A SPECIAL PREPARED SYSTEM FOR TWO QUADRATICS IN N VARIABLES.

A thesis

respectfully submitted to the

Faculty of Science of the University of Edinburgh

in candidacy for the degree of

Doctor of Science

by

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C O N T E N T S.

INTRODUCTION.

A complete system of concomitants for two quadratics in n variables is a system of concomitants, in terms of which every rational integral concomitant of the two quadratics may be expressed rationally and integrally. The existence of such a finite complete system is a particular case of Gordan's theorem.[1] There are three possible types of complete systems;

(1) $G = G(a, r, u_1, u_2, \dots u_{n-2}, x)$,

(2) $H = H(a, r, x_1, x_2, \dots x_{n-1}, x_n)$,

(3) $K = K(a, r, \pi_1, \pi_2, \dots \pi_{n-1}, x)$,

where each contains co-efficients a, r of the two quadratic ground forms, u_i denote plane co-ordinates, x and x_i denote point co-ordinates, and π_i denote general line co-ordinates (See part I. § 1). From a geometrical point of view the K system is the most important but also it is the largest and most difficult to determine. This system K is known for the cases $n = 2$,[2] $n = 3$,[3] and $n = 4$.[4] The number of

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Footnote 1. Grace and Young. Algebra of Invariants, chap 6.

Footnote 2. Grace and Young. Loc. cit. page 161.

Footnote 3. Grace and Young. Loc. cit pages 280-286.

Footnote 4. Turnbull. The simuktaneous system of two quadratic quaternary forms. Proceedings of the London Mathematical Society, Ser. 2, Vol. 18, Parts 1 and 2. For brevity we shall refer to this paper as Turnbull's paper.

irreducible concomitants for the cases $n=2$, $n=3$, and $n=4$ are respectively 6, 20 and 122. When $n=2$ or 3, the system is strictly irreducible and thus the complete system in these two cases is also the minimum system.[1]

The G system has been determined,[2] and in a joint paper by H. W. Turnbull and the author, to be published shortly in the Proceedings of the Royal Society of Edinburgh, a complete system is found, when the co-ordinates π_i (or p_i) are decomposed into their components u_i and x_i.

In this paper we are interested in the K system and, though no complete system is determined in the general case, a distinct step is made in that direction. If every concomitant of the two quadratics may be expressed as a product of symbolic factors, where each symbol occurs an even number of times in each product and distinguishing marks on equivalent symbols may be neglected, the totality of such factors is said to form a prepared system for the two quadratics. In part I we determine a prepared system, in terms of which every concomitant of the two quadratics, if multiplied by a suitable invariant factor, may be expressed. This prepared

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Footnote 1. Van de Waerden, Amsterdam Ak. Versl., (1923), 138-147.

Footnote 2. H. W. Turnbull. The irreducible concomitants of two quadratics in n variables. Trans. Cam. Phil. Soc., Vol.XXI, No VII, pp 197-240, July 1909.

system is comparatively simple and consists of 2^n-1 factors. In obtaining a prepared system, in terms of which every concomitant can be expressed, without being multiplied by an invariant factor, new more complicated bracket factors must be introduced, when $n \geq 4$. The number of such new bracket factors is 1, 8, and 52 for the cases n=4, 5, and 6 respectively.

The method used in attacking this problem shows the important rôle played by the n quadratic covariants of the two ground forms and in fact the prepared system, determined in part I, involves no symbols apart from those representing the n quadratic covariants.[1]

In part II the complete K system for two quadratics in two, three and four variables is determined. Though the results are not new, the method of obtaining them is simple and straightforward and so justifies the inclusion of this section. In part III the prepared system for two quadratics in five variables is determined and a complete list of several types of irreducible concomitants is found.[2] In part IV the prepared system for two quadratics in six variables is obtained.

Footnote 1. The idea of trying to express all concomitants of two quadratics in terms of the symbols of the n quadratic covariants was suggested by a paper of H. W. Turnbull. The invariant theory of the quaternary quadratic complex. Proc. Roy. Soc. Edin., Vol.XLVII, PartI,pp 70-91. 1927.

Footnote 2. The prepared system for the case of five variables has also been determined by W. Saddler but has not been published. The results obtained here agree with his.

Part I.

General theory for two quadratics in n variables.

§ 1. Co-ordinate Systems.

Let

$$(u) \equiv u^1 \; u^2 \; \ldots\ldots u^n$$

denote a set of n independent variables and suppose that (u) represents hyperplane co-ordinates in a space of n - 1 dimensions. If

$$(u_1) = (u), \; (u_2), \; (u_3), \ldots\ldots, (u_n)$$

are n sets of cogredient variables, such that the sets (u_i) are linearly independent, we have an n-rowed square matrix

$$U = \| u_i^j \|$$

such that the determinant of U is different from zero; i.e.

$$D = |U| \neq 0. \qquad (1)$$

A complete co-ordinate system for this space of n - 1 dimensions is given by

$$\pi_r = (u_1 u_2 \ldots . u_r) \qquad r = 1, 2, \ldots, n-1,$$

where π_r denotes the set of $\frac{n!}{r!\,(n-r)!}$ determinants formed from the first r columns of U.

If

$$X = \| x_i^j \|$$

is the n-rowed square matrix where x_i^j is the cofactor of u_{n+1-i}^{n+1-j} in D, then by Jacobi's Ratio Theorem

$$\pi_r = (u_1 u_2 \ldots u_r) = D^{r+1-n} (x_1 x_2 \ldots x_{n-r})$$
$$= D^{r+1-n} \, p_{n-r},$$

where p_{n-r} is the set of $\frac{n!}{r!\,(n-r)!}$ determinants formed from the first n-r columns of X.

Hence we have the two dual co-ordinate systems:[1]

$$(u) = \Pi_1 \qquad = D^{2-n} p_{n-1},$$
$$(u_1 u_2) = \Pi_2 \qquad = D^{3-n} p_{n-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(u_1 u_2 \ldots u_{n-2}) = \Pi_{n-2} \qquad = D^{-1} p_2,$$
$$(u_1 u_2 \ldots u_{n-1}) = \Pi_{n-1} \qquad = (x_1) = (x),$$

and finally

$$|X| = D^{n-1}.$$

§ 2. Notation.

Let

$$\begin{aligned} f &= a_x^2 = b_x^2 = c_x^2 \quad \ldots\ldots \\ g &= r_x^2 = s_x^2 = t_x^2 \quad \ldots\ldots \end{aligned} \qquad (2)$$

be the symbolic forms of two quadratics in n variables, where

$$a_x = \sum_{i=1}^{i=n} a_i x^i .$$

Further let $A_i = B_i = C_i$ denote a matrix of n rows and i columns formed by i equivalent symbols of f, and $R_i = S_i = T_i$ a matrix of n rows and i columns formed by i equivalent symbols of g. We then say that A_i or R_i is of currency i. If, also, X_k denotes a matrix of n rows and k columns formed by k sets of variables cogredient with (x), and U_k a similar matrix formed by k sets of cogredient variables, each contragredient to the set (x), we may denote the or-

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Footnote [1]. For a more detailed account of dual co-ordinate systems, see Turnbull. Determinants, Matrices and Invariants. Page 86 .

dinary determinantal bracket factor by,

$$(A_i R_j U_k), \quad i+j+k=n, \quad i\geqq o, \; j\geqq o, \; k\geqq o.$$

We shall however require other types of bracket factors:

I. Compound Inner Products. II. Generalised Outer Products.

III. Generalised Compound Inner Products.

I. <u>Compound Inner Products.</u>

The compound inner product

$$(A_i R_j \mid X_{i+j}) \qquad i+j\leqq n \; , \; i\geqq o, j\geqq o \tag{3}$$

is a determinant of i+j rows and columns and is equal to

$$\begin{aligned} &\left| a_x b_y \ldots\ldots r_z s_w \right| \\ &= \Sigma \pm a_x b_y \ldots\ldots r_z s_w \, , \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_i &= ab\ldots. \\ R_j &= rs\ldots. \\ X_{i+j} &= xy\ldots.zw \; . \end{aligned}$$

For example

$$\begin{aligned} (ab \mid xy) &= \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\ &= a_x b_y - b_x a_y \; . \end{aligned}$$

The determinantal permutation indicated by $\Sigma\pm$ in (4) is often written

$$\dot{a}_x \dot{b}_y \ldots . \dot{r}_z \dot{s}_w \quad ,$$

or

$$a_{\dot{x}} b_{\dot{y}} \ldots . r_{\dot{z}} s_{\dot{w}} \; ,$$

where the dots mean that the letters beneath them must be permuted in every possible way, a change in sign being made with each transposition. If $i+j=n$ in (3),

$$(A_i R_j | X_n) = (A_i R_j) | X_n |,$$

and if $i+j=1$, this inner product reduces to the ordinary simple $(a|x) = a_x$. Since the inner product (3) is a determinant, it may be expanded by Laplacian developments in various ways. In fact,

$$(A_i R_j | X_i Y_j) = (\dot{A}_i | X_i)(\dot{R}_j | Y_j),$$
$$= (A_i | \dot{X}_i)(R_j | \dot{Y}_j),$$

where again the dots indicate a series of $\binom{i+j}{i}$ terms obtained by interchanging the symbols of A_i with those of R_j in every possible way (or else those of X_i with those of Y_j). Once again each interchange is accompanied by a change of sign. For example,

$$(abc | xyz) = (\dot{a}\dot{b} | xy)(\dot{c} | z),$$
$$= (ab | xy)(c | z) - (ac | xy)(b | z) - (cb | xy)(a | z).$$

It should be noted, that the dot notation does not mean that symbols appearing in the same bracket factor should be interchanged.[1]

More over

$$(A_i R_j | p_{i+j}) = D^{n-i-j-1}(A_i R_j \pi_{n-i-j}). \tag{5}$$

II. Generalised Outer Products.

The generalised outer product

$$(A_i R_j | X_k) \qquad i+j \geqq n, \; k = i+j-n \tag{6}$$

is defined as

$$(\dot{B}_{n-j} R_j)(\dot{C}_k | X_k),$$

where $B_{n-j} C_k = A_i$ and the dots indicate a series of $\binom{i}{k}$

Footnote. [1] See Turnbull. Loc. cit. Pages 43 sq.

terms. By the elementary fundamental identities,

$$(\dot{B}_{n-j} R_j)(\dot{C}_k | X_k) = (A_i \dot{S}_{n-i})(\dot{T}_k | X_k)$$

where $T_k S_{n-i} = R_j$.

The simplest type of such a factor is

$$(A_i R_{n+1-i} x) = (\dot{A}_{i-1} R_{n+1-i}) \dot{a}_x \qquad i=2,3,..,n-1 \quad (7)$$

where $A_i = A_{i-1} a$.

III. Generalised Compound Inner Products.

If for brevity in (7) we denote $(A_i R_{n+1-i} x)$ symbolically by i_x, we can form a third type of factor, a generalised compound inner product, which is defined in a manner analogous to the definition of a compound inner product. The generalised compound inner product

$$(ij...k | xy...z)$$

is a determinant of m rows and columns, where m is the number of symbols i,j,..,k. It is equal to

$$\begin{aligned}(i_x j_y ... k_z) &= \dot{i}_x \dot{j}_y .... \dot{k}_z, \\ &= i_{\dot{x}} j_{\dot{y}} .... k_{\dot{z}} .\end{aligned}$$

§ 3. Determination of a Prepared System.

By the fundamental theorem of invariants, every concomitant T of the two quadratics (2) can be represented as a sum of terms, where each term is a product of factors of the types $(d_1 d_2 .... d_n)$, $(d_i d_2 ... d_i \pi_{n-i})$, and d_x, together with D, and each symbol d represents a symbol of f or g and must occur exactly twice in every product. But by (5)

$$D^{n-1}(d_1 d_2 ... d_n) = |X|(d_1 d_2 ... d_n),$$

$$= (d_1 d_2 \ldots d_n / X), \tag{8}$$

and

$$D^{n-i-1}(d_1 d_2 \ldots d_i \pi_{n-i}) = (d_1 d_2 \ldots d_i | p_i). \tag{9}$$

Hence if M be any such term of T,

$$D^k M = K \tag{10}$$

where K is a product of factors of the types on the right of equations (8) and (9) and k is the total number of variables x_i distinct from x, which occur in M. We now prove:

Lemma I. If K has the equivalent symbols a,b,..,c, i in number, convolved together as A_i in the same factor, the complementary symbols a,b,..,c appearing in K may also be convolved toghether as A_i.

Since every factor, occurring in K, is a compound inner product, we may write

$$K = \Sigma (A_i | X_i) a_x b_y \ldots c_z N.$$

As a,b,..,c are equivalent symbols, by permuting them determinantally we get

$$i!K = \Sigma (A_i | X_i)(A_i | Y_i)N,$$

where $Y_i = (xy..z)$. This proves the lemma.

we apply the lemma successively for the cases i=n, n-1, etc and finally have the result :

Lemma II. The concomitant K is zero, or else it can be expressed as a sum of terms, where each term is a product of factors of the two types $(A_i | Y_i)$, $(R_j | Y_j)$ and each symbol A_i, R_j occurs exactly twice in each product.

Further, if K has the factor $(A_n | X_n)$ or $(R_n | X_n)$, K has the invariant factor $(A_n|X_n)^2$ or $(R_n|X_n)^2$, or else is zero, since there are only n sets of variables (x). In either case we say that K is reducible.

Now two quadratics in n variables have the n+1 invariants

$$(A_i R_{n-i})^2 \qquad i = o,1,\ldots,n \tag{11}$$

which form a complete system of invariants,[1] and n quadratic covariants;

the two original quadratics a_x^2 and r_x^2 ,

and the n-2

$$(A_i R_{n+1-i} x)^2 \qquad i = 2,3,\ldots,n-1 \tag{12}$$

defined by (7).[2]

Let us denote the quadratic covariants symbolically by 1_x^2 and n_x^2 for a_x^2 and r_x^2 respectively, and by i_x^2 for $(A_i R_{n+1-i} x)^2$ $i=2,3,\ldots,n-1$. We require the following lemma:

<u>Lemma III.</u> <u>The generalised inner product</u>

$$(i,i+1,\dot{i}+2,\ldots,i+k\,|X_{k+1})$$
$$\equiv\left\{\prod_{j=i}^{j=i+k-1}(A_j R_{n-j})\right\}(A_{i+k}R_{n+1-i}\,|X_{k+1})$$

<u>together with additional terms.</u>

For, if $s \geq o$ and $A_{i+k} = A_k A_i$,

$$(\dot{A}_i R_{n-i})(\dot{A}_k B_{i-s} U_{n-i+s-k})$$
$$= (B_{i-s}\dot{U}_s R_{n-i})(A_{i+k}\dot{U}_{n-i-k})$$

Footnote.[1] Turnbull. Loc.cit. page 304.

Footnote.[2] Turnbull and Williamson. The minimum system of two quadratics in n variables. Proc. Roy. Soc. Edin. Vol. 45. Part 2, 1924-25, pp 149-165.

$$+\sum_{t=1}^{t=i-s}\lambda_t(\dot{B}_{i-s-t}\dot{U}_{s+t}R_{n-i})(A_{i+k}\dot{B}_t\dot{U}_{n-i-k-t})$$
$$=(B_{i-s}\dot{U}_sR_{n-i})(A_{i+k}\dot{U}_{n-i-k})$$
$$+\sum_{t=1}^{i-s}\lambda_t(B_{i-s-t}\dot{U}_{s+t}R_{n-i})(A_{i+k+t}\dot{U}_{n-i-k-t}),$$

where $A_{i+k}B_t = A_{i+k+t}$

$$=(B_{i-s}\dot{A}_sR_{n-i})(\dot{A}_{i+k-s}U_{n-i-k+s})$$
$$+\sum_{t=1}^{i-s}\lambda_t(B_{i-s-t}\dot{A}_{s+t}R_{n-i})(\dot{A}_{i+k-s}U_{n-i-k+s}).[1]$$

In all but the first term the currency[2] of A_i has been increased at the expense of the currency of B_{i-s}, which is less than or equal to that of A_i. By (5) we may change the variables throughout and write

$$(\dot{A}_iR_{n-i})(\dot{A}_kB_{i-s}|X_{i+k-s})\equiv(B_{i-s}\dot{A}_sR_{n-i})(\dot{A}_{i+k-s}|X_{i+k-s}) \quad (13)$$

mod increased currency.

We use $\equiv$ to denote equal to, except for terms which may be neglected. We proceed to prove lemma III by induction assuming its truth for the value k. Hence

$$(i,i+1,\ldots,i+k,i+k+1|X_{k+2})$$
$$\equiv\left\{\prod_{j=i}^{j=i+k-1}(A_jR_{n-j})\right\}(\dot{A}_{i-1}R_{n+1-i})(\bar{B}_{i+k}R_{n-i-k})(\dot{A}_{k+1}\bar{b}|X_{k+2}),$$

where the bar over the B's is used in place of a dot,

$$\equiv\left\{\prod_{j=i}^{j=i+k-1}(A_jR_{n-j})\right\}(\dot{A}_{i-1}R_{n+1-i})(\bar{B}_{i-1}\dot{A}_{k+1}R_{n-i-k})(\bar{B}_{k+2}|X_{k+2}),$$

by (13)

$$\equiv\left\{\prod_{j=i}^{j=i+k-1}(A_jR_{n-j})\right\}(\bar{B}_{i-1}R_{n+1-i})(A_{i+k}R_{n-i-k})(\bar{B}_{k+2}|X_{k+2})$$

mod increased currency of R_{n+1-i} by transferring A_{i-1} from the second to the third factor,

$$\equiv\left\{\prod_{j=i}^{j=i+k}(A_jR_{n-j})\right\}(\dot{B}_{i-1}R_{n+1-i})(\dot{B}_{k+2}|X_{k+2}).$$

In addition

$$(i,i+1|xy)=(\dot{A}_{i-1}R_{n-i+1})(\bar{B}_iS_{n-i})(\dot{a}\bar{b}|xy),$$

Footnote 1. λ_t is a numerical constant $\binom{i-s}{t}$.

Footnote 2. For a more detailed explanation of the meaning of "increased currency" see page 17.

$$\equiv (A_i S_{n-i})(\bar{B}_{i-1} R_{n-i+1})(\bar{B}_2 | xy)$$

by putting $k = o$ in the previous work. Hence the proof by induction is complete except when i=1 or n. But, if i=1, the term $(A_{i-1} R_{n+1-i})$ vanishes and by an otherwise similar proof

$$(1,2,\ldots,k|X_k) \equiv \left\{\prod_{i=1}^{i=k-1} (A_i R_{n-i})\right\}(A_k|X_k), \tag{14}$$

and

$$(n,n-1,\ldots,n-k+1) \equiv \left\{\prod_{i=1}^{i=k-1} (R_i A_{n-i})\right\}(R_k|X_k). \tag{15}$$

Though not necessary at this stage, it is important to notice for future applications that the terms neglected in lemma III have all had their currency raised.

It follows from lemma III that, if K be multiplied by the invariant factor $\prod_{i=1}^{i=k-1} (A_i R_{n-i})^2$ for every factor $(A_k|X_k)(A_k|Y_k)$ occurring in it and by $\prod_{i=1}^{i=k-1}(R_i A_{n-i})^2$ for every factor $(R_k|X_k)(R_k|Y_k)$ occurring in it, and if I denote this total invariant factor, IK can be expressed as a sum of terms, where each term is a product of factors of the types $(1,2,\ldots,k/X_k)$ and $(n,n-1,\ldots,n-k+1/X_k)$. What we have done is to replace each symbol a,b,..,r,s,.. occurring in K by a symbol i occurring in IK. We can therefore convolve the variables back again exactly as they were originally convolved in K and have the final result:

<u>The concomitant $IK = \Sigma N$, where N is a product of factors of the type $(i,j,\ldots,k|b_m)$ and each symbol i,j,...,k[1] must occur exactly twice in each product.</u>

But each $N = \Sigma I_i G_i$, where I_i is an invariant factor composed of powers of $D_j = (A_j R_{n-j})^2$, and G_i is a concomitant

Footnote 1. The symbols i,j,k etc. are not necessarily successive integers.

involving the symbols A_i, R_j convolved in pairs. Hence, if we determine the complete system for the G_i, we have determined a system in terms of which every concomitant of the two quadratics, if multiplied by an invariant factor, composed of powers of D_i, $i = 1, .., n$, can be expressed.

Order in which the concomitants G_i are considered.

Let $\mu_{i,j}$ denote the number of pairs of symbols A_j occurring in G_i and $\nu_{i,j}$ the number of symbols R_j occurring in G_i. We consider G_1 before G_2, if G_1 is of less total degree in the coefficients of the two quadratics (2). But, if G_1 and G_2 are of the same degree in the coefficients of the two quadratics, we consider G_1 before G_2, if

$$\mu_{1,i+k} = \mu_{2,i+k} \qquad k = 1, 2, .., n-1-i,$$
$$\nu_{1,i+k} = \nu_{2,i+k}$$

and

$$\mu_{1,i} > \mu_{2,i}, \qquad \nu_{1,i} \nless \nu_{2,i}$$

or

$$\nu_{1,i} > \nu_{2,i}, \qquad \mu_{1,i} \nless \mu_{2,i}.$$

In other words, G_1 is considered before G_2, if the currency of G_1 is greater than the currency of G_2.

That this is a legitimate procedure follows from the fact, that each G_i may be treated as a concomitant K and may therefore be expressed in terms of the symbols $i, j, .., k$, in which the symbols A_k, R_m convolved in G_i will also be convolved. For a similar reason, if in G_i a new convolution of symbols A or R is made, the complementary symbols may also be convolved together. Finally, if A_n or R_n appears in G_i, A_n^2 or R_n^2 is a factor of G_i and G_i is said to be reducible.

As a direct consequence of the order in which the concomitants G_i are considered, we may use formulae (14) and (15), in determining the G_i, since in lemma III the neglected terms are all of greater currency.

Bracket Factors with equivalent symbols.

If Θ denote a convolution of symbols i, j, .. k and Φ a similar convolution, while no symbol i occurs in Φ equivalent to a symbol j occurring in Θ and no symbol in Φ is a symbol successive to one in Θ, we may write

$$(\Theta\Phi p) = (\Theta \dot{p}_\theta)(\Phi \dot{p}_\phi)$$

without disturbing the invariant factors D_i, which may have been introduced by (14) and (15). Hence in considering factors with equivalent symbols we need only consider factors of the one type

$$(i, i+1, \ldots i+k, t, s, \ldots, m \,|\, X),$$

where

$$i \leqq t,\ s,\ m \leqq i+k.$$

If the symbols t, s, m are not successive integers, we may break the factor up again into a series of products of factors without disturbing the invariant factors D_i. Therefore all factors involving equivalent symbols may be reduced to the type

$$T = (i, i+1, \ldots, i+k, t, t+1, \ldots, t+s \,|\, X)$$

$$i \leqq t,\ i+k \geqq t+s,$$

where of course the variables p have been decomposed. Writing for i its value $(A_i R_{n+1-i})$ we see that T involves a convolution of

$$k+1+s+1 = k+s+2$$

equivalent symbols a. Since A_{i+k} is the symbol A of greatest currency appearing in T and since none of the symbols R have been disturbed, if

$$k+s+2 > i+k,$$

the currency of the resulting G_i has been increased. Hence, according to our scheme of order, G_i is reducible, if G_i contain the factor T, in which $s+2 > i$. We now consider the case, in which $s+2 \leqq i$.

If I denote the invariant factors, which appear in T by lemma III,

$$T = I\,(\dot{A}_{i-1}R_{n+1-i})(\bar{B}_{t-1}S_{n+1-t})(\dot{A}_{k+1}\bar{B}_{s+1}\,|\,X_{k+s+2})$$
$$= I\,(\bar{B}_{s+1}\dot{A}_{i-2-s}R_{n+1-i})(\bar{B}_{t-1}S_{n+1-t})(\dot{A}_{k+s+2}\,|\,X_{k+s+2})$$

mod. increased currency, since $i-1 \geqq s+1$, by (13).

But $n+1-i \geqq n+1-t$ and by transferring $\bar{B}_{s+1}$ from the first to the second factor we displace some of S_{n+1-t} and so increase the currency of R_{n+1-i}. This proof fails, if $i-1 < 1$, that is if $i<2$ or $i=1$. But in this case $s+2 > i$, since s is not negative. Therefore in every case, if G_i contain a factor T involving equivalent symbols, G_i is reducible.

Since $i_x i'_y - i_y i'_x = (ii' \,|\, xy) \equiv o$, we may remove distinguishing marks on the equivalent symbols $\dot{i}$, for when i and i' are interchanged the invariant factors will not be disturbed. We have now proved the theorem:

Theorem I. *Every concomitant K of the two quadratics (2) is reducible or else, if multiplied by an invariant factor, composed of powers of $(A_i R_{n-i})^2$, $i=1,2,\ldots,n-1$, is a sum of terms, where each term is a ~~sum~~ product of the symbolic factors i_x, $(ij|p_2)$, $(ijk|p_3)$, ..., $(ijk\ldots m|p_{n-1})$, $(ijk\ldots n)$, where $i,j,k,m = 1,2,\ldots,n$, and each symbol i occurs an even number of times in each product and no equivalent symbols occur in the same factor. If the concomitant K is reducible, $K = IK'$, where I is an invariant factor composed of powers of $(A_n)^2$ and $(R_n)^2$ and K' is not reducible.*

Since the variables are now convolved back again into the form p_j, we may replace the variables p by the variables π, thus removing the powers or D, with which we originally multiplied the concomitants. In fact, since the co-ordinates are homogeneous, it is immaterial which set of variables is used, as long as the variables are properly convolved. Henceforward we shall drop the variables and write (ij) for $(ij\pi_2)$, (ijk) for $(ijk\pi_3)$ etc.

§ 4. Identities.

If I_m, J_m, K_m etc denote convolutions of m symbols i,j,k,... the following identities exist.

$$
\begin{aligned}
&(I_i \dot{J}_j K_k)(I_i \dot{J}_r M_m) && i+j+k \geqq i+r+m \\
&\equiv (I_i J_{j+r} \dot{K}_{k-r})(I_i \dot{K}_r M_m) && k>r, \\
&\equiv (I_i J_{j+r})(I_i K_k M_m) && k=r, \qquad (16)\\
&\equiv 0 && k<r.
\end{aligned}
$$

For, if the variables are taken as π, in transferring J_r to the first factor we cannot turn out any of the variables, since otherwise the second factor would be zero.

$$
\begin{aligned}
&(I_i \bar{J}_j K_k)(I_i \dot{J}_r M_m), \quad i+j+k=i+r+m-1 \\
&\equiv (I_i J_{j+r} \acute{K}_{k-r})(I_i \dot{K}_r M_m) + (I_i J_{j+r} \dot{K}_{k-r+1})(I_i \dot{K}_{r-1} M_m), \quad k>r-1, \\
&\equiv (I_i J_{j+r})(I_i K_{r-1} M_m) \quad k=r-1, \qquad (17) \\
&\equiv 0 \quad k<r-1.
\end{aligned}
$$

The two terms in this identity occur, because one of the variables in the first factor can be displaced. Other identities exist, when $i+j+k < i+r+m-1$, but they disturb the proper convolution of the variables. These identities (16) and (17) are true whether the variables p or π are used.[1]

§ 5. <u>Principle of Duality</u>.

Let K be a concomitant $\Pi(ij..k)$. Then, if $i', j', .., k'$ are the symbols complementary to $i, j, .., k$, ie, if $(ij..k)(i'j'..k')$ involves each of the symbols $1, 2, .., n$ once and only once, $(ij..k)$ and $(i'j'..k')$ are said to be dual factors. With this notation $\Pi(ij..k)\Pi(i'j'..k')$ involves all the symbols $1, 2, .., n$ exactly m times if there are m factors in K. If m is even , all the symbols occur an even number of times and therefore $\Pi(i'j'..k')$ is also a concomitant. But, if m is odd $(123..n)\Pi(i'j'..k')$ is a concomitant. Hence from a concomitant K we can form a dual concomitant by writing the dual factor of each factor in K and multiplying by the factor $(123..n)$, if the number of factors in K is odd.

Footnote.[1] Turnbull. Loc. cit. Pages 92 sq.

If we consider identity (16) and write the dual of each factor in it, we get,

$$(TM_m\dot{J}_r)(T\dot{J}_jK_k) \equiv (T\dot{K}_rM_m)(TJ_{j+r}\dot{K}_{k-r}),$$

where $(TM_mJ_{j+r}K_kI_i) = (12\ldots n)$, and this identity interchanges exactly the same symbols as did the original one. Similarily there is an identity dual to (17) acting on the same symbols as (17). Hence we have the important lemma:

<u>Lemma IV.</u> <u>Corresponding to any identity there is a dual identity operating on the same symbols.</u>

Since, if i and j are successive integers and ij is convolved an even number of times in a concomitant, an identity separating i and j cannot be used without disturbing the invariant factors, we must now prove a further lemma.

<u>Lemma V.</u> <u>If K and K′ are dual concomitants, the symbols i and j are convolved an even or an odd number of times in K′ according as they are convolved an even or an odd number of times in K.</u>

Let the symbol i occur alone in m factors of K,

" " " j " " " r " "

" " symbols i,j occur together in t factors of K,

" neither of the symbols i or j occur in q factors of K.

Then the total number of factors in K is

$$m+r+t+q=s.$$

Since t+m and t+r are both even,

$$s+t=q+m+t+r+t,$$

$$\equiv q \mod 2,$$

or

$$q-t\equiv s \mod 2.$$

Hence

$$t\equiv q \mod 2\text{, if s is even,}$$

$$\equiv q+1 \mod 2\text{, if s is odd.}$$

But, if s is even, q is the number of factors in K' containing both i and j, while, if s is odd, q+1 is the number of factors in K' containing both i and j, since , if s is odd, in determining K' from K we introduce the factor (12..n). Accordingly, this proves our lemma.

§ 6. Incompleteness of the Prepared System.

We should like to determine a prepared system, in terms of which every concomitant can be expressed, without being multiplied by an invariant factor. To do this, we must see, if ever in convolving the variables to give the factors (ij..k) we have broken up convolutions of successive symbols. We shall find in parts II and III that this is the case for $n>3$, and that accordingly we must introduce other more complicated factor types. With the introduction of these new factor types, once we have found a complete system for the concomitants multiplied by invariant factors, we may remove the invariant factors from each concomitant and so obtain the actual complete system, provided that no identity is used, which disturbs a convolution of two successive symbols, if that convolution occurs an even number of times in a concomitant. These new factor types are of the nature

$$(\dot{i}\dot{j}..t)(\dot{k}..\bar{s})(\bar{m}...)(\bar{n}...)....(\dot{r}...),$$

or, in other words, combinations of simpler bracket factors, in which successive symbols are implicitly convolved. The dual of such a factor is obtained by taking the duals of the component bracket factors and permuting the same symbols. Accordingly, if one of these new factor types is counted as equivalent to q simple factors, where q is the number of component factors in the new factor type, the dual of a concomitant K cantaining such a factor can be written down as in § 4. Further the introduction of these new factor types does not vitiate the proof of lemma V, since any symbols implicitly convolved in K will also be implicitly convolved in K'. Hence from a list of irreducible concomitants we can immediately write down a list of the dual irreducible concomitants. (see part II §4 ,part III §3).

Since the concomitants f and g enter symmetrically into the discussion, the actual labour of determining the irreducible concomitants can be shortened still further. Two factors, concomitants or identities are said to be <u>similar</u>, if the first can be derived from the second by writing $n+1-i$ for every symbol i that occurs. From the symmetry mentioned above it follows immediately that if a concomitant is reducible so is the concomitant similar to it.

Part II.

Applications of the general theory to two quadratics in two, three and four variables.

§ 1. Complete System for two quadratics in 2 variables.

In this case $n=2$ and we require two variables, only one of which occurs explicitly. It is obvious that the prepared system consists of the three simple factors:

$$1_x = a_x, \quad 2_x = r_x, \quad (12) = (ar) .$$

The complete system then consists of six forms:

three invariants $(A_2)^2$, $(R_2)^2$, $(ar)^2$;

three covariants 1_x^2, 2_x^2, $(12)1_x 2_x$.

In addition there is the actual concomitant of the field $D = (uv)$.[1]

§ 2. Complete system for two quadratics in 3 variables.

In this case $n = 3$ and we require two co-ordinates x and $u = (xy)$. The prepared system consists of the six simple factors:

$$1_x = a_x,$$
$$3_x = r_x,$$
$$2_x = (ARx) = (\dot{a}R)\dot{b}_x,$$
$$(12) = (aR)(Au),$$
$$(32) = (rA)(Ru),$$
$$(13) = (aru),$$
$$(123) = (aR)(Ar),$$

where A and R are written for A_2 and R_2 respectively.

Footnote [1]. Weitzenböck. Invariantentheorie. Chap 2. pages 42 and 43.

The complete system consists of 20 forms:

four invariants $(A_3)^2$, $(Ar)^2$, $(aR)^2$, $(R_3)^2$;

four covariants 1_x^2, 2_x^2, 3_x^2, $(123)1_x2_x3_x$;

four contravariants $(12)^2$, $(23)^2$, $(13)^2$, $(23)(31)(12)$;

eight mixed forms;

two linear in u and x $(123)(23)1_x$, $(123)(12)3_x$,

three linear in u, quadratic in x $(ij)i_xj_x$,

three linear in x, quadratic in u $(123)(ij)(ik)i_x$,

where $i,j,k = 1,2,3$. In addition there is the actual concomitant of the field $D = (uvw)$.

The actual concomitants are obtained from the above list by dropping the invariant factors, e.g.

$$(23)^2 = (rA)^2(Ru)^2$$

gives the actual concomitant $(Ru)^2$.[1]

<u>Determination of the complete system.</u>

No new types of bracket factors are necessary, since the only chance of 12 or 23 being separated would be in the formation of the factor (123). But, as in (123) both 12 and 23 are convolved, no invariant factor would be disturbed. Accordingly the prepared system is that quoted above.

<u>Covariants.</u> If (123) does not occur, the only possibilities are the squares of the factor i_x, that is i_x^2, $i = 1,2,3$. If (123) occurs, the only possibility is $(123)1_x2_x3_x$.

<u>Contravariants.</u> By the principle of duality we obtain the

---

Footnote. [1] Weitzenböck. Loc. cit. chap.2, § 16, page 61.

contravariants immediately from the list of covariants.

Mixed Concomitants. If the factor (123) does not occur, by analogy with binary forms the only possibilities are the three forms $(ij)i_x j_x \quad i,j = 1,2,3$.

From these we get the dual forms

$$(123)(kj)(ki)k_x \quad i,j,k = 1,2,3 .$$

If the factor (123) occurs, we may have the forms

$$(123)(23)1_x,$$
$$(123)(21)3_x,$$
$$(123)(13)2_x,$$

of which the last can be expressed in terms of the other two. It is obvious that three u factors cannot occur, since then we would have the concomitant

$$(123)(12)(23)(31)1_x 2_x 3_x$$

and this has the concomitant factor $(13)1_x 3_x$. Hence we have obtained the complete list of irreducible concomitants.

§ 3. Two quadratics in four variables.

List A. The prepared system.

$$1_x = a_x, \quad 2_x = (A\rho x), \quad 3_x = (R\alpha x), \quad 4_x = r_x,$$

$(12) = a_\rho(Ap)$, $\qquad (23) = (AR)(\alpha\rho p) \quad (AR)\dot{a}_\rho(\dot{b}\dot{c}p)$,

$(43) = r_\alpha(Rp)$, $\qquad (14) = (arp)$,

$(13) = (aR\alpha p) = \dot{r}_\alpha(a\dot{s}p)$, $\qquad (42) = (rA\rho p)$,

$(123) = -a_\rho(AR)u_\alpha$, $\qquad (432) = -r_\alpha(AR)u_\rho$,

$(124) = a_\rho(Aru)$, $\qquad (431) = r_\alpha(Rau)$,

$(1234) = -a_\rho(AR)r_\alpha$, $\qquad (12,43) = a_\rho r_\alpha(ARux)$,

where A, α , R, ρ , are written for A_2, A_3, R_2, R_3 respectively.

List B. The 122 unreduced forms.

5 invariants	a_α^2, r_α^2, $(AR)^2$, a_ρ^2, r_ρ^2.
5 covariants	i_x^2 4, $(1234)1_x 2_x 3_x 4_x$.
5 contravariants	$(ijk)^2$ 4, $(1234)(123)(234)(124)(134)$.
16 complexes	$(ij)^2$ 6, $(ij)(jk)(ki)$ 4,
	$(1234)(12)(34)$, $(1234)(14)(23)$,
	$(1234)(ij)(ik)(im)$. 4
19 mixed forms containing x and u	$(1234)(123)4_x$, $(1234)(234)1_x$,
	$(ijk)(ijm)(ikm)i_x$ 4,
	$(ijk)i_x j_x k_x$ 4,
	$(1234)(ijk)(ijm)i_x j_x$ 6,
	$(12,43)^2$, $(1234)(12,43)$,
	$(12,43)(124)(134)1_x 4_x$.
14 mixed forms containing x and p	$(ij)i_x j_x$ 6, $(1234)(23)1_x 4_x$,
	$(1234)(21)(14)1_x 3_x$, $(1234)(23)(34)1_x 3_x$, $(1234)(34)1_x 2_x$, and three similar forms,
	$(1234)(12)(23)(34)2_x 3_x$.
14 mixed forms containing u and p	$(1234)(ijk)(ijm)(ij)$ 6, $(123)(234)(14)$,
	$(134)(342)(12)$, $(1234)(123)(134)(12)(23)$, $(1234)(123)(134)(14)(43)$, and three similar forms,
	$(124)(134)(12)(14)(34)$.
44 mixed forms containing x, p and u	$(123)(12)3_x$, $(123)23)1_x$, $(124)(12)4_x$,
	$(124)(14)2_x$, $(1234)(123)(41)1_x$,
	$(1234)(124)(34)4_x$, $(1234)(123)(43)3_x$,
	$(1234)(124)(32)2_x$, $(123)(23)(14)2_x$,

and nine similar forms,

$(143)(23)(12)4_x$,

$(ijk)(ij)(ik)i_x$, 12,

$(1234)(124)(14)(34)1_x$,
$(1234)(123)(23)(34)2_x$,
$(1234)(124)(12)(23)1_x$,
$(1234)(234)(42)(21)4_x$,
$(1234)(124)(24)(43)2_x$,
} and five similar forms,

$(12,43)(12)(34)$,

$(12,43)(124)(34)4_x$ and a similar form.

In this list i,j,k,m take the values 1,2,3,4. The number appearing on the right of a concomitant is the number of distinct concomitants of that type. A similar form means a form in which the symbols 1,4 and 2,3 are interchanged. The actual complete system is obtained by removing from each form any invariant factor , which may appear. For example,

$$(12)^2 = a_\rho^2 (Ap)^2 :$$

we therefore drop the factor a_ρ^2 and take $(Ap)^2$ as the unreduced form.[1]

Footnote.[1] Gordan in the Mathematische Annalen, Bd.56, determined a complete system of 580 forms. Turnbull in the proceedings of the London Mathematical Society, Ser.2, Vol. 18, Parts 1 and 2, pages 69-93, reduced the system to 125 forms. Throughout we shall refer to this paper briefly as Turnbull's paper.

§ 4. New type of bracket factor.

When $n=4$, there are three types of co-ordinates x, $p=xy$, $u=xyz$. In forming the variable u, that is in convolving x, y z together, we might have at some stage,

$$(12\,|\dot{x}\dot{y})(34\,|x\dot{z})$$

where x,y,z are to be convolved together to give u. But the invariant factor r_α will be disturbed in writing,

$$(12\,|xy)(34|\,xz) = (124)3_x - (123)4_x.$$

Hence we require the new bracket factor,

$$(12,43) = (12\dot{4})\dot{3}_x = (\dot{1}34)\dot{2}_x,$$
$$= a_\rho r_\alpha (ARux),$$
$$= a_\rho r_\alpha (A\dot{s}u)\dot{r}_x = a_\rho r_\alpha(\dot{b}Ru)\dot{a}_x\,.$$

In the factor $(12,43)$ both 12 and 43 are convolved. (see part I, § 6).

§ 5. Determination of the complete system.

Covariants and contravariants.

The covariants must be

$$i_x^2 \quad i=1,2,3,4, \text{ and } (1234)1_x2_x3_x4_x,$$

and we obtain the contravariants

$$(kjm)^2 \quad k,j,m=1,2,3,4, \qquad (1234)(234)(134)(124)(123),$$

from the covariants by the principle of duality.

Complexes.

The only factors which may occur are the six (ij) factors and (1234). If the factor (1234) does not occur, the problem of determining the irreducible complexes is strictly analogous to that of binary forms, and accord-

Footnote 1. See appendix A page 112.

ingly the result is,

the six quadratic complexes $(ij)^2$,

the four cubic complexes $(ij)(jk)(ki)$.

If the factor (1234) occurs and the form is a quadratic complex, there are three possible cases,

$$(1234)(12)(34),$$

$$(1234)(23)(14),$$

$$(1234)(13)(24),$$

and of these the third is expressible in terms of the other two. If the form is a cubic complex, it must be one of the four $\quad (1234)(ij)(ik)(im) \qquad i,j,k,m = 1,2,3,4.$

The only other possibility is

$$(1234)(12)(23)(34)(41)(13)(24)$$

and this is obviously reducible.

Identities. For a complete discussion we require the identities:

$$(24)1_x = (21)4_x + (14)2_x, \tag{1}$$

$$(24)3_x = (23)4_x + (34)2_x, \tag{2}$$

$$(13)2_x = (12)3_x + (23)1_x, \tag{3}$$

$$(13)4_x = (14)3x + (43)1_x, \tag{4}$$

together with the dual identities

$$(13)(324) = (43)(321) + (23)(314) \quad \text{etc},$$

$$(24)(13) = (21)(43) + (23)(14), \tag{5}$$

$$(123)4_x = (124)3_x + (143)2_x + (423)1_x. \tag{6}$$

Also from the definition of (12,43),

$$\begin{aligned}(12,43) &= (124)3_x - (423)4_x, \\ &= (134)2_x - (234)1_x.\end{aligned} \tag{7}$$

We deduce four other identities.

$$(13)(12,43) = (13)[(134)2_x-(234)1_x],$$
$$= (134)(23)1_x+(134)(12)3_x-(134)(23)1_x-(43)(231)1_x$$
$$= (134)(12)3_x-(43)(231)1_x . \qquad (8)$$

By the principle of duality,

$$(24)(12,43) = (124)(34)2_x-(21)(234)4_x. \qquad (9)$$

Also,

$$(23)(12,43) = (23)[(134)2_x-(234)1_x],$$
$$= (234)(13)2_x +(43)(132)2_x$$
$$-(234)(13)2_x- (43)(123)2_x,$$
$$= (12)(234)3_x- (41)(143)2_x. \qquad (10)$$

By the principle of duality,

$$(14)(12,43) = (34)(124)1_x- (12)(143)2_x. \qquad (11)$$

These identities are all obtained in Turnbull's paper, but the first six are simply particular cases of identity (16) part I and so require no proof.

Reductions. If we regard (24) and (13) as the most complicated factors, it follows from identities (1)-(7) that

$$(24)(13)\equiv 0 \quad , \qquad (13)(234)\equiv 0, \qquad (a)$$
$$(24)1_x \equiv 0 \quad , \qquad (13)(234)\equiv 0, \qquad (b)$$
$$(24)3_x \equiv 0 \quad , \qquad (13)(124)\equiv 0, \qquad (c)$$
$$(13)4_x \equiv 0 \quad , \qquad (24)(123)\equiv 0, \qquad (d)$$
$$(13)2_x \equiv 0 \quad , \qquad (24)(134)\equiv 0, \qquad (e)$$
$$(24)2_x4_xM\equiv 0 \quad , \qquad (13)(134)(123)M\equiv 0, \qquad (f)$$
$$(13)1_x3_xM\equiv 0 \quad , \qquad (24)(234)(124)M\equiv 0, \qquad (g)$$
$$(14)1_x4_xM\equiv 0 \quad , \qquad (23)(123)(234)M\equiv 0, \qquad (h)$$

$$(124)(12)4_x M \equiv 0, \qquad (34)(123)3_x M \equiv 0, \qquad (i)$$

$$(431)(43)1_x M \equiv 0, \qquad (12)(234)4_x M \equiv 0. \qquad (j)$$

The notation $K \equiv 0$ means that the factors in K can be expressed in simpler forms, or in other words that K is equivalent to another product of factors, which has already been considered. The formulae (f) to (j)are true, because each product of factors has an actual concomitant factor. Further

$$(123)4_x M \equiv 0, \text{ if 34 is convolved an odd number of times in M.} \qquad (k)$$

For,under these circumstances, there is the concomitant factor $u_\alpha r_x r_\alpha$.

Similarily,

$$(234)1_x M \equiv 0, \text{ if M contains 12 convolved an odd number of times.} \qquad (m)$$

For the same reasons,

$$(234)(12)2_x \equiv 0, \qquad (123)(34)3_x \equiv 0. \qquad (n)$$

In writing out a list of concomitants, the mark R (a) after a concomitant will mean that it is reducible by (a); R by $3_x(124)$ will mean that it is reducible by the identity, which transfers the three from 3_x to the bracket factor (124).

Mixed concomitants containing u and x, but not the factor (12,43).

If only one u factor occur, since (123),(432) and (124),(431) are similar factors, we need only consider the four types,

$(123)1_x 2_x 3_x$,

$(1234)(123)4_x$,

$(124)1_x 2_x 4_x$,

$(1234)(124)3_x$ R.

The last of these is equivalent to $(1234)[(12,43)+(123)4_x]$

If two u factors occur, we need only consider the types, (123)(124), (123)(134), (123)(234), (124)(134). These give the concomitants,

$(123)(124)3_x 4_x$ R by squaring (6),

$(123)(134)2_x 4_x$ R by $2_x(134)$,

$(123)(234)1_x 4_x$ R (m),

$(124)(134)2_x 3_x$ R by $3_x(124)$,

$(1234)(123)(124)1_x 2_x$,

$(1234)(123)(134)1_x 3_x$,

$(1234)(123)(234)2_x 3_x$,

$(1234)(124)(134)1_x 4_x$.

If three factors u occur, we need only consider the case in which there are three x factors, since the concomitants with three u factors and one x factor are the duals of those with one u factor and three x factors. Therefore we have the types,

$(1234)(123)(124)(134)2_x 3_x 4_x$ R by $(124)3_x$ and (k),

$(1234)(123)(234)(124)1_x 3_x 4_x$ R (m).

We are accordingly left with,

$(1234)(123)4_x$ and $(1234)(432)1_x$,

$(ijk)i_x j_x k_x$ 4, and the duals $(ijk)(ijm)(imk)i_x$ 4,

$(1234)(ijk)(ijm)i_x j_x$ 6.

The concomitants containing the factor (12,43) will be treated later.

<u>Mixed concomitants containing p, u and x, but not the factor (12,43).</u>

It follows immediately from (a) that any concomitant containing six p factors is reducible. Hence a concomitant may contain at most five p factors, and the only possible type of such a concomitant is,

(12)(23)(34)(13)(14)M.

By (b),(c),(d), and (e) M may not contain 4_x,2_x,(342)or (421). Further 3_x may not occur by 3_x(14)nor may (341) by (23)(341), Hence M must contain 1_x or (123) or both. The symbols cannot now be paired off, and so a concomitant may contain at most four p factors.

Since (13)(24) may not occur together, the possibilities for four p factors are,

(12)(23)(34)(41),

(13)(34)(32)(21),

(13)(32)(21)(14),

(13)(34)(41)(23),

together with similar and dual products. We consider each possibility in turn.

<u>(12)(23)(34)(41)M.</u>

If M is non-existent, the form is reducible by analogy with binary types. If (23) is not convolved an odd number of times in M, the form has the factor (AR)(Ap)(Rp) and so is reducible.

There are two cases to be considered.

Case I. <u>M does not contain the factor (1234).</u>

M cannot contain only x factors or by duality only u factors, since we are dealing with a self-dual product of p-factors. If M contains one u factor, it must be (123) with $1_x 2_x 3_x$ or the similar product and the form has the factor $(34)(41)3_x 1_x$. If M contains two u factors M must be

$(123)(124)3_x 4_x$ or the similar form,

$(123)(134)2_x 4_x$ or the similar form.

The first of these has the factor $(34)3_x 4_x$, and the second $\equiv (12)(23)(34)(42)(123)(134)1_x 4_x$, which has the factor $(34)(134)1_x$.

If three u factors occur in M, only 1_x may occur and by duality such a form is reducible.

Case 2. <u>M does contain the factor (1234).</u>

Since 23 is now convolved in M the only possibilities are,

$(1234)(12)(23)(34)(41)(124)3_x$ R by $(41)3_x$ and (h),

$(123)(234)2_x 3_x$ R (h),

$(124)(134)1_x 4_x$ R (h),

$(123)(124)(234)1_x 3_x 4_x$ R (h).

<u>(13)(34)(32)(21)M.</u>

As in the previous case, 23 must be convolved an even number of times. By (b),(c),(d) and (e), the factors (234), (124), 2_x and 4_x may not appear in M. Hence we have the possibilities,

$$(13)(34)(32)(21)(123)1_x 2_x 3_x \quad R \ (g),$$

$$(123)(134)2_x 4_x \quad R \ (e),$$

$$(1234)(134)2_x \quad R \ (e).$$

(13)(32)(21)(14)M.

The factors (342), (421), 2_x and 4_x may not appear in M, as in the previous case, nor may 3_x by $(14)3_x$. Hence the only possible x factor is 1_x and this yields,

$$(13)(32)(21)(14)(1234)(123)1_x \quad R \text{ by } (32)1_x.$$

(13)(34)(41)(23)M.

Again 1_x is the only x factor , which may occur in M, and the resulting form is,

$$(13)(34)(41)(23)(123)1_x \quad R \text{ by } (34)1_x.$$

Accordingly a concomitant may contain at most three p factors. The possibilities for three p factors are,

(12)(23)(34),

(41)(12)(23),

(13)(32)(21),

(21)(13)(34),

(23)(31)(14),

(31)(32)(34),

together with dual and similar products. We consider each possibility in turn.

(12)(23)(34).

This is a self similar product, and also 23 must be convolved an even number of times. Hence we must consider,

$$(12)(23)(34)2_x 3_x 4_x \quad R \text{ by } (34)3_x 4_x,$$

$(12)(23)(34)(123)(124)1_x3_x$ R by $(34)1_x$ and (j),

$(123)(134)1_x2_x$ R (j),

$(123)(134)(124)4_x$ R (j),

$(1234)(124)1_x4_x3_x$ R (j),

$(1234)(124)(134) = T$,

$(1234)(124)(123)(234)3_x$ R (i),

$(1234)(123)(234)1_x2_x3_x4_x$ R (k),

$(1234)3_x2_x$.

The dual of T is

$(34)(41)(12)3_x2_x$,

which is reducible by analogy with binary forms, and so by the principle of duality T is also reducible.

<u>(41)(12)(23)M.</u>

By (h) the products 1_x4_x, (123)(234) may not appear in M. Also, since

$$(23)(124) = (21)(324) + (24)(123),$$

M cannot contain the product (124)(231) and similarily M cannot contain the product (134)(234). Hence we must consider

$(41)(12)(23)(124)1_x2_x3_x$ R by $(23)1_x$,

$(134)1x$ R by (23)(134) and (m),

$(234)2_x$ R (j),

$(123)(134)2_x3_x$ R by $(134)2_x$,

$(124)(134)2_x4_x$ R (i),

$(1234)(123)3_x$ R by $(41)3_x$ and (k),

$(1234)(124)4_x$ R (i),

$(1234)(41)(12)(23)(134)2_x3_x4_x$ R by $(41)3_x$,

$(124)(134)1_x3_x$ R by $(41)3_x$ and (i),

$(124)(234)2_x3_x$ R by $(23)2_x3_x$,

1_x2_x R by $(41)2_x$.

<u>(13)(32)(21)M.</u>

By (b), (c), (d), and (e), M cannot contain the factors (124), (234), 4_x, and 2_x, nor by (g) and (f) the products 1_x3_x and (134)(123). Accordingly we are left with

(13)(32)(21)

alone.

<u>(21)(13)(34)M.</u>

Since the factor (13) occurs, the same factors are prohibited as in the previous case. Further 23 must be convolved an even number of times, and so we have no concomitants of this type.

Since the factor (13) also occurs in the two remaining types, (23)(31)(14) and (31)(32)(34), it is easy to show that no irreducible concomitants arise from them.

We are therefore left with,

$(12)(23)(34)(1234)3_x2_x$,

(34)(14)(12)(124)(134),

(13)(32)(21),

(42)(41)(43)(1234),

(43)(31)(14),

(21)(24)(23)(1234),

together with similar forms. Of these only the first two are new.

In determining the concomitants, which contain two or less p factors, we first consider those containing x but not u factors.

Concomitants containing p and x factors.

If the factor (1234) does not appear, by analogy with binary forms we have only the six concomitants

$$(ij)i_x j_x \quad i,j = 1,2,3,4.$$

But, if (1234) does appear, the concomitants with one p factor are, by (b), (c), (d), and (e),

$(1234)(12)3_x 4_x$,

$(1234)(23)1_x 4_x$,

$(1234)(14)2_x 3_x$ R by $(14)2_x$ and (c),

together with similar forms.

If two p factors occur we have,

$(1234)(12)(23)2_x 4_x$

$(1234)(12)(34)$,

$(1234)(12)(14)1_x 3_x$,

$(1234)(23)(14)$,

together with similar forms. The complete list of concomitants involving both p and x factors is thus seen to be that given in list B. By the principle of duality the mixed concomitants containing p and u can be written down.

Concomitants containing one p factor, with x and u factors.

Since x and u are dual co-ordinates, we need only consider three types of p factor, (12), (23), (13)

occurring alone. Let us consider each in turn.

(12)M. We have the possible forms,

$(12)(123)3_x$,

$(124)4_x$,

$(234)1_x 3_x 4_x$ R (m),

$(134)2_x 3_x 4_x$ R by $(134)2_x$ and (m),

$(123)(124)3_x 4_x$ R (i),

$(123)(134)1_x 4_x$ R (k),

$(123)(234)2_x 4_x$ R by (k),

$(124)(134)1_x 3_x$,

$(124)(234)2_x 3_x$ R (j),

(134)(234) Already considered,

$(123)(124)(234)1_x$ R (m),

$(123)(124)(134)2_x$ R by $(134)2_x$ and (m),

$(123)(134)(234)1_x 2_x 3_x$

$\equiv (123)(12,43)(234)1_x 3_x$ R by factors,

$(124)(134)(234)1_x 2_x 4_x$ R (i),

$(123)(124)(134)(234)3_x 4_x$ R (i).

$(1234)(12)(123)1_x 2_x 4_x$ R by $(12)1_x 2_x$,

$(124)1_x 2_x 3_x$ R by $(12)1_x 2_x$,

$(134)1_x$,

$(234)2_x$,

(123)(124) already considered,

$(123)(134)2_x 3_x$,

$(123)(234)1_x 3_x$ R (m),

$(1234)(12)(124)(134)2_x4_x$ R (i),

$(124)(234)1_x4_x$ R (i),

$(134)(234)1_x2_x3_x4_x$ R by $(134)1_x3_x4_x$,

$(123)(124)(134)1_x3_x4_x$ R (i),

$(123)(124)(234)2_x3_x4_x$ R (i).

Those concomitants with ~~with~~ three u factors and one x factor are the duals of those with one u factor and three x factors and so do not require to be considered. Neither do those containing four u factors need to be considered.

<u>(23)M.</u> By (h) M cannot contain the product (123)(234), and (23)(124)(123) reduces to the previous case. Since (23) is a self similar factor we need only consider,

$(23)(123)1_x$,

$(124)3_x1_x4_x$, R mod (12,43),

$(123)(134)3_x4_x$ R (k),

(124)(134) already considered,

$(1234)(23)(123)4_x2_x3_x$ R by $(23)2_x3_x$,

$(124)2_x$,

$(123)(134)1_x2_x$ R mod (12,43) since 23 is convolved an odd number of times,

$(124)(134)1_x2_x3_x4_x$ R by (23)(134),

<u>(13)M.</u> It is easily shown by identities (b), (c), (d), (e), (f), (g), that there are no irreducible concomitants of this type.

We are accordingly left with,

$(12)(123)3_x$,

$(12)(124)4_x$,

$(12)(124)(134)1_x3_x$ R,

$(23)(123)1_x$,

$(23)(1234)(124)2_x$,

together with duals and similar forms. Of these

$$(12)(124)(134)1_x3_x \equiv (12)(12,43)(134)1_x$$

and as the latter appears in list B we may take the former as reducible. Each form in the list above yields three other forms, and so we have sixteen forms linear in p, x and u.

Two p factors. With dual and similar products, all that need be considered are,

(12)(34), (12)(23), (23)(14), (12)(13), (34)(13), (23)(13).

(12)(34)M. In M 23 must not be convolved an odd number of times, and by (i) and (j) the products $(124)4_x$, $(123)3_x$, $(431)1_x$ and $(431)2_x$ are prohibited. Hence we have only the single concomitant of this type,

$$(12)(34)(124)3_x \quad R \bmod (12,43).$$

(12)(23)M. As before, $(124)4_x$ and $(234)2_x$ are prohibited and so is (123)(234) by (h). In addition (123)(124) and (234)(134) cannot occur by (23)(124) and (23)(134). Accordingly we have the types,

$(12)(23)(123)2_x$,

$(12)(23)(134)4_x$,

$(12)(23)(123)(134)1_x3_x2_x4_x$ R by $(134)1_x3_x4_x$, (12)(23)(124)(134)$1_x 2_x$ R by (134)2_x,

(12)(23)(124)(234) already considered,

(12)(23)(1234)(123)$1_x 3_x 4_x$ R by (23)4_x,

(124)1_x,

(134)$1_x 2_x 3_x$ R by (23)(134) and (g),

(234)3_x,

(123)(134) already considered,

<u>(23)(14)M.</u> As before the product (123)(234) is prohibited and so is (124)(123) by (23)(124) and similarily the product (134)(234). Since (23)(14) is a self similar product we need only consider the types,

(23)(14)(123)4_x,

(23)(14)(124)3_x, R to above mod (12,43)(23),

(23)(14)(123)(134)$1_x 3_x$ R by (23)(134),

(23)(14)(124)(134)$1_x 4_x$ R (h),

(23)(14)(1234)(123)$1_x 2_x 3_x$ R by (23)$2_x 3_x$,

(23)(14)(1234)(124)$1_x 2_x 4_x$ R (h),

(23)(14)(1234)(123)(134)$2_x 4_x$ R by (23)(134) and (d),

(23)(14)(1234)(124)(134)$2_x 3_x$ R by (23)(134) and (e).

In the remaining cases, since (13) occurs, 2_x, 4_x, (234),(124), and the products $1_x 3_x$, (134)(123) are prohibited. It is now an easy matter to show , that the possible concomitants of these types are,

(12)(13)(123)1_x,

(12)(13)(1234)(134)3_x,

(13)(34)(143)3_x,

$(13)(34)(1234)(123)1_x$,

$(13)(32)(123)3_x$,

$(13)(32)(1234)(134)1_x$ R by $(32)(134)$.

We are accordingly left with,

$(12)(23)(123)2_x$,

$(12)(23)(134)4_x$,

$(12)(23)(1234)124)1_x$,

$(12)(23)(1234)(234)3_x$,

$(23)(14)(123)4_x$,

$(12)(13)(123)1_x$,

$(12)(13)(1234)(134)3_x$

$(13)(34)(134)3_x$,

$(13)(34)(1234)(123)1_x$,

$(13)(32)(123)3_x$,

together with dual and similar forms.

This list includes 12 of type $(ijk)(ij)(ik)i_x$.

$(12)(23)(134)4_x$ yields the three other forms,

$(34)(14)(123)2_x$ R,

$(34)(32)(124)1_x$ R,

$(12)(14)(234)3_x$ R,

$(32)(14)(123)4_x$ yields the similar form,

$(32)(14)(432)1_x$.

$(12)(23)(1234)(124)1_x$ yields the three other forms,

$(34)(14)(1234)(234)3_x$ R,

$(34)(23)(1234)(134)4_x$,

$(12)(14)(1234)(123)2_x$ R,

$(12)(23)(1234)(234)3_x$ yields the three other forms,

$(34)(14)(1234)(124)1_x$,

$(34)(23)(1234)(123)2_x$,

$(12)(14)(1234)(134)4_x$,

$(12)(13)(1234)(134)3_x$, yields the similar form,

$(43)(42)(1234)(421)2_x$,

$(13)(34)(1234)(123)1_x$ yields the similar form,

$(42)(21)(1234)(234)4_x$.

The concomitants marked R in the above list are equivalent to others in the list. For,

$$\begin{aligned}(34)(14)(123)2_x &\equiv (34)(12)(123)4_x + (34)(24)(123)1_x,\\ &\equiv (34)(24)(123)1_x \text{ by } (k),\\ &\equiv (34)(23)(124)1_x + (34)(12)(423)1_x,\\ &\equiv (34)(23)(124)1_x \text{ by } (m),\end{aligned}$$

and similarily,

$$(21)(41)(432)3_x \equiv (21)(32)(432)4_x.$$

But also,

$$\begin{aligned}(34)(23)(124)1_x &\equiv (31)(23)(124)4_x + (14)(23)(124)3_x,\\ &\equiv (21)(23)(134)4_x + (41)(23)(123)4_x\\ &\quad + (14)(23)(124)3_x,\\ &\equiv (21)(23)(134)4_x + (14)(23)(12,43),\end{aligned}$$

where the term $(14)(23)(12,43)$ has the factor $(AR)(ARux)$ and so is reducible. In addition,

$$\begin{aligned}(34)(14)(1234)(234)3_x &\equiv (34)(13)(1234)(234)4_x,\\ &\equiv (34)(23)(1234)(134)4_x.\end{aligned}$$

Similarily we may consider $(12)(14)(1234)(123)2_x$ to be reducible.

Concomitants containing the factor (12,43).

If we consider a concomitant M involving the factor (12,43) to be reducible, when (12,43) can be expressed in terms of factors $(ijk)m_x$, any irreducible concomitant M must have (12) and (23) convolved an even number of times, (formulae §7)).

Obviously M may be,

$$(12,43)(1234),$$
$$(12,43)^2.$$

If M $\neq$ (12,43)(1234), 23 must be convolved an even number of times in M, for otherwise M would have the the factor (AR)(ARux). It follows immediately that, if M does not contain the variable p nor the factor (1234), there are only two possibilities for M,

$$(12,43)(123)(234)3_x 2_x,$$
$$(12,43)(124)(134)1_x 4_x,$$

where in the first 12,23,34 are all convolved twice and in the second 12 and 34 are convolved twice. If M = (1234)N, 23 must be convolved once in N for it cannot be convolved three times. Hence N contains (123) or (234) and N

$= (123)(124)3_x 4_x$ or the similar product,

$(123)\ [(123)4_x - (12,43)4_x]$ by (7),

and accordingly M is reducible.

By formulae (8)- (11), M is reducible if it contain (13), (24), (23) or (14), and so if two p factors occur in M, they must be (12) and (34). In this case we get only the one concomitant,

$$(12,43)(12)(34).$$

If only one p factor occur, we may consider it to be (12) and so 34 must be convolved a second time and we have,

$$(12,43)(12)(341)1_x,$$
$$(12,43)(1234)(12)(231)3_x,$$
$$(12,43)(1234)(12)(234)(341)(124)1_x2_x4_x \quad R\ (i),$$

together with similar forms. The complete list of concomitants involving (12,43) is,

$$(12,43)^2,$$
$$(12,43)(1234),$$
$$(12,43)(123)(234)3_x2_x, \quad R$$
$$(12,43)(124)(134)1_x4_x,$$
$$(12,43)(12)(34),$$
$$(12,43)(124)(34)4_x,$$
$$(12,43)(134)(12)1_x,$$
$$(12,43)(1234)(231)(12)3_x, \quad R$$
$$(12,43)(1234)(234)(34)2_x, \quad R.$$

§ 6. <u>Special reductions</u>.

The concomitants above, which are marked R, appear in Turnbull's list but are reducible. In fact, $(12,43)(234)2_x$ is always reducible, for after removing the invariant factors,

$$(12,43)(234)2_x = (ARux)u_\rho(A\rho x),$$
$$= 2(aRu)b_xu_\rho[a_\rho b_x - a_xb_\rho].$$

Here each term has a concomitant factor b_x^2 or $b_\rho b_xu_\rho$.

A similar reduction holds for $(12,43)(123)3_x$. It is interesting to note that the duals of these products do not reduce. In this reduction we have broken up our symbols 2_x etc. and so cannot use our principle of duality. No other concomitants seem to reduce by this method, but an exactly similar identity applies in the case of five variables, part III, § 6.[1]

This completes the determination of the unreduced concomitants 122 in number, which together with the concomitant D form a complete system for two quadratics in four variables.[2]

Footnote. 1. Note on the Simultaneous System of Two Quadratic Quaternary Forms. Journal of the London Mathematical Society, Volume 4, Part 3, pages 182-183.

Footnote 2. In a paper entitled "The Complete System of Two Quaternary Quadratics", Amer. Jour. Math. Vol.LI, No 4, October 1929, pages 565-576, I have determined this system of 122 forms. In this paper Theorem I of part I is proved only for the case when $n = 4$.

Part III.

Applications of the general theory to two quadratics in five variables.

§ 1. Prepared system.

$1_x = a_x,$ $5_x = r_x,$

$2_x = (A\rho x) = \dot{a}_\rho \dot{b}_x,$ $4_x = (R\alpha x) = \dot{r}_\alpha \dot{s}_x,$

$3_x = (A_3 R_3 x) = (\dot{a}\dot{b}R_3)\dot{c}_x,$

$(12) = a_\rho (AP),$

$(54) = r_\alpha (RP),$

$(13) = (aA_3R_3P) = (\dot{a}\dot{b}R_3)(a\dot{c}P),$

$(53) = (rA_3R_3P) = (\dot{a}\dot{b}R_3)(r\dot{c}P),$

$(14) = (aR\alpha P) = \dot{r}_\alpha (a\dot{s}P),$

$(52) = (rA\rho P) = \dot{a}_\rho (r\dot{b}P),$

$(15) = (arP),$

$(23) = (AR_3)(A_3\rho P) = (AR_3)\dot{a}_\rho(\dot{b}\dot{c}P),$

$(43) = (RA_3)(R_3\alpha P) = (RA_3)\dot{r}_\alpha(\dot{s}\dot{t}P),$

$(24) = (A\rho R\alpha P) = \dot{a}_\rho \bar{r}_\alpha (\dot{b}\bar{s}P),$

$(123) = a_\rho (AR_3)(A_3 p),$

$(543) = r_\alpha (RA_3)(R_3 p),$

$(124) = a_\rho (AR\alpha p) = a_\rho \dot{r}_\alpha (A\dot{s}p),$

$(542) = r_\alpha (RA\rho p) = r_\alpha \dot{a}_\rho (R\dot{b}p),$

$(125) = a_\rho (Arp),$

$(541) = r_\alpha (Rap),$

$(143) = (RA_3)(aR_3\alpha p) = (RA_3)\dot{r}_\alpha (a\dot{s}\dot{t}p),$

$(523) = (AR_3)(rA_3\rho p) = (AR_3)\dot{a}_\rho (r\dot{b}\dot{c}p),$

$(135) = (aA_3R_3rp) = (\dot{a}\dot{b}R_3)(a\dot{c}rp),$

$(234) = (AR_3)(RA_3)(\alpha\rho p) = (AR_3)(RA_3)\dot{a}_\rho(\dot{b}\dot{c}\dot{d}p),$

$(1234) = a_\rho(AR_3)(RA_3)u_\alpha,$

$(5432) = r_\alpha(RA_3)(AR_3)u_\rho,$

$(1235) = a_\rho(AR_3)(A_3ru),$

$(5431) = r_\alpha(RA_3)(R_3au),$

$(1254) = a_\rho r_\alpha(ARu),$

$(12345) = a_\rho(AR_3)(RA_3)r_\alpha,$

$(12,54) = \dot{1}_x(\dot{2}54) = (12\dot{5})\dot{4}_x,$

$\quad = a_\rho r_\alpha(ARpx) = a_\rho r_\alpha \dot{a}_x(\dot{b}Rp),$

$\quad = a_\rho r_\alpha \dot{s}_x(A\dot{r}p),$

$(12,43) = \dot{1}_x(\dot{2}43) = (12\dot{4})\dot{3}_x,$

$\quad = a_\rho(RA_3)(AR_3\alpha px),$

$\quad = a_\rho(RA_3)\dot{a}_x\bar{r}_\alpha(\dot{b}\bar{s}\bar{t}p),$

$\quad = a_\rho(RA_3)\dot{r}_x\dot{s}_\alpha(A\dot{t}p),$

$(54,23) = \dot{5}_x(\dot{4}23) = (54\dot{2})\bar{3}_x,$

$\quad = r_\alpha(AR_3)(RA_3\rho px),$

$\quad = r_\alpha(AR_3)\dot{r}_x\bar{a}_\rho(\dot{s}\bar{b}\bar{c}p),$

$\quad = r_\alpha(AR_3)\dot{a}_x\dot{b}_\rho(R\dot{c}p),$

$(123,543) = (\dot{2}3)(\dot{1}543) = (\dot{4}3)(123\dot{5}),$

$\quad = a_\rho(AR_3)r_\alpha(RA_3)(A_3R_3Pu),$

$\quad = a_\rho(AR_3)r_\alpha(RA_3)(\dot{b}\dot{c}P)(\bar{a}R_3u),$

$\quad = a_\rho(AR_3)r_\alpha(RA_3)(\dot{r}\dot{s}P)(A_3\dot{t}u),$

$(123,154) = (1\dot{2})(154\dot{3}) = (1\dot{4})(123\dot{5}),$

$\quad = a_\rho(AR_3)r_\alpha(A_3aRPu),$

$\quad = a_\rho(AR_3)r_\alpha(\dot{a}\dot{b}P)(aR\dot{c}u),$

$\quad = a_\rho(AR_3)r_\alpha(a\dot{s}P)(A_3\dot{r}u),$

$(543,512) = (\dot{5}4)(512\dot{3}) = (5\dot{2})(543\dot{1}),$

$= r_\alpha (RA_3) a_\rho (R_3 rAPu),$

$= r_\alpha (RA_3) a_\rho (\dot{r}sP)(rA\dot{t}u),$

$= r_\alpha (RA_3) a_\rho (r\dot{b}P)(R_3 \bar{a}u),$

$(12,543) = \dot{1}_x(543\dot{2}) = \dot{3}_x(\dot{5}412),$

$= a_\rho r_\alpha (RA_3)(R_3 Aux),$

$= a_\rho r_\alpha (RA_3)\dot{a}_x(R_3 \dot{b}u),$

$= a_\rho r_\alpha (RA_3)\acute{t}_x(\dot{r}\dot{s}Au),$

$(54,123) = \dot{5}_x(123\dot{4}) = \acute{3}_x(\dot{1}254),$

$= r_\alpha a_\rho (AR_3)(A_3 Rux),$

$= r_\alpha a_\rho (AR_3)\dot{r}_x(A_3 \dot{s}u),$

$= r_\alpha a_\rho (AR_3)\dot{c}_x(\dot{a}\dot{b}Ru).$

In the above A, R, α, ρ, have been written for A_2, R_2, A_4, R_4 respectively and A=ab, A_3=abc, R=rs, R_3=rst.

§ 2. Complete list of irreducible concomitants of several types.

6 invariants: $(a\alpha)^2$, $(a\rho)^2$, $(AR_3)^2$, $(RA_3)^2$, $(r\alpha)^2$, $(r\rho)^2$.

6 covariants: 5 quadratics i_x^2 $i = 1,2,3,4,5,$

1 quintic $(12345)1_x 2_x 3_x 4_x 5_x$.

6 contravariants: 5 quadratics $(ijkm)^2$ $i,j,m,k = 1,2,3,4,5,$

1 quintic $(1234)(1235)(1245)(1345)(2345)$.

20 complexes containing the variable P:

10 quadratics $(ij)^2$,

10 cubics $(ij)(jk)(ki)$, $i,j,k = 1,2,3,4,5.$

20 complexes containing the variable p:

10 quadratics $(ijk)^2$,

10 cubics (12345)(ijk)(ijm)(ijn).

44 mixed forms containing u and x:

5 of orders 1 in u and 1 in x,

$(12345)(1234)5_x$ and a similar form,

$(12345)(1245)3_x$,

(12345)(54,123) and a similar form.

5 of orders 1 in u and 4 in x,

$(ijkm)i_x j_x k_x m_x$.

5 of orders 4 in u and 1 in x,

$(12345)(mijk)(mijn)(mikn)(mjkn)m_x$.

10 of orders 2 in u and 3 in x,

$(12345)(ijkm)(ijkn)i_x j_x k_x$.

10 of orders 3 in u and 2 in x,

$(mnkj)(mnij)(mnik)m_x n_x$,

2 of orders 3 in u and 3 in x,

$(12345)(1245)(1345)(2345)1_x 2_x 3_x$,

and a similar form.

2 of orders 3 in u and 4 in x,

$(12,543)(1245)(1345)1_x 4_x 5_x$,

and a similar form.

4 of orders 4 in u and 3 in x,

$(12,543)(12345)(1234)(1245)(1345)1_x 4_x$,

$(12,543)(12345)(1235)(1245)(1345)1_x 5_x$,

and two similar forms.

1 of orders 5 in u and 4 in x,

$(12,543)(54,123)(1235)(1245)(1345)1_x 5_x$.

67 mixed forms containing P and x:

10 of orders 1 in P and 2 in x,

$(ij)i_x j_x$.

4 of orders 1 in P and 3 in x,

$(12345)(12)3_x 4_x 5_x$,

$(12345)(23)1_x 5_x 4_x$ and two similar forms.

5 of orders 2 in P and 1 in x,

$(12345)(23)(45)1_x$ and a similar form,

$(12345)(43)(15)2_x$ and a similar form,

$(12345)(12)(45)3_x$.

5 of orders 2 in P and 3 in x,

$(12345)(21)(15)1_x 3_x 4_x$ and a similar form,

$(12345)(12)(23)2_x 4_x 5_x$ and a similar form,

$(12345)(23)(34)3_x 1_x 5_x$.

20 of orders 3 in P and 1 in x,

$(12345)(12)(14)(15)3_x$ and a similar form,

$(12345)(21)(23)(25)4_x$ and a similar form,

$(12345)(31)(32)(34)5_x$,

$(12345)(12)(13)(45)1_x$ and a similar form,

$(12345)(12)(15)(43)1_x$ and a similar form,

$(12345)(14)(15)(23)1_x$ and a similar form,

$(12345)(21)(23)(54)2_x$ and a similar form,

$(12345)(24)(23)(51)2_x$ and a similar form,

$(12345)(25)(21)(43)2_x$ and a similar form,

$(12345)(34)(35)(21)3_x$ and a similar form,

$(12345)(34)(32)(15)3_x$.

2 of orders 3 in P and 3 in x,

$(12345)(34)(32)(45)1_x 3_x 4_x$ and a similar form.

18 of orders 4 in P and 3 in x,

$(12345)(ij)(ik)(im)(in)i_x$, 5

$(12345)(13)(15)(32)(34)1_x$ and a similar form,

$(12345)(14)(12)(43)(45)1_x$ and a similar form,

$(12345)(15)(12)(53)(54)1_x$,

$(12345)(21)(23)(14)(15)2_x$, and a similar form,

$(12345)(24)(21)(43)(45)2_x$ and a similar form,

$(12345)(31)(34)(12)(15)3_x$ and a similar form,

$(12345)(32)(34)(21)(25)3_x$ and a similar form.

1 of orders 4 in P and 3 in x,

$(12345)(12)(23)(34)(45)2_x 3_x 4_x$.

2 of orders 5 in P and 1 in x,

$(12345)(31)(14)(23)(34)(45)1_x$ and a similar form.

67 mixed forms containing p and u:

These forms are the duals of the mixed forms containing P and u and can be written down immediately. For example from the 5 forms

$(12345)(ij)(ik)(im)(in)i_x$,

we obtain the 5 dual forms

(kmn)(jmn)(jkn)(jkm)(jkmn).

In the above list i,j,k,m,n take the values 1,2,3,4,5, with the understanding that in any form i,j,k,m,n are all distinct.

§ 3. Determination of the prepared system.

If n=5, there are six invariants

a_α^2, a_ρ^2, $(AR_3)^2$, $(RA_3)^2$, r_α^2, r_ρ^2,

and five quadratic covariants

1_x^2, 2_x^2, 3_x^2, 4_x^2, 5_x^2. (Part I. § 3.)

By theorem I every concomitant , multiplied by a suitable invariant factor , can be expressed in terms of the symbolic factors

i_x, (ij), (ijk), (ijkm), (12345), i,j,k,m = 1,2,3,4,5.

We must now determine, if ever in forming these bracket factors we have disturbed an invariant factor, which appears when 12, 23, 34, or 45 are convolved together. Originally we have five variables x,y,z,t,w, which are to be convolved as

Δ = (xyztw),

u = xyzt,

p = xyz,

P = xy,

x = x.

Since the only factor involving Δ is (12345) and since 12, 23, 34, 45 are all convolved in this, no invariant factor has been disturbed in forming it. When all the variables w have been convolved to form Δ, we are left to consider 4-factors, 3-factors, and 2-factors, where an i-factor is a factor involving i symbols. We may neglect all 4-factors, as the variables then can only be xyzt = u.

Let us now consider the possible cases, in which x, y, z, t may be convolved to form u. If one of these variables occur in a 3-factor, we may assume that three of them occur in this 3-factor, for

$$(ijk|xy\dot{z})(rs|\dot{t}z)$$

$$=(ijk\dot{r}|xyzt)(s|z)+\text{terms in which } tz \text{ are convolved}$$

together, and

$$(ijk|x\dot{y}\dot{z})(rs|\dot{t}z)$$

$$=(ijk\dot{r}|xyzt)(\dot{s}|z)+\text{ terms in which } yzt \text{ are convolved}$$

together.

Hence we must consider the cases when,

(a) Three variables occur in a 3-factor and the fourth occurs in another 3-factor,

(b) Three variables occur in a 3-factor and the fourth occurs in a 2-factor.

Case (a) gives the possibility,

$$(ijk\dot{n}|u)(i\dot{m}|\xi\eta),$$

and case (b)

$$(ijk\dot{n}u)(\dot{m}|\xi),$$

Where ξ, η may be any of x,y,z,t.

In (a) and (b), neither of m, n is the same as any of i, j,k or else no convolution of successive integers is disturbed. At first sight it would appear that $(ijk\dot{\bar{r}}|u)(\dot{m}|\xi)(\bar{n}/\eta)$ is a possibility, arising from three 2-factors, but i,j,k, r,m,n, must all be distinct, which is impossible and so this type cannot occur. But if the variable t does not appear, we might have the single new type

(c) $\quad (ij\dot{k}|p)(\dot{m}/\varsigma)$

arising from two 2-factors.

We now write the factors for simplicity without the variables, since no confusion can arise. There are no further types of factors, as we shall see.

Type (a) cannot occur with an other u factor, as

$$(ijk\dot{n})(i\dot{m}rs),$$

$$(ijk\dot{n})(i\dot{m}\bar{r}\bar{s})\bar{t}$$

are the only possibilities. In the first rs cannot contain i, m or n, and so must be jk. But by the fundamental identities this is impossible. In the second case none of r,s,t can be i, therefore two of them must be either ij or nm and in either case no invariant factor is disturbed. Further since

$$(ijk\dot{n})(i\acute{m}r) = (ijkr)(imn) + (ijk)(inmr)$$

by identity (17) part I, type (a) cannot occur with a further p factor. Hence type (a) gives solely the new factor type,

$$(ijk\dot{n})(i\dot{m}).$$

Similarily it may be shown that type (b) cannot occur with a further u or p factor. Type (c) cannot occur with another p factor, for

$$(ij\grave{k})(\dot{n}m) = (ijm)(nk) + (ij)(knm) \quad \text{by (17) part I,}$$

and in both terms on the right ij and nk are convolved.

There are three other possible cases to consider,

$(ijk\dot{n})(\dot{m}a)$ form (b) and a 1-factor,

$(ijk\dot{n})(\dot{m}\bar{a})(\bar{b}cde)$, from two (b)factors,

$(ij\dot{k})(\acute{m}ab)$ from two (c) factors.

Of these, the first reduces to type (a) since a=one of ijk; the third gives nothing new, since one of ab must be i or j; the second is more easily treated later.

In type (a), m,n must be consecutive integers and so must i,j. Hence we have

$$(123\dot{5})(1\dot{4}) = (123,154),$$

$$(32\dot{1}5)(3\dot{4}) = (123,543),$$

$$(543\dot{2})(5\dot{2}) = (543,512).$$

In type (b) m,n must be consecutive integers and so we have,

$$\dot{1}_x(\dot{2}345),$$

$$\dot{2}_x(\dot{3}145),$$

$$\dot{3}_x(\dot{4}125),$$

$$\dot{5}_x(\dot{4}321).$$

But

$$\dot{2}_x(\dot{3}145) \equiv 1_x(3245) + (321\dot{4})\dot{5}_x,$$

$$\equiv \dot{5}_x(321\dot{4}),$$

since in (3245) both 23 and 45 are convolved.

Similarily

$$\dot{3}_x(\dot{4}125) \equiv \dot{1}_x(432\dot{5}).$$

Accordingly type (b) yields only two new factor types,

$$\dot{1}_x(543\dot{2}) = (12,543),$$

$$\dot{5}_x(123\dot{4}) = (54,123).$$

In type (c) both i,j and m,n must be successive integers and so we have,

$$\dot{1}_x(\dot{2}43) = (12,43),$$

$$\dot{5}_x(\dot{4}23) = (54,23),$$

Footnote 1 See appendix A page 112.

$\dot{1}_x(\dot{2}54) = (12,54)$.

If a new type of factor arises from two (b) factors , it must be

$$(\dot{1}4)(\dot{2}345)(\bar{5}321),$$
$$\equiv (12\dot{3}\bar{4})(\dot{4}\dot{5})(\bar{5}321),$$
$$\equiv (1254)(\dot{4}\bar{5})(\dot{3}321),$$
$$\equiv (1254)(3\dot{4})(\dot{5}321),$$

and so is expressible in terms of simpler factors.
We thus have the eight new factor types; three of type (a), three of type (c) and three ~~of~~ type (b). The factors of type (c) are the duals of those of type (a), while the factors of type (b) are self dual.

Now ,since,with the addition of these new factor types, no invariant factors, which were originally introduced in part I § 3, have been disturbed, we can work with the symbols i, j etc and at the end remove all actual invariant factors and obtain the actual irreducible concomitants, provided that no identity is used ~~is used~~, which breaks up successive integers convolved an even number of times. An alternative method is to use as a prepared system the factors (AP), for (12) etc. This prepared system was actually found by Dr. Wm. Saddler, but has never been published.
He determined the prepared system by methods analogous to those used by Professor Turnbull in his paper on two quadratics in four variables. To find any of the irreducible concomitants by this method is cumbersome, as all identities

have to be worked out in detail and in addition the ten symbols $a, r, A, R, A_3, R_3, \alpha, \rho$, must be paired off instead of the five symbols 1,2,3,4,5. This, together with the great simplification of the identities, more than compensates for the addition of the extra factor (12345) and the fact, that the identities cannot be applied blindly.

§ 4. Determination of the irreducible covariants and contravariants.

The factors which may occur in a covariant are the five i_x factors and (12345). The irreducible covariants are then six in number;

the five quadratics i_x^2,

and the quintic $(12345)1_x 2_x 3_x 4_x 5_x$.

By duality the contravariants are six in number;

the five quadratics $(kmnj)^2$,

and the quintic (1234)(1235)(1245)(1345)(2345).

§ 5. Determination of the irreducible complexes.

The possible factors, which may occur, are the 10 factors (ij) and (12345). But, as a product of (12345) by factors of the type (ij) always involves an odd number of symbols, the factor (12345) cannot appear in such a concomitant. Since the factors (ij) are strictly analogous to simple bracket factors of binary forms, we have only 20 possible complexes,

the 10 quadratic complexes $(ij)^2$,

and the 10 cubic complexes (ij)(jk)(ki),

for a product of four or more factors (ij) is reducible.

In fact,

$$(ij)(km) = (ik)(jm) + (kj)(im),$$

$$(ij)(kn) = (ik)(jn) + (kj)(in).$$

By multiplying these two equations together and neglecting the terms, which involve a factor squared, we have

$$(ni)(ik)(kj)(jm) + (mi)(ik)(kj)(jn) \equiv 0,$$

or

$$2(ni)(ik)(kj)(jm) + (ji)(ik)(kj)(mn) \equiv 0,$$

by applying the identity (mi)(jn) to the second term.

But, as (ji)(ik)(kj) is itself a concomitant,

(ni)(ik)(kj)(jm) is reducible.[1]

By the principle of duality we see that there are only 20 irreducible complexes involving the variable p,

the ten quadratic complexes $(ijk)^2$,

and the ten cubic complexes (12345)(ijk)(ijm)(ijn).

§ 6. Determination of the mixed concomitants containing u and x.

Reductions. Since $(12,543) \equiv 1'_x(543\dot{2}) \equiv \dot{3}_x(\dot{5}\dot{4}12)$, any concomitant containing the factor (12,543) is reducible, if 12 or both of 34,45 are convolved an odd number of times. In addition such a ~~factor~~ concomitant has a factor $(AR_3)(AR_3ux)$ if 23 is convolved an odd number of times. (a)

Footnote 1. Grace and Young. Algebra of Invariants. Chap.XV. Page 322.

Further,

$$(12,543) \equiv \dot{3}_x(\ddot{5}412),$$
$$\equiv 3_x(5412) - \dot{5}_x(\dot{3}412),$$
$$\equiv 3_x(5412) - (54,123).$$

Hence,

$$(12,543)(1254)3_x \equiv (12,543)(54,123)$$
$$\equiv 0 \text{ by (a).} \qquad \text{(b)}$$

There also exists a reduction similar to that of part II, § 8.

$$(12,543)(5432)2_x \equiv 0. \qquad \text{(c)}$$

For neglecting the invariant factors we have,

$$(AR_3ux)(A\rho x)u_\rho = 2(\dot{a}R_3u)\dot{b}_x[a_\rho b_x - a_x b_\rho]u_\rho,$$

and each term on the right has a factor b_x^2 or $b_\rho u_\rho b_x$. Once again the dual product is not reducible.

Moreover,

$$(1234)5_xM \equiv 0,$$

if 45 is convolved an odd number of times in M, and (d)

$$(2345)1_xM \equiv 0,$$

if 12 is convolved an ~~even~~ odd number of times in M.

Since,

$$(12,543) \equiv (5432)1_x - (5431)2_x,$$

by squaring this identity

$$(5432)(5431)1_x 2_x \equiv 0. \qquad \text{(e)}$$

If now we consider (5432) as simpler than (5431),

$$(5431)2_xM \equiv 0,$$

if 23 is convolved an odd number of times in M. (f)

Again by squaring the identity

$$(12,543) - (2431)5_x \equiv (5231)4_x + (5421)3_x,$$

we have,

$$(5231)(5421)4_x3_x \equiv 0 \text{ by (a).} \qquad (g)$$

In the above reductions we may replace each factor by its similar factor and in most cases obtain a new reduct ion. We now consider the possible forms in the following order, first those without the factor (12,543) and in ascending order in u, then those with one factor (12,543) and finally those with both the factors (12,543) and (54,123).. We only write down one of each pair of forms, which are similar, and those forms which are reducible are marked R. The method of reduction in each case is indicated shortly at the side.

No factor (12,543).

One u factor. We have the five concomitants

$(ijkm)i_xj_xk_xm_x$,

and the types

$(12345)(1234)5_x$,

$(12345)(1235)4_x$ R (f),

$(12345)(1245)3_x$.

Two u factors. We have the types

$(1234)(1235)4_x5_x$ R (e),

$(1234)(1245)3_x5_x$ R (d),

$(1234)(2345)1_x5_x$ R (d),

$(1234)(1345)2_x5_x$ R (d) and (f),

$(1235)(1245)3_x4_x$ R (g),

$(1235)(1345)2_x4_x$ R (f) and (a),

and the ten

$$(12345)(ijkm)(ijkn)i_xj_xk_x.$$

Three u factors.

We have the ten

$$(mnjk)(mnij)(mnik)m_xn_x,$$

the duals of the previous case and the types

$$(12345)(1245)(1345)(2345)1_x2_x3_x,$$

$$(12345)(1234)(1235)(1345)2_x4_x5_x \quad R\ (f),$$

$$(12345)(1423)(1425)(1435)2_x3_x5_x \quad R\ (d),$$

$$(12345)(1523)(1524)(1534)2_x3_x4_x \quad R \text{ by } (1524)3_x,$$

$$(12345)(2314)(2315)(2345)1_x4_x5_x \quad R\ (d),$$

$$(12345)(2415)(2413)(2435)1_x3_x5_x \quad R\ (d).$$

Four u factors. We have the types

$$(1234)(2315)(1245)(1345)2_x3_x4_x5_x \quad R\ (e),$$

$$(1234)(1235)(1245)(2345)1_x3_x4_x5_x \quad R\ (d),$$

$$(1234)(1253)(1345)(2345)1_x2_x4_x5_x \quad R\ (f),$$

and the five

$$(12345)(ijkm)(ijkn)(ijmn)(ikmn)i_x,$$

the duals of the concomitants with one u and four x factors.

Five u factors. Either no x factors or five x factors occur. The first has been already considered and the second is reducible.

One factor (12,543). We have the simple forms

$$(12345)(12,543),$$

$$(12,543)^2,$$

together with two similar forms.

<u>One u factor.</u> By using reduction (a) we see that there is only one possibility,

$(12,543)(1245)3_x$ R (b).

Two u factors. We have the types

$(12,543)(1234)(2345)2_x 3_x 4_x$ R (c),

$(12,543)(1235)(2345)5_x 3_x 2_x$ R (c),

$(12,543)(1245)(1345)1_x 4_x 5_x$,

$(12,543)(12345)(1245)(1234)3_x 5_x$ R (d),

$(12,543)(12345)(1245)(1235)3_x 4_x$ R (f) mod (12,543)(54,123)

$(12,543)(51234)(2345)(1345)1_x 2_x$ R (f) mod $(12,543)^2$,

Three u factors. We have the types

$(12,543)(1234)(1235)(1245)3_x 4_x 5_x$ R mod (12,543)(54,123),

$(12,543)(1234)(2345)(1345)1_x 2_x 5_x$ R mod $(12,543)^2$.

$(12,543)(12345)(1234)(1235)(2345)2_x 3_x$ R (c),

$(12,543)(12345)(1234)(1245)(1345)1_x 4_x$,

$(12,543)(12345)(1235)(1245)(1345)1_x 5_x$.

The only possibility with four u factors is

$(12,543)(12345)(1345)(2345)(1234)(1235)1_x 2_x 4_x 5_x$,

and this is reducible mod $(12,543)^2$.

There is no possibility with five u factors. There are thus six irreducible concomitants containing one factor of type (12,543), the three in the list above and three similar forms. It is important to notice that each of these six is irreducible but that their duals reduce by (c).

<u>Both factors (12,543) and (54,123).</u>

If both the factors (12,543) and (54,123) appear in a concomitant, since 12, 54, 23, and 34 must all be convolved an even number of times, there are very few possibilities and finally we are left with

$$(12,543)(54,123)(1235)(1245)(1345)1_x 5_x,$$

and its dual

$$(12,543)(54,123)(12345)(1234)(2345)2_x 3_x 4_x \quad R\ (c).$$

§ 7. Determination of the mixed concomitants containing P and x.

Reductions. If we consider the P factors in the order of simplicity,

(12), (54), (23), (34),

(15),

(14), (52), (13), (53),

(24),

by identities of the type $(\dot{i}\dot{j})\dot{k}_x \equiv 0$, we see that

2_x cannot occur with (13) or (53),			
4_x " " " (53) or (13),			
3_x " " " (24),			(h)
1_x " " " (24) or (25),			
5_x " " " (42) or (41).			

Further by identities of the type $(\dot{i}\dot{j}')(\dot{k}'m) \equiv 0$, we see that the products

(13)(24),

(53)(24),

(13)(52),

(53)(14), (i)

(14)(25)(15),

(24)(14)(15),

(42)(52)(51),

(14)(24)(25),

are reducible. The concomitant

$$(13)(35)(51)M \equiv 0 \qquad (j)$$

also, since (13)(35)(51) is an actual concomitant containing no invariant factors.

We first consider those concomitants which do not contain the factor (12345).

No factor (12345). In this case the only irreducible concomitants that appear are the 10 mixed concomitants

$$(ij)i_x j_x.$$

This follows as a result of the analogy with binary forms (See § 5).

Forms containing the factor (12345). If the factor (12345) appears in a concomitant M, M must contain five x factors, three x factors, or one x factor, since the number of symbols in a P factor is even. Five x factors cannot occur in M, for in that case the symbols appearing in the P factors must be paired off. Accordingly M contains at least one factor (ij), in which i,j are not successive integers, and as a result the concomitant factor $(ij)i_x j_x$.

By considering list (i), we see that there can be no irreducible concomitants involving eight or more P factors. But if seven P factors occur and (24) occurs, (13) and (53) cannot appear nor can any of the products,

(14)(25)(15),

(24)(15)(14),

(24)(51)(52),

(24)(14)(25),

and so in this case seven P factors cannot occur. But , if (24) does not occur, the products

(13)(52),

(53)(14),

(13)(35)(52),

(14)(25)(15),

(13)(35)(51)

are prohibited. Hence the only possible from involving seven P factors is

(12)(23)(34)(45)(25)(15)(35)M,

or the similar form. Since 1_x must occur among the x factors in M, this form is reducible by (h). Hence there are no irreducible concomitants containing seven P factors.

We shall now consider the remaining concomitants in ascending order in P. Since (12), (54); (13), (53); (23), (43); (25), (41); are similar factors and (15), (24) self similar factors, we need write down only one of every two similar forms.

One P factor. There is only one type

$(12345)(ij)k_x m_x n_x$,

but all concomitants of this type are equivalent to

$(12345)(12)3_x 4_x 5_x$,

$(12345)(23)1_x 4_x 5_x$,

and two similar forms by reductions (h).

Two P factors, one x factor. There is only one type

$(12345)(ij)(km)n_x$.

If we let n=1,2,3 in succession and use reductions (h), we are left with,

$(12345)(23)(45)1_x$,

$(12345)(43)(15)2_x$,

$(12345)(12)(45)3_x$,

and two similar forms.

Two P factors, three x factors. There is only one type

$(12345)(ij)(ik)m_x n_x i_x$.

Since

$$(12345)(ij)(ik)m_x n_x i_x \equiv (12345)(ij)(im)k_x n_x i_x \text{ etc.},$$

if i,k are not successive integers, by letting i =1,2,3 in turn we see that all concomitants of this type reduce to

$(12345)(12)(15)1_x 3_x 4_x$,

$(12345)(21)(23)2_x 4_x 5_x$,

$(12345)(32)(34)3_x 1_x 5_x$,

and two similar forms.

Three P factors, one x factor. There is only one type

$(12345)(ij)(ik)(im)n_x$,

$(12345)(ij)(ik)(mn)i_x$.

The concomitants of the first type reduce by reductions (h) to

$(12345)(12)(14)(15)3_x$,

$(12345)(21)(23)(25)4_x$,

$(12345)(31)(32)(34)5_x$,

and two similar forms. For, since the form

$$X \equiv (12345)(31)(32)(34)5_x$$

$$\equiv (12345)(35)(34)(32)1_x + (12345)(15)(32)(34)3_x,$$

$$= Y + Z,$$

and the form Z appears in the list of irreducible forms of the second type, we may neglect Y the form similar to X.

The concomitants of the second type are equivalent to

$(12345)(12)(13)(45)1_x$,

$(12345)(12)(15)(43)1_x$,

$(12345)(14)(15)(23)1_x$,

$(12345)(21)(23)(54)2_x$,

$(12345)(24)(23)(51)2_x$,

$(12345)(25)(23)(14)2_x$ R,

$(12345)(25)(21)(43)2_x$,

$(12345)(34)(35)(21)3_x$,

$(12345)(34)(32)(15)3_x$,

and seven similar forms. The form marked R reduces by the identity $(\dot{2}\dot{5})(\dot{1}4) \equiv 0$.

Three P factors, three x factors. There is only one type

$$(12345)(mj)(ji)(ik)i_x j_x n_x.$$

In this i,j; i,k; j,m must all be successive integers, or else the form reduces by the identities

$$(\dot{i}\dot{j})\dot{n}_x \equiv 0, \quad (\dot{i}\dot{k})\dot{j}_x \equiv 0, \quad (\dot{j}\dot{m})\dot{i}_x \equiv 0.$$

Accordingly we ~~we~~ are left with

$$(12345)(34)(32)(45)1_x 3_x 4_x,$$

and the similar form

$$(12345)(32)(34)(21)5_x 3_x 2_x.$$

Four P factors ,one x factor. There are two types

$$(12345)(ij)(ik)(im)(in)i_x, \qquad 5$$

$$(12345)(ij)(ik)(jm)(jn)i_x.$$

The first type yields five concomitants. In the second type, if $i=1$ and $j=2$, one of (24) or (25) must occur and so the form is reducible. If $i=1$ and $j=3$, (35) cannot occur and so we have the possible form

$$(12345)(13)(15)(32)(34)1_x.$$

If $i=1$ and $j=4$, (42) cannot occur and if $i=1$ and $j=5$, (52) cannot occur and so we have the two forms

$$(12345)(14)(12)(43)(45)1_x,$$

$$(12345)(15)(12)(53)(54)1_x,$$

the second of which is equivalent to its similar form. By a similar treatment for the cases $i=2$ and 3, we have as a final list of concomitants of the second type

$$(12345)(13)(15)(32)(34)1_x,$$

$$(12345)(14)(12)(43)(45)1_x,$$

$$(12345)(15)(12)(53)(54)1_x \equiv \text{its similar form},$$

$$(12345)(21)(23)(41)(15)2_x,$$

$$(12345)(24)(21)(43)(45)2_x,$$

$$(12345)(31)(34)(12)(15)3_x,$$

$$(12345)(32)(34)(21)(25)3_x,$$

and six similar forms.

Four P factors, three x factors. There are three types

$$(12345)(mn)(ij)(jk)(ki)i_xj_xk_x,$$

$$(12345)(mi)(ij)(jk)(kn)i_xj_xk_x,$$

$$(12345)(mn)(nj)(jk)(kn)i_x j_x k_x.$$

The first type is not possible, since all of i,j; j.k; k,i cannot be successive integers and so the concomitant resolves into factors. In the second type m,i; i,j; j,k; k,n must be successive integers and so we have only the one irreducible form

$$(12345)(12)(23)(34)(45)2_x 3_x 4_x.$$

In the third type, n,j; j,k; k,n must all be successive integers and this is impossible.

Five P factors and one x factor. There is only one type

$$(12345)(ij)(ik)(mj)(jk)(kn)i_x.$$

In such a concomitant, by identities of the type $(\dot{j}\dot{k})\dot{i}_x \equiv 0$, we see that j,k must be successive integers;

that, if k n are not successive integers, ij must be;

that, if m,j are not successive integers, i,k must be.

Since j,k are successive integers, both i,j and m,j cannot be successive integers. If i,j are successive integers, it follows then that i,k must be successive integers. But this is impossible, since i,j; j,k; k,i cannot all be successive integers. Therefore i,j must not be successive integers and so k,n must be successive integers. Since k,j are also successive integers, i,k cannot be successive integers and so m,j must be. As a result we have only two concomitants of this type

$$(12345)(13)(14)(23)(34)(45)1_x,$$

and the similar form

$(12345)(53)(52)(43)(32)(21)1_x$.

Five P factors and three x factors. There are four types

$(12345)(ij)(jk)(kn)(ni)(mn)i_x j_x k_x$,

$(12345)(in)(nk)(km)(mi)(mn)i_x j_x k_x$,

$(12345)(ij)(jk)(ki)(mi)(in)i_x j_x k_k$,

$(12345)(km)(mj)(jk)(mi)(in)i_x j_x k_x$.

In the first type i,j; j,k; k,m; n, i must all be successive integers, which is impossible and similarily the last two types are also impossible. In the second type, if m, n are not successive integers, the form is reducible by the identity $(\dot{m}\dot{n})i_x$ 0. Therefore both of k,m and m, i cannot be successive integers. Further either i,n or k,m must be successive integers and also one of the pairs n,k and m,i. If i,n are successive integers, m,i cannot be, since m,n are successive integers, and therefore n,k must be successive integers while km cannot be. Hence the form has the factor $(km)(mi)k_x i_x$. If i,n are not successive integers, k,m must be and so m,i cannot be successive integers. But, since one of the pairs n,k and m,i must be successive integers, n,k must be successive integers. Hence k,m; m,n; n,k must all be successive integers and this is impossible. Accordingly there are no irreducible concomitants containing five P factors and three x factors.

Six P factors and one x factor. There are two types

$(12345)(jk)(jm)(jn)(km)(kn)(nm)i_x$,

$(12345)(ji)(ik)(mj)(jk)(km)(nm)i_x$.

Of these two types we need only consider the second, for, since one of k,m; k,n; n,m cannot be successive integers, we can apply an identity of the type $(\dot{\dot{m}}\dot{n})\dot{i}_x \equiv 0$ to the first type and reduce it to two forms of the second type. In the second type we see that k,j must be successive integers, that either m,j or i,k must be successive integers and that either m,k or i,j must be successive integers. But it is impossible for this to be the case, since the product of (kj) with one of both pairs (mk),(ij) and (mj),(ik) consists of three factors with a symbol in common or else is of the type (ij)(jk)(ki).

Six P factors and three x factors.[1] There are two types

$$(12345)(im)(mj)(jk)(kn)(ni)(mn)i_x j_x k_x,$$

$$(12345)(ij)(jm)(mk)(ki)(mi)(in)i_x j_x k_x.$$

In the first type, m,j; j,k; k ,n must be successive integers. Accordingly m,n cannot be successive integers and the form reduces by $(mn)i_x$. In the second type m,i; m,j; m,k must all be successive integers and this is impossible. Hence there are no irreducible concomitants containing six P factors. We have already shown that there are no irreducible concomitants with more than six P factors and so the list given in § 2 is complete.

By the principle of duality we can write down the irreducible concomitants involving p and u.

The determination of the concomitants containing P and x has not been attempted. The list of irreducible forms

Footnote 1. There is a third possible type $(12345)(im)(mj)(jn)(ni)(mk)(kn)i_x j_x k_x$ but this type reduces to the first type by identities of the type $(\dot{i}\dot{m})\dot{k}_x \equiv 0$.

would be considerably longer but could be obtained by the methods that we have used. Since the complete system for the case of four variables contains 122 forms, we should expect to obtain at least 700 or 800 forms in the complete system for two quadratics in five variables, as the labour involved in the latter case is at least five times as heavy as that of the former.

Part IV.

Applications of the general theory to two quadratics in six variables.

§ 1. The prepared system.

§ 2. Summary of results.

§ 3. Determination of the prepared system.

§ 1. The prepared system.

There are five sets of variables

$$x,$$
$$Q = xy,$$
$$P = xyz,$$
$$p = xyzt,$$
$$u = xyztw.$$

The invariant factors are the six

$$(a\alpha),\ (a\rho),\ (AR_4),\ (A_3R_3),\ (A_4R),\ (r\alpha),\ (r\rho),$$

where A, R, α, and ρ are written for A_2, R_2, A_5 and R_5 respectively.

There are six x factors of the type i_x;

$$1_x = a_x,$$
$$2_x = (A\rho x) = \dot{a}_\rho \dot{b}_x,$$
$$3_x = (A_3R_3x) = (\dot{a}\dot{b}R_4)\dot{c}_x,$$
$$4_x = (A_4R_3x) = (\dot{a}\dot{b}\dot{c}R_3)\dot{d}_x = (A_4\dot{r}\dot{s})\dot{t}_x,$$
$$5_x = (R\alpha x) = (\dot{a}\dot{b}\dot{c}\dot{d}R)\dot{e}_x = \dot{r}_\alpha \dot{s}_x,$$
$$6_x = r_x.$$

There are fifteen Q factors of the type (ij);

$$(12) = a_\rho (AQ),$$
$$(65) = r_\alpha (RQ),$$
$$(13) = (aA_3R_4Q) = (\dot{a}\dot{b}R_4)(a\dot{c}Q),$$
$$(64) = (rR_3A_4Q) = (\dot{r}\dot{s}A_4)(r\dot{t}Q),$$
$$(14) = (aA_4R_3Q) = (\dot{a}\dot{b}\dot{c}R_3)(a\dot{d}Q),$$
$$(63) = (rR_4A_3Q) = (\dot{r}\dot{s}\dot{t}A_3)(r\dot{k}Q),$$
$$(15) = (aR\alpha Q) = r_\alpha (a\dot{s}Q),$$

$(62) = (rA\rho Q) = \dot{a}_\rho(r\dot{b}Q),$

$(16) = (arQ),$

$(23) = (AR_4)(A_3\rho Q) = (AR_4)\dot{a}_\rho(\dot{b}\dot{c}Q),$

$(54) = (RA_4)(R_3\alpha Q) = (RA_4)\dot{r}_\alpha(\dot{s}\dot{t}Q),$

$(34) = (A_3R_3)(A_4R_4Q) = (A_3R_3)(\dot{a}\dot{b}R_4)(\dot{c}\dot{d}Q),$

$(24) = (A\rho A_4R_3Q) = (A_4\dot{r}\dot{s})(\bar{b}\dot{t}Q)\bar{a}_\rho,$

$(53) = (R\alpha R_4A_3Q) = (R_4\dot{a}\dot{b})(\bar{s}\dot{c}Q)\bar{r}_\alpha,$

$(25) = (A\rho R\alpha Q) = \dot{a}_\rho\bar{r}_\alpha(\dot{b}\bar{s}Q).$

There are fifteen p factors of the type (ijkm), the duals of the fifteen Q factors. For brevity in the remainder of the list a product of invariant factors has been indicated by I. If 12 is convolved, I includes a_ρ, if 23, (AR_4) etc. and in any particular case the value of I may be written down immediately.

$(6543) = I(R_4p),$

$(1234) = I(A_4p),$

$(6542) = I(R_3A\rho p) = I(R_3\dot{b}p)\dot{a}_\rho,$

$(1235) = I(A_3R\alpha p) = I(A_3\dot{s}p)\dot{r}_\alpha,$

$(6523) = I(RA_3\rho p) = I\dot{a}_\rho(R\dot{b}\dot{c}p),$

$(1254) = I(AR_3\alpha p) = I\dot{r}_\alpha(A\dot{s}\dot{t}p),$

$(6234) = I(rA_4\rho p) = I\dot{a}_\rho(r\dot{b}\dot{c}\dot{d}p),$

$(1543) = I(aR_4\alpha p) = I\dot{r}_\alpha(a\dot{s}\dot{t}\dot{k}p),$

$(2345) = I(\alpha\rho p) = I\dot{a}_\rho(\dot{b}\dot{c}\dot{d}\dot{e}p),$

$(1654) = I(aR_3p),$

$(6123) = I(rA_3p),$

$(1365) = I(aA_3R_4Rp) = I(\dot{a}\dot{b}R_4)(a\dot{c}Rp),$

$(6412) = I(rR_3A_4Ap) = I(\dot{r}\dot{s}A_4)(r\dot{t}Ap),$

$(1346) = I(aA_4R_4rp) = I(\dot{r}\dot{s}A_4)(a\dot{k}\bar{t}rp)$,

$(1265) = I(ARp)$.

There are twenty P factors of the type (ijk);

$(123) = I(A_3P)$,

$(654) = I(R_3P)$,

$(124) = I(AA_4R_3P) = I(A_4\dot{r}\dot{s})(A\dot{t}P)$,

$(653) = I(RR_4A_3P) = I(R_4\dot{a}\dot{b})(R\dot{c}P)$,

$(125) = I(AR\alpha P) = I\dot{r}_\alpha(A\dot{s}P)$,

$(652) = I(RA\rho P) = I\dot{a}_\rho(R\dot{b}P)$,

$(126) = I(ArP)$,

$(651) = I(RaP)$,

$(134) = I(aA_4R_4P) = I(\dot{a}\dot{b}R_4)(a\dot{c}\dot{d}P)$,

$(643) = I(rR_4A_4P) = I(\dot{r}\dot{s}A_4)(r\dot{t}\dot{k}P)$,

$(135) = (aA_3R_4R\alpha P) = (\dot{a}\bar{b}R_4)\bar{r}_\alpha(a\dot{c}\bar{s}P)$,

$(642) = (rR_3A_4A\rho P) = (\dot{r}\dot{s}A_4)\bar{a}_\rho(r\dot{t}\bar{b}P)$,

$(136) = (aA_3R_4rP) = (\dot{a}\bar{b}R_4)(a\dot{c}rP)$,

$(641) = (rR_3A_4aP) = (\dot{r}\dot{s}A_4)(r\dot{t}aP)$,

$(154) = I(aR_3\alpha P) = I\dot{r}_\alpha(a\dot{s}\dot{t}P)$,

$(623) = I(rA_3\rho P) = I\dot{a}_\rho(r\dot{b}\dot{c}P)$,

$(234) = I(A_4\rho P) = I\dot{a}_\rho(\dot{b}\dot{c}\dot{d}P)$,

$(543) = I(R_4\alpha P) = I\dot{r}_\alpha(\dot{s}\dot{t}\dot{k}P)$,

$(254) = I(A\rho R_3\alpha P) = I\dot{a}_\rho\bar{r}_\alpha(\dot{b}\bar{s}\bar{t}P)$,

$(523) = I(R\alpha A_3\rho P) = I\dot{r}_\alpha\bar{a}_\rho(\dot{s}\bar{b}\bar{c}P)$.

There are six u factors, the duals of the six x factors, of the type (ijkm);

$(12345) = a_\rho(AR_4)(A_3R_3)(RA_4)(\alpha u)$,

$(65432) = r_\alpha (RA_4)(R_3A_3)(R_4A)(\rho u)$,

$(12346) = a_\rho (AR_4)(A_3R_3)(A_4ru)$,

$(65431) = r_\alpha (RA_4)(R_3A_3)(R_4au)$,

$(12365) = a_\rho (AR_4) r_\alpha (A_3Ru)$,

$(65412) = r_\alpha (RA_4) a_\rho (R_3Au)$.

There is one factor of the type (ijkmnt),

$(123456) = a_\rho (AR_4)(A_3R_3)(A_4R)(\alpha r)$.

There are three factors involving u and x;

$(12,6543) = \dot{1}_x(\dot{2}6543) = - \dot{6}_x(12\dot{5}\dot{4}\dot{3})$,

$= I(AR_4ux)$,

$= I\dot{a}_x(\dot{b}R_4u)$;

$(65,1234) = \dot{6}_x(\dot{5}1234) = - \dot{1}_x(65\dot{2}\dot{3}\dot{4})$,

$= I(RA_4ux)$,

$= I\dot{r}_x(\dot{s}A_4u)$;

$(123,654) = \dot{1}_x(\dot{2}\dot{3}654) = \dot{6}_x(123\dot{5}\dot{4})$,

$= I(A_3R_3ux)$,

$= I\dot{a}_x(\dot{b}\dot{c}R_3u)$;

There are eight factors involving Q and u;

$(123,1654) = (1\dot{2})(\dot{3}1654) = (1\dot{6})(123\dot{5}\dot{4})$,

$= I(A_3aR_3Qu)$,

$= I(\dot{a}\dot{b}Q)(\dot{c}aR_3u)$;

$(654,6123) = (6\dot{5})(\dot{4}6123) = (6\dot{1})(654\dot{2}\dot{3})$,

$= I(R_3rA_3Qu)$,

$= I(\dot{r}\dot{s}Q)(\dot{t}rA_3u)$;

$(165,1234) = (1\dot{6})(\dot{5}1234) = - (1\dot{3})(65\dot{2}\dot{4})$,

$= I(aRA_4Qu)$,

$= I(a\dot{r}Q)(\dot{s}A_4u)$;

$$(612,6543) = (6\dot{1})(\dot{2}6543) = -\ (6\dot{4})(12\dot{5}\dot{3}),$$
$$= I(rAR_4Qu),$$
$$= I(r\dot{a}Q)(\dot{b}R_4u);$$
$$(265,1234) = (2\dot{6})(\dot{5}1234) = -\ (2\dot{3})(6512\dot{4}),$$
$$= I(A\rho RA_4Qu),$$
$$= I(A\rho\dot{r}Q)(\dot{s}A_4u);$$
$$(512,6543) = (5\dot{1})(\dot{2}6543) = -\ (5\dot{4})(1265\dot{3}),$$
$$= I(R\alpha AR_4Qu),$$
$$= I(R\alpha\dot{a}Q)(\dot{b}R_4u);$$
$$(123,6543) = (\dot{1}3)(\dot{2}6543) = (\dot{6}3)(21\dot{5}\dot{4}3),$$
$$= I(A_3R_4Qu),$$
$$= I(\dot{a}cQ)(\dot{b}R_4u);$$
$$(654,1234) = (\dot{6}4)(\dot{5}1234) = (\dot{1}4)(562\dot{3}4),$$
$$= I(R_3A_4Qu),$$
$$= I(\dot{r}tQ)(\dot{s}A_4u).$$

There are six factors involving P and u;

$$(1265,6543) = (\dot{1}65)(\dot{2}6543) = (\dot{4}65)(2165\dot{3}),$$
$$= I(ARR_4Pu),$$
$$= I(\dot{a}RP)(\dot{b}R_4u);$$
$$(6512,1234) = (\dot{6}12)(\dot{5}1234) = (\dot{3}12)(5612\dot{4}),$$
$$= I(RAA_4Pu),$$
$$= I(\dot{r}AP)(\dot{s}A_4u);$$
$$(1236,6543) = (\dot{1}36)(\dot{2}6543) = (\dot{5}36)(126\dot{4}3),$$
$$= I(A_3rR_4Pu),$$
$$= I(\dot{a}crP)(\dot{b}R_4u);$$

$$(6541,1234) = (\dot{6}41)(\dot{5}1234) = (\dot{2}41)(651\dot{3}4),$$
$$= I(R_3aA_4Pu),$$
$$= I(\dot{r}\dot{t}aP)(\dot{s}A_4u);$$
$$(1234,6543) = (\dot{1}34)(\dot{2}6543) = (\dot{5}34)(12\dot{6}43),$$
$$= I(A_4R_4Pu),$$
$$= I(\dot{a}\dot{c}dP)(\dot{b}R_4u);$$
$$(1236,6541) = (1\dot{2}6)(\dot{3}6541) = (1\dot{5}6)(362\dot{4}1),$$
$$= I(A_3rR_3aPu),$$
$$= I(\dot{a}\dot{b}rP)(\dot{c}R_3au).$$

There are sixteen factors involving p and Q:

$$(123,654) = (\dot{1}\dot{2})(\dot{3}654) = (\dot{6}\dot{5})(123\dot{4}),$$
$$= I(A_3R_3Qp),$$
$$= I(\dot{a}\dot{b}Q)(\dot{c}R_3p);$$
$$(126,543) = (\dot{1}6)(\dot{2}543) = (\dot{5}6)(21\dot{4}\dot{3}),$$
$$= I(ArR_4\alpha Qp),$$
$$= I(\dot{a}rQ)(\dot{b}R_4\alpha p);$$
$$(651,234) = (\dot{6}1)(\dot{5}234) = (\dot{2}1)(56\dot{3}\dot{4}),$$
$$= I(RaA_4\rho Qp),$$
$$= I(\dot{r}aQ)(\dot{s}A_4\rho p);$$
$$(123,154) = (1\dot{2})(\dot{3}154) = (1\dot{5})(123\dot{4}),$$
$$= I(A_3aR_3\alpha Qp),$$
$$= I(\dot{a}\dot{b}Q)(\dot{c}aR_3\alpha p);$$
$$= I(\dot{a}\dot{b}Q)(\dot{c}as\bar{t}p)\bar{r}_\alpha;$$
$$(654,623) = (6\dot{5})(\dot{4}623) = (6\dot{2})(654\dot{3}),$$
$$= I(R_3rA_3\rho Qp),$$
$$= I(\dot{r}\dot{s}Q)(\dot{t}rA_3\rho Qp),$$
$$= I(\dot{r}\dot{s}Q)(\dot{t}r\bar{b}\bar{c}p)\bar{a}_\rho;$$

$$
\begin{aligned}
(123,165) &= (1\dot{3})(\dot{2}165) = (1\dot{5})(123\dot{6}), \\
&= I(A_3 aRQp), \\
&= I(\dot{a}\dot{c}Q)(\dot{b}aRp), \\
&= -\ I(a\dot{r}Q)(A_3\dot{s}p); \\
(654,612) &= (6\dot{4})(\dot{5}612) = (6\dot{2})(654\dot{1}), \\
&= I(R_3 rAQp), \\
&= I(\dot{r}\dot{s}Q)(\dot{t}rAp), \\
&= -\ I(r\dot{a}Q)(R_3\dot{b}p); \\
(134,165) &= (1\dot{3})(\dot{4}165) = (1\dot{6})(413\dot{5}), \\
&= I(aA_4R_4 aRQp), \\
&= I(a\dot{r}Q)(aA_4R_4\dot{s}p); \\
(643,612) &= (6\dot{4})(\dot{3}612) = (6\dot{1})(364\dot{2}), \\
&= I(rA_4R_4 rAQp), \\
&= I(r\dot{a}Q)(rA_4R_4\dot{b}p); \\
(234,265) &= (2\dot{3})(\dot{4}265) = (2\dot{6})(234\dot{5}), \\
&= I(A_4\rho A\rho RQp), \\
&= I(\dot{b}\dot{c}Q)(\dot{d}A\rho Rp)\dot{a}_\rho, \\
&= I(\dot{b}\dot{c}Q)(\dot{d}A\bar{b}p)\dot{a}_\rho\bar{a}_\rho; \\
(543,512) &= (5\dot{4})(\dot{3}512) = (5\dot{1})(543\dot{2}), \\
&= I(R_4\alpha R\alpha AQp), \\
&= I(\dot{s}\dot{t}Q)(\dot{k}R\alpha Ap)\dot{r}_\alpha, \\
&= I(\dot{s}\dot{t}Q)(\dot{k}\bar{s}Ap)\dot{r}_\alpha\bar{r}_\alpha; \\
(123,543) &= (1\dot{3})(\dot{2}543) = (\dot{5}3)(21\dot{4}3), \\
&= I(A_3R_4\alpha Qp), \\
&= I(\dot{a}\dot{c}Q)(\dot{b}R_4\alpha p);
\end{aligned}
$$

$$(654,234) = (\dot{6}4)(5\dot{2}34) = (\dot{2}4)(563\dot{4}),$$
$$= I(R_3A_4\rho Qp),$$
$$= I(\dot{r}\dot{t}Q)(\dot{s}A_4\rho p);$$
$$(123,365) = (\dot{1}3)(\dot{2}365) = (\dot{6}3)(123\dot{5}),$$
$$= I(A_3A_3R_4RQp),$$
$$= I(\dot{a}\dot{c}Q)(\dot{b}A_3R_4Rp),$$
$$= I(\dot{r}A_3R_4Q)(A_3\bar{s}p);$$
$$(654,412) = (\dot{6}4)(\dot{5}412) = (\dot{1}4)(654\dot{2}),$$
$$= I(R_3R_3A_4AQp),$$
$$= I(\dot{a}R_3A_4Q)(R_3\dot{b}p),$$
$$(12,34,65) = (12\dot{3}\bar{6})(\dot{4}\bar{5}) = (\dot{1}43\bar{6})(\dot{2}\bar{5}) = (\dot{1}5\bar{3}6)(\dot{2}\bar{4}),$$
$$= I(AA_4R_4RQp),$$
$$= I(A\dot{c}\bar{r}p)(\dot{d}\bar{s}Q)(\dot{a}bR_4).$$

There are eight factors involving p and x;

$$(12,543) = \dot{1}_x(\dot{2}543) = \dot{4}_x(12\dot{5}\dot{3}),$$
$$= I(AR_4\alpha px),$$
$$= I\dot{a}_x(\dot{b}R_4\alpha p);$$
$$(65,234) = \dot{6}_x(\dot{5}234) = \dot{3}_x(65\dot{2}\dot{4}),$$
$$= I(RA_4\rho px),$$
$$= I\dot{r}_x(\dot{s}A_4\rho p);$$
$$(12,346) = \dot{1}_x(\dot{2}346) = \dot{4}_x(12\dot{3}6),$$
$$= I(AA_4R_4rpx),$$
$$= I\dot{a}_x(\dot{b}A_4R_4rp);$$
$$(65,431) = \dot{6}_x(\dot{5}431) = \dot{3}_x(65\dot{4}1),$$
$$= I(RR_4A_4apx),$$
$$= I\dot{r}_x(\dot{s}R_4A_4ap);$$

$$(12,654) = \dot{1}_x(\dot{2}654) = (12\dot{6}\dot{4})5_x,$$
$$= I(AR_3px),$$
$$= I\dot{a}_x(\dot{b}R_3p);$$
$$(65,123) = \dot{6}_x(\dot{5}123) = \dot{2}_x(65\dot{1}\dot{3}),$$
$$= I(RA_3px),$$
$$= I\dot{r}_x(\dot{s}A_3p);$$
$$(23,654) = \dot{2}_x(\dot{3}654) = \dot{5}_x(23\dot{6}\dot{4}),$$
$$= I(A_3\rho R_3px),$$
$$= I\dot{a}_\rho \dot{b}_x(\dot{c}R_3p);$$
$$(54,123) = \dot{5}_x(\dot{4}123) = \dot{2}_x(54\dot{1}\dot{3}),$$
$$= I(R_3\alpha A_3px),$$
$$= I\dot{r}_\alpha \dot{s}_x(\dot{t}A_3p).$$

There are six factors involving P and x;

$$(12,34) = \dot{1}_x(\dot{2}34) = \dot{4}_x(12\dot{3}),$$
$$= I(AA_4R_4Px),$$
$$= I\dot{a}_x(\dot{b}A_4R_4P);$$
$$(65,43) = \dot{6}_x(\dot{5}43) = \dot{3}_x(65\dot{4}),$$
$$= I(RR_4A_4Px),$$
$$= I\dot{r}_x(\dot{s}R_4A_4P);$$
$$(12,65) = \dot{1}_x(\dot{2}65) = \dot{5}_x(12\dot{6}),$$
$$= I(ARPx),$$
$$= I\dot{a}_x(\dot{b}RP);$$
$$(23,54) = \dot{2}_x(\dot{3}54) = \dot{4}_x(23\dot{5}),$$
$$= I(A_3\rho R_3\alpha Px),$$
$$= Ia_\rho b_x(cR_3\alpha P);$$

$$(12,54) = \dot{1}_x(254) = \dot{4}_x(12\dot{5}),$$
$$= I(AR_4\alpha Px),$$
$$= I\dot{a}_x(\dot{b}R_3\alpha P);$$
$$(65,23) = \dot{6}_x(\dot{5}23) = \dot{3}_x(5\dot{2}6),$$
$$= I(RA_3\rho Px),$$
$$= I\dot{r}_x(\dot{s}A_3\rho P).$$

There is one factor involving p, x and x;

$$(12,34,65)' = (12\dot{3}\bar{6})\dot{4}_x\bar{5}_x = (\dot{1}\bar{4}3\bar{6})\dot{2}_x\bar{5}_x = (\dot{1}\bar{4}56)\dot{2}_x\bar{3}_x,$$
$$= I(A\ A_4R_4Rpxx),$$
$$= I(A\dot{c}\bar{r}p)(\dot{a}bR_4)\dot{d}_x\bar{s}_x.$$

There is one factor involving Q, u and u;

$$(1265,6543,1234) = (\dot{1}\bar{6})(\dot{2}6543)(\bar{5}1234),$$
$$= I(AR_4RA_4Quu),$$
$$= I(\dot{a}\bar{r}Q)(\dot{b}R_4u)(\bar{s}A_4u).$$

There is one factor involving P and P;

$$(12,34,65)'' = (12\dot{3})(\dot{4}65) = (\dot{1}43)(\dot{2}65) = (12\dot{6})(43\dot{5}),$$
$$= I(AA_4R_4RPP),$$
$$= I(A\dot{c}P)(\dot{d}RP)(\dot{a}bR_4).$$

There are two factors involving P, u and x;

$$(123,6543,65) = (\dot{1}\bar{6}3)(\dot{2}6543)\bar{5}_x,$$
$$= I(A_3R_4Rpux),$$
$$= I(\dot{a}\bar{r}\dot{c}P)(\dot{b}R_4u)\bar{s}_x;$$
$$(654,1234,12) = (\dot{6}\bar{1}4)(\dot{5}1234)\bar{2}_x,$$
$$= I(R_3A_4APux),$$
$$= I(\dot{r}\bar{a}\dot{t}P)(\dot{s}A_4u)\bar{b}_x.$$

In finding the values of I in the above list symbols separated by a comma are not to be counted as convolved; thus in (12,34) the value of I is $a_\rho(A_3R_4)$,and in (12,34,65) is $a_\rho(A_3R_3)r_\alpha$.

Throughout we have written,

$$A = ab,\quad A_3 = abc,\quad A_4 = abcd,$$

$$R = rs,\quad R_3 = rst,\quad R_4 = rstk.$$

In the definitions of the bracket factors, it is sometimes necessary to use previous definitions; for example,

$$(12,34) = I\dot{a}_x(\dot{b}A_4R_4P),$$

$$I\dot{a}_x(\bar{a}\bar{b}R_4)(\dot{b}\bar{c}\bar{d}P)$$

by the definition of (134).

§ 2. Summary of results.

The prepared system thus consists of 115 factors. Of these 63 are simple factors;

6 x factors of type i_x, } duals
6 u factors of type (jkmnt), }

15 Q factors of type (ij), } duals
15 p factors of type (kmnt), }

20 P factors of type (ijk),

1 factor (123456);

47 are linear in two sets of variables;

3 u,x factors of type $\dot{i}_x(\dot{j}kmnt)$,

8 Q,u factors of type $(i\dot{j})(imnt\dot{k})$, } duals
8 p,x factors of type $(mnt\dot{k})\dot{j}_x$ }

6 P,u factors of type $(ij\dot{k})(ijtn\dot{m})$, } duals
6 P,x factors of type $(\dot{m}nt)\dot{k}_x$, }
3 p,Q factors of type $(\dot{i}\dot{j})(\dot{k}mnt)$,
12 p,Q factors of type $(i\dot{j})(\dot{k}imn)$,
1 p,Q factor $(12\dot{3}\bar{5})(\dot{4}\bar{6})$;

1 is quadratic in the variable P; $(12\dot{3})(\dot{4}65)$;
1 is quadratic in x and linear in p; $(12\dot{3}\bar{5})\dot{4}_x\bar{6}_x$; } duals
1 is quadratic in u and linear in Q; $(\dot{4}\bar{6})(12\dot{3}56)(1234\bar{5})$; }
2 are linear in the three variables P,u and x;

$(\dot{1}\bar{6}3)(\dot{2}6543)\bar{5}_x$, } duals
$(\bar{5}4\dot{2})(1234\bar{6})\dot{1}_x$. }

These factors illustrate very clearly how the principle of duality applies to the non-simple bracket factors. (See Part I § 6). Corresponding to every non-simple factor is a dual factor formed by taking the duals of the component factors and permuting the same symbols. For example, $\dot{1}_x(\dot{2}34)$ yields the dual factor $(\dot{2}3456)(\dot{1}56)$. A factor may of course be self dual, as is the case with $(12\dot{3})(\dot{4}65)$.

§ 3. Determination of the prepared system.

When $n = 6$, there are six sets of cogredient variables which we may take as x,y,z,t,w,k. These variables must be convolved to give,

$(xyztwk) = \Delta$,

$(xyztw) = u$,

$(xyzt) = p$,

$(xyz) = P$,

$(xy) = Q$.

Hence by theorem I, Part I, we have the 63 simple factors given in § 2. We must now determine, if ever in forming the simple factors (ij) etc we have broken up a convolution of successive integers, thus disturbing the invariant factors introduced in § 3, Part I. Since the only factor involving all six variables is (123456), and in this 12,23,34,45,56 are all convolved, it follows that no invariant factor is disturbed in forming Δ. Let us consider the formation of the factors involving the variable u first of all. For simplicity we call a factor containing m symbols an m-factor. If one of the variables x,y,z,t,w, convolved to form u, appear in a four-factor, we may take four of these variables as appearing in that factor. For

$$(ijkr|xyz\dot{t})(mn\ \dot{w}y) = (ijkr|\dot{x}\dot{y}wt)(mn|\dot{z}y) + (ijkr\dot{m}|u)\dot{n}_y.$$

Proceeding in this manner, we see that we lose nothing by assuming that four of the variables occur in the four-factor. We have then to consider the cases, in which the fifth variable occurs in a two-factor, a three-factor or a four-factor. The case with the fifth variable in a one-factor obviously does not require to be considered. If the four-factor is (ijkr|xyzt) or more shortly (ijkr), the two-factor cannot contain any of the symbols i,j,k,r,; the three-factor can have at most one and the four-factor at most two of the symbols i,j,k,r, for otherwise in the formation of u no invariant factors would be disturbed. Since there are only six possible values for i,j,k,r, we must consider,

$(ijkr\dot{n})\dot{m}$,

$(ijkr\dot{n})(i\dot{m})$,

$(ijkr\dot{n})(ij\dot{m})$,

where i,j,k,r,n,m are all distinct and we have not written in the variables.

If none of the variables, convolved to form u, occur in a four-factor, one may occur in a three-factor. In this case, as before, we may assume that three of the variables forming u occur in this factor and we have to consider the cases,

(a) Three variables in one three-factor, two in another,

(b) Three variables in one three factor, one in each of two three-factors,

(c) Three variables in one three-factor, one in another three-factor, one in a two-factor,

(d) Three variables in one three-factor, one in each of two two-factors.

In case (a) the same symbol cannot appear in both three-factors, and accordingly we have the sole possibility

$(ijk\dot{r}\dot{m})\dot{n}$.

In case (b) we have

$(ijk\dot{r}\bar{a})(\dot{m}\dot{n})(\bar{b}\bar{c})$,

where no two of i,j,k are the same as two of r,m,n or of a,b,c nor two of a,b,c are the same as two of r,m,n. For, if $rmn \equiv ijn$, (b) becomes

$(ijkn\bar{a})(ij)(\bar{b}\bar{c})$

and here ijn are still convolved. The rest follows from the fact that we might have started with the factor (rmn) or (abc) in place of (ijk).

In case (c) we have $(\dot{i}jk\bar{r}\dot{a})(\bar{m}\bar{n})\dot{b}$,
where ijkrmn must involve at least five distinct symbols. But neither of a,b can be the same as one of i,j,k or the same as one of r,m,n and hence this type is impossible.

In case (d) we have $(ijk\dot{r}\bar{a})\dot{m}\bar{b}$, where r,m and a,b are distinct from i,j,k. If $ab = rm$, this type is obviously reducible and so we must consider the case

$$(ijk\dot{r}\bar{r})\dot{m}\bar{n} = (ijknr)\dot{m}\dot{r} + (\dot{i}jmnr)\dot{k}\, r$$

and each term on the right reduces to simpler bracket factors.

The further case, in which only two-factors can occur, is easily seen to be impossible.

If the variable w does not occur but the variable t does, ie if the variable p appears but not the variable u, and one of the three variables convolved to form p occur in a three-factor, three of them may be considered to occur in that factor. Hence we have the types,

$(ijk\dot{r})(\dot{m}\acute{n})$ from two three-factors,

$(ijk\dot{r})(i\grave{m})$ from two three-factors,

$(ijk\dot{r})\dot{m}$ from one three-factor and one two-factor.

But, if no three-factor occur, we have the type

$(ij\dot{k}\bar{m})\dot{r}\,\bar{n}$ from three two-factors.

If no variables w or t occur, but z occurs, we have the single type

$$(ij\dot{k})\dot{r}\ .$$

We have thus the following types to consider,

A. $(ijkr\dot{m})\dot{n}$,

B. $(ijkr\dot{m})(i\dot{n})$,

C. $(ijkr\dot{m})(ij\dot{n})$,

D. $(ijk\dot{r}\dot{m})\dot{n}$,

E. $(ijk\dot{r}\bar{n})(i\dot{m})(j\bar{m})$,

F. $(ijk\dot{r})(\dot{m}\dot{n})$,

G. $(ijk\dot{r})(i\dot{m})$,

H. $(ijk\dot{r})\dot{m}$,

I. $(ij\dot{k}\bar{m})\dot{r}\ \bar{n}$,

J. $(ij\dot{k})\dot{m}$.

We now consider these types in detail.

<u>Type A.</u>[1] In A, mn must be successive integers and so we have the types,

$$\dot{1}(\dot{2}3456),$$
$$\dot{2}(\dot{3}1456) \equiv \dot{6}(123\dot{4}\dot{5}) + 1(32456),$$
$$\dot{3}(\dot{4}1256) \equiv \dot{1}(\dot{2}3456) + \dot{6}(1234\dot{5}),$$
$$\dot{4}(\dot{5}1236) \equiv \dot{3}(456\dot{1}\dot{2}) + 6(51234),$$
$$\dot{5}(\dot{6}1234).$$

We are accordingly left with only two of this type,

$$\dot{1}(\dot{2}3456) \quad \text{and} \quad \dot{5}(\dot{6}1234),$$

if we include type D. For example the term 1(32456), on the right of $\dot{2}(\dot{3}1456)$ can be neglected, since (32456) has 32,45,56 all convolved.

Footnote 1. See appendix A page 112.

<u>Type B.</u> In type B m,n must be successive integers and by letting i=1,2,3 in turn we have the possibilities,

$(1\dot{2})(1456\dot{3})$,

$(1\dot{3})(1256\dot{4}) \equiv (1\dot{5})(1234\dot{6})$,

$(1\dot{4})(1236\dot{5}) \equiv (1\dot{2})(1456\dot{3})$,

$(1\dot{5})(234\dot{6})$,

$(2\dot{3})(2156\dot{4}) \equiv (2\dot{5})(1234\dot{6})$,

$(2\dot{4})(2136\dot{5}) \equiv (26)(21345)$,

$(2\dot{5})(2134\dot{6})$,

$(3\dot{2})(3456\dot{1})$,

$(3\dot{4})(3216\dot{5}) \equiv (3\dot{2})(3456\dot{1})$,

$(3\dot{5})(3214\dot{6}) \equiv (3\dot{2})(3456\dot{1})$,

together with similar types. These types reduce as indicated above to the eight,

$(1\dot{2})(1456\dot{3})$,	$(6\dot{5})(6321\dot{4})$,
$(1\dot{5})(1234\dot{6})$,	$(6\dot{2})(6543\dot{1})$,
$(2\dot{5})(2134\dot{6})$,	$(5\dot{2})(5643\dot{1})$,
$(3\dot{1})(3456\dot{2})$,	$(4\dot{6})(4321\dot{5})$.

<u>Type C.</u> In type C, m,n must be successive integers and so must k,r. Accordingly m,n and k,r may have the following values,

m,n = 1,2; k,r = 3,4 or 4,5 or 5,6;

m,n = 2,3; k,r = 4,5 or 5,6;

m,n = 3,4; k,r = 5,6.

From these values we get six types,

$(\dot{1}56)(5634\dot{2})$,

$(\dot{1}63)(6345\dot{2})$,

$(\dot{1}34)(3456\dot{2})$,

$(\dot{2}16)(1645\dot{3})$,

$(\dot{2}14)(1456\dot{3})$,

$(\dot{4}12)(1256\dot{3})$.

Type D. Since r,m,n and i,j,k must be successive integers, there is only one factor of this type,

$(123\dot{4}\dot{5})\dot{6}$.

Type E. Since

$$(ijk\dot{r}\bar{n})(i\dot{m})(j\bar{m}) \equiv (ijkrm)(i\bar{n})(j\bar{m}) + (i\dot{j}mr\bar{n})(i\dot{k})(j\bar{m}),$$

$$\equiv (i\dot{j}mr\bar{n})(i\dot{k})(j\bar{m}) \text{ mod factors already considered,}$$

we may interchange the rôles of ijk and imr and similarily the rôles of ijk and jmn. If we consider the factors ijk, imr, jmn in turn as the foundation for the u factor, we see that ijk, imr, jmn must all be sets of three successive integers. They must be chosen from 123, 234, 345, 456 and any three of these include two sets with two symbols the same. Accordingly a factor of type E would simplify.

Type F. In this type at least two of i,j,k and at least two of r,m,n must be successive integers and so we have the possible cases,

$(123\dot{4})(\dot{5}\dot{6})$,

$(124\dot{3})(\dot{5}\dot{6}) \equiv (124\dot{5})(3\dot{6})$,

$(125\dot{3})(\dot{4}\dot{6}) \equiv (125\dot{3})(\dot{4}6) \equiv (\dot{1}345)(6\dot{2})$,

$(126\dot{3})(\dot{4}\dot{5}) \equiv (\dot{1}345)(6\dot{2})$,

$$(134\dot{2})(\dot{5}\dot{6}) \equiv (134\dot{5})(2\dot{6}) \equiv (156\dot{3})(\dot{4}2) \equiv (156\dot{2})(\dot{3}\dot{4}),$$

$$(145\dot{2})(\dot{3}\dot{6}) \equiv (145\dot{2})(\dot{3}6) \equiv (123\dot{4})(\dot{5}6),$$

$$\equiv (123\dot{4})(\dot{5}\dot{6}) - (1236)(45),$$

$$(156\dot{2})(\dot{3}4) \equiv (234\dot{5})(1\dot{6}).$$

These factors reduce as indicated above to the three,

$$(123\dot{4})(\dot{5}\dot{6}),$$

$$(345\dot{2})(6\dot{1}),$$

$$(432\dot{5})(1\dot{6}).$$

Type G. In this type, m,n must be successive integers and so must j,k. We let $i = 1,2,3$ in turn and so get the six factors,

$$(1\dot{2})(1\dot{3}45),$$

$$(1\dot{2})(1\dot{3}56),$$

$$(1\dot{3})(1\dot{4}56),$$

$$(2\dot{3})(2\dot{4}56),$$

$$(3\dot{1})(3\dot{2}45),$$

$$(3\dot{1})(3\dot{2}56),$$

and six similar factors making twelve in all.

Type H. In this type, r,m must be successive integers and so must at least two of i,j,k. We accordingly have,

$$\dot{1}(\dot{2}345),$$

$$\dot{1}(\dot{2}346),$$

$$\dot{1}(\dot{2}356) \equiv \dot{6}(123\dot{5}),$$

$$\dot{1}(\dot{2}456),$$

$$\dot{2}(\dot{3}145) \equiv (123\dot{5})\dot{4},$$

$$\dot{2}(\dot{3}156) \equiv (123\dot{6})\dot{5},$$

$\dot{2}(\dot{3}456)$,

$\dot{3}(\dot{4}126) \equiv \dot{1}(\dot{2}346)$,

$\dot{3}(\dot{4}125) \equiv \dot{1}(\dot{2}345)$,

$\dot{3}(\dot{4}156) \equiv \dot{6}(134\dot{5})$,

$\dot{3}(\dot{4}256) \equiv \dot{6}(234\dot{5})$,

and similar factors. But these reduce as indicated above to the four factors,

$\dot{1}(\dot{2}345)$,

$\dot{1}(\dot{2}346)$,

$\dot{1}(\dot{2}456)$,

$\dot{2}(\dot{3}456)$,

and four similar factors.

Type I. In this type, i,j; k,r; m,n must all be pairs of successive integers and must all be distinct and so we have only the one possibility

$(12\dot{3}\bar{5})\dot{4}\ \bar{6}$.

Type J. In this type i,j and k,m must both be pairs of successive integers and so we have the types,

$\dot{1}(\dot{2}34)$,

$\dot{1}(\dot{2}45)$,

$\dot{1}(\dot{2}56)$,

$\dot{2}(\dot{3}45)$,

$\dot{2}(\dot{3}56)$,

$\dot{3}(\dot{4}56)$.

Further u factors. Since types F,G,H,I, and J only arise when no variable u is present, we need only consider types A, B, C, and D in the formation of new u factors. Let us first consider $C = (ijkr\dot{m})(ij\dot{n}|Y)$.

If one of the variables of Y is to be convolved to form a variable u, and none of the components of this u occur in a four-factor, then all three variables in Y may be taken as forming part of u. Thus we have the possibilities,[1]

$$(ijkr\dot{m})(ij\dot{n}ab),$$
$$(ijkr\dot{m})(ij\dot{n}\bar{a}b),$$
$$(ijkr\dot{m})(ij\dot{n}\bar{\dot{a}}\bar{b})\bar{c},$$
$$(ijkr\dot{m})(ij\dot{n}\bar{\dot{a}}\overset{\times}{b}).$$

In the first case $ab \neq ij$ or m or n and so must be kr. By the fundamental identities this reduces. Similarily the second and the fourth obviously reduce; for example the second

$$\equiv (ijkr\bar{a})(ijnmb),$$

which is a factor of the types already considered multiplied by the simple factor (ijnmb). In the third case, none of a,b,c is i or j; hence two of them are m,n or k,r and if $ab = mn$, $c = k$. We have then the type

$$(ijkr\dot{m})(ij\dot{n}\bar{m}\bar{r})\bar{n},$$

which is reducible. Since in C k,r and n,m are interchangeable this type reduces in every case. The only other case to be considered is that, in which one of the variables of u appears in a four-factor. We thus have

$$(ijkrm)(\dot{i}\dot{j})(\dot{n}abcd).$$

This is obviously reducible, if abcd includes ij. Let now

Footnote 1. The symbols $\cdot, -, \times$ over a letter mean that the letter below belongs to a convolution of letters even though the other members of the convolution are omitted.

abcd involve i but not j, then , since k,r and m,n are interchangeable in C, $(ijkr\dot{m})(i\dot{j})(\dot{n}imnr)$ is typical. But this is equivalent to

$$(ijkr\dot{m})(i\dot{n})(jimnr),$$

which is a product of factor types already considered. We are left to consider

$$(ijkr\dot{m})(\dot{i}\dot{j})(\dot{n}mnkr),$$

$$\equiv (ijkr\dot{m})(\bar{i}\dot{n})(\bar{j}mnkr),$$

and this latter can be obtained from two A factors and appears in the consideration of the Q factors.

We now consider type $B \equiv (ijkr\dot{m})(i\dot{n}|Y)$. If one of the component variables of Y is convolved into u, we have the types,

$$(ijkr\dot{m})\dot{i}(\dot{n}abcd),$$

$$(ijkr\dot{m})(\dot{i}\dot{n}abc),$$

$$(ijkr\dot{m})(\dot{i}\dot{n}\bar{a}\bar{f}s)(abcd\bar{e}).$$

The first two are obviously reducible to simpler types and the last, formed from two B factors, is also reducible. For $a \neq i$, m, or n and so $a = j$, k or r, and

$$(ijkr\dot{m}).(i\dot{n}a\bar{f}s)(abcd\bar{e})$$

$$\equiv (ijkr\bar{f})(inams)(abcd\bar{e}) + (ijkrs)(inaf\bar{m})(abcd\bar{e}).$$

Each of the last two terms has a simple bracket factor as an actual factor and hence this type is also reducible.

Since, in the formation of new u factors, A and D can only occur with simple factors of the type (ij), (ijk)etc. it is easily seen that A and D do not give rise to any new u factors.

Thus there are no new u factors.

New p factors. Type C cannot occur with a p factor , for

$$(ijkr\dot{m})(ij\dot{n}s) \equiv (ijkrs)(ijnm) + (ijkr)(ijnms),$$

by identity (17) Part I. Thus C yields only six factors of the type $(ijkr\dot{m}u)(ij\dot{n}P)$.

Further type B cannot occur with a p factor, for

$$(ijkr\dot{m})(i\dot{n}\bar{a}\overset{\times}{b})$$

is reducible by identity (17), where a and b appear from factors of types A, D, F, G, H, or I. Also

$$(ijkr\dot{m})(i\dot{n}ab)$$

is reducible, since $a, b =$ two of j, k, r. We are left to consider

$$(ijkr\dot{m})(i\dot{n}\bar{a}\bar{b})(\bar{c}def),$$

arising from a B and an H factor. But this is impossible, since i cannot be one of a, b, c or one of d, e, f, since a, b, c and d, e, f are interchangeable. If on the other hand one of the components of u occur in a three-factor, we have the type,

$$(ijkr\dot{m})i(\dot{n}abc),$$

where abc cannot involve i or both of m, n. If a=m, we have

$$(ijkr\dot{m})i(\dot{n}mkr) \equiv (ijkrm)\dot{i}(\dot{n}\dot{m}kr),$$

and this latter is reducible, being the product of (ijkrm) and $i(\dot{n}\dot{m}kr) \equiv \dot{k}(nmi\dot{r})$. Hence abc = jkr and

$$(ijkr\dot{m})i(\dot{n}jkr) \equiv (ijkr)i(\dot{n}\dot{m}jkr),$$

where the latter is a product of two simpler factor types. Type $F = (ijk\grave{r})(\dot{m}\dot{n})$ cannot occur with a p factor, for as before

$(ijk\dot{r})(\dot{m}\dot{n}ab)$, or

$(ijk\dot{r})(\dot{m}\dot{n}\bar{a}\overset{\times}{b})$

are both reducible, and

$(ijk\dot{r})(\dot{m}\dot{n}\bar{a}\bar{b})(\bar{c}def)$,

from two F factors is impossible, since two of a,b,c cannot be the same as two of r,m,n or two of i,j,k. Hence we must consider

$(ijk\dot{r})(\dot{n}abc)\dot{m}$.

In this a,b,c cannot contain two of r,m,n and must therefore contain two of i,j,k, which is impossible.

Type G cannot appear with a G factor, for the types

$(ijk\dot{r})(i\dot{m}\bar{s}\overset{\times}{a})$, and

$(ijk\dot{r})(i\dot{m}\bar{s}a)$

are reducible by identity (17),Part I. In the type

$(ijk\dot{r})(i\dot{m}a\bar{e})(abc\bar{d})$,

formed from two G factors, a is distinct from i, r,m, j and k and must therefore be equal to n. Similarily i is not equal to any of a,b,c,d,e. Accordingly we must consider

$(ijk\hat{r})(i\dot{m}n\bar{m})(n\bar{r}jk)$

$\equiv (ijk\dot{r})(imnr)(n\dot{m}jk)$,

which is reducible, and

$(ijk\dot{r})(i\dot{m}n\bar{r})(n\bar{k}mj)$

$\equiv (ijkn)(imr\bar{r})(n\bar{k}mj) + (ijk\bar{r})(imnr)(n\bar{k}mj)$,

and this is reducible, unless n,r,k are successive integers. Similarily i,j,k; i,m,r; n,m,j must all be sets of successive integers and on trial this is seen to be impossible. We have still to consider the type

$(ijk\dot{r})(\acute{m}abc)\dot{i}$,

where a,b,c cannot contain i or both of r,m or both of j,k, since r,m and j,k are interchangeable in G. Hence we have

$(ijk\dot{r})(\dot{m}njr)\dot{i}$,

and this type is reducible, if i,m,n; or i,j,k are not successive integers. But

$(ijk\dot{r})(\dot{m}njr)\dot{i} \equiv (ijk\bar{n})(\dot{m}\dot{r}j\bar{r})\dot{i}$,

and this latter is reducible, unless n,r are successive integers. Similarily j,r must be successive integers. Since the only type of G factor now is

$(312\acute{4})(3\dot{5})$,

$n \simeq 6$ and accordingly n,j,r cannot be successive integers. Since factors of the types A, F, H, and I can only apppear with factors of the type (ij), (ijk), or with factors in which the symbols are not convolved, it is easy to show that no new p factors arise from considering them. We take type H as an illustration. In the type

$(ijk\dot{r})(\dot{m}abc)$

a,b,c cannot include r or m and must be i,j,k or i,j,n. But

$(ijk\acute{r})(\dot{m}ijk) \equiv 0$, and

$(ijk\dot{r})(\dot{m}ijn) \equiv (ijkn)(mrij)$,

where n is still convolved with i and j. The only other possible types are

$(ijk\dot{r})(\dot{m}ab\bar{s})$, and

$$(ijk\dot{r})(\dot{m}a\bar{b}\overset{\times}{c}),$$

both of which are obviously reducible. Accordingly there are no new p factors introduced.

<u>New P factors.</u> The types F and G cannot occur with a P factor, for

$$(ijk\dot{r})(i\dot{m}s) \equiv (ijks)(imr) + (ijk)(imsr)$$

and no convolutions of successive symbols have been disturbed. A similar proof holds for the type F. Accordingly F yields only three factors of the type $(ijk\dot{r}p)(\hat{m}\dot{n}Q)$, and G twelve of the type $(ijk\dot{r}p)(i\dot{m}Q)$.

If type B occur with a further P factor, it may occur with a single symbol, thus yielding

$$(ijkr\dot{m})(i\dot{n}a),$$

where a is not equal to i, m, or n, and so reduces to type C. But B may occur with a simple two-factor giving the type

$$(ijkr\dot{m})(i\dot{n}\bar{a})\bar{b},$$

where neither of a,b is equal to i or one of n,m, since

$$(ijkr\dot{m})(i\dot{n}\bar{a})\bar{b} \equiv (i\bar{j}\bar{k})(\bar{r}imn\hat{a})\dot{b}.$$

Therefore we may take

$$(ijkr\dot{m})(i\dot{n}\bar{j})\bar{k}$$

as typical. But this reduces to

$$(ijkr\dot{m})(k\hat{n}j)i,$$

unless i,n,m are successive integers. From the list of B factors we have only four possible types,

$$(\dot{3}1456)(1\dot{2}\bar{4})\bar{5},$$

$$(\dot{3}1456)(1\dot{2}\bar{5})\bar{6},$$

$(3456\dot{2})(3\dot{1}\bar{4})\bar{5}$,

$(3456\dot{2})(3\dot{1}\bar{5})\bar{6}$.

Of these, the first is equivalent to

$(1\bar{4}\bar{5})(123\bar{6}\dot{4})\dot{5}$,

$\equiv (1\bar{4}\bar{5})(12345)\bar{6} + (1\bar{4}\bar{5})(\overset{\times}{1}\overset{\times}{2}\dot{5}\bar{6}\bar{4})\overset{\times}{3}$,

$\equiv (145)(\dot{1}\dot{2}564)\dot{3}$,

which is reducible; similarily the second is reducible; the third is equivalent to

$(3\dot{4}\dot{5})(312\bar{6}\dot{4})\bar{5}$,

$\equiv (345)(456\dot{1}\dot{2})\dot{3}$,

and is reducible; the fourth is equivalent to

$(3\dot{4}\dot{5})(312\bar{5}\dot{6})\bar{6}$,

$\equiv (356)(\dot{1}\dot{2}456)\dot{3}$,

but is not reducible, since 3,4, which was originally convolved, is no longer convolved. Thus we have the new factor type

$(3456\dot{2})(3\dot{1}\bar{5})6$,

and the similar factor type. This type need not be considered farther, for, if the extra variable appear in a P factor, it is obviously reducible except in the case

$(3456\dot{2})(3\dot{1}\bar{5})(\bar{6}ij)$,

where i,j are successive integers. But neither of i, j can be 5 or 3, and so ij must be 12, and

$(3456\dot{2})(3\dot{1}\bar{5})(\bar{6}12)$

$(3456\dot{2})(312)(56\dot{1})$,

which is reducible. Further

$$(31\dot{5})(\dot{6}a) = (31a)(65) + (31)(65a),$$

and so, if the extra variable is convolved to form Q, no new factor type is obtained.

If a new P factor is formed from two B factors we have the general type,

$$(ijkr\dot{m})(abcd\bar{e})(i\dot{n}\bar{a})\bar{f}.$$

This type is reducible unless i, n, m and a, e, f are both sets of successive integers and also if these two sets coincide. From the list of B factors we see that we must consider

$$(1\dot{2}\bar{6})(1456\dot{3})(6321\bar{4})\bar{5},$$

$$(1\dot{2}\bar{5})(4321\bar{6})(1456\dot{3})\bar{4},$$

$$(3\dot{1}\bar{4})(3456\dot{2})(4321\bar{6})\bar{5}.$$

Of these, the first is equivalent to

$$(1\dot{2}6)(1456\dot{3})(\bar{6}321\bar{4})\bar{5} + (1\dot{2}\overset{\times}{3})(1456\dot{3})(6\bar{6}\overset{\times}{2}\overset{\times}{1}\bar{4})\bar{5},$$

$$\equiv (1\dot{2}6)(1456\dot{3})(123,456) + (123)(1456\dot{3})(654\dot{2}\dot{1})6,$$

$$\equiv (126)(14563)(123,456);$$

similarily the second and the third are reducible.

We now consider the possibility of a new factor type formed by a B factor and one other factor of the types A, D, etc in turn.

Types A and B. $B = (ijkr\dot{m})(i\dot{n})$, $A = (abcd\dot{e})\dot{f}$.

In such a factor, e, f is not the same as m, n, nor is one of e, f equal to i. If $e = m$, we have

$$(ijkr\dot{m})(ijkn\bar{m})(\bar{r}i\dot{n}),$$

$$\equiv (\bar{r}imn\dot{j})(i\dot{k}\dot{r})(ijkn\bar{m}),$$

$$\equiv (rimn\dot{j})(ir\dot{k})(ijknm),$$

and the last of these is reducible. Hence $e, f = j, k$ and we must consider

$$(ijkr\dot{m})(inmr\bar{j})(i\dot{n}\bar{k}).$$

But there is only one type of A factor and so, from the list of B factors, we are left with

$$(\dot{2}3456)(\dot{1}6\bar{5})(\dot{6}321\bar{4}),$$

$$\equiv (\dot{2}3456)(\dot{1}456\bar{2})(6\bar{3}\bar{1}),$$

$$\equiv (\bar{2}3456)(14562)(6\bar{3}\bar{1}),$$

$$\equiv (\bar{2}3456)(14562)(63\bar{1}),$$

and

$$(\dot{2}3456)(4321\bar{5})(4\bar{6}\dot{1}),$$

$$\equiv (\dot{2}3456)(4\bar{3}\bar{2})(465\dot{1}\bar{1}),$$

$$\equiv (\bar{1}3456)(43\bar{2})(46512),$$

both of which are reducible.

Types B and D. $B = (ijkr\grave{m})(i\dot{n})$, $D = (abc\dot{d}\dot{e})\dot{f}$.

Since abc and def are interchangeable in D, i may be taken equal to e, and we have the type

$$(ijkr\dot{m})(i\dot{n}\bar{f})(abci\bar{d}),$$

and this is of the type A, B and has already been considered.

Types B and H. $H \equiv (cde\dot{b})\dot{a}$. We have the factor type

$$(ijkr\dot{m})(i\dot{n}\bar{a})(bcd\bar{e}),$$

which reduces as before, if either of a,b is one of i,m,n. Farther, since

$$(in\dot{a})(\dot{b}cde) \equiv (iba)(ncde) + (inab)(cde),$$

this type reduces, unless i,m,n are consecutive integers and also if i is equal to one of c,d,e. Accordingly $i = f$, and we have

$$(fabc\dot{d})(f\dot{e}\bar{a})(\bar{b}cde),$$

$$\equiv (fabcd)(bea)(fcde) + (fabc\dot{d})(f\acute{e}ab)(cde),$$

and both terms on the right are products of simpler factor types.

Types B and I. $I = (ab\dot{c}\bar{e})\dot{d}\,\bar{f}$.

This type is not possible, since ab, cd, and ef are interchangeable in I and neither of c,d can equal i in B.

Types B and J. $J = (b\dot{c}\dot{d})\dot{a}$.

As in previous cases i,n,m must be consecutive integers and none of a,b,c,d can belong to inm and this is impossible.

P factors involving type A and not B.

If A occurs with an ordinary two-factor, the resulting factor is obviously reducible. The only other possibility is

$$(ijkr\dot{m})(\dot{n}\bar{a}\overset{\times}{b}),$$

$$\equiv (ijkr\bar{a})(mn\overset{\times}{b}) + (ijkr\overset{\times}{b})(mn\bar{a}) + (\dot{i}\dot{j}\dot{k})(\dot{r}mn\bar{a}\overset{\times}{b}),$$

$$\equiv (\dot{i}\dot{j}\dot{k})(\dot{r}mnab).$$

Accordingly, if a,b arise from two other A factors, this type is reducible, since there are only two distinct A factors. Thus there is no new P factor formed by three A factors. Further, since in A mn=12 or 56, the combination of an A factor, an F factor and any other factor cannot occur. Moreover, since

$$(ijk\dot{r})(\dot{m}ts) \equiv (ijk)(rmts) + (ijkt)(rms) + (ijks)(mtr),$$

any factor formed from factors of types A, H, I, or J is reducbile. Similarily type D cannot appear with types H, I or J. If D occur with an ordinary two-factor, we have

$(ijk\dot{r}m)(\dot{n}mj)$,

and this is of type C. We must now consider the types H, I, and J with ordinary simple factors (ab). In type H $\equiv (ijk\dot{r})(\dot{m}ab)$, neither of a,b is equal to one of r,m and so we have

$$(ijk\dot{r})(\dot{m}ab) = (ijk)(rmab) + (ijk\bar{a})(rm\bar{b}),$$

which is reducible, since at least one of a,b is the same as one of i, j. Since neither of k,r in I can equal one of a,b and since k,r; m,n; i,j are interchangeable in I, the factor I with the simple factor (ab) is impossible But the type J and the simple factor (ab) yields the new type

$$(ij\dot{k})(\dot{m}ab) \equiv (12\dot{3})(\dot{4}56).$$

The only type, which we have neglected, is

$$(ij\dot{k}\bar{m})(\dot{r}\bar{n}a),$$

and this is reducible. For

$$(ij\dot{k}\bar{m})(\dot{r}\bar{n}a) \equiv (ijkr)(mna) + (ij\dot{k}a)(\dot{r}mn) + (ij\dot{k})(mn\dot{r}a),$$

and each term on the right has at least one less broken convolution.

<u>New Q factors.</u> Type J cannot occur with a new Q factor, for

$$(ij\dot{k})(\dot{m}a) = (ija)(km) + (ij)(mka).$$

Two factors of the type **A** yield the new factor type

$$(1234\dot{5})(\dot{6}\bar{1})(\bar{2}3456).$$

The only possibility from one A factor and one D factor is

$$(1234\dot{5})(\dot{6}\bar{1})(\bar{2}\bar{3}456),$$

$$\equiv (1234\dot{5})(\dot{6}\bar{6})(123\bar{4}\bar{5}),$$

$$\equiv (\bar{1}\bar{2})(\bar{3}\bar{4}56\dot{6})(123\dot{4}\dot{5}),$$

$$\equiv (\bar{1}\bar{2})(\bar{3}\bar{4}456)(12356),$$

$$\equiv (\dot{1}4)(\dot{3}\dot{2}456)(12356),$$

and this last is reducible.

From A and H we have the type

$$(ijkr\dot{m})(\dot{n}\bar{a})(\bar{b}cde),$$

$$\equiv (ijkr\dot{m})(\dot{n}ab\bar{c})(\bar{d}\bar{e}),$$

$$\equiv (ijkr\dot{a})(mn\dot{b}\bar{c})(\bar{d}\bar{e}),$$

where we have neglected terms which are reducible. From the first identity we see that neither of n,m can be the same as one of a,b and from the second, that neither of a,b can be the same as one of i,j,k,r. But this is impossible and so the type reduces.

Since in I i,j; k,r; m,n are interchangeable, no new factor arises from A and I.

From two D factors we have the type

$$(123\dot{4}\dot{5})(\dot{6}\bar{6})(123\bar{4}\bar{5}),$$

$$\equiv (123\bar{4}\bar{5})(123\bar{6} \mid \dot{x}\dot{y}\dot{z}\dot{w})(456 \mid xy\dot{t}),$$

$$\equiv 0, \quad \text{by}$$

the convolution of 4,5,6 in the first factor.

From D and H we have the possibility

$$(ijkr\dot{m})(\dot{n}\bar{a})(\bar{b}cde),$$

$$\equiv (ijk\dot{r}\dot{m})(\dot{n}ab\bar{c})(\bar{d}\bar{e}).$$

This latter is reducible if a and b appear among r,m,n or among i,j,k. Accordingly $a,b = 3,4$ and cde is of the type 126 or 125. Therefore a,b and c,d can be interchanged and since cd $\equiv$ 12, this new factor reduces.

Similarily the combinations of D with I, and H with I are impossible.

From two H factors we have the type

$$(ijk\dot{\bar{r}})(\dot{m}\bar{a})(\bar{b}cde),$$

$$\equiv (ijk\dot{r})(\dot{m}ab\bar{c})(\bar{d}\bar{e}),$$

$$\equiv (ijk\bar{c})(rmab)(\bar{d}\bar{e}) + (ijk\dot{a})(rm\dot{b}\bar{c})(\bar{d}\bar{e}).$$

From the first identity we see that neither of a,b is the same as one of r,m and therefore one of a,b must equal one of i,j,k. Accordingly this type reduces by the second identity.

No new type arises from two I factors, but the single I factor yields the new type

$$(12\dot{3}\bar{5})(\dot{4}\bar{6}).$$

We have now found all possible cases in which a convolution of successive symbols has been disturbed. The complete list of factor types is given in § 2. In § 1 these factors are defined in terms of the symbols a, r, A, R, etc, and if the factors I are removed, we are left with a prepared system similar to that used by Turnbull in his paper on Quaternary Quadratic Forms.

Appendix A.

At first sight it might appear that other factors of the type (ij,km) should be introduced. But

$$(ij,km) \equiv (ijk)m_x - (ijm)k_x,$$

$$\equiv (jkm)i_x - (ikm)j_x.$$

If k,m are not successive integers, the factor (ij,km) can be expressed in terms of the simple factors (ijk), (ijm), m_x and k_x, without disturbing any invariant factors. Thus, if (ij,km) is to be retained as a new factor k,m must be successive integers. Similarily i,j must be successive integers. Further if k is a symbol equivalent to i or j, the factor (ijk) is reducible and $(ij,km) \equiv - (ijm)k_x$. But in the factor (ijm), i,j and m,k are still convolved, for k is a symbol equivalent to i or j. Hence once again no invariant factors have been disturbed. Accordingly (12,34) is the only new factor of this type that must be introduced.

For similar reasons, factors of the types (13,245), (125,234), (13,456), etc. do not require to be introduced into the prepared system for two quadratics in five and six variables respectively.