

The algebraic theory of Kreck surgery

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Jörg Sixt)

Abstract

Surgery theory is a classification technique for high-dimensional manifolds which was developed in the 1960s. The traditional Browder-Novikov-Sullivan-Wall-theory decides whether an $(n + 1)$ -dimensional normal cobordism $(e, f, f'): (W, M, M') \rightarrow X$ with f and f' homotopy equivalences is cobordant rel ∂ to an s -cobordism. There is an obstruction in a group $L_{n+1}(\mathbb{Z}[\pi_1(X)])$ which vanishes if and only if this is possible. Algebraic L -groups have been extensively studied and computed. For a ring Λ with involution, $L_n(\Lambda)$ is a Witt group of quadratic forms if n is even and a Witt group of quadratic formations if n is odd.

In the 1980s M. Kreck generalized Wall's original approach by dealing with cobordisms $(e, f, f'): (W, M, M') \rightarrow X$ of *normal smoothings* in which f and f' are only $[\frac{n+1}{2}]$ -equivalences. There is an obstruction in a monoid $l_{n+1}(\mathbb{Z}[\pi_1(X)])$ which is *elementary* if and only if that cobordism is cobordant rel ∂ to an s -cobordism. The l -monoids are little understood algebraically and there are no computations of them.

This thesis studies the algebraic properties of $l_{2q}(\Lambda)$ (i.e. $n + 1 = 2q$). l -monoids are equivalence classes of generalized formations which we call *preformations*. We introduce three obstructions to an element $z \in l_{2q}(\Lambda)$ being elementary. Firstly, it is shown that every elementary $z \in l_{2q}(\Lambda)$ has a stable *flip-isomorphism*. In certain cases there is a close relationship between flip-isomorphisms and isometries of the topological linking forms of M and M' . Secondly, every flip-isomorphism of z determines an asymmetric form which vanishes in the asymmetric Witt group $LAsy^0(\Lambda)$ if z is elementary. At last, a quadratic signature can be defined for certain kinds of flip-isomorphisms. z is elementary if and only if one of these quadratic signatures is zero in $L_{2q}(\Lambda)$.

There are also computations of $l_{2q}(\Lambda)$ for skew fields Λ and it will be shown that $l_{4m+2}(\mathbb{Z}[\pi])$ cannot be embedded in a group.

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A dissertation is not only the product of scientific discussion - psychology plays an important role as well.

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Chapter 1

Introduction

Throughout this thesis all manifolds are compact and smooth.

Surgery theory was pioneered in the famous paper of Kervaire and Milnor [KM63] on the classification of homotopy spheres. Surgery on high-dimensional manifolds was then developed by Browder, Novikov, Sullivan and Wall, culminating in the general theory of Wall's book [Wal99]. A modified theory which needs weaker prerequisites has been presented and applied by M. Kreck (see [Kre99]). It assigns to any cobordism of **normal smoothings** an element in the monoid $l_{2q}(\Lambda)$ which is **elementary** if and only if that cobordism is cobordant $\text{rel } \partial$ to an s -cobordism.

The main aim of this treatise is to provide obstructions which can help to decide whether an element in $l_{2q}(\Lambda)$ is elementary or not.

The first step is to determine whether such an element has a **flip-isomorphism** which, in certain simply-connected cases, is the same as the existence of an isometry of linking forms of the manifolds one wants to classify.

The second step is to compute **asymmetric signatures** i.e. certain elements in the asymmetric Witt-group $L\text{Asy}^0(\Lambda)$ for each flip-isomorphism. In the case of an elementary element all these signatures vanish. If the element in $l_{2q+2}(\Lambda)$ allows linking forms it turns out that the asymmetric signatures only depends on a choice of isometry of those linking forms.

Alternatively one can define **quadratic signatures** for a certain class of flip-isomorphisms. An element in the l -monoid is elementary if it allows such a special kind of flip-isomorphism and if the quadratic signature in the quadratic Witt-group $L_{2q+2}(\Lambda)$ vanishes for at least one of them.

The quadratic signature is technically more difficult to handle than its asymmetric sister but they are related via the canonical map $L_{2q+2}(\Lambda) \longrightarrow L\text{Asy}^0(\Lambda)$.

Nicer results can be obtained when one deals with all those elements in $l_{2q}(\Lambda)$ that are represented by non-singular formations (the objects which help to define the odd-dimensional L -groups). They are the obstructions to a Kreck surgery problem where e.g. all normal smoothings are in fact normal maps and if M and M' are closed. Then

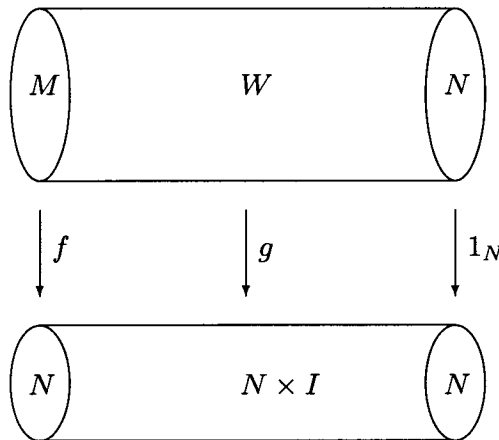
the definition of asymmetric signatures still requires the existence of a flip-isomorphism but they will be independent of the particular choice. One can even tame the quadratic signatures: they exist for all flip-isomorphisms and considerable simplifications can be achieved.

All these construction involve a lot of algebraic surgery theory. Independently, linear algebra allows us to compute $l_{2q+2}(\Lambda)$ for skew fields. As a by-product, it will be shown that certain naïve attempts of replacing the l -monoids by groups are not successful.

1.1 Classical and Kreck's Surgery Theory

In the following the surgery theory of Browder-Novikov-Sullivan-Wall and its modification by Kreck are outlined. A more elaborate account can be found in Section 2.1.

In the 1960s C.T.C. Wall and others developed surgery theory as a tool to find out when a (normal) homotopy equivalence $f: M \rightarrow N$ of n -dimensional manifolds is homotopic to a diffeomorphism. There is a first obstruction which decides whether f can be extended to a degree 1 normal cobordism



$$(g, f, 1_N) : (W, M, N) \rightarrow N \times (I, \{0\}, \{1\})$$

i.e. f , g and 1_N are covered by maps of the stable normal bundles and they map fundamental class to fundamental class. A second obstruction determines - in a more general setting - whether a normal cobordism into a finite geometric Poincaré space X

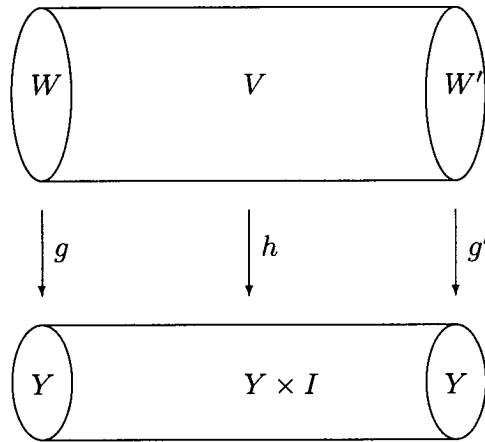
$$(g, f_0, f_1) : (W, M_0, M_1) \rightarrow X \times (I, 0, 1) \tag{1.1}$$

with homotopy equivalences f_i is cobordant rel ∂ to a homotopy equivalence i.e. an s -cobordism. In that case and for $\pi_1(X)$ with vanishing Whitehead groups¹ the s -cobordism theorem tells us that M_0 and M_1 are diffeomorphic. In the following we will concentrate on the second obstruction.

¹Of course there is also a version for other fundamental groups. Then we have to replace homotopy equivalences by simple homotopy equivalences and use the simple L -groups.

Assume the dimension of W is even ($\dim W = 2q + 2$). Surgery below the middle dimension allows us to replace g by a $(q + 1)$ -equivalence. Then define $(K_{q+1}(W), \lambda, \nu)$ with $K_{q+1}(W)$ the **kernel module** which is the homology of the induced map $\tilde{g}: \tilde{W} \rightarrow \tilde{X}$ of the universal covers with twisted coefficients. λ is induced by the Poincaré duality on W and X and ν is the self-intersection map. Together they are a non-singular $(-)^{q+1}$ -quadratic form which vanishes in the Witt group $L_{2q+2}(\mathbb{Z}[\pi_1(X)])$ if and only if (1.1) is cobordant rel ∂ to an s -cobordism.

If the dimension of W is odd ($\dim W = 2q + 1$), the construction of an obstruction is slightly more complicated. One way is to extend $g: W \rightarrow Y = X \times I$ to a **presentation** (see [Ran01]), that is a $(2q + 2)$ -dimensional degree 1 normal cobordism (V, W, W')



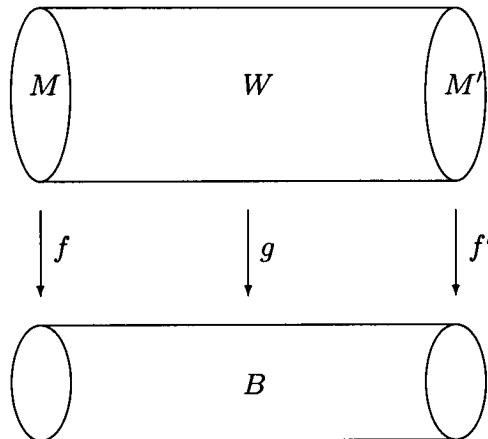
such that also h and g' are highly-connected. Define a **non-singular formation** $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ with

$$\begin{aligned} \gamma: G = K_{q+1}(V) &\longrightarrow F = K_{q+1}(V, W) \\ \mu: G = K_{q+1}(V) &\longrightarrow K_{q+1}(V, W') \cong K^{q+1}(V, W) = F^* \end{aligned}$$

A non-singular formation is a tuple $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ such that $(\gamma_\mu): G \rightarrow H_{(-)q}(F)$ is an inclusion of a lagrangian into a hyperbolic form and $(G, \gamma^* \mu, \bar{\theta})$ is a $(-)^{q+1}$ -quadratic form. A formation determines a class of automorphisms of the hyperbolic form sending F to G which represents the obstruction in Wall's original version of odd-dimensional surgery theory (see [Ran01] for details).

The obstruction formation lives in some kind of Witt-group $L_{2q+1}(\mathbb{Z}[\pi_1(X)])$ of non-singular formations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$. Again the obstruction vanishes if and only if surgery is successful in producing an s -cobordism i.e. if and only if there exists a presentation as before with $f': W' \rightarrow Y$ a homotopy equivalence (i.e. an s -cobordism). Matthias Kreck modified traditional surgery theory in the early 1980s (see [Kre99]) for the odd- and even-dimensional case such that it requires much weaker topological input.

In this treatise we shall only focus on the even-dimensional case. There are two major differences to the classical even-dimensional surgery programme. First of all Kreck can replace the normal maps by a considerably weaker notion called **normal smoothings**, that is, a lift of the stable normal bundle $M \rightarrow BO$ to a fibration $B \rightarrow BO$. As pointed out in his paper normal maps are a special case of this concept. The second main difference is that Kreck just needs $[n/2]$ -equivalences on the boundary of a normal cobordism whereas in Wall's theory we started with full homotopy equivalences. Hence we look at a $(2q + 2)$ -dimensional cobordism



with f and f' being q -equivalences and (after the usual surgery below the middle dimension) g being a $(q + 1)$ -equivalence. The obstruction here is an object

$$\begin{aligned} & (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) \\ &= (H_{q+1}(W, M_0) \longleftarrow (\text{im}(d: \pi_{q+2}(B, W) \rightarrow \pi_{q+1}(W))) \longrightarrow H_{q+1}(W, M_1), \psi) \\ &\in l_{2q+2}(\mathbb{Z}[\pi_1(B)]) \end{aligned}$$

$(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is called a **preformation**. It is basically a tuple of homomorphisms of (free) f.g. modules over $\mathbb{Z}[\pi_1(B)]$ such that $(G, \gamma^* \mu, \theta)$ is a $(-)^{q+1}$ -quadratic form. The obstruction lives in a monoid $l_{2q+2}(\Lambda)$ and is **elementary** if and only if (W, M, M') is cobordant rel ∂ to an s -cobordism.

One observes that on the one hand side the classical even-dimensional case at the beginning is a special case of Kreck's surgery setting. On the other hand the obstruction looks quite similar to the obstruction formation in the odd-dimensional case.

The theory was successfully applied by M. Kreck and others (see also Introduction of [Kre99]) to the classification of 4-manifolds (see e.g. [Kre01], [HKT94]), 7-dimensional homogeneous spaces (see e.g. [KS88], [KS91]), of complete intersections (see [Kre99]) or of classification of higher dimensional manifolds in general (see e.g. [KT91])

Despite its successes, there are disadvantages of this programme: the complicated algebra. The obstructions do not lie in the L -groups anymore but in monoids $l_{2q+2}(\Lambda)$

with Λ a ring with involution. The criterion that surgery is successful in creating an s -cobordism is not that the obstruction vanishes but that it fulfils certain complicated conditions (being **elementary**). M. Kreck himself writes: “The obstruction l -monoids are very complicated and algebraically - in contrast to the L -groups - not understood.” ([Kre00] p.135) No-one has been able to compute $l_{2q+2}(\Lambda)$ so far - except for skew fields by the author. Hardly any relations to the L -groups are known. The aim of this treatise is to bring some more insight into the structure of $l_{2q+2}(\Lambda)$, find relations to the quadratic and asymmetric Witt-groups and give obstructions for elements in $l_{2q+2}(\Lambda)$ to be elementary.

1.2 The Main Strategy

For a moment let us forget about normal maps and smoothings and consider a cobordism of manifolds (W, M, M') with $\dim W \geq 6$ and even. Let $\Lambda = \mathbb{Z}[\pi_1(W)]$. The surgery theory question is: can we surger W (without affecting the boundary) in such a way that the result will be an s -cobordism? Let us assume that we already know that there is a diffeomorphism $h: M \xrightarrow{\cong} M'$. Using that diffeomorphism, the boundary of W can be changed to a **twisted double** $\partial W = M \cup_{h|\partial M} M$ i.e. two copies of M glued together along its boundary by the automorphism on ∂M induced by h .

Twisted doubles play an important rôle in topology e.g. if one wants to compute the cobordism group of diffeomorphisms or investigate open book decompositions. Kreck computed the cobordism of automorphisms first (see e.g. [Kre84]), followed by Quinn who offered an alternative approach. In [Qui79] he develops a theory about open books decompositions which are strongly related to twisted doubles. Twisted doubles were studied by Winkelnkemper in [Win73] and both results were connected in [Ran98] Chapter 30.

The main question in this context is to decide whether manifold with a twisted double on the boundary is cobordant $\text{rel} \partial$ to another manifold that itself carries a compatible twisted double structure (see Section 5.1). In the even-dimensional case, he constructed a non-singular **asymmetric** form which vanishes in the **asymmetric Witt-group** $LA_{sy}^0(\Lambda)$ if and only if the twisted double on the boundary can be extended (up to cobordism) to the whole manifold. If (W, M, M') is an s -cobordism (i.e. a tube) any diffeomorphism $M \xrightarrow{\cong} M'$ will transform W into a twisted double. Hence, if (W, M, M') is cobordant $\text{rel} \partial$ to an s -cobordism, the asymmetric signature will vanish for any diffeomorphism $M \xrightarrow{\cong} M'$.

This approach does not lead to a practical method helping in the classification of manifolds. After all, it starts with the assumption that the manifolds, we want to classify, are already diffeomorphic!

The asymmetric signature becomes workable though if we use algebraic surgery theory.

This theory provides us with constructions that imitate Quinn’s asymmetric signature for **symmetric Poincaré pairs** (see [Ran98] Chapter 30 or Section 5.2 of this thesis). Symmetric pairs are purely algebraic objects but arise naturally from topology. As an example, the symmetric Poincaré pair associated to a manifold W with the twisted double $M \cup_h M$ as a boundary consists of singular chain complexes of the universal covers of W , M and M together with chain equivalences that induce Poincaré duality on those manifolds and further maps which guarantee the symmetry properties of Poincaré duality. In addition, one needs the chain maps induced by the inclusion of the boundary into W and the diffeomorphism $h: \partial M \xrightarrow{\cong} \partial M$.

The algebraic surgery version of the asymmetric signature gives another way of finding out whether W is cobordant to a twisted double (see e.g. [Ran98] Corollary 30.12). It also shows that the answer to this question only depends on the homotopy type of the ingredients.

But it does even more than that. It is a purely algebraic calculus and can be used to help us testing the elementariness of elements in $l_{2q+2}(\Lambda)$.

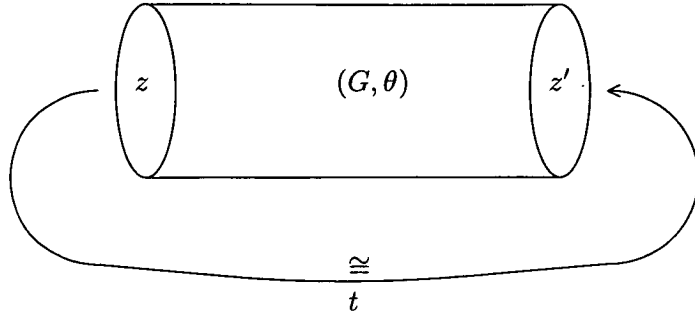
It will be shown that an element in $l_{2q+2}(\Lambda)$ (or to be more precise a **preformation** $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ representing that element) gives rise to a **quadratic Poincaré pair** x . A quadratic Poincaré pair is an algebraic model of a normal map. Every quadratic Poincaré pair induces a symmetric Poincaré pair. For such quadratic Poincaré pairs there are notions of surgery, cobordism $\text{rel} \partial$ and h -cobordisms. In particular, x is cobordant $\text{rel} \partial$ to an algebraic h -cobordism if and only if the element $l_{2q+2}(\Lambda)$ at the beginning was elementary. Hence x behaves just like an “algebraic Kreck surgery problem”. This gives us some hope that a purely algebraic version of an asymmetric signature, as presented for manifolds before, is feasible and helps to test elementariness. As a first step we need to transform the boundary of the quadratic Poincaré pair x into some kind of algebraic equivalent of a twisted double. Instead of a diffeomorphism $M \xrightarrow{\cong} M'$ we only need an equivalence of the quadratic complexes which are the two algebraic “boundary components” of x . For the preformation z this means that there exists a (stable weak) isomorphism t between z and its **flip** $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\theta)$: a **flip-isomorphism**.

There is also a more geometrical reason why flip-isomorphisms are the correct algebraic substitute for a diffeomorphism. Let

$$(e, f, f'): (W, M, M') \longrightarrow X \times (I, 0, 1)$$

be a presentation i.e. a degree 1 normal cobordism such that e , f and f' are highly-connected and assume for simplicity that M and M' are closed. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be the Kreck obstruction formation. We encountered such a cobordism before when we defined the odd-dimensional Wall obstruction for f . By Corollary 2.4.6

both obstructions can be assumed to be the same. The Wall obstruction of the odd-dimensional highly-connected normal map f' is the flip $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\theta)$ of z . An “algebraic” isomorphism from the L -theory point of view is hence a (weak) isomorphism t between z and z' - a flip-isomorphism:



In any case we can prove that if z is (stably) elementary it has indeed (stable) flip-isomorphisms.

It is easy to determine when a preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ over \mathbb{Z} with finite coker γ and coker μ allows a flip-isomorphism. The preformation induces **linking forms** i.e. \pm -symmetric forms on those cokernels. If z is an obstruction to an even-dimensional Kreck surgery problem $(W, M, M') \rightarrow B$ then those algebraic linking forms of z are induced by the topological linking forms of M and M' . Any isometry of those linking form gives rise to flip-isomorphisms and conversely any flip-isomorphism induces an isometry of the linking forms of z .

But let us return to the quadratic Poincaré complex x we constructed out of z . Every flip-isomorphism t transforms the boundary of the Poincaré pair x into an algebraic twisted double. Now the algebraic surgery version of the asymmetric signature yields an asymmetric signature $\sigma^*(z, t) \in LAsy^0(\Lambda)$ depending only on the preformation z and a flip-isomorphism t . (These constructions do not only work for group rings or \mathbb{Z} but for any weakly finite ring Λ with 1 and involution). If $[z] \in l_{2q+2}(\Lambda)$ is elementary then the asymmetric signatures vanish for all (stable) flip-isomorphism.

As mentioned before, isometries of linking forms are a good source for flip-isomorphisms. It turns out that the asymmetric signatures only depend on those linking forms.

Obviously we are interested in whether these asymmetric signatures will be complete obstructions to elementariness. The evidence does not look all too optimistic. First of all asymmetric signatures completely ignore all quadratic information simply because - unlike in other surgery problems - their definition and application does only require manifolds but not necessarily normal maps/smoothings. The computation of l -monoids for skew fields will provide further counter-examples.

We again resort to the manifold world for some inspiration for a stronger obstruction. Assume again we have an even-dimensional cobordism (W, M, M') . For a moment let

us assume that M and M' are closed. Any diffeomorphism $h: M \xrightarrow{\cong} M'$ allows us to glue both ends of W together. Alternatively we obtain the same manifold if we replace M by M' using h (then W is a manifold with an (un-)twisted double $M + M$ on the boundary) and then glue the s -cobordism $M \times (I, 0, 1)$ on it. The resulting closed manifold V is null-cobordant if and only if (W, M, M') is cobordant to an s -cobordism. If M and M' are not closed, h again turns (W, M, M') into a manifold with a twisted double as a boundary. But we have to be careful now: not every twisted double is the boundary of an s -cobordism. If, however, we demand that $h|_{\partial M}$ is isotopic to $1_{\partial M}$ we can glue (W, M, M') onto $M \times (I, 0, 1)$ and again get a closed manifold which is null-cobordant if and only if (W, M, M') is cobordant to an s -cobordism. Similar constructions also work for a normal cobordism and a compatible diffeomorphism h .

Again we follow our philosophy that anything that can be done for manifolds can also be done in the algebraic world of complexes. We imitate the procedures for the quadratic Poincaré pair x that we created out of a preformation z . The case $\partial M = \emptyset$ corresponds to the case where z is a **non-singular formation**. As an example, the obstruction of a Kreck surgery problem $(W, M, M') \rightarrow X \times (I, 0, 1)$ is a formation when all maps involved are normal maps, X is a finite geometric Poincaré space and the induced map $\partial M \rightarrow \partial X$ is a homotopy equivalence. Just like with manifolds and normal maps, it is possible to glue quadratic Poincaré pairs together and there is a notion of cobordism. All we need is an equivalent for the diffeomorphism $h: M \xrightarrow{\cong} M'$. Any chain equivalence of the quadratic complexes which constitute the two “boundary components” of x will do the job. It is nothing else than the flip-isomorphism, we have encountered before. So we use a choice of flip-isomorphism to glue the “ends” of the Poincaré pair together and the resulting Poincaré complex is null-cobordant if and only if x is cobordant $\text{rel} \partial$ to an algebraic h -cobordism and this is the case if and only if $[z] \in l_{2q+2}(\Lambda)$ is elementary. One of the fundamental facts of algebraic surgery theory is that the set of cobordism classes of Poincaré complexes are Wall’s L -groups. Hence our construction leads to an obstructions in $L_{2q+2}(\Lambda)$ - the **quadratic signatures**.

Unfortunately, the generalization to arbitrary preformations is much more unpleasant. In the case of manifolds or normal maps not any diffeomorphism could be used for the glueing operation. The same is true in the algebraic surgery world. Not every flip-isomorphism is suitable to produce a quadratic signature. We have to introduce a new class of flip-isomorphisms called **flip-isomorphisms $\text{rel} \partial$** . The quadratic signatures of such a flip-isomorphism $\text{rel} \partial$ will also depend on other choices and hence are rather difficult to handle.

In any case, one can show that quadratic and asymmetric signatures are connected via

the canonical map

$$\begin{aligned} L_{2q+2}(\Lambda) &\longrightarrow LAsy^0(\Lambda) \\ (K, \psi) &\longmapsto (K, \psi_0 - \epsilon\psi_0^*) \end{aligned}$$

1.3 The Results

Let Λ be a weakly finite ring with 1 and an involution. Let $\epsilon = (-)^q$ and let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an ϵ -quadratic split preformation, that is a tuple consisting of a free f.g. Λ -module F , a f.g. Λ -module G , a Λ -homomorphism $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}) : G \rightarrow F \oplus F^*$ and a map $\theta : G \rightarrow Q_{-\epsilon}(\Lambda)$ such that $(G, \gamma^*\mu, \theta)$ is a $(-\epsilon)$ -quadratic form. The obstruction to a surgery problem in Kreck's theory is such an object.

We say that two preformations are **stably strongly isomorphic** if they are isometric after one adds "hyperbolic elements" of the form

$$((P \oplus P^*) \xleftarrow{1} (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix}} (P \oplus P^*)^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$$

The equivalence classes form the l -monoid $l'_{2q+2}(\Lambda)$ and the equivalence classes of regular preformations (i.e. G is free) define the l -monoid $l_{2q+2}(\Lambda)$ (compare Definition 2.2.13 on p.28). The main theorem in Kreck's modified surgery theory states that surgery leads to an s -cobordism if the obstruction is **stably elementary**. Hence, our aim is to find obstructions for z to be stably elementary. By Corollary 2.4.7 (p.37) we can replace z by a **regular** preformation i.e. one with a free G .

In Kreck's original theory all isometries and isomorphisms were simple and all modules involved were based. We will ignore the Whitehead obstruction in the following and hence only deal with h -cobordisms. For $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{Z}[\mathbb{Z}^m]$ there is no difference.

There are certain obvious primitive obstruction for a preformation to be elementary.

Proposition (See Corollary 2.4.3 on p.36). *If $[z] \in l_{2q+2}(\Lambda)$ is elementary then $\ker \gamma \cong \ker \mu$, $\operatorname{coker} \gamma \cong \operatorname{coker} \mu$, $\ker \gamma^*\mu \cong \ker \gamma \oplus \ker \gamma^*$, $\operatorname{coker} \gamma^*\mu \cong \operatorname{coker} \gamma \oplus \operatorname{coker} \gamma^*$ and $\operatorname{rk} F$ is even*

The first really important obstruction we will discuss is the existence of a **flip-isomorphism**. A flip-isomorphism is a weak isomorphism (α, β, χ) of z with its flip $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\theta)$. (Compare Definition 4.1.1 on p.61). Weak isomorphisms are a generalization of isomorphisms of formations as they are used in the definition of the odd-dimensional L -groups (see Definition 2.2.14 on p.29). A strong isomorphism (which is used in the definition of the even-dimensional l -monoids) is also a weak isomorphism (see Remark 2.2.17 on p.30). Every elementary preformation has a flip-isomorphism:

Proposition (See Corollary 4.1.3 on p.62). *Let z be a regular ϵ -quadratic split preformation. If z is (stably) elementary there is a (stable) strong flip-isomorphism*

$(\alpha, \beta, 0)$ such that $\alpha: F \rightarrow F^*$ is ϵ -symmetric and zero in $L^{2q}(\Lambda)$ (and hence also in $L\text{Asy}^0(\Lambda)$) and $\beta^2 = 1_G$.

Now we use the flip-isomorphism to define asymmetric and quadratic signatures.

Flip-isomorphisms and Asymmetric signatures

For any flip-isomorphism $t = (\alpha, \beta, \chi)$ (even those that do not fulfil the stronger conditions of the previous proposition) there is an **asymmetric signature** $\sigma^*(z, t) = (M, \rho) \in L\text{Asy}^0(\Lambda)$ of a **flip-isomorphism** t of z given by

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha(\chi - \epsilon\chi^*)^*\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \rightarrow M^*.$$

(Definition 5.3.1 on p.80). We define an abelian monoid $fl_{2q+2}(\Lambda)$ as a kind of l -monoid of preformations with a choice of flip-isomorphism (Definition 5.5.1 on p.84). Then the asymmetric signatures define a map from that monoid into the asymmetric Witt-group which vanishes for all stably elementary preformations.

Theorem (See Theorem 5.5.3 and Remark 5.5.2 on p.85). *The asymmetric signatures give rise to a well-defined homomorphism of abelian monoids*

$$\begin{aligned} \sigma^*: fl_{2q+2}(\Lambda) &\rightarrow L\text{Asy}^0(\Lambda) \\ [(z, t)] &\mapsto \sigma^*(z, t) \end{aligned}$$

If $[z'] \in l_{2q+2}(\Lambda)$ is elementary then $\sigma^(z, t) = 0$ for all flip-isomorphisms t of all preformations z with $[z] = [z'] \in l_{2q+2}(\Lambda)$.*

We will give two proofs of this Theorem (or rather of the underlying Theorem 5.4.1 on p.81): one using algebraic surgery theory and an explicit stable lagrangian of our asymmetric signature.

In Section 9.6, asymmetric signatures are studied if Λ is a field. The converse of the theorem turns out to be true e.g. if $\Lambda = \mathbb{Q}$ or for fields of characteristic different from 2 endowed with a non-identity involution and q even. Unfortunately there are also counterexamples.

A test case for asymmetric signatures are boundaries of quadratic forms in particular the submonoid of $l_{2q+2}(\Lambda)$ given by the injection

$$\begin{aligned} L_{2q+2}(\Lambda) &\hookrightarrow l_{2q+2}(\Lambda) \\ (K, \theta) &\mapsto \partial(K, \theta) = (K \xleftarrow{1_K} K \xrightarrow{\theta - \epsilon\theta^*} K^*, \theta) \end{aligned}$$

Here live the obstructions of traditional surgery theory interpreted as a special case of Kreck's modified theory.

Corollary (Corollary 7.3.2, p.110). *Let (K, θ) be a non-singular $(-\epsilon)$ -quadratic form. Then $z = \partial(K, \theta)$ is a non-singular ϵ -quadratic split formation.*

- i) z has a (stable) flip-isomorphism.*
- ii) $[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if $[(K, \theta)] = 0 \in L_{2q+2}(\Lambda)$.*
- iii) All asymmetric signatures equal $[(K, \theta - \epsilon\theta^*)] \in LAsy^0(\Lambda)$.*
- iv) Assume that either Λ is a field of characteristic different to 2 or that $\Lambda = \mathbb{Z}$ and $\epsilon = -1$. z is elementary if and only if the asymmetric signatures vanish.*

Asymmetric signatures ignore any “quadratic split” information. Hence non-singular skew-quadratic forms over \mathbb{Z} and $\mathbb{Z}/2\mathbb{Z}$ with non-trivial Arf-invariant have vanishing asymmetric signatures but their boundaries aren’t stably elementary (see Example 7.3.3 on p.111).

Linking forms

What is an easy source for flip-isomorphisms? For a certain class of preformations the answer is linking forms. Assume that $S \subset \Lambda$ is a central multiplicative subset, e.g. $\Lambda = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$. We call a map an S -isomorphism if tensoring with $S^{-1}\Lambda$ makes it an isomorphism. If μ is an S -isomorphism the preformation determines a linking form L_μ and if γ is an S -isomorphism it defines a linking form L_γ .

Proposition (See Proposition 8.2.3 on p.118). *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular split ϵ -quadratic preformation with either μ or γ an S -isomorphism.*

- i) If z allows a flip-isomorphism then both γ and μ are S -isomorphisms. Every flip-isomorphism $t = (\alpha, \beta, \chi)$ induces an isomorphism of split $(-\epsilon)$ -quadratic linking forms $[\alpha^{-*}]: L_\mu \xrightarrow{\cong} L_\gamma$.*
- ii) Assume γ and μ are both S -isomorphisms and L_γ and L_μ are isomorphic. Every isomorphism $l: L_\mu \xrightarrow{\cong} L_\gamma$ induces a stable flip-isomorphism (α, β, χ) of z such that $[\alpha^{-*}] = l: L_\mu \xrightarrow{\cong} L_\gamma$.*

We introduce sub-monoids of the l -monoids $l^{2q+2}(\Lambda)$ and $fl^{2q+2}(\Lambda)$ of preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ where γ and μ are S -isomorphisms and the quadratic refinement is omitted:

$$\begin{aligned} l_S^{2q+2}(\Lambda) &= \left\{ [(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)] \in l_{2q+2}(\Lambda) \mid \mu \text{ and } \gamma \text{ are } S\text{-isomorphisms} \right\} \\ fl_S^{2q+2}(\Lambda) &= \left\{ [(z, t)] : [z] \in l_S^{2q+2}(\Lambda) \right\} \end{aligned}$$

Similarly we define an l -monoid $ll_S^{2q+2}(\Lambda)$ of preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ with a choice of isometry $L_\mu \cong L_\gamma$. The previous proposition can be interpreted as the existence of a surjection $fl_S^{2q+2}(\Lambda) \longrightarrow ll_S^{2q+2}(\Lambda)$ (see Section 8.3).

For preformations in $l_S^{2q}(\Lambda)$ the asymmetric signature $\sigma^*(z, t)$ of a flip-isomorphism t does only depend on the preformation and the isometry of linking forms induced by t .

Theorem (See Theorem 8.4.3 on p.121). *There is a lift of the asymmetric signature map of Theorem 5.5.3*

$$\begin{array}{ccc} fl_S^{2q+2}(\Lambda) & \xrightarrow{\sigma^*} & LAsy^0(\Lambda) \\ L \downarrow & \nearrow \sigma^* & \\ ll_S^{2q+2}(\Lambda) & & \end{array}$$

If $[z'] \in l_S^{2q+2}(\Lambda)$ is elementary then $\sigma^*(z, l) = 0 \in LAsy^0(\Lambda)$ for all isomorphisms $l: L^\mu \xrightarrow{\cong} L^\gamma$.

This theorem is quite an improvement. Instead of checking the asymmetric signature for all flip-isomorphisms of all stably strongly isomorphic preformations, we only have to go through all isometries of linking forms of one representative. In the case $\Lambda = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$ they are only finitely many of them. We can do even more. In certain circumstances we can read off the linking forms from a simply-connected manifold directly.

Proposition (See Propositions 8.5.6 on p.123). *Let $p: B \rightarrow BO$ be a fibration with $\pi_1(B) = 0$. Let M_i be $(2q+1)$ -dimensional manifolds with a $(q-1)$ -smoothings in B i.e. a lift of the stable normal bundle over p which is q -connected. Let $f: \partial M_0 \rightarrow \partial M_1$ be a diffeomorphism compatible with the smoothings. Let W be a cobordism of $M_0 \cup_f M_1$ with a compatible q -smoothing over B . As in Corollary 2.4.6 we define an obstruction*

$$\begin{aligned} x(W) &= (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) \\ &= (H_{q+1}(W, M_0) \longleftarrow H_{q+2}(B, W) \longrightarrow H_{q+1}(W, M_1), \theta) \\ &\in l'_{2q+2}(\mathbb{Z}) \end{aligned}$$

Let $l_{M_i}^B$ be the linking form on $H_{q+1}(B, M_i)$ which is induced by the topological linking form of M_i .

If $\text{coker } \gamma = H_{q+1}(B, M_0)$ is finite then $L^\gamma = -l_{M_0}^B$.

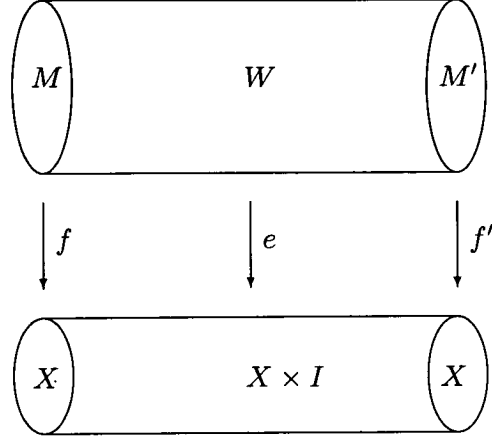
If $\text{coker } \mu = H_{q+1}(B, M_1)$ is finite then $L^\mu = -l_{M_1}^B$.

Assume both cokernels are finite. If W is cobordant rel ∂ to an h -cobordism then there exist isomorphisms $l: L^\mu = -l_{M_1}^B \xrightarrow{\cong} L^\gamma = \epsilon - l_{M_0}^B$ and their asymmetric signatures $\sigma^*(x(W), l) \in LAsy^0(\mathbb{Z})$ will all vanish.

Non-singular formations

In general asymmetric signatures are not strong enough to show elementariness, therefore we look out for a stronger obstruction - **quadratic signatures**.

Quadratic signatures turn out to be rather complicated objects so we will deal first with a special class of preformations for which they behave nicely. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a non-singular ϵ -quadratic split formation i.e. that means the map $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}) : G \rightarrow H_\epsilon(F)$ is an inclusion of a lagrangian. Let $(e, f, f') : (W, M, M') \rightarrow X \times (I, 0, 1)$



be a degree 1 normal cobordism (i.e. e and f and f' are normal maps and $(X, \partial X)$ is a finite geometric Poincaré pair) and $f| : \partial M \rightarrow \partial X$ is homotopy equivalence. Then the modified Kreck surgery obstruction of Corollary 2.4.6 (p.37) is such a non-singular formation.

By [Ran80a] Proposition 2.2, the map $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}) : G \rightarrow H_\epsilon(F)$ can be extended to an isomorphism of hyperbolic ϵ -quadratic forms

$$\left(f = \begin{pmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \tilde{\gamma}^* \mu & \tilde{\theta} \end{pmatrix} \right) : H_\epsilon(G) \xrightarrow{\cong} H_\epsilon(F)$$

For any $\tau : G^* \rightarrow G$ the maps $\tilde{\gamma}' = \tilde{\gamma} + \gamma(\tau - \epsilon\tau^*)$, $\tilde{\mu}' = \tilde{\mu} + \mu(\tau - \epsilon\tau^*)$, $\tilde{\theta}' = \tilde{\theta} + (\tau - \epsilon\tau^*)^* \theta (\tau - \epsilon\tau^*) + \tilde{\gamma}^* \mu (\tau - \epsilon\tau^*)^* - \epsilon\tau$ define another extension to an isomorphism of hyperbolic forms. Conversely any other extension has this form.

For any flip-isomorphism t of a non-singular ϵ -quadratic split formation z and a choice of extensions $\tilde{\gamma}$, $\tilde{\mu}$, $\tilde{\theta}$ we can define a **quadratic signature** $\tilde{\rho}^*(z, t, \tilde{\gamma}, \tilde{\mu}, \tilde{\theta}) = (M, \xi') \in L_{2q+2}(\Lambda)$ given by

$$\xi' = \begin{pmatrix} \tilde{\gamma}^* \tilde{\mu} + \tilde{\gamma}^* \alpha \nu \alpha^* \tilde{\gamma} & -\tilde{\gamma}^* \alpha \gamma & 0 \\ 0 & \epsilon \theta^* & 0 \\ \epsilon (\alpha^* \tilde{\gamma} - \tilde{\mu}) & -\mu & 0 \end{pmatrix} : M = G^* \oplus G \oplus F^* \rightarrow M^*$$

(see Definition 7.2.1 on p.108).

Theorem (See Theorem 7.2.3 on p.108). *Let z' be a non-singular ϵ -quadratic split formation. $[z'] \in l_{2q+2}(\Lambda)$ is elementary if and only if there is a stably strongly isomorphic $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$, a flip-isomorphism t and $\tilde{\gamma}$, $\tilde{\mu}$ and $\tilde{\theta}$ such that*

$$\left(f = \begin{pmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \tilde{\gamma}^* \mu & \tilde{\theta} \end{pmatrix} \right) : H_\epsilon(G) \xrightarrow{\cong} H_\epsilon(F)$$

is an isomorphism of hyperbolic ϵ -quadratic forms and $\tilde{\rho}^(z, t, \tilde{\gamma}, \tilde{\mu}, \tilde{\theta}) = 0 \in L_{2q+2}(\Lambda)$.*

There is also a surprise about asymmetric signatures of non-singular formations: they are independent of the choice of flip-isomorphisms.

Theorem (See Theorem 7.3.1 on p.110). *Let z be a non-singular formation. Let t and t' be two flip-isomorphisms. Then $\sigma^*(z, t) = \sigma^*(z, t') \in LAsy^0(\Lambda)$.*

Quadratic signatures for arbitrary preformations

The general definition of a quadratic signature demands much more preparation and we will only sketch it here. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation and let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism of z . z and t and a choice of representatives for $\bar{\nu}$ and $\bar{\theta}$, etc. define a self-equivalence $(h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ of $2q$ -dimensional quadratic Poincaré complexes (see (4.2) on p.64). Assume there exists a homotopy $(\Delta, \eta): (1, 0) \simeq (h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ (see Definition 6.1.4 on p.94) then t is called a **flip-isomorphism rel** ∂ (see Definition 6.2.1 on p.96). Those ingredients define a quadratic signature in $L_{2q+2}(\Lambda)$ which vanishes for a choice of Δ, η, θ , etc. if and only if z is stably elementary. The construction is described in Section 6.3 on p.98 and the relation to elementariness in Theorem 6.4.1 on p.101.

The quadratic signature of z, t, Δ , etc. is mapped to the asymmetric signature $\sigma^*(z, t)$ via the map

$$\begin{aligned} L_{2q+2}(\Lambda) &\longrightarrow LAsy^0(\Lambda) \\ (K, \psi) &\longmapsto (K, \psi_0 - \epsilon\psi_0^*) \end{aligned}$$

(see Theorem 6.5.1 on p.103). Its kernel can be computed in terms of cobordism classes of automorphisms of quadratic Poincaré complexes (see Remark 6.5.2 on p.104).

In the case of $\Lambda = \mathbb{Z}$ and $\epsilon = -1$ i.e. $q = 2m - 1$ the map is an injection:

Proposition (See Corollary 6.5.3 on p.104). *Let $q = 2m - 1$ i.e. $\epsilon = -1$. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular skew-quadratic split preformation over \mathbb{Z}*

i) $[z] \in l_{4m}(\mathbb{Z})$ is elementary if and only if there is a flip-isomorphism rel ∂ t such that $\sigma^(z, t) = 0 \in LAsy^0(\mathbb{Z})$.*

ii) The quadratic signature $\rho^(z, t, \nu, \theta, \kappa, \Delta, \eta) \in L_{4m}(\mathbb{Z})$ only depends on z and t .*

l -monoids for skew fields

The following results are independent from the theory about asymmetric and quadratic signatures and answer some questions about the structure of the monoid $l_{2q+2}(\Lambda)$. First of all it is possible to compute the l -monoid in case of skew fields. For simplicity we just present the non-split case. Note that $l_{2q+2}(\Lambda) = l^{2q+2}(\Lambda)$ for skew fields Λ with $\text{char } \Lambda \neq 2$.

Theorem (See Corollary 9.1.9, Remark 9.1.13, Corollary 9.2.2). *For any weakly finite ring Λ with 1 and involution there is a monomorphism*

$$\begin{aligned} \eta: L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^5 &\hookrightarrow l^{2q+2}(\Lambda) \\ ([K, \lambda], s, t, x, z, y) &\longmapsto [\partial(K, \lambda) + \partial(\Lambda^s, 0) \\ &+ (\Lambda^t \xleftarrow{0} \Lambda^t \xrightarrow{1} \Lambda^t) \Lambda^t \\ &+ (\Lambda^{x+x} \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Lambda^x \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \Lambda^{x+x}) \Lambda^{x+x} \\ &+ (\Lambda^y \xleftarrow{0} \Lambda^z \xrightarrow{0} \Lambda^y) \Lambda^y] \end{aligned}$$

with $\partial(K, \lambda) = (K \xleftarrow{1} K \xrightarrow{\lambda} K^*)$.

Let Λ be a skew field. Then η is an isomorphism. In that case the map

$$\begin{aligned} \nu: l^{2q+2}(\Lambda) &\xrightarrow{\cong} L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^4 \oplus \mathbb{Z} \\ (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*) &\longmapsto ([G/\ker \gamma^* \mu, \gamma^* \mu], \dim \ker \gamma, \dim \ker \mu, \\ &\dim \ker(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}), \dim \ker \gamma^* \mu, \dim F - \dim G) \end{aligned}$$

is also an isomorphism.

$[z] \in l^{2q+2}(\Lambda)$ is elementary if and only if $\nu(z) = (0, a, a, b, c + 2a, c)$ for some $a, b \in \mathbb{N}_0$ and $c \in \mathbb{Z}$ i.e. if and only if $[(G/\ker \gamma^* \mu, \gamma^* \mu)] = 0 \in L^{2q+2}(\Lambda)$, $\dim \ker \gamma = \dim \ker \mu$ and $\dim \ker \gamma^* \mu = \dim \ker \gamma + \dim \ker \gamma^*$.

Can we make groups out of the l -monoids?

The theory of monoids is rather difficult, so one would like to replace or “approximate” the l -monoids by groups. There are two naïve ways: one tries to find a group into which $l_{2q+2}(\Lambda)$ can be embedded or one looks at $l_{2q+2}(\Lambda)/l_{2q+2}^{el}(\Lambda)$. There is not much hope for either attempt:

Proposition (See Proposition 9.5.2 on p.143f.). $l_{4q+2}(\mathbb{Z}[\pi])$ cannot be embedded into a group.

Proposition (See Corollary 9.2.3 and 9.2.4 on p.137f.). $l^{2q}(\Lambda)/l_{elem}^{2q}(\Lambda)$ is not a group for a field or a principal ideal domain Λ .

1.4 The Contents

Chapter 2 gives an introduction into the topology and algebra of traditional and modified surgery theories. We will define forms, preformations and elementariness.

The next two chapters build up the foundation for the application of algebraic surgery theory to the study of preformations. In **Chapter 3** we translate preformations into quadratic complexes and pairs. We define algebraic versions of surgery and cobordism

$\text{rel}\partial$ and h -cobordisms for quadratic pairs. In **Chapter 4** we define the important concept of flip-isomorphisms and discuss how they fit into the algebraic chain models we constructed in the preceding chapter.

The following chapters discuss asymmetric and quadratic signatures in all generality. **Chapter 5** presents the theory of asymmetric forms and complexes and how one can define asymmetric signatures for Poincaré pairs that have an algebraic twisted double as a boundary. These general constructions are applied to the Poincaré pairs defined in Chapter 4 and produce the asymmetric signatures of flip-isomorphisms. **Chapter 6** deals with the definition of quadratic signatures for general preformations.

We continue with the treatment of special classes of preformations. **Chapter 7** covers the quadratic signatures for the easier case of non-singular formations. It also contains a proof for the fact their asymmetric signatures do not depend on the choice of flip-isomorphism. For preformations with linking forms the theory of asymmetric signatures becomes particularly elegant as will be seen in **Chapter 8**.

Finally, in **Chapter 9** we will compute $l_{2q+2}(\Lambda)$ for skew fields Λ and investigate the relation between the existence of flip-isomorphisms, vanishing asymmetric signatures and elementariness for preformations over fields. We will give examples of non-elementary preformations over certain fields that have flip-isomorphisms and vanishing asymmetric signatures. We also show that naïve ways of replacing the monoid $l_{2q+2}(\Lambda)$ by a group won't work in general.

The **Appendix A** contains a compilation of constructions and formulae from algebraic surgery theory.

Chapter 2

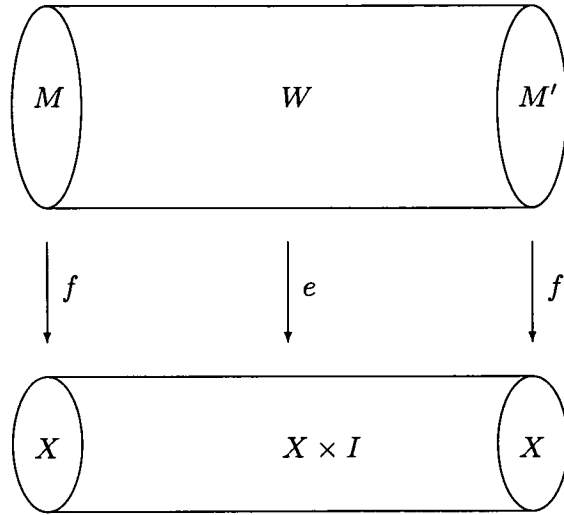
Preformations

Section 2.1 presents the algebraic and geometric concepts behind Kreck's surgery theory and its relation to traditional surgery theories. In **Section 2.2** we introduce the language of forms, formations and preformations - the building blocks of all our various surgery obstruction groups and monoids. Preformations are the objects that appear as obstructions in Kreck's surgery theory; its main theorem states that surgery succeeds in producing an s -cobordism if and only if that obstruction preformation has a certain property: stable elementariness. **Section 2.3** will present a heuristic way from topology to a definition of an elementary preformation. Then (**Section 2.4**) various equivalent definitions and some simple properties of that important concept are given, e.g. simple invariants which are obstructions to elementariness using cokernels and kernels of γ , μ and $\gamma^*\mu$ of a preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ (see Corollary 2.4.3).

An obstacle to transferring preformations the into algebraic surgery world in the next chapters is the fact that the module G in a preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ does not need to be free. By Corollary 2.4.7 any preformation can be replaced by a **regular** preformation (i.e. a preformation with *free* G).

2.1 Forms, Preformations and Formations: the Geometry

In this section we want to compare even- and odd-dimensional traditional surgery theory and Kreck's even-dimensional theory. We start with the **traditional even-dimensional theory** as developed by C.T.C. Wall and others. Let



$$(e, f, f'): (W, M, M') \rightarrow X \times (I, 0, 1)$$

be a $(2q + 2)$ -dimensional degree 1 normal cobordism with a finite $(2q + 2)$ -dimensional geometric Poincaré space¹ X such that f and f' are homotopy equivalences.

Our aim is to perform surgery on $W \text{ rel } \partial$ such that the result is an s -cobordism. Then, by the s -cobordism theorem, M and M' are diffeomorphic. Surgery theory works only in higher dimensions, hence we assume $q \geq 2$.

After having made e highly-connected by surgery below the middle dimension, we can define a $(-1)^{q+1}$ -dimensional quadratic form $(K_{q+1}(W), \lambda, \mu)$ with $K_{q+1}(W) = H_{q+2}(\tilde{e})$ the homology of the induced map of the universal covers of W and X with local coefficients and λ and ν the intersection and self-intersection numbers on W . This form is zero in the Witt group $L_{2q+2}(\mathbb{Z}[\pi_1(X)])$ of non-singular $(-)^{q+1}$ -quadratic forms over $\mathbb{Z}[\pi_1(X)]$ if and only if $e: W \rightarrow X \times I$ is cobordant $\text{rel } \partial$ to a homotopy equivalence, i.e. if and only if we can do surgery on the inside of W to obtain an s -cobordism. A quadratic form vanishes in the L -group if (after addition of hyperbolic forms) it has a lagrangian (i.e. a free direct summand of half dimension on which the quadratic form vanishes). If there is a lagrangian for $(K_{q+1}(W), \lambda, \mu)$ one simply kills its generators by surgery and the result will be an s -cobordism.

Before introducing Kreck's even-dimensional approach we have a look at the **traditional odd-dimensional case**. Let $(X, \partial X)$ be a finite $(2q + 1)$ -dimensional geometric Poincaré pair. Let N and N' be two $2q$ -dimensional manifolds such that $\partial M = N \cup N'$ and let $f: (M, \partial M) \rightarrow (X, \partial X)$ be a degree 1 normal map such that its restriction to the boundary $\partial M \rightarrow \partial X$ is a homotopy equivalence. Surgery below the middle-dimension makes $f \rightarrow X$ highly-connected. We are interested in the question whether N and N' are diffeomorphic, that is, whether $f: M \rightarrow X$ is cobordant $\text{rel } \partial$ to a

¹ A Poincaré space (or pair) is a topological space (or a pair of spaces) for which there exists a Poincaré (or Poincaré-Lefschetz) duality. A Poincaré space or pair is finite if it is a finite CW -complex. All closed manifolds are finite Poincaré spaces.

homotopy equivalence. One can construct an obstruction by looking at a so-called **presentation of f** . A presentation is a $(2q + 2)$ -dimensional normal cobordism

$$(e, f, f'): (W, M, M') \longrightarrow X \times (I, 0, 1)$$

such that e is $(q + 1)$ -connected and f and f' is q -connected. Such presentations exist for any such f with the above properties: one chooses a set of generators $\{x_1, \dots, x_k\}$ of the kernel module $K_q(M)$. Because we are below the middle dimension, the generators can be realized by disjoint framed embeddings $g_i: S^q \times D^{q+1} \longrightarrow M$. The trace W of the surgeries performed on them will be a presentation.

There is a purely algebraic way to test whether there's any presentation that contains a homotopy equivalence $f': M' \longrightarrow X$ on the other end (see also [Ran02] Chapter 12.2): Let U be the union of all the images of all g_i and $M_0 = \overline{M} \setminus U$. Such a decomposition is called a **Heegaard splitting** (see [Ran02] Definition 12.6). Then the self-intersection form on $K_q(\partial U) = K_q(\#_k S^q \times S^q)$ is the hyperbolic form on $\mathbb{Z}[\pi_1(X)]^{2k}$. Because $\partial M = \partial U$, the images of $K_{q+1}(U, \partial U)$ and $K_{q+1}(M_0, \partial U)$ in $K_q(\partial U)$ are lagrangians. A non-singular quadratic form with a pair of lagrangians is called a **non-singular $(-)^q$ -quadratic formation**. It turns out that the formation $(K_q(\partial U); K_{q+1}(U, \partial U), K_{q+1}(M_0, \partial U))$ provides enough data to decide our surgery problem.

We can read off the same information from our presentation. We define

$$\begin{aligned} F &= K_{q+1}(W, M) \cong K_{q+1}(U, \partial U) \\ G &= K_{q+1}(W) \cong K_{q+1}(M_0, U) \end{aligned}$$

Let $\gamma: G = K_{q+1}(W) \longrightarrow F = K_{q+1}(W, M)$ and $\mu: G = K_{q+1}(W) \longrightarrow K_{q+1}(W, M') \cong F^*$ be the maps induced by the long exact sequences of (W, M) and (W, M') and Poincaré-Lefschetz duality. Then $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix})$ is the inclusion of one lagrangian G and F is obviously another lagrangian of the hyperbolic form $H_{(-)^q}(F) \cong K_q(\partial U)$. Hence the obstruction formation can also be written as $(H_{(-)^q}(F); F, G) = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$. Different choices of presentations for $f: M \longrightarrow X$ change the formations by a **stable isomorphism**. A stable isomorphism class of formations is Wall's algebraic model for an odd-dimensional normal map. We will introduce this kind of isomorphism in Definition 2.2.14 as a **weak isomorphism**, since we need to distinguish it from another kind of isomorphism which we will encounter in the discussion of the modified theory later. A presentation yields a homotopy equivalence $f': M \longrightarrow X$ if and only if its obstruction is a **boundary** (see Definition 2.2.12). From this discussion equivalence relations can be derived which are used to define the algebraic odd-dimensional surgery obstruction groups $L_{2q+1}(\mathbb{Z}[\pi_1(X)])$. z vanishes in that L -group if and only if $f: M \longrightarrow X$ is cobordant rel ∂ to a homotopy equivalence.

In the 1980s Kreck generalized Wall's results. This thesis will only deal with his **even-dimensional modified theory** (compare [Kre99] p.724-732). Surprisingly, certain aspects of it resemble the traditional odd-dimensional theory we have just discussed. Kreck's theory starts off, not with normal maps, but the weaker notion of **normal smoothings**. Let $p: B \rightarrow BO$ be a fibration. A normal B -smoothing of a manifold is a factorization through B of the classifying map for its stable normal map. (Under certain connectivity assumptions the homotopy type of B actually depends only on M but we will not need this fact. See [Kre99] p. 711.) Assume that there is a $(2q + 2)$ -dimensional cobordism $(e, f, f'): (W, M, M') \rightarrow B$ of normal smoothings such that f and f' are q -equivalences.

If B is a finite geometric Poincaré space and if $p: B \rightarrow BO$ is its Spivak bundle, a normal B -smoothing is nothing but a normal map. In addition, if f and f' are homotopy equivalences, the situation is exactly the one of Wall's even-dimensional case. The geometrical input of the modified case is considerably weaker than in Wall's original theory. There we started by comparing the complete homotopy and normal bundle information of M and M' whereas in the modified theory only "half" of that information is needed.

Surgery below the middle dimension yields a $(q + 1)$ -connected map $e': W' \rightarrow B$. Now we can read off the obstruction which is a tuple $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ with $G := \ker(e'_*: \pi_{q+1}(W') \rightarrow \pi_{q+1}(B))$, $\bar{\theta}: G \rightarrow Q_{(-)^q}(\mathbb{Z}[\pi_1(B)])$ the self-intersection form on W' , $F := H_q(W', M')$ and γ and μ are the compositions of the obvious maps in homology and the Hurewicz homomorphism. Note that $\bar{\theta} + (-)^{q+1}\bar{\theta}^* = \gamma^*\mu$. Such a tuple $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ with $(G, \gamma^*\mu, \bar{\theta})$ a \pm -quadratic form will be called a **preformation**.

There is of course an ambiguity as there may be many ways to make $e: W \rightarrow B$ highly-connected. But the resulting manifolds W' will only differ by a couple of tori $S^{q+1} \times S^{q+1}$ (see [Kre99] p.729) and algebraically, the obstruction preformations will differ only by hyperbolic elements

$$(P \oplus P^* \xleftarrow{1} P \oplus P^* \begin{pmatrix} 0 & 1 \\ (-)^{q+1} & 0 \end{pmatrix} P \oplus P^{**}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$$

There is also a notion of isomorphism namely **strong isomorphisms** (see Definition 2.2.13). If $(W, M, M') \rightarrow B$ is changed by a diffeomorphism compatible with the normal B -smoothings. the obstructions will change by such a strong isomorphism. The stable strong isomorphism classes (using hyperbolic preformations for stabilization) define the monoid $l_{2q+2}(\mathbb{Z}[\pi_1(B)])$. $e: W \rightarrow X$ is cobordant rel ∂ to an s -cobordism if and only if the class of the obstruction preformation in the l -monoid is **elementary**. We will discuss elementariness later in Sections 2.3 and 2.4.

The modified theory appeals by its ability to digest much simpler geometrical input than the traditional case:

- i) The normal maps on M and M' are not necessarily homotopy equivalences but only q -equivalences.
- ii) The space B with which we compare our cobordism does not need to be a finite Poincaré space.

However, these generalizations come with a price tag: the difficult algebra. In fact, very little is known about the algebraic structure of those l -monoids compared to the extensive literature that exists about the L -groups.

There is a striking similarity between the obstructions of the modified even-dimensional case and the traditional odd-dimensional case. In both cases one studies a cobordism $(e, f, f'): (W, M, M') \rightarrow X$ of highly-connected normal maps/smoothings such that $H_i(W, M) = H_i(W, M') = 0$ for $i \neq q + 1$. To a certain degree the discussion about formations can be extended to preformations. In both cases $(\gamma_\mu): (G, 0) \rightarrow H_{(-)q}(F)$ defines a map from a zero form to a hyperbolic form.

The situation in the even-dimensional modified theory is of course more general: X is not necessarily a finite Poincaré complex, the maps involved don't need to be normal maps and $\partial M \rightarrow X$ might not be a homotopy equivalence. Therefore the map (γ_μ) is not always an inclusion of a lagrangian.

There are more differences if one looks at the equivalence relations in the obstruction groups/monoids. In the modified even-dimensional case, the equivalence relations for the l -monoids are very strict, because they seek to preserve all algebraic data of the whole cobordism $(e, f, f'): (W, M, M') \rightarrow X$. On the other hand, in the traditional odd-dimensional case that cobordism is just used to write down the obstruction data of $f: M \rightarrow X$. It is not important which cobordism is chosen if only it is highly-connected. The equivalence relations for $L_{2q+1}(\Lambda)$ are designed such that only the surgery-relevant information of $f: M \rightarrow X$ is retained.

Hence, philosophically, a preformation can be interpreted as an algebraic model for a cobordism $(e, f, f'): (W, M, M') \rightarrow X$ or as a model for the map $f: M \rightarrow X$ only. In the first case, one identifies preformations via strong isomorphisms (the ones used to define the l -monoids) in the second case one uses weak isomorphisms (the ones used to define the odd-dimensional L -groups). We will come back to this issue when we introduce flip-isomorphisms in Chapter 4.

In any case, the similarity of the objects in odd-dimensional L -theory and l -monoids enable us to use algebraic surgery theory to investigate $l_{2q}(\Lambda)$. There are standard ways of translating quadratic and symmetric complexes into forms and formations and vice versa. These procedures extend to (regular) preformations as we will see in Chapter 3.

2.2 Forms, Preformations and Formations: the Algebra

We begin by defining a class of rings for which the rank of any f.g. free module is well-defined. These rings are called **weakly finite**. All the rings which we are interested in (like fields, principal ideal domains, group rings, etc.) have this property.

Definition and Lemma 2.2.1 ([Coh89] p.143). A ring Λ is **weakly finite** if for any $n \in \mathbb{N}_0$ and Λ -module K , $\Lambda^n \cong \Lambda^n \oplus K$ implies $K = 0$. \square

Proposition 2.2.2. *i) All commutative rings, artinian rings, noetherian rings, sub-rings of weakly finite rings, skew fields and group rings $\mathbb{C}[\pi]$ and $\mathbb{Z}[\pi]$ are weakly finite.*

ii) All free modules over weakly finite rings have a well-defined rank.

iii) Let $f, g : \Lambda^n \rightarrow \Lambda^n$ with $fg = 1_{\Lambda^n}$ and let Λ be weakly finite. Then $gf = 1_{\Lambda^n}$ i.e. f and g are isomorphisms.

iv) If $f : \Lambda^n \rightarrow \Lambda^n$ is surjective or a split injection then f is an isomorphism.

*v) Let A be an $r \times r$ -matrix s.t. $\begin{pmatrix} * & 0 \\ * & A \end{pmatrix}$ is an isomorphism. Then A is an isomorphism.*

Proof. For i) - iii) see [Coh89] pp.143-4 and [Mon69]. iv) is very easy to show. There is an exact sequence

$$0 \rightarrow \ker f \rightarrow \Lambda^n \xrightarrow{f} \Lambda^n \rightarrow 0$$

which splits because the last module is free. Then $\ker f \oplus \Lambda^n \cong \Lambda^n$. Hence the kernel is trivial and f injective. A similar argument covers the case of a split injection. The last statement follows from the fact that A^* must be an epimorphism, hence it is an isomorphism and so is A . \square

Let $\epsilon = (-)^q$. (All constructions would equally work for $\epsilon \in \Lambda$ such that $\epsilon^{-1} = \bar{\epsilon}$.) Let Λ be a weakly finite ring with 1 and an involution $x \mapsto \bar{x}$ (i.e. an anti-automorphism $\Lambda \xrightarrow{\cong} \Lambda^{op}$, $x \mapsto \bar{x}$ with Λ^{op} the opposite ring). All Λ -modules are left modules.

Lemma 2.2.3. *For a projective Λ -module M the map $M \rightarrow M^{**}$, $x \mapsto (f \mapsto \overline{f(x)})$ is an isomorphism of left Λ -modules.*

Remark 2.2.4. In this section all surgery obstruction groups and monoids of the various surgery theories presented before are defined. Strictly speaking, if we want to decide whether or whether not a cobordism can be turned into an s -cobordism all the equivalence relations below must only use **simple isomorphisms** (i.e. isomorphisms for which the torsion in the Whitehead group vanishes). We will ignore this condition in

this thesis, hence the results will only deal with h -cobordisms instead of s -cobordisms. A careful analysis of the proofs and constructions given in this thesis, will certainly lead to similar results for the simple l -monoids.

In the cases $\Lambda = \mathbb{Z}[\mathbb{Z}^m]$ and $\Lambda = \mathbb{Z}$ the Whitehead groups vanish and every h -cobordism is an s -cobordism.

Forms and even-dimensional L -groups

Definition 2.2.5. i) Let M be a Λ -module. Using the canonical homomorphism $M \longrightarrow M^{**}$ of Lemma 2.2.3 we define the ϵ -**duality involution map**

$$\begin{aligned} T_\epsilon: \text{Hom}_\Lambda(M, M^*) &\longrightarrow \text{Hom}_\Lambda(M, M^*) \\ \phi &\longmapsto (\epsilon\phi^*: x \longmapsto (y \longmapsto \overline{\epsilon\phi(y)(x)})) \end{aligned}$$

and the abelian groups

$$\begin{aligned} Q^\epsilon(M) &= \ker(1 - T_\epsilon) & Q_\epsilon(M) &= \text{coker}(1 - T_\epsilon) \\ Q^\epsilon(\Lambda) &= \{b \in \Lambda \mid b = \epsilon\bar{b}\} & Q_\epsilon(\Lambda) &= \Lambda / \{b - \epsilon\bar{b} \mid b \in \Lambda\} \end{aligned}$$

ii) An ϵ -**symmetric form** (M, ϕ) **over** Λ is a Λ -module M together with a $\phi \in Q^\epsilon(M)$. It is **non-singular** if $\phi: M \longrightarrow M^*$ is an isomorphism of Λ -modules.

iii) A **sublagrangian** L of an ϵ -symmetric form (M, ϕ) is a direct summand $j: L \hookrightarrow M$ such that $j^*\phi j = 0$. Then the **annihilator** $L^\perp = \ker(j^*\phi: M \longrightarrow L^*)$ contains L . L is a **lagrangian** if $L = L^\perp$. A form that allows a lagrangian is called **metabolic**.

iv) An ϵ -**quadratic form** (M, λ, ν) **over** Λ is an ϵ -symmetric form (M, λ) together with a map $\nu: M \longrightarrow Q_\epsilon(\Lambda)$ such that for all $x, y \in M$ and $a \in \Lambda$

$$(a) \quad \nu(x + y) - \nu(x) - \nu(y) = \lambda(x, y) \in Q_\epsilon(\Lambda)$$

$$(b) \quad \nu(x) + \overline{\epsilon\nu(x)} = \lambda(x, x) \in Q^\epsilon(\Lambda)$$

$$(c) \quad \nu(ax) = a\nu(x)\bar{a} \in Q_\epsilon(\Lambda)$$

ν is called a **quadratic refinement** of the ϵ -symmetric form (M, λ) . It is **non-singular** if the underlying symmetric form is non-singular.

v) A **sublagrangian** L of an ϵ -quadratic form (M, λ, ν) is a direct summand $j: L \hookrightarrow M$ such that $j^*\lambda j = 0$ and $\nu j = 0$. Then the annihilator L^\perp of the underlying ϵ -symmetric form (M, λ) contains L . L is a **lagrangian** if $L = L^\perp$. A form which allows a lagrangian is called **metabolic**.

vi) A **morphism** $f: (M, \lambda) \longrightarrow (M', \lambda')$ of ϵ -symmetric forms is a map $f \in \text{Hom}_\Lambda(M, M')$ such that $f^* \lambda' f = \lambda$. It is an **isomorphism** if $f: M \longrightarrow M'$ is an isomorphism of Λ -modules.

A **morphism** $f: (M, \lambda, \mu) \longrightarrow (M', \lambda', \mu')$ of ϵ -quadratic forms is a morphism $f: (M, \lambda) \longrightarrow (M', \lambda')$ of ϵ -symmetric forms such that $\mu' f = \mu$. It is an **isomorphism** if $f: M \longrightarrow M'$ is an isomorphism of Λ -modules. \square

Remark 2.2.6. For a f.g. projective Λ -module M there is no difference between the definition of ϵ -quadratic forms and the following alternative (see e.g. [Ran80a] p.117ff): An ϵ -quadratic form (M, ψ) over Λ is a tuple consisting of a f.g. projective Λ -module M together with an element $\psi \in Q_\epsilon(M)$. It is **non-singular** if $(1 + T_\epsilon)\psi: M \longrightarrow M^*$ is an isomorphism of Λ -modules.

A **sublagrangian** L of an ϵ -quadratic form (M, ψ) is a direct summand $j: L \hookrightarrow M$ such that $j^* \psi j = 0 \in Q_\epsilon(L)$. Then the **annihilator** $L^\perp = \ker(j^*(1 + T_\epsilon)\psi: M \longrightarrow L^*)$ contains L . L is a **lagrangian** if $L = L^\perp$. A form which allows a lagrangian is called **metabolic**.

A **morphism** $f: (M, \psi) \longrightarrow (M', \psi')$ is a map $f \in \text{Hom}_\Lambda(M, M')$ such that $f^* \psi' f = \psi \in Q_\epsilon(M)$. It is an **isomorphism** if $f: M \longrightarrow M'$ is an isomorphism of Λ -modules. An ϵ -quadratic form $(M, \psi \in Q^\epsilon(M))$ defines an ϵ -quadratic form $(M, (1 + T_\epsilon)\psi, \nu)$ with $\nu(x) = \psi(x)(x)$. Conversely any ϵ -quadratic form (M, λ, ν) gives rise to an ϵ -quadratic form $(M, \psi \in Q^\epsilon(M))$ (see [Ran02] Proposition 11.9).

If $\frac{1}{2} \in \Lambda$ then quadratic and symmetric forms are the same. More generally:

Proposition 2.2.7. *Assume there is a central $s \in \Lambda$ such that $s + \bar{s} = 1$. Then there is a one-to-one correspondence between ϵ -quadratic and ϵ -symmetric forms over f.g. free (or projective) Λ -modules given by $(G, \theta \in Q_\epsilon(G)) \longmapsto (G, (1 + T_\epsilon)\theta \in Q^\epsilon(G))$. Its inverse is $(G, \lambda \in Q^\epsilon(G)) \longmapsto (G, [s\lambda] \in Q_\epsilon(G))$.*

It can be shown ([Ran80a] Proposition 2.2) that any metabolic form is isometric to a hyperbolic form.

Definition 2.2.8. For any $(-\epsilon)$ -symmetric form (L^*, ϕ) over a f.g. free Λ -module L we define the non-singular **hyperbolic ϵ -symmetric form**

$$H^\epsilon(L, \phi) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \epsilon & \phi \end{pmatrix} \in Q_\epsilon(L \oplus L^*) \right)$$

We abbreviate $H^\epsilon(L) = H^\epsilon(L, 0)$.

For any f.g. free Λ -module L we define the non-singular **quadratic hyperbolic ϵ -quadratic form**

$$H_\epsilon(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_\epsilon(L \oplus L^*) \right)$$

\square

Definition and Lemma 2.2.9. The even-dimensional quadratic L -group $L_{2q}(\Lambda)$ is the set of equivalence class of all non-singular ϵ -quadratic forms on f.g. free modules over Λ where two forms are equivalent if they are isometric up to the addition of metabolic (i.e. hyperbolic) forms. It is also called the **Witt-group of quadratic forms**.

Similarly we can define the even-dimensional symmetric L -group $L^{2q}(\Lambda)$. \square

Preformations and even-dimensional l -monoids

The building blocks of the even-dimensional l -monoids are **preformations**. A special case are **formations** which help to define the odd-dimensional L -groups (see below). We also introduce the notion of **regular preformations**. They are preformations where all modules involved are f.g. free. Only they can be fed into the algebraic surgery machine which we will present in later chapters. Corollary 2.4.7 shows that the restriction to regular preformations is not a serious limitation of the scope of our theory.

Definition 2.2.10. i) An ϵ -quadratic preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is a tuple with

- (a) a free f.g. Λ -module F ,
- (b) a f.g. Λ -module G ,
- (c) $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}) \in \text{Hom}_{\Lambda}(G, F \oplus F^*)$ such that $(G, \gamma^* \mu)$ is a $(-\epsilon)$ -symmetric form.

An ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is an ϵ -quadratic preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and a map $\bar{\theta}: G \rightarrow Q_{-\epsilon}(\Lambda)$ such that $(G, \gamma^* \mu, \bar{\theta})$ is a $(-\epsilon)$ -quadratic form.

- ii) An ϵ -quadratic preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is called **regular** if G is free. An ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is regular if G is a free. In that case we interpret $\bar{\theta} \in Q_{-\epsilon}(G)$ as in Remark 2.2.6.

An ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is called an ϵ -quadratic **split formation** if $(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}) G$ is a sublagrangian of the ϵ -quadratic hyperbolic form $H_{\epsilon}(F)$. It is called **non-singular** if the sublagrangian is indeed a lagrangian. Similar for the non-split case. \square

Remark 2.2.11. In Andrew Ranicki's work the notation for ϵ -quadratic formations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is $(H_{\epsilon}(F), F, G)$ and for ϵ -quadratic split formations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ it is $(F, ((\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}), \bar{\theta}) G)$.

In an obvious way all symmetric and quadratic forms are non-singular formations:

- Definition 2.2.12.** i) Let (K, λ) be a $(-\epsilon)$ -symmetric form on a free f.g. Λ -module K . Then the **boundary of (K, λ)** is the non-singular ϵ -quadratic formation $\partial(K, \lambda) = (K \xleftarrow{1_K} K \xrightarrow{\lambda} K^*)$.
- ii) Let (K, θ) be a $(-\epsilon)$ -quadratic form on a free f.g. Λ -module K . Then the **boundary of (K, θ)** is the non-singular ϵ -quadratic split formation $\partial(K, \theta) = (K \xleftarrow{1_K} K \xrightarrow{\theta - \epsilon \theta^*} K^*, \theta)$.
- iii) A **trivial formation** is a non-singular ϵ -quadratic split formation of the form $(P, P^*) = (P \xleftarrow{0} P \xrightarrow{1} P^*, 0)$ with P a free f.g. Λ -module. Similar for the non-split case. \square

Now we define strong isomorphisms and stable strong isomorphism for preformations.

- Definition 2.2.13.** i) The **sum of two ϵ -quadratic split preformations** $x = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $x' = (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \bar{\theta}')$ is the ϵ -quadratic split preformations

$$x + x' := ((F \oplus F') \xleftarrow{\gamma \oplus \gamma'} G \oplus G' \xrightarrow{\mu \oplus \mu'} (F \oplus F')^*, \bar{\theta} \oplus \bar{\theta}').$$

Similar for the non-split case.

- ii) A **strong isomorphism of two ϵ -quadratic split preformations** $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \bar{\theta}')$ is a tuple (α, β) of isomorphisms $\alpha \in \text{Hom}_\Lambda(F, F')$ and $\beta \in \text{Hom}_\Lambda(G, G')$ such that

$$\begin{array}{ccccc} F & \xleftarrow{\gamma} & G & \xrightarrow{\mu} & F^* \\ \cong \downarrow \alpha & & \cong \downarrow \beta & & \cong \downarrow \alpha^{-*} \\ F' & \xleftarrow{\gamma'} & G' & \xrightarrow{\mu'} & F'^* \end{array} \quad (2.1)$$

commutes and $\bar{\theta} = \bar{\theta}'\beta$.

- iii) Two (regular) ϵ -quadratic split preformations x and x' are **stably strongly isomorphic** if there are boundaries of hyperbolic forms h and h' such that there is a strong isomorphism between $x + h$ and $x' + h'$. Similar for the non-split case.
- iv) **The l -monoid $l'_{2q+2}(\Lambda)$** is the set of stably strongly isomorphism classes of ϵ -quadratic split preformations. The “simple” version of this monoid is Kreck’s original l -monoid.

The l -monoid $l_{2q+2}(\Lambda)$ is the set of stably strongly isomorphism classes of regular ϵ -quadratic split preformations.

The l -monoid $l'^{2q+2}(\Lambda)$ is the set of stably strongly isomorphism classes of ϵ -quadratic preformations.

The l -monoid $l^{2q+2}(\Lambda)$ is the set of stably strongly isomorphism classes of regular ϵ -quadratic preformations.

All l -monoids are abelian monoids with zero. □

Formations and odd-dimensional L -groups

In Section 2.1 we explained that odd-dimensional traditional surgery theory and the modified even-dimensional case use similar obstructions but that the equivalence relations used in the construction of the obstruction groups/monoids are very different. In both cases the obstruction associated to a $(2q + 2)$ -dimensional cobordism $(e, f, f'): (W, M, M') \longrightarrow X \times (I, 0, 1)$ of normal maps/smoothings with f and f' and e highly-connected is some ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ with $F = H_{q+1}(W, M)$ and $(G, \gamma^* \mu, \bar{\theta})$ containing the self-intersection form on some homology or homotopy group related to e .

In Kreck's surgery theory this cobordism is the very surgery problem in question. The equivalence relations (strong isomorphism, stabilization with boundaries of hyperbolic forms) are very rigid and preserve the important data of the whole cobordism.

In contrast, traditional odd-dimensional surgery theory uses the cobordism

$$(e, f, f'): (W, M, M') \longrightarrow X \times (I, 0, 1)$$

just as a prop to define an obstruction to the odd-dimensional surgery problem given by $f: M \longrightarrow X$. Hence the equivalence relations we present below are much more flexible - they need to filter out ambiguities which arise by the choice of a different presentation i.e. another $(2q + 2)$ -dimensional normal cobordism

$$(\hat{e}, \hat{f}, \hat{f}'): (\hat{W}, M, \hat{M}') \longrightarrow X \times (I, 0, 1)$$

with \hat{e} and \hat{f}' highly-connected.

This leads to the unfortunate situation that there exist two notions of (stable) isomorphisms for preformations. Stable strong isomorphism classes of preformations are algebraic models for diffeomorphism classes of $(2q + 2)$ -dimensional cobordisms $(e, f, f'): (W, M, M') \longrightarrow X \times (I, 0, 1)$ whereas stable weak isomorphism classes are models for diffeomorphism classes of $(2q + 1)$ -dimensional normal maps $f: M \longrightarrow X$. Weak isomorphisms will reappear in Chapter 4 which deals with **flip-isomorphisms**. We also have to be careful about stabilization. In the l -monoids we stabilize with "hyperbolics" (i.e. boundaries of hyperbolics) and in L -theory we use trivial preformations.

Definition 2.2.14. i) A (weak) isomorphism (α, β, σ) of two regular ϵ -quadratic preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*)$ is a triple consisting of an isomorphism $\alpha \in \text{Hom}_\Lambda(F, F')$, an isomorphism $\beta \in \text{Hom}_\Lambda(G, G')$ and $\sigma \in Q^{-\epsilon}(F^*)$ such that

$$(a) \alpha\gamma + \alpha\sigma\mu = \gamma'\beta \in \text{Hom}_\Lambda(G, F')$$

$$(b) \alpha^{-*}\mu = \mu'\beta \in \text{Hom}_\Lambda(G, F'^*).$$

ii) A **(weak) isomorphism** (α, β, ν) of two regular ϵ -quadratic split preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \bar{\theta}')$ is a triple consisting of an isomorphism $\alpha \in \text{Hom}_\Lambda(F, F')$, an isomorphism $\beta \in \text{Hom}_\Lambda(G, G')$ and $\nu \in Q_{-\epsilon}(F^*)$ such that

$$(a) \alpha\gamma + \alpha(\nu - \epsilon\nu^*)\mu = \gamma'\beta \in \text{Hom}_\Lambda(G, F')$$

$$(b) \alpha^{-*}\mu = \mu'\beta \in \text{Hom}_\Lambda(G, F'^*)$$

$$(c) \bar{\theta} + \mu^*\nu\mu = \beta^*\bar{\theta}'\beta \in Q_{-\epsilon}(G)$$

iii) A **stable weak isomorphism** of two regular ϵ -quadratic split preformations z and z' is a weak isomorphism $z + t \cong z' + t'$ for trivial formations t, t' . \square

An odd-dimensional normal map is cobordant to a homotopy-equivalence if and only if its obstruction (pre-)formation is stably isomorphic to a boundary. It can be shown that for any form there is another cobordant odd-dimensional map whose obstruction preformation differs from the original one by the boundary of a form (see [Ran02] Proposition 12.13, Theorem 12.29). This motivates the definition of the odd-dimensional surgery obstruction groups.

Definition and Lemma 2.2.15 ([Ran02] Definition 12.23, Proposition 12.33).

We call two non-singular ϵ -quadratic (split) formations z and z' equivalent if there is a stable weak isomorphism between $z + b$ and $z' + b'$ for some boundaries b and b' . In both cases (split and not-split) the equivalence classes form the **odd-dimensional L -groups** $L_{2q+1}(\Lambda)$. \square

Remark 2.2.16. There are also odd-dimensional symmetric L -groups which are defined as the Witt-group of ϵ -symmetric formations (see [Ran80a] Chapter 5).

Remark 2.2.17. i) Any strong isomorphism between regular preformations is also a weak isomorphism.

ii) Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $z' = (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \bar{\theta}')$ be regular ϵ -quadratic split preformations and $t = (\alpha, \beta, \nu): z \rightarrow z'$ a weak isomorphism. Then $(\alpha, \beta, (\nu - \epsilon\nu^*)^*)$ is a weak isomorphism of the underlying regular ϵ -quadratic preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*)$.

iii) If z is a formation, weak isomorphisms are nothing but isomorphisms of quadratic (split) formations as defined in [Ran80a] p.122 and p.128. For example,

an isomorphism (α, β, σ) of ϵ -quadratic formations $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $z' = (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F')$ is an isomorphism of ϵ -quadratic hyperbolic forms

$$\begin{pmatrix} \alpha & \alpha\sigma \\ 0 & \alpha^{-*} \end{pmatrix} : H^\epsilon(F) \xrightarrow{\cong} H^\epsilon(F')$$

which maps the (sub-)lagrangians F and G onto F' and G' respectively.

- iv) Despite the different ways of stabilizing, every stable strong isomorphism is also a stable weak isomorphism. That's because there is a weak isomorphism between an even-dimensional trivial formation and a boundary of hyperbolic forms:

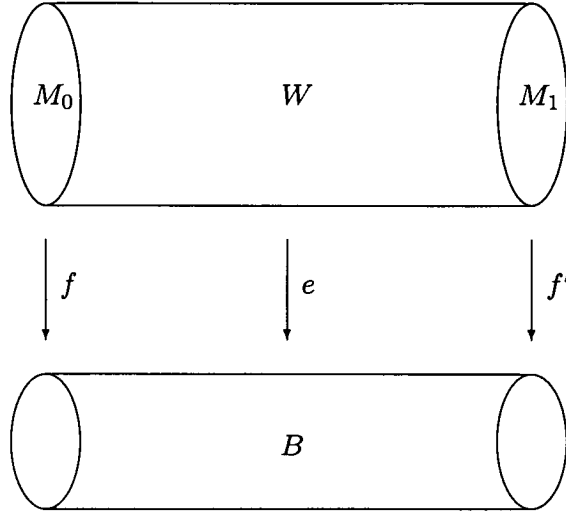
$$\left(1, \begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix} \right) : \partial H_\epsilon(P) \longrightarrow (P \oplus P^*, (P \oplus P^*)^*)$$

The converse is not true: Let Q be a free Λ -module of rank 1. Let $y = (Q, Q^*)$ and $z = \partial H_{-\epsilon}(Q)$. By the above, both preformations are stably weakly isomorphic, but for rank reasons they cannot be stably strongly isomorphic.

2.3 Elementariness: the Geometry

We haven't quite explained yet how the obstruction preformation in Kreck's surgery theory can tell us whether surgery is able to turn our cobordism into an h -cobordism. Again let $q \geq 2$. In the case of the **traditional even-dimensional case** we only have to check that the obstruction $(K_{q+1}(W), \lambda, \mu)$ of a highly-connected $(2q + 2)$ -dimensional normal cobordism $(W, M, M') \longrightarrow X \times (I, 0, 1)$ is zero in the Witt-group $L_{2q+2}(\mathbb{Z}[\pi_1(X)])$. Then we know that there is a lagrangian L of $K_{q+1}(W) \oplus H_{(-)q+1}(K)$. The stable lagrangian L is a recipe for successful surgery: we perform $\text{rk } K$ trivial surgeries on W (with the result $W \#_{\text{rk } K} S^{q+1} \times S^{q+1}$) and then kill a basis of L via surgery. The result will be an h -cobordism.

In the modified case the criteria for success or failure are more complicated. The starting point of Kreck's **modified surgery theory** is the situation we described on p.21f: Let $p: B \rightarrow BO$ be a fibration. Let M_0 and M_1 be $(2q + 1)$ -dimensional manifolds with $(q-1)$ -smoothings f and f' in B . Let $f: \partial M_0 \xrightarrow{\cong} \partial M_1$ be a diffeomorphism compatible with the smoothings. There is a normal smooth cobordism i.e.



$$(e, f, f'): (W, M_0, M_1) \longrightarrow B$$

Surgery below the middle dimension on W is possible and yields a q -smoothing $e': W' \rightarrow B$. Then we can define a $(-)^{q+1}$ -quadratic split preformation.

Definition 2.3.1.

$$\begin{aligned} y(W) &= (F \xleftarrow{\sigma} H \xrightarrow{\tau} F^*, \psi) \\ &= (H_{q+1}(W', M_0) \leftarrow \text{im}(d: \pi_{q+2}(B, W') \rightarrow \pi_{q+1}(W')) \rightarrow H_{q+1}(W', M_1), \psi) \\ &\in l'_{2q+2}(\mathbb{Z}[\pi_1(X)]) \end{aligned}$$

is the Kreck surgery obstruction of W . □

It turns out that killing low-dimensional homotopy classes by surgery in a different manner will not change the class $y(W) \in l'_{2q+2}(\Lambda)$. The obstruction contains all the data to find out whether W can be made an h -cobordism due to the main theorem:

Theorem 2.3.2 ([Kre99] Theorem 3 and Remark p.730ff). *W is B -cobordant rel ∂ to an h -cobordism if and only if $y(W) \in l'_{2q+2}(\Lambda)$ is elementary.*

Before we give a strict definition of **elementariness** (see the next section), a heuristical argument provides some geometric motivation for this new concept. We will later show (Corollary 2.4.6) that in Definition 2.3.1 $\text{im}(d: \pi_{q+2}(B, W') \rightarrow \pi_{q+1}(W'))$ can be replaced by $H_{q+2}(B, W')$. Without loss of generality we assume that $W \rightarrow B$ is $(q+1)$ -connected and $W = W'$.

From the long exact sequences for (B, W, M_i) we learn that $H_j(W, M_i) = 0$ for $j \leq q$, $i = 0, 1$ and that there is an exact sequence

$$0 \longrightarrow H_{q+2}(B, M_i) \longrightarrow H_{q+2}(B, W) \longrightarrow H_{q+1}(W, M_i) \longrightarrow H_{q+1}(B, M_i) \longrightarrow 0$$

and that the canonical maps $H_j(B, M_i) \longrightarrow H_j(B, W)$ are isomorphisms for $j \geq q + 3$. Then a (possibly non-free) chain complex model of the cobordism $(W, M_0, M_1) \longrightarrow B$ looks like

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 \downarrow & & 0 \downarrow & & 0 \downarrow \\
C_j(B, M_0) = H_j(B, M_0) & \xrightarrow{\cong} & C_j(B, W) = H_j(B, W) & \xleftarrow{\cong} & C_j(B, M_1) = H_j(B, M_1) \\
0 \downarrow & & 0 \downarrow & & 0 \downarrow \\
\vdots & & \vdots & & \vdots \\
0 \downarrow & & 0 \downarrow & & 0 \downarrow \\
C_{q+2}(B, M_0) = G & \xrightarrow{1} & C_{q+2}(B, W) = G & \xleftarrow{1} & C_{q+2}(B, M_1) = G \\
\gamma \downarrow & & & & \mu \downarrow \\
C_{q+1}(B, M_0) = F & & & & C_{q+1}(B, M_1) = F^*
\end{array}$$

Assume it is possible to do simultaneous surgery on W killing some homology classes $x_1, \dots, x_k \in H_{q+1}(W)$ without changing the boundary. Assume further that they are the basis of a f.g. free submodule $j: U = \langle x_1, \dots, x_k \rangle \hookrightarrow H_{q+1}(W)$. A chain complex model for the resulting cobordism $(V, M_0, M_1) \longrightarrow B$ is

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 \downarrow & & 0 \downarrow & & 0 \downarrow \\
C_{q+3}(B, V) = U & & & & \\
j \downarrow & & & & \\
C_{q+2}(B, M_0) = G & \xrightarrow{1} & C_{q+2}(B, V) = G & \xleftarrow{1} & C_{q+2}(B, M_1) = G \\
\gamma \downarrow & & j^* \gamma^* \mu \downarrow & & \mu \downarrow \\
C_{q+1}(B, M_0) = F & \xrightarrow{\pm j^* \mu^*} & C_{q+1}(B, V) = U^* & \xleftarrow{j^* \gamma^*} & C_{q+1}(B, M_1) = F^*
\end{array}$$

(compare with the proof of Theorem 3.3.3). We observe that the relative middle-dimensional homology groups of the new cobordism are

$$\begin{aligned}
H_{q+1}(V, M_0) &= \frac{\text{im}(j^* \gamma^*)}{\ker(\mu j)} \\
H_{q+1}(V, M_1) &= \frac{\text{im}(j^* \mu^*)}{\ker(\gamma j)}
\end{aligned}$$

Using Poincaré-Lefschetz duality this means that (V, M_0, M_1) is an h -cobordism if and only if these homologies vanish or, equivalently, the mapping cones of either map $C(B, M_i) \longrightarrow C(B, V)$ (that is

$$0 \longrightarrow U \xrightarrow{\gamma j} F \xrightarrow{(\mu j)^*} U^* \longrightarrow 0$$

and its dual) are exact sequences. This is in fact one way of defining that $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is elementary (see Proposition 2.4.2iii).

2.4 Elementariness: the Algebra

First we will repeat the original definition of an elementary preformation before we present alternative ways of looking at this concept.

Definition 2.4.1 ([Kre99] p.730). An ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is **elementary (in respect to U)** if there is a f.g. free submodule $j: U \hookrightarrow G$ with

- i) $j^* \gamma^* \mu j = 0$ and $\bar{\theta} j = 0$,
- ii) γj and μj are injective and their images U_0 and U_1 are direct summands in F and F^* respectively,
- iii) $R_1 = F^*/U_1 \rightarrow U_0^*$, $f \mapsto f|_{U_0}$ is an isomorphism.

Such an U is called an **h -lagrangian** of the preformation.

An **element** in $l'_{2q+2}(\Lambda)$ is **elementary** if it has an elementary representative. All elementary elements form a submonoid $l'^{el}_{2q+2}(\Lambda)$ of $l'_{2q+2}(\Lambda)$.

An ϵ -quadratic split preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is **stably elementary** if $[z] \in l'_{2q+2}(\Lambda)$ is elementary.

Similar for non-split and regular preformations. □

Proposition 2.4.2. *Let $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an ϵ -quadratic split preformation. Then the following statements are equivalent:*

- i) *The preformation is elementary in respect to U .*
- ii) *There is a f.g. free submodule $j: U \hookrightarrow G$ such that*

$$0 \longrightarrow U \xrightarrow{\mu j} F^* \xrightarrow{(\gamma j)^*} U^* \longrightarrow 0$$

is an exact sequence and $\bar{\theta}|_U = 0$.

- iii) *There is a f.g. free submodule $j: U \hookrightarrow G$ such that the two horizontal chain maps*

$$\begin{array}{ccccc}
 & & U & & \\
 & & \downarrow j & & \\
 G & \xrightarrow{1} & G & \xleftarrow{1} & G \\
 \gamma \downarrow & & \downarrow j^* \gamma^* \mu & & \downarrow \mu \\
 F & \xrightarrow{-\epsilon j^* \mu^*} & U^* & \xleftarrow{j^* \gamma^*} & F^*
 \end{array}$$

are chain equivalences (i.e. this is a “chain complex model of an h -cobordism”) and $\bar{\theta}|_U = 0$.

iv) The preformation is strongly isomorphic to a preformation of the form

$$(U \oplus U^* \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \leftarrow U \oplus R \begin{pmatrix} 0 & -\epsilon\sigma \\ 1 & \tau \end{pmatrix} \rightarrow U^* \oplus U, \bar{\theta})$$

for some maps $\sigma: R \rightarrow U^*$ and $\tau: R \rightarrow U$ such that $\tau^*\sigma = -\epsilon\sigma^*\tau$ and a quadratic refinement $\theta': R \rightarrow Q_{-\epsilon}(\Lambda)$ of $\sigma^*\tau$ such that

$$\begin{aligned} \bar{\theta}: U \oplus R &\longrightarrow Q_{-\epsilon}(\Lambda) \\ (u, r) &\longmapsto \theta'(r) - \epsilon\sigma(r)(u) \end{aligned}$$

Similar for the non-split case.

Proof. The first two assertions are obviously equivalent. One observes that

$$0 \longrightarrow U \xrightarrow{\mu_j} F^* \xrightarrow{(\gamma_j)^*} U^* \longrightarrow 0$$

and its dual are the mapping cones of the chain maps in iii). So iii) is equivalent to ii). Finally, we concentrate on iv). Every preformation of the form described in there is elementary in respect to U . On the other hand let $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be an elementary ϵ -quadratic preformation. First we show that U splits, in particular $G = U \oplus \ker(\pi\gamma)$ with an arbitrary projection $\pi: F \rightarrow U_0$ along some complement $R_0 \subset F$. Let $x \in U \cap \ker(\pi\gamma)$. Since π is a projection and $\gamma(U) = U_0$ we see that $\gamma(x) = \pi\gamma(x) = 0$ and therefore $x = 0$. Now let $z \in G$. Then there is an $x \in U$ such that $\pi\gamma(z) = \gamma(x)$. It follows that $z = x + (z - x)$ with $x \in U$ and $\pi\gamma(z - x) = 0$.

Let $R := \ker(\pi\gamma)$ and let $R_1 \subset F^*$ be some complement of $U_1 = \mu(U)$. We write

$$\begin{aligned} \gamma &= \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}: U \oplus R \longrightarrow U_0 \oplus R_0 \\ \mu' &= \begin{pmatrix} \mu'_1 & \mu'_2 \\ \mu'_3 & \mu'_4 \end{pmatrix}: U \oplus R \longrightarrow U_1 \oplus R_1 \\ \Phi &= \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}: U_1 \oplus R_1 \longrightarrow U_0^* \oplus R_0^* \\ & f \longmapsto (f|_{U_0}, f|_{R_0}) \\ \mu &= \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}: U \oplus R \longrightarrow U_0^* \oplus R_0^* \\ & x \longmapsto \Phi\mu'(x) \end{aligned}$$

By assumption, γ_1 and μ'_1 are isomorphisms and γ_3 and μ'_3 are vanishing. We can apply the strong isomorphism $(1_F, \begin{pmatrix} \gamma_1 & \gamma_2 \\ 0 & 1 \end{pmatrix})$ to achieve the simpler situation of $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_4 \end{pmatrix}$ and $U_0 = U$.

We compute $\gamma^*\mu = \begin{pmatrix} \Phi_1\mu'_1 & * \\ * & * \end{pmatrix}$ and see that $\Phi_1 = 0$. The last criterion of elementariness implies that Φ_2 is an isomorphism and therefore Φ_3 is bijective as well. We use these facts to see that

$$\mu = \Phi\mu' = \begin{pmatrix} 0 & \Phi_2\mu'_4 \\ \Phi_3\mu'_1 & \Phi_3\mu'_2 + \Phi_4\mu'_4 \end{pmatrix} = \begin{pmatrix} 0 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$$

Hence μ_3 is an isomorphism. Because $\gamma^*\mu = \begin{pmatrix} 0 & \mu_2 \\ \gamma_4^*\mu_3 & \gamma_4^*\mu_4 \end{pmatrix}$ is $(-\epsilon)$ -symmetric, $\mu_2 = -\epsilon\mu_3^*\gamma_4$. We apply the strong isomorphism $\left(\begin{pmatrix} 1 & 0 \\ 0 & \mu_3^* \end{pmatrix}, 1_G\right)$ and get a preformation with the properties we want.

At last we deal with the case of quadratic split preformation. The same steps as before yield a strong isomorphism between an ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and

$$(U \oplus U^* \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}} U \oplus R \xrightarrow{\begin{pmatrix} 0 & -\epsilon\sigma \\ 1 & \tau \end{pmatrix}} U^* \oplus U, \bar{\theta})$$

We define $\theta' := \bar{\theta}|_R$ which is obviously a quadratic refinement of $\sigma^*\tau$. Because $\bar{\theta}$ is a quadratic refinement of $\gamma^*\mu = \begin{pmatrix} 0 & -\epsilon\sigma \\ \sigma^* & \sigma^*\tau \end{pmatrix} : U \oplus R \rightarrow U^* \oplus R^*$ it follows for $u \in U$ and $r \in R$: $\bar{\theta}(u, r) = \bar{\theta}(u) + \bar{\theta}(r) + \gamma^*\mu(u, r) = 0 + \theta'(r) - \epsilon\sigma(r)(u)$. \square

The proposition allows us to derive some quite simple obstructions for elementariness.

Corollary 2.4.3. *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be a regular ϵ -quadratic preformation.*

- i) *The isomorphism classes of kernels and cokernels of γ , μ , $\begin{pmatrix} \gamma \\ \mu \end{pmatrix}$, $\gamma^*\mu$ as well as $\text{rk } G - \text{rk } F \in \mathbb{Z}$ and $\text{rk } F \in \mathbb{Z}/2\mathbb{Z}$ are invariants of $[z] \in l^{2q+2}(\Lambda)$.*
- ii) *If $[z] \in l^{2q+2}(\Lambda)$ is elementary then $\ker \gamma \cong \ker \mu$, $\text{coker } \gamma \cong \text{coker } \mu$, $\ker \gamma^*\mu \cong \ker \gamma \oplus \ker \gamma^*$, $\text{coker } \gamma^*\mu \cong \text{coker } \gamma \oplus \text{coker } \gamma^*$ and $\text{rk } F$ is even*

Similar for the split and non-regular case.

Corollary 2.4.4.

$$\begin{aligned} l'_{2q+2}(\Lambda) &\longrightarrow l^{elem}_{2q+2}(\Lambda) \\ \left[(U^* \xleftarrow{\sigma} R \xrightarrow{\tau} U, \theta') \right] &\longmapsto \left[(U \oplus U^* \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}} U \oplus R \xrightarrow{\begin{pmatrix} 0 & -\epsilon\sigma \\ 1 & \tau \end{pmatrix}} U^* \oplus U, \bar{\theta}) \right] \\ &\text{with } \bar{\theta}(u, r) = \theta'(r) - \epsilon\sigma(r)(u) \end{aligned}$$

is a well-defined surjective morphism of abelian monoids with kernel

$$\{[\partial(K, \lambda, \mu)] \mid (K, \lambda, \mu) \text{ is a } (-)^{q+1}\text{-quadratic form}\}$$

Similar for the non-split and regular case.

Finally we prove a little lemma about elementariness which has two interesting applications.

Lemma 2.4.5. *Let $x = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $y = (F \xleftarrow{\sigma} H \xrightarrow{\tau} F^*, \bar{\psi})$ be two ϵ -quadratic split preformations and $\pi : G \rightarrow H$ a surjective homomorphism such that*

$$\begin{array}{ccccc} F & \xleftarrow{\gamma} & G & \xrightarrow{\mu} & F^* \\ & \searrow \sigma & \downarrow \pi & \nearrow \tau & \\ & & H & & \end{array} \quad (2.2)$$

commutes and $\bar{\theta} = \bar{\psi}\pi$. Then x is (stably) elementary if and only if y is (stably) elementary. Similar for the non-split case.

Proof. Let $j: U \hookrightarrow G$ be the inclusion of an h -lagrangian of x . Then $\sigma\pi j = \gamma j$ is injective and so is πj . Hence $i = \pi j: U \hookrightarrow G$ is the inclusion of a free f.g. submodule. It is in fact an h -lagrangian of y because $\sigma i = \sigma\pi j = \gamma j$ and $\tau i = \tau\pi j = \mu j$.

On the other hand let $j: U \hookrightarrow H$ be the inclusion of an h -lagrangian of y . Let $\{e_1, \dots, e_n\}$ be a basis of U . For any k choose a $b_k \in \pi^{-1}(j(e_k))$. This defines a homomorphism $i: U \rightarrow G, e_k \mapsto b_k$. Obviously $\pi i = j$ and i is an inclusion of a f.g. free submodule. It is in fact an h -lagrangian for x because $\gamma i = \sigma\pi i = \sigma j$ and $\mu i = \tau\pi i = \tau j$. \square

As a first application we can slightly improve the elegance of the obstruction in Definition 2.3.1.

Corollary 2.4.6. *In the situation of Definition 2.3.1 and Theorem 2.3.2 we can define an alternative $(-)^q$ -quadratic split preformation*

$$\begin{aligned} x(W) &= (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) \\ &= (H_{q+1}(W', M_0) \leftarrow H_{q+2}(B, W') \rightarrow H_{q+1}(W', M_1), \bar{\theta}) \\ &\in l'_{2q+2}(\mathbb{Z}[\pi_1(X)]) \end{aligned}$$

with $\bar{\theta}$ being induced by the self-intersection form on W and maps γ and μ from the long exact sequence of the triads (B, W, M_i) .

Then W is B -cobordant to an h -cobordism if and only if $x(W) \in l'_{2q+2}(\mathbb{Z}[\pi_1(X)])$ is elementary.

The long exact sequences of (B, W, M_i) yields

$$\begin{array}{ll} \ker \gamma = H_{q+2}(B, M_0) & \text{coker } \gamma = H_{q+1}(B, M_0) \\ \ker \mu = H_{q+2}(B, M_1) & \text{coker } \mu = H_{q+1}(B, M_1) \end{array}$$

(Compare with Corollary 2.4.3)

The second application is a more theoretical: the decision whether a preformation is elementary can always be replaced by checking that a related regular preformation is elementary

Corollary 2.4.7. *Let $x = (F \xleftarrow{\sigma} H \xrightarrow{\tau} F^*, \psi)$ be an ϵ -quadratic split preformation and let G be a free f.g. module with an epimorphism $\pi: G \twoheadrightarrow H$. Then there is an ϵ -quadratic split preformation $y = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ which makes the diagram 2.2 commute. x is regular and it is elementary if and only if y is.*

Chapter 3

Translating Kreck's Surgery into Algebraic Surgery Theory

For the whole chapter let $q \geq 2$, $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

The asymmetric and quadratic signatures which will be defined in the next chapters are obstructions to the elementariness of a preformation. Constructions and proofs will use results from the vast theory of algebraic surgery.

This section will provide the first step in the programme by translating preformations into the language of algebraic surgery theory: quadratic Poincaré pairs and complexes (see **Section 3.1**).

Preformations arise as obstructions when we ask whether a cobordism $(W, M, M') \rightarrow B$ of normal smoothings is cobordant rel ∂ to an h -cobordism. As there is no realization result for preformations, we cannot be sure whether they all arise from a surgery problem. The constructions in Section 3.1 can be thought of as an “algebraic realization result”: any preformation appears as an “obstruction” of a certain Poincaré pair to be cobordant rel ∂ to an algebraic h -cobordism. However, we will not try to develop a general l -obstruction theory for Poincaré pairs simply because we do not need it. It suffices to create a quadratic chain complex model for a preformation and to apply algebraic surgery theory to it.

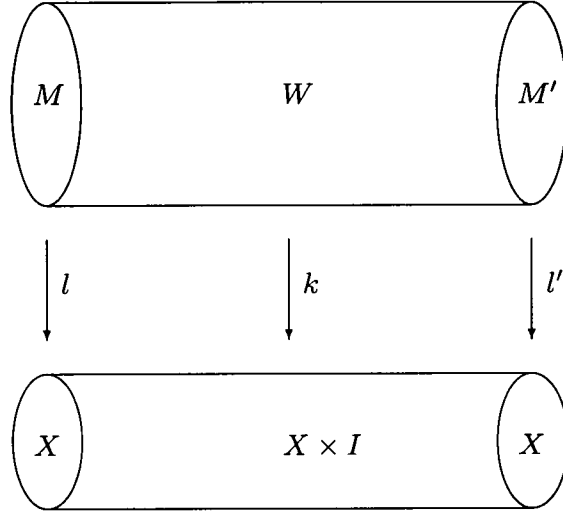
Algebraic equivalents of concepts like cobordisms rel ∂ and surgery inside a manifold will be needed to model Kreck's surgery theory. **Section 3.2** deals with this rather technical issue and confirms our expectations, namely that those notions exist and that they behave similarly to their geometric equivalents (e.g. that two Poincaré pairs are cobordant if and only if one is the result of a surgery of the other).

In **Section 3.3** we prove some kind of algebraic equivalent of Theorem 2.3.2: the Poincaré pair constructed in Section 3.1 is cobordant rel ∂ to an algebraic h -cobordism if and only if the preformation is (stably) elementary.

3.1 From Preformations to Quadratic Pairs

If we want to use the tools of algebraic surgery theory, we will need to translate preformations into the language of quadratic chain complexes and pairs. Readers can brush up their knowledge of algebraic surgery theory by reading [Ran80a] or the appendix (Chapter A, p. 148).

The translation is easier for non-singular formations. They can always be realized ([Ran02] Proposition 12.17) as an obstruction of a **presentation** i.e. a $(2q + 2)$ -dimensional cobordism of degree 1 normal maps



$$(k, l, l'): (W, M, M') \longrightarrow X \times (I, 0, 1)$$

into a finite geometric Poincaré pair $(X, \partial X)$ such that l and l' are q -connected, k is $(q + 1)$ -connected and $l|: \partial M \rightarrow \partial X$ is a homotopy equivalence. We note that a presentation is a special case of a Kreck surgery situation but also a way to find the L -obstruction of the odd-dimensional normal map $l: (M, \partial M) \rightarrow (X, \partial X)$ (see also Section 2.1). In both cases (see Corollary 2.4.6 and [Ran02] Chapter 12) the obstruction is the non-singular ϵ -quadratic split formation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ with $F = K_{q+1}(W, M)$, $G = K_{q+1}(W)$, etc. It is elementary in $l_{2q+2}(\mathbb{Z}[\pi_1(X)])$ if and only if $k: W \rightarrow X \times I$ is cobordant rel ∂ to an h -cobordism and it vanishes in $L_{2q+1}(\mathbb{Z}[\pi_1(X)])$ if and only if $l: (M, \partial M) \rightarrow (X, \partial X)$ is cobordant rel ∂ to a homotopy equivalence i.e. an h -cobordism (see also Section 2.1).

Algebraic surgery theory presents an alternative surgery obstruction for $l: (M, \partial M) \rightarrow (X, \partial X)$: the **quadratic kernel** (D, ν) of l . It is a $(2q + 1)$ -dimensional **quadratic Poincaré complex** over $\mathbb{Z}[\pi_1(X)]$ where $D = \mathcal{C}(l^!)$ is the mapping cone of the so-called **Umkehr chain map**

$$l^!: C(\tilde{X}) \xrightarrow{\cong} C(\tilde{X}, \widetilde{\partial X})^{2q+1-*} \xrightarrow{\tilde{l}^*} C(\widetilde{M}, \widetilde{\partial M})^{2q+1-*} \xrightarrow{\cong} C(\widetilde{M})$$

with \widetilde{X} and \widetilde{M} the universal covers. The homology modules of D are the kernel modules $K_*(M)$. The quadratic structure $\nu \in W_{\%}(D)_{2q+1}$ is a family of maps $\nu_s \in \text{Hom}(D^{2q+1-r-s}, D_r)$ which generalizes the self-intersection number. It contains a chain equivalence $(1+T)\nu_0: D^{2q+1-*} \xrightarrow{\cong} D$ inducing the Poincaré duality $K^{2q+1-*}(M) \xrightarrow{\cong} K_*(M)$. (For the details of the construction see [Ran80b] Chapter 1 and 4.)

The algebraic surgery approach has two main advantages to the traditional obstruction theory:

- i) it works for normal maps $l: (M, \partial M) \rightarrow (X, \partial X)$ which are **not** highly-connected,
- ii) it provides a uniform obstruction theory for the odd- **and** even-dimensional case.

There are notions of algebraic surgery and cobordism for Poincaré complexes. In the case of quadratic kernels they correspond to geometric surgery and normal cobordism of the normal maps for which they were defined. The set of cobordism classes of n -dimensional quadratic Poincaré complexes over a ring Λ with involution are isomorphic to Wall's $L_n(\Lambda)$. The instant surgery obstruction provides an easy formula to distill the traditional surgery obstruction form or formation out of a quadratic kernel. (See [Ran80a] Chapter 4 and [Ran80b] Chapter 7 for details.)

The quadratic kernel construction can be generalized to (odd- or even-dimensional) normal maps which are not a homotopy equivalence on the boundary (nor on the whole manifold). The result will be a **quadratic Poincaré pair** of the same dimension. Assume for example that $\partial l = l|: \partial M \rightarrow \partial X$ is no longer necessarily a homotopy equivalence. Then there is chain homotopy commutative diagram

$$\begin{array}{ccc} C(\widetilde{\partial X}) & \xrightarrow{i_X} & C(\widetilde{X}) \\ \partial l! \downarrow & & \downarrow l! \\ C(\widetilde{\partial M}) & \xrightarrow{i_M} & C(\widetilde{M}) \end{array}$$

with i_X and i_M the inclusions of the boundary of W and X . It induces a map of the mapping cones

$$f = i_l: C = \mathcal{C}(\partial l!) \rightarrow D = \mathcal{C}(l!)$$

The quadratic kernel of l is the $(2q+1)$ -dimensional quadratic Poincaré pair $c = (f: C \rightarrow D, (\delta\psi, \psi) \in W_{\%}(f)_{2q+1})$. The quadratic structure contains again self-intersection information and the maps

$$(\delta\phi_0, f\phi_0) : \mathcal{C}(f)^{2q+1-*} \rightarrow D$$

induce the Poincaré-Lefschetz duality maps $K^{2q+1-*}(M, \partial M) \xrightarrow{\cong} K_*(M)$. Its boundary (C, ψ) is a $2q$ -dimensional quadratic Poincaré complex and it is by construction the

quadratic kernel of the normal map $l|: \partial M \longrightarrow \partial X$. (See also [Ran80b] Proposition 6.5.)

By Proposition A.2.10, there is a one-to-one correspondence between homotopy classes of quadratic Poincaré pairs and quadratic complexes (the latter are not necessarily Poincaré). It is induced by the **Thom construction** which assigns to every quadratic Poincaré pair $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ a quadratic complex $(N = \mathcal{C}(f), \zeta = \delta\psi/\psi)$ of the same dimension. The homology of N are the relative kernel modules $K_*(M, \partial M)$. The chain map $(1 + T)\zeta_0: N^{2q+1-*} \xrightarrow{\simeq} N$ induces the maps

$$K^{2q+1-*}(M, \partial M) \longrightarrow K^{2q+1-*}(M) \xrightarrow{\cong} K_*(M, \partial M)$$

If $l|: \partial M \xrightarrow{\simeq} \partial X$ is a homotopy equivalence as in the beginning, then $C \simeq 0$, $(N, \zeta) \simeq (D, \nu)$ and the Poincaré pair c is homotopy equivalent to $(0 \longrightarrow D, (0, \nu))$.

In the same fashion we can translate the $(2q+2)$ -dimensional normal map $k: (W, \partial W) \rightarrow (X \times I, X \cup_{\partial X} X)$ into a $(2q+2)$ -dimensional quadratic Poincaré pair $x = (g: \partial E \longrightarrow E, (\delta\omega, \omega))$. The boundary ∂W is the union of M and M' glued together along their common boundary. Similarly, the quadratic kernel $(\partial E, \omega)$ of the normal map $k|_{\partial W}$ is the algebraic union of the quadratic kernels of l and l' in the sense of Definition A.3.1. Maps/diffeomorphisms of manifolds that are compatible with normal maps on them give rise to morphisms/isomorphisms between the quadratic kernels (like e.g. $\partial M \xrightarrow{\cong} \partial M'$). All in all, the constructions above yield a translation of normal maps (on manifolds) to quadratic complexes as the table below illustrates:

| Topology | Algebraic surgery theory |
|--|--|
| ∂M | $2q$ -dimensional quadratic Poincaré complex (C, ψ) |
| $\partial M \hookrightarrow M$ | $(2q+1)$ -dim. quadratic Poincaré pair $(f: C \longrightarrow D, (\delta\psi, \psi))$ |
| $M/\partial M$ | $(2q+1)$ -dimensional quadratic complex $(N = \mathcal{C}(f), \zeta)$ |
| $\partial M'$ | $2q$ -dimensional quadratic Poincaré complex (C', ψ') |
| $\partial M' \hookrightarrow M'$ | $(2q+1)$ -dim. quadratic Poincaré pair $(f': C' \longrightarrow D', (\delta\psi', \psi'))$ |
| $M'/\partial M'$ | $(2q+1)$ -dimensional quadratic complex $(N' = \mathcal{C}(f'), \zeta')$ |
| $\partial M \xrightarrow{\cong} \partial M'$ | equivalence $(h, \chi): (C, \psi) \xrightarrow{\simeq} (C', \psi')$ |
| $M \cup_{\partial M} M'$ | $(2q+1)$ -dimensional quadratic Poincaré complex $(\partial E, \omega)$ |
| $M \cup_{\partial M} M' \hookrightarrow W$ | $(2q+2)$ -dim. quadratic Poincaré pair $(g: \partial E \longrightarrow E, (\delta\omega, \omega))$ |

Unfortunately, a generalization of this procedure to all preformations will not work for two simple reasons: firstly, there is no generalization of quadratic kernels to normal smoothings, secondly, there is no geometric realization result known for general preformations.

There is however a purely algebraic translation method which enables us to construct the quadratic pairs and complexes by just using the data given by the formation. It turns out that this method extends without a problem to ϵ -quadratic (split) preformations as long as they are regular i.e. all modules in it are f.g. free.

One has to be cautious. For arbitrary preformations and arbitrary Kreck surgery problems the relationship between geometry and algebra is not as straightforward as for presentations. It can happen e.g. that there is a non-contractible algebraic boundary (C, ψ) although M is closed. Nevertheless the philosophy remains the same. We can think of the complexes and pairs as vague algebraic models of the manifolds or normal smoothings as in the table above - but only to boost our intuition! If we want to prove statements about those quadratic complexes and pairs we will not be able to use geometry but we have to resort to the methods of algebraic surgery theory alone.

One example for this strategy is Theorem 3.3.3. It states that a preformation is (stably) elementary if and only if the quadratic Poincaré pair x associated to it is cobordant rel ∂ to an algebraic h -cobordism. It is an almost word by word translation of the proof of Kreck's Theorem 2.3.2. Nevertheless, it is not an automatic consequence because there is no mathematically rigid connection between the algebraic model x and the original (geometric) Kreck surgery problem.

By [Ran01] Proposition 9.4 there is a one-to-one correspondence between certain equivalence classes of non-singular formations and **short odd complexes**. A similar result can be found in [Ran80a] Proposition 2.3 and 2.5. We will not need those theorems or a generalization in detail. We just use it as a motivation for translating a regular ϵ -quadratic split preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ into a connected $(2q+1)$ -dimensional quadratic chain complex (N, ζ)

$$\begin{aligned} d_N &= \mu^*: N_{q+1} = F \longrightarrow N_q = G^* \\ \zeta_0 &= \gamma: N^q = G \longrightarrow N_{q+1} = F \\ \zeta_1 &= \epsilon\theta: N^q = G \longrightarrow N_q = G^* \end{aligned} \tag{3.1}$$

with θ a representative of $\bar{\theta} \in Q_{-\epsilon}(G)$. Obviously (N, ζ) depends on the choice of θ . We will deal with this issue the end of this section (see Remark 3.1.3).

An obstruction preformation is an algebraic model for a $(2q+2)$ -dimensional cobordism $(W, M, M') \longrightarrow X$ of highly-connected normal smoothings. But the results from algebraic surgery theory that we are using were proven with odd-dimensional traditional surgery theory in mind. As explained in Section 2.1, in that context an obstruction formation for a $(2q+2)$ -dimensional normal cobordism $(W, M, M') \longrightarrow X$ is thought of as an obstruction for $M \longrightarrow X$ only. Hence (N, ζ) is a quadratic complex model for the normal map $(M, \partial M) \longrightarrow (X, \partial X)$.

Now we turn around the cobordism to derive the quadratic chain complex given by the normal map $(M', \partial M' = \partial M) \longrightarrow (X, \partial X)$. The obstruction of the "new" cobordism can easily be constructed out of z and is called the **flip of z** . Again we take a relationship which holds in the world of formations and presentation and generalize it to all preformations:

Definition 3.1.1. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an ϵ -quadratic split preformation. Its **Flip** is the preformation

$$z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\bar{\theta})$$

Similar for the non-split case. □

As in (3.1) we use the flip preformation to define a connected (N', ζ') be the $(2q + 1)$ -dimensional complex

$$\begin{aligned} d_{N'} &= \gamma^*: N'_{q+1} = F^* \longrightarrow N'_q = G^* \\ \zeta'_0 &= \epsilon\mu: N'^q \longrightarrow N'_{q+1} \\ \zeta'_1 &= -\epsilon\theta: N'^q \longrightarrow N'_q \end{aligned} \quad (3.2)$$

The next step is to thicken up (N, η) and (N', η') to $(2q + 1)$ -dimensional quadratic Poincaré pairs

$$\begin{aligned} c &= (f: C = \partial N \longrightarrow D = N^{2q+1-*}, (\delta\psi = 0, \psi = \partial\zeta)) \\ c' &= (f: C' = \partial N' \longrightarrow D' = N'^{2q+1-*}, (\delta\psi' = 0, \psi' = \partial\zeta')) \end{aligned}$$

(using the constructions in Definition A.2.7). They are given by

$$\begin{array}{ccc} C_{q+1} = G & \xrightarrow{f_{q+1}=1_G} & D_{q+1} = G \\ \begin{pmatrix} -\epsilon\gamma \\ -\epsilon\mu \end{pmatrix} \downarrow & & \downarrow -\epsilon\mu \\ C_q = F \oplus F^* & \xrightarrow{f_q=(0 \ 1)} & D_q = F^* \\ \begin{pmatrix} \mu^* & \epsilon\gamma^* \end{pmatrix} \downarrow & & \\ C_{q-1} = G^* & & \end{array} \quad (3.3)$$

$$\delta\psi = 0$$

$$\psi_0 = 1_G: C^{q-1} = G \longrightarrow C_{q+1} = G$$

$$\widetilde{\psi}_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}: C^q = F^* \oplus F \longrightarrow C_q = F \oplus F^*$$

$$\psi_1 = (\epsilon\gamma^* \ 0): C^q = F^* \oplus F \longrightarrow C_{q-1} = G^*$$

$$\psi_2 = -\epsilon\theta^*: C^{q-1} = G \longrightarrow C_{q-1} = G^*$$

and

$$\begin{array}{ccc} C'_{q+1} = G & \xrightarrow{f'_{q+1}=1_G} & D'_{q+1} = G \\ \begin{pmatrix} -\mu \\ -\epsilon\gamma \end{pmatrix} \downarrow & & \downarrow -\epsilon\gamma \\ C'_q = F^* \oplus F & \xrightarrow{f'_q=(0 \ 1)} & D'_q = F \\ \begin{pmatrix} \gamma^* & \mu^* \end{pmatrix} \downarrow & & \\ C'_{q-1} = G^* & & \end{array} \quad (3.4)$$

$$\begin{aligned}
\delta\psi' &= 0 \\
\psi'_0 &= 1_G: C'^{q-1} = G \longrightarrow C'_{q+1} = G \\
\widetilde{\psi}'_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}: C'^q = F \oplus F^* \longrightarrow C'_q = F^* \oplus F \\
\psi'_1 &= (\mu^* \ 0): C'^q = F \oplus F^* \longrightarrow C'_{q-1} = G^* \\
\psi'_2 &= \epsilon\theta^*: C'^{q-1} = G \longrightarrow C'_{q-1} = G^*
\end{aligned}$$

In the geometric situation we obviously find that $\partial M \cong \partial M'$. We expect a chain complex analogue and indeed

$$\begin{array}{ccc}
C_{q+1} = G & \xrightarrow{h_{q+1}=1_G} & C'_{q+1} = G \\
\downarrow \begin{pmatrix} -\epsilon\gamma \\ -\epsilon\mu \end{pmatrix} & & \downarrow \begin{pmatrix} -\mu \\ -\epsilon\gamma \end{pmatrix} \\
C_q = F \oplus F^* & \xrightarrow{h_q = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}} & C'_q = F^* \oplus F \\
\downarrow \begin{pmatrix} \mu^* & \epsilon\gamma^* \end{pmatrix} & & \downarrow \begin{pmatrix} \gamma^* & \mu^* \end{pmatrix} \\
C_{q-1} = G^* & \xrightarrow{h_{q-1}=1_{G^*}} & C'_{q-1} = G^*
\end{array} \tag{3.5}$$

$$\begin{aligned}
\chi_1 &= \begin{pmatrix} 0 & -\epsilon \\ 0 & 0 \end{pmatrix}: C'^q = F \oplus F^* \longrightarrow C'_q = F^* \oplus F \\
\chi_2 &= \begin{pmatrix} -\mu \\ 0 \end{pmatrix}: C'^{q-1} = G \longrightarrow C'_q = F^* \oplus F \\
\chi_3 &= \theta: C'^{q-1} = G \longrightarrow C'_{q-1} = G^*
\end{aligned}$$

defines an isomorphism $(h, \chi): (C, \psi) \xrightarrow{\cong} (C', \psi')$. We glue c and c' together along (h, χ) i.e. by Definition A.3.1 and Lemma A.2.6 we compute the union

$$(\partial E, \omega) = (f'h: C \longrightarrow D', ((-)^{2q} f' \chi f'^* = 0, \psi)) \cup (f: C \longrightarrow D, (0, -\psi))$$

given by $(\partial E_r = D'_r \oplus C_{r-1} \oplus D_r)$

$$\begin{array}{c}
\partial E_{q+2} = 0 \oplus G \oplus 0 \\
\downarrow -\epsilon \begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix} \\
\partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G \\
\downarrow \begin{pmatrix} -\epsilon\gamma & \epsilon & 0 & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \end{pmatrix} \\
\partial E_q = F \oplus G^* \oplus F^*
\end{array}$$

$$\begin{aligned}
\omega_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} : \partial E^{q+1} = G^* \oplus (F^* \oplus F) \oplus G^* \longrightarrow \partial E_q = F \oplus G^* \oplus F^* \\
\tilde{\omega}_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\gamma & 0 \\ \epsilon & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} : \partial E^q = F^* \oplus G \oplus F \longrightarrow \partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G \\
\omega_1 &= \begin{pmatrix} 0 & 0 & 0 \\ -\gamma^* & -\epsilon\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial E^q = F^* \oplus G \oplus F \longrightarrow \partial E_q = F \oplus G^* \oplus F^*
\end{aligned}$$

We will try to simplify the quadratic Poincaré complex $(\partial E, \omega)$ and it is already clear what the result will be if we look at the special case of formations and presentations. In this case the long exact sequence of $(W, \partial W)$ shows that $\gamma^* \mu: G = K_{q+1}(W) \longrightarrow G^* = K_{q+1}(W, \partial W)$ is a chain complex model for $K_{q+1}(\partial W)$. So it makes sense to expect that the $(2q+1)$ -dimensional quadratic Poincaré complex (A, τ) (arising from the regular ϵ -quadratic split preformation $\partial(G, \theta)$ by the same process as in (3.1))

$$\begin{aligned}
d_A &= ((1 + T_{-\epsilon})\theta)^*: A_{q+1} = G \longrightarrow A_q = G^* & (3.6) \\
\tau_0 &= 1: A^q = G \longrightarrow A_{q+1} = G \\
\tau_1 &= \epsilon\theta: A^q = G \longrightarrow A_q = G^*
\end{aligned}$$

will be isomorphic to $(\partial E, \omega)$. But if $(\partial E, \omega)$ looks so simple, why did we go through all the complicated procedures of thickening and glueing in the first place? Well, the aim of Kreck's surgery theory is to decide whether (W, M, M') is cobordant to an h -cobordism i.e. whether the inclusions of M and M' into some W' cobordant to W are homotopy equivalences. In our algebraic model, we will have to check whether the chain maps of D and D' into some algebraic cobordism are chain equivalences. Hence, we have to keep track where exactly D and D' are hidden in the boundary ∂E .

The isomorphism of chain complexes

$$\begin{array}{ccc}
\partial E_{q+2} & \xrightarrow{-\epsilon} & G \\
\downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
\partial E_{q+1} & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\epsilon\gamma & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \\ -1 & 0 & 0 & 1 \end{pmatrix}} & G \oplus F \oplus F^* \oplus G \\
\downarrow & & \downarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mu^* \gamma \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
\partial E_q & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ \epsilon\mu^* & -1 & \gamma^* \\ 0 & 0 & 1 \end{pmatrix}} & F \oplus G^* \oplus F^*
\end{array}$$

induces the equivalence $(a, \kappa): (\partial E, \omega) \longrightarrow (A, \tau)$ given by

$$\begin{array}{ccc} \partial E_{q+1} & \xrightarrow{a_{q+1}=(-1 \ 0 \ 0 \ 1)} & A_{q+1} = G \\ \downarrow & & \downarrow (\gamma^* \mu)^* \\ \partial E_q & \xrightarrow{a_q=(\epsilon \mu^* \ -1 \ \gamma^*)} & A_q = G^* \end{array} \quad (3.7)$$

$$\kappa_2 = \epsilon \theta: A^q = G \longrightarrow A_q = G^*$$

Every boundary of a form can easily be expressed as a Poincaré pair as the following lemma suggests:

Lemma 3.1.2. *Let (G, θ) be an $(-\epsilon)$ -quadratic form. Then the $(2q + 1)$ -dimensional quadratic Poincaré complex (A, τ) defined in (3.6) is the boundary of the $(2q + 2)$ -dimensional quadratic Poincaré pair $y = (p: A \longrightarrow E, (\delta\tau = 0, \tau))$ given by $p = 1: A_{q+1} = G \longrightarrow E_{q+1} = G$.*

Using Lemma A.2.6 we find that

$$x = (g: \partial E \longrightarrow E, (\delta\omega = 0, \omega))$$

with $g = (1 \ 0 \ 0 \ -1): \partial E_{q+1} \longrightarrow E_{q+1} = G$ is a $(2q + 2)$ -dimensional quadratic Poincaré pair.

Remark 3.1.3. It is time to investigate the effect of a choice of representative θ for $\bar{\theta} \in Q_{-\epsilon}(G)$ on the construction of x . Let $\hat{\theta} = \theta + \tilde{\theta} + \epsilon\tilde{\theta}^*$ be another representative. Let $(C, \hat{\psi})$ and $(C', \hat{\psi}')$ be the $2q$ -dimensional quadratic Poincaré complex given by (3.3) and (3.4) using the representative $\hat{\theta}$.

Then $\hat{\psi} - \psi = d(\tilde{\psi})$ and $\hat{\psi}' - \psi' = d(-\tilde{\psi})$ with $\tilde{\psi} \in W_{\%}(C)_{2q+1}$ given by $\tilde{\psi}_3 = -\epsilon\tilde{\theta}^*: C^{q-1} \longrightarrow C_{q-1}$.

In the language of the Q -groups (compare Definition A.1.2) this means that $[\hat{\psi}] = [\psi] \in Q_{2q}(C)$ and $[\hat{\psi}'] = [\psi'] \in Q_{2q}(C')$. For the $(2q+1)$ -dimensional quadratic Poincaré pairs c and c' follows that $[(0, \psi)] = [(0, \hat{\psi})] \in Q_{2q+1}(f)$ and $[(0, \psi')] = [(0, \hat{\psi}')] \in Q_{2q+1}(f'h)$. The union construction is designed such that different representatives lead to the same element $[\omega] \in Q_{2q+1}(\partial E)$ (compare [Ran80a] p.135f). It is not difficult to verify that both representatives of $\bar{\theta}$ lead to the same element $[(0, \omega)] \in Q_{2q+2}(g)$.

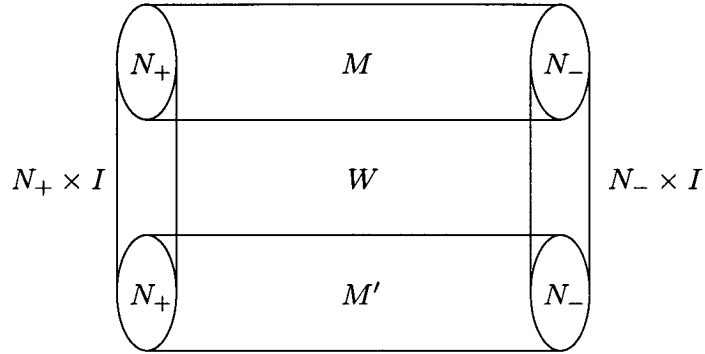
In any case, if we define concepts which are based on the chain complex models presented here but which are concepts about preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ (that is **asymmetric signatures**, **flip-isomorphism rel** ∂ and **quadratic signatures**) we still have to take care of the effect of choices of representatives. See also Remarks 4.2.1, 6.2.2 and Lemma 6.4.2.

3.2 Algebraic Surgery and Cobordisms of Pairs

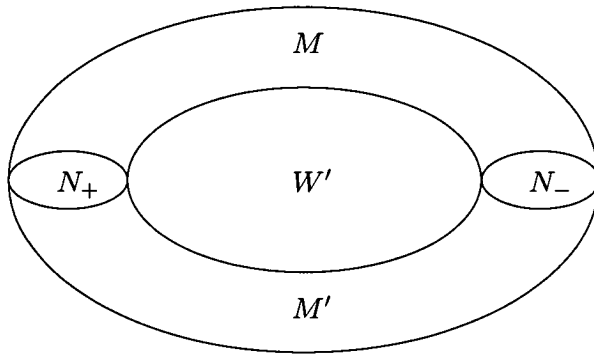
In Kreck's surgery theory we look at a cobordism $(e, f, f'): (W, M, M') \rightarrow B$ of normal smoothings and wonder whether it is cobordant $\text{rel}\partial$ to an h -cobordism or equivalently whether surgery inside of W will produce an h -cobordism. This is the case if and only if the obstruction in $l'_{2q+2}(\Lambda)$ is elementary. This section introduces algebraic versions of cobordisms and surgery for Poincaré pairs.

The first step will be a purely technical namely to define the notion of algebraic cobordism of quadratic pairs $\text{rel}\partial$ and algebraic surgery inside a quadratic pair.

In geometry a cobordism $\text{rel}\partial$ between two manifolds M and M' with the same boundary N is often thought of as a manifold with corners $(W, \partial W = M \cup N \times I \cup M')$.



By “collapsing” the $N \times I$ -part of ∂W and glueing together the boundaries of M and M' we produce a new manifold W' with the boundary $\partial W' = M \cup_N M'$.



Differential topology shows us that the existence of a null-cobordism of $M \cup_N M'$ is in fact equivalent to (M, N) and (M', N) being cobordant $\text{rel}\partial$. We will use this picture in order to define algebraic cobordisms $\text{rel}\partial$ of quadratic pairs.

Definition 3.2.1. Two $(n + 1)$ -dimensional ϵ -quadratic Poincaré pairs $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \rightarrow D', (\delta\psi', \psi))$ are **cobordant $\text{rel}\partial$** if there is an $(n + 2)$ -dimensional ϵ -quadratic Poincaré pair $(h: D' \cup_C D \rightarrow E, (\delta\omega, \delta\psi' \cup_\psi -\delta\psi))$. \square

An easy example for such cobordisms are homotopy equivalences.

Lemma 3.2.2. *Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \rightarrow D', (\delta\psi', \psi))$ be $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs. Let $(1, h; k): c \rightarrow c'$ be a homotopy equivalence. Then c and c' are cobordant rel ∂ .*

Proof. There is a $(\delta\chi, \chi) \in W_{\%}(f', \epsilon)_{n+2}$ such that

$$(1, h; k)_{\%}(\delta\psi, \psi) - (\delta\psi', \psi') = d(\delta\chi, \chi).$$

Define the $(n+2)$ -dimensional ϵ -quadratic Poincaré pair

$$(b: D \cup_C D' \rightarrow D', ((-)^n \delta\chi, \delta\psi \cup_{\psi} -\delta\psi')) \quad (3.8)$$

by $b = (h, (-)^{r-1}k, -1): (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_{r-1} \rightarrow D'_r$. \square

The next lemma proves the useful fact that changing the common boundary of two Poincaré pairs c and c' doesn't change anything about their cobordism relationship.

Lemma 3.2.3. *Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \rightarrow D', (\delta\psi', \psi))$ be two ϵ -quadratic $(n+1)$ -dimensional Poincaré pairs. Let $(h, \chi): (\widehat{C}, \widehat{\psi}) \xrightarrow{\cong} (C, \psi)$ be an equivalence. Define the $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs (using Lemma A.2.6)*

$$\begin{aligned} \widehat{c} &= (\widehat{f} = fh: \widehat{C} \rightarrow D, (\widehat{\delta\psi} = \delta\psi + (-)^n f\chi f^*, \widehat{\psi})) \\ \widehat{c}' &= (\widehat{f}' = f'h: \widehat{C} \rightarrow D', (\widehat{\delta\psi}' = \delta\psi' + (-)^n f'\chi f'^*, \widehat{\psi})) \end{aligned}$$

Then c and c' are cobordant rel ∂ if and only if \widehat{c} and \widehat{c}' are.

Proof. If c is cobordant rel ∂ to c' then there is an $(n+2)$ -dimensional ϵ -quadratic Poincaré pair

$$(e: D \cup_C D \rightarrow E, (\delta\omega, \omega = \delta\psi \cup_{\psi} -\delta\psi'))$$

By Lemma A.3.3 there is an equivalence

$$(a, \kappa): \widehat{c} \cup -\widehat{c}' = (D \cup_{\widehat{C}} D', \widehat{\delta\psi} \cup_{\widehat{\psi}} -\widehat{\delta\psi}') \xrightarrow{\cong} c \cup -c' = (D \cup_C D', \delta\psi \cup_{\psi} -\delta\psi')$$

Hence, by Lemma A.2.6 there is an $(n+2)$ -dimensional ϵ -quadratic Poincaré pair

$$(ea: D \cup_{\widehat{C}} D \rightarrow E, (\delta\omega + (-)^{n+1} e\kappa e^*, \widehat{\delta\psi} \cup_{\widehat{\psi}} -\widehat{\delta\psi}'))$$

\square

It is a well-known fact that two manifolds are cobordant if and only if one manifold is derived from the other by a finite sequence of surgeries and diffeomorphisms. There is an algebraic equivalent for Poincaré complexes (Proposition A.4.5). We will establish the same relationship in the case of Poincaré pairs. First we need to define a surgery on the inside of a pair:

Definition 3.2.4. Let $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair and $d = (g: \mathcal{C}(f) \longrightarrow B, (\delta\sigma, \delta\psi/\psi))$ a connected $(n+2)$ -dimensional ϵ -quadratic pair. Write $g = (a, b): \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow B_r$. **The result of the surgery d on the inside of c** is the $(n+1)$ -dimensional ϵ -quadratic Poincaré pair $c' = (f': C \longrightarrow D', (\delta\psi', \psi))$ given by

$$\begin{aligned}
d_{D'} &= \begin{pmatrix} d_D & 0 & (-)^n(1+T_\epsilon)\delta\psi_0a^* + (-)^nf(1+T_\epsilon)\psi_0b^* \\ (-)^ra & d_B & (-)^r(1+T_\epsilon)\delta\sigma_0 + (-)^{n+1}b\psi_0b^* \\ 0 & 0 & (-)^rd_B^* \end{pmatrix} : \\
D'_r &= D_r \oplus B_{r+1} \oplus B^{n+2-r} \longrightarrow D'_{r-1} = D_{r-1} \oplus B_r \oplus B^{n+3-r} \\
f' &= \begin{pmatrix} f \\ -b \\ 0 \end{pmatrix} : C_r \longrightarrow D'_r = D_r \oplus B_{r+1} \oplus B^{n+2-r} \\
\delta\psi'_0 &= \begin{pmatrix} \delta\psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \\
D'^{n+1-r} &= D^{n+1-r} \oplus B^{n+2-r} \oplus B_{r+1} \longrightarrow D'_r = D_r \oplus B_{r+1} \oplus B^{n+2-r} \\
\delta\psi'_s &= \begin{pmatrix} \delta\psi_s & (-)^sT_\epsilon\delta\psi_{s-1}a^* - fT_\epsilon\psi_{s-1}b^* & 0 \\ 0 & (-)^{n-r-s+1}T_\epsilon\delta\sigma_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \\
D'^{n+1-r-s} &= D^{n+1-r-s} \oplus B^{n+2-r-s} \oplus B_{r+s+1} \\
&\longrightarrow D'_r = D_r \oplus B_{r+1} \oplus B^{n+2-r} \quad (s > 0)
\end{aligned}$$

□

The following proposition will justify the formulae above by showing that surgery inside of a pair is nothing but the composition of the following standard procedures of algebraic surgery theory: Thom complex, algebraic surgery and thickening. (The latter is the inverse operation to the Thom complex. See Proposition A.2.10.)

Proposition 3.2.5. *We use the terminology of the previous definition.*

- i) *If $C = 0$ then $(D', \delta\psi')$ is the result of the surgery $(a: D \longrightarrow B, (\delta\sigma, \delta\psi))$ as in Definition A.4.1.*
- ii) *The result of the surgery $d = (g: \mathcal{C}(f) \longrightarrow B, (\delta\sigma, \delta\psi/\psi))$ on the Thom complex $(\mathcal{C}(f), \delta\psi/\psi)$ of c is isomorphic to the Thom complex $(\mathcal{C}(f'), \delta\psi'/\psi)$ of c' .*

Proof. The first part is trivial. So we turn our attention to the second claim. The

Thom-complex of c' is given by

$$\begin{aligned}
d' &= \begin{pmatrix} d_D & 0 & (-)^n(1+T_\epsilon)\delta\psi_0a^* + (-)^nf(1+T_\epsilon)\psi_0b^* & (-)^{r-1}f \\ (-)^ra & d_B & (-)^r(1+T_\epsilon)\delta\sigma_0 + (-)^{n+1}b\psi_0b^* & (-)^rb \\ 0 & 0 & (-)^rd_B^* & 0 \\ 0 & 0 & 0 & d_C \end{pmatrix} : \\
\mathcal{C}(f')_r &= (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \\
&\longrightarrow \mathcal{C}(f')_{r-1} = (D_{r-1} \oplus B_r \oplus B^{n+3-r}) \oplus C_{r-2} \\
(\delta\psi'/\psi)_0 &= \begin{pmatrix} \delta\psi_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (-)^{n+1-r}\psi_0f^* & (-)^{n-r}\psi_0b^* & 0 & (-)^{n-r}T_\epsilon\psi_1 \end{pmatrix} : \\
\mathcal{C}(f')^{n+1-r} &= (D^{n+1-r} \oplus B^{n+2-r} \oplus B_{r+1}) \oplus C^{n-r} \\
&\longrightarrow \mathcal{C}(f')_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \\
(\delta\psi'/\psi)_s &= \begin{pmatrix} \delta\psi_s & (-)^sT_\epsilon\delta\psi_{s-1}a^* - fT_\epsilon\psi_{s-1}b^* & 0 & 0 \\ 0 & (-)^{n-r-s+1}T_\epsilon\delta\sigma_{s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (-)^{n+1-r}\psi_s f^* & (-)^{n-r}\psi_s b^* & 0 & (-)^{n-r-s}T_\epsilon\psi_{s+1} \end{pmatrix} : \\
\mathcal{C}(f')^{n+1-r-s} &= (D^{n+1-r-s} \oplus B^{n+2-r-s} \oplus B_{r+s+1}) \oplus C^{n-r-s} \\
&\longrightarrow \mathcal{C}(f')_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \quad (s > 0)
\end{aligned}$$

The result of the surgery d on the other hand is the $(n+1)$ -dimensional ϵ -quadratic complex (M, τ) given as

$$\begin{aligned}
d_M &= \begin{pmatrix} d & (-)^{r-1}f & 0 & (-)^n(1+T_\epsilon)\delta\psi_0a^* + (-)^nfT_\epsilon\psi_0b^* \\ 0 & d & 0 & (-)^n\psi_0b^*d^* + (-)^{n+r}d\psi_0b^* \\ (-)^ra & (-)^rb & d & (-)^r(1+T_\epsilon)\delta\sigma_0 \\ 0 & 0 & 0 & (-)^rd^* \end{pmatrix} : \\
M_r &= (D_r \oplus C_{r-1}) \oplus B_{r+1} \oplus B^{n+2-r} \\
&\longrightarrow M_{r-1}(D_{r-1} \oplus C_{r-2}) \oplus B_r \oplus B^{n+3-r} \\
\tau_0 &= \begin{pmatrix} \delta\psi_0 & 0 & 0 & 0 \\ (-)^{n+1-r}\psi_0f^* & (-)^{n-r}T_\epsilon\psi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \\
M^{n+1-r} &= (D^{n+1-r} \oplus C^{n-r}) \oplus B^{n-r+2} \oplus B_{r+1} \\
&\longrightarrow M_r = (D_r \oplus C_{r-1}) \oplus B_{r+1} \oplus B^{n+2-r} \\
\tau_s &= \begin{pmatrix} \delta\psi_s & 0 & (-)^sT_\epsilon\delta\psi_{s-1}a^* - fT_\epsilon\psi_{s-1}b^* & 0 \\ (-)^{n+1-r}\psi_s f^* & (-)^{n-r-s}T_\epsilon\psi_{s+1} & (-)^{n-r}\psi_s b^* & 0 \\ 0 & 0 & (-)^{n+1-r-s}T_\epsilon\delta\sigma_{s-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \\
M^{n+1-r-s} &= (D^{n+1-r-s} \oplus C^{n-r-s}) \oplus B^{n-r-s+2} \oplus B_{r+s+1} \\
&\longrightarrow M_r = (D_r \oplus C_{r-1}) \oplus B_{r+1} \oplus B^{n+2-r} \quad (s > 0)
\end{aligned}$$

The isomorphisms

$$u_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & (-)^{n-r} \psi_0 b^* \end{pmatrix} :$$

$$M_r = (D_r \oplus C_{r-1}) \oplus B_{r+1} \oplus B^{n+2-r}$$

$$\longrightarrow \mathcal{C}(f')_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1}$$

define an isomorphism $(u, 0) : (M, \tau) \xrightarrow{\cong} (\mathcal{C}(f'), (\delta\psi'/\psi))$ of connected $(n+1)$ -dimensional ϵ -quadratic complexes. \square

At last we prove the expected relationship between cobordisms and surgery.

Proposition 3.2.6. *Let $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \longrightarrow D', (\delta\psi', \psi))$ be $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs. They are cobordant rel ∂ if and only if one can be obtained from the other by surgeries and homotopy equivalences of the type $(1, h; k)$.*

One direction of the proof is covered by the following lemma.

Lemma 3.2.7. *Let $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair and $d = (g = (a \ b) : \mathcal{C}(f) \longrightarrow B, (\delta\sigma, \delta\psi/\psi))$ a connected $(n+2)$ -dimensional ϵ -quadratic pair. Let $c' = (f': C \longrightarrow D', (\delta\psi', \psi))$ be the result of the surgery d on the inside of c .*

i) Let¹ $(D \cup_C D, \delta\psi \cup_\psi - \delta\psi) = c \cup -c$ be the union of c with itself along its boundary C . Then

$$\tilde{d} = (\tilde{g}: D \cup_C D \longrightarrow B, (\delta\sigma, \delta\psi \cup_\psi - \delta\psi))$$

given by $\tilde{g} = (a \ b \ 0) : (D \cup_C D)_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow B_r$ is a connected $(n+2)$ -dimensional ϵ -quadratic complex. The result of the surgery \tilde{d} is isomorphic to $(D' \cup_C D, \delta\psi' \cup_\psi - \delta\psi) = c' \cup -c$.

ii) $(h: D \cup_C D \longrightarrow D, (0, \delta\psi \cup_\psi - \delta\psi))$ with $h = (1 \ 0 \ -1) : D_r \oplus C_{r-1} \oplus D_r \longrightarrow D_r$ is an $(n+2)$ -dimensional ϵ -quadratic Poincaré pair.

iii) $(D' \cup_C D, \delta\psi' \cup_\psi - \delta\psi) = c' \cup -c$ is null-cobordant i.e. c and c' are cobordant rel ∂ .

Proof. i) The philosophy of this proof is that in some sense we can transfer everything we did for the Thom complexes in the proof of Lemma 3.2.5 to the union $D \cup_C D'$ using the morphism

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, 0 \right) : (D \cup_C D, \delta\psi \cup_\psi - \delta\psi) \longrightarrow (\mathcal{C}(f), \delta\psi/\psi)$$

¹Define $-c = (f: C \longrightarrow D, (-\delta\psi, -\psi))$ for a pair $c = (f: C \longrightarrow D, (\delta\psi, \psi))$.



In particular we can apply it to Lemma A.2.6 and show that \tilde{d} is an $(n+2)$ -dimensional ϵ -quadratic pair.

The $(n+1)$ -dimensional ϵ -quadratic Poincaré complex $(V, \sigma) = (D \cup_C D, \delta\psi \cup_\psi -\delta\psi) = c \cup -c$ is given by

$$d_V = \begin{pmatrix} d_D & (-)^{r-1}f & 0 \\ 0 & d_C & 0 \\ 0 & (-)^{r-1}f & d_D \end{pmatrix} :$$

$$V_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow V_{r-1} = D_{r-1} \oplus C_{r-2} \oplus D_{r-1}$$

$$\sigma_s = \begin{pmatrix} \delta\psi_s & 0 & 0 \\ (-)^{n-r+1}\psi_s f^* & (-)^{n-r-s}T_\epsilon\psi_{s+1} & 0 \\ 0 & (-)^{s-1}f\psi_s & -\delta\psi_s \end{pmatrix} :$$

$$V^{n+1-r-s} = D^{n+1-r-s} \oplus C^{n-r-s} \oplus D^{n+1-r-s} \longrightarrow V_r = D_r \oplus C_{r-1} \oplus D_r$$

The result of the surgery \tilde{d} is the $(n+1)$ -dimensional ϵ -quadratic Poincaré complex $(\tilde{V}, \tilde{\sigma})$:

$$d_{\tilde{V}} = \begin{pmatrix} d_D & (-)^{r-1}f & 0 & 0 & (-)^n(1+T_\epsilon)\delta\psi_0 a^* + (-)^n f T_\epsilon \psi_0 b^* \\ 0 & d_C & 0 & 0 & (-)^n \psi_0 b^* d^* + (-)^{n+r} d \psi_0 b^* \\ 0 & (-)^{r-1}f & d_D & 0 & (-)^{n+1} f \psi_0 b^* \\ (-)^r a & (-)^r b & 0 & d_B & (-)^r (1+T_\epsilon) \delta \sigma_0 \\ 0 & 0 & 0 & 0 & (-)^r d_B^* \end{pmatrix} :$$

$$\tilde{V}_r = (D_r \oplus C_{r-1} \oplus D_r) \oplus B_{r+1} \oplus B^{n+2-r}$$

$$\longrightarrow \tilde{V}_{r-1} = (D_{r-1} \oplus C_{r-2} \oplus D_{r-1}) \oplus B_r \oplus B^{n+3-r}$$

$$\tilde{\sigma}_0 = \begin{pmatrix} \delta\psi_0 & 0 & 0 & 0 & 0 \\ (-)^{n-r+1}\psi_0 f^* & (-)^{n-r}T_\epsilon\psi_1 & 0 & 0 & 0 \\ 0 & -f\psi_0 & -\delta\psi_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} :$$

$$\tilde{V}^{n+1-r} = (D^{n+1-r} \oplus C^{n-r} \oplus D^{n+1-r}) \oplus B^{n+2-r} \oplus B_{r+1}$$

$$\longrightarrow \tilde{V}_r = (D_r \oplus C_{r-1} \oplus D_r) \oplus B_{r+1} \oplus B^{n+2-r}$$

$$\tilde{\sigma}_s = \begin{pmatrix} \delta\psi_s & 0 & 0 & (-)^s T_\epsilon \delta\psi_{s-1} a^* + (-)^{n-r-s} f T_\epsilon \psi_{s-1} b^* & 0 \\ (-)^{n-r+1}\psi_s f^* & (-)^{n-r-s} T_\epsilon \psi_{s+1} & 0 & (-)^{n-r} \psi_s b^* & 0 \\ 0 & (-)^{s-1} f \psi_s & -\delta\psi_s & 0 & 0 \\ 0 & 0 & 0 & (-)^{n-r-s} T_\epsilon \delta\sigma_{s-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} :$$

$$\tilde{V}^{n+1-r-s} = (D^{n+1-r-s} \oplus C^{n-r-s} \oplus D^{n+1-r-s}) \oplus B^{n+2-r-s} \oplus B_{r+s+1}$$

$$\longrightarrow \tilde{V}_r = (D_r \oplus C_{r-1} \oplus D_r) \oplus B_{r+1} \oplus B^{n+2-r} \quad (s > 0)$$

On the other hand we have the union $(V', \sigma') = (D' \cup_C D, \delta\psi' \cup_\psi -\delta\psi) = c' \cup -c$

given by

$$d_{V'} = \begin{pmatrix} d_D & 0 & (-)^n(1+T_\epsilon)\delta\psi_0a^* + (-)^nf(1+T_\epsilon)\psi_0b^* & (-)^{r-1}f & 0 \\ (-)^ra & d_B & (-)^r(1+T_\epsilon)\delta\sigma_0 + (-)^{n+1}b\psi_0b^* & (-)^rb & 0 \\ 0 & 0 & (-)^rd_B^* & 0 & 0 \\ 0 & 0 & 0 & d_C & 0 \\ 0 & 0 & 0 & (-)^{r-1}f & d_D \end{pmatrix} :$$

$$V'_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \oplus D_r$$

$$\longrightarrow V'_{r-1} = (D_{r-1} \oplus B_r \oplus B^{n+3-r}) \oplus C_{r-2} \oplus D_{r-1}$$

$$\sigma'_0 = \begin{pmatrix} \delta\psi_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ (-)^{n+1-r}\psi_0f^* & (-)^{n-r}\psi_0b^* & 0 & (-)^{n-r}T_\epsilon\psi_1 & 0 \\ 0 & 0 & 0 & -f\psi_0 & -\delta\psi_0 \end{pmatrix} :$$

$$V'^{n+1-r} = (D^{n+1-r} \oplus B^{n+2-r} \oplus B_{r+1}) \oplus C^{n-r} \oplus D^{n+1-r}$$

$$\longrightarrow V'_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \oplus D_r$$

$$\sigma'_s = \begin{pmatrix} \delta\psi_s & (-)^sT_\epsilon\delta\psi_{s-1}a^* - fT_\epsilon\psi_{s-1}b^* & 0 & 0 & 0 \\ 0 & (-)^{n-r-s+1}T_\epsilon\delta\sigma_{s-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (-)^{n+1-r}\psi_sf^* & (-)^{n-r}\psi_sb^* & 0 & (-)^{n-r-s}T_\epsilon\psi_{s+1} & 0 \\ 0 & 0 & 0 & (-)^{s+1}f\psi_s & -\delta\psi_s \end{pmatrix}$$

$$V'^{n+1-r-s} = (D^{n+1-r-s} \oplus B^{n+2-r-s} \oplus B_{r+s+1}) \oplus C^{n-r-s} \oplus D^{n+1-r-s}$$

$$\longrightarrow V'_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \oplus D_r \quad (s > 0)$$

The isomorphisms

$$u_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & (-)^{n-r}\psi_0b^* \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} :$$

$$\tilde{V}_r = (D_r \oplus C_{r-1} \oplus D_r) \oplus B_{r+1} \oplus B^{n+2-r}$$

$$\longrightarrow V'_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \oplus D_r$$

define an isomorphism $(u, 0): (\tilde{V}, \tilde{\sigma}) \xrightarrow{\cong} (V', \sigma')$.

ii) Exercise.

iii) Follows from Propositions A.4.3 and A.4.5.

□

Proof of Proposition 3.2.6. It remains to show that cobordant pairs can be obtained from each other by surgery and homotopy equivalences which leave the boundary untouched.

Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \rightarrow D', (\delta\psi', \psi))$ be $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs which are cobordant $\text{rel}\partial$ so that there exists an $(n+2)$ -dimensional ϵ -quadratic Poincaré pair

$$e = (h: D \cup_C D' \rightarrow E, (\delta\omega, \omega = \delta\psi \cup_\psi -\delta\psi'))$$

with $h = \begin{pmatrix} j_0 & k & j_1 \end{pmatrix}: D_r \oplus C_{r-1} \oplus D_r \rightarrow E_r$. We define the connected $(n+2)$ -dimensional ϵ -quadratic pair

$$\begin{aligned} d &= (g: \mathcal{C}(f) \rightarrow B = \mathcal{C}(j_1), (\delta\sigma, \sigma = \delta\psi/\psi)) \\ g &= \begin{pmatrix} j_0 & k \\ 0 & -f \end{pmatrix}: \mathcal{C}(f)_r = D_r \oplus C_{r-1} \rightarrow B_r = E_r \oplus D'_{r-1} \\ \sigma_s &= \begin{pmatrix} \delta\omega_s & 0 \\ (-)^{n-r-1}(\delta\psi'_s j_1^* + (-)^s f' \psi_s k^*) & (-)^{n-r-s} T_\epsilon \delta\psi'_{s+1} \end{pmatrix}: \\ & B^{n+2-r-s} = E^{n+2-r-s} \oplus D'^{n+1-r-s} \rightarrow B_r = E_r \oplus D'_{r-1} \end{aligned}$$

The result of the surgery d inside of c is the $(n+1)$ -dimensional ϵ -quadratic Poincaré pair $c'' = (f'': C \rightarrow D'', (\delta\psi'', \psi))$. There is a homotopy equivalence $m: D'' \rightarrow D'$ given by

$$\begin{aligned} m &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \delta\psi'_0 \end{pmatrix}: \\ D''_{r+1} &= D_r \oplus (E_{r+1} \oplus D'_r) \oplus (E^{n+2-r} \oplus D'^{n+1-r}) \rightarrow D'_r \end{aligned}$$

such that $m\delta\psi''m^* = \delta\psi'$. Hence $(1, m; 0)$ defines a homotopy equivalence from c'' to c' . \square

Remark 3.2.8. Definitions 3.2.4 and 3.2.1 can be rephrased in the language of ϵ -quadratic triads. (For a thorough introduction into triads the reader can consult [Ran81] 1.3 and 2.1. and [Ran98] 20C.) Using the notation of Definition 3.2.4, a surgery d inside of c can be thought of as an $(n+2)$ -dimensional ϵ -quadratic triad (Γ, Ψ) with

$$\Gamma: \begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & \searrow^{\pm b} & \downarrow a \\ 0 & \longrightarrow & B \end{array}$$

and $\Psi = (\delta\sigma, 0, \delta\psi, \psi) \in Q^{n+2}(\Gamma)$.

A cobordism $(h: D' \cup_C D \rightarrow E, (\delta\omega, \delta\psi' \cup_\psi -\delta\psi))$ of the $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f': C \rightarrow D', (\delta\psi', \psi))$ $\text{rel}\partial$ can be interpreted as an $(n+2)$ -dimensional ϵ -quadratic Poincaré triad (Γ', Ψ') . If we write $h = \begin{pmatrix} a' & b & a \end{pmatrix}: (D' \cup_C D)_r = D'_r \oplus C_{r-1} \oplus D_r \rightarrow E_r$ then the triad is given by

$$\Gamma': \begin{array}{ccc} C & \xrightarrow{f} & D \\ f' \downarrow & \searrow^{\pm b} & \downarrow a \\ D' & \xrightarrow{a'} & E \end{array}$$

and $\Psi' = (\delta\omega, \delta\psi', \delta\psi, \psi) \in Q^{n+2}(\Gamma')$ (see also [Ran81] Proposition 2.1.1).

3.3 Elementariness in Algebraic Surgery Theory

In this section we want to understand elementariness in the context of algebraic surgery theory by reproving Theorem 2.3.2 for Poincaré pairs: a preformation is stably elementary if and only if the Poincaré pair x defined in Section 3.1 is cobordant $\text{rel}\partial$ to an **algebraic h -cobordism**. This theorem is the key to applying algebraic surgery theory to preformations in this treatise.

Definition 3.3.1. Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f: C \rightarrow D', (\delta\psi', \psi))$ be ϵ -quadratic n -dimensional Poincaré pairs whose union is the boundary of an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair $d = (g: D \cup_C D' \rightarrow E, (\delta\omega, \delta\psi \cup_\psi - \delta\psi'))$. Write $g = (j_0 \quad k \quad j_1) : (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D_r \rightarrow E_r$. d is an **algebraic h -cobordism** if the chain maps j_0 and j_1 are chain equivalences. \square

Example 3.3.2. The cobordism (3.8) constructed in the proof of Lemma 3.2.2 and the Poincaré pair defined in Lemma 3.2.7ii) are algebraic h -cobordisms.

Theorem 3.3.3. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation. Let $x = (g: \partial E \rightarrow E, (\delta\omega = 0, \omega))$ be the $(2q+2)$ -dimensional quadratic Poincaré pair constructed in Section 3.1 for an arbitrary representative θ of $\bar{\theta} \in Q_{-\epsilon}(G)$.

- i) If z is elementary then x is cobordant $\text{rel}\partial$ to an algebraic h -cobordism.*
- ii) If x is cobordant $\text{rel}\partial$ to an algebraic h -cobordism then $[z] \in l_{2q+2}(\Lambda)$ is elementary.*

We need some technical results before we can move on to the proof of this theorem.

Lemma 3.3.4. Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f: C \rightarrow D', (\delta\psi', \psi))$ be ϵ -quadratic n -dimensional Poincaré pairs. Let $d = (g: D \cup_C D' \rightarrow E, (\delta\omega, \delta\psi \cup_\psi - \delta\psi'))$ and $d' = (g': D \cup_C D' \rightarrow E', (\delta\omega', \delta\psi \cup_\psi - \delta\psi'))$ be two $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs. Assume there is a homotopy equivalence² between them that is the identity on the boundary. If d is an algebraic h -cobordism then so is d' .

Lemma 3.3.5. Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ and $c' = (f: C \rightarrow D', (\delta\psi', \psi))$ be ϵ -quadratic n -dimensional Poincaré pairs. Let $d = (g: D \cup_C D' \rightarrow E, (\delta\omega, \delta\psi \cup_\psi - \delta\psi'))$ be an $(n+1)$ -dimensional algebraic h -cobordism. Then d is homotopy equivalent to an algebraic h -cobordism

$$d' = (g': D \cup_C D' \rightarrow D, (\delta\omega', \delta\psi \cup_\psi - \delta\psi'))$$

such that $g' = (1 \quad l \quad h) : (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_r \rightarrow D_r$ with $h: D' \xrightarrow{\cong} D$ a chain equivalence.

²See Definition A.2.5.

Proof. Write $g = (j_0 \quad k \quad j_1) : (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow E_r$. Let $i_0 : E \longrightarrow D$ the chain homotopy inverse of j_0 and let $\Delta : i_0 j_0 \simeq 1 : D \longrightarrow D$ be a chain homotopy. Then there is a homotopy equivalence $(1, i_0; (-\Delta \quad 0 \quad 0)) : d \longrightarrow d'$ with $l = i_0 k + (-)^{r-1} \Delta f$ and $h = i_0 j_1$. \square

Lemma 3.3.6. *Let $V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \dots \rightarrow V_0 \longrightarrow 0$ be an exact sequence of free f.g. Λ -modules. Then $\text{im } d_i \subset V_{i-1}$ and $\ker d_i \subset V_i$ are stably f.g. free direct summands for all $i \in \{1, \dots, n\}$.*

Proof. For $i = 1$ observe that $0 \longrightarrow \ker d_1 \longrightarrow V_1 \xrightarrow{d_1} V_0 \longrightarrow 0$ is an exact sequence and since V_0 is free it splits. Hence $\text{im } d_1 = V_0$ and $\ker d_1$ are stably f.g. free and a direct summand in the respective V_i . Now assume the claim is true for $i \in \{1, \dots, n-1\}$. Then we look at the exact sequence $0 \longrightarrow \ker d_{i+1} \longrightarrow V_{i+1} \xrightarrow{d_{i+1}} \ker d_i \longrightarrow 0$. Again the sequence is exact and splits because by assumption $\ker d_i$ is projective. Now the claim follows for $i+1$. \square

Proof of Theorem 3.3.3. i) Let $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be elementary and $i : U \hookrightarrow G$ be the inclusion of an h -lagrangian. We define the chain map $m : \mathcal{C}(g) \longrightarrow B$ (with $\mathcal{C}(g)_r = E_r \oplus \partial E_{r-1}$)

$$\begin{array}{ccc}
\mathcal{C}(g)_{q+3} = 0 \oplus G & & \\
\downarrow -\epsilon \begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix} & & \\
\mathcal{C}(g)_{q+2} = 0 \oplus (G \oplus F \oplus F^* \oplus G) & & \\
\downarrow \begin{pmatrix} -\epsilon & 0 & 0 & \epsilon \\ -\epsilon\gamma & \epsilon & 0 & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \end{pmatrix} & & \\
\mathcal{C}(g)_{q+1} = G \oplus (F \oplus G^* \oplus F^*) & \xrightarrow{m=(a \ b)} & B_{q+1} = U^*
\end{array}$$

with $a = -i^* \gamma^* \mu$ and $b = (-\epsilon i^* \mu^* \quad i^* \quad -i^* \gamma^*)$.

Because of $i^* \theta i = 0 \in Q_{-\epsilon}(U)$ there is a $\delta \chi \in \text{Hom}_\Lambda(U, U^*)$ such that $i^* \theta i = \delta \chi + \epsilon \delta \chi^* \in \text{Hom}_\Lambda(U, U^*)$. We can check that $(\delta \sigma, \sigma = \delta \omega / \omega) \in W_{\%}(m)_{2q+3}$ with $\delta \sigma_1 = \epsilon \delta \chi : B^{q+1} \longrightarrow B_{q+1}$ is a cycle. Hence we have a connected $(2q+3)$ -dimensional quadratic pair $d = (m : \mathcal{C}(g) \longrightarrow B, (\delta \sigma, \sigma))$. The result of the surgery d on the inside of x is the $(2q+2)$ -dimensional quadratic Poincaré pair $x' = (g' : \partial E \longrightarrow E', (\delta \omega', \omega))$ given by $(\partial E_r = D'_r \oplus C_{r-1} \oplus D_r, E'_r = E_r \oplus B_{r+1} \oplus$

B^{2q+3-r}):

$$\begin{array}{ccc}
\partial E_{q+2} = 0 \oplus G \oplus 0 & \xrightarrow{0} & E'_{q+2} = 0 \oplus 0 \oplus U \\
\downarrow -\epsilon \begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix} & & \downarrow -i \\
\partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G & \xrightarrow{(1 \ 0 \ 0 \ -1)} & E'_{q+1} = G \oplus 0 \oplus 0 \\
\downarrow \begin{pmatrix} -\epsilon\gamma & \epsilon & 0 & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \end{pmatrix} & & \downarrow \epsilon i^* \gamma^* \mu \\
\partial E_q = F \oplus G^* \oplus F^* & \xrightarrow{(\epsilon i^* \mu^* \ -i^* \ i^* \gamma^*)} & E'_q = 0 \oplus U^* \oplus 0
\end{array}$$

$$\delta\omega'_0 = 1_U: E'^q = U \longrightarrow E'_{q+2} = U$$

Using the inclusion $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}: D' \oplus D \longrightarrow \partial E$ and with the help of Proposition 2.4.2iii) we see that x' is an algebraic h -cobordism.

- ii) Now we assume that x is cobordant rel ∂ to an algebraic h -cobordism $x' = (g': \partial E \longrightarrow E', (\delta\omega', \omega))$. In order to simplify our calculations we remember that the boundary of x and x' can be reduced in size by using the equivalence $(a, \kappa): (\partial E, \omega) \longrightarrow (A, \tau)$ defined in (3.7) on page 46. Let $y' = (p': A \longrightarrow E', (\delta\tau', \tau))$ be the $(2q+2)$ -dimensional quadratic Poincaré pair induced by x' and the inverse of (a, κ) . Let $y = (p: A \longrightarrow E, (\delta\tau = 0, \tau))$ be the $(2q+2)$ -dimensional quadratic Poincaré pair from Lemma 3.1.2. (It is also induced by x and the inverse of (a, κ)).

By Lemma 3.2.3, y is cobordant rel ∂ to y' and by (the proof of) Proposition 3.2.6 and Lemma 3.3.4 we can assume that y' is the result of a surgery $d = (m: \mathcal{C}(p) \longrightarrow B, (\delta\sigma, \sigma = \partial\tau/\tau))$ inside of y with

$$\begin{array}{ccc}
\mathcal{C}(p)_{q+2} = G & \xrightarrow{m_{q+2}=b_{q+2}} & B_{q+2} \\
\downarrow \begin{pmatrix} -\epsilon \\ -\epsilon\gamma^*\mu \end{pmatrix} & & \downarrow d \\
\mathcal{C}(p)_{q+1} = G \oplus G^* & \xrightarrow{m_{q+1}=(a_{q+1} \ b_{q+1})} & B_{q+1}
\end{array}$$

$$\sigma_0 = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon\theta^* \end{pmatrix}: \mathcal{C}(p)^{q+1} = E^{q+1} \oplus A^q \longrightarrow \mathcal{C}(p)_{q+1} = E_{q+1} \oplus A_q$$

Our next step will be the analysis of the complex E' . If $r \geq q+3$ or $r \leq q$ the differential is given by

$$d'_r = \begin{pmatrix} d & (-)^r(1+T)\delta\sigma_0 \\ 0 & (-)^r d^* \end{pmatrix}: E'_r = B_{r+1} \oplus B^{2q+3-r} \longrightarrow E'_{r-1} = B_r \oplus B^{2q+4-r}$$

The top differentials are dual to the bottom ones, i.e.

$$\begin{pmatrix} 0 & (-)^r \\ 1 & 0 \end{pmatrix} d_r^* \begin{pmatrix} 0 & (-)^{r-1} \\ 1 & 0 \end{pmatrix} = d_{2q+3-r}$$

for $r \geq q+3$ and $r \leq q$.

Because $E' \simeq D$ the homology groups $H_r(E')$ vanish for $r \neq q+1, q$. Hence by Lemma 3.3.6 there is a f.g. free submodule $X \subset E'_q$ such that $\ker d'_q \oplus X = E'_q$. Therefore $E'_{q+2}/\ker d'_{q+2} = \text{coker } d'_{q+3} = U$ is stably f.g. free and $U^* = \ker d'_q = \text{im } d'_{q+1}$. This observation gives us the chance to cut away the higher and lower parts of E' and establish a chain equivalence $E' \xrightarrow{\simeq} E''$ using the maps

$$\begin{array}{ccc} E'_{q+2} = B_{q+3} \oplus B^{q+1} & \xrightarrow{\text{proj}_X} & E''_{q+2} = U \\ d'_{q+2} \downarrow & & \downarrow i \\ E'_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ b_{q+2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & E''_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} \\ d'_{q+1} \downarrow & & \downarrow p \\ E'_q = B_{q+1} \oplus B^{q+3} & \xrightarrow{\left[\begin{pmatrix} 0 & -1 \\ -\epsilon & 0 \end{pmatrix} \right]} & E''_q = U^* \end{array} \quad (3.9)$$

with

$$\begin{aligned} d'_{q+2} &= \begin{pmatrix} 0 & -b_{q+1}^* \\ d & \epsilon(1+T)\delta\sigma_0 + b_{q+2}b_{q+1}^* \\ 0 & \epsilon d^* \end{pmatrix} \\ d'_{q+1} &= \begin{pmatrix} -\epsilon a_{q+1} & d & -\epsilon(1+T)\delta\sigma_0 \\ 0 & 0 & -\epsilon d^* \end{pmatrix} \\ i &= \left[\begin{pmatrix} 0 & -b_{q+1}^* \\ d & \epsilon(1+T)\delta\sigma_0 \\ 0 & \epsilon d^* \end{pmatrix} \right] \\ p &= \left[\begin{pmatrix} 0 & 0 & \epsilon d^* \\ -b_{q+1}\gamma^*\mu & -\epsilon d & (1+T)\delta\sigma_0 \end{pmatrix} \right] \end{aligned}$$

Let's define a regular ϵ -quadratic split preformation

$$z' = (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \bar{\theta}') = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) \oplus \partial \left(B_{q+2} \oplus B^{q+2}, \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix} \right)$$

which is another representative of $[z] \in l_{2q+2}(\Lambda)$. One can easily compute that $p = i^* \gamma'^* \mu'$.

Now we have a look at the boundary of y' . The map $g': A \rightarrow E'$ is given by

$$\begin{array}{ccc} A_{q+1} = G & \xrightarrow{\begin{pmatrix} 1 \\ -b_{q+2} \\ 0 \end{pmatrix}} & E'_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} \\ (\gamma^*\mu)^* \downarrow & & \downarrow d'_{q+1} \\ A_q = G^* & \xrightarrow{\begin{pmatrix} -b_{q+1} \\ 0 \end{pmatrix}} & E'_q = B_{q+1} \oplus B^{q+3} \end{array}$$

Applying the chain equivalence (3.9) and the map $D' \oplus D \rightarrow \partial E \xrightarrow{a} A$ to $g': A \rightarrow E'$ we find two chain maps $D' \rightarrow E'' \leftarrow D$

$$\begin{array}{ccccc}
 & & U & & \\
 & & \downarrow i & & \\
 G & \xrightarrow{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}} & G' & \xleftarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} & G \\
 \downarrow -\epsilon\gamma & & \downarrow i^*\gamma'^*\mu' & & \downarrow -\epsilon\mu \\
 F & \xrightarrow{\begin{pmatrix} 0 \\ b_{q+1}\mu^* \end{pmatrix}} & U^* & \xleftarrow{\begin{pmatrix} 0 \\ \epsilon b_{q+1}\gamma^* \end{pmatrix}} & F^*
 \end{array}$$

which by assumption are chain equivalences. From $(\delta\sigma, \sigma) \in Q_{2q+2}(m)$ one can deduce that $i^*\bar{\theta}i = 0 \in Q_{-\epsilon}(U)$. Now it is not very difficult to verify that the preformation z' fulfils the assumption of Proposition 2.4.2 iii) in respect to the stably f.g. free submodule U . Further stabilization of z' by boundaries of hyperbolic forms helps to replace U by a f.g. free submodule. Hence $[z] = [z'] \in l_{2q+2}(\Lambda)$ is elementary.

□

Chapter 4

Flip-Isomorphisms

For the whole chapter let $q \geq 2$, $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

Obviously, a $(2q + 2)$ -dimensional normal cobordism $(W, M, M') \rightarrow X$ only stands a chance to be cobordant $\text{rel} \partial$ to an s -cobordism if there is a compatible diffeomorphism between M and M' . Some kind of “algebraic isomorphism” between M and M' can be produced by just using preformations. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be the obstruction preformation. We can interpret z also as an algebraic model for the normal map $M \rightarrow X$ and the flip $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\bar{\theta})$ of z as a model for $M' \rightarrow X$. Following that philosophy, we hope that z and z' are weakly isomorphic if z is elementary. We shall call such an isomorphism **flip-isomorphism**. In **Section 4.1** we motivate and define flip-isomorphisms and show that, indeed, any elementary preformation - even those that aren't obstructions of the above or any surgery problem - has at least one. Just like we translated preformations into quadratic pairs and complexes in **Section 3.1**, we translate flip-isomorphisms into isomorphisms of those quadratic complexes in **Section 4.2**. Those isomorphisms can be applied to the quadratic Poincaré pair x from **Section 3.1**. They transform x into a Poincaré pair with an algebraic **twisted double** on the boundary (see **Sections 4.3 and 4.4**). This is a necessary preparation before we can define asymmetric and quadratic signatures in the following chapters.

4.1 Flip-isomorphism

The idea behind flip-isomorphisms is inspired by an observation in geometry: a cobordism $(W, M, M') \rightarrow X$ of highly-connected normal maps/normal smoothings can only be cobordant to an s -cobordism if M and M' are diffeomorphic. This hardly seems to be a very revealing insight. After all it is the aim of any surgery theory to establish existence or non-existence of such a diffeomorphisms.

But in the context of preformations we can produce a notion of some kind of “algebraic isomorphism” of M and M' : **flip-isomorphisms**. First of all we remember from

Section 2.1 that there are actually two ways of looking at preformations. We can think of them as algebraic vehicles for surgery-relevant data of an even-dimensional cobordism $(W, M, M') \longrightarrow X \times (I, 0, 1)$ of normal smoothings/maps. In that case we identify preformations by the very rigid equivalence relation of (stable) strong isomorphisms in order to preserve the essential information of the whole cobordism. This is the view of l -theory. But we have also learnt that in traditional odd-dimensional surgery theory, formations encode the information of the normal map $M \longrightarrow X$ only. That is why the odd-dimensional obstruction groups $L_{2q+1}(\Lambda)$ have a much more flexible equivalence relation which includes the use of weak isomorphisms (Definition 2.2.14).

So, philosophically, if we have a cobordism $(W, M, M') \longrightarrow X$ of normal smoothings/maps and define its obstruction z and we think of it as a description of the whole cobordism we use strong isomorphisms. If we treat z only as a description of the map $M \longrightarrow X$ we use weak isomorphisms. As alluded to in Section 3.1, turning around the new cobordism leads to the flip of z as the obstruction for the cobordism $(W, M', M) \longrightarrow X$. The heuristics so far suggests that the preformation-equivalent of a diffeomorphism between $M \longrightarrow X$ and $M' \longrightarrow X$ is a weak isomorphism between z and its flip. Such an isomorphism will be called a **flip-isomorphism**.

For fans of algebraic surgery theory the importance of flip-isomorphisms as an obstruction to elementariness is even more evident: in Section 3.1, a regular ϵ -quadratic split preformation z defined a $(2q + 2)$ -dimensional quadratic Poincaré pair $x = (g: \partial E \longrightarrow E, (\delta\omega = 0, \omega))$. By Theorem 3.3.3 we know that if z is elementary, x is cobordant rel ∂ to an h -cobordism. Then the $(2q + 1)$ -dimensional Poincaré pairs $c = (f: C \longrightarrow D, (0, \psi))$ and $c' = (f'h: C \longrightarrow D', ((-)^{2q} f\chi f^* = 0, \psi))$, whose union constitutes the boundary of x (that is $(\partial E, \omega) = c' \cup -c$), have to be homotopy equivalent. Both pairs are thickening-ups of the quadratic complexes (N, ζ) and (N', ζ') from (3.1) and (3.2) which themselves stem from the preformation z and its flip z' . Hence a homotopy equivalence of the pairs c and c' will eventually lead to some kind of equivalence between z and z' . [Ran80a] Proposition 2.3 and 2.5, which provided a recipe for translating the preformations z, z' to quadratic complexes $(N, \zeta), (N', \zeta')$ respectively, states that there is a natural bijection between equivalences of those quadratic complexes and (stable) weak isomorphism classes of formations. A generalization of those Propositions shows that z and z' are (stably) weakly isomorphic. The existence of a flip-isomorphism for elementary preformations can also be shown quite easily without algebraic surgery theory (see the proof of the Proposition 4.1.2 below).

Definition 4.1.1. i) A **flip-isomorphism of a regular ϵ -quadratic preformations** $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is a weak isomorphism of z with its flip i.e. a triple (α, β, σ) consisting of isomorphisms $\alpha \in \text{Hom}_\Lambda(F, F^*)$ and $\beta \in \text{Hom}_\Lambda(G, G)$ and an element $\sigma \in Q^{-\epsilon}(F^*)$ such that

$$(a) \alpha\gamma + \alpha\sigma\mu = \epsilon\mu\beta \in \text{Hom}_\Lambda(G, F^*)$$

$$(b) \alpha^{-*}\mu = \gamma\beta \in \text{Hom}_\Lambda(G, F)$$

It is **strong** if $\sigma = 0$.

ii) A **flip-isomorphism of a regular ϵ -quadratic split preformation** $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is a weak isomorphism of z with its flip i.e. a triple $(\alpha, \beta, \bar{\nu})$ consisting of isomorphisms $\alpha \in \text{Hom}_\Lambda(F, F^*)$ and $\beta \in \text{Hom}_\Lambda(G, G)$ and an element $\bar{\nu} \in Q_{-\epsilon}(F^*)$ such that

$$(a) \alpha\gamma + \alpha(\bar{\nu} - \epsilon\bar{\nu}^*)\mu = \epsilon\mu\beta \in \text{Hom}_\Lambda(G, F^*)$$

$$(b) \alpha^{-*}\mu = \gamma\beta \in \text{Hom}_\Lambda(G, F)$$

$$(c) \bar{\theta} + \mu^*\bar{\nu}\mu + \beta^*\bar{\theta}\beta = 0 \in Q_{-\epsilon}(G)$$

It is **strong** if $\bar{\nu} = 0$.

iii) A **stable flip-isomorphism of a regular ϵ -quadratic (split) preformation** z is a flip-isomorphism of $z + h$ with h a boundary of a hyperbolic form. (Note that $[z] = [z + h]$ in the l -monoid). \square

Proposition 4.1.2. *Let z be a regular ϵ -quadratic (split) preformation. If z is elementary then z has a flip-isomorphism.*

Proof. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an elementary regular ϵ -quadratic split preformation. We assume that our preformation has the form described in Proposition 2.4.2 iv). There is a flip isomorphism $(\alpha, \beta, \bar{\nu})$ of z given by

$$\begin{aligned} \alpha &= \begin{pmatrix} 0 & -1 \\ -\epsilon & 0 \end{pmatrix} : F = U \oplus U^* \longrightarrow F^* = U^* \oplus U \\ \beta &= \begin{pmatrix} -1 & -\tau \\ 0 & 1 \end{pmatrix} : G = U \oplus R \longrightarrow G = U \oplus R \\ \bar{\nu} &= 0 : F^* \longrightarrow F \end{aligned}$$

In the non-split case the flip-isomorphism is $(\alpha, \beta, 0)$. \square

Corollary 4.1.3. *Let z be a regular ϵ -quadratic split preformation. If z is (stably) elementary there is a (stable) strong flip-isomorphism $(\alpha, \beta, 0)$ such that $\alpha : F \longrightarrow F^*$ is ϵ -symmetric and zero in $L^{2q}(\Lambda)$ (and hence also in $L\text{Asy}^0(\Lambda)$) and $\beta^2 = 1_G$. Similar for the non-split case.*

Corollary 4.1.4. *Let z be a regular ϵ -quadratic split preformation and z' its flip. If z is stably elementary then it has a stable flip-isomorphism and $[z] = [z'] \in l_{2q+2}(\Lambda)$.*

4.2 Translating Flip-Isomorphisms into Algebraic Surgery Theory

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation. Let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism of z .

In the following we fix representatives θ and ν for $\bar{\theta}$ and $\bar{\nu}$ and let $\kappa \in \text{Hom}(G, G^*)$ such that $\beta^*\theta\beta + \theta + \mu^*\nu\mu = \kappa + \epsilon\kappa^* \in \text{Hom}_\Lambda(G, G^*)$. We must of course be aware that once we leave the realm of chain complexes and define concepts about preformations and flip-isomorphisms (that is asymmetric signatures, flip-isomorphism $\text{rel}\partial$ and quadratic signatures) we have to check to what extent they depend on the choice of representatives. (Compare Remark 4.2.1)

In Section 3.1 we translated z into a Poincaré pair x . The first step of this construction was to use the proofs of [Ran80a] Proposition 2.3. and 2.5 to create $(2q+1)$ -dimensional quadratic complexes (N, ζ) and (N', ζ') out of z and its flip z' . Those proofs also suggest that an isomorphism of two preformations gives rise to an isomorphism of those quadratic complexes. Both propositions just cover formations but without problems we can generalize the construction for all regular preformations. Hence t induces an isomorphism $(e_t, \rho_t): (N, \zeta) \longrightarrow (N', \zeta')$ of $(2q+1)$ -dimensional quadratic complexes given by

$$\begin{array}{ccc} N_{q+1} = F & \xrightarrow{e_{t,q+1}=\alpha} & N'_{q+1} = F^* \\ \mu^* \downarrow & & \gamma^* \downarrow \\ N_q = G^* & \xrightarrow{e_{t,q}=\beta^{-*}} & N'_q = G^* \end{array} \quad (4.1)$$

$$\begin{aligned} \zeta_0 &= \gamma: N^q \longrightarrow N_{q+1} \\ \zeta_1 &= \epsilon\theta: N^q \longrightarrow N_q \\ \zeta'_0 &= \epsilon\mu: N'^q \longrightarrow N'_{q+1} \\ \zeta'_1 &= -\epsilon\theta: N'^q \longrightarrow N'_q \\ \rho_{t,0} &= \alpha\nu\alpha^*: N'^{q+1} \longrightarrow N'_{q+1} \\ \rho_{t,1} &= \gamma^*\alpha\nu\alpha^*: N'^{q+1} \longrightarrow N'_q \\ \rho_{t,2} &= -\beta^{-*}\kappa^*\beta^{-1}: N'^q \longrightarrow N'_q \end{aligned}$$

The Poincaré pairs c and c' defined in (3.3) and (3.4) are thickenings of (N, ζ) and (N', ζ') and the isomorphism (e_t, ρ_t) leads to a homotopy equivalence of those two pairs in particular to an isomorphism of the $2q$ -dimensional quadratic Poincaré complexes $(\partial e_t, \partial \rho_t): (C, \psi) = \partial(N, \zeta)$ and $(C', \psi') = \partial(N', \zeta')$ (see Lemma A.2.9). Composing this isomorphism with the inverse of the canonical isomorphism $(h, \chi): (C, \psi) \xrightarrow{\cong} (C', \psi')$ defined in (3.5) yields a self-equivalence $(h_t, \chi_t) = (h, \chi)^{-1}(\partial e_t, \partial \rho_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$

(C, ψ) given by

$$\begin{array}{ccc}
C_{q+1} = G & \xrightarrow{\beta} & C_{q+1} = G \\
\downarrow \begin{pmatrix} -\epsilon\gamma \\ -\epsilon\mu \end{pmatrix} & & \downarrow \begin{pmatrix} -\epsilon\gamma \\ -\epsilon\mu \end{pmatrix} \\
C_q = F \oplus F^* & \xrightarrow{\begin{pmatrix} 0 & \alpha^{-*} \\ \epsilon\alpha & \epsilon\alpha(\nu^* - \epsilon\nu) \end{pmatrix}} & C_q = F \oplus F^* \\
\downarrow \begin{pmatrix} \mu^* & \epsilon\gamma^* \end{pmatrix} & & \downarrow \begin{pmatrix} \mu^* & \epsilon\gamma^* \end{pmatrix} \\
C_{q-1} = G^* & \xrightarrow{\beta^{-*}} & C_{q-1} = G^*
\end{array} \tag{4.2}$$

$$\begin{aligned}
\chi_{t,1} &= \begin{pmatrix} 0 & 0 \\ 1 & -\alpha\nu^*\alpha^* \end{pmatrix} : C^q = F^* \oplus F \longrightarrow C_q = F \oplus F^* \\
\chi_{t,2} &= \begin{pmatrix} 0 \\ \epsilon\mu - \alpha\nu^*\alpha^*\gamma \end{pmatrix} : C^{q-1} = G \longrightarrow C_q = F \oplus F^* \\
\chi_{t,3} &= -\theta + \beta^{-*}\kappa\beta^{-1} : C^{q-1} = G \longrightarrow C_{q-1} = G^*
\end{aligned}$$

(see Definition A.1.5 for inverses and compositions of isomorphisms of quadratic complexes).

Remark 4.2.1. Let $\widehat{\nu}, \widehat{\theta}$ be other representatives for $\nu \in Q_{-\epsilon}(z, \beta)$ and $\theta \in Q_{-\epsilon}(G)$. Then there are $\tilde{\theta} \in \text{Hom}_\Lambda(G, G^*)$, $\tilde{\nu} \in \text{Hom}_\Lambda(F^*, F)$ such that $\widehat{\nu} - \nu = \tilde{\nu} + \epsilon\tilde{\nu}^* \in \text{Hom}_\Lambda(F^*, F)$ and $\widehat{\theta} - \theta = \tilde{\theta} + \epsilon\tilde{\theta}^* \in \text{Hom}_\Lambda(G, G^*)$. Define $\widehat{\kappa} = \kappa + \tilde{\kappa} - \epsilon\tilde{\kappa}^* + \beta^*\tilde{\theta}\beta + \tilde{\theta} + \mu^*\tilde{\nu}\mu \in \text{Hom}_\Lambda(G, G^*)$ for some $\tilde{\kappa} \in \text{Hom}_\Lambda(G, G^*)$.

$\widehat{\theta}, \widehat{\theta}$ and $\widehat{\kappa}$ induce an isomorphism $(h_t, \widehat{\chi}_t) : (C, \widehat{\psi}) \xrightarrow{\cong} (C, \widehat{\psi})$. (Note that h_t and the chain complex C are not affected by the choice of representative.) From Remark 3.1.3 we know that there is a $\tilde{\psi} \in W_{\%}(C)_{2q+1}$ such that $\widehat{\psi} - \psi = d(\tilde{\psi})$. Then $\widehat{\chi}_t - \chi_t = \tilde{\psi} - h_t\tilde{\psi}h_t^* + d\tilde{\chi}$ with $\tilde{\chi} \in W_{\%}(C)_{2q+2}$ given by

$$\begin{aligned}
\tilde{\chi}_2 &= \begin{pmatrix} 0 & 0 \\ 0 & -\alpha\tilde{\nu}^*\alpha^* \end{pmatrix} : C^q = F^* \oplus F \longrightarrow C_q = F \oplus F^* \\
\tilde{\chi}_3 &= -\tilde{\chi}_2^*d^* : C^{q-1} = G \longrightarrow C_q = F \oplus F^* \\
\tilde{\chi}_4 &= \beta^{-*}(\tilde{\theta} + \tilde{\kappa})\beta^{-1} : C^{q-1} = G \longrightarrow C_{q-1} = G^*
\end{aligned}$$

See also Remarks 3.1.3, 6.2.2 and Lemma 6.4.2.

4.3 Quadratic Twisted Doubles

A preformation with a flip-isomorphism is some kind of algebraic equivalent to a $(2q+2)$ -dimensional normal cobordism $(W, M, M') \longrightarrow X$ with a diffeomorphism $h : M \xrightarrow{\cong} M'$ compatible with the highly-connected normal maps $M \longrightarrow X$ and $M' \longrightarrow X$. The

boundary of $W \rightarrow X$ can be replaced by a normal map on the **twisted double** $M \cup_{h|\partial M} M$ into X . As sketched in the introduction this is the starting point for constructions which yield obstructions for (W, M, M') to be an h -cobordism: asymmetric signatures and, if $h|\partial M \simeq 1_{\partial M}$, quadratic signatures. These constructions can be imitated for quadratic Poincaré pairs as we will find out in Chapters 5 and 6. A prerequisite for those constructions is to turn the boundary of the $(2q+2)$ -dimensional ϵ -quadratic Poincaré pair x from Section 3.1 into an algebraic twisted double.

Definition 4.3.1 ([Ran98] 30.8(ii)). The **twisted double** of an n -dimensional ϵ -quadratic Poincaré pair $c = (f: C \rightarrow D, (\delta\psi, \psi))$ over Λ with respect to a self-equivalence $(h, \chi): (C, \psi) \xrightarrow{\cong} (C, \psi)$ is the n -dimensional ϵ -quadratic Poincaré complex over Λ

$$\begin{aligned} c \cup_{(h, \chi)} -c &= (D \cup_h D, \delta\psi \cup_\chi -\delta\psi) \\ &= (fh: C \rightarrow D, (\delta\psi + (-)^{n-1} f\chi f^*, \psi)) \cup (f: C \rightarrow D, (-\delta\psi, -\psi)) \end{aligned}$$

with

$$\begin{aligned} D \cup_h D &= \mathfrak{e} \left(\begin{pmatrix} fh \\ f \end{pmatrix} : C \rightarrow D \oplus D \right), \\ (\delta\psi \cup_\chi \delta\psi)_s &= \begin{pmatrix} \delta\psi_s + (-)^{n-1} f\chi_s f^* & 0 & 0 \\ 0 & -\delta\psi_s & (-)^{s-1} f\psi_s \\ (-)^{n-r} \psi_s h^* f^* & 0 & (-)^{n-r+s+1} T_\epsilon \psi_{s-1} \end{pmatrix} : \\ & (D \cup_h D)^{n-r-s} = D^{n-r-s} \oplus D^{n-r-s} \oplus C^{n-r-s-1} \\ & \rightarrow (D \cup_h D)_r = D_r \oplus D_r \oplus C_{r-1} \end{aligned}$$

□

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation. Let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism of z . Again we pick representatives θ and ν for $\bar{\theta}$ and $\bar{\nu}$ and let $\kappa \in \text{Hom}(G, G^*)$ such that $\beta^* \theta \beta + \theta + \mu^* \nu \mu = \kappa + \epsilon \kappa^* \in \text{Hom}_\Lambda(G, G^*)$.

The twisted double of $c = (f: C \rightarrow D, (\partial\psi, \psi))$ of (3.3) with respect to the self-equivalence $(h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ from (4.2) is the $(2q+1)$ -dimensional quadratic Poincaré complex $(\partial E_t, \omega_t)$ given by $(\partial E_r = D_r \oplus D_r \oplus C_{r-1})$

$$\begin{aligned} \partial E_{t, q+2} &= 0 \oplus 0 \oplus G & (4.3) \\ & \downarrow -\epsilon \begin{pmatrix} \beta \\ 1 \\ \gamma \\ \mu \end{pmatrix} \\ \partial E_{t, q+1} &= G \oplus G \oplus (F \oplus F^*) \\ & \downarrow \begin{pmatrix} -\epsilon\mu & 0 & \alpha & \alpha(\nu^* - \epsilon\nu) \\ 0 & -\epsilon\mu & 0 & \epsilon \\ 0 & 0 & \mu^* & \epsilon\gamma^* \end{pmatrix} \\ \partial E_{t, q} &= F^* \oplus F^* \oplus G^* \end{aligned}$$

$$\omega_{t,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\gamma \\ \alpha^* & 0 & 0 \end{pmatrix} : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^*$$

$$\omega_{t,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \partial E_t^{q+1} = G^* \oplus G^* \oplus (F^* \oplus F) \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^*$$

$$\omega_{t,1} = \begin{pmatrix} -\alpha\nu^*\alpha^* & 0 & 0 \\ 0 & 0 & 0 \\ -\epsilon\gamma^*\alpha^* & 0 & -\epsilon\theta \end{pmatrix} : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^*$$

There is an equivalence $(a_t, \sigma_t) : (\partial E_t, -\omega_t) \xrightarrow{\cong} (\partial E, \omega)$ of $(2q+1)$ -dimensional quadratic Poincaré complexes given by

$$\begin{aligned} a_{t,q+2} &= \beta : \partial E_{t,q+2} = G \longrightarrow \partial E_{q+2} = G \\ a_{t,q+1} &= \begin{pmatrix} 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \alpha^{-*} \\ 0 & 0 & \epsilon\alpha & \epsilon\alpha(\nu^* - \epsilon\nu) \\ 1 & 0 & 0 & 0 \end{pmatrix} : \\ &\partial E_{t,q+1} = G \oplus G \oplus (F \oplus F^*) \longrightarrow \partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G \\ a_{t,q} &= \begin{pmatrix} 0 & \alpha^{-*} & 0 \\ 0 & 0 & \beta^{-*} \\ 1 & 0 & 0 \end{pmatrix} : \partial E_{t,q} = F^* \oplus F^* \oplus G^* \longrightarrow \partial E_q = F \oplus G^* \oplus F^* \\ \sigma_{t,0} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\nu\alpha^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \\ &\partial E^{q+1} = G^* \oplus (F^* \oplus F) \oplus G^* \longrightarrow \partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G \\ \sigma_{t,0} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \partial E^{q+2} = G^* \longrightarrow \partial E_q = F \oplus G^* \oplus F^* \\ \sigma_{t,1} &= \begin{pmatrix} 0 & 0 & \epsilon & 0 \\ -1 & -\gamma^* & -\mu^* + \epsilon\gamma^*\alpha\nu\alpha^* & -1 \\ 0 & 1 & \epsilon\alpha\nu\alpha^* & 0 \end{pmatrix} : \\ &\partial E^{q+1} = G^* \oplus (F^* \oplus F) \oplus G^* \longrightarrow \partial E_q = F \oplus G^* \oplus F^* \\ \sigma_{t,2} &= \begin{pmatrix} 0 & 0 & 0 \\ -\gamma^* & -\epsilon\beta^{-*}\kappa\beta^{-1} & 0 \\ -1 & -\epsilon\mu - \epsilon\alpha\nu\alpha^*\gamma & -\epsilon\alpha\nu\alpha^* \end{pmatrix} : \\ &\partial E^q = F^* \oplus G \oplus F \longrightarrow \partial E_q = F \oplus G^* \oplus F^* \end{aligned}$$

Applying Lemma A.2.6 to the Poincaré pair $x = (g : \partial E \longrightarrow E, (\delta\omega = 0, \omega))$ from Section 3.1 yields a $(2q+2)$ -dimensional quadratic Poincaré pair given by

$$\begin{aligned} x_t &= (g_t = ga_t : \partial E_t \longrightarrow E, (\delta\omega_t, \omega_t)) \\ g_{t,q+1} &= (1 \quad -\beta \quad 0 \quad 0) : \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^* \longrightarrow E_{q+1} = G \\ \delta\omega_t &= -(\delta\omega + (-)^{2q+1}g\sigma g^*) = 0 \end{aligned}$$

and a homotopy equivalence of pairs $(a_t, -1; 0): x_t \longrightarrow -x$.

The equivalence a_t maps each copy of D in ∂E_t isomorphically onto a copy of D and D' in ∂E . Hence x is cobordant rel ∂ to an h -cobordism if and only if x_t is.

4.4 Symmetric Twisted Doubles

The computation of asymmetric signature of a flip-isomorphism (which we will accomplish in Chapter 5) demands a symmetric version of the constructions of the previous section.

Definition 4.4.1 ([Ran98] 30.8(ii)). The **twisted double** of an n -dimensional ϵ -symmetric Poincaré pair $c = (f: C \longrightarrow D, (\delta\phi, \phi))$ over Λ with respect to a self-equivalence $(h, \chi): (C, \phi) \xrightarrow{\cong} (C, \phi)$ is the n -dimensional ϵ -symmetric Poincaré complex over Λ

$$\begin{aligned} c \cup_{(h, \chi)} -c &= (D \cup_h D, \delta\phi \cup_\chi -\delta\phi) \\ &= (fh: C \longrightarrow D, (\delta\phi + (-)^{n-1} f\chi f^*, \phi)) \cup (f: C \longrightarrow D, (-\delta\phi, -\phi)) \end{aligned}$$

with

$$\begin{aligned} D \cup_h D &= \mathfrak{c} \left(\begin{pmatrix} fh \\ f \end{pmatrix} : C \longrightarrow D \oplus D \right), \\ (\delta\phi \cup_\chi -\delta\phi)_s &= \begin{pmatrix} \delta\phi_s + (-)^{n-1} f\chi_s f^* & 0 & 0 \\ 0 & -\delta\phi_s & (-)^{s-1} f\phi_s \\ (-)^{n-r} \phi_s h^* f^* & 0 & (-)^{n-r+s} T_\epsilon \phi_{s-1} \end{pmatrix} : \\ &(D \cup_h D)^{n-r+s} = D^{n-r+s} \oplus D^{n-r+s} \oplus C^{n-r+s-1} \\ &\longrightarrow (D \cup_h D)_r = D_r \oplus D_r \oplus C_{r-1} \end{aligned}$$

□

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation. Let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism of z . Let ν, θ and κ chosen as in the previous section.

We symmetrize our ingredients (h_t, χ_t) and c from the previous section:

The $(2q + 1)$ -dimensional symmetric Poincaré pair which is the product of the symmetrization of c defined in (3.3) is given by

$$\begin{aligned} (1 + T)c &= (f: C \longrightarrow D, (\delta\phi = 0, \phi) = (1 + T)(\delta\psi, \psi)) & (4.4) \\ \phi_0 &= 1: C^{q-1} = G \longrightarrow C_{q+1} = G \\ \phi_0 &= \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} : C^q = F^* \oplus F \longrightarrow C_q = F \oplus F^* \\ \phi_0 &= -\epsilon: C^{q+1} = G^* \longrightarrow C_{q-1} = G^* \end{aligned}$$

The symmetrization of (h_t, χ_t) defined in (4.2) leads to a self-equivalence

$$(h_t, 0): (C, \phi) \xrightarrow{\cong} (C, \phi)$$

of the $2q$ -dimensional symmetric Poincaré complex (C, ϕ) .

The symmetric twisted double $(\partial E_t, \theta_t)$ is given by the chain complex ∂E_t from (4.3) and

$$\begin{aligned}
\theta_{t,0} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -(\nu^* - \epsilon\nu)\alpha^* & 0 & 0 \\ \alpha^* & 0 & 0 \end{pmatrix} : \partial E_t^q = F \oplus F \oplus G \\
&\longrightarrow \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^* \\
\theta_{t,0} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \beta^* & 0 & 0 & 0 \end{pmatrix} : \partial E_t^{q+1} = G^* \oplus G^* \oplus F^* \oplus F \\
&\longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^* \\
\theta_{t,1} &= (0 \ 0 \ \epsilon) : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q+2} = G \\
\theta_{t,1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -\epsilon & 0 \end{pmatrix} : \partial E_t^{q+1} = G^* \oplus G^* \oplus F^* \oplus F \\
&\longrightarrow \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^* \\
\theta_{t,1} &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} : \partial E_t^{q+2} = G^* \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^*
\end{aligned}$$

The twisted double construction and symmetrization are commutative operations up to an equivalence.

Lemma 4.4.2. *Let $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ be an n -dimensional quadratic Poincaré pair and let $(h, \chi): (C, \psi) \longrightarrow (C, \psi)$ be a self-equivalence. Then there is a chain equivalence*

$$(1, \sigma): (1+T)c \cup_{(h, (1+T)\chi)} - (1+T)c \xrightarrow{\cong} (1+T)(c \cup_{(h, \chi)} - c)$$

with $(1+T)c = (f: C \longrightarrow D, (1+T)(\delta\psi, \psi))$ and

$$\begin{aligned}
\sigma_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-)^{r-1} T\psi_0 \end{pmatrix} : \\
&(D \cup_h D)^{n+1-r} = D^{n+1-r} \oplus D^{n+1-r} \oplus C^{n-r} \\
&\longrightarrow (D \cup_h D)_r = D_r \oplus D_r \oplus C_{r-1}
\end{aligned}$$

We apply this lemma and Lemma A.2.6 to the symmetrization of the $(2q+2)$ -dimensional quadratic Poincaré pair x_t defined in the previous section and obtain a $(2q+2)$ -dimensional symmetric Poincaré pair

$$x^t = (g_t: \partial E_t \longrightarrow E, (\delta\theta_t = 0, \theta_t))$$

If x is cobordant rel ∂ to an h -cobordism then so is x^t .

Remark 4.4.3. The construction of x^t and $(h_t, 0): (C, \psi) \longrightarrow (C, \psi)$ just depends on the “symmetrization” of z and t i.e. $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $(\alpha, \beta, (\nu^* - \epsilon\nu)^*)$ (compare Remark 2.2.17). Every choice of representative for $\bar{\nu}$ leads to the same Poincaré pair x^t .

For an ϵ -quadratic preformation $y = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and a flip-isomorphism $s = (\alpha, \beta, \sigma)$, x^s can be constructed in the same way if, in the definition of θ_t and h_t , $(\nu^* - \epsilon\nu)^*$ is replaced by σ .

Chapter 5

Asymmetric Signatures of Flip-Isomorphisms

For the whole chapter let $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

Let (W, M, M) be a cobordism with boundary $M \cup_h M$ for some automorphism h of ∂M . Then an asymmetric signature can be defined which vanishes in the asymmetric Witt-group if and only if W is cobordant $\text{rel}\partial$ to a twisted double. An s -cobordism (W, M, M) is a twisted double, so the asymmetric signature provides an obstruction for W to be cobordant $\text{rel}\partial$ to an s -cobordism.

The previous chapter explained how a flip-isomorphism of a preformation defines a symmetric Poincaré pair with an algebraic twisted double as the boundary. For such a Poincaré pair we can define an asymmetric signature, too. Like in geometry it vanishes if the pair is cobordant $\text{rel}\partial$ to an h -cobordism i.e. if the preformation is stably elementary. Asymmetric signatures of manifolds do not require normal maps or smoothings (see e.g. [Ran98] Corollary 30.12.). Similarly symmetric complexes and pairs suffice as input for the algebraic asymmetric signature, e.g. the symmetric Poincaré pair x^t defined in Section 4.4.

In **Section 5.1** we will give a short introduction into the origins of asymmetric forms and twisted doubles in geometry. Then **Section 5.2** will present the algebraic chain complex analogues. We define asymmetric forms, complexes and pairs and show how a Poincaré pair with a twisted double on the boundary determines an asymmetric complex. We have already seen in Section 4.4 how a preformation and a flip-isomorphism can be used to construct a symmetric Poincaré pair with an algebraic twisted double as the boundary. In **Section 5.3** we compute its asymmetric signature and show in **Section 5.4** that it vanishes for every flip-isomorphism of an elementary preformation. The asymmetric signatures depend on the choice of flip-isomorphism. Hence in **Section 5.5** we define an l -monoid of flip-isomorphism $fl^{2q}(\Lambda)$. The asymmetric signatures define a monoid homomorphism $fl^{2q}(\Lambda) \longrightarrow LAsy^0(\Lambda)$. It turns out that the asymmetric

signatures vanish for any flip-isomorphism of any stably elementary preformation.

In **Section 5.6** we investigate the relationship between the asymmetric signature of strong flip-isomorphisms in special cases.

5.1 Geometric Twisted Doubles and Asymmetric Forms

One of the successes of asymmetric forms in surgery theory was the open book obstruction theory of Quinn ([Qui79]) which led to a second computation of the cobordism ring of automorphism. (The first computation was achieved by Kreck (see e.g. [Kre84]) using an early version of his l -surgery theory.)

We will outline Quinn's approach to the cobordism of automorphism. For simplicity we assume that all manifolds are simply-connected. Let $h: N \xrightarrow{\cong} N$ be a diffeomorphism of a closed $2q$ -dimensional manifold ($q > 1$). The first obvious obstruction for h to be null-cobordant is the cobordism class of the mapping torus $T(h)$.

So let us assume that $T(h)$ has a $(2q + 2)$ -dimensional null-cobordism V and after surgery on V we can make $H_i(V, N)$ vanish for $i < q$. The isomorphism

$$K = H_{q+1}(V, N) \longrightarrow H^{q+1}(V, \partial V - N) \longrightarrow H^{q+1}(V, N \times I) \longrightarrow H^{q+1}(V, N) \quad (5.1)$$

gives us a non-singular asymmetric form $\lambda: K \longrightarrow K^*$ which is zero in the asymmetric Witt group $LAsy^0(\mathbb{Z})$ if and only if there is a diffeomorphism $H: Q \longrightarrow Q$ with $V = T(H)$ and hence $\partial H = h$.

More generally, an exact sequence describes the connection between the cobordism ring of automorphisms and the asymmetric Witt group. Let $\Omega_i(X)$ be the cobordism group of continuous maps from i -dimensional manifolds to X and $\Delta_i(X)$ the group of cobordism classes of triples (F, g, h) with F a closed i -dimensional manifold, $g: F \longrightarrow X$ a map, $h: F \longrightarrow F$ an automorphism together with a homotopy $g \simeq gh$ such that there is an induced map

$$T(g): T(h) \longrightarrow T(1: X \longrightarrow X) = X \times S^1$$

Then for any $k > 2$ and topological space X there is an exact sequence

$$\begin{aligned} 0 &\longrightarrow \Delta_{2k+1}(X) \xrightarrow{T} \Omega_{2k+2}(X \times S^1) \longrightarrow LAsy^0(\mathbb{Z}[\pi_1(X)]) \\ &\longrightarrow \Delta_{2k}(X) \xrightarrow{T} \Omega_{2k+1}(X \times S^1) \longrightarrow 0 \end{aligned}$$

with $T: \Delta_i(X) \longrightarrow \Omega_{i+1}(X \times S^1), (F, g, h) \longmapsto (T(g), T(h))$ (see also [Ran98] 30.6 (iv) or [Qui79]).

In our case we start with an $(n + 1)$ -dimensional cobordism (W, M, M') such that $M' \cong M$ and M may have a boundary. Then there is an isomorphism $h: \partial M \longrightarrow \partial M$ such that the boundary of W is the **twisted double** $M \cup_h M$ (see [Win73] and [Ran98])

Chapter 30). By glueing the ends of the cobordism together we obtain a manifold V with boundary $T(h)$. One can do surgery below the middle dimension to make (V, N) highly-connected and read off an asymmetric form as before. It vanishes in the asymmetric Witt group if and only if V is cobordant to a mapping torus of an automorphism and that is the case if and only if W is cobordant rel ∂ to a twisted double.

It is also possible to define a chain complex version of that construction: an asymmetric Poincaré complex consisting of the singular chain complex $C(\tilde{V}, \partial\tilde{M})$ together with a chain equivalence inducing the isomorphisms $\lambda: H^{n+1-*}(\tilde{V}, \partial\tilde{M}) \xrightarrow{\cong} H_*(\tilde{V}, \partial\tilde{M})$. The maps fit into a diagram of exact sequences

$$\begin{array}{ccccc} H^{n+1-r}(\tilde{V}, \tilde{W}) \cong H^{n-r}(\tilde{M}) & \longrightarrow & H^{n+1-r}(\tilde{V}, \partial\tilde{M}) & \longrightarrow & H^{n+1-r}(\tilde{W}, \partial\tilde{M}) & (5.2) \\ \downarrow \cong & & \downarrow \lambda \cong & & \downarrow \cong & \\ H_r(\tilde{M}, \partial\tilde{M}) & \longrightarrow & H_r(\tilde{V}, \partial\tilde{M}) & \longrightarrow & H_r(\tilde{W}, \tilde{M} + \tilde{M}) & \end{array}$$

In particular, if $\partial M = \emptyset$, the asymmetric complex is $C_*(\tilde{V})$ together with the Poincaré duality on V . One can find a a twisted double cobordant rel ∂ to W if and only if that asymmetric complex is zero in the asymmetric L -group $LAsy^{n+1}(\mathbb{Z}[\pi_1(V)])$ (see e.g. [Ran98] 30.12).

5.2 Asymmetric Forms, Complexes and Pairs

We present the algebraic equivalents of the geometric constructions of the previous section. Note that the asymmetric signatures of manifolds do not require normal maps. Therefore it is not surprising that the asymmetric signatures only require symmetric complexes and pairs and not quadratic ones. (Obviously we can always symmetrize any quadratic complexes, etc. and feed that information into the asymmetric signature construction. Compare Section 4.4.)

For the following compare with [Ran98] Chapter 28F.

Definition 5.2.1. An **asymmetric form** (M, λ) over Λ is a f.g. free Λ -module M and a $\lambda \in \text{Hom}_\Lambda(M, M^*)$. It is **non-singular** if and only if λ is an isomorphism of Λ -modules.

A **lagrangian** L of an **asymmetric form** (M, λ) is a direct summand $L \subset M$ such that $L = L^\perp$ with $L^\perp = \{x \in L \mid \lambda(x)(K) = 0\}$. If an asymmetric form has a lagrangian we call it **metabolic**.

An **isomorphism** $f: (M, \lambda) \xrightarrow{\cong} (M', \lambda')$ of **asymmetric forms** is an isomorphism of Λ -modules $f: M \xrightarrow{\cong} M'$ such that $\lambda' = f^* \lambda f$.

The **asymmetric Witt-group** $LAsy^0(\Lambda)$ is the abelian group of equivalence classes of non-singular asymmetric forms where

$$(N_1, \lambda_1) \sim (N_2, \lambda_2) \Leftrightarrow \exists (M_1, \kappa_1) \oplus (M_2, \kappa_2) \xrightarrow{\cong} (N_2, \lambda_2) \oplus (M_2, \kappa_2)$$

for some non-singular metabolic forms (M_i, κ_i) . \square

Definition 5.2.2. An n -dimensional asymmetric complex (C, λ) over Λ is a chain complex C together with a chain map $\lambda: C^{n-*} \rightarrow C$. (C, λ) is **Poincaré** if λ is a chain equivalence.

A **morphism** $f: (C, \lambda) \rightarrow (C', \lambda')$ of n -dimensional asymmetric complexes is a chain map $f: C \rightarrow C'$ such that there is a chain homotopy $\lambda' \simeq f\lambda f^*: C'^{n-*} \rightarrow C'$. The morphism is an **equivalence** if $f: C \rightarrow C'$ is a chain equivalence.

An $(n+1)$ -dimensional asymmetric pair $(f: C \rightarrow D, (\delta\lambda, \lambda))$ is an n -dimensional asymmetric complex (C, λ) , a chain map $f: C \rightarrow D$ and a chain homotopy $\delta\lambda: f\lambda f^* \simeq 0: D^{n-*} \rightarrow D$. It is **Poincaré** if the chain maps given by

$$\begin{aligned} \begin{pmatrix} \delta\lambda \\ (-)^{r+1}\lambda f^* \end{pmatrix} &: D^{n+1-r} \rightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1} \\ (\delta\lambda \quad (-)^n f\lambda) &: \mathcal{C}(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \rightarrow D_r \end{aligned}$$

are chain equivalences, in which case (C, λ) is Poincaré as well.

Asymmetric Poincaré complexes (C, λ) and (C', λ') are **cobordant** if $(C, \lambda) \oplus (C', -\lambda')$ is the boundary of an asymmetric Poincaré pair.

The **asymmetric L -groups** $L\text{Asy}^n(\Lambda)$ is the cobordism group of n -dimensional Poincaré complexes. \square

Remark 5.2.3. A 0-dimensional asymmetric complex is an asymmetric form. It is Poincaré if the form is non-singular. For details see the errata to [Ran98]. [Ran98] (errata) Proposition 28.34 shows that any $2m$ -dimensional asymmetric Poincaré complex is cobordant to an m -connected $2m$ -dimensional asymmetric Poincaré complex which again is nothing but a 0-dimensional asymmetric Poincaré complex i.e. $L\text{Asy}^{2n}(\Lambda) \cong L\text{Asy}^0(\Lambda)$. (The odd-dimensional asymmetric Witt groups are all trivial.) Hence we will identify asymmetric Poincaré complexes with asymmetric forms.

We explained before that there is a geometric construction to assign an asymmetric form to manifold with a twisted double on a boundary. We will state the algebraic analogue. For that reason we need to define a chain equivalence of a Poincaré pair with a twisted double on the boundary (which we shall call **b-duality map**) modelling the Lefschetz-duality map $H^{n+1-*}(W, \partial M) \xrightarrow{\cong} H_*(W, M + M)$. It is mimicking the diagram of exact sequences with the ordinary Poincaré dualities of our various manifolds

$$\begin{array}{ccccccc} \longrightarrow & H^{n-r}(\partial M) & \longrightarrow & H^{n+1-r}(W, \partial M) & \longrightarrow & H^{n+1-r}(W) & \longrightarrow & (5.3) \\ & \text{P.D.} \downarrow \cong & & \pm & \text{P.D.} \downarrow \cong & & \text{h.P.D.} \downarrow \cong & \\ \longrightarrow & H_{r-1}(\partial M) & \longrightarrow & H_r(W, M + M) & \longrightarrow & H_r(W, \partial W) & \longrightarrow & \end{array}$$

The rules for the cap product show that the first square commutes up to an alternating sign.

Definition and Lemma 5.2.4 ([Ran98] 30.10). Let $x = (g: \partial E \rightarrow E, (\theta, \partial\theta))$ be an $(n+1)$ -dimensional symmetric Poincaré pair such that the boundary $(\partial E, \partial\theta)$ is a twisted double of an n -dimensional symmetric Poincaré pair with respect to a self-equivalence $(h, \chi): (C, \phi) \xrightarrow{\cong} (C, \phi)$ (compare Definition 4.4.1). We write

$$g = (j_0 \ j_1 \ k) : \partial E_r = D_r \oplus D_r \oplus C_{r-1} \rightarrow E_r$$

- i) The **b-duality map** $\kappa: \mathcal{C}(j_0 f)^{n+1-r} \rightarrow \mathcal{C}(j_0, j_1)$ of x is defined (up to chain homotopy) as the chain equivalence which fits into the chain homotopy commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^{n-*} & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & \mathcal{C}(j_0 f)^{n+1-*} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & E^{n+1-r} \longrightarrow 0 \\ & & \downarrow \xi \cong & & \downarrow \kappa \cong & & \downarrow \nu \cong \\ 0 & \longrightarrow & C_{*-1} & \xrightarrow{\beta} & \mathcal{C}(j_0, j_1) & \xrightarrow{\alpha} & \mathcal{C}(g) \longrightarrow 0 \end{array} \quad (5.4)$$

(compare (5.3)) such that

- (a) the first square commutes up to an alternating sign

(b) $\alpha_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} :$

$$\mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1} \rightarrow \mathcal{C}(g)_r = E_r \oplus D_{r-1} \oplus D_{r-1} \oplus C_{r-2}$$

(c) $\beta_r = \begin{pmatrix} k \\ -fh \\ -f \end{pmatrix} : C_{r-1} \rightarrow \mathcal{C}(j_0, j_1)_r$

- (d) $\nu_r = \begin{pmatrix} \theta_0 \\ (-)^{n+1-r} \partial\theta_0 g^* \end{pmatrix} : E^{n+1-r} \rightarrow \mathcal{C}(g)_r = E_r \oplus \partial E_{r-1}$ is the Poincaré duality map of x (see Definition A.2.3)

(e) $\xi_r = \phi_0 h : C^{n-r} \rightarrow C_{r-1}$.

- ii) The **asymmetric complex** (B, λ) of x is (up to chain homotopy) the $(n+1)$ -dimensional asymmetric Poincaré complex with

$$B = \mathcal{C}(j_0 - j_1 : D \rightarrow \mathcal{C}(j_0 f : C \rightarrow E))$$

and $\lambda: B^{n+1-*} \rightarrow B$ a chain equivalence which fits into the chain homotopy commutative diagram of exact sequences (compare (5.2))

$$\begin{array}{ccccccc} 0 & \longrightarrow & D^{n-*} & \xrightarrow{\pi} & B^{n+1-*} & \xrightarrow{\iota} & \mathcal{C}(j_0 f)^{n+1-*} \longrightarrow 0 \\ & & \downarrow \zeta \cong & & \downarrow T\lambda \cong & & \downarrow \kappa \cong \\ 0 & \longrightarrow & \mathcal{C}(f) & \xrightarrow{\tau} & B & \xrightarrow{\sigma} & \mathcal{C}(j_0, j_1) \longrightarrow 0 \end{array} \quad (5.5)$$

with

- (a) $\pi_r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : D^{n-r} \longrightarrow B^{n+1-r} = E^{n+1-r} \oplus C^{n-r} \oplus D^{n-r}$
- (b) $\iota_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : B^{n+1-r} \longrightarrow \mathcal{C}(j_0 f)^{n+1-r} = E^{n+1-r} \oplus C^{n-r}$
- (c) $\tau_r = \begin{pmatrix} j_0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow B_r = E_r \oplus C_{r-1} \oplus D_{r-1}$
- (d) $\sigma_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 1 \\ 0 & 0 & -1 \end{pmatrix} : B_r \longrightarrow \mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1}$
- (e) $\zeta_r = \begin{pmatrix} \delta\phi_0 \\ (-)^{n-r} \phi_0 f^* \end{pmatrix} : D^{n-r} \longrightarrow \mathcal{C}(f)_r$ the Poincaré duality map of $(f: C \longrightarrow D, (\delta\phi, \phi))$ (see Definition A.2.3)
- (f) $T\lambda_r = (-)^{rn} \lambda^* : B^{n+1-r} \longrightarrow B_r$ the duality involution of λ
- (g) κ the b-duality map.

iii) **The asymmetric signature of x** is the asymmetric cobordism class

$$\sigma^*(x) = [(B, \lambda)] \in LAsy^{n+1}(\Lambda)$$

Proof. All we need to do is to show that the vertical sequences of Diagram (5.4) and (5.5) are exact. First we notice that $\mathcal{C}(\beta: C_{*-1} \longrightarrow \mathcal{C}(j_0, j_1)) = \mathcal{C}(g)$. Hence the bottom sequence of Diagram (5.4) is the mapping cone sequence of β and therefore exact.

In the case of the bottom sequence of Diagram (5.5) we note that every element of $\ker \tau_r$ is in the image of σ_r . On the other side $\sigma_r \circ \tau_r$ is null-homotopic:

$$\begin{aligned} \sigma_r \circ \tau_r = d_{\mathcal{C}(j_0, j_1)} \Delta_{r+1} + \Delta_r d_{\mathcal{C}(f)} & : \mathcal{C}(f)_r \longrightarrow \mathcal{C}(j_0, j_1)_r \\ \Delta_r = \begin{pmatrix} 0 & 0 \\ (-)^{r-1} & 0 \\ 0 & 0 \end{pmatrix} & : \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2} \\ & \longrightarrow \mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1} \end{aligned}$$

□

Remark 5.2.5. The asymmetric signature is vanishing if and only if one can extend the twisted double structure on the boundary to the whole Poincaré pair x and it is invariant under cobordism (see [Ran98] Proposition 30.11). We will only need certain properties: Proposition 5.4.4 states that two Poincaré pairs which are cobordant rel ∂ have the same asymmetric signature and Proposition 5.4.5 shows that the asymmetric signature of an algebraic h -cobordism is zero.

Now we will present an explicit formula for the asymmetric complex (B, λ) .

Proposition 5.2.6. *The chain maps κ and λ in the previous definition are given (up to chain homotopy) by*

$$\kappa_r = \begin{pmatrix} \theta_0 & (-)^{n-r} k\phi_0 h^* \\ (-)^{n-r+1}(\delta\phi_0 + (-)^{n-1} f\chi_0 f^*)j_0^* & (-)^{n-r+1} fh\phi_0 h^* \\ (-)^{n-r}(\delta\phi_0 j_1^* + f\phi_0 k^*) & (-)^{n-r+1} f\phi_0 h^* \end{pmatrix} : \\ \mathcal{C}(j_0 f)^{n+1-r} = E^{n+1-r} \oplus C^{n-r} \longrightarrow \mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1}$$

$$T\lambda_r = \begin{pmatrix} \theta_0 & (-)^{n-1} j_0 f\chi_0 + (-)^{n-r} k\phi_0 h^* & j_1 \delta\phi_0 \\ (-)^{n-r} \phi_0 k^* & (-)^{n-r+1} \phi_0(1 + h^*) & (-)^{n-r} \phi_0 f^* \\ (-)^{n-r+1}(\delta\phi_0 j_0^* + f\phi_0 k^*) & (-)^{n-r} f\phi_0 h^* & (-)^{n-r+1} f\phi_0 f^* \end{pmatrix} : \\ B^{n+1-r} = E^{n+1-r} \oplus C^{n-r} \oplus D^{n-r} \longrightarrow B_r = E_r \oplus C_{r-1} \oplus D_{r-1}$$

Proof. First we check that our definition of κ makes the right square of Diagram (5.4) commute i.e. whether $Z = \alpha\kappa - \nu(1 \ 0) : \mathcal{C}(j_0 f)^{n+1-*} \longrightarrow \mathcal{C}(g)$ is null-homotopic. We define

$$\Delta_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & (-)^n \phi_0 h^* \end{pmatrix} : \mathcal{C}(j_0 f)^{n+2-r} = E^{n+2-r} \oplus C^{n+1-r} \\ \longrightarrow \mathcal{C}(g)_r = E_r \oplus D_{r-1} \oplus D_{r-1} \oplus C_{r-2}$$

and find that $Z_r = d_{\mathcal{C}(g)}\Delta_{r+1} + \Delta_r d_{\mathcal{C}(j_0 f)^{n+1-*}} \mathcal{C}(j_0 f)^{n+1-r} \longrightarrow \mathcal{C}(g)_r$.

The left square commutes up to an alternating sign because

$$(-)^{n-r} \beta_r \xi_r = \kappa_r \begin{pmatrix} 0 \\ 1 \end{pmatrix} : C^{n-r} \longrightarrow \mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1}.$$

This choice of κ helps us to confirm that our formula of λ fits into Diagram (5.5).

$Y = T\lambda\pi - \tau\zeta : D^{n-*} \longrightarrow B$ is null-homotopic with chain homotopy

$$\Delta'_r : \begin{pmatrix} 0 \\ 0 \\ (-)^r \delta\phi_0 \end{pmatrix} : D^{n-r+1} \longrightarrow B_r = E_r \oplus C_{r-1} \oplus D_{r-1}$$

which fulfils

$$Y_r = d_B \Delta'_{r+1} + \Delta'_r d_{D^{n-*}} : D^{n-r} \longrightarrow B_r$$

Hence the left square of Diagram (5.5) commutes up to homotopy. For the other square define $Z = \kappa\iota - \sigma T\lambda : B^{n+1-*} \longrightarrow \mathcal{C}(j_0, j_1)$. Then

$$\Delta''_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r+1} f\chi_0 & 0 \\ 0 & 0 & (-)^r \delta\phi_0 \end{pmatrix} : B^{n+1-r} = E^{n+1-r} \oplus C^{n-r} \oplus D^{n-r} \\ \longrightarrow \mathcal{C}(j_0, j_1)_r = E_r \oplus D_{r-1} \oplus D_{r-1}$$

defines a chain null homotopy of Z i.e.

$$Z_r = d_{\mathcal{C}(j_0, j_1)} \Delta''_{r+1} + \Delta''_r d_{B^{n+1-*}}$$

Finally, one has to check that λ is a chain map. Because Z is obviously a chain map and ι consists of injective module homomorphisms it follows easily that κ is a chain map as well. By the five-lemma the chain maps λ and κ have to be chain equivalences. \square

In the case of $\partial E = 0$ or $C = 0$ the asymmetric complex (B, λ) is equivalent to the obvious symmetric complexes.

Lemma 5.2.7. *Let (C, ϕ) be an ϵ -symmetric n -dimensional Poincaré complex. Then the identity induces an equivalence $1: (C, \phi_0) \xrightarrow{\simeq} (C, T_\epsilon \phi_0)$ of n -dimensional asymmetric complexes. (i.e. $(C, \phi_0) = (C, T_\epsilon \phi_0) \in LAsy^n(\Lambda)$)*

Proof. The symmetric structure ϕ of C fulfils the relation

$$d\phi_1 + (-)^r \phi_1 d^* + (-)^n (\phi_0 - T_\epsilon \phi_0) = 0: C^{n-r} \longrightarrow C_r$$

Hence

$$\phi_0 - T_\epsilon \phi_0 = d((-)^n \phi_1) + ((-)^n \phi_1) \underbrace{((-)^r d^*)}_{d_{C^{n-r}}}$$

shows that the identity on C induces a chain equivalence between (C, ϕ_0) and (C', ϕ_1) (and also defines an asymmetric cobordism $((1, 1): C \oplus C \longrightarrow C, ((-)^n \phi_1, \phi_0 \oplus -T_\epsilon \phi_0))$.) \square

Corollary 5.2.8. *We use the notation of Definition 5.2.4.*

i) *Let $\partial E = 0$. Then (B, λ) and (E, θ_0) are equivalent.*

ii) *Let $C = 0$. Let (V, σ) be the union of the fundamental n -dimensional symmetric Poincaré pair $(g = (j_0, j_1): D \oplus D \longrightarrow E, (\theta, \partial\theta = \delta\phi \oplus -\delta\phi))$ (see Definition A.3.4). Then $(B, \lambda) = (V, \sigma_0) \in LAsy^{n+1}(\Lambda)$.*

Proof. i) If $\partial E = 0$ then $T\lambda_r = \theta_0: B^{n+1-r} = E^{n+1-r} \longrightarrow B_r = E_r$.

ii) In the case of $C = 0$ we compute

$$T\lambda_r = \begin{pmatrix} \theta_0 & j_1 \delta \phi_0 \\ (-)^{n+1-r} \delta \phi_0 j_0^* & 0 \end{pmatrix} : B^{n+1-r} = E^{n+1-r} \oplus D^{n-r} \\ \longrightarrow B_r = E_r \oplus D_{r-1}$$

and $B = \mathcal{C}(j_0 - j_1: D \longrightarrow E)$. Then use the previous Lemma. \square

5.3 The Asymmetric Signature of a Flip-Isomorphism

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation. Let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism of z . Let ν be a representative of $\bar{\nu}$. As an abbreviation, define $\sigma = (\nu - \epsilon\nu^*)^*$. σ is independent of the choice of representative for $\bar{\nu}$.

The $(2q+2)$ -dimensional symmetric Poincaré pair $x^t = (g^t: \partial E_t \rightarrow E, (0, \theta_t))$ of Section 4.4 has a twisted double structure on its boundary which enables us to apply the asymmetric signature construction from Proposition 5.2.6. The result is the $(2q+2)$ -dimensional asymmetric complex (B, λ) . ($B_r = E_r \oplus C_{r-1} \oplus D_{r-1}$):

$$\begin{array}{ccc}
 B^q = 0 \oplus G \oplus 0 & \xrightarrow{\lambda_{q+2}} & B_{q+2} = 0 \oplus G \oplus G & (5.6) \\
 \downarrow \epsilon d_0^* & & \downarrow d_1 = -\epsilon \begin{pmatrix} 1 & 1+\beta \\ \gamma & 0 \\ \mu & 0 \\ 0 & \mu \end{pmatrix} \\
 B^{q+1} = G^* \oplus (F^* \oplus F) \oplus F & \xrightarrow{\lambda_{q+1}} & B_{q+1} = G \oplus (F \oplus F^*) \oplus F^* \\
 \downarrow -\epsilon d_1^* & & \downarrow d_0 = (0 \ \mu^* \ \epsilon\gamma^* \ 0) \\
 B^{q+2} = 0 \oplus G^* \oplus G^* & \xrightarrow{\lambda_q} & B_q = 0 \oplus G^* \oplus 0
 \end{array}$$

with

$$\begin{aligned}
 \lambda_{q+2} &= \begin{pmatrix} -\epsilon(1+\beta) \\ \epsilon \end{pmatrix} \\
 \lambda_{q+1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon\alpha^{-*} & 1 & 0 \\ 0 & \epsilon + \alpha\sigma & \epsilon\alpha & -\epsilon\alpha \\ 0 & -\epsilon & 0 & 0 \end{pmatrix} \\
 \lambda_q &= (1 + \beta^{-*} \quad -\beta^{-*})
 \end{aligned}$$

We can reduce this complex to a smaller $(2q+2)$ -dimensional asymmetric complex (B', λ') via the chain equivalence

$$\begin{array}{ccccc}
 B_{q+2} = G \oplus G & \xrightarrow[\Psi_2]{\cong} & G \oplus G & \xrightarrow{(0 \ 1)} & B'_{q+2} = G \\
 \downarrow d_1 & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & d'_1 \end{pmatrix} & & \downarrow d'_1 \\
 B_{q+1} = G \oplus (F \oplus F^*) \oplus F^* & \xrightarrow[\Psi_1]{\cong} & G \oplus (F \oplus F^*) \oplus F^* & \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} & B'_{q+1} = F \oplus F^* \oplus F^* \\
 \downarrow d_0 & & \downarrow (0 \ d'_0) & & \downarrow d'_0 \\
 B_q = G^* & \xrightarrow[\Psi_0]{\cong} & G^* & \xrightarrow{1} & B'_q = G^*
 \end{array}$$

with

$$\begin{aligned}
\Psi_2 &= \begin{pmatrix} 1 & 1+\beta \\ 0 & 1 \end{pmatrix} \\
\Psi_1 &= -\epsilon \begin{pmatrix} 1 & 0 & 0 & 0 \\ \gamma & -1 & 0 & -\alpha^{-*} \\ 0 & 0 & 0 & 1 \\ \mu - \epsilon\alpha\gamma & \epsilon\alpha & -1 & \epsilon\alpha\alpha^{-*} - 1 - \epsilon\alpha\sigma \end{pmatrix} \\
\Psi_0 &= 1 \\
d'_1 &= \begin{pmatrix} \gamma \\ \mu \\ 0 \end{pmatrix} \\
d'_0 &= (\epsilon(1 + \beta^{-*})\mu^* \quad (1 + \beta^{-*})\gamma^* \quad \gamma^*) \\
\lambda'_{q+2} &= \epsilon: B'^q \longrightarrow B'_{q+2} \\
\lambda'_{q+1} &= \begin{pmatrix} 0 & 0 & 1 \\ \epsilon & 0 & -\alpha^* \\ -\epsilon & \epsilon\alpha & \alpha^* - \epsilon\alpha + \epsilon\alpha\sigma\alpha^* \end{pmatrix}: B'^{q+1} \longrightarrow B'_{q+1} \\
\lambda'_q &= -\beta^{-*}: B'^{q+2} \longrightarrow B'_q
\end{aligned}$$

(All λ_r are in fact isomorphisms of Λ -modules.)

With the help of [Ran98] (errata) 28.34, we compute a highly-connected $(2q + 2)$ -dimensional asymmetric complex (B'', λ'') which is cobordant to the asymmetric complex (B', λ') :

$$\begin{array}{ccc}
B''^q & \xrightarrow{\lambda''_{q+2}} & B''_{q+2} = G \\
\downarrow cd_0''^* & & \downarrow d_1'' \\
B''^{q+1} & \xrightarrow{\lambda''_{q+1}} & B''_{q+1} = F \oplus F^* \oplus F^* \oplus G^* \oplus G \\
\downarrow -\epsilon d_1''^* & & \downarrow d_0'' \\
B''^{q+2} & \xrightarrow{\lambda''_q} & B''_q = G^*
\end{array}$$

with

$$\begin{aligned}
d''_1 &= \begin{pmatrix} \gamma \\ \mu \\ 0 \\ 0 \\ \beta^{-1} \end{pmatrix} \\
d''_0 &= (\epsilon(1 + \beta^{-*})\mu^* \quad (1 + \beta^{-*})\gamma^* \quad \gamma^* \quad -\epsilon\beta^{-*} \quad 0) \\
\lambda''_{q+1} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \epsilon & 0 & -\alpha^* & 0 & 0 \\ -\epsilon & \epsilon\alpha & \alpha^* - \epsilon\alpha + \epsilon\alpha\sigma\alpha^* & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon\beta^{-*} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

We can simplify this asymmetric complex to gain the asymmetric form we were looking for using the isomorphism of chain complexes

$$\begin{array}{ccc}
B''_{q+2} = G & \xrightarrow{\Phi_2} & G \\
\downarrow d''_1 & & \downarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
B''_{q+1} = F \oplus F^* \oplus F^* \oplus G^* \oplus G & \xrightarrow{\Phi_1} & F^* \oplus F \oplus F^* \oplus G^* \oplus G \\
\downarrow d''_0 & & \downarrow (0 \ 0 \ 0 \ 1 \ 0) \\
B''_q = G^* & \xrightarrow{\Phi_0} & G^*
\end{array}$$

with

$$\begin{aligned}
\Phi_2 &= 1 \\
\Phi_1 &= \begin{pmatrix} \alpha & 0 & 0 & 0 & -\alpha\gamma\beta \\ 0 & \epsilon\alpha^{-*} & 0 & 0 & -\epsilon\alpha^{-*}\mu\beta \\ 0 & 1 & 1 & 0 & -\mu\beta \\ \epsilon(1 + \beta^{-*})\mu^* & (1 + \beta^{-*})\gamma^* & \gamma^* & -\epsilon\beta^{-*} & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix} \\
\Phi_0 &= 1
\end{aligned}$$

Thus the asymmetric signature is given by the asymmetric form

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha\sigma\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^*$$

It is clear that ρ does not depend on the choice of representative for $\bar{\nu}$. Hence we can define

Definition 5.3.1. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an ϵ -quadratic split preformation. The asymmetric signature $\sigma^*(z, t)$ of a flip-isomorphism $t = (\alpha, \beta, \bar{\nu})$ of z is an element $(M, \rho) \in LAsy^0(\Lambda)$ given by

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha(\nu^* - \epsilon\nu)\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^*$$

□

Obviously we can define an asymmetric signature also for the non-split case

Definition 5.3.2. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be a regular ϵ -quadratic preformation. The asymmetric signature $\sigma^*(z, t)$ of a flip-isomorphism $t = (\alpha, \beta, \sigma)$ of z is an element $(M, \rho) \in LAsy^0(\Lambda)$ given by

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha\sigma\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^*$$

□

5.4 Asymmetric Signatures and Elementariness

In this section we show that the asymmetric signatures are an obstruction to elementariness

Theorem 5.4.1. *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation which allows flip-isomorphisms. If z is elementary then the asymmetric signature $\sigma^*(z, t) \in LAsy^0(\Lambda)$ vanishes for all flip-isomorphisms t .*

Remark 5.4.2. Theorem 5.5.3 will present a version for stably elementary preformations. The converse is not true in general. Counterexamples are presented in Example 7.3.3 and in Section 9.6.

The asymmetric signatures are generally not-trivial as Corollary 7.3.2, Propositions 9.6.7 and 9.6.8 will show.

We will give two proofs for this theorem. The first one is based on algebraic surgery theory whereas the second proof is a low-level calculation of asymmetric forms.

The Definition 5.3.1 shows that the asymmetric signatures only depend on the underlying non-split preformation. It ignores the quadratic structures of both preformation and flip-isomorphism. A generalization of Theorem 5.4.1 for non-split preformations comes without surprise:

Corollary 5.4.3. *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be a regular ϵ -quadratic preformation which allows flip-isomorphisms. If z is elementary then the asymmetric signature $\sigma^*(z, t) \in LAsy^0(\Lambda)$ vanishes for all flip-isomorphisms t .*

Proof. The second proof for Theorem 5.4.1 works also for the non-split case. There should be no problem in using algebraic surgery again - one “only” needs to prove symmetric versions of the previous two chapters. We leave this as an exercise to the reader. \square

The first proof of Theorem 5.4.1 needs some preparation. In the next two propositions we show algebraic equivalents of the following facts from the world of manifolds:

- i) Two manifolds with a twisted double on their boundary have the same asymmetric signature if they are cobordant rel ∂ .
- ii) An s -cobordism (W, M, M) is in fact a twisted double and hence its asymmetric signature must vanish.

Proposition 5.4.4. *Let $x = (g: \partial E \longrightarrow E, (\theta, \partial\theta))$ and $x' = (g': \partial E \longrightarrow E', (\theta', \partial\theta'))$ be two $(n + 1)$ -dimensional symmetric Poincaré pairs such that the boundary $(\partial E, \partial\theta)$ is a twisted double of an n -dimensional symmetric Poincaré pair $(f: C \longrightarrow D, (\delta\phi, \phi))$ with respect to a homotopy self-equivalence $(h, \chi): (C, \phi) \xrightarrow{\simeq} (C, \phi)$ (compare Definition 4.4.1).*

i) If x and x' are cobordant $\text{rel}\partial$, then $\sigma^*(x) = \sigma^*(x') \in L\text{Asy}^{n+1}(\Lambda)$.

ii) $\sigma^*(x) - \sigma^*(x') = \sigma^*(x \cup -x') \in L\text{Asy}^{n+1}(\Lambda)$.

Proof. i) This is a special case of [Ran98] Proposition 30.11(iii).

ii) By using the union construction one can easily verify that $((x \cup -x') + x') \cup x = (x \cup -x') + (-x \cup -x')$. In this formula the sum $(C, \phi) + (f: D \rightarrow E, (\partial\theta, \theta))$ of an $(n+1)$ -dimensional ϵ -symmetric (Poincaré) complex with an $(n+1)$ -dimensional ϵ -symmetric (Poincaré) pair is the $(n+1)$ -dimensional ϵ -symmetric (Poincaré) pair $((\begin{smallmatrix} f \\ 0 \end{smallmatrix}): D \rightarrow E \oplus C, (\partial\theta \oplus \phi, \theta))$.

For any Poincaré complex (C, ϕ) (and in particular for $x \cup -x'$) $((1 \ -1): C \oplus C \rightarrow C, (0, \phi \oplus -\phi))$ defines a null-cobordism of $(C, \phi) \oplus (C, -\phi)$. Hence the pairs x and $(x \cup -x') + x'$ are cobordant $\text{rel}\partial$. Therefore $\sigma^*(x) = \sigma^*((x \cup -x') + x') \in L\text{Asy}^{n+1}(\Lambda)$. It is not hard to see that the latter expression is the same as $\sigma^*(x \cup x') + \sigma^*(x')$.

□

Proposition 5.4.5. *Let $x = (g: \partial E \rightarrow E, (\theta, \partial\theta))$ be an $(n+1)$ -dimensional symmetric Poincaré pair such that the boundary $(\partial E, \partial\theta)$ is a twisted double of an n -dimensional symmetric Poincaré pair $(f: C \rightarrow D, (\delta\phi, \phi))$ with respect to a self-equivalence $(h, \chi): (C, \phi) \rightarrow (C, \phi)$ (compare Definition 4.4.1). We write*

$$g = (j_0 \ j_1 \ k): \partial E_r = D_r \oplus D_r \oplus C_{r-1} \rightarrow E_r$$

Additionally assume that x is an h -cobordism i.e. that $j_0, j_1: D \rightarrow E$ are chain equivalences. Then $\sigma^(x) = 0 \in L\text{Asy}^{n+1}(\Lambda)$.*

Proof. We could refer to [Ran98] Proposition 30.11(ii) but instead we give a quick and direct proof of the claim. Obviously it is enough to construct an asymmetric null-cobordism for the asymmetric Poincaré complex $(B, T\lambda)$ given in Proposition 5.2.6.¹ We define the $(n+2)$ -dimensional asymmetric Poincaré pair

$$\begin{aligned} (s: B &\rightarrow D_{*-1}, (\delta\lambda, \lambda)) \\ s &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}: B_r = E_r \oplus C_{r-1} \oplus D_{r-1} \rightarrow D_{r-1} \\ \delta\lambda &= (-)^{r+1} \delta\phi_0: D^{(n+2)-r-1} \rightarrow D_{r-1} \end{aligned}$$

In order to proof that it is Poincaré one observes that there is a chain equivalence $\mathcal{C}(s) \simeq \mathcal{C}(j_0 f)_{*-1}$ given by

$$\begin{aligned} \begin{pmatrix} j_0 - j_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} &: \mathcal{C}(s)_r = D_{r-1} \oplus E_{r-1} \oplus C_{r-2} \oplus D_{r-2} \\ &\longrightarrow \mathcal{C}(j_0 f)_{r-1} = E_{r-1} \oplus C_{r-2} \end{aligned}$$

¹In general, if $(f: C \rightarrow D, (\delta\lambda, \lambda))$ is an n -dimensional asymmetric (Poincaré) pair, so is $(f: C \rightarrow D, (T_e \delta\lambda, T_e \lambda))$.

□

First proof of Theorem 5.4.1. By Theorem 3.3.3 the Poincaré pair $x = (g: \partial E \rightarrow E, (\delta\omega = 0, \omega))$ from Section 3.1 is cobordant rel ∂ to an algebraic h -cobordism. By Section 4.4 the Poincaré pair $x_t = (g: \partial E_t \rightarrow E, (0, \partial\theta_t))$ is cobordant rel ∂ to an algebraic h -cobordism. By Proposition 5.4.5 its asymmetric signature (which is $\sigma^*(z, t) \in LAsy^0(\Lambda)$ by Section 5.3) is vanishing. □

Second proof of Theorem 5.4.1. We can also give a proof of the theorem without algebraic surgery theory.² For simplicity we assume that $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ has the nice form presented in Proposition 2.4.2iv) and that $i: U \hookrightarrow G$ is the inclusion of the h -lagrangian. We define a metabolic asymmetric form (M', ρ') :

$$\begin{aligned} a &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow F = U \oplus U^* \\ b &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow F^* = U^* \oplus U \\ t &= \begin{pmatrix} 1 & 0 \\ \tau^* & 1 \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow G^* = U^* \oplus R^* \\ k &= \begin{pmatrix} \alpha a \\ 0 \\ -\epsilon\alpha a - \epsilon b \end{pmatrix} : G^* \longrightarrow F^* \oplus F \oplus F^* \\ \rho' &= \begin{pmatrix} \rho & k & 0 \\ 0 & 0 & -\epsilon\beta \\ 0 & t & 0 \end{pmatrix} : M' = M \oplus G^* \oplus G \longrightarrow M'^* = M^* \oplus G \oplus G^* \end{aligned}$$

A lagrangian for (M', ρ') is given by

$$j = \begin{pmatrix} \epsilon\gamma\beta i & \epsilon\gamma(1+\beta) & 0 \\ \epsilon\alpha\gamma i & \epsilon\alpha\gamma\beta^{-1} & \epsilon\alpha \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma^*\alpha \\ -\epsilon i & -\epsilon & 0 \end{pmatrix} : U \oplus G \oplus F \longrightarrow M' = (F \oplus F^* \oplus F) \oplus G^* \oplus G$$

Then $(M, \rho) \oplus (M', -\rho')$ has a lagrangian

$$l = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} : M \oplus G \longrightarrow M \oplus M' = M \oplus M \oplus G^* \oplus G$$

□

²Here is the recipe for how the rather abstract theory of asymmetric complexes and pairs can be used to produce the following explicit stable lagrangian for our asymmetric signature: One can express the explicit algebraic h -cobordism $x' = (g': \partial E \rightarrow E', (\omega', \partial\omega))$ from the proof of Theorem 3.3.3 as a twisted double using the isomorphism $(a, \kappa): (\partial E, -(1+T)\partial\omega) \xrightarrow{\cong} (\partial E_t, \partial\theta_t)$ from Section 4.4. One can compute its asymmetric signature in much the same way as in Section 5.3 and surprisingly one gets the same result. Now one has to keep track of all the manipulations one did: equivalences of asymmetric complexes and the use of [Ran98] Proposition 28.34 leads to an asymmetric cobordism. Finally one glues the asymmetric null-cobordism one gets from Proposition 5.4.5 onto it. This leads to an asymmetric null-cobordism of the asymmetric form (M, ρ) (see Definition 5.3.1). Using a method presented in the Errata to [Ran98] one finally computes a stable lagrangian of (M, ρ) .

5.5 The Flip- l -Monoids

The asymmetric signature of a flip-isomorphism suggests the definition of an extension of Kreck's l -monoids which includes a choice of a stable flip-isomorphisms.

Definition 5.5.1. i) Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*)$ be regular ϵ -quadratic preformations and let $t = (\alpha, \beta, \sigma)$ and $t' = (\alpha', \beta', \sigma')$ be flip-isomorphisms of z and z' respectively.

An isomorphism (η, ζ) of the tuples (z, t) and (z', t') is a strong isomorphism $(\eta, \zeta): z \xrightarrow{\cong} z'$ of preformations such that $\alpha' = \eta^{-*} \alpha \eta^{-1}$, $\beta' = \zeta \beta \zeta^{-1}$ and $\sigma' = \eta \sigma \eta^*$.

The sum $(z, t) + (z', t')$ is the well-defined tuple $(z \oplus z', t \oplus t')$.

Let $y^k = \partial H^{-\epsilon}(\Lambda^k)$ be a hyperbolic preformation and $t^k = \left(\begin{pmatrix} 0 & -\epsilon \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$ a (strong) flip-isomorphism of y^k .³

A stable isomorphism of the tuples (z, t) and (z', t') is an isomorphism of $(z, t) + (y^k, t^k)$ with $(z', t') + (y^l, t^l)$ for some $k, l \in \mathbb{N}_0$. The stable isomorphism classes form an abelian monoid namely the **flip- l -monoid** $fl^{2q+2}(\Lambda)$.

ii) Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \theta')$ be regular ϵ -quadratic split preformations and let $t = (\alpha, \beta, \bar{\nu})$ and $t' = (\alpha', \beta', \bar{\nu}')$ be flip-isomorphisms of z and z' respectively.

An isomorphism (η, ζ) of the tuples (z, t) and (z', t') is a strong isomorphism $(\eta, \zeta): z \xrightarrow{\cong} z'$ of preformations such that $\alpha' = \eta^{-*} \alpha \eta^{-1}$, $\beta' = \zeta \beta \zeta^{-1}$ and $\bar{\nu}' = \eta \bar{\nu} \eta^*$.

The sum $(z, t) + (z', t')$ is the well-defined tuple $(z \oplus z', t \oplus t')$.

Let $y_k = \partial H_{-\epsilon}(\Lambda^k)$ be a hyperbolic preformation and $t_k = \left(\begin{pmatrix} 0 & -\epsilon \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$ a strong flip-isomorphism of y_k .⁴

A stable isomorphism of the tuples (z, t) and (z', t') is an isomorphism of $(z, t) + (y_k, t_k)$ with $(z', t') + (y_l, t_l)$ for some $k, l \in \mathbb{N}_0$. The stable isomorphism classes form an abelian monoid namely the **flip- l -monoid** $fl_{2q+2}(\Lambda)$. \square

Remark 5.5.2. i) There are well-defined morphisms of abelian monoids

$$\begin{aligned} \pi: fl^{2q+2}(\Lambda) &\longrightarrow l^{2q+2}(\Lambda), & [(z, t)] &\longmapsto [z] \\ \pi: fl_{2q+2}(\Lambda) &\longrightarrow l_{2q+2}(\Lambda), & [(z, t)] &\longmapsto [z] \end{aligned}$$

³Compare Corollary 4.1.3.

⁴Compare Corollary 4.1.3.

ii) There is a well-defined morphism of abelian monoids

$$\begin{aligned} fl_{2q+2}(\Lambda) &\longrightarrow fl^{2q+2}(\Lambda) \\ ((F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}), (\alpha, \beta, \bar{\nu})) &\longmapsto ((F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*), (\alpha, \beta, \bar{\nu}^* - \epsilon \bar{\nu}^*)) \end{aligned}$$

Theorem 5.5.3. *The asymmetric signature of Definition 5.3.1 gives rise to a well-defined homomorphism of abelian monoids*

$$\begin{aligned} \sigma^*: fl^{2q+2}(\Lambda) &\longrightarrow LAsy^0(\Lambda) \\ [(z, t)] &\longmapsto \sigma^*(z, t) \end{aligned}$$

If $[z'] \in fl^{2q+2}(\Lambda)$ is elementary then $\sigma^*(\pi^{-1}[z']) = \{0\}$ (with π as in Remark 5.5.2) i.e. $\sigma^*(z, t) = 0$ for all flip-isomorphisms t of all preformations z with $[z] = [z'] \in fl_{2q+2}(\Lambda)$ (i.e. for all stable flip-isomorphisms t of z).

Proof. Using the notation of Definition 5.5.1 assume that there is an isomorphism (η, ζ) of (z, t) and (z', t') . Let

$$\begin{aligned} \rho &= \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha(\nu^* - \epsilon\nu)\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^* \\ \rho' &= \begin{pmatrix} 0 & 0 & \alpha' \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha'(\nu'^* - \epsilon\nu')\alpha'^* \end{pmatrix} : M' = F' \oplus F'^* \oplus F' \longrightarrow M'^* \end{aligned}$$

be the asymmetric forms whose image in $LAsy^0(\Lambda)$ are by definition the asymmetric signatures of (z, t) and (z', t') . Then $f = \begin{pmatrix} \eta^{-1} & 0 & 0 \\ 0 & \eta^* & 0 \\ 0 & 0 & \eta^{-1} \end{pmatrix} : M \xrightarrow{\cong} M'$ is an isometry of the asymmetric forms (M, ρ) and (M', ρ') . So $\sigma^*(z, t) = \sigma^*(z', t')$.

Clearly $\sigma^*(y_k, t_k) = 0$ by Proposition 5.6.2i) and it is obvious that $\sigma^*(z, t) + \sigma^*(z', t') = \sigma^*(z \oplus z', t \oplus t')$. Hence the asymmetric signature doesn't change under stable isomorphisms of tuples (z, t) and it is compatible with the actions of both monoids. The rest follows from Corollary 5.4.3. \square

5.6 Strong Flip-Isomorphism and Asymmetric Signatures

In the following we restrict our attention to strong flip-isomorphisms $t = (\alpha, \beta, 0)$. We want to bring light into the relationship between α and $\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & 0 \end{pmatrix}$. It is not as straightforward as one might expect. For instance, α and ρ are not necessarily equal in $LAsy^0(\Lambda)$ as we will show for the case $\Lambda = \mathbb{C}^+$ (complex field with conjugation involution). The map $\alpha \longmapsto \rho$ defines an endomorphism of the asymmetric Witt group which but it is not always injective.

Definition 5.6.1 ([Ran98] Definition 39.25). For an asymmetric form (F, α) over a commutative ring Λ define $ch_\alpha(z) = \det(z - \alpha^{-1}\alpha^*) \in \Lambda[z]$. \square

Proposition 5.6.2. i) There is a group homomorphism

$$\begin{aligned} \Phi_\epsilon: LAsy^0(\Lambda) &\longrightarrow LAsy^0(\Lambda) \\ (F, \alpha) &\longmapsto \left(M = F \oplus F^* \oplus F, \rho_\alpha = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & 0 \end{pmatrix} \right) \end{aligned}$$

ii) Let (α, β) be the strong flip isomorphism of the boundary of a non-singular $(-\epsilon)$ -symmetric form (K, λ) . Then $\Phi_\epsilon(K, \alpha) = (K, \lambda)$.

iii) Let $(F, \alpha) \in \ker \Phi_\epsilon$. Let $(F \oplus N \oplus N^*, \alpha') \in LAsy^0(\Lambda)$ of the form $\alpha' := \begin{pmatrix} \alpha & 0 & \rho_1 \\ 0 & 0 & \rho_2 \\ \rho_3 & \rho_4 & \rho_5 \end{pmatrix}$. Then $\alpha' \in \ker \Phi_\epsilon$

iv) For an asymmetric form (F, α) over a commutative ring Λ with $k = \text{rk}(F)$

$$ch_{\rho_\alpha}(z) = (z + \epsilon)^k \det(z^2 - \epsilon \alpha^{-1} \alpha^*) = (z + \epsilon)^k \epsilon^k ch_\alpha(\epsilon z^2)$$

Proof. i) Let $g \in GL(F)$ and $\alpha: F \longrightarrow F^*$ an isomorphism. Then

$$\begin{pmatrix} g^* & 0 & 0 \\ 0 & g^{-1} & 0 \\ 0 & 0 & g^* \end{pmatrix} \rho_\alpha \begin{pmatrix} g & 0 & 0 \\ 0 & g^{-*} & 0 \\ 0 & 0 & g \end{pmatrix} = \rho_{g^* \alpha g}.$$

Now let α' be another non-degenerate asymmetric form on F' . Define

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$P^* \begin{pmatrix} \rho_\alpha & 0 \\ 0 & \rho_{\alpha'} \end{pmatrix} P = \rho_{\alpha \oplus \alpha'}.$$

Finally we have to show that ρ_α is metabolic for metabolic α . We can assume that $L \subset F$ is a lagrangian of α hence

$$\alpha = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} : L \oplus L^\perp \longrightarrow L^* \oplus L^{\perp*}$$

Therefore

$$\rho_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 & B & C \\ 1 & 0 & 0 & 0 & -\epsilon & 0 \\ 0 & 1 & 0 & 0 & 0 & -\epsilon \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

has the lagrangian

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

ii) The map

$$i = \begin{pmatrix} \epsilon\beta & 1 \\ 0 & \epsilon\lambda \\ 1 & 0 \\ -\epsilon\beta & \beta \end{pmatrix} : K \oplus K \longrightarrow K \oplus K^* \oplus K \oplus K$$

is obviously a split injection and turns out to be a lagrangian of $\begin{pmatrix} \rho & 0 \\ 0 & \lambda \end{pmatrix}$.

iii) Let $i = \begin{pmatrix} x \\ y \\ z \end{pmatrix} : L \longrightarrow F \oplus F^* \oplus F$ be the inclusion of a lagrangian of $\Phi_\epsilon(\alpha)$. Then

$$j = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} : L \longrightarrow (F \oplus N \oplus N^*) \oplus (F \oplus N \oplus N^*)^* \oplus (F \oplus N \oplus N^*)$$

is the inclusion of a lagrangian of $\Phi_\epsilon(\alpha')$.

iv) Computation. □

Remark 5.6.3. Proposition 5.6.2ii) will be generalized in Corollary 7.3.2.

In special cases we can compute Φ_ϵ or at least decompose it into smaller bits. From the following calculations we can deduce how Φ_ϵ looks like for algebraically closed fields of characteristic $\neq 2$. In the following we use some facts from [Ran98] 39D. In particular we define $L^0(\Lambda, a)$ for $a \in U(\Lambda) = \{x \in \Lambda : x\bar{x} = 1\}$ as the Witt-group of all non-singular a -symmetric forms over Λ .

Lemma 5.6.4. *i) Let $(F, \alpha) \in L^0(\Lambda, \epsilon)$ and assume that $2 \in \Lambda^\times$.*

Then $\Phi_\epsilon(F, \alpha) = (F, \epsilon\alpha)$.

ii) Let $(F, \alpha) \in LAsy^0(\Lambda)$ with $ch_\alpha(\epsilon) \in \Lambda^\times$.

Then $\Phi_\epsilon(F, \alpha) = (F, \alpha^ - \epsilon\alpha) + (F^* \oplus F, \begin{pmatrix} \epsilon\alpha^{-*} & \epsilon \\ 1 & \epsilon\alpha \end{pmatrix})$.*

iii) Let $(F, \alpha) \in L^0(\Lambda, \epsilon b^2)$ with $b\bar{b} = -1$ and $1 - b^2 \in \Lambda^\times$.

Then $\Phi_\epsilon(F, \alpha) = (F, \alpha^* - \epsilon\alpha)$.

iv) Let $(F, \alpha) \in L^0(\Lambda, \epsilon b^2)$ with $b\bar{b} = 1$ and $2, 1 - b^2 \in \Lambda^\times$.

Then $\Phi_\epsilon(F, \alpha) = (F, \alpha^* - \epsilon\alpha) + (F^*, 2(\epsilon + b)\alpha) + (F^*, 2(\epsilon - b)\alpha)$.

Proof. i) Let $Q := \begin{pmatrix} 0 & -\alpha^{-1} & \epsilon \\ -\alpha & 0 & \frac{1}{2}\epsilon\alpha \\ \epsilon & \epsilon\alpha^{-1} & \frac{1}{2} \end{pmatrix}$. The matrix equation

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \epsilon & \epsilon\alpha^{-1} & 1 \end{pmatrix} Q = \begin{pmatrix} \epsilon & \epsilon\alpha^{-1} & \frac{1}{2} \\ 0 & -\alpha^{-1} & \epsilon \\ 0 & 0 & 2 \end{pmatrix}$$

shows that Q is invertible. Computing $Q^* \rho_\alpha Q$ yields the claim.

ii) Let $Q := \begin{pmatrix} 0 & \epsilon\alpha^{-*} & \epsilon \\ \alpha & 1 & 0 \\ -\epsilon & 0 & 1 \end{pmatrix}$. The matrix equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & \epsilon\alpha \\ -\epsilon\alpha^{-1}\alpha^* & \alpha^{-1} & \epsilon \end{pmatrix} Q = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & 1 & \epsilon\alpha \\ 0 & 0 & \epsilon - \alpha^{-1}\alpha^* \end{pmatrix}$$

shows that Q is invertible. Computing $Q^* \rho_\alpha Q$ yields the claim.

iii) By assumption, $ch_\alpha(\epsilon) = (\epsilon(1 - b^2))^{\text{rk } F} \in \Lambda^\times$. So we can use ii). We observe that $\begin{pmatrix} b \\ \alpha^{-1} \end{pmatrix}$ is a lagrangian of $\begin{pmatrix} \epsilon\alpha^{-*} & \epsilon \\ 1 & \epsilon\alpha \end{pmatrix}$.

iv) As before we can use ii) or we use the matrix $P := \begin{pmatrix} 0 & \epsilon + b & \epsilon - b \\ \alpha & \alpha^* & \alpha^* \\ -\epsilon & \epsilon b & -\epsilon b \end{pmatrix}$. Then

$P^* \rho_\alpha P = (\alpha^* - \epsilon\alpha) \oplus (2(\epsilon + b)\alpha) \oplus (2(\epsilon - b)\alpha)$. Now we have to show that this matrix is invertible: $\alpha^* - \epsilon\alpha = \epsilon(b^2 - 1)\alpha$ obviously is invertible. With $1 - b^2$ also $1 \pm b$ is invertible and so is $2(\epsilon \pm b)\alpha$.

□

This is enough to compute Φ_ϵ for some fields.

Corollary 5.6.5. *Let Λ be an algebraically closed field with $\text{char } \Lambda \neq 2$. Then by [Ran98] 39.27 and 39.22, $L\text{Asy}^0(\Lambda) = \bigoplus_{a \in U(\Lambda)} L\text{Asy}_{(z-a)\infty}^0(\Lambda)$ with $U(\Lambda) = \{x \in \Lambda \mid x\bar{x} = 1\}$. Let $(F, \alpha) \in L\text{Asy}_{(z-a)\infty}^0(\Lambda)$ and $b \in \Lambda$ with $a = \epsilon b^2$. Then*

$$\Phi_\epsilon(F, \alpha) = \begin{cases} (F, \epsilon\alpha) \in L\text{Asy}_{(z-\epsilon)\infty}^0(\Lambda) & \text{if } b = \pm 1 \\ (F, \alpha^* - \epsilon\alpha) \in L\text{Asy}_{(z+\epsilon)\infty}^0(\Lambda) & \text{if } b\bar{b} = -1 \\ (F, \alpha^* - \epsilon\alpha) + (F, 2(\epsilon + b)\alpha) + (F, 2(\epsilon - b)\alpha) \\ \in L\text{Asy}_{(z+\epsilon)\infty}^0(\Lambda) \oplus L\text{Asy}_{(z-b)\infty}^0(\Lambda) \oplus L\text{Asy}_{(z+b)\infty}^0(\Lambda) & \text{else} \end{cases}$$

Proof. Use Lemma 5.6.4 and the isomorphism $L^0(\Lambda, a) \xrightarrow{\cong} LAsy_{(z-a)^\infty}^0(\Lambda)$, $x \mapsto x$ of [Ran98] 39.27 (iii). \square

Example 5.6.6. Let $\Lambda = \mathbb{C}^-$ be the field of complex numbers with conjugation. Using the isomorphism $LAsy^0(\mathbb{C}^+) \cong \mathbb{Z}[S^1]$ of [Ran98] 40B, Φ_ϵ looks like

$$\begin{aligned} \Phi_1: \mathbb{Z}[S^1] &\longrightarrow \mathbb{Z}[S^1] \\ \mathbf{1}_a &\longmapsto \begin{cases} \mathbf{1}_1 & \text{if } a = 1, \\ -\mathbf{1}_{-1} - \mathbf{1}_b + \mathbf{1}_{-b} & \text{if } a \neq 1 \text{ with } b^2 = a, \Im b > 0. \end{cases} \\ \Phi_{-1}: \mathbb{Z}[S^1] &\longrightarrow \mathbb{Z}[S^1] \\ \mathbf{1}_a &\longmapsto \begin{cases} \mathbf{1}_1 - \mathbf{1}_i - \mathbf{1}_{-i} & \text{if } a = 1, \\ -\mathbf{1}_{-1} & \text{if } a = -1, \\ -\text{sgn } \Im a (\mathbf{1}_1 - \mathbf{1}_b + \mathbf{1}_{-b}) & \text{if } a \neq \pm 1 \text{ with } b^2 = -a, \Im b > 0. \end{cases} \end{aligned}$$

They are injective but not surjective and obviously not the identity map.

At last we deal with the question whether Φ_ϵ is always an injection.

Proposition 5.6.7. *Let Λ be a field with $\text{char } \Lambda \neq 2$. If there is a $b \in \Lambda$ with $b\bar{b} = -1$, $L^0(\Lambda, -\epsilon) \neq 0$, $|U(\Lambda)| > 2$ then Φ_ϵ is not injective.*

If Λ is algebraically closed, the converse is true as well.

Lemma 5.6.8. *Let $\epsilon, \eta \in U(\Lambda)$ with $\epsilon + \eta \in \Lambda^\times$. Then*

$$\begin{aligned} L^0(\Lambda, \epsilon) &\longrightarrow L^0(\Lambda, \eta) \\ (K, \lambda) &\longmapsto (K, \lambda^* + \bar{\eta}\lambda = (\bar{\eta} + \bar{\epsilon})\lambda) \end{aligned}$$

is an isomorphism.

Proof of Proposition (5.6.7). Let $x \in U(\Lambda)$ with $x \neq \pm 1$. Define $b_1 := b$ and $b_2 := bx$. Obviously $b_i\bar{b}_i = -1$ and $b_1 \neq b_2$. By the preceding lemma we know that $\kappa_i: L^0(\Lambda, \epsilon b_i^2) \rightarrow L^0(\Lambda, -\epsilon)$, $(K, \lambda) \mapsto (K, \bar{\epsilon}(\bar{b}_i^2 - 1)\lambda)$ is an isomorphism. Choose a non-zero $\alpha \in L^0(\Lambda, -\epsilon)$ and define $\alpha_i := \kappa_i^{-1}((-)^i \alpha)$. Then by Proposition 5.6.4 iii) $\Phi_\epsilon(\alpha_1 \oplus \alpha_2) = \kappa_1(\alpha_1) \oplus \kappa_2(\alpha_2) = 0$. On the other side $\alpha_i \in LAsy_{(z-\epsilon b_i^2)^\infty}^0(\Lambda)$ is not zero and $b_1^2 \neq b_2^2$ and therefore $\alpha_1 \oplus \alpha_2 \neq 0$ in $LAsy^0(\Lambda)$.

Now we assume that Λ is algebraically closed. By [Ran98] 39.27 and 39.22 we know that

$$LAsy^0(\Lambda) = \bigoplus_{a \in U(\Lambda)} L^0(\Lambda, a).$$

Let $(F, \alpha) \in \ker \Phi_\epsilon$. We can write $(F, \alpha) = \sum_{a \in U(\Lambda)} (F_a, \alpha_a) \in \bigoplus_{a \in U(\Lambda)} L^0(\Lambda, a)$.

Let $\pi': LAsy^0(\Lambda) \rightarrow L^0(\Lambda, \epsilon)$ be the projection. Then by Corollary 5.6.5, $0 = \pi' \Phi_\epsilon(F, \alpha) = (F_\epsilon, \epsilon \alpha_\epsilon)$. Hence $(F_\epsilon, \alpha_\epsilon) = 0$.

Now let $a \in U(\Lambda)$ such that there is a $b \neq \pm 1$ with $a = \epsilon b^2$, $b\bar{b} = 1$. Let $\pi_a: LAsy^0(\Lambda) \rightarrow L^0(\Lambda, b)$ be the projection. By Corollary 5.6.5, $0 = \pi_a \Phi_\epsilon(F, \alpha) = (F_a, 2(\epsilon + b)\alpha_a)$. Therefore $(F_a, \alpha_a) = 0$. So far we have shown that

$$\ker \Phi_\epsilon \subset A := \bigoplus_{a \in \{\epsilon b^2 | b\bar{b} = -1\}} L^0(\Lambda, a)$$

If either $b\bar{b} \neq -1$ for all $b \in \Lambda$ or if $L^0(\Lambda, \epsilon b^2) \cong L^0(\Lambda, -\epsilon) = 0$ for a $b \in \Lambda$ with $b\bar{b} = -1$ then obviously $A = 0$ and Φ_ϵ is injective. If $U(\Lambda)$ has only two elements (i.e. $U(\Lambda) = \{\pm 1\}$) and there is a $b \in \Lambda$ with $b\bar{b} = -1$ then $\epsilon b^2 = -\epsilon$ and $A = L^0(\Lambda, -\epsilon)$. By Corollary 5.6.5, $\Phi_\epsilon|_A$ is injective. So again $\ker \Phi_\epsilon = 0$. \square

Example 5.6.9. Here is an explicit example of a ring for which Φ_ϵ is not injective. Let $s, y, v, w \in \mathbb{Q}^\times$ such that $s^2 + v^2$ is not a square but $s^2 + v^2 + y^2 = t^2$ (e.g. $s = 12, v = 4, y = 3, t = 13$). Define $x := \frac{v}{s}$, $r := \frac{t}{y}$ and $d := \frac{x^2 + 1}{y^2} = \frac{r^2 - 1}{s^2}$. As d is not a square, $\Lambda := \mathbb{Q}(\sqrt{d})$ is a genuine field extension of \mathbb{Q} . We introduce the standard involution $\overline{q + p\sqrt{d}} = q - p\sqrt{d}$ on Λ . Then $b = x + y\sqrt{d}$ has the property $b\bar{b} = -1$ and $c = r + s\sqrt{d}$ is not ± 1 but $c\bar{c} = 1$. So $|U(\Lambda)| > 2$. Finally we observe that $(\Lambda, 1)$ is a non-trivial element in $L^0(\Lambda, 1)$ and (Λ, \sqrt{d}) is non-trivial in $L^0(\Lambda, -1)$. Hence the maps $\Phi_{\pm 1}$ are not injective.

Chapter 6

Quadratic Signatures of Flip-Isomorphisms

For the whole chapter, let $q \geq 2$, $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

Let (W, M, M') be a cobordism such that $\partial M = \partial M' = \emptyset$. If there is an automorphism $H: M \xrightarrow{\cong} M'$, we can glue M on M' along H in order to obtain a closed manifold V_H . If V_H is null-cobordant then (W, M, M') is cobordant rel ∂ to the h -cobordism $(M \times I \cup_{H^{-1}} M', M, M')$.

As usual we try to transfer the above into the world of algebraic surgery theory. Let $x = (g: \partial E = D' \cup_C D \rightarrow E, (\delta\omega, \omega))$ be a $(2q + 2)$ -dimensional quadratic Poincaré pair e.g. the one we constructed out of a preformation z in Section 3.1. Assume that C is zero or at least contractible. For the Poincaré pair x from Section 3.1 that is only the case if z is a non-singular formation (see also Chapter 7). So we deal in fact with a Poincaré pair of the form $(g: D \oplus D' \rightarrow E, (\delta\omega', \nu \oplus \nu'))$. A flip-isomorphism induces an isomorphism of D and D' , so that x transforms to a fundamental pair $x' = (D \oplus D \rightarrow E, (\delta\omega', \nu \oplus -\nu))$ which we can glue together along D (see Definition A.3.4). The result is a $(2q + 2)$ -dimensional Poincaré complex. It is (algebraically) null-cobordant if and only if x' is cobordant rel ∂ to $((1, 1): D \oplus D \rightarrow D, (0, \nu \oplus -\nu))$ which is the case if and only if x is cobordant to an h -cobordism. Using standard surgery theory (e.g. Lemma A.4.6) this Poincaré complex corresponds to a non-singular quadratic form and that form vanishes in the even-dimensional L -group if and only if the Poincaré complex is null-cobordant. Hence we expect to be able to define an element in $L_{2q+2}(\Lambda)$ for each flip-isomorphism of z such that z is elementary if and only if such a quadratic signature vanishes for a flip-isomorphism.

The manifold case requires more care if the boundary of M and M' is non-empty. Again we go through all automorphisms $H: M \xrightarrow{\cong} M'$ and replace M' by M using H . The original cobordism becomes (W, M, M) and the boundary of W turns into a twisted double $M \cup_h M$ with $h = H|: \partial M \rightarrow \partial M' \cong \partial M$. But not every twisted

double $M \cup_h M$ is a boundary of an h -cobordism. If we want to follow the strategy of the closed case in the beginning we have to assume that e.g. h is isotopic to the identity. Then (W, M, M) can be glued onto $M \times (I, 0, 1)$. If the result, a closed manifold, is null-cobordant then (W, M, M) and hence (W, M, M') is cobordant rel ∂ to an h -cobordism.

Similarly, for general preformations the situation is more complicated. In Section 4.3 a flip-isomorphism t of z replaces the boundary ∂E of x by an algebraic twisted double. This yields a $(2q + 2)$ -dimensional quadratic Poincaré pair $x_t = (g_t: D' \cup_{h_t} D \rightarrow E, (\delta\omega_t, \omega_t = \delta\psi \cup_{\chi_t} -\delta\psi))$ with a twisted double at the boundary. It is not always possible to find an algebraic h -cobordism with that boundary except e.g. if (h_t, χ_t) is homotopic to the identity. This involves a concept of homotopies of morphisms between quadratic complexes which we develop in the rather technical **Section 6.1**. We define **flip-isomorphisms rel ∂** in **Section 6.2** as flip-isomorphisms for which (h_t, χ_t) is homotopic to $(1, 0): (C, \psi) \xrightarrow{\cong} (C, \psi)$.

Then we deviate slightly from the example in geometry. We use the homotopy to change x_t such that it looks like $(D \cup_C D \rightarrow E, (*, \delta\psi \cup_\psi -\delta\psi))$ and then stick the standard algebraic h -cobordism $(D \cup_C D \rightarrow D, (0, \delta\psi \cup_\psi -\delta\psi))$ on it. As before, the result will be a $(2q + 2)$ -dimensional Poincaré complex which corresponds to a non-singular quadratic form. This is the quadratic signature which will be constructed in **Section 6.3**. In **Section 6.4** it is proven that a preformation z is stably elementary if and only if one of its quadratic signatures is vanishing.

The disadvantage of the quadratic signatures is that they not only depend on the preformation and the flip-isomorphism, but e.g. also the explicit homotopy of $(h_t, \chi_t) \simeq (1, 0)$. Hence we do not have something like a map $fl_{2q}(\Lambda) \rightarrow L_{2q}(\Lambda)$. In certain cases, though, we can restrict the effect of those choices on the quadratic signature (see Lemma 6.4.4).

Curiously, the quadratic and asymmetric signatures are related by the canonical map $L_{2q}(\Lambda) \rightarrow LAsy^0(\Lambda)$, $(K, \psi) \rightarrow (K, \psi - \epsilon\psi^*)$ as we will show in **Section 6.5**.

6.1 Homotopy and Twisted Doubles

This section deals with a very technical issue, the extension of the concept of chain homotopies of chain maps

$$\Delta: f \simeq f': C \rightarrow C'$$

to a homotopy of morphisms of quadratic (or symmetric) complexes

$$(\Delta, \eta): (f, \chi) \simeq (f', \chi'): (C, \psi) \rightarrow (C', \psi')$$

Obviously the chain homotopy Δ will affect the quadratic structures χ and χ' . Their difference is determined by an operation $\Delta\% \psi$ (and a boundary $d\eta$). In Lemma 6.1.5

we show that the homotopy of morphisms is an equivalence relation.

We will only be able to define the quadratic signature if the chain map (h_t, χ_t) defined in (4.2) is homotopic to $(1, 0)$. This makes it necessary to keep track how a twisted double changes if the self-equivalence used is changed by a homotopy. Lemma 6.1.6 deals with this case.

Section 6.5 discusses the relationship between quadratic and asymmetric signatures. We will need Lemma 6.1.8 which shows that changing the self-equivalence involved by a homotopy will not affect the asymmetric signature.

Definition 6.1.1. Let $\Delta: f \simeq f': C \rightarrow C'$ be a chain homotopy of two chain maps. Let $\phi \in W^\% (C, \epsilon)_n$. Define $\Delta^\% \phi \in W^\% (C', \epsilon)_{n+1}$ by

$$(\Delta^\% \phi)_s = -\Delta \phi_s f^* + (-)^{r+1} (f' \phi_s + (-)^{n+1} \Delta T_\epsilon \phi_{s-1}) \Delta^*: C'^{n+1-r+s} \rightarrow C'_r$$

Let $\psi \in W_\% (C', \epsilon)_n$. Define $\Delta_\% \psi \in W_\% (C', \epsilon)_{n+1}$ by

$$(\Delta_\% \psi)_s = -\Delta \psi_s f^* + (-)^{r+1} (f' \psi_s + (-)^n \Delta T_\epsilon \psi_{s+1}) \Delta^*: C'^{n+1-r-s} \rightarrow C'_r$$

□

Lemma 6.1.2. Let $\Delta: f \simeq f': C \rightarrow C'$ be a chain homotopy of two chain maps.

- i) Let $\phi \in W^\% (C, \epsilon)_n$. Then $d(\Delta^\% \phi) = -\Delta^\% (d\phi) + f\phi f^* - f'\phi f'^* \in W^\% (C, \epsilon)_n$.
- ii) Let $\psi \in W_\% (C, \epsilon)_n$. Then $d(\Delta_\% \psi) = -\Delta_\% (d\psi) + f\psi f^* - f'\psi f'^* \in W_\% (C, \epsilon)_n$.
- iii) Let $\psi \in W_\% (C, \epsilon)_n$. Then

$$\begin{aligned} & (1 + T_\epsilon)(\Delta_\% \psi) - \Delta^\%((1 + T_\epsilon)\psi) \\ &= d\xi + \begin{cases} (-)^r \Delta d(T_\epsilon \psi)_0 \Delta^*: C'^{n+1-r} \rightarrow C'_r & : s = 0 \\ 0 & : s > 0 \end{cases} \end{aligned}$$

with $\xi \in W^\% (C', \epsilon)_{n+2}$ given by $\xi_0 = (-)^{r+1} \Delta T_\epsilon \psi_0 \Delta^*: C'^{n+2-r} \rightarrow C'_r$.

- iv) Let $g: C' \rightarrow D$ be a chain map. Then $g\Delta: gf \simeq gf': C \rightarrow D$ is a chain homotopy.

Let $\phi \in W^\% (C, \epsilon)_n$. Then $(g\Delta)^\% \phi = g(\Delta^\% \phi)g^* \in W^\% (D, \epsilon)_{n+1}$.

Let $\psi \in W_\% (C, \epsilon)_n$. Then $(g\Delta)_\% \psi = g(\Delta_\% \psi)g^* \in W_\% (D, \epsilon)_{n+1}$.

Lemma 6.1.3. Let $(f, \chi): (C, \psi) \rightarrow (C', \psi')$ be a morphism of n -dimensional ϵ -quadratic complexes. Let $\Delta: f \simeq f': C \rightarrow C$ be a chain homotopy.

Then $(f', \chi + \Delta_\% \psi): (C, \psi) \rightarrow (C', \psi')$ is also a morphism of n -dimensional ϵ -quadratic complexes. Similar in the symmetric case.

Definition 6.1.4. A homotopy (Δ, η) of two morphisms of n -dimensional ϵ -quadratic complexes $(f, \chi), (f', \chi') : (C, \psi) \longrightarrow (C', \psi')$ is a chain homotopy $\Delta : f \simeq f' : C \longrightarrow C'$ and an element $\eta \in W_{\%}(C')_{n+2}$ such that

$$\chi' - \chi = \Delta_{\%}\psi + d(\eta) \in W_{\%}(C')_{n+1}$$

Similar in the symmetric case. □

Lemma 6.1.5. Let $(\Delta, \eta) : (f, \chi) \simeq (f', \chi') : (C, \psi) \longrightarrow (C', \psi')$ be a homotopy of two morphisms of n -dimensional ϵ -quadratic complexes.

- i) Then there is a homotopy $(\Delta' = -\Delta, \eta') : (f', \chi') \simeq (f, \chi) : (C, \psi) \longrightarrow (C', \psi')$.
- ii) Let $(\Delta', \eta') : (f', \chi') \simeq (f'', \chi'') : (C, \psi) \longrightarrow (C', \psi')$ be another homotopy. Then also (f, χ) and (f'', χ'') are homotopic.
- iii) Homotopy induces an equivalence relation on all morphisms $(C, \psi) \longrightarrow (C', \psi')$.
- iv) Let $(g, \rho) : (C', \psi') \longrightarrow (D, \theta)$ be a morphism. Then there is a homotopy

$$(g\Delta, g\eta g^*) : (g, \rho)(f, \chi) \longrightarrow (g, \rho)(f', \chi') : (C, \psi) \longrightarrow (D, \theta)$$

with $(g, \rho)(f, \chi) = (gf, \rho + g\chi g^*)$ as in Definition A.1.5.

Similar in the symmetric case.

Proof. i) Use $\eta'_s = -\eta_s + (-)^{r+1} \Delta \psi_s \Delta^* : C'^{n+2-r-s} \longrightarrow C'_r$.

ii) Define $\Delta'' = \Delta + \Delta'$ and $\eta''_s = \eta_s + \eta'_s + (-)^r \Delta' \psi_s \Delta^* : C'^{n+2-r-s} \longrightarrow C'_r$. Then $(\Delta'', \eta'') : (f, \chi) \simeq (f'', \chi'') : (C, \psi) \longrightarrow (C', \psi')$ is a homotopy.

iii) Clear from the previous two previous statements.

iv) Obvious. □

Lemma 6.1.6. Let $c = (f : C \longrightarrow D, (\delta\psi, \psi))$ be an n -dimensional ϵ -quadratic Poincaré pair. Let $(\Delta, \eta) : (h, \chi) \simeq (h', \chi') : (C, \psi) \xrightarrow{\cong} (C, \psi)$ be a homotopy of self-equivalences. Then there is an isomorphism $(a, \sigma) : c \cup_{(h, \chi)} -c \xrightarrow{\cong} c \cup_{(h', \chi')} -c$ of the corresponding

quadratic twisted doubles given by

$$\begin{aligned}
a_r &= \begin{pmatrix} 1 & 0 & (-)^r f \Delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \\
&(D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow (D \cup_{h'} D)_r = D_r \oplus C_{r-1} \oplus D_r \\
\sigma_s &= \begin{pmatrix} (-)^{n-1} f \eta_s f^* & 0 & 0 \\ 0 & 0 & 0 \\ (-)^n \psi_s \Delta^* f^* & 0 & 0 \end{pmatrix} : \\
&(D \cup_{h'} D)^{n+1-r-s} = D^{n+1-r-s} \oplus C^{n-r-s} \oplus D^{n+1-r-s} \\
&\longrightarrow (D \cup_{h'} D)_r = D_r \oplus C_{r-1} \oplus D_r
\end{aligned}$$

Lemma 6.1.7. Let $c = (f: C \longrightarrow D, (\delta\phi, \phi))$ be an n -dimensional ϵ -symmetric Poincaré pair. Let $(\Delta, \eta): (h, \chi) \simeq (h', \chi'): (C, \phi) \xrightarrow{\cong} (C, \phi)$ be a homotopy of self-equivalences.

Then there is an isomorphism $(a, \sigma): c \cup_{(h, \chi)} -c \xrightarrow{\cong} c \cup_{(h', \chi')} -c$ of the corresponding symmetric twisted doubles given by

$$\begin{aligned}
a_r &= \begin{pmatrix} 1 & 0 & (-)^r f \Delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \\
&(D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow (D \cup_{h'} D)_r = D_r \oplus C_{r-1} \oplus D_r \\
\sigma_s &= \begin{pmatrix} (-)^{n-1} f \eta_s f^* & 0 & 0 \\ 0 & 0 & 0 \\ (-)^n \phi_s \Delta^* f^* & 0 & 0 \end{pmatrix} : \\
&(D \cup_{h'} D)^{n+1-r+s} = D^{n+1-r+s} \oplus C^{n-r+s} \oplus D^{n+1-r+s} \\
&\longrightarrow (D \cup_{h'} D)_r = D_r \oplus C_{r-1} \oplus D_r
\end{aligned}$$

Lemma 6.1.8. Let $c = (f: C \longrightarrow D, (\delta\phi, \phi))$ be an n -dimensional ϵ -symmetric Poincaré pair. Let $(\Delta, \eta): (h, \chi) \simeq (h', \chi'): (C, \phi) \xrightarrow{\cong} (C, \phi)$ be a homotopy of self-equivalences. Then there is an isomorphism

$$(a, \sigma): (\partial E, \theta) = c \cup_{(h, \chi)} -c \xrightarrow{\cong} (\partial E', \theta') = c \cup_{(h', \chi')} -c$$

of the corresponding symmetric twisted doubles given in Lemma 6.1.7.

Let $x' = (g': \partial E' \longrightarrow E, (\delta\theta', \theta'))$ be an $(n+1)$ -dimensional ϵ -symmetric Poincaré pair. Then (by Lemma A.2.6) $x = (g = g'a: \partial E \longrightarrow E, (\delta\theta = \delta\theta' + (-)^n g' \sigma g'^*, \theta))$ is also an $(n+1)$ -dimensional ϵ -symmetric Poincaré pair and the asymmetric signatures $\sigma^*(x) = \sigma^*(x') \in LAsy^0(\Lambda)$.

Proof. Use Definition 5.2.4 and show that the b-duality maps of x and x' are homotopic. Then it follows that the asymmetric complexes of x and x' are equivalent. \square

6.2 Flip-Isomorphisms $\text{rel}\partial$

As explained in the introduction we can only produce a quadratic signature if $(h_t, \chi_t) \simeq (1, 0)$. Inconveniently, (h_t, χ_t) depends on the choice of representatives for $\bar{\theta}$ and $\bar{\nu}$ and a map κ . This is the reason for the next, rather awkward, definition.

Definition 6.2.1. A flip-isomorphism $t \text{ rel}\partial$ of a regular ϵ -quadratic split preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is a flip-isomorphism $t = (\alpha, \beta, \bar{\nu})$ of z such that there is a representative θ of $\bar{\theta} \in Q_{-\epsilon}(G)$ and a representative ν for $\bar{\nu} \in Q_{-\epsilon}(F^*)$ and $\kappa \in \text{Hom}_{\Lambda}(G, G^*)$ such that $\beta^* \theta \beta + \theta + \mu^* \nu \mu = \kappa + \epsilon \kappa^*$ and such that the isomorphism $(h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ defined in (4.2) and (3.3) is homotopic to the identity $(1, 0): (C, \psi) \xrightarrow{\cong} (C, \psi)$. \square

Remark 6.2.2. To a certain extent the “ $\text{rel}\partial$ ”-property is independent of the choices of θ and ν (but not necessarily of the choice of κ): Assume that for ν, θ and κ as before, there exists a homotopy

$$(\Delta, \eta): (1, 0) \simeq (h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$$

Let $\widehat{\nu}, \widehat{\theta}, \widehat{\kappa}, \tilde{\nu}, \tilde{\theta}, \tilde{\kappa}$ as in Remark 4.2.1. They induce a new isomorphism $(h_t, \widehat{\chi}_t): (C, \widehat{\psi}) \xrightarrow{\cong} (C, \widehat{\psi})$ which is also homotopic to the identity by

$$(\Delta, \eta + \tilde{\chi}): (1, 0) \simeq (h_t, \widehat{\chi}_t): (C, \widehat{\psi}) \xrightarrow{\cong} (C, \widehat{\psi})$$

with $\tilde{\chi}$ defined in Remark 4.2.1. See also Remarks 3.1.3, 4.2.1 and Lemma 6.4.2.

The “ $\text{rel}\partial$ ”-property is invariant under the equivalence relations of $fl_{2q+2}(\Lambda)$ and any elementary preformation has such a flip-isomorphism.

Proposition 6.2.3. *i) Every elementary preformation has a flip-isomorphism $\text{rel}\partial$.*

ii) Let t and t' be flip-isomorphisms of ϵ -quadratic split preformations z and z' respectively. If t and t' are flip-isomorphisms $\text{rel}\partial$ then so is $t \oplus t'$.

iii) Let $[(z, t)] = [(z', t')] \in fl_{2q+2}(\Lambda)$. If t is a flip-isomorphism $\text{rel}\partial$ then so is t' .

Proof. i) Let z be of the form describe in Proposition 2.4.2iv). Then the flip-isomorphism defined in Proposition 4.1.2 is a flip-isomorphism $\text{rel}\partial$ with representatives

$$\begin{aligned} \theta &= \begin{pmatrix} 0 & -\epsilon\sigma \\ 0 & \theta' \end{pmatrix} : G = U \oplus R \longrightarrow G^* = U^* \oplus R^* \\ \nu &= 0 \\ \kappa &= \begin{pmatrix} 0 & 0 \\ 0 & \theta' \end{pmatrix} : G = U \oplus R \longrightarrow G^* = U^* \oplus R^* \end{aligned}$$

and with a homotopy $(\Delta, \eta): (1, 0) \simeq (h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ given by

$$\begin{aligned} \Delta_{q+1} &= \begin{pmatrix} \epsilon & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} : C_q = (U \oplus U^*) \oplus (U^* \oplus U) \longrightarrow C_{q+1} = U \oplus R \\ \Delta_q &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ -\epsilon & 0 \\ 0 & 0 \end{pmatrix} : C_{q-1} = U^* \oplus R^* \longrightarrow C_q = (U \oplus U^*) \oplus (U^* \oplus U) \\ \eta_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : C^q = (U^* \oplus U) \oplus (U \oplus U^*) \longrightarrow C_{q+1} = U \oplus R \\ \eta_1 &= \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} : C^{q+1} = U^* \oplus R^* \longrightarrow C_q = (U \oplus U^*) \oplus (U^* \oplus U) \\ \eta_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : C^{q-1} = U \oplus R \longrightarrow C_{q+1} = U \oplus R \\ \eta_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \epsilon & 0 \end{pmatrix} \\ &\quad : C^q = (U^* \oplus U) \oplus (U \oplus U^*) \longrightarrow C_q = (U \oplus U^*) \oplus (U^* \oplus U) \\ \eta_3 &= \begin{pmatrix} -\epsilon & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : C^{q-1} = U \oplus R \longrightarrow C_q = (U \oplus U^*) \oplus (U^* \oplus U) \\ \eta_4 &= \begin{pmatrix} 0 & 0 \\ -\sigma^* & 0 \end{pmatrix} : C^{q-1} = U \oplus R \longrightarrow C_{q-1} = U^* \oplus R^* \end{aligned}$$

ii) Obvious.

iii) We only sketch the proof. Let t and t' be flip-isomorphisms of ϵ -quadratic split preformations z and z' respectively. t and t' induce self-equivalences $(h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$ and $(h'_t, \chi'_t): (C', \psi') \xrightarrow{\cong} (C', \psi')$. Let t and t' be flip-isomorphisms of ϵ -quadratic split preformations z and z' respectively. Let (η, ζ) be an isomorphism between (z, t) and (z', t') in the sense of Definition 5.5.1. Then define an isomorphism $(\tilde{h}, 0): (C, \psi) \longrightarrow (C', \psi')$ by

$$\begin{aligned} \tilde{h}_{q+1} &= \zeta : C_{q+1} = G \xrightarrow{\cong} C'_{q+1} = G' \\ \tilde{h}_q &= \begin{pmatrix} \eta & 0 \\ 0 & \eta'^* \end{pmatrix} : C_q = F \oplus F^* \xrightarrow{\cong} C'_q = F' \oplus F'^* \\ \tilde{h}_{q-1} &= \zeta'^* : C_{q-1} = G \xrightarrow{\cong} C'_{q-1} = G' \end{aligned}$$

Then $\tilde{h}h_t\tilde{h}^{-1} = h'_t : C' \xrightarrow{\cong} C'$ and $\tilde{h}\chi_t\tilde{h}^* = \chi'_t$. Assume

$$(\Delta, \eta): (1, 0) \simeq (h_t, \chi_t): (C, \psi) \longrightarrow (C, \psi)$$

is a homotopy, then

$$(\bar{h}\Delta\bar{h}^{-1}, \bar{h}\eta\bar{h}^*): (1, 0) \simeq (h'_t, \chi'_t): (C', \psi') \longrightarrow (C', \psi')$$

is a homotopy as well.

Assume now that there is a $k \in \mathbb{N}_0$ such that the flip-isomorphism $t + t_k$ of $z + y_k$ is a flip-isomorphism $\text{rel}\partial$ (t_k and y_k are defined in Definition 5.5.1.) Now observe that the $2q$ -dimensional quadratic complex C induced by y_k is contractible. It follows that t is a flip-isomorphism $\text{rel}\partial$. □

6.3 Construction of the Quadratic Signature

Let $t = (\alpha, \beta, \bar{\nu})$ be a flip-isomorphism $\text{rel}\partial$ of a regular ϵ -quadratic split preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$. Choose θ, ν and κ as in Definition 6.2.1 Then there exists a homotopy $(\Delta, \eta): (1, 0) \simeq (h_t, \chi_t): (C, \psi) \xrightarrow{\cong} (C, \psi)$. Write $\Delta_{q+1} = \begin{pmatrix} R & S \end{pmatrix}: C_q = F \oplus F^* \longrightarrow C_{q+1} = G$ and $\Delta_q = \begin{pmatrix} U \\ V \end{pmatrix}: C_{q-1} = G^* \longrightarrow C_q = F \oplus F^*$.

We use the homotopy to change the boundary of $x_t = (g_t: \partial E_t = D \cup_{h_t} D \longrightarrow E, (\delta\omega_t, \omega_t))$ to the simpler quadratic Poincaré complex

$$(\partial E', \omega') = c \cup -c = (D \cup_C D, \delta\psi \cup_\psi -\delta\psi) = (D \cup_1 D, \delta\psi \cup_0 -\delta\psi)$$

by applying Lemma 6.1.6. We obtain an isomorphism $(a, \sigma): (\partial E', \omega') \xrightarrow{\cong} (\partial E_t, \omega_t)$ given by $(\partial E'_r = D_r \oplus C_{r-1} \oplus D_r, \partial E_{t,r} = D_r \oplus D_r \oplus C_{r-1})$

$$\begin{array}{ccc} \partial E'_{q+2} = 0 \oplus G \oplus 0 & \xrightarrow{1} & \partial E_{t,q+2} = 0 \oplus 0 \oplus G \\ \downarrow -\epsilon \begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix} & & \downarrow -\epsilon \begin{pmatrix} \beta \\ 1 \\ \gamma \\ \mu \end{pmatrix} \\ \partial E'_{q+1} = G \oplus (F \oplus F^*) \oplus G & \xrightarrow{\begin{pmatrix} 1 & -\epsilon R & -\epsilon S & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}} & \partial E_{t,q+1} = G \oplus G \oplus (F \oplus F^*) \\ \downarrow \begin{pmatrix} -\epsilon\mu & 0 & \epsilon & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \end{pmatrix} & & \downarrow \begin{pmatrix} -\epsilon\mu & 0 & \alpha & \alpha(\nu^* - \epsilon\nu) \\ 0 & -\epsilon\mu & 0 & \epsilon \\ 0 & 0 & \mu^* & \epsilon\gamma^* \end{pmatrix} \\ \partial E'_q = F^* \oplus G^* \oplus F^* & \xrightarrow{\begin{pmatrix} 1 & \epsilon V & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} & \partial E_{t,q} = F^* \oplus F^* \oplus G^* \end{array}$$

$$\begin{aligned}
\omega'_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} : \partial E'^q = F \oplus G \oplus F \longrightarrow \partial E'_{q+1} = G \oplus (F \oplus F^*) \oplus G \\
\omega'_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} : \partial E'^{q+1} = G^* \oplus F^* \oplus F \oplus G^* \longrightarrow \partial E'_q = F^* \oplus G^* \oplus F^* \\
\omega'_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial E'^q = F \oplus G \oplus F \longrightarrow \partial E'_q = F^* \oplus G^* \oplus F^* \\
\sigma_0 &= (-V^* \ 0 \ 0) : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q+2} = G \\
\sigma_0 &= \begin{pmatrix} \eta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -R^* & 0 & 0 & 0 \end{pmatrix} : \partial E_t^{q+1} = G^* \oplus G^* \oplus (F^* \oplus F) \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^* \\
\sigma_1 &= \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q+1} = G \oplus G \oplus (F \oplus F^*) \\
\sigma_1 &= \begin{pmatrix} \eta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon\gamma^* R^* & 0 & 0 & 0 \end{pmatrix} : \partial E_t^{q+1} = G^* \oplus G^* \oplus (F^* \oplus F) \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^* \\
\sigma_2 &= \begin{pmatrix} \eta_2 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon\theta^* V^* & 0 & 0 \end{pmatrix} : \partial E_t^q = F \oplus F \oplus G \longrightarrow \partial E_{t,q} = F^* \oplus F^* \oplus G^*
\end{aligned}$$

Then we change the boundary of the $(2q+2)$ -dimensional quadratic Poincaré complex

$$x_t = (g_t : \partial E_t \longrightarrow E, (\delta\omega_t, \omega_t))$$

from Section 4.3 using Lemma A.2.6 and the above isomorphism. We obtain

$$\begin{aligned}
w_t &= (g'_t = g_t a : \partial E' \longrightarrow E, (\delta\omega' = \delta\omega_t + (-)^{2q+1} g_t \sigma g_t^*, \omega')) \tag{6.1} \\
g'_{t,q+1} &= (1 \ -\epsilon R \ -\epsilon S \ -\beta) : \partial E'_{q+1} = G \oplus (F \oplus F^*) \oplus G \longrightarrow E_{q+1} = G \\
\delta\omega'_{t,0} &= -\eta_0 : E^{q+1} = G^* \longrightarrow E_{q+1} = G
\end{aligned}$$

The next step is to glue w_t to an algebraic h -cobordism $y = (m : \partial E' = D \cup_C D \longrightarrow D, (0, \omega'))$ with $m_r = (-1 \ 0 \ 1) : \partial E'_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow D_r$ (compare Example 3.3.2). Let the result be the $(2q+2)$ -dimensional quadratic Poincaré complex $(V, \tau) =$

$w_t \cup -y$. Using the isomorphism ($V_r = E_r \oplus \partial E'_{r-1} \oplus D_r$)

$$\begin{array}{ccc}
V_{q+3} = 0 \oplus G \oplus 0 & \xrightarrow{-\epsilon} & G \\
\downarrow -\epsilon \begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
V_{q+2} = 0 \oplus (G \oplus F \oplus F^* \oplus G) \oplus 0 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\gamma & 1 & 0 & 0 \\ -\mu & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}} & G \oplus F \oplus F^* \oplus G \\
\downarrow \begin{pmatrix} -\epsilon & R & S & \epsilon\beta \\ -\epsilon\mu & 0 & \epsilon & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \\ \epsilon & 0 & 0 & -\epsilon \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \mu^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
V_{q+1} = G \oplus (F^* \oplus G^* \oplus F^*) \oplus G & \xrightarrow{\begin{pmatrix} 1 & -\epsilon S & 0 & 0 & \beta \\ 0 & \epsilon & 0 & 0 & 0 \\ 0 & -\gamma^* & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -\mu \\ 0 & 0 & 0 & 0 & -\epsilon \end{pmatrix}} & G \oplus F^* \oplus G^* \oplus F^* \oplus G \\
\downarrow (0 \ -\epsilon \ 0 \ \epsilon \ -\epsilon\mu) & & \downarrow (0 \ 0 \ 0 \ 1 \ 0) \\
V_q = 0 \oplus 0 \oplus F^* & \xrightarrow{\epsilon} & F^*
\end{array}$$

we can shrink the chain complex V to a smaller chain complex V' via the chain equivalence given by

$$\begin{array}{ccc}
V_{q+2} = G \oplus F \oplus F^* \oplus G & \xrightarrow{(-\gamma \ 1 \ 0 \ 0)} & V'_{q+2} = F \\
\downarrow & & \downarrow \begin{pmatrix} R \\ \mu^* \end{pmatrix} \\
V_{q+1} = G \oplus F^* \oplus G^* \oplus F^* \oplus G & \xrightarrow{\begin{pmatrix} 1 & -\epsilon S & 0 & 0 & \beta \\ 0 & -\gamma^* & 1 & 0 & 0 \end{pmatrix}} & V'_{q+1} = G \oplus G^*
\end{array}$$

The induced quadratic structure on V' is given by

$$\tau'_0 = \begin{pmatrix} -\eta_0 & \beta \\ 0 & -\epsilon\theta^* \end{pmatrix} : V'^{q+1} = G^* \oplus G \longrightarrow V'_{q+1} = G \oplus G^*$$

Using Lemma A.4.6 we can define the quadratic signature

Definition 6.3.1. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation and $t = (\alpha, \beta, \bar{\nu})$ a flip-isomorphism $\text{rel} \partial$ of z .

Choose representatives θ of $\bar{\theta} \in Q_{-\epsilon}(G)$ and a representative ν for $\bar{\nu} \in Q_{-\epsilon}(F^*)$ and a $\kappa \in \text{Hom}_\Lambda(G, G^*)$ such that $\beta^* \theta \beta + \theta + \mu^* \nu \mu = \kappa + \epsilon \kappa^*$ and such that there is a homotopy $(\Delta, \eta) : (h_t, \chi_t) \simeq (1, 0) : (C, \psi) \xrightarrow{\cong} (C, \psi)$ of the isomorphism defined in (4.2) and (3.3). The **quadratic signature** $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) = [(M, \xi)] \in L_{2q+2}(\Lambda)$ is given by the non-singular $(-\epsilon)$ -quadratic form

$$\xi = \begin{pmatrix} -\eta_0 & \beta & 0 \\ 0 & -\epsilon\theta^* & 0 \\ R^* & \mu & 0 \end{pmatrix} : M = G^* \oplus G \oplus F \longrightarrow M^*$$

□

6.4 Properties of the Quadratic Signatures

First we show that the quadratic signatures detect whether an element in $l_{2q+2}(\Lambda)$ is elementary or not.

Theorem 6.4.1. *$[z'] \in l_{2q+2}(\Lambda)$ is elementary if and only if there is a flip-isomorphism $\text{rel}\partial$ t of z with $[z] = [z'] \in l_{2q+2}(\Lambda)$ and $\Delta, \kappa, \theta, \nu$ and η as in Definition 6.3.1 such that $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) = 0 \in L_{2q+2}(\Lambda)$.*

Proof. Let z be a regular ϵ -quadratic split preformation. If z is elementary then use the flip-isomorphism and the choices for Δ, ν , etc. made in the proof of Proposition 6.2.3i).

On the other hand assume that there is t, Δ , etc. such that $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) = 0 \in L_{2q+2}(\Lambda)$. Then $(V, \tau) = y \cup -w_t$ constructed in the previous Section is null-cobordant. Hence the Poincaré pairs y and w_t from the previous section are cobordant $\text{rel}\partial$. y is an algebraic h -cobordism. The Poincaré pair x constructed in Section 4.3 is cobordant $\text{rel}\partial$ to an algebraic h -cobordism. By Theorem 3.3.3 $[z] \in l_{2q+2}(\Lambda)$ is elementary. \square

Quadratic signature depend on a lot of choices. We can restrict that dependency to a certain degree.

Lemma 6.4.2. *Let $z, t, \nu, \theta, \kappa, \Delta$ and η as in Definition 6.3.1.*

Let $\tilde{\theta} \in \text{Hom}_\Lambda(G, G^), \tilde{\nu} \in \text{Hom}_\Lambda(F^*, F), \tilde{\kappa} \in \text{Hom}_\Lambda(G, G^*)$ and $\tilde{\eta} \in W_{\%}(C, \epsilon)_{2q+3}$.*

Define

$$\begin{aligned}\hat{\nu} &= \nu + \tilde{\nu} + \epsilon\tilde{\nu}^* \in \text{Hom}_\Lambda(F^*, F) \\ \hat{\theta} &= \theta + \tilde{\theta} + \epsilon\tilde{\theta}^* \in \text{Hom}_\Lambda(G, G^*) \\ \hat{\kappa} &= \kappa + \tilde{\kappa} - \epsilon\tilde{\kappa}^* + \beta^*\tilde{\theta}\beta + \tilde{\theta} + \mu^*\tilde{\nu}\mu \in \text{Hom}_\Lambda(G, G^*) \\ \hat{\eta} &= \eta + \tilde{\chi} + d(\tilde{\eta})\end{aligned}$$

with $\tilde{\chi}$ defined in Remark 4.2.1.

Then $\rho^(z, t, \hat{\nu}, \hat{\theta}, \hat{\kappa}, \Delta, \hat{\eta})$ exists and is equal to $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta)$.*

Proof. Straightforward. See also Remarks 3.1.3, 4.2.1 and 6.2.2. \square

Lemma 6.4.3. *Let $z, t, \nu, \theta, \kappa, \Delta$ and η as in Definition 6.3.1. Let $z', t', \nu', \theta', \kappa', \Delta'$ and η' another set of data as in Definition 6.3.1. Then*

$$\begin{aligned}\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) &+ \rho^*(z', t', \nu', \theta', \kappa', \Delta', \eta') \\ &= \rho^*(z \oplus z', t \oplus t', \nu \oplus \nu', \theta \oplus \theta', \kappa \oplus \kappa', \Delta \oplus \Delta', \eta \oplus \eta')\end{aligned}$$

In a special case the quadratic signatures only depend on z, t and Δ . Later it will be shown that the quadratic signature does only depend on the preformation and flip-isomorphism $\text{rel}\partial$ if $\Lambda = \mathbb{Z}$ and $\epsilon = -1$ (see Corollary 6.5.3).

Lemma 6.4.4. *Let $z, t, \nu, \theta, \kappa, \Delta$ and η as in Definition 6.3.1. Let $z, t, \nu', \theta', \kappa', \Delta$ and η' another set of data as in Definition 6.3.1. Assume that the map $Q_{-\epsilon}(G^*) \rightarrow Q_{-\epsilon}(F)$, $\theta \mapsto \mu\theta\mu^*$ is injective. Then $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) = \rho^*(z, t, \nu', \theta', \kappa', \Delta, \eta')$. Hence the quadratic signature only depends on z, t and Δ .*

If $\Lambda = \mathbb{Z}$ this is the case if μ is injective and, for $\epsilon = 1$, if additionally $|\text{Tor coker } \mu|$ is odd.

Proof. Using Lemma 6.4.2 we can reduce the problem to the case where $\theta = \theta', \nu = \nu'$. Then we have two homotopies

$$\begin{aligned} (\Delta, \eta) & : (1, 0) \simeq (h_t, \chi_t) : (C, \psi) \longrightarrow (C, \psi) \\ (\Delta, \eta') & : (1, 0) \simeq (h_t, \chi'_t) : (C, \psi) \longrightarrow (C, \psi) \end{aligned}$$

Hence $\chi'_t - \chi_t = d(\eta' - \eta)$. It follows that $d(\tilde{\eta})_0 = 0 : C^{q+1} \rightarrow C_q$ and $d(\tilde{\eta})_1 = 0 : C^q \rightarrow C_q$ for $\tilde{\eta} = \eta' - \eta$. Combining the equations yields $d\tilde{\eta}_0 d^* = (1 + T_\epsilon)(-\epsilon d\tilde{\eta}_1 - \epsilon\tilde{\eta}_2) : C^q \rightarrow C_q$. Hence $\mu\tilde{\eta}_0\mu^* = 0 \in Q_{-\epsilon}(F)$. Hence $\eta' = \eta \in Q_{-\epsilon}(G)$ and the claim follows.

Now let $\Lambda = \mathbb{Z}$. We need to prove that μ induces an injection between the Q -groups. By the Smith Normal Form Theorem 9.4.2, we can assume that

$$\mu = \begin{pmatrix} d_1 & 0 \cdots 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 \cdots 0 & d_n \\ 0 & 0 \cdots 0 & 0 \end{pmatrix} : G = \mathbb{Z}^n \longrightarrow F^* = \mathbb{Z}^m$$

with all $d_i > 0$ and $m \geq n$. By assumption, all d_i are odd if $\epsilon = 1$. Let $\theta \in \text{Mat}(n \times n, \mathbb{Z})$ such that $\mu\theta\mu^* = \kappa + \epsilon\kappa^*$ for some $\kappa \in \text{Mat}(m \times m, \mathbb{Z})$. It follows that $\theta_{ij} = \epsilon\theta_{ji}$ and $d_i^2\theta_{ii} = (1 + \epsilon)\kappa_{ii}$ for all $i, j \in \{1, \dots, n\}$. If $\epsilon = -1$ then $\theta_{ii} = 0$ for all i . Hence

$$\kappa'_{ij} = \begin{cases} \theta_{ij} & \text{if } i < j \\ 0 & \text{else} \end{cases} \text{ fulfils } \theta = \kappa' + \epsilon\kappa'^* \text{ and } [\theta] = 0 \in Q_{-\epsilon}(G^*).$$

If $\epsilon = 1$ then $d_i^2\theta_{ii} = 2\kappa_{ii}$. By assumption $d_i^2 | \kappa_{ii}$ and hence we can define $\kappa' \in \text{Mat}(n \times n, \mathbb{Z})$ by $\kappa'_{ij} = \begin{cases} \theta_{ij} & \text{if } i < j \\ \kappa_{ii}/d_{ii}^2 & \text{if } i = j \\ 0 & \text{else} \end{cases}$. It follows that $\theta = \kappa' + \epsilon\kappa'^*$ and $[\theta] = 0 \in Q_{-\epsilon}(G^*)$. □

6.5 Quadratic and Asymmetric Signatures

There is a close relationship of asymmetric and quadratic signatures which is not that surprising if we re-examine their construction. For simplicity let (W, M, M) be a $(2q + 2)$ -dimensional cobordism with $\partial M = \emptyset$. By glueing W together along M and after surgery below the middle dimension we obtain a $(2q + 2)$ -dimensional closed manifold V . Poincaré duality induces a non-singular $(-)^q$ -symmetric form

$\lambda: H^{q+1}(V) \xrightarrow{\cong} H_{q+1}(V)$. The asymmetric signature of (W, M, M) is the image of that form in $LAsy^0(\Lambda)$ (compare Section 5.1).

Similarly, in the algebraic surgery world we can glue an $(2q+2)$ -dimensional quadratic Poincaré pair $d = (D \oplus D \longrightarrow E, (\delta\nu, \nu \oplus -\nu))$ together along D and obtain a $(2q+2)$ -dimensional quadratic Poincaré pair (V, τ) (which is basically the quadratic signature in this simple situation). By Lemma 5.2.8 the image of its symmetrization $(V, (1+T)\tau)$ in $LAsy^0(\Lambda)$ is the asymmetric signature of d . This fact generalizes to the case of all quadratic signatures.

Theorem 6.5.1. *Let $z, t, \nu, \theta, \kappa, \Delta$ and η as in Definition 6.3.1. The image of the quadratic signature $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta)$ under the map*

$$\begin{aligned} L_{2q+2}(\Lambda) &\longrightarrow LAsy^0(\Lambda) \\ (K, \psi) &\longmapsto (K, \psi_0 - \epsilon\psi^*) \end{aligned}$$

is the asymmetric signature $\sigma^*(z, t)$.

Proof. Let $(V, \tau) = w_t \cup -y$ be the union of $(2q+2)$ -dimensional quadratic Poincaré pairs defined in Section 6.3. (Using standard algebraic surgery theory - e.g. Lemma A.4.6 - (V, τ) is a $(2q+2)$ -dimensional quadratic Poincaré complex representing the quadratic signature $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) \in L_{2q+2}(\Lambda)$).

By Lemma A.3.2, $(V, (1+T)\tau) \cong (1+T)w_t \cup -(1+T)y$. Again by Lemma A.3.2 there is an isomorphism

$$(1, \bar{\sigma}): (\partial E', \theta') = (1+T)c \cup -(1+T)c \xrightarrow{\cong} (\partial E', (1+T)\omega') = (1+T)(c \cup -c)$$

with $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ (compare (4.4) and (3.3)) which - applied to $(1+T)w_t$ - yields a new $(2q+2)$ -dimensional symmetric Poincaré pair

$$\begin{aligned} w^t &= (g'_t: \partial E' \longrightarrow E, (\delta\theta' = (1+T)\delta\omega' + (-)^{2q-1}g'_t\bar{\sigma}g'^*_t, \theta')) \\ \delta\theta'_0 &= -((1+T_\epsilon)\eta_0 + \epsilon\Delta_{q+1}T\psi_0\Delta_{q+1}^*): E^{q+1} = G^* \longrightarrow E_{q+1} = G \end{aligned}$$

(we use Lemma A.2.6 here).

By Lemma A.3.3 $(1+T)w_t \cup -(1+T)y \cong w^t \cup -y'$ with some Poincaré pair y' which is - like y and $(1+T)y$ - an algebraic h -cobordism.

All in all this means that $(V, (1+T)\tau) \cong w^t \cup -y'$. Note that w^t is a Poincaré pair with a (trivial) twisted double on its boundary. Hence by Proposition 5.4.4, 5.4.5 and Corollary 5.2.8

$$\begin{aligned} (1+T_{-\epsilon})\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) &= (V, (1+T)\tau) = \sigma^*(V, (1+T)\tau) \\ &= \sigma^*(w^t) - \sigma^*(y') = \sigma^*(w^t) \in LAsy^0(\Lambda) \end{aligned}$$

Now we verify that $(\Delta, (1+T)\eta + \xi): (h_t, 0) \simeq (1, 0): (C, \phi) \xrightarrow{\cong} (C, \phi)$ is a well-defined homotopy with ξ as in Lemma 6.1.2 which transforms w^t to x^t using Lemma 6.1.7. Then Lemma 6.1.8 shows that $\sigma^*(w^t) = \sigma^*(x^t) = \sigma^*(x, t) \in LAsy^0(\Lambda)$. \square

Remark 6.5.2. The relationship between quadratic and asymmetric signatures can be made more precise. By [Ran98] 30.29 there is an exact sequence

$$0 \longrightarrow DBL_{2q+1}(\Lambda) \xrightarrow{t} L_{2q+2}(\Lambda) \xrightarrow{(1+T-\epsilon)} LAsy^0(\Lambda) \\ \rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) \longmapsto \sigma^*(z, t)$$

with $DBL_{2q+1}(\Lambda)$ the kernel of the map

$$LAut_{2q+1}(\Lambda) \longrightarrow L_{2q+1}^p(\Lambda) \\ [(h, \chi): (C, \psi) \xrightarrow{\cong} (C, \psi)] \longmapsto [(C, \psi)]$$

The map t is induced by the **algebraic mapping torus**: Let $(h, \chi): (C, \psi) \xrightarrow{\cong} (C, \psi)$ be a self-equivalence of an n -dimensional quadratic Poincaré complex. Then the algebraic mapping torus is the union of the fundamental $(n+1)$ -dimensional quadratic Poincaré pair

$$((h, 1): C \oplus C \longrightarrow C, ((-)^n \chi, \psi \oplus -\psi))$$

as described in Definition A.3.4.

In the case of $\Lambda = \mathbb{Z}$ and $q = 2k$, the map $L_{4k+2}(\mathbb{Z}) \longrightarrow LAsy^0(\mathbb{Z})$ factors through $L^{4k+2}(\mathbb{Z}) = 0$ hence the above sequence boils down to

$$0 \longrightarrow DBL_{4k+1}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} L_{4k+2}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} LAsy^0(\mathbb{Z})$$

In the case $\Lambda = \mathbb{Z}$ and $q = 2k - 1$, the composition $L_{4k}(\mathbb{Z}) = \mathbb{Z} \longrightarrow LAsy^0(\mathbb{Z}) \longrightarrow LAsy^0(\mathbb{C})$ is an injection (because composition with the asymmetric multi-signatures $LAsy^0(\mathbb{C}) \cong \mathbb{Z}[S^1]$ from [Ran98] Proposition 40.6 and the projection $\mathbb{Z}[S^1] \longrightarrow \mathbb{Z}$, $\sum_{g \in S^1} n_g \longmapsto n_1$ gives the signature on $L_{4k}(\mathbb{Z}) = \mathbb{Z}$). The exact sequence becomes

$$0 \longrightarrow DBL_{4k-1}(\mathbb{Z}) = 0 \xrightarrow{0} L_{4k}(\mathbb{Z}) = \mathbb{Z} \xrightarrow{(1+T)} LAsy^0(\mathbb{Z})$$

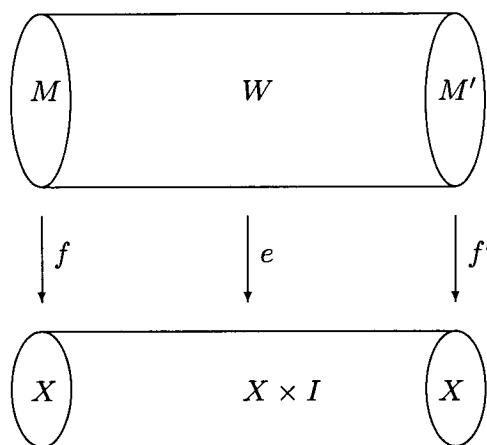
Corollary 6.5.3. *Let $q = 2m - 1$ i.e. $\epsilon = -1$. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular skew-quadratic split preformation over \mathbb{Z}*

- i) $[z] \in l_{4m}(\mathbb{Z})$ is elementary if and only if there is a flip-isomorphism rel ∂ t such that $\sigma^*(z, t) = 0 \in LAsy^0(\mathbb{Z})$.*
- ii) The quadratic signature $\rho^*(z, t, \nu, \theta, \kappa, \Delta, \eta) \in L_{4m}(\mathbb{Z})$ only depends on z and t .*

Chapter 7

Non-Singular Formations

Throughout this chapter “formation” will mean a non-singular ϵ -quadratic split formation for $\epsilon = (-)^q$, $q \geq 2$. Let Λ be a weakly finite ring with involution and 1.



Let $(e, f, f'): (W, M, M') \rightarrow X \times (I, 0, 1)$ be a special kind of Kreck surgery situation (i.e. e, f and f' are highly-connected): all maps are normal maps and $(X, \partial X)$ is a finite geometric Poincaré pair such that $\partial M \xrightarrow{\cong} \partial X$ is a homotopy equivalence. Such a normal cobordism is called a **presentation** of f . Presentations are also used to define obstructions to odd-dimensional traditional surgery problems (for details see Section 2.1). Hence the Kreck surgery obstruction $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ with $F = K_{q+1}(W, M)$ and $G = K_{q+1}(W)$ in this case is a formation i.e. $(\gamma_\mu): G \rightarrow H_\epsilon(F)$ is an inclusion of a lagrangian.

It is possible to prove much stronger results about formations. In **Section 7.1** we deal with some useful technical observations about formations and their flip-isomorphism. If one applies the construction of Section 3.1 to a formation one obtains a $(2q + 2)$ -dimensional quadratic Poincaré pair $x = (g: \partial E = D' \cup_C D \rightarrow E, (\delta\omega = 0, \omega))$ for which C is contractible.

The definition of quadratic signatures was rather awkward because we had to make sure that a flip-isomorphism induces a self-equivalence (h_t, χ_t) on (C, ψ) which is homotopic

to the identity. For formations C is contractible and, hence, every flip-isomorphism is a flip-isomorphism $\text{rel}\partial$. This also leads to a simpler version of the quadratic signatures in **Section 7.2**.

We investigate the behaviour of the asymmetric signatures of formations in **Section 7.3**. They turn out to be independent from the choice of flip-isomorphism. As an application we will construct non-elementary preformations for which all asymmetric signatures are vanishing. We will also show how the asymmetric signatures relate to traditional even-dimensional surgery theory i.e. how they behave for boundaries of non-singular forms.

7.1 Flip-Isomorphisms

We will need to discuss some very technical properties of non-singular formations in order to make the computations in the following sections.

A formation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is an ϵ -quadratic split preformation such that the map

$$\begin{pmatrix} \gamma \\ \mu \end{pmatrix} : G \longrightarrow H_\epsilon(F)$$

is an inclusion of a lagrangian. By [Ran80a] Proposition 2.2., this map can be extended to an isomorphism of hyperbolic ϵ -quadratic forms

$$\left(f = \begin{pmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \tilde{\gamma}^* \mu & \tilde{\theta} \end{pmatrix} \right) : H_\epsilon(G) \xrightarrow{\cong} H_\epsilon(F)$$

For any $\tau : G^* \longrightarrow G$ the maps $\tilde{\gamma}' = \tilde{\gamma} + \gamma(\tau - \epsilon\tau^*)$, $\tilde{\mu}' = \tilde{\mu} + \mu(\tau - \epsilon\tau^*)$, $\tilde{\theta}' = \tilde{\theta} + (\tau - \epsilon\tau^*)^* \theta (\tau - \epsilon\tau^*) + \tilde{\gamma}^* \mu (\tau - \epsilon\tau^*)^* - \epsilon\tau$ define another extension to an isomorphism of hyperbolic forms. Conversely any such extension can be derived from $\tilde{\gamma}$, $\tilde{\mu}$, $\tilde{\theta}$.

The relationship between those maps and a flip-isomorphism of the formation can be described as follows:

Lemma 7.1.1. *Let $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a formation and (α, β, ν) a flip isomorphism. Let f , $\tilde{\gamma}$, $\tilde{\mu}$ and $\tilde{\theta}$ as explained before.*

$$i) f^{-1} = \begin{pmatrix} \tilde{\mu}^* & \epsilon\tilde{\gamma}^* \\ \epsilon\mu^* & \gamma^* \end{pmatrix}.$$

ii) *There is an isomorphism*

$$\begin{aligned} & \left(\begin{pmatrix} \tilde{\mu}^* & \tilde{\gamma}^* \\ \mu^* & \gamma^* \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ \epsilon\tilde{\gamma}^* \mu & \tilde{\mu}^* \gamma \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right) : \\ & (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) + (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, \theta) \xrightarrow{\cong} \partial(G, \theta) + (G^*, G) \end{aligned}$$

$$iii) (a) \alpha(\gamma + (\nu - \epsilon\nu^*)^* \mu) = \epsilon\mu\beta$$

$$(b) \alpha^{-*} \mu = \gamma\beta$$

$$(c) \beta^* \theta \beta + \theta + \mu^* \nu \mu = 0 \in Q_{-\epsilon}(G)$$

iv) There is a $\xi \in Q_{-\epsilon}(G^*)$ and $Y = \xi - \epsilon \xi^*$ such that

$$(a) \epsilon \alpha (\tilde{\gamma} + (\nu - \epsilon \nu^*)^* \tilde{\mu}) \beta^* = \mu Y + \tilde{\mu}$$

$$(b) \alpha^{-*} \tilde{\mu} \beta^* = \gamma Y + \tilde{\gamma}$$

Proof. i) From $f^* H_\epsilon f = H_\epsilon$ it follows that $f^{-1} = H_\epsilon^{-1} f^* H_\epsilon$.

ii) Follows straight from the Definition 2.2.13.

iii) One can compute the composition of isometries of hyperbolic forms

$$f^{-1} H_\epsilon \begin{pmatrix} \alpha & \alpha(\nu - \epsilon \nu^*)^* \\ 0 & \alpha^{-*} \end{pmatrix} f \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta^* \end{pmatrix} = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$$

with

$$\begin{aligned} Y &= \tilde{\mu}^* \alpha^{-*} \tilde{\mu} \beta^* + \tilde{\gamma}^* \alpha \tilde{\gamma} \beta^* + \tilde{\gamma}^* \alpha (\nu - \epsilon \nu^*)^* \tilde{\mu} \beta^* \\ \xi &= -\beta \tilde{\theta}^* \beta^* - \beta \tilde{\mu}^* \nu^* \tilde{\mu} \beta^* + \epsilon Y^* \theta Y + \epsilon \tilde{\gamma}^* \mu Y + \epsilon \tilde{\theta} \end{aligned}$$

□

7.2 Quadratic Signatures

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a formation and $t = (\alpha, \beta, \nu)$ be a flip-isomorphism. there is a representative θ of $\bar{\theta} \in Q_{-\epsilon}(G)$ and a representative ν for $\bar{\nu} \in Q_{-\epsilon}(F^*)$ and a $\kappa \in \text{Hom}_\Lambda(G, G^*)$ such that $\beta^* \theta \beta + \theta + \mu^* \nu \mu = \kappa + \epsilon \kappa^*$.

As described in the previous section we can extend $\begin{pmatrix} \gamma \\ \mu \end{pmatrix}$ to an isomorphism $\begin{pmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{pmatrix}$ of hyperbolic forms. A choice of $\tilde{\gamma}, \tilde{\mu}, \tilde{\theta}$ (compare previous section) defines a homotopy $\Delta_C: 1 \simeq 0: C \rightarrow C$ with

$$\begin{aligned} \Delta_{C, q+1} &= (\epsilon \tilde{\mu}^* \quad \tilde{\gamma}^*) : C_q = F \oplus F^* \rightarrow C_{q+1} = G \\ \Delta_{C, q} &= -\epsilon \begin{pmatrix} \tilde{\gamma} \\ \tilde{\mu} \end{pmatrix} : C_{q-1} = G^* \rightarrow C_q = F \oplus F^* \end{aligned}$$

Then all flip-isomorphisms of a z are flip-isomorphisms rel ∂ with a homotopy

$$(\Delta = \Delta_C(1 - h_t), \eta = \Delta_{C, \%}(\chi_t - \Delta_{\%} \psi)) : (1, 0) \simeq (h_t, \chi_t) : (C, \psi) \rightarrow (C, \psi)$$

In particular¹

$$\begin{aligned} \eta_0 &= \tilde{\gamma}^* \alpha \nu \alpha^* \tilde{\gamma} - \epsilon (\tilde{\mu}^* \alpha^{-*} \tilde{\mu} + \tilde{\gamma}^* \alpha (\nu^* - \epsilon \nu) \tilde{\mu} - \tilde{\gamma}^* \alpha (\nu^* - \epsilon \nu) \alpha^* \tilde{\gamma}) : \\ &C^{q+1} = G^* \rightarrow C_{q+1} = G \\ R &= \epsilon \tilde{\mu}^* - \epsilon \tilde{\gamma}^* \alpha \end{aligned}$$

¹The computations are easier if one uses the fact that $\Delta^2 = 0$.

The non-singular $(-\epsilon)$ -quadratic form (M, ξ) from the definition of quadratic signatures (6.3.1) is given by

$$\xi = \begin{pmatrix} -\eta_0 & \beta & 0 \\ 0 & -\epsilon\theta^* & 0 \\ R^* & \mu & 0 \end{pmatrix} : M = G^* \oplus G \oplus F^* \longrightarrow M^*$$

Using the isomorphism

$$f = \begin{pmatrix} 1 & 0 & -\tilde{\gamma}^*\alpha(\nu^* - \epsilon\nu) - \tilde{\mu}^*\alpha^{-*} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : M^* \xrightarrow{\cong} M^*$$

we obtain a prettier non-singular $(-\epsilon)$ -quadratic form $(M, \xi' = -f\xi f^*)$

$$\xi' = \begin{pmatrix} \tilde{\gamma}^*\tilde{\mu} + \tilde{\gamma}^*\alpha\nu\alpha^*\tilde{\gamma} & -\tilde{\gamma}^*\alpha\gamma & 0 \\ 0 & \epsilon\theta^* & 0 \\ \epsilon(\alpha^*\tilde{\gamma} - \tilde{\mu}) & -\mu & 0 \end{pmatrix} : M = G^* \oplus G \oplus F^* \longrightarrow M^*$$

Definition 7.2.1. The quadratic signature $\tilde{\rho}^*(z, t, \tilde{\gamma}, \tilde{\mu}, \tilde{\theta})$ is the element $(M, \xi') \in L_{2q+2}(\Lambda)$. \square

Remark 7.2.2. For any $\tau: G^* \longrightarrow G$ the maps $\tilde{\gamma}' = \tilde{\gamma} + \gamma(\tau - \epsilon\tau^*)$, $\tilde{\mu}' = \tilde{\mu} + \mu(\tau - \epsilon\tau^*)$, $\tilde{\theta}' = \tilde{\theta} + (\tau - \epsilon\tau^*)^*\theta(\tau - \epsilon\tau^*) + \tilde{\gamma}^*\mu(\tau - \epsilon\tau^*)^* - \epsilon\tau$ define another extension to an isomorphism of hyperbolic forms. Conversely any such extension can be derived from $\tilde{\gamma}, \tilde{\mu}, \tilde{\theta}$.

Theorem 7.2.3. Let z' be a formation. $[z'] \in l_{2q+2}(\Lambda)$ is elementary if and only if there is a stably strongly isomorphic $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \tilde{\theta})$, a flip-isomorphism t and $\tilde{\gamma}, \tilde{\mu}$ and $\tilde{\theta}$ such that

$$\left(f = \begin{pmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \tilde{\gamma}^*\mu & \tilde{\theta} \end{pmatrix} \right) : H_\epsilon(G) \xrightarrow{\cong} H_\epsilon(F)$$

is an isomorphism of hyperbolic ϵ -quadratic forms and $\tilde{\rho}^*(z, t, \tilde{\gamma}, \tilde{\mu}, \tilde{\theta}) = 0 \in L_{2q+2}(\Lambda)$.

Proof. If there exist $t, \tilde{\gamma}, \tilde{\mu}$ and $\tilde{\theta}$ as above then by construction $0 = \tilde{\rho}^*(z, t, \tilde{\gamma}, \tilde{\mu}) = -\rho^*(z, t, \Delta, \rho)$. By Theorem 6.4.1 $[z] \in l_{2q+2}(\Lambda)$ is elementary.

Assume now that $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \tilde{\theta})$ is elementary and let it have the special form described in Proposition 2.4.2iv). Clearly $(\sigma_\tau) : U \longrightarrow U \oplus U^*$ is a split injection and even a lagrangian. As in Section 7.1 the map can be extended to an isometry

$$\left(\begin{pmatrix} \sigma & \tilde{\sigma} \\ \tau & \tilde{\tau} \end{pmatrix}, \begin{pmatrix} \theta' & 0 \\ \tilde{\sigma}^*\tau & \tilde{\theta}' \end{pmatrix} \right) : H_\epsilon(R) \longrightarrow H_\epsilon(U^*)$$

of hyperbolic forms. Then the maps

$$\tilde{\gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma} \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow F = U \oplus U^*$$

$$\tilde{\mu} = \begin{pmatrix} 1 & -\epsilon\tilde{\sigma} \\ 0 & \tilde{\tau} \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow F^* = U^* \oplus U$$

$$\tilde{\theta} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta}' \end{pmatrix} : G^* = U^* \oplus R^* \longrightarrow G = U \oplus R$$

are completing $(\tilde{\mu})$ to an isometry of hyperbolic forms (compare Section 7.1).

Define a strong flip-isomorphism $t = (\alpha, \beta, \nu = 0)$ of z by

$$\begin{aligned}\alpha &= \begin{pmatrix} 0 & -1 \\ -\epsilon & 0 \end{pmatrix} : F = U \oplus U^* \longrightarrow F^* = U^* \oplus U \\ \beta &= \begin{pmatrix} -1 & -\tau \\ 0 & 1 \end{pmatrix} : G = U \oplus R \longrightarrow G = U \oplus R\end{aligned}$$

Then $\tilde{\rho}^*(z, t, \tilde{\mu}, \tilde{\theta})$ is given by the non-singular form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\sigma}^* \tilde{\tau} & \epsilon \tilde{\sigma}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma^* & \epsilon \theta'^* & 0 & 0 \\ -\epsilon & 0 & 0 & \epsilon \sigma & 0 & 0 \\ 0 & -\epsilon \tilde{\tau} & -1 & -\tau & 0 & 0 \end{pmatrix} : M = U^* \oplus R^* \oplus U \oplus R \oplus U^* \oplus U \\ \longrightarrow M^* = U \oplus R \oplus U^* \oplus R^* \oplus U \oplus U$$

with a lagrangian

$$i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \epsilon \tilde{\sigma} & 0 \end{pmatrix} : U^* \oplus R^* \oplus U \longrightarrow M = U^* \oplus R^* \oplus U \oplus R \oplus U^* \oplus U$$

□

7.3 Asymmetric Signatures

The asymmetric signature of formations has one surprising property: it is independent of the choice of flip-isomorphism (although the existence of a flip-isomorphism is still a necessity to define it). We illustrate this fact by showing an analogy in the world of manifolds. Let (W, M, M') be a manifold with $\partial M = \emptyset$ and let $H: M \xrightarrow{\cong} M'$ be a diffeomorphism. Glueing the cobordism along H yields a closed manifold W_H . Different choices of H lead to different manifolds which however are in the same *Schneiden-und-Kleben-cobordism class* (= cut-and-paste-cobordism class). These cobordism groups were e.g. studied in [KKNO73] (see also [Ran98] Remark 30.30) and are quotients of the ordinary cobordism groups using the equivalence relation $P \cup_f N \sim P \cup_g N$ for manifolds with boundary $(P, \partial P)$ and $(N, \partial N)$ and homeomorphisms $f, g: \partial N \xrightarrow{\cong} \partial P$. The SKL-cobordism group of an $(n+1)$ -dimensional manifold V with $n > 5$ is discovered by an asymmetric signature similar to the one used for twisted doubles in Section 5.1: one takes a singular chain complex $C = C(V)$ with a chain equivalence $\lambda = [M] \cap -: C^{n+1-*} \xrightarrow{\cong} C$ and looks at the image of $(C, \lambda) \in LAsy^{n+1}(\mathbb{Z})$. The SKL-bordism group is isomorphic of the image of $L^{n+1}(\mathbb{Z}) \longrightarrow LAsy^{n+1}(\mathbb{Z})$, hence \mathbb{Z}

if $n \equiv 0 \pmod{4}$ and zero else. Our proof will use equivalent facts about symmetric Poincaré pairs (see [Ran98] 30.30).

Theorem 7.3.1. *Let z be a formation. Let t and t' be two flip-isomorphisms. Then $\sigma^*(z, t) = \sigma^*(z, t') \in LAsy^0(\Lambda)$. Hence the map σ^* defined in Theorem 5.5.3 induces a map*

$$\sigma^*: \tilde{l}_{2q+2}(\Lambda) \longrightarrow LAsy^0(\Lambda)$$

with $\tilde{l}_{2q+2}(\Lambda) = \{[z] \in l_{2q+2}(\Lambda) : z \text{ allows stable flip-isomorphisms}\}$ a submonoid of $l_{2q+2}(\Lambda)$.

This theorem can be applied to the boundary of non-singular forms. They are the obstructions of Wall's surgery theory interpreted as a special case of Kreck's surgery theory i.e. they live in the image of the inclusion

$$\begin{aligned} L_{2q+2}(\Lambda) &\hookrightarrow l_{2q+2}(\Lambda) \\ (K, \theta) &\longmapsto \partial(K, \theta) = (K \xleftarrow{1_K} K \xrightarrow{\theta - \epsilon\theta^*} K^*, \theta) \end{aligned}$$

Corollary 7.3.2. *Let (K, θ) be a $(-\epsilon)$ -quadratic form. Then $z = \partial(K, \theta)$ is a formation.*

- i) z has a (stable) flip-isomorphism (i.e. $[z] \in \tilde{l}_{2q+2}(\Lambda)$) if and only if (K, θ) is non-singular.
- ii) z has a stable strong flip-isomorphism if and only if (K, θ) is non-singular and $2 \cdot [(K, \theta)] = 0 \in L_{2q+2}(\Lambda)$.
- iii) $[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if (K, θ) is non-singular and $[(K, \theta)] = 0 \in L_{2q+2}(\Lambda)$.
- iv) If (K, θ) is non-singular, $\sigma^*([z, t]) = [(K, \theta - \epsilon\theta^*)] \in LAsy^0(\Lambda)$ for any stable flip-isomorphism t .
- v) Assume that either Λ is a field of characteristic different to 2 or that $\Lambda = \mathbb{Z}$ and $\epsilon = -1$. z is elementary if and only if the asymmetric signature vanishes.

Proof. i) Obviously (K, θ) must be non-singular if z allows a stable flip-isomorphism. If (K, θ) is non-singular then $t = (\lambda^*, 1, \epsilon\lambda^{-1})$ is a flip-isomorphism of z .

- ii) If (K, θ) has a strong flip-isomorphism (α, β) then $\beta^*\theta\beta = -\theta \in Q_{-\epsilon}(K)$. Hence $[(K, \theta)] = [(K, -\theta)] \in L_{2q+2}(\Lambda)$. On the other hand if $2 \cdot [(K, \theta)] = 0 \in L_{2q+2}(\Lambda)$ then, after stabilization, there is an isomorphism $\beta^*\theta\beta = -\theta \in Q_{-\epsilon}(K)$ and $(\epsilon\lambda\beta, \beta)$ is a strong flip-isomorphism.

iii) Easy.

iv) Let $t = (\alpha, \beta, \nu)$ be the flip-isomorphism of i). Let $\lambda = \theta - \epsilon\theta^*$ and let

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha(\nu^* - \epsilon\nu)\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^*$$

as in Definition 5.3.1. Then $\rho \oplus -\lambda$ has the lagrangian

$$\begin{pmatrix} \epsilon & 1 \\ 0 & -\epsilon\lambda \\ 1 & 0 \\ -\epsilon & 1 \end{pmatrix}$$

Hence the asymmetric signature $\sigma^*(z, t) = (K, \lambda) \in LAsy^0(\Lambda)$. By Theorem 7.3.1 the asymmetric signature is independent of the choice of stable flip-isomorphism.

v) In all these cases the maps $(1 + T_{-\epsilon}) : L_{2q+2}(\Lambda) \longrightarrow L^{2q+2}(\Lambda)$ and $L^{2q+2}(\Lambda) \longrightarrow LAsy^0(\Lambda)$ are injective (see e.g. [Ran98] Chapter 39D and Remark 6.5.2).

□

Here is a counter-example for the converse of Theorem 5.5.3.

Example 7.3.3. Let (K, θ) be a non-singular $(-\epsilon)$ -quadratic form such that

- i) $[(K, \theta)] \neq 0 \in L_{2q+2}(\Lambda)$
- ii) $[(K, \lambda = \theta - \epsilon\theta^*)] = 0 \in LAsy^0(\Lambda)$

Any non-singular skew-quadratic form over \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$ with non-trivial Arf-invariant fulfils these properties. By Corollary 7.3.2 $z = \partial(K, \theta)$ has stable flip-isomorphisms and all asymmetric signatures vanish although it is not stably elementary.

Back to the proof of Theorem 7.3.1. We recall that in Section 5.3 the asymmetric signature $\sigma^*(z, t) \in LAsy^0(\Lambda)$ of a flip-isomorphism t of a formation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ was defined as the asymmetric signature of the $(2q + 2)$ -dimensional symmetric Poincaré pair $x_t = (g_t : \partial E_t \longrightarrow E, (0, \partial\theta_t))$. For our purposes, we need to analyze the asymmetric signature of the case that the chain complex C is contractible and therefore that $D' \oplus D$ (or $D \oplus D$) and ∂E (or ∂E_t respectively) are chain equivalent. The following two lemmas treat this situation in general.

Lemma 7.3.4. *Let $(f : C \longrightarrow D, (\delta\phi, \phi))$ be an n -dimensional symmetric Poincaré pair and $(h, \chi) : (C, \phi) \xrightarrow{\cong} (C, \phi)$ a self-equivalence. Assume that C is contractible with $\Delta : 1 \simeq 0 : C \longrightarrow C$.*

Define $\nu = \delta\phi + (-)^{n-1} f \Delta^{\%} \phi f^$ and $\bar{\rho} = \Delta^{\%} (\Delta^{\%} \phi - \chi - h \Delta^{\%} \phi h^*)$.*

There is an equivalence

$$\begin{aligned}
(a, \sigma) &: (D \oplus D, \nu \oplus -\nu) \xrightarrow{\cong} (D \cup_h D, \delta\phi \cup_\chi -\delta\phi) \\
a &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : D_r \oplus D_r \longrightarrow (D \cup_h D)_r = D_r \oplus D_r \oplus C_{r-1} \\
\sigma_s &= \begin{pmatrix} (-)^n f \bar{\rho}_s f^* & 0 & 0 \\ 0 & 0 & (-)^{s-1} f \Delta^{\%} \phi_s \\ (-)^{n+1-r} \Delta^{\%} \phi_s h^* f^* & 0 & (-)^{n+1-r+s} T_\epsilon \Delta^{\%} \phi_{s-1} \end{pmatrix} : \\
& (D \cup_h D)^{n+1-r+s} = D^{n+1-r+s} \oplus D^{n+1-r+s} \oplus C^{n-r+s} \\
& \longrightarrow (D \cup_h D)_r = D_r \oplus D_r \oplus C_{r-1}
\end{aligned}$$

of n -dimensional ϵ -symmetric Poincaré complexes.

Lemma 7.3.5. *Let $x = (g: \partial E \longrightarrow E, (\theta, \partial\theta))$ be an $(n+1)$ -dimensional symmetric Poincaré pair such that the boundary $(\partial E, \partial\theta)$ is a twisted double of an n -dimensional symmetric Poincaré pair $(f: C \longrightarrow D, (\delta\phi, \phi))$ with respect to a self-equivalence $(h, \chi): (C, \phi) \xrightarrow{\cong} (C, \phi)$. We write*

$$g = (j_0 \ j_1 \ k) : \partial E_r = D_r \oplus D_r \oplus C_{r-1} \longrightarrow E_r$$

Assume that C is contractible with $\Delta: 1 \simeq 0: C \longrightarrow C$. Hence there is a chain equivalence $\Delta: 0 \simeq 1: C \longrightarrow C$ (i.e. $d\Delta + \Delta d = 1_C$). Applying Lemmas 7.3.4, A.2.6 to x yields an $(n+1)$ -dimensional symmetric Poincaré pair

$$x' = ((j_0 \ j_1) : D \oplus D \longrightarrow E, (\theta' = \theta + (-)^n g \sigma g^*, \nu \oplus -\nu))$$

Let (B, λ) be the asymmetric complex of x and (B', λ') the asymmetric complex of x' as given in Proposition 5.2.6. Then there is an equivalence

$$\begin{aligned}
(b, \xi) &: (B, T\lambda) \longrightarrow (B', T\lambda') \\
b &= \begin{pmatrix} 1 & (-)^r j_0 f \Delta & 0 \\ 0 & 0 & 1 \end{pmatrix} : B_r = E_r \oplus C_{r-1} \oplus D_r \longrightarrow B'_r = E_r \oplus D_r \\
\xi &= \begin{pmatrix} (-k \Delta \phi_0 h^* + (-)^{r+1} j_0 f \bar{\rho}_0) \Delta^* f^* j_0^* & 0 \\ f \Delta \phi_0 ((-)^{n-r} k^* - h^* \Delta^* f^* j_0^*) & (-)^{n-r} f \Delta \phi_0 f^* \end{pmatrix} : \\
& B^{n+2-r} = E^{n+2-r} \oplus D^{n+1-r} \longrightarrow B'_r = E_r \oplus D_r
\end{aligned}$$

of $(n+1)$ -dimensional asymmetric complexes with ρ and $\bar{\rho}$ as defined in Lemma 7.3.4.

Proof of Theorem 7.3.1. By Lemma 7.3.5, $\sigma^*(z, t) \in LAsy^0(\Lambda)$ is the asymmetric signature of the $(2q+2)$ -dimensional symmetric Poincaré pair

$$\begin{aligned}
x'^t &= (g'^t: D \oplus D \longrightarrow E, (\delta\theta', \nu \oplus -\nu)) \\
g'_{q+1} &= (1 \ -\beta) : D_{q+1} \oplus D_{q+1} = G \oplus G \longrightarrow E_{q+1} = G \\
\delta\theta'_0 &= -\epsilon Y : E^{q+1} = G^* \longrightarrow E_{q+1} = G \\
\nu_0 &= -\tilde{\mu}^* : D^q = F \longrightarrow D_{q+1} = G \\
\nu_0 &= -\tilde{\mu} : D^{q+1} = G^* \longrightarrow D_q = F^*
\end{aligned}$$

By Corollary 5.2.8 $\sigma^*(x'^t)$ is the image of the union of x'^t in $LAsy^0(\Lambda)$. But there is another way to construct x'^t : there is a $(2q+2)$ -dimensional quadratic Poincaré pair

$$\begin{aligned}\tilde{x} &= (\tilde{g}: D \oplus D' \longrightarrow E, (0, \nu \oplus -\nu')) \\ \tilde{g} &= \begin{pmatrix} 1 & -1 \end{pmatrix}: D_{q+1} \oplus D_{q+1} = G \oplus G \longrightarrow E_{q+1} = G \\ \nu'_0 &= -\tilde{\gamma}^*: D'^q = F^* \longrightarrow D'_{q+1} = G \\ \nu'_0 &= -\tilde{\gamma}: D'^{q+1} = G^* \longrightarrow D'_q = F\end{aligned}$$

and an isomorphism $(\bar{e}_t, \bar{\chi}_t): (D, \nu) \xrightarrow{\cong} (D', \nu')$ given by

$$\begin{aligned}\bar{e}_{t,q+1} &= \beta: D_{q+1} = G \longrightarrow D'_{q+1} = G \\ \bar{e}_{t,q} &= \alpha^{-*}: D_q = F^* \longrightarrow D'_q = F \\ \bar{\chi}_{t,0} &= -\epsilon Y: D'^{q+1} = G^* \longrightarrow D'_{q+1} = G\end{aligned}$$

Lemma A.2.6 and the isomorphism can be used to replace the “boundary component” (D', ν') by (D, ν) . The result will be x'^t . So, glueing both ends (i.e. the “boundary components” D and D') of \tilde{x} together using $(\bar{e}_t, \bar{\chi}_t)$ yields the union of x'^t . Hence the unions of x'^t for different choices of t are all in the same **algebraic Schneiden-und-Kleben-cobordism** class. By [Ran98] 30.30(ii) their images in $LAsy^{2q+2}(\Lambda)$ coincide. Those images are precisely the asymmetric signatures $\sigma^*(x'^t) = \sigma^*(z, t)$. \square

Chapter 8

Preformations with Linking Forms

For the whole chapter let $q \geq 2$, $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

We consider a special group of preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ namely those for which μ becomes an isomorphism after localization. The most important examples are probably preformations over \mathbb{Z} with injective μ and $\text{rk } G = \text{rk } F$. For those classes of preformations one can use the theory of linking forms developed for formations in [Ran81] 3.4. and improve our results for asymmetric signatures.

In **Section 8.1** we quickly repeat the concept of localization and define linking forms following [Ran81]. **Section 8.2** defines linking forms on preformations and discusses the relationship between isometries of those linking forms and flip-isomorphisms. It turns out that every flip-isomorphism induces an isometry of linking forms and in turn every isometry of linking forms gives rise to a stable flip-isomorphism.

Similar to the flip- l -monoids, in **Section 8.3** we define linking- l -monoids of preformations with a choice of isometry of their linking forms

In **Section 8.4** we show that the asymmetric signatures we defined in Section 5.3 only depend on the effect of the flip-isomorphism on the linking forms of the preformation. If a preformation is stably elementary then all those asymmetric signatures vanish (see Theorem 8.4.3). This theorem is an improvement to Theorem 5.4.1 because isometries of linking forms are easier to handle than flip-isomorphisms. For \mathbb{Z} there are only finitely many isometries of a given linking form. Also, it is enough to look at one representative of a class in $l_{2q+2}(\Lambda)$.

We will use these results to define asymmetric signatures for certain simply-connected Kreck surgery problems using the topological linking forms of the manifolds involved (see **Section 8.5**).

8.1 Localization

Although we could generalize our results for Ore-localization, we will focus on the easier case of localization away from a central multiplicative subset.

We repeat some definitions from [Ran81] Chapter 3.1. and 3.4.

Definition 8.1.1. A subset $S \subset \Lambda$ is called **central and multiplicative** if

- i) $st \in S$ for all $s, t \in S$,
- ii) $\bar{s} \in S$ for all $s \in S$,
- iii) if $sa = 0 \in \Lambda$ for some $s \in S$ and $a \in \Lambda$ then $a = 0 \in \Lambda$,
- iv) $sa = as \in \Lambda$ for all $s \in S$ and $a \in \Lambda$. □

Definition 8.1.2. Let $S \subset \Lambda$ be a central and multiplicative subset. The **localization** $S^{-1}\Lambda$ of Λ away from S is the ring with involution and 1 defined by the equivalence classes of pairs $(a, s) \in \Lambda \times S$ under the relation

$$(a, s) \sim (a', s') \iff as' = a's \in \Lambda$$

with

$$\begin{aligned} (a, s) + (a', s') &= (as' + a's, ss') \\ (a, s) \cdot (a', s') &= (aa', ss') \\ \overline{(a, s)} &= (\bar{a}, \bar{s}) \end{aligned}$$

□

Example 8.1.3. Let π be a group and $w: \pi \rightarrow \mathbb{Z}/2\mathbb{Z}$ be a group morphism. Let $\Lambda = \mathbb{Z}[\pi]$ be its group ring endowed with the w -twisted involution $n \cdot 1_g \mapsto w(g)n \cdot 1_{g^{-1}}$. Then $S = \mathbb{Z} \setminus \{0\}$ is a central multiplicative subset of Λ . The localization of Λ away from S is canonically isomorphic to the group ring $\mathbb{Q}[\pi]$ with the obvious involution.

Definition 8.1.4. Let $S \subset \Lambda$ be a central and multiplicative subset. A morphism $f: M \rightarrow N$ of Λ -modules is called an S -isomorphism if the induced $S^{-1}\Lambda$ -module morphism

$$\begin{array}{ccc} S^{-1}f: S^{-1}M & \longrightarrow & S^{-1}N \\ \frac{x}{s} & \longmapsto & \frac{f(x)}{s} \end{array}$$

is an isomorphism. □

Definition 8.1.5. Let $S \subset \Lambda$ be a central and multiplicative subset. A chain complex C over Λ is S -acyclic if the chain complex $S^{-1}C = C \otimes_{\Lambda} S^{-1}\Lambda$ is acyclic. □

Definition 8.1.6. Let $S \subset \Lambda$ be a central and multiplicative subset. A (Λ, S) -**module** M is an Λ -module such that there is an exact sequence of Λ -modules

$$0 \longrightarrow P \xrightarrow{d} Q \longrightarrow M \longrightarrow 0$$

with P and Q f.g. projective and d an S -isomorphism. \square

Definition 8.1.7. Let $S \subset \Lambda$ be a central and multiplicative subset.

i) Define the **relative Q -groups**

$$\begin{aligned} Q^\epsilon(\Lambda, S) &= \{b \in S^{-1}\Lambda \mid b - \epsilon\bar{b} = a - \epsilon\bar{a}, a \in \Lambda\} / \Lambda \\ &\subset Q^\epsilon(S^{-1}\Lambda / \Lambda) = \{b \in S^{-1}\Lambda \mid b - \epsilon\bar{b} \in \Lambda\} / \Lambda \\ Q_\epsilon(\Lambda, S) &= \{b \in S^{-1}\Lambda \mid b = \epsilon\bar{b}\} / \{a + \epsilon\bar{a} : a \in \Lambda\} \\ &\subset Q_\epsilon(S^{-1}\Lambda / \Lambda) = S^{-1}\Lambda / \{a + b - \epsilon\bar{b} \mid a \in \Lambda, b \in S^{-1}\Lambda\} \end{aligned}$$

ii) An ϵ -**symmetric linking form** (M, λ) **over** (Λ, S) is an (Λ, S) -module M together with a pairing $\lambda: M \times M \longrightarrow S^{-1}\Lambda / \Lambda$ such that $\lambda(x, -): M \longrightarrow S^{-1}\Lambda / \Lambda$ is Λ -linear for all $x \in M$ and $\lambda(x, y) = \overline{\epsilon\lambda(y, x)}$ for all $x, y \in M$.

iii) A **split ϵ -quadratic linking form** (M, λ, ν) **over** (Λ, S) is an ϵ -symmetric linking form (M, λ) over (Λ, S) together with a map $\nu: M \longrightarrow Q_\epsilon(S^{-1}\Lambda / \Lambda)$ such that for all $x, y \in M$ and $a \in \Lambda$

$$\begin{aligned} \text{(a)} \quad &\nu(ax) = a\nu(x)\bar{a} \in Q_\epsilon(S^{-1}\Lambda / \Lambda) \\ \text{(b)} \quad &\nu(x + y) - \nu(x) - \nu(y) = \lambda(x, y) \in Q_\epsilon(S^{-1}\Lambda / \Lambda) \\ \text{(c)} \quad &(1 + T_\epsilon)\nu(x) = \lambda(x, x) \in Q^\epsilon(\Lambda, S) \end{aligned}$$

\square

Example 8.1.8. Let $\Lambda = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$. Then $S^{-1}\Lambda = \mathbb{Q}/\mathbb{Z}$. A (Λ, S) -module is nothing but a finite abelian group.

An ϵ -symmetric linking form (M, λ) over (Λ, S) is a finite abelian group M together with a bilinear ϵ -symmetric pairing $\lambda: M \times M \longrightarrow \mathbb{Q}/\mathbb{Z}$ on it.

For $\epsilon = 1$ a split quadratic linking form (M, λ, ν) over (Λ, S) is nothing but symmetric linking form (M, λ) with $\nu: M \longrightarrow Q_1(S^{-1}\Lambda / \Lambda) = \mathbb{Q}/\mathbb{Z}$ given by $\nu(x) = \frac{1}{2}\lambda(x, x)$.

For $\epsilon = -1$, $Q_{-1}(\mathbb{Q}/\mathbb{Z}) = 0$ and a split skew-quadratic linking form (M, λ, ν) over (Λ, S) is a skew-symmetric linking form (M, λ) with $\lambda(x, x) = 0$ for all $x \in M$.

8.2 Flip-Isomorphisms and Linking Forms

Let $S \subset \Lambda$ be a central multiplicative subset of Λ .

As in the proof of [Ran81] p. 242ff we define the linking forms of preformations for which γ or μ are S -isomorphisms.

Definition 8.2.1. Let $x = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation.

- i) If μ is an S -isomorphism there is a split $(-\epsilon)$ -quadratic linking form $L_\mu = (\text{coker } \mu, \lambda_\mu, \nu_\mu)$ over (Λ, S) given by

$$\begin{aligned} \lambda_\mu: \text{coker } \mu \times \text{coker } \mu &\longrightarrow S^{-1}\Lambda/\Lambda, & (x, y) &\longmapsto \frac{1}{s}\gamma^*(x)(g) \\ \nu_\mu: \text{coker } \mu &\longrightarrow Q_{-\epsilon}(S^{-1}\Lambda/\Lambda), & y &\longmapsto \frac{1}{s}\theta(g)(g)\frac{\bar{1}}{s} \end{aligned}$$

for $x, y \in F^*, g \in G, s \in S$ such that $sy = \mu(g)$.

- ii) If γ is an S -isomorphism there is a split $(-\epsilon)$ -quadratic linking form $L_\gamma = (\text{coker } \gamma, \lambda_\gamma, \nu_\gamma)$ over (Λ, S) given by

$$\begin{aligned} \lambda_\gamma: \text{coker } \gamma \times \text{coker } \gamma &\longrightarrow S^{-1}\Lambda/\Lambda, & (x, y) &\longmapsto \epsilon\frac{1}{s}\mu^*(x)(g) \\ \nu_\gamma: \text{coker } \gamma &\longrightarrow Q_{-\epsilon}(S^{-1}\Lambda/\Lambda), & y &\longmapsto -\frac{1}{s}\theta(g)(g)\frac{\bar{1}}{s} \end{aligned}$$

for $x, y \in F, g \in G, s \in S$ such that $sy = \gamma(g)$.

- iii) If $\gamma^*\mu$ is an S -isomorphism there is a split $(-\epsilon)$ -quadratic linking form $L_{\gamma^*\mu} = (\text{coker } \gamma^*\mu, \lambda_{\gamma^*\mu}, \nu_{\gamma^*\mu})$ over (Λ, S) given by

$$\begin{aligned} \lambda_{\gamma^*\mu}: \text{coker } \gamma^*\mu \times \text{coker } \gamma^*\mu &\longrightarrow S^{-1}\Lambda/\Lambda, & (x, y) &\longmapsto \frac{1}{s}x(g) \\ \nu_{\gamma^*\mu}: \text{coker } \gamma^*\mu &\longrightarrow Q_{-\epsilon}(S^{-1}\Lambda/\Lambda), & y &\longmapsto \frac{1}{s}\theta(g)(g)\frac{\bar{1}}{s} \end{aligned}$$

for $x, y \in G^*, g \in G, s \in S$ such that $sy = \gamma^*\mu(g)$.

Similarly, there exist $(-\epsilon)$ -symmetric linking forms L^γ, L^μ and $L^{\gamma^*\mu}$ for ϵ -quadratic preformations with γ respectively μ S -isomorphisms. \square

Remark 8.2.2. The definitions are taken from the proof of [Ran81] Proposition 3.4.3 which establishes a correspondence between S -formations and linking forms. It is easy to verify that a regular split ϵ -quadratic preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ and its flip z' are S -formations if μ or γ are S -isomorphisms. Under that correspondence z is mapped to L_μ and z' is mapped to L_μ . We will exploit this fact in the proof of Proposition 8.2.3.

Let $\widetilde{M}_{-\epsilon}(\Lambda, S)$ be the abelian monoid of isomorphism classes of split $(-\epsilon)$ -quadratic linking forms over (Λ, S) . Then the previously defined linking forms give rise to the following homomorphisms of abelian monoids with zero.

$$\begin{aligned} L_\mu: \{[(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})] \in l_{2q+2}(\Lambda) | \mu \text{ is an } S\text{-isomorphism}\} &\longrightarrow \widetilde{M}_{-\epsilon}(\Lambda, S) \\ L_\gamma: \{[(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})] \in l_{2q+2}(\Lambda) | \gamma \text{ is an } S\text{-isomorphism}\} &\longrightarrow \widetilde{M}_{-\epsilon}(\Lambda, S) \\ L_{\gamma^*\mu}: \{[(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})] \in l_{2q+2}(\Lambda) | \gamma^*\mu \text{ is an } S\text{-isomorphism}\} &\longrightarrow \widetilde{M}_{-\epsilon}(\Lambda, S) \end{aligned}$$

Linking forms can tell us something about elementariness and the existence of flip-isomorphisms.

Proposition 8.2.3. *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular split ϵ -quadratic preformation with either μ or γ an S -isomorphism.*

- i) If z allows a flip-isomorphism then both γ and μ are S -isomorphisms. Every flip-isomorphism $t = (\alpha, \beta, \chi)$ induces an isomorphism of split $(-\epsilon)$ -quadratic linking forms $[\alpha^{-*}]: L_\mu \xrightarrow{\cong} L_\gamma$.*
- ii) Assume γ and μ are both S -isomorphisms and L_γ and L_μ are isomorphic. Every isomorphism $l: L_\mu \xrightarrow{\cong} L_\gamma$ induces a stable flip-isomorphism (α, β, χ) of z such that $[\alpha^{-*}] = l: L_\mu \xrightarrow{\cong} L_\gamma$.*
- iii) If z is stably elementary then*
 - (a) γ and μ are S -isomorphism,*
 - (b) $L_\mu \cong L_\gamma$,*
 - (c) $(G, \gamma^* \mu, \theta)$ is S -hyperbolic i.e. $L_{\gamma^* \mu}$ vanishes in the Witt group $\tilde{L}_\epsilon(\Lambda, S)$ of non-singular ϵ -quadratic split linking forms (see [Ran81] p.271).*

Similar for the non-split case.

Proof. For the proof it is necessary to remember the definition of the $(2q+1)$ -dimensional quadratic complexes (N, ζ) and (N', ζ') associated with z and its flip z' ((3.1) and (3.2) on p. 42).

- i) Direct calculation.*
- ii) z and its flip z' are split ϵ -quadratic S -formations in the sense of [Ran81] p.240. Hence we can apply [Ran81] Proposition 3.4.3. The proof shows that there exists a stable isomorphism of split ϵ -quadratic S -formations between z and z' . Using the isomorphism in Remark 2.2.17 iv) it is not difficult to show that this leads to a stable (weak) flip-isomorphism of z .*
- iii) Obvious from Proposition 2.4.2 iv) and [Ran81] Proposition 3.4.6ii).*

□

8.3 The Linking- l -Monoid $ll_{2q}^S(\Lambda)$

Let $S \subset \Lambda$ be a central multiplicative subset.

Proposition 8.2.3 shows that there is a strong relationship between flip-isomorphisms of preformations which allow linking forms and the isomorphisms of those linking forms.

Similar to Section 5.5 we define a monoid of preformations with linking forms and a choice of isomorphism between them.

Definition 8.3.1.

$$\begin{aligned} l_{2q+2}^S(\Lambda) &= \left\{ [(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})] \in l_{2q+2}(\Lambda) \mid \mu \text{ and } \gamma \text{ are } S\text{-isomorphisms} \right\} \\ fl_{2q+2}^S(\Lambda) &= \left\{ [(z, t)] : [z] \in l_{2q+2}^S(\Lambda) \right\} \end{aligned}$$

are sub-monoids of $l_{2q+2}(\Lambda)$ and $fl_{2q+2}(\Lambda)$. Similarly one defines $l_S^{2q+2}(\Lambda)$ and $fl_S^{2q+2}(\Lambda)$ in the non-split case. \square

Definition 8.3.2. We consider tuples (z, l) with $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ a regular ϵ -quadratic preformation such that γ and μ are S -isomorphisms and $l: L_\mu \xrightarrow{\cong} L_\gamma$ an isomorphism of linking forms.

An isomorphism (η, ζ) of such tuples (z, l) and (z', l') is a strong isomorphism $(\eta, \zeta): z \xrightarrow{\cong} z'$ of preformations such that $l' = \eta^* l \eta$.

Define the hyperbolic elements $(y_k, 0)$ with $y_k = \partial H_{-\epsilon}(\Lambda^k)$.

The stable isomorphism classes of such tuples form the **linking- l -monoid** $ll_{2q+2}^S(\Lambda)$. Similarly we can define $ll_S^{2q+2}(\Lambda)$ for the non-split case. \square

Proposition 8.3.3. *There is a commuting diagram of surjective morphisms of abelian monoids with zero*

$$\begin{array}{ccc} fl_{2q+2}^S(\Lambda) & \xrightarrow{\pi_f} & l_{2q+2}^S(\Lambda) \\ L \downarrow & \nearrow \pi_l & \\ ll_{2q+2}^S(\Lambda) & & \end{array}$$

with $L(z, t = (\alpha, \beta, \nu)) = (z, [\alpha^{-*}])$, $\pi_f(z, t) = z$ and $\pi_l(z, l) = z$. The fibre $\pi^{-1}([z])$ of a preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is either empty or the set of all isometries $L_\mu \xrightarrow{\cong} L_\gamma$. Similar for the non-split case.

Proof. Use Proposition 8.2.3 \square

8.4 Asymmetric Signatures

Let $S \subset \Lambda$ be a central multiplicative subset. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a regular ϵ -quadratic split preformation such that γ and μ are S -isomorphisms. In other words $[z] \in l_{2q+2}^S(\Lambda)$. Proposition 8.2.3 shows that any flip-isomorphism gives rise to an isomorphism of the linking forms L_γ and L_μ and vice versa. The relationship between flip-isomorphism and linking form isomorphism goes even further. We will prove that the asymmetric signature of two flip-isomorphisms which induce the same isomorphism $L_\mu \xrightarrow{\cong} L_\gamma$ are identical.

In the case of $\Lambda = \mathbb{Z}$, this drastically reduces the amount of work one has to put into checking all the asymmetric signatures (see Theorem 8.4.3). Instead of going through all flip-isomorphisms of all the preformations $z + \partial H_{-\varepsilon}(\Lambda^n)$, we only have to compute them for the finite number of isomorphisms of two given linking forms of one arbitrary representative of $[z] \in l_{2q+2}(\Lambda)$.

First we need to check that the asymmetric signature of an isomorphism $L^\mu \xrightarrow{\cong} L^\gamma$ is well-defined.

Proposition 8.4.1. *Let $t = (\alpha, \beta, \chi)$ and $t' = (\alpha', \beta', \chi')$ two flip-isomorphisms of z which induce the same isomorphism of linking forms*

$$[\alpha^{-*}] = [\alpha'^{-*}]: L^\mu \xrightarrow{\cong} L^\gamma$$

Then $\sigma^(z, t) = \sigma^*(z, t') \in LAsy^0(\Lambda)$. Similar in the non-split case.*

We need a little lemma which shows that for 1-dimensional S -acyclic complexes quasi-isomorphisms and chain-equivalences are the same. This is not true for arbitrary (even free) chain complexes.

Lemma 8.4.2. *Let N and N' be free 1-dimensional S -acyclic chain complexes and $f: N \rightarrow N'$ a chain map which induces the zero map in homology. Then there is a chain homotopy $\Delta: f \simeq 0$.*

Proof. From $f_* = 0: H_0(N) \rightarrow H_0(N')$ follows that $\text{im } f_0 \subset \text{im}(d': N'_1 \rightarrow N'_0)$. Let $\{e_1, \dots, e_n\}$ be a basis for N_0 and choose $b_i \in N'_1$ such that $f_0(e_i) = d'(b_i)$. There is a $\Delta \in \text{Hom}_\Lambda(N_0, N'_1)$ such that $\Delta(e_i) = b_i$ and therefore $f_0 = d'\Delta$. On the other hand $d'f_1 = f_0d = d'\Delta d$. d' is injective (because N' is S -acyclic) and therefore $f_1 = \Delta d$. \square

Proof of Proposition 8.4.1. The asymmetric signature of a flip-isomorphism is constructed in Section 5.3 as the asymmetric signature of the $(2q+2)$ -dimensional symmetric Poincaré pair $x^t = (g^t: \partial E_t \rightarrow E, (0, \theta_t))$. Its boundary is a twisted double of the symmetric Poincaré pair $(f: C \rightarrow D, (0, \phi))$ in respect to the self-equivalence $(h_t, 0): (C, \phi) \xrightarrow{\cong} (C, \phi)$. We will show that the two flip-isomorphisms t and t' lead to homotopic equivalences $(h_t, 0) \simeq (h_{t'}, 0)$ and that therefore the twisted doubles $(\partial E_t, \theta_t)$ and $(\partial E_{t'}, \theta_{t'})$ are equivalent and, finally, that the asymmetric signatures of x^t and $x^{t'}$ are the same.

As described in Section 4.2, t and t' induce two isomorphisms $(e, \rho), (e', \rho'): (N, \zeta) \xrightarrow{\cong} (N', \zeta')$ of the $(2q+1)$ -dimensional quadratic complexes defined in (3.1) and (3.2) on p. 42ff. The fact that t and t' induce the same linking form isomorphisms translates into $e^* = e'^*: H^*(N') \xrightarrow{\cong} H^*(N)$.

By Lemma 8.4.2, e and e' are chain homotopic. Let $\Delta: e \simeq e': N \rightarrow N'$ be a chain homotopy.

By the proof of [Ran80a] Proposition 3.4. (see also Lemma A.2.9) (e, ρ) and (e', ρ') induce isomorphisms

$$(\partial e, 0), (\partial e', 0): (C, \phi) = (\partial N, (1+T)\partial\zeta) \xrightarrow{\cong} (C', \phi') = (\partial N', (1+T)\partial\zeta')$$

of $2q$ -dimensional symmetric Poincaré complexes given by

$$\partial e = \begin{pmatrix} e & (-)^{r+1}(1+T)\rho \\ 0 & e^{-*} \end{pmatrix}: \partial N_r = N_{r+1} \oplus N^{2q+1-r} \longrightarrow \partial N'_r = N'_{r+1} \oplus N'^{2q+1-r}$$

Using the fact that N and N' are short and S -acyclic one can show that there is a chain equivalence $(\partial\Delta, 0): (\partial e, 0) \simeq (\partial e', 0): (C, \phi) \xrightarrow{\cong} (C', \phi')$ given by

$$\begin{aligned} \partial\Delta_{q+1} &= \begin{pmatrix} 0 & \epsilon\beta'\Delta^*\alpha^{-*} \end{pmatrix}: C_q = F \oplus F^* \longrightarrow C'_{q+1} = G^* \\ \partial\Delta_q &= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}: C_{q-1} = G^* \longrightarrow C'_q = F^* \oplus F \end{aligned}$$

As explained in Section 4.2 we compose ∂e with the inverse $(h, 0): (C, \phi) \xrightarrow{\cong} (C', \phi')$ from (3.5) on page 44 in order to get the self-equivalence $(h_t, 0): (C, \phi) \xrightarrow{\cong} (C, \phi)$. Using Lemma 6.1.2 or by direct calculation one finds a homotopy of the chain maps

$$(h^{-1}\partial\Delta, 0): (h_t, 0) \simeq (h'_t, 0): (C, \phi) \xrightarrow{\cong} (C, \phi)$$

which can be fed into Lemma 6.1.8. Hence $\sigma^*(z, t') = \sigma^*(x^{t'}) = \sigma^*(x'') \in LAsy^0(\Lambda)$ with $x'' = (g'': \partial E_t \longrightarrow E, (0, \theta_t))$ given by

$$p = \begin{pmatrix} 1 & \beta' & 0 & -\beta'\Delta^*\alpha^{-*} \end{pmatrix}: \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^* \longrightarrow E_{q+1} = G$$

Finally, there is a homotopy equivalence $(1, 1; l): x'' \longrightarrow x^t$ given by

$$l = \begin{pmatrix} 0 & \epsilon\beta'\Delta^*\alpha^{-*} & 0 \end{pmatrix}: \partial E_{t,q} = F^* \oplus F^* \oplus G^* \longrightarrow E_{q+1} = G$$

Hence x'' and x^t are cobordant rel ∂ by Lemma 3.2.2. By Proposition 5.4.4 their asymmetric signatures coincide. \square

This proposition shows that the asymmetric signature of a flip-isomorphism in $l_{2q+2}^S(\Lambda)$ only depends on its induced isomorphism of linking forms.

Theorem 8.4.3. *There is a lift of the asymmetric signature map of Theorem 5.5.3*

$$\begin{array}{ccc} fl_S^{2q+2}(\Lambda) & \xrightarrow{\sigma^*} & LAsy^0(\Lambda) \\ L \downarrow & \nearrow \sigma^* & \\ ll_S^{2q+2}(\Lambda) & & \end{array}$$

Similar in the non-split case.

Let π_l be as in Proposition 8.3.3. If $[z] \in l_S^{2q+2}(\Lambda)$ is elementary then $\sigma^*(\pi_l^{-1}[z]) = \{0\}$ i.e. there exist isomorphisms $l: L^\mu \xrightarrow{\cong} L^\gamma$ and the asymmetric signatures $\sigma^*(z, l) \in LAsy^0(\Lambda)$ vanish for all of them.

8.5 Asymmetric Signatures of Simply-Connected Manifolds

We will now concentrate our efforts to simply-connected manifold i.e. the case of $\Lambda = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$. First we observe that results of Section 8.4 can be generalized - they remain true even if γ and μ are not injective and if G is not free (i.e. if the preformation is not regular). We can also show that the algebraic linking forms of an obstruction preformation are induced by the topological linking forms in certain simply-connected Kreck surgery problems.

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be a ϵ -quadratic preformation over \mathbb{Z} such that $\text{coker } \gamma$ and $\text{coker } \mu$ are finite. (G may have torsion and γ and μ might not be injective). We will show that the asymmetric signature for flip-isomorphisms of z also only depend on their behaviour on the linking forms L^γ and L^μ by constructing a new preformation z' which is closely linked to z but fulfils all the requirements of Section 8.4 (i.e. it is regular and its maps are S -isomorphisms).

Lemma 8.5.1. *i) $\ker \gamma = \ker \mu$*

ii) $G/\ker \gamma$ is f.g. free and of rank $\text{rk } F$.

iii) There is another regular split ϵ -quadratic preformation z' over \mathbb{Z} defined by $z' = (F \xleftarrow{[\gamma]} G/\ker \gamma \xrightarrow{[\mu]} F^, \psi)$ for which the following diagram commutes*

$$\begin{array}{ccccc} F & \xleftarrow{\gamma} & G & \xrightarrow{\mu} & F^* \\ & \searrow [\gamma] & \downarrow \pi & \nearrow [\mu] & \\ & & G/\ker \gamma & & \end{array}$$

iv) z is elementary if and only if z' is elementary

v) Every flip-isomorphism $t = (\alpha, \beta, \sigma)$ of the ϵ -quadratic formation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^)$ induces a flip-isomorphism $t' = (\alpha, \beta', \sigma)$ of z' . Then $\sigma^*(z, t) = \sigma^*(z', t') \in LAsy^0(\mathbb{Z})$.*

Proof. There is a free f.g. submodule G' such that $G = G' \oplus \ker \gamma$. There can be no torsion in G' because γ is a homomorphism into a free module. We write

$$\begin{aligned} \gamma &= (\gamma_1 \ 0) : G = G' \oplus \ker \gamma \longrightarrow F \\ \mu &= (\mu_1 \ \mu_2) : G = G' \oplus \ker \gamma \longrightarrow F^* \end{aligned}$$

γ_1 is obviously injective and hence induces an isomorphism over \mathbb{Q} . Therefore G' has the same rank as F . As $\gamma^* \mu$ is $(-\epsilon)$ -symmetric, μ_2 must vanish. Then μ_1 must be injective as well. Hence $z' = (F \xleftarrow{\gamma_1} G' \xrightarrow{\mu_1} F^*, \theta|_{G'})$ is a well-defined preformation with all the claimed properties.

By Lemma 2.4.5 z is elementary if and only if z' is elementary.

Let $t = (\alpha, \beta, \chi)$ be a flip-isomorphism of z . Obviously $\beta(\ker \gamma) = \ker \mu = \ker \gamma$. So we can write

$$\beta = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \beta_3 \end{pmatrix} : G = G' \oplus \ker \gamma \longrightarrow G = G' \oplus \ker \gamma$$

It follows that $t' = (\alpha, \beta_1, \sigma)$ is a flip-isomorphism of z' . The Definition 5.3.1 of the asymmetric signature is independent of β and G , hence the signatures of t and t' are the same. \square

The lemma justifies the following generalization of asymmetric signatures:

Definition 8.5.2. Let $l: L^\mu \xrightarrow{\cong} L^\gamma$ be an isomorphism of split $(-\epsilon)$ -quadratic linking forms. Let z' be the regular ϵ -quadratic split preformation constructed in Lemma 8.5.1. Then $\tilde{\sigma}^*(z, l) = \sigma^*(z', l) \in LAsy^0(\mathbb{Z})$ is the **asymmetric signature of z and the isomorphism $l: L^\mu \xrightarrow{\cong} L^\gamma$ of linking forms**. \square

Theorem 8.5.3. *If $[z] \in l^{2q+2}(\mathbb{Z})$ is elementary then the asymmetric signature $\tilde{\sigma}^*(z, l) \in LAsy^0(\mathbb{Z})$ vanishes for all isomorphisms $l: L^\mu \xrightarrow{\cong} L^\gamma$ of linking forms.*

Proof. Follows from Lemma 8.5.1, Lemma 2.4.5 and Theorem 8.4.3. \square

Definition 8.5.4 ([Ran02] Example 12.44). i) Let M be a $(2q+1)$ -dimensional manifold. The **linking form on M** is the bilinear form on the torsion submodule of $H_q(M)$ and $H_q(M, \partial M)$ given by

$$\begin{aligned} l_M: TH_q(M) \times TH_q(M, \partial M) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ (x, y) &\longmapsto \frac{1}{s} \langle z, y \rangle \end{aligned}$$

with $z \in C^q(M, \partial M)$ and $s \in \mathbb{Z} \setminus \{0\}$ such that $sx = d(z \cap [M]) \in C_q(M)$.

ii) Let $M \longrightarrow B$ be a map of $(2q+1)$ -dimensional manifold in a topological space. The **B-linking form on M** is the $(-\epsilon)$ -symmetric form on the torsion submodule of $H_{q+1}(B, M)$ given by

$$\begin{aligned} l_M^B: TH_{q+1}(B, M) \times TH_{q+1}(B, M) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ (x, y) &\longmapsto l_M(p(x), p(y)) \end{aligned}$$

with $p: H_{q+1}(B, M) \longrightarrow H_q(M)$. \square

Remark 8.5.5. If $\partial M = \emptyset$ then l_M is a non-singular $(-\epsilon)$ -symmetric linking form on $TH_q(M)$.

Proposition 8.5.6. *We repeat the assumptions of Kreck's surgery theory in the simply-connected case: Let $p: B \rightarrow BO$ be a fibration with $\pi_1(B) = 0$. Let M_i be $(2q+1)$ -dimensional manifolds with a $(q-1)$ -smoothings in B i.e. a lift of the stable normal*

bundle over p which is q -connected. Let $f: \partial M_0 \rightarrow \partial M_1$ be a diffeomorphism compatible with the smoothings. Let W be a cobordism of $M_0 \cup_f M_1$ with a compatible q -smoothing over B . As in Corollary 2.4.6 we define an obstruction

$$\begin{aligned} x(W) &= (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) \\ &= (H_{q+1}(W, M_0) \longleftarrow H_{q+2}(B, W) \longrightarrow H_{q+1}(W, M_1), \theta) \\ &\in l'_{2q+2}(\mathbb{Z}) \end{aligned}$$

If $\text{coker } \gamma = H_{q+1}(B, M_0)$ is finite then $L^\gamma = -l_{M_0}^B$.

If $\text{coker } \mu = H_{q+1}(B, M_1)$ is finite then $L^\mu = -l_{M_1}^B$.

Assume both cokernels are finite. If W is bordant rel ∂ to an h -cobordism then there exist isomorphisms $l: L^\mu = -l_{M_1}^B \xrightarrow{\cong} L^\gamma = \epsilon - l_{M_0}^B$ and their asymmetric signatures $\bar{\sigma}^*(x(W), l) \in L\text{Asy}^0(\mathbb{Z})$ will all vanish.

Proof. The complex $\tilde{C}_{q+2} = H_{q+2}(B, W) \xrightarrow{\gamma} \tilde{C}_{q+1} = H_{q+1}(W, M_0)$ has homology $H_*(\tilde{C}) = H_*(B, M_0)$. There is a homotopy equivalence $m: \tilde{C} \rightarrow C(B, M_0)$ and there is a chain map $C(B, M_0) \rightarrow C_{*-1}(M_0)$ which induces the connecting homomorphism $\partial_*: H_*(B, M_0) \rightarrow H_{*-1}(M_0)$. Both maps together yield a chain map

$$\begin{array}{ccc} C_{q+1}(M_0) & \xrightarrow{d} & C_q(M_0) \\ p \uparrow & & \uparrow p \\ \tilde{C}_{q+2} & \xrightarrow{\gamma} & \tilde{C}_{q+1} \end{array}$$

which induces the connecting map $p: H_{q+1}(B, M_0) \rightarrow H_q(M_0)$.

Let $a, b \in \text{coker } \gamma = H_{q+1}(B, M_0) = H_{q+1}(\tilde{C})$. Represent both homology classes by chains $\bar{a}, \bar{b} \in \tilde{C}_{q+1}$. Then there is a $g \in \tilde{C}_{q+2}$ and an $s \in \mathbb{Z} \setminus \{0\}$ such that $s\bar{a} = \gamma(g)$. Let $z \in C^q(M_0, \partial M_0)$ such that $p(g) = z \cap [M_0]$. Then $sp(\bar{a}) = d(z \cap [M_0])$. Hence $l_{M_0}^B(a, b) = \frac{1}{s} \langle z, p(\bar{b}) \rangle$. Let $b' \in H^{q+1}(W, M_0')$ such that $b' \cap [W] = \bar{b}$. Then $l_{M_0}^B(a, b) = \frac{1}{s} \langle p^*z, b' \cap [W] \rangle = -\epsilon \frac{1}{s} \langle b', p^*(z) \cap [W] \rangle$. Since p is a connecting homomorphism $p^*(z) \cap [W] = -\epsilon i(z \cap [M_0]) = -\epsilon i p(g) = -\epsilon \mu(g)$. Hence $l_{M_0}^B(a, b) = \frac{1}{s} \langle b', \mu(g) \rangle = -\epsilon \frac{1}{s} \mu^*(b)(g) = -L^\gamma(a, b)$. \square

Chapter 9

Preformations over Skew Fields

For the whole chapter let $\epsilon = (-)^q$ and let Λ be a weakly finite ring with 1 and involution.

At first sight the l -monoids seem to be uncomputable but with a little linear algebra we can calculate them in the case of skew fields (see **Section 9.1**). If the characteristic of Λ is not 2 the result is very simple: $l_{2q+2}(\Lambda) \cong L_{2q+2}(\Lambda) \oplus \mathbb{N}_0^5$. Any element $[z] = [(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})] \in l_{2q+2}(\Lambda)$ is determined by the dimension of the kernels of $\gamma, \mu, \gamma^* \mu, (\frac{\gamma}{\mu})$ and $\dim F - \dim G$ and the Witt-class of $(G/\ker \gamma^* \mu, \gamma^* \mu)$ (see Corollary 9.1.12 and Remark 9.1.13).

In **Section 9.2** we compute all its elementary elements. Again for the case $\text{char } \Lambda \neq 2$ we find a rather simple relationship: $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ is stably elementary if and only if $(G/\ker \gamma^* \mu, \gamma^* \mu)$ is stably metabolic, $\dim \ker \gamma = \dim \ker \mu$ and $\dim \ker \gamma^* \mu = \dim \ker \gamma + \dim \ker \gamma^*$ (see Corollary 9.2.2). As a by-product we prove that the quotient monoid $l_{2q+2}(\Lambda)/l_{2q+2}^{el}(\Lambda)$ is not group if Λ is a skew field or a principal ideal domain (see Corollaries 9.2.3 and 9.2.4).

The case $\text{char } \Lambda = 2$ demands more work. The simplest example is $l_{2q}(\mathbb{Z}/2\mathbb{Z})$ of which we will give a complete description in **Section 9.3**.

Then we apply our findings about fields to $l_{2q+2}(\mathbb{Z})$ (see **Section 9.4**). Unfortunately the results are rather meagre: we get the “trivial” invariants described in Corollary 2.4.3 and, if q is odd, invariants in $L_0(\mathbb{Q})$ and $L_0(\mathbb{Z}/p\mathbb{Z})$ for all primes p .

We would have much less trouble in understanding the l -monoids if they were groups. The computation of the l -monoids for skew fields of characteristic $\neq 2$ shows that they at least can be embedded into $L_{2q}(\Lambda) \oplus \mathbb{Z}^5$. Unfortunately **Section 9.5** shatters our hopes by revealing that $l_{4m+2}(\mathbb{Z}[\pi])$ violates the cancellation rule and that therefore it is not a sub-monoid of a group.

We are obviously interested whether vanishing asymmetric signatures suffice in order to show that a preformation is stably elementary. The computations of the l -monoids for skew fields gives us the opportunity to falsify that conjecture in **Section 9.6**.

9.1 Computation of the l -Monoids

We start with the definition of the \widehat{Q} -modules (see also [Ran02] 11.1) and the monoid $\widehat{M}^\epsilon(\Lambda)$ of isomorphism classes of quadratic forms (G, θ) with underlying symmetric form $(G, 0)$. We will use them later to describe our candidate for the l -monoid of a skew field - the m -monoid. They reappear in Section 9.5 when we deal with the cancellation rule for $l_{2q+2}(\Lambda)$. All these objects can be defined over any weakly finite ring Λ with 1 and involution.

Definition 9.1.1 ([Ran02] Definition 11.1). For a free f.g. Λ -module K define

$$\begin{aligned}\widehat{Q}^\epsilon(K) &= \{\nu \in \text{Hom}_\Lambda(K, K^*) \mid \nu - \epsilon\nu^* = 0\} / \{\omega + \epsilon\omega^* \mid \omega \in \text{Hom}_\Lambda(K, K^*)\} \\ \widehat{Q}^\epsilon(\Lambda) &= \{a \in \Lambda \mid a - \epsilon\bar{a} = 0\} / \{x + \epsilon\bar{x} \mid x \in \Lambda\}\end{aligned}$$

□

Remark 9.1.2. They fit into an exact sequence of abelian groups

$$0 \longrightarrow \widehat{Q}^\epsilon(K) \longrightarrow Q_{-\epsilon}(K) \xrightarrow{1+T_{-\epsilon}} Q^{-\epsilon}(K) \longrightarrow \widehat{Q}^{-\epsilon}(K) \longrightarrow 0$$

Definition 9.1.3. The isomorphism classes of $(-\epsilon)$ -quadratic forms $(G, \theta \in \widehat{Q}^\epsilon(G))$ over free f.g. modules G form an abelian monoid $\widehat{M}^\epsilon(\Lambda)$ with operation $(G, \theta) + (G', \theta') = (G \oplus G', \theta \oplus \theta')$ and zero represented by the zero module. □

There is another way of characterizing $\widehat{M}^\epsilon(\Lambda)$.

Lemma 9.1.4.

$$\bigcup_{k \in \mathbb{N}_0} \widehat{Q}^\epsilon(\Lambda)^k / \sim \xrightarrow{\cong} \widehat{M}^\epsilon(\Lambda), \quad [a_1, \dots, a_k] \mapsto \left[\Lambda^k, \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_k \end{pmatrix} \right]$$

with $(a_1, \dots, a_k) \sim (b_1, \dots, b_k)$ if and only if there exists an $f \in GL_k(\Lambda)$ such that $\sum_l \bar{f}_{li} a_l f_{li} = b_i \in \widehat{Q}^\epsilon(\Lambda)$ for all $1 \leq i \leq k$.

Proof. The map $\widehat{Q}^\epsilon(\Lambda)^k \rightarrow \widehat{Q}^\epsilon(\Lambda^k)$, $([a_i])_{1 \leq i \leq k} \mapsto \left[\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_k \end{pmatrix} \right]$ is bijective. □

Example 9.1.5. i) There is a well-defined rank map

$$\text{rk}: \widehat{M}^\epsilon(\Lambda) \longrightarrow \mathbb{N}_0, \quad [M, \theta] \mapsto \text{rk}(M)$$

Assume that $(1 + T_{-\epsilon}): Q_{-\epsilon}(\Lambda) \longrightarrow Q^{-\epsilon}(\Lambda)$ is an injection. This is true e.g. for rings which contain a central element $x \in \Lambda$ with $x + \bar{x} = 1$ (if Λ is a skew field of characteristic $\neq 2$ take $x = \frac{1}{2}$). Then the group $\widehat{Q}^\epsilon(\Lambda)$ vanishes and the rank map becomes an isomorphism. Hence $\widehat{M}^\epsilon(\Lambda) = \mathbb{N}_0$.

- ii) Let Λ be a field such that $\widehat{Q}^\epsilon(\Lambda) \neq 0$ (that means $\text{char } \Lambda = 2$) and such that the map $\Lambda \longrightarrow \widehat{Q}^\epsilon(\Lambda)$, $x \longmapsto \bar{x}x$ is a surjection.

Finite fields of characteristic 2 with the identity involution fulfil those conditions: their Frobenius homomorphism $x \longmapsto x^2$ is surjective and $\widehat{Q}^\epsilon(\Lambda) = \Lambda \neq 0$.

We use Lemma 9.1.4 to compute $\widehat{M}^\epsilon(\Lambda)$. Let $k \in \mathbb{N}$. Let $a_i \in \widehat{Q}^\epsilon(\Lambda)$ for $1 \leq i \leq k$ with $a_j \neq 0 \in \widehat{Q}^\epsilon(\Lambda)$ for some $j \in \{1, \dots, k\}$. Without loss of generality we can assume $j = 1$. Let $c_i \in \Lambda$ such that $\bar{c}_i c_i = a_i$. Define an invertible matrix $f \in GL_k(\Lambda)$ by $f_{li} = \begin{cases} 1 & \text{if } i = l \text{ and } l \neq 1, \\ c_i & \text{if } l = 1, \\ 0 & \text{else,} \end{cases}$. Then $\sum_l \bar{f}_{li} b_l f_{li} = a_i$ with $b =$

$(1, 0, \dots, 0)$. Hence $a = b \in \widehat{M}^\epsilon(\Lambda)$. b does not vanish because $1 \neq 0 \in \widehat{Q}^\epsilon(\Lambda)$. Therefore $\widehat{M}^\epsilon(\Lambda)$ contains precisely two elements of rank k : o_k represented by $(0, \dots, 0) \in \Lambda^k$ and e_k represented by $b = (1, 0, \dots, 0) \in \Lambda^k$.

Therefore, the abelian monoid $\widehat{M}^\epsilon(\Lambda)$ is given by the set $\{0 = o_0, o_1, e_1, o_2, e_2, \dots\}$ with 0 represented by the trivial zero module and the operation $o_k + o_l = o_{k+l}$ and $o_k + e_l = e_k + e_l = e_{k+l}$. $(G, \theta \in \widehat{Q}^\epsilon(G))$ with $\text{rk } G = k$ represents $e_k \in \widehat{M}^\epsilon(\Lambda)$ if $\theta \neq 0 \in \widehat{Q}^\epsilon(G)$ and $o_k \in \widehat{M}^\epsilon(\Lambda)$ if $\theta = 0 \in \widehat{Q}^\epsilon(G)$.

- iii) $\widehat{Q}^1(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and as in the previous example we get the same monoid $\widehat{M}^1(\mathbb{Z}) = \widehat{M}^1(\mathbb{Z}/2\mathbb{Z})$. $\widehat{Q}^{-1}(\mathbb{Z}) = 0$ and $\widehat{M}^{-1}(\mathbb{Z}) = \mathbb{N}_0$.

The m -monoids are our candidates for the l -monoid of skew fields. Their definition is rather cumbersome but for skew fields with $\text{char } \Lambda \neq 2$ it is rather easy (see Remark 9.1.7). Again we can define them over any ring. For arbitrary rings they are submonoids of the l -monoids (see Proposition 9.1.7).

Definition 9.1.6. We consider tuples

$$\begin{aligned} &((K, \theta_K \in Q_{-\epsilon}(K)), (S, \theta_S \in \widehat{Q}^\epsilon(S)), (T, \theta_T \in \widehat{Q}^\epsilon(T)), \\ &(X, \theta_X \in \widehat{Q}^\epsilon(X)), (Z, \theta_Z \in \widehat{Q}^\epsilon(Z)), y \in \mathbb{N}_0) \end{aligned}$$

such that (K, θ_K) is non-singular i.e. $(1 + T_{-\epsilon})\theta_K = \theta_K - \epsilon\theta_K^*: K \longrightarrow K^*$ is an isomorphism. We introduce an equivalence relation on those tuples.

$$\begin{aligned} &((K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y) \sim \\ &((K', \theta'_K), (S', \theta'_S), (T', \theta'_T), (X', \theta'_X), (Z', \theta'_Z), y') \end{aligned}$$

if and only if there exist $t_z \in \text{Hom}_\Lambda(T, Z)$, $s_z \in \text{Hom}_\Lambda(S, Z)$, $x_z \in \text{Hom}_\Lambda(X, Z)$, $x_s \in \text{Hom}_\Lambda(X, S)$, $x_t \in \text{Hom}_\Lambda(X, T)$, $k_z \in \text{Hom}_\Lambda(K, Z)$, $k_s \in \text{Hom}_\Lambda(K, S)$, $k_t \in$

$\text{Hom}_\Lambda(K, T)$, $k_x \in \text{Hom}_\Lambda(K, X)$ such that

$$\begin{aligned}
[(K', \theta'_K)] &= [(K, \theta_K + k_z^* \theta_Z k_z + k_s^* \theta_S k_s + k_t^* \theta_T k_t + k_x^* \theta_X k_x = \theta''_K)] \in L_{2q+2}(\Lambda) \\
[(X', \theta'_X)] &= [(X, \theta_X + x_z^* \theta_Z x_z + x_s^* \theta_S x_s + x_t^* \theta_T x_t)] \in \widehat{M}^\epsilon(\Lambda) \\
[(S', \theta'_S)] &= [(S, \theta_S + s_z^* \theta_Z s_z)] \in \widehat{M}^\epsilon(\Lambda) \\
[(T', \theta'_T)] &= [(T, \theta_T + t_z^* \theta_Z t_z)] \in \widehat{M}^\epsilon(\Lambda) \\
[(Z', \theta'_Z)] &= [(Z, \theta_Z)] \in \widehat{M}^\epsilon(\Lambda) \\
y' &= y \in \mathbb{N}_0
\end{aligned}$$

(Note that because of $\theta = -\theta \in \widehat{Q}^\epsilon(M)$, the equivalence relation is symmetric. The form (K, θ''_K) is non-singular because $(1 + T_{-\epsilon})\theta''_K = (1 + T_{-\epsilon})\theta_K$.)

The set of equivalence classes forms the abelian m -**Monoid** $m_{2q+2}(\Lambda)$. \square

Remark 9.1.7. i) There is a well-defined map of abelian monoids

$$\begin{aligned}
m_{2q+2}(\Lambda) &\rightarrow \mathbb{N}_0^5 \\
[(K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y] &\mapsto (\text{rk } S, \text{rk } T, \text{rk } X, \text{rk } Z, y)
\end{aligned}$$

ii) There is a well-defined surjection of abelian monoids

$$\begin{aligned}
L_{2q+2}(\Lambda) \oplus \widehat{M}^\epsilon(\Lambda)^4 \oplus \mathbb{N}_0 &\rightarrow m_{2q+2}(\Lambda) \\
([(K, \theta_K)], [(S, \theta_S)], [(T, \theta_T)], [(X, \theta_X)], [(Z, \theta_Z)], y] &\mapsto [(K, \theta_K), (S, \theta_S), (T, \theta_T), \\
&\quad (X, \theta_X), (Z, \theta_Z), y]
\end{aligned}$$

iii) There is a well-defined injection of abelian monoids

$$\begin{aligned}
L_{2q+2}(\Lambda) \oplus \mathbb{N}_0^5 &\rightarrow m_{2q+2}(\Lambda) \\
([(K, \theta)], s, t, x, z, y) &\mapsto [(K, \theta), (\Lambda^s, 0), (\Lambda^t, 0), (\Lambda^x, 0), (\Lambda^z, 0), y]
\end{aligned}$$

It is an isomorphism if $\widehat{Q}^\epsilon(\Lambda) = 0$.

We show that there is an injection of monoids $m_{2q+2}(\Lambda) \hookrightarrow l_{2q+2}(\Lambda)$ for any ring Λ .

Proposition 9.1.8. *For any set of $(-\epsilon)$ -quadratic forms $(K, \lambda \in Q_{-\epsilon}(K))$, $(S, \theta_S \in \widehat{Q}^\epsilon(S))$, $(T, \theta_T \in \widehat{Q}^\epsilon(T))$, $(X, \theta_X \in \widehat{Q}^\epsilon(X))$, $(Z, \theta_Z \in \widehat{Q}^\epsilon(Z))$ and $y \in \mathbb{N}_0$ one can define an ϵ -quadratic split preformation*

$$\begin{aligned}
\eta((K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y) &= \partial(K, \theta_K) \\
&+ (T^* \xleftarrow{0} T \xrightarrow{1} T, \theta_T) \\
&+ (X \oplus X^* \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} X^* \oplus X, \theta_X) \\
&+ (Y \xleftarrow{0} Z \xrightarrow{0} Y^*, \theta_Z)
\end{aligned}$$

with $Y = \Lambda^y$. η induces a monomorphism $m_{2q+2}(\Lambda) \hookrightarrow l_{2q+2}(\Lambda)$ of abelian monoids.

Corollary 9.1.9. *There is a monomorphism of abelian monoids*

$$\begin{aligned} \eta: L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^5 &\hookrightarrow l^{2q+2}(\Lambda) \\ ([K, \lambda], s, t, x, z, y) &\longmapsto [\partial(K, \lambda) + \partial(\Lambda^s, 0) + (\Lambda^t, \Lambda^t) \\ &\quad + (\Lambda^{x+x} \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Lambda^x \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \Lambda^{x+x}) \Lambda^{x+x} \\ &\quad + (\Lambda^y \xleftarrow{0} \Lambda^z \xrightarrow{0} \Lambda^y) \Lambda^y] \end{aligned}$$

Proof. Use Remark 9.1.7. □

Lemma 9.1.10. *Let G_i be free modules over Λ . Let $(G = G_1 \oplus G_2, \theta \in Q_{-\epsilon}(G))$ be an $(-\epsilon)$ -quadratic form such that $\theta - \epsilon\theta^* = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} : G \oplus H \rightarrow G^* \oplus H^*$. Then there exist unique $\theta_1 \in Q_{-\epsilon}(G_1)$ and $\theta_2 \in \widehat{Q}^\epsilon(G_2)$ such that $\theta = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \in Q_{-\epsilon}(G)$.*

Proof. Let $A := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : G \oplus H \rightarrow G^* \oplus H^*$ be a representative of θ and define $B := \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}$. $\theta - \epsilon\theta^* = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ implies that $A_2 - \epsilon A_3^* = 0$. Then

$$\theta = [A] = \left[\begin{pmatrix} A_1 & A_2 \\ \epsilon A_3^* & A_4 \end{pmatrix} \right] = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} + B + \epsilon B^* \right]$$

Therefore $\theta_G = [A_1]$ and $\theta_H = [A_4]$ do the job. □

Proof of Proposition 9.1.8. Let $(K, \lambda_K), (K', \lambda'_K)$ be non-singular $(-\epsilon)$ -quadratic forms and $(S, \theta_S \in \widehat{Q}^\epsilon(S)), (T, \theta_T \in \widehat{Q}^\epsilon(T)), (X, \theta_X \in \widehat{Q}^\epsilon(X)), (Z, \theta_Z \in \widehat{Q}^\epsilon(Z)), (S', \theta'_S \in \widehat{Q}^\epsilon(S')), (T', \theta'_T \in \widehat{Q}^\epsilon(T')), (X', \theta'_X \in \widehat{Q}^\epsilon(X')), (Z', \theta'_Z \in \widehat{Q}^\epsilon(Z'))$ and $y, y' \in \mathbb{N}_0$ such that

$$\begin{aligned} \eta((K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y) &= \\ \eta((K', \theta'_K), (S', \theta'_S), (T', \theta'_T), (X', \theta'_X), (Z', \theta'_Z), y') &\in l_{2q+2}(\Lambda) \end{aligned}$$

After adding enough hyperbolic forms to (K, λ) and (K', λ') we can assume that there is an (unstable) strong isomorphism of preformations

$$\begin{array}{ccccc} F & \xleftarrow{\gamma} & G & \xrightarrow{\mu} & F^* \\ \alpha \cong \downarrow & & \beta \cong \downarrow & & \alpha^{-*} \cong \downarrow \\ F' & \xleftarrow{\gamma'} & G' & \xrightarrow{\mu'} & F'^* \end{array}$$

with

$$\begin{aligned}
F &= K \oplus S \oplus T^* \oplus X \oplus X^* \oplus Y \\
G &= K \oplus S \oplus T \oplus X \oplus Z \\
\gamma &= 1_K \oplus 1_S \oplus 1_X : G \longrightarrow F \\
\mu &= \lambda \oplus 1_T \oplus 1_X : G \longrightarrow F^* \\
\theta &= \begin{pmatrix} \theta_K & 0 & 0 & 0 & 0 \\ 0 & \theta_S & 0 & 0 & 0 \\ 0 & 0 & \theta_T & 0 & 0 \\ 0 & 0 & 0 & \theta_X & 0 \\ 0 & 0 & 0 & 0 & \theta_Z \end{pmatrix} \\
F' &= K' \oplus S' \oplus T'^* \oplus X' \oplus X'^* \oplus Y' \\
G' &= K' \oplus S' \oplus T' \oplus X' \oplus Z' \\
\gamma' &= 1_{K'} \oplus 1_{S'} \oplus 1_{X'} : G' \longrightarrow F' \\
\mu' &= \lambda' \oplus 1_{T'} \oplus 1_{X'} : G' \longrightarrow F'^* \\
\theta &= \begin{pmatrix} \theta_{K'} & 0 & 0 & 0 & 0 \\ 0 & \theta_{S'} & 0 & 0 & 0 \\ 0 & 0 & \theta_{T'} & 0 & 0 \\ 0 & 0 & 0 & \theta_{X'} & 0 \\ 0 & 0 & 0 & 0 & \theta_{Z'} \end{pmatrix}
\end{aligned}$$

Because of Corollary (2.4.3) we see that

$$\begin{aligned}
Z \oplus T &= \ker \gamma \cong \ker \gamma' = Z' \oplus T' & (9.1) \\
Z \oplus S &= \ker \mu \cong \ker \mu' = Z' \oplus S' \\
K \oplus S \oplus X &= \operatorname{im} \gamma \cong \operatorname{im} \gamma' = K' \oplus S' \oplus X' \\
K^* \oplus T \oplus X &= \operatorname{im} \mu \cong \operatorname{im} \mu' = K'^* \oplus T' \oplus X' \\
K^* &= \operatorname{im} \gamma^* \mu \cong \operatorname{im} \gamma'^* \mu' = K'^* \\
Z &= \ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \cong \ker \begin{pmatrix} \gamma' \\ \mu' \end{pmatrix} = Z' \\
K \oplus S \oplus T^* \oplus X \oplus X^* \oplus Y &\cong K' \oplus S' \oplus T'^* \oplus X' \oplus X'^* \oplus Y'
\end{aligned}$$

and therefore $K \cong K'$, $S \cong S'$, $T \cong T'$, $X \cong X'$, $y = y'$ and $Z \cong Z'$. If $\widehat{Q}^\epsilon(\Lambda)$ was zero we would be done now. The general case demands more work. We need to evaluate the equations $\alpha\gamma = \gamma'\beta$, $\alpha^*\mu'\beta = \mu$ and $\beta^*\theta'\beta = \theta \in Q_{-\epsilon}(G)$.

$$\begin{aligned}
\alpha\gamma &= \begin{pmatrix} \alpha_{11} & \dots & \alpha_{16} \\ \vdots & & \vdots \\ \alpha_{61} & \dots & \alpha_{66} \end{pmatrix} \begin{pmatrix} \operatorname{id}_K & 0 & 0 & 0 & 0 \\ 0 & \operatorname{id}_S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{id}_X & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} & 0 \\ \alpha_{31} & \alpha_{32} & 0 & \alpha_{34} & 0 \\ \alpha_{41} & \alpha_{42} & 0 & \alpha_{44} & 0 \\ \alpha_{51} & \alpha_{52} & 0 & \alpha_{54} & 0 \\ \alpha_{61} & \alpha_{62} & 0 & \alpha_{64} & 0 \end{pmatrix} \\
\gamma'\beta &= \begin{pmatrix} \operatorname{id}_{K'} & 0 & 0 & 0 & 0 \\ 0 & \operatorname{id}_{S'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{id}_{X'} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_{11} & \dots & \beta_{15} \\ \vdots & & \vdots \\ \beta_{51} & \dots & \beta_{55} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} \\ 0 & 0 & 0 & 0 & 0 \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} & \beta_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Comparing both sides shows that $\alpha\gamma = \gamma'\beta$ if and only if

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ 0 & 0 & \alpha_{33} & 0 & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ 0 & 0 & \alpha_{53} & 0 & \alpha_{55} & \alpha_{56} \\ 0 & 0 & \alpha_{63} & 0 & \alpha_{65} & \alpha_{66} \end{pmatrix}$$

$$\beta = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{35} \\ \alpha_{41} & \alpha_{42} & 0 & \alpha_{44} & 0 \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} \end{pmatrix}$$

Now we look at $\mu = \alpha^* \mu' \beta$. First compute:

$$J := \alpha^* \mu' \beta = \begin{pmatrix} \alpha_{11}^* & \alpha_{21}^* & 0 & \alpha_{41}^* & 0 & 0 \\ \alpha_{12}^* & \alpha_{22}^* & 0 & \alpha_{42}^* & 0 & 0 \\ \alpha_{13}^* & \alpha_{23}^* & \alpha_{33}^* & \alpha_{43}^* & \alpha_{53}^* & \alpha_{63}^* \\ \alpha_{14}^* & \alpha_{24}^* & 0 & \alpha_{44}^* & 0 & 0 \\ \alpha_{15}^* & \alpha_{25}^* & \alpha_{35}^* & \alpha_{45}^* & \alpha_{55}^* & \alpha_{65}^* \\ \alpha_{16}^* & \alpha_{26}^* & \alpha_{36}^* & \alpha_{46}^* & \alpha_{56}^* & \alpha_{66}^* \end{pmatrix} \begin{pmatrix} \lambda' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{id}_{T'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{id}_{X'} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \beta =$$

$$\begin{pmatrix} \alpha_{11}^* \lambda' & 0 & 0 & 0 & 0 \\ \alpha_{12}^* \lambda' & 0 & 0 & 0 & 0 \\ \alpha_{13}^* \lambda' & 0 & \alpha_{33}^* & \alpha_{53}^* & 0 \\ \alpha_{14}^* \lambda' & 0 & 0 & 0 & 0 \\ \alpha_{15}^* \lambda' & 0 & \alpha_{35}^* & \alpha_{55}^* & 0 \\ \alpha_{16}^* \lambda' & 0 & \alpha_{36}^* & \alpha_{56}^* & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{35} \\ \alpha_{41} & \alpha_{42} & 0 & \alpha_{44} & 0 \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} \end{pmatrix} =$$

$$\begin{pmatrix} \alpha_{11}^* \lambda' \alpha_{11} & \alpha_{11}^* \lambda' \alpha_{12} & 0 & \alpha_{11}^* \lambda' \alpha_{14} & 0 \\ \alpha_{12}^* \lambda' \alpha_{11} & \alpha_{12}^* \lambda' \alpha_{12} & 0 & \alpha_{12}^* \lambda' \alpha_{14} & 0 \\ \alpha_{13}^* \lambda' \alpha_{11} + \alpha_{33}^* \beta_{31} + \alpha_{53}^* \alpha_{41} & \alpha_{13}^* \lambda' \alpha_{12} + \alpha_{33}^* \beta_{32} + \alpha_{53}^* \alpha_{42} & \alpha_{33}^* \beta_{33} & \alpha_{13}^* \lambda' \alpha_{14} + \alpha_{33}^* \beta_{34} + \alpha_{53}^* \alpha_{44} & \alpha_{33}^* \beta_{35} \\ \alpha_{14}^* \lambda' \alpha_{11} & \alpha_{14}^* \lambda' \alpha_{12} & 0 & \alpha_{14}^* \lambda' \alpha_{14} & 0 \\ \alpha_{15}^* \lambda' \alpha_{11} + \alpha_{35}^* \beta_{31} + \alpha_{55}^* \alpha_{41} & \alpha_{15}^* \lambda' \alpha_{12} + \alpha_{35}^* \beta_{32} + \alpha_{55}^* \alpha_{42} & \alpha_{35}^* \beta_{33} & \alpha_{15}^* \lambda' \alpha_{14} + \alpha_{35}^* \beta_{34} + \alpha_{55}^* \alpha_{44} & \alpha_{35}^* \beta_{35} \\ \alpha_{16}^* \lambda' \alpha_{11} + \alpha_{36}^* \beta_{31} + \alpha_{56}^* \alpha_{41} & \alpha_{16}^* \lambda' \alpha_{12} + \alpha_{36}^* \beta_{32} + \alpha_{56}^* \alpha_{42} & \alpha_{36}^* \beta_{33} & \alpha_{16}^* \lambda' \alpha_{14} + \alpha_{36}^* \beta_{34} + \alpha_{56}^* \alpha_{44} & \alpha_{36}^* \beta_{35} \end{pmatrix}$$

We evaluate the matrix equation

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{id}_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{id}_X & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = J$$

and find that $\alpha^{-*}\mu = \mu'\beta$ (and $\alpha\gamma = \gamma'\beta$) if and only if

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 & \alpha_{15} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ 0 & 0 & \beta_{33}^{-*} & 0 & 0 & 0 \\ \alpha_{41} & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ 0 & 0 & \alpha_{53} & 0 & \alpha_{55}^{-*} & 0 \\ 0 & 0 & \alpha_{63} & 0 & \alpha_{65} & \alpha_{66} \end{pmatrix} \quad (9.2)$$

$$\beta = \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} & 0 \\ \beta_{31} & 0 & \beta_{33} & \beta_{34} & 0 \\ \alpha_{41} & 0 & 0 & \alpha_{44} & 0 \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} \end{pmatrix}$$

with $\alpha_{11}^* \lambda' \alpha_{11} = \lambda$, $\alpha_{53} = -\alpha_{55} \beta_{34}^* \alpha_{33}^*$, $\alpha_{15} = \epsilon \lambda'^{-1} \alpha_{11}^{-*} \alpha_{41}^* \alpha_{55}$, $\alpha_{13} = \epsilon \lambda'^{-1} \alpha_{11}^{-*} (\beta_{31}^* - \alpha_{41}^* \alpha_{55} \beta_{34}^*) \alpha_{33}$ and β_{33} as well as α_{44} isomorphisms. Because α and β are isomorphisms it follows (by Proposition 2.2.2) that α_{22} , α_{66} and β_{55} are isomorphisms as well. Conversely, any map α and β of the form (9.2) with the above properties defines an isomorphism of the ϵ -quadratic preformations $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F')$ (i.e. ignoring quadratic refinements).

At last we analyze what (α, β) does to the quadratic refinement i.e. we evaluate $\theta = \beta^* \theta' \beta \in Q_{-\epsilon}(G)$. First we compute

$$\begin{aligned} \tilde{\theta} &:= \beta^* \theta' \beta = \begin{pmatrix} \alpha_{11}^* & \alpha_{21}^* & \beta_{31}^* & \alpha_{41}^* & \beta_{51}^* \\ 0 & \alpha_{22}^* & 0 & 0 & \beta_{52}^* \\ 0 & 0 & \beta_{33}^* & 0 & \beta_{53}^* \\ 0 & \alpha_{24}^* & \beta_{34}^* & \alpha_{44}^* & \beta_{54}^* \\ 0 & 0 & 0 & 0 & \beta_{55}^* \end{pmatrix} \begin{pmatrix} \theta_{K'} & 0 & 0 & 0 & 0 \\ 0 & \theta_{S'} & 0 & 0 & 0 \\ 0 & 0 & \theta_{T'} & 0 & 0 \\ 0 & 0 & 0 & \theta_{X'} & 0 \\ 0 & 0 & 0 & 0 & \theta_{Z'} \end{pmatrix} \beta \\ &= \begin{pmatrix} \alpha_{11}^* \theta_{K'} & \alpha_{21}^* \theta_{S'} & \beta_{31}^* \theta_{T'} & \alpha_{41}^* \theta_{X'} & \beta_{51}^* \theta_{Z'} \\ 0 & \alpha_{22}^* \theta_{S'} & 0 & 0 & \beta_{52}^* \theta_{Z'} \\ 0 & 0 & \beta_{33}^* \theta_{T'} & 0 & \beta_{53}^* \theta_{Z'} \\ 0 & \alpha_{24}^* \theta_{S'} & \beta_{34}^* \theta_{T'} & \alpha_{44}^* \theta_{X'} & \beta_{54}^* \theta_{Z'} \\ 0 & 0 & 0 & 0 & \beta_{55}^* \theta_{Z'} \end{pmatrix} \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} & 0 \\ \beta_{31} & 0 & \beta_{33} & \beta_{34} & 0 \\ \alpha_{41} & 0 & 0 & \alpha_{44} & 0 \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} \end{pmatrix} \end{aligned}$$

Using Lemma 9.1.10

$$\tilde{\theta} = \begin{pmatrix} \theta_K & 0 & 0 & 0 & 0 \\ 0 & \theta_S & 0 & 0 & 0 \\ 0 & 0 & \theta_T & 0 & 0 \\ 0 & 0 & 0 & \theta_X & 0 \\ 0 & 0 & 0 & 0 & \theta_Z \end{pmatrix} \in Q_{-\epsilon}(G)$$

if and only if

$$\begin{aligned} \theta_Z &= \beta_{55}^* \theta_{Z'} \beta_{55} \in \widehat{Q}^\epsilon(Z) \\ \theta_T &= \beta_{33}^* \theta_{T'} \beta_{33} + \beta_{53}^* \theta_{Z'} \beta_{53} \in \widehat{Q}^\epsilon(T) \\ \theta_S &= \alpha_{22}^* \theta_{S'} \alpha_{22} + \beta_{52}^* \theta_{Z'} \beta_{52} \in \widehat{Q}^\epsilon(S) \\ \theta_X &= \alpha_{24}^* \theta_{S'} \alpha_{24} + \beta_{34}^* \theta_{T'} \beta_{34} + \alpha_{44}^* \theta_{X'} \alpha_{44} + \beta_{54}^* \theta_{Z'} \beta_{54} \in \widehat{Q}^\epsilon(X) \\ \theta_K &= \alpha_{11}^* \theta_{K'} \alpha_{11} + \alpha_{21}^* \theta_{S'} \alpha_{21} + \beta_{31}^* \theta_{T'} \beta_{31} + \alpha_{41}^* \theta_{X'} \alpha_{41} + \beta_{51}^* \theta_{Z'} \beta_{51} \in Q_{-\epsilon}(K) \end{aligned}$$

These are exactly the equivalence relations of the Proposition. □

Theorem 9.1.11. *If Λ is a skew field then η is an isomorphism.*

Corollary 9.1.12. *Let $\text{char } \Lambda \neq 2$. Then the inverse of η is given by*

$$\begin{aligned} \eta^{-1}: l_{2q+2}(\Lambda) &\xrightarrow{\cong} L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^5 \\ \left[(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*) \right] &\longmapsto \left([(G/\ker \gamma^* \mu, \gamma^* \mu)], \right. \\ & \quad s := \dim \ker \mu - \dim \ker \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \\ & \quad t := \dim \ker \gamma - \dim \ker \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \\ & \quad x := \dim \ker \gamma^* \mu - \dim \ker \mu - \dim \ker \gamma + \dim \ker \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \\ & \quad z := \dim \ker \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \\ & \quad \left. y := \dim \text{coker } \gamma - \dim \ker \gamma^* \mu + \dim \ker \mu \right) \end{aligned}$$

Remark 9.1.13. i) Let $\text{char } \Lambda \neq 2$. In regard to Corollary 2.4.3 and because of $\dim \text{coker } \gamma - \dim \ker \gamma = \dim F - \dim G$, one can simplify η^{-1} and find an injection of abelian monoids

$$\begin{aligned} \nu: l_{2q+2}(\Lambda) &\xrightarrow{\cong} L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^4 \oplus \mathbb{Z} \\ (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) &\longmapsto \left([(G/\ker \gamma^* \mu, \gamma^* \mu)], \dim \ker \gamma, \dim \ker \mu, \right. \\ & \quad \left. \dim \ker \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \dim \ker \gamma^* \mu, \dim F - \dim G \right) \end{aligned}$$

ii) For symmetric l -monoids $l^{2q+2}(\Lambda)$ there is an uniform result for *all* skew fields regardless of their characteristic: Corollary 9.1.12 and i) give rise to an isomorphism $\eta^{-1}: l^{2q+2}(\Lambda) \xrightarrow{\cong} L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^5$ and an injection $\nu: l^{2q+2}(\Lambda) \xrightarrow{\cong} L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^4 \oplus \mathbb{Z}$

Proof of Theorem 9.1.11. Let $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be an ϵ -quadratic split preformation.

Step 1: Due to Lemma 9.1.10 we get

$$\begin{aligned} (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) &= (F \xleftarrow{\bar{\gamma}} G/(\ker \gamma \cap \ker \mu) \xrightarrow{\bar{\mu}} F^*, \bar{\omega}) \\ &\oplus (0 \xleftarrow{0} \ker \gamma \cap \ker \mu \xrightarrow{0} 0^*, \bar{\theta}'). \end{aligned}$$

with some $\bar{\theta}' \in \hat{Q}^\epsilon(\ker \gamma \cap \ker \mu)$.

Step 2: We can now start with a preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ such that $\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right)$ is injective.

Claim 1: For any subspace $R \subset G$ with $\ker \gamma \oplus \ker \mu \oplus R = G$ there is some subspace $A \subset F$ such that

$$\gamma(\ker \mu) \oplus \mu(\ker \gamma)^* \oplus \gamma(R) \oplus A = F.$$

Proof of Claim 1: Surely $\gamma|\bar{R}$ and $\mu|\ker\gamma$ are injective for $\bar{R} := R \oplus \ker\mu$. It is enough to show that $\gamma(\bar{R}) \cap \mu(\ker\gamma)^* = 0$. So let $r \in \bar{R}$ and $\gamma(r) \in \mu(\ker\gamma)^*$. Using the canonical isomorphism $F \longrightarrow F^{**}$, the map

$$\Phi : F^* \longrightarrow \Lambda, \quad f \longmapsto f(\gamma(r))$$

is in $\mu(\ker\gamma)^*$. Hence $r = 0 \Leftrightarrow \Phi = 0 \Leftrightarrow \Phi|\mu(\ker\gamma) = 0$. But for any $s \in \ker\gamma$, $\Phi(\mu(s)) = \mu(s)(\gamma(r)) = \epsilon\mu(r)(\gamma(s)) = 0$.

Claim 2: There is a subspace $R' \subset G$ such that $\ker\gamma \oplus \ker\mu \oplus R' = G$ and $\mu(R') \subset \gamma(R')^* \oplus A^*$ with A' chosen as in Claim 1.

Proof of Claim 2: First we choose some arbitrary complement R of $\ker\mu \oplus \ker\gamma$ in G and A as in Claim 1. Then $\mu(R) \subset F^* = \gamma(\ker\mu)^* \oplus \mu(\ker\gamma) \oplus \gamma(R)^* \oplus A^*$. But since for any $r \in R, s \in \ker\gamma$, $\mu(r)(\gamma(s)) = \epsilon\mu(s)(\gamma(r)) = 0$, we can conclude that $\mu(R) \subset \mu(\ker\gamma) \oplus \gamma(R)^* \oplus A^*$. Hence for any $r \in R$ there's an $s_r \in \ker(\gamma)$ and elements $g_r \in \gamma(R)^*$ and $a_r \in A^*$ with $\mu(r) = \mu(s_r) + g_r + a_r$. The map $r \mapsto s_r$ is in fact a vector space homomorphism

$$R \xrightarrow{\mu} \mu(R) \xrightarrow{\text{incl.}} F^* \xrightarrow{\text{proj.}} \mu(\ker\gamma) \xrightarrow{(\mu|\ker\gamma)^{-1}} \ker\gamma$$

Hence $R' := \{r - s_r | r \in R\}$ is a well-defined subspace of $R \oplus \ker\gamma$. We have to check now whether R' has the two properties we want. Let $x \in R' \cap (\ker\gamma \oplus \ker\mu)$. We show that x vanishes necessarily. There are $r \in R, a \in \ker\gamma$ and $b \in \ker\mu$ with $x = r - s_r = a + b$. It follows that $r = 0$ and $s_r = 0$ and finally $x = 0$. Obviously $\dim R = \dim R'$. Hence $\ker\gamma \oplus \ker\mu \oplus R' = G$. It is clear that $\gamma(R) = \gamma(R')$. This means we can use the same A for R' and Claim 1 holds. At last, we want to show that $\mu(R') \subset \gamma(R')^* \oplus A^*$: For any $x \in R'$ there is an $r \in R$ with $x = r - s_r$ and $\mu(x) = g_r + a_r \in \gamma(R')^* \oplus A^*$ by construction. Hence Claim 2 is proven.

From Claim 1 and 2 and Lemma 9.1.10 we conclude that

$$\begin{aligned} (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) &= (\gamma(\ker\mu) \xleftarrow{\gamma} \ker\mu \xrightarrow{0} \gamma(\ker\mu)^*, \theta_M) \\ &\oplus (\mu(\ker\gamma)^* \xleftarrow{0} \ker\gamma \xrightarrow{\mu} \mu(\ker\gamma), \theta_G) \\ &\oplus ((\gamma(R') \oplus A') \longleftarrow R' \longrightarrow (\gamma(R') \oplus A')^*, \theta'_R). \end{aligned}$$

The first summand is strongly isomorphic to the boundary of the form $(\ker\mu, \theta_M \in \hat{Q}^\epsilon(\ker\mu))$ and the second one is of the type $(T^* \xleftarrow{0} T \xrightarrow{1_T} T^{**}, \theta_T)$.

Step 3: At last we deal with the remaining preformation $((\gamma(R) \oplus A) \xleftarrow{\gamma} R \xrightarrow{\mu} (\gamma(R) \oplus A)^*, \theta_R)$. $\gamma = \begin{pmatrix} \gamma_1 \\ 0 \end{pmatrix}$ and $\mu = \begin{pmatrix} \mu_R \\ \mu_A \end{pmatrix}$ are injective by construction in step 2. Without loss of generality $\gamma_1 = 1_R$. Let $R = \ker\mu_R \oplus K$. We rewrite our formation therefore as

$$((\ker\mu_R \oplus K \oplus A) \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} \ker\mu_R \oplus K \xrightarrow{\begin{pmatrix} 0 & \mu_1 \\ 0 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}} (\ker\mu_R \oplus K \oplus A)^*, \theta_R).$$

Now $\gamma^*\mu$ must be $(-\epsilon)$ -symmetric, hence $\mu_1 = 0$. It is also clear that μ_2 must be an isomorphism.

As μ is injective so is $\mu_3: \ker \mu_R \rightarrow A^*$. Therefore there are vector spaces X and Y and isomorphisms $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}: A^* \xrightarrow{\cong} X \oplus Y^*$ and $\tau: X \xrightarrow{\cong} \ker \mu_R$ such that $\sigma\mu_3\tau = \begin{pmatrix} 1_X \\ 0 \end{pmatrix}$. This leads to the commuting diagram

$$\begin{array}{ccccc} \ker \mu_R \oplus K \oplus A & \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & \ker \mu_R \oplus K & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}} & (\ker \mu_R)^* \oplus K^* \oplus A^* \\ \alpha \uparrow \cong & & \beta \uparrow \cong & & \alpha^{-*} \uparrow \cong \\ X \oplus X^* \oplus Y \oplus K & \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}} & X \oplus K & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \mu_2 \end{pmatrix}} & X^* \oplus X \oplus Y^* \oplus K^* \end{array}$$

with

$$\alpha = \begin{pmatrix} \tau & 0 & 0 & 0 \\ 0 & -\mu_2^{-*} \mu_4^* \sigma_1^* & -\mu_2^{-*} \mu_4^* \sigma_2^* & 1 \\ 0 & \sigma_1^* & \sigma_2^* & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}.$$

□

9.2 Computation of Elementary Elements

Even more important than the computation of $l_{2q+2}(\Lambda)$ for skew fields is probably to provide an easy test for elementariness of its elements.

Theorem 9.2.1. *Let Λ be a skew field with involution.*

$$\eta([(K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y]) \in l_{2q+2}(\Lambda)$$

is elementary if and only if

$$\begin{aligned} [(K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y] &= \\ [0, (S, \theta_S), (S, \theta_S), (X, 0), (Z, \theta_Z), 0] &\in m_{2q+2}(\Lambda) \end{aligned}$$

Again the case of $\text{char } \Lambda \neq 2$ is much more digestible.

Corollary 9.2.2. *Let Λ be a skew field with involution. Assume that $\text{char } \Lambda \neq 2$. Then $\eta([(K, \lambda)], s, t, x, z, y)$ is elementary if and only if $s = t$, $[(K, \lambda)] = 0 \in L^{2q+2}(\Lambda)$ and $y = 0$.*

$[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if $\nu(z) = (0, a, a, b, c + 2a, c)$ for some $a, b \in \mathbb{N}_0$ and $c \in \mathbb{Z}$, i.e. if and only if $[(G/\ker \gamma^*\mu, \gamma^*\mu)] = 0 \in L^{2q+2}(\Lambda)$, $\dim \ker \gamma = \dim \ker \mu$ and $\dim \ker \gamma^*\mu = \dim \ker \gamma + \dim \ker \gamma^*$.

The same statements hold for the symmetric l -monoids $l^{2q+2}(\Lambda)$ for all skew fields of any characteristic.

Proof. See Corollaries 9.1.9, 2.4.3 and Remark 9.1.13. □

Proof of Theorem 9.2.1. It is obvious that the boundary of a stably metabolic form, $((X \oplus X^*) \xleftarrow{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} X \xrightarrow{\begin{pmatrix} 0 \\ 1_X \end{pmatrix}} (X \oplus X^*)^*, 0)$ and $(0 \longleftarrow Z \longrightarrow 0, \theta_Z)$ are elementary. The preformation $((S \oplus S^*) \xleftarrow{1_S \oplus 0} S \oplus S \xrightarrow{0 \oplus 1_S} (S \oplus S^*)^*, \theta_S \oplus \theta_S)$ is elementary in respect to $U := \{(s, s) | s \in S\}$.

Assume that after adding hyperbolic forms to (K, θ_K) we can assume that the ϵ -quadratic split preformation

$$(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta}) = \eta((K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), y, (Z, \theta_Z))$$

is elementary in respect to the h -lagrangian

$$j = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} : U \longrightarrow G = K \oplus S \oplus T \oplus X \oplus Z$$

Then $\gamma j = \begin{pmatrix} a \\ b \\ 0 \\ d \\ 0 \end{pmatrix}$ and $\mu j = \begin{pmatrix} \lambda a \\ c \\ 0 \\ d \\ 0 \end{pmatrix}$. We rephrase what it means to be elementary:

i) $a^* \theta_K a + b^* \theta_S b + c^* \theta_T c + d^* \theta_X d + e^* \theta_Z e = 0 \in Q_{-\epsilon}(\Lambda)$

ii) $\begin{pmatrix} a \\ b \\ d \end{pmatrix}$ and $\begin{pmatrix} a \\ c \\ d \end{pmatrix}$ are injective

iii) $2 \dim U = \dim K + \dim S + \dim T + 2 \dim X + \dim Y$

Criterion i) implies $a^* \lambda a = 0$ and therefore $\text{rk } a \leq \frac{1}{2} \dim K$. That means that $\dim U = \text{rk} \begin{pmatrix} a \\ b \\ d \end{pmatrix} \leq \frac{1}{2} \dim K + \dim S + \dim X$ and $\dim U = \text{rk} \begin{pmatrix} a \\ c \\ d \end{pmatrix} \leq \frac{1}{2} \dim K + \dim T + \dim X$. Together with criterion iii) this leads to $y = \dim Y = 0$ and $\dim S = \dim T$. Therefore $\dim K$ is even and $\dim U = \frac{1}{2} \dim K + \dim S + \dim X$. The ranks tell us even more:

$$\begin{aligned} \frac{1}{2} \dim K + \dim S + \dim X &= \dim U = \text{rk} \begin{pmatrix} a \\ b \\ d \end{pmatrix} \\ &\leq \text{rk } a + \text{rk } b + \text{rk } d \leq \frac{1}{2} \dim K + \dim S + \dim X \end{aligned}$$

Therefore the inequalities $\text{rk } a \leq \frac{1}{2} \dim K$ and $\text{rk } b \leq \dim S$ and $\text{rk } d \leq \dim X$ are in fact identities. Similarly we deduce $\text{rk } c = \dim T$. Together with the fact that $a^* \lambda a = 0$ this means that the $(-\epsilon)$ -symmetric form (K, λ) has the lagrangian $L = \text{im } a$. We write $K = L^* \oplus L$. There are $\theta_1 \in Q_{-\epsilon}(L^*)$ and $\theta_2 \in \widehat{Q}^\epsilon(L)$ such that $\theta_K = \begin{pmatrix} \theta_1 & 1 \\ 0 & \theta_2 \end{pmatrix} \in Q_{-\epsilon}(L^* \oplus L)$. Criterion ii) is equivalent to $V \cap \ker b = V \cap \ker c = 0$ with $V := \ker a \cap \ker d$. This implies

$$\begin{aligned} \dim S &= \text{rk } b = \dim U - \dim \ker b \geq \dim V \\ &= \dim \ker a + \dim \ker d - \dim(\ker a + \ker d) \\ &\geq (\dim U - \text{rk } a) + (\dim U - \text{rk } d) - \dim U = \dim S \end{aligned}$$

which means that $\dim U - \dim \ker b = \dim V = \dim S$ and therefore $V \oplus \ker b = U$.

Using this decomposition, j looks like this:

$$j = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \\ b_1 & 0 \\ c_1 & c_2 \\ 0 & d_2 \\ e_1 & e_2 \end{pmatrix} : V \oplus \ker b \rightarrow (L^* \oplus L) \oplus S \oplus T \oplus X \oplus Z$$

Now $b_1 = b|V$ and $c_1 = c|V$ are injective and due to dimension reasons they are in fact isomorphisms. On the other hand $\begin{pmatrix} a_2 \\ d_2 \end{pmatrix}$ is injective and again by comparing dimensions, the map turns out to be an isomorphism. We feed this information into criterion i):

$$\begin{aligned} & \begin{pmatrix} 0 & 0 \\ 0 & a_2^* \end{pmatrix} \begin{pmatrix} \theta_1 & 1 \\ 0 & \theta_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} b_1^* \\ 0 \end{pmatrix} \theta_S (b_1 \ 0) + \begin{pmatrix} c_1^* \\ c_2^* \end{pmatrix} \theta_T (c_1 \ c_2) \\ + & \begin{pmatrix} 0 \\ d_2^* \end{pmatrix} \theta_X (0 \ d_2) + \begin{pmatrix} e_1^* \\ e_2^* \end{pmatrix} \theta_Z (e_1 \ e_2) = 0 \in \widehat{Q}^\epsilon(U) \\ \Rightarrow & \begin{pmatrix} b_1^* \theta_S b_1 + c_1^* \theta_T c_1 + e_1^* \theta_Z e_1 & c_1^* \theta_T c_2 + e_1^* \theta_Z e_2 \\ c_2^* \theta_T c_1 + e_2^* \theta_Z e_1 & a_2^* \theta_2 a_2 + c_2^* \theta_T c_2 + d_2^* \theta_X d_2 + e_2^* \theta_Z e_2 \end{pmatrix} = 0 \in \widehat{Q}^\epsilon(U) \end{aligned}$$

This is the case if and only if

$$\begin{aligned} \theta_S &= f^*(\theta_T + g^* \theta_Z g) f = 0 \in \widehat{Q}^\epsilon(S) \\ \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_X \end{pmatrix} &= h^* \theta_T h + k^* \theta_Z k \in \widehat{Q}^\epsilon(L \oplus X) \end{aligned}$$

with $f = c_1 b_1^{-1}$, $g = e_1 c_1^{-1}$, $h = c_2 \begin{pmatrix} a_2 \\ d_2 \end{pmatrix}^{-1}$ and $k = e_2 \begin{pmatrix} a_2 \\ d_2 \end{pmatrix}^{-1}$. (Note that signs do not matter because $2\theta = 0 \in \widehat{Q}^\epsilon(M)$).

Define $h_1 = h \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $k_1 = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $h_2 = h \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $k_2 = k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$\begin{aligned} \theta_X &= h_2^* \theta_T h_2 + k_2^* \theta_Z k_2 \in \widehat{Q}^\epsilon(X) \\ \theta_K &= \begin{pmatrix} \theta_1 & 1 \\ 0 & \theta_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (0 \ h_1)^* \theta_T (0 \ h_1) + (0 \ k_1)^* \theta_Z (0 \ k_1) \in Q_{-\epsilon}(L^* \oplus L) \end{aligned}$$

□

As an approximation to the question whether a class in $l_{2q+2}(\Lambda)$ is elementary or not one can look at its image in the quotient monoid $l_{2q+2}(\Lambda)/l_{2q+2}^{el}(\Lambda)$. Much would be won if this object was a group. Unfortunately the following corollaries show that in the case of skew fields and principal ideal domains they are not.

Corollary 9.2.3. *Let Λ be a skew field. Then the quotient monoid $l_{2q+2}(\Lambda)/l_{2q+2}^{elem}(\Lambda)$ is not a group.*

Proof. Let $y \neq 0$. There is no $x \in l_{2q+2}(\Lambda)$ such that $x + \eta(0, 0, 0, 0, 0, y) \in l_{2q+2}^{elem}(\Lambda)$. □

Corollary 9.2.4. *Let Λ be a principal ideal domain. Then the quotient monoid $l_{2q+2}(\Lambda)/l_{2q+2}^{elem}(\Lambda)$ is not a group.*

Proof. Let $F = \text{Quot}(\Lambda)$. Let $y \neq 0$. Assume there is an $x \in l_{2q+2}(\Lambda)$ such that $x + \eta(0, 0, 0, 0, 0, y) \in l_{2q+2}^{elem}(\Lambda)$. Tensoring with F yields the existence of an $x' = x \otimes F \in l_{2q+2}(F)$ such that $x' + \eta(0, 0, 0, 0, 0, y) \in l_{2q+2}^{elem}(F)$. This is a contradiction. \square

9.3 The l -Monoid $l_{2q}(\mathbb{Z}/2\mathbb{Z})$

The easiest non-trivial example (i.e. $\text{char } \Lambda = 2$) for $m_{2q+2}(\Lambda)$ and hence $l_{2q+2}(\Lambda)$ is the case $\Lambda = \mathbb{Z}/2\mathbb{Z}$.

We know that the Arf-invariant induces an isomorphism $L_{2q+2}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. By Example 9.1.5, $\widehat{M}(\mathbb{Z}/2\mathbb{Z}) = \{0 = o_0, o_1, e_1, o_2, e_2, \dots\}$ with a commutative operation given by $o_l + o_k = o_{k+l}$ and $e_{k+l} = e_k + e_l = e_k + o_l$. $(G, \theta \in \widehat{Q}(G)) = e_{\text{rk } G} \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ if $\theta \neq 0 \in \widehat{Q}(G)$ and $(G, \theta \in \widehat{Q}(G)) = o_{\text{rk } G} \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ if $\theta = 0 \in \widehat{Q}(G)$.

A little lemma helps us to disentangle the equivalence relations in the definition of $m_{2q+2}(\mathbb{Z}/2\mathbb{Z})$.

Lemma 9.3.1. *Let $(M = \mathbb{Z}/2\mathbb{Z}^m, \theta \in \widehat{Q}(M))$ and $(N = \mathbb{Z}/2\mathbb{Z}^n, \nu \in \widehat{Q}(N))$ be quadratic forms over $\mathbb{Z}/2\mathbb{Z}$.*

There exists a $b \in \text{Hom}_{\mathbb{Z}/2\mathbb{Z}}(M, N)$ such that $(M, \theta' = \theta + b^\nu b) = o_m \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ if and only if $(M, \theta) = o_m \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ or $(N, \nu) = e_n \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$.*

There exists a $b \in \text{Hom}_{\mathbb{Z}/2\mathbb{Z}}(M, N)$ such that $(M, \theta' = \theta + b^\nu b) = e_m \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ if and only if $(M, \theta) = e_m \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$ or $(N, \nu) = e_n \in \widehat{M}(\mathbb{Z}/2\mathbb{Z})$.*

Proof. If $\theta \neq 0 \in \widehat{Q}(M)$ then there is a $g \in GL(M)$ such that $g^*\theta g = \text{diag}(1, 0, \dots, 0) = b \in \widehat{Q}(M)$. If $\nu \neq 0 \in \widehat{Q}(N)$ then there is an $h \in GL(N)$ such that $h^*\nu h = \text{diag}(1, 0, \dots, 0) = c \in \widehat{Q}(N)$. Let $A \in \text{Hom}_{\mathbb{Z}/2\mathbb{Z}}(N, M)$ such that $A^*cA = b$ (i.e. $A_{11} = 1$ all other $A_{ij} = 0$).

If $\theta = 0 \in \widehat{Q}(M)$, $b = 0$ gives $\theta' = 0 \in \widehat{Q}(M)$.

If $\theta \neq 0 \in \widehat{Q}(M)$ and $\nu \neq 0 \in \widehat{Q}(N)$, $b = hAg^{-1}$ gives $\theta' = 0 \in \widehat{Q}(M)$.

If $\theta \neq 0 \in \widehat{Q}(M)$, $b = 0$ gives $\theta' \neq 0 \in \widehat{Q}(M)$.

If $\theta = 0 \in \widehat{Q}(M)$ and $\nu \neq 0 \in \widehat{Q}(N)$, $b = hA$ gives $\theta' \neq 0 \in \widehat{Q}(M)$.

If $\theta \neq 0 \in \widehat{Q}(M)$ and $\nu = 0 \in \widehat{Q}(N)$ then $\theta' \neq 0 \in \widehat{Q}(M)$ for all choices of b .

Conversely if $\nu = 0 \in \widehat{Q}(N)$ then $\theta' = \theta \in \widehat{Q}(M)$ for all choices of b . \square

Proposition 9.3.2. *Let $(\mathbb{Z}/2\mathbb{Z}^2, \theta_i)$ be non-singular quadratic preformation with Arf-invariant $i \in \mathbb{Z}/2\mathbb{Z}$. There is an isomorphism*

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} \oplus \widehat{M}(\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{N}_0 / \sim &\xrightarrow{\cong} m_{2q+2}(\mathbb{Z}/2\mathbb{Z}) \\ [k, s, t, x, z, y] &\longmapsto [(\mathbb{Z}/2\mathbb{Z}^2, \theta_k), s, t, x, z, y] \end{aligned}$$

with $(k, s, t, x, z, y) \sim (k', s', t', x', z', y')$ if and only if

$$\begin{cases} y' = y \in \mathbb{N}_0 \\ z' = z \in \widehat{M}(\mathbb{Z}/2\mathbb{Z}) \\ t' = t \in \widehat{M}(\mathbb{Z}/2\mathbb{Z}) & \text{if } z = o_a \text{ for an } a \in \mathbb{N}_0 \\ s' = s \in \widehat{M}(\mathbb{Z}/2\mathbb{Z}) & \text{if } z = o_a \text{ for an } a \in \mathbb{N}_0 \\ x' = x \in \widehat{M}(\mathbb{Z}/2\mathbb{Z}) & \text{if } z = o_a, s = o_b \text{ and } t = o_c \text{ for some } a, b, c \in \mathbb{N}_0 \\ k' = k \in \mathbb{Z}/2\mathbb{Z} & \text{if } z = o_a, s = o_b, t = o_c \text{ and } x = o_d \text{ for some } a, b, c, d \in \mathbb{N}_0 \end{cases}$$

Define $m(o_k) = (\mathbb{Z}/2\mathbb{Z}^k, 0 \in \widehat{Q}(\mathbb{Z}/2\mathbb{Z}^k))$ for $k \in \mathbb{N}_0$. For any $l \in \mathbb{N}$ choose some $\nu_l \neq 0 \in \widehat{Q}(\mathbb{Z}/2\mathbb{Z}^l)$ (for example one represented by the identity matrix) and define $m(e_l) = (\mathbb{Z}/2\mathbb{Z}^l, \nu_l)$. Then there is an isomorphism

$$\begin{aligned} \eta' : \mathbb{Z}/2\mathbb{Z} \oplus \widehat{M}(\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{N}_0 / \sim & \xrightarrow{\cong} l_{2q+2}(\mathbb{Z}/2\mathbb{Z}) \\ [k, s, t, x, y, z] & \longmapsto [\eta((\mathbb{Z}/2\mathbb{Z}^2, \theta_k), m(s), m(t), m(x), m(y), m(z))] \end{aligned}$$

Proposition 9.3.3. Define the subsets $A_i \subset \mathbb{Z}/2\mathbb{Z} \oplus \widehat{M}(\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{N}_0$

$$\begin{aligned} A_0 &= \{(0, o_s, o_t, o_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0\} \\ A_1 &= \{(1, o_s, o_t, o_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0\} \\ A_2 &= \{(0, o_s, o_t, e_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0, x \neq 0\} \\ A_3 &= \{(0, e_s, o_t, o_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0, s \neq 0\} \\ A_4 &= \{(0, o_s, e_t, o_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0, t \neq 0\} \\ A_5 &= \{(0, e_s, e_t, o_x, o_z, y) : s, t, x, y, z \in \mathbb{N}_0, s, t \neq 0\} \\ A_6 &= \{(0, o_s, o_t, o_x, e_z, y) : s, t, x, y, z \in \mathbb{N}_0, z \neq 0\} \end{aligned}$$

Then $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ is a complete set of representatives.

$A_{elem} = \{(k, s, t, x, 0, z) \in A_0 \cup A_5 \cup A_6\}$ are all the elements of A which η' of Proposition 9.3.2 maps to elementary elements.

Proposition 9.3.4. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \bar{\theta})$ be a quadratic split preformation over $\mathbb{Z}/2\mathbb{Z}$. Define $B_i = \eta'(A_i) \subset l_{2q+2}(\mathbb{Z}/2\mathbb{Z})$. Let

$$\begin{aligned} s &= \dim \ker \mu - \dim \ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \\ t &= \dim \ker \gamma - \dim \ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \\ x &= \dim \ker \gamma^* \mu - \dim \ker \mu - \dim \ker \gamma + \dim \ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \\ z &= \dim \ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \\ y &= \dim \text{coker } \gamma - \dim \ker \gamma^* \mu + \dim \ker \mu \end{aligned}$$

(compare Corollary 9.1.12). Then

i) If $\bar{\theta}(\ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix}) \neq 0 \in \widehat{Q}(\ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix})$ then $[z] = \eta'(0, o_s, o_t, o_x, e_z, y) \in B_6$.

- ii) If $\bar{\theta}|(\ker(\gamma_\mu)) = 0 \in \widehat{Q}(\ker(\gamma_\mu))$ and $\bar{\theta}|(\ker \gamma) \neq 0 \in \widehat{Q}(\ker \gamma)$ and $\bar{\theta}|(\ker \mu) \neq 0 \in \widehat{Q}(\ker \mu)$ then $[z] = \eta'(0, e_s, e_t, o_x, o_z, y) \in B_5$.
- iii) If $\bar{\theta}|(\ker(\gamma_\mu)) = 0 \in \widehat{Q}(\ker(\gamma_\mu))$ and $\bar{\theta}|(\ker \gamma) \neq 0 \in \widehat{Q}(\ker \gamma)$ and $\bar{\theta}|(\ker \mu) = 0 \in \widehat{Q}(\ker \mu)$ then $[z] = \eta'(0, o_s, e_t, o_x, o_z, y) \in B_4$.
- iv) If $\bar{\theta}|(\ker(\gamma_\mu)) = 0 \in \widehat{Q}(\ker(\gamma_\mu))$ and $\bar{\theta}|(\ker \gamma) = 0 \in \widehat{Q}(\ker \gamma)$ and $\bar{\theta}|(\ker \mu) \neq 0 \in \widehat{Q}(\ker \mu)$ then $[z] = \eta'(0, e_s, o_t, o_x, o_z, y) \in B_3$.
- v) If $\bar{\theta}|(\ker(\gamma_\mu)) = 0 \in \widehat{Q}(\ker(\gamma_\mu))$ and $\bar{\theta}|(\ker \gamma) = 0 \in \widehat{Q}(\ker \gamma)$ and $\bar{\theta}|(\ker \mu) = 0 \in \widehat{Q}(\ker \mu)$ and $\bar{\theta}|(\ker \gamma^* \mu) \neq 0 \in \widehat{Q}(\ker \gamma^* \mu)$ then $[z] = \eta'(0, o_s, o_t, e_x, o_z, y) \in B_2$.
- vi) If $\bar{\theta}|(\ker \gamma^* \mu) = 0 \in \widehat{Q}(\ker \gamma^* \mu)$ then $(G/\ker \gamma^* \mu, \bar{\theta})$ is a well-defined non-singular quadratic form. If its Arf-invariant vanishes then $z = \eta'(0, o_s, o_t, o_x, o_z, y) \in B_0$ else $[z] = \eta'(1, o_s, o_t, o_x, o_z, y) \in B_1$.

$[z] \in l_{2q+2}(\mathbb{Z}/2\mathbb{Z})$ is elementary if and only if $y = 0$ and $[z] \in B_0 \cup B_5 \cup B_6$

Proof. By the proof of Proposition 9.1.11 there is an (unstable) strong isomorphism (α, β) of z with $\eta((K, \theta_K), (S, \theta_S), (T, \theta_T), (X, \theta_X), (Z, \theta_Z), y)$. It is easy to check (compare with (9.1)) that β induces isomorphisms of the quadratic forms

$$\begin{aligned}
(\ker(\gamma_\mu), \theta|(\ker(\gamma_\mu))) &\cong (Z, \theta_Z) \\
(\ker \gamma, \theta|(\ker \gamma)) &\cong (Z \oplus T, \theta_Z \oplus \theta_T) \\
(\ker \mu, \theta|(\ker \mu)) &\cong (Z \oplus S, \theta_Z \oplus \theta_S) \\
(\ker \gamma^* \mu, \theta|(\ker \gamma^* \mu)) &\cong (Z \oplus S \oplus T \oplus X, \theta_Z \oplus \theta_S \oplus \theta_T \oplus \theta_X)
\end{aligned}$$

Using the equivalence relations laid down in Proposition 9.3.2 proves the claims. \square

9.4 Application to Preformations over \mathbb{Z}

Let Λ be a field. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be a regular ϵ -quadratic preformation over \mathbb{Z} . Tensoring with Λ yields a preformation $z \otimes \Lambda$ over Λ . If z is elementary obviously also $z \otimes \Lambda$ is elementary. Therefore it makes sense to wonder whether additional obstructions to the (stable) elementariness of z can be found by applying the results of the previous section to $z \otimes \Lambda$.

Let $P \subset \mathbb{N}_0$ be the set of all primes. Define for any prime $p \in P$

$$\begin{aligned}
\phi_p: l^{2q+2}(\mathbb{Z}) &\longrightarrow l^{2q+2}(\mathbb{Z}/p\mathbb{Z}), & z &\longmapsto z \otimes \mathbb{Z}/p\mathbb{Z} \\
\phi_0: l^{2q+2}(\mathbb{Z}) &\longrightarrow l^{2q+2}(\mathbb{Q}), & z &\longmapsto z \otimes \mathbb{Q}
\end{aligned}$$

Any map $l^{2q+2}(\mathbb{Z}) \longrightarrow l^{2q+2}(\Lambda)$, $z \longmapsto z \otimes \Lambda$ factors through $\phi_{\text{char } \Lambda}$ because \mathbb{Q} and $\mathbb{Z}/p\mathbb{Z}$ are the prime fields and isomorphic to the images of the map $\mathbb{Z} \longrightarrow \Lambda$. Hence we can concentrate on those fields.

Note that $l_{2q+2}(\mathbb{Q}) = l^{2q+2}(\mathbb{Q})$ and $l_{2q+2}(\mathbb{Z}/p\mathbb{Z}) = l^{2q+2}(\mathbb{Z}/p\mathbb{Z})$ for $p \neq 2$. In order to transfer the following results to $l_{2q+2}(\mathbb{Z})$ the reader simply needs to replace $l^{2q+2}(\mathbb{Z}/2\mathbb{Z}) \cong L^{2q+2}(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{N}_0^5$ (see Remark 9.1.13) by the map for the more cumbersome $l_{2q+2}(\mathbb{Z}/2\mathbb{Z})$ in Section 9.3.

It is well-known that in L -theory there is an exact sequence

$$0 \longrightarrow L^0(\mathbb{Z}) \longrightarrow L^0(\mathbb{Q}) \longrightarrow \bigoplus_p L^0(\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

The following observation shows that a similar result cannot be expected from the l -monoids.

Proposition 9.4.1. *For all $p \in P \cup \{0\}$*

- i) ϕ_p is surjective unless $p = 2$ and q even,*
- ii) ϕ_p is not injective,*
- iii) there is a non-elementary $z \in l^{2q+2}(\mathbb{Z})$ such that $\phi_p(z)$ is elementary.*

We need some important facts about maps and skew-symmetric forms over \mathbb{Z} .

Theorem 9.4.2 (Smith Normal Form). *Let $A \in \text{Mat}(n \times m, \mathbb{Z})$ with $r = \text{rk } A$. Then there are $g \in \text{GL}_n(\mathbb{Z})$ and $h \in \text{GL}_m(\mathbb{Z})$ such that $gAh = \text{diag}(d_1, \dots, d_r, 0, \dots, 0)$ with $d_i | d_{i+1}$ for all $1 \leq i < r$ and all $d_i > 0$.*

Theorem 9.4.3 (Skew Smith Normal Form ([New72] pp. 57)). *Let $A \in \text{Mat}(n \times n, \mathbb{Z})$ be skew-symmetric with $r = \text{rk } A$. Then r is even and there is a $g \in \text{GL}_n(\mathbb{Z})$ and a diagonal matrix $D = \text{diag}(d_1, \dots, d_k)$ with $d_i | d_{i+1}$ for all $1 \leq i < k$ and all $d_i > 0$ such that $r = 2k$ and $g^*Ag = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.*

Proof of Proposition 9.4.1. i) Let $p = 0$. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be an ϵ -quadratic preformation over \mathbb{Q} . There is an $N \in \mathbb{Z}$ such that $\gamma' = N\gamma$ and $\mu' = N\mu$ are matrices over the integers. Let F', G' be free modules over \mathbb{Z} of the same rank as F, G respectively. Then there is a strong isomorphism $(1, \frac{1}{N})$ between z and $(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F') \otimes \mathbb{Q}$. Hence ϕ_0 is surjective.

Let $H = \begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix} \in \text{Mat}(2k \times 2k, \mathbb{Z})$ and $\gamma \in \text{Mat}(2k \times 2k, \mathbb{Z})$ of full rank. Define $\mu = H\gamma$. Then $z_\gamma = (\mathbb{Z}^{2k} \xleftarrow{\gamma} \mathbb{Z}^{2k} \xrightarrow{\mu} \mathbb{Z}^{2k})$ is an ϵ -quadratic preformation over \mathbb{Z} . There is a strong isomorphism $(1, \gamma): z_\gamma \otimes \mathbb{Q} \longrightarrow \partial(H^{-\epsilon}(\mathbb{Q}^k))$ over \mathbb{Q} . Hence $\phi_0(z_\gamma) = 0 \in l^{2q+2}(\mathbb{Q})$ for all choices of γ . However $z_\gamma \neq z_{\gamma'} \in l^{2q+2}(\mathbb{Z})$ if e.g. $\text{coker } \gamma \neq \text{coker } \gamma'$ (See Corollary 2.4.3). So ϕ_0 cannot be injective. If e.g. $\gamma = 2$ then z_γ cannot be elementary (because γj is not a split injection for any map $j: U \longrightarrow \mathbb{Z}^{2k}$) but $\phi_0(z_\gamma)$ is.

ii) Let $p \in P$ be a prime. Let z be an ϵ -quadratic preformation over $\mathbb{Z}/p\mathbb{Z}$. By Corollary 9.1.9 and Remark 9.1.13, z is strongly isomorphic to $\eta((K, \lambda), s, t, x, y, z)$ for some $s, t, x, y, z \in \mathbb{N}_0$ and $(K, \lambda) \in L^{2q+2}(\mathbb{Z}/p\mathbb{Z})$. If q is odd $L^{2q+2}(\mathbb{Z}/p\mathbb{Z})$ is generated by elements $(\mathbb{Z}/p\mathbb{Z}, [n])$ for $n \in \{1, \dots, p-1\}$. Tensoring the symmetric form (\mathbb{Z}, n) over \mathbb{Z} with $\mathbb{Z}/p\mathbb{Z}$ obviously gives $(\mathbb{Z}/p\mathbb{Z}, [n])$. Hence there is a $(-\epsilon)$ -symmetric form (K', λ') over \mathbb{Z} such that $(K', \lambda') \otimes \mathbb{Z}/p\mathbb{Z} = (K, \lambda)$. If q is even and $p \neq 2$, $L^{2q+2}(\mathbb{Z}/p\mathbb{Z}) = 0$ and we choose $(K', \lambda') = 0$. Obviously the image of the quadratic preformation

$$\eta((K', \lambda'), s, t, x, y, z)$$

under ϕ_p is $z \in l^{2q+2}(\mathbb{Z}/p\mathbb{Z})$. Hence ϕ_p is surjective.

If q is even and $p = 2$ then $[\partial(\mathbb{Z}/2\mathbb{Z}, 1)] \in l^{2q+2}(\mathbb{Z}/2\mathbb{Z})$ cannot be in the image of ϕ_2 .

$z_1 = \partial(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \partial H^{-\epsilon}(\mathbb{Z})$ and $z_2 = \partial(\mathbb{Z}, \begin{pmatrix} 0 & 1+p \\ 0 & 0 \end{pmatrix})$ are different in $l^{2q+2}(\mathbb{Z})$ but $\phi_p(z_1) = \phi_p(z_2) = 0 \in l^{2q+2}(\mathbb{Z}/p\mathbb{Z})$. So ϕ_p cannot be injective. Note that z_2 is not elementary but $\phi_p(z_2)$ is.

□

Now we analyze the combination of all ϕ_p .

Proposition 9.4.4. *Define*

$$\phi = \phi_0 \oplus \prod_{p \in P} \phi_p: l^{2q+2}(\mathbb{Z}) \longrightarrow l^{2q+2}(\mathbb{Q}) \oplus \prod_{p \in P} l^{2q+2}(\mathbb{Z}/p\mathbb{Z})$$

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ and $z' = (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F')$ be ϵ -quadratic preformation over \mathbb{Z} . Assume that

$$\ker \gamma \cong \ker \gamma'$$

$$\operatorname{coker} \gamma \cong \operatorname{coker} \gamma'$$

$$\ker \mu \cong \ker \mu'$$

$$\operatorname{coker} \mu \cong \operatorname{coker} \mu'$$

$$\ker \gamma^* \mu \cong \ker \gamma'^* \mu'$$

$$\operatorname{coker} \gamma^* \mu \cong \operatorname{coker} \gamma'^* \mu'$$

$$\ker \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \cong \ker \begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}$$

$$\operatorname{coker} \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \cong \operatorname{coker} \begin{pmatrix} \gamma' \\ \mu' \end{pmatrix}$$

$$[(G / \ker \gamma^* \mu, \gamma^* \mu)] = [(G' / \ker \gamma'^* \mu', \gamma'^* \mu')] \in L^{2q+2}(\mathbb{Q})$$

$$[(G \otimes \mathbb{Z}/p\mathbb{Z} / \ker(\gamma^* \mu \otimes \mathbb{Z}/p\mathbb{Z}), \gamma^* \mu \otimes \mathbb{Z}/p\mathbb{Z})] =$$

$$[(G' \otimes \mathbb{Z}/p\mathbb{Z} / \ker(\gamma'^* \mu' \otimes \mathbb{Z}/p\mathbb{Z}), \gamma'^* \mu' \otimes \mathbb{Z}/p\mathbb{Z})] \in L^{2q+2}(\mathbb{Z}/p\mathbb{Z}) \quad (p \in P)$$

Then $\phi(z) = \phi(z')$.

Assume that

$$\begin{aligned} \ker \gamma &\cong \ker \mu \\ \operatorname{coker} \gamma &\cong \operatorname{coker} \mu \\ \ker \gamma^* \mu &\cong \ker \gamma \oplus \ker \gamma^* \\ \operatorname{coker} \gamma^* \mu &\cong \operatorname{coker} \gamma \oplus \operatorname{coker} \gamma^* \\ [(G / \ker \gamma^* \mu, \gamma^* \mu)] &= 0 \in L^{2q+2}(\mathbb{Q}) \\ [(G \otimes \mathbb{Z}/p\mathbb{Z} / \ker(\gamma^* \mu \otimes \mathbb{Z}/p\mathbb{Z}), \gamma^* \mu \otimes \mathbb{Z}/p\mathbb{Z})] &= 0 \in L^{2q+2}(\mathbb{Z}/p\mathbb{Z}) \quad (p \in P) \end{aligned}$$

Then $\phi_p(z)$ is elementary for all $p \in P$.

For $\epsilon = 1$ one can ignore all the conditions about the L -groups since they all vanish. The proof of the Proposition follows from Remark 9.1.13, Corollary 9.2.2 and the following observation.

Lemma 9.4.5. *Let $f: M \rightarrow N$ be a homomorphism of free \mathbb{Z} -modules. Then*

$$\begin{aligned} \operatorname{rk}(\operatorname{coker}(f \otimes \Lambda)) &= \operatorname{rk}(\operatorname{coker}(f) \otimes \Lambda) \\ \operatorname{rk}(\ker(f \otimes \Lambda)) &= \operatorname{rk}(\ker(f)) + \operatorname{rk}(\operatorname{Tor}(\operatorname{coker}(f), \Lambda)) \end{aligned}$$

Proof. Apply the Universal Coefficient Theorem to the “complex” $f: M \rightarrow N$. \square

9.5 The Cancellation Rule and $l_{2q+2}(\Lambda)$

On the algebraic side, one obvious disadvantage of Kreck’s approach is that the obstructions do not live in a group but in a monoid. One can easily see that $l_{2q+2}(\Lambda)$ is not a group. The set of all its invertible elements is the image of

$$L_{2q+2}(\Lambda) \hookrightarrow l_{2q+2}(\Lambda), \quad (K, \theta) \mapsto \partial(K, \theta)$$

One can still wonder whether $l_{2q+2}(\Lambda)$ can at least be embedded into a group. A necessary and sufficient criteria for the existence of such an embedding is the validity of the cancellation rule.

Definition 9.5.1. A monoid M fulfils the **cancellation rule** if for all $x, y, z \in M$ with $x + z = y + z$ follows that $x = y$. \square

Proposition 9.1.9 shows that $l_{2q+2}(\Lambda)$ is embeddable into a group if Λ is a skew field of characteristic not equal 2. In a lot of other cases we are less fortunate.

Proposition 9.5.2. *i) If Λ is a weakly finite ring with involution such that $x + \bar{x} \neq 1$ for all $x \in \Lambda$ then $l_{4m+2}(\Lambda)$ does not fulfil the cancellation rule.*

ii) Let $\mathbb{Z}[\pi]$ a group ring as in Example 8.1.3. $l_{4m+2}(\mathbb{Z}[\pi])$ does not fulfil the cancellation rule.

iii) For any skew field Λ with $\text{char } \Lambda = 2$ and identity involution $l_{2q+2}(\Lambda)$ does not fulfil the cancellation rule.

Proof. Let $\epsilon = (-)^q$. One can easily prove from scratch (or using Proposition 9.1.8) that the map

$$\begin{aligned} \widehat{M}^\epsilon(\Lambda) &\longrightarrow l_{2q+2}(\Lambda) \\ (S, \theta) &\longmapsto \partial(S, \theta) = (S \xleftarrow{1} S \xrightarrow{0} S^*, \theta) \end{aligned}$$

is an injection. We will show that $\widehat{M}^1(\Lambda)$ does not fulfil the cancellation rule for the rings described above.

i) Assume that there is no $x \in \Lambda$ such that $x + \bar{x} = 1$. Define $a = (\Lambda, 1), b = (\Lambda, 0) \in \widehat{Q}^1(\Lambda)$. Then

$$\begin{aligned} a + b &= (\Lambda^2, \theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \in \widehat{Q}^1(\Lambda^2) \\ a + a &= (\Lambda^2, \theta' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \in \widehat{Q}^1(\Lambda^2) \\ f &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL_2(\Lambda) \end{aligned}$$

Then $f^* \theta' f = \theta \in \widehat{Q}^1(\Lambda^2)$ and hence $a + b = a + a \in \widehat{M}^1(\Lambda)$. Now we have to show that $b \neq a \in \widehat{M}^1(\Lambda)$. Assume $b = a \in \widehat{M}^1(\Lambda)$ i.e. assume there is a $g \in \Lambda^\times = GL_1(\Lambda)$ such that $g^* 1 g = 0 \in \widehat{Q}^1(\Lambda)$. Then there is a $y \in \Lambda$ such that $\bar{g} g = y + \bar{y}$. Then $x = \bar{g}^{-1} y g^{-1}$ fulfils $x + \bar{x} = 1$ which is a contradiction.

ii) Assume that $x = \sum_g n_g \cdot 1_g \in \mathbb{Z}[\pi]$ is such that $x + \bar{x} = 1$. Then $2n_1 = n_1 + w(1)n_1 = 1 \in \mathbb{Z}$ which is not possible. Hence $x + \bar{x} = 1$ for no $x \in \Lambda$. Hence $l_{4m+2}(\mathbb{Z}[\pi])$ does not fulfil the cancellation rule.

iii) If Λ is a skew field of characteristic 2 with identity involution then $\bar{x} + x = 0$ for all $x \in \Lambda$.

□

9.6 Asymmetric Signatures over Fields

Let Λ be a field with involution. By Remark 9.1.13 there is an isomorphism

$$\eta: L^{2q+2}(\Lambda) \oplus \mathbb{N}_0^5 \xrightarrow{\cong} l^{2q+2}(\Lambda)$$

. Using this computation, we will investigate whether it is enough for a preformation to have flip-isomorphisms and vanishing asymmetric forms in order to be elementary. The proofs will use properties of the asymmetric Witt-groups of fields. The reader is referred to [Ran98] Chapter 39D for any details.

Proposition 9.6.1. *Let $w = \eta((K, \lambda), s, t, x, y, z)$ be an ϵ -quadratic preformation. Then w allows (stable) weak flip-isomorphisms if and only if $s = t$. w has stable strong flip-isomorphisms if also $2 \cdot (K, \lambda) = 0 \in L^{2q+2}(\Lambda)$.*

Proof. Let $w = \eta((K, \lambda), s, s, x, y, z)$. The proof of Proposition 9.1.11 reveals that even without stabilization we can split w up into three ϵ -quadratic preformations $w = w_1 \oplus w_2 \oplus w_3$ such that $w_1 = \eta((K, \lambda), 0, 0, 0, 0, 0)$, $w_2 = \eta(0, s, s, x, 0, z)$ and $w_3 = \eta(0, 0, 0, 0, y, 0)$. w_2 is elementary and therefore has a strong flip-isomorphism by Corollary 4.1.3. Any isomorphism $f: Y \xrightarrow{\cong} Y^*$ yields a strong flip-isomorphism $(f, 0)$ of the ϵ -quadratic preformation $(Y \xleftarrow{0} 0 \xrightarrow{0} Y)$. Hence w_3 has a strong flip-isomorphism. For any $(-\epsilon)$ -symmetric form (K, λ) , there is a flip-isomorphism $(\lambda^*, 1_K, -2\lambda^{-1})$ of $\partial(K, \lambda)$.

On the other hand if w has a flip isomorphism then obviously $\text{rk } \gamma = \text{rk } \mu$. Using (9.1) one easily deduces $s = t$.

Now we consider the case of strong stable flip-isomorphisms. The flip of w is an strongly isomorphic to $\eta((K, -\lambda), t, s, x, y, z)$. Hence w has - after some stabilization - a strong flip-isomorphism if and only if $s = t$ and $2 \cdot (K, \lambda) = 0 \in L^{2q+2}(\Lambda)$. \square

Definition 9.6.2. A ϵ -quadratic preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is nice if it is either stably elementary or if it has a stable flip-isomorphisms for which the asymmetric signature is not zero.

A field with involution is nice if all its ϵ -quadratic preformations are nice i.e. if the converse of Theorem 5.5.3 is true. \square

Proposition 9.6.3. *A field Λ is nice if*

i) *for all $(K, \lambda) \in L^{2q+2}(\Lambda)$ there is an $a \in \Lambda^\times$ and $s \in \Lambda$ such that $\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & -\epsilon \\ 0 & 1 & s \end{pmatrix} \neq (K, \lambda) \in LAsy^0(\Lambda)$,*

ii) *there is an $\tilde{\alpha} \in GL_2(\Lambda)$ and $\tilde{\sigma} \in Mat(2 \times 2, \Lambda)$ such that $\begin{pmatrix} 0 & 0 & \tilde{\alpha} \\ 1 & 0 & -\epsilon \\ 0 & 1 & \tilde{\sigma} \end{pmatrix} \neq 0 \in LAsy^0(\Lambda)$.*

If the second criterion is violated then Λ is not nice.

Proof. Let Λ be a field with involution for which the two criteria are fulfilled. Let $w = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be an ϵ -quadratic preformation for which there exists (stable) flip-isomorphisms and for which all asymmetric signatures vanish. Without loss of generality we can assume that $w = w_1 + w_2 + w_3$ with $w_1 \in \eta((K, \lambda), 0, 0, 0, 0, 0)$,

$w_2 \in \eta(0, s, s, x, 0, z)$ and $w_3 \in \eta(0, 0, 0, 0, y, 0)$ (compare with the proofs of Propositions 9.6.1 and 9.1.11). We need to show that $y = 0$ and $[(K, \lambda)] = 0 \in L^{2q+2}(\Lambda)$.

$f_1 = (\lambda^*, 1, -2\lambda^{-1})$ is a flip-isomorphism of w_1 . Its asymmetric signature $\sigma^*(w_1, f_1) = (K, \lambda) \in LAsy^0(\Lambda)$ because $\rho \oplus -\lambda$ has the lagrangian

$$\begin{pmatrix} \epsilon & 1 \\ 0 & -\epsilon\lambda \\ 1 & 0 \\ -\epsilon & 1 \end{pmatrix}$$

with ρ as in Definition 5.3.2.

w_2 is elementary and hence has a flip-isomorphism f_2 for which $\sigma^*(w_2, f_2) = 0 \in LAsy^0(\Lambda)$.

Assume that $y = 2m + 1$. Let $a \in \Lambda^\times$ and $s \in \Lambda$ such that $\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & -\epsilon \\ 0 & 1 & s \end{pmatrix} \neq (K, -\lambda) \in LAsy^0(\Lambda)$. Define a flip-isomorphism $f_3 = \left(\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, 0, \begin{pmatrix} \epsilon a^{-1} s \bar{a}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$ for w_3 . Then $\sigma^*(w, f_1 + f_2 + f_3) \neq 0$ which is a contradiction.

So we assume $y = 2m$. Define a new flip-isomorphism $f'_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right)$. Then $0 = \sigma^*(w, f_1 + f_2 + f'_3) = (K, \lambda) \in LAsy^0(\Lambda)$. The natural map $L^{2q+2}(\Lambda) \rightarrow LAsy^0(\Lambda)$ is injective for fields, so $(K, \lambda) = 0 \in L^{2q+2}(\Lambda)$. Assume that $m > 0$. Then $f''_3 = \left(\begin{pmatrix} \bar{\alpha} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, 0, \begin{pmatrix} \epsilon \bar{\alpha}^{-1} \bar{\sigma} \bar{\alpha}^{-*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$ is another flip-isomorphism of w_3 such that $\sigma^*(w, f_1 + f_2 + f''_3) \neq 0$. This is again a contradiction, hence $m = 0$.

If the last criterion is not true then $(\Lambda^2 \xleftarrow{0} 0 \xrightarrow{0} \Lambda^2)$ is not nice. \square

Corollary 9.6.4. *Every non-singular ϵ -quadratic formation over a field is nice.*

Proof. Similar to the proof of the preceding Proposition using the fact that $[z] \in \eta(L^{2q}(\Lambda), \mathbb{N}_0, \mathbb{N}_0, \mathbb{N}_0, 0, 0)$. \square

Example 9.6.5. Fields with $LAsy^0(\Lambda) = \mathbb{Z}/2\mathbb{Z}$ are not nice. Let $\overline{\mathcal{M}}(\Lambda)$ be defined as in [Ran98] 39.19.

- i) Let \mathbb{C}^+ be the complex field with the identity involution. Then $\overline{\mathcal{M}}(\Lambda) = \{z \pm 1\}$.
By [Ran98] 39.27

$$LAsy^0(\mathbb{C}^+) = L^0(\mathbb{C}^+, 1) \oplus L^0(\mathbb{C}^+, -1) = L^0(\mathbb{C}^+, 1) = \mathbb{Z}/2\mathbb{Z}$$

- ii) Let Λ be an algebraically closed field of characteristic 2 with the identity involution. Then $\overline{\mathcal{M}}(\Lambda) = \{z - 1\}$ and $LAsy^0(\Lambda) = L^0(\Lambda) = \mathbb{Z}/2\mathbb{Z}$ (Every symmetric form over Λ can be diagonalized. Because Λ is algebraically closed the diagonal elements can be chosen to be 1. Obviously such forms are metabolic if and only if the rank is even).

But there are also positive results:

Lemma 9.6.6. *If $\epsilon = 1$ and $\text{char } \Lambda \neq 2$ then the first condition of Proposition 9.6.3 is fulfilled.*

Proof. Let $q = 2m$. Let $(K, \lambda) \in L^{2q+2}(\Lambda) = L^{4m+2}(\Lambda)$. Then by [Ran98] Proposition 39.27 $L^{4m+2}(\Lambda) = L^0(\Lambda, -1)$ and $L^{4m}(\Lambda) = L^0(\Lambda, 1)$ are both subgroups of $L\text{Asy}^0(\Lambda)$ whose intersection is trivial.

By Proposition 5.6.4, $\Phi_1(\Lambda, 1) = (\Lambda, 1) \in L^{4m}(\Lambda)$. It cannot be the image of any $(-\epsilon)$ -symmetric form in $L\text{Asy}^0(\Lambda)$. \square

Proposition 9.6.7. *Let $\epsilon = 1$. Every field Λ with $\text{char } \Lambda \neq 2$ and an involution that is not the identity is nice.*

Proof. We have to show that the second condition of Proposition 9.6.3 is fulfilled. Let $b \in \Lambda$ such that $b \neq \bar{b}$. Let $a = \frac{\bar{b}}{b} \neq 1$. By Proposition 5.6.4, $\Phi_1(\Lambda, 1) = (\Lambda, 1) \in L^{4m+2}(\Lambda) = L^0(\Lambda, 1) \subset L\text{Asy}^0(\Lambda)$ and $\Phi_1(\Lambda, b) = (\Lambda, \bar{b} - b) + x \in L^0(\Lambda, -1) \oplus L\text{Asy}_{(z^2 - a)^\infty}^0(\Lambda) \subset L\text{Asy}^0(\Lambda)$ (see also [Ran98] Chapter 39D). For rank reasons $\Phi_1(\Lambda, 1)$ and $\Phi_1(\Lambda, b)$ are not zero in $L\text{Asy}^0(\Lambda)$ and because the intersection of $L^{4m+2}(\Lambda) = L^0(\Lambda, -1)$ and $L^{4m}(\Lambda) = L^0(\Lambda, 1)$ in $L\text{Asy}^0(\Lambda)$ is zero, $\Phi_1\left(\Lambda^2, \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\right) \neq 0 \in L\text{Asy}^0(\Lambda)$. \square

Proposition 9.6.8. *\mathbb{Q} is nice.*

Proof. Let $\epsilon = 1$. $\Phi_1(\mathbb{Q}^2, 1) = (\mathbb{Q}^2, 1) \neq 0 \in L^{4m}(\mathbb{Q}) \subset L\text{Asy}^0(\mathbb{Q})$ by Proposition 5.6.4. By Lemma 9.6.6 and Proposition 9.6.3, \mathbb{Q} is nice.

Now let $\epsilon = -1$. By Proposition 5.6.4 and [Ran98] Chapter 39D,

$$\begin{aligned} \Phi_{-1}(\mathbb{Q}^n, a) &= (\mathbb{Q}^n, 2a) + \left(\mathbb{Q}^n \oplus \mathbb{Q}^n, \begin{pmatrix} -\frac{1}{a} & -1 \\ 1 & -a \end{pmatrix} \right) \in L^0(\Lambda, 1) \oplus L\text{Asy}_{(z^{2n+1})^\infty}^0(\Lambda) \\ &\subset L\text{Asy}^0(\Lambda) \end{aligned}$$

for any $a \in \mathbb{Q} \setminus \{0\}$. Hence e.g. $\Phi_{-1}(\mathbb{Q}^2, 1) \neq 0 \in L\text{Asy}^0(\Lambda)$.

Let $(K, \lambda) \in L^{4m}(\Lambda) = L^0(\Lambda, 1)$. Let $a = 1$ if $\text{sgn}(K, \lambda) > 0$ and $a = -1$ else. Using the signature it is obvious that $(K, \lambda) + \Phi_{-1}(\mathbb{Q}, a) \neq 0 \in L\text{Asy}^0(\Lambda)$ because their projection onto $L^0(\Lambda, 1)$ is not vanishing. Again Proposition 9.6.3 finishes off the proof. \square

Appendix A

A Crash Course in Algebraic Surgery Theory

This chapter is a compilation of the main theorems and constructions of algebraic surgery theory, taken from [Ran80a] or [Ran81]. **Section A.1** repeats basic definition of algebraic topology and defines quadratic and symmetric complexes. In **Section A.2** we define quadratic and symmetric pairs, boundaries, Thom-complexes and thickenings. The union of Poincaré pairs are treated in **Section A.3**. The last **Section A.4** discusses surgery on complexes.

Throughout this chapter let Λ be a weakly finite ring with 1 and an involution and let $\epsilon \in \Lambda$ such that $\bar{\epsilon} = \epsilon^{-1}$ (e.g. $\epsilon = \pm 1$).

A.1 Quadratic and Symmetric Complexes

Let X be a finite CW complex with universal cover \tilde{X} . Let $C = C(\tilde{X})$ be its chain complex of f.g. finite $\mathbb{Z}[\pi_1(X)]$ -modules. A generalization of the diagonal chain map that is used to define the Steenrod squares yields a natural transformation

$$\Delta: C \longrightarrow W^{\%}(C) = \{\phi_s: C^{n-r+s} \longrightarrow C_r \mid r \in \mathbb{Z}, s \geq 0\}$$

For any homology class $x \in C_n(X)$ we can define an n -dimensional symmetric Poincaré complex $(C, \phi = \Delta(x) \in W^{\%}(C)_n)$ over $\mathbb{Z}[\pi_1(X)]$ such that $\phi_0 = x \cap -: C^{n-*} \longrightarrow C$. In particular if (and only if) X is an n -dimensional geometric Poincaré space with fundamental class $x = [X]$ this construction gives an n -dimensional symmetric Poincaré complex.

There is a similar construction, the **quadratic kernel**, for a degree 1 normal map $f: M \longrightarrow X$ between a closed n -dimensional manifold M and an n -dimensional finite geometric Poincaré space. It is an n -dimensional quadratic Poincaré complex (C, ψ) with C the mapping cone of the **Umkehr chain map**

$$f^!: C(\tilde{X}) \xrightarrow{\simeq} C(\tilde{X})^{n-*} \xrightarrow{f^*} C(\tilde{M})^{n-*} \xrightarrow{\simeq} C(\tilde{M})$$

with \tilde{X} and \tilde{M} the universal covers. The homology of C are the kernel modules $K_*(M)$ of f . The chain map

$$(1 + T)\psi_0: C^{n-*} \xrightarrow{\cong} C$$

induces the Poincaré duality on the kernel modules. One can relate symmetric construction and quadratic kernel. In the category of symmetric Poincaré complexes

$$(C(\tilde{M}), \Delta[M]) \simeq (C, (1 + T)\psi) \oplus (C(\tilde{X}), \Delta[X])$$

See [Ran80b] for details.

Both constructions can be extended to geometric Poincaré pairs (e.g. manifolds with boundary) and relative normal maps $f: (M, \partial M) \rightarrow (X, \partial X)$ from manifolds with boundary to geometric Poincaré pairs. The result will be a symmetric or quadratic Poincaré pair. If the boundary is contractible or $f|: \partial M \rightarrow \partial X$ is a homotopy equivalence this Poincaré pair will be again a Poincaré complex.

Algebraic surgery theory was originally invented to give an alternative way of finding the obstruction to a Wall surgery problem. Let $(e, f, f'): (W, M, M') \rightarrow X \times (I, 1, 0)$ be an n -dimensional normal cobordism such that f and f' are homotopy equivalences and $n > 5$. Wall asked in [Wal99] whether it is possible to write down a surgery obstruction without making e highly-connected beforehand.

The answer is the quadratic kernel (C, ψ) of $e: W \rightarrow X$. It has the following remarkable properties: If one performs surgery on the cobordism, the quadratic kernel of the result will change from the old kernel (C, ψ) by an **algebraic surgery** for which explicit formulae exist (see Definition A.4.1). If $e: W \rightarrow X \times I$ is cobordant rel ∂ to a homotopy equivalence then the relative quadratic kernel of that cobordism will be an $(n + 1)$ -dimensional Poincaré pair $(f: C \rightarrow D, (\delta\psi, \psi))$. Its boundary is the quadratic kernel (C, ψ) . Hence (C, ψ) is **algebraically null-cobordant**.

It is not surprising to suspect that the converse is true as well: $e: W \rightarrow X \times I$ is cobordant rel ∂ to a homotopy equivalence if and only if (C, ψ) is algebraically null-cobordant. In fact this is true and even more. Algebraic cobordism of Poincaré pairs is an equivalence relation and its cobordism classes form a groups which are isomorphic to Wall's L -groups. As in geometry, there is a close relationship between cobordism and surgeries: every cobordism is realized as a trace of surgeries (Proposition A.4.3). Using the instant surgery obstruction it is very easy to read off the conventional obstruction form and formation from (C, ψ) (Lemma A.4.6 is a special case of it). For further details see [Ran80a] and [Ran80b].

Definition A.1.1. A **chain complex C (over Λ)** is a collection of homomorphisms of f.g. free Λ -modules $\{d_r: C_r \rightarrow C_{r-1} | r \in \mathbb{Z}\}$ such that $d_r d_{r+1} = 0: C_{r+1} \rightarrow C_{r-1}$ for all r . C is called n -dimensional if $C_r = 0$ for $r < 0$ and $r > n$.

Its homology Λ -modules $H_*(C)$ are defined by

$$H_r(C) = \ker(d: C_r \longrightarrow C_{r-1}) / \text{im}(d: C_{r+1} \longrightarrow C_r).$$

Its cohomology Λ -modules $H^*(C)$ are defined by

$$H^r(C) = \ker(d^*: C^r \longrightarrow C^{r+1}) / \text{im}(d^*: C^{r-1} \longrightarrow C^r).$$

A **chain map** $f: C \longrightarrow D$ of chain complexes over Λ is a collection of Λ -module morphisms $\{f_r: C_r \longrightarrow D_r | r \in \mathbb{Z}\}$ such that $d_D f_r = f_{r-1} d_C: C_r \longrightarrow D_{r-1}$ for all r .

A **chain homotopy** $g: f \simeq f': C \longrightarrow D$ of two chain maps f and f' is a collection of Λ -module morphisms $\{g_r: C_{r-1} \longrightarrow D_r | r \in \mathbb{Z}\}$ such that $f'_r - f_r = d_D g_{r+1} + g_r d_C: C_r \longrightarrow D_r$.

A chain map is a **chain equivalence** if it has a chain homotopy inverse. It is an **isomorphism** if it consists of isomorphism of modules only.

The **mapping cone** $\mathcal{C}(f)$ of a chain map $f: C \longrightarrow D$ is the chain complex given by

$$d_{\mathcal{C}} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}$$

□

Definition A.1.2. Let C be a chain complex. The ϵ -**duality involution** T_ϵ is defined by

$$\begin{aligned} T_\epsilon: \text{Hom}_\Lambda(C^p, C^q) &\longrightarrow \text{Hom}_\Lambda(C^q, C^p) \\ \psi &\longmapsto (-)^{pq} \epsilon \psi^* \end{aligned}$$

We define new chain complexes $W^\% (C, \epsilon)$ and $W_\% (C, \epsilon)$ by

$$\begin{aligned} W^\% (C, \epsilon)_n &= \{\phi_s: C^{n-r+s} \longrightarrow C_r | r \in \mathbb{Z}, s \geq 0\} \\ d^\%: W^\% (C, \epsilon)_n &\longrightarrow W^\% (C, \epsilon)_{n-1} \\ \{\phi_s\} &\longmapsto \{d\phi_s + (-)^r \phi_s d^* + (-)^{n+s-1} (\phi_{s-1} + (-)^s T_\epsilon \phi_{s-1}) : \\ &\quad C^{n-r+s-1} \longrightarrow C_r | r \in \mathbb{Z}, s \geq 0\} \\ &\quad \text{where we set } \phi_{-1} = 0. \\ W_\% (C, \epsilon)_n &= \{\psi_s: C^{n-r-s} \longrightarrow C_r | r \in \mathbb{Z}, s \geq 0\} \\ d_\%: W_\% (C, \epsilon)_n &\longrightarrow W_\% (C, \epsilon)_{n-1} \\ \{\psi_s\} &\longmapsto \{d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1} (\psi_{s+1} + (-)^{s+1} T_\epsilon \psi_{s+1}) : \\ &\quad C^{n-r-s-1} \longrightarrow C_r | r \in \mathbb{Z}, s \geq 0\} \end{aligned}$$

Their homology groups are the ϵ -**symmetric Q -groups** $Q^n(C, \epsilon) = H_n(W^\% (C, \epsilon))$ and the ϵ -**quadratic Q -groups** $Q_n(C, \epsilon) = H_n(W_\% (C, \epsilon))$. They are related by the

ϵ -symmetrization map

$$\begin{aligned} Q_n(C, \epsilon) &\longrightarrow Q^n(C, \epsilon) \\ \{\psi_s\} &\longmapsto \begin{cases} \{(1 + T_\epsilon)\psi_0\} & : \text{if } s = 0, \\ 0 & : \text{if } s \neq 0. \end{cases} \end{aligned}$$

□

Remark A.1.3. In the case of $\epsilon = 1$ we omit the ϵ and we will simply speak of symmetric complexes, $Q_n(C)$, $W^\%_n(C)$, T etc.

Definition A.1.4. Let C be a chain complex and $n \in \mathbb{N}$. Define the chain complex C^{n-*} by

$$d_{C^{n-*}} = (-)^r d_C^* : (C^{n-*})_r = C^{n-r} = C_{n-r}^* \longrightarrow (C^{n-*})_{r-1}.$$

An ϵ -symmetric n -dimensional complex (C, ϕ) is a chain complex C together with a cycle $\phi \in W^\%(C, \epsilon)_n$. It is called **Poincaré** if the **Poincaré duality map**

$$\phi_0 : C^{n-*} \longrightarrow C$$

is a chain equivalence.

An ϵ -quadratic n -dimensional complex (C, ψ) is a chain complex C together with a cycle $\psi \in W_\%(C, \epsilon)_n$. It is called **Poincaré** if the **Poincaré duality map**

$$(1 + T_\epsilon)\psi_0 : C^{n-*} \longrightarrow C$$

is a chain equivalence.

A **morphism of ϵ -symmetric n -dimensional complexes** $f = (f, \rho) : (C, \phi) \longrightarrow (C', \phi')$ is a chain map $f : C \longrightarrow C'$ and a $\rho \in W^\%(C', \epsilon)_{n+1}$ such that $\phi' - f\phi f^* = d^\%(\rho)$ i.e.

$$\phi'_s - f\phi_s f^* = d\rho_s + (-)^r \rho_s d^* + (-)^{n+s}(\rho_{s-1} + (-)^s T_\epsilon \rho_{s-1}) : C^{n-r+s} \longrightarrow C_r$$

i.e. $f\phi f^* = \phi' \in Q^n(C)$. It is an **equivalence** if $f : C \longrightarrow C'$ is a chain equivalence. It is an **isomorphism** if $f : C \longrightarrow C'$ is an isomorphism.

A **map of ϵ -quadratic n -dimensional complexes** $f = (f, \sigma) : (C, \psi) \longrightarrow (C', \psi')$ is a chain map $f : C \longrightarrow C'$ and a $\sigma \in W_\%(C', \epsilon)_{n+1}$ such that $\psi' - f\psi f^* = d_\%(\sigma)$ i.e.

$$\psi'_s - f\psi_s f^* = d\sigma_s + (-)^r \sigma_s d^* + (-)^{n-s}(\sigma_{s+1} + (-)^{s+1} T_\epsilon \sigma_{s+1}) : C^{n-r-s} \longrightarrow C_r$$

i.e. $f\psi f^* = \psi' \in Q_n(C)$. It is an **equivalence** if $f : C \longrightarrow C'$ is a chain equivalence. It is an **isomorphism** if $f : C \longrightarrow C'$ is an isomorphism. □

We can define compositions and inverses of morphisms. One can also define inverses for homotopy equivalences but we will not need such a construction in this treatise.

Definition A.1.5. The composition of two morphisms of ϵ -symmetric n -dimensional complexes $(f, \rho): (C, \phi) \longrightarrow (C', \phi')$ and $(f', \rho'): (C', \phi') \longrightarrow (C'', \phi'')$ is the morphism $(f'f, \rho' + f'\rho f'^*): (C, \phi) \longrightarrow (C'', \phi'')$.

The composition of two morphisms of ϵ -quadratic n -dimensional complexes $(f, \sigma): (C, \psi) \longrightarrow (C', \psi')$ and $(f', \sigma'): (C', \psi') \longrightarrow (C'', \psi'')$ is the morphism $(f'f, \sigma' + f'\sigma f'^*): (C, \psi) \longrightarrow (C'', \psi'')$.

The inverse of an isomorphism $(f, \rho): (C, \phi) \xrightarrow{\cong} (C', \phi')$ of ϵ -symmetric n -dimensional complexes is the isomorphism $(f, \rho)^{-1} = (f^{-1}, -f^{-1}\rho f^{-*}): (C', \phi') \xrightarrow{\cong} (C, \phi)$.

The inverse of an isomorphism $(f, \sigma): (C, \psi) \xrightarrow{\cong} (C', \psi')$ of ϵ -quadratic n -dimensional complexes is the isomorphism $(f, \sigma)^{-1} = (f^{-1}, -f^{-1}\sigma f^{-*}): (C', \psi') \xrightarrow{\cong} (C, \psi)$. \square

Remark A.1.6. For any closed n -dimensional manifold M it is possible to construct an n -dimensional symmetric complex (C, ϕ) with $C = C(\widetilde{M})$ the chain complex over $\Lambda = \mathbb{Z}[\pi_1(M)]$ of the universal cover (See [Ran80b] for details). The Poincaré duality map of (C, ϕ) is a chain level representative of the Poincaré duality isomorphisms $H^{n-r}(\widetilde{M}, \Lambda) \longrightarrow H_r(\widetilde{M}, \Lambda), x \longmapsto x \cap [M]$. ϕ_1 is a chain homotopy between ϕ_0 and $T_\epsilon \phi_0$ making sure that the homology isomorphism induced by ϕ_0 has the symmetry properties we are expecting from the Poincaré duality.

Let $f: M \longrightarrow X$ be an n -dimensional degree 1 normal map of the manifold into an n -dimensional Poincaré space X . The geometry gives rise to an n -dimensional quadratic complex $(C = C(\widetilde{f}: \widetilde{M} \longrightarrow \widetilde{X}), \psi)$. The Poincaré duality map is a chain level representative of the duality of Kernel groups $K^{n-r}(M) \longrightarrow K_r(M), x \longmapsto x \cap [M]$ with $K_r(M) = H_{r-1}(\widetilde{f})$.

Remark A.1.7. In the standard references about Algebraic Surgery Theory ([Ran80a], [Ran81], etc.) an ϵ -symmetric {quadratic} complex is defined as a tuple $(C, [\phi] \in Q^n(C, \epsilon)) \{(C, [\psi] \in Q_n(C, \epsilon))\}$. Some of the constructions we are using however need a specific representative $\phi \{\psi\}$. Therefore we follow [Ran98] by defining the complexes slightly differently as tuples $(C, \phi \in \ker d_\%^n) \{(C, \psi \in \ker d_\%^n)\}$.

A.2 Quadratic and Symmetric Pairs

Whereas the algebraic equivalent of closed manifolds (respectively normal maps of closed manifolds) are symmetric Poincaré complexes (resp. quadratic Poincaré complexes), the analogues of manifolds with boundaries or normal maps are symmetric and quadratic pairs.

Definition A.2.1. Let $f: C \longrightarrow D$ be a chain map. We define chain complexes

$W^\%(f, \epsilon)$ and $W_\% (f, \epsilon)$ by

$$\begin{aligned} W^\%(f, \epsilon)_{n+1} &= \{(\delta\phi_s: D^{n-p+s+1} \longrightarrow D_p, \phi_s: C^{n-r+s} \longrightarrow C_r) | p, r \in \mathbb{Z}, s \geq 0\} \\ d^\%: W^\%(f, \epsilon)_{n+1} &\longrightarrow W^\%(f, \epsilon)_n \\ \{(\delta\phi_s, \phi_s)\} &\longmapsto \{(d(\delta\phi_s) + (-)^r(\delta\phi_s)d^* + (-)^{n+s}(\delta\phi_{s-1} + (-)^s T_\epsilon(\delta\phi_{s-1})) \\ &\quad + (-)^n f\phi_s f^*: D^{n-r+s} \longrightarrow D_r, \\ &\quad d\phi_s + (-)^r \phi_s d^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T_\epsilon \phi_{s-1}): \\ &\quad C^{n-r+s-1} \longrightarrow C_r) | r \in \mathbb{Z}, s \geq 0\} \end{aligned}$$

where we set $\phi_{-1} = 0$ and $\delta\phi_{-1} = 0$.

$$\begin{aligned} W_\%(f, \epsilon)_{n+1} &= \{(\delta\psi_s: D^{n-p-s+1} \longrightarrow D_p, \psi_s: C^{n-r-s} \longrightarrow C_r) | p, r \in \mathbb{Z}, s \geq 0\} \\ d_\%: W_\%(f, \epsilon)_{n+1} &\longrightarrow W_\%(f, \epsilon)_n \\ \{(\delta\psi_s, \psi_s)\} &\longmapsto \{(d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1} T_\epsilon(\delta\psi_{s+1})) \\ &\quad + (-)^n f\psi_s f^*: D^{n-r-s} \longrightarrow D_r, \\ &\quad d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\epsilon \psi_{s+1}): \\ &\quad C^{n-r-s-1} \longrightarrow C_r) | r \in \mathbb{Z}, s \geq 0\} \end{aligned}$$

Their homology groups are the ϵ -symmetric Q -groups $Q^n(f, \epsilon) = H_n(W^\%(f, \epsilon))$ and the ϵ -quadratic Q -groups $Q_n(f, \epsilon) = H_n(W_\%(f, \epsilon))$. They are related by the ϵ -symmetrization map

$$\begin{aligned} Q_n(f, \epsilon) &\longrightarrow Q^n(f, \epsilon) \\ (\delta\psi_s, \psi_s) &\longmapsto \begin{cases} ((1 + T_\epsilon)\delta\psi_0, (1 + T_\epsilon)\psi_0) & : \text{if } s = 0, \\ 0 & : \text{if } s \neq 0. \end{cases} \end{aligned}$$

□

Remark A.2.2. Let $f: C \longrightarrow D$ be a chain map. The rather complicated differential of $W^\%(f, \epsilon)$ can be understood in terms of $W^\%(C, \epsilon)$ and $W^\%(D, \epsilon)$. Let $x = (\delta\psi, \psi) \in W^\%(f, \epsilon)_{n+1}$. Then obviously $\delta\psi \in W^\%(D, \epsilon)_{n+1}$ and $\psi \in W^\%(C, \epsilon)_n$. The differential is alternatively given by

$$\begin{aligned} d: W^\%(f, \epsilon)_{n+1} &\longrightarrow W^\%(f, \epsilon)_n \\ (\delta\psi, \psi) &\longmapsto (d_1(\delta\psi) + (-)^n f\psi f^*, d_2(\psi)) \end{aligned}$$

with $d_1: W^\%(D, \epsilon)_{n+1} \longrightarrow W^\%(D, \epsilon)_n$ and $d_2: W^\%(C, \epsilon)_n \longrightarrow W^\%(C, \epsilon)_{n-1}$ the differentials we discussed above. x is a cycle if and only if

$$(f, (-)^n \delta\psi): (C, \psi) \longrightarrow (D, 0)$$

is a map of ϵ -quadratic n -dimensional complexes. Same for the symmetric case.

Definition A.2.3. An $(n+1)$ -dimensional ϵ -symmetric pair $(f: C \rightarrow D, (\delta\phi, \phi))$ is a chain map $f: C \rightarrow D$ together with a cycle $(\delta\phi, \phi) \in W_{\%}(f, \epsilon)_{n+1}$. It is called **Poincaré** if the **Poincaré duality map** $D^{n+1-*} \rightarrow \mathcal{C}(f)$ given by

$$\begin{pmatrix} \delta\phi_0 \\ (-)^{n+1-r}\phi_0 f^* \end{pmatrix}: D^{n+1-r} \rightarrow \mathcal{C}(f)_r$$

is a chain equivalence.

An $(n+1)$ -dimensional ϵ -quadratic pair $(f: C \rightarrow D, (\delta\psi, \psi))$ is a chain map $f: C \rightarrow D$ together with a cycle $(\delta\psi, \psi) \in W_{\%}(f, \epsilon)_{n+1}$. It is called **Poincaré** if the **Poincaré duality map** $D^{n+1-*} \rightarrow \mathcal{C}(f)$ given by

$$\begin{pmatrix} (1+T_\epsilon)\delta\psi_0 \\ (-)^{n+1-r}(1+T_\epsilon)\psi_0 f^* \end{pmatrix}: D^{n+1-r} \rightarrow \mathcal{C}(f)_r$$

is a chain equivalence. □

Remark A.2.4. In the above definitions the Poincaré duality maps can be replaced by the chain maps

$$(\delta\phi_0, f\phi_0): \mathcal{C}(f)^{n+1-*} \rightarrow D$$

in the symmetric case and by

$$((1+T_\epsilon)\delta\psi_0, f(1+T_\epsilon)\psi_0): \mathcal{C}(f)^{n+1-*} \rightarrow D$$

in the quadratic case.

Definition A.2.5. A homotopy equivalence of $(n+1)$ -dimensional ϵ -quadratic pairs

$$(g, h; k): (f: C \rightarrow D, (\delta\psi, \psi)) \rightarrow (f': C' \rightarrow D', (\delta\psi', \psi'))$$

is a triple $(g, h; k)$ consisting of chain equivalences

$$g: C \rightarrow C', \quad h: D \rightarrow D'$$

and a chain homotopy

$$k: f'g \simeq hf: C \rightarrow D'$$

such that

$$(g, h; k)_{\%}(\delta\psi, \psi) = (\delta\psi', \psi') \in Q_{n+1}(f', \epsilon)$$

with

$$\begin{aligned} (g, h; k)_{\%}(\delta\psi, \psi)_s &= (h\delta\psi_s h^* + (-)^n k\psi_s (hf)^* + (-)^{r+1} kT_\epsilon\psi_{s+1} k^* + \\ &(-)^{n-r} f'g\psi_s k^*: D'^{n+1-s-r} \rightarrow D'_r, \\ g\psi_s g^*: C'^{n-s-q} &\rightarrow C'_q) \end{aligned}$$

□

Here are some useful lemmas about changing the boundary of a pair and examples for homotopy equivalences of pairs.

Lemma A.2.6. *Let $c = (f: C \rightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional ϵ -quadratic pair and $(g, \sigma): (C', \psi') \rightarrow (C, \psi)$ be a map of n -dimensional ϵ -quadratic complexes. Then*

$$c' = (f' = fg: C' \rightarrow D, (\delta\psi' = \delta\psi + (-)^n f\sigma f^*, \psi'))$$

is an $(n+1)$ -dimensional ϵ -quadratic pair. If g is a chain equivalence, then c and c' are homotopy equivalent pairs. Same for the symmetric case.

Proof. First we show that c' is an ϵ -quadratic pair. We use Remark A.2.2 to compute

$$\begin{aligned} d(\delta\psi + (-)^n f\sigma f^*, \psi') &= (d\delta\psi + (-)^n fd\sigma f^* + (-)^n fg\psi'g^*f^*, d\psi') \\ &= (d\delta\psi + (-)^n f(\psi - g\psi'g^*)f^* + (-)^n fg\psi'g^*f^*, d\psi') = 0 \end{aligned}$$

There is a homotopy equivalence $(g, 1; 0): c' \rightarrow (f: C \rightarrow D(\delta\psi', g\psi'g^*))$. By definition $\sigma \in W_{\%}(C, \epsilon)_{n+1}$, hence $y = (0 \ \sigma) \in W_{\%}(f, \epsilon)_{n+2}$. Its boundary is $d(y) = ((-)^{n+1} f\sigma f^*, d\sigma) = ((-)^{n+1} f\sigma f^*, \psi - g\psi'g^*)$ and $(\delta\psi', g\psi'g^*) + d(y) = (\delta\psi, \psi)$. \square

Complexes and pairs are in a one-to-one correspondence.

Definition A.2.7. An n -dimensional ϵ -symmetric complex (C, ϕ) is **connected** if $H_0(\phi_0: C^{n-*} \rightarrow C) = 0$. An n -dimensional ϵ -quadratic complex (C, ψ) is **connected** if $H_0((1 + T\epsilon)\psi_0: C^{n-*} \rightarrow C) = 0$.

The **boundary** $(\partial C, \partial\phi)$ of a **connected** n -dimensional ϵ -symmetric complex (C, ϕ) is the $(n-1)$ -dimensional ϵ -symmetric Poincaré complex defined by

$$\begin{aligned} d_{\partial C} &= \begin{pmatrix} d_C & (-)^r \phi_0 \\ 0 & (-)^r d_C^* \end{pmatrix} : \partial C_r = C_{r+1} \oplus C^{n-r} \rightarrow \partial C_{r-1} = C_r \oplus C^{n-r+1} \\ \partial\phi_0 &= \begin{pmatrix} (-)^{n-r-1} T_\epsilon \phi_1 & (-)^{rn} \epsilon \\ 1 & 0 \end{pmatrix} : \partial C^{n-r-1} = C^{n-r} \oplus C_{r+1} \rightarrow \partial C_r = C_{r+1} \oplus C^{n-r} \\ \partial\phi_s &= \begin{pmatrix} (-)^{n-r+s-1} T_\epsilon \phi_{s+1} & 0 \\ 0 & 0 \end{pmatrix} : \partial C^{n-r+s-1} = C^{n-r+s} \oplus C_{r-s+1} \rightarrow \partial C_r \quad (s > 0) \end{aligned}$$

The **boundary** $(\partial C, \partial\psi)$ of a **connected** n -dimensional ϵ -quadratic complex (C, ψ) is the $(n-1)$ -dimensional ϵ -quadratic Poincaré complex defined by

$$\begin{aligned} d_{\partial C} &= \begin{pmatrix} d_C & (-)^r (1 + T_\epsilon)\psi_0 \\ 0 & (-)^r d_C^* \end{pmatrix} : \partial C_r = C_{r+1} \oplus C^{n-r} \rightarrow \partial C_{r-1} = C_r \oplus C^{n-r+1} \\ \partial\psi_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \partial C^{n-r-1} = C^{n-r} \oplus C_{r+1} \rightarrow \partial C_r = C_{r+1} \oplus C^{n-r} \\ \partial\psi_s &= \begin{pmatrix} (-)^{n-r-s-1} T_\epsilon \psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} : \partial C^{n-r-s-1} = C^{n-r-s} \oplus C_{r+s+1} \rightarrow \partial C_r \quad (s > 0) \end{aligned}$$

The **thickening of a connected n -dimensional ϵ -symmetric complex** (C, ϕ) is the ϵ -symmetric n -dimensional Poincaré pair $(i_C: \partial C \longrightarrow C^{n-*}, (0, \partial\phi))$ with $i_C = \begin{pmatrix} 0 & 1 \end{pmatrix}: \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow (C^{n-*})_r = C^{n-r}$.

The **thickening of a connected n -dimensional ϵ -quadratic complex** (C, ψ) is the ϵ -quadratic n -dimensional Poincaré pair $(i_C: \partial C \longrightarrow C^{n-*}, (0, \partial\psi))$ with $i_C = \begin{pmatrix} 0 & 1 \end{pmatrix}: \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow (C^{n-*})_r = C^{n-r}$.

The **Thom complex of an $(n+1)$ -dimensional ϵ -symmetric Poincaré pair** $(f: C \longrightarrow D, (\delta\phi, \phi))$ is the connected $(n+1)$ -dimensional ϵ -symmetric complex $(\mathcal{C}(f), \delta\phi/\phi)$ given by

$$(\delta\phi/\phi)_s = \begin{pmatrix} \delta\phi_s & 0 \\ (-)^{n+1-r}\phi_s f^* & (-)^{n-r+s+1}T_\epsilon\phi_{s-1} \end{pmatrix}: \\ \mathcal{C}(f)^{n+1-r+s} = D^{n+1-r+s} \oplus C^{n-r+s} \longrightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

The **Thom complex of an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair** $(f: C \longrightarrow D, (\delta\psi, \psi))$ is the connected n -dimensional ϵ -quadratic complex $(\mathcal{C}(f), \delta\psi/\psi)$ given by

$$(\delta\psi/\psi)_s = \begin{pmatrix} \delta\psi_s & 0 \\ (-)^{n+1-r}\psi_s f^* & (-)^{n-r-s}T_\epsilon\psi_{s+1} \end{pmatrix}: \\ \mathcal{C}(f)^{n+1-r-s} = D^{n+1-r-s} \oplus C^{n-r-s} \longrightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

□

Lemma A.2.8. *Let $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair. Then the operations “Symmetrization” and “Thom-complex” are commuting i.e. the identity map on $\mathcal{C}(f)$ induces an isomorphism*

$$(1, \chi) : (\mathcal{C}(f), ((1 + T_\epsilon)\delta\psi)/((1 + T_\epsilon)\psi)) \longrightarrow (\mathcal{C}(f), (1 + T_\epsilon)(\delta\psi/\psi)) \\ \chi_0 = \begin{pmatrix} 0 & 0 \\ 0 & (-)^r T_\epsilon\psi_0 \end{pmatrix} : \mathcal{C}(f)^{n+2-r} = D^{n+2-r} \oplus C^{n+1-r} \longrightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

of connected $(n+1)$ -dimensional ϵ -symmetric complexes.

Lemma A.2.9 ([Ran80a] Proposition 3.4.). *Let $(f, \chi): (C, \psi) \xrightarrow{\cong} (C', \psi')$ be an isomorphism of n -dimensional ϵ -quadratic complexes. Then there is an isomorphism*

$$(\partial f, \partial\chi) : (\partial C, \partial\psi) \xrightarrow{\cong} (\partial C', \partial\psi') \\ \partial f = \begin{pmatrix} f & (-)^{r-1}(1 + T_\epsilon)\chi_0 f^{-*} \\ 0 & f^{-*} \end{pmatrix} : \partial C_r = C_{r+1} \oplus C^{n-r} \xrightarrow{\cong} \partial C'_r = C'_{r+1} \oplus C'^{n-r} \\ \partial\chi_0 = 0 : \partial C'^{n-r} \longrightarrow \partial C'_r \\ \partial\chi_s = \begin{pmatrix} (-)^{n-r-s}T_\epsilon\chi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} : \partial C'^{n-r-s} = C'^{n-r-s+1} \oplus C'_{r+s} \\ \longrightarrow \partial C'_r = C'_{r+1} \oplus C'^{n-r} \quad (s > 0)$$

and a homotopy equivalence $(\partial f, f^{-*}; 0)$ between the thickening-ups of (C, ψ) and (C', ψ') .

Proposition A.2.10 ([Ran80a] Proposition 3.4.). *The Thom complex operation induces an natural one-to-one correspondence between the homotopy equivalence classes of n -dimensional ϵ -symmetric Poincaré pairs and the homotopy equivalence classes of connected n -dimensional ϵ -symmetric complexes. Poincaré pairs with contractible boundaries correspond to Poincaré complexes. Thickening is the inverse operation. Similar for the quadratic case.*

Lemma A.2.11 ([Ran80a] Prop 3.4.). *Let $(f: C \rightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional ϵ -quadratic Poincaré pair. Then the maps*

$$(0 \ 1 \ 0 \ -\psi_0) : C_r \rightarrow \partial\mathcal{C}(f)_r = D_{r+1} \oplus C_r \oplus D_{n+1-r} \oplus C^{n-r}$$

defines a chain equivalence $C \simeq \partial\mathcal{C}(f)$.

A.3 Unions of Pairs

The union-construction is an algebraic analogue of glueing two $(n+1)$ -dimensional cobordisms (W, M, M') and (W', M', M'') together at M' .

Definition A.3.1 ([Ran80a] p.135). **The union of two adjoining ϵ -symmetric $(n+1)$ -dimensional cobordisms**

$$\begin{aligned} c &= ((f_C \ f_{C'}) : C \oplus C' \rightarrow D, (\delta\phi, \phi \oplus -\phi')) \\ c' &= ((f'_{C'} \ f'_{C''}) : C' \oplus C'' \rightarrow D', (\delta\phi', \phi' \oplus -\phi'')) \end{aligned}$$

is the ϵ -symmetric $(n+1)$ -dimensional cobordism

$$c \cup c' = ((f''_{C'} \ f''_{C''}) : C \oplus C'' \rightarrow D'', (\delta\phi'', \phi \oplus -\phi''))$$

given by

$$\begin{aligned} d_{D''} &= \begin{pmatrix} d_D & (-)^{r-1} f_{C'} & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-)^{r-1} f'_{C'} & d_{D'} \end{pmatrix} : \\ D''_r &= D_r \oplus C'_{r-1} \oplus D'_r \rightarrow D''_{r-1} = D_{r-1} \oplus C'_{r-2} \oplus D'_{r-1} \\ f''_C &= \begin{pmatrix} f_C \\ 0 \\ 0 \end{pmatrix} : C_r \rightarrow D''_r \\ f''_{C''} &= \begin{pmatrix} 0 \\ 0 \\ f'_{C''} \end{pmatrix} : C''_r \rightarrow D''_r \\ \delta\phi''_s &= \begin{pmatrix} \delta\phi_s & 0 & 0 \\ (-)^{n-r} \phi'_s f_{C'}^* & (-)^{n-r+s} T_\epsilon \phi'_{s-1} & 0 \\ 0 & (-)^s f'_{C'} \phi'_s & \delta\phi'_s \end{pmatrix} : \\ D''^{n-r+s+1} &= D^{n-r+s+1} \oplus C'^{n-r+s} \oplus D'^{n-r+s+1} \rightarrow D''_r = D_r \oplus C'_{r-1} \oplus D'_r \end{aligned}$$

We shall normally write

$$D'' = D \cup_{C'} D', \quad \delta\phi'' = \delta\phi \cup_{\phi'} \delta\phi'$$

The union of two adjoining ϵ -quadratic $(n+1)$ -dimensional cobordisms

$$\begin{aligned} c &= ((f_C \ f_{C'}) : C \oplus C' \longrightarrow D, (\delta\psi, \psi \oplus -\psi')) \\ c' &= ((f'_{C'} \ f'_{C''}) : C' \oplus C'' \longrightarrow D', (\delta\psi', \psi' \oplus -\psi'')) \end{aligned}$$

is the ϵ -quadratic $(n+1)$ -dimensional cobordism

$$c \cup c' = ((f''_{C'} \ f''_{C''}) : C \oplus C'' \longrightarrow D'', (\delta\psi'', \psi \oplus -\psi''))$$

given by the same complex D'' and the same maps $f''_C, f''_{C''}$ as in the symmetric case and

$$\delta\psi''_s = \begin{pmatrix} \delta\psi_s & 0 & 0 \\ (-)^{n-r}\psi'_s f_{C'}^* & (-)^{n-r-s+1}T_\epsilon\psi'_{s+1} & 0 \\ 0 & (-)^s f'_{C'}\psi'_s & \delta\psi'_s \end{pmatrix} :$$

$$D''^{n-r-s+1} = D^{n-r-s+1} \oplus C'^{n-r-s} \oplus D'^{n-r-s+1} \longrightarrow D''_r = D_r \oplus C'_{r-1} \oplus D'_r$$

We shall normally write

$$D'' = D \cup_{C'} D', \quad \delta\psi'' = \delta\psi \cup_{\psi'} \delta\psi'$$

□

Glueing and symmetrizing cobordisms are commutative operations as the following example illustrates.

Lemma A.3.2. *Let*

$$\begin{aligned} c &= (f_{C'} : C' \longrightarrow D, (\delta\psi, -\psi')) \\ c' &= (f'_{C'} : C' \longrightarrow D', (\delta\psi', \psi')) \end{aligned}$$

be two ϵ -quadratic $(n+1)$ -dimensional Poincaré pairs. Then

$$(1, \chi) : (D'' = D \cup_{C'} D', (1 + T_\epsilon)(\delta\psi \cup_{\psi'} \delta\psi'))$$

$$\longrightarrow (D'', (1 + T_\epsilon)(\delta\psi) \cup_{(1+T_\epsilon)\psi'} (1 + T_\epsilon)(\delta\psi'))$$

$$\chi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{r-1}T\psi_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : D''^{n+2-r} = D^{n+2-r} \oplus C'^{n+1-r} \oplus D'^{n+2-r}$$

$$\longrightarrow D''_r = D_r \oplus C'_{r-1} \oplus D'_r$$

is an isomorphism of $(n+2)$ -dimensional ϵ -symmetric Poincaré complexes.

Next we show that changing the common boundary of two pairs doesn't change their union

Lemma A.3.3. *Let*

$$\begin{aligned} c &= (f: C \longrightarrow D, (\delta\psi, \psi)) \\ c' &= (f': C \longrightarrow D', (\delta\psi', \psi)) \end{aligned}$$

be two ϵ -quadratic $(n+1)$ -dimensional Poincaré pairs. Let $(h, \chi): (\widehat{C}, \widehat{\psi}) \xrightarrow{\cong} (C, \psi)$ be an equivalence of quadratic complexes. Using Lemma A.2.6, define the $(n+1)$ -dimensional ϵ -quadratic Poincaré pairs

$$\begin{aligned} \widehat{c} &= (\widehat{f} = fh: \widehat{C} \longrightarrow D, (\widehat{\delta\psi} = \delta\psi + (-)^n f\chi f^*, \widehat{\psi})) \\ \widehat{c}' &= (\widehat{f}' = f'h: \widehat{C} \longrightarrow D', (\widehat{\delta\psi}' = \delta\psi' + (-)^n f'\chi f'^*, \widehat{\psi})) \end{aligned}$$

Then there is a chain equivalence of $(n+2)$ -dimensional ϵ -quadratic Poincaré complexes

$$\begin{aligned} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma \right) &: \widehat{c} \cup \widehat{c}' = (D \cup_{\widehat{C}} D', \widehat{\delta\psi} \cup_{\widehat{\psi}} -\widehat{\delta\psi}') \\ &\xrightarrow{\cong} c \cup c' = (D \cup_C D', \delta\psi \cup_{\psi} -\delta\psi') \\ \sigma_s &= \begin{pmatrix} 0 & 0 & 0 \\ (-)^{n-r} \chi_s f^* & (-)^{n+1-r-s} T_{\epsilon} \chi_{s+1} & 0 \\ 0 & (-)^{s-1} f' \chi_s & 0 \end{pmatrix} : \\ &(D \cup_C D')^{n+2-r-s} = D^{n+2-r-s} \oplus C^{n+1-r-s} \oplus D'^{n+2-r-s} \\ &\longrightarrow (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_{r-1} \end{aligned}$$

Another construction we will use is the union of a **fundamental pair**. The geometrical analogue can be described as such: Let (W, M, M) be an $(n+1)$ -dimensional cobordism and glue it together along its boundaries. Using Mayer-Vietoris one sees that the resulting manifold V has the chain complex $C(f-g)$ with $(f, g): M + M \longrightarrow W$ being the inclusion of the boundary.

Definition and Lemma A.3.4 ([Ran98] **Definition 24.1.**). An $(n+1)$ -dimensional ϵ -symmetric pair is called **fundamental** if it is of the form $((f, g): C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -\phi))$. Similar for the quadratic case.

The **union of a fundamental $(n+1)$ -dimensional ϵ -symmetric Poincaré pair $((f, g): C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -\phi))$ over Λ** is the $(n+1)$ -dimensional ϵ -symmetric Poincaré (U, ρ) complex over Λ given by

$$\begin{aligned} U &= \mathcal{C}(f-g: C \longrightarrow D) \\ \rho_s &= \begin{pmatrix} \delta\phi_s & (-)^s g\phi_s \\ (-)^{n-r+1} \phi_s f^* & (-)^{n-r+s+1} T_{\epsilon} \phi_{s-1} \end{pmatrix} : \\ &U^{n+1-r+s} = D^{n+1-r+s} \oplus C^{n-r+s} \longrightarrow U_r = D_r \oplus C_{r-1} \end{aligned}$$

The **union of a fundamental $(n + 1)$ -dimensional ϵ -quadratic Poincaré pair $((f, g): C \oplus C \longrightarrow D, (\delta\psi, \psi \oplus -\psi))$ over Λ is the $(n + 1)$ -dimensional ϵ -symmetric Poincaré (U, σ) complex over Λ given by**

$$\begin{aligned} U &= \mathcal{C}(f - g: C \longrightarrow D) \\ \sigma_s &= \begin{pmatrix} \delta\psi_s & (-)^s g\psi_s \\ (-)^{n-r+1}\psi_s f^* & (-)^{n-r-s} T_\epsilon \psi_{s+1} \end{pmatrix}: \\ U^{n+1-r-s} &= D^{n+1-r-s} \oplus C^{n-r-s} \longrightarrow U_r = D_r \oplus C_{r-1} \end{aligned}$$

Proof. Compute the union (D'', ϕ'') of the two cobordism over Λ

$$\begin{aligned} c &= ((f, g): C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -\phi)) \\ c' &= ((1, 1): C \oplus C \longrightarrow C, (0, -\phi \oplus \phi)) \end{aligned}$$

(or the quadratic analogue). Then use the isomorphism of chain complexes

$$a_r = \begin{pmatrix} 1 & 0 & 0 & -g \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & (-)^{r+1}d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} D_r'' = D_r \oplus (C_{r-1} \oplus C_{r-1}) \oplus C_r & \xrightarrow{a_r} & D_r'' \\ \downarrow \begin{pmatrix} d & (-)^{r-1}f & (-)^{r-1}g & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & (-)^{r-1} & (-)^{r-1} & d \end{pmatrix} & & \downarrow \begin{pmatrix} d & (-)^{r-1}(f-g) & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (-)^{r-1} & 0 \end{pmatrix} \\ D_{r-1}'' = D_{r-1} \oplus (C_{r-2} \oplus C_{r-2}) \oplus C_{r-1} & \xrightarrow{a_{r-1}} & D_{r-1}'' \end{array}$$

□

A.4 Surgery on Complexes

Definition A.4.1. Let (C, ψ) be a connected n -dimensional ϵ -quadratic complex and $c = (f: C \longrightarrow D, (\delta\psi, \psi))$ an ϵ -quadratic $(n + 1)$ -dimensional pair.

c is **connected** if the zeroth homology of its Poincaré duality map vanishes.

The **result of an ϵ -quadratic surgery on a connected pair c** is the connected

n -dimensional ϵ -quadratic complex (C', ψ') given by

$$\begin{aligned}
d_{C'} &= \begin{pmatrix} d_C & 0 & (-)^{n+1}(1+T_\epsilon)\psi_0 f^* \\ (-)^r f & d_D & (-)^r(1+T_\epsilon)\delta\psi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix} : \\
C'_r &= C_r \oplus D_{r+1} \oplus D^{n-r+1} \longrightarrow C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2} \\
\psi'_0 &= \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \\
C'^{n-r} &= C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \\
\psi'_s &= \begin{pmatrix} \psi_s & (-)^s T_\epsilon \psi_{s-1} f^* & 0 \\ 0 & (-)^{n-r-s} T_\epsilon \delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \\
C'^{n-r-s} &= C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \\
&\longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \quad (s > 0)
\end{aligned}$$

Similar for the symmetric case. □

In an obvious way we can introduce the notion of an cobordism of complexes. It turns out to be an equivalence relation and its equivalence classes are the surgery L -groups.

Definition A.4.2. A cobordism of two n -dimensional ϵ -quadratic Poincaré complexes (C, ψ) and (C', ψ') is an $(n+1)$ -dimensional ϵ -symmetric Poincaré pair $(f: C \oplus C' \longrightarrow D, (\delta\psi, \phi \oplus -\psi'))$. Similar for the symmetric case. □

The well-known relations between surgery and cobordism hold also in the algebraic world

Proposition A.4.3 ([Ran80a] Proposition 4.1.). *i) Algebraic surgery preserves the homotopy type of the boundary, sending Poincaré complexes to Poincaré complexes.*

ii) Two Poincaré complexes are cobordant if and only if the one can be obtained from the other by surgery and homotopy equivalence.

Lemma A.4.4 ([Ran80a] Proposition 1.4i). *With C and C' as in the Definition A.4.1 the map*

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \bar{\epsilon}(-)^{(n+r)r} & 0 \end{pmatrix} : \\
\partial C_r &= C_{r+1} \oplus C^{n-r} \longrightarrow \partial C'_r = C_{r+1} \oplus D_{r+2} \oplus D^{n-r} \oplus C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1}
\end{aligned}$$

defines a homotopy equivalence.

Proposition A.4.5 ([Ran80a] Proposition 3.2). *Cobordism is an equivalence relation on n -dimensional Poincaré complexes. Homotopy equivalent Poincaré complexes are cobordant.*

The cobordism classes of Poincaré complexes are groups and are the preferred definition for the L -groups amongst algebraic surgeons because they are related to the L -groups defined as Witt-groups of forms and formations. See [Ran80a] Chapter 4 and 5 for more details. For our purposes the only fact we really need is the following lemma:

Lemma A.4.6. *Let (C, ψ) be an $2m$ -dimensional quadratic Poincaré complex with $C_i = 0$ for $i \notin \{m+1, m\}$. Then*

$$\left(M = C^m \oplus C_{m+1}, \theta = \begin{pmatrix} \psi_0 & 0 \\ d^* & 0 \end{pmatrix} \in Q_{(-)^m}(M) \right)$$

is a non-singular $(-)^m$ -quadratic form.

(C, ψ) is null-cobordant if and only if $(M, \theta) = 0 \in L_{2m+2}(\Lambda)$. Homotopic or cobordant ϵ -quadratic complexes lead to the same element in $L_{2m+2}(\Lambda)$.

Remark A.4.7. This is a special case of the instant surgery obstruction given in [Ran80a]. A similar result does not hold in general for symmetric complexes.

Proof. Define the connected $(2m+1)$ -dimensional quadratic pair $(f: C \rightarrow D, (0, \psi))$ with $f = 1: C_{m+1} \rightarrow D_{m+1} = C_{m+1}$ and $D_i = 0$ for $i \neq m+1$. We simplify the result C' of the surgery on C using the homotopy equivalence

$$\begin{array}{ccc} C'_{m+1} = C_{m+1} & & \\ \left(\begin{array}{c} d \\ (-)^{m+1} \\ 0 \end{array} \right) \downarrow & & \\ C'_m = C_m \oplus D_{m+1} \oplus D^{m+1} & \xrightarrow{\begin{pmatrix} (-)^m & d & 0 \\ 0 & 0 & 1 \end{pmatrix}} & M^* \end{array}$$

[Ran80a] Proposition 4.3. and Proposition 5.1. finish the proof □

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