I| \ge 1$ and suppose $|J| = 2m + 1$ where $m \in \mathbb{N}$. For $|I| + |J| = \ell \le N_0 + 1$, again using lemma 6.11, we have

$$\begin{aligned}
\|t^{-1} Z^I \nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} &\lesssim \| \|t^{-1} Z^I u\|_{H^{2m+1}(\mathcal{K})} \|_{L^2(\mathcal{H}_s)} \\
&\lesssim \sum_{i=1}^n \|Y_i Z^{I-1} u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \sum_{i=1}^n \|Y_i Z^{I-1} (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \\
&\quad + \|\nabla[\gamma](Z^I (\Delta_\gamma)^m u)\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \qquad (6.4.22) \\
&\lesssim \mathcal{E}[0; Z^{I-1} u; s]^{1/2} + \mathcal{E}[0; Z^{I-1} (\Delta_\gamma)^m u; s]^{1/2} \\
&\quad + \mathcal{E}[0; Z^I (\Delta_\gamma)^m u; s]^{1/2} \\
&\lesssim \mathcal{E}_\ell(s)^{1/2}.
\end{aligned} \tag{6.4.23}$$

We now turn to the case $|I| = 0$. Again we split into the cases of $|J|$ being even and odd. Start with $|J| = 2m$ for $m \in \mathbb{N}$. Note that N_0 is chosen even so that we have the strict inequality $2m < N_0 + 1$. Applying the Hardy estimate from lemma 6.19, and recalling that $t \ge r$ on the hyperboloid, yields

$$\begin{aligned}
\|t^{-1} \nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} &\lesssim \Big\| \|r^{-1} u\|_{H^{2m}(\mathcal{K})} \Big\|_{L^2(\mathcal{H}_s)} \\
&\lesssim \|r^{-1} (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \|r^{-1} u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \\
&\lesssim \sum_{i=1}^n \|Y_i (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \sum_{i=1}^n \|Y_i u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} \\
&\lesssim \mathcal{E}[0, (\Delta_\gamma)^m u; s]^{1/2} + \mathcal{E}[0, u; s]^{1/2} \\
&\lesssim \mathcal{E}_{N_0+1}(s)^{1/2}.
\end{aligned} \tag{6.4.24}$$

Finally we consider the case $|I| = 0$ and $|J| = 2m + 1 \le N_0 + 1$ for $m \in \mathbb{N}$. Again using lemma 6.19 we obtain

$$\begin{aligned}
\|t^{-1} \nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} &\lesssim \Big\| \|r^{-1} u\|_{H^{2m+1}(\mathcal{K})} \Big\|_{L^2(\mathcal{H}_s)} \\
&\lesssim \|r^{-1} \nabla[\gamma] (\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s \times \mathcal{K})} + \|r^{-1} u\|_{L^2(\mathcal{H}_s \times \mathcal{K})}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla[\gamma](\Delta_\gamma)^m u\|_{L^2(\mathcal{H}_s\times\mathcal{K})} + \sum_{i=1}^n \|Y_i u\|_{L^2(\mathcal{H}_s\times\mathcal{K})} \\
&\lesssim \mathcal{E}[0,(\Delta_\gamma)^m u; s]^{1/2} + \mathcal{E}[0,u;s]^{1/2} \\
&\lesssim \mathcal{E}_{|J|}(s)^{1/2}. \qquad (6.4.25)
\end{aligned}
$$

Adding together the above estimates over all appropriate multi-indices gives the required result. □

Corollary 6.21 (L^2 estimate for eventually prescribed functions on hyperboloids foliating product spacetimes)**.** *Let $n \geq 4$. Let $u_{\mu\nu}, f_{\mu\nu}$ be tensors defined on $\mathbb{R}^{1+n}\times\mathcal{K}$ with f depending only on the Minkowski coordinates. Suppose $u = f$ for $|x| \geq t-1$. Let $f \in C^\infty(\mathbb{R}^{1+n}\times\mathcal{K})$ be smooth and such that for all $I \in \mathbb{N}$, there is a C_I such that*[3]

$$
|\nabla[\hat{g}]^I f|_E \leq C_{|I|}|x|^{-(n+1)/2-|I|}. \qquad (6.4.26)
$$

Suppose N_0 is even, $\ell \in \mathbb{N}$ and $\ell \leq N_0+1$, then

$$
\sum_{|I|+|J|\leq\ell} \|(s/t)Z^I\nabla[\gamma]^J u\|_{L^2(\mathcal{H}_s\times\mathcal{K})} \lesssim s\mathcal{E}_{N_0+1}(s)^{1/2} + \sum_{|I|+|J|\leq\ell} C_{|I|,|J|}. \qquad (6.4.27)
$$

Proof. We will consider separately the regions $|x| \leq t-1$ and $|x| > t-1$. The estimate in the region $|x| \leq t-1$ follows by applying Lemma 6.20 with an additional factor of s. Next consider the region $|x| > t-1 \geq t_0 - 1$ where we let $t_0 = (s^2+1)/2$ be the value of t at which $\mathcal{H}_s$ intersects $|x| = t-1$. Using assumption (6.4.26) we find

$$
\begin{aligned}
&\|(s/t)Z^I\nabla[\gamma]^J u\|^2_{L^2(\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\})} \\
&\leq \int_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t_0-1\}} |Z^I\nabla[\gamma]^J u|_E^2 \mathrm{d}x\mathrm{d}\mu_\gamma \\
&\leq C\int_{\mathcal{H}_s\cap\{|x|>t_0-1\}} |Z^I\nabla[\gamma]^J f|_E^2 \mathrm{d}x \\
&\leq CC^2_{|I|,|J|}\int_{\mathbb{S}^{n-1}}\int_{\mathcal{H}_s\cap\{|x|\geq t_0-1\}} (|r|^{-(n+1)/2})^2|r|^{n-1}\mathrm{d}r\mathrm{d}\omega_{\mathbb{S}^{n-1}} \\
&\leq CC^2_{|I|,|J|}\int_{\mathbb{S}^{n-1}}\int_{\mathcal{H}_s\cap\{|x|\geq t_0-1\}} r^{-2}\mathrm{d}r\mathrm{d}\omega_{\mathbb{S}^{n-1}} \\
&\leq CC^2_{|I|,|J|}. \qquad (6.4.28)
\end{aligned}
$$

Adding together the above estimate over all appropriate multi-indices yields (6.4.27). □

We next use lemma 6.15 to obtain L^∞ estimates for terms which appear as factors in the right hand side of (6.4.19).

Corollary 6.22 (Higher-order Sobolev estimates)**.** *Let $n \geq 7$. Let $\tilde{d}, \tilde{\nu}, u_{\mu\nu}, f_{\mu\nu}$ be as defined in lemma 6.15. Then for $|I|+|J| = \ell \in \mathbb{N}$ there is a constant C such that*

$$
\begin{aligned}
&\sup_{\mathcal{H}_s\times\mathcal{K}}\left(s^{4\delta(n)}|Z^I\nabla[\gamma]^J u|_E^2 + s^{4\delta(n)-2}|(t/s)Z^I\nabla[\gamma]^J u|_E^2\right) \\
&\leq C\sum_{|I|+2j\leq\tilde{\nu}+\ell+1}\mathcal{E}[0;Z^I(\Delta_\gamma)^j u;s] + C\sum_{|I|\leq\tilde{\nu}+\ell-1} C^2_{|I|}.
\end{aligned} \qquad (6.4.29)
$$

[3]Note that decay assumption on f is stronger here than the assumption (6.3.2) in lemma 6.15.

Proof. We consider the left most term in (6.4.29) first. Let $\tilde{\jmath}$ be the smallest even integer such that $\tilde{\jmath} \geq |J|$. In particular this means $|I| + |J| \leq |I| + \tilde{\jmath} \leq \ell + 1$. Recall that $\tilde{d}$ is the smallest even integer larger than $d/2$ and $\tilde{\nu}$ is the smallest integer greater than $n/2 + \tilde{d}$. Applying lemma 6.11 yields

$$\sup_{\mathcal{K}} |\nabla[\gamma]^J u|_E \leq \|u\|_{H^{\tilde{d}+\tilde{\jmath}}(\mathcal{K})} \leq \|(\Delta_\gamma)^{(\tilde{d}+\tilde{\jmath})/2} u\|_{L^2(\mathcal{K})} + \|u\|_{L^2(\mathcal{K})}.$$

Thus, using in particular (6.3.6), we have

$$\begin{aligned}
&\sup_{(t,x,\omega)\in\mathcal{H}_s\times\mathcal{K}} s^{4\delta(n)} |Z^I \nabla[\gamma]^J u(t,x^i,\omega)|_E^2 \\
&\lesssim \sum_{|I_1|\leq\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \|\sup_{\mathcal{K}}(Y_i Z^{I_1} Z^I \nabla[\gamma]^J u)\|_{L^2(\mathcal{H}_s)}^2 + \sum_{|I_1|\leq\tilde{\nu}-1} C_{I_1}^2 \\
&\lesssim \sum_{|I_1|\leq\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \Big(\|Y_i Z^{I+I_1} u\|_{L^2(\mathcal{H}_s\times\mathcal{K})}^2 + \|Y_i Z^{I+I_1} (\Delta_\gamma)^{(\tilde{d}+\tilde{\jmath})/2} u\|_{L^2(\mathcal{H}_s\times\mathcal{K}))}^2 \Big) \\
&\quad + C \sum_{|I_1|\leq\tilde{\nu}-1} C_{I_1}^2 \\
&\lesssim \sum_{|I|+2j\leq\tilde{\nu}+\ell+1} \mathcal{E}[0; Z^I(\Delta_\gamma)^j u; s] + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2.
\end{aligned} \tag{6.4.30}$$

To complete the proof for the second term of (6.4.29) we observe that $s \geq Ct^{1/2}$ in the region $|x| \leq t-1$ while we only have $s \leq t \leq r$ in the region $|x| > t-1$. Since $n \geq 7$ we have $\delta(n) \geq 1$ and thus

$$\begin{aligned}
\sup_{\mathcal{H}_s\times\mathcal{K}} s^{4\delta(n)-2} |(t/s) Z^I \nabla[\gamma]^J u|_E^2 &\lesssim \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|\leq t-1\}} (t^2/s^4) s^{4\delta(n)} |Z^I \nabla[\gamma]^J u|_E^2 \\
&\quad + \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\}} s^{4\delta(n)-4} r^2 |Z^I \nabla[\gamma]^J f|_E^2 \\
&\lesssim \sum_{|I|+2j\leq\tilde{\nu}+\ell+1} \mathcal{E}[0; Z^I(\Delta_\gamma)^j u; s] + \sum_{|I|\leq\tilde{\nu}+\ell-1} C_I^2 \\
&\quad + C_I^2 \sup_{\mathcal{H}_s\times\mathcal{K}\cap\{|x|>t-1\}} r^{(n-2)-2} r^{-(n-1)}.
\end{aligned}$$

Note in the final line we applied (6.3.2) and the first estimate of (6.4.29). □

6.5 Proof of stability

6.5.1 Stability for the reduced Einstein equations

We now restate our main theorem 6.1 in terms of the reduced Einstein equations. For convenience we translate the initial data of theorem 6.1 to $\{t = 4\}$.

Theorem 6.23 (Stability for the reduced Einstein equations)**.** *Let $n, d \in \mathbb{Z}^+$ be such that $n \geq 9$ and let $N_0 \in \mathbb{N}$ be an even integer strictly larger than $(n + d + 8)/2$. Let $(\mathbb{R}^{1+n} \times \mathcal{K}, \hat{g} = \eta_{\mathbb{R}^{1+n}} + \gamma)$ be a spacetime with a supersymmetric compactification.*

Let $(\{t = 4\} \times \mathbb{R}^n \times \mathcal{K}, g_0, g_1)$ be Cauchy data for the reduced Einstein equations (1.1.5). Assume that, for $|x| \geq 1$ with respect to Minkowski coordinates on $\mathbb{R}^{1+n}$, $(g_0, g_1) = (g_S + \gamma, 0)$ where g_S is the Schwarzschild metric in the $\eta_{\mathbb{R}^{1+n}}$-wave gauge with *parameter* $C_S \in [0,\infty)$.

There is an $\varepsilon > 0$ *such that, if the initial data satisfies*

$$\sum_{|I|\le N_0} \|\nabla[g_0]^I(g_0 - \hat{g}|_{t=4})\|^2_{L^2(\mathbb{R}^n\times\mathcal{K})} + \sum_{|I|\le N_0-1} \|\nabla[g_0]^I g_1\|^2_{L^2(\mathbb{R}^n\times\mathcal{K})} + C_S^2 \le \varepsilon, \quad (6.5.1)$$

then there is a future global solution $g_{\mu\nu}$ *of the reduced Einstein equations* (1.1.5) *with initial data* $(h, \partial_t h)|_{t=4} = (g_0, g_1)$. *Furthermore, there is the bound*

$$\sup_{(t,x,\omega)\in\mathcal{H}_s\times\mathcal{K}} s^{4\delta(n)}|g(t,x^i,\omega) - \hat{g}(t,x^i,\omega)|^2_E \lesssim \varepsilon, \quad (6.5.2)$$

where $\delta(n)$ *was defined in* (6.1.7).

Proof. We start by expressing the reduced Einstein equations 1.5.18 in terms of the perturbation and inverse perturbation:

$$h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}, \quad (6.5.3)$$
$$H^{\mu\nu} = g^{\mu\nu} - \hat{g}^{\mu\nu}. \quad (6.5.4)$$

Since g is a solution of the reduced Einstein equation (1.1.5), it follows that

$$\begin{aligned}&(\hat{g}^{\alpha\beta} + H^{\alpha\beta})\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta h_{\mu\nu} + 2(R[\hat{g}]\circ h)_{\mu\nu}\\ &= \mathcal{F}_{\mu\nu}[g](\nabla[\hat{g}]h, \nabla[\hat{g}]h) + F_{\mu\nu}(H,h),\end{aligned} \quad (6.5.5)$$

where $\mathcal{F}_{\mu\nu}$ and $F_{\mu\nu}$ are defined by

$$\begin{aligned}\mathcal{F}_{\mu\nu}[g](\nabla[\hat{g}]g, \nabla[\hat{g}]g) = g^{\gamma\delta}g^{\alpha\beta}\Big(&\nabla[\hat{g}]_\nu h_{\delta\beta}\nabla[\hat{g}]_\alpha h_{\mu\gamma} + \nabla[\hat{g}]_\mu h_{\gamma\alpha}\nabla[\hat{g}]_\beta h_{\nu\delta} - \tfrac{1}{2}\nabla[\hat{g}]_\nu h_{\delta\beta}\nabla[\hat{g}]_\mu h_{\gamma\alpha}\\ &+ \nabla[\hat{g}]_\gamma h_{\mu\alpha}\nabla[\hat{g}]_\rho h_{\mu\alpha} - \nabla[\hat{g}]_\gamma h_{\mu\alpha}\nabla[\hat{g}]_\beta h_{\nu\delta}\Big), \quad (6.5.6)\\ F_{\mu\nu}(H,h) = H^{\alpha\beta}&\left(h_{\alpha\delta}\,\mathrm{Riem}[\hat{g}]^\delta{}_{\mu\nu\beta} + h_{\alpha\delta}\,\mathrm{Riem}[\hat{g}]^\delta{}_{\nu\mu\beta}\right)\\ + H^{\alpha\beta}&\left(h_{\mu\delta}\,\mathrm{Riem}[\hat{g}]^\delta{}_{\alpha\nu\beta} + h_{\nu\delta}\,\mathrm{Riem}[\hat{g}]^\delta{}_{\alpha\mu\beta}\right). \quad (6.5.7)\end{aligned}$$

By commuting the symmetries $Z^I(\Delta_\gamma)^\ell$ through the system (6.5.5) we obtain

$$(\hat{g}^{\alpha\beta} + H^{\alpha\beta})\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta(Z^I(\Delta_\gamma)^\ell h_{\mu\nu}) - 2(R[\hat{g}]\circ Z^I(\Delta_\gamma)^\ell h)_{\mu\nu} = \sum_{i=1}^{3} F^{i,I,\ell}_{\mu\nu}, \quad (6.5.8)$$

where

$$\begin{aligned}F^{1,I,\ell}_{\mu\nu} &= Z^I(\Delta_\gamma)^\ell\mathcal{F}_{\mu\nu}[g](\nabla[\hat{g}]h, \nabla[\hat{g}]h)\,,\\ F^{2,I,\ell}_{\mu\nu} &= Z^I(\Delta_\gamma)^\ell F_{\mu\nu}(H,h)\,,\\ F^{3,I,\ell}_{\mu\nu} &= [Z^I(\Delta_\gamma)^\ell, H^{\alpha\beta}\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta]h_{\mu\nu}\,.\end{aligned} \quad (6.5.9)$$

The symmetry boosted energy is given by

$$\mathcal{E}_{k+1}(s) = \sum_{|I|+2\ell\le k} \mathcal{E}[H; Z^I(\Delta_\gamma)^\ell g; s]. \quad (6.5.10)$$

From lemma 6.17 and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\mathcal{E}_{N_0+1}(s')^{1/2} &\leq \mathcal{E}_{N_0+1}(4)^{1/2} \\ &\quad + \sum_{|I|+2\ell\leq N_0} \int_4^{s'} \left(\int_{\mathcal{H}_s\times\mathcal{K}} \Big(\sum_{i=1}^{3} |F^{i,I,\ell}|_E^2 + |G^{I,\ell}|_E^2 \Big) \mathrm{d}y\mathrm{d}\mu_\gamma \right)^{1/2} \mathrm{d}s,\end{aligned} \tag{6.5.11}$$

where the $G^{I,\ell}$ terms arise from applying $Z^I(\Delta_\gamma)^j$ to the terms involving $\nabla[\hat{g}]v$ or $\partial_t v$ on the right side of the energy equality (6.4.5). In particular, these can be bounded by

$$|G^{I,j}|_E^2 \leq C|\nabla[\hat{g}]H|_E^2 |Z^I(\Delta_\gamma)^\ell \nabla[\hat{g}]h|_E^2. \tag{6.5.12}$$

The reduced field equations (6.5.5) are a system of quasilinear, quasidiagonal wave equations for the spacetime metric $h_{\mu\nu}$. The existence of unique local solutions emanating from Cauchy data is standard [CB09, Theorem 4.6 Appendix III].

The proof then follows a boot-strap argument (or continuous induction). The goal is to prove that there exists a constant $C > 0$ and $\varepsilon > 0$ such that: if $\mathcal{E}_{N_0+1}(4) + C_S^2 < \varepsilon$ and $\forall s : \mathcal{E}_{N_0+1}(s) \leq C\varepsilon$, then $\forall s : \mathcal{E}_{N_0+1}(s) \leq \varepsilon + C\varepsilon^2$ and hence $\mathcal{E}_{N_0+1}(s) \leq C\varepsilon/2$. Note that there is clearly no loss of generality in placing our initial data at $t = 4$.

We consider the integral term on the right-hand-side in (6.5.11) as the sum of integrals over $\mathcal{H}_s \cap \{|x| \leq t-1\}$ and over $\mathcal{H}_s \cap \{|x| > t-1\}$. Our approach is that, for sufficiently small C_S, in the latter exterior region the solution is identically the product of Schwarzschild with the internal manifold. Thus in the region $|x| \geq t-1$ the perturbation $h_{\mu\nu}$ is only nonzero on its Minkowski indices and on these indices it is identically Schwarzschild. We note that sufficiently small compactly supported initial data on $\{t = 4\} \cap \{|x| \leq 1\}$ can be extended to compactly supported initial data on Σ_4 [LM14, Chapter 39].

Recall from section 6.2.3 that the difference between components of the Minkowski metric and the Schwarzschild metric in wave coordinates decay as $C_S r^{-n+2}$ and the Christoffel symbols decay as $C_S r^{-n+1}$. Along a geodesic parametrised by λ, one has $\mathrm{d}^2x^i/\mathrm{d}\lambda^2 = \Gamma^i_{jk}(\mathrm{d}x^j/\mathrm{d}\lambda)(\mathrm{d}x^k/\mathrm{d}\lambda)$. Since $C_S r^{-n+1}$ is integrable in r, there are geodesics along which t and r grow linearly and the $(\mathrm{d}x^j/\mathrm{d}\lambda)$ approach constant values, not all of which are vanishing. In particular, $\mathrm{d}r/\mathrm{d}t$ asymptotically approaches a constant, and this constant is 1 for null geodesics. The next-to-leading order term in the geodesic equation arises from the metric, so it is of the form Cr^{-n+2}, which is again integrable. Furthermore, the smaller the mass C_S the sooner this asymptotic behaviour comes to dominate. In particular, if C_S is sufficiently small, then any causal curve launched from within $\Sigma_4 \cap \{|x| \leq t-2\}$ can never reach the region where $|x| \geq t-1$. Furthermore, by uniqueness of solutions to quasilinear wave equations, since the initial data on Σ_4 is identically Schwarzschild for $|x| > t-2$, the solution is identically Schwarzschild for $|x| > t-1$. In particular, when estimating the components of the solution to (6.5.8), we can use the Sobolev lemma 6.15 and corollary 6.21 on hyperboloids with eventually prescribed functions. (The conclusion of this paragraph is essentially proposition 2.3 of [LM16a].)

The estimate (6.4.2) required by lemma 6.17 is established by combining (6.3.4) with the bootstrap assumptions and noting that since $n \geq 9$ we certainly have $\delta(n) > 1$. Similarly since $n \geq 9$ the decay assumptions (6.4.26) in corollary 6.21 and (6.3.2) in lemma 6.15 are satisfied.

We are now in a position to apply the results from section 6.4.2 to the nonlinearities in (6.5.11). In general we will distribute $(s/t)(t/s) = 1$ across the terms and estimate high-derivative terms with a factor of (s/t) using corollary 6.21 and low-derivative terms with a factor of (t/s) using corollary 6.22. We begin by estimating the term $G^{I,j}$. Using (6.5.12) we find

$$\begin{aligned}
&\sum_{|I|+2\ell\leq N_0}\|G^{I,\ell}\|_{L^2(\mathcal{H}_s\times\mathcal{K})}\\
&\lesssim \sum_{|I|+|J|\leq N_0}\left(\int_{\mathcal{H}_s\times\mathcal{K}}|(t/s)\nabla[\hat{g}]H|_E^2|(s/t)Z^I\nabla[\gamma]^J\nabla[\hat{g}]h|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}\\
&\leq \sup_{\mathcal{H}_s\times\mathcal{K}}\left(\big|(t/s)\nabla[\hat{g}]h\big|_E\right)\left(\int_{\mathcal{H}_s\times\mathcal{K}}\big|(s/t)Z^I\nabla[\gamma]^J\nabla[\hat{g}]h\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}\\
&\lesssim \frac{1}{s^{2\delta(n)-1}}\Big(\mathcal{E}_{\tilde{\nu}+3}(s)^{1/2}+C_S\Big)\Big(s\mathcal{E}_{N_0+1}(s)^{1/2}+C_S\Big).
\end{aligned}\tag{6.5.13}$$

The term $F^1_{\mu\nu}$ involves the standard quadratic derivative nonlinearities of the Einstein equations. Their weak-null structure is of course not relevant here since the Minkowski dimension is taken so high. We first look at what type of terms are contained in $F^1_{\mu\nu}$:

$$\begin{aligned}
&\sum_{|I|+2\ell\leq N_0}\|F^{1,I,\ell}_{\mu\nu}\|_{L^2(\mathcal{H}_s\times\mathcal{K})}\\
&\lesssim \sum_{|I|+|J|\leq N_0}\left(\int_{\mathcal{H}_s\times\mathcal{K}}\big|(\hat{g}+H)^{-1}\big|_E^2\big|Z^I\nabla[\gamma]^J(\nabla[\hat{g}]h\nabla[\hat{g}]h)\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}\\
&+\sum_{\substack{|I_i|+|J_i|\leq N_0\\|I_1|+|J_1|\geq 1}}\left(\int_{\mathcal{H}_s\times\mathcal{K}}\big|Z^{I_1}\nabla[\gamma]^{J_1}h\big|_E^2\big|Z^{I_2}\nabla[\gamma]^{J_2}(\nabla[\hat{g}]h\nabla[\hat{g}]h)\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}.
\end{aligned}\tag{6.5.14}$$

We treat the first term on the right hand side of (6.5.14) since the second term is higher-order and thus easier to estimate. Once again we estimate high-derivative terms with a factor of (s/t) using corollary 6.21 and low-derivative terms with a factor of (t/s) using corollary 6.22. This yields

$$\begin{aligned}
&\sum_{|I|+|J|\leq N_0}\left(\int_{\mathcal{H}_s\times\mathcal{K}}\big|(\hat{g}+H)^{-1}\big|_E^2\big|Z^I\nabla[\gamma]^J(\nabla[\hat{g}]h\nabla[\hat{g}]h)\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}\\
&\lesssim \sum_{\substack{|I_i|+|J_i|\leq N_0\\|I_2|+|J_2|\leq\frac{N_0}{2}+1}}\left(\int_{\mathcal{H}_s\times\mathcal{K}}C\big|Z^{I_1}\nabla[\gamma]^{J_1}\nabla[\hat{g}]h\big|\big|Z^{I_2}\nabla[\gamma]^{J_2}\nabla[\hat{g}]h)\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2},
\end{aligned}\tag{6.5.15}$$

where by symmetry we can assume $|I_2|+|J_2|\leq\frac{N_0}{2}+1$. After using $(s/t)(t/s)=1$ we find

$$\begin{aligned}
&\sum_{\substack{|I_i|+|J_i|\leq N_0\\|I_2|+|J_2|\leq\frac{N_0}{2}+1}}\left(\int_{\mathcal{H}_s\times\mathcal{K}}C\big|(s/t)Z^{I_1}\nabla[\gamma]^{J_1}\nabla[\hat{g}]h\big|\big|(t/s)Z^{I_2}\nabla[\gamma]^{J_2}\nabla[\hat{g}]h)\big|_E^2\mathrm{d}y\mathrm{d}\mu_\gamma\right)^{1/2}\\
&\lesssim \sup_{\mathcal{H}_s\times\mathcal{K}}\Bigg(\sum_{|I_2|+|J_2|\leq\frac{N_0}{2}+1}|(t/s)Z^{I_2}\nabla[\gamma]^{J_2}\nabla[\hat{g}]h|_E\Bigg)
\end{aligned}$$

$$\times \sum_{|I_1|+|J_1|\le N_0} \Big(\int_{\mathcal{H}_s\times\mathcal{K}} \Big|(s/t)Z^{I_1}\nabla[\gamma]^{J_1}\nabla[\hat g]h\Big|^2_E \mathrm{d}y\mathrm{d}\mu_\gamma\Big)^{1/2}$$
$$\lesssim \frac{1}{s^{2\delta(n)-1}}\Big(\sum_{|I|+2j\le\tilde\nu+\frac{N_0}{2}+3} \mathcal{E}[0;Z^I(\Delta_\gamma)^j u;s]^{1/2} + C_S \sum_{|I|\le\tilde\nu+\frac{N_0}{2}} C_I^2\Big)\Big(s\mathcal{E}_{N_0+1}(s)^{1/2}+C_S\Big)$$
$$\lesssim \frac{1}{s^{2\delta(n)-2}}\Big(\mathcal{E}_{\tilde\nu+\frac{N_0}{2}+4}(s)^{1/2}+C_S\Big)\Big(\mathcal{E}_{N_0+1}(s)^{1/2}+C_S\Big). \tag{6.5.16}$$

The term $F^2_{\mu\nu}$ involves the new nonlinearities which are only nonzero when both $\mu,\nu\in\{A,\dots,B\}$. This means we can control $F^2_{\mu\nu}$ by the following

$$\sum_{|I|+2\ell\le N_0} \|F^{2,I,\ell}_{\mu\nu}\|_{L^2(\mathcal{H}_s\times\mathcal{K})} \lesssim \sup_{\mathcal{H}_s\times\mathcal{K}}\Big(\sum_{|I_0|\le N}\big|\nabla[\gamma]^{I_0}\,\mathrm{Riem}[\gamma]\big|\Big)$$
$$\times \sum_{|I_i|+|J_i|\le N_0}\Big(\int_{\mathcal{H}_s\times\mathcal{K}} \big|Z^{I_1}\nabla[\gamma]^{J_1}h\big|^2_E\big|Z^{I_2}\nabla[\gamma]^{J_2}h\big|^2_E \mathrm{d}y\mathrm{d}\mu_\gamma\Big)^{1/2}. \tag{6.5.17}$$

The Riemann curvature components of γ are bounded (since $\mathcal{K}$ is compact) which allows us to control the first factor in (6.5.17). To estimate the second factor in (6.5.17) we follow the same procedure as in $F^1_{\mu\nu}$, by controlling high-derivatives with a factor of (s/t) using corollary 6.21 and low-derivatives with a compensating factor of (t/s) using corollary 6.22. The result of this procedure leads to a term controlled by (6.5.16).

The final term $F^3_{\mu\nu}$ is a commutator involving the quasilinear perturbation of the principal part of the differential operator. Note first the identity

$$\sum_{|I|+2\ell\le N_0} |F^{3,I,\ell}_{\mu\nu}|_E \le C \sum_{\substack{|I_i|+|J_i|\le N_0\\ |I_2|+|J_2|\le N_0-1}} |Z^{I_1}\nabla[\gamma]^{J_1}H|_E|Z^{I_1}\nabla[\gamma]^{J_1}\nabla[\hat g]\nabla[\hat g]h|_E. \tag{6.5.18}$$

Once again we distribute the product $(s/t)(t/s)=1$ across the two terms appearing here depending on where the derivatives land. The term with high-derivatives gains a factor of (s/t) and is controlled using corollary 6.21 while the term with low-derivatives absorbs a compensating factor of (t/s) and is estimated using corollary 6.22. Note that when the term $Z^{I_2}\nabla[\gamma]^{J_2}(\nabla[\hat g]\nabla[\hat g]h)$ is estimated in L^∞ the Sobolev inequality will lead to a symmetry boosted energy at order $\tilde\nu+\frac{N_0}{2}+5$. We eventually obtain

$$\sum_{|I|+2\ell\le N_0} \|F^{3,I,\ell}_{\mu\nu}\|_{L^2(\mathcal{H}_s\times\mathcal{K})} \lesssim \frac{1}{s^{2\delta(n)-2}}\Big(\mathcal{E}_{\tilde\nu+\frac{N_0}{2}+5}(s)^{1/2}+C_S\Big)\Big(\mathcal{E}_{N_0+1}(s)^{1/2}+C_S\Big).$$

Putting these all together, inserting the bootstrap assumptions and using also $C_S^2<\varepsilon$ we find

$$\sum_{|I|+2\ell\le N_0}\int_4^{s'}\Big(\int_{\mathcal{H}_s\times\mathcal{K}}\Big(\sum_{i=1}^3 |F^{i,I,\ell}|^2_E+|G^{I,\ell}|^2_E\Big)\mathrm{d}y\mathrm{d}\mu_\gamma\Big)^{1/2}\mathrm{d}s \lesssim \varepsilon\int_4^{s'}\frac{1}{s^{2\delta(n)-2}}\mathrm{d}s. \tag{6.5.19}$$

For integrability we require $2\delta(n)-2>1$, which is equivalent to each of the following

$$\delta(n)>\frac{3}{2}, \tag{6.5.20}$$

$$n > 8. \tag{6.5.21}$$

This implies $n \geq 9$. For the Sobolev estimates we require

$$\tilde{\nu} + N_0/2 + 4 \leq N_0. \tag{6.5.22}$$

Recalling the definition of $\tilde{\nu}$ given in lemma 6.15 this certainly holds provided $N_0 > (n + d + 8)/2$ and N_0 is even.

Consequently for sufficiently small ε and by Grönwall's inequality applied to the energy estimate (6.5.11) we find $\mathcal{E}_{\nu+1}(s) \leq \frac{1}{2}C_1\varepsilon$. We have thus obtained a future global solution $h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}$ to the reduced Einstein equations and which clearly satisfies the decay bounds given in theorem 6.23. □

Remark 6.24. The system (6.5.5) contains quadratic nonlinearities F_{AB}, F_{iA} that are new compared to the weak-null terms identified in the proof of Minkowski stability in [LR03, LR10] and the proof of zero-mode Kaluza-Klein stability in [Wya18]. These terms did however appear in the proof of Milne stability in [AM11].

6.5.2 Proof of Theorem 6.1

We are now in a position to use the results from theorem 6.23 in order to prove our main result. Take an initial data set $(\mathbb{R}^n \times \mathcal{K}, \bar{g}, \bar{K})$ as specified in theorem 6.1 with smallness conditions (6.1.5). We now transform this data into the form required by theorem 6.23, which is a standard procedure, see for example [LR05]. We first set $((g_0)_{ab}, (g_1)_{ab}) = (\bar{g}_{ab}, \bar{K}_{ab})$. Diffeomorphism invariance allow us the freedom to choose the lapse and shift. We set the shift to be zero $X_a = 0$. We choose the lapse to be a smooth function satisfying

$$\begin{aligned} N(r) &= 1, \quad r \leq 1/2, \\ |N - 1| &\lesssim C_S, \quad 1/2 \leq r \leq 1, \\ N(r) &= \left(1 - \frac{h_{00}(r^{-1})}{r^{n-2}}\right)^{1/2}, \quad r \geq 1. \end{aligned} \tag{6.5.23}$$

We relate the lapse and shift with the Cauchy data for the reduced equations in theorem 6.23 by setting $(g_0)_{00} = -N^2$ and $(g_0)_{0a} = X_a$. The initial data for $(\partial_t N, \partial_t X_a) = ((g_1)_{00}, (g_1)_{0a})$ is chosen by satisfying $V^\gamma = 0$, see (1.1.19). We have now brought the initial data of theorem 6.1 into the form of theorem 6.23. It remains to check that our assumptions on the lapse and shift are compatible with smallness conditions (6.5.1). To do this, recall the final sentence of theorem 6.13. This implies the following

$$\begin{aligned} \int_{\{r\geq 1\}\cap\mathbb{R}^n} |\nabla[g_0]^I(-N^2 - \eta_{00})|^2 \mathrm{d}x &\leq \int_{\{r\geq 1\}\cap\mathbb{R}^n} C_S^2 (r^{-(n-2)-|I|})^2 r^{n-1} \mathrm{d}r \mathrm{d}^{n-1}\omega_{\mathbb{S}^{n-1}} \\ &\leq C_S^2 \int_{\{r\geq 1\}\cap\mathbb{R}^n} r^{-(n-3)-2|I|} \mathrm{d}r \mathrm{d}^{n-1}\omega_{\mathbb{S}^{n-1}} \\ &\leq C C_S^2. \end{aligned}$$

By inverting the expressions (1.1.19) for $(\partial_t N, \partial_t X_a)$ it is clear that the smallness conditions (6.5.1) are satisfied. Furthermore as discussed in Chapter 1, the future global solution constructed in theorem 6.23 is in fact also a solution to the full Einstein equations.

Finally, note that the solution found in theorem 6.23 is only defined to the future $t \geq 4$. Nonetheless, by time translation, we can treat the initial data as being on $\{t = 0\}$ instead of $\{t = 4\}$, so that theorem 5.1 ensures the existence of a solution for $t \geq 0$. By time reversibility for the Einstein equation (and the reduced Einstein equation), we similarly obtain a solution for $t \leq 0$. Thus, we can construct the global solution required in theorem 6.1.

It now remains to prove the causal geodesic completeness of $(\mathbb{R}^{1+n} \times \mathcal{K}, g)$.

Globally, the metrics g and $\hat{g}$ are very close, in the sense that, with respect to a basis constructed from the X_i and an orthonormal basis on $\mathcal{K}$, their components vanish to order ε globally. Denote from now onwards $T = \mathrm{d}t$. This is a globally timelike one-form such that $|g(T, T) + 1| \lesssim \varepsilon$. Thus, $g + 2TT$ defines a Riemannian metric. (Note that in the introduction, we used the slightly different Euclidean metric $\hat{g} + 2TT$.) Within this proof, we define, for a vector u, the Euclidean length to be

$$|u|^2 = u^\alpha u^\beta (g_{\alpha\beta} + 2T_\alpha T_\beta). \tag{6.5.24}$$

Note that the fact that g and $\hat{g}$ are very close implies the equivalence $|u|_E \sim |u|$.

Consider a causal geodesic γ that is affinely parameterised by λ. For the remainder of this paragraph, let $t = t(\lambda)$ denote the value of the Cartesian coordinate t at the point $\gamma(\lambda)$. By rescaling, we may assume that $\mathrm{d}t/\mathrm{d}\lambda = 1$ at $t = 0$. Let v be the (artificial, Euclidean) speed defined by $v \geq 0$ and

$$v^2 = \left| \frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\lambda} \right|^2 . \tag{6.5.25}$$

Since g and $\hat{g}$ are very close, the rate of change in the t direction is not much greater than the Euclidean speed, i.e. $|\frac{\mathrm{d}t}{\mathrm{d}\lambda}| = |\frac{\mathrm{d}\gamma^0}{\mathrm{d}\lambda}| \lesssim v$. On the other hand, since γ is causal, the component of $\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}$ in the T direction cannot vanish faster than the the length of the component in the orthogonal spatial directions, and the square of Euclidean velocity is the sum of the squares of the lengths of the T components and the orthogonal spatial component (up to order ε multiplicative errors); thus $|\frac{\mathrm{d}t}{\mathrm{d}\lambda}| = |\frac{\mathrm{d}\gamma^0}{\mathrm{d}\lambda}| \gtrsim v$. In particular, there is the equivalence $|\frac{\mathrm{d}t}{\mathrm{d}\lambda}| \sim v$.

The rate of change of the velocity is, since $\nabla[g]g = 0$ and $\nabla[g]_{\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}} \frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} v^2 = 4 \left(\frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\lambda} T_\alpha \right) \left(\frac{\mathrm{d}\gamma^\beta}{\mathrm{d}\lambda} \nabla[g]_{\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}} T_\beta \right). \tag{6.5.26}$$

Since the absolute value of $\frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\lambda} T_\alpha = \frac{\mathrm{d}t}{\mathrm{d}\lambda}$ and the Euclidean length of $\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}$ are dominated by v

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} \lesssim \left| \nabla[g]_{\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}} T \right| v. \tag{6.5.27}$$

The $\nabla[g]T$ can be expanded in terms of g and $\nabla[\hat{g}]g$. Both of these have norms that decay as $t^{-\delta(n)}$ due to (6.5.20). Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} \lesssim \varepsilon t^{-\delta(n)} v^2, \tag{6.5.28}$$

and using the equivalence $|\frac{\mathrm{d}t}{\mathrm{d}\lambda}| \sim v$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} \lesssim \varepsilon t^{-\delta(n)} v. \tag{6.5.29}$$

Thus, for ε sufficiently small, a simple bootstrap argument shows that $v \sim 1$ along all of γ. Thus, $\frac{\mathrm{d}t}{\mathrm{d}\lambda} \sim 1$. In particular, t is monotone along γ.

Let t_{sup} be the supremum of the t values that are achieved along γ. For contradiction, suppose $t_{\mathrm{sup}} < \infty$. Since the length of the spatial component of $\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}$ is also uniformly equivalent to v, and hence to $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$, it follows that, as $t \nearrow t_{\mathrm{sup}}$, the curve γ has a limit in $\mathbb{R}^{1+n} \times \mathcal{K}$. Because of the global bounds on g and its derivatives, by the standard Picard-Lindelöf theorem for ODEs, the curve γ must smoothly extend through this limiting point, contradicting the definition of t_{sup}. Thus, $t_{\mathrm{sup}} = \infty$. The only other way in which γ can be future incomplete is if t diverges to ∞ in a finite λ interval, but this is also impossible, since $\frac{\mathrm{d}t}{\mathrm{d}\lambda} \sim 1$. By time symmetry, the same argument holds in the past. Thus, any causal geodesic is complete.

The previous construction shows that every causal geodesic goes through each level set of t. Thus, the level sets of t are Cauchy surfaces, and $(\mathbb{R}^{1+n} \times \mathcal{K}, g)$ is globally hyperbolic.

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