The Donaldson-Thomas theory and cohomological Hall algebras of character stacks

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Vivek Mistry*)

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Abstract

Given a smooth finitely generated algebra with a potential one can study the refined Donaldson-Thomas theory of its moduli stack of representations via motivic or cohomological methods. In this thesis we focus on fundamental group algebras whose stacks of representations are known as character varieties or character stacks. These arise naturally in the realm of algebraic geometry and Donaldson-Thomas theory via the non-abelian Hodge correspondence which relates the study of Higgs bundles to character varieties.

In the first part of this thesis we consider approaches to studying the motivic Donaldson-Thomas invariants of fundamental group algebras over mapping tori of Riemann surfaces by constructing an isomorphism between the fundamental group algebra and the Jacobi algebra of a so-called brane tiling on the Riemann surface. Using the critical locus structure of a Jacobi algebra this presents us with a natural way to study the motivic Donaldson-Thomas invariants of the character varieties of mapping tori and we present ideas on how this can be accomplished.

In the second part of this thesis we focus on the cohomological Donaldson-Thomas theory of fundamental group algebras over Riemann surfaces. Again utilising brane tilings we prove that the cohomological Hall algebra of the character variety of a Riemann surface has a natural 2 Calabi-Yau structure arising from a 2D Jacobi algebra, and hence can be obtained by dimensional reduction of the corresponding 3D cohomological Hall algebra of the 3D Jacobi algebra.

Lay summary

Algebraic geometry is often concerned with classifying geometric objects. There are often many ways of describing the same geometric space, and so the fundamental question we ask is if given two geometric spaces are they the same or different?

In general this is a very difficult question to answer and therefore tools have been developed to help solve this question in particular cases. One broad category of such tools are known as "invariants". An invariant of a geometric space is a much simpler mathematical object, often just a number or perhaps a vector space, which has the property that if two geometric spaces are indeed the same then so are their respective invariants. Since the invariants are simpler objects comparing them is generally much easier to do, and therefore they may allow us, at least partially, to answer the above question. Note however that by design this process necessitates losing information- this is beneficial as it allows us to perform comparisons, but come at a cost. This cost is that we cannot use an invariant to decide if two geometric objects are the same, they can only tell us if they are different. In particular there may exist different geometric spaces which have the same invariants, but by definition there cannot be the same geometric spaces written in different ways with different invariants. To help get around this problem mathematicians have developed a number of different invariants.

This thesis focuses on one such group of invariants knows as Donaldson-Thomas invariants, which were defined by Thomas in [65] for a particular class of geometric spaces known as Calabi-Yau threefolds. We investigate and calculate these invariants for some specific geometric spaces in order to understand and reveal more information about them.

Contents

Introduction

Donaldson-Thomas (DT) theory first developed by Thomas in [65] is a mechanism for generating invariants for Calabi-Yau threefolds. They were originally defined as the virtual counts of stables sheaves with fixed determinant and Chern character. Since then it has undergone a number of changes and upgrades both in terms of the objects that we can study and also the invariants being calculated. The first of the these upgrades was provided by Behrend where in [2] he extended the definition of DT invariants to non-proper schemes X with symmetric obstruction theories by the use of the eponymous constructible function $\nu_X : X \to \mathbb{Z}$. Later, this was further generalised by Joyce in a series of papers [33], [37], [32], [34], [35], [38], [36] and Joyce-Song in [40]. They addressed issues with the choice of Chern character needed allowing one to define DT invariants, named "generalised DT invariants", for any element of the numerical Grothendieck group of $\mathrm{Coh}(X)$.

At the same time Kontsevich-Soibelman developed a theory of *motivic* DT invariants in [44] and [45], for which the output was no longer a number but instead a class inside the Grothendieck ring of motives. When taking the Euler characteristic one recovers the usual numerical DT invariants. These invariants were defined for non-commuative compact Calabi-Yau threefolds and in [45] Kontsevich-Soibelman showed that these agreed with the generalised DT invariants of Joyce-Song when the two cases overlapped.

Also in [45] the theory of *cohomological* DT invariants was developed, whereby the invariants were now the cohomology groups of sheaves on the threefold with the property that the Euler characteristic of these cohomology groups would return the numerical DT invariants. Furthermore they endowed this cohomology with a multiplication giving rise to cohomological Hall algebras (CoHAs).

In this thesis we study both the motivic and cohomological DT theory of a specific type of varieties, namely *character varieties* or *character stacks*. In the following we define a a character stack to be the stack of representations $\mathrm{Rep}_n(\mathbb{C}[\pi_1(M)])$ of the $\mathbb{C}$-algebra over the fundamental group of a real manifold M. We focus on two cases- when M is the mapping torus $M_{g,\varphi}$ of a genus g Riemann surface Σ_g under a finite order automorphism $\varphi : \Sigma_g \xrightarrow{\sim} \Sigma_g$, and when M is just the Riemann surface Σ_g.

In the first of these cases for the mapping torus $M_{g,\varphi}$ we aim to study its motivic DT theory. This first step in doing this is the following theorem proved in chapter 6

Theorem 0.0.1 (Theorem 6.3.17)**.** *Let Σ_g be a Riemann surface of genus g and let φ*

be an orientation-preserving automorphism of Σ_g of order n. Let Δ be a brane tiling of Σ_g which is preserved under φ such that the size of the order of each vertex in the dual quiver $Q = Q_\Delta$ is n. Choose generating arrows from Q and isomorphism arrows to construct the quiver Q' with potential W', such that the W' is homogeneous of degree n and does not contain both an isomorphism arrow $r_{i_u,t}$ and its inverse $r^{-1}_{i_u,t}$. Then we have an isomorphism of algebras

$$\mathrm{Jac}(\widetilde{Q'}, W') \xrightarrow[\sim]{\Psi} \mathrm{Mat}_{m\times m}\big(\mathbb{C}[\pi_1(M_{g,\varphi})]\big)$$

where $m = |Q'_0|$.

Theorem 0.0.1 gives us a Jacobi algebra presentation for the fundamental group algebra of the mapping torus. The theory of motivic DT invariants for quivers with potential is perhaps the simplest case and has been extensively studied in the literature, see for example [45], [16], [54], [55], [8]. The critical locus structure of the stack of representations of a Jacobi algebra therefore allows us to easily define the motivic DT invariants for the fundamental group algebra of the mapping torus, and we explore methods to calculate these motives.

We then move on to looking at the fundamental group algebra of the Riemann surface Σ_g. This has been studied by Davison in [14] where he found a 2D Jacobi algebra presentation of $\mathbb{C}[\pi_1(\Sigma_g)]$ and then used dimensional reduction to relate the cohomological DT invariants of $C[\pi_1(\Sigma_g \times S^1)]$ to the standard cohomology of the character stack of $C[\pi_1(\Sigma_g)]$. Note that $C[\pi_1(\Sigma_g)]$ is 2-dimensional whilst DT invariants are only defined for 3-dimensional objects and hence why the algebra $C[\pi_1(\Sigma_g \times S^1)]$ was needed. In chapter 7 we study a 2D analogue of the cohomological Hall algebra of Kontsevich-Soibelman, first developed by Shciffman-Vasserot in [61] for the space of commuting matrices. Our main result is the following theorem

Theorem 0.0.2 (Theorem 7.4.6). *Let Σ_g be a Riemann surface of genus g with brane tiling Δ giving dual quiver Q_Δ and potential W_Δ. Fix a cut E for W_Δ and a maximal tree T in $Q_\Delta \setminus E$, and let Q be the quiver obtained by contracting T in $Q_\Delta \setminus E$ with corresponding potential W. Then the 2D CoHA*

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\pi_1(\Sigma_g)), \mathbb{Q})^\vee$$

is isomorphic as an algebra to the 2D CoHA

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)), \mathbb{Q})^\vee.$$

which gives an isomorphism of algebras between the 2D CoHA of the character stack $\mathbb{C}[\pi_1(\Sigma_g)]$ and the 2D CoHA of the 2D Jacobi algebra $\mathrm{Jac}(\widetilde{Q}, W, E)$. We can then utilise Davison's result from [14] to relate the 2D CoHA of the character stack to the 3D CoHA of the Jacobi algebra and hence to cohomological DT theory.

We briefly outline the thesis below.

Chapter 1 gives an introduction to moduli problems and stacks which underpin the spaces on which we do DT theory.

Chapter 2 is an overview on quivers with potential and the 2D version of Jacobi algebras.

Chapter 3 introduces the constructible derived category, operations on that derived category, perverse sheaves, and vanishing cycles sheaves which forms the basis of cohomological DT theory.

Chapter 4 concerns Calabi-Yau algebras motivating the existence of the algebra isomorphism given in Theorem 0.0.1.

Chapter 5 covers the basics of motivic DT theory, introducing the Grothendieck ring of varieties and operations ond that ring, as well as motivic vanishing cycles which are used analogously to the sheaf case in defining motivic DT invariants.

Chapter 6 contains our first main results. Brane tilings are defined providing the link between fundamental group algebras and Jacobi algebras. We then construct the appropriate quiver with potential that provides us with the isomorphism to the fundamental group algebra as stated in Theorem 0.0.1. The chapter ends with a discussion and ideas on how this isomorphism can be used to calculate the motivic DT invariants of the character stack of the mapping torus.

Chapter 7 focuses on our other new results. It begins with the definition of the 2D CoHA, and then explains the different versions of this construction that can be obtained on the character stack of the Riemann surface from the work presented in [14]. We then prove that these multiplications give the same algebra structure on the cohomology of the character stack as in Theorem 0.0.2.

Chapter 1

Moduli problems and stacks

Stacks will play a central role in this thesis, being the geometric objects that we are interested in calculating (refined) Donaldson-Thomas invariants of, so we motivate and rigorously define them in this preliminary chapter. The main references for this chapter are [48] for Grothendieck topologies and sheaves, and [26] and [67] for stacks.

1.1 Moduli problems, Gronthendieck topologies and sheaves

In algebraic geometry we are often concerned with counting geometric objects with certain properties. A *moduli problem* consists of trying to classify isomorphism classes of certain geometric objects on schemes. More formally we can write it as a functor $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ where

$$X \mapsto \big\{ \text{isomorphism classes of objects over } X \big\}.$$

If we have a scheme S such that there exists a natural isomorphism of functors

$$F \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sch}}(-, S)$$

then we say that the moduli problem F is *representable* and that S is a *fine moduli space* for F. Fine moduli spaces will not always exist but there may be a scheme T with a natural transformation

$$\eta : F \to \mathrm{Hom}_{\mathrm{Sch}}(-, T)$$

such that the following two properties hold

i) For all schemes Y with a natural transformation $\xi : F \to \mathrm{Hom}_{\mathrm{Sch}}(-, Y)$, there exists a unique natural transformation $\nu : \mathrm{Hom}_{\mathrm{Sch}}(-, T) \to \mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ such that $\xi = \nu \circ \eta$.

ii) For all algebraically closed fields k we have a bijection

$$\eta(\mathrm{Spec}(k)) : F(\mathrm{Spec}(k)) \to \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(k), T).$$

We call such a T a *coarse moduli space* for F. Moduli spaces are useful because they often allow one to parametrise objects in the moduli problem (i.e. the isomorphism classes of geometric objects we are trying to classify) by much simpler parameters that come from coordinates on the moduli space. However most moduli problems do not have moduli spaces and problems with the existence of a moduli space is often due to automorphisms of the objects being parametrised. One way of resolving this issue is by enlarging the category of schemes and thereby allow more complicated spaces to represent our moduli functor. To do this we need the notion of a topology on a category.

Definition 1.1.1. Let $\mathcal{C}$ be a category with fibred products. A *Grothendieck pre-topology* on $\mathcal{C}$ consists of collections of morphisms $\left\{U_i \xrightarrow{f_i} X\right\}_{i\in I}$ called *coverings* for every object $X \in \mathrm{Obj}(\mathcal{C})$ such that

i) Every isomorphism $U \xrightarrow{\sim} X$ is a covering.

ii) If we have a morphism $Y \to X$ and $\{U_i \to X\}_{i\in I}$ is a covering of X then $\{U_i \times_X Y \to Y\}_{i\in I}$ is a covering of Y.

iii) If $\{U_i \to X\}_{i\in I}$ is a covering of X and for each i we have a covering $\{U_{ij} \to U_i\}_{j\in J}$ of U_i then $\{U_{ij} \to X\}_{i\in I, j\in J}$ is a covering of X.

Remark 1.1.2. The axioms of a pre-topology are reminiscent of the axioms of a topological space whereby axiom i) relates to condition that the whole set is open, ii) corresponds to how finite intersections of opens are open and iii) corresponds to how arbitrary unions of opens are open.

Remark 1.1.3. A pre-topology on $\mathcal{C}$ uniquely defines a *Grothendieck topology* on $\mathcal{C}$ [[48] Section III.2] but for our purposes the data of the pre-topology is sufficient.

A category together with a (pre)-topology is called a *site*.

Example 1.1.4. The category of topological spaces Top has a canonical pre-topology given by letting $\left\{U_i \xrightarrow{f_i} X\right\}_{i\in I}$ be a covering if and only if f_i is an open immersion for all i and $\bigcup_i U_i = X$ i.e. if and only if $\{U_i\}_{i\in I}$ is a covering of X in the standard definition of the word.

Since our focus is on the category of schemes we describe a few pre-topologies specifically for Sch. First we have *Zar* the Zariski site where, as for Top, the coverings are just the usual coverings with the Zarsiki topology on each scheme. Next we have *smth* the smooth site where coverings $\{U_i \xrightarrow{f_i} X\}$ are finite collections given by the conditions that f_i is smooth for all i and that $\bigcup_i f_i(U_i) = X$. The main topology we shall consider is *ét* the étale topology. This has the same conditions as smth except we require that the f_i are étale morphisms as well. We then have the *fppf* site in which the f_i are finitely presented flat morphisms. Finally we have the *fpqc* site where the f_i are flat and quasi-compact morphisms.

Remark 1.1.5. These topologies on Sch can be ordered by coarseness as

$$\text{Zar} \subset \text{ét} \subset \text{smth} \subset \text{fppf} \subset \text{fpqc}.$$

The finer topologies are needed because in general the Zariski site does not have "enough" open sets. For example when working with varieties over $\mathbb{C}$ we can also use the Euclidean topology to form a site, and we would like our scheme-theoretic approach to mimic an analytical one as closely as possible. But this would not be possible using the Zariski site; for example we cannot define a (local) inverse to the regular map $x^2 : \mathbb{A}^1 \to \mathbb{A}^1$ in the Zariski topology as the square root function is not an algebraic map. In these finer topologies we can remedy this situation and for our purposes the étale site is close enough to the ideal analytic case; for example an étale local inverse to x^2 is given by the étale morphism induced by the ring map $\mathbb{C}[x] \to \mathbb{C}[x, y]/(x - y^2)$.

Definition 1.1.6. Let $F : \text{Sch}^{\text{op}} \to \text{Set}$ be a functor and fix a topology on Sch. Then F is a *sheaf* with respect to that topology if for every cover $\left\{U_i \xrightarrow{f_i} X\right\}$ we have

i) If M and N are elements in $F(X)$ such that $M|_i = N|_i$ for all i then $M = N$.

ii) For each i let $M_i \in F(U_i)$. If $M_i|_{ij} = M_j|_{ij}$ for all i, j then there exists some $M \in F(X)$ such that $M|_i = M_i$ for all i.

where $M|_i$ denotes $F(f_i)(M)$ and $M_i|_{ij}$ denotes $F(\pi_{ij,i})(M_i)$ for the projection $\pi_{ij,i} : U_i \times_X U_j \to U_i$.

Sheaves on a site and natural transformations of those sheaves form a category in the obvious way. This category has all the nice properties that one would expect from a category of sheaves on a topological space such as having limits and colimits [[48] Section III.4 Proposition 4].

Definition 1.1.7. Let $F \to J$ and $H \to J$ be morphisms of sheaves. The *fibre product* $F \times_J H$ is defined to be the sheaf that sends the scheme X to

$$F \times_J H(X) = F(X) \times_{J(X)} H(X).$$

Note there is a small amount of work needed to verify that the fibre product of sheaves is indeed a sheaf itself (see [Maclane Moedrijk]...).

Lemma 1.1.8 ([67] Theorem 2.55)**.** *For any scheme Y the functor* $\text{Hom}_{\text{Sch}}(-, Y)$ *is a sheaf in the fpqc topology.*

In particular $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ is a sheaf in the étale topology too. We often use the notation Y to denote both the scheme Y and the sheaf $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$. A sheaf F is said to be *representable* by a scheme Y if F is isomorphic to the sheaf $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$.

Definition 1.1.9. Let $f : F \to H$ be a morphism of sheaves on Sch. Then f is *representable* if for any morphism $Y \to H$ where Y is a scheme viewed as a sheaf, there exists a scheme S_Y such that the fibre product of sheaves

$$F \times_G Y$$

is representable by S_Y.

Definition 1.1.10. A sheaf $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ is called an *algebraic space* if the diagonal morphism

$$\Delta_F : F \to F \times F$$

is representable and there is a surjective étale morphism from a scheme Y

$$Y \to F.$$

Algebraic spaces are a kind of mid-ground between schemes and stacks. They can be thought of as affine schemes that have been glued together using the étale topology, as opposed to the Zariski topology. Let Sp denote the category of algebraic spaces. We have an inclusion of categories $\mathrm{Sch} \hookrightarrow \mathrm{Sp}$.

Remark 1.1.11. The condition that the diagonal Δ_F is representable is similar to that of a scheme being separated. In particular if the diagonal is representable then for any morphism of sheaves $S \to F$ where S is a scheme is also representable [[58] Lemma 64.5.10]. This allows us to transfer properties of morphisms of schemes onto algebraic spaces by saying that a morphism $S \to F$ has the property if for all schemes $T \to F$ the pullback $S \times_F T \to T$ (which is a morphism of schemes as $S \times_F T$ is representable) has that property.

1.2 Stacks

Stacks were constructed as a further enlargement of the category of schemes to help deal with the representability of an even larger class of moduli problems that algebraic spaces could not handle. Defining a stack in a similar manner to a sheaf is not sufficient as we run into issues with automorphisms.

Example 1.2.1. We work over $\mathbb{C}$. Fix a variety Y and suppose our moduli problem asks us to classify vector bundles (of fixed rank and Chern class) over Y. Then our functor is given by $\mathfrak{M}_Y : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}$ that sends an arbitrary scheme X to the set of isomorphism classes of vector bundles on $Y \times X$ (i.e. isomorphism classes of families of vector bundles on Y parametrised by X) and sends a morphism $f : X \to Z$ to the usual pullback morphism of vector bundles. Now suppose $Y = \mathrm{Spec}(\mathbb{C})$. Then our functor $\mathfrak{M}_{\mathrm{Spec}(\mathbb{C})}$ sends a scheme X to the set of isomorphism classes of vector bundles (with a fixed rank and Chern class) that are flat over X. Take a non-trivial vector bundle $V \to X$ and let $\{U_i\}$ be a trivialising cover of X for V. If $\mathfrak{M}_{\mathrm{Spec}(\mathbb{C})}$ was a sheaf then because $V|_{U_i} \cong U_i \times \mathbb{A}^r$ for all i the first property of a sheaf would imply that $V \cong X \times \mathbb{A}^r$ which contradicts V being non-trivial. The issue here with $\mathfrak{M}_{\mathrm{Spec}(\mathbb{C})}$ being a sheaf arises from the fact that non-trivial automorphisms of the trivial vector bundles $(U_i \cap U_j) \times \mathbb{A}^r$ allow us to glue the trivial pieces together into something which is not globally trivial. If this moduli problem had a fine moduli space M then the map $X \to M$ corresponding to the vector bundle V must send every point $x \in X$ to the same point in M as the fibre of V over any x is just a copy of $\mathbb{A}^r$. Hence the map $X \to M$ factors through a point and so every vector bundle on X must be trivial, which is obviously not true for arbitrary X.

To help deal with this the notion of 2-categories and 2-functors are introduced.

Definition 1.2.2. A *2-category* is a category $\mathcal{C}$ in which the Hom-sets are themselves categories, subject to the following conditions:

i) For all objects $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ there exists a composition functor

$$\mu_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z).$$

ii) For all objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ there exists an object $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that we have the equalities of functors

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}.$$

iii) For all objects $X, Y, Z, W \in \mathrm{Obj}(\mathcal{C})$ we have

$$\mu_{X,Z,W} \circ \big(\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(Z,W)}\big) = \mu_{X,Y,W} \circ \big(\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \times \mu_{Y,Z,W}\big).$$

We call objects $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *1-morphisms* and morphisms in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *2-morphisms.*

Example 1.2.3. The archetypal example of a 2-category is the category of categories Cat. Its objects are categories, its 1-morphisms are functors, and its 2-morphisms are natural transformations between these functors. This 2-category also contains the sub-2-category of groupoids Grpd. A *groupoid* is category in which all the morphisms are isomorphisms. Note that any 1-category can be upgraded to a 2-category simply by declaring that the only 2-morphisms are the identity morphisms.

Definition 1.2.4. A *commutative diagram* of 1-morphisms f, g, h in a 2-category is a diagram of the form

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$

where α is a 2-isomorphism from $g \circ f$ to h.

Definition 1.2.5. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$ associates to each object X, 1-morphism f, and 2-morphism α in $\mathcal{C}$ an object $F(X)$, 1-morphism $F(f)$, and 2-morphism $F(\alpha)$ in $\mathcal{D}$ respectively, such that certain technical conditions hold (see [[26] Appendix B] for further details).

We now extend the notion of a 2-functor to the notion of a "2-sheaf", just as a sheaf extends the notion of a presheaf/ (contravariant) functor.

Definition 1.2.6. Let $\mathcal{F} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$ be a 2-functor where Sch is given the trivial 2-category structure, and fix a topology on Sch. Then $\mathcal{F}$ is a *stack* with respect to that topology if for every cover $\left\{U_i \xrightarrow{f_i} X\right\}$ we have

i) If M and N are two objects in $\mathcal{F}(X)$ where for all i we have morphisms $\varphi_i : M|_i \to N|_i$ such that for all i, j $\varphi_i|_{ij} = \varphi_j|_{ij}$, then there exists a morphism $\varphi : M \to N$ such that $\varphi|_i = \varphi_i$ for all i.

ii) If M and N are two objects in $\mathcal{F}(X)$, and $\varphi : M \to N$ and $\psi : M \to N$ are two morphisms such that $\varphi|_i = \psi|_i$ for all i, then $\varphi = \psi$.

iii) For each i let M_i be an object in $\mathcal{F}(U_i)$ and suppose we have morphisms $\varphi_{ij} : M_j|_{ij} \to M_i|_{ij}$ satisfying the cocycle condition

$$\varphi_{ij}|_{ijk} \circ \varphi_{jk}|_{ijk} = \varphi_{ik}|_{ijk}$$

for all i, j, k. Then there exists an object M in $\mathcal{F}(X)$ and isomorphisms $\varphi_i : M|_i \to M_i$ such that $\varphi_{ji} \circ \varphi_i|_{ij} = \varphi_j|_{ij}$ for all i, j.

So stacks are sheaves evaluated on groupoids in the setting of 2-categories with the additional condition that there always exists an object that will allow us to glue together pieces using isomorphisms that satisfy the cocycle condition. This alleviates the issue we saw in Example 1.2.1 where for vector bundles, if we had just used the standard notion of a sheaf, we had no element that could be the glueing of trivial pieces along non-trivial automorphisms.

Definition 1.2.7. Let G be an algebraic group. A *principal G-bundle* (or *G-bundle*) over a scheme X is a scheme E with a G-action and a morphism $E \xrightarrow{\pi} X$ such that there exists an étale cover $\left\{U_i \xrightarrow{f_i} X\right\}$ where

$$U_i \times_X E \cong U_i \times G$$

with G acting only on the second factor in $U_i \times G$ (i.e. E is étale locally trivial with fibres G). We say an algebraic group G is *special* if any principal G-bundle is also Zariski locally trivial.

Remark 1.2.8. The general linear group GL_n is a special algebraic group. Heuristically, to see this take a trivialising cover of the GL_n-bundle, replace the copies of GL_n with copies of $\mathbb{A}^n$, and then glue everything back together using the same glueing data from the GL_n-bundle. This gives us an étale locally trivial vector bundle. But étale locally trivial vector bundles are Zariski locally trivial i.e. just normal vector bundles (see [[29] Exercise III.10.5]) which implies that the GL_n-bundle is also Zariski locally trivial.

For most of the time from now on we work in the étale site.

Example 1.2.9.

a) For an arbitrary scheme Y the sheaf $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ is clearly a stack. This gives us an inclusion (using the Yoneda lemma) from the category of schemes into the category of stacks. We write Y to mean both the scheme Y and the stack $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ when the context is clear. A stack that is isomorphic to Y is called *representable by a scheme.*

b) Similarly if F is an algebraic space then $\mathrm{Hom}_{\mathrm{Sp}}(-, F)$ is also a stack. A stack that is isomorphic to F is called *representable by an algebraic space.*

c) Returning to the example of vector bundles let Y be a variety over $\mathbb{C}$ and let $\mathcal{M}_Y : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Grpd}$ send a scheme X to the groupoid $\mathcal{M}_Y(X)$ of vector bundles on $Y \times X$ along with vector bundle isomorphisms. Then $\mathcal{M}_Y$ is a stack.

d) For an algebra A define the stack $\mathrm{Rep}_n(A)$ of n-dimensional representations of A by sending a scheme X to the groupoid of vector bundles V of rank n over X with an algebra homomorphism $A \to \mathrm{End}(V)$, with morphisms the vector bundle isomorphisms that commute with the prescribed algebra homomorphisms.

e) The key example of a stack for us is that of a *quotient stack.* Let G be an affine smooth algebraic group acting on a scheme Y. The stack Y/G is defined by sending a scheme X to the groupoid of principal G-bundles over X with a G-equivariant map to Y, namely diagrams of the form

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "f"'] & X \\ Y & \end{tikzcd}$$

where π is a principal G-bundle and f is G-equivariant, with morphisms consisting of G-bundle isomorphisms that are compatible with the G-equivariant maps.

Definition 1.2.10. Given morphisms of stacks $f : \mathcal{F} \to \mathcal{H}$ and $g : \mathcal{G} \to \mathcal{H}$ the *fibre product* $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is defined as the stack that sends a scheme X to the groupoid whose objects are triples (M, N, α) where $M \in \mathcal{F}(X)$, $N \in \mathcal{G}(X)$ and

$$\alpha : f(X)(M) \xrightarrow{\sim} g(X)(N) \in \mathrm{Hom}_{\mathcal{H}(X)}\big(f(X)(M), g(X)(N)\big)$$

and whose morphisms $(M_1, N_1, \alpha_1) \to (M_2, N_2, \alpha_2)$ are given by pairs

$$\begin{aligned}\varphi &: M_1 \to M_2 \in \mathrm{Hom}_{\mathcal{F}(X)}(M_1, M_2)\\ \psi &: N_1 \to N_2 \in \mathrm{Hom}_{\mathcal{G}(X)}(N_1, N_2)\end{aligned}$$

such that

$$\alpha_2 \circ f(X)(\varphi) = g(X)(\psi) \circ \alpha_1.$$

One can check that this definition satisfies the usual definition of a categorical fibre product.

Definition 1.2.11. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is called *representable* if for any morphism $Y \to \mathcal{G}$ where Y is a scheme, there exists an algebraic space S_Y such that the fibre product of stacks

$$\mathcal{F} \times_{\mathcal{G}} Y$$

is representable by S_Y. Given any representable morphism f and a property of morphisms of schemes or algebraic spaces that is local with respect to the chosen topology on Sch and is stable under base-change, for example separated, quasi-compact, flat, smooth, surjective, finite type, etc. we say that f has this property if for every morphism $Y \to \mathcal{G}$ where Y is a scheme, the pullback $\mathcal{F} \times_{\mathcal{G}} Y \to Y$ as a morphism of algebraic spaces has this property.

Definition 1.2.12. An *atlas* for a stack $\mathcal{F}$ is a scheme Y with a smooth, surjective morphism

$$Y \to \mathcal{F}.$$

Definition 1.2.13. Let $\mathcal{F}$ be a stack in the fppf topology. Then $\mathcal{F}$ is an *Artin stack* if the diagonal $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable, separated and quasi-compact, and $\mathcal{F}$ has an atlas.

Example 1.2.14. Let G be a smooth affine group acting on a scheme Y. Then the quotient stack Y/G is an Artin stack [[57] Section 8.4.1]. The natural quotient map $Y \to Y/G$ is an atlas for Y/G. To see this we must show that this morphism is representable, smooth and surjective. So let $S \to Y/G$ be a morphism of stacks where S is a scheme. By the Yoneda lemma this is equivalent to an object in $Y/G(S)$ i.e. a diagram of the form

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "f"'] & S \\ Y & \end{tikzcd}$$

where π is a principal G-bundle and f is G-equivariant. Similarly the quotient map $Y \to Y/G$ corresponds to the diagram

$$\begin{tikzcd}
Y \times G \arrow[r, "p_Y"] \arrow[d, "\theta"'] & Y \\
Y &
\end{tikzcd}$$

where p_Y is the projection onto Y and θ is the group action of G on Y. Hence for a scheme X objects in $Y \times_{Y/G} S(X)$ are morphisms $X \xrightarrow{\varphi} Y$ and $X \xrightarrow{\psi} S$ along with an isomorphism between the outsides of the diagrams

$$\begin{tikzcd}
X \times G \arrow[r, "p_X"] \arrow[d, "\varphi \times \mathrm{id}_G"'] & X \arrow[d, "\varphi"] & & E' \arrow[r, "\pi'"] \arrow[d] & X \arrow[d, "\psi"] \\
Y \times G \arrow[r, "p_Y"] \arrow[d, "\theta"'] & Y \arrow[r, phantom, ""] & {} \arrow[r, "\sim"] & E \arrow[r, "\pi"] \arrow[d, "f"'] & S \\
Y & & & Y &
\end{tikzcd}$$

in which both upper squares are Cartesian. This data is equivalent to a diagram of the form

$$\begin{tikzcd}
X \times G \arrow[r, "p_X"] \arrow[d] & X \arrow[d, "\psi"] \\
E \arrow[r, "\pi"] \arrow[d, "f"'] & S \\
Y &
\end{tikzcd}$$

in which the upper square is Cartesian and the composition of the left-hand column is equal to $\theta \circ (\varphi \times \mathrm{id}_G) : X \times G \to Y$.

This is turn is equivalent to the data of a morphism of schemes $X \to E$. Indeed given such a diagram we can define a morphism $X \to E$ by considering the composition $X \hookrightarrow X \times G \to E$ where the inclusion $X \hookrightarrow X \times G$ sends $x \in X \mapsto (x, 1_G)$. Conversely given a morphism $\alpha : X \to E$ we can construct such a diagram by taking $\psi = \pi \circ \alpha : X \to Y$ and $\varphi = f \circ \alpha : X \to S$. Therefore the stack $Y \times_{Y/G} S$ is representable by the

G-bundle E and we have a Cartesian square

$$\begin{tikzcd}
Y \times_{Y/G} S \cong E \arrow[r, "\pi"] \arrow[d, "f"'] & S \arrow[d] \\
Y \arrow[r] & Y/G
\end{tikzcd}$$

The morphism $E \xrightarrow{\pi} S$ is clearly smooth and surjective for all S and hence so is the quotient map $Y \to Y/G$.

Chapter 2

Quivers and Jacobi algebras

Quivers and their representations will form much of the basis of the geometric objects this thesis will focus on. They have been studied in great detail across the subject and have been used extensively in DT theory, see [5], [10], [40], [45], [8], [14], [15], [16], [17].

2.1 Quivers and their representations

Definition 2.1.1. A *quiver* is finite directed graph i.e. a pair $Q = (Q_0, Q_1)$ where Q_0 is a finite set of vertices and Q_1 is a finite set of directed edges or *arrows* between those vertices. For an arrow $a \in Q_1$ we denote by $s(a)$ the vertex at its *source*, $t(a)$ the vertex at its *target* and we write $a : s(a) \to t(a)$. A *path* in Q is a finite string of arrows written as $p = a_r \dots a_1$ such that $s(a_{i+1}) = t(a_i)$ for all $i = 1, \dots, r-1$. A *loop* is a an arrow a such that $s(a) = t(a)$ and a *cycle* is a path $p = a_r \dots a_1$ such that $s(p) := s(a_1) = t(a_r) =: t(p)$. For $i \in Q_0$ let e_i denote the constant path in Q at the vertex i.

Definition 2.1.2. A *representation* of a quiver Q is a family of vector spaces V_i for all $i \in Q_0$ along with a choice of linear maps $f_a : V_{s(a)} \to V_{t(a)}$ for all $a \in Q_1$. A *morphism* between representations $M = (V_i, f_a)$ and $N = (W_i, g_a)$ of a quiver Q is a family of linear maps $\phi_i : V_i \to W_i$ for all $i \in Q_0$ such that $\phi_{t(a)} \circ f_a = g_a \circ \phi_{s(a)}$ for all $a \in Q_1$.

We can compose morphisms of representations of Q and therefore we obtain a category of representations of the quiver Q which we denote by $\mathfrak{Rep}(Q)$. We will focus on *finite dimensional* representations of Q i.e. those where the dimension of V_i is finite for all $i \in Q_0$. If we let $n \in \mathbb{N}^{Q_0}$ denote a *dimension vector* we can consider the subcategory of n-dimensional representations of Q

$$\mathfrak{Rep}_n(Q) \subset \mathfrak{Rep}(Q).$$

Example 2.1.3. Two quivers with an example of a representation of each below.

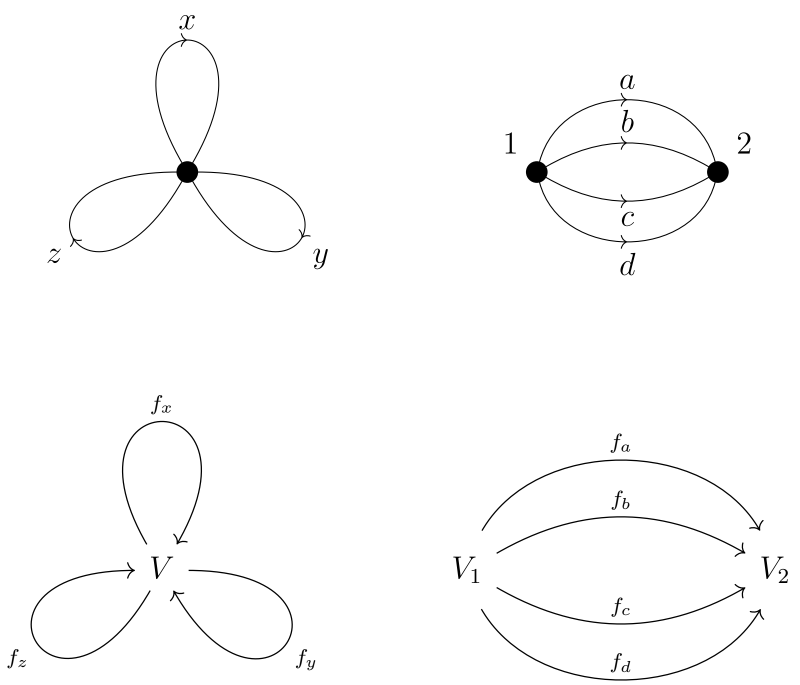


Figure 2.1: On the left we have the 3-loop quiver $Q_{(3)}$ and on the right we have the conifold quiver.

Definition 2.1.4. For a field k the *path algebra* kQ of the quiver Q is the k-algebra whose generators are the constant paths e_i for $i \in Q_0$ and the arrows $a \in Q_1$ with the relations

$$\begin{aligned} e_i^2 &= e_i && \text{for all } i \in Q_0 \\ e_i e_j &= 0 && \text{if } i \neq j \\ a e_{s(a)} &= a = e_{t(a)} a && \text{for all } a \in Q_1. \end{aligned}$$

Multiplication in the path algebra is therefore given by

$$q \cdot p = \begin{cases} qp\,, & s(q) = t(p) \\ 0\,, & \text{otherwise} \end{cases}$$

where qp is the concatentation of the paths p and q.

The unit in kQ is the sum of all the constant paths $\sum_{i \in Q_0} e_i$.

Proposition 2.1.5 (cf. [6] Proposition 1.2.2)**.** *The category*

$$\mathfrak{Rep}_{\mathrm{f.d.}}(Q) := \bigsqcup_{n \in \mathbb{N}^{Q_0}} \mathfrak{Rep}_n(Q)$$

of finite-dimensional representations of Q is equivalent to the category $\mathrm{Mod}_{\mathrm{f.d.}}$-$kQ$ *of unital finite-dimensional left kQ-modules.*

Proof. Fix a dimension vector $n \in \mathbb{N}^{Q_0}$ and consider an n-dimensional representation (V_i, f_a) of Q. Let $V = \bigoplus_i V_i$ and let kQ act on V via the maps f_a i.e. for an arrow $a : i \to j \in Q_1$ let $a : V \to V$ be given by the composition $V \to V_i \xrightarrow{f_a} V_j \hookrightarrow V$ then extend to all paths in the natural way. This gives us a kQ-module V of finite dimension $N = \sum_i n_i$.

Conversely, given a kQ-module V for each $i \in Q_0$ let V_i be the image of V under the action of $e_i \in kQ$. Then for an arrow $a : i \to j$, since we have $e_j \cdot a \cdot e_i = a$ in kQ the action of a on V must factor through $V_i \to V_j$ which gives us the map f_a. □

The data of a representation of Q can be summarised as follows. For each $i \in Q_0$ fix vector spaces $V_i = k^{n_i}$ of dimension n_i. Then a representation of Q for these vector spaces is a choice of maps $f_a : k^{n_i} \to k^{n_j}$ for all $a : i \to j \in Q_1$. Hence it is an element in the space

$$M_n(Q) := \prod_{a:i\to j\in Q_1} \mathrm{Hom}(k^{n_i}, k^{n_j}) = \prod_{a:i\to j\in Q_1} \mathrm{Mat}_{n_i\times n_j}(k).$$

However we can also vary the choices of vector spaces which amounts to choosing different bases. Therefore if $g_i \in \mathrm{GL}_{n_i}(k)$ is the change of basis matrix from k^{n_i} to the new choice of vector space W_i then a representation of Q using the vector spaces W_i is given by $(g_j f_a g_i^{-1})_{a\in Q_1}$ where $(f_a) \in M_n(Q)$.

Hence to classify representations of Q up to isomorphism we take the quotient of $M_n(Q)$ by the gauge group $G_n = \prod_{i\in Q_0} \mathrm{GL}_{n_i}(k)$ that acts via simultaneous conjugation. Define the *stack of n-dimensional representations of Q* as the quotient stack

$$\mathrm{Rep}_n(Q) = M_n(Q)/G_n.$$

Proposition 2.1.6 (cf. [50] Example 2.25)**.** *For a quiver Q the disjoint union of stacks*

$$\bigsqcup_{n\,:\,\sum_i n_i = N} \mathrm{Rep}_n(Q)$$

of finite dimensional representations whose total dimension is $N \in \mathbb{N}$ is isomorphic to the stack $\mathrm{Rep}_N(kQ)$ *of N-dimensional representations of the algebra kQ.*

Definition 2.1.7. Let $k\widetilde{Q}$ denote the *localised path algebra of* Q in which we add formal inverses a^{-1} to the algebra kQ for each $a \in Q_1$ with the property that $a^{-1}a = e_{s(a)}$ and $aa^{-1} = e_{t(a)}$.

We also have the corresponding notion of a *partially localised path algebra* in which we only localise with respect to a chosen subset of arrows $S \subset Q_1$.

2.2 Quivers with potential

To get more interesting stacks of representations we now look at quivers with relations. We focus on relations that come from a potential on the quiver due to the nice properties such algebras and their representations have.

Definition 2.2.1. A *potential* W on a quiver Q is an element of the k-vector space $kQ/[kQ, kQ]$, which has as a basis the set of cycles in Q up to cyclic permutation.

In order to get relations in the path algebra kQ we consider the "derivatives" of a potential W. There is a natural map $\sigma : kQ/[kQ, kQ] \to kQ$ from the vector space of potentials to the path algebra, given on basis elements by sending a cycle to the sum of all its possible cyclic permutations

$$[p = a_r \dots a_1] \mapsto \sum_i a_i \dots a_1 a_r \dots a_{i+1}.$$

Then we define the derivative $\partial_a : kQ/[kQ, kQ] \to kQ$ with respect to the arrow $a \in Q_1$ as the map that sends

$$[p] \mapsto a^{-1}\sigma([p])$$

where for a path $q = b_s \dots b_1 \in kQ$ we define $a^{-1}q = b_{s-1} \dots b_1$ if $a = b_s$ and $a^{-1}q = 0$ otherwise. In other words, for a potential W on Q we take all the cycles in W that contain the arrow a, cyclically permute a to the front of those cycles and then delete a. We write the derivative of W with respect to $a \in Q_1$ as $\partial W/\partial a = \partial_a(W)$. We obtain an ideal in kQ

$$I_W = (\partial W/\partial a \,:\, a \in Q_1)$$

generated by the derivatives of W with respect to all the arrows in Q.

Definition 2.2.2. The *Jacobi algebra* of the quiver Q with potential W is the quotient algebra

$$\mathrm{Jac}(Q, W) = kQ/I_W.$$

Example 2.2.3. Consider the 3-loop quiver $Q_{(3)}$ from Example 2.1.3. We take the potential

$$\begin{aligned} W &= xyz - xzy \\ &= yzx - yxz \\ &= zxy - zyx \;=\; \text{etc}\dots \end{aligned}$$

Then

$$\begin{aligned}\partial W/\partial x &= yz - zy\\ \partial W/\partial y &= zx - xz\\ \partial W/\partial z &= xy - yx\end{aligned}$$

hence

$$\begin{aligned}\mathrm{Jac}(Q_{(3)}, W) &= k\langle x, y, z\rangle/(yz - zy, zx - xz, xy - yx)\\ &\cong k[x, y, z].\end{aligned}$$

For the quiver Q fix a dimension vector $n \in \mathbb{N}^{Q_0}$ and vector spaces $V_i = k^{n_i}$. A potential W defines a regular map

$$\mathrm{Tr}(W)_n : M_n(Q) \to k$$

that sends a representation (f_a) to the trace of the matrix $W(f_a)$. This is well-defined as the trace of a product of matrices is invariant under the cyclic permutation of those matrices.

Proposition 2.2.4 (cf. [50] Proposition 3.8)**.** *Taking the standard coordinates x_i on $M_n(Q)$, the critical locus of the map $\mathrm{Tr}(W)_n$ (i.e. the locus on which $\partial_{x_i}\mathrm{Tr}(W)_n = 0$ for all i) is exactly the set of representations that vanish on I_W.*

Proof. Consider the representation $f = (f_a) \in M_n$ with coordinates $f_a = (x_{r_a,s_a})$. To calculate $\partial Tr(W)_n/\partial x_{r_a,s_a}$ we only care about the cycles in the potential W that contain the arrow a. Isolating this part of the potential gives exactly $a \cdot \partial W/\partial a$. The entry of the matrix $\partial W/\partial a(f)$ that appears as the coefficient of x_{r_a,s_a} in the trace function is $(\partial W/\partial a(f))_{s_a,r_a}$. Hence the collection of matrices that satisfy $\partial Tr(W)_n/\partial x_{r_a,s_a} = 0$ for all $a : i \to j \in Q_1$ and all $1 \leq r_a \leq n_i,\ 1 \leq s_a \leq n_j$ is equivalent to the zero-locus of all the derivatives $\partial W/\partial a$ for all $a \in Q_1$. □

It follows that an element in $\mathrm{crit}(\mathrm{Tr}(W)_n)$ gives an n-dimensional $\mathrm{Jac}(Q, W)$-module. So for the dimension vector $n \in \mathbb{N}^{Q_0}$ we can consider the quotient stack

$$\mathrm{crit}(\mathrm{Tr}(W)_n)/G_n.$$

Proposition 2.2.5 (cf. [63] Proposition 3.8 and [50] Example 2.25)**.** *For a quiver Q with potential W the disjoint union of stacks*

$$\bigsqcup_{n:\sum_i n_i = N} \mathrm{crit}(\mathrm{Tr}(W)_n)/G_n$$

is isomorphic to the stack $\mathrm{Rep}_N(\mathrm{Jac}(Q, W))$ of N-dimensional representations of the algebra $\mathrm{Jac}(Q, W)$.

Proof. For convenience we restrict to the case in which Q has only one vertex and so $n \in \mathbb{N}$ and $G_n = \mathrm{GL}_n$. The general case follows in similar manner.

Let X be a scheme then an object in $\mathrm{Rep}_n(\mathrm{Jac}(Q,W))(X)$ is a vector bundle V of rank n over X with an algebra homomorphism $\mathrm{Jac}(Q,W) \to \mathrm{End}(V)$. Consider the framing bundle $Fr(V)$ of V over X which parametrises all possible choices of bases in the fibres of V. This can be defined locally by taking a trivialising cover $\{U_i\}$ of V over X where

$$\begin{aligned} Fr(V)|_{U_i} &= \big\{(x,\tau) \,:\, x \in U_i,\, \tau \in \mathrm{Hom}(k^n, V_x) \text{ is invertible}\big\} \\ &\cong U_i \times \mathrm{GL}_n. \end{aligned}$$

giving $Fr(V)$ as a principal GL_n-bundle. For each $a \in Q_1$ the homomorphism $\mathrm{Jac}(Q,W) \to \mathrm{End}(V)$ gives us elements $\hat{a} \in \mathrm{End}(V)$ that satisfy the relations in I_W. Hence we obtain a GL_n-equivariant map $Fr(V) \to \mathrm{crit}(\mathrm{Tr}(W)_n)$ that sends

$$(x,\tau) \longmapsto (\tau^{-1} \circ \hat{a} \circ \tau).$$

This gives us an object in the quotient stack $\mathrm{crit}(\mathrm{Tr}(W)_n)/\mathrm{GL}_n$.

Conversely given a principal GL_n-bundle $P \to X$ with GL_n-equivariant map $\phi : P \to \mathrm{crit}(\mathrm{Tr}(W)_n)$ consider the trivial vector bundle $P \times \mathbb{A}^n \to P$ with the natural GL_n-action. For each $a \in Q_1$ we have an endomorphism $\hat{a}$ of $P \times \mathbb{A}^n$ given by

$$(p,z) \longmapsto \big(p, (\pi_a \circ \phi(p))(z)\big)$$

where $\pi_a : \prod_{a \in Q_1} \mathrm{Mat}_{n\times n}(k) \to \mathrm{Mat}_{n \times n}(k)$ is the projection onto the a-component. The GL_n-equivariance of ϕ ensures that these endomorphisms commute with the GL_n-action on $P \times \mathbb{A}^n$. Then because GL_n is special taking the quotient by GL_n gives a vector bundle of rank n

$$P \times_{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n \to X$$

and the endomorphisms $\hat{a}$ of the vector bundle over P descend to endomorphisms of this vector bundle over X. By construction these endomorphisms satisfy the relations in $\mathrm{Jac}(Q,W)$ hence we get a map $\mathrm{Jac}(Q,W) \to \mathrm{End}(P \times_{\mathrm{GL}_n} \mathbb{A}^n)$ and therefore an object in $\mathrm{Rep}_n(\mathrm{Jac}(Q,W))(X)$.

It remains to show that these two operations are inverses. This can be done locally and so it suffices to show this for the trivial bundle over X. Our first map, over a trivialising cover of the vector bundle V, essentially replaces the copies of $\mathbb{A}^n$ with copies of GL_n while the second map similarly replaces the copies of GL_n with copies of $\mathbb{A}_n$ over a trivialising cover of the GL_n-bundle P. Therefore we get that

$$\begin{aligned} X \times \mathbb{A}^n &\longmapsto Fr(X \times \mathbb{A}^n) = X \times \mathrm{GL}_n \\ &\longmapsto (X \times \mathrm{GL}_n) \times_{\mathrm{GL}_n} \mathbb{A}^n = X \times \mathbb{A}^n \end{aligned}$$

and

$$\begin{aligned} X \times \mathrm{GL}_n \longmapsto & (X \times \mathrm{GL}_n) \times_{\mathrm{GL}_n} \mathbb{A}^n = X \times \mathbb{A}^n \\ \longmapsto & Fr(X \times \mathbb{A}^n) = X \times \mathrm{GL}_n. \end{aligned}$$

Finally the algebra map $\mathrm{Jac}(Q, W) \to \mathrm{End}(X \times \mathbb{A}^n)$ that corresponds to the elements $\hat{a} \in \mathrm{End}(X \times \mathbb{A}^n)$ is sent to the GL_n-equivariant map $m : X \times \mathrm{GL}_n \to \mathrm{crit}(\mathrm{Tr}(W)_n)$ given by

$$(x, g) \mapsto (g^{-1} \circ \hat{a} \circ g)$$

which in turn is sent to the endomorphisms $\mathrm{id}_{X \times \mathbb{A}^n}^{-1} \circ \hat{a} \circ \mathrm{id}_{X \times \mathbb{A}^n} = \hat{a}$ of $X \times \mathbb{A}^n$. On the other hand, given a GL_n-equivariant map $\phi : X \times \mathrm{GL}_n \to \mathrm{crit}(\mathrm{Tr}(W)_n)$ we get endomorphisms

$$\hat{a} : (x, z) \longmapsto \big(x, (\pi_a \circ \phi(x, \mathrm{id}_{\mathrm{GL}_n}))(z)\big)$$

of $X \times \mathbb{A}^n$ which are then sent to the GL_n-equivariant map that sends

$$\begin{aligned} (x, g) \longmapsto & (g^{-1} \circ \hat{a} \circ g) \\ = & (\pi_a \circ \phi(x, g)) \\ = & \phi(x, g). \end{aligned}$$

Hence these operations are indeed inverse. $\square$

We have a corresponding notion of the *localised Jacobi algebra* denoted by $\mathrm{Jac}(\widetilde{Q}, W)$ which is defined as the quotient algebra

$$\mathrm{Jac}(\widetilde{Q}, W) = k\widetilde{Q}/I_W.$$

2.3 2D Jacobi algebras

A variant on the Jacobi algebras described above are what's known as 2D Jacobi algebras. These algebras will be defined in a similar fashion to the standard Jacobi algebras albeit with some restrictions on what generators and relations we end up using. Their name will be explained in the fourth chapter on Calabi-Yau algebras.

Definition 2.3.1. A *cut* $E \subset Q_1$ for the quiver and potential (Q, W) is a choice of arrows such that W is homogeneous of degree 1 with respect to the chosen arrows. Given the triple (Q, W, E) consider the ideal

$$\widehat{I}_{W,E} = \big(\partial W/\partial e\,,\, e \;\big|\; e \in E\big)$$

in kQ. Then define the *2D Jacobi algebra* as

$$\mathrm{Jac}(Q, W, E) = kQ/\widehat{I}_{W,E}.$$

We can also consider the ideal

$$I_{W,E} = \big(\partial W/\partial e \;\big|\; e \in E\big)$$

in $k(Q \setminus E)$, which makes sense since $\partial W/\partial e$ is a sum of paths in $Q \setminus E$ for any $e \in E$ due to the definition of a cut. We write

$$\mathrm{Jac}(Q \setminus E, W, E) = k(Q \setminus E)/I_{W,E}$$

then clearly

$$\mathrm{Jac}(Q \setminus E, W, E) \cong \mathrm{Jac}(Q, W, E)$$

as alternative notation we will use later on.

For the dimension vector $n \in \mathbb{N}^{Q_0}$ let G_n act on $\prod_{e \in E} \mathrm{Mat}_{n_{s(e)} \times n_{t(e)}}(k)$ via simultaneous conjugation. Then define the G_n-equivariant map

$$\partial W/\partial E : M_n(Q \setminus E) \to \prod_{e \in E} \mathrm{Mat}_{n_{s(e)} \times n_{t(e)}}(k)$$

by

$$(R_a)_{a \in Q_1 \setminus E} \longmapsto \left(\frac{\partial W}{\partial e}(R_a)\right)_{e \in E}.$$

Proposition 2.3.2. *We have an isomorphism of stacks*

$$\mathrm{Rep}_n(\mathrm{Jac}(Q, W, E)) \cong (\partial W/\partial E)^{-1}(0)/G_n.$$

Proof. This follows in exactly the same fashion as Proposition 2.2.5. □

We again have the corresponding notion of the *localised 2D Jacobi algebra* denoted by $\mathrm{Jac}(\widetilde{Q}, W, E)$ defined as the quotient

$$\mathrm{Jac}(\widetilde{Q}, W, E) = k(\widetilde{Q \setminus E})/I_{W,E}.$$

Chapter 3

Perverse sheaves and vanishing cycles

In this chapter we cover perverse sheaves and vanishing cycles with the goal to state the "dimensional reduction" theorem of [15] that relates the vanishing cycle cohomology of a $\mathbb{C}^*$-equivariant function on a trivial vector bundle $f = \sum_{i=1}^n f_i y_i : X \times \mathbb{A}^n \to \mathbb{A}^1$ to the compactly supported cohomology of $Z = \{x \in X \ : \ f_i(x) = 0 \text{ for all } i\}$. This will motivate the results we present in Chapter 7 by relating the 2D picture to the 3D picture.

For simplicity all spaces in this chapter will be complex algebraic varieties. For a good introduction to the derived category of sheaves see [30]. The main references for this chapter are [41], [23], and [62].

3.1 The derived category of constructible complexes

Let X be a complex algebraic variety and denote by $\mathrm{Sh}(X) = \mathrm{Sh}_{\mathbb{Q}}(X)$ the abelian category of sheaves of $\mathbb{Q}$-vector spaces on X. For a $\mathbb{Q}$-vector space V we denote by $V_X \in \mathrm{Sh}(X)$ the constant sheaf associated to V. In particular $\mathbb{Q}_X \in \mathrm{Sh}(X)$ (often denoted just by $\mathbb{Q}$ when the context is clear) is the constant sheaf associated to $\mathbb{Q}$. For a map of algebraic varieties $f : X \to Y$ we have the standard pushforward and pullback morphisms

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

and

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X).$$

For sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$ the tensor product

$$\mathcal{F} \otimes \mathcal{G}$$

is also a sheaf on X and we have an internal hom

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

We also have the *pushforward with compact support*

$$f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

that takes a sheaf $\mathcal{F}$ to the sheaf $f_!\mathcal{F}$ given on an open subset $U \subset Y$ by

$$f_!\mathcal{F}(U) := \left\{ s \in \mathcal{F}(f^{-1}(U)) \;\middle|\; f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U \text{ is proper} \right\}.$$

If f is proper then $f_! = f_*$. $f_!$ just like f_* is left-exact.

Proposition 3.1.1 ([41] Proposition 2.5.11)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "\varphi"] \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd}$$

be a Cartesian square. Then we have a natural isomorphism of functors

$$\psi^* f'_! \xrightarrow{\sim} f_! \varphi^*.$$

In particular if φ and ψ are proper this isomorphism is induced from the unit and counit of the adjunction $f^* \dashv f_*$.

Definition 3.1.2. A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is called a *local system* if for all $x \in X$ there exists an open neighbourhood $x \in U \subset X$ and a vector space V such that

$$\mathcal{F}|_U \cong V_U$$

i.e. the sheaf $\mathcal{F}$ is locally constant.

Theorem 3.1.3 ([64] Theorem 2.5.15)**.** *For X path-connected a local system $\mathcal{F}$ on X with stalk V is equivalent to a V-representation of the fundamental group $\pi_1(X)$.*

The idea behind this theorem is fairly straightforward. From a local system we get a representation of the fundamental group by considering a representative loop of each class in $\pi_1(X, x)$ and then calculating the automorphism of $V = V_x$ given by tracing the loop through $\mathcal{F}$. Conversely given a representation $\pi_1(X, x) \xrightarrow{\rho} \mathrm{GL}(V)$ we can consider the sections of the constant sheaf $V_{\widetilde{X}}$ on the universal cover $\widetilde{X}$ of X that are invariant under $\rho(\gamma)$ for all $\gamma \in \pi_1(X, x)$, which gives a local system on X.

Definition 3.1.4. A *stratification* of a variety X is a collection of disjoint locally closed subspaces $X_\alpha \subset X \,:\, \alpha \in A$ such that

$$\bigsqcup_{\alpha\in A} X_\alpha = X$$

and if X_α intersects the closure $\overline{X}_\beta$ then $X_\alpha \subset \overline{X}_\beta$.

Definition 3.1.5. A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is *constructible* if there exists a finite stratification $\bigsqcup_{\alpha=1}^n X_\alpha$ of X such that for all α the restriction $\mathcal{F}|_{X_\alpha}$ is a local system.

Local systems are not preserved by the operations given above, for example if X is connected and $f : X \to \{0,1\}$ sends X to 0 then the stalk of the sheaf $f_*\mathbb{Q}_X$ at 0 is $\mathbb{Q}$ but the stalk at 1 is 0 and hence $f_*\mathbb{Q}_X$ is not a local system. Constructible sheaves are a generalisation of local systems which are preserved under the operations $f_*, f^*, f_!, \otimes$.

Let $\mathrm{D}(X) = \mathrm{D}(\mathrm{Sh}(X))$ denote the derived category of sheaves of $\mathbb{Q}$-vector spaces on X. As the functor f^* is exact it descends to a derived functor

$$f^* : \mathrm{D}(Y) \to \mathrm{D}(X).$$

The left-exact functors f_* and $f_!$ have right-derived functors

$$Rf_* : \mathrm{D}(X) \to \mathrm{D}(Y)$$
$$Rf_! : \mathrm{D}(X) \to \mathrm{D}(Y).$$

The left-exact internal hom has a right-derived functor $R\mathcal{H}om(-,-)$ and the right-exact tensor product has a left-derived functor $\otimes^L$. Henceforth, because we shall only work in the derived category, we drop the derived notation from these functors. The derived functors f^* and f_* remain an adjoint pair $f^* \dashv f_*$ and we denote the unit and counit of this adjunction by

$$\eta^f : \mathrm{id}_{\mathrm{D}(Y)} \to f_*f^*$$

and

$$\sigma^f : f^*f_* \to \mathrm{id}_{\mathrm{D}(X)}$$

respectively.

Theorem 3.1.6 ([41] Proposition 3.1.9, cf. Proposition 3.1.1)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "\varphi"] \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd}$$

be a Cartesian square. Then we have a natural isomorphism of derived functors

$$\epsilon^{\psi,f'} : \psi^* f'_! \xrightarrow{\sim} f_! \varphi^*.$$

In particular if φ and ψ are proper we can write

$$\epsilon^{\psi,f'} = (\sigma^{\psi}) f_! \varphi^* \circ \psi^* f'_!(\eta^{\varphi}). \tag{3.1}$$

Definition 3.1.7. A complex $\mathcal{F} \in \mathrm{D}(X)$ is *constructible* if its cohomology sheaves $\mathcal{H}^i(\mathcal{F}) \in \mathrm{Sh}(X)$ are constructible for all i. The *derived category of constructible complexes* $\mathrm{D}_c(X)$ is the full subcategory of constructible complexes.

3.2 Verdier duality

A right-adjoint also exists for the functor $f_!$. In most cases this is only possible at the level of derived categories. Thus we want a functor

$$f^! : \mathrm{D}(Y) \to \mathrm{D}(X)$$

such that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{D}(Y)}(f_! \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{F}, f^! \mathcal{G})$$

natural in $\mathcal{F} \in \mathrm{D}(X)$ and $\mathcal{G} \in \mathrm{D}(Y)$.

Theorem 3.2.1 ([41] Theorem 3.1.5)**.** *A right-adjoint $f^!$ exists for $f_!$.*

Remark 3.2.2. As $f^!$ is defined to be a right-adjoint it is only unique up to isomorphism.

We use

$$\nu^f : f_! f^! \to \mathrm{id}_{\mathrm{D}(Y)}$$

and

$$\theta^f : \mathrm{id}_{\mathrm{D}(X)} \to f^! f_!$$

to denote the unit and counit for the adjunction $f_! \dashv f^!$.

The construction of the functor $f^!$ is quite complicated and we will only need its values on a few complexes so instead we describe properties of $f^!$ that will allow us to use it in calculations later on.

Proposition 3.2.3 ([41] Section 3.1)**.**

a) Let $f : X \to Y$ be an inclusion of a locally closed subspace. Define $f^\circ : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ by sending a sheaf $\mathcal{G}$ to the sheaf $f^* \mathcal{G}_X$ where $\mathcal{G}_X \subset \mathcal{G}$ is the subsheaf of sections that are supported on the subspace X. Then

$$f^! = Rf^\circ.$$

b) Let $f : X \to Y$ be a map between smooth varieties X and Y. Then there is a canonical choice of $f^!$ such that

$$f^!\mathcal{L} = \mathcal{L}[2\dim(f)]$$

for any local system $\mathcal{L}$ on X.

c) Let $f : X \to Y$ be smooth. Then we can canonically choose

$$f^! = f^*[2\dim(f)].$$

Hence if i is an open inclusion $i^! = i^*$.

d) Let $i : U \hookrightarrow X$ be open and $j : Z \hookrightarrow X$ be its closed complement. Then we have functorial distinguished triangles

$$i_!i^! \to \mathrm{id}_{\mathrm{D}(X)} \to j_*j^* \xrightarrow{[1]}$$
$$j_!j^! \to \mathrm{id}_{\mathrm{D}(X)} \to i_*i^* \xrightarrow{[1]} .$$

e) Let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "\varphi"] \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd}$$

be a Cartesian square. Then we have a natural transformation of functors

$$\varphi^* f'^! \to f^!\psi^*$$

and a natural isomorphism of functors

$$f'^!\psi^* \xrightarrow{\sim} \varphi_* f^!.$$

Definition 3.2.4. For $f : X \to Y$ define the *dualising sheaf of f* to be

$$\omega_{X/Y} := f^!\mathbb{Q}_Y.$$

The *dualising sheaf of X ω_X* is the dualising sheaf of $X \to \mathrm{pt}$.

As Proposition 3.2.3 b) states, when X and Y are smooth we can canonically choose $f^!$ so that the dualising sheaf is $\mathbb{Q}_X[2\dim(f)]$ and from now on we take this choice.

From [[41] equation (3.1.6)] we have a natural transformation of functors

$$\kappa^f : f^* \otimes \omega_{X/Y} \longrightarrow f^!.$$

Hence when X and Y are smooth this gives us the natural transformation

$$\kappa^f : f^*[2\dim(f)] \longrightarrow f^!.$$

If f itself is smooth then this is in fact an equality

$$\kappa^f : f^*[2\dim(f)] \xrightarrow{=} f^! \tag{3.2}$$

by [[41] Proposition 3.3.2 (ii)].

Definition 3.2.5. Define the *Verdier duality* functor $\mathbb{D} : \mathrm{D}(X) \to \mathrm{D}(X)$ to be the contravariant functor

$$\mathbb{D} = \mathcal{H}om(-, \omega_X).$$

Proposition 3.2.6.

a) Let $f : X \to Y$ then

$$\mathbb{D}_Y \circ f_! \cong f_* \circ \mathbb{D}_X$$
$$f^! \circ \mathbb{D}_Y \cong \mathbb{D}_X \circ f^*.$$

b) If X is smooth then

$$\mathbb{D}\,\mathbb{Q}_X \cong \mathbb{Q}_X[2\dim(X)].$$

$\mathbb{D}$ although called a duality is not in fact a duality on $\mathrm{D}(X)$. However it is so when restricted to the subcategory of constructible complexes $\mathrm{D}_c(X)$.

Theorem 3.2.7 ([23] Theorem 4.1.5 and [41] Proposition 3.4.3)**.** *The full subcategory $\mathrm{D}_c(X) \subset \mathrm{D}(X)$ is preserved by the operations $f_*, f^*, f_!, f^!, \otimes, \mathcal{H}om(-,-), \mathbb{D}$. In addition $\mathbb{D}$ is a duality on $\mathrm{D}_c(X)$ i.e.*

$$\mathbb{D}^2 \cong \mathrm{id}_{\mathrm{D}_c(X)}.$$

3.3 Perverse sheaves, vanishing cycles and dimensional reduction

Perverse sheaves form an abelian, Noetherian, Artinian subcategory inside $\mathrm{D}_c(X)$ with many nice properties. When X is smooth as we saw in Proposition 3.2.3 the category of local systems shifted by [dim(X)] is a subcategory of $\mathrm{D}_c(X)$ preserved by Verdier duality. Perverse sheaves can be thought of an analogue to this in the singular case.

Definition 3.3.1. A *t-structure* on a triangulated category T is a pair $(D^{\leq 0}, D^{\geq 0})$ of full subcategories of T such that

i) For all $A \in D^{\leq 0}$ and $B \in D^{\geq 0}$ we have

$$\mathrm{Hom}_T(A, B[-1]) = 0.$$

ii) $D^{\leq 0}$ is closed under $[1]$ and $D^{\geq 0}$ is closed under $[-1]$.

iii) For all $C \in T$ there exists $A \in D^{\leq 0}$ and $B \in D^{\geq 0}$ such that

$$A \to C \to B[-1] \xrightarrow{[1]}$$

is a distinguished triangle.

The *heart* of a t-structure $(D^{\leq 0}, D^{\geq 0})$ is the full subcategory

$$D^{\circ} = D^{\leq 0} \cap D^{\geq 0}.$$

The heart of any t-structure is an abelian subcategory of the triangulated category.

Example 3.3.2. Let $T = \mathrm{D}(\mathcal{A})$ be the derived category of some abelian category $\mathcal{A}$. The *standard t-structure* on $\mathrm{D}(A)$ is given by

$$D(\mathcal{A})^{\leq 0} := \big\{A \in \mathrm{D}(\mathcal{A}) \bigm| \mathrm{H}^i(A) = 0 \text{ for all } i > 0\big\}$$
$$D(\mathcal{A})^{\geq 0} := \big\{A \in \mathrm{D}(\mathcal{A}) \bigm| \mathrm{H}^i(A) = 0 \text{ for all } i < 0\big\}.$$

The heart of this t-structure is then isomorphic to $\mathcal{A}$.

Definition 3.3.3. Let T_1 be a triangulated category with t-structure $(D_1^{\leq 0}, D_1^{\geq 0})$ and T_2 be a triangulated category with t-structure $(D_2^{\leq 0}, D_2^{\geq 0})$. An exact functor $F : T_1 \to T_2$ is called *t-exact* if both

$$F\big(D_1^{\leq 0}\big) \subset D_2^{\leq 0}$$
$$F\big(D_1^{\geq 0}\big) \subset D_2^{\geq 0}.$$

Clearly a t-exact functor $F : T_1 \to T_2$ descends to a functor on the hearts $F^{\circ} : D_1^{\circ} \to D_2^{\circ}$.

We shall be interested in the *perverse t-structure* on $\mathrm{D}_c(X)$. This is given by

$${}^pD^{\leq 0} := \big\{\mathcal{F} \in \mathrm{D}_c(X) \bigm| \dim\big(\mathrm{supp}\big(\mathcal{H}^i(\mathcal{F})\big)\big) \leq -i \text{ for all } i \in \mathbb{Z}\big\}$$
$${}^pD^{\geq 0} := \big\{\mathcal{F} \in \mathrm{D}_c(X) \bigm| \mathbb{D}\mathcal{F} \in {}^pD^{\leq 0}\big\}.$$

Definition 3.3.4. The heart of the perverse t-structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ is called the category of *perverse sheaves* and is denoted by $\mathrm{Perv}(X)$.

It is clear from the definition that the category of perverse sheaves is preserved by $\mathbb{D}$.

Example 3.3.5.

1. For X smooth the sheaves $\mathcal{L}[\dim(X)]$ and $\mathbb{D}\mathcal{L}[\dim(X)]$ are perverse for any local system $\mathcal{L}$. Note the shifts here which are needed to ensure that the cohomology lies in the correct dimension. In particular $\mathbb{Q}[\dim(X)]$ is perverse (and Verdier self-dual).

2. Let Z be smooth and $i : Z \hookrightarrow X$ be a closed immersion. Then $i_*\mathcal{L}[\dim(Z)]$ is perverse in $\mathrm{D}_c(X)$ for any local system $\mathcal{L}$ on Z. This follows from the fact that because i is proper then $i_* = i_!$.

3. Skyscraper sheaves are perverse because they are Verdier self-dual.

For Donaldson-Thomas theory we will be mainly interested in a particular perverse sheaf- the sheaf of vanishing cycles. We shall work in the analytic setting to construct these perverse sheaves.

Let X be a smooth complex variety of dimension d and $f : X \to \mathbb{C}$ a regular function. Let $X_0 = f^{-1}(0) \overset{i}{\hookrightarrow} X$ denote the zero-fibre of f and $\mathrm{crit}(f)$ the critical locus of f. We can assume (by shrinking and translating) that $\mathrm{crit}(f) \subset X_0$. Let $X^* = X \setminus X_0 \overset{j}{\hookrightarrow} X$. We get the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, hook', "j"'] \arrow[d, "f|_{X^*}"] & \widetilde{X^*} \arrow[l, two heads, "\widetilde{\pi}"'] \arrow[d, "\widetilde{f^*}"] \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook'] & \widetilde{\mathbb{C}^*} \arrow[l, two heads, "\pi"']
\end{tikzcd}$$

where $\widetilde{\mathbb{C}^*}$ is the cyclic universal cover of $\mathbb{C}^*$, the map π sends $z \mapsto \exp(2\pi i z)$, and $\widetilde{X^*}$ is the pullback of π and $f|_{X^*}$ and hence is a cyclic cover of X^*.

Definition 3.3.6. The *nearby cycle functor* of f is the functor $\psi_f : \mathrm{D}_c(X) \to \mathrm{D}_c(X_0)$ that sends a complex $\mathcal{F} \in \mathrm{D}_c(X)$ to

$$\psi_f(\mathcal{F}) := i^*(j \circ \widetilde{\pi})_*(j \circ \widetilde{\pi})^*\mathcal{F}.$$

The *nearby cycle sheaf* or *sheaf of nearby cycles* of f is the complex $\psi_f\mathbb{Q}_X[\dim(X)]$.

The group $\mathbb{Z}$ acts via deck transformations on $\widetilde{\mathbb{C}^*}$ and therefore on $\widetilde{X^*}$. This induces an automorphism on $\psi_f\mathcal{F}$ called the *monodromy automorphism*.

Example 3.3.7. Consider the function $\mathbb{C} \xrightarrow{z^d} \mathbb{C}$. Then the nearby cycle sheaf of z^d is the shifted vector space $\psi_{z^d}\mathbb{Q}_{\mathbb{C}}[1] = \mathbb{Q}^d[1]$ on the point $\{0\}$.

The nearby cycle functor encodes the data of the function f on a generic nearby fibre $f^{-1}(t)$, $t \neq 0$ which is achieved when pulling back to $\widetilde{X^*}$. Formally let $B_\epsilon(x)$ be the open ball of radius ϵ centred around $x \in X \cap U \cong \mathbb{C}^d$.

Definition 3.3.8. The *Milnor fibre* of f at $X \in X$ is the intersection

$$\mathrm{MF}_{f,x} := B_\epsilon(X) \cap X_t$$

where $X_t = f^{-1}(t)$ is the fibre of f over some generic value $t \in \mathbb{C}^*$.

Proposition 3.3.9 ([23] Proposition 4.2.2)**.** *For every $x \in X_0$ we have a natural isomorphism of vector spaces*

$$\mathcal{H}^i(\psi_f \mathcal{F})_x \cong \mathrm{H}^i(\mathrm{MF}_{f,x}, \mathcal{F}|_{X_t}).$$

Consider the adjunction natural transformation

$$\eta^{j\circ\widetilde{\pi}} : \mathrm{id}_{\mathrm{D}_c(X)} \to (j \circ \widetilde{\pi})_*(j \circ \widetilde{\pi})^*.$$

Applying i^* gives a natural transformation

$$i^* \to \psi_f.$$

Evaluating this at a complex $\mathcal{F} \in \mathrm{D}_c(X)$ we can then take the cone of this morphism

$$\phi_f \mathcal{F} := \mathrm{cone}\big(i^*\mathcal{F} \to \psi_f \mathcal{F}\big)$$

giving a distinguished triangle

$$i^*\mathcal{F} \to \psi_f \mathcal{F} \to \phi_f \mathcal{F} \xrightarrow{[1]} \tag{3.3}$$

Definition 3.3.10. The *vanishing cycle functor* of f is the functor $\phi_f : \mathrm{D}_c(X) \to \mathrm{D}_c(X_0)$ that sends a complex $\mathcal{F} \in \mathrm{D}_c(X)$ to the complex $\phi_f \mathcal{F}$. The *vanishing cycle sheaf* or *sheaf of vanishing cycles* of f is the complex $\phi_f\, \mathbb{Q}_X[\dim(X)]$.

Remark 3.3.11. One must take care when defining the vanishing cycle functor as the cone construction is not functorial. For a functorial definition see [[41] Definition 8.6.2]. For practical use however our description of ϕ_f is sufficient.

Example 3.3.12. The vanishing cycle sheaf of $\mathbb{C} \xrightarrow{z^d} \mathbb{C}$ is the shifted vector space $\phi_{z^d}\mathbb{Q}_{\mathbb{C}}[1] = \mathbb{Q}^{d-1}[1]$ on the point $\{0\}$.

Similarly to the nearby cycle functor, the vanishing cycle functor encodes data about the fibres of f. Specifically it measures the difference between a nearby generic fibre $f^{-1}(t)$ and the central fibre $f^{-1}(0)$ which is achieved when taking the cone.

Corollary 3.3.13. *For every $x \in X_0$ we have a natural isomorphism of vector spaces*

$$\mathcal{H}^i(\phi_f \mathcal{F})_x \cong \mathrm{H}^{i+1}\big(B_\epsilon(x),\, \mathrm{MF}_{f,x};\, \mathcal{F}|_{X_t}\big)$$

for the relative cohomology of $\mathrm{MF}_{f,x} \subset B_\epsilon(x)$.

Proof. This follows from Proposition 3.3.9, the fact that

$$\mathrm{H}^i\big(B_\epsilon(x) \cap X_0,\, \mathcal{F}|_{X_0}\big) \cong \mathcal{H}^i(i^*\mathcal{F})_x \cong \mathcal{H}^i(\mathcal{F})_x \cong \mathrm{H}^i\big(B_\epsilon(x),\, \mathcal{F}\big)$$

the long exact sequence in cohomology induced from the distinguished triangle (3.3), and the long exact sequence of relative cohomology for $\mathrm{MF}_{f,x} \subset B_\epsilon(x)$. □

Proposition 3.3.14 ([62] Lemma 6.0.2, [49], and [41] (8.6.12))**.**

1. *The functors $\psi_f[-1]$ and $\phi_f[-1]$ are exact and t-exact and hence descend to functors of perverse sheaves*

$$\begin{aligned}\psi_f[-1] &: \mathrm{Perv}(X) \to \mathrm{Perv}(X_0)\\ \phi_f[-1] &: \mathrm{Perv}(X) \to \mathrm{Perv}(X_0).\end{aligned}$$

2. *There are natural isomorphisms of functors*

$$\begin{aligned}\psi_f[-1] \circ \mathbb{D} &\cong \mathbb{D} \circ \psi_f[-1]\\ \phi_f[-1] \circ \mathbb{D} &\cong \mathbb{D} \circ \phi_f[-1].\end{aligned}$$

3. *The vanishing cycle sheaf $\phi_f \mathbb{Q}_X[\dim(X)]$ is supported on the critical locus* $\mathrm{crit}(f)$.

Example 3.3.15. Let Q be a quiver with potential W. Then for each dimension vector $n \in \mathbb{N}^{|Q_0|}$ we have a regular function $f_n = \mathrm{Tr}(W)_n : M_n(Q) \to \mathbb{C}$ on the space $M_n(Q)$ of n-dimensional representations of Q. Proposition 2.2.5 tells us that the critical locus of this function is the space of n-dimensional representations of the Jacobi algebra

$$M_n(\mathrm{Jac}(Q, W)).$$

It follows that for each n the vanishing cycle sheaf $\phi_{f_n} \mathbb{Q}_{M_n(Q)}[\dim(M_n(Q))]$ is a complex in $\mathrm{D}_c(M_n(\mathrm{Jac}(Q, W)))$.

Theorem 3.3.16 (Thom-Sebastiani isomorphism, [62] Corollary 1.3.4)**.** *Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be regular functions on smooth complex varieties X and Y. Define the function*

$$f + g : X \times Y \xrightarrow{f \times g} \mathbb{C} \times \mathbb{C} \xrightarrow{+} \mathbb{C}$$

and note that $X_0 \times Y_0 = f^{-1}(0) \times g^{-1}(0) \subset (f + g)^{-1}(0) = (X \times Y)_0$. Let $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$, $\pi_{X_0} : X_0 \times Y_0 \to X_0$, and $\pi_{Y_0} : X_0 \times Y_0 \to Y_0$ denote *the projections. Then for $\mathcal{F} \in \mathrm{D}_c(X)$ and $\mathcal{G} \in \mathrm{D}_c(Y)$ we have a natural isomorphism in $\mathrm{D}_c(X_0 \times Y_0)$*

$$\phi_{f+g}[-1]\big(\pi_X^*\mathcal{F} \otimes \pi_Y^*\mathcal{G}\big)\big|_{X_0\times Y_0} \cong \pi_{X_0}^*\phi_f[-1]\big(\mathcal{F}\big) \otimes \pi_{Y_0}^*\phi_g[-1]\big(\mathcal{G}\big).$$

We shall mainly be interested in the cohomology vector spaces

$$\begin{aligned}\mathrm{H}^i(X,f) :&= \mathrm{H}^i\big(X, \phi_f[-1]\mathbb{Q}_X[d]\big)\\ &= \mathrm{H}^i\big(X, \phi_f\mathbb{Q}_X[\dim(f)]\big)\end{aligned}$$

where $d = \dim(X)$ and, more importantly for Donaldson-Thomas theory, the compactly supported cohomology

$$\mathrm{H}^i_c(X,f) := \mathrm{H}^i_c\big(X, \phi_f\mathbb{Q}_X[\dim(f)]\big).$$

They can be thought of as cohomology theories for smooth varieties over $\mathbb{C}$ that also takes into account the data of a function on the variety. For example we have the following properties for $\mathrm{H}^i(X,f)$:

i) $\mathrm{H}^i(X,0) = \mathrm{H}^i(X)[d-1]$.

ii) Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$

be a commutative triangle. Then we get a pullback on cohomology

$$u^\star : \mathrm{H}^\bullet(Y,g)[\dim(u)] \longrightarrow \mathrm{H}^\bullet(X,f).$$

iii) There is a Künneth isomorphism

$$\mathrm{H}^i\big(X\times Y, f+g\big) \cong \bigoplus_{j+k=i} \mathrm{H}^j(X,f)\otimes \mathrm{H}^k(Y,g).$$

However, except in a few cases, this cohomology is often very difficult to compute due to the complicated definition of vanishing cycle. The following theorem, so-called the "dimensional reduction" theorem because in the 3 Calabi-Yau case it takes cohomology of an object in a 3CY category to the cohomology of an object in 2CY category, presented in [[15] Appendix A] gives a powerful tool to help tackle this problem.

Theorem 3.3.17. *Let X be a smooth complex variety and consider the function*

$$f = \sum_{i=1}^{n} f_i y_i : X \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

where $f_i : X \to \mathbb{C}$ is a regular function for all i and y_i are coordinates on $\mathbb{C}^n$. Let

$$Z = \left\{ x \in X \,:\, f_i(x) = 0 \text{ for all } i \right\}$$

with closed inclusion $i : Z \hookrightarrow X$, and let $\pi : X \times \mathbb{C}^n \to X$ be the projection. Then there is a natural isomorphism of functors

$$\pi_! \phi_f \pi^*[-1] \cong \pi_! \pi^* i_* i^*$$

and so in particular

$$\begin{aligned} \mathrm{H}_c^\bullet(X \times \mathbb{C}^n, f) &\cong \mathrm{H}_c^\bullet(Z \times \mathbb{C}^n, \mathbb{Q}) \\ &\cong \mathrm{H}_c^{\bullet - 2n}(Z, \mathbb{Q}). \end{aligned} \tag{3.4}$$

Example 3.3.18 ([14] Proposition 5.3)**.** The most relevant application of this for us will be in the context of quivers with a potential and cut. Recall that a cut E for (Q, W) is a collection of arrows such that W is homogeneous of degree 1 with respect to the grading on Q given by arrows in E having degree 1 and the rest having degree 0. It follows that the function $f_n = \mathrm{Tr}(W)_n : M_n(Q) \to \mathbb{C}$ splits into

$$f_n = \sum_i f_{n,i} x_{e_i}$$

where $f_{n,i} : M_n(Q \setminus E) \to \mathbb{C}$ and x_{e_i} are coordinates for the matrices for the arrows in E. Using dimensional reduction we get for all n that

$$\mathrm{H}_c^\bullet\big(M_n(Q), f_n\big) \cong \mathrm{H}_c^{\bullet - 2N}\big(M_n(\mathrm{Jac}(Q, W, E)), \mathbb{Q}\big)$$

for the 2D Jacobi algebra $\mathrm{Jac}(Q, W, E)$, where $N = \dim(M_n(E))$.

Chapter 4

Calabi-Yau algebras

Calabi-Yau structures are essential for defining DT invariants so we formally describe these structures on algebras, following [25] and [13].

Fix a field k of characteristic 0 and let A be a k-algebra. In this chapter all unadorned tensor products will be taken over the base field k while tensor products over any other ring will be adorned with the ring unless otherwise specifically stated.

4.1 Calabi-Yau algebras and Ginzburg differential graded algebras

Definition 4.1.1. The *enveloping algebra* A^e of A is the algebra

$$A^e = A \otimes A^{\mathrm{op}}.$$

The category A-Bimod of A-bimodules is naturally isomorphic to A^e-Mod the category of left A^e-modules and we will often swap between the two. We shall work in derived categories from now on so in order to reduce notational clutter we drop the derived notation on relevant derived functors. Define a functor

$$-^\vee : \mathrm{D}(A\text{-Bimod}) \to \mathrm{D}(A\text{-Bimod})$$

by sending the bimodule M to $\mathrm{Hom}_{A^e}(M, A \otimes A)$ where $A \otimes A$ has the outer A-bimodule structure and $M^\vee$ has the A-bimodule structure induced by the inner bimodule structure on $A \otimes A$.

Definition 4.1.2. An object $M^\bullet$ in the derived category $\mathrm{D}(A - \mathrm{Mod})$ is called *perfect* if it is isomorphic to a bounded complex of projective A-modules.

Definition 4.1.3. An algebra A is called *homologically finite* if it is perfect as an A^e-module.

The subsequent definition for Calabi-Yau algebras is due to Ginzburg [25].

Definition 4.1.4. A *Calabi-Yau structure of dimension* d on a homologically finite algebra A is an isomorphism in $D(A\text{-}\mathbf{Bimod})$

$$f : A \xrightarrow{\sim} A^\vee[d] \quad \text{such that} \quad f = f^\vee[d].$$

If A is homologically finite we have an isomorphism

$$\mathrm{HH}_d(A) \cong \mathrm{Ext}^{-d}_{A^e}(A^\vee, A) = \mathrm{Hom}_{A^e}(A^\vee[d], A)$$

where $\mathrm{HH}_d(A)$ is the dth Hochschild homology of A. So an isomorphism $A \cong A^\vee[d]$ corresponds to an element in $\mathrm{HH}_d(A)$ (we call such elements *non-degenerate*), and the self-duality $f = f^\vee[d]$ corresponds to the element being fixed by the induced map on homology of the flip isomorphism $\beta : A \otimes_{A^e} A \xrightarrow{\sim} A \otimes_{A^e} A$ that swaps the copies of A in the tensor product. However it turns out that the map on Hochschild homology induced by β is just the identity (see [[66] Proposition C.1]) and so any non-degenerate element in $\mathrm{HH}_d(A)$ or indeed any isomorphism $A \cong A^\vee[d]$ gives rise to a d-dimensional Calabi-Yau structure on A.

For an algebra A recall we have the following long exact sequence in cyclic and Hochschild homology

$$\cdots \to \mathrm{HC}_{n+1}(A) \to \mathrm{HC}_{n-1}(A) \xrightarrow{\partial} \mathrm{HH}_n(A) \to \mathrm{HC}_n(A) \to \cdots$$

Definition 4.1.5. An *exact Calabi-Yau structure of dimension* d on an algebra A is a non-degenerate element $\nu \in \mathrm{HH}_d(A)$ that is in the image of the boundary map ∂.

Now let A be a differential graded algebra (dga). Its *bimodule of 1-forms* is

$$\Omega^1 A = \mathrm{Ker}(A \otimes A \xrightarrow{m} A)$$

where m is the multiplication map. $\Omega^1 A$ inherits a grading from A making it a differential graded A-bimodule.

Definition 4.1.6. A finitely generated dga A is *smooth* if $\Omega^1 A$ is projective as an A-bimodule.

Given a finitely generated negatively graded bimodule V over a smooth algebra A we define $T_A(V)$ to be the tensor-algebra generated by V over A. If V is free as a bimodule then we call $T_A(V)$ a *noncommutative vector bundle* over A.

Lemma 4.1.7 ([11] Proposition 5.3 (3))**.** *Let A be smooth and let V be a finitely generated negatively graded projective bimodule over A. Then the algebra $T_A(V)$ is smooth.*

For a bimodule M over a dga A let $\mathrm{Der}(A, M)$ denote the graded vector space of super-derivations from A to M and set $\mathrm{Der}(A) = \mathrm{Der}(A, A)$. Giving $A \otimes A$ the outer A-bimodule structure we let $\mathbb{D}\mathrm{er}(A) = \mathrm{Der}(A, A \otimes A)$ be the bimodule of *double* *derivations* on A, where the A-bimodule structure on $\mathbb{D}\mathrm{er}(A)$ is induced via the inner bimodule structure on $A \otimes A$. Note there is a natural isomorphism $\mathbb{D}\mathrm{er}(A) \cong (\Omega^1 A)^\vee$. Let $D : A \to \Omega^1 A$ be the canonical derivation that sends $a \mapsto a \otimes 1 - 1 \otimes a$. Then we can define the dga $(\Omega^\bullet A, D)$ of *noncommutative differential forms* of A by

$$\Omega^\bullet A = T_A(\Omega^1 A)$$

with differential induced by D. We next define the super-commutator quotient

$$\mathrm{DR}(A) = \Omega^\bullet A / [\Omega^\bullet A, \Omega^\bullet A]$$

called the *cyclic quotient* of $\Omega^\bullet A$, which is a differential graded vector space. Note that both $\Omega^\bullet A$ and $\mathrm{DR}(A)$ are bigraded- they have the usual tensor grading as well as a grading induced from A. Denote by $\Omega^i A$ and $\mathrm{DR}^i(A)$ the ith graded parts with respect to the tensor grading.

Lemma 4.1.8. *Let A be a finitely generated algebra over k and let V be a finitely generated A-bimodule. Then $\Omega^1 T_A(V)$ is generated by homogeneous elements of degree 0 and 1 as a graded $T_A(V)$-bimodule.*

Proof. Because $T_A(V)$ is a free algebra $\Omega^1 T_A(V)$ is generated as a vector space by homogeneous elements of the form

$$x = (x_1 \otimes_A \ldots \otimes_A x_i) \otimes (x_{i+1} \otimes_A \ldots \otimes_A x_n) - (x_1 \otimes_A \ldots \otimes_A x_{i-1}) \otimes (x_i \otimes_A \ldots \otimes_A x_n) \in (\Omega^1 T_A(V))^n$$

for $x_m \in V$ and $1 \leq i \leq n$. Then

$$(x_1 \otimes_A \ldots \otimes_A x_{i-1}) \cdot (a_i \otimes 1 - 1 \otimes a_i) \cdot (x_{i+1} \otimes_A \ldots \otimes_A x_n) = x$$

and so because $(x_1 \otimes_A \ldots \otimes_A x_{i-1})$, $(x_{i+1} \otimes_A \ldots \otimes_A x_n) \in T_A(V)$ and $(a_i \otimes 1 - 1 \otimes a_i) \in (\Omega^1 T_A(V))^1$ we get the result. $\square$

For $\theta \in \mathrm{Der}(A)$ define the degree 0 derivation $L_\theta : \Omega^\bullet \to \Omega^\bullet$ by

$$A \ni a \mapsto \theta(a)$$
$$\Omega^1 A \ni D(a) \mapsto D(\theta(a))$$

and extend this to the rest of $\Omega^\bullet A$ using the Leibniz rule and linearity. We also define the *contraction mapping* $i_\theta : \Omega^\bullet \to \Omega^\bullet$ as the super-derivation given by

$$A \ni a \mapsto 0$$
$$\Omega^1 A \ni D(a) \mapsto \theta(a).$$

i_θ has degree -1 with respect to the tensor grading and degree 0 with respect to the induced grading by A. The derivations L_θ and i_θ are related by the *Cartan identity*

$$L_\theta = D \circ i_\theta + i_\theta \circ D\,.$$

Also both derivations descend to maps $\mathrm{DR}(A) \to \mathrm{DR}(A)$; indeed if $x \in \Omega^m A$ and $y \in \Omega^n A$ then we have

$$\begin{aligned} L_\theta([x,y]) &= L_\theta\big(xy - (-1)^{mn}\, yx\big) \\ &= L_\theta(x)y + x\, L_\theta(y) - (-1)^{mn}\, L_\theta(y)x - (-1)^{mn}\, y\, L_\theta(x) \\ &= L_\theta(x)y - (-1)^{mn}\, y\, L_\theta(x) + x\, L_\theta(y) - (-1)^{mn}\, L_\theta(y)x \\ &= [L_\theta(x), y] + [x, L_\theta(y)] \end{aligned}$$

and

$$\begin{aligned} i_\theta([x,y]) &= i_\theta\big(xy - (-1)^{mn}\, yx\big) \\ &= i_\theta(x)y + (-1)^m\, x\, i_\theta(y) - (-1)^{mn}\, i_\theta(y)x - (-1)^{mn}(-1)^n\, y\, i_\theta(x) \\ &= i_\theta(x)y - (-1)^{(m+1)n}\, y\, i_\theta(x) + (-1)^m\, x\, i_\theta(y) - (-1)^{mn}\, i_\theta(y)x \\ &= i_\theta(x)y - (-1)^{(m-1)n}\, y\, i_\theta(x) + (-1)^m \Big(x\, i_\theta(y) - (-1)^{m(n-1)}\, i_\theta(y)x\Big) \\ &= [i_\theta(x), y] + (-1)^m [x, i_\theta(y)]. \end{aligned}$$

For $\lambda \in \mathbb{D}\mathrm{er}(A)$ we can also define a contraction mapping $i_\lambda : \Omega^\bullet A \to \Omega^\bullet A \otimes \Omega^\bullet A$ as the double super-derivation given by

$$\begin{aligned} a \in\ A &\mapsto 0 \\ D(a) \in\ \Omega^1 A &\mapsto \lambda(a). \end{aligned}$$

The *reduced contraction mapping* $\iota_\lambda : \Omega^\bullet A \to \Omega^\bullet A$ is then defined as

$$\iota_\lambda = m(\beta \circ i_\lambda)$$

where $m : \Omega^\bullet A \otimes \Omega^\bullet A \to \Omega^\bullet A$ is the multiplication map and β is the flip isomorphism on $\Omega^\bullet A \otimes \Omega^\bullet A$ which sends homogeneous elements $x \in \Omega^m A$ and $y \in \Omega^n A$ to

$$x \otimes y \longmapsto (-1)^{mn} y \otimes x.$$

Definition 4.1.9. Let $\omega \in \mathrm{DR}^2(A)$ be closed with respect to the differential D. Define a map $i^\omega : \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ by sending

$$\mathrm{Der}(A) \ni \theta \mapsto i_\theta(\omega).$$

We call ω *symplectic* if i^ω is an isomorphism. Similarly we can define a map ι^ω : $\mathbb{D}\mathrm{er}(A) \to \Omega^1 A$ and call ω *bisymplectic* if ι^ω is an isomorphism.

From [[10] Section 4.2] we have the following result:

Lemma 4.1.10. *Let A be smooth and ω be a bisymplectic 2-form. Then ω is symplectic.*

For ω bisymplectic and any $a \in A$ we can define a derivation $\{a, -\}_\omega : A \to A$ by

$$b \mapsto m((\iota^\omega)^{-1}(D(a))(b))$$

which makes sense, because as ω is bisymplectic, $(\iota^\omega)^{-1} : \Omega^1 A \to \mathbb{D}\mathrm{er}(A)$ is a well-defined map. Similarly if ω is symplectic then for $W \in A/[A, A]$ we can define a derivation in $\mathrm{Der}(A)$ by

$$\{W, -\}_\omega := (i^\omega)^{-1}(D(W)).$$

Definition 4.1.11. An algebra A is *connected* if the sequence

$$0 \to k \to \mathrm{DR}^0(A) \to \mathrm{DR}^1(A)$$

is exact.

Let $\Delta : A \to A \otimes A$ be the distinguished double derivation that sends

$$a \mapsto a \otimes 1 - 1 \otimes a.$$

Then there exists a map (see [[10] Section 4])

$$\mu_{\mathrm{nc}} : \overline{\mathrm{DR}^2(A)} \to A/k$$

where $\overline{\mathrm{DR}^2(A)} \subset \mathrm{DR}^2(A)$ are the closed 2-forms, having the property that if $\mathrm{w} \in A$ is a representative of $\mu_{\mathrm{nc}}(\omega)$ then

$$D(\mathrm{w}) = \iota_\Delta(\omega).$$

Definition 4.1.12. Let (A, ω, ξ) be a triple of a non-positively graded smooth algebra A such that $\Omega^1 A$ is generated by homogeneous elements of degree $0, -1, \dots, c$ as a graded A-bimodule, a bisymplectic 2-form $\omega \in \mathrm{DR}^2(A)$ which is homogeneous of degree c with respect to the induced grading of A, and a super-derivation $\xi \in \mathrm{Der}(A)$ of degree 1 such that $\xi^2 = 0, \quad L_\xi(\omega) = 0$ and $\xi(\mathrm{w}) = 0$. We define the *Ginzburg differential graded algebra* (Gdga) $\mathfrak{D}(A, \omega, \xi)$ of this triple to be the following data:

- The underlying algebra of $\mathfrak{D}(A, \omega, \xi)$ is the free product of algebras $A * k[t]$ where t has degree $c - 1$.
- The differential d is given by

$$\begin{aligned} A \ni a &\mapsto \xi(a) \\ t &\mapsto \mathrm{w}. \end{aligned}$$

Remark 4.1.13.

1. As $\mu_{\mathrm{nc}}(\omega) \in A/k$ the representative w is only determined up to some constant in k. Thus we choose w to be homogeneous (of degree c) so it is unique. This can

cause a small issue when $c = 0$ but won't be relevant for us; see [[13] Remark 4.3.1] for details.

2. It's clear that $d^2 = 0$ follows from the assumptions $\xi^2 = 0$ and $\xi(\mathrm{w}) = 0$. One way of ensuring $\xi(\mathrm{w}) = 0$ is to assume the additional condition $k \cap \xi(A) = 0$. By [[10] Proposition 4.1.3] we have for any $\theta \in \mathrm{Der}(A)$

$$\mu_{\mathrm{nc}}(L_\theta(\omega)) = L_\theta(\mu_{\mathrm{nc}}(\omega))$$

hence

$$\xi(\mu_{\mathrm{nc}}(\omega)) = L_\xi(\mu_{\mathrm{nc}}(\omega)) = \mu_{\mathrm{nc}}(L_\xi(\omega)) = \mu_{\mathrm{nc}}(0) = 0.$$

It follows that $\xi(\mathrm{w}) \in k$ and so if $k \cap \xi(A) = 0$ we get $\xi(\mathrm{w}) = 0$.

Because A is smooth ω is also symplectic by Lemma 4.1.10.

Definition 4.1.14. A *superpotential algebra* B is an algebra (viewed as a dga concentrated in degree 0) that is quasi-isomorphic to a Gdga $\mathfrak{D}(A, \omega, \xi)$ in which A is connected and $\xi = \{W, -\}_\omega$ for some $W \in A/[A, A]$. W is called a *potential.*

We shall see some examples of Gdgas when we look at quivers and Jacobi algebras.

Theorem 4.1.15 ([25] Theorem 3.6.4)**.** *Let $\mathfrak{D}(A, \omega, \xi)$ be a Gdga with ω of degree c and suppose that $H^i(\mathfrak{D}(A, \omega, \xi)) = 0$ for all $i \neq 0$. Then $H^0(\mathfrak{D}(A, \omega, \xi))$ is Calabi-Yau of dimension $-c + 2$.*

Theorem 4.1.16 ([13] Theorem 4.3.8)**.** *Let $\mathfrak{D}(A, \omega, \xi)$ be as in Theorem 4.1.15 and additionally assume that A is connected. Then $H^0(\mathfrak{D}(A, \omega, \xi))$ is exact Calabi-Yau of dimension $-c + 2$. In particular a superpotential algebra is exact Calabi-Yau.*

4.2 Ginzburg differential graded algebras in the relative case

We can strengthen Theorem 4.1.15 by looking at a smaller class of Gdgas arising in dimension 3. First we must extend the previous story of noncommutative algebras over a field k to noncommutative algebras over a finite dimensional semisimple k-algebra R. Everything follows in a natural way to the relative case. We introduce notation with "R" in the subscript to distinguish this change. Recall unadorned tensor products are to be taken over the base field k.

Let A be an R-algebra, then

$$\Omega^1_R A = \mathrm{Ker}(A \otimes_R A \xrightarrow{m} A)$$

is the bimodule of *relative 1-forms*, and similarly

$$\Omega^\bullet_R A = T_A(\Omega^1_R A)$$

is the dga of *relative noncommutative differential forms*, with its cyclic quotient

$$\mathrm{DR}_R(A) = \Omega_R^\bullet A/[\Omega_R^\bullet A, \Omega_R^\bullet A].$$

An important difference is in the bimodule of *relative double derivations* $\mathbb{D}\mathrm{er}_R(A)$ which are derivations $\lambda : A \to A \otimes A$ such that $\lambda \circ (R \to A) = 0$. This is defined so that we again have a natural isomorphism $\mathbb{D}\mathrm{er}_R(A) \cong (\Omega_R^1 A)^\vee$. From now on we let $R = kI$ be the algebra of k-valued functions on a finite indexing set I. For each $i \in I$ let $1_i \in kI$ be the function that sends $i \mapsto 1$ and everything else to 0, and let $e = \sum_{i\in I} 1_i \otimes 1_i \in R \otimes R$. Define $\delta \in \mathbb{D}\mathrm{er}_R(A)$ as the distinguished relative double derivation given by

$$A \ni a \longmapsto a \cdot e - e \cdot a \in A \otimes A.$$

Then δ is the appropriate relative counterpart to the distinguished double derivation Δ we had in Section 4.1.

As in the non-relative case we have a map $\mu_{\mathrm{nc}} : \overline{\mathrm{DR}_R^2(A)} \to A/R$ which is defined by the property that if $\mathrm{w} \in A$ is a representative of $\mu_{\mathrm{nc}}(\omega)$ then

$$D(\mathrm{w}) = \iota_\delta(\omega).$$

We can then define the Gdga analogously to the non-relative case.

Definition 4.2.1. Let A be a non-positively graded smooth R-algebra such that $\Omega_R^1 A$ is generated by homogeneous elements of degree $0, -1, \ldots, c$ as a graded A-bimodule, $\omega \in \mathrm{DR}_R^2(A)$ a bisymplectic 2-form of degree c, and $\xi \in \mathrm{Der}_R(A)$ a super-derivation of degree 1 such that $\xi^2 = 0$, $L_\xi(\omega) = 0$ and $\xi(\mathrm{w}) = 0$. The *Ginzburg differential graded algebra* $\mathfrak{D}(A, \omega, \xi)$ is the free product of R-algebras $A *_R R[t]$ with t in degree $c - 1$ and differential d given by

$$\begin{aligned} A \ni a &\mapsto \xi(a) \\ t &\mapsto \mathrm{w}. \end{aligned}$$

Following [[25] Section 5] let F be a smooth algebra over R and let $\alpha \in \mathrm{DR}_R^1(F)$ be a cyclic 1-form such that

$$D(\alpha) = 0 \quad \text{and} \quad \iota_\delta(\alpha) = 0. \tag{4.1}$$

Set A to be the tensor-algebra $T_F(\mathbb{D}\mathrm{er}_R(F))$ with non-positive grading given by negating the natural tensor grading. A is smooth by [[10] Theorem 5.1.1]. There is a canonical closed cyclic bisymplectic 2-form $\omega \in \mathrm{DR}_R^2(A)$ of degree -1 (see [[10] Theorem 5.1.1 and Proposition 5.4.1]) and by Lemma 4.1.8 $\Omega_R^1 A$ is generated by homogeneous elements of degree 0 and -1. Let ξ_α be the super-derivation which is the image of α under the following map

$$\mathrm{DR}_R^1(F) \hookrightarrow \mathrm{DR}_R^1(A) \xrightarrow[\sim]{(i^\omega)^{-1}} \mathrm{Der}_R(A)$$

where the first inclusion is induced by the map $F \xrightarrow{\sim} A^0 \hookrightarrow A$.

Lemma 4.2.2. *With the algebra A, bisymplectic 2-form ω and super-derivation ξ_α as above we have*

$$\xi_\alpha^2 = 0 \quad \text{and} \quad L_{\xi_\alpha}(\omega) = 0.$$

Proof. In [[25] the proof of Proposition 5.2.4] we are given that ξ_α sends

$$A^0 = F \ni f \mapsto 0$$
$$A^1 = \mathbb{D}\mathrm{er}_R(F) \ni \theta \mapsto \iota_\theta(\alpha) \in A^0.$$

It follows that (omitting the F subscript in the tensor product for clarity)

$$A^n \ni \theta_1 \otimes \ldots \otimes \theta_n \xmapsto{\xi_\alpha} \sum_{i=1}^{n} (-1)^{i-1} \theta_1 \otimes \ldots \otimes \theta_{i-1} \otimes \iota_{\theta_i}(\alpha)\theta_{i+1} \otimes \ldots \otimes \theta_n \in A^{n-1}$$

hence

$$\begin{aligned}
\xi_\alpha^2(\theta_1 \otimes \ldots \otimes \theta_n) &= \sum_{i=1}^{n} (-1)^{i-1} \Bigg(\sum_{j<i} (-1)^{j-1}\, \theta_1 \otimes \ldots \iota_{\theta_j}(\alpha)\theta_{j+1} \ldots \iota_{\theta_i}(\alpha)\theta_{i+1} \ldots \otimes \theta_n \\
&\qquad + \sum_{j>i} (-1)^{j}\, \theta_1 \otimes \ldots \iota_{\theta_i}(\alpha)\theta_{i+1} \ldots \iota_{\theta_j}(\alpha)\theta_{j+1} \ldots \otimes \theta_n \Bigg) \\
&= \sum_{i=1}^{n-1} \Bigg(\sum_{j>i}^{n} (-1)^{i+j-1}\, \theta_1 \otimes \ldots \iota_{\theta_i}(\alpha)\theta_{i+1} \ldots \iota_{\theta_j}(\alpha)\theta_{j+1} \ldots \otimes \theta_n \\
&\qquad + (-1)^{i+j-2}\, \theta_1 \otimes \ldots \iota_{\theta_i}(\alpha)\theta_{i+1} \ldots \iota_{\theta_j}(\alpha)\theta_{j+1} \ldots \otimes \theta_n \Bigg) \\
&= 0.
\end{aligned}$$

Then because the elements $\theta_1 \otimes \ldots \otimes \theta_n$ generate A^n over F we have that $\xi_\alpha^2(x) = 0$ for any $x \in A^n$ and any $n \in \mathbb{N}$.

To show $L_{\xi_\alpha}(\omega) = 0$, by definition we have $\xi_\alpha = (i^\omega)^{-1}(\alpha)$ and so $i_{\xi_\alpha}(\omega) = \alpha$. Then by the Cartan identity we get

$$L_{\xi_\alpha}(\omega) = D(i_{\xi_\alpha}(\omega)) + i_{\xi_\alpha}(D(\omega)) = D(\alpha) = 0$$

by the assumption (4.1) that $D(\alpha) = 0$ and the fact that ω is closed. □

Definition 4.2.3. Let F be a smooth R-algebra and $\alpha \in \mathrm{DR}^1_R(F)$ a cyclic 1-form satisfying the conditions in (4.1). Define the *Gdga associated to the data* F, α to be

$$\mathfrak{D}(F, \alpha) = \mathfrak{D}(A, \omega, \xi_\alpha)$$

where $A = T_F(\mathbb{D}\mathrm{er}_R(F))$, and ω and ξ_α are as above.

Remark 4.2.4. As per the definition of the Gdga we still need to establish the condition $\xi_\alpha(\mathrm{w}) = 0$. From [[25] Proposition 5.2.4 and Remark 5.2.5] we do not necessarily have the additional condition $R \cap \xi_\alpha(A) = 0$ analogous to $k \cap \xi(A) = 0$ we saw in the non-relative setting in Remark 4.1.13. However $\xi_\alpha(\mathrm{w}) = 0$ instead follows from the assumption $\iota_\delta(\alpha) = 0$ in (4.1). Indeed from the proof of Lemma 4.2.2 we have that $\xi_\alpha(\delta) = \iota_\delta(\alpha) = 0$. Then by [[10] Theorem 5.1.1] δ is a representative of $\mu_{\mathrm{nc}}(\omega)$ giving the required result.

This smaller class of Gdgas have some nice properties which were studied by Ginzburg, see [[25] Section 5.3]. In particular he showed that the zeroth cohomology $H^0(\mathfrak{D}(F,\alpha))$ of the Gdga is Calabi-Yau of dimension 3 if and only if the so-called *completed* dga $\widehat{\mathfrak{D}}(F,\alpha)$ is acyclic. Alternatively if $\mathfrak{D}(F,\alpha)$ has an additional strictly positive grading (i.e. $(\mathfrak{D}(F,\alpha))_0 = k$) which is preserved by the differential d, then $H^0(\mathfrak{D}(F,\alpha))$ is Calabi-Yau of dimension 3 if and only if $\mathfrak{D}(F,\alpha)$ is acyclic. However, later on it was shown by Keller and Van den Bergh that $\mathfrak{D}(F,\alpha)$ is in fact itself Calabi-Yai of dimension 3.

Theorem 4.2.5 ([42] Theorem A.12)**.** *The Gdga $\mathfrak{D}(F,\alpha)$ is Calabi-Yau of dimension 3.*

4.3 Quivers with potential

Let Q be a quiver with potential W and consider the Jacobi algebra $\mathrm{Jac}(Q,W)$ defined in section 2.2. Unfortunately not all Jacobi algebras are superpotential algebras or even Calabi-Yau algebras but there is a natural Gdga associated to them. The following is developed from [[25] Section 4.2]. We work in the relative setting and let $R = kQ_0$ be the k-algebra over the vertex set of the quiver Q or equivalently the k-algebra over the constant paths. The constant paths e_i then take the role of the functions 1_i mentioned in Section 4.2.

Lemma 4.3.1. *Let V be the vector space kQ_1 viewed in the natural way as an R-bimodule. Then we have an isomorphism of algebras*

$$kQ \cong T_R(V). \tag{4.2}$$

In particular kQ is a smooth R-algebra.

Proof. The isomorphism is immediate from the definition of the multiplication on the path algebra versus on the tensor algebra. For smoothness, clearly R is smooth and as V is finite dimensional it is projective as an R-bimodule hence by Lemma 4.1.7 $T_R(V)$ is smooth. □

Given a quiver Q with potential W consider the new quiver $\widehat{Q}$ which has vertex set

$\widehat{Q}_0 = Q_0$ and arrows

$$\begin{array}{rll} a : i \to j & \text{of degree 0 for any} & a : i \to j \in Q_1 \\ a^* : j \to i & \text{of degree -1 for any} & a : i \to j \in Q_1 \\ \text{loops } t_i : i \to i & \text{of degree -2 for any} & i \in Q_0. \end{array}$$

Definition 4.3.2. The *Gdga associated to* (Q, W), denoted by $\Gamma_{Q,W}$, is the path algebra $k\widehat{Q}$ along with the differential d that sends

$$\begin{aligned} &a \mapsto 0 \\ &a^* \mapsto \partial W/\partial a \\ &t_i \mapsto e_i \Big(\sum_{a \in Q_1} [a, a^*] \Big) e_i. \end{aligned}$$

Proposition 4.3.3. *$\Gamma_{Q,W}$ is a Gdga in the sense of Definition 4.2.1.*

Proof. Let $\overline{Q}$ be the double quiver of Q, then the underlying algebra of $\Gamma_{Q,W}$ is $k\overline{Q} *_R R[t]$ where $\sum_{i \in Q_0} t_i = t$. As $k\overline{Q}$ is a path algebra of a quiver it is smooth by Lemma 4.3.1. If p is a path in $k\overline{Q}$ and we write $p = p_1 x p_2$ for some arrow $x \in \overline{Q}_1$, then $\Omega^1_R k\overline{Q}$ is generated as a R-bimodule by

$$p_1 x \otimes_R p_2 - p_1 \otimes_R x p_2$$

for all paths p and arrows x. Hence, because

$$p_1 \cdot (x \otimes_R 1 - 1 \otimes_R x) \cdot p_2 = p_1 x \otimes_R p_2 - p_1 \otimes_R x p_2$$

$\Omega^1_R k\overline{Q}$ is generated as a $k\overline{Q}$-bimodule by the homogeneous elements of degree 0 and -1. From [[10] Section 8.1 and Proposition 8.1.1] we have that the 2-form ω is [1]

$$-\sum_{a \in Q_1} D(a) \cdot D(a^*) = \sum_{a \in Q_1} D(a^*) \cdot D(a)$$

which has degree -1, and the super-derivation is $\xi = \{W, -\}_\omega$.

We first show that ω is bisymplectic. Let $\lambda \in \mathbb{D}\mathrm{er}_R(k\overline{Q})$ be a double derivation defined on arrows $x \in \overline{Q}_1$ by

$$\lambda(x) = \sum_r c^x_r p^x_r \otimes q^x_r$$

where $c^x_r \in k$ and p^x_r, q^x_r are paths in $k\overline{Q}$. Note that for each term in $\lambda(x)$ we have $s(x) = s(q^x_r)$ and $t(x) = t(p^x_r)$ for the paths p^x_r, q^x_r. We consider the map ι^ω that sends

[1] note this is $-1\cdot$ the 2-form given in [10] due to a sign error in [[10] Lemma 3.1.1 (ii)]

$\lambda \mapsto \iota_\lambda(\omega)$. Then

$$\begin{aligned}
\iota_\lambda(\omega) &= m(\beta \circ i_\lambda(\omega)) \\
&= m\Big(\beta\Big(\sum_{a\in Q_1} -\lambda(a)\cdot D(a^*) + D(a)\cdot \lambda(a^*)\Big)\Big) \\
&= m\Big(\beta\Big(\sum_{a\in Q_1} -\Big(\sum_r c_r^a p_r^a \otimes q_r^a\Big)\cdot D(a^*) + D(a)\cdot\Big(\sum_s c_s^{a^*} p_s^{a^*}\otimes q_s^{a^*}\Big)\Big)\Big) \\
&= m\Big(\beta\Big(\sum_{a\in Q_1} -\Big(\sum_r c_r^a p_r^a \otimes (q_r^a D(a^*)) + \sum_s c_s^{a^*} (D(a)\, p_s^{a^*})\otimes q_s^{a^*}\Big)\Big)\Big) \\
&= \sum_{a\in Q_1} -\Big(\sum_r c_r^a q_r^a D(a^*)\, p_r^a - \sum_s c_s^{a^*} q_s^{a^*} D(a)\, p_s^{a^*}\Big) \\
&= \sum_{a\in Q_1} -\Big(\sum_r c_r^a (q_r^a a^* \otimes_R p_r^a - q_r^a \otimes_R a^* p_r^a) - \sum_s c_s^{a^*}(q_s^{a^*} a \otimes_R p_s^{a^*} - q_s^{a^*}\otimes_R a p_s^{a^*})\Big).
\end{aligned} \tag{4.3}$$

For surjectivity of ι^ω, as $\Omega^1_R k\overline{Q}$ is generated by $p_1 x \otimes_R p_2 - p_1 \otimes_R x p_2$ for all paths $p = p_1 x p_2$ and arrows $x \in k\overline{Q}_1$, if we define $\lambda_{p,x} \in \mathbb{D}\mathrm{er}_R(k\overline{Q})$ by $\lambda_{p,x}(x^*) = p_2 \otimes p_1$ and $\lambda(y) = 0$ for all other $y \in \overline{Q}_1$ (*notation:* $(x^*)^* = x$) then we can see that $\iota_{\lambda_{p,x}}(\omega) = p_1 x \otimes_R p_2 - p_1 \otimes_R x p_2$ using equation (4.3). For injectivity suppose that $\iota_\lambda(\omega) = 0$. Each simple tensor element in the sum given in equation (4.3) is non-zero since $s(q_r^x) = s(x) = t(x^*)$ and $t(p_r^x) = t(x) = s(x^*)$. Then because simple tensor elements give a basis of $k\overline{Q} \otimes_R k\overline{Q}$, for each $x_0 \in \overline{Q}_1$ and each r_0 with paths $p_{r_0}^{x_0}, q_{r_0}^{x_0}$ in $k\overline{Q}$ given by $\lambda(x_0)$ there must exist at least one $x_1 \in \overline{Q}_1$ and some r_1 with paths $p_{r_1}^{x_1}, q_{r_1}^{x_1}$ in $k\overline{Q}$ given by $\lambda(x_1)$ such that

$$q_{r_0}^{x_0} x_0^* \otimes_R p_{r_0}^{x_0} = q_{r_1}^{x_1} \otimes_R x_1^* p_{r_1}^{x_1}$$

in order for the overall coefficient of the term $q_{r_0}^{x_0} x_0^* \otimes_R p_{r_0}^{x_0}$ in equation (4.3) to be 0. Let the length of the path $p_{r_0}^{x_0}$ be l_1 and the length of $q_{r_0}^{x_0}$ be l_2. Then the length of $p_{r_1}^{x_1} = l_1 - 1$ and the length of $q_{r_1}^{x_1} = l_2 + 1$. Now equation (4.3) also contains the non-zero term $c_{r_1}^{x_1} q_{r_1}^{x_1} x_1^* \otimes_R p_{r_1}^{x_1}$. Hence we repeat this for x_1 to find a $x_2 \in \overline{Q}_1$ and r_2 such that $q_{r_1}^{x_1} x_1^* \otimes_R p_{r_1}^{x_1} = q_{r_2}^{x_2} \otimes_R x_2^* p_{r_2}^{x_2}$ where the length of $p_{r_2}^{x_2} = l_1 - 2$ and the length of $q_{r_2}^{x_2} = l_2 + 2$. We continue to repeat this argument and after l_1 successive iterations we find an arrow $x_{l_1} \in \overline{Q}_1$ and some r_{l_1} such that the length of $p_{r_{l_1}}^{x_{l_1}} = 0$ i.e. $p_{r_{l_1}}^{x_{l_1}}$ is just a constant path in $\overline{Q}$. However now we have the non-zero term $c_{r_{l_1}}^{x_{l_1}} q_{r_{l_1}}^{x_{l_1}} x_{l_1}^* \otimes_R p_{r_{l_1}}^{x_{l_1}}$ in equation (4.3) but there cannot exist some $y \in \overline{Q}_1$ some s and paths p_s^y, q_s^y in $k\overline{Q}$ such that

$$q_{r_{l_1}}^{x_{l_1}} x_{l_1}^* \otimes_R p_{r_{l_1}}^{x_{l_1}} = q_s^y \otimes_R y^* p_s^y$$

because $y^* p_s^y$ cannot be a constant path. It follows that the coefficient of the term $q_{r_{l_1}}^{x_{l_1}} x_{l_1}^* \otimes_R p_{r_{l_1}}^{x_{l_1}}$ in equation (4.3) is $c_{r_{l_1}}^{x_{l_1}} \neq 0$ which contradicts our initial assumption that $\iota_\lambda(\omega) = 0$.

Next we check the differential d. In the definition of the Gdga the differential is given on A by the super-derivation ξ, so we want to explicitly verify that for all $x \in \overline{Q}_1$ we have $d(x) = \xi(x) = \{W, x\}_\omega$. ξ is defined as the unique super-derivation such that $i_\xi(\omega) = D(W)$ so write $W = \sum_{t=1}^m a_{t,n_t} \dots a_{t,1}$ for arrows $a_{t,j} \in \overline{Q}_1$. Then

$$\frac{\partial W}{\partial a} = \sum_{\substack{t,j| \\ a_{t,j}=a}} a_{t,j-1} \dots a_{t,1}\, a_{t,n_t} \dots a_{t,j+1}.$$

Now because we are in the quotient $\mathrm{DR}^1_R(k\overline{Q})$ we can cyclically permute terms (up to a sign, which in this case is always $+1$ since we are dealing with terms of degree 0 and a term of degree 1 in $\Omega^\bullet_R k\overline{Q}$), hence we get that

$$\begin{aligned}
D(W) &= \sum_{t=1}^m \left(\sum_{j=1}^{n_t} a_{t,n_t} \dots a_{t,j+1}\, D(a_{t,j})\, a_{t,j-1} \dots a_{t,1} \right) \\
&= \sum_{t=1}^m \left(\sum_{j=1}^{n_t} D(a_{t,j})\, a_{t,j-1} \dots a_{t,1}\, a_{t,n_t} \dots a_{t,j+1} \right) \\
&= \sum_{a \in Q_1} \left(\sum_{\substack{t,j| \\ a_{t,j}=a}} D(a_{t,j})\, a_{t,j-1} \dots a_{t,1}\, a_{t,n_t} \dots a_{t,j+1} \right) \\
&= \sum_{a \in Q_1} D(a) \frac{\partial W}{\partial a}.
\end{aligned}$$

Then as

$$i_\xi(\omega) = \sum_{a \in Q_1} -\xi(a) D(a^*) + D(a)\xi(a^*)$$

by comparing coefficients between these two expressions we must have that $\xi(a) = 0$ and $\xi(a^*) = \partial W/\partial a$ for any $a \in Q_1$, as required.

We must also check the differential on t. To do this we calculate a representative w of $\mu_{\mathrm{nc}}(\omega)$. Recall $\mu_{\mathrm{nc}}(\omega)$ was defined such that $D(\mu_{\mathrm{nc}}(\omega)) = \iota_\delta(\omega)$ where $\delta : k\overline{Q} \to k\overline{Q} \otimes k\overline{Q}$

was the distinguished double derivation introduced in Section 4.2. So

$$\begin{aligned}
\iota_\delta(\omega) &= m(\beta \circ i_\delta(\omega)) \\
&= m\Big(\beta\Big(-\sum_{a\in Q_1} \delta(a)\cdot D(a^*) - D(a)\cdot\delta(a^*)\Big)\Big) \\
&= m\Big(\beta\Big(-\sum_{a\in Q_1} (a\otimes e_{s(a)} - e_{t(a)}\otimes a)\cdot(a^*\otimes_R 1 - 1\otimes_R a^*) \\
&\qquad\qquad - (a\otimes_R 1 - 1\otimes_R a)\cdot(a^*\otimes e_{s(a^*)} - e_{t(a^*)}\otimes a^*)\Big)\Big) \\
&= m\Big(\beta\Big(-\sum_{a\in Q_1} a\otimes(a^*\otimes_R 1 - e_{s(a)}\otimes_R a^*) - e_{t(a)}\otimes(aa^*\otimes_R 1 - a\otimes_R a^*) \\
&\qquad\qquad - (a\otimes_R a^* - 1\otimes_R aa^*)\otimes e_{t(a)} + (a\otimes_R e_{t(a^*)} - 1\otimes_R a)\otimes a^*\Big)\Big) \\
&= -\sum_{a\in Q_1} a^*\otimes_R a - e_{s(a)}\otimes_R a^*a - aa^*\otimes_R e_{t(a)} + a\otimes_R a^* \\
&\qquad - a\otimes_R a^* + e_{t(a)}\otimes_R aa^* + a^*a\otimes_R e_{s(a)} - a^*\otimes_R a \\
&= \sum_{a\in Q_1} e_{s(a)}\otimes_R a^*a - a^*a\otimes_R e_{s(a)} + aa^*\otimes_R e_{t(a)} - e_{t(a)}\otimes_R aa^*
\end{aligned}$$

while on the other hand

$$\begin{aligned}
D\big([a,a^*]\big) &= D(a)a^* + aD(a^*) - D(a^*)a - a^*D(a) \\
&= (a\otimes_R e_{s(a)} - e_{t(a)}\otimes_R a)a^* + a(a^*\otimes_R e_{s(a^*)} - e_{t(a^*)}\otimes_R a^*) \\
&\quad - (a^*\otimes_R e_{s(a^*)} - e_{t(a^*)}\otimes_R a^*)a - a^*(a\otimes_R e_{s(a)} - e_{t(a)}\otimes_R a) \\
&= a\otimes_R a^* - e_{t(a)}\otimes_R aa^* + aa^*\otimes_R e_{t(a)} - a\otimes_R a^* \\
&\quad - a^*\otimes_R a + e_{s(a)}\otimes_R a^*a - a^*a\otimes_R e_{s(a)} + a^*\otimes_R a \\
&= e_{s(a)}\otimes_R a^*a - a^*a\otimes_R e_{s(a)} + aa^*\otimes_R e_{t(a)} - e_{t(a)}\otimes_R aa^*.
\end{aligned}$$

It follows that a homogeneous representative of degree -1 of $\mu_{\mathrm{nc}}(\omega)$ is $\sum_{a\in Q_1}[a,a^*]$ hence

$$d(t) = \mathrm{w} = \sum_{a\in Q_1}[a,a^*] = \sum_{i\in Q_0} d(t_i) = d\Big(\sum_{i\in Q_0} t_i\Big).$$

Finally we must check the conditions $\xi^2 = 0$, $L_\xi(\omega) = 0$ and $\xi(\mathrm{w}) = 0$. The first equality is clear from the explicit description given above of how ξ acts on arrows. For the second equality we have

$$\begin{aligned}
L_\xi(\omega) &= \sum_{a\in Q_1} L_\xi(D(a))\cdot D(a^*) + D(a)\cdot L_\xi(D(a^*)) \\
&= \sum_{a\in Q_1} D(\xi(a))\cdot D(a^*) + D(a)\cdot D(\xi(a^*)) \\
&= \sum_{a\in Q_1} D(a)\cdot D\left(\frac{\partial W}{\partial a}\right) \\
&= \sum_{a\in Q_1} D^2(a)\cdot \frac{\partial W}{\partial a} + D(a)\cdot D\left(\frac{\partial W}{\partial a}\right) \\
&= D^2(W) = 0.
\end{aligned}$$

For the third equality again write $W = \sum_{t=1}^m a_{t,n_t}\dots a_{t,1}$. Then

$$\begin{aligned}
\xi(\mathrm{w}) &= \xi\Bigg(\sum_{a\in Q_1}[a,a^*]\Bigg) = \sum_{a\in Q_1}\xi(aa^*) - \xi(a^*a) \\
&= \sum_{a\in Q_1}\xi(a)a^* + a\,\xi(a^*) - \xi(a^*)a - (-1)a^*\,\xi(a) \\
&= \sum_{a\in Q_1} a\,\frac{\partial W}{\partial a} - \frac{\partial W}{\partial a}\,a \\
&= \sum_{a\in Q_1}\Bigg(\sum_{\substack{t,j|\\ a_{t,j}=a}} a_{t,j}\,a_{t,j-1}\dots a_{t,1}\,a_{t,n_t}\dots a_{t,j+1} \\
&\qquad - \sum_{\substack{s,i|\\ a_{s,i}=a}} a_{s,i-1}\dots a_{s,1}\,a_{s,n_s}\dots a_{s,i+1}\,a_{s,i}\Bigg) \\
&= \sum_{t,j} a_{t,j}\,a_{t,j-1}\dots a_{t,1}\,a_{t,n_t}\dots a_{t,j+1} \\
&\quad - \sum_{s,i} a_{s,i-1}\dots a_{s,1}\,a_{s,n_s}\dots a_{s,i+1}\,a_{s,i} \\
&= 0.
\end{aligned}$$

□

Proposition 4.3.4. $\mathrm{Jac}(Q,W) = H^0(\Gamma_{Q,W})$.

Proof. Since $(\Gamma_{Q,W})_0 = kQ$ and $(\Gamma_{Q,W})_{-1} = kQ\{a^* : a \in Q_1\}$ we get

$$\begin{aligned} H^0(\Gamma_{Q,W}) &= \frac{\text{Ker } d_0}{\text{Im } d_{-1}} \\ &= \frac{kQ}{(\partial W/\partial a : a \in Q_1)} \\ &= \text{Jac}(Q, W). \end{aligned}$$

□

There is a natural identification of graded algebras $k\overline{Q} \cong T_{kQ}(\mathbb{D}\text{er}_R(kQ))$ that sends the arrow a^* to the double derivation λ_{a^*} given on arrows by

$$Q_1 \ni x \longmapsto \begin{cases} 1 \otimes 1 & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that under this identification the distinguished relative double derivation $\delta \in \mathbb{D}\text{er}_R(kQ)$ is sent to $\sum_{a \in Q_1}[a, a^*] \in k\overline{Q}$. Setting the cyclic 1-form $\alpha \in \text{DR}^1_R(kQ)$ to be $D(W)$ we have that the super-derivation $\{W, -\}_\omega \in \text{Der}_R(k\overline{Q})$ corresponds to the super-derivation $\xi_\alpha \in \text{Der}_R(T_{kQ}(\mathbb{D}\text{er}_R(kQ)))$. Indeed it suffices to check this for degree 0 and -1 elements. The degree 0 parts of $k\overline{Q}$ and $T_{kQ}(\mathbb{D}\text{er}_R(kQ))$ are both isomorphic to kQ and both super-derivations send degree 0 elements to 0. Then in degree -1 we have $\{W, a^*\}_\omega = \partial W/\partial a$ which under the identification above corresponds to sending $\lambda_{a^*} \mapsto \partial W/\partial a$. Hence we must show that $\xi_\alpha(\lambda_{a^*}) = \iota_{\lambda_{a^*}}(D(W))$ is equal to $\partial W/\partial a$. Then

$$\begin{aligned} \iota_{\lambda_{a^*}}(D(W)) &= m \circ \beta\big(i_{\lambda_{a^*}}(D(W))\big) \\ &= m \circ \beta\left(i_{\lambda_{a^*}}\left(\sum_{b \in Q_1} D(b)\frac{\partial W}{\partial b}\right)\right) \\ &= m \circ \beta\left(\sum_{b \in Q_1} \lambda_{a^*}(b)\frac{\partial W}{\partial b}\right) \\ &= m \circ \beta\left(1 \otimes_R 1 \cdot \frac{\partial W}{\partial a}\right) \\ &= \frac{\partial W}{\partial a}. \end{aligned}$$

We also have $\iota_\delta(\alpha) = 0$ from Remark 4.2.4. It follows that the Gdga $\Gamma_{Q,W}$ is naturally isomorphic to the Gdga $\mathfrak{D}(kQ, D(W))$ described in Definition 4.2.3. The proceeding result then follows immediately from Theorem 4.2.5.

Proposition 4.3.5. *The Gdga $\Gamma_{Q,W}$ is Calabi-Yau of dimension 3.*

Proposition 4.3.6 ([1] Proposition 2.3)**.** *There exists a natural t-structure for the derived category* $\mathrm{D}(\Gamma_{Q,W})$ *whose heart is equivalent to* $\mathrm{Jac}(Q,W)$-Mod.

Proposition 4.3.6 allows us to view the moduli stack of representations $\mathrm{Rep}_n(\mathrm{Jac}(Q,W))$ as a moduli stack in a 3 Calabi-Yau category regardless of whether the algebra $\mathrm{Jac}(Q,W)$ is 3 Calabi-Yau or not. This is important for us to be able to define DT invariants. If however $\mathrm{Jac}(Q,W)$ is in fact Calabi-Yau and W is homogeneous we get something stronger. There is a natural grading on the path algebra kQ of a quiver given by the path length, which makes the isomorphism (4.2) an isomorphism of graded algebras. If W is homogeneous of degree m then this descends to a grading on $\mathrm{Jac}(Q,W)$ too. This gives us a second grading on $\Gamma_{Q,W}$, which is strictly positive and preserved by the differential d if we set the degree of the dual arrows a^* to be $m-1$ and degree of the loops t_i to be m. By [[10] Proposition 8.1.1] and its proof we have that the path algebra $k\overline{Q}$ of the doubled quiver is connected. Then by Proposition 4.3.6, and [[25] Proposition 3.7.7 and Theorem 5.3.1] we get the following:

Theorem 4.3.7 (cf. [25] Corollary 5.4.3)**.** *Let W be a homogeneous potential with respect to the path grading on kQ. Then* $\mathrm{Jac}(Q,W)$ *is Calabi-Yau of dimension 3 if and only if it is a superpotential algebra.*

Chapter 5

Motivic DT Theory

5.1 The Grothendieck ring of motives

To motivate the theory of motives we first take a look at constructible functions. This section is based on [50] and [16].

Definition 5.1.1. Let X be a scheme over $\mathbb{C}$ and let $X_{\mathbb{C}}$ denote its set of closed points. A *constructible function* is a map $c : X_{\mathbb{C}} \to \mathbb{Z}$ such that c has only finitely many values on each connected component of X and for all $n \in \mathbb{Z}$ the fibres $c^{-1}(n)$ are the closed points of locally closed subsets of X. Let $\mathrm{Con}(X)$ denote the set of constructible functions on X.

There are many operations that can be performed on the set $\mathrm{Con}(X)$. For a regular map $f : X \to Y$ over $\mathbb{C}$ we have a natural *pullback* $f^* : \mathrm{Con}(Y) \to \mathrm{Con}(X)$ defined in the obvious way. We also have the *pointwise product* of constructible functions $c, d : X_{\mathbb{C}} \to \mathbb{Z}$ given by

$$c \cap d(x) := c(x)d(x).$$

The constant function 1_X that sends all points to $1 \in \mathbb{Z}$ is the unit of this product. We also have an *exterior product* on $X \times Y$ for constructible functions $c : X_{\mathbb{C}} \to \mathbb{Z}$ and $d : Y_{\mathbb{C}} \to \mathbb{Z}$ given by

$$c \boxtimes d = p_X^*(c) \cap p_Y^*(d)$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. This product has unit $1_{\mathrm{Spec}(\mathbb{C})}$. Given a map $g : X \to Y$ of finite type there exists a *pushforward* $g_! : \mathrm{Con}(X) \to \mathrm{Con}(Y)$ given by

$$g_!(c)(y) = \sum_{n \in \mathbb{Z}} n \cdot \chi_c\big(\{x \in X : g(x) = y,\, c(x) = n\}\big)$$

where χ_c denotes the Euler characteristic with compact support.

Define X to be *monoidal* if there exist maps $0 : \mathrm{Spec}(\mathbb{C}) \to X$ and $+ : X \times X \to X$ of finite type satisfying the usual rules of a monoid structure. We can define a third product on $\mathrm{Con}(X)$ when $(X, +, 0)$ is monoidal called the *convolution product*. For $c, d : X_{\mathbb{C}} \to \mathbb{Z}$ it is given by

$$c \cdot d = +_!(c \boxtimes d)$$

with unit $0_!(1_{\mathrm{Spec}(\mathbb{C})})$. The convolution product makes $\mathrm{Con}(X)$ into a unital ring and if $f : X \to Y$ is a homomorphism of monoidal schemes then f^* and $f_!$ (if f is of finite type) are ring homomorphisms.

For any scheme X we can consider the scheme

$$\mathrm{Sym}(X) = \cup_{n \in \mathbb{N}} \mathrm{Sym}^n(X)$$

where $\mathrm{Sym}^n(X)$ is the GIT quotient $X^n /\!\!/ S_n$. This has a natural monoidal structure with $0 : \mathrm{Spec}(\mathbb{C}) = \mathrm{Sym}^0(X) \hookrightarrow \mathrm{Sym}(X)$ and commutative product $+ : \mathrm{Sym}(X) \times \mathrm{Sym}(X) \to \mathrm{Sym}(X)$ that concatenates unordered tuples of points in X. The convolution product then makes $\mathrm{Con}(\mathrm{Sym}(X))$ into a commutative ring. Finally we have natural maps $\sigma^n : \mathrm{Con}(X) \to \mathrm{Con}(\mathrm{Sym}^n(X))$ defined as follows; given a constructible function $c : X_{\mathbb{C}} \to \mathbb{Z}$ we define $\sigma^n(c) : \mathrm{Sym}^n(X) \to \mathbb{Z}$ by sending

$$[x_1, \ldots, x_n] \longmapsto \prod_{i=1}^{p} \binom{f(x_{r_i}) + m_i - 1}{m_i}$$

where p is the number of distinct values of the x_i and m_i denotes the number of x_j that equal x_{r_i}.

Proposition 5.1.2 ([50] Proposition 4.3)**.**

1. *For all schemes X over $\mathbb{C}$ the morphism $\mathrm{Con}(X) \to \prod_{X_i \in \pi_0(X)} \mathrm{Con}(X_i)$, given by pullback along all the connected components of X, is an isomorphism.*

2. *Given $f : X \to Z$ of finite type and $g : Y \to Z$ let $\tilde{f} : X \times_Z Y \to Y$ and $\tilde{g} : X \times_Z Y \to X$ be the projection morphisms, then the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Con}(X \times_Z Y) \arrow[r, "\tilde{f}_!"] & \mathrm{Con}(Y) \\
\mathrm{Con}(X) \arrow[u, "\tilde{g}^*"] \arrow[r, "f_!"] & \mathrm{Con}(Z) \arrow[u, "g^*"']
\end{tikzcd}$$

3. *Let $f : X \to X'$, $g : Y \to Y'$ and $c \in \mathrm{Con}(X')$, $d \in \mathrm{Con}(Y')$. Then*

$$(f \times g)^*(c \boxtimes d) = f^*(c) \boxtimes g^*(d).$$

If in addition f and g are of finite type then for $c \in \mathrm{Con}(X)$, $d \in \mathrm{Con}(Y)$ we *have*

$$(f \times g)_!(c \boxtimes d) = f_!(c) \boxtimes g_!(d)$$

and

$$\mathrm{Sym}^n(f)_!(\sigma^n(c)) = \sigma^n(f_!(c)).$$

4. *For all* $c, d \in \mathrm{Con}(X)$ *we have*

$$\sigma^n(c + d) = \sum_{i=0}^{n} \sigma^i(c)\sigma^{n-i}(d).$$

5. *Let* $i : Z \hookrightarrow X$ *be a closed subscheme with complement* $j : X - Z \hookrightarrow X$*. Then*

$$\begin{aligned} i^* i_! &= \mathrm{id}_{\mathrm{Con}(Z)} \\ j^* j_! &= \mathrm{id}_{\mathrm{Con}(X-Z)} \\ i^* j_! &= j^* i_! = 0 \end{aligned}$$

and for all $c \in \mathrm{Con}(X)$ *we have*

$$c = i_! i^*(c) + j_! j^*(c).$$

6. *For any scheme* X *and any* $n \in \mathbb{N}$ *we have*

$$\sigma^n(1_X) = 1_{\mathrm{Sym}^n(X)}.$$

The set $\mathrm{Con}(X)$ with the exterior and convolution products, pullbacks, pushforwards for finite type morphisms, the maps σ^n, and the relationships all these structures have intertwined are what we will base a motivic theory off of.

Definition 5.1.3. A *motivic theory* M associates to each scheme X over $\mathbb{C}$ an abelian group $M(X)$ such that the following holds:

i) For every $f : X \to Y$ we have a pullback map $f^* : M(Y) \to M(X)$.

ii) For every $g : X \to Y$ of finite type we have a pushforward map $g_! : M(X) \to M(Y)$.

iii) There is an associative, symmetric exterior product $\boxtimes : M(X) \times M(Y) \to M(X \times Y)$ which has unit $1 \in M(\mathrm{Spec}(\mathbb{C}))$.

iv) For all $n \in \mathbb{N}$ there exist maps $\sigma^n : M(X) \to M(\mathrm{Sym}^n(X))$.

v) Constructing the convolution product for $(X, +, 0)$ monoidal as

$$ab = +_!(a \boxtimes b)$$

with $a, b \in M(X)$, the equivalent properties as given in Proposition 5.1.2 hold.

Remark 5.1.4. If X is of finite type over $\mathbb{C}$ then we will always have a pushforward map $M(X) \to M(\mathrm{Spec}(\mathbb{C}))$. This particular map will be denoted as an integral $\int_X$.

The main example of a motivic theory that we will study in detail and build upon throughout the rest of this thesis is the *Grothendieck ring of motives.* Let X be a scheme over $\mathbb{C}$ then define the group $K_0(\mathrm{Var}/X)$ to be the abelian group generated by isomorphism classes of morphisms of finite type $S \xrightarrow{a} X$ where S is a variety over $\mathbb{C}$, up to the relations

$$S \xrightarrow{a} X \sim \left(Z \xrightarrow{a|_Z} X\right) + \left(S \setminus Z \xrightarrow{a|_{S\setminus Z}} X\right)$$

where $Z \subset S$ is any closed and reduced subscheme. These are often called the *cut and paste* or *scissor* relations. We denote the motive of $S \xrightarrow{a} X$ by

$$[S \xrightarrow{a} X]$$

and motives of this form are called *effective.*

We can define the pullback for $f : X \to Y$ via base-change

$$f^*([S \xrightarrow{a} Y]) = [X \times_Y S \xrightarrow{\tilde{a}} X]$$

and if $g : X \to Y$ is of finite type the obvious map

$$[S \xrightarrow{a} X] \mapsto [S \xrightarrow{a} X \xrightarrow{g} Y]$$

defines a pushforward $g_! : K_0(\mathrm{Var}/X) \to K_0(\mathrm{Var}/Y)$. The exterior product is simply given by

$$[S \xrightarrow{a} X] \boxtimes [T \xrightarrow{b} Y] = [S \times T \xrightarrow{a\times b} X \times Y].$$

For $(X, +, 0)$ monoidal we can define a convolution product on $K_0(\mathrm{Var}/X)$ by

$$[S \xrightarrow{a} X] \cdot [T \xrightarrow{b} X] = [S \times T \xrightarrow{a\times b} X \times X \xrightarrow{+} X]$$

which has unit $[\mathrm{Spec}(\mathbb{C}) \xrightarrow{0} X]$. This gives $K_0(\mathrm{Var}/X)$ a unital ring structure. Let $\mathbb{L}$ denote the motive $[\mathbb{A}^1 \to \mathrm{Spec}(\mathbb{C}) \xrightarrow{0} X]$ and $\mathbb{L}^r$ denote $[\mathbb{A}^r \to X]$. Finally we define the maps $\sigma^n : K_0(\mathrm{Var}/X) \to K_0(\mathrm{Var}/\mathrm{Sym}^n(X))$ by

$$\sigma^n([S \xrightarrow{a} X]) = [\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(a)} \mathrm{Sym}^n(X)]$$

extending linearly using Proposition 5.1.2 4.

Remark 5.1.5 (cf. [16] Remark 2.2)**.**

1. We can replace the category Var/X of varieties of finite type over X by the categories Sch/X of schemes over X, or Sp/X of algebraic spaces over X, or

$\mathrm{St}^{\mathrm{aff}}/X$ of Artin stacks with affine stabilisers over X, or St/X of Artin stacks over X (all of finite type over X).

2. We can also replace the scheme X with an Artin stack $\mathcal{X}$ which is locally of finite type over $\mathbb{C}$ giving us a motivic theory for Artin stacks (locally of finite type over $\mathbb{C}$). In this case we replace $\mathrm{Sym}^n(X)$ with the stacky symmetric product given by the quotient stack $\mathcal{S}ym^n(\mathcal{X}) = \mathcal{X}^n/S_n$.

3. There is a useful completion $\hat{K}_0(-/\mathcal{X})$ given by completing $K_0(-/\mathcal{X})$ with respect to the topology where $K_0(-/\mathcal{X} \setminus \mathcal{U}) \subset K_0(-/\mathcal{X})$ are the open neighbourhoods of 0 for $\mathcal{U} \subset \mathcal{X}$ an open substack of finite type over $\mathbb{C}$. In particular we have that for a locally finite stratification $(\mathcal{X}_n)_{n\in\mathbb{N}}$ of a connected stack $\mathcal{X}$, where each $\mathcal{X}_n$ is a locally closed substack of finite type, $[\mathcal{X} \xrightarrow{\mathrm{id}} \mathcal{X}] = \sum_{n\geq 0}[\mathcal{X}_n \hookrightarrow \mathcal{X}]$.

4. When $X = \mathrm{Spec}(\mathbb{C})$ we write the Grothendieck ring as $K_0(-/\mathbb{C})$.

It remains to check that Proposition 5.1.2 holds for this setup in order for us to actually have a motivic theory.

Proposition 5.1.6. *Let X be a scheme over $\mathbb{C}$ and define the ring $K_0(\mathrm{Var}/X)$ along with the operations as above. This gives a motivic theory for schemes.*

Proof.

1. Let $\{X_i\}_{i\in I}$ be the connected components of X. Then our morphism $K_0(\mathrm{Var}/X) \to \prod_{i\in I} K_0(\mathrm{Var}/X_i)$ sends $[S \xrightarrow{a} X] \mapsto \big([S \times_X X_i \xrightarrow{a_i} X_i]\big)_{i\in I}$. Since the X_i are closed subsets of X we can write

$$[S \xrightarrow{a} X] = \sum_{i\in I}[S_i \xrightarrow{a|_{S_i}} X]$$

where $S_i = a^{-1}(X_i)$. But S_i is exactly $S \times_X X_i$ and so as this decomposition is unique we get injectivity. For surjectivity suppose we are given $[S_i \xrightarrow{a_i} X_i]$ for each i. Define $S = \bigsqcup_{i\in I} S_i$ and let $S \xrightarrow{a} X$ be defined piecewise by the a_i. Then clearly $[S \xrightarrow{a} X]$ maps to $\big([S_i \xrightarrow{a_i} X_i]\big)_{i\in I}$.

2. Given $f : X \to Z$ of finite type and $g : Y \to Z$ we have

$$\begin{aligned} g^* \circ f_!([S \xrightarrow{a} X]) &= g^*([S \xrightarrow{a} X \xrightarrow{f} Z]) \\ &= [S \times_Z Y \xrightarrow{\widetilde{f\circ a}} Y] \end{aligned}$$

whilst

$$\begin{aligned}\widetilde{f_!} \circ \widetilde{g}^*([S \xrightarrow{a} X]) &= \widetilde{f_!}([S \times_X (X \times_Z Y) \xrightarrow{\tilde{a}} X \times_Z Y]) \\ &= \widetilde{f_!}([S \times_Z Y \xrightarrow{\tilde{a}} X \times_Z Y]) \\ &= [S \times_Z Y \xrightarrow{\widetilde{f} \circ \tilde{a}} Y]\end{aligned}$$

hence the result follows.

3. Let $f : X \to X'$ and $g : Y \to Y'$. Then

$$\begin{aligned}(f \times g)^*([S \xrightarrow{a} X'] \boxtimes [T \xrightarrow{b} Y']) &= (f \times g)^*([S \times T \xrightarrow{a \times b} X' \times Y']) \\ &= [(S \times T) \times_{X' \times Y'} (X \times Y) \xrightarrow{\widetilde{a \times b}} X \times Y] \\ &= [(S \times_{X'} X) \times (T \times_{Y'} Y) \xrightarrow{\tilde{a} \times \tilde{b}} X \times Y]\end{aligned}$$

and

$$\begin{aligned}f^*([S \xrightarrow{a} X']) \boxtimes g^*([T \xrightarrow{b} Y']) &= [S \times_{X'} X \xrightarrow{\tilde{a}} X] \boxtimes [T \times_{Y'} Y \xrightarrow{\tilde{b}} Y] \\ &= [(S \times_{X'} X) \times (T \times_{Y'} Y) \xrightarrow{\tilde{a} \times \tilde{b}} X \times Y].\end{aligned}$$

If in addition f and g are of finite type then

$$\begin{aligned}(f \times g)_!([S \xrightarrow{a} X] \boxtimes [T \xrightarrow{b} Y]) &= (f \times g)_!([S \times T \xrightarrow{a \times b} X \times Y]) \\ &= [S \times T \xrightarrow{a \times b} X \times Y \xrightarrow{f \times g} X' \times Y'].\end{aligned}$$

and

$$\begin{aligned}f_!([S \xrightarrow{a} X]) \boxtimes g_!([T \xrightarrow{b} Y]) &= [S \xrightarrow{a} X \xrightarrow{f} X'] \boxtimes [T \xrightarrow{b} Y \xrightarrow{g} Y'] \\ &= [S \times T \xrightarrow{(f \circ a) \times (g \circ b)} X' \times Y'].\end{aligned}$$

Also

$$\begin{aligned}\mathrm{Sym}^n(f)_!(\sigma^n([S \xrightarrow{a} X])) &= \mathrm{Sym}^n(f)_!([\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(a)} \mathrm{Sym}^n(X)]) \\ &= [\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(a)} \mathrm{Sym}^n(X) \xrightarrow{\mathrm{Sym}^n(f)} \mathrm{Sym}^n(X')]\end{aligned}$$

whilst

$$\begin{aligned}\sigma^n(f_!([S \xrightarrow{a} X])) &= \sigma^n([S \xrightarrow{a} X \xrightarrow{f} X']) \\ &= [\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(f \circ a)} \mathrm{Sym}^n(X')].\end{aligned}$$

4. This is by definition.
5. Let $i : Z \hookrightarrow X$ be a closed subset with complement $j : X \setminus Z \hookrightarrow X$. The first 4 equalities are clear. Then given $[S \xrightarrow{a} X]$ we have

$$\begin{aligned} i_! i^*([S \xrightarrow{a} X]) + j_! j^*([S \xrightarrow{a} X]) &= [a^{-1}(Z) \to Z \hookrightarrow X] + [a^{-1}(X \setminus Z) \to X \setminus Z \hookrightarrow X] \\ &= [S \to X] \end{aligned}$$

by the cut and paste relations in $K_0(\mathrm{Var}/X)$.
6.

$$\begin{aligned} \sigma^n(1_X) &= \sigma^n([\operatorname{Spec}\mathbb{C} \xrightarrow{0} X]) \\ &= [\operatorname{Sym}^n(\operatorname{Spec}(\mathbb{C})) = \operatorname{Spec}(\mathbb{C}) \xrightarrow{\operatorname{Sym}^n(0)} \operatorname{Sym}^n(X)] \\ &= 1_{\operatorname{Sym}^n(X)}. \end{aligned}$$

□

For an affine group G there a G-equivariant version of the Grothendieck ring of motives which we now describe.

Definition 5.1.7. The action of an algebraic group G on a scheme X is called *good* if every point in X has an affine neighbourhood which is invariant under the G-action.

Fix an affine group G and let X be monoidal with a good G-action. Then consider just the classes of G-equivariant morphisms $S \xrightarrow{a} X$ in $K_0(\mathrm{Sch}/X)$. We denote this ring of motives by $K_0^G(\mathrm{Sch}/X)$. We also consider the quotient of $K_0^G(\mathrm{Sch}/X)$ by the subgroup generated by

$$[S \xrightarrow{\pi} T \xrightarrow{a} X] - \mathbb{L}^r[T \xrightarrow{a} X] \tag{5.1}$$

where $S \xrightarrow{\pi} T$ is a G-equivariant vector bundle of rank r. This subgroup is in fact an ideal and so the quotient, which is denoted by $K^G(\mathrm{Sch}/X)$, is a ring as well. If the G-action on X is trivial we also have a natural homomorphism of $K_0(\mathrm{Sch}/X)[\mathbb{L}^{-\frac{1}{2}}]$-modules

$$q_G : K_0^G(\mathrm{Sch}/X)[\mathbb{L}^{-\frac{1}{2}}] \to K_0(\mathrm{Sch}/X)[\mathbb{L}^{-\frac{1}{2}}] \tag{5.2}$$

that sends

$$[S \to X] \longmapsto [S/G \to X].$$

Remark 5.1.8. It is also possible to define a G-equivariant theory for $K_0(\mathrm{St}/\mathcal{X})$ where $\mathcal{X}$ is a monoidal stack. However we will not need that level of generality and so stick with the simpler case of schemes.

Lemma 5.1.9 ([24] Proposition 1.1 and [16] Lemma 2.6). *Let $S \xrightarrow{\pi} T$ be a G-equivariant GL_n-principal bundle on T. Then for any motive $[T \xrightarrow{a} X]$ we have*

$$[S \xrightarrow{\pi} T \xrightarrow{a} X] = [\mathrm{GL}_n][T \xrightarrow{a} X]$$

in $K^G(\mathrm{Sch}/X)$.

Proof. We first calculate the motive $[\mathrm{GL}_n]$. We claim that

$$[\mathrm{GL}_n] = \prod_{i=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^i).$$

Proceeding by induction the base case is immediate. Suppose it is true for $n-1$. Consider the map

$$\mathrm{GL}_n \xrightarrow{p} \mathbb{A}^n \setminus \{0\}$$

that sends an invertible matrix to its first column. This is a Zariski trivial fibration with fibre $\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}$ and a finite trivialising cover. Indeed a trivialising cover consists of the open sets $U_i = \{z_i \neq 0\} \subset \mathbb{A}^n \setminus \{0\}$ for coordinates $(z_1, \ldots, z_n)$ on $\mathbb{A}^n$. On U_i the local trivialisation sends the matrix $A = (a_{jk}) \in p^{-1}(U_i)$ to

$$\left((a_{ji})_{j=1}^n,\ \big(\det(A_{ji})\big)_{j=2}^n,\ \sum_{j=1}^n a_{ji}A_{ji}\right) \in U_i \times (\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1})$$

where A_{jk} is the (j,k)-minor of A. Writing V_i as the complement to U_i in $\mathbb{A}^n \setminus \{0\}$ we get the finite stratification

$$\bigsqcup_{i=1}^n Z_i$$

of $\mathbb{A}^n \setminus \{0\}$ where

$$Z_i = \bigcap_{j=1}^{i-1} V_j \cap U_i$$

such that

$$Z_i \times (\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}) \cong p^{-1}(Z_i) \xrightarrow{p} Z_i$$

for all i. Therefore using the cut and paste relations we get that

$$\begin{aligned}[\mathrm{GL}_n] &= [\mathrm{GL}_n \xrightarrow{p} \mathbb{A}^n \setminus \{0\} \to \mathrm{Spec}(\mathbb{C})] \\ &= \left[\mathrm{GL}_n \xrightarrow{p} \bigsqcup_{i=1}^{n} Z_i \to \mathrm{Spec}(\mathbb{C})\right] \\ &= \sum_{i=1}^{n} \left[p^{-1}(Z_i) \xrightarrow{p} Z_i \to \mathrm{Spec}(\mathbb{C})\right] \\ &= \sum_{i=1}^{n} [Z_i \times (\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}) \xrightarrow{p} Z_i \to \mathrm{Spec}(\mathbb{C})] \\ &= \sum_{i=1}^{n} [Z_i \times (\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1})] \\ &= \sum_{i=1}^{n} [Z_i][\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}] \\ &= \left[\bigsqcup_{i=1}^{n} Z_i\right][\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}] \\ &= [\mathbb{A}^n \setminus \{0\}][\mathbb{A}^{n-1} \times \mathrm{GL}_{n-1}].\end{aligned}$$

Applying the induction hypothesis we get

$$\begin{aligned}[\mathrm{GL}_n] &= [\mathbb{A}^n \setminus \{0\}][\mathbb{A}^{n-1}][\mathrm{GL}_{n-1}] \\ &= (\mathbb{L}^n - 1)\mathbb{L}^{n-1} \prod_{i=0}^{n-2} (\mathbb{L}^{n-1} - \mathbb{L}^i) \\ &= \prod_{i=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^i).\end{aligned}$$

Now let

$$B = S \times^{\mathrm{GL}_n} \mathbb{A}^n := (S \times \mathbb{A}^n)/\mathrm{GL}_n \longrightarrow T$$

be the G-equivariant vector bundle associated to the G-equivariant principal GL_n-bundle $S \xrightarrow{\pi} T$ i.e. if $\{W_j\}$ is a local trivialisation of $S \xrightarrow{\pi} T$ then B is locally given by $W_j \times \mathbb{A}^n$ and is glued using the same data as $S \xrightarrow{\pi} T$. For $1leqi \leq n$ write $S_i \to T$ for the bundle of i linearly independent vectors in B and $S_0 = T$. Then $S_n = S$ since on a local trivialisation we can bijectively identify the n-independent vectors $\{v_1, \ldots, v_n\}$ with $g \in \mathrm{GL}_n$ such that g is the change of basis matrix from the standard basis $\{e_1, \ldots, e_n\}$ to $\{v_1, \ldots, v_n\}$. We have natural maps $S_{i+1} \to S_i$ making S_{i+1} into a bundle over S_i which is the complement in $B \times_T S_i \to S_i$ of the sub-bundle $E_i \to S_i$ of rank i generated by the vectors in S_i. Since we are in the quotient ring $K^G(\mathrm{Sch}/X)$

and $B \times_T S_i$ is a vector bundle of rank n over S_i we get that

$$\begin{aligned}
[S_{i+1} \to S_i \to T \xrightarrow{a} X] &= [(B \times_T S_i) \setminus E_i \to S_i \to T \xrightarrow{a} X] \\
&= [B \times_T S_i \to S_i \to T \xrightarrow{a} X] - [E_i \to S_i \to T \xrightarrow{a} X] \\
&= \mathbb{L}^n [S_i \to T \xrightarrow{a} X] - \mathbb{L}^i [S_i \to T \xrightarrow{a} X] \\
&= (\mathbb{L}^n - \mathbb{L}^i)[S_i \to T \xrightarrow{a} X].
\end{aligned}$$

Hence by induction it follows that

$$\begin{aligned}
[S \to T \xrightarrow{a} X] &= [S_n \to S_{n-1} \to T \xrightarrow{a} X] \\
&= (\mathbb{L}^n - \mathbb{L}^{n-1})[S_{n-1} \to T \xrightarrow{a} X] \\
&= (\mathbb{L}^n - \mathbb{L}^{n-1})[S_{n-1} \to S_{n-2} \to T \xrightarrow{a} X] \\
&= \ldots = \prod_{i=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^i)[T \xrightarrow{a} X] \\
&= [\mathrm{GL}_n][T \xrightarrow{a} X].
\end{aligned}$$

□

Proposition 5.1.10 ([24] Theorem 1.2)**.** *Let X be a monoidal scheme of finite type over $\mathbb{C}$. Then for any affine group G the ring homomorphism*

$$K^G(\mathrm{Var}/X)\big[[\mathrm{GL}_n]^{-1} \,:\, n \in \mathbb{N}\big] \to K^G(\mathrm{St}^{\mathrm{aff}}/X)$$

is an isomorphism.

Proposition 5.1.10 says that to study the ring of motives of stacks with affine stabilisers over X it is enough to study the ring of motives of varieties over X localised only at the classes $[\mathrm{GL}_n]$ for $n \in \mathbb{N}$. This is a huge reduction in the complexity of the objects that are being dealt with the in stack case.

5.2 λ-rings and power structures

Our main references for this section are [43], [16], [7], [3], and [20].

Definition 5.2.1. A *λ-ring* is a commutative ring R with a map $\lambda : R \to R[[t]]$ written as

$$a \longmapsto \lambda_a(t) = \sum_{n \geq 0} \lambda^n(a) t^n$$

such that for all $a, b \in R$

i) $\lambda_a(t) = 1 + at \bmod t^2$.

ii) $\lambda_0(t) = 1$.

iii) $\lambda_{a+b}(t) = \lambda_a(t)\lambda_b(t)$.

The motivation of the definition of a λ-ring comes from the different operations one can perform on finite dimensional vector spaces (over $\mathbb{C}$) to form other finite dimensional vector spaces; namely for vector spaces V and W we can make the direct sum $V \oplus W$, the tensor product $V \otimes W$ and the n-th exterior power $\Lambda^n(V)$. A λ-ring abstracts the properties these three constructions have when they interact with each other.

Definition 5.2.2. A *λ-ring homomorphism* $\phi : (R, \lambda) \to (S, \delta)$ is a ring homomorphism such that

$$\phi(\lambda^n(a)) = \delta^n(\phi(a))$$

for all $a \in R$ and $n \in \mathbb{N}$.

Definition 5.2.3. A *λ-ideal* $I \subset R$ is an ideal such that for all $a \in I$ and $n \in N$

$$\lambda^n(a) \in I.$$

The quotient of a λ-ring by a λ-ideal is again a λ-ring.

Remark 5.2.4.

1. Some authors often include two additional conditions in their definition of a λ-ring, namely that for $n, m \in \mathbb{N}$ there are specific universal polynomials P_n and $P_{n,m}$ such that for $a, b \in R$

$$\lambda^n(ab) = P_n\big(\lambda^1(a), \dots, \lambda^n(a), \lambda^1(b), \dots, \lambda^n(b)\big)$$

and

$$\lambda^n(\lambda^m(a)) = P_{n,m}\big(\lambda^1(a), \dots, \lambda^m n(a)\big)$$

(for a description of these polynomials see [43]). We will instead refer to λ-rings which satisfy these extra conditions as *special* λ-rings.

2. From now on we will tend to use σ to denote the λ-ring map $\sigma : R \to R[[t]]$ as the λ-ring structures we will consider are derived from symmetric powers instead of exterior powers.

Definition 5.2.5. An element $a \in R$ is called a *line element* if

$$\sigma_a(t) = \sum_{n \geq 0} a^n t^n = \frac{1}{1 - at}.$$

Definition 5.2.6. The *opposite λ-ring* (R, σ^{op}) is given by

$$\sigma_a^{\text{op}}(t) = \sigma_a(-t)^{-1}.$$

Example 5.2.7 (cf. [16] Example 3.5)**.**

1. Consider the ring $\mathbb{Z}$ with λ-ring structure given by $\sigma^n(a) = \binom{a+n-1}{n}$. This has the opposite λ-ring structure given by $(\sigma^{\mathrm{op}})^n(a) = \binom{a}{n}$.

2. For a scheme X recall the set $\mathrm{Con}(X)$ of constructible functions $c : X_{\mathbb{C}} \to Z$ which had a natural ring structure for monoidal schemes $(X, +, 0)$ given by

$$c \cdot d = +_!(c \boxtimes d).$$

   We can make $\mathrm{Con}(X)$ into a λ-ring using the maps $\sigma^n : \mathrm{Con}(X) \to \mathrm{Con}(\mathrm{Sym}^n(X))$ described in Section 5.1 and then pushing forward using $+$ to return a constructible function in $\mathrm{Con}(X)$. This indeed gives a λ-ring structure by Proposition 5.1.2. If we take $X = \mathrm{Spec}(\mathbb{C})$ this gives us the λ- ring structure described previously on $\mathbb{Z}$. Moreover this in fact defines a special λ-ring structure and for a morphism $f : X \to Y$ of finite type the pushforward $f_!$ gives a λ-ring homomorphism.

3. Again let $(X, +, 0)$ be a monoidal scheme. We get a (not special) λ-ring structure on $K_0(\mathrm{Var}/X)$ using the maps

$$\sigma^n([S \xrightarrow{a} X]) := [\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(a)} \mathrm{Sym}^n(X) \xrightarrow{+} X].$$

   It follows that this gives a λ-ring structure by Proposition 5.1.6. $\mathbb{L}$ is a line element for this λ-ring structure since

$$\begin{aligned}
\sigma_{\mathbb{L}}(t) &= \sum_{n\geq 0} \sigma_n(\mathbb{L})t^n \\
&= \sum_{n\geq 0} [\mathrm{Sym}^n(\mathbb{A}^1) \to \mathrm{Sym}^n(\mathrm{Spec}(\mathbb{C})) \to X]\, t^n \\
&= \sum_{n\geq 0} [\mathbb{A}^n \to \mathrm{Spec}(\mathbb{C}) \to X]\, t^n \\
&= \sum_{n\geq 0} \mathbb{L}^n t^n.
\end{aligned}$$

   This λ-ring structure can be extended to certain localisations of $K_0(\mathrm{Var}/X)$ including $K_0(\mathrm{Var}/X)\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big]$. The λ-ring structure is also present in the G-equivariant case. It can be shown that the ideal generated by $[S \xrightarrow{\pi} T \xrightarrow{a} X] - \mathbb{L}^r[T \xrightarrow{a} X]$ where π is a G-equivariant vector bundle of rank r is in fact a λ-ideal and hence the λ-ring structure descends to the ring $K^G(\mathrm{Sch}/X)$.

4. If $(\mathcal{X}, +, 0)$ is a commutative monoid in the category of Artin stacks locally of finite type over $\mathbb{C}$ then we can get a (not special) λ-ring structure on $K_0(\mathrm{St}^{\mathrm{aff}}/\mathcal{X})$

and $K_0(\mathrm{St}/\mathcal{X})$ using the maps

$$\sigma^n([\mathcal{S} \xrightarrow{a} \mathcal{X}]) := [\mathcal{S}ym^n(\mathcal{S}) \xrightarrow{\mathcal{S}ym^n(a)} \mathcal{S}ym^n(\mathcal{X}) \xrightarrow{+} \mathcal{X}].$$

Proposition 5.2.8 ([16] Proposition 3.6)**.** *Let X be a monoidal scheme and G be an affine group. Then we have a sequence of λ-ring homomorphisms*

$$K^G(\mathrm{Var}/\mathbb{C}) \to K^G(\mathrm{Var}/X) \to K^G(\mathrm{Sch}/X) \to K^G(\mathrm{Sp}/X) \to K^G(\mathrm{St}^{\mathrm{aff}}/X) \to K^G(\mathrm{St}/X)$$

and the isomorphism

$$K^G(\mathrm{Var}/X)\big[[\mathrm{GL}_n]^{-1} \,:\, n \in \mathbb{N}\big] \xrightarrow{\sim} K^G(\mathrm{St}^{\mathrm{aff}}/X)$$

of Proposition 5.1.10 is an isomorphism of λ-rings.

Definition 5.2.9. For a λ-ring R we define the *plethystic exponential* as the map

$$\mathrm{Exp} : R[[t_1, \dots, t_r]]_+ \to 1 + R[[t_1, \dots, t_r]]_+$$

where $R[[t_1, \dots, t_r]]_+ \subset R[[t_1, \dots, t_r]]$ is the ideal generated by $t_1, \dots, t_r$, that sends

$$a\, t_1^{k_1} \dots\, t_r^{k_r} \longmapsto \sum_{n \geq 0} \sigma^n(a)\, t_1^{nk_1} \dots\, t_r^{nk_r}$$

extending to a series $\sum_{m \geq 1} a_m t_1^{k_{1m}} \dots\, t_r^{k_{rm}}$ using the normal exponential rules.

Importantly this defines a plethystic exponential for us on $K_0(\mathrm{Var}/X)$ for X monoidal, which will be useful when defining motivic DT invariants.

Definition 5.2.10. A *power structure* on a ring R is a map

$$\big(1 + R[[t]]_+\big) \times R \to 1 + R[[t]]_+$$

often denoted as

$$(A(t), r) \longmapsto A(t)^r$$

that satisfies

i) $A(t)^0 = 1$ and $A(t)^1 = A(t)$

ii) $A(t)^{r+s} = A(t)^r A(t)^s$

iii) $A(t)^{rs} = (A(t)^r)^s$

iv) $(A(t)B(t))^r = A(t)^r B(t)^r$

v) $(1+t)^r = 1 + rt \mod t^2$

vi) $(A(t^r))^s = (A(t))^s|_{t \mapsto t^r}$.

It was shown in [[27] Proposition 1] that to determine a power structure on R it is enough to define the series

$$(1-t)^{-r}$$

for each $r \in R$, so that $(1-t)^{-r} = 1 + rt \mod t^2$, $(1-t)^{-r-s} = (1-t)^{-r}(1-t)^{-s}$, and $(1-t)^{-1} = \sum_{n \geq 0} t^n$ all hold. The link between λ-rings and power structures is therefore as follows. Given a λ-ring R we can define a power structure on R by declaring

$$\begin{aligned}(1-t)^{-r} &= \mathrm{Exp}(rt)\\ &= \sum_{n\geq 0} \sigma^n(r) t^n.\end{aligned}$$

In particular we get a power structure on $K_0(\mathrm{Var}/\mathbb{C})$ uniquely defined by the equation

$$(1-t)^{-[X]} = \sum_{n \geq 0} \left[\mathrm{Sym}^n(X)\right] t^n.$$

Letting $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a partition of n i.e. $\sum_{i=1}^n i \cdot \alpha_i = n$ and $S_\alpha = \prod_i S_{\alpha_i}$ be the product of symmetric groups, then for a series $A(t) = 1 + \sum_{i \geq 1} A_i t^i$ where the coefficients A_i are all effective motives we have that (see [27] or [[3] (1.6)])

$$A(t)^{[X]} = 1 + \sum_{\alpha \vdash n} q_{S_\alpha} \left(\prod_{i=1}^n A_i^{\alpha_i} \cdot \left[\prod_{i=1}^n X^{\alpha_i} \setminus \Delta \right] \right) t^n \tag{5.3}$$

where $\Delta \subset \prod_i X^{\alpha_i}$ is the big diagonal consisting of the tuples such that $x_j = x_k$ for some $j \neq k$, and S_α acts on both the terms $\prod_{i=1}^n A_i^{\alpha_i}$ and $\prod_{i=1}^n X^{\alpha_i} \setminus \Delta$ by permuting the factors. If the coefficients of $A(t)$ are not effective then we can always write $A(t)B(t) = C(t)$ for series $B(t)$ and $C(t)$ with effective coefficients, and so we define

$$A(t)^{[X]} = \left(B(t)^{[X]}\right)^{-1} C(t)^{[X]}.$$

Note that on $K_0(\mathrm{Var}/\mathbb{C})$ we have for effective motives A_n

$$\mathrm{Exp}\Big(\sum_n A_n t^n\Big) = \prod_n (1-t^n)^{-A_n}.$$

This power structure on $K_0(\mathrm{Var}/\mathbb{C})$ extends to $K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-\frac{1}{2}}]$ and to $K_0(\mathrm{St}^{\mathrm{aff}}/\mathbb{C})$ (see [28]). In particular if the coefficients of $A(t) = 1 + \sum_{i \geq 1} \mathbb{L}^{-\frac{c_i}{2}} A_i t^i$ lie in $K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-\frac{1}{2}}]$ and are effective then we generalise the formula (5.3) to

$$A(t)^{[X]} = 1 + \sum_{\alpha \vdash n} \mathbb{L}^{-\frac{\sum_i c_i \alpha_i}{2}} q_{S_\alpha} \left(\prod_{i=1}^n A_i^{\alpha_i} \cdot \left[\prod_{i=1}^n X^{\alpha_i} \setminus \Delta \right] \right) t^n \tag{5.4}$$

and if the coefficients of $A(t)$ are effective and lie in $K_0(\mathrm{St}^{\mathrm{aff}}/\mathbb{C})$ then the formula (5.3) continues to hold (see [[7] Lemma 5]).

5.3 Motivic vanishing cycles

Motivic vanishing cycles will form the basis of how we define motivic DT invariants, just as the sheaf of vanishing cycles is intimately related to normal DT invariants. This section is based off of [50], [16], and [21].

Definition 5.3.1. A *vanishing cycle* for a motivic theory of schemes M associates to each smooth scheme X and regular function $f : X \to \mathbb{A}^1$ an element

$$[\phi_f] \in M(X_0)$$

where $X_0 = f^{-1}(0)$ is the zero-fibre of f. More generally we can define a vanishing cycle for a motivic theory of stacks and for a relative motivic theory i.e. instead of having varieties/schemes/stacks over $\mathbb{C}$ we take them over an Artin stack locally of finite type over $\mathbb{C}$.

We first tackle the equivariant case. Let $\mathcal{X}$ be a commutative monoidal Artin stack and consider $\mathbb{A}^1_{\mathcal{X}} := \mathcal{X} \times \mathbb{A}^1$. Let $\mathbb{G}_m$ denote the smooth affine algebraic group $\mathbb{A}^1 \setminus \{0\}$ under multiplication. Then $\mathbb{A}^1_{\mathcal{X}}$ is itself a commutative monoid with a $\mathbb{G}_m$-action of weight d given by

$$g \cdot (x, z) = (x, g^d z).$$

Therefore we may consider the λ-ring $K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$ of $\mathbb{G}_m$-equivariant motives. This has the subgroup $K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathrm{GL}_1(\mathcal{X}))$ of motives supported on $\mathrm{GL}_1(\mathcal{X}) := \mathcal{X} \times \mathrm{GL}_1 = \mathcal{X} \times \mathbb{G}_m$.

This subgroup is isomorphic to $K_0^{\mu_d}(\mathrm{Var}/\mathcal{X})$ where μ_d denotes the group of the d-th roots of unity under multiplication which acts trivially on $\mathcal{X}$. Indeed given a $\mathbb{G}_m$-equivariant motive $[S \xrightarrow{a} \mathrm{GL}_1(\mathcal{X})]$ we can take the fibre over $\mathcal{X} \times \{1\} \subset \mathcal{X} \times \mathbb{G}_m$ giving a μ_d-equivariant motive on $\mathcal{X}$, while a motive $[T \xrightarrow{b} \mathcal{X}]$ over $\mathcal{X}$ is sent to the motive $[T \times^{\mu_d} \mathbb{G}_m \xrightarrow{b\times(-)^d} \mathcal{X} \times \mathbb{G}_m]$. If $a(s) = (x, z) \in \mathcal{X} \times \mathbb{G}_m$ and we take $g \in \mathbb{G}_m$ such that $g^d = z$, then we get an identification

$$S \xrightarrow{\sim} a^{-1}\big(\mathcal{X} \times \{1\}\big) \times^{\mu_d} \mathbb{G}_m$$

via

$$s \longmapsto (gs,\, g^{-1})$$

with a commutative triangle

$$\begin{tikzcd} S \arrow[r, "\sim"] \arrow[dr, "a"'] & a^{-1}\big(\mathcal{X} \times \{1\}\big) \times^{\mu_d} \mathbb{G}_m \arrow[d, "a \times (-)^d"] \\ & \mathcal{X} \times \mathbb{G}_m \end{tikzcd}$$

Similarly

$$T \xrightarrow{\sim} \big(b \times (-)^d\big)^{-1}\big(\mathcal{X} \times \{1\}\big)$$

via

$$t \longmapsto [t, 1]$$

and we have the commutative triangle

$$\begin{tikzcd} T \arrow[r, "\sim"] \arrow[dr, "b"'] & \big(b \times (-)^d\big)^{-1}\big(\mathcal{X} \times \{1\}\big) \arrow[d, "b \times (-)^d"] \\ & \mathcal{X} \cong \mathcal{X} \times \{1\} \end{tikzcd}$$

hence these two operations are inverse to each other.

Suppose that d divides d' then we get a ring homomorphism

$$K_0^{\mu_d}(\mathrm{Var}/\mathcal{X}) \longrightarrow K_0^{\mu_{d'}}(\mathrm{Var}/\mathcal{X})$$

that takes a μ_d-equivariant motive and considers it as a $\mu_{d'}$-equivariant motive via the surjection $\mu_{d'} \to \mu_d$. Then let

$$K_0^{\hat{\mu}}(\mathrm{Var}/\mathcal{X}) = \varinjlim_{d} K_0^{\mu_d}(\mathrm{Var}/\mathcal{X})$$

be the limit under these homomorphisms. This limit is compatible with the above isomorphisms

$$K_0^{\mu_d}(\mathrm{Var}/\mathcal{X}) \cong K_0^{\mathbb{G}_m, d}(\mathrm{Var}/\mathrm{GL}_1(\mathcal{X}))$$

where for d dividing d', given a $\mathbb{G}_m, d$-equivariant motive $[T \xrightarrow{b} \mathrm{GL}_1(\mathcal{X})]$ we obtain a $\mathbb{G}_m, d'$-equivariant motive $[T \xrightarrow{b} \mathrm{GL}_1(\mathcal{X})]$ by adapting the $\mathbb{G}_m$-action on T so that

$$g \cdot t := g^{d'/d} t$$

for $g \in \mathbb{G}_m$ and $t \in T$.

$K_0^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$ can be equipped with a more exotic λ-ring structure than the one given by Example 5.2.7. To construct this consider the injective homomorphism

$$K_0^{\mathbb{G}_m}(\mathrm{Var}/\mathcal{X}) \longrightarrow K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$$

that sends

$$[S \xrightarrow{a} \mathcal{X}] \longmapsto [S \times \mathbb{A}^1 \xrightarrow{a \times \mathrm{id}_{\mathbb{A}^1}} \mathbb{A}^1_{\mathcal{X}}]$$

where $\mathbb{G}_m$ acts on $S \times \mathbb{A}^1$ by $g \cdot (s, z) = (g \cdot s, g^d z)$, and let $\mathfrak{J}_d$ be the image of this homomorphism. Therefore

$$\mathfrak{J}_d \cong K_0^{\mathbb{G}_m}(\mathrm{Var}/\mathcal{X})$$

and this is again compatible with the limit $K_0^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$.

Lemma 5.3.2 ([16] Lemma 4.1)**.** *$\mathfrak{J}_d \subset K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$ is a λ-ideal and we have an isomorphism of rings*

$$\begin{aligned} K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})/\mathfrak{J}_d &\cong K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathrm{GL}_1(\mathcal{X})) \\ &\cong K_0^{\mu_d}(\mathrm{Var}/\mathcal{X}). \end{aligned}$$

In particular the λ-ring quotient $K_0^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})/\mathfrak{J}_d$ induces a different λ-ring structure on $K_0^{\mu_d}(\mathrm{Var}/\mathcal{X})$. This structure is compatible with the directed system $K_0^{\mu_d}(\mathrm{Var}/\mathcal{X}) \to K_0^{\mu_{d'}}(\mathrm{Var}/\mathcal{X})$ giving a λ-ring structure on $K_0^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$.

Everything above generalises to $\mathrm{St}^{\mathrm{aff}}/\mathcal{X}$ and $\mathrm{St}/\mathcal{X}$ and this new λ-ring structure is compatible with the vector bundle relation (5.1) hence this exotic λ-ring structure descends to $K^{\hat{\mu}}(-/\mathcal{X})$ too.

Example 5.3.3 (cf. [16] Example 4.3)**.** We calculate the image of the motive

$$[\mathbb{A}^1 \xrightarrow{f=z^2} \mathbb{A}^1] \in K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)$$

in $K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$. First using cut and paste we have that

$$\begin{aligned} [\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1] &= [f^{-1}(0) \xrightarrow{f} \mathbb{A}^1] + [\mathbb{A}^1 \setminus f^{-1}(0) \xrightarrow{f} \mathbb{A}^1] \\ &= [\{0\} \to \mathbb{A}^1] + [\mathbb{A}^1 \setminus \{0\} \xrightarrow{f} \mathbb{A}^1]. \end{aligned}$$

Moving to the quotient $K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)/\mathfrak{J}_2$ means that

$$[\{0\} \to \mathbb{A}^1] = -[\{0\} \times \mathbb{G}_m \xrightarrow{p_2} \mathbb{A}^1].$$

This follows from the relation

$$[S \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1] = [S \times \{0\} \xrightarrow{p_2} \mathbb{A}^1] + [S \times \mathbb{G}_m \xrightarrow{p_2} \mathbb{A}^1]$$

for an arbitrary variety S where $p_2 : S \times \mathbb{A}^1 \to \mathbb{A}^1$ denotes the projection, and the fact that
$$[S \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1] = 0$$
in the quotient by $\mathfrak{J}_2$. Therefore
$$[\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1] \longmapsto [\mathbb{A}^1 \setminus \{0\} \xrightarrow{f} \mathbb{A}^1] - [\{0\} \times \mathbb{G}_m \xrightarrow{p_2} \mathbb{A}^1] \in K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)/\mathfrak{J}_2.$$

Then using the isomorphism between $K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)/\mathfrak{J}_2$ and $K^{\mu_2}(\mathrm{Var}/\mathbb{C})$ (and multiplying by -1 which we do for technical reasons), we get that
$$\begin{aligned}[\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1] \longmapsto & [\{0\} \times \{1\}] - [f^{-1}(1)] \\ &= [f^{-1}(0)] - [f^{-1}(1)] \\ &= 1 - [\mu_2] \in K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})\end{aligned}$$

where $\hat{\mu}$ acts via μ_2 on $[\mu_2]$.

Next we calculate the image of
$$[\mathbb{A}^2 \xrightarrow{g=z_1^2+z_2^2} \mathbb{A}^1] \in K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)$$

in $K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$. We make the coordinate change
$$\begin{aligned} w_1 &= z_1 + iz_2 \\ w_2 &= z_1 - iz_2 \end{aligned}$$

giving
$$g = w_1 w_2.$$

Using the same argument as for f we get that
$$[\mathbb{A}^2 \xrightarrow{g} \mathbb{A}^1] \longmapsto [g^{-1}(0)] - [g^{-1}(1)].$$

Now $g^{-1}(0) = w_1 w_2^{-1}(0)$ is the 2 coordinate axes in the plane $\mathbb{A}^2$ and therefore via cut and paste we get
$$[g^{-1}(0)] = 2\mathbb{L} - 1.$$

Also $g^{-1}(1) = \big\{(w_1, 1/w_1) : w_1 \in \mathbb{A}^1 \setminus \{0\}\big\}$ so has motive
$$[g^{-1}(1)] = \mathbb{L} - 1.$$

Hence
$$[\mathbb{A}^2 \xrightarrow{g} \mathbb{A}^1] \longmapsto \mathbb{L}.$$

Using the convolution product in $K^{\mathbb{G}_m,2}(\mathrm{Var}/\mathbb{A}^1)$ it follows that

$$\left([\mathbb{A}^1 \xrightarrow{f=z^2} \mathbb{A}^1]\right)^2 = [\mathbb{A}^2 \xrightarrow{g=z_1^2+z_2^2} \mathbb{A}^1]$$

and so because the morphism to $K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$ is a ring homomorphism we get that

$$1 - [\mu_2] = \mathbb{L}^{\frac{1}{2}} \in K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}).$$

Definition 5.3.4. Let $\mathcal{X}$ be a commutative monoidal Artin stack locally of finite type over $\mathbb{C}$ and let S be a variety of pure dimension n which is locally of finite type over $\mathcal{X}$. Assume S carries a good $\mathbb{G}_m$-action, $\mathbb{G}_m$ acts on $\mathbb{A}^1_{\mathcal{X}}$ with degree d and let $f : S \to \mathbb{A}^1_{\mathcal{X}}$ be $\mathbb{G}_m$-equivariant. The *equivariant vanishing cycle* $\int_S[\phi^{eq}_f] \in K^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})\big[\mathbb{L}^{-\frac{1}{2}}\big]$ is defined to be

$$\int_S [\phi^{eq}_f] := \mathbb{L}^{-\frac{\dim(S)}{2}}\left([f^{-1}(0) \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}} \to \mathcal{X}] - [f^{-1}(1) \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}} \to \mathcal{X}]\right)$$

where $\hat{\mu}$ acts trivially on $f^{-1}(0)$ and on $[f^{-1}(1)]$ via μ_d. In other words $\int_S[\phi^{eq}_f]$ is the normalised image of the motive $[S \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}}] \in K^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$ in $K^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$.

Remark 5.3.5.

1. As defined $\int_S[\phi^{eq}_f]$ takes values in $M(\mathcal{X})$ rather than $M(S_0)$ so it is not a vanishing cycle for M per se. However it imitates the pushforward of a vanishing cycle (as the integral notation suggests) which is what will be really interested in.

2. We can also define the equivariant vanishing cycle for some $\mathbb{G}_m$-invariant locally closed subscheme $S' \subset S$ by taking

$$\int_{S'} [\phi^{eq}_f] := \mathbb{L}^{-\frac{\dim(S)}{2}}\left([f^{-1}(0) \cap S'] - [f^{-1}(1) \cap S']\right).$$

3. The normalisation factor enforces for example that for $z^2 : \mathbb{A}^1 \to \mathbb{A}^1$ we have

$$\int_{\mathbb{A}^1} [\phi^{eq}_{z^2}] = 1$$

   which makes certain calculations easier and more natural, and allows us to not have to renormalise in the future.

Proposition 5.3.6 ([16] Proposition 4.5)**.** *Let S, T, W be pure dimensional schemes locally of finite type over $\mathcal{X}$ with each having a good $\mathbb{G}_m$-action. Let S', T', W' be locally closed $\mathbb{G}_m$-invariant subschemes and let $\pi : W \to S$ be $\mathbb{G}_m$-equivariant, and $f : S \to \mathbb{A}^1_{\mathcal{X}}$ and $g : T \to \mathbb{A}^1_{\mathcal{X}}$ be $\mathbb{G}_m$-equivariant regular functions. Choose the $\mathbb{G}_m$-*

action on $\mathbb{A}^1_{\mathcal{X}}$ *so that*

$$f+g : S \times T \xrightarrow{f\times g} \mathbb{A}^1_{\mathcal{X}} \times \mathbb{A}^1_{\mathcal{X}} \xrightarrow{+} \mathbb{A}^1_{\mathcal{X}}$$

and

$$\mathrm{Sym}^n_+(f) : \mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n(f)} \mathrm{Sym}^n(\mathbb{A}^1_{\mathcal{X}}) \xrightarrow{+} \mathbb{A}^1_{\mathcal{X}}$$

are $\mathbb{G}_m$*-equivariant. Then*

a) If $f = 0$ *then*

$$\int_{S^s} [\phi^{eq}_f] = \mathbb{L}^{-\frac{\dim(S)}{2}} [S^s \to \mathcal{X}].$$

b) If $S'' \subset S$ *is some other* $\mathbb{G}_m$*-invariant locally closed substack that does not intersect* S' *then*

$$\int_{S' \cup S''} [\phi^{eq}_f] = \int_{S'} [\phi^{eq}_f] + \int_{S''} [\phi^{eq}_f].$$

c)

$$\int_{S' \times T'} \phi^{eq}_{f+g} = \int_{S'} [\phi^{eq}_f] \cdot \int_{T'} [\phi^{eq}_g].$$

d) For all $n \geq 0$

$$\mathbb{L}^{\frac{n\dim(S)}{2}} \int_{\mathrm{Sym}^n(S')} [\phi^{eq}_{\mathrm{Sym}^n_+(f)}] \; = \; \sigma^n\Big(\mathbb{L}^{\frac{\dim(S)}{2}} \int_{S'} [\phi^{eq}_f]\Big).$$

e) If $\pi|_{W'} : W' \to S'$ *is a Zariski locally trivial fibration with fibre* F *of dimension* r *then*

$$\int_{W'} [\phi^{eq}_{f\circ\pi}] = \mathbb{L}^{-\frac{r}{2}} [F] \int_{S'} [\phi^{eq}_f]$$

where $[F] \in K^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$ *denotes the image of* $[F \to 0 \hookrightarrow \mathbb{A}^1_{\mathcal{X}}] \in K^{\mathbb{G}_m,1}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$ *under the isomorphism from Lemma 5.3.2.*

Proof.

a) This is clear.

b) Using the cut and paste relations this is also clear.

c) The morphism $K^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}}) \xrightarrow{\theta} K^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$ induced by Lemma 5.3.2 is a ring homomorphism, and by definition $\int_{S'\times T'}[\phi^{eq}_{f+g}]$ is the normalised image of the motive

$$[S' \times T' \xrightarrow{f+g} \mathbb{A}^1_{\mathcal{X}}] \in K^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}})$$

under this homomorphism. Hence we get that

$$\begin{aligned}
\int_{S^s\times T^s}[\phi^{eq}_{f+g}] &= \mathbb{L}^{-\frac{\dim(S\times T)}{2}}\theta\Big([S'\times T' \xrightarrow{f+g} \mathbb{A}^1_{\mathcal{X}}]\Big)\\
&= \mathbb{L}^{-\frac{\dim(S\times T)}{2}}\theta\Big([S' \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}}]\cdot[T' \xrightarrow{g} \mathbb{A}^1_{\mathcal{X}}]\Big)\\
&= \mathbb{L}^{-\frac{\dim(S)}{2}}\mathbb{L}^{-\frac{\dim(T)}{2}}\theta\Big([S' \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}}]\Big)\cdot\theta\Big([T' \xrightarrow{g} \mathbb{A}^1_{\mathcal{X}}]\Big)\\
&= \int_{S'}[\phi^{eq}_f]\cdot\int_{T'}[\phi^{eq}_g].
\end{aligned}$$

d) Additionally, the morphism $K^{\mathbb{G}_m,d}(\mathrm{Var}/\mathbb{A}^1_{\mathcal{X}}) \xrightarrow{\theta} K^{\hat{\mu}}(\mathrm{Var}/\mathcal{X})$ is by construction a λ-ring homomorphism. Hence $\sigma^n\Big(\mathbb{L}^{\frac{\dim(S)}{2}}\int_{S'}[\phi^{eq}_f]\Big)$ is the image of

$$\sigma^n\Big([S \xrightarrow{f} \mathbb{A}^1_{\mathcal{X}}]\Big) = [\mathrm{Sym}^n(S) \xrightarrow{\mathrm{Sym}^n_+(f)} \mathbb{A}^1_{\mathcal{X}}]$$

under θ i.e. $\int_{\mathrm{Sym}^n(S')}[\phi^{eq}_{\mathrm{Sym}^n_+(f)}]$ normalised appropriately.

e) $\int_{W'}[\phi^{eq}_{f\circ\pi}]$ is equal to

$$\mathbb{L}^{-\frac{\dim(W)}{2}}\big([(f\circ\pi)^{-1}(0)] - [(f\circ\pi)^{-1}(1)]\big).$$

Taking a finite Zariski trivialising cover of $W' \xrightarrow{\pi} S'$ we can use the cut and paste relations to show that

$$[(f\circ\pi)^{-1}(0)] = [F][f^{-1}(0)]$$

and

$$[(f\circ\pi)^{-1}(1)] = [F][f^{-1}(1)].$$

Then since $\dim(W) = r + \dim(S)$ the result follows.

□

We can now move onto defining general vanishing cycles. Recalling the definition of the sheaf of vanishing cycles for a regular function $f : X \to \mathbb{A}^1$, morally it measures the difference between the topology of a nearby fibre $f^{-1}(\epsilon)$ and the central fibre $f^{-1}(0)$. In the $\mathbb{G}_m$-equivariant case the choice of ϵ does not matter, hence why the prior definition of the equivariant motivic vanishing cycle was relatively simple and used the fibre $f^{-1}(1)$. However without the $\mathbb{G}_m$-action, in general the fibres $f^{-1}(\epsilon)$ will depend upon ϵ. Therefore we must find a better way of defining the motivic vanishing cycle $[\phi_f] \in K^{\hat{\mu}}(\mathrm{Var}/X_0)$ to remedy this issue.

To begin with let X be a smooth variety of finite type over $\mathbb{C}$ of dimension d and let $f : X \to \mathbb{A}^1$ be a regular function. Write $X_0 = f^{-1}(0)$ for the central fibre and $\mathrm{crit}(f)$ for the critical locus of f. By taking a change of coordinates we may assume that $\mathrm{crit}(f) \subset X_0$.

Definition 5.3.7. The *space of arcs of length n in X*, denoted by $\mathfrak{L}_n(X)$, is the functor

$$\mathrm{Hom}_{\mathrm{Sch}}(-, \mathfrak{L}_n(X)) : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Set}$$

that sends

$$Y \longmapsto \mathrm{Hom}_{\mathrm{Sch}}\big(Y \times \mathrm{Spec}\big(\mathbb{C}[t]/(t^{n+1})\big), X\big).$$

Points γ of $\mathfrak{L}_n(X)$ are therefore given by maps

$$\mathrm{Spec}\big(\mathbb{C}[t]/(t^{n+1})\big) \xrightarrow{\gamma} X$$

and we have that $\mathfrak{L}_0(X) = X$.

Lemma 5.3.8. $\mathfrak{L}_n(\mathbb{A}^1) \cong \mathbb{A}^{n+1}$.

Proof. Initially suppose that $Y = \mathrm{Spec}(A)$ is affine for some $\mathbb{C}$-algebra A. Then we have a bijection between $\mathbb{C}$-algebra homomorphisms

$$\mathbb{C}[x] \longrightarrow A[t]/(t^{n+1})$$

and $\mathbb{C}$-algebra homomorphisms

$$\mathbb{C}[x_0, \ldots, x_n] \longrightarrow A.$$

Indeed given $\alpha : \mathbb{C}[x_0, \ldots, x_n] \to A$ we can define $\widehat{\alpha} : \mathbb{C}[x] \to A[t]/(t^{n+1})$ by

$$\widehat{\alpha}(x) = \alpha(x_0) + \alpha(x_1)t + \ldots + \alpha(x_n)t^n.$$

This is injective because if $\widehat{\alpha} = 0$ then $\widehat{\alpha}(x) = 0$ hence $\alpha(x_i) = 0$ for all i and so $\alpha = 0$. This is surjective because an arbitrary homomorphism $\beta : \mathbb{C}[x] \to A[t]/(t^{n+1})$ is determined by where x is sent i.e. if $x \mapsto a_0 + \ldots + a_n t^n$ then β is determined by the elements $a_0, \ldots, a_n \in A$. But for arbitrary $a_0, \ldots, a_n \in A$ we can define a homomorphism $\alpha : \mathbb{C}[x_0, \ldots, x_n] \to A$ by the property that

$$\alpha(x_i) = a_i$$

for all i. Then it is clear that $\widehat{\alpha} = \beta$.

For general Y take an affine cover $\{U_i = \mathrm{Spec}(A_i)\}$ of Y. Letting $|_{ij}$ denote restriction to the intersection $U_i \cap U_j$, a map

$$\alpha : Y \times \mathrm{Spec}\big(\mathbb{C}[t]/(t^{n+1})\big) \longrightarrow \mathbb{A}^1$$

is equivalent to maps

$$\alpha_i : \mathrm{Spec}\big((A_i[t]/(t^{n+1})\big) \longrightarrow \mathbb{A}^1$$

for each i such that $\alpha_i|_{ij} = \alpha_j|_{ij}$ for all i, j. Similarly a map

$$\beta : Y \to \mathbb{A}^n$$

is equivalent to maps

$$\beta_i : U_i \to \mathbb{A}^n$$

for each i such that $\beta_i|_{ij} = \beta_j|_{ij}$ for all i, j. So it suffices to show that given maps α_i for each i such that $\alpha_i|_{ij} = \alpha_j|_{ij}$ for all i, j we get maps $\widehat{\alpha}_i$ such that $\widehat{\alpha}_i|_{ij} = \widehat{\alpha}_j|_{ij}$ for all i, j by using our bijection from the affine case. But this is clear simply by taking affine covers of $U_i \cap U_j$ for all i, j. $\square$

By functoriality the function $f : X \to \mathbb{A}^1$ induces a map

$$\mathfrak{L}_n(f) : \mathfrak{L}_n(X) \to \mathfrak{L}_n(\mathbb{A}^1) \cong \mathbb{A}^{n+1}.$$

Let

$$\mathfrak{L}_{n,\mathbb{G}_m}(X) := \mathfrak{L}_n(f)^{-1}\Big(\{0\} \times \ldots \times \{0\} \times \mathbb{G}_m\Big) \subset \mathfrak{L}_n(X)$$

be the subvariety of arcs γ such that $f \circ \gamma$ is induced by an algebra homomorphism $\mathbb{C}[x] \to \mathbb{C}[t]/(t^{n+1})$ of the form $x \mapsto zt^n$ for some $z \in \mathbb{G}_m$. For $n \geq m$ we have projection maps

$$\pi^n_m : \mathfrak{L}_n(X) \to \mathfrak{L}_m(X)$$

induced by the homomorphisms $\mathbb{C}[t]/(t^{n+1}) \to \mathbb{C}[t]/(t^{m+1})$. Let

$$\mathfrak{L}_n(X)|_{X_0} = (\pi^n_0)^{-1}(X_0).$$

Then an arc γ in $\mathfrak{L}_n(X)|_{X_0}$ will be such that either $f \circ \gamma$ is induced by the zero homomorphism and so is fully contained in X_0, or $f \circ \gamma$ is induced from a homomorphism of the form $z_i t^i + \ldots + z_n t^n$ for some smallest $i \neq 0$ such that z_i is non-zero. Note that i cannot be 0 here otherwise the map $f \circ \pi^n_0(\gamma)$ would be induced from the homomorphism $\mathbb{C}[x] \to \mathbb{C}[t]/(t)$ that sends $x \mapsto z_0 \neq 0$ and so will never hit $0 \in \mathbb{A}^1$. Therefore $\mathfrak{L}_n(X)|_{X_0}$ is a Zariski locally-trivial affine fibration over X_0 with fibre $\mathbb{A}^{nd}$ which parametrises the coefficients $z_1, \ldots, z_n \in \mathbb{A}^1$. This also gives us the following stratification of $\mathfrak{L}_n(X)|_{X_0}$

$$\mathfrak{L}_n(X)|_{X_0} = \mathfrak{L}_n(X_0) \sqcup \bigsqcup_{m=1}^{n} (\pi^n_m)^{-1}\big(\mathfrak{L}_{m,\mathbb{G}_m}(X)\big).$$

Similarly $(\pi^n_m)^{-1}\big(\mathfrak{L}_{m,\mathbb{G}_m}(X)\big)$ is also a Zariski locally-trivial affine fibration over X_0, this time with fibre $\mathbb{A}^{(n-m)d}$ because it parametrises the different homomorphisms $\mathbb{C}[x] \to \mathbb{C}[t]/(t^{n+1})$ given by $x \mapsto z_m t^m + \ldots + z_n t^n$.

Taking motives over X_0 we get that

$$\big[\mathfrak{L}_n(X)|_{X_0} \to X_0\big] = \mathbb{L}^{nd}[X_0 \to X_0]$$

which implies that

$$\begin{aligned}
[X_0 \to X_0] &= \mathbb{L}^{-nd}\big[\mathfrak{L}_n(X)|_{X_0} \to X_0\big] \\
&= \mathbb{L}^{-nd}\Bigg(\big[\mathfrak{L}_n(X_0) \to X_0\big] + \sum_{m=1}^{n} \big[(\pi^n_m)^{-1}\big(\mathfrak{L}_{m,\mathbb{G}_m}(X)\big) \to X_0\big]\Bigg) \\
&= \mathbb{L}^{-nd}\big[\mathfrak{L}_n(X_0) \to X_0\big] + \sum_{m=1}^{n} \mathbb{L}^{-md}\big[\mathfrak{L}_{m,\mathbb{G}_m}(X) \to X_0\big].
\end{aligned}$$

The $\mathbb{G}_m$-action on $\mathrm{Spec}\big((\mathbb{C}[t]/(t^{n+1})\big)$ given by $t \mapsto gt$ for $g \in \mathbb{G}_m$ induced a $\mathbb{G}_m$-action on $\mathfrak{L}_n(X)$ given by

$$\gamma \overset{g}{\mapsto} \gamma \circ g.$$

Let

$$f_n : \mathfrak{L}_n(X) \xrightarrow{\mathfrak{L}_n(f)} \mathbb{A}^{n+1} \xrightarrow{p_{n+1}} \mathbb{A}^1$$

where p_{n+1} is projection onto the last factor. Using the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ and supposing that $f \circ \gamma$ is induced by the algebra homomorphism

$$x \longmapsto z_0 + \ldots + z_n t^n$$

we get that

$$f_n(g \cdot \gamma) = z_n g^n = g^n f_n(\gamma)$$

and so f_n is equivariant of degree n. Writing

$$\widetilde{f}_n = (f_n, \pi^n_0) : \mathfrak{L}_n(X) \longrightarrow \mathbb{A}^1 \times X = \mathbb{A}^1_X$$

we define the following generating series in the formal variable T

$$\begin{aligned}
Z^{eq}_f(T) :&= \sum_{n\geq 1} \int_{\mathfrak{L}_n(X)|_{X_0}} [\phi^{eq}_{\widetilde{f}_n}]\, T^n \\
&= \sum_{n\geq 1} \mathbb{L}^{-\frac{(n+1)d}{2}} \Big(\big[\widetilde{f}_n^{-1}(0) \cap \mathfrak{L}_n(X)|_{X_0} \to X_0\big] - \big[\widetilde{f}_n^{-1}(1) \cap \mathfrak{L}_n(X)|_{X_0} \to X_0\big]\Big)\, T^n \\
&= \sum_{n\geq 1} \mathbb{L}^{-\frac{(n+1)d}{2}} \Big(\big[f_n^{-1}(0) \cap \mathfrak{L}_n(X)|_{X_0} \to X_0\big] - \big[f_n^{-1}(1) \cap \mathfrak{L}_n(X)|_{X_0} \to X_0\big]\Big)\, T^n.
\end{aligned}$$

This whole series may in fact be seen as a motive in $\hat{K}^{\hat{\mu}}(\mathrm{Var}/X_0 \times \mathbb{N}_{>0})\big[\mathbb{L}^{-\frac{1}{2}}\big]$ where $\hat{\mu}$ acts non-trivially only on the components $f_n^{-1}(1) \cap \mathfrak{L}_n(X)|_{X_0}$. Indeed, in general if $(Y, +, 0)$ is monoidal then $Y \times \mathbb{N}_{>0}$ has a natural monoidal structure as a scheme given by

$$(y, i) \cdot (y', j) = \big(+(y, y'), ij\big)$$

and so a motive

$$[S \xrightarrow{f} Y \times \mathbb{N}_{>0}]$$

decomposes in the completion $\hat{K}^G(\mathrm{Var}/Y \times \mathbb{N}_{>0})$ as

$$\sum_{n>0}[S_n \xrightarrow{f} Y]$$

using the stratification $\{S_n = f^{-1}(Y \times \{n\})\}_{n\in\mathbb{N}_{>0}}$ of S (see Remark 5.1.5).

It was shown in [[22] Theorem 3.2] that $Z_f^{eq}(T)$ rational power series that is regular at ∞ (see also [[19] Section 4]).

Definition 5.3.9. For a smooth variety X of finite type over $\mathbb{C}$ of dimension d and a regular function $f : X \to \mathbb{A}^1$ the *motivic vanishing cycle* of f is

$$[\phi_f] = -Z_f^{eq}(\infty) \in K^{\hat{\mu}}(\mathrm{Var}/X_0)\left[\mathbb{L}^{-\frac{1}{2}}\right].$$

For a more explicit description of Z_f^{eq} and the motivic vanishing cycle $[\phi_f]$ we have the following result.

Theorem 5.3.10 ([16] Theorem 5.2, cf. [21] Theorem 3.3.1)**.** *Choose an embedded resolution of $X_0 \hookrightarrow X$ i.e. a smooth variety Y and a proper map $\pi : Y \to X$ such that $Y_0 = \pi^{-1}(X_0)$ is a normal crossing divisor in Y and $\pi : Y \setminus Y_0 \xrightarrow{\sim} X \setminus X_0$. Let $\{E_i\}_{i\in J}$ be the irreducible components of Y_0 where the multiplicity of E_i is given by m_i. For any $\emptyset \neq I \subset J$ the map $f \circ \pi$ induces a regular function $f_I : N_I \to \mathbb{A}^1$ where $N_I = \prod_{i\in I} \left(N_{E_i|Y} - E_i\right)\big|_{\tilde{E}_I}$ and $\tilde{E}_I = \cap_{i\in I} E_i \setminus \cup_{j\notin I} E_j$. Then we have that*

$$[\phi_f] = \mathbb{L}^{-\frac{d}{2}}\left([X_0 \to X_0] + \sum_{\emptyset\neq I\subset J}(-1)^{|I|}\left[f_I^{-1}(1) \to X_0\right]\right)$$

where $\hat{\mu}$ acts nontrivially only on $f_I^{-1}(1)$ via μ_{m_I} for $m_I = \sum_{i\in I} m_i$.

Proposition 5.3.11 ([16] Proposition 5.3)**.**

a) If $g : X' \to X$ is smooth of relative dimension r then

$$[\phi_{f\circ g}] = \mathbb{L}^{-\frac{r}{2}} g^*([\phi_f]).$$

b) If $f = 0$ then

$$[\phi_f] = \mathbb{L}^{-\frac{d}{2}}[X \to X].$$

c) The motive $[\phi_f]$ is supported on $\mathrm{crit}(f) \subset X_0$.

Proof. (of *c)*.) We show that for all $x \notin \mathrm{crit}(f)$ there exists an open neighbourhood $x \in U$ such that $[\phi_f]|_U = 0$, which therefore implies that $[\phi_f]$ is supported on $\mathrm{crit}(f)$. We claim that étale locally we can write $f : X \to \mathbb{A}^1$ as projection onto the first coordinate $\mathbb{A}^d \to \mathbb{A}^1$. To see this note that because $f : X \setminus \mathrm{crit}(f) \to \mathbb{A}^1$ is smooth for all $x \in X \setminus \mathrm{crit}(f)$ there exists an open affine neighbourhood $U \xrightarrow{i} X$ such that the following diagram is commutative

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "\pi"] & X \arrow[d, "f"] \\ \mathbb{A}^n \arrow[r, "\widetilde{f}"] & \mathbb{A}^1 \end{tikzcd}$$

where π is étale, $U = \mathrm{Spec}(A)$ with $A = \mathbb{C}[x, x_1, \ldots, x_r]/(g_1, \ldots, g_s)$ for polynomials g_i such that

$$\det \begin{pmatrix} \partial g_1/\partial x_1 & \ldots & \partial g_s/\partial x_1 \\ \vdots & \ddots & \vdots \\ \partial g_1/\partial x_r & \ldots & \partial g_s/\partial x_r \end{pmatrix}$$

is invertible in A, the map $U \to \mathbb{A}^1$ is induced by the natural ring homomorphism $\mathbb{C}[x] \to A$, and the map $U \xrightarrow{\pi} \mathbb{A}^n$ is induced by the quotient homomorphism $\mathbb{C}[x, x_{s+1}, \ldots, x_r] \to A$ (see [[59] Lemma 28.34.20]). Given this it is clear that the map $\mathbb{A}^n \xrightarrow{\widetilde{f}} \mathbb{A}^1$ is induced by $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, x_{s+1}, \ldots, x_r]$ and therefore is the projection onto the first coordinate.

It is enough to consider things étale locally because from the commutative diagram (5.3) and part i) of this proposition we have that

$$\begin{aligned} [\phi_f]|_U &= [\phi_{f\circ i}] \\ &= [\phi_{\widetilde{f}\circ\pi}] \\ &= \pi^*[\phi_{\widetilde{f}}] \end{aligned}$$

since π is étale.

Hence it remains to show that $[\phi_{\widetilde{f}}] = 0$ where $\widetilde{f}$ is the projection onto the first coordinate. Now

$$\widetilde{f}^{-1}(0) = \{0\} \times \mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^n$$

is an embedded resolution, and so because $\{0\} \times \mathbb{A}^{n-1}$ is irreducible we only have the one map

$$\widetilde{f}_J : N_{\{0\}\times\mathbb{A}^{n-1}|\mathbb{A}^n} \setminus \left(\{0\} \times \mathbb{A}^{n-1}\right) \longrightarrow \mathbb{A}^1$$

which is easily calculated to be the projection onto the first coordinate

$$\mathbb{A}^n \setminus \left(\{0\} \times \mathbb{A}^{n-1}\right) \longrightarrow \mathbb{A}^1.$$

Hence

$$\widetilde{f}_J^{-1}(1) = \{1\} \times \mathbb{A}^{n-1}$$

and $\widetilde{f}_J^{-1}(1) \to \{0\} \times \mathbb{A}^{n-1}$ sends

$$(1, x_2, \ldots, x_n) \longmapsto (0, x_2, \ldots, x_n).$$

It therefore follows directly from Theorem 5.3.10 that

$$[\phi_{\widetilde{f}}] = 0.$$

$\square$

Remark 5.3.12. Since Proposition 5.3.11 part c) says that the motivic vanishing cycle is supported on the critical locus of the function f we often instead consider the pullback of $[\phi_f]$ to $\mathrm{crit}(f)$ and view the motivic vanishing cycle as a motive in $K^{\hat{\mu}}(\mathrm{Var}/\mathrm{crit}(f))$.

We can generalise the definition of the motivic vanishing cycle to Artin stacks that are locally certain types of quotient stacks. Let $\mathcal{X}$ be a smooth Artin stack locally of finite type over $\mathbb{C}$ such that every closed point in $\mathcal{X}$ has an open neighbourhood $\mathcal{U}$ of finite type over $\mathbb{C}$ that is isomorphic to a quotient stack of the form U/GL_n where U is a smooth connected variety. Let $f : \mathcal{X} \to \mathbb{A}^1$ be regular with zero-locus $\mathcal{X}_0$ and write $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{X}_0$. Thus we have local atlases $\pi : U \to U/\mathrm{GL}_n$ for $\mathcal{X}$ and we define the motivic vanishing cycle $[\phi_f] \in K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_0)$ locally via

$$[\phi_f]|_{\mathcal{U}_0} = \mathbb{L}^{\frac{n^2}{2}}[\mathrm{GL}_n]^{-1}\pi_![\phi_{f\circ\pi}] \tag{5.5}$$

in $K^{\hat{\mu}}(\mathrm{Var}/\mathcal{U}_0)\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big] \cong K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{U}_0)$.

Proposition 5.3.13. *Let $\mathcal{X}$ be a smooth Artin stack locally of finite type over $\mathbb{C}$ such that it is locally a quotient stack of the form U/GL_n as above. Let $f : \mathcal{X} \to \mathbb{A}^1$ be a regular function. Then the local definition* (5.5) *of $[\phi_f]|_{\mathcal{U}_0}$ is independent upon the choice of local atlas $\pi : U \to U/\mathrm{GL}_n$ and so in particular the motives $[\phi_f]|_{\mathcal{U}_0}$ glue to a well-defined element $[\phi_f] \in K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{X}_0)$.*

Proof. Consider a principal GL_n-bundle $B \xrightarrow{\psi} T$ for an arbitrary monoidal scheme $(T, +, 0)$. Let $[S \xrightarrow{a} T]$ be a motive over T. Using Lemma 5.1.9 we get that

$$\begin{aligned}
\psi_!\psi^*[S \xrightarrow{a} T] &= \psi_![S \times_T B \xrightarrow{\widetilde{a}} B] \\
&= [S \times_T B \xrightarrow{\widetilde{a}} B \xrightarrow{\psi} T] \\
&= [S \times_T B \xrightarrow{\widetilde{\psi}} S \xrightarrow{a} T] \\
&= [\mathrm{GL}_n] \cdot [S \xrightarrow{a} E]
\end{aligned} \tag{5.6}$$

because $\widetilde{\psi} : S \times_T B \to S$ is a principal GL_n-bundle.

So suppose we are given another local description for $\mathcal{X}$ where $\theta : Y \to Y/\mathrm{GL}_m$ is such that

$$\alpha : Y/\mathrm{GL}_m \xrightarrow{\sim} U/\mathrm{GL}_n \cong \mathcal{U}.$$

The fibre product diagram

$$\begin{tikzcd}
U \times_{\mathcal{U}} Y \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\alpha\circ\theta}"'] & Y \arrow[d, "\alpha\circ\theta"] \\
U \arrow[r, "\pi"] & \mathcal{U}
\end{tikzcd}$$

makes $\widetilde{\pi} : U \times_{\mathcal{U}} Y \to Y$ a GL_n-bundle and $\widetilde{\alpha \circ \theta} : U \times_{\mathcal{U}} Y \to U$ a GL_m-bundle. By Proposition 5.1.2 2. we have that

$$\pi^*(\alpha \circ \theta)_! = \widetilde{\alpha \circ \theta}_! \widetilde{\pi}^*.$$

Therefore by Proposition 5.3.11 i) we get

$$\begin{aligned}
\pi^*\alpha_![\phi_{f\circ\alpha}]|_{\mathcal{U}_0} &= \mathbb{L}^{\frac{m^2}{2}}[\mathrm{GL}_m]^{-1}\pi^*\alpha_!\theta_![\phi_{f\circ\alpha\circ\theta}] \\
&= \mathbb{L}^{\frac{m^2}{2}}[\mathrm{GL}_m]^{-1}\widetilde{\alpha \circ \theta}_!\widetilde{\pi}^*[\phi_{f\circ\alpha\circ\theta}] \\
&= \mathbb{L}^{\frac{m^2}{2}}[\mathrm{GL}_m]^{-1}\mathbb{L}^{\frac{n^2}{2}}\widetilde{\alpha \circ \theta}_![\phi_{f\circ\alpha\circ\theta\circ\widetilde{\pi}}] \\
&= \mathbb{L}^{\frac{n^2}{2}}[\mathrm{GL}_m]^{-1}\mathbb{L}^{\frac{m^2}{2}}\widetilde{\alpha \circ \theta}_![\phi_{f\circ\pi\circ\widetilde{\alpha\circ\theta}}] \\
&= \mathbb{L}^{\frac{n^2}{2}}[\mathrm{GL}_m]^{-1}\widetilde{\alpha \circ \theta}_!\widetilde{\alpha \circ \theta}^*[\phi_{f\circ\pi}] \\
&= \mathbb{L}^{\frac{n^2}{2}}[\phi_{f\circ\pi}]
\end{aligned}$$

where the last line follows from (5.6). This implies that

$$\pi_!\pi^*\alpha_![\phi_{f\circ\alpha}]|_{\mathcal{U}_0} = \pi_!\mathbb{L}^{\frac{n^2}{2}}[\phi_{f\circ\pi}]$$

and so using (5.6) once more we get

$$[\mathrm{GL}_n]\alpha_![\phi_{f\circ\alpha}]|_{\mathcal{U}_0} = \mathbb{L}^{\frac{n^2}{2}}\pi_![\phi_{f\circ\pi}]$$

and therefore

$$\begin{aligned}
\alpha_![\phi_{f\circ\alpha}]|_{\mathcal{U}_0} &= \mathbb{L}^{\frac{n^2}{2}}[\mathrm{GL}_n]^{-1}\pi_![\phi_{f\circ\pi}] \\
&= [\phi_f]|_{\mathcal{U}_0}
\end{aligned}$$

as required. $\square$

As before let X be a smooth variety of finite type over $\mathbb{C}$ of dimension d and let $f : X \to \mathbb{A}^1$ be a regular function. Consider the stacky symmetric product of X

$$\mathcal{S}ym^n(X) = X^n/S_n$$

with regular function

$$\mathcal{S}ym^n_+(f) : \mathcal{S}ym^n(X) \xrightarrow{\mathcal{S}ym^n(f)} \mathcal{S}ym^n(\mathbb{A}^1) \xrightarrow{+} \mathbb{A}^1$$

and note that $\mathcal{S}ym^n(X_0) \subset \mathcal{S}ym^n(X)_0$.

Theorem 5.3.14 ([16] Theorem 5.4)**.** *Viewing $[\phi_f] \in K^{\hat{\mu}}(\mathrm{Var}/X_0)[\mathbb{L}^{-\frac{1}{2}}]$ and $[\phi_{\mathcal{S}ym^n_+(f)}]|_{\mathcal{S}ym^n(X_0)} \in K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{S}ym^n(X_0))$ as motives in $\hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathcal{S}ym(X_0))$ we have that for any $n \geq 0$*

$$\mathbb{L}^{\frac{nd}{2}}[\phi_{\mathcal{S}ym^n_+(f)}]|_{\mathcal{S}ym^n(X_0)} = \sigma^n\Big(\mathbb{L}^{\frac{d}{2}}[\phi_f]\Big).$$

Proposition 5.3.15 ([16] Proposition 5.8, cf. [20] Theorem 5.2.2)**.** *Let X, Y, Z be pure dimensional smooth varieties of finite type over $\mathbb{C}$. Let X', Y', Z' be locally closed subvarieties and let $\pi : Z \to X$ be smooth, and $f : X \to \mathbb{A}^1$ and $g : Y \to \mathbb{A}^1$ be regular functions. Then*

a) If $f = 0$ then

$$\int_{X'} [\phi_f] = \mathbb{L}^{-\frac{\dim(X)}{2}}[X'].$$

b) If $X'' \subset X$ is some other locally closed subvariety that does not intersect X' then

$$\int_{X' \cup X''} [\phi_f] = \int_{X'} [\phi_f] + \int_{X''} [\phi_f].$$

c)

$$\int_{X' \times Y'} [\phi_{f+g}] = \int_{X'} [\phi_f] \cdot \int_{Y'} [\phi_g].$$

d) For all $n \geq 0$

$$\mathbb{L}^{\frac{n\dim(X)}{2}} \int_{\mathcal{S}ym^n(X')} [\phi_{\mathcal{S}ym^n_+(f)}] = \sigma^n\Big(\mathbb{L}^{\frac{\dim(X)}{2}} \int_{X'} [\phi_f]\Big).$$

e) If $\pi|_{Z'} : Z' \to X'$ is a Zariski locally trivial fibration with fibre F of dimension r then

$$\int_{Z'} [\phi_{f\circ\pi}] = \mathbb{L}^{-\frac{r}{2}}[F] \int_{X'} [\phi_f].$$

The following result is an incredibly useful tool for computing motivic vanishing cycles which we shall leverage later on. First proved in [3] then generalised in [16] and further generalised in [56] the theorem links the motivic vanishing cycle to the equivariant vanishing cycle for equivariant functions of a particular form. This reduces the complexity of calculating the motivic vanishing cycle to calculating the motives of the central fibre and the fibre over 1. As before take X be a smooth variety of finite type over $\mathbb{C}$. Suppose that X is equipped with a $\mathbb{G}_m$-action such that for all $x \in X$ there exists a neighbourhood $x \in U \subset X$ and

$$U \cong \mathbb{A}^{r_x} \times U^{\mathbb{G}_m}$$

where $U^{\mathbb{G}_m}$ is the $\mathbb{G}_m$-invariant locus of U and $g \in \mathbb{G}_m$ acts on the right-hand side as

$$g \cdot (z_1, \ldots, z_{r_x}, u) = (g^{w_1} z_1, \ldots, g^{w_{r_x}} z_{r_x}, u)$$

for strictly positive weights $w_1, \ldots, w_{r_x}$.

Theorem 5.3.16 ([56] Theorem 4.1.1, cf. [16] Theorem 5.9)**.** *Let X be a smooth variety of finite type over $\mathbb{C}$ with a $\mathbb{G}_m$-action such that the above condition holds. Let $f : X \to \mathbb{A}^1$ be a regular function that is equivariant of degree t. Then*

$$\int_X [\phi_f] = \int_X [\phi_f^{eq}] = \mathbb{L}^{-\frac{d}{2}}\big([f^{-1}(0)] - [f^{-1}(1)]\big)$$

in $K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-\frac{1}{2}}]$, where $\hat{\mu}$ acts non-trivially only on $f^{-1}(1)$ via μ_t.

5.4 Motivic DT invariants

We shall define motivic DT invariants for the Jacobi algebra of an algebra with potential in the style of Kontsevich-Soibelman [45], following [16] and [3].

Let X be a smooth variety of finite type over $\mathbb{C}$ with regular function $f : X \to \mathbb{A}^1$. Let $Z = \mathrm{crit}(f)$ be the critical locus of f.

Definition 5.4.1. The *relative virtual motive* of Z is the motive

$$[Z]_{\mathrm{relvir}} = [\phi_f]|_Z \in K^{\hat{\mu}}(\mathrm{Var}/Z).$$

The *virtual motive* of Z is the motive

$$\begin{aligned}[Z]_{\mathrm{vir}} &= \int_Z [Z]_{\mathrm{relvir}} \\ &= \int_Z [\phi_f]|_Z \in K^{\hat{\mu}}(\mathrm{Var}/\mathbb{C}).\end{aligned}$$

Remark 5.4.2. The (relative) virtual motive of Z is very much dependent upon its presentation as a critical locus (X, f). A priori there is nothing stopping two critical loci presentations (X_1, f_1) and (X_2, f_2) of Z giving different virtual motives.

The virtual motive of a critical locus can be thought of as a motivic refinement of Behrend's virtual Euler characteristic (see [2]) in that

$$\chi\big([Z]_{\mathrm{vir}}\big) = \chi_{\mathrm{vir}}(Z)$$

where χ sends a effective motive $[Y]$ to the Euler characteristic of Y (see [[3] Proposition 1.15]). If Z is the moduli space of stable sheaves on a Calabi-Yau threefold then its virtual Euler characteristic gives the DT invariants of Z.

Now consider a finitely generated $\mathbb{C}$-algebra B with potential $W \in B/[B, B]$. Let $A = \mathrm{Jac}(B, W)$ be the Jacobi algebra. For each $n \in N$ we have the stack $\mathrm{Rep}_n(B)$ of n-dimensional representations of B and we can then consider the stack of finite-dimensional representations of B

$$\mathrm{Rep}(B) = \bigsqcup_{n\in\mathbb{N}} \mathrm{Rep}_n(B).$$

The stack $\mathrm{Rep}(B)$ is monoidal where multiplication is given by the direct sum of representations and 0 is the zero-representation, and we have a monoidal-stack morphism $\dim : \mathrm{Rep}(B) \to \mathbb{N}$. The potential W gives a regular function

$$f_n = \mathrm{Tr}(W)_n : \mathrm{Rep}_n(B) \to \mathbb{A}^1$$

which in turn defines a regular function $f : \mathrm{Rep}(B) \to \mathbb{A}^1$ given on $\mathrm{Rep}_n(B)$ by f_n. The critical locus of f_n is isomorphic to the stack $\mathrm{Rep}_n(A)$ of n-dimensional representations of the Jacobi algebra A (see. Proposition 2.2.5). Hence we get that

$$\begin{aligned}\mathrm{crit}(f) &= \bigsqcup_{n\in\mathbb{N}} \mathrm{crit}(f_n)\\ &\cong \bigsqcup_{n\in\mathbb{N}} \mathrm{Rep}_n(A)\\ &= \mathrm{Rep}(A).\end{aligned}$$

Definition 5.4.3. Let $T = [\mathrm{Spec}(\mathbb{C}) \to 1] \in \hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{N})$. The *motivic DT partition function* or *universal DT series* of the Jacobi algebra $A = \mathrm{Jac}(B, W)$ in the formal variable T is

$$\begin{aligned}\Phi_A(T) := \dim_![\mathrm{Rep}(A)]_{\mathrm{relvir}} &= \dim_![\phi_f]|_{\mathrm{Rep}(A)}\\ &= \sum_{n\in\mathbb{N}} \int_{\mathrm{Rep}_n(A)} [\phi_{f_n}]|_{\mathrm{Rep}_n(A)}\, T^n\\ &= \sum_{n\in\mathbb{N}} [\mathrm{Rep}_n(A)]_{\mathrm{vir}}\, T^n\end{aligned}$$

in $\hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{N})$.

Definition 5.4.4. The *motivic DT invariants* of the Jacobi algebra $A = \mathrm{Jac}(B, W)$ are the motives $\Omega_n \in \hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{C})$ such that

$$\Phi_A(T) = \mathrm{Exp}\left(\frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \sum_{n \geq 1} \Omega_n T^n\right)$$

where the plethystic exponential Exp is taken with respect to the exotic λ-ring structure on $\hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{N})$ given in Lemma 5.3.2.

In the case of a quiver with potential (Q, W) we adapt this definition. Let $n \in \mathbb{N}^{Q_0}$ now be a dimension vector, then we have the stack of n-dimensional representations of Q

$$\begin{aligned}\mathrm{Rep}_n(Q) &\cong M_n/G_n \\ &= \prod_{a:i\to j\in Q_1} \mathrm{Mat}_{n_i\times n_j}(\mathbb{C})/\prod_{k\in Q_0} \mathrm{GL}_{n_k}(\mathbb{C}).\end{aligned}$$

The stack of all finite-dimensional representations of Q

$$\mathrm{Rep}(Q) = \bigsqcup_{n\in\mathbb{N}^{Q_0}} \mathrm{Rep}_n(Q)$$

is again monoidal and we have a monoidal-stack morphism $\dim : \mathrm{Rep}(Q) \to \mathbb{N}^{Q_0}$.

The potential W defines a regular function for each n

$$f_n := \mathrm{Tr}(W)_n : M_n \to \mathbb{A}^1$$

which descends to maps

$$\widetilde{f_n} : M_n/G_n \to \mathbb{A}^1$$

which defines a regular function $f : \mathrm{Rep}(Q) \to \mathbb{A}^1$. Similarly using Proposition 2.2.5 we have that

$$\mathrm{crit}(\mathcal{W}) \cong \mathrm{Rep}(\mathrm{Jac}(Q, W)).$$

Let $e_i \in \mathbb{N}^{Q_0}$ be the tuple with a 1 in the i-th position and zeroes everywhere else and write

$$T_i = [\mathrm{Spec}(\mathbb{C}) \to e_i] \in \hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{N}^{Q_0}).$$

Definition 5.4.5. The *motivic DT partition function* or *universal DT series* of the quiver and potential (Q, W) in the formal multivariable $T = (T_i)_{i \in Q_0}$ is

$$\begin{aligned}
\Phi_{Q,W}(T) &:= \dim_! \big[\mathrm{Rep}(\mathrm{Jac}(Q, W))\big]_{\mathrm{relvir}} \\
&= \sum_{n \in \mathbb{N}^{Q_0}} \int_{\mathrm{Rep}_n(\mathrm{Jac}(Q,W))} [\phi_{\widetilde{f}_n}]\, T^n \\
&= \sum_{n \in \mathbb{N}^{Q_0}} \big[\mathrm{Rep}_n(\mathrm{Jac}(Q, W))\big]_{\mathrm{vir}}\, T^n \\
&= \sum_{n \in \mathbb{N}^{Q_0}} \mathbb{L}^{\frac{N}{2}} [G_n]^{-1} \int_{M_n(\mathrm{Jac}(Q,W))} [\phi_{f_n}]\, T^n
\end{aligned}$$

in $\hat{K}^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{N}^{Q_0})$, where $N = \sum_{i \in Q_0} n_i^2$.

The motivic DT invariants for (Q, W) are defined analogously to the general Jacobi algebra case.

Example 5.4.6 ([16] Section 6)**.** Consider the one loop quiver $Q_{(1)}$ with potential W. Following the above theory we can write the motivic DT partition function of $(Q_{(1)}, W)$ as

$$\Phi_{Q,W}(T) = \sum_{n \in \mathbb{N}} \mathbb{L}^{\frac{n^2}{2}} [\mathrm{GL}_n]^{-1} \int_{M_n(\mathrm{Jac}(Q_{(1)},W))} [\phi_{f_n}]\, T^n.$$

Now $\mathrm{Rep}(Q_{(1)})$ carries a good $\mathbb{G}_m$-action given by scalar multiplication so if W is homogeneous the function f is $\mathbb{G}_m$-equivariant. Hence we can instead consider the equivariant motivic DT partition function

$$\Phi^{eq}_{Q_{(1)},W}(T) := \dim_! [\phi^{eq}_f] = \sum_{n \in \mathbb{N}} \mathbb{L}^{\frac{n^2}{2}} [\mathrm{GL}_n]^{-1} \int_{\mathrm{Rep}_n(\mathrm{Jac}(Q_{(1)},W)} [\phi^{eq}_{f_n}]\, T^n.$$

By Theorem 5.3.16 this is equal to $\Phi_{Q_{(1)},W}(T)$. In the case of the potential $W = x^d$ this was calculated in [16] using the Hilbert schemes of $\mathbb{A}^1$. They concluded [[16] Theorem 6.2] that the motivic DT invariants for $(Q_{(1)}, x^d)$ are

$$\Omega_n = \begin{cases} \mathbb{L}^{-\frac{1}{2}}(1 - [\mu_d])\,, & n = 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Chapter 6

A superpotential description of the fundamental group algebra of the mapping torus of a Riemann Surface

Given a d-dimensional manifold X we can define its fundamental group algebra over a field k as the group ring $k[\pi_1(X)]$. In [25] it is stated that if X is compact, orientable and has a contractible universal cover then $k[\pi_1(X)]$ is a Calabi-Yau algebra of dimension d (see [[25] Corollary 6.1.4] and [[13] Proposition 5.2.6]). Many Calabi-Yau algebras turn out to be what are known as superpotential algebras and so it was conjectured in [25] that in the dimension 3 case these fundamental group algebras which were Calabi-Yau were also superpotential algebras.

However in [13] Davison showed that this was not the case in general (for $d \geq 2$), and that in order to have a superpotential structure the algebra $k[\pi_1(X)]$ (and hence the manifold X) had to have certain specific properties. In particular a superpotential structure implies a stronger notion of an *exact* Calabi-Yau structure which Davison showed required non-trivial central units in $k[\pi_1(X)]$. Therefore Davison proposed an updated conjecture [[13] Conjecture 7.1.1] regarding the possibility of a superpotential structure on $k[\pi_1(X)]$ when X is a circle bundle, precisely because in such a manifold X non-trivial central units in $k[\pi_1(X)]$ are easily found. In this chapter we focus on the 3-dimensional case of this conjecture and specifically we consider the case when X is the mapping torus $M_{g,\varphi}$ of a Riemann surface Σ_g of genus g under a finite-order, orientation-preserving automorphism $\varphi : \Sigma_g \xrightarrow{\sim} \Sigma_g$.

This chapter and the new results contained within are adapted from the author's own work in [52].

6.1 Brane Tilings

Definition 6.1.1. A *brane tiling* Δ of a genus g Riemann surface Σ_g is an embedding $\Gamma \hookrightarrow \Sigma_g$ of a bipartite graph Γ such that each connected component of $\Sigma_g \setminus \Gamma$ is simply connected. We choose a partition of the vertex set of Γ into 2 disjoint subsets of black and white vertices, such that every edge in Γ goes between a black vertex and a white vertex.

From a brane tiling we can obtain a quiver with potential (Q_Δ, W_Δ). The underlying graph of Q_Δ is the dual graph to Γ in Σ_g and it is directed so that arrows in Q_Δ go clockwise around a white vertex and anticlockwise around a black vertex. For a vertex $v \in \Gamma$ let c_v denote the minimal cycle in Q_Δ that goes around v, i.e. c_v consists of all the arrows that are dual to the edges that come out of v. Then we take the potential to be

$$W_\Delta = \sum_{v \,\text{white}} c_v - \sum_{u \,\text{black}} c_u.$$

Example 6.1.2. The following genus 2 example will be used throughout this chapter and will be useful to keep in mind to picture what's going on. It can also easily be extended to higher genus surfaces. A brane tiling Δ of Σ_2 and its dual quiver Q_Δ are given in Fig. 6.1.

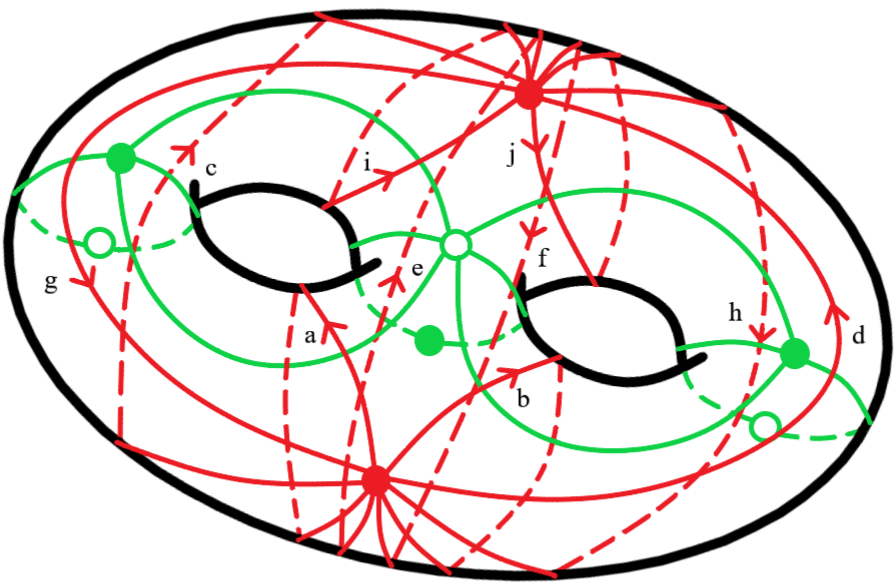
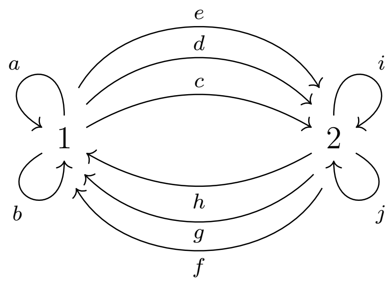


Figure 6.1: A brane tiling Δ of Σ_2 in green with its dual quiver Q_Δ in red.

Explicitly the dual quiver Q_Δ is



with potential

$$W_\Delta = abfjie + gc + hd - agic - bhjd - fe.$$

Looking at the relations we get from this potential, we have for example

$$\frac{\partial W_\Delta}{\partial a} = bfjie - gic\,, \quad \frac{\partial W_\Delta}{\partial d} = h - bhj$$

and it is not so hard to see from Figure 6.1 that indeed the path $bfjie$ is homotopic to gic and h is homotopic to bhj.

The idea behind using brane tilings in the context of fundamental groups of Riemann surfaces is as follows (from [14]). If the arrow $a \in Q_{\Delta,1}$ is dual to the edge between the white vertex v and black vertex u in Δ then $a \cdot \partial W_\Delta / \partial a = c_v - c_u$. This tells us that in $\mathrm{Jac}(Q_\Delta, W_\Delta)$ we can identify the paths c_v and c_u, which are homotopic in Σ_g via a homotopy which is spanned by the edge from the brane tiling. This allows us to link the algebra over the fundamental group of the Riemann surface Σ_g with a Jacobi algebra. However not all homotopic paths in $\mathbb{C}Q_\Delta$ are identified in $\mathrm{Jac}(Q_\Delta, W_\Delta)$, for example the minimal cycles c_v are null-homotopic but are not equal to any constant paths e_i in $\mathrm{Jac}(Q_\Delta, W_\Delta)$.

Definition 6.1.3. A dimer D for a brane tiling Δ is a choice of edges in Δ such that every vertex in Δ is an endpoint of exactly one edge in D.

Dimers are useful because after dualising they give us a collection of arrows in Q_Δ with the property that every cycle in W_Δ contains exactly one arrow from this set i.e. the dual of a dimer is a cut for (Q_Δ, W_Δ). This is straightforward from the definition of the dimer and the potential W_Δ; if a is dual to the edge in D that goes between the vertices v and u then a will appear in the minimal cycles c_v and c_u in W_Δ, and since vertices in Δ are endpoints to exactly one edge in D no other arrow in the dual of D will appear in c_v or c_u and every term in W contains one of these arrows.

6.2 Fundamental group algebras

We give an overview of existing results relating algebras over fundamental groups to the aforementioned Calabi-Yau and superpotential properties, motivating the conjecture we aim to provide evidence for. This section is based on [[13] Section 5].

Definition 6.2.1. A manifold X is called *acyclic* if its universal cover is contractible.

For the rest of this section let X be a compact, acyclic, orientable manifold of dimension n and let $k[\pi_1(X)]$ be its fundamental group algebra.

Theorem 6.2.2 ([13] Theorem 5.2.2)**.** *The fundamental group algebra $k[\pi_1(X)]$ is homologically finite.*

Let LX denote the *free loop-space* of X i.e. the space of continuous maps $S^1 \to X$. As path components of LX are in one-to-one correspondence with conjugacy classes $c \in \pi_1(X)$, let $(LX)_c$ denote the path component corresponding to c. Denote by *const* : $X \to LX$ the map that sends a point to the constant loop at that point, let $bp : LX \to X$ denote the map that sends a loop to its basepoint, and let $const_*$ and bp_* denote the respective pushforwards in homology.

Theorem 6.2.3 ([31] Theorem 6.2)**.** *For all $d \in \mathbb{N}$ we have a natural isomorphism*

$$\mathrm{HH}_d(k[\pi_1(X)]) \cong H_d(LX).$$

Recall for an algebra A that an element $\nu \in \mathrm{HH}_d(A)$ is called non-degenerate if it induces a (quasi-) isomorphism $A \cong A^\vee[d]$. There is an action of the centre $Z(A)$ on $\mathrm{HH}_*(A)$ induced by $Z(A)$ acting on the first copy of A in $A \otimes_{A^e} A$.

Proposition 6.2.4 ([13] Proposition 5.2.6)**.** *An element $\nu \in \mathrm{HH}_n(k[\pi_1(X)])$ is non-degenerate if and only if $\nu = z \cdot const_*([X])$ for some central unit $z \in k[\pi_1(X)]$. In particular $k[\pi_1(X)]$ is Calabi-Yau of dimension n.*

Theorem 6.2.5 ([13] Theorem 6.1.3)**.** *$k[\pi_1(X)]$ is exact Calabi-Yau of dimension n only if it contains a non-trivial central unit z.*

The idea behind the proof of Theorem 6.2.5 is as follows. First, from [31], we have an isomorphism of long exact sequences (where $A = k[\pi_1(X)]$)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{HC}_n(A) \arrow[r] \arrow[d] & \mathrm{HC}_{n-2}(A) \arrow[r, "\partial"] \arrow[d] & \mathrm{HH}_{n-1}(A) \arrow[r] \arrow[d] & \mathrm{HC}_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n^{S^1}(LX) \arrow[r] & H_{n-2}^{S^1}(LX) \arrow[r, "\partial'"] & H_{n-1}(LX) \arrow[r] & H_{n-1}^{S^1}(LX) \arrow[r] & \cdots
\end{tikzcd}$$

Hence an exact Calabi-Yau structure of dimension d on A is equivalent to a non-degenerate element $\eta \in H_d(LX)$ which is in the image of ∂'. By Proposition 6.2.4 a non-degenerate η must be of the form $\eta = z \cdot const_*([X])$ for a central unit $z \in A$. Now there exists a natural grading of both $H_{d-1}^{S^1}(LX)$ and $H_d(LX)$ by conjugacy classes in $\pi_1(X)$ (the grading exists at the level of chains) which is preserved by ∂', so the existence of some $\lambda \in H_{d-1}^{S^1}(LX)$ such that $\partial'(\lambda) = \eta$ is equivalent to equalities

$\partial(\lambda_c) = a_c s_c \cdot const_*([X])$ where c is a conjugacy class in $\pi_1(X)$, $a_c \in k$, $s_c = \sum_{g\in c} g$, and $\lambda_c \in H^{S^1}_{d-1}((LX)_c)$ such that $\eta = \sum_c a_c s_c \cdot const_*([X])$. If we were to take the central unit $z = 1$ then we would need to look for some $\lambda_0 \in H^{S^1}_{d-1}((LX)_0)$ such that $\partial'(\lambda_0) = const_*([X])$. But, from [[13] Theorem 6.1.3 and its proof], it turns out the map $H^{S^1}_{d-1}((LX)_0) \to H_d(LX)$ is 0 hence 1 does not give an exact Calabi-Yau structure.

Example 6.2.6 ([13] Corollary 6.2.3 and Corollary 6.2.4)**.** Let X be a compact hyperbolic manifold of dimension > 1. Then $k[\pi_1(X)]$ is not a superpotential algebra. Indeed one can show that $k[\pi_1(X)]$ has trivial centre, hence by Theorem 6.2.5 it is not exact Calabi-Yau. But Theorem 4.1.16 says that every superpotential algebra is exact Calabi-Yau.

We saw earlier that when A is connected a Gdga with cohomology concentrated in degree 0 will have an exact Calabi-Yau cohomology algebra. Theorem 6.2.5 describes the manifolds which cannot have an exact Calabi-Yau fundamental group algebra. This gives a rudimentary justification for looking for superpotential descriptions for the fundamental group algebras of circle bundles; often circle bundles have a natural non-trivial central element in their fundamental groups. In particular for the dimension 3 case we will be looking for isomorphisms between the fundamental group algebra and the Jacobi algebra of a quiver with potential. Then because our fundamental group algebras are always Calabi-Yau we get by Theorem 4.3.7 that the Jacobi algebra presentation is a superpotential algebra (provided the potential is homogeneous). As a starting point of this conjecture we have:

Theorem 6.2.7 ([14] Proposition 4.2)**.** *Let Δ be a brane tiling of a Riemann surface Σ_g and let $\mathbb{C}\widetilde{Q}_\Delta$ denote the localisation of the path algebra $\mathbb{C}Q_\Delta$ with respect to all the arrows in $Q_{\Delta,1}$. Then we have an isomorphism of algebras*

$$\mathrm{Jac}(\widetilde{Q}_\Delta, W_\Delta) \cong \mathrm{Mat}_{r\times r}(\mathbb{C}[\pi_1(\Sigma_g \times S^1)])$$

where r is the number of vertices in Q_Δ.

As remarked earlier there is a natural way to think about how the relations in the Jacobi algebra of the brane tiling identify homotopic paths in the surface Σ_g, but also not all relations in the fundamental group can be obtained from the potential. Adding in this extra S^1 direction fully rectifies this issue because paths in the Jacobi algebra that would be null-homotopic in Σ_g are instead sent to loops around the circle in $\Sigma_g \times S^1$. Explicitly we grade the arrows in Q_Δ such that the potential W_Δ is homogeneous of degree 1 (note this is entirely separate to the path grading we already have on the path algebra) and then define a new embedding $Q_\Delta \hookrightarrow \Sigma_g \times S^1$ where this grading determines how far the arrows go around the S^1 direction. In particular, for any genus g surface it is possible to find a tiling Δ such that the potential W_Δ is homogeneous (e.g. see [[13] Figure 2]) and so because Theorem 6.2.7 holds for any brane tiling we get a superpotential description for the fundamental group algebra $\mathbb{C}[\pi_1(\Sigma_g \times S^1)]$.

Conjecture 6.2.8 ([12] Conjecture 7.1.1)**.** *Let X be a compact, acyclic, orientable circle bundle of dimension n. Then $k[\pi_1(X)]$ is a superpotential algebra.*

Guided by Theorem 6.2.7 we shall provide a Jacobi algebra description for the mapping torus of a Riemann surface by a finite-order, orientation-preserving automorphism. Thereby we provide more evidence for Conjecture 6.2.8 if a path-graded homogeneous potential can be found.

6.3 A Jacobi algebra presentation for $\mathbb{C}[\pi_1(M_{g,\varphi})]$

From now on we work over the field $\mathbb{C}$.

Let Σ_g be a genus g Riemann surface and let φ be an orientation-preserving automorphism of Σ_g of order n. Let $M_{g,\varphi} = \Sigma_g \times [0,1]/\sim_\varphi$ denote the mapping torus of φ.

Consider a brane tiling Δ of Σ_g giving rise to a dual quiver $Q = Q_\Delta$ and potential $W = W_\Delta$ as described in section 6.1. We assume that the brane tiling and the chosen colouring of the vertices in the tiling are preserved by φ. This implies that φ induces an automorphism on Q and hence also on the path algebra $\mathbb{C}Q$ (we denote this automorphism by φ as well). We focus on the case in which the size of the orbit of every vertex in Q under φ is n.

Lemma 6.3.1 ([9] Proposition 3.3 1.)**.** *Let $f : D^2 \xrightarrow{\sim} D^2$ be an orientation-preserving automorphism of the disc D^2 of order n. Then*

a) the fixed point set of f is a single point in the interior of D^2.

b) the fixed point set of the composition f^i is equal to the fixed point set of f for all $i \neq 0 \bmod n$.

Lemma 6.3.2. *Let φ be an orientation-preserving automorphism of Σ_g of order n. Then there exists a brane tiling Δ which is preserved by φ such that every vertex in the dual quiver $Q = Q_\Delta$ has an orbit of size n under the induced action of φ on Q.*

Proof. A point in $p \in \Sigma_g$ will have orbit size strictly less than n under φ if and only if p is a fixed point of φ^d for some $0 < d < n$ that divides n. Let Δ_0 be a brane tiling of Σ_g with colouring that is preserved by φ. Then a vertex i in the dual quiver Q_{Δ_0} will have orbit size strictly less than n if and only if the tile S_i in Δ_0 that is dual to i is such that $\varphi^d(S_i) = S_i$ for some $0 < d < n$ that divides n. Suppose that there exists a vertex $i \in Q_{\Delta_0,0}$ with orbit size less than n and suppose that d is the smallest positive integer such that $\varphi^d(S_i) = S_i$. Then $\varphi^d|_{S_i} : S_i \xrightarrow{\sim} S_i$ is an (orientation-preserving) automorphism of S_i. Therefore by Lemma 6.3.1 a) it has a single fixed point p in the interior of S_i, and because by Lemma 6.3.1 b) the fixed point set of φ^{dj} is the same point for all $j = 1, \dots, n/d - 1$ every other point in S_i has orbit size n under φ.

We add new brane tiling vertices of the same colour at $p, \varphi(p), \dots, \varphi^{d-1}(p)$, then choose an opposite colour vertex v on the boundary of S_i and add an edge between the new vertex at p and v. We then add the images of this edge under φ^j for all $j = 1, \ldots, n-1$, which go between the vertices $\varphi^j(p)$ and $\varphi^j(v)$, to the brane tiling. This gives us a new brane tiling Δ_1 that is also preserved by φ, and in which the tile S_i in Δ_0 has been subdivided into tiles $S_{i_1}, \ldots, S_{i_{n/d}}$ (see Fig. 6.2). Indeed because the vertex v has orbit size n we have added n/d distinct edges into the tile S_i that only intersect at the point p and so we do in fact end up with a brane tiling (each of the distinct tiles $S_i, \varphi(S_i), \ldots, \varphi^{d-1}(S_i)$ each have n/d edges added, giving us a total of n new edges added to Δ_0 as expected).

We claim that $\varphi^k(S_{i_j}) \neq S_{i_j}$ for all i_j and all $0 < k < n$, and hence the vertex $i \in Q_{\Delta_0,0}$ which had orbit size less than n has been replaced by the vertices $i_j \in Q_{\Delta_1,0}$ all of orbit size n. So suppose not i.e. there exists some j and k such that $\varphi^k(S_{i_j}) = S_{i_j}$. Note that k must be multiple of d since otherwise

$$\varphi^{md+l}(S_{i_j}) \subset \varphi^l(S_i) \neq S_i$$

for $0 < l < d$. If $k = md$ then we have that $\varphi^{md}|_{S_{i_j}} : S_{i_j} \xrightarrow{\sim} S_{i_j}$ is an automorphism of S_{i_j} and so again by Lemma 6.3.1 a) it has a fixed point in the interior of S_{i_j}. But this would also be a fixed point of $\varphi^{md}|_{S_i}$ and it would be distinct from p contradicting Lemma 6.3.1 b).

We repeat this procedure for all such vertices in the dual quiver with orbit size less than n giving us the required brane tiling Δ. $\square$

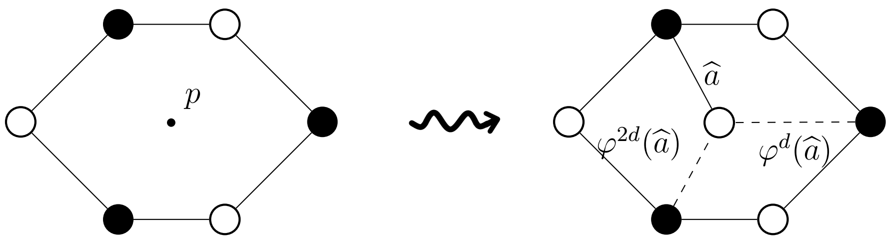


Figure 6.2: An example of a tile in Δ_0 for which we suppose the induced action of φ^d is clockwise rotation by 120° and the tile has orbit size $d = n/3 < n$. The fixed point of this rotation is the point p. We add a vertex at this point (in this case a white vertex), and then attach this to the brane tiling via an edge $\widehat{a}$. We then add all the images under $\varphi, \ldots, \varphi^{n-1}$ of both the new vertex and edge to Δ_0. In the resulting brane tiling the three new tiles are no longer preserved by φ^d.

Remark 6.3.3. Take a brane tiling Δ such that every vertex in the dual quiver has an orbit of size n under the induced action of φ on $Q = Q_\Delta$. Then we may freely add additional vertices and edges to Δ along with their images under $\varphi, \ldots, \varphi^{n-1}$ and still maintain a brane tiling which is preserved by φ such that the orbit of every vertex in the dual quiver has size n. Indeed, given such a Δ adding extra vertices and edges will amount to subdividing existing tiles in Δ. From the proof of Lemma 6.3.2 we see that if one of these new tiles S_{i_j}, which subdivides the old tile S_i, gives a dual vertex with orbit size less than n then we must have that $\varphi^d(S_{i_j}) = S_{i_j}$ for some $d < n$, and hence $\varphi^d|_{S_{i_j}}$ has a fixed point in S_{i_j}. Then as φ and hence φ^d preserves Δ we know that $\varphi^d(S_i)$ is also a tile in Δ. But φ^d has a fixed point inside $S_{i_j} \subset S_i$ and so $\varphi^d(S_i)$ must equal S_i which contradicts Δ not having any dual vertices of orbit size less than n.

Consider the semidirect product $S := \mathbb{C}Q \rtimes_\varphi \mathbb{Z}/n\mathbb{Z}$, which has the following description

$$\mathbb{C}Q \rtimes_\varphi \mathbb{Z}/n\mathbb{Z} = \bigoplus_{l=0}^{n-1} \mathbb{C}Q \times \{l\}$$

with multiplication given by

$$(a, l) \cdot (b, m) = (a\,\varphi^l(b),\ l + m)$$

where a, b are paths in $\mathbb{C}Q$ and l, m and $l + m$ are elements in $\mathbb{Z}/n\mathbb{Z}$.

Lemma 6.3.4. *S is Morita equivalent to a localised path algebra of a quiver.*

Proof. A representation of S is a representation of Q with the extra data of the action of $\mathbb{1} \in \mathbb{Z}/n\mathbb{Z}$, subject to the multiplication relations in S. Let $V = \bigoplus_{i \in Q_0} V_i$ and $(f_a)_{a \in Q_1}$ be the data of a representation of Q. The action of $\mathbb{1}$ is then given by isomorphisms $V_i \xrightarrow{r_i} V_{\varphi(i)}$ for each $i \in Q_0$ such that if we let $r : V \xrightarrow{\sim} V$ be the isomorphism given component-wise on V_i by r_i, we have $r^n = \mathrm{id}_V$. The data $(V, (f_a), (r_i))$ then gives a representation of S if we also have that for all $a : i \to j \in Q_1$, $f_{\varphi(a)} = r_j \circ f_a \circ r_i^{-1}$. Equivalently if we partition the orbits of φ in Q as follows; for the vertices we have orbits $O_1 = \{i_1, \varphi(i_1), \ldots, \varphi^{n-1}(i_1)\}, \ldots, O_y = \{i_y, \ldots, \varphi^{n-1}(i_y)\}$ and for the arrows we have orbits $P_1 = \{a_1, \ldots, \varphi^{n-1}(a_1)\}, \ldots, P_z = \{a_z, \ldots, \varphi^{n-1}(a_z)\}$, then the data of a representation of S is given by the vector space $V = \bigoplus_{i \in Q_0} V_i$, linear maps f_{a_t} for $t \in \{1, \ldots, z\}$ and isomorphisms $r_{i_u,0} : V_{i_u} \xrightarrow{\sim} V_{\varphi(i_u)},\ r_{i_u,1} : V_{\varphi(i_u)} \xrightarrow{\sim} V_{\varphi^2(i_u)}, \ldots, r_{i_u,n-2} : V_{\varphi^{n-2}(i_u)} \xrightarrow{\sim} V_{\varphi^{n-1}(i_u)}$ for $u \in \{1, \ldots, y\}$.

Let $Q^\#$ denote the quiver with vertices $Q_0^\# = Q_0$ and arrows $Q_1^\# = \{a_1, \ldots, a_z,\ r_{i_u,t} : u = 1, \ldots, y \text{ and } t = 0, \ldots, n-2\}$, and let $\mathbb{C}Q'$ denote the localisation of the path algebra $\mathbb{C}Q^\#$ at the arrows $\{r_{i_u,t}\}$. Then it is clear that the data of a representation of S is then exactly the same as the data of a representation of $\mathbb{C}Q'$. □

We shall use the notation Q' for the "localised" quiver associated to the partially localised path algebra $\mathbb{C}Q'$. We call the arrows $a_1, \ldots, a_z \in Q_1^\#$ *generating arrows* and the arrows $r_{i_u,t} \in Q_1^\#$ *isomorphism arrows.*

Example 6.3.5. Consider a Riemann surface of genus 2 and let the automorphism φ be rotation by 180° around the z-axis through the centre of the surface. We take the brane tiling from Example 6.1.2.

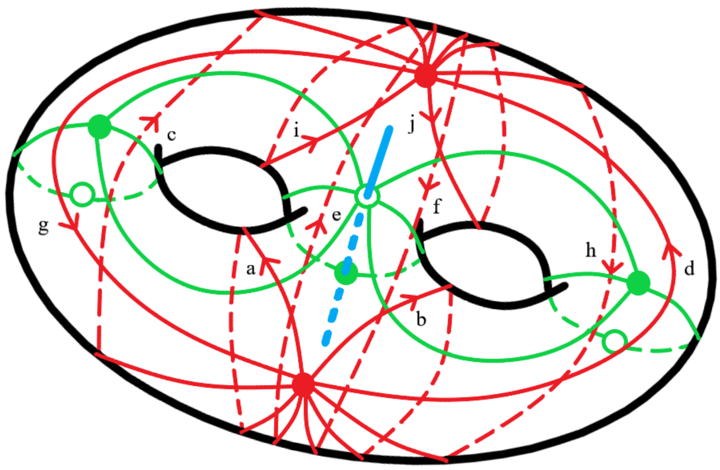
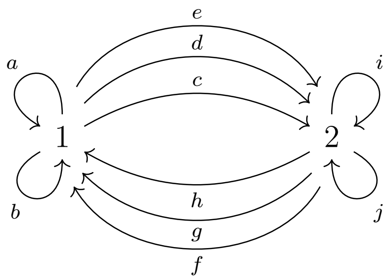
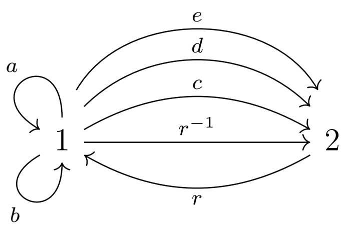


Figure 6.3: Let φ be the rotation by 180° around the blue axis.

Recall this gives us the following dual quiver Q



and potential $W = abfjie + gc + hd - agic - bhjd - fe$. φ then swaps the two vertices 1 and 2 and also the arrows a and j, b and i, c and h, d and g, e and f. The quiver Q' corresponding to the algebra $\mathbb{C}Q \rtimes \mathbb{Z}/2\mathbb{Z}$ is



as the arrows a, b, c, d, e generate Q under φ and the arrows r, r^{-1} give the isomorphism between the vector spaces at the two vertices.

Lemma 6.3.6. *There is an inclusion of algebras $\xi : \mathbb{C}Q \hookrightarrow \mathbb{C}Q'$.*

Proof. The map ξ is induced by the natural inclusion $\mathbb{C}Q \times \{0\} \hookrightarrow S$. As per the proof of Lemma 6.3.4 we make a choice of a generating set of arrows $\{a_1, \ldots, a_z\}$ of Q under φ as well as a choice of isomorphism arrows $r_{i_u,t} : \varphi^t(i_u) \to \varphi^{t+1}(i_u)$ for $u = 1, \ldots, y$

and $t = 0, \dots, n-2$. Then define the map ξ as follows; since $Q_0 = Q'_0$ send the constant paths $e_i \mapsto e_i$, and then map the arrow $a \in Q_1$ to the path $p_a a_j q_a$ where a is in the orbit under φ of the generating arrow a_j and $p_a : t(a_j) \to t(a)$, $q_a : s(a) \to s(a_j)$ are paths comprised solely of the isomorphism arrows or their inverses (see Fig. 6.4 for an example).

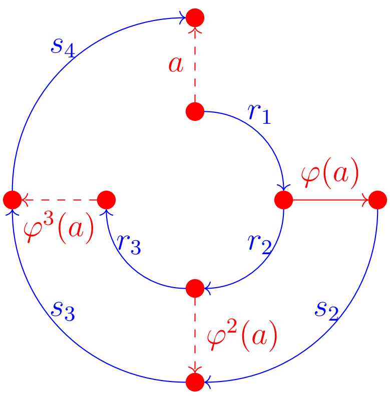


Figure 6.4: An example of how Q fits inside Q'. Let the red vertices and (dashed) arrows be in Q with φ being clockwise rotation by 90°. If we take the generating arrow of the orbit of a to be $\varphi(a)$ and let the blue arrows denote the chosen isomorphism arrows, then $\xi(a) = s_4 s_3 s_2 \varphi(a) r_1$, $\xi(\varphi(a)) = \varphi(a)$, $\xi(\varphi^2(a)) = s_2 \varphi(a) r_2^{-1}$, and $\xi(\varphi^3(a)) = s_3 s_2 \varphi(a) r_2^{-1} r_3^{-1}$.

ξ is well-defined because by construction of Q' all such paths a_j, p_a, q_a are unique and because both a and $\xi(a)$ are paths $s(a) \to t(a)$ all the trivial relations in $\mathbb{C}Q$ are satisfied.

To see that ξ is an inclusion it suffices to show that ξ is injective on paths and that

$$\{\xi(p) : p \text{ is a path in } \mathbb{C}Q\}$$

is linearly independent. So first suppose we have paths p, $q \in \mathbb{C}Q$ such that $\xi(p) = \xi(q)$. Write

$$p = b_n b_{n-1} \dots b_2 b_1$$
$$q = c_m c_{m-1} \dots c_2 c_1$$

for arrows $b_i, c_j \in Q_1$. Then the equality $\xi(p) = \xi(q)$ implies that both b_1 and c_1 lie in the same orbit of some generating arrow a_1 i.e.

$$b_1 = \varphi^{i_1}(a_1) \quad \text{and} \quad c_1 = \varphi^{j_1}(a_1)$$

for some i_1, j_1. But as paths in $\mathbb{C}Q'$ they must have the same source, and so by the assumption that the size of the orbit of each vertex in Q under φ has size n we know that the source of $\varphi^{i_1}(a_1)$ and $\varphi^{j_1}(a_1)$ are equal only when $i_1 = j_1$ and so $b_1 = c_1$.

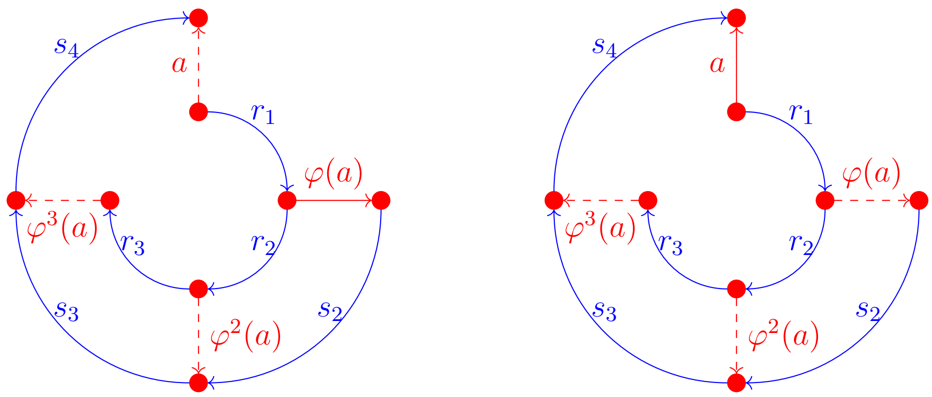


Figure 6.5: Let φ be clockwise rotation by 90° on the quiver. The left-hand side is the same setup as we had in Fig. 6.4 with generating arrow for the orbit $\varphi(a)$. As we saw $\xi(a) = s_4 s_3 s_2 \varphi(a) r_1$ has degree 4 whilst $\xi(\varphi^2(a)) = s_2 \varphi(a) r_2^{-1}$ has degree 0 due to the choices of isomorphism arrows between the sources and targets of the arrows in this orbit. On the right-hand side we instead take the generating arrow of this orbit to be a and choose the same isomorphism arrows. In this case we would have, for example, $\xi(\varphi(a)) = s_2^{-1} s_3^{-1} s_4^{-1} a r_1^{-1}$ which has degree -4.

We can then repeat this argument for the rest of the arrows in p and q giving us that $m = n$ and $b_i = c_i$ for all i, hence $p = q$ and ξ is injective on paths. Then certainly we have

$$\{\xi(p) : p \text{ is a path in } \mathbb{C}Q\} \subseteq \{p' : p' \text{ is a path in } \mathbb{C}Q'\}$$

and since $\{p' : p' \text{ is a path in } \mathbb{C}Q'\}$ is a $\mathbb{C}$-basis of $\mathbb{C}Q'$ we get the required result. $\square$

We grade the isomorphism arrows $r_{i_u,t}$ in Q' with degree 1, their inverses $r_{i_u,t}^{-1}$ with degree -1, and all other arrows in Q' with degree 0.

Lemma 6.3.7. *For all arrows $a \in Q_1$ the element $\xi(a) \in \mathbb{C}Q'$ either has degree 0, degree n or degree $-n$.*

Proof. This is clear from the assumption that each orbit of vertices/arrows under φ has size n. Indeed if the arrow a is in the orbit of the generating arrow b then we can write $a = \varphi^k(b)$ for some k. Therefore, depending upon the choice of the isomorphism arrows $r_{i_u,t}$ for each orbit of vertices, between the source vertices of a and b there will either be k or $n-k$ isomorphism arrows and similarly between the target vertices of a and b there will either be k or $n-k$ isomorphism arrows. Hence $\xi(a)$ is either going to be the composition of k (resp. $n-k$) isomorphism arrows followed by b followed by k (resp. $n-k$) inverses and hence it will have degree 0, or it will be the composition of k (resp. $n-k$) isomorphism arrows followed by b followed by $n-k$ (resp. k) isomorphism arrows and hence it will have degree n or it will be the composition of k (resp. $n-k$) inverses arrows followed by b followed by $n-k$ (resp. k) more inverse arrows and hence it will have degree $-n$ (see Fig. 6.5 for an illustration of each case). $\square$

From now on when we say a path $p \in \mathbb{C}Q'$ is made from isomorphism arrows we mean that the only arrows that appear in p are the isomorphism arrows $r_{i_u,t}$ or their inverses $r_{i_u,t}^{-1}$.

Lemma 6.3.8. *For all paths $p' \in \mathbb{C}Q'$ we can write*

$$p' = q\,\xi(p)$$

for some path $p \in \mathbb{C}Q$ and a path $q \in \mathbb{C}Q'$ made from isomorphism arrows. In particular if $\deg(p') \equiv 0 \bmod n$ then $p' \in \operatorname{Im}(\xi)$.

Remark 6.3.9. Having the path of isomorphism arrows q at the end is somewhat arbitrary, it is also possible to write $p' = \xi(p)q'$ or $p' = q''\xi(p)q'''$.

Proof. Write

$$p' = r_{m+1}a_m \dots a_2 r_2 a_1 r_1$$

for arrows $a_i \in Q_1'$ coming from the generating set of Q_1 under φ, and paths of isomorphism arrows r_i. Let $\widetilde{a_1}$ be the arrow in Q_1 such that r_1 is the unique path of isomorphism arrows in $\mathbb{C}Q'$ between $s(a_1)$ and $s(\widetilde{a_1})$ (in particular $\widetilde{a_1} = \varphi^{k_1}(a_1)$ for some k_1). Then $\xi(\widetilde{a_1}) = s_1 a_1 r_1$ where s_1 is the unique path of isomorphism arrows in $\mathbb{C}Q'$ between $t(a_1)$ and $t(\widetilde{a_1})$. Then let $\widetilde{a_2}$ be the arrow in Q_1 such that $r_2 s_1^{-1}$ is the unique path of isomorphism arrows in $\mathbb{C}Q'$ between $s(a_2)$ and $s(\widetilde{a_2})$, and write $\xi(\widetilde{a_2}) = s_2 a_2 r_2 s_1^{-1}$. Repeating this procedure for all the generating arrows a_i in p' it is clear that

$$p' = r_{m+1}s_m^{-1}\xi(\widetilde{a_m \dots} \widetilde{a_1}).$$

If $\deg(p') \equiv 0 \bmod n$ this implies that

$$\deg(r_{m+1}s_m^{-1}) + \deg(\xi(\widetilde{a_m \dots} \widetilde{a_1})) \equiv 0 \bmod n.$$

By Lemma 6.3.7 $\deg(\xi(\widetilde{a_m \dots} \widetilde{a_1})) \equiv 0 \bmod n$ and so the path of isomorphism arrows $r_{m+1}s_m^{-1}$ has degree a multiple of n. Now a path made of isomorphism arrows must go between vertices in Q' in the same orbit, but in any orbit of a vertex there are only $n-1$ isomorphism arrows that connect the whole orbit. Hence the degree of $r_{m+1}s_m^{-1}$ is 0 and so it is just a constant path. $\square$

It is clear that ξ descends to a map $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q'/[\mathbb{C}Q', \mathbb{C}Q']$ so let W' be the image of W under this morphism, giving a potential on Q'. Therefore we can construct the Jacobi algebra $\operatorname{Jac}(Q', W')$. In our running example Example 6.3.5 ξ

sends

$$\begin{aligned} a &\longmapsto a, & f &\longmapsto rer \\ b &\longmapsto b, & g &\longmapsto rdr \\ c &\longmapsto c, & h &\longmapsto rcr \\ d &\longmapsto d, & i &\longmapsto r^{-1}br \\ e &\longmapsto e, & j &\longmapsto r^{-1}ar \end{aligned}$$

and

$$W' = abreabre + 2rdrc - 2ardbrc - rere. \tag{6.1}$$

There is a choice involved in taking generators of Q under φ and the isomorphism arrows $r_{i_u,t}$ (or equivalently a choice of ξ) and hence the algebra $\mathbb{C}Q'$ is not unique (but it is unique up to canonical isomorphism). We choose ξ so that W' does not contain both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$, so that we may view it as a potential on the un-localised quiver algebra and hence the noncommutative derivative description of $\mathrm{Jac}(Q', W')$ makes sense.

Lemma 6.3.10. *There exists a choice of ξ such that for each isomorphism arrow $r_{i_u,t} \in Q'_1$ the potential W' does not contain both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$.*

Proof. Consider the orbit of vertices $\{i_u, \varphi(i_u), \ldots, \varphi^{n-1}(i_u)\}$ in Q', where $r_{i_u,t} : \varphi^t(i_u) \to \varphi^{t+1}(i_u)$. If W' were to contain both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$ then there would need to be arrows a, b, c, $d \in Q_1$ such that ba and dc are part of cycles in W and we would need to choose ξ so that if we let a_0, b_0, c_0, d_0 be the corresponding generators of the orbits of the arrows then $t(a_0) = \varphi^{k_1}(i_u)$, $s(b_0) = \varphi^{k_2}(i_u)$, and $t(c_0) = \varphi^{l_1}(i_u)$, $s(d_0) = \varphi^{l_2}(i_u)$ with $k_1, l_2 \leq t$, $k_2, l_1 \geq t + 1$. Indeed if this was the case then

$$\xi(ba) = p_4\, b_0\, p_3\, r_{i_u,t}\, p_2\, a_0\, p_1$$

and

$$\xi(dc) = q_4\, d_0\, q_3\, r_{i_u,t}^{-1}\, q_2\, c_0\, q_1$$

for paths of isomorphism arrows p_i, q_i (see Fig. 6.6 for an illustration of this). Hence if we simply choose ξ so that the generators of arrows of two different paths in the cycles in W that cross in the same orbit of vertices do not overlap in this way we remove instances of both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$ appearing in W'. One way of ensuring this is for each orbit of vertices in Q when choosing generators of the orbits of arrows whose sources lie in this orbit of vertices, we have that the sources of each of the generators are equal. This would amount to taking $s(b_0) = s(d_0)$ and hence at most either $r_{i_u,t}$ or $r_{i_u,t}^{-1}$ can appear in W'. $\square$

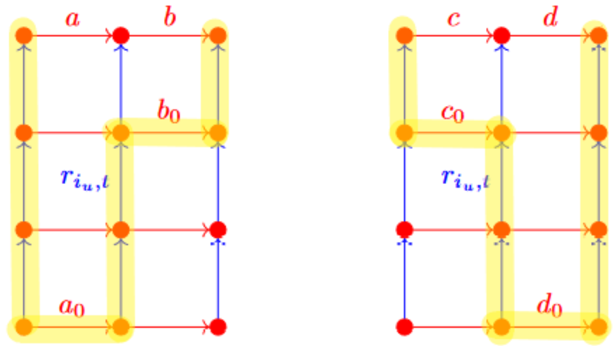


Figure 6.6: Two parts ba and dc of cycles that appear in W that cross at the orbit of the vertex i_u. The chosen generators for the orbits are a_0, b_0, c_0, d_0 giving $\xi(ba)$ and $\xi(dc)$ as the paths highlighted in yellow. We can see that, due to these choices of generators, $r_{i_u,t}$ appears in $\xi(ba)$ and $r_{i_u,t}^{-1}$ appears in $\xi(dc)$ and so both appear in $W' = \xi(W)$. A choice of ξ where this will not occur would be for example to take b and d to be the generators of their orbits. As the sources of b and d are equal this means no matter what choice we take for the generators of the orbits for a and c, we can never see both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$ in W' (assuming no other parts of cycles in W go through the orbit of i_u).

Lemma 6.3.11.

a) *Let $a \in Q_1'$ be a generating arrow which is dual to the edge in Δ that goes between the white vertex v and the black vertex u. Then*

$$a \cdot \frac{\partial W'}{\partial a} = n\left(\xi(c_v) - \xi(c_u)\right) \tag{6.2}$$

for the minimal cycles c_v and c_u in W.

b) *The relations $\partial W'/\partial r_{i_u,t}$ (or if relevant $\partial W'/\partial r_{i_u,t}^{-1}$) can be derived from the relations $\partial W'/\partial a$ for generating arrows $a \in Q_1'$.*

Proof. a) Write $\varphi^j(a) = a_j \in Q_1$ (so $a_0 = a$). Then $\xi(a_j) = p_j a\, q_j$ where p_j, q_j are paths in $\mathbb{C}Q'$ made from isomorphism arrows. For each j write c_{v_j} and c_{u_j} for the minimal cycles in W such that a_j is dual to the edge in Δ between the white vertex v_j and black vertex u_j; as these cycles can be written up to cyclic permutation we write them so that a_j appears at the end of these cycles. As φ preserves the tiling Δ we have for any vertex $v \in \Delta$ that $\varphi(c_v) = c_{\varphi(v)}$. Hence $c_{v_j} = \varphi^j(c_{v_0})$ and $c_{u_j} = \varphi^j(c_{u_0})$. Then because we wrote the cycles so that a_j appears at the end it follows that $\xi(c_{v_j}) = p_j \xi(c_{v_0}) p_j^{-1}$ and $\xi(c_{u_j}) = p_j \xi(c_{u_0}) p_j^{-1}$, and hence up to cyclic permutation $\xi(c_{v_j}) = \xi(c_{v_0})$ and $\xi(c_{u_j}) = \xi(c_{u_0})$. This is because $\xi(c_{v_0})$ (resp. $\xi(c_{u_0})$) ends with a so is a cycle at the vertex $t(a)$ whilst c_{v_j} (resp. c_{u_j}) is a cycle at the vertex $t(a_j)$, and p_j is the unique path of isomorphism arrows between $t(a)$ and $t(a_j)$.

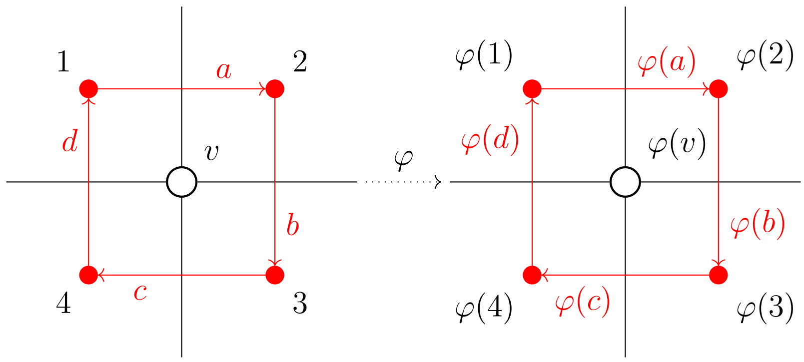


Figure 6.7: An example of part of a brane tiling with the action of φ. Let φ send the part of the brane tiling on the left to the part on the right. Take the generators of the orbits to be a, b, c, d and write p_i for the path in $\mathbb{C}Q'$ made of isomorphism arrows between the vertices i and $\varphi(i)$. Hence $\xi(\varphi(a)) = p_2ap_1^{-1}$, $\xi(\varphi(b)) = p_3bp_2^{-1}$, $\xi(\varphi(c)) = p_4cp_3^{-1}$, and $\xi(\varphi(d)) = p_1dp_4^{-1}$. Therefore $\xi(c_{\varphi(v)}) = \xi(\varphi(d))\xi(\varphi(c))\xi(\varphi(b))\xi(\varphi(a)) = p_1dcbap_1^{-1} = p_1\xi(c_v)p_1^{-1}$ and all the intermediary paths of isomorphism arrows cancel, as required.

To calculate $\partial W'/\partial a$, the relevant terms in W' are

$$\xi(c_{v_j}) - \xi(c_{u_j}) = \xi(c_{v_0}) - \xi(c_{u_0}) \text{ (up to cyclic permutation)}$$

for $j \in \{0, \dots, n-1\}$. Writing the cycles $\xi(c_{v_0})$ and $\xi(c_{u_0})$ so that a is at the end, it follows that

$$a \cdot \frac{\partial W'}{\partial a} = n\left(\xi(c_{v_0}) - \xi(c_{u_0})\right).$$

b) Morally this is true because $r_{i_u,t} \cdot \partial W'/\partial r_{i_u,t}$ is a sum whose terms are just the cycles in W' that contain $r_{i_u,t}$ with $r_{i_u,t}$ cyclically permuted to the front. But the cycles in W' can be made from the relations $\partial W'/\partial a$ for generating arrows a and so because $r_{i_u,t}$ is invertible in $\mathbb{C}Q'$, $\partial W'/\partial r_{i_u,t}$ is $r_{i-u,t}^{-1}$ multiplied by terms that can be obtained from the relations $\partial W'/\partial a$. We just need to check which generating arrows a are needed to obtain the relevant cycles in W' which contain $r_{i_u,t}$.

So consider the relation $\partial W'/\partial r_{i_u,t}$ and without loss of generality we assume that W' does not contain $r_{i_u,t}^{-1}$. For clarity write $r = r_{i_u,t}$. If $a \in Q_1$ has r in its image under ξ and if $b \in Q_1$ has r^{-1} in its image under ξ then we can write

$$\xi(a) = p'_a\, r^{x_a}\, p''_a\, a_0\, q'_a\, r^{y_a}\, q''_a$$

and

$$\xi(b) = p'_b\, r^{x_b}\, p''_b\, b_0\, q'_b\, r^{y_b}\, q''_b$$

for a_0 and b_0 the chosen generators of the orbits, p'_a, p''_a, q'_a, q''_a and p'_b, p''_b, q'_b, q''_b paths of isomorphism arrows that do not contain r or r^{-1}, and x_a, $y_a \in \{0, 1\}$ and x_b, $y_b \in \{-1, 0\}$. If a is dual to the edge in Δ between the white vertex v_a and black vertex u_a and we write the minimal cycles c_{v_a} and c_{u_a} so that a is at the end, then $r \cdot \partial W'/\partial r$

will contain terms like

$$r\,p''_a\,a_0\,q'_a\,r^{y_a}\,q''_a\Big(\xi(a^{-1}c_{v_a})-\xi(a^{-1}c_{u_a})\Big)p'_a \tag{6.3}$$

and

$$r\,q''_a\Big(\xi(a^{-1}c_{v_a})-\xi(a^{-1}c_{u_a})\Big)p'_a\,r^{x_a}\,p''_a\,a_0\,q'_a. \tag{6.4}$$

However since W' does not contain r^{-1} whenever an arrow b appears in W the r^{-1} it gives under ξ must be cancelled out by an r given by some a under ξ. Hence we must remove those terms from $r\cdot\partial W'/\partial r$ that are cancelled. What is removed is exactly of the form

$$r\,r^{-1}\,p''_b\,b_0\,q'_b\,r^{y_b}\,q''_b\Big(\xi(b^{-1}c_{v_b})-\xi(b^{-1}c_{u_b})\Big)p'_b \tag{6.5}$$

or

$$r\,r^{-1}\,q''_b\Big(\xi(b^{-1}c_{v_b})-\xi(b^{-1}c_{u_b})\Big)p'_b\,r^{x_b}\,p''_b\,b_0\,q'_b \tag{6.6}$$

and so $r\cdot\partial W'/\partial r$ is a sum of terms like (6.3) and (6.4) minus terms like (6.5) and (6.6). Now (6.3) can be written as

$$\begin{aligned}
&r\,p''_a\,a_0\,q'_a\,r^{y_a}\,q''_a\Big(\xi(a^{-1}c_{v_a})-\xi(a^{-1}c_{u_a})\Big)p'_a=\\
&\qquad=p'^{-1}_a\Big(p'_a\,r\,p''_a\,a_0\,q'_a\,r^{y_a}\,q''_a\Big(\xi(a^{-1}c_{v_a})-\xi(a^{-1}c_{u_a})\Big)\Big)p'_a\\
&\qquad=p'^{-1}_a\Big(\xi(a)\xi(a^{-1})\Big(\xi(c_{v_a})-\xi(c_{u_a})\Big)\Big)p'_a\\
&\qquad=\frac{1}{n}\,p'^{-1}_a\Big(p_a\,a_0\cdot\partial W'/\partial a_0\,p_a^{-1}\Big)p'_a
\end{aligned}$$

where $p_a=p'_a r p''_a$ and the last line follows from part a). Similarly (6.4), (6.5), (6.6) can all be written in terms of the derivatives of W' with respect to the generating arrows a_0 or b_0. This gives the result. □

Lemma 6.3.12. *ξ induces an inclusion $\widetilde{\xi}:\mathrm{Jac}(Q,W)\hookrightarrow\mathrm{Jac}(Q',W')$.*

Proof. We first show the induced map is well-defined. Let $b\in Q_1$ and suppose $b=\varphi^k(a)$ for some arrow a in the chosen generating set of Q_1. As in the proof of Lemma 6.3.11 we write $\xi(b)=p_k a\,q_k$ where p_k, q_k are paths of isomorphism arrows in $\mathbb{C}Q'$. Then we see that

$$\begin{aligned}
\xi\left(b\cdot\frac{\partial W}{\partial b}\right)&=\xi(c_{v_b})-\xi(c_{u_b})\\
&=p_k\big(\xi(c_{v_a})-\xi(c_{u_a})\big)p_k^{-1}.
\end{aligned} \tag{6.7}$$

Combining (6.7) with (6.2) gives

$$\xi\left(b\cdot\frac{\partial W}{\partial b}\right)=\frac{1}{n}\,p_k\,a\cdot\frac{\partial W'}{\partial a}p_k^{-1}$$

and so from the equation $\xi(b)=p_k a\,q_k$ we get that

$$\xi\left(\frac{\partial W}{\partial b}\right)=\frac{1}{n}q_k^{-1}\frac{\partial W'}{\partial a}\,p_k^{-1}. \tag{6.8}$$

As for injectivity, we must show that given some $x\in\mathbb{C}Q$ such that $\xi(x)\in I_{W'}$ then $x\in I_W$. Write

$$\xi(x)=\sum_a p'_a\frac{\partial W'}{\partial a}\,q'_a$$

for paths p'_a, $q'_a\in\mathbb{C}Q'$. Using Lemma 6.3.8 we can write this as

$$\xi(x)=\sum_a p'_a\frac{\partial W'}{\partial a}\,r_a\,\xi(q_a)$$

for paths $q_a\in\mathbb{C}Q$ and paths of isomorphism arrows $r_a\in\mathbb{C}Q'$. Let $\varphi^{k_a}(a)$ be the arrow in Q_1 such that r_a^{-1} is the unique path of isomorphism arrows in $\mathbb{C}Q'$ between $t(a)$ and $t(\varphi^{k_a}(a))$, then using (6.8) we can write

$$\xi(x)=\sum_a p''_a\,\xi\left(\frac{\partial W}{\partial\varphi^{k_a}(a)}\,q_a\right)$$

where $p''_a=p'_a s_a$ and s_a is the unique path of isomorphism arrows between $s(a)$ and $s(\varphi^{k_a}(a))$. Taking degrees we see from Lemma 6.3.7 that $\deg(p''_a)\equiv 0 \bmod n$ and so Lemma 6.3.8 gives us a path $p_a\in\mathbb{C}Q$ such that $\xi(p_a)=p''_a$. As ξ is injective we get

$$x=\sum_a p_a\,\frac{\partial W}{\partial\varphi^{k_a}(a)}\,q_a$$

giving the result. $\square$

We make the further assumption that we can choose generators and isomorphism arrows such that additionally W' will be homogeneous of degree $n=\deg(\varphi)$. We can see that in the case of the running example Example 6.3.5 that $W'=abreabre+2rdrc-2ardbrc-rere$ indeed has degree 2 in the isomorphism arrow r. We give some justification about why this assumption can be made.

Lemma 6.3.13. *Let φ be an automorphism of order n of the Riemann surface Σ_g and let Δ_0 be a brane tiling of Σ_g which is preserved by φ. Then we can extend Δ_0 into a brane tiling Δ which is also preserved by φ such that there exists a dimer for Δ.*

Proof. In a brane tiling define the *distance* between two distinct vertices u and v, denoted by $\operatorname{dist}(u, v)$, to be 1 if there exists a tile in the brane tiling for which both u and v are in the perimeter of it. Then recursively define the distance of two vertices to be $x+1$ if there exists another vertex u' such that $\operatorname{dist}(u, u') = 1$ and $\operatorname{dist}(u', v) = x$ or vice versa.

For our brane tiling Δ_0 we first add vertices and their images under $\varphi, \dots, \varphi^{n-1}$ until the number of black vertices equals the number of white vertices. Next we add edges and their images under $\varphi, \dots, \varphi^{n-1}$ until we end up with a brane tiling which we call Δ_0'. We then begin to construct a dimer D for Δ_0' by choosing pairs of vertices in Δ_0' (one black and one white) which are connected by an edge. Once a vertex has been chosen in a pair it is then removed from further consideration for future pairings.

In this way we can see that if we are able to have all vertices in Δ_0' in one of these pairs then we will indeed have a dimer for Δ_0'. However it might turn out that due to our choices there exists a (w.l.o.g) black vertex u such that all the white vertices adjacent to it are already paired off.

To remedy such an issue with constructing our dimer, we note that because the number of black and white vertices is equal there must exist at least one white vertex v that has not been paired off. If $\operatorname{dist}(u, v) = 1$ we add an edge to Δ_0' (along with all its images under $\varphi, \dots, \varphi^{n-1}$) connecting u to v and then take this to be in D. If $\operatorname{dist}(u, v) > 1$ we consider all the white vertices v_1 such that $\operatorname{dist}(u, v_1) = 1$. We then choose a white vertex v_1 such that v_1 is paired with the black vertex u_1 and $\operatorname{dist}(u_1, v) < \operatorname{dist}(u, v)$. Such a vertex u_1 must exist due to our definition of distance in the brane tiling. We then add an edge between u and v_1 to the brane tiling (along with its images under $\varphi, \dots, \varphi^{n-1}$) unless one already exists in Δ_0'. Then we replace the pair (u_1, v_1) in D with (u, v_1), thereby shifting the issue we are having with constructing our dimer onto u_1. We continue to repeat this, each time noting that the distance between the black vertex u_i and our white vertex v is decreasing. Hence after a finite number of steps we obtain a black vertex u_y such that $\operatorname{dist}(u_y, v) = 1$ and so we can add an edge between u_y and v into the brane tiling and then add the pair (u_y, v) into D. See Fig. 6.8 for an illustrative example.

This will fully rectify the issue we had of not being able to include the vertex u as a pair into D and also not introduce any further issues since all vertices that had been paired up in D beforehand still remain in D (albeit some will be paired with different vertices now). Continuing to do this for each vertex will ensure that all vertices end up in D and hence D will be a dimer for the brane tiling Δ we end up with. $\square$

We must always ensure that our brane tilings are preserved by φ and that the orbits of the vertices in the dual quiver Q have size n. By Remark 6.3.3 we can add to a brane tiling with this property and retain the property, so if we take a brane tiling Δ_0 that is preserved by φ then apply Lemma 6.3.2 and then Lemma 6.3.13 to it we end up with a brane tiling Δ for which both the orbit size of every vertex in $Q = Q_\Delta$ is n and for which there exists a dimer D for Δ. To try to make W' homogeneous of degree n in the isomorphism arrows we consider the cut E consisting of the arrows in

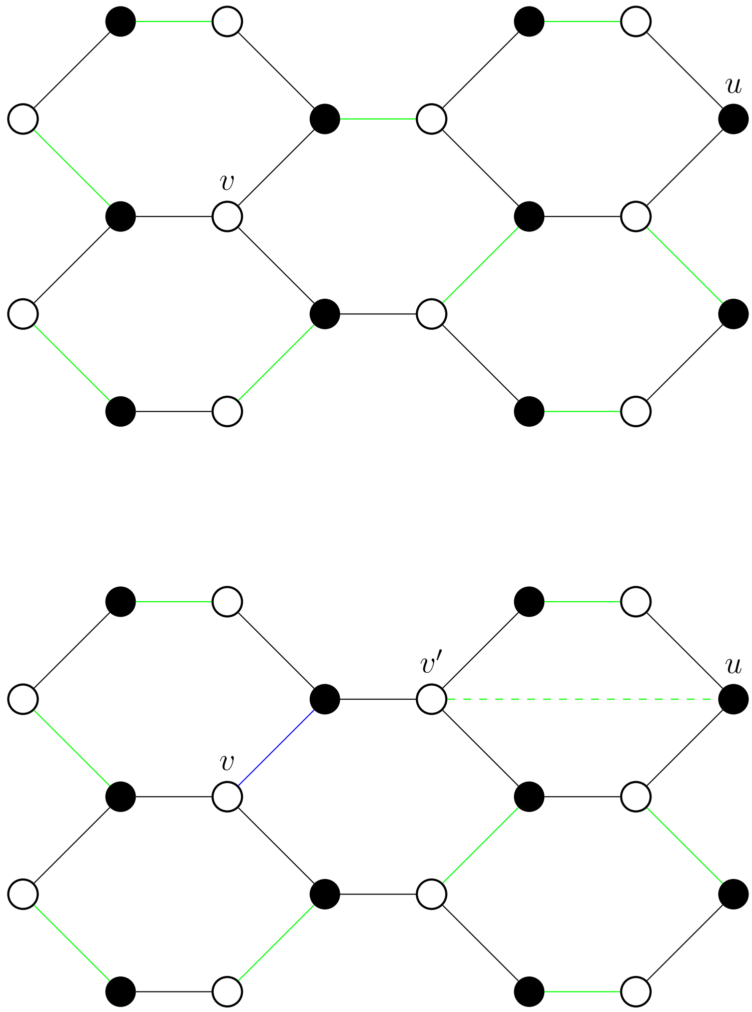


Figure 6.8: The top picture shows an example of part of a brane tiling with the partial dimer D in green. There are two vertices v and u which are not contained by an edge in D with $\mathrm{dist}(u, v) = 2$. With the current setup it is not possible to include v in D to get a dimer. We rectify this in the second picture by changing the edges we include in D (with the new edge in D shown in blue). This shifts the problem onto the vertex v' which is such that $\mathrm{dist}(u, v') = 1$. We can then add an edge that connects v' and u into Δ_0' (along with all its images under $\varphi, \ldots, \varphi^{n-1}$), and then include this new edge into D.

Q that are dual to the edges in D. It is then hoped that there is a choice of ξ for which the conditions explained in the proof of in Lemma 6.3.10 hold and for which $\xi(e)$ has degree n for each $e \in E$ and $\xi(a)$ has degree 0 for every other arrow $a \in Q_1 \setminus E$. This will then give a potential W' which has the properties of not containing both $r_{i_u,t}$ and $r^{-1}_{i_u,t}$, and having degree n in the isomorphism arrows since W is homogeneous of degree 1 in the arrows in E.

Given these assumptions on W' hold, similarly to [[14] Proposition 4.2 and the preceding discussion], we define a homomorphism

$$\mathrm{Jac}(\widetilde{Q'}, W') \xrightarrow{\Psi} \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(M_{g,\varphi})]) \tag{6.9}$$

where m is the number of vertices in Q' (which is also the number of vertices in Q), and recall from Chapter 2 that the tilde denotes that we are localising the path algebra $\mathbb{C}Q'$ with respect to every arrow $a \in Q'_1$ before taking the quotient. To construct Ψ explicitly consider a maximal tree $T \subset Q$ such that $\xi(t)$ has degree 0 for all $t \in T$. With regards to the dimer D discussed above, this means that the intersection of T and the set of arrows dual to edges in D is empty. Fix a basepoint $\widetilde{\mathrm{bp}} \in \Sigma_g$ that is invariant under φ (note such a basepoint will be a vertex in the brane tiling Δ as per Lemma 6.3.2), fix a vertex $\mathrm{bp} \in Q_0$, and fix a path $\delta : \widetilde{\mathrm{bp}} \to \mathrm{bp}$ in Σ_g. Then we can view

$$\pi_1\big(M_{g,\varphi},\, [\widetilde{\mathrm{bp}}, 0]\big) \cong \pi_1\big(\Sigma_g,\, \widetilde{\mathrm{bp}}\big) \rtimes_\varphi \mathbb{Z}.$$

From now on, although technically all loops in $M_{g,\varphi}$ and Σ_g have basepoint $\widetilde{\mathrm{bp}}$, we work with loops at the basepoint $\mathrm{bp} \in Q_0 \subset \Sigma_g$ and implicitly use δ to formally view them as loops at $\widetilde{\mathrm{bp}}$. For each $i \in Q_0$ let $i_s = \varphi^s(i)$ and let t_i be the unique path $\mathrm{bp} \to i$ in $\mathbb{C}\widetilde{Q'}$ comprised solely from arrows in T or their inverses. Let $E_{x,y}(z)$ denote the matrix with entries $z \in \mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}]$ in the the (x, y)-th position and zeroes everywhere else. Then define $\Psi : \mathrm{Jac}(\widetilde{Q'}, W') \to \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}])$ by sending

$$\begin{aligned} a : i \to j &\longmapsto E_{j,i}\big(([t_j^{-1} a\, t_i], 0)\big) \\ r_{i_u,s} : i_{u,s} \to i_{u,s+1} &\longmapsto E_{i_{u,s+1}, i_{u,s}}\big(([\varphi^{-1}(t^{-1}_{i_{u,s+1}})\, t_{i_{u,s}}], -1)\big) \end{aligned}$$

where $a \in Q'_1$ is a generating arrow, $r_{i_u,s} \in Q'_1$ is an isomorphism arrow, $[l] \in \pi_1(\Sigma_g, \widetilde{\mathrm{bp}})$ denotes the class of the loop $l \in \Sigma_g$ and we view paths $p \in \mathbb{C}Q$ as paths in Σ_g via the natural inclusion $Q \hookrightarrow \Sigma_g$.

Remark 6.3.14. It is not immediate that Ψ as given above is well-defined.

Remark 6.3.15. The assignment of an arrow $a : i \to j$ to $t_j^{-1} a\, t_i$ in the above homomorphism is equivalent to the contraction of the tree T seen in [[14] Proposition 4.2].

Remark 6.3.16. $\varphi^{-1}(t^{-1}_{i_{q,p+1}})\, t_{i_{q,p}}$ is not a loop in Σ_g but in fact is a path $\mathrm{bp} \to \varphi^{-1}(\mathrm{bp})$. So, first fixing some path $\gamma : \mathrm{bp} \to \varphi(\mathrm{bp})$ in Σ_g, what we actually mean by $[\varphi^{-1}(t_{i_{q,p+1}}^{-1})t_{i_{q,p}}] \in \pi_1(\Sigma_g, \mathrm{bp})$ is the class of the loop $\varphi^{-1}(\gamma) \circ \varphi^{-1}(t_{i_{q,p+1}}^{-1})t_{i_{q,p}}$. Since $\mathrm{bp} \in Q_0$ we take $\gamma = t_{\varphi(\mathrm{bp})}$. In a similar vein when multiplying in the semi-direct product we technically should write for $k > 0$

$$([\beta], m) \cdot ([\alpha], k) = \big([\gamma^{-1}\,\varphi(\gamma^{-1}) \cdots \varphi^{k-1}(\gamma^{-1})\,\varphi^k(\beta)\,\varphi^{k-1}(\gamma) \cdots \varphi(\gamma)\,\gamma\,\alpha],\, k+m\big)$$

and for $k < 0$

$$([\beta], m) \cdot ([\alpha], k) = \big([\varphi^{-1}(\gamma) \cdots \varphi^{k}(\gamma)\,\varphi^k(\beta)\,\varphi^{k}(\gamma^{-1}) \cdots \varphi^{-1}(\gamma^{-1})\alpha],\, k+m\big)$$

in order to ensure we are multiplying loops at bp. However it will turn out that in the multiplications we present later these intermediary paths, that are needed to correct for the basepoints of the loops involved, will mostly all cancel and so we omit them from the proceeding discussion to aid in notational clarity. This would not be necessary if we worked with the φ-invariant basepoint $\widetilde{\mathrm{bp}}$, but it must be done to allow us to work within the quiver Q'.

Note that Ψ sends

$$r_{i_{u,s}}^{-1} \longmapsto E_{i_{u,s}, i_{u,s+1}}\big(([\varphi(t_{i_{u,s}}^{-1})\, t_{i_{u,s+1}}], 1)\big).$$

We can now prove the main theorem of this chapter.

Theorem 6.3.17. *Let Σ_g be a Riemann surface of genus g and let φ be an orientation-preserving automorphism of Σ_g of order n. Let Δ be a brane tiling of Σ_g which is preserved under φ such that the size of the order of each vertex in $Q = Q_\Delta$ is n. Choose generating arrows from Q and isomorphism arrows to construct the quiver Q' as before, such that the potential W' is homogeneous of degree n and does not contain both an isomorphism arrow $r_{i_u,t}$ and its inverse $r_{i_u,t}^{-1}$. Then the homomorphism of algebras* (6.9)

$$\mathrm{Jac}(\widetilde{Q'}, W') \xrightarrow[\sim]{\Psi} \mathrm{Mat}_{m\times m}\big(\mathbb{C}[\pi_1(M_{g,\varphi})]\big)$$

is well-defined and an isomorphism.

Proof. Using the definition of Ψ we have that

$$\Psi \circ \widetilde{\xi}(p) = E_{y,x}\big(([t_y^{-1} p\, t_x], -\deg(\widetilde{\xi}(p)))\big) \tag{6.10}$$

for any path $p : x \to y \in \mathrm{Jac}(\widetilde{Q}, W)$. Indeed for an arrow $b \in Q_1$ we have that $b = \varphi^l(a)$ for some generating arrow $a \in Q_1$ and $l \in \{0, \ldots, n-1\}$, and hence we can write $\widetilde{\xi}(b)$

as equal to

$$
\begin{aligned}
& s_1^{-1}\dots s_k^{-1}\,a\,r_k\dots r_1 \\
\text{or}\quad & s_{l-k}\dots s_1\,a\,r_k\dots r_1 \\
\text{or}\quad & s_k\dots s_1\,a\,r_1^{-1}\dots r_k^{-1} \\
\text{or}\quad & s_1^{-1}\dots s_{l-k}^{-1}\,a\,r_1^{-1}\dots r_k^{-1}
\end{aligned}
$$

for isomorphism arrows $r_i : x_{i-1} \to x_i$, $s_i : y_{i-1} \to y_i$, and some $0 \le k \le l$. In the first of these cases we get that $a : x_k \to y_k$ and $b : x_0 \to y_0$, and therefore

$$
\begin{aligned}
\Psi(\widetilde{\xi}(b)) &= \Psi(s_1^{-1}\dots s_k^{-1}\,a\,r_k\dots r_1) \\
&= E_{y_0,y_1}\big(([\varphi(t_{y_0}^{-1})t_{y_1}],1)\big)\ \cdots\ E_{y_{k-1},y_k}\big(([\varphi(t_{y_{k-1}}^{-1})t_{y_k}],1)\big)\ E_{y_k,x_k}\big(([t_{y_k}^{-1}a\,t_{x_k}],0)\big) \\
&\quad\ E_{x_k,x_{k-1}}\big(([\varphi^{-1}(t_{x_k}^{-1})t_{x_{k-1}}],-1)\big)\ \cdots\ E_{x_1,x_0}\big(([\varphi^{-1}(t_{x_1}^{-1})t_{x_0}],-1)\big) \\
&= E_{y_0,x_0}\big(([\varphi(t_{y_0}^{-1})t_{y_1}],1)\cdot\ldots\cdot([\varphi(t_{y_{k-1}}^{-1})t_{y_k}],1)\cdot([t_{y_k}^{-1}a\,t_{x_k}],0)\cdot \\
&\qquad\qquad \cdot([\varphi^{-1}(t_{x_k}^{-1})t_{x_{k-1}}],-1)\cdot\ldots\cdot([\varphi^{-1}(t_{x_1}^{-1})t_{x_0}],-1)\big) \\
&= E_{y_0,x_0}\big(([t_{y_0}^{-1}\varphi^k(a)\,t_{x_0}],0)\big) \\
&= E_{y_0,x_0}\big(([t_{y_0}^{-1}b\,t_{x_0}],-\deg(\widetilde{\xi}(b)))\big)
\end{aligned}
$$

as required. The other cases follow in a similar manner and since we have shown the statement for all arrows in Q it readily extends to all paths in $\mathbb{C}\widetilde{Q}$.

By Lemma 6.3.11, to show that Ψ is well-defined it suffices to check that

$$\Psi\left(\frac{\partial W'}{\partial a}\right) = 0$$

for all generating arrows $a \in Q_1$. From the proof of Lemma 6.3.12 it follows that

$$\frac{\partial W'}{\partial a} = n\,\xi\left(\frac{\partial W}{\partial a}\right)$$

and hence we just need to show that $\Psi(\widetilde{\xi}(\partial W/\partial a)) = 0$. Writing $\partial W/\partial a = p - q$ for $a : i \to j$, (6.10) tells us that

$$
\begin{aligned}
\Psi\left(\widetilde{\xi}\left(\frac{\partial W}{\partial a}\right)\right) &= E_{i,j}\big(([t_i^{-1}pt_j^{-1}],-\deg(\widetilde{\xi}(p))\big) - E_{i,j}\big(([t_i^{-1}qt_j^{-1}],-\deg(\widetilde{\xi}(q))\big) \\
&= E_{i,j}\big(([t_i^{-1}pt_j^{-1}],-\deg(\widetilde{\xi}(p)) - ([t_i^{-1}qt_j^{-1}],-\deg(\widetilde{\xi}(q))\big).
\end{aligned}
$$

But [[14] Proposition 4.2 and Proposition 5.4] implies that $[p] = [q] \in \pi_1(\Sigma_g)$ and since W' is homogeneous $\deg(\widetilde{\xi}(p)) = \deg(\widetilde{\xi}(q))$, hence

$$E_{i,j}\big(([t_i^{-1}pt_j^{-1}],-\deg(\widetilde{\xi}(p)) - ([t_i^{-1}qt_j^{-1}],-\deg(\widetilde{\xi}(q))\big) = E_{i,j}\big(0\big) = 0$$

as required.

We now move onto showing that Ψ is an isomorphism. For surjectivity we choose $\mathrm{bp} \in Q'_0$ such that there exists an isomorphism arrow $r : \mathrm{bp} \to \varphi(\mathrm{bp})$ and recall we take $\gamma = t_{\varphi(\mathrm{bp})}$. Then r has image under Ψ given by

$$\begin{aligned} E_{\varphi(\mathrm{bp}),\mathrm{bp}}\big(([\varphi^{-1}(\gamma) \circ \varphi^{-1}(t^{-1}_{\varphi(\mathrm{bp})})], -1)\big) &= E_{\varphi(\mathrm{bp}),\mathrm{bp}}\big(([\varphi^{-1}(t_{\varphi(\mathrm{bp})}) \circ \varphi^{-1}(t^{-1}_{\varphi(\mathrm{bp})})], -1) \\ &= E_{\varphi(\mathrm{bp}),\mathrm{bp}}\big(([1], -1)\big) \end{aligned}$$

where $[1] \in \pi_1(\Sigma_g)$ is the class of the constant path. For $i, j \in Q'_0$ consider the path

$$p_{ij} = \widetilde{\xi}(t_i\, t^{-1}_{\varphi(\mathrm{bp})})\, r\, \widetilde{\xi}(t_j^{-1})$$

in $\widetilde{Q}'$. Since $\deg(t) = 0$ for all $t \in T$ we have that $\deg(t_i) = 0$ for all $i \in Q'_0$, and so by (6.10) applying Ψ to p_{ij} gives

$$\begin{aligned} &E_{i,\varphi(\mathrm{bp})}\big(([t_i^{-1} t_i t^{-1}_{\varphi(\mathrm{bp})} t_{\varphi(\mathrm{bp})}], 0)\big) \cdot E_{\varphi(\mathrm{bp}),\mathrm{bp}}\big(([1], -1)\big) \cdot E_{\mathrm{bp},j}\big(([t^{-1}_{\mathrm{bp}} t_j^{-1} t_j], 0)\big) \\ &= E_{i,j}\big(([\varphi^{-1}(\gamma) \cdot \varphi^{-1}(t_i^{-1} t_i t^{-1}_{\varphi(\mathrm{bp})} t_{\varphi(\mathrm{bp})}) \cdot \varphi^{-1}(\gamma^{-1}) \cdot t^{-1}_{\mathrm{bp}} t_j^{-1} t_j], -1)\big) \\ &= E_{i,j}\big(([1], -1)\big) \end{aligned}$$

because by definition $t_{\mathrm{bp}} = 1$. In a similar way replacing the r with r^{-1} gives the matrices $E_{i,j}\big(([1], 1)\big)$ in the image of Ψ too. Hence these paths, varying over all vertices i, j in Q', will generate the $\mathbb{Z}$ summand in the semi-direct product $\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}$ for all coordinates (i, j). Next let

$$\pi : \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}]) \to \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g)])$$

be the projection map, and note that π is not an algebra homomorphism but becomes one when we restrict to the subalgebra $\mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi n\mathbb{Z}]) = \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \times n\mathbb{Z}])$, since $\varphi^n = \mathrm{id}_{\Sigma_g}$. Then (6.10) tells us that the composition

$$\mathrm{Jac}(\widetilde{Q}, W) \xrightarrow{\widetilde{\xi}} \mathrm{Jac}(\widetilde{Q}', W') \xrightarrow{\Psi} \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}]) \xrightarrow{\pi} \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g)])$$

is the homomorphism that sends a path $p : x \to y \in \mathrm{Jac}(\widetilde{Q}, W)$ to

$$E_{y,x}([t_y^{-1} p\, t_x]) \in \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g)])$$

since $\mathrm{Im}(\Psi \circ \widetilde{\xi}) \subset \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi n\mathbb{Z}])$. This homomorphism is surjective by Theorem 6.2.7. Therefore Ψ is surjective.

To show injectivity suppose $\Psi(\alpha) = \Psi(\beta)$ for paths $\alpha, \beta : i \to j$ in $\mathbb{C}Q'$. Then

$$\Psi(\beta^{-1}\alpha) = E_{i,i}\big(([1], 0)\big).$$

Let $\pi_{\mathbb{Z}} : \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}]) \to \mathrm{Mat}_{m\times m}(\mathbb{C}[\mathbb{Z}])$ then from the definition of Ψ it is clear that the composition $\pi_{\mathbb{Z}} \circ \Psi$ sends a path $p : i \to j$ in $\mathbb{C}\widetilde{Q}'$ to

$$E_{j,i}(-\deg(p)).$$

Hence $\deg(\beta^{-1}\alpha) = 0$ and so by Lemma 6.3.8 $\beta^{-1}\alpha$ lies in the image of $\widetilde{\xi}$. So we can view $\beta^{-1}\alpha$ as a path in $\mathrm{Jac}(\widetilde{Q}, W)$ and so by [[12] Lemma 2.7] we have that $\beta^{-1}\alpha = c_v^d$ for some minimal cycle c_v around a vertex $v \in \Delta$ and some $d \in \mathbb{Z}$. Since $\deg(\widetilde{\xi}(c_v)) = n$ from our assumption that the potential W' is homogeneous of degree n, this implies that $\pi_{\mathbb{Z}}(\Psi(\widetilde{\xi}(c_v)) \neq 0$. Therefore d must be equal to 0 and hence $\beta^{-1}\alpha = c_v^0 = e_i$ the constant path at the vertex $i \in Q_0'$, and so $\alpha = \beta$ in $\mathbb{C}\widetilde{Q}'$. □

Example 6.3.18. We apply the homomorphism Ψ to our running example of the genus 2 surface with φ the rotation by 180° from Example 6.3.5, using the brane tiling given in Example 6.1.2.

Note that a finite presentation for the algebra $\mathbb{C}[\pi_1(\Sigma_2) \rtimes_\varphi \mathbb{Z}]$ is given by the generators $\{x, y, z\}$ and the relations

$$\begin{aligned} &xyx^{-1}y^{-1}z^{-1}xyx^{-1}y^{-1}z = 1 \\ &xz^2 = z^2x \\ &yz^2 = z^2y. \end{aligned} \tag{6.11}$$

We saw from (6.1) that for our brane tiling and that particular choice of generators and isomorphism arrows we get the potential

$$W' = abreabre + 2rdrc - 2ardbrc - rere.$$

This satisfies the conditions that W' does not contain both r and r^{-1} and that W' is homogeneous of degree $2 = \deg(\varphi)$ in the isomorphism arrow r. We choose the maximal tree $T = \{e\} \subset Q$ since $\xi(e) = e$ has degree 0. We have the following relations in $\mathrm{Jac}(\widetilde{Q}', W')$

$$\frac{\partial W'}{\partial a} = 2breabre - 2rdbrc \qquad \frac{\partial W'}{\partial b} = 2reabrea - 2rcard$$
$$\frac{\partial W'}{\partial c} = 2rdr - 2ardbr \qquad \frac{\partial W'}{\partial d} = 2rcr - 2brcar$$
$$\frac{\partial W'}{\partial e} = 2abreabr - 2rer$$

and recall from Lemma 6.3.11 b) that we do not need to consider $\partial W'/\partial r$.

Since the arrow e is in our maximal tree T, we can think of e as a means to go between the vertices in $\widetilde{Q}'$ and therefore as a means to go between the different coordinates in $\mathrm{Mat}_{2\times 2}(\mathbb{C}[\pi_1(\Sigma_2) \rtimes_\varphi \mathbb{Z}])$. Equivalently we can simply contract the arrow e in $\widetilde{Q}'$ giving a quiver with 1 vertex and with all the arrows becoming loops at this vertex. So to simplify things, in the above relations we set $e = 1$.

Then we have

$$\begin{aligned}\frac{\partial W'}{\partial a} &= 0\\ \implies brabr &= rdbrc\end{aligned}$$

which combined with $\partial W'/\partial e = 0$ gives

$$ardbrc = rr$$

and then combining with $\partial W'/\partial c = 0$ gives

$$\begin{aligned}rdrc &= rr\\ \implies d &= rc^{-1}r^{-1}.\end{aligned}\tag{6.12}$$

Additionally

$$\begin{aligned}\frac{\partial W'}{\partial b} &= 0\\ \implies b &= a^{-1}carda^{-1}r^{-1}\\ \implies b &= a^{-1}carrc^{-1}r^{-1}a^{-1}r^{-1}.\end{aligned}\tag{6.13}$$

Using these substitutions we get

$$\begin{aligned}\frac{\partial W'}{\partial e} &= 0\\ \implies carrc^{-1}r^{-1}a^{-1}carrc^{-1}r^{-1}a^{-1} &= rr\end{aligned}$$

and

$$\begin{aligned}\frac{\partial W'}{\partial a} &= 0\\ \implies a^{-1}carrc^{-1}r^{-1}a^{-1}carrc^{-1}r^{-1}a^{-1} &= rrc^{-1}r^{-1}a^{-1}carrc^{-1}r^{-1}a^{-1}c\end{aligned}$$

which combined give

$$\begin{aligned}a^{-1}rr &= a^{-1}c^{-1}rrc\\ \implies crr &= rrc.\end{aligned}\tag{6.14}$$

Then combining $\partial W'/\partial b = 0$ and $\partial W'/\partial d = 0$ gives

$$\begin{aligned}rabra &= b^{-1}rcrd\\ \implies brabra &= rcrd\\ \implies a^{-1}carrrc^{-1}r^{-1}a^{-1}carrc^{-1}r^{-1} &= rcrrc^{-1}r^{-1}\end{aligned}$$

and then using $rrc = crr$ implies that

$$a^{-1}carrrc^{-1}r^{-1}a^{-1}carrc^{-1}r^{-1} = rr.$$

Combining this with $\partial W'/\partial a = 0$ and $\partial W'/\partial e = 0$ gives

$$rra^{-1} = a^{-1}c^{-1}rrc$$

so again using $rrc = crr$ implies that

$$arr = rra. \tag{6.15}$$

Finally we have

$$\begin{aligned} &\frac{\partial W'}{\partial d} = 0 \\ \implies\ & rcr = a^{-1}carrrc^{-1}r^{-1}a^{-1}r^{-1}rcar \\ \implies\ & rc = a^{-1}carrrc^{-1}r^{-1}a^{-1}ca \\ \implies\ & 1 = a^{-1}carc^{-1}ra^{-1}cac^{-1}r^{-1}. \end{aligned} \tag{6.16}$$

again using $rrc = crr$ in the last step.

The relations (6.12) and (6.13) tell us that the generators b and d depend upon a, c, r, and the relations (6.14), (6.15), and (6.16) are exactly the relations given in (6.11). Hence the map $a^{-1} \mapsto x$, $c \mapsto y$, $r \mapsto z$ is an isomorphism. It is easy to check that the map Ψ described in Theorem 6.3.17 is this isomorphism; for example at the $(1,1)$-coordinate $\Psi(a) = ([a], 0)$, $\Psi(c) = ([e^{-1}c], 0)$ and $\Psi(r) = ([1], -1)$ and the loops $[a]$, $[e^{-1}c] \in \Sigma_2$ can be seen using Fig. 6.2 as the first pair of the standard generators of $\pi_1(\Sigma_2)$, i.e. x_1^{-1} and y_1, as required.

6.4 Calculating motivic DT invariants for $\mathbb{C}[\pi_1(M_{g,\varphi})]$

The overarching goal for us is to calculate the motivic DT invariants of the algebra $\mathbb{C}[\pi_1(M_{g,\varphi})]$. A Jacobi algebra (or potentially a superpotential) description of an algebra gives a more approachable way of doing this, as described in Chapter 5.

For a finitely generated algebra A, because we have an isomorphism of stacks

$$\mathrm{Rep}_d(A) \cong \mathrm{Rep}_{dm}\big(\mathrm{Mat}_{m\times m}(A)\big)$$

(see [[14] Proposition 5.5]) it follows from Theorem 6.3.17 and Proposition 2.2.5 that we get a description of $\mathrm{Rep}_d(\mathbb{C}[\pi_1(M_{g,\varphi})])$ as a critical locus, and thus it is straightforward to define its motivic DT partition function. Explicitly, if we let $\mathrm{Rep}_d(\widetilde{Q}')$ denote the stack of d-dimensional representations of the algebra $\mathbb{C}\widetilde{Q}'$ for $d = (d, \ldots, d) \in \mathbb{N}^{|Q_0|} =$

$\mathbb{N}^m$ and we let $\widetilde{g}_d = \mathrm{Tr}(W')_d : \mathrm{Rep}_d(\widetilde{Q}') \to \mathbb{C}$, then Theorem 6.3.17 implies that

$$\mathrm{Rep}_d(\mathbb{C}[\pi_1(M_{g,\varphi})]) \cong \mathrm{crit}(\widetilde{g}_d).$$

$M_d(\widetilde{Q}')$ the space of d-dimensional representations of $\mathbb{C}\widetilde{Q}'$ gives a smooth atlas of $\mathrm{Rep}_d(\widetilde{Q}')$ and therefore the motivic DT partition function of $\mathbb{C}[\pi_1(M_{g,\varphi})])$ is

$$\begin{aligned}\Phi_{\pi_1(M_{g,\varphi})}(t) &= \sum_{n=0}^{\infty} \Big[\mathrm{Rep}_d(\mathbb{C}[\pi_1(M_{g,\varphi})])\Big]_{\mathrm{vir}} t^n \\ &= \sum_{n=0}^{\infty} \mathbb{L}^{\frac{r}{2}}[G]^{-1} \int_{\mathrm{crit}(g_d)} [\phi_{g_d}]\, t^n\end{aligned}$$

where $G = \mathrm{GL}_d^m$, $r = \dim(G)$, and $g_d : M_d(\widetilde{Q}') \to \mathbb{C}$ is a lift of $\widetilde{g}_d$.

Hence the problem of calculating motivic DT partition function and therefore the motivic DT invariants for $\mathbb{C}[\pi_1(M_{g,\varphi})]$ comes down to calculating the motivic vanishing cycles $\int[\phi_{g_d}]$.

To approach this we utilise power structures from Section 5.2 and the powerful "motivic dimensional reduction" theorem Theorem 5.3.16. Unfortunately we can't apply Theorem 5.3.16 directly as the variety $M_d(\widetilde{Q}')$ cannot locally be written in the form $\mathbb{A}^r \times Z$ with a $\mathbb{G}_d$-action such that g_d is equivariant and the induced action on Z is trivial. To remedy this we present a conjecture that utilises power structures to overcome this.

Let B be a finitely generated algebra with potential W and let $A = \mathrm{Jac}(B, W)$ be its Jacobi algebra. Let $\widetilde{\omega} \in B$ be such that its image $\omega \in A$ is central. Then define two new partition functions $\Phi_A^{\omega-\mathrm{nilp}}(t)$ and $\Phi_A^{\omega-\mathrm{inv}}(t)$ for the substacks $\mathrm{Rep}^{\omega-\mathrm{nilp}}(A)$ of representations of A for which ω acts nilpotently and $\mathrm{Rep}^{\omega-\mathrm{inv}}(A)$ of representations of A for which ω acts invertibly, by

$$\begin{aligned}\Phi_A^{\omega-\mathrm{nilp}}(t) &= \sum_{d=0}^{\infty} \Big[\mathrm{Rep}_d(A)\Big]_{\mathrm{vir}}^{\omega-\mathrm{nilp}} t^d \\ \Phi_A^{\omega-\mathrm{inv}}(t) &= \sum_{d=0}^{\infty} \Big[\mathrm{Rep}_d(A)\Big]_{\mathrm{vir}}^{\omega-\mathrm{inv}} t^d\end{aligned}$$

where $\big[\mathrm{Rep}_d(A)\big]_{\mathrm{vir}}^{\omega-\mathrm{nilp}}$ (resp. $\big[\mathrm{Rep}_d(A)\big]_{\mathrm{vir}}^{\omega-\mathrm{inv}}$) denotes the pushforward to $K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathbb{C})$ of the pullback of $\big[\mathrm{Rep}_d(A)\big]_{\mathrm{relvir}}$ to $\mathrm{Rep}_d^{\omega-\mathrm{nilp}}(A)$ (resp. $\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A)$).

Remark 6.4.1. If we have a function $f : X \to \mathbb{C}$ with critical locus $Z = \mathrm{crit}(f)$ and an open subset $U \subset X$ then

$$[\phi_f]|_{Z\cap U} = [\phi_{f|_U}].$$

Then $\widetilde{\omega}$ acting invertibly on representations of B is an open condition and

$$\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A) = \mathrm{crit}(\mathrm{Tr}(W)_d) \cap \mathrm{Rep}_d^{\widetilde{\omega}-\mathrm{inv}}(B).$$

Hence defining the virtual motive

$$\left[\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A)\right]_{\mathrm{vir}} := \int_{\mathrm{crit}(\mathrm{Tr}(W)_d)} [\phi_{f_d^{\mathrm{inv}}}]$$

where $f_d^{\mathrm{inv}} = \mathrm{Tr}(W)_d|_{\mathrm{Rep}_d^{\widetilde{\omega}-\mathrm{inv}}(B)}$, we get that

$$\left[\mathrm{Rep}_d(A)\right]_{\mathrm{vir}}^{\omega-\mathrm{inv}} = \left[\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A)\right]_{\mathrm{vir}}.$$

If we let A_ω denote the localisation of A with respect to ω then it is clear that $\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A) \cong \mathrm{Rep}_d(A_\omega)$ hence

$$\left[\mathrm{Rep}_d(A)\right]_{\mathrm{vir}}^{\omega-\mathrm{inv}} = \left[\mathrm{Rep}_d^{\omega-\mathrm{inv}}(A)\right]_{\mathrm{vir}} = \left[\mathrm{Rep}_d(A_\omega)\right]_{\mathrm{vir}}$$

and so

$$\Phi_A^{\omega-\mathrm{inv}}(t) = \Phi_{A_\omega}(t). \tag{6.17}$$

We construct a new space of representations for our fundamental group algebras which will arise from a partially localised quiver algebra. As before consider a brane tiling Δ for a Riemann surface Σ_g with automorphism $\varphi : \Sigma_g \xrightarrow{\sim} \Sigma_g$ of order n, giving dual quiver $Q_\Delta = Q$ and potential $W_\Delta = W$. Choose generating arrows $a \in Q_1$ for the action of φ and choose isomorphism arrows $r_{i_u,t} : \varphi^t(i_u) \to \varphi^{t+1}(i_u)$ for all orbits of vertices and $t = 0, \ldots, n-2$ giving the quiver $Q^\#$ as explained in the proof of Lemma 6.3.4. Then define the algebra $\mathbb{C}\widehat{Q}'$ to be the localisation of $\mathbb{C}Q^\#$ with respect to all the generating arrows a (recall that $\mathbb{C}Q'$ was the localisation of $\mathbb{C}Q^\#$ with respect to the isomorphism arrows $r_{i_u,t}$). Since we assumed the potential W' does not contain any inverse arrows (more correctly it did not contain both $r_{i_u,t}$ and $r_{i_u,t}^{-1}$ so if it contained an inverse then we simply change the arrow $r_{i_u,t}$ in $Q^\#$ to its inverse $r_{i_u,t}^{-1}$) then W' also gives a potential on $\mathbb{C}\widehat{Q}'$.

Conjecture 6.4.2. *Let $B = \mathbb{C}\widehat{Q'}$ be the path algebra of $\widehat{Q}'$ with potential W' and let $A = \mathrm{Jac}(B, W')$ be the Jacobi algebra of (B, W'). Let $\omega \in A$ be a central element and* *consider the three partition functions*

$$\Phi_A(t) = \sum_{d=0}^{\infty} \Big[\mathrm{Rep}_d(A)\Big]_{\mathrm{vir}} t^d$$
$$\Phi_A^{\omega-\mathrm{nilp}}(t) = \sum_{d=0}^{\infty} \Big[\mathrm{Rep}_d(A)\Big]_{\mathrm{vir}}^{\omega-\mathrm{nilp}} t^d$$
$$\Phi_A^{\omega-\mathrm{inv}}(t) = \sum_{d=0}^{\infty} \Big[\mathrm{Rep}_d(A)\Big]_{\mathrm{vir}}^{\omega-\mathrm{inv}} t^d\,.$$

Then

$$\Phi_A(t) = \left(\Phi_A^{\omega-\mathrm{nilp}}(t)\right)^{\mathbb{L}}$$

and

$$\Phi_A^{\omega-\mathrm{inv}}(t) = \left(\Phi_A^{\omega-\mathrm{nilp}}(t)\right)^{\mathbb{L}-1}.$$

Since $\mathrm{Jac}(\widetilde{Q}', W')$ is a localisation of A (with respect to the isomorphism arrows $\{r_{i_u,t}\}$) we try find a central element $\omega \in A$ such that $A_\omega = \mathrm{Jac}(\widetilde{Q}', W')$. We expect to be able to do this because $\mathbb{C}[\pi_1(M_{g,\varphi})] \cong \mathbb{C}[\pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}]$ and so $\omega = ([1], n) \in \pi_1(\Sigma_g) \rtimes_\varphi \mathbb{Z}$ is central, then using Theorem 6.3.17 we have $\mathrm{Jac}(\widetilde{Q}', W') \cong \mathrm{Mat}_{m\times m}(\mathbb{C}[\pi_1(M_{g,\varphi})])$ so by construction A should have such elements. We can then apply Conjecture 6.4.2 to write $\Phi_{\mathrm{Jac}(\widetilde{Q}',W')}(t) = \Phi_{A_\omega}(t) = \Phi_A^{\omega-\mathrm{inv}}(t)$ in terms of $\Phi_A(t)$. We get that

$$\Phi_{\mathrm{Jac}(\widetilde{Q}',W')}(t) = \big(\Phi_A(t)\big)^{\frac{\mathbb{L}-1}{\mathbb{L}}}.$$

Then because Theorem 6.3.17 implies that

$$\Phi_{\mathbb{C}[\pi_1(M_{g,\varphi})]}(t^m) = \Phi_{\mathrm{Jac}(\widetilde{Q}',W')}(t)$$

where $m = |Q_0|$, it follows that

$$\Phi_{\mathbb{C}[\pi_1(M_{g,\varphi})]}(t^m) = \big(\Phi_A(t)\big)^{\frac{\mathbb{L}-1}{\mathbb{L}}}\,. \tag{6.18}$$

Writing $d = (d, \dots, d) \in \mathbb{N}^m$ we have that $\mathrm{Rep}_d(B) \cong M_d(B)/G$ with

$$M_d(B) = \prod_{r_{i_u,t}} \mathrm{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{GL}_d^{|Q_1^{\#}|}$$

which is globally of the form $\mathbb{A}^r \times Z$. Let $f_d = \mathrm{Tr}(W')_d : M_d(B) \to \mathbb{C}$ and define a $\mathbb{G}_m$-action on $M_d(B)$ by scaling the non-invertible matrices only. This gives us a $\mathbb{G}_m$-action such that the induced action on Z is trivial, and because W' is homogeneous of degree n in the isomorphism arrows $\{r_{i_u,t}\}$ it follows that f_d is equivariant of degree n. Hence we can apply Theorem 5.3.16 to the virtual motives in the partition function $\Phi_A(t)$ reducing the problem to calculating the motives $[f_d^{-1}(0)]$ and $[f_d^{-1}(1)]$ for each

d.

To see this in action let us consider our running example, namely Σ_2 with φ being the rotation by 180° and with the brane tiling given in Example 6.1.2 and quivers Q, Q' and potentials W, W' given in Example 6.3.5 and Example 6.3.18. Let $B = \mathbb{C}\widehat{Q}'$ and $A = \mathrm{Jac}(\widehat{Q}', W')$ be the algebras described above. In particular the generators in B are $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}, r\}$.

Lemma 6.4.3. *The element $\omega = rere + erer \in A$ is central and $A_\omega = \mathrm{Jac}(\widetilde{Q'}, W')$.*

Proof. We first show that $A_\omega = \mathrm{Jac}(\widetilde{Q'}, W')$. It suffices to show that r^{-1} exists in A_ω. And so because

$$er\omega^{-1}e = er(e^{-1}r^{-1}e^{-1}r^{-1} + r^{-1}e^{-1}r^{-1}e^{-1})e = err^{-1}e^{-1}r^{-1}e^{-1}e = r^{-1}$$

and e, r, ω^{-1} are all elements of A_ω, then $r^{-1} \in A_\omega$.

Then to show $\omega \in A$ is central, since ω is not a 0-divisor it suffices to show that $\omega \in A_\omega$ is central. Now

$$A_\omega = \mathrm{Jac}(\widetilde{Q'}, W') \underset{\sim}{\xrightarrow{\Psi}} \mathrm{Mat}_{2\times 2}(\mathbb{C}[\pi_1(M_{2,\varphi})])$$

where, viewing $\pi_1(M_{2,\varphi}) \cong \pi_1(\Sigma_2) \rtimes_\varphi \mathbb{Z}$, ω is sent under this isomorphism to the matrix

$$\omega' = \begin{pmatrix} ([1], -2) & 0 \\ 0 & ([1], -2) \end{pmatrix}.$$

Since $\varphi^2 = \varphi^{-2} = \mathrm{id}_{\Sigma_2}$ we have for $([\alpha], m) \in \pi_1(\Sigma_2) \rtimes_\varphi \mathbb{Z}$

$$([\alpha], m) \cdot ([1], -2) = ([\varphi^{-2}(\alpha)], m - 2) = ([\alpha], -2 + m) = ([1], -2) \cdot ([\alpha], m).$$

Hence ω' is central and therefore so is ω. $\square$

It follows that we can apply the conjecture in this case for the particular choice of $\omega = rere + erer$. We have that $\mathrm{Rep}_{(d,d)}(B) \cong \big(\mathrm{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{GL}_d^5\big)/\mathrm{GL}_d^2$ and

$\operatorname{Rep}_{(d,d)}(A) \cong \operatorname{crit}(\widetilde{f}_d)$ where $\widetilde{f}_d = \operatorname{Tr}(W')_d : \operatorname{Rep}_{(d,d)}(B) \to \mathbb{C}$. Therefore

$$\begin{aligned}
\Phi_{\mathbb{C}[\pi_1(M_{2,\varphi})]}(t^2) &= \sum_{d=0}^{\infty} \Big[\operatorname{Rep}_d(\mathbb{C}[\pi_1(M_{2,\varphi})])\Big]_{\mathrm{vir}} t^{2d} \\
&= \sum_{d=0}^{\infty} \Big[\operatorname{Rep}_{2d}(A_\omega)\Big]_{\mathrm{vir}} t^{2d} \\
&= \sum_{d=0}^{\infty} \Big[\operatorname{Rep}_{(d,d)}(A_\omega)\Big]_{\mathrm{vir}} t^{2d} \\
&= \sum_{d=0}^{\infty} \Big[\operatorname{Rep}_{(d,d)}(A)\Big]_{\mathrm{vir}}^{\omega-\mathrm{inv}} t^{2d} \\
&= \Phi_A^{\omega-\mathrm{inv}}(t)
\end{aligned}$$

where we take the series in the variable t^2 because a d-dimensional representation of $\mathbb{C}[\pi_1(M_{2,\varphi})]$ corresponds to a (d,d)-dimensional representation of the Jacobi algebra $\operatorname{Jac}(\widetilde{Q'}, W)$ which has two vertices. So by Conjecture 6.4.2

$$\begin{aligned}
\Phi_{\mathbb{C}[\pi_1(M_{2,\varphi})]}(t^2) &= (\Phi_A(t))^{\frac{\mathbb{L}-1}{\mathbb{L}}} \\
&= \left(\Phi_A(t)^{\mathbb{L}^{-1}}\right)^{[\mathbb{C}^*]} \\
&= \Phi_A(\mathbb{L}^{-1}t)^{[\mathbb{C}^*]} \\
&= \Phi_A(t)^{[\mathbb{C}^*]}|_{t\mapsto \mathbb{L}^{-1}t}.
\end{aligned}$$

We can then apply Theorem 5.3.16 to find the motivic vanishing cycles for A because our regular function $\widetilde{f}_d$ lifts to $f_d : \operatorname{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{GL}_d^5 \to \mathbb{C}$ where $\operatorname{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{GL}_d^5$ is of the form $\mathbb{A}^r \times Z$ and f_d is equivariant of degree 2 when $\mathbb{G}_m$ acts by scaling on $\operatorname{Mat}_{d\times d}(\mathbb{C})$. We get that

$$\begin{aligned}
\Big[\operatorname{Rep}_{(d,d)}(A)\Big]_{\mathrm{vir}} &= \mathbb{L}^{2d^2}[\mathrm{GL}_d]^{-2} \int_{\operatorname{crit}(f_d)} [\phi_{f_d}] \\
&= \mathbb{L}^{2d^2}[\mathrm{GL}_d]^{-2} \int_{\operatorname{crit}(f_d)} [\phi_{f_d}^{\mathrm{eq}}] \\
&= \mathbb{L}^{-d^2}[\mathrm{GL}_d]^{-2}\Big([f_d^{-1}(0)] - [f_d^{-1}(1)]\Big)
\end{aligned}$$

and so we can find the motivic DT partition function of $\mathbb{C}[\pi_1(M_{2,\varphi})]$ by studying the fibres of f_d over 0 and 1 for each $d \in \mathbb{N}$.

We end this chapter with some remarks on why Conjecture 6.4.2 should be true, taking inspiration from [[3] Section 2.4 and Proposition 2.6] and [[18] Section 3]. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a partition of d (i.e. $\sum_{i=1}^d \alpha_i \cdot i = d$), let $\omega \in A$ be a central element and let $\widetilde{\omega} \in B$ be a lift of ω. Denote by $\operatorname{Rep}_d^\alpha(B)$ (resp. $\operatorname{Rep}_d^\alpha(A)$) the closed substack of d-dimensional representations of B for which $\widetilde{\omega}$ has generalised eigenvalues of shape α (resp. the closed substack of d-dimensional representations of A for which ω has generalised eigenvalues of shape α). Also denote by $\mathrm{Rep}_d^\alpha(B)'$ (resp. $\mathrm{Rep}_d^\alpha(A)'$) the open substack of d-dimensional representations of B such that the generalised eigenvalues of $\widetilde{\omega}$ are distinct when split according to α (resp. the open substack of d-dimensional representations of A such that the generalised eigenvalues of ω are distinct when split according to α). Put another way we have

$$\mathrm{Rep}_d^\alpha(B)' = \bigsqcup_{\alpha' \leq \alpha} \mathrm{Rep}_d^{\alpha'}(B)$$
$$\mathrm{Rep}_d^\alpha(A)' = \bigsqcup_{\alpha' \leq \alpha} \mathrm{Rep}_d^{\alpha'}(A)$$

where $\alpha' \leq \alpha$ runs over all sub-partitions of α. We obtain stratifications

$$\mathrm{Rep}_d(B) = \bigsqcup_{\alpha \vdash d} \mathrm{Rep}_d^\alpha(B)$$
$$\mathrm{Rep}_d(A) = \bigsqcup_{\alpha \vdash d} \mathrm{Rep}_d^\alpha(A). \tag{6.19}$$

Fixing a presentation of B and of A, a d-dimensional representation ρ of A can be characterised by a vector space V of dimension d along with $d \times d$-matrices $\rho(a_i)$ for each of the finite number of generators $a_i \in A$ that act on V subject to the relations in A. Consider the generalised eigenspace decomposition of V with respect to ω

$$V = \bigoplus_{i=1}^r V_i$$

where

$$V_i = \mathrm{Im}\Big(\prod_{j \neq i} (\rho(\omega) - \lambda_j \mathrm{Id}_d)^{s_j}\Big)$$

with $\prod_j (x - \lambda_j)^{s_j}$ the characteristic polynomial of $\rho(\omega)$. Since $\omega \in A$ is central it follows that $\rho(a)$ respects this decomposition for any $a \in A$ i.e. $\rho(a)(V_i) \subset V_i$ for all i. Hence the matrix $\rho(a)$ is a block-diagonal matrix, and so for a partition $\alpha \vdash d$ we get an isomorphism

$$\mathrm{Rep}_d^\alpha(A) \cong Z \subset \prod_{i=1}^d \big(\mathrm{Rep}_i^{(0,\dots,0,1)}(A)\big)^{\alpha_i} / S_\alpha \tag{6.20}$$

where $(0, \dots 0, 1) \vdash i$ is the partition consisting of the whole of i and so $\mathrm{Rep}_i^{(0,\dots,0,1)}(A)$ is the stack of i-dimensional representations of A for which ω has a single generalised eigenvalue, $S_\alpha = \prod_{i=1}^a S_i^{\alpha_i}$ is a product of symmetric groups that acts on $\prod_{i=1}^d \big(\mathrm{Rep}_i^{(0,\dots,0,1)}(A)\big)^{\alpha_i}$ by permuting the factors, and Z is the substack for which the generalised eigenvalue of ω is distinct for each factor in the product.

We now make the following two assumptions. The first is that we have an isomorphism of stacks

$$\mathrm{Rep}_d^{\alpha}(A) \xrightarrow[\sim]{\xi} \left(\prod_{i=1}^{d} \left(\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)\right)^{\alpha_i} \times \left(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta \right) \right) / S_\alpha \tag{6.21}$$

where $\Delta \subset \prod_{i=1}^{d} \mathbb{A}^{\alpha_i}$ is the big diagonal i.e. the set of tuples (x_j) with $x_j \in \mathbb{A}^1$ such that $x_{j_1} = x_{j_2}$ for some $j_1 \neq j_2$, and S_α acts on both $\prod_{i=1}^{d} \left(\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)\right)^{\alpha_i}$ and $\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta$ by permuting the factors. For the second assumption consider the projections

$$\left(\prod_{i=1}^{d} \left(\mathrm{Rep}_i(B)\right)^{\alpha_i} \times \left(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta \right) \right) / S_\alpha \xrightarrow{\pi_B} \prod_{i=1}^{d} \left(\mathrm{Rep}_i(B)\right)^{\alpha_i} / S_\alpha$$

and

$$\left(\prod_{i=1}^{d} \left(\mathrm{Rep}_i(A)\right)^{\alpha_i} \times \left(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta \right) \right) / S_\alpha \xrightarrow{\pi_A} \prod_{i=1}^{d} \left(\mathrm{Rep}_i(A)\right)^{\alpha_i} / S_\alpha$$

and the natural inclusion

$$\prod_{i=1}^{d} \left(\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)\right)^{\alpha_i} / S_\alpha \xhookrightarrow{j'} \prod_{i=1}^{d} \left(\mathrm{Rep}_i(A)\right)^{\alpha_i} / S_\alpha.$$

Then we have the non-commuting triangle

$$\begin{tikzcd}
\mathrm{Rep}_d^{\alpha}(A) \arrow[r, hook, "\iota"] \arrow[d, "\xi", "\wr"'] & \prod_{i=1}^{d} \left(\mathrm{Rep}_i(A)\right)^{\alpha_i} / S_\alpha \\
Z_\alpha / S_\alpha \arrow[ur, "j"] &
\end{tikzcd} \tag{6.22}$$

where $V_\alpha = \prod_{i=1}^{d} \left(\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)\right)^{\alpha_i}$ and $Z_\alpha = V_\alpha \times \left(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta \right)$, and j is the composition $\pi_A \circ (j' \times \mathrm{id}_{\prod_i \mathbb{A}^{\alpha_i} \setminus \Delta})$. We assume that we have the equality of motives over $\mathrm{Spec}(\mathbb{C})$

$$\int_{\mathrm{Rep}_d^{\alpha}(A)} \iota^* [\phi_{\overline{\sum_i \alpha_i \mathrm{Tr}(W')_i}}] = \int_{\mathrm{Rep}_d^{\alpha}(A)} \xi^* j^* [\phi_{\overline{\sum_i \alpha_i \mathrm{Tr}(W')_i}}] \tag{6.23}$$

where the bar denotes that these functions are taken on the quotients modulo S_α. Note that we have

$$
\begin{aligned}
\xi^* j^*[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}] &= \xi^*(j' \times \mathrm{id})^* \pi_A^*[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}] \\
&= \xi^*(j' \times \mathrm{id})^* \mathbb{L}^{\frac{\sum_i \alpha_i}{2}}[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i \circ \pi_B}] \\
&= \xi^*\Big([\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}]|_{V_\alpha / S_\alpha} \cdot \Big[\Big(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta\Big)/S_\alpha\Big]\Big)
\end{aligned}
$$

where the second equality follows from Proposition 5.3.11 a), and the third equality follows from the motivic Thom-Sebastiani isomorphism Proposition 5.3.15 c) and Proposition 5.3.15 e). Hence the assumption (6.23) is equivalent to

$$
\int_{\mathrm{Rep}_d^\alpha(A)} \iota^*[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}] = \int_{V_\alpha/S_\alpha} \Big([\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}]|_{V_\alpha/S_\alpha}\Big) \cdot \Big[\Big(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta\Big)/S_\alpha\Big]. \quad (6.24)
$$

Consider the closed substack of $\mathrm{Rep}_d^\alpha(B)'$

$$
X_\alpha = \prod_{i=1}^{d} \Big(\mathrm{Rep}_i(B)\Big)^{\alpha_i}/S_\alpha \cap \mathrm{Rep}_d^\alpha(B)'.
$$

Then by using the following holomorphic Morse-Bott style lemma we have that locally around each point in $\mathrm{Rep}_d^\alpha(A)' \subset \mathrm{Rep}_d^\alpha(B)'$ the function $f_d^\alpha = \mathrm{Tr}(W')_d^\alpha : \mathrm{Rep}_d^\alpha(B)' \to \mathbb{C}$ can be written as

$$
f_d^\alpha = f_d^\alpha|_{X_\alpha} + \sum_j x_j^2 \quad (6.25)
$$

where j runs over all the coordinates that cut out X_α as a subspace of $\mathrm{Rep}_d^\alpha(B)'$, and recall for functions $f : X \to \mathbb{A}^1$ and $g : Y \to \mathbb{A}^1$ we write $f + g$ for the composition $X \times Y \xrightarrow{f \times g} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1$.

Lemma 6.4.4 ([39] Proposition 2.22)**.** *Let Y be a smooth complex variety of dimension d, $f : Y \to \mathbb{C}$ a regular function, and $X \subset Y$ a smooth subvariety of dimension m such that* $\mathrm{crit}(f) = \mathrm{crit}(f|_X)$*. Then analytically locally around any $x \in \mathrm{crit}(f)$ we may write*

$$
f = f|_X + \sum_{i=m+1}^{d} x_i^2
$$

for local coordinates $x_1, \ldots, x_d$.

We can apply this lemma since

$$\begin{aligned}
\operatorname{crit}(f_d^\alpha|_{X_\alpha}) &= \left(\prod_{i=1}^{d}\Big(\operatorname{Rep}_i(A)\Big)^{\alpha_i}\right)/S_\alpha \cap \operatorname{Rep}_d^\alpha(A)' \\
&= \operatorname{Rep}_d^\alpha(A)' \\
&= \operatorname{crit}(f_d^\alpha)
\end{aligned}$$

because $\operatorname{Rep}_d^\alpha(A)' \subset \prod_{i=1}^d \big(\operatorname{Rep}_i(A)\big)^{\alpha_i}/S_\alpha$ as $\omega \in A$ is central.

Now consider the Cartesian diagram

$$\begin{tikzcd}
\operatorname{Rep}_d^\alpha(B) \arrow[r, hook, "\widetilde{\iota}''_\alpha"] & \operatorname{Rep}_d^\alpha(B)' \arrow[r, hook, "\widetilde{\iota}'_\alpha"] & \operatorname{Rep}_d(B) \\
\operatorname{Rep}_d^\alpha(A) \arrow[u, hook] \arrow[r, hook, "\iota''_\alpha"] \arrow[rr, hook, bend right, "\iota_\alpha"'] & \operatorname{Rep}_d^\alpha(A)' \arrow[u, hook] \arrow[r, hook, "\iota'_\alpha"] & \operatorname{Rep}_d(A) \arrow[u, hook]
\end{tikzcd}$$

where recall $\widetilde{\iota}'_\alpha$ and ι'_α are open inclusions and $\widetilde{\iota}''_\alpha$, ι_α and ι''_α are closed inclusions. Also consider the Cartesian square

$$\begin{tikzcd}
X_\alpha \arrow[r, hook, "\widetilde{j}_\alpha"] & \prod_{i=1}^d \big(\operatorname{Rep}_i(B)\big)^{\alpha_i}/S_\alpha \\
\operatorname{Rep}_d^\alpha(A)' \arrow[u, hook] \arrow[r, hook, "j_\alpha"] & \prod_{i=1}^d \big(\operatorname{Rep}_i(A)\big)^{\alpha_i}/S_\alpha \arrow[u, hook]
\end{tikzcd}$$

where $\widetilde{j}_\alpha$ and j_α are open inclusions. Note that $j_\alpha \circ \iota''_\alpha = \iota$ from (6.22). Then using

(6.25) it follows, at least locally, that

$$
\begin{aligned}
[\mathrm{Rep}_d(A)]_{\mathrm{vir}} &= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha^{*}[\phi_{\mathrm{Tr}(W')_d}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*} \iota_\alpha'^{*}[\phi_{\mathrm{Tr}(W')_d}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}[\phi_{\mathrm{Tr}(W')_d \circ \widetilde{\iota}'_\alpha}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}[\phi_{\mathrm{Tr}(W')_d^{\alpha}}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}[\phi_{f_d^{\alpha}|_{X_\alpha} + \sum_j x_j^2}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}\Big([\phi_{f_d^{\alpha}|_{X_\alpha}}] \cdot [\phi_{\sum_j x_j^2}]\Big) \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}[\phi_{f_d^{\alpha}|_{X_\alpha}}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*}[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i \circ \widetilde{j}_\alpha}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota_\alpha''^{*} j_\alpha^{*}[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}] \\
&= \sum_{\alpha \vdash d} \int_{\mathrm{Rep}_d^{\alpha}(A)} \iota^{*}[\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}] \\
&= \sum_{\alpha \vdash d} \int_{V_\alpha / S_\alpha} \Big([\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')}_i}]|_{V_\alpha/S_\alpha}\Big) \cdot \Big[\Big(\prod_{i=1}^{d} \mathbb{A}^{\alpha_i} \setminus \Delta\Big)/S_\alpha\Big]. \qquad (6.26)
\end{aligned}
$$

where the first equality follows from the cut and paste relations and the stratification (6.19), the third equality from the fact that $\widetilde{\iota}'_\alpha$ and ι'_α are open and the square is Cartesian, the fifth equality from (6.25), the sixth equality from the motivic Thom-Sebastiani theorem, the seventh equality from the fact that the motivic vanishing cycle of a quadratic term $\int_{\mathbb{C}}[\phi_{x^2}]$ is equal to 1, the eight equality from the fact that the function $\overline{\mathrm{Tr}(W')}_d$ when restricted to block-diagonal matrices of the form α is equal to the function $\sum_{i=1}^{d} \alpha_i \overline{\mathrm{Tr}(W')}_i$, the ninth equality from the fact that $\widetilde{j}_\alpha$ and j_α are open and the square is Cartesian, and the final equality from (6.24).

Let $q_\alpha : K_0^{S_\alpha}(\mathrm{Var}/\mathbb{C})\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big] \to K_0(\mathrm{Var}/\mathbb{C})\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big]$ be the quotient map (5.2) from the ring of S_α-equivariant motives over $\mathrm{Spec}(\mathbb{C})$ to the ring of motives over $\mathrm{Spec}(\mathbb{C})$. Then if the function $g : Y \to \mathbb{A}^1$ is S_α-invariant and acts freely on Y [[4] Proposition 8.6] says that

$$q_\alpha \int_Y [\phi_g] = \int_{Y/S_\alpha} [\phi_{\overline{g}}]$$

where $\overline{g} : Y/S_\alpha \to \mathbb{A}^1$ is the induced function on the quotient. It follows that

$$\begin{aligned}
&\int_{V_\alpha/S_\alpha} \Big([\phi_{\sum_i \alpha_i \overline{\mathrm{Tr}(W')_i}}]|_{V_\alpha/S_\alpha} \Big) \cdot \Big[\Big(\prod_{i=1}^d \mathbb{A}^{\alpha_i} \setminus \Delta \Big) / S_\alpha \Big] = \\
&\qquad = q_\alpha \bigg(\int_{V_\alpha} \Big([\phi_{\sum_i \alpha_i \mathrm{Tr}(W')_i}]|_{V_\alpha} \Big) \cdot \Big[\prod_{i=1}^d \mathbb{A}^{\alpha_i} \setminus \Delta \Big] \bigg) \\
&\qquad = q_\alpha \Bigg(\prod_{i=1}^d \bigg(\int_{\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)} [\phi_{\mathrm{Tr}(W')_i}]|_{\omega-\mathrm{nilp}} \bigg)^{\alpha_i} \cdot \Big[\prod_{i=1}^d \mathbb{A}^{\alpha_i} \setminus \Delta \Big] \Bigg). \qquad (6.27)
\end{aligned}$$

Hence combining (6.26) and (6.27) gives us that locally

$$\begin{aligned}
[\mathrm{Rep}_d(A)]_{\mathrm{vir}} &= \sum_{\alpha \vdash d} q_\alpha \Bigg(\prod_{i=1}^d \bigg(\int_{\mathrm{Rep}_i^{\omega-\mathrm{nilp}}(A)} [\phi_{\mathrm{Tr}(W')_i}]|_{\omega-\mathrm{nilp}} \bigg)^{\alpha_i} \cdot \Big[\prod_{i=1}^d \mathbb{A}^{\alpha_i} \setminus \Delta \Big] \Bigg) \\
&= \sum_{\alpha \vdash d} q_\alpha \Bigg(\prod_{i=1}^d \Big([\mathrm{Rep}_i(A)]_{\mathrm{vir}}^{\omega-\mathrm{nilp}} \Big)^{\alpha_i} \cdot \Big[\prod_{i=1}^d \mathbb{A}^{\alpha_i} \setminus \Delta \Big] \Bigg). \qquad (6.28)
\end{aligned}$$

Since power structures are linked to λ-ring structures, and the λ-ring structure on $K_0^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})$ is the exotic one from Lemma 5.3.2 we additionally require that $\int [\phi_{\mathrm{Tr}(W')_i}]|_{\omega-\mathrm{nilp}}$ lies in the λ-subring

$$K_0(\mathrm{Var}/\mathbb{C})\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big] \subset K_0^{\hat{\mu}}(\mathrm{Var}/\mathbb{C})\big[[\mathrm{GL}_n]^{-1} : n \in \mathbb{N}\big].$$

Then if (6.28) can be upgraded to a global statement, from the explicit description of the power structure (5.3) for the standard λ-ring structure on $K_0(\mathrm{Var}/\mathbb{C})$, this tells us exactly that

$$\big(\Phi_A^{\omega-\mathrm{nilp}}(t)\big)^{\mathbb{L}} = \Phi_A(t)$$

and restricting to the ω-invertible locus implies that

$$\big(\Phi_A^{\omega-\mathrm{nilp}}(t)\big)^{\mathbb{L}-1} = \Phi_A^{\omega-\mathrm{inv}}(t)$$

as per Conjecture 6.4.2.

Chapter 7

2D Cohomological Hall Algebras

Instead of refining DT invariants motivically we can also refine them cohomologically. This means finding cohomological vector spaces whose dimensions are the numerical DT invariants. These vector spaces can often be endowed with a multiplication giving rise to *cohomological Hall algebras* or CoHAs. Studying these algebras can offer further insights to the DT theory of a space. For Jacobi algebras and when doing DT theory in general, these CoHAs will be 3-dimensional objects because we take the cohomology of sheaves in a 3-Calabi-Yau category (see Proposition 4.3.5 and Proposition 4.3.6). The dimensional reduction theorem Theorem 3.3.17 can then be used to turn a 3D CoHA of a sheaf of vanishing cycles into a 2D CoHA with $\mathbb{Q}$-coefficients.

Practically the underlying cohomology spaces on the 2D side are easier to compute as we do not have to worry about the vanishing cycle sheaf. However this comes at the cost of having a significantly more complicated definition for the multiplication (see [[60] Theorem 4.4]).

In this chapter we study the 2D CoHA for the character varieties $\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)])$ with the goal to compare two - a priori - different multiplications that arise from two different presentations of the fundamental group algebra $\mathbb{C}[\pi_1(\Sigma_g)]$. The first of these comes from the *standard presentation*

$$\mathbb{C}[\pi_1(\Sigma_g)] \cong \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle/(\lambda - 1)$$

where $\lambda = \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}$, and the second from 2D Jacobi algebra presentation given in Theorem 7.2.2. Whilst the first of these presentations is more natural and hence the multiplication it induces can therefore also be deemed more natural, the second presentation has an obvious 3D enhancement that is obtained via dimensional reduction. This allows us to relate the 2D CoHA from the standard presentation to DT theory. Also, coming from a quiver with potential, the second presentation has other benefits such as being able to apply the PBW isomorphism from [17].

This chapter and the new results contained within are adapted from the author's own work in [53].

7.1 The 2D CoHA multiplication

Building on Schiffmann and Vasserot's work from [61], and in the same vein as the appendix in [60], we shall describe a version of the 2D CoHA construction for

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_{c,G}(Z_n, \mathbb{Q})^\vee$$

the dual of the G-equivariant compactly supported cohomology of the spaces Z_n which can be written as the zero-loci of a specific type of function $f_n : X_n \to \mathbb{A}^m$ on smooth ambient spaces X_n.

Let us recall some notation from Chapter 3 for various morphisms and natural transformations of sheaves. All functors will be derived unless otherwise stated and when underived functors are necessary the same notation will be used as the derived versions, but we shall make it clear from the context which we mean.

So let X and Y be varieties over $\mathbb{C}$ and let $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ denote the derived categories of sheaves of $\mathbb{Q}$-vector spaces on X and Y respectively. For a map $f : X \to Y$ write

$$\eta^f : \mathrm{id}_{\mathrm{D}(Y)} \to f_* f^*$$
$$\sigma^f : f^* f_* \to \mathrm{id}_{\mathrm{D}(X)}$$

and

$$\nu^f : f_! f^! \to \mathrm{id}_{\mathrm{D}(Y)}$$
$$\theta^f : \mathrm{id}_{\mathrm{D}(X)} \to f^! f_!$$

for the canonical unit-counit pairs for the adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$ respectively. Let

$$\kappa^f : f^* \otimes \omega_{X/Y} \to f^!$$

where $\omega_{X/Y} := f^! \mathbb{Q}_Y$ is the dualising sheaf, and for the Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "\varphi"] \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd}$$

let

$$\epsilon^{\psi, f'} : \psi^* f'_! \xrightarrow{\sim} f_! \varphi^*$$

be the base-change natural isomorphism.

Notation. We fix some notation of common spaces and objects used throughout this chapter. We shall always work over the field of complex numbers $\mathbb{C}$. We denote by

$\mathrm{Mat}_{m\times n} := \mathrm{Mat}_{m\times n}(\mathbb{C})$	the space of all $m \times n$-matrices
$\mathrm{Mat}_n := \mathrm{Mat}_{n\times n}(\mathbb{C})$	the space of all $n \times n$ square matrices
$\mathrm{Mat}_{m,n} \subset \mathrm{Mat}_{m+n}$	the space of all $(m+n)$ square matrices whose lower-left $n \times m$-block is 0
$\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$	the space of all $n \times n$ invertible matrices
$\mathrm{GL}_{m,n} \subset \mathrm{GL}_{m+n}$	the space of all $(m+n)$ invertible matrices whose lower-left $n \times m$-block is 0.

Note that $\mathrm{GL}_{m,n}$ naturally acts on $\mathrm{Mat}_{m,n}$ by conjugation via the inclusions $\mathrm{GL}_{m,n} \subset \mathrm{GL}_{m+n}$ and $\mathrm{Mat}_{m,n} \subset \mathrm{Mat}_{m+n}$.

To begin the description of the 2D CoHA multiplication take smooth finitely generated $\mathbb{C}$-algebras A and B and an algebra homomorphism $\widehat{f} : B \to A$. Fix presentations of A and B

$$\begin{aligned} A &\cong \mathbb{C}\langle a_1 \dots a_k\rangle / I \\ B &\cong \mathbb{C}\langle b_1 \dots b_l\rangle / J \end{aligned} \tag{7.1}$$

and a vector space V of dimension n, then consider the spaces

$$\begin{aligned} M_n(A) &= \left\{ (R_i) \in \bigoplus_{i=1}^{k} \mathrm{End}(V) \;\middle|\; p(R_i) = 0 \ \text{ for all } p \in I \right\} \\ M_n(B) &= \left\{ (S_i) \in \bigoplus_{i=1}^{l} \mathrm{End}(V) \;\middle|\; q(S_i) = 0 \ \text{ for all } q \in J \right\} \end{aligned}$$

of representations with underlying vector space V of A and B respectively, along with their natural $\mathrm{GL}(V)$-actions. Let $\mathrm{Rep}_n(A)$ and $\mathrm{Rep}_n(B)$ denote the stack of representations with underlying vector space V of A and B respectively i.e.

$$\begin{aligned} \mathrm{Rep}_n(A) &\cong M_n(A)/\mathrm{GL}(V) \\ \mathrm{Rep}_n(B) &\cong M_n(B)/\mathrm{GL}(V). \end{aligned}$$

From now on take $V = \mathbb{C}^n$, hence $M_n(A)$ and $M_n(B)$ will be spaces of matrices. We obtain natural maps $f_n : M_n(A) \to M_n(B)$ induced from $\widehat{f}$ and so let $Z_n = f_n^{-1}(0)$, where $0 \in M_n(B)$ is the representation for which every element of B acts on $\mathbb{C}^n$ as zero.

We want to define a multiplication on the dual of the vector space of GL_n-equivariant compactly supported cohomology of Z_n, or equivalently on the dual of the compactly supported cohomology of the stack Z_n/GL_n

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_{c,\mathrm{GL}_n}(Z_n,\mathbb{Q})^\vee = \bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(Z_n/\mathrm{GL}_n,\mathbb{Q})^\vee.$$

Let $M_{m,n}(A)$ denote the space of short-exact sequences of representations of A

$$0 \to \rho' \to \rho \to \rho'' \to 0$$

such that $\dim(\rho') = m$ and $\dim(\rho'') = n$. Then $M_{m,n}(A)$ naturally embeds into $M_{m+n}(A)$. Let $Z_{m,n} = Z_{m+n} \cap M_{m,n}(A)$ and let $\mathrm{GL}_{m,n} \subset \mathrm{GL}_{m+n}$ denote those automorphisms that preserve the subspace $\mathbb{C}^m \subset \mathbb{C}^{m+n}$. Then $\mathrm{GL}_{m,n}$ acts naturally via conjugation on $M_{m,n}(A)$. Viewing a square matrix $R \in \mathrm{Mat}_{m+n}$ in block-form as

$$R = \begin{pmatrix} R^{(1)} & R^{(3)} \\ R^{(4)} & R^{(2)} \end{pmatrix}$$

where $R^{(1)}$ is an $m \times m$-matrix, $R^{(2)}$ is an $n \times n$-matrix, $R^{(3)}$ is an $m \times n$-matrix, and $R^{(4)}$ is an $n \times m$ matrix, let

$$\begin{aligned}
\pi^A_{m+n,m} &: M_{m+n}(A) \to M_m(A) \\
\pi^A_{m+n,n} &: M_{m+n}(A) \to M_n(A) \\
\pi^B_{m+n,m} &: M_{m+n}(B) \to M_m(B) \\
\pi^B_{m+n,n} &: M_{m+n}(B) \to M_n(B)
\end{aligned}$$

denote the projections

$$\begin{aligned}
(R_i) &\longmapsto (R_i^{(1)}) \\
(R_i) &\longmapsto (R_i^{(2)}) \\
(S_i) &\longmapsto (S_i^{(1)}) \\
(S_i) &\longmapsto (S_i^{(2)})
\end{aligned}$$

and we use the same notation for their restrictions to $M_{m,n}(A)$ and $M_{m,n}(B)$. Then let $Y_{m,n} \subset M_m(A) \times M_n(A) \times M_{m,n}(B)$ denote the space

$$Y_{m,n} = \Big\{(\rho',\rho'',\sigma) : f_m(\rho') = \pi^B_{m+n,m}(\sigma),\ f_n(\rho'') = \pi^B_{m+n,n}(\sigma)\Big\}.$$

We can draw the following diagram of $\mathrm{GL}_{m,n}$-equivariant maps

$$\begin{tikzcd}
M_{m+n}(A) \arrow[d, hook, "i_{m+n}"'] & M_{m,n}(A) \arrow[l, hook', "h"'] \arrow[r, "f"] \arrow[d, hook, "i_{m,n}"'] & Y_{m,n} \arrow[d, hook, "i"'] \\
Z_{m+n} & Z_{m,n} \arrow[l, hook', "\widetilde{h}"'] \arrow[r, "\widetilde{f}"] & Z_m \times Z_n
\end{tikzcd} \tag{7.2}$$

where h and i_{m+n} and $i_{m,n}$ are the natural inclusions, $i = (i_m, i_n, 0)$, the tilde denotes restriction to $Z_{m,n}$, and

$$f(\rho) = \big(\pi^A_{m+n,m}(\rho),\ \pi^A_{m+n,n}(\rho),\ f_{m+n}(\rho)\big).$$

Remark 7.1.1. From the assumption that A and B are smooth, every space on the top row of (7.2) is smooth. Also both squares in (7.2) are Cartesian; the left-hand square is clear, and for the right-hand square this is why we needed to define $Y_{m,n}$ as above instead of taking $M_m(A) \times M_n(A)$ as one might expect from the standard convolution product on (cohomological) Hall algebras.

Remark 7.1.2. When normally defining a cohomological Hall algebra one would usually try to use just the bottom line of (7.2) as the correspondence diagram and use the pullback along $\widetilde{h}$ and the pushforward along $\widetilde{f}$ to induce the multiplication. Unfortunately however, in our case this will not work because the spaces Z_n are not smooth. In particular this implies that $\widetilde{f}^!\mathbb{Q}_{Z_m\times Z_n}$ is not necessarily equal to $\mathbb{Q}_{Z_{m,n}}[2\dim(\widetilde{f})]$ and so there is no natural morphism we can take that would induce a pushforward along $\widetilde{f}$ to complete the definition of the multiplication. The following will describe how we alleviate this issue using the top row in (7.2) along with the facts that each space in the top row is smooth and the right-hand square is Cartesian.

Let $p : Z_{m+n}/\mathrm{GL}_{m+n} \to \mathrm{pt}$ and $p' : (Z_m \times Z_n)/(\mathrm{GL}_m \times \mathrm{GL}_n) \to \mathrm{pt}$ be the respective structure maps. Recall that compactly supported cohomology is defined by taking the compactly supported pushforward of the constant sheaf along the structure map. The multiplication on the dual of the equivariant compactly supported cohomology is then given as follows:

Firstly the inclusion $\mathrm{GL}_{m,n} \hookrightarrow \mathrm{GL}_{m+n}$ induces the quotient map $q : Z_{m+n}/\mathrm{GL}_{m,n} \to Z_{m+n}/\mathrm{GL}_{m+n}$ which gives the natural morphism of sheaves

$$\eta^q(\mathbb{Q}) : \mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m+n}} \to q_*q^*\mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m+n}}.$$

Because q is proper and so $q_* = q_!$ we have that $q_*q^*\mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m+n}} = q_!\mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m,n}}$ and so applying $p_!$ to $\eta^q(\mathbb{Q})$ gives the pullback morphism

$$q^\star : \mathrm{H}_{c,\mathrm{GL}_{m+n}}(Z_{m+n}, \mathbb{Q}) \to \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_{m+n}, \mathbb{Q}). \tag{7.3}$$

Next we obtain a natural pullback induced by $\widetilde{h}$. Because $\widetilde{h}$ is $\mathrm{GL}_{m,n}$-equivariant we get an induced map on quotient stacks $\overline{\overline{h}} : Z_{m,n}/\mathrm{GL}_{m,n} \to Z_{m+n}/\mathrm{GL}_{m,n}$. We then have the natural morphism

$$\eta^{\overline{\overline{h}}}(\mathbb{Q}) : \mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m,n}} \to \overline{\overline{h}}_* \overline{\overline{h}}^* \mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m,n}}$$

where again, because $\overline{\overline{h}}$ is a closed immersion and hence proper, we have $\overline{\overline{h}}_* = \overline{\overline{h}}_!$ and therefore $\overline{\overline{h}}_* \overline{\overline{h}}^* \mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m,n}} = \overline{\overline{h}}_! \mathbb{Q}_{Z_{m,n}/\mathrm{GL}_{m,n}}$. So after pushing forward along $p_!$ this gives a morphism

$$\overline{\overline{h}}^\star : \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_{m+n}, \mathbb{Q}) \to \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_{m,n}, \mathbb{Q}). \tag{7.4}$$

Then we have a natural pushforward induced by f (here we utilise the fact that in (7.2) the right-hand square is Cartesian and the top row consists of smooth spaces). As f is $\mathrm{GL}_{m,n}$-equivariant we get the induced map $\overline{f} : M_{m,n}(A)/\mathrm{GL}_{m,n} \to Y_{m,n}/\mathrm{GL}_{m,n}$ and so have the morphism of sheaves

$$\nu^{\overline{f}}(\mathbb{Q}) : \overline{f}_! \overline{f}^! \mathbb{Q}_{Y_{m,n}/\mathrm{GL}_{m,n}} \to \mathbb{Q}_{Y_{m,n}/\mathrm{GL}_{m,n}}.$$

Because $M_{m,n}(A)/\mathrm{GL}_{m,n}$ is smooth we have $\overline{f}_! \overline{f}^! \mathbb{Q}_{Y_{m,n}/\mathrm{GL}_{m,n}} = \overline{f}_! \mathbb{Q}_{M_{m,n}(A)/\mathrm{GL}_{m,n}}[2d]$ where $d = \dim(\overline{f}) = \dim(f)$. Restricting this morphism to $(Z_m \times Z_n)/\mathrm{GL}_{m,n}$ and precomposing with the base-change isomorphism $(\epsilon^{\overline{i},\overline{f}})^{-1} : \overline{\overline{f}}_! \,\overline{i}^*_{m,n} \xrightarrow{\sim} \overline{i}^* \,\overline{f}_!$ (where $\overline{i}$ and $\overline{i}_{m,n}$ denote the maps on the quotient stacks induced by i and $i_{m,n}$ respectively) gives

$$\begin{aligned}
&\overline{\overline{f}}_! \mathbb{Q}_{Z_{m,n}/\mathrm{GL}_{m,n}}[2d] = \overline{\overline{f}}_! \overline{i}^*_{m,n} \mathbb{Q}_{M_{m,n}(A)/\mathrm{GL}_{m,n}}[2d] \xrightarrow{(\epsilon^{\overline{i},\overline{f}}(\mathbb{Q})[2d])^{-1}} \overline{i}^* \overline{f}_! \mathbb{Q}_{M_{m,n}(A)/\mathrm{GL}_{m,n}}[2d] \\
&= \overline{i}^* \overline{f}_! \overline{f}^! \mathbb{Q}_{Y_{m,n}/\mathrm{GL}_{m,n}} \xrightarrow{\overline{i}^*(\nu^{\overline{f}}(\mathbb{Q}))} \overline{i}^* \mathbb{Q}_{Y_{m,n}/\mathrm{GL}_{m,n}} = \mathbb{Q}_{(Z_m \times Z_n)/\mathrm{GL}_{m,n}}
\end{aligned}$$

and so pushing forward along $p'_!$ gives us a morphism

$$\overline{f}_\star : \mathrm{H}^{+2d}_{c,\mathrm{GL}_{m,n}}(Z_{m,n}, \mathbb{Q}) \to \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_m \times Z_n, \mathbb{Q}). \tag{7.5}$$

Finally we have the isomorphisms

$$\begin{aligned}
\mathrm{H}^{+2mn}_{c,\mathrm{GL}_{m,n}}(Z_m \times Z_n, \mathbb{Q}) &\cong \mathrm{H}_{c,\mathrm{GL}_m \times \mathrm{GL}_n}(Z_m \times Z_n, \mathbb{Q}) \\
&\cong \mathrm{H}_{c,\mathrm{GL}_m}(Z_m, \mathbb{Q}) \otimes \mathrm{H}_{c,\mathrm{GL}_n}(Z_n, \mathbb{Q})
\end{aligned} \tag{7.6}$$

- the first induced by the affine fibration $(Z_m \times Z_n)/\mathrm{GL}_{m,n} \to (Z_m \times Z_n)/(\mathrm{GL}_m \times \mathrm{GL}_n)$

with fibre $\mathrm{Mat}_{m\times n} \cong \mathbb{A}^{mn}$ which comes from the affine fibration $\mathrm{GL}_{m,n} \to \mathrm{GL}_m \times \mathrm{GL}_n$, and the second isomorphism is the Künneth isomorphism for equivariant compactly supported cohomology.

Definition 7.1.3. The *(2D) cohomological Hall algebra* of the data $\widehat{f} : B \to A$ is the vector space

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_{c,\mathrm{GL}_n}(Z_n, \mathbb{Q})^\vee$$

where $Z_n = f_n^{-1}(0)$ and $f_n : M_n(A) \to M_n(B)$ is induced from $\widehat{f}$, with the multiplication being the dual of the composition of (7.3), (7.4) and (7.5) with the isomorphism (7.6)

$$(\overline{f}_\star \circ \overline{\widetilde{h}}^\star \circ q^\star)^\vee : \mathrm{H}_{c,\mathrm{GL}_m}(Z_m, \mathbb{Q})^\vee \otimes \mathrm{H}_{c,\mathrm{GL}_n}(Z_n, \mathbb{Q})^\vee \longrightarrow \mathrm{H}^{-2(d+mn)}_{c,\mathrm{GL}_{m+n}}(Z_{m+n}, \mathbb{Q})^\vee.$$

Remark 7.1.4. The diagram (7.2) depends upon our initial choices (7.1) of presentations of A and B, and hence so does this multiplication. This underpins the central question we are trying to answer- when do different presentations result in the same multiplication?

Remark 7.1.5. We have defined things above using sheaves on quotient stacks. Now if G is an algebraic group acting on a variety X then

$$X \to X/G$$

is an atlas for the quotient stack X/G, and the derived category of sheaves on X/G is equivalent to the derived category of G-equivariant sheaves on X. Because the morphisms $\overline{\widetilde{h}}^\star$ and $\overline{f}_\star$ above are induced from the equivariant maps $\widetilde{h}$ and f on the atlases of the quotient stacks, we in turn have induced morphisms $\widetilde{h}^\star$ and $f_\star$ in the derived category of equivariant sheaves on the atlases that are compatible with $\overline{\widetilde{h}}^\star$ and $\overline{f}_\star$. Hence it is enough to instead view everything at the level of equivariant sheaves on the atlases of the quotient stacks and check the commutativity of diagrams there. Therefore in the following sections we work only at the level of sheaves on varieties and do not work explicitly with the compactly supported cohomology of the quotient stacks/equivariant compactly supported cohomology. In particular instead of the morphism on equivariant compactly supported cohomology $\overline{\widetilde{h}}^\star : \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_{m+n}, \mathbb{Q}) \to \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_{m,n}, \mathbb{Q})$ defined above, we consider the morphism of sheaves

$$\widetilde{h}^\star := \eta^{\widetilde{h}}(\mathbb{Q}) : \mathbb{Q}_{Z_{m+n}} \to \widetilde{h}_* \widetilde{h}^* \mathbb{Q}_{Z_{m+n}} = \widetilde{h}_! \mathbb{Q}_{Z_{m,n}} \tag{7.7}$$

and instead of $\overline{f}_\star : \mathrm{H}^{+2d}_{c,\mathrm{GL}_{m,n}}(Z_{m,n}, \mathbb{Q}) \to \mathrm{H}_{c,\mathrm{GL}_{m,n}}(Z_m \times Z_n, \mathbb{Q})$ we consider $f_\star$ defined as the composition

$$\widetilde{f}_!\mathbb{Q}_{Z_{m,n}}[2d] = \widetilde{f}_! i^*_{m,n}\mathbb{Q}_{M_{m,n}(A)}[2d] \xrightarrow{\epsilon^{i,f}(\mathbb{Q}[2d])^{-1}} i^* f_!\mathbb{Q}_{M_{m,n}(A)}[2d] \tag{7.8}$$
$$= i^* f_! f^! \mathbb{Q}_{Y_{m,n}} \xrightarrow{i^*(\nu^f(\mathbb{Q}))} i^*\mathbb{Q}_{Y_{m,n}} = \mathbb{Q}_{Z_m \times Z_n}$$

Since the definition of the pushforward along f is slightly unorthodox we verify that the pushforward of a composition is the composition of pushforwards.

Lemma 7.1.6. *Suppose we have a diagram of Cartesian squares*

$$\begin{tikzcd}
X_1 \arrow[r, "f"] & X_2 \arrow[r, "g"] & X_3 \\
Z_1 \arrow[u, hook, "i_1"] \arrow[r, "\widetilde{f}"] & Z_2 \arrow[u, hook, "i_2"] \arrow[r, "\widetilde{g}"] & Z_3 \arrow[u, hook, "i_3"]
\end{tikzcd}$$

where X_1, X_2, X_3 are smooth, i_1, i_2, i_3 are closed immersions, and $d = \dim(f)$, $e = \dim(g)$. Then the morphism

$$(g \circ f)_\star : (\widetilde{g} \circ \widetilde{f})_!\mathbb{Q}_{Z_1}[2(d+e)] \to \mathbb{Q}_{Z_3}$$

equals the morphism

$$g_\star \circ \widetilde{g}_!(f_\star) : \widetilde{g}_!\widetilde{f}_!\mathbb{Q}_{Z_1}[2(d+e)] \to \mathbb{Q}_{Z_3}.$$

Proof. From the definition (7.8) we see that $g_\star \circ \widetilde{g}_!(f_\star)$ is the composition

$$(\widetilde{g} \circ \widetilde{f})_! i_1^* (g \circ f)^! \mathbb{Q}_{X_3} = \widetilde{g}_!\widetilde{f}_! i_1^* f^! g^! \mathbb{Q}_{X_3} \xrightarrow{\widetilde{g}_!(\epsilon^{i_2,f}(f^! g^! \mathbb{Q}))^{-1}} \widetilde{g}_! i_2^* f_! f^! g^! \mathbb{Q}_{X_3} \xrightarrow{\widetilde{g}_! i_2^*(\nu^f(g^!\mathbb{Q}))} \tag{7.9}$$
$$\widetilde{g}_! i_2^* g^! \mathbb{Q}_{X_3} \xrightarrow{\epsilon^{i_3,g}(g^!\mathbb{Q})^{-1}} i_3^* g_! g^! \mathbb{Q}_{X_3} \xrightarrow{i_3^*(\nu^g(\mathbb{Q}))} i_3^*\mathbb{Q}_{X_3} = \mathbb{Q}_{Z_3}.$$

Applying the natural transformation $\epsilon^{i_3,g}$ to the morphism

$$i_3^* g_! f_! f^! g^! \mathbb{Q}_{X_3} \xrightarrow{i_3^* g_!(\nu^f(g^!\mathbb{Q}))} i_3^* g_! g^! \mathbb{Q}_{X_3}$$

gives the commutative square

$$\begin{tikzcd}
i_3^* g_! f_! f^! g^! \mathbb{Q}_{X_3} \arrow[r, "i_3^* g_!(\nu^f(g^!\mathbb{Q}_{X_3}))"] \arrow[d, "\epsilon^{i_3,g}(f_! f^! g^! \mathbb{Q})"] & i_3^* g_! g^! \mathbb{Q}_{X_3} \arrow[d, "\epsilon^{i_3,g}(g^!\mathbb{Q})"] \\
\widetilde{g}_! i_2^* f_! f^! g^! \mathbb{Q}_{X_3} \arrow[r, "\widetilde{g}_! i_2^*(\nu^f(g^!\mathbb{Q}_{X_3}))"] & \widetilde{g}_! i_2^* g^! \mathbb{Q}_{X_3}
\end{tikzcd}$$

and so (7.9) is equal to

$$(\widetilde{g} \circ \widetilde{f})_! i_1^* (g \circ f)^! \mathbb{Q}_{X_3} = \widetilde{g}_! \widetilde{f}_! i_1^* f^! g^! \mathbb{Q}_{X_3} \xrightarrow{\widetilde{g}_!(\epsilon^{i_2,f}(f^! g^! \mathbb{Q}))^{-1}} \widetilde{g}_! i_2^* f_! f^! g^! \mathbb{Q}_{X_3} \xrightarrow{(\epsilon^{i_3,g}(f_! f^! g^! \mathbb{Q}))^{-1}} \tag{7.10}$$

$$i_3^* g_! f_! f^! g^! \mathbb{Q} \xrightarrow{i_3^* g_!(\nu^f(g^! \mathbb{Q}))} i_3^* g_! g^! \mathbb{Q}_{X_3} \xrightarrow{i_3^*(\nu^g(\mathbb{Q}))} i_3^* \mathbb{Q}_{X_3} = \mathbb{Q}_{Z_3}.$$

Now since

$$i_3^* g_! f_! = i_3^* (g \circ f)_! \underset{\sim}{\xrightarrow{\epsilon^{i_3, g\circ f}}} (\widetilde{g} \circ \widetilde{f})_! i_1^* = \widetilde{g}_! \widetilde{f}_! i_1^*$$

equals

$$i_3^* g_! f_! \underset{\sim}{\xrightarrow{\widetilde{g}_!(\epsilon^{i_2,f})}} \widetilde{g}_! i_2^* f_! \underset{\sim}{\xrightarrow{(\epsilon^{i_3,g}) f_!}} \widetilde{g}_! \widetilde{f}_! i_1^*$$

because these natural transformations are defined using the underlying maps f, g, i_1, i_2, i_3 directly, we have that (7.10) equals

$$\begin{aligned}(\widetilde{g} \circ \widetilde{f})_! i_1^* (g \circ f)^! \mathbb{Q}_{X_3} &\xrightarrow{\epsilon^{i_3, g\circ f}(f^! g^! \mathbb{Q}))^{-1}} i_3^* (g \circ f)_! f^! g^! \mathbb{Q}_{X_3} = i_3^* g_! f_! f^! g^! \mathbb{Q}_{X_3} \\ &\xrightarrow{i_3^* g_!(\nu^f(g^! \mathbb{Q}))} i_3^* g_! g^! \mathbb{Q}_{X_3} \xrightarrow{i_3^*(\nu^g(\mathbb{Q}))} i_3^* \mathbb{Q}_{X_3} = \mathbb{Q}_{Z_3}.\end{aligned} \tag{7.11}$$

Then as

$$g_! f_! f^! g^! = (g \circ f)_! (g \circ f)^! \xrightarrow{\nu^{g \circ f}} \mathrm{id}_{X_3}$$

equals

$$g_! f_! f^! g^! \xrightarrow{g_!(\nu^f) g^!} g_! g^! \xrightarrow{\nu^g} \mathrm{id}_{X_3}$$

we get (7.11) is equal to

$$(\widetilde{g} \circ \widetilde{f})_! i_1^* (g \circ f)^! \mathbb{Q}_{X_3} \xrightarrow{\epsilon^{i_3, g\circ f}(f^! g^! \mathbb{Q}))^{-1}} i_3^* (g \circ f)_! (g \circ f)^! \mathbb{Q}_{X_3} \xrightarrow{i_3^*(\nu^{g \circ f}(\mathbb{Q}))} i_3^* \mathbb{Q}_{X_3} = \mathbb{Q}_{Z_3} \tag{7.12}$$

which is exactly $(g \circ f)_\star$. $\square$

7.2 2D CoHA multiplications for the character variety

Definition 7.2.1. A *character variety* or *character stack* is the stack of n-dimensional representations $\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)])$ of the fundamental group algebra of a genus g Riemann surface Σ_g.

We seek to compare two 2D CoHAs for the character variety, i.e. to compare two multiplications induced by diagrams as in (7.2) on the dual of the compactly supported cohomology of the stack $\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)])$.

The first of these comes from the standard presentation of the fundamental group algebra of a Riemann surface

$$\mathbb{C}[\pi_1(\Sigma_g)] \cong \frac{\mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle}{(\lambda - 1)}$$

where $\lambda = \prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}$. This presentation allows us to define the space $M_n(\mathbb{C}[\pi_1(\Sigma_g)])$ of n-dimensional representations of $\mathbb{C}[\pi_1(\Sigma_g)]$ as the zero-locus of the function $\lambda_n : \mathrm{GL}_n^{2g} \to \mathrm{Mat}_n$ given by

$$(A_i, B_i) \longmapsto \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} - \mathrm{Id}_n$$

which has a natural GL_n-action via simultaneous conjugation. Then we have that

$$\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)]) \cong M_n(\mathbb{C}[\pi_1(\Sigma_g)])/\mathrm{GL}_n.$$

Recall that $\mathrm{Mat}_{m,n}$ is the space of all $(m+n)$-dimensional matrices whose lower-left $n \times m$-block is 0. We shall call such matrices m,n*-block matrices.* For such an m,n-block matrix R we write

$$R = \begin{pmatrix} R^{(1)} & R^{(3)} \\ 0 & R^{(2)} \end{pmatrix}$$

where $R^{(1)}$ is an $m \times m$-matrix, $R^{(2)}$ is an $n \times n$-matrix and $R^{(3)}$ is an $m \times n$-matrix. Define $Y_{m,n} \subset \mathrm{GL}_m^{2g} \times \mathrm{GL}_n^{2g} \times \mathrm{Mat}_{m,n}$ by

$$Y_{m,n} := \left\{ \Big((A_i, B_i), (C_i, D_i), R \Big) \;\Big|\; \big(R^{(1)}, R^{(2)} \big) = \big(\lambda_m(A_i, B_i), \lambda_n(C_i, D_i) \big) \right\}$$

then this space is defined analogously to the $Y_{m,n}$ from diagram (7.2). Let

$$V_n = \lambda_n^{-1}(0) = M_n(\mathbb{C}[\pi_1(\Sigma_g)]).$$

We get a new diagram, in the form of (7.2), of $\mathrm{GL}_{m,n}$-equivariant maps

$$\begin{tikzcd}
\mathrm{GL}_{m+n}^{2g} & \mathrm{GL}_{m,n}^{2g} \arrow[l, hook', "h"'] \arrow[r, "f"] & Y_{m,n} \\
V_{m+n} \arrow[u, hook, "j_{m+n}"] & V_{m,n} \arrow[l, hook', "\widetilde{h}"'] \arrow[r, "\widetilde{f}"] \arrow[u, hook, "j_{m,n}"] & V_m \times V_n \arrow[u, hook, "i"]
\end{tikzcd} \tag{7.13}$$

where $j_{m+n} : V_{m+n} \hookrightarrow \mathrm{GL}_{m+n}^{2g}$ is the natural inclusion and

$$f(A_i, B_i) = \Big((A_i^{(1)}, B_i^{(1)}),\ (A_i^{(2)}, B_i^{(2)}),\ \lambda_{m+n}(A_i, B_i)\Big).$$

The diagram (7.13) induces the first multiplication on $\bigoplus_n \mathrm{H}_c(\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)]), \mathbb{Q})^\vee$ as described in the previous section.

For our second multiplication let Δ be a brane tiling of the surface Σ_g. We saw in Section 6.1 that brane tilings provide a link between Jacobi algebras and fundamental group algebras on Riemann surfaces. However there was a problem with this because not all homotopic paths in the surface were identified in the Jacobi algebra. In Theorem 6.2.7 we saw one way of rectifying this issue by adding an extra S^1 to our manifold thereby giving an additional degree of freedom in the fundamental group algebra and thus allowing the two algebras to be isomorphic. However there is another way of solving this issue by using cuts. Recall a cut for a quiver with potential (Q, W) is a choice of arrows such that W is homogeneous of degree 1 with respect to the arrows in the cut.

So suppose there exists a cut E for (Q_Δ, W_Δ). Also recall from Section 6.1 that such a cut E corresponds exactly to a dimer D of the brane tiling Δ. For the fundamental group algebra of Σ_g, as we noted in Section 6.2, issues with an isomorphism to the Jacobi algebra only arise from the minimal cycles c_v (for v a vertex in Δ) not being trivial paths in the Jacobi algebra and so by taking a selection of arrows such that W_Δ is homogeneous of degree 1 in those arrows, when considering the 2D Jacobi algebra we end up removing the minimal cycles. This entirely gets rid of the problem of having non-trivial paths in the Jacobi algebra which are not null-homotopic when viewed as paths in Σ_g. This leads us to the following result from [14].

Theorem 7.2.2 ([14] Proposition 5.4). *Let Δ be a brane tiling of a Riemann surface Σ_g with localised path algebra $\mathbb{C}\widetilde{Q}_\Delta$ and potential W_Δ. Suppose there exists a cut E for*

(Q_Δ, W_Δ). *Then we have an isomorphism of algebras*

$$\mathrm{Jac}(\widetilde{Q}_\Delta, W_\Delta, E) \cong \mathrm{Mat}_{r\times r}(\mathbb{C}[\pi_1(\Sigma_g)])$$

where r is the number of vertices in Q_Δ.

To tidy up the fact that we must take matrices over the fundamental group algebra in Theorem 7.2.2 consider a maximal tree $T \subset Q_\Delta \setminus E$. Let Q denote the quiver with one vertex and arrow set indexed by elements in $Q_{\Delta,1} \setminus (E \cup T)$ (equivalently Q is obtained by contracting the tree T in the quiver $Q_\Delta \setminus E$). Then for all $i, j \in Q_{\Delta,0}$ let $t_{i,j} : i \to j$ be the unique path in the localised path algebra $\mathbb{C}(\widetilde{Q_\Delta \setminus E})$ comprised solely from arrows in T or their inverses. We can define an algebra homomorphism

$$\sigma : \mathbb{C}(\widetilde{Q_\Delta \setminus E}) \longrightarrow \mathrm{Mat}_r(\mathbb{C}\widetilde{Q})$$

by

$$\begin{aligned} \mathrm{const}_i &\longmapsto E_{i,i}(1) \\ a : i \to j \in Q_{\Delta,1} \setminus (E \cup T) &\longmapsto E_{j,i}(a) \\ t_{i,j} &\longmapsto E_{j,i}(1) \end{aligned}$$

where const_i is the constant path in $\mathbb{C}(\widetilde{Q_\Delta \setminus E})$ at the vertex $i \in Q_{\Delta,0}$, and $E_{i,j}(p)$ is the matrix with entries $p \in \mathbb{C}\widetilde{Q}$ in the (i,j)-th place and zeroes everywhere else. Let W be the potential on $Q_\Delta \setminus T$ obtained by removing all instances of the arrows that belong to T from the cycles in W_Δ. It is clear that E remains as a cut for $(Q_\Delta \setminus T, W)$. Then σ descends to a homomorphism

$$\overline{\sigma} : \mathrm{Jac}(\widetilde{Q}_\Delta, W_\Delta, E) \longrightarrow \mathrm{Mat}_r(\mathrm{Jac}(\widetilde{Q}, W, E))$$

since $\sigma(\partial W_\Delta / \partial a) = E_{i,j}(\partial W / \partial a)$.

Lemma 7.2.3. *The homomorphisms σ and $\overline{\sigma}$ are isomorphisms.*

Proof. We define a homomorphism $\sigma^{-1} : \mathrm{Mat}_r(\mathbb{C}\widetilde{Q}) \to \mathbb{C}(\widetilde{Q_\Delta \setminus E})$ by

$$E_{i,j}(a) \longmapsto t_{t(a),i}\, a\, t_{j,s(a)}.$$

For a path $p = a_m \dots a_1 \in \mathbb{C}\widetilde{Q}$, σ^{-1} will send $E_{i,j}(p)$ to

$$t_{t(a_m),i}\, a_m\, t_{t(a_{m-1}),s(a_m)}\, a_{m-1}\, t_{t(a_{m-2}),s(a_{m-1})} \dots t_{t(a_1),s(a_2)}\, a_1\, t_{j,s(a_1)}.$$

Hence σ^{-1} will descend to the 2D Jacobi algebras since

$$\sigma^{-1}(E_{i,j}(\partial W / \partial a)) = t_{s(a),i}\, (\partial W_\Delta / \partial a)\, t_{j,t(a)}.$$

It is clear that σ and σ^{-1} are inverses. $\square$

In particular Lemma 7.2.3 and Theorem 7.2.2 imply that

$$\mathrm{Jac}(\widetilde{Q}, W, E) \cong \mathbb{C}[\pi_1(\Sigma_g)]. \tag{7.14}$$

Example 7.2.4. For each genus g we have the following brane tiling from [[13] Section 7] containing four tiles. We have drawn this tiling for the genus 2 case in Figure 7.1. Informally one can think of it as being constructed by taking the surface and first cutting it in half down the middle through all the holes, leaving you with two cylinders with $g+1$ holes in each (including the two end holes). Then cut each of those cylinders in half, once again cutting through the holes, leaving you with two tiles per cylinder.

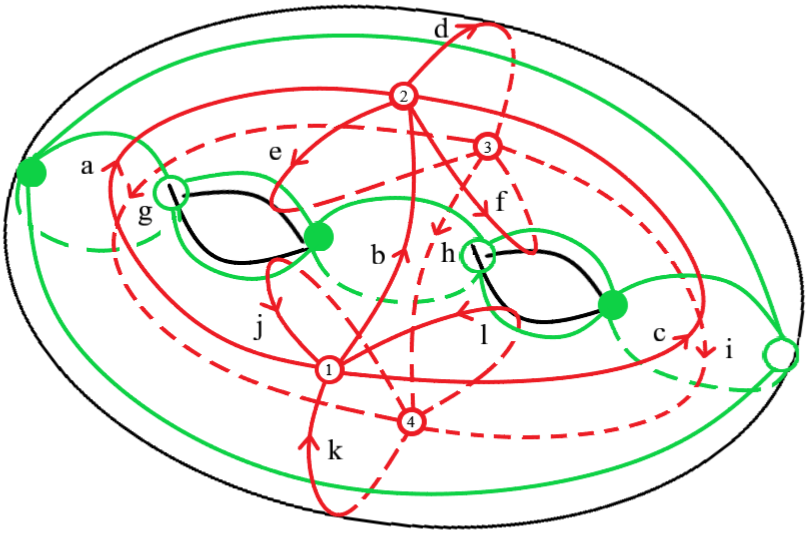


Figure 7.1: The tiling from [13] for a genus 2 surface. The tiling Δ is in green and the dual quiver Q_Δ is in red.

We get that Q_Δ is the quiver

$$\begin{tikzcd}
1 \arrow[r, bend left, "a"] \arrow[r, "b"] \arrow[r, bend right, "c"'] & 2 \arrow[d, bend left, "d"] \arrow[d, "e"] \arrow[d, bend right, "f"'] \\
4 \arrow[u, bend left, "j"] \arrow[u, "k"] \arrow[u, bend right, "l"'] & 3 \arrow[l, bend right, "i"'] \arrow[l, "h"] \arrow[l, bend left, "g"]
\end{tikzcd}$$

and the potential is

$$W_\Delta = jgea + lhfb + kidc - kgda - jheb - lifc.$$

We could take $E = \{a, b, c\}$ as a cut for W_Δ (we could also take $\{d, e, f\}$ or $\{g, h, i\}$ or $\{j, k, l\}$ among many others). Using the cut E we can choose $T = \{e, h, k\}$ for our maximal tree, then we have that the localised 2D Jacobi algebra of (Q, W, E) is

$$\mathrm{Jac}(\widetilde{Q}, W, E) = \frac{\mathbb{C}\langle d^{\pm 1}, f^{\pm 1}, g^{\pm 1}, i^{\pm 1}, j^{\pm 1}, l^{\pm 1}\rangle}{(jg - gd,\ lf - j,\ id - lif)}.$$

Note

$$\begin{aligned} jg - gd = 0 \quad &\Rightarrow \quad j = gdg^{-1} \\ id - lif = 0 \quad &\Rightarrow \quad f = i^{-1}l^{-1}id. \end{aligned}$$

Therefore $lf - j = 0$ becomes

$$li^{-1}l^{-1}id - gdg^{-1} = 0$$

or equivalently

$$li^{-1}l^{-1}idgd^{-1}g^{-1} - 1 = 0$$

and hence

$$\begin{aligned} \mathrm{Jac}(\widetilde{Q}, W, E) &= \frac{\mathbb{C}\langle d^{\pm 1}, f^{\pm 1}, g^{\pm 1}, i^{\pm 1}, j^{\pm 1}, l^{\pm 1}\rangle}{(j - gdg^{-1},\ li^{-1}l^{-1}idgd^{-1}g^{-1} - 1,\ f - i^{-1}l^{-1}id)} \\ &= \frac{\mathbb{C}\langle d^{\pm 1}, g^{\pm 1}, i^{\pm 1}, l^{\pm 1}\rangle}{(li^{-1}l^{-1}idgd^{-1}g^{-1} - 1)} \\ &\cong \mathbb{C}[\pi_1(\Sigma_2)]. \end{aligned}$$

Let Q be the quiver with one vertex obtained by contracting T in $Q_\Delta \setminus E$ and let W be corresponding potential on $Q_\Delta \setminus T$. For $n \in \mathbb{N}$ let

$$M_n = M_n(\mathbb{C}\widetilde{Q}) = \bigoplus_{a \in Q_1} \mathrm{GL}_n$$

be the space of n-dimensional representations of $\mathbb{C}\widetilde{Q}$, where recall $\mathbb{C}\widetilde{Q}$ is the localised path algebra of Q. Let $M_{m,n} = M_{m,n}(\mathbb{C}\widetilde{Q})$ be the space of short exact sequences of m-dimensional and n-dimensional representations. Then $M_{m,n}$ can be viewed as the space of m,n-block representations of $\mathbb{C}\widetilde{Q}$. Define $Y'_{m,n} \subset M_m \times M_n \times \mathrm{Mat}_{m,n}^{|E|}$ by

$$Y'_{m,n} := \left\{ \big(\rho', \rho'', (R_e)\big) \;\middle|\; \big(R_e^{(1)}, R_e^{(2)}\big) = \left(\frac{\partial W}{\partial e}(\rho'), \frac{\partial W}{\partial e}(\rho'')\right) \text{ for all } e \in E \right\}.$$

Let

$$Z_n = \left\{ \rho \in M_n \;\middle|\; \frac{\partial W}{\partial e}(\rho) = 0 \text{ for all } e \in E \right\} \subset M_n$$

then clearly $Z_n = M_n(\mathrm{Jac}(\widetilde{Q}, W, E))$ and so

$$\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)) \cong Z_n/\mathrm{GL}_n.$$

As in (7.2), we get the following diagram of $\mathrm{GL}_{m,n}$-equivariant maps

$$\begin{tikzcd}
M_{m+n} & M_{m,n} \arrow[l, hook', "h'"'] \arrow[r, "f''"] & Y'_{m,n} \\
Z_{m+n} \arrow[u, hook, "j'_{m+n}"] & Z_{m,n} \arrow[l, hook', "\widetilde{h'}"'] \arrow[r, "\widetilde{f''}"] \arrow[u, hook, "j'_{m,n}"] & Z_m \times Z_n \arrow[u, hook, "i'"]
\end{tikzcd} \tag{7.15}$$

where $j'_{m+n} : Z_{m+n} \hookrightarrow M_{m+n}$ is the natural inclusion and

$$f''(\rho) = \left(\rho^{(1)}, \rho^{(2)}, \left(\frac{\partial W}{\partial e}(\rho) \right)_{e \in E} \right).$$

The diagram (7.15) then induces the second 2D CoHA multiplication on

$$\begin{aligned}
\bigoplus_{n \in \mathbb{N}} \mathrm{H}_{c,\mathrm{GL}_n}(Z_n, \mathbb{Q})^\vee &= \bigoplus_{n \in \mathbb{N}} \mathrm{H}_{c,\mathrm{GL}_n}(M_n(\mathrm{Jac}(\widetilde{Q}, W, E)), \mathbb{Q})^\vee \\
&\cong \bigoplus_{n \in \mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)), \mathbb{Q})^\vee \\
&\cong \bigoplus_{n \in \mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)]), \mathbb{Q})^\vee
\end{aligned}$$

where the final isomorphism is induced from the isomorphism of algebras (7.14). Note, a priori, this multiplication depends upon the choices for the cut E and the maximal tree T.

7.3 Comparing morphisms on compactly supported cohomology

In this section we accumulate results that will allow us to compare the morphisms on compactly supported cohomology used in defining our two prior multiplications on the 2D CoHA.

Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X \times S \arrow[u, "\pi_X"] \arrow[r, "f'"] & Y \times S \arrow[u, "\pi_Y"'] \\
X \arrow[u, hook, "\varphi"] \arrow[r, "f"] & Y \arrow[u, hook, "\psi"']
\end{tikzcd} \tag{7.16}$$

where

- X, Y, S are smooth
- π_X and π_Y are the respective projections
- φ and ψ are closed immersions such that $\pi_X \circ \varphi = \mathrm{id}_X$ and $\pi_Y \circ \psi = \mathrm{id}_Y$
- both squares are Cartesian (and hence $\dim(f) = \dim(f') = d$).

Remark 7.3.1. This last condition is satisfied if and only if we can write f' as $(f \times \mathrm{id}_S) \circ \zeta$ where $\zeta : X \times S \xrightarrow{\sim} X \times S$ is an isomorphism.

For convenience we write $X' = X \times S$ and $Y' = Y \times S$ and let $m = \dim(S)$. From (3.2) we have that $\pi_X^*[-2m] = \pi_X^!$ and $\pi_Y^*[-2m] = \pi_Y^!$ since π_X and π_Y are smooth, and so

$$f^! = \mathrm{id}_{\mathrm{D}(X)}^* f^! = \varphi^* \pi_X^* f^! = \varphi^* \pi_X^! [-2m] f^! = \varphi^* f'^! \pi_Y^! [-2m] = \varphi^* f'^! \pi_Y^*.$$

We refer back to the beginning of Section 3.1 for a reminder on the notation of various natural transformations between functors of sheaves.

Lemma 7.3.2. *The pair of compositions*

$$\mathrm{id}_{\mathrm{D}(X)} = \varphi^* \pi_X^* \xrightarrow{\varphi^*(\theta^{f'})\pi_X^*} \varphi^* f'^! f'_! \pi_X^* \xrightarrow{\varphi^* f'^! (\epsilon^{\pi_Y, f})^{-1}} \varphi^* f'^! \pi_Y^* f_! = f^! f_! \tag{7.17}$$

and

$$f_! f^! = f_! \varphi^* f'^! \pi_Y^* \xrightarrow{(\epsilon^{\psi, f'})^{-1} f'^! \pi_Y^*} \psi^* f'_! f'^! \pi_Y^* \xrightarrow{\psi^*(\nu^{f'})\pi_Y^*} \psi^* \pi_Y^* = \mathrm{id}_{\mathrm{D}(Y)} \tag{7.18}$$

form a unit-counit adjunction for $f_! \dashv f^!$.

Proof. Denote (7.17) by α and (7.18) by β. Then we must show that the following compositions

$$\begin{aligned} &\text{a)} \qquad f_! \xrightarrow{f_!(\alpha)} f_! f^! f_! \xrightarrow{(\beta) f_!} f_! \\ &\text{b)} \qquad f^! \xrightarrow{(\alpha) f^!} f^! f_! f^! \xrightarrow{f^!(\beta)} f^! \end{aligned}$$

are the identity natural transformations.

For a) we have that

$$f_! \xrightarrow{f_!(\alpha)} f_! f^! f_! \xrightarrow{(\beta) f_!} f_!$$

is

$$\begin{aligned} f_! = f_!\varphi^*\pi_X^* &\xrightarrow{f_!\varphi^*(\theta^{f'})\pi_X^*} f_!\varphi^* f'^! f'_! \pi_X^* \xrightarrow{f_!\varphi^* f'^!(\epsilon^{\pi_Y, f})^{-1}} f_!\varphi^* f'^! \pi_Y^* f_! = f_! f^! f_! \\ = f_!\varphi^* f'^! \pi_Y^* f_! &\xrightarrow{(\epsilon^{\psi, f'})^{-1} f'^! \pi_Y^* f_!} \psi^* f'_! f'^! \pi_Y^* f_! \xrightarrow{\psi^*(\nu^{f'})\pi_Y^* f_!} \psi^* \pi_Y^* f_! = f_!. \end{aligned} \tag{7.19}$$

Then because these are natural transformations (7.19) is equal to

$$\begin{aligned} &f_! = f_!\varphi^*\pi_X^* \xrightarrow{(\epsilon^{\psi, f'})^{-1}\pi_X^*} \psi^* f'_! \pi_X^* \xrightarrow{\psi^* f'_!(\theta^{f'})\pi_X^*} \psi^* f'_! f'^! f'_! \pi_X^* \xrightarrow{\psi^*(\nu^{f'}) f'_! \pi_X^*} \\ &\psi^* f'_! \pi_X^* \xrightarrow{\psi^*(\epsilon^{\pi_Y, f})^{-1}} \psi^* \pi_Y^* f_! = f_!. \end{aligned} \tag{7.20}$$

Indeed by evaluating at some object $\mathcal{F} \in \mathrm{D}(X)$ we can first apply the commutativity property of the natural transformation $(\epsilon^{\psi, f'})^{-1}$ to the morphism $f_!\varphi^*(\theta^{f'}(\pi_X^*\mathcal{F}))$ giving us the commutative square

$$\begin{tikzcd} f_!\varphi^*\pi_X^*\mathcal{F} \arrow[r, "f_!\varphi^*(\theta^{f'}(\pi_X^*\mathcal{F}))"] \arrow[d, "(\epsilon^{\psi, f'})^{-1}(\pi_X^*\mathcal{F})"] & f_!\varphi^* f'^! f'_! \pi_X^*\mathcal{F} \arrow[d, "(\epsilon^{\psi, f'})^{-1}(f'^! f'_! \pi_X^*\mathcal{F})"] \\ \psi^* f'_! \pi_X^*\mathcal{F} \arrow[r, "\psi^* f'_!(\theta^{f'}(\pi_X^*\mathcal{F}))"] & \psi^* f'_! f'^! f'_! \pi_X^*\mathcal{F} \end{tikzcd} \tag{7.21}$$

We get similar commutative squares for the natural transformation $(\epsilon^{\psi, f'})^{-1}$ and morphism $f_!\varphi^* f'^!(\epsilon^{\pi_Y, f}(\mathcal{F}))^{-1}$, and the natural transformation $\nu^{f'}$ and morphism $f'_! f'^!(\epsilon^{\pi_Y, f}(\mathcal{F}))^{-1}$

resulting in commutative squares

$$\begin{tikzcd} f_!\varphi^* f'^! f'_! \pi_X^* \mathcal{F} \arrow[r, "f_!\varphi^* f'^!(\epsilon^{\pi_Y, f}(\mathcal{F})^{-1})"] \arrow[d, "(\epsilon^{\psi,f'})^{-1}(f'^! f'_! \pi_X^* \mathcal{F})"] & f_!\varphi^* f'^! \pi_Y^* f_! \mathcal{F} \arrow[d, "(\epsilon^{\psi,f'})^{-1}(f'^! \pi_Y^* f_! \mathcal{F})"] \\ \psi^* f'_! f'^! f'_! \pi_X^* \mathcal{F} \arrow[r, "\psi^* f'_! f'^!(\epsilon^{\pi_Y, f}(\mathcal{F})^{-1})"] & \psi^* f'_! f'^! \pi_Y^* f_! \mathcal{F} \end{tikzcd} \tag{7.22}$$

and

$$\begin{tikzcd} f'_! f'^! f'_! \pi_X^* \mathcal{F} \arrow[r, "f'_! f'^!(\epsilon^{\pi_Y, f}(\mathcal{F})^{-1})"] \arrow[d, "\nu^f(f'_! \pi_X^* \mathcal{F})"] & f'_! f'^! \pi_Y^* f_! \mathcal{F} \arrow[d, "\nu^f(\pi_Y^* f_! \mathcal{F})"] \\ f'_! \pi_X^* \mathcal{F} \arrow[r, "\epsilon^{\pi_Y, f}(\mathcal{F})^{-1}"] & \pi_Y^* f_! \mathcal{F} \end{tikzcd} \tag{7.23}$$

Combining these three squares we can see that (7.20) equals (7.19).

Now by the adjunction $f'_! \dashv f'^!$ (7.20) just equals

$$f_! = f_!\varphi^*\pi_X^* \xrightarrow{(\epsilon^{\psi,f'})^{-1}\pi_X^*} \psi^* f'_! \pi_X^* \xrightarrow{\psi^*(\epsilon^{\pi_Y,f})^{-1}} \psi^*\pi_Y^* f_! = f_!. \tag{7.24}$$

Then because $(\epsilon^{\psi,f'})\pi_X^* \circ \psi^*(\epsilon^{\pi_Y,f}) = \epsilon^{\pi_Y\circ\psi,f} = \epsilon^{\mathrm{id}_X,f} = \mathrm{id}$ (7.24) is the identity as required.

For b) we have that

$$f^! \xrightarrow{(\alpha)f^!} f^! f_! f^! \xrightarrow{f^!(\beta)} f^!$$

is

$$\begin{aligned} f^! = \varphi^*\pi_X^* f^! &\xrightarrow{\varphi^*(\theta^{f'})\pi_X^* f^!} \varphi^* f'^! f'_! \pi_X^* f^! \xrightarrow{\varphi^* f'^!(\epsilon^{\pi_Y,f})^{-1} f^!} \varphi^* f'^! \pi_Y^* f_! f^! = f^! f_! f^! \\ = f^! f_! \varphi^* f'^! \pi_Y^* &\xrightarrow{f^!(\epsilon^{\psi,f'})^{-1} f'^! \pi_Y^*} f^! \psi^* f'_! f'^! \pi_Y^* \xrightarrow{f^! \psi^*(\nu^{f'})\pi_Y^*} f^! \psi^* \pi_Y^* = f^!. \end{aligned} \tag{7.25}$$

Now because

$$\pi_Y^* f_! \xrightarrow{\epsilon^{\pi_Y,f}} f'_! \pi_X^*$$

equals

$$\pi_Y^* f_! = \pi_Y^* \psi^* \pi_Y^* f_! \xrightarrow{\pi_Y^* \psi^*(\epsilon^{\pi_Y, f}))} \pi_Y^* \psi^* f'_! \pi_X^* \xrightarrow{\pi_Y^*(\epsilon^{\psi, f'})\pi_X^*} \pi_Y^* f_! \varphi^* \pi_X^*$$
$$\xrightarrow{(\epsilon^{\pi_Y, f})\varphi^* \pi_X^*} f'_! \pi_X^* \varphi^* \pi_X^* = f'_! \pi_X^*$$

this implies that

$$f'_! \pi_X^* \xrightarrow{(\epsilon^{\pi_Y, f})^{-1}} \pi_Y^* f_! \xrightarrow{\epsilon^{\pi_Y, f}} f'_! \pi_X^*$$

equals

$$f'_! \pi_X^* \xrightarrow{(\epsilon^{\pi_Y, f})^{-1}} \pi_Y^* f_! = \pi_Y^* \psi^* \pi_Y^* f_! \xrightarrow{\pi_Y^* \psi^*(\epsilon^{\pi_Y, f}))} \pi_Y^* \psi^* f'_! \pi_X^*$$
$$\xrightarrow{\pi_Y^*(\epsilon^{\psi, f'})\pi_X^*} \pi_Y^* f_! \varphi^* \pi_X^* \xrightarrow{(\epsilon^{\pi_Y, f})\varphi^* \pi_X^*} f'_! \pi_X^* \varphi^* \pi_X^* = f'_! \pi_X^*$$

i.e.

$$f'_! \pi_X^* \xrightarrow{\mathrm{id}} f'_! \pi_X^*$$

equals

$$\pi_Y^* \psi^* f'_! \pi_X^* \xrightarrow{\pi_Y^*(\epsilon^{\psi, f'})\pi_X^*} \pi_Y^* f_! \varphi^* \pi_X^* \xrightarrow{(\epsilon^{\pi_Y, f})\varphi^* \pi_X^*} f'_! \pi_X^* \varphi^* \pi_X^* = f'_! \pi_X^*$$

and therefore (7.25) equals

$$f^! = \varphi^* f'^! \pi_Y^* \xrightarrow{\varphi^*(\theta^{f'}) f'^! \pi_Y^*} \varphi^* f'^! f'_! f'^! \pi_Y^* \xrightarrow{\varphi^* f'^!(\nu^{f'})\pi_Y^*} \varphi^* f'^! \pi_Y^* = f^! \tag{7.26}$$

which is once again the identity by the adjunction $f'_! \dashv f'^!$.

□

Lemma 7.3.3. *The natural isomorphism*

$$\epsilon^{\pi_Y, f} : \pi_Y^* f_! \xrightarrow{\sim} f'_! \pi_X^*$$

is the identity.

Proof. By Remark 7.3.1 we have that $f' = (f \times \mathrm{id}_S) \circ \gamma$ where $\gamma : X \times S \xrightarrow{\sim} X \times S$ is an isomorphism. For the moment assume that $f' = f \times \mathrm{id}_S$.

Working at the level of sheaves (so for the moment everything is underived) the isomorphism $\pi_Y^* f_! \xrightarrow{\sim} f'_! \pi_X^*$ is given in [[41] Proposition 2.5.11]. It is enough to check this is an equality on a basis for the topology of $Y \times S$. Let $\mathcal{F} \in \mathrm{Sh}(X)$ and $U \times W \subset Y \times S$

for $U \subset Y$ and $W \subset S$ open. Then because the maps π_X and π_Y are open we have

$$\begin{aligned}
\pi_Y^* f_! \mathcal{F}(U \times W) &= f_! \mathcal{F}(\pi_Y(U \times W)) \\
&= f_! \mathcal{F}(U) \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; f|_{\mathrm{supp}(t)} \text{ is proper} \big\} \\
\pi_Y^* f_! \pi_{X*} \pi_X^* \mathcal{F}(U \times W) &= f_! \pi_{X*} \pi_X^* \mathcal{F}(\pi_Y(U \times W)) \\
&= f_! \pi_{X*} \pi_X^* \mathcal{F}(U) \\
&= \big\{ r \in \pi_{X*} \pi_X^* \mathcal{F}(f^{-1}(U)) \;\big|\; f|_{\mathrm{supp}(r)} \text{ is proper} \big\} \\
&= \big\{ s \in \pi_X^* \mathcal{F}((\pi_X \circ f)^{-1}(U)) \;\big|\; f|_{\pi_X(\mathrm{supp}(s))} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}((\pi_X(f^{-1}(U) \times S)) \;\big|\; f|_{\mathrm{supp}(t)} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; f|_{\mathrm{supp}(t)} \text{ is proper} \big\} \\
\pi_Y^* \pi_{Y*} f'_! \pi_X^* \mathcal{F}(U \times W) &= \pi_{Y*} f'_! \pi_X^* \mathcal{F}(\pi_Y(U \times W)) \\
&= f'_! \pi_X^* \mathcal{F}(\pi_Y^{-1}(U)) \\
&= \big\{ s \in \pi_X^* \mathcal{F}(f'^{-1}(U \times S)) \;\big|\; f'|_{\mathrm{supp}(s)} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; (f \times \mathrm{id}_S)|_{\mathrm{supp}(t) \times S} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; f|_{\mathrm{supp}(t)} \text{ is proper} \big\} \\
f'_! \pi_X^* \mathcal{F}(U \times W) &= \big\{ s \in \pi_X^* \mathcal{F}(f'^{-1}(U \times W)) \;\big|\; f'|_{\mathrm{supp}(s)} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; (f \times \mathrm{id}_S)|_{\mathrm{supp}(t) \times S} \text{ is proper} \big\} \\
&= \big\{ t \in \mathcal{F}(f^{-1}(U)) \;\big|\; f|_{\mathrm{supp}(t)} \text{ is proper} \big\}
\end{aligned}$$

where the final equalities for both $\pi_Y^* \pi_{Y*} f'_! \pi_X^* \mathcal{F}(U \times W)$ and $f'_! \pi_X^* \mathcal{F}(U \times W)$ arise due to the fact that properness is preserved under base extension. The isomorphism given in [[41] Proposition 2.5.11] is then the composition

$$\pi_Y^* f_! \longrightarrow \pi_Y^* f_! \pi_{X*} \pi_X^* \longrightarrow \pi_Y^* \pi_{Y*} f'_! \pi_X^* \longrightarrow f'_! \pi_X^* \tag{7.27}$$

where the first morphism comes from the natural transformation $\mathrm{id}_{\mathrm{Sh}(X)} \to \pi_{X*} \pi_X^*$ for the adjunction $\pi_X^* \dashv \pi_{X*}$, the second morphism becomes the identity following the description in the proof of [[41] Proposition 2.5.11], and the third morphism comes from the natural transformation $\pi_Y^* \pi_{Y*} \to \mathrm{id}_{\mathrm{Sh}(Y \times S)}$ for the adjunction $\pi_Y^* \dashv \pi_{Y*}$.

Note that for a projection $\pi : P \times Q \to P$ the natural transformation $\mathrm{id}_{\mathrm{Sh}(P)} \to \pi_* \pi^*$ is the identity, since for $\mathcal{F} \in \mathrm{Sh}(P)$ and $U \subset P$ open

$$\begin{aligned}
\pi_* \pi^* \mathcal{F}(U) &= \pi^* \mathcal{F}(\pi^{-1}(U)) \\
&= \pi^* \mathcal{F}(U \times T) \\
&= \mathcal{F}(\pi(U \times T)) \\
&= \mathcal{F}(U)
\end{aligned}$$

with restriction maps for $V \subset U$

$$\begin{aligned}\mathrm{res}^{\pi_*\pi^*\mathcal{F}}_{U,V} &= \mathrm{res}^{\pi^*\mathcal{F}}_{U\times S, V\times S} \\ &= \mathrm{res}^{\mathcal{F}}_{U,V}.\end{aligned}$$

Hence the first morphism in (7.27) is also the identity. For $\mathcal{G} \in \mathrm{Sh}(Y \times S)$ and $U \times W \subset Y \times S$ open we have

$$\pi_Y^*\pi_{Y*}\mathcal{G}(U \times W) = \mathcal{G}(U \times S)$$

and so

$$\sigma^{\pi_Y}(\mathcal{G}(U \times W)) = \mathrm{res}^{\mathcal{G}}_{U\times S, U\times W}$$

hence

$$\begin{aligned}\sigma^{\pi_Y}(f'_!\pi_X^*(\mathcal{F}(U \times W)) &= \mathrm{res}^{f'_!\pi_X^*\mathcal{F}}_{U\times S, U\times W} \\ &= \mathrm{res}^{\pi_X^*\mathcal{F}}_{f^{-1}(U)\times S, f^{-1}(U)\times W}|_{\text{sections with proper support}} \\ &= \mathrm{res}^{\mathcal{F}}_{f^{-1}(U), f^{-1}(U)}|_{\text{sections with proper support}} \\ &= \mathrm{id}_{\mathcal{F}(f^{-1}(U))}|_{\text{sections with proper support}}\end{aligned}$$

showing that on a basis (7.27) is the identity morphism. Therefore, since by [[41] Proposition 2.6.7] the natural isomorphism $\epsilon^{\pi_Y,f}$ of derived functors is induced by the natural isomorphism (7.27) at the level of sheaves, upgrading to derived functors implies that $\epsilon^{\pi_Y,f}$ is the identity natural isomorphism too.

In the case that $f' = (f \times \mathrm{id}_S) \circ \gamma$ we have

$$f'_!(\pi_X \circ \gamma)^* = (f \times \mathrm{id}_S)_!\gamma_!\gamma^*\pi_X^* = (f \times \mathrm{id}_S)_!\pi_X^* = \pi_Y^* f_!$$

because for an isomorphism $\gamma : X \times S \xrightarrow{\sim} X \times S$ it is clear that the functor

$$\gamma_!\gamma^* : \mathrm{D}(X \times S) \to \mathrm{D}(X \times S)$$

is simply the identity functor $\mathrm{id}_{\mathrm{D}(X\times S)}$. □

Lemma 7.3.4. *The composition of natural transformations*

$$\varphi^! f'^! \Big(f'_!\pi_X^! \xrightarrow{(\theta^{\pi_Y})f'_!\pi_X^!} \pi_Y^!\pi_{Y!}f'_!\pi_X^! = \pi_Y^! f_!\pi_{X!}\pi_X^! \xrightarrow{\pi_Y^! f_!(\nu^{\pi_X})} \pi_Y^! f_! \Big) \tag{7.28}$$

is the identity.

Proof. To show this natural transformation is the identity it suffices to show its Verdier dual

$$\varphi^* f'^* \Big(\pi_Y^* f_* \xrightarrow{\pi_Y^* f_*(\eta^{\pi_X})} \pi_Y^* f_*\pi_{X*}\pi_X^* = \pi_Y^*\pi_{Y*}f'_*\pi_X^* \xrightarrow{(\sigma^{\pi_Y})f'_*\pi_X^*} f'_*\pi_X^* \Big). \tag{7.29}$$

is the identity. But showing this can be done using almost exactly the same method as the proof of Lemma 7.3.3, with the only changes being that we do not consider proper supports and the middle morphism here is explicitly the identity. □

Proposition 7.3.5. *The canonical natural transformation* $\nu^f : f_! f^! \to \mathrm{id}_{\mathrm{D}(Y)}$ *from the adjunction* $f_! \dashv f^!$ *is equal to the composition* (7.18) *i.e.*

$$f_! f^! = f_! \varphi^* f'^! \pi_Y^* \xrightarrow{(\epsilon^{\psi,f'})^{-1} f'^! \pi_Y^*} \psi^* f'_! f'^! \pi_Y^* \xrightarrow{\psi^*(\nu^{f'})\pi_Y^*} \psi^* \pi_Y^* = \mathrm{id}_{\mathrm{D}(Y)}$$

Proof. We shall show that (7.17) is equal to the unit θ^f implying that the counits must also be equal, namely (7.18) equals ν^f.

First note the following: from the adjunction $\pi_{X!} \dashv \pi_X^!$ we have that

$$\pi_X^! \xrightarrow{\pi_X^!(\theta^{\pi_X})} \pi_X^! \pi_{X!} \pi_X^! \xrightarrow{(\nu^{\pi_X})\pi_X^!} \pi_X^!$$

is the identity natural transformation. Applying $\varphi^!$ then post-composing with θ^f gives

$$\mathrm{id}_{\mathrm{D}(X)} = \varphi^! \pi_X^! \xrightarrow{\varphi^! \pi_X^!(\theta^{\pi_X})} \varphi^! \pi_X^! \pi_{X!} \pi_X^! \xrightarrow{\varphi^!(\nu^{\pi_X})\pi_X^!} \varphi^! \pi_X^! = \mathrm{id}_{\mathrm{D}(X)} \xrightarrow{\theta^f} f^! f_!. \tag{7.30}$$

Because these are natural transformations we can rearrange the order in which we do them (using a similar argument as in the proof of Lemma 7.3.2) making (7.30) equal to

$$\begin{aligned}\mathrm{id}_{\mathrm{D}(X)} = \varphi^! \pi_X^! &\xrightarrow{\varphi^! \pi_X^!(\theta^{\pi_X})} \varphi^! \pi_X^! \pi_{X!} \pi_X^! \xrightarrow{\varphi^! \pi_X^!(\theta^f)\pi_{X!}\pi_X^!} \varphi^! \pi_X^! f^! f_! \pi_{X!} \pi_X^! \\ &\xrightarrow{\varphi^! \pi_X^! f^! f_!(\nu^{\pi_X})} \varphi^! \pi_X^! f^! f_! = f^! f_!.\end{aligned} \tag{7.31}$$

Then because

$$\pi_X^!(\theta^f)\pi_{X!} \circ \theta^{\pi_X} = \theta^{f \circ \pi_X} = \theta^{\pi_Y \circ f'} = f'^!(\theta^{\pi_Y}) f'_! \circ \theta^{f'}$$

we get that (7.31) equals

$$\begin{aligned}\mathrm{id}_{\mathrm{D}(X)} = \varphi^! \pi_X^! &\xrightarrow{\varphi^!(\theta^{f'})\pi_X^!} \varphi^! f'^! f'_! \pi_X^! \xrightarrow{\varphi^! f'^!(\theta^{\pi_Y}) f'_! \pi_X^!} \varphi^! f'^! \pi_Y^! \pi_{Y!} f'_! \pi_X^! \\ &= \varphi^! f'^! \pi_Y^! f_! \pi_{X!} \pi_X^! \xrightarrow{\varphi^! f'^! \pi_Y^! f_!(\nu^{\pi_X})} \varphi^! \pi_X^! f^! f_! = f^! f_!.\end{aligned} \tag{7.32}$$

so in particular θ^f equals (7.32).

We construct the following diagram whose top row is (7.17) and whose bottom row is (7.32).

$$
\begin{tikzcd}
\mathrm{id}_{\mathrm{D}(X)} = \varphi^*[-2m]\pi_X^*[2m] \arrow[r, "\varphi^*(\theta^{f'})\pi_X^*"] \arrow[d, "(\kappa^{\varphi})\pi_X^*[2m]"'] & \varphi^*[-2m]f'^! f'_! \pi_X^*[2m] \arrow[r, "\varphi^* f'^!(\epsilon^{\pi_Y, f})^{-1}"] \arrow[d, "(\kappa^{\varphi})f'^! f'_! \pi_X^*[2m]"'] & \varphi^*[-2m]f'^! \pi_Y^*[2m]f_! \arrow[d, "(\kappa^{\varphi})f'^! \pi_Y^*[2m]f_!"'] \\
\varphi^! \pi_X^*[2m] \arrow[r, "\varphi^!(\theta^{f'})\pi_X^*[2m]"] \arrow[d, "\varphi^!(\kappa^{\pi_X})"', "\|"] & \varphi^! f'^! f'_! \pi_X^*[2m] \arrow[r, "\varphi^! f'^!(\epsilon^{\pi_Y, f}[2m])^{-1}"] \arrow[d, "\varphi^! f'^! f'_!(\kappa^{\pi_X})"', "\|"] & \varphi^! f'^! \pi_Y^*[2m]f_! \arrow[d, "\varphi^! f'^!(\kappa^{\pi_Y})f_!"', "\|"] \\
\mathrm{id}_{\mathrm{D}(X)} = \varphi^! \pi_X^! \arrow[r, "\varphi^!(\theta^{f'})\pi_X^!"] & \varphi^! f'^! f'_! \pi_X^! \arrow[r, "\varphi^! f'^!(\xi^{\pi_X, f'})"] & \varphi^! f'^! \pi_Y^! f_!
\end{tikzcd}
\tag{7.33}
$$

where $\xi^{\pi_X, f'} : f'_! \pi_X^! \to \pi_Y^! f_!$ is the composition

$$f'_! \pi_X^! \xrightarrow{(\theta^{\pi_Y}) f'_! \pi_X^!} \pi_Y^! \pi_{Y!} f'_! \pi_X^! = \pi_Y^! f_! \pi_{X!} \pi_X^! \xrightarrow{\pi_Y^! f_!(\nu^{\pi_X})} \pi_Y^! f_!.$$

The composition of the natural transformations in the left-hand column is the identity since

$$\varphi^!(\kappa^{\pi_X}) \circ (\kappa^{\varphi})\pi_X^*[2m] = \kappa^{\pi_X \circ \varphi} = \kappa^{\mathrm{id}_X} = \mathrm{id}.$$

For the right-hand column we have that

$$(\kappa^{\varphi}) f'^! \pi_Y^*[2m] f_! = (\kappa^{\varphi}) \pi_X^*[2m] f^! f_!$$

and

$$\varphi^! f'^! (\kappa^{\pi_Y}) f_! = \varphi^! (\kappa^{\pi_X}) f^! f_!$$

(simply using the fact that $\kappa^{\pi_X} = \mathrm{id}$ and $\kappa^{\pi_Y} = \mathrm{id}$) and so, for the same reason as the left-hand column, the composition in the right-hand column is also the identity. Therefore it suffices to show this diagram commutes in order to prove that the two units are equal and hence conclude the proof of the proposition. Commutativity of the top two squares and bottom left square in (7.33) are clear because once evaluated on some $\mathcal{F} \in \mathrm{D}(X)$ these once again become the squares obtained by applying the natural transformations to some morphism, and hence they must commute by definition of a natural transformation. Finally for the bottom right square; we already know that κ^{π_X} and κ^{π_Y} are just the identity natural transformations, and by Lemma 7.3.3 and Lemma 7.3.4 we have that the two horizontal arrows in this square are the identity as well, and hence it also commutes. $\square$

Now consider the following commutative diagram

$$
\begin{tikzcd}
& X' \arrow[rr, "f'"] & & Y' \\
X \arrow[ur, hook, "\varphi"] \arrow[rr, "f"] & & Y \arrow[ur, hook, "\psi"] & \\
& V' \arrow[uu, hook, "j'"'] \arrow[rr, "\widetilde{f'}"] & & Z' \arrow[uu, hook, "i'"'] \\
V \arrow[uu, hook, "j"'] \arrow[ur, "\widetilde{\varphi}", "\sim"'] \arrow[rr, "\widetilde{f}"] & & Z \arrow[uu, hook, "i"'] \arrow[ur, "\widetilde{\psi}", "\sim"'] &
\end{tikzcd}
\tag{7.34}
$$

where

- the top face of this cube fits as the bottom square into a diagram as in (7.16) and satisfies all the conditions proceeding (7.16)

- i, i', j, j' are closed immersions

- $\widetilde{\varphi}$ and $\widetilde{\psi}$ are isomorphisms

- all squares are Cartesian.

We wish to show that the morphisms

$$\widetilde{\psi}_!(f_\star) : \widetilde{\psi}_! \widetilde{f}_! \mathbb{Q}_V[2d] \longrightarrow \widetilde{\psi}_! \mathbb{Q}_Z$$

and

$$f'_\star : \widetilde{f}'_! \mathbb{Q}_{V'}[2d] \longrightarrow \mathbb{Q}_{Z'}$$

are equal via the isomorphisms on sheaves between V and V' and between Z and Z' induced by pulling-back along $\widetilde{\varphi}$ and $\widetilde{\psi}$ respectively. Using the definition (7.8) from Section 3.1 we have that $\widetilde{\psi}_!(f_\star)$ is given by

$$
\begin{aligned}
\widetilde{\psi}_! \widetilde{f}_! \mathbb{Q}_V[2d] = \widetilde{\psi}_! \widetilde{f}_! j^* \mathbb{Q}_X[2d] &\xrightarrow[\sim]{\widetilde{\psi}_!(\epsilon^{i,f}(\mathbb{Q}_X[2d])^{-1})} \widetilde{\psi}_! i^* f_! \mathbb{Q}_X[2d] \\
= \widetilde{\psi}_! i^* f_! f^! \mathbb{Q}_Y &\xrightarrow{\widetilde{\psi}_! i^*(\nu^f(\mathbb{Q}_Y))} \widetilde{\psi}_! i^* \mathbb{Q}_Y = \widetilde{\psi}_! \mathbb{Q}_Z
\end{aligned}
\tag{7.35}
$$

and $f'_\star$ is given by

$$\begin{aligned}\widetilde{f}'_!\mathbb{Q}_{V'}[2d] = \widetilde{f}'_! j'^*\mathbb{Q}_{X'}[2d] &\xrightarrow[\sim]{\epsilon^{i',f'}(\mathbb{Q}_{X'}[2d])^{-1}} i'^* f'_!\mathbb{Q}_{X'}[2d] \\ = i'^* f'_! f'^! \mathbb{Q}_{Y'} &\xrightarrow{i'^*(\nu^{f'}(\mathbb{Q}_{Y'}))} i'^*\mathbb{Q}_{Y'} = \mathbb{Q}_{Z'}.\end{aligned} \tag{7.36}$$

Theorem 7.3.6. *The following diagram of sheaves in* $\mathrm{D}(Z')$ *commutes*

$$\begin{tikzcd} \widetilde{f}'_!\mathbb{Q}_{V'}[2d] \arrow[r, "\widetilde{f}'_!(\widetilde{\varphi}^\star)", "\sim"'] \arrow[d, "f'_\star"] & \widetilde{\psi}_!\widetilde{f}_!\mathbb{Q}_V[2d] \arrow[d, "\widetilde{\psi}_!(f_\star)"] \\ \mathbb{Q}_{Z'} \arrow[r, "\widetilde{\psi}^\star", "\sim"'] & \widetilde{\psi}_!\mathbb{Q}_Z \end{tikzcd} \tag{7.37}$$

Using the descriptions (7.35) and (7.36) and the description of the pullback on compactly supported cohomology given in Section 3.1, we get that diagram (7.37) is equal to the outer square of the following diagram and hence to prove Theorem 7.3.6 it suffices to show that (7.38) commutes.

$$\begin{tikzcd} \widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \arrow[r, "\widetilde{f}'_!(\eta^{\widetilde{\varphi}}(f'^!\mathbb{Q}_{Y'}))", "\sim"'] \arrow[d, "\epsilon^{i',f'}(f'^!\mathbb{Q}_{Y'})^{-1}", "\wr"'] & \widetilde{f}'_!\widetilde{\varphi}_!\widetilde{\varphi}^* j'^* f'^! \mathbb{Q}_{Y'} = \widetilde{\psi}_!\widetilde{f}_! j^* f^! \mathbb{Q}_Y \arrow[d, "\widetilde{\psi}_!(\epsilon^{i,f}(f^!\mathbb{Q}_Y)^{-1})", "\wr"'] \\ i'^* f'_! f'^! \mathbb{Q}_{Y'} \arrow[r, "\alpha", "\sim"'] \arrow[d, "i'^*(\nu^{f'}(\mathbb{Q}_{Y'}))"] & \widetilde{\psi}_! i^* f_! f^! \mathbb{Q}_Y \arrow[d, "\widetilde{\psi}_! i^*(\nu^f(\mathbb{Q}_Y))"] \\ i'^*\mathbb{Q}_{Y'} = \mathbb{Q}_{Z'} \arrow[r, "\eta^{\widetilde{\psi}}(\mathbb{Q}_{Z'})", "\sim"'] & \widetilde{\psi}_!\widetilde{\psi}^*\mathbb{Q}_{Z'} = \widetilde{\psi}_! i^* \mathbb{Q}_Y \end{tikzcd} \tag{7.38}$$

where α is the isomorphism

$$i'^* f'_! f'^! \mathbb{Q}_{Y'} \xrightarrow[\sim]{\eta^{\widetilde{\psi}}(i'^* f'_! f'^! \mathbb{Q}_{Y'})} \widetilde{\psi}_! \widetilde{\psi}^* i'^* f'_! f'^! \mathbb{Q}_{Y'} = \widetilde{\psi}_! \widetilde{\psi}^* (\widetilde{\psi}^{-1})^* i^* \psi^* f'_! f'^! \mathbb{Q}_{Y'} =$$
$$\widetilde{\psi}_! i^* \psi^* f'_! f'^! \mathbb{Q}_{Y'} \xrightarrow[\sim]{\widetilde{\psi}_! i^*(\epsilon^{\psi, f'}(f'^! \mathbb{Q}_{Y'}))} \widetilde{\psi}_! i^* f_! \varphi^* f'^! \mathbb{Q}_{Y'} = \widetilde{\psi}_! i^* f_! \varphi^* f'^! \pi_Y^* \mathbb{Q}_Y = \widetilde{\psi}_! i^* f_! f^! \mathbb{Q}_Y.$$

Lemma 7.3.7. *The top square in diagram* (7.38) *commutes.*

Proof. This is just a simple matter of rearranging the order we do the morphisms that comprise either side of the square and cancelling. In particular by applying the natural transformation $\eta^{\widetilde{\psi}}$ to the morphism $\epsilon^{i', f'}(f'^! \mathbb{Q}_{Y'})^{-1}$ we get the following commutative square

$$\begin{tikzcd}
\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \arrow[r, "\epsilon^{i', f'}(f'^! \mathbb{Q}_{Y'})^{-1}"] \arrow[d, "\eta^{\widetilde{\psi}}(\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'})"] & i'^* f'_! f'^! \mathbb{Q}_{Y'} \arrow[d, "\eta^{\widetilde{\psi}}(i'^* f'_! f'^! \mathbb{Q}_{Y'})"] \\
\widetilde{\psi}_! \widetilde{\psi}^* \widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \arrow[r, "\widetilde{\psi}_! \widetilde{\psi}^*(\epsilon^{i', f'}(f'^! \mathbb{Q}_{Y'})^{-1})"] & \widetilde{\psi}_! \widetilde{\psi}^* i'^* f'_! f'^! \mathbb{Q}_{Y'} = \widetilde{\psi}_! i^* \psi^* f'_! f'^! \mathbb{Q}_{Y'}
\end{tikzcd} \tag{7.39}$$

Hence (7.39) tells us that going along the left-hand and bottom sides of the top square in (7.38) equals

$$\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \xrightarrow{\eta^{\widetilde{\psi}}(\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'})} \widetilde{\psi}_! \widetilde{\psi}^* \widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \xrightarrow{\widetilde{\psi}_! \widetilde{\psi}^*(\epsilon^{i', f'}(f'^! \mathbb{Q}_{Y'})^{-1})} \tag{7.40}$$
$$\widetilde{\psi}_! \widetilde{\psi}^* i'^* f'_! f'^! \mathbb{Q}_{Y'} = \widetilde{\psi}_! i^* \psi^* f'_! f'^! \mathbb{Q}_{Y'} \xrightarrow{\widetilde{\psi}_! i^*(\epsilon^{\psi, f'}(f'^! \mathbb{Q}_{Y'}))} \widetilde{\psi}_! i^* f_! \varphi^* f'^! \mathbb{Q}_{Y'}.$$

Then because

$$\epsilon^{i', f'} = \epsilon^{\psi \circ i \circ \widetilde{\psi}^{-1}, f'} = (\epsilon^{\widetilde{\psi}^{-1}, \widetilde{f}}) j^* \varphi^* \circ (\widetilde{\psi}^{-1})^* (\epsilon^{i, f}) \varphi^* \circ (\widetilde{\psi}^{-1})^* i^* (\epsilon^{\psi, f'})$$

we get that (7.40) equals

$$\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} \xrightarrow{\eta^{\widetilde{\psi}}(\widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'})} \widetilde{\psi}_! \widetilde{\psi}^* \widetilde{f}'_! j'^* f'^! \mathbb{Q}_{Y'} = \tag{7.41}$$
$$\widetilde{\psi}_! \widetilde{\psi}^* \widetilde{f}'_! (\widetilde{\varphi}^{-1})^* j^* \varphi^* f'^! \mathbb{Q}_{Y'} \xrightarrow{\widetilde{\psi}_! \widetilde{\psi}^*(\epsilon^{\widetilde{\psi}^{-1}, \widetilde{f}}(j^* \varphi^* f'^! \mathbb{Q}_{Y'})^{-1})} \widetilde{\psi}_! \widetilde{\psi}^* (\widetilde{\psi}^{-1})^* \widetilde{f}_! j^* \varphi^* f'^! \mathbb{Q}_{Y'} =$$
$$\widetilde{\psi}_! \widetilde{f}_! j^* \varphi^* f'^! \mathbb{Q}_{Y'} \xrightarrow{\widetilde{\psi}_!(\epsilon^{i, f}(\varphi^* f'^! \mathbb{Q}_{Y'})^{-1})} \widetilde{\psi}_! i^* f_! i^* \varphi^* f'^! \mathbb{Q}_{Y'}$$

Now consider the description of $\epsilon^{\widetilde{\psi}^{-1},\widetilde{f}}$ given in (3.1), namely it equals

$$(\widetilde{\psi}^{-1})^*\widetilde{f}_! \xrightarrow{(\widetilde{\psi}^{-1})^*\widetilde{f}_!(\eta^{\widetilde{\varphi}^{-1}})} (\widetilde{\psi}^{-1})^*\widetilde{f}_!(\widetilde{\varphi}^{-1})_!(\widetilde{\varphi}^{-1})^* = (\widetilde{\psi}^{-1})^*(\widetilde{\psi}^{-1})_!\widetilde{f}'_!(\widetilde{\varphi}^{-1})^*$$
$$\xrightarrow{(\sigma^{\widetilde{\psi}^{-1}})\widetilde{f}'_!(\widetilde{\varphi}^{-1})^*} \widetilde{f}'_!(\widetilde{\varphi}^{-1})^*.$$

But since

$$(\widetilde{\varphi}^{-1})_!(\widetilde{\varphi}^{-1})^* \xrightarrow{(\eta^{\widetilde{\varphi}^{-1}})^{-1}} \mathrm{id}_{\mathrm{D}(V)}$$

equals

$$(\widetilde{\varphi}^{-1})_!(\widetilde{\varphi}^{-1})^* \xrightarrow{(\widetilde{\varphi}^{-1})_!(\eta^{\widetilde{\varphi}})(\widetilde{\varphi}^{-1})^*} (\widetilde{\varphi}^{-1})_!\widetilde{\varphi}_!\widetilde{\varphi}^*(\widetilde{\varphi}^{-1})^* = \mathrm{id}_{\mathrm{D}(V)}$$

and similarly

$$\mathrm{id}_{\mathrm{D}(Z')} \xrightarrow{(\sigma^{\widetilde{\psi}^{-1}})^{-1}} (\widetilde{\psi}^{-1})^*(\widetilde{\psi}^{-1})_!$$

equals

$$\mathrm{id}_{\mathrm{D}(Z')} = (\widetilde{\psi}^{-1})^*\widetilde{\psi}^*\widetilde{\psi}_!(\widetilde{\psi}^{-1})_! \xrightarrow{(\widetilde{\psi}^{-1})^*(\sigma^{\widetilde{\psi}})(\widetilde{\psi}^{-1})_!} (\widetilde{\psi}^{-1})^*(\widetilde{\psi}^{-1})_!$$

we get that (7.41) equals

$$\widetilde{f}'_!j'^*f'^!\mathbb{Q}_{Y'} \xrightarrow{\eta^{\widetilde{\psi}}(\widetilde{f}'_!j'^*f'^!\mathbb{Q}_{Y'})} \widetilde{\psi}_!\widetilde{\psi}^*\widetilde{f}'_!j'^*f'^!\mathbb{Q}_{Y'} = \widetilde{\psi}_!\widetilde{\psi}^*\widetilde{\psi}_!(\widetilde{\psi}^{-1})_!\widetilde{f}'_!(\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'} \tag{7.42}$$
$$\xrightarrow{\widetilde{\psi}_!(\sigma^{\widetilde{\psi}}((\widetilde{\psi}^{-1})_!\widetilde{f}_!(\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'}))} \widetilde{\psi}_!(\widetilde{\psi}^{-1})_!\widetilde{f}'_!(\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'} = \widetilde{\psi}_!\widetilde{f}_!(\widetilde{\varphi}^{-1})_!(\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'}$$
$$\xrightarrow{\widetilde{\psi}_!\widetilde{f}_!(\widetilde{\varphi}^{-1})_!(\eta^{\widetilde{\varphi}}((\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'}))} \widetilde{\psi}_!\widetilde{f}_!(\widetilde{\varphi}^{-1})_!\widetilde{\varphi}_!\widetilde{\varphi}^*(\widetilde{\varphi}^{-1})^*j^*\varphi^*f'^!\mathbb{Q}_{Y'}$$
$$= \widetilde{\psi}_!\widetilde{f}_!j^*\varphi^*f'^!\mathbb{Q}_{Y'} \xrightarrow{\widetilde{\psi}_!i^*(\epsilon^{i,f}(\varphi^*f'^!\mathbb{Q}_{Y'})^{-1})} \widetilde{\psi}_!i^*f_!\varphi^*f'^!\mathbb{Q}_{Y'}.$$

The composition of first two morphisms in (7.42) is the identity due to the adjunction $\widetilde{\psi}^* \dashv \widetilde{\psi}_*$ and so we are left with the top and right-hand sides of the top square in (7.38). ☐

Lemma 7.3.8. *The bottom square in diagram* (7.38) *commutes.*

Proof. This follows from Proposition 7.3.5 and more rearranging. Indeed Proposition 7.3.5 says that

$$\nu^f = \psi^*(\nu^{f'})\pi_Y^* \circ (\epsilon^{\psi,f'})^{-1}f'^!\pi_Y^*$$

and so we can immediately see that the composition of top and right-hand sides of the bottom square in (7.38) gives

$$i'^*f'_!f'^!\mathbb{Q}_{Y'} \xrightarrow{\eta^{\widetilde{\varphi}}(i'^*f'_!f'^!\mathbb{Q}_{Y'})} \widetilde{\psi}_!\widetilde{\psi}^*i'^*f'_!f'^!\mathbb{Q}_{Y'} = \widetilde{\psi}_!i^*\psi^*f'_!f'^!\mathbb{Q}_{Y'} \tag{7.43}$$
$$\xrightarrow{\widetilde{\psi}_!i^*\psi^*(\nu^{f'}(\mathbb{Q}_{Y'}))} \widetilde{\psi}_!i^*\psi^*\mathbb{Q}_{Y'} = \widetilde{\psi}_!\mathbb{Q}_Z.$$

Now applying the natural transformation $\eta^{\widetilde{\psi}}$ to the morphism $i'^* f'_! f'^! \mathbb{Q}_{Y'} \xrightarrow{i'^*(\nu^{f'}(\mathbb{Q}_{Y'}))} i'^* \mathbb{Q}_{Y'}$ gives the commutative square

$$\begin{tikzcd}
i'^* f'_! f'^! \mathbb{Q}_{Y'} \arrow[r, "\eta^{\widetilde{\psi}}(i'^* f'_! f'^! \mathbb{Q}_{Y'})"] \arrow[d, "i'^*(\nu^{f'}(\mathbb{Q}_{Y'}))"] & \widetilde{\psi}_! \widetilde{\psi}^* i'^* f'_! f'^! \mathbb{Q}_{Y'} \arrow[d, "\widetilde{\psi}_! i^* \psi^* (\nu^{f'}(\mathbb{Q}_{Y'}))"] \\
i'^* \mathbb{Q}_{Y'} \arrow[r, "\eta^{\widetilde{\psi}}(i'^* \mathbb{Q}_{Y'})"] & \widetilde{\psi}_! \widetilde{\psi}^* i'^* \mathbb{Q}_{Y'}
\end{tikzcd} \tag{7.44}$$

and we can see that the left-hand and bottom sides of (7.44) are exactly those in the bottom square in (7.38) whilst the top and right-hand sides are (7.43) proving the result. □

The two preceding lemmas then directly imply Theorem 7.4.6. We end this section with a more general reformulation of Theorem 7.4.6, in which we slightly generalise what the inclusions ψ and φ can look like, and more importantly we only require the squares in diagram (7.16) to be Cartesian up to an isomorphism of $Y \times S$. This will allow us to more easily apply the results in this section to our character variety CoHAs.

Corollary 7.3.9. *Consider diagram* (7.34) *and let $\alpha : Y \xrightarrow{\sim} Y$ and $\beta : X \xrightarrow{\sim} X$ be isomorphisms, and let $\delta : Y \times S \xrightarrow{\sim} Y \times S$ be an isomorphism such that the restriction of δ to Z' via i' is $\mathrm{id}_{Z'}$. Define*

$$\begin{aligned}
\psi' &= \psi \circ \alpha \\
\varphi' &= \varphi \circ \beta \\
\pi'_Y &= \alpha^{-1} \circ \pi_Y \\
\pi'_X &= \beta^{-1} \circ \pi_X \\
f'' &= \delta \circ f'.
\end{aligned}$$

Suppose that in diagram (7.16) *we replace ψ with ψ', φ with φ', π_Y with π'_Y, and π_X with π'_X, and in* (7.34) *we replace ψ with ψ', and φ with φ'. Assume that all of the* *conditions subsequent to* (7.34) *hold with these replacements. Then the diagram*

$$
\begin{tikzcd}
\widetilde{f}'_! \mathbb{Q}_{V'}[2d] \arrow[r, "\widetilde{f}'_!(\widetilde{\varphi}'^{\star})", "\sim"'] \arrow[d, "f''_{\star}"] & \widetilde{\psi}'_! \widetilde{f}_! \mathbb{Q}_V[2d] \arrow[d, "\widetilde{\psi}'_!(f_{\star})"] \\
\mathbb{Q}_{Z'} \arrow[r, "\widetilde{\psi}'^{\star}", "\sim"'] & \widetilde{\psi}'_! \mathbb{Q}_Z
\end{tikzcd}
\tag{7.45}
$$

commutes.

Remark 7.3.10. Note that in (7.45) we are using the morphism induced by f'' instead of f'. Also if we let $\widetilde{f}''$ denote the restriction of f'' to V' then since $\delta|_{Z'} = \mathrm{id}_{Z'}$ we have that

$$\widetilde{f}'' = (\delta \circ f')|_{V'} = \delta_{Z'} \circ f'_{V'} = \widetilde{f}'.$$

Proof. First consider the case in which $\delta = \mathrm{id}_{Y \times S}$. Then the proofs for Lemma 7.3.2, Lemma 7.3.3, Lemma 7.3.4, Proposition 7.3.5, Lemma 7.3.7 and Lemma 7.3.8 all follow when replacing ψ and φ with $\psi \circ \alpha$ and $\varphi \circ \beta$ and π_Y and π_X with $\alpha^{-1} \circ \pi_Y$ and $\beta^{-1} \circ \pi_X$.

If δ is arbitrary then for $f'' = \delta \circ f'$ the pushforward $f''_{\star}$ is given by the composition

$$
\begin{aligned}
&\widetilde{f}''_! j'^* f''^! \mathbb{Q}_{Y'} = \mathrm{id}_{Z'!} \widetilde{f}'_! j'^* f''^! \mathbb{Q}_{Y'} \xrightarrow{\mathrm{id}_{Z'!}(\epsilon^{i',f'}(f''^! \mathbb{Q}_{Y'})^{-1})} \mathrm{id}_{Z'!} i'^* f'_! f''^! \mathbb{Q}_{Y'} \xrightarrow{\epsilon^{i',\delta}(f'_! f''^! \mathbb{Q}_{Y'})^{-1}} \\
&i'^* \delta_! f'_! f''^! \mathbb{Q}_{Y'} = i'^* \delta_! f'_! f'^! \delta^! \mathbb{Q}_{Y'} \xrightarrow{i'^* \delta_!(\nu^{f'}(\delta^! \mathbb{Q}_{Y'}))} i'^* \delta_! \delta^! \mathbb{Q}_{Y'} \xrightarrow{i'^*(\nu^{\delta}(\mathbb{Q}_{Y'}))} i'^* \mathbb{Q}_{Y'}.
\end{aligned}
$$

Hence if we can show that $\epsilon^{i',\delta}$ and ν^{δ} are equal to the identity natural transformations then we are done by Proposition 7.3.5 and the proofs of Lemma 7.3.7 and Lemma 7.3.8, as we have reduced to the case of f'. Clearly ν^{δ} is the identity transformation because for an isomorphism δ all the units and counits for both adjunctions can be taken to be the identity. As for $\epsilon^{i',\delta}$, we again initially work at the level of sheaves and underived functors and consult Proposition 3.1.1. The natural isomorphism $i'^* \delta_! \xrightarrow{\sim} \mathrm{id}_{Z'!} i'^*$ is the composition

$$i'^* \delta_! \longrightarrow i'^* \delta_! i'_* i'^* \longrightarrow i'^* i'_* \mathrm{id}_{Z'!} i'^* \longrightarrow \mathrm{id}_{Z'!} i'^*$$

where the first morphism comes from the natural transformation $\mathrm{id}_{\mathrm{Sh}(Y \times S)} \to i'_* i'^*$ and the third morphism comes from the natural transformation $i'^* i'_* \to \mathrm{id}_{\mathrm{Sh}(Z')}$ for the adjunction $i'^* \dashv i'_*$. For the second morphism, since δ is an isomorphism and so $\delta_! = \delta_*$, the natural transformation

$$\delta_! i'_* \to i'_* \mathrm{id}_{Z'!}$$

described in the proof of [[41] Proposition 2.5.11] is the identity. Similarly to the derived case, the natural transformation $i'^* i'_* \to \mathrm{id}_{\mathrm{Sh}(Z')}$ is the identity natural isomorphism since i' is a closed immersion. Hence it remains to consider $i'^* \delta_!(\eta^{i'})$. We have that for

$\mathcal{F} \in \mathrm{Sh}(Y')$

$$(i'_* i'^* \mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in Z' \\ 0 & \text{if } p \in Y' \setminus Z' \end{cases}$$

and so

$$\eta^{i'}(\mathcal{F})_p = \begin{cases} \mathrm{id}_{\mathcal{F}_p} & \text{if } p \in Z' \\ 0 & \text{if } p \in Y' \setminus Z'. \end{cases}$$

Therefore, because $\delta|_{Z'} = \mathrm{id}_{Z'}$, by taking stalks at $p \in Z'$ we get that $i'^* \delta_!(\eta^{i'})$ is also the identity. Upgrading to the level of derived categories then implies that the natural isomorphism $\epsilon^{i',\delta}$ is also the identity. □

7.4 An isomorphism of 2D CoHAs for the character variety

We now utilise the results from the previous section to show that the two multiplications on the dual of the compactly supported cohomology of the character variety described in Section 7.2 are equal, and in the process show that the multiplication derived from the brane tiling picture is independent of the choices of the cut E and maximal tree T. We first explain the setup that will allow us to apply the results from Section 7.3.

Lemma 7.4.1. *Let G be the free group $F(y_1, \dots, y_m)$ and G' be the free group $F(x_1 \dots, x_m)$, and let $\lambda \in G$ and $\lambda' \in G'$ be words. Suppose we have the following commutative diagram of algebras*

$$\begin{tikzcd} \mathbb{C}[G'] \arrow[r, "\tau", "\sim"'] \arrow[d, two heads, "p"] & \mathbb{C}[G] \arrow[d, two heads, "q"] \\ \mathbb{C}[G']/\big(\lambda' - 1\big) \arrow[r, "\widetilde{\tau}", "\sim"'] & \mathbb{C}[G]/\big(\lambda - 1\big) \end{tikzcd} \tag{7.46}$$

where λ and λ' exist as elements of the group algebras in the natural way, p and q are the obvious quotient maps, τ is induced from an isomorphism of the free groups

$$F(x_1, \dots, x_m) \xrightarrow[\sim]{\epsilon} F(y_1, \dots, y_m),$$

and $\widetilde{\tau}$ is also an isomorphism. Then

$$\tau(\lambda') = u\, \lambda^c\, u^{-1}$$

for some monomial $u \in \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle$ and $c \in \{1, -1\}$.

Proof. First we claim that

$$\big(\tau(\lambda') - 1\big) = \big(\lambda - 1\big)$$

as ideals in $\mathbb{C}\langle y_1^{\pm 1}, \ldots, y_m^{\pm 1}\rangle$. Suppose not, then either there exists some element $a \in \big(\tau(\lambda') - 1\big)$ such that $a \notin \big(\lambda - 1\big)$ or there exists some $a \in \big(\lambda - 1\big)$ such that $a \notin \big(\tau(\lambda') - 1\big)$. Assume the first case holds. Write $a = \tau(b)$ for some $b \in \big(\lambda' - 1\big)$. Then $p(b) = 0$ hence $\widetilde{\tau}(p(b)) = 0$ and so by the commutativity of (7.46) we get that $q(\tau(b)) = q(a) = 0$. But this implies that $a \in \big(\lambda - 1\big)$ a contradiction to the initial assumption. The second case follows from a similar argument.

As τ is induced from the group isomorphism ϵ consider the elements λ, $\epsilon(\lambda') \in F(y_1, \ldots, y_m)$. Let $N(\lambda)$ and $N(\epsilon(\lambda'))$ denote the normal closures in $F(y_1, \ldots, y_m)$ of λ and $\epsilon(\lambda')$ respectively i.e. the smallest normal subgroups of $F(y_1, \ldots, y_m)$ that contain λ or $\epsilon(\lambda')$ respectively. Now for an arbitrary group H we have an injective mapping from the set of all normal subgroups of H to the set of two-sided ideals of the group algebra $\mathbb{C}[H]$ that sends a normal subgroup $N \lhd H$ to the ideal generated by all elements of the form $h - 1$ for $h \in N$ (see [[51] Proposition 3.3.6]). Then clearly the ideal generated by $N(\lambda)$ is $\big(\lambda - 1\big)$ and the ideal generated by $N(\epsilon(\lambda'))$ is $\big(\tau(\lambda') - 1\big)$, and hence the equality of ideals $\big(\tau(\lambda') - 1\big) = \big(\lambda - 1\big)$ in $\mathbb{C}[G]$ implies the equality of the normal closures

$$N(\epsilon(\lambda')) = N(\lambda)$$

in $F(y_1, \ldots, y_m)$.

Then [[47] Proposition 5.8] says that two elements of a free group have equal normal closures if and only if one is conjugate to the other or its inverse, i.e. we get that

$$\epsilon(\lambda') = u\lambda^c u^{-1}$$

for some $c \in \{1, -1\}$ and some word $u \in F(x_1, y_1, \ldots, x_g, y_g)$ which immediately implies the result. $\square$

For an arbitrary quiver with one vertex Q note that the localised path algebra $\mathbb{C}\widetilde{Q}$ is the group algebra of the free group $F(Q) := F(a : a \in Q_1)$ whose generators are the arrows in Q.

Proposition 7.4.2. *For a Riemann surface Σ_g fix a brane tiling Δ, a cut E for (Q_Δ, W_Δ) and a maximal tree $T \subset Q_\Delta \setminus E$. Let Q denote the quiver given by contracting T in $Q_\Delta \setminus E$. Then there exists a subquiver $Q' \subset Q$ such that*

i) *There exists an equality of ideals in $\mathbb{C}\widetilde{Q}$*

$$\Big(h - p_h \,:\, h \in Q_1 \setminus Q_1'\Big) = \left(\frac{\partial W}{\partial e} \,:\, e \neq e_0\right)$$

where p_h is a path in the localised path algebra $\mathbb{C}\widetilde{Q}'$ and e_0 is some nominated element of the cut E

ii) $|Q_1'| = 2g$

iii) *There exists an isomorphism of free groups*

$$\epsilon : F(Q_1') \xrightarrow{\sim} F(x_1, y_1, \dots, x_g, y_g)$$

which induces the isomorphism of group algebras

$$\tau'' : \mathbb{C}\widetilde{Q}' \xrightarrow{\sim} \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle$$

such that the surjection

$$\tau' : \mathbb{C}\widetilde{Q} \xrightarrow{\tau} \mathbb{C}\widetilde{Q}' \xrightarrow[\sim]{\tau''} \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle$$

induces the isomorphism

$$\mathrm{Jac}(\widetilde{Q}, W, E) \xrightarrow[\sim]{\widetilde{\tau}'} \mathbb{C}[\pi_1(\Sigma_g)]$$

from Theorem 7.2.2, where

$$\tau(a) = \begin{cases} a & \text{if } a \in Q_1', \\ p_h & \text{if } a = h \in Q_1 \setminus Q_1'. \end{cases}$$

Proof. For an arrow $a \in Q_{\Delta,1}$ let $\widehat{a} \in \Delta$ denote its dual edge. Let

$$\widehat{E} = \big\{\widehat{e} \in \Delta \mid e \in E\big\}$$

be the dimer corresponding to the cut E, i.e. the set of edges in the brane tiling which are dual to the arrows in E. Extend $\widehat{E}$ to a maximal tree $\widehat{F}$ inside Δ. Let $\widehat{H}$ denote the set of edges in the complement $\widehat{F} \setminus \widehat{E}$ and let $r = |\widehat{H}|$. Note that $r = |E| - 1$, since to extend $|\widehat{E}| = |E|$ disjoint edges that touch every vertex into a maximal tree we require $|E| - 1$ additional edges. Let

$$H = \big\{h \in Q_1 \mid \widehat{h} \in \widehat{H}\big\}.$$

then take $Q' \subset Q$ to be the quiver with arrow set $Q_1' = Q_1 \setminus H$.

Let us consider what a relation $\partial W/\partial e$ for an arrow $e \in E$ means. Since E is a cut we have that $\partial W/\partial e$ identifies two paths in Q. For any $h \in H$ there exists some $e \in E$ (specifically e is dual to $\widehat{e}$ where $\widehat{e}$ and $\widehat{h}$ are connected in Δ) such that we can use the relation $\partial W/\partial e$ to identify h with a path in $\mathbb{C}(\widetilde{Q \setminus \{h\}})$. As $\widehat{H}$ turns $\widehat{E}$ into a tree for each $h \in H$ we can choose a distinct $e \in E$ giving pairs (h, e) such that $\partial W/\partial e$ allows us to identify h with a path in $\mathbb{C}(\widetilde{Q \setminus \{h\}})$, and since $|H| = r = |E| - 1$

we will always have one extra arrow $e_0 \in E$ that has not been paired with any $h \in H$. The edges on the ends of the tree $\widehat{F}$ must be in $\widehat{E}$, so if we take $\widehat{e}_1$ to be an edge on the end of $\widehat{F}$ such that it is only connected to one $\widehat{h}_1 \in \widehat{H}$ we get that the relation $\partial W/\partial e_1$ identifies a path in $\mathbb{C}(Q \setminus H)$ with a path in $\mathbb{C}Q$, and so in particular we can use $\partial W/\partial e_1$ to identify h_1 with a path in $\mathbb{C}\widetilde{Q}'$. This can be repeated for all $h \in H$ by following the edge $\widehat{h}$ to the ends of the tree $\widehat{F}$ and utilising the relations $\partial W/\partial e$ for any $\widehat{e}$ we encounter along this route (see (7.2), (7.3) and (7.4) for an illustration of this).

To be more precise we go by induction. First fix pairs (h_i, e_i) where $\widehat{e_i}$ is connected to $\widehat{h}_i$ in Δ, for $i = 1, \dots, r$. As $|E| = r+1 = |H|+1$ we have one remaining element of the cut which we shall denote by e_0. Define the *route length* of h_i to be 1 plus the sum of the route lengths of h_j where $\widehat{e}_i$ is connected to $\widehat{h}_j$ for $i \neq j$ (see (7.4)). Note that there must exist some h_i of route length 1 as otherwise this means that every $\widehat{e} \in \widehat{E}$ is connected to at least two $\widehat{h} \in \widehat{H}$, but as $\widehat{E}$ and $\widehat{H}$ together form a tree this would mean that $|\widehat{H}| > |\widehat{E}|$ which is a contradiction. We go by induction on the route length of the h_i. For the base case suppose that h_i has route length 1. Then up to a sign we have that

$$\frac{\partial W}{\partial e_i} = q_i' h_i q_i'' - p_i'$$

for paths p_i', q_i', q_i'' in Q', since from the definition of the potential W the route length of h_i being 1 implies that no other arrow $h \in H$ appears in $\partial W/\partial e_i$. Therefore in $\mathbb{C}\widetilde{Q}$ we can write

$$h_i = p_i + q_i'^{-1}\left(\frac{\partial W}{\partial e_i}\right)q_i''^{-1}$$

where $p_i = q_i'^{-1} p_i' q_i''^{-1}$ is a path in $\mathbb{C}\widetilde{Q}'$, and

$$\frac{\partial W}{\partial e_i} = q_i'\big(h_i - p_i\big)q_i''$$

as required. So now suppose it is true that for all h_j of route length at most m we have the equality of ideals

$$\begin{aligned} I_m &:= \Big(h_j - p_j \ : \ \text{route length}(h_j) \leq m\Big) \\ &= \left(\frac{\partial W}{\partial e_k} \ : \ e_k \text{ paired with } h_k \text{ for route length}(h_k) \leq m\right). \end{aligned}$$

Then for h_i of route length $m+1$ consider (again up to a sign)

$$\frac{\partial W}{\partial e_i} = q_i' h_i q_i'' - p_i'$$

where this time p_i', q_i', q_i'' may now contain arrows in H too. However any arrow $h_j \in H$ that appears in either of p_i' or q_i' or q_i'' must have route length strictly less than the route length of h_i. Hence if we suppose that

$$q_i'^{-1} p_i' q_i''^{-1} = \prod_j q_j' h_j^{b_j} q_j''$$

where q_j' and q_j'' are paths in $\mathbb{C}\widetilde{Q}'$ and $b_j \in \{-1, 1\}$ (h_j can appear at most once in $\partial W/\partial e_i$ due to how W_Δ was defined), by the induction hypothesis we can write in $\mathbb{C}\widetilde{Q}/I_m$ (where $\overline{p} \in \mathbb{C}\widetilde{Q}/I_m$ denotes the image of $p \in \mathbb{C}\widetilde{Q}$ under the natural quotient map)

$$\overline{h}_i = \prod_j \overline{q}_j' \overline{p}_j^{b_j} \overline{q}_j'' + \overline{q}_i'^{-1} \left(\overline{\frac{\partial W}{\partial e_i}} \right) \overline{q}_i''^{-1}$$

with $\prod_j q_j' p_j^{b_j} q_j'' = p_i$ a path in $\mathbb{C}\widetilde{Q}'$, and

$$\overline{\frac{\partial W}{\partial e_i}} = \overline{q}_i' \Big(\overline{h}_i - \overline{p}_i \Big) \overline{q}_i''.$$

It follows that that

$$\frac{\mathbb{C}\widetilde{Q}}{\big(h_i - p_i, I_m\big)} = \frac{\mathbb{C}\widetilde{Q}}{\big(\partial W/\partial e_i, I_m\big)}$$

and hence as ideals in $\mathbb{C}\widetilde{Q}$

$$\big(h_i - p_i, I_m\big) = \left(\frac{\partial W}{\partial e_i}, I_m \right)$$

as required for i).

This implies the following isomorphism of algebras

$$\mathrm{Jac}(\widetilde{Q}, W, E) = \frac{\mathbb{C}\widetilde{Q}}{\big(\partial W/\partial e : e \in E\big)} \xrightarrow[\sim]{\tilde{\tau}} \frac{\mathbb{C}\widetilde{Q}'}{\big(\tau(\partial W/\partial e_0)\big)}$$

where $\tau : \mathbb{C}\widetilde{Q} \longrightarrow \mathbb{C}\widetilde{Q}'$ is the surjective homomorphism

$$a \in Q_1 \mapsto \begin{cases} a & \text{if } a \in Q_1', \\ p_i & \text{if } a = h_i \in H \end{cases}$$

and recall $e_0 \in E$ is the remaining element of the cut not paired with any $h \in H$. Let

$$\tau \left(\frac{\partial W}{\partial e_0} \right) = p_0 - q_0$$

then write

$$\frac{\mathbb{C}\widetilde{Q}'}{\big(\tau(\partial W/\partial e_0)\big)} = \mathbb{C}[G'] \tag{7.47}$$

where G' is the group with presentation $\langle a \in Q'_1 \mid p_0 q_0^{-1} \rangle$. We get the following diagram of algebras in which the left-hand square commutes

$$\begin{tikzcd}
\mathbb{C}\widetilde{Q} \arrow[r, two heads, "\tau"] \arrow[d, two heads, "p"] & \mathbb{C}\widetilde{Q}' \arrow[r, dashed, "\tau''", "\sim"'] \arrow[d, two heads, "q'"] & \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1} \rangle \arrow[d, two heads, "q"] \\
\mathrm{Jac}(\widetilde{Q}, W, E) \arrow[r, "\widetilde{\tau}", "\sim"'] & \mathbb{C}[G'] \arrow[r, "\widetilde{\tau}''", "\sim"'] & \mathbb{C}[\pi_1(\Sigma_g)]
\end{tikzcd} \tag{7.48}$$

where $\widetilde{\tau}''$ is chosen such that $\widetilde{\tau}'' \circ \widetilde{\tau} = \widetilde{\tau}'$ is the isomorphism given in Theorem 7.2.2, and the vertical maps are the quotients. We aim to find an isomorphism τ'' which makes the right-hand square commute.

We first note that the description of the isomorphism $\widetilde{\tau}' : \mathrm{Jac}(\widetilde{Q}, W, E) \xrightarrow{\sim} \mathbb{C}[\pi_1(\Sigma_g)]$ from [[14] Sections 4 and 5] tells us that $\widetilde{\tau}'$ is induced from a homomorphism of groups

$$G_{Q,W,E} \longrightarrow \pi_1(\Sigma_g)$$

where $G_{Q,W,E}$ is the group with presentation

$$\langle a \in Q_1 \mid p_e q_e^{-1} \,:\, e \in E \rangle$$

for $\partial W/\partial e = p_e - q_e$. Indeed the quiver Q embeds into the Riemann surface Σ_g as explained in [14] and the multiplication of paths in the quiver Q (which are in fact loops as Q only has one vertex) corresponds to concatenation of loops in the surface. The relations from the cut and potential then correspond to the relation in $\pi_1(\Sigma_g)$. This implies that the isomorphism $\widetilde{\tau}'' : \mathbb{C}[G'] \xrightarrow{\sim} \mathbb{C}[\pi_1(\Sigma_g)]$ is induced from a homomorphism of groups

$$G' \xrightarrow{\xi} \pi_1(\Sigma_g)$$

since $\widetilde{\tau}$ is induced from the isomorphism of groups

$$G_{Q,W,E} \xrightarrow{\sim} G'$$

that sends $a \in Q_1 \setminus H \mapsto a$ and $h \in H \mapsto p_h$.

We claim that the group homomorphism ξ is an isomorphism. Suppose ξ is not surjective, then there exists some $y \in \pi_1(\Sigma_g)$ not in the image of ξ. An arbitrary element of $\mathbb{C}[G']$ is of the form $x = \sum_{i=1}^m c_i x_i$ and therefore

$$\widetilde{\tau}''(x) = \sum_{i=1}^{m} c_i \, \xi(x_i).$$

As $\widetilde{\tau}''$ is surjective suppose that $\widetilde{\tau}''(x) = y$. Hence for at least one i we must have that $\xi(x_i) = y$ giving a contradiction to the assumption that y does not lie in the image of ξ. Now suppose ξ is not injective. Then there exists an $x \in G'$ such that $\xi(x)$ is null-homotopic. But then $\widetilde{\tau}''(x) = \xi(x)$ is null-homotopic and hence equal to 1 in the fundamental group algebra $\mathbb{C}[\pi_1(\Sigma_g)]$ which gives a contradiction to the injectivity of $\widetilde{\tau}''$.

So we have an isomorphism of one-relator groups $G' \underset{\sim}{\xrightarrow{\xi}} \pi_1(\Sigma_g)$ and hence [[47] Proposition 5.11] tells us that the number of generators of G' must be equal to $2g$, i.e. $|Q_1'| = 2g$ as required by ii).

Finally we show the existence of the isomorphism τ''. Let $F(z_1, \ldots, z_m)$ be the free group on the generators z_i. Define a *marking* of $\pi_1(\Sigma_g)$ of size m to be a surjective group homomorphism $\pi : F(z_1, \ldots, z_m) \to \pi_1(\Sigma_g)$, hence a marking of size m is equivalent to a set $\{\alpha_1, \ldots, \alpha_m\} \subset \pi_1(\Sigma_g)$ of generators for $\pi_1(\Sigma_g)$. Then [[46] Theorem 1.1] says that any two markings π and π' of $\pi_1(\Sigma_g)$ of the same size are Nielsen equivalent, i.e. that there exists an automorphism ϵ of the free group $F(z_1, \ldots, z_m)$ such that the following diagram commutes

$$\begin{tikzcd}
F(z_1, \ldots, z_m) \arrow[rr, "\epsilon", "\sim"'] \arrow[drr, "\pi"] & & F(z_1, \ldots, z_m) \arrow[d, "\pi'"] \\
& & \pi_1(\Sigma_g)
\end{tikzcd}$$

In our case we take the generators $\{\xi(a) : a \in Q_1'\}$ to define the marking π of size $2g$ and the standard generators $\{x_1, y_1, \ldots, x_g, y_g\}$ to define π', giving us the automorphism ϵ. Upgrading ϵ into an isomorphism of group algebras provides us with τ''. $\square$

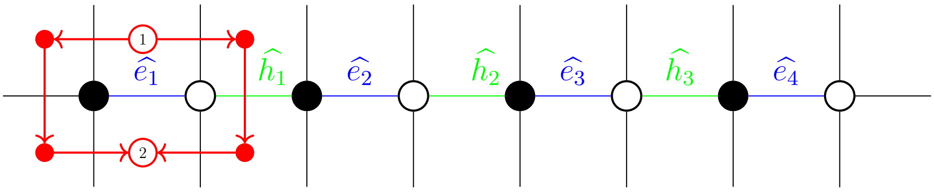


Figure 7.2: An example of a brane tiling with black and white vertices, with the edges in the dimer $\widehat{E}$ in blue and the edges in $\widehat{H}$ in green. The tree $\widehat{F}$ then consists of all the blue and green edges and so its ends are $\widehat{e}_1$ and $\widehat{e}_4$. For illustrative purposes we do not contract the tree T in Q_Δ and so our quivers will have more than one vertex. We can see that the relation $\partial W/\partial e_1$ identifies the two paths in the quiver (in red) that go from vertex 1 to vertex 2 in Q_Δ. Since $\widehat{e_1}$ is at an end of the tree one path does not contain any arrows which are dual to edges in $\widehat{H}$.

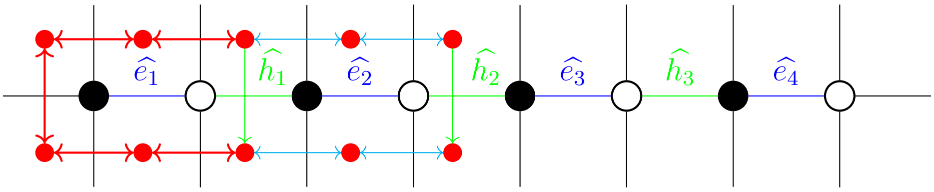


Figure 7.3: By adding inverses to the arrows in Q_Δ we can use the relation $\partial W/\partial e_1$ to write the arrow in the quiver dual to $\widehat{h_1}$ (in green) in terms of the red path and so in particular as a path in $\mathbb{C}\widetilde{Q}'_\Delta = \mathbb{C}(\widetilde{Q_\Delta \setminus H})$. Similarly for the arrow in the quiver dual to $\widehat{h_2}$ (also in green), by following the edge $\widehat{h_2}$ to an end of the tree in the brane tiling (here we go to the left and end up at $\widehat{e_1}$) we can also write it as a path in $\mathbb{C}\widetilde{Q}'_\Delta$; first by using $\partial W/\partial e_2$ we can write h_2 in terms of the cyan arrows and h_1, then using $\partial W/\partial e_1$ we can write it as the concatenation of the cyan path along the top, then the red path, then the cyan path along the bottom.

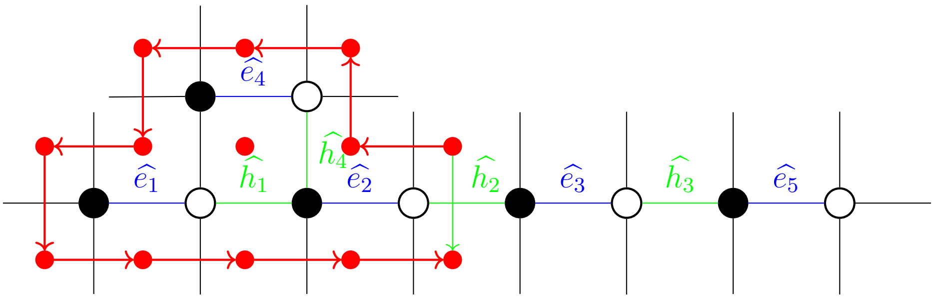


Figure 7.4: Fixing the pairs (e_1, h_1), (e_2, h_2), (e_3, h_3), (e_4, h_4) we get that route length$(h_1) = 1$, route length$(h_2) = 3$, route length$(h_3) = 4$, route length$(h_4) = 1$. Route length 1 arrows in H exist as those paired to arrows $e \in E$ with the property that $\widehat{e}$ only connects to a single $\widehat{h} \in \widehat{H}$. We can use the relations $\partial W/\partial e_j$, where $\widehat{e_j}$ is on the route from $\widehat{h_i}$ to an end of the tree in the brane tiling, to write h_i as a path in $\mathbb{C}\widetilde{Q}'_\Delta$ e.g. we see that the arrow h_2 in green is equal to the red path using $\partial W/\partial e_2$ then $\partial W/\partial e_1$ and $\partial W/\partial e_4$.

We are now finally in a position to begin applying the results from Section 7.3 to the character variety. We fix the brane tiling Δ on our Riemann surface Σ_g, as well as the cut E, maximal tree T, and subquiver $Q' \subset Q$ as in Proposition 7.4.2. Recall that $|E| = r+1$, $|Q'_1| = 2g$ and $|Q_1| = 2g + r$. Let $e_i \in E$ for $i = 0, \ldots, r$. Also note that

$$\begin{aligned}
M_n &= M_n(\mathbb{C}\widetilde{Q}) = \mathrm{GL}_n^{2g+r}\\
Y'_{m,n} &\cong M_m \times M_n \times \mathrm{Mat}_{m\times n}^{|E|}\\
&= \mathrm{GL}_m^{2g+r} \times \mathrm{GL}_n^{2g+r} \times \mathrm{Mat}_{m\times n}^{r+1}\\
&= \left(\mathrm{GL}_m^{2g} \times \mathrm{GL}_n^{2g} \times \mathrm{Mat}_{m\times n}\right) \times \left(\mathrm{GL}_m^{r} \times \mathrm{GL}_n^{r} \times \mathrm{Mat}_{m\times n}^{r}\right)\\
&\cong Y_{m,n} \times \mathrm{GL}_{m,n}^{r}\\
M_{m,n} &= M_{m,n}(\mathbb{C}\widetilde{Q}) = \mathrm{GL}_{m,n}^{2g+r}\\
&= \mathrm{GL}_{m,n}^{2g} \times \mathrm{GL}_{m,n}^{r}\\
&= M_{m,n}(\mathbb{C}\widetilde{Q}') \times \mathrm{GL}_{m,n}^{r}.
\end{aligned}$$

It will be much more convenient to view $Y_{m,n}$ as $\mathrm{GL}_m^{2g} \times \mathrm{GL}_n^{2g} \times \mathrm{Mat}_{m\times n}$ and $Y'_{m,n}$ as $M_m \times M_n \times \mathrm{Mat}_{m\times n}^{r+1}$. We shall index matrices in the $M_{m,n}(\widetilde{Q}') = \mathrm{GL}_{m,n}^{2g}$ part using $a \in Q'_1$ and matrices in the $\mathrm{GL}_{m,n}^{r}$ part using h or $h_i \in Q_1 \setminus Q'_1$ for $i = 1, \ldots, r$. Then recalling our notation for m, n-block matrices

$$R = \begin{pmatrix} R^{(1)} & R^{(3)} \\ 0 & R^{(2)} \end{pmatrix}$$

we have that in diagrams (7.2) and (7.15) for

$(A_i, B_i) \in \mathrm{GL}^{2g}_{m,n} = M_{m,n}(\mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle)$ and for $\rho \in M_{m,n}$

$$\begin{aligned} f(A_i, B_i) &= \Big((A_i^{(1)}, B_i^{(1)}),\, (A_i^{(2)}, B_i^{(2)}),\; \lambda_{m+n}(A_i, B_i)^{(3)} \Big) \\ f''(\rho) &= \left(\rho^{(1)},\, \rho^{(2)},\, \left(\frac{\partial W}{\partial e_i}(\rho)^{(3)} \right)_{i=1,\dots,r} \right). \end{aligned}$$

So consider the following diagram whose front face is (7.13) and whose back face is (7.15):

$$\begin{tikzcd}
 & M_{m+n} & & M_{m,n} \arrow[ll, hook', "h'"'] \arrow[rr, "f''"] & & Y'_{m,n} \\
\mathrm{GL}^{2g}_{m+n} \arrow[ur, hook, "\varphi'_{m+n}"] & & \mathrm{GL}^{2g}_{m,n} \arrow[ll, hook', "h"'] \arrow[ur, hook, "\varphi'_{m,n}"] \arrow[rr, "f"] & & Y_{m,n} \arrow[ur, hook, "\psi'_{m,n}"] & \\
 & Z_{m+n} \arrow[uu, hook, "j'_{m+n}"] & & Z_{m,n} \arrow[ll, hook', "\widetilde{h'}"'] \arrow[uu, hook, "j'_{m,n}"] \arrow[rr, "\widetilde{f''}"] & & Z_m \times Z_n \arrow[uu, hook, "i'"'] \\
V_{m+n} \arrow[uu, hook, "j_{m+n}"] \arrow[ur, "\widetilde{\varphi}'_{m+n}", "\sim"'] & & V_{m,n} \arrow[ll, hook', "\widetilde{h}"'] \arrow[uu, hook, "j_{m,n}"] \arrow[ur, "\sim", "\widetilde{\varphi}'_{m,n}"'] \arrow[rr, "\widetilde{f}"] & & V_m \times V_n \arrow[uu, hook, "i"] \arrow[ur, "\sim", "\widetilde{\psi}'_{m,n}=\widetilde{\varphi}'_m \times \widetilde{\varphi}'_n"'] &
\end{tikzcd} \tag{7.49}$$

where the inclusion $\varphi'_n : \mathrm{GL}^{2g}_n \to M_n$ is induced from the algebra homomorphism τ' given in Proposition 7.4.2, the inclusion $\psi'_{m,n} : Y_{m,n} \to Y'_{m,n}$ is given by

$$\big(\rho',\, \rho'',\, R\big) \longmapsto \big(\varphi'_m(\rho'),\, \varphi'_n(\rho''),\, R,\, 0, \dots, 0\big)$$

and the isomorphism $\widetilde{\varphi}'_n : V_n \xrightarrow{\sim} Z_n$ is induced from the algebra isomorphism $\widetilde{\tau}'$ from Theorem 7.2.2.

All the squares that go into the page are Cartesian due to the commutativity of diagram (7.48). The rest of the squares, except for the top-right square, are all easily checked to be Cartesian as well. The top-right square may not even be commutative in fact, since we have that

$$\begin{aligned} f'' \circ \varphi'_{m,n}(A_i, B_i) &= \left(\varphi'_{m,n}(A_i, B_i)^{(1)},\, \varphi'_{m,n}(A_i, B_i)^{(2)},\, \left(\frac{\partial W}{\partial e_j}(\varphi'_{m,n}(A_i, B_i))^{(3)} \right) \right) \\ &= \left(\varphi'_m(A_i^{(1)}, B_i^{(1)}),\, \varphi'_n(A_i^{(2)}, B_i^{(2)}),\, \left(\frac{\partial W}{\partial e_j}(\varphi'_{m,n}(A_i, B_i))^{(3)} \right) \right) \end{aligned}$$

whilst

$$\psi_{m,n} \circ f(A_i, B_i) = \Big(\varphi'_m(A_i^{(1)}, B_i^{(1)}),\, \varphi'_n(A_i^{(2)}, B_i^{(2)}),\; \lambda_{m+n}(A_i, B_i)^{(3)},\, 0, \dots, 0 \Big)$$

and therefore this square is commutative if and only if (up to a re-ordering of the arrows in E) we have $\frac{\partial W}{\partial e_0}(\varphi'_{m,n}(A_i,B_i))^{(3)} = \lambda_{m+n}(A_i,B_i)^{(3)}$ and $\frac{\partial W}{\partial e_j}(\varphi'_{m,n}(A_i,B_i))^{(3)} = 0$ for $j = 1,\dots,r$. This means that we cannot apply Corollary 7.3.9 to the right-hand cube in (7.49) because there most likely will not exist an isomorphism $\delta : Y'_{m,n} \xrightarrow{\sim} Y'_{m,n}$ that will make this square Cartesian after replacing f'' with $f' = \delta^{-1} \circ f''$.

Example 7.4.3. Continuing the example of the brane tiling for the genus 2 surface from Example 7.2.4, we took the cut as $E = \{a,b,c\}$ and the maximal tree as $T = \{e,h,k\}$. To extend the dimer $\widehat{E}$ to a maximal tree in the tiling we can take $\widehat{H} = \{\widehat{f},\widehat{j}\}$ and so $H = \{f,j\}$. Pairing f with c and j with a we can use $\partial W/\partial c$ and $\partial W/\partial a$ to write

$$\begin{aligned} f &= i^{-1}l^{-1}id \\ j &= gdg^{-1} \end{aligned}$$

as we saw before. Then $e_0 = b$ and $\partial W/\partial b$ becomes

$$li^{-1}l^{-1}idgd^{-1}g^{-1} - 1$$

and so defining τ'' by sending $l \mapsto x_1$, $i^{-1} \mapsto y_1$, $d \mapsto x_2$, $g \mapsto y_2$ gives us the isomorphism and surjection of algebras from Proposition 7.4.2. It follows that φ'_n sends

$$\big(A_1,B_1,A_2,B_2\big) \longmapsto \big(A_2,\ B_1A_1^{-1}B_1^{-1}A_2,\ B_2,\ B_1^{-1},\ B_2A_2B_2^{-1},\ A_1\big)$$

and we see that

$$\begin{aligned} f'' \circ \varphi'_{m,n}(A_1,B_1,A_2,B_2) &= \Big(B_2A_2B_2^{-1}B_2 - B_2A_2, \quad A_1B_1A_1^{-1}B_1^{-1}A_2 - B_2A_2B_2^{-1}, \\ &\qquad B_1^{-1}A_2 - A_1B_1^{-1}B_1A_1^{-1}B_1^{-1}A_2\Big) \\ &= \Big(0,\ A_1B_1A_1^{-1}B_1^{-1}A_2 - B_2A_2B_2^{-1},\ 0\Big) \end{aligned}$$

whilst

$$\psi_{m,n} \circ f(A_1,B_1,A_2,B_2) = \Big(0,\ A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2B_2^{-1} - \mathrm{Id}_{m+n},\ 0\Big).$$

Hence in this case the top-right square in (7.49) does not commute and there does not exist an isomorphism δ that can rectify this.

To deal with the non-commutativity of the top-right square in (7.49) we extend the right-hand cube into:

$$
\begin{tikzcd}
& M_{m,n} \arrow[rr, "f_0''"] & & X'_{m,n} \arrow[rr, "\pi_1'"] & & Y'_{m,n} \\
\mathrm{GL}^{2g}_{m,n} \arrow[ur, hook, "\varphi'_{m,n}"] \arrow[rr, "f_0"] & & X_{m,n} \arrow[ur, hook, "\psi'_{m,n}\times \mathrm{id}"] \arrow[rr, "\pi_1"] & & Y_{m,n} \arrow[ur, hook, "\psi'_{m,n}"] & \\
& Z_{m,n} \arrow[uu, hook, "j'_{m,n}"'] \arrow[rr, "\widetilde{f_0''}"] & & Z \arrow[uu, hook, "j'"'] \arrow[rr, "\widetilde{\pi}_1'"] & & Z_m \times Z_n \arrow[uu, hook, "i'"'] \\
V_{m,n} \arrow[uu, hook, "j_{m,n}"] \arrow[ur, hook, "\widetilde{\varphi}'_{m,n}"'] \arrow[rr, "\widetilde{f}_0"'] & & V \arrow[uu, hook, "j"] \arrow[ur, hook, "\widetilde{\psi}'_{m,n}\times \mathrm{id}"'] \arrow[rr, "\widetilde{\pi}_1"'] & & V_m \times V_n \arrow[uu, hook, "i"] \arrow[ur, hook, "\widetilde{\psi}'_{m,n}"'] &
\end{tikzcd}
\tag{7.50}
$$

where

$$
\begin{aligned}
X_{m,n} &= Y_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \\
X'_{m,n} &= Y'_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \\
V &= V_m \times V_n \times \mathrm{Mat}^{2g+r}_{m\times n} \\
Z &= Z_m \times Z_n \times \mathrm{Mat}^{2g+r}_{m\times n}
\end{aligned}
$$

and

$$
\begin{aligned}
f_0(A_i, B_i) &= \Big(f(A_i, B_i),\ \varphi'_{m,n}(A_i, B_i)^{(3)}\Big) \\
f_0''(\rho) &= \Big(f''(\rho),\ \rho^{(3)}\Big)
\end{aligned}
$$

the maps π_1, π_1', $\widetilde{\pi}_1$, $\widetilde{\pi}_1'$ are the respective projections, and j, j' are the obvious inclusions.

Hence (7.50) is indeed just the right-hand cube of (7.49) since $\pi_1 \circ f_0 = f$, $\pi_1' \circ f_0'' = f''$ and $\widetilde{\pi}_1 \circ \widetilde{f}_0 = \widetilde{f}$, $\widetilde{\pi}_1' \circ \widetilde{f}_0'' = \widetilde{f}''$. It is also clear that all the squares in diagram (7.50) except the top-left square are Cartesian. The top-left square of (7.50) will fail to be commutative due to the top-right square of (7.49) not being commutative.

Proposition 7.4.4. *The right-hand cube in diagram* (7.50) *satisfies all the conditions needed to apply Corollary 7.3.9.*

Proof. Looking at the conditions from (7.34) needed to apply Corollary 7.3.9, all are easily verified apart from the condition that the top face of this cube can be fitted into a diagram of the form (7.16) using maps $\psi' = \psi \circ \alpha$, $\varphi' = \varphi \circ \beta$ and $f'' = \delta \circ f'$ such that $\pi_Y \circ \psi = \mathrm{id}_{Y_{m,n}}$ and $(\pi_Y \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}}) \circ \varphi = \mathrm{id}_{X_{m,n}}$, where ψ, α, φ, β, f', δ and π_Y need to be determined.

For this cube, we have $\varphi' = \psi'_{m,n} \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}}$ and $\psi' = \psi'_{m,n}$. The inclusion $\psi'_{m,n}$ is defined using φ'_m and φ'_n which are in turn determined by the algebra surjection τ'

given by Proposition 7.4.2. τ' was constructed such that $\tau' = \tau'' \circ \tau$ which gives us the factorisation of φ' and of ψ' we are looking for. In particular, we write $\varphi'_n = \varphi_n \circ \alpha_n$ where $\varphi_n : \mathrm{GL}_n^{2g} \hookrightarrow \mathrm{GL}_n^{2g+r}$ is the inclusion

$$\big(A_a\big)_{a\in Q'_1} \longmapsto \big(A_a, p_i(A_a)\big)_{a\in Q'_1,\ i=1,\ldots,r}$$

which is induced from the surjection τ, and $\alpha_n : \mathrm{GL}_n^{2g} \xrightarrow{\sim} \mathrm{GL}_n^{2g}$ is the isomorphism induced from the isomorphism τ''. Also consider the projection $\pi_n : \mathrm{GL}_n^{2g+r} \longrightarrow \mathrm{GL}_n^{2g}$

$$\big(A_a, B_{h_i}\big)_{a\in Q'_1,\, i=1,\ldots r} \longmapsto \big(A_a\big)_{a\in Q'_1}.$$

Then we take $\psi = \psi_{m,n} : Y_{m,n} \longrightarrow Y'_{m,n}$ to be

$$\big(\rho',\, \rho'',\, R\big) \longmapsto \big(\varphi_m(\rho'),\, \varphi_n(\rho''),\, R,\ 0,\ldots,0\big)$$

and $\pi_Y : Y'_{m,n} \longrightarrow Y_{m,n}$ to be

$$\big(\rho',\, \rho'',\, R_0,\ldots,R_r\big) \longmapsto \big(\pi_m(\rho'),\, \pi_n(\rho''),\, R_0\big)$$

and

$$\begin{aligned}
\alpha &= \alpha_m \times \alpha_n \times \mathrm{id}_{\mathrm{Mat}_{m\times n}} \\
\varphi &= \psi_{m,n} \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}} \\
\beta &= \alpha \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}} \\
\delta &= \mathrm{id}_{Y'_{m,n}} \\
f' &= f'' = \pi'_1.
\end{aligned}$$

The squares from diagram (7.16) then become

$$\begin{tikzcd}
Y'_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \arrow[r, "\pi'_1"] & Y'_{m,n} \\
Y_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \arrow[r, "\pi_1"] \arrow[u, hook, "\psi_{m,n}\times\mathrm{id}"] & Y_{m,n} \arrow[u, hook, "\psi_{m,n}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \arrow[rr, "\pi_1"] & & Y_{m,n} \\
Y'_{m,n} \times \mathrm{Mat}^{2g+r}_{m\times n} \arrow[rr, "\pi'_1"] \arrow[u, "\pi_Y \times \mathrm{id}"] & & Y'_{m,n} \arrow[u, "\pi_Y"']
\end{tikzcd}$$

both of which are clearly Cartesian, and we have that $\pi_Y \circ \psi = \mathrm{id}_{Y_{m,n}}$ and $(\pi_Y \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}}) \circ \varphi = \mathrm{id}_{X_{m,n}}$ as required. □

The reason we have constructed diagram (7.50) is that, whilst for (7.49) there most likely will not exist an isomorphism $\delta_Y : Y'_{m,n} \xrightarrow{\sim} Y'_{m,n}$ that will make the top-right square Cartesian when replacing f'' with $f' = \delta_Y^{-1} \circ f''$, there will exist an isomorphism $\delta_X : X'_{m,n} \xrightarrow{\sim} X'_{m,n}$ such that the top-left square will become Cartesian after replacing f''_0 with $f'_0 = \delta_X^{-1} \circ f''_0$. This is because due to the definitions of $X'_{m,n}$ and f''_0 we retain all the information of the representations in $M_{m,n}$. Informally we can see this happening in Example 7.4.3. It is not too hard to check that we can fully recover the data of $(A_1, B_1, A_2, B_2) \in \mathrm{GL}^4_{m,n}$ in the image of $f''_0 \circ \varphi'_{m,n}$. So if $(\rho', \rho'', R_1, R_2, R_3, (D, F, G, I, J, L))$ is in the image of $f''_0 \circ \varphi'_{m,n}$ our isomorphism δ_X will then send this to

$$\big(\rho',\, \rho'',\, R_1,\, R_2B_2A_2^{-1}B_2^{-1},\, R_3,\, (D,\, F,\, G,\, I,\, J,\, L)\big)$$

and we can see that from the calculation in Example 7.4.3 this will make the square commutative (and in fact it will be Cartesian).

Proposition 7.4.5. *The left-hand cube in diagram* (7.50) *satisfies all the conditions needed to apply Corollary 7.3.9.*

Proof. As in Proposition 7.4.4 all the conditions are obvious bar the fact that the top face of the cube can be fitted into a diagram of the form (7.16). For this cube we have $\varphi' = \varphi'_{m,n}$ and $\psi' = \psi'_{m,n} \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}}$. We utilise the same factorisation $\tau' = \tau'' \circ \tau$ to factorise ψ' and φ'. Therefore $\varphi = \varphi_{m,n}$ and $\psi = \psi_{m,n} \times \mathrm{id}_{M_{m,n}}$ with φ_n and $\psi_{m,n}$ described in the proof of Proposition 7.4.4. Then f_0 becomes the map that sends $(A_a)_{a\in Q'_1} \in \mathrm{GL}^{2g}_{m,n}$ to

$$\Big((A_a^{(1)})_{a\in Q'_1}\,,\ (A_a^{(2)})_{a\in Q'_1}\,,\ \big(\tau''^{-1}(\lambda)(A_a) - \mathrm{Id}_{m+n}\big)^{(3)}\,,\ (A_a^{(3)}, p_i(A_a)^{(3)})_{a\in Q'_1,\, i=1,\dots,r} \Big) \tag{7.51}$$

for $\lambda = \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} \in \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1}\rangle$.

Let $a \in Q_1'$ and $h \in Q_1 \setminus Q_1'$ index the matrices in a representation of Q, then $X'_{m,n} \cong X_{m,n} \times \mathrm{GL}^r_{m,n}$ with the isomorphism given by

$$\Big((A'_a, B'_h),\, (A''_a, B''_h),\, R_0, R_1, \ldots, R_r,\, (A'''_a, B'''_h)\Big) \longmapsto \Big((A'_a),\, (A''_a),\, R_0,\, (A'''_a, B'''_h), (C_{h_i})\Big)$$

where

$$C_{h_i} = \begin{pmatrix} B'_{h_i} & R_i \\ 0 & B''_{h_i} \end{pmatrix}$$

for $i = 1, \ldots r$.

So as per Remark 7.3.1 it suffices to show that there exists some isomorphism $\gamma : M_{m,n} \xrightarrow{\sim} M_{m,n}$ such that

$$f_0' = \Big(f_0 \times \mathrm{id}_{\mathrm{Mat}^{2g+r}_{m\times n}}\Big) \circ \gamma$$

where $f_0'' = \delta \circ f_0'$ for some isomorphism $\delta : X_{m,n} \times \mathrm{GL}^r_{m,n} \xrightarrow{\sim} X_{m,n} \times \mathrm{GL}^r_{m,n}$.

Write

$$\frac{\partial W}{\partial e_0} = \prod_{i=r}^{1} s_i h_i^{c_i} s_0 - \prod_{i=r}^{1} t_i h_i^{d_i} t_0 \tag{7.52}$$

where $h_i \in Q_1 \setminus Q_1'$, s_i, t_i are paths in $\mathbb{C}Q'$ for $i = 0, \ldots, r$, and c_i, $d_i \in \{0, 1\}$ such that for each i c_i and d_i cannot both be 1 (any $h \in Q_1 \setminus Q_1'$ can appear at most once in $\partial W/\partial e_0$, since for a brane tiling each arrow from the dual quiver must appear exactly twice in the potential and we already know that h appears at least once in the relations $\partial W/\partial e_i$ for $i = 1, \ldots r$). Using Proposition 7.4.2, for each $i = 1, \ldots, r$ we can write

$$h_i - p_i = \sum_j l_{i,j}\, u_{i,j} \left(\frac{\partial W}{\partial e_j}\right) v_{i,j} \tag{7.53}$$

where $l_{i,j} \in \mathbb{C}$, p_i is the path in $\mathbb{C}\widetilde{Q}'$ related to h_i, and $u_{i,j}$ and $v_{i,j}$ are paths in $\mathbb{C}\widetilde{Q}$.

Write

$$\tau\left(\frac{\partial W}{\partial e_0}\right) = \prod_{i=r}^{1} s_i p_i^{c_i} s_0 - \prod_{i=r}^{1} t_i p_i^{d_i} t_0 = p_0 - q_0$$

for p_0 and q_0 paths in $\mathbb{C}\widetilde{Q}'$. Then we can apply Lemma 7.4.1 to $F(Q')$ and $F(x_1, y_1, \ldots, x_g, y_g)$, and $\lambda = \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}$ and $\lambda' = p_0 q_0^{-1}$, giving us the unit $u \in \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1}\rangle$ such that

$$\tau''(\lambda) = u\, \lambda^c\, u^{-1}$$

for $c \in \{1, -1\}$. Define $u' \in \mathbb{C}\widetilde{Q}'$ to be

$$u' = \tau''^{-1}(u)$$

hence

$$p_0 q_0^{-1} = u' \, \tau''^{-1}(\lambda^c) \, u'^{-1}. \tag{7.54}$$

For the moment assume that $c = 1$. Then for $(A_a)_{a \in Q_1'} \in \mathrm{GL}_{m,n}^{2g}$ we have

$$\begin{aligned} p_0(A_a) - q_0(A_a) &= \tau\left(\frac{\partial W}{\partial e_0}\right)(A_a) \\ &= \frac{\partial W}{\partial e_0}(\varphi_{m,n}(A_a)) \\ &= \prod_{i=r}^{1} s_i(A_a)\, p_i(A_a)^{c_i}\, s_0(A_a) - \prod_{i=r}^{1} t_i(A_a)\, p_i(A_a)^{d_i}\, t_0(A_a). \end{aligned} \tag{7.55}$$

Then define $\delta^{-1} : X_{m,n} \times \mathrm{GL}_{m,n}^r \xrightarrow{\sim} X_{m,n} \times \mathrm{GL}_{m,n}^r$ by sending

$$\Big((A_a'),\, (A_a''),\, R_0,\, (A_a''', B_h'''), (C_h)\Big)_{a \in Q_1',\, h \in Q_1 \setminus Q_1'}$$

to

$$\bigg((A_a'),\, (A_a''),\, \Big(u'^{-1}(X_a)\,\big(R - S - T\big)\, q_0^{-1}(X_a)\, u'(X_a)\Big)^{(3)},\, (A_a''', L_h'''),\, (D_h)\bigg)_{a \in Q_1',\, h \in Q_1 \setminus Q_1'}$$

where for $a \in Q_1'$ and $h \in Q_1 \setminus Q_1'$

$$\begin{aligned} X_a &= \begin{pmatrix} A_a' & A_a''' \\ 0 & A_a'' \end{pmatrix} \\ Y_h &= \begin{pmatrix} C_h^{(1)} & B_h''' \\ 0 & C_h^{(2)} \end{pmatrix} \\ F_{h_i} &= C_{h_i} - \begin{pmatrix} C_{h_i}^{(1)} - \partial W/\partial e_i(A_a', C_h^{(1)}) & 0 \\ 0 & C_{h_i}^{(2)} - \partial W/\partial e_i(A_a'', C_h^{(2)}) \end{pmatrix} \\ G_{h_i} &= \sum_j l_{i,j}\, u_{i,j}(X_a, Y_h)\, F_{h_j}\, v_{i,j}(X_a, Y_h) \\ L_{h_i}''' &= B_{h_i}''' - G_{h_i}^{(3)} \\ D_{h_i} &= \begin{pmatrix} B_{h_i}' & G_{h_i}^{(3)} \\ 0 & B_{h_i}'' \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
R &= R\big((A'_a),(A''_a),(C_h),R_0\big)\\
&= \begin{pmatrix} \partial W/\partial e_0(A'_a, C_h^{(1)}) & R_0 \\ 0 & \partial W/\partial e_0(A''_a, C_h^{(2)}) \end{pmatrix}\\
S &= S\big((X_a),(Y_h),(G_{h_i})\big)\\
&= \sum_{i:c_i=1}\left(\left(\prod_{j=r}^{i+1} s_j(X_a) Y_{h_j}^{c_j}\right) s_i(X_a)\, G_{h_i}\left(\prod_{j=i-1}^{1} s_j(X_a)\, p_j(X_a)^{c_j}\right) s_0(X_a)\right)\\
T &= T\big((X_a),(Y_h),(G_{h_i})\big)\\
&= \sum_{i:d_i=1}\left(\left(\prod_{j=r}^{i+1} t_j(X_a) Y_{h_j}^{d_j}\right) t_i(X_a)\, G_{h_i}\left(\prod_{j=i-1}^{1} t_j(X_a)\, p_j(X_a)^{d_j}\right) t_0(X_a)\right).
\end{aligned}$$

We can therefore calculate where $f'_0 = \delta^{-1} \circ f''_0$ sends $\big((A_a, B_h)_{a\in Q'_1,\, h\in Q_1\setminus Q'_1}\big)$:

Using the above notation and (7.53) we get

$$\begin{aligned}
C_{h_i} &= \begin{pmatrix} B_{h_i}^{(1)} & (\partial W/\partial e_i(A_a,B_h))^{(3)} \\ 0 & B_{h_i}^{(2)} \end{pmatrix}\\
X_a &= A_a\\
Y_h &= B_h\\
F_{h_i} &= C_{h_i} - \begin{pmatrix} B_{h_i}^{(1)} - \partial W/\partial e_i(A_a^{(1)}, B_h^{(1)}) & 0 \\ 0 & B_{h_i}^{(2)} - \partial W/\partial e_i(A_a^{(2)}, B_h^{(2)}) \end{pmatrix}\\
&= \frac{\partial W}{\partial e_i}(A_a, B_h)\\
G_{h_i} &= \sum_j l_{i,j}\, u_{i,j}(A_a,B_h)\, \frac{\partial W}{\partial e_j}(A_a,B_h)\, v_{i,j}(A_a,B_h)\\
&= B_{h_i} - p_i(A_a)\\
L'''_{h_i} &= B_{h_i}^{(3)} - \big(B_{h_i} - p_i(A_a)\big)^{(3)}\\
&= p_i(A_a)^{(3)}\\
D_{h_i} &= \begin{pmatrix} B_{h_i}^{(1)} & \big(B_{h_i} - p_i(A_a)\big)^{(3)} \\ 0 & B_{h_i}^{(2)} \end{pmatrix}\\
&= B_{h_i} - \begin{pmatrix} 0 & p_i(A_a)^{(3)} \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
R &= R\big((A_a^{(1)}), (A_a^{(2)}), (C_h), (\partial W/\partial e_0(A_a, B_h))^{(3)}\big) \\
&= \begin{pmatrix} \partial W/\partial e_0(A_a^{(1)}, C_h^{(1)}) & (\partial W/\partial e_0(A_a, B_h))^{(3)} \\ 0 & \partial W/\partial e_0(A_a^{(2)}, C_h^{(2)}) \end{pmatrix} \\
&= \begin{pmatrix} \partial W/\partial e_0(A_a^{(1)}, B_h^{(1)}) & (\partial W/\partial e_0(A_a, B_h))^{(3)} \\ 0 & \partial W/\partial e_0(A_a^{(2)}, B_h^{(2)}) \end{pmatrix} \\
&= \frac{\partial W}{\partial e_0}(A_a, B_h) \\
S &= S\big((X_a), (Y_h), (G_h)\big) = S\big((A_a), (B_h), (B_{h_i} - p_i(A_a))\big) \\
&= \sum_{i:c_i=1} \left(\left(\prod_{j=r}^{i+1} s_j(A_a) B_{h_j}^{c_j} \right) s_i(A_a) \big(B_{h_i} - p_i(A_a)\big) \left(\prod_{j=i-1}^{1} s_j(A_a)\, p_j(A_a)^{c_j} \right) s_0(A_a) \right) \\
&= \prod_{j=r}^{1} s_j(A_a) B_{h_j}^{c_j} s_0(A_a) - \prod_{j=r}^{1} s_j(A_a) p_j(A_a)^{c_j} s_0(A_a) \\
T &= T\big((X_a), (Y_h), (G_h)\big) = T\big((A_a), (B_h), (B_{h_i} - p_i(A_a))\big) \\
&= \sum_{i:d_i=1} \left(\left(\prod_{j=r}^{i+1} t_j(A_a) B_{h_j}^{d_j} \right) t_i(A_a) \big(B_{h_i} - p_i(A_a)\big) \left(\prod_{j=i-1}^{1} t_j(A_a)\, p_j(A_a)^{d_j} \right) t_0(A_a) \right) \\
&= \prod_{j=r}^{1} t_j(A_a) B_{h_j}^{d_j} t_0(A_a) - \prod_{j=r}^{1} t_j(A_a) p_j(A_a)^{d_j} t_0(A_a)
\end{aligned}
$$

since the sums in S and T are telescopic. Hence from (7.52) and (7.55) we see that

$$
\begin{aligned}
R - S - T = \frac{\partial W}{\partial e_0}(A_a, B_h) - \Bigg[&\prod_{j=r}^{1} s_j(A_a) B_{h_j}^{c_j} s_0(A_a) - \prod_{j=r}^{1} s_j(A_a) p_j(A_a)^{c_j} s_0(A_a) + \\
&\prod_{j=r}^{1} t_j(A_a) B_{h_j}^{d_j} t_0(A_a) - \prod_{j=r}^{1} t_j(A_a) p_j(A_a)^{d_j} t_0(A_a) \Bigg] \\
= &\prod_{i=r}^{1} s_i(A_a)\, p_i(A_a)^{c_i}\, s_0(A_a) - \prod_{i=r}^{1} t_i(A_a)\, p_i(A_a)^{d_i}\, t_0(A_a) \\
= &\,\tau \left(\frac{\partial W}{\partial e_0} \right) (A_a) \\
= &\, p_0(A_a) - q_0(A_a)
\end{aligned}
$$

and so using (7.54) we get

$$\Big(u'^{-1}(X_a)\,\big(R-S-T\big)\,q_0^{-1}(X_a)\,u'(X_a)\Big)^{(3)} =$$
$$\begin{aligned}
&= \Big(u'^{-1}(A_a)\,\big(p_0(A_a)-q_0(A_a)\big)\,q_0^{-1}(A_a)\,u'(A_a)\Big)^{(3)}\\
&= \Big(u'^{-1}(A_a)\,p_0(A_a)\,q_0^{-1}(A_a)\,u'(A_a)-\mathrm{Id}_{m+n}\Big)^{(3)}\\
&= \Big(\tau''^{-1}(\lambda)(A_a)-\mathrm{Id}_{m+n}\Big)^{(3)}.
\end{aligned}$$

Therefore we have that $f_0'\big((A_a,B_h)_{a\in Q_1',\,h\in Q_1\setminus Q_1'}\big)$ equals

$$\left((A_a^{(1)}),\,(A_a^{(2)}),\,\Big(\tau''^{-1}(\lambda)(A_a)-\mathrm{Id}_{m+n}\Big)^{(3)},\,(A_a^{(3)},p_i(A_a)^{(3)}),\,\left(B_{h_i}-\begin{pmatrix}0 & p_i(A_a)^{(3)}\\ 0 & 0\end{pmatrix}\right)\right)_{a,\,h}$$

and so if we take $\gamma : M_{m,n} \xrightarrow{\sim} M_{m,n}$ to be the isomorphism

$$(A_a,B_h)_{a\in Q_1',\,h\in Q_1\setminus Q_1'} \longmapsto \left(A_a,\,B_{h_i}-\begin{pmatrix}0 & p_i(A_a)^{(3)}\\ 0 & 0\end{pmatrix}\right)_{a\in Q_1',\,h\in Q_1\setminus Q_1'}$$

comparing with (7.51) we see that

$$f_0' = \Big(f_0\times \mathrm{id}_{\mathrm{GL}^r_{m,n}}\Big)\circ\gamma$$

as required.

Lastly we must check that $\delta|_Z = \mathrm{id}_Z$. So $\big((A_a',B_h'),\,(A_a'',B_h''),\,(A_a''',B_h''')\big)\in Z$ is sent by j' to $\big((A_a',B_h'),\,(A_a'',B_h''),\,0,\ldots,0,\,(A_a''',B_h''')\big)\in X'_{m,n}$ which corresponds to $\big((A_a'),\,(A_a''),\,0,\,(A_a''',B_h'''),\,(C_h)\big)\in X_{m,n}\times \mathrm{GL}^r_{m,n}$ where

$$C_h = \begin{pmatrix}B_h' & 0\\ 0 & B_h''\end{pmatrix}.$$

Then, using the fact that since $(A_a',B_h')\in Z_m$ and $(A_a'',B_h'')\in Z_m$ the evaluation of

$\partial W/\partial e_i$ on these matrices is 0 for $i = 0, \ldots r$, we have

$$\begin{aligned}
X_a &= \begin{pmatrix} A'_a & A'''_a \\ 0 & A''_a \end{pmatrix} \\
Y_h &= \begin{pmatrix} B'_h & B'''_h \\ 0 & B''_h \end{pmatrix} \\
F_{h_i} &= \begin{pmatrix} B'_{h_i} & 0 \\ 0 & B''_{h_i} \end{pmatrix} - \begin{pmatrix} B'_{h_i} - \partial W/\partial e_i(A'_a, B'_h) & 0 \\ 0 & B''_{h_i} - \partial W/\partial e_i(A''_a, B''_h) \end{pmatrix} \\
&= \begin{pmatrix} B'_{h_i} & 0 \\ 0 & B''_{h_i} \end{pmatrix} - \begin{pmatrix} B'_{h_i} & 0 \\ 0 & B''_{h_i} \end{pmatrix} \\
&= 0 \\
G_{h_i} &= \sum_j l_{i,j}\, u_{i,j}(X_a, Y_h)\, F_{h_j}\, v_{i,j}(X_a, Y_h) \\
&= 0 \\
L'''_{h_i} &= B'''_{h_i} \\
D_{h_i} &= \begin{pmatrix} B'_{h_i} & 0 \\ 0 & B''_{h_i} \end{pmatrix} \\
&= C_{h_i}
\end{aligned}$$

and

$$\begin{aligned}
R &= R\big((A'_a), (A''_a), (C_h), 0\big) \\
&= \begin{pmatrix} \partial W/\partial e_0(A'_a, C_h^{(1)}) & 0 \\ 0 & \partial W/\partial e_0(A''_a, C_h^{(2)}) \end{pmatrix} \\
&= \begin{pmatrix} \partial W/\partial e_0(A'_a, B'_h) & 0 \\ 0 & \partial W/\partial e_0(A''_a, B''_h) \end{pmatrix} \\
&= 0 \\
S &= S\big((X_a), (Y_h), (0)\big) \\
&= \sum_{i:c_i=1} \left(\left(\prod_{j=r}^{i+1} s_j(X_a) Y_{h_j}^{c_j} \right) s_i(X_a) \cdot 0 \cdot \left(\prod_{j=i-1}^{1} s_j(X_a)\, p_j(X_a)^{c_j} \right) s_0(X_a) \right) \\
&= 0 \\
T &= T\big((X_a), (Y_h), (0)\big) \\
&= \sum_{i:d_i=1} \left(\left(\prod_{j=r}^{i+1} t_j(X_a) Y_{h_j}^{d_j} \right) t_i(X_a) \cdot 0 \cdot \left(\prod_{j=i-1}^{1} t_j(X_a)\, p_j(X_a)^{d_j} \right) t_0(X_a) \right) \\
&= 0.
\end{aligned}$$

Hence $\big((A'_a),\, (A''_a),\, 0,\, (A'''_a, B'''_h),\, (C_h)\big)$ is sent by δ^{-1} to

$$\Big((A'_a),\, (A''_a),\, 0,\, (A'''_a, B'''_h),\, (C_h)\Big)$$

and so we do indeed get that

$$\delta|_Z = \mathrm{id}_Z.$$

Finally if $c = -1$ i.e. if

$$\lambda' = u'\tau''^{-1}(\lambda^{-1})\, u'^{-1}$$

then we modify δ to send $\Big((A'_a),\, (A''_a),\, R_0,\, (A'''_a, B'''_h), (C_h)\Big)_{a\in Q'_1,\, h\in Q_1\setminus Q'_1}$ to

$$\bigg((A'_a),\, (A''_a),\, -\Big(\tau''^{-1}(\lambda)(X_a)\, u'^{-1}(X_a)\, \big(R{-}S{-}T\big)\, q_0^{-1}(X_a)\, u'(X_a)\Big)^{(3)},\, (A'''_a, L'''_h),\, (D_h)\bigg)_{a,\,h}.$$

Following the above calculations we get that for $f'_0 = \delta^{-1}\circ f''_0$

$$\begin{aligned}
-\Big(\tau''^{-1}(\lambda)(X_a)\, u'^{-1}(X_a)\, \big(R - S - T\big)\, q_0^{-1}(X_a)\, u'(X_a)\Big)^{(3)} = \\
&= -\Big(\tau''^{-1}(\lambda)(A_a)\, u'^{-1}(A_a)\, \big(p_0(A_a) - q_0(A_a)\big)\, q_0^{-1}(A_a)\, u'(A_a)\Big)^{(3)} \\
&= -\Big(\tau''^{-1}(\lambda)(A_a)\, u'^{-1}(A_a)\, p_0(A_a)\, q_0^{-1}(A_a)\, u'(A_a) - \tau''^{-1}(\lambda)(A_a)\Big)^{(3)} \\
&= -\Big(\tau''^{-1}(\lambda)(A_a)\, \tau''^{-1}(\lambda^{-1})(A_a) - \tau''^{-1}(\lambda)(A_a)\Big)^{(3)} \\
&= \Big(\tau''^{-1}(\lambda)(A_a) - \mathrm{Id}_{m+n}\Big)^{(3)}
\end{aligned}$$

as required. □

We can now put all the pieces together allowing us to prove the main result of this chapter.

Theorem 7.4.6. *Let Σ_g be a Riemann surface of genus g with brane tiling Δ giving dual quiver Q_Δ and potential W_Δ. Fix a cut E for W_Δ and a maximal tree T in $Q_\Delta\setminus E$, and let Q be the quiver obtained by contracting T in $Q_\Delta\setminus E$ with corresponding potential W. Then the 2D CoHA*

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\pi_1(\Sigma_g)), \mathbb{Q})^\vee$$

with multiplication induced by the diagram (7.13) *is isomorphic as an algebra to the 2D CoHA*

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)), \mathbb{Q})^\vee$$

with multiplication induced by the diagram (7.15).

Proof. We use diagram (7.49) to construct a bridge between the two multiplications. This gives us the following diagram on cohomology; where $p_m : Z_m/\mathrm{GL}_m \to \mathrm{pt}$, $p_n : Z_n/\mathrm{GL}_n \to \mathrm{pt}$, $p_{m+n} : Z_{m+n}/\mathrm{GL}_{m+n} \to \mathrm{pt}$ and $p : (Z_m \times Z_n)/(\mathrm{GL}_m \times \mathrm{GL}_n) \to \mathrm{pt}$ are the respective structure morphisms, and we use an overline on a map to denote the respective induced map on the quotient stack.

$$\begin{tikzcd}
\mathrm{H}_{c,\mathrm{GL}_{m+n}}\big(Z_{m+n},\mathbb{Q}\big) \arrow[r, "p_{m+n!}\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})", "\sim"'] \arrow[d, "q_Z^\star"] & \mathrm{H}_{c,\mathrm{GL}_{m+n}}\big(V_{m+n},\mathbb{Q}\big) \arrow[d, "q_V^\star"] \\
\mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(Z_{m+n},\mathbb{Q}\big) \arrow[r, "p_{m+n!}\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})", "\sim"'] \arrow[d, "\overline{\widetilde{h}}'^\star"] & \mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(V_{m+n},\mathbb{Q}\big) \arrow[d, "\overline{\widetilde{h}}^\star"] \\
\mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(Z_{m,n},\mathbb{Q}\big) \arrow[r, "p_!\overline{\widetilde{f}}''_!\,\eta^{\overline{\widetilde{\varphi}}'_{m,n}}(\mathbb{Q})", "\sim"'] \arrow[d, "\overline{f}''_\star"] & \mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(V_{m,n},\mathbb{Q}\big) \arrow[d, "\overline{f}_\star"] \\
\mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(Z_m \times Z_n,\mathbb{Q}\big) \arrow[r, "p_!\eta^{\overline{\widetilde{\varphi}}'_m \times \overline{\widetilde{\varphi}}'_n}(\mathbb{Q})", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}_{c,\mathrm{GL}_{m,n}}\big(V_m \times V_n,\mathbb{Q}\big) \arrow[d, "\wr"] \\
\mathrm{H}_{c,\mathrm{GL}_m}\big(Z_m,\mathbb{Q}\big) \otimes \mathrm{H}_{c,\mathrm{GL}_n}\big(Z_n,\mathbb{Q}\big) \arrow[r, "p_{m!}\eta^{\overline{\widetilde{\varphi}}'_m}(\mathbb{Q}) \otimes p_{n!}\eta^{\overline{\widetilde{\varphi}}'_n}(\mathbb{Q})", "\sim"'] & \mathrm{H}_{c,\mathrm{GL}_m}\big(V_m,\mathbb{Q}\big) \otimes \mathrm{H}_{c,\mathrm{GL}_n}\big(V_n,\mathbb{Q}\big)
\end{tikzcd} \tag{7.56}$$

Recalling that $\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)) \cong Z_n/\mathrm{GL}_n$ and $\mathrm{Rep}_n(\mathbb{C}[\pi_1(\Sigma_g)]) \cong V_n/\mathrm{GL}_n$, the dual of the left-hand side of (7.56) exactly gives the multiplication on $\mathrm{H}_c(\mathrm{Rep}_n(\mathrm{Jac}(\widetilde{Q}, W, E)), \mathbb{Q})$ and the dual of the right-hand side gives the multiplication on $\mathrm{H}_c(\mathrm{Rep}_n(\pi_1(\Sigma_g)), \mathbb{Q})$, as described in Section 3.1. As $\widetilde{\varphi'_n}$ is an isomorphism each of the horizontal arrows in (7.56) are isomorphisms, hence to prove the two 2D CoHAs are isomorphic as algebras it suffices to show that (7.56) commutes.

Commutativity of the bottom square follows from the naturality of isomorphisms involved. For the top square note we have the following commutative diagram of maps

$$\begin{tikzcd} V_{m+n}/\mathrm{GL}_{m,n} \arrow[r, "\overline{\overline{\varphi}}'_{m+n}", "\sim"'] \arrow[d, "q_V"] & Z_{m+n}/\mathrm{GL}_{m,n} \arrow[d, "q_Z"] \\ V_{m+n}/\mathrm{GL}_{m+n} \arrow[r, "\overline{\overline{\varphi}}'_{m+n}", "\sim"'] & Z_{m+n}/\mathrm{GL}_{m+n} \end{tikzcd} \tag{7.57}$$

Now for a general commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"] & Y \arrow[d, "g"] \\ W \arrow[r, "g'"] & Z \end{tikzcd}$$

we have that the composition

$$\mathbb{Q}_Z \xrightarrow{\eta^g(\mathbb{Q})} g_*g^*\mathbb{Q}_Z \xrightarrow{g_*(\eta^f(g^*\mathbb{Q}))} g_*f_*f^*g^*\mathbb{Q}_Z$$

equals

$$\mathbb{Q}_Z \xrightarrow{\eta^{g\circ f}(\mathbb{Q})} (g\circ f)_*(g\circ f)^*\mathbb{Q}_Z = g_*f_*f^*g^*\mathbb{Q}_Z$$

which equals

$$\mathbb{Q}_Z \xrightarrow{\eta^{g'\circ f'}(\mathbb{Q})} (g'\circ f')_*(g'\circ f')^*\mathbb{Q}_Z = g'_*f'_*f'^*g'^*\mathbb{Q}_Z$$

which equals

$$\mathbb{Q}_Z \xrightarrow{\eta^{g'}(\mathbb{Q})} g'_*g'^*\mathbb{Q}_Z \xrightarrow{g'_*(\eta^{f'}(g'^*\mathbb{Q}))} g'_*f'_*f'^*g'^*\mathbb{Q}_Z$$

i.e.

$$g_*(\eta^f(g^*\mathbb{Q}_Z))\circ\eta^g(\mathbb{Q}_Z) = g'_*(\eta^{f'}(g'^*\mathbb{Q}_Z))\circ\eta^{g'}(\mathbb{Q}_Z). \tag{7.58}$$

Applying this to (7.57) and recalling the definitions of the morphisms from Section 7.1

$$\begin{aligned} q_V^\star &:= p_{m+n!}\overline{\overline{\varphi}}'_{m+n!}(\eta^{q_V}(\mathbb{Q}_{V_{m+n}/\mathrm{GL}_{m+n}})) \\ q_Z^\star &:= p_{m+n!}(\eta^{q_Z}(\mathbb{Q}_{Z_{m+n}/\mathrm{GL}_{m+n}})) \end{aligned}$$

it follows that

$$
\begin{aligned}
q_V^\star \circ p_{m+n!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})) &= p_{m+n!}\overline{\widetilde{\varphi}}'_{m+n!}(\eta^{q_V}(\mathbb{Q}) \circ p_{m+n!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})) \\
&= p_{m+n!}\overline{\widetilde{\varphi}}'_{m+n!}(\eta^{q_V}(\overline{\widetilde{\varphi}}'^{*}_{m+n}\mathbb{Q}) \circ p_{m+n!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})) \\
&= p_{m+n!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n} \circ q_V}(\mathbb{Q})) \\
&= p_{m+n!}(\eta^{q_Z \circ \overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})) \\
&= p_{m+n!}q_{Z!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(q_Z^*\mathbb{Q})) \circ p_{m+n!}(\eta^{q_Z}(\mathbb{Q})) \\
&= p_{m+n!}q_{Z!}(\eta^{\overline{\widetilde{\varphi}}'_{m+n}}(\mathbb{Q})) \circ q_Z^\star .
\end{aligned}
$$

The middle two squares in diagram (7.56) are induced by the respective morphisms of sheaves in $\mathrm{D}(Z_{m+n})$, namely

$$
\begin{tikzcd}
\mathbb{Q}_{Z_{m+n}} \arrow[r, "\eta^{\widetilde{\varphi}'_{m+n}}(\mathbb{Q})", "\sim"'] \arrow[d, "\widetilde{h}'^\star"] & \widetilde{\varphi}'_{m+n!}\mathbb{Q}_{V_{m+n}} \arrow[d, "\widetilde{\varphi}'_{m+n!}(\widetilde{h}^\star)"] \\
\widetilde{h}'_!\mathbb{Q}_{Z_{m,n}} \arrow[r, "\widetilde{h}'_!\eta^{\widetilde{\varphi}'_{m,n}}(\mathbb{Q})", "\sim"'] & \widetilde{h}'_!\widetilde{\varphi}'_{m,n!}\mathbb{Q}_{V_{m,n}} = \widetilde{\varphi}'_{m+n!}\widetilde{h}_!\mathbb{Q}_{V_{m,n}}
\end{tikzcd}
\tag{7.59}
$$

and

$$
\begin{tikzcd}
\widetilde{f}''_!\mathbb{Q}_{Z_{m,n}}[2d] \arrow[r, "\widetilde{f}''_!\eta^{\widetilde{\varphi}'_{m,n}}(\mathbb{Q}[2d])", "\sim"'] \arrow[d, "f''_\star"] & \widetilde{f}''_!\widetilde{\varphi}'_{m,n!}\mathbb{Q}_{V_{m,n}} = (\widetilde{\varphi}'_m \times \widetilde{\varphi}'_n)_!\widetilde{f}_!\mathbb{Q}_{V_{m,n}}[2d] \arrow[d, "(\widetilde{\varphi}'_m \times \widetilde{\varphi}'_n)_!(f_\star)"] \\
\mathbb{Q}_{Z_m \times Z_n} \arrow[r, "\eta^{\widetilde{\varphi}'_m \times \widetilde{\varphi}'_n}(\mathbb{Q})", "\sim"'] & (\widetilde{\varphi}'_m \times \widetilde{\varphi}'_n)_!\mathbb{Q}_{V_m \times V_n}
\end{tikzcd}
\tag{7.60}
$$

hence showing (7.59) and (7.60) commute is enough. Commutativity of (7.59) follows from the fact that the bottom-left square in diagram (7.49) is commutative. Indeed recall the definition of the morphisms $\widetilde{h}^\star$ and $\widetilde{h}'^\star$ from (7.7)

$$\begin{aligned}\widetilde{h}^\star &= \eta^{\widetilde{h}}(\mathbb{Q}_{V_{m+n}})\\ \widetilde{h}'^\star &= \eta^{\widetilde{h}'}(\mathbb{Q}_{Z_{m+n}})\end{aligned}$$

then applying (7.58) we get that

$$\begin{aligned}\widetilde{\varphi}'_{m+n!}(\widetilde{h}^\star) \circ \eta^{\widetilde{\varphi}'_{m+n}}(\mathbb{Q}) &= \widetilde{\varphi}'_{m+n!}(\eta^{\widetilde{h}}(\mathbb{Q})) \circ \eta^{\widetilde{\varphi}'_{m+n}}(\mathbb{Q})\\ &= \widetilde{\varphi}'_{m+n!}(\eta^{\widetilde{h}}(\widetilde{\varphi}'^*_{m+n}\mathbb{Q})) \circ \eta^{\widetilde{\varphi}'_{m+n}}(\mathbb{Q})\\ &= \eta^{\widetilde{\varphi}'_{m+n}\circ\widetilde{h}}(\mathbb{Q})\\ &= \eta^{\widetilde{h}'\circ\widetilde{\varphi}'_{m,n}}(\mathbb{Q})\\ &= \widetilde{h}'_!(\eta^{\widetilde{\varphi}'_{m,n}}(\widetilde{h}'^*\mathbb{Q})) \circ \eta^{\widetilde{h}'}(\mathbb{Q})\\ &= \widetilde{h}'_!(\eta^{\widetilde{\varphi}'_{m,n}}(\mathbb{Q})) \circ \widetilde{h}'^\star.\end{aligned}$$

It remains to show the commutativity of (7.60). Again recall the definition of the morphisms $f_\star$ and $f''_\star$ from (7.8)

$$\begin{aligned}f_\star &= i^*(\nu^f(\mathbb{Q}_{Y_{m,n}})) \circ \epsilon^{i,f}(f^!\mathbb{Q}_{Y_{m,n}})^{-1}\\ f''_\star &= i'^*(\nu^{f''}(\mathbb{Q}_{Y'_{m,n}})) \circ \epsilon^{i',f''}(f''^!\mathbb{Q}_{Y'_{m,n}})^{-1}.\end{aligned}$$

Since $f = \pi_1 \circ f_0$ and $f'' = \pi'_1 \circ f''_0$ we have that $f_\star = \pi_{1\star} \circ \widetilde{\pi}_{1!}(f_{0\star})$ and $f''_\star = \pi'_{1\star} \circ \widetilde{\pi}'_{1!}(f''_{0\star})$ from Lemma 7.1.6. Hence we can use diagram (7.50) to split (7.60) into the following diagram

$$
\begin{tikzcd}
\widetilde{\pi}'_{1!}\widetilde{f}''_{0!}\mathbb{Q}_{Z_{m,n}}[2d] \arrow[r, "\widetilde{\pi}'_{1!}\widetilde{f}''_{0!}\eta^{\widetilde{\varphi}'_{m,n}}(\mathbb{Q}[2d])", "\sim"'] \arrow[d, "\widetilde{\pi}'_{1!}(f''_{0\star}[2e])"] & \widetilde{\pi}'_{1!}\widetilde{f}''_{0!}\widetilde{\varphi}'_{m,n!}\mathbb{Q}_{V_{m,n}} = \widetilde{\psi}'_{m,n!}\widetilde{\pi}_{1!}\widetilde{f}_{0!}\mathbb{Q}_{V_{m,n}}[2d] \arrow[d, "\widetilde{\psi}'_{m,n!}\widetilde{\pi}_{1!}(f_{0\star}[2e])"] \\
\widetilde{\pi}'_{1!}\mathbb{Q}_Z[2e] \arrow[r, "\widetilde{\pi}'_{1!}\eta^{\widetilde{\psi}'_{m,n}\times\mathrm{id}}(\mathbb{Q}[2e])", "\sim"'] \arrow[d, "\pi'_{1\star}"] & \widetilde{\pi}'_{1!}(\widetilde{\psi}'_{m,n}\times\mathrm{id})_!\mathbb{Q}_V[2e] = \widetilde{\psi}'_{m,n!}\widetilde{\pi}_{1!}\mathbb{Q}_V[2e] \arrow[d, "\widetilde{\psi}'_{m,n!}(\pi_{1\star})"] \\
\mathbb{Q}_{Z_m\times Z_n} \arrow[r, "\eta^{\widetilde{\psi}'_{m,n}}(\mathbb{Q})", "\sim"'] & \widetilde{\psi}'_{m,n!}\mathbb{Q}_{V_m\times V_n}
\end{tikzcd}
\tag{7.61}
$$

where $\dim(\pi_1) = \dim(\pi'_1) = e$ and $\dim(f_0) = \dim(f''_0) = d - e$, and recall $\widetilde{\psi}'_{m,n} = \widetilde{\varphi}'_m \times \widetilde{\varphi}'_n$. Then Proposition 7.4.4 and Proposition 7.4.5 tell us that we can apply Corollary 7.3.9 to both cubes in diagram (7.50), which in turn tells us that the both squares in (7.61) commute and hence (7.60) commutes and therefore so does (7.56). $\square$

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