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QCD Scattering Amplitudes in the High-Energy Limit

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Abstract

In perturbative quantum field theory, logarithmic divergences generically occur whenever there is a large disparity between physical scales. In this thesis we are interested in the so-called high-energy logarithms, L , that arise in quantum chromodynamics (QCD). These logarithms occur when the centre-of-mass energy of a collision is much larger than the transverse scales of the produced particles. Even if we are within the perturbative regime, where the strong coupling constant, α_s , is much less than unity, at sufficiently large centre-of-mass energy the combination $\alpha_s L$ may be of order unity. These terms therefore need to be summed to all orders in α_s to ensure the stability of perturbative predictions. In contrast to a fixed-order expansion, where accuracy is stated in terms of leading order (LO), next-to-leading order (NLO) etc., we can also consider an expansion of the amplitude in terms of leading logarithmic (LL) terms, next-to-leading logarithmic (NLL) terms etc.

The first all-orders summation of high-energy logarithms in QCD was obtained via the LL BFKL equation. This programme was then extended to NLL accuracy. There has been a recent renewal of interest in the high energy asymptotics of QCD, and in particular there has been a recent effort to extend the BFKL framework to next-to-next-to-leading logarithmic (NNLL) accuracy. A necessary ingredient is the one-loop central emission vertex (CEV) for two gluons which are not strongly ordered in rapidity. In the first part of this thesis we consider the one-loop six-gluon amplitude in $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in a central next-to-multi-Regge kinematic (NMRK) limit. In this limit, the dispersive part of the amplitude factorises and we extract the one-loop two-gluon CEV for any helicity configuration within this theory. This is a component of the two-gluon CEV in QCD. We show that the absorptive part of the one-loop six-gluon amplitude does not contribute to NNLL accuracy in this limit.

The primary motivation of the work presented in this thesis is the inclusion of

such logarithmic terms to all orders in α_s for stable and high-precision predictions for the Large Hadron Collider (LHC). At the LHC, the centre-of-mass energy is large, but finite, and so some of the asymptotic approximations of the high-energy literature are too severe to be applicable to our current experiments. The High Energy Jets (HEJ) framework was developed to solve these issues. In particular, it uses Monte Carlo event generation for phase-space integration, which allows the use of arbitrary kinematic cuts, the freedom to use more involved factorised expressions to approximate the hard matrix element, and the ability to match to full fixed-order results. The factorised expressions that enter into the kernel of the BFKL equation have been re-derived with minimal approximations in order to capture significantly more of the fixed-order physics. In the second part of this thesis we derive two new factorised expressions, which were the last two-parton components required to bring the HEJ framework to NLL accuracy.

Prior to the work presented here, HEJ has been matched to LO fixed-order results, and incorporated LL resummation, as well as providing a NLL description of some key processes. In the final part of this thesis we present the method by which HEJ was further matched to a NLO fixed-order calculation, namely the production of a W boson in association with two jets.

Lay Summary

The Standard Model (SM) of particle physics is a model of how our universe behaves at very small distances. Ironically, we need very large machines to investigate what happens at these small distances. The Large Hadron Collider (LHC) is the largest machine humans have ever built. It collides protons together at very large energies, and it lets us test how well the SM describes these collisions. In a sense, the SM is working *too well*. This confirms that we have a good understanding of how the universe works at very small scales, but the remarkable success of the SM leaves us with very few clues as to what might be happening at even smaller scales. In order to look for subtle deviations from experimental results, we need to improve the precision of our SM predictions, and we need to make sure we are not neglecting any important effects. To explain how my thesis helps with this effort, I will explain what each term in the title means:

- Quantum chromodynamics (QCD): This is one of the three main sectors of the SM. Protons are made up of smaller particles called quarks and gluons. QCD is a model of how quarks and gluons interact with each other.
- Scattering amplitudes: These are mathematical expressions with which we can compute the probability a certain collision process will occur. They act as a Rosetta stone with which we can compare a theory (e.g. the SM) to an experiment (e.g. the LHC).
- The high-energy limit: This is a mathematical limit of scattering amplitudes, where we consider the collision of particles with infinite energy. In this limit we are able to compute an infinite number of terms that are not typically included when we compare the SM to experiment. The higher the collision energy, the more important these missing terms become.

The main motivation of this thesis is to help determine how important these missing terms are at the high, but not infinitely high, energy of the LHC.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have been published in [1–4].

(Emmet Piers Byrne, July 2022)

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Chapter 1

Introduction

The underlying motivation for this thesis is to help answer the question of whether the Standard Model of particle physics provides a sufficient description of nature at the highest energy scales we have experimental access to.

The Large Hadron Collider (LHC) provides us with an unprecedented window into the behaviour of matter at small scales. With the discovery of the Higgs boson, the Standard Model was shown to be a self-consistent description of nature at this microscopic level, and has continued to give an unrivalled description of experimental results. However, for a host of experimental and theoretical reasons (including the existence of gravity, dark matter and dark energy; the non-zero masses of neutrinos; the Hierarchy problem and the strong CP problem), we know that the Standard Model is not the end of the story. It was the expectation of the theoretical physics community that physics beyond the content of the Standard Model, in the form of supersymmetry, technicolor or extra dimensions would be discovered at the ~ 10 TeV energy scale accessible by the LHC. So far there has been no direct evidence of these phenomena at the LHC, and the search for physics beyond the Standard Model has shifted from searching for model-dependent signatures to searching for small deviations between experiment and the Standard Model.

Steady advances in experimental techniques have continually reduced systematic experimental errors and the vast amount of data collected at the LHC so far is perhaps just a twentieth of what we can look forward to obtaining across the lifetime of the LHC. We as a theoretical community are being challenged to match these improvements in precision. Subtle deviations from the predictions

of the Standard Model may provide clues to the form of new physics, so we need to state these predictions as precisely and accurately as possible. Not only are we challenged to form a better understanding of the Standard Model itself, but we must also make this improved understanding accessible to the experimental community in the form of robust, automated and generally applicable computer programmes.

This thesis represents the attempt of the author to provide a modest contribution to both of these complimentary goals. In chapter 3 we perform an investigation of a kinematic limit of a one-loop six-particle amplitude. Not only do we obtain a compact, factorised, expression for the amplitude in this limit, but it is hoped that we also demonstrate how the investigation of such kinematic limits provides more general clues about the structure of amplitudes which may aid our understanding of amplitudes even outwith these kinematic limits. This project can be seen as one small piece in a research programme into the investigation of the nature of amplitudes at asymptotically high energy that extends over four decades, and we therefore give an extended introduction to this research direction in chapter 2, focusing on the building blocks and techniques we will use in the investigation performed in chapter 3.

In chapters 4 and 5 we take a step back from the frontier of theoretical investigations, and ask whether the factorised picture of scattering that emerges at asymptotically high energy is relevant at the high, but finite energy scales accessible by the LHC. The High Energy Jets (HEJ) framework has shown that this high-energy description is indeed relevant for the LHC and that it is essential to include these all-orders corrections for many key processes in order to restore stability to the standard theoretical approach. The HEJ framework is implemented as a Monte Carlo event generator which bypasses many of the limitations of the traditional approach to the high-energy limit reviewed in chapter 2. In particular, it allows much more fixed-order physics to be captured within the factorised expressions which comprise amplitudes at high energy. In chapter 4 we derive two new factorised expressions of this kind which are required to systematically increase the accuracy of HEJ, and which are of critical phenomenological relevance.

Of course, the ultimate goal is a synthesis of the most precise available fixed-order results and the most precise available all-orders results. In chapter 5 we review the method by which the results of HEJ are automatically matched to the results of leading-order (LO) perturbation theory, and we present results of the first matching of a HEJ calculation with a next-to-leading order (NLO) calculation.

This approach is seen to give remarkable agreement with LHC data.

As mentioned above, the study of the high-energy limit of scattering processes has a long and rich history. Many of the concepts and the terminology we will use throughout this thesis arose from the study of the asymptotic limit of quantum mechanical scattering amplitudes. To introduce these terms and ideas, and to provide some historical context of the present work, we give a very brief overview of these ideas in section 1.1. This is followed by a discussion of high-energy scattering in the LHC era in section 1.2, where we place the work of this thesis within the broader landscape of current developments in collider phenomenology. We then give a brief introduction to perturbative quantum chromodynamics (QCD) in section 1.3. In section 1.4 we review the connection between the outgoing partons of a perturbative QCD amplitude and the collimated jets of hadrons observed in collisions at the LHC. In sections 1.5–1.7 we review some key modern developments in the organisation of QCD scattering amplitudes which have enabled the progress that we detail in chapters 3–5.

1.1 The S -matrix approach and Regge theory

Since the early 1970s, QCD has been our unrivalled theory of the strong interaction. However, the study of the high energy limit of scattering amplitudes predates the acceptance of QCD, and hence also much of the terminology used in this field today has its origin in the language of the pre-QCD era. In this section we give a brief sketch of some key ideas of the high-energy limit of scattering in the decades preceding the establishment of QCD as a successful theory of the strong interaction. This will firstly provide some broader historical context for the new results presented in this thesis, and secondly will provide an overview of the origin of several terms and concepts that we will use throughout this thesis. The model-independent predictions of Regge theory that we will encounter in this section will furthermore provide a simple interpretation of many results later obtained in QCD, and will provide us with an indispensable heuristic for predicting the scaling of QCD amplitudes in various kinematic limits. Finally, it is hoped, for readers like the author who had little exposure to these ideas during their graduate lectures, that this section will help to bridge the gap to the vast and rich literature of this era.

Let us begin by sketching the ideas of S -matrix theory, following the classic

textbook ref. [7]. This theory-independent approach was advocated shortly after the development of quantum mechanics [8, 9], but gained popularity in the 1950s and 60s due to the difficulties of applying the techniques of perturbative QFT to the strong force.

Drawing lessons from quantum mechanics, the S -matrix approach worked directly with physically observable quantities, namely transition probabilities. Consider the collision of particles in some initial state, $|i\rangle$, which after interacting becomes a final configuration of particles $|f\rangle$. We would like to know, given some $|i\rangle$, what is the probability of obtaining the result $|f\rangle$. Using the language of quantum mechanics, we suppose there is an operator S , such that its matrix element between an initial and final state $\langle f|S|i\rangle$ gives the probability amplitude that $|f\rangle$ will result from $|i\rangle$:

$$P_{i\rightarrow f} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle. \quad (1.1)$$

Here $S^\dagger$ is the hermitian conjugate of S .

Rather than specifying an underlying field theory to compute these probabilities, which is the approach we will take in this thesis, the S -matrix approach aimed to extract as much information as possible about the about $\langle f|S|i\rangle$ itself based on general and physically motivated postulates, which we briefly review.

We begin with the requirement that the interaction between our particles must be sufficiently short-range:

Postulate 1: The interactions between the particles we are considering are sufficiently short-range so that we can consider the initial and final states to consist of free particles.

In the development of this subject, this requirement was met by, e.g. the short range Yukawa potential of massive pion exchange. The requirement appears troublesome from the point of view of QCD, where quarks and gluons are never directly observed as free particles, but in section 1.2 we will see that within the framework of collinear factorisation we can treat the partonic matrix element as the collision of free quarks and gluons.

The next postulate adopts some more ideas from quantum mechanics:

Postulate 2: Free particle states, containing any number of particles

satisfy the superposition principle of quantum mechanics. These states are assumed to be orthonormal and complete.

In other words, if $|a\rangle$ and $|b\rangle$ are physical states, then so is $(|a\rangle + |b\rangle)$, and we assume that $\sum_n |n\rangle\langle n| = 1$.

The third postulate builds on the other pillar of 20th century physics:

Postulate 3: The S -matrix is invariant under Lorentz transformations.

The squared matrix elements must consist of Lorentz invariant products of the 4-momenta, $p^\mu = (p^0, p^x, p^y, p^z)$, of the particles in $|i\rangle$ and $|f\rangle$. We use the metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We will often write the Lorentz invariant inner product of two 4-momenta p_i and p_j as $p_i \cdot p_j = \eta_{\mu\nu} p_i^\mu p_j^\nu$, and we write $\eta_{\mu\nu} p_i^\mu p_i^\nu = p_i^2$. Energy and momentum will be conserved in the interactions. If we have a scattering with n external particles, there will be $4n$ 4-momentum components. However, each external particle will be on its mass-shell, $p_i^2 = m_i^2$, which introduces 4 constraints on these quantities. The symmetries of the Poincaré group gives us another 10 constraints, and therefore an n particle interaction will be described by $3n - 10$ independent variables. For example, a $2 \rightarrow 2$ collision, say $p_1 + p_2 \rightarrow p_3 + p_4$, will be described by 2 independent variables. We will most commonly describe such collisions by the Mandelstam variables $s \equiv (p_1 + p_2)^2$, $t \equiv (p_1 - p_3)^2$ and $u \equiv (p_1 - p_4)^2$. Only two of these are independent as momentum conservation gives us the relation $s + u + t = \sum_{i=1}^4 m_i^2$. In addition to the conservation of energy and momentum, it is assumed that probabilities are conserved:

Postulate 4: Given an initial state $|i\rangle$, the probability that there will be a final state is 1.

Using the above notation, this statement reads

$$\begin{aligned} \sum_f P_{i \rightarrow f} &= 1 \\ \Rightarrow \sum_f \langle i | S^\dagger | f \rangle \langle f | S | i \rangle &= 1 \\ \Rightarrow \langle i | S^\dagger S | i \rangle &= 1 \end{aligned} \tag{1.2}$$

where in the final line we have used the completeness property of postulate 2. As this must be true for any initial state, we conclude

$$S^\dagger S = 1 = S S^\dagger, \quad (1.3)$$

i.e. the S -matrix is *unitary*.

It is useful to separate the S -matrix into its connected and disconnected parts,

$$S = 1 + iT \quad (1.4)$$

in terms of which eq. (1.3) becomes

$$T T^\dagger = -i(T - T^\dagger) \quad (1.5)$$

Factoring out a ubiquitous momentum conserving delta function, we define a scattering amplitude $\mathcal{M}$ via

$$\langle f|T|i\rangle = i(2\pi)^4 \delta^4(P_i - P_f) \mathcal{M}_{i \rightarrow f}, \quad (1.6)$$

where P_i and P_f are the sum of momenta in the initial and final state respectively. In terms of scattering amplitudes, eq. (1.5) becomes

$$\mathcal{M}_{i \rightarrow f} - \mathcal{M}_{f \rightarrow i}^* = i \sum_c \int d\Phi_c \mathcal{M}_{f \rightarrow c}^* \mathcal{M}_{i \rightarrow c}, \quad (1.7)$$

where the sum runs over all final states c (including a sum over all quantum numbers such as helicity or colour) that are allowed by 4-momentum conservation, and the measure $d\Phi_c$ represents the standard Lorentz invariant measure for the momenta in state c . For the production of n particles in the scattering of particles with the summed momenta P_{in} , this Lorentz invariant measure is

$$d\Phi_n = \prod_{i=1}^n \left[\frac{d^4 p_i}{(2\pi)^4} (2\pi) \delta(p_i^2 - m_i^2) \Theta(p_i^{(0)}) \right] (2\pi)^4 \delta^4(P_{\text{in}} - \sum_{i=1}^n p_i). \quad (1.8)$$

We will refer to eq. (1.7) as the s -channel unitarity equation. In the case of forward scattering eq. (1.7) simplifies to the optical theorem,

$$2\text{Im} [\mathcal{M}_{i \rightarrow i}] = \sum_c \int d\Phi_c |\mathcal{M}_{i \rightarrow c}|^2. \quad (1.9)$$

Postulate 3 implies that scattering amplitudes will be functions of Lorentz invariant products of the external momenta. Postulate 5 specifies what kind of function:

Postulate 5: Scattering amplitudes are the real boundary values of analytic functions of the Mandelstam variables, regarded as complex variables, with only those singularities that are required by the unitarity equation.

While the previous postulates had quite clear physical consequences, the physical meaning of postulate 5 is more obscure. An overview of the connection between analyticity and causality, in both classical and quantum contexts, is provided by ref. [10]. The wording of postulate 5 suggests that the unitarity equation demands that scattering amplitudes are not entire functions of Mandelstam variables. A detailed explanation of how the unitarity equation gives rise to singularities of amplitudes is given in [7]. For example, for the $2 \rightarrow 2$ scattering of particles of equal mass m , the unitarity equation demands that the amplitude has a pole in the complex s plane at $s=m^2$, and threshold branch points at $s = (nm)^2$ for $n = 2, 3, \dots$. This analytic structure is sketched in blue in fig. 1.1.

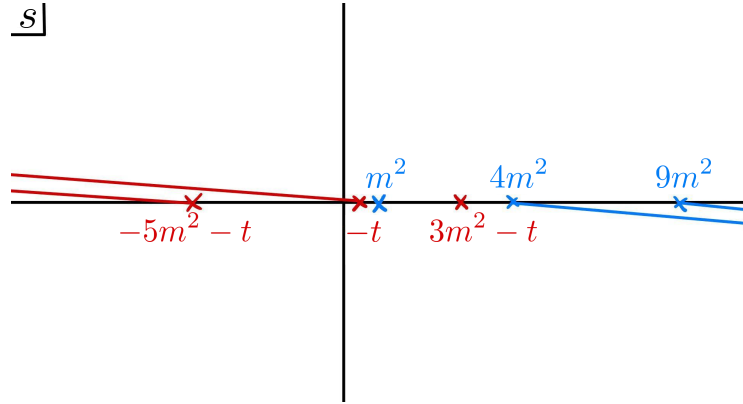


Figure 1.1 *Singularities in the complex s -plane of the $2 \rightarrow 2$ scattering amplitude of particles of equal-mass particles of mass m . The singularities required by unitarity are shown in blue. Assuming crossing relations, the amplitude also possesses the singularities shown in red.*

Postulate 5 suggests that the amplitude $\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}(s, t, u)$ for the scattering of particles with ‘flavour’ f_i , with physical region

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0,$$

may be analytically continued to the region

$$u \geq 4m^2, \quad s \leq 0, \quad t \leq 0.$$

It is assumed that (taking a suitable limit to the latter region) this is the physical scattering amplitude for the process $\mathcal{M}_{f_1 \bar{f}_3 \rightarrow \bar{f}_2 f_4}(s, t, u)$, where $\bar{f}_i$ is the antiparticle of f_i . Such crossing relations were taken as assumptions in the S -matrix approach, but are true in perturbative QFT. The above crossing relation implies that the amplitude $\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}(s, t, u)$ must have a pole at $s = 3m^2 - t$ and branch points at $s = (4 - n^2)m^2 - t$, which have also been depicted in fig. 1.1, in red.

Crucially, postulate 5 allows us to utilise the machinery of complex analysis to great effect. For our massive $2 \rightarrow 2$ example, the singularity structure depicted in fig. 1.1 allows us to use Cauchy's integral formula

$$\mathcal{M}(s, t) = \frac{1}{2\pi i} \oint \frac{ds'}{s' - s} \mathcal{M}(s', t), \quad (1.10)$$

to rewrite the large s limit of the amplitude as an integral along its real s -axis,

$$\begin{aligned} \mathcal{M}(s, t) = & \frac{\text{Res}_{s=m^2}[\mathcal{M}(s, t)]}{m^2 - s} + \frac{\text{Res}_{s=3m^2-t}[\mathcal{M}(s, t)]}{3m^2 - t - s} \\ & + \frac{1}{\pi} \int_{s_T}^{\infty} ds' \frac{\text{Disc}_s[\mathcal{M}(s', t)]}{s' - s} + \frac{1}{\pi} \int_{-\infty}^{u_T} du' \frac{\text{Disc}_u[\mathcal{M}(u', t)]}{u' - u}, \end{aligned} \quad (1.11)$$

given that the contribution from the contour at infinity vanishes at large $|s|$. Here we have labelled s and u threshold branch points by s_T and u_T respectively, and we have used the notation

$$\text{Disc}_z[f(z)] = \lim_{\epsilon \rightarrow 0} [f(z + i\epsilon) - f(z - i\epsilon)]. \quad (1.12)$$

If the amplitude does not vanish as $|s| \rightarrow 0$, it is necessary to use subtracted dispersion relations, which are reviewed in e.g. ref. [11].

We now turn to the ideas of Regge theory. In his seminal papers [12, 13], Regge introduced the idea of analytically continuing amplitudes in the complex angular momentum plane. To begin, we assume the scattering process has axial symmetry, and we decompose the amplitude as a sum over ‘partial-wave’ amplitudes

$$\mathcal{M}(s, t) = 16\pi \sum_{l=0}^{\infty} (2l+1) m_l(t) P_l(z_t), \quad (1.13)$$

with $z_t = 1 + \frac{2s}{t}$, and where, due to the orthogonality of the Legendre polynomials, P_l , the coefficients $m_l(t)$ are given by

$$m_l(t) = \frac{1}{16\pi} \frac{1}{2} \int_{-1}^1 dz_t P_l(z_t) \mathcal{M}(s(z_t, t), t). \quad (1.14)$$

Using eq. (1.11), an alternative form for the partial wave amplitudes can be obtained in terms of Legendre polynomials of the second kind

$$\begin{aligned} m_l(t) = & \frac{1}{16\pi^2} \int_{z_t(s_T, t)}^{\infty} dz'_t \text{Disc}_s[\mathcal{M}(s(z'_t, t), t)] Q_l(z'_t) \\ & + \frac{1}{16\pi^2} \int_{z_t(u_T, t)}^{-\infty} dz'_t \text{Disc}_u[\mathcal{M}(u(z'_t, t), t)] Q_l(z'_t), \end{aligned} \quad (1.15)$$

which, if the large z limit of the discontinuities of the amplitude scales as z^N , is valid for $l > N$. The form of the partial wave given in eq. (1.15) is known as the Froissart-Gribov projection. The large l behaviour of $Q_l(z)$ is given by

$$Q_l(z) \xrightarrow{l \rightarrow \infty} l^{-\frac{1}{2}} e^{-(l+\frac{1}{2})\xi(z)}, \quad \xi(z) = \log \left(z + \sqrt{z^2 - 1} \right). \quad (1.16)$$

The first line of eq. (1.15) vanishes in the large l limit, but the second line is unbounded due to the imaginary part of ξ . Below, we would like to work with quantities that converge in the large l limit. This was the motivation to introduce partial waves of odd (+) and even (−) signature,

$$m_l^{(\pm)}(t) = \frac{1}{16\pi^2} \int_{z_T}^{\infty} dz'_t D_s^{(\pm)}(s(z'_t), t) Q_l(z'_t), \quad (1.17)$$

where

$$D_s^{(\pm)}(s(z'_t), t) = \text{Disc}_s[\mathcal{M}(s(z'_t), t), t] \pm \text{Disc}_u[\mathcal{M}(s(z'_t), t), t], \quad (1.18)$$

such that both odd and even $m_l^{(\pm)}$ are exponentially suppressed in the large l limit. In terms of signature, the amplitude eq. (1.13) is written

$$\mathcal{M}^{(\pm)}(s, t) = 16\pi \sum_{l=0}^{\infty} (2l+1) m_l^{(\pm)}(t) P_l(z_t), \quad (1.19)$$

where the concept of odd (even) signature translates into odd (even) behaviour

with respect to the interchange of $s \leftrightarrow u$

$$\mathcal{M}(s, t) = \mathcal{M}^{(-)}(s, t) + \mathcal{M}^{(+)}(s, t), \quad \mathcal{M}^{(\pm)}(s, t) = \frac{1}{2} (\mathcal{M}(s, t) \pm \mathcal{M}(u, t)) . \quad (1.20)$$

As an aside, we note that the s -channel analogue of eq. (1.17) can be used in conjunction with the optical theorem, eq. (1.9), to place a bound on the total cross section at asymptotically large s [14–16],

$$\sigma_{\text{total}}(s) \underset{s \rightarrow \infty}{\leq} k \log^2(s), \quad (1.21)$$

for some constant k . Equation (1.21) is known as the Froissart-Martin bound. In chapter 2 we will compare results obtained in perturbative QCD with this general bound.

Now we consider the bold step of Regge and analytically continue l in eq. (1.17). To proceed further we must make the critical assumption of Regge theory:

Maximal analyticity of the second kind: The amplitude $m_l(t)$ is an analytic function of l with only isolated singularities.

If this is true then we can use Cauchy's theorem, eq. (1.10), to write eq. (1.13) as the integral

$$\mathcal{M}^{(\pm)}(s, t) = -\frac{16\pi}{2i} \int_{\mathcal{C}_1} dl (2l + 1) m_l^{(\pm)}(t) \frac{P_l(-z_t)}{\sin(\pi l)}, \quad (1.22)$$

where the contour $\mathcal{C}_1$ encircles the non-negative integer poles of the integrand as depicted in fig. 1.2.

Let us consider the simple example where the partial-wave amplitude only has simple poles at $l = \alpha_i(t)$, with residues $\beta_i(t)$. Then opening the contour to $\mathcal{C}_2$ of fig. 1.2a, the amplitude becomes

$$\begin{aligned} \mathcal{M}^{(\pm)}(s, t) = & -\frac{16\pi}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl (2l + 1) m_l^{(\pm)}(t) \frac{P_l(-z_t)}{\sin(\pi l)} \\ & - 16\pi^2 \sum_i (2\alpha_i(t) + 1) \beta_i(t) \frac{P_{\alpha_i(t)}(-z_t)}{\sin(\pi \alpha_i(t))}, \end{aligned} \quad (1.23)$$

where we have used the fact that $m_l(t)$ vanishes in the large l limit. In the

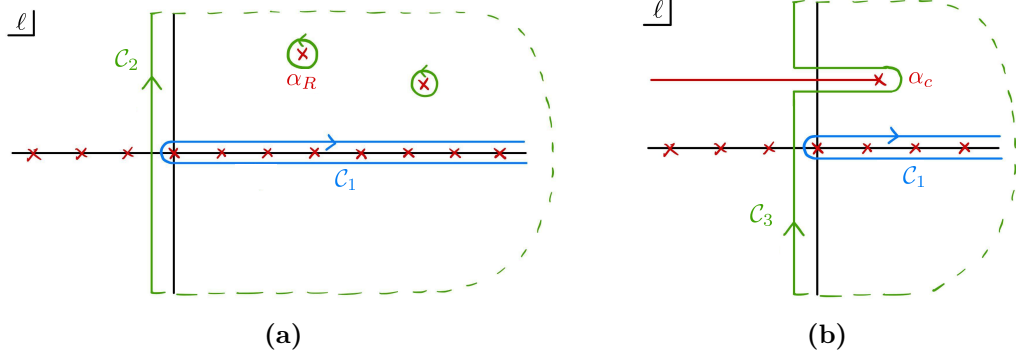


Figure 1.2 *Deforming the Sommerfeld-Watson contour $\mathcal{C}_\infty$ in the situations where: (a) the leftmost singularity is a simple pole at $l = \alpha_R(t)$; (b) the leftmost singularity is a branch point at $l = \alpha_c(t)$.*

high-energy, or Regge limit of $s \gg |t|$, the Legendre polynomial behaves as

$$P_l \left(1 + \frac{2s}{t} \right) \xrightarrow{s \gg |t|} \frac{\Gamma(2l+1)}{\Gamma^2(l+1)} \left(\frac{s}{2t} \right)^l, \quad (1.24)$$

and therefore the first term in eq. (1.23), called the ‘background integral’ vanishes. Furthermore, only the rightmost pole, $\alpha_R(t)$, will contribute to the sum over residues, and the amplitude will have the high-energy behaviour

$$\mathcal{M} \sim s^{\alpha_R(t)} \gamma(t). \quad (1.25)$$

We can interpret this amplitude as the t -channel exchange of an object with t -dependent ‘angular momentum’ $\alpha_R(t)$. This object is referred to as a Reggeon. In chapter 1, in the context of perturbative QCD, we will interpret the Reggeon as the superposition of the exchange of all possible particles in the t -channel. From eq. (1.25) we find that the total $2 \rightarrow 2$ cross section scales with s as

$$\sigma \sim s^{2\alpha_R(t)-2}. \quad (1.26)$$

In the ‘double-Regge’ limit, the 5-particle amplitude similarly scales as [17]

$$\mathcal{M} \sim s_1^{\alpha(t_1)} s_2^{\alpha(t_2)} \gamma(t_1, t_2, \eta_{12}), \quad (1.27)$$

where $\eta_{12} = s_1 s_2 / s$ is a finite scale, and where we again assume that the only singularities in the angular momentum plane are simple poles. This leads us to expect that the $2 \rightarrow n-2$ amplitude, in the Multi-Regge kinematic limit scales

as

$$\mathcal{M} \sim s_1^{\alpha(t_1)} s_2^{\alpha(t_2)} \dots s_{n-3}^{\alpha(t_{n-3})} \gamma(\{t_i, \eta_{ij}\}), \quad (1.28)$$

where γ is some function that only depends on the finite degrees of freedom.

We emphasise that the scaling given by eqs. (1.25), (1.27) and (1.28) is only valid if the leading behaviour is determined by poles in the angular momentum plane. This is not necessarily the case. Let us instead assume that the right-most singularity is a branch point, as depicted in fig. 1.2b. To be concrete, let us also assume that $m_\ell(t)$ behaves as $(\ell - \alpha_c(t))^{-1-\beta(t)}$. Then, instead of eq. (1.25), we find that the amplitude scales as

$$\mathcal{M} \sim s^{\alpha_c(t)} (\log(s))^{\beta(t)} \gamma(t). \quad (1.29)$$

This logarithmic behaviour is called a Regge cut contribution.

The ideas sketched here apply to the simple scattering of spinless particles and many assumptions were made along the way. Decades later, QCD has been established as our uncontested theory of the strong interaction. We need not make such general statements as the ones of S -matrix theory, and we have the luxury of using the machinery of perturbative QFT. In chapter 2 we will embrace QCD as our microscopic theory of the strong interaction, and we will see to what extent the asymptotic behaviour of QCD corresponds to the results of Regge theory. To facilitate this discussion, in the next section we will review some key features of QCD, and we will review some modern developments which we will utilise throughout this thesis.

1.2 Collisions at the LHC

While the previous section discussed general aspects of scattering amplitudes, from this point we specify our interest to the proton-proton collisions at the LHC. Protons are not considered the fundamental fields of a theory, but bound states that cannot be understood through the framework of perturbative QFT. Nevertheless, in deep inelastic scattering (DIS) experiments at SLAC, it was revealed that in energetic collisions the proton could be modelled as a collection of free constituents termed partons. In this thesis we take the term parton to mean a quark, antiquark or gluon.

That we can treat the partons within colliding hadrons as free particles can be motivated by the following heuristic argument [18, 19]. Consider the collision of two protons A and B , of mass ~ 1 GeV and collision energy $\sim 14,000$ GeV. In the centre-of-mass frame, the protons will be heavily Lorentz contracted in the direction of the collision, and therefore the time taken for parton a (b) within hadron A (B) to go through hadron B (A) will be heavily Lorentz contracted. Conversely, the parton-parton interactions within each hadron will be time dilated. The picture that emerges is that parton a (b) sees proton B (A) as a frozen distribution of partons. If the squared transfer of momentum of the collision, $-q^2$, is sufficiently large, parton a (b) will probe proton B (A) in a region of transverse area $\sim 1/q^2$. Unless the hadron is densely packed, it will meet only parton b (a). Fortunately, this simple picture appears to generally give a good description of collisions at the LHC, and the cross section for the scattering of two protons to a final state X is given by the factorised formula

$$\sigma_{AB \rightarrow X} = \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a, \mu_F) f_{b/B}(x_b, \mu_F) \hat{\sigma}_{ab}(\mu_F, \alpha_s(\mu_R)). \quad (1.30)$$

The objects $f_{a/A}(x_a, \mu_F)$ denote *parton distribution functions* which depend only on the longitudinal momentum fraction x_a that parton a has with respect to the hadron A . The factorisation scale is chosen to be a scale on the order $-q^2$. In the context of DIS, where PDFs were originally defined and measured, this scale was the virtuality of the probing photon.

Collinear factorisation has been proven for the cases of deep-inelastic scattering and Drell-Yan production [20, 21], but not for proton-proton collisions in general. Indeed the study of factorisation-breaking effects is a growing field, and alternatives to the collinear factorisation picture, such as the use of ‘unintegrated’, or transverse-momentum dependent PDFs [22–25], is an active area of research.

The simple picture of a single parton from each proton interacting is not always sufficient at the LHC however, and as theorists are pushed to increase theoretical precision, the study of so called multi-parton interactions (MPI) will become increasingly important. The simplest example of MPI is where four incoming partons interact pairwise, termed hard double parton scattering (DPS). In this thesis we only consider ‘single-parton scattering’ (SPS), but to illustrate the importance of MPI at the LHC, we mention that, at 13 TeV, the DPS four-jet cross section is 34% of the SPS four-jet cross section [18].

The object $\hat{\sigma}_{ab}(\mu_F, \alpha_s(\mu_R))$ denotes the partonic cross section, given by

$$\int d\Phi_X \frac{|\mathcal{M}_{ab \rightarrow X}(\Phi, \mu_F, \alpha_s(\mu_R))|^2}{\mathcal{F}_{ab}}. \quad (1.31)$$

Here $\mathcal{F}_{ab}$ denotes the incoming partonic flux

$$\mathcal{F}_{ab} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \xrightarrow{m_a^2 m_b^2 \rightarrow 0} 2\hat{s}. \quad (1.32)$$

Throughout this thesis we will assume that we can neglect the masses of the partons, so that we can simply take the flux factor to be twice the partonic centre-of-mass, $\hat{s} = (p_a + p_b)^2 = x_a x_b s$. The integral is over the available phase space of the final state particles. For n particles, the Lorentz invariant phase space measure is given in eq. (1.8).

As indicated by eq. (1.30), the partonic cross section depends on the strong coupling constant, $\alpha_s(\mu_R)$, which itself depends on the renormalisation scale μ_R . The amplitude is typically computed as a power expansion in $\alpha_s(\mu_R)$. This perturbative approach requires the coupling to be small. In the next section we will review some key ideas required to treat QCD perturbatively, and the nature of α_s .

1.3 Perturbative QCD

Quantum chromodynamics (QCD) is a quantum field theory describing the interactions of quarks and gluons. The QCD Lagrangian density is given by¹

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{\psi}_{r\bar{i}}^f [i\gamma_{rs}^\mu D_{\mu\bar{i}} - \mathbb{1}_{rs}\delta_{\bar{i}i} m_f] \psi_{s i}^f \quad (1.33)$$

where $r, s = 1, \dots, 4$ are spinor indices and $\mu, \nu = 1, \dots, 4$ are Lorentz indices. The field $\psi_s^f(x)$ is a spin- $\frac{1}{2}$ Dirac field with flavour f , mass m_f and with colour index i ($\bar{i}$) in the (anti-)fundamental representation of SU(3). The field $A_\mu^a(x)$ is a spin-1 massless vector field with colour index a in the adjoint representation of SU(3).

The field A and the coupling g_s enter eq. (1.33) through the covariant derivative. When acting on a field of SU(3) representation R , the covariant derivative is given

¹In the notation of ref. [26], we take $\eta_s = -1$ and $\eta_G = +1$.

by

$$D_\mu = \partial_\mu - ig_s A_\mu^a t_R^a, \quad (1.34)$$

where t_R^a is a generator of the representation R . In particular, when acting on a field in the fundamental representation, as in eq. (1.33), the covariant derivative reads

$$D_{\mu \bar{u}} = \delta_{\bar{u}} \partial_\mu - ig_s A_\mu^a T_{\bar{u}}^a, \quad (1.35)$$

where T^a are generators of the fundamental representation, i.e. a basis for the set of traceless and hermitian 3×3 matrices. We write the normalisation of these generators as $\text{tr}(T^a T^b) = T_F \delta^{ab}$. When acting on a field in the adjoint representation, the covariant derivative reads

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - ig_s A_\mu^c f^{abc}, \quad (1.36)$$

where f^{abc} are the structure constants of the Lie Algebra,

$$[t_R^a, t_R^b] = i f^{abc} t_R^c. \quad (1.37)$$

From this point we will often suppress colour and spinor indices, and we will sometimes use a matrix notation for the gauge field, $A_\mu(x) = A_\mu^a(x) t_R^a$. From the covariant derivative we can obtain the field strength tensor

$$F_{\mu\nu} = \frac{1}{g_s} [D_\mu, D_\nu] = F_{\mu\nu}^a t_R^a, \quad (1.38)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c. \quad (1.39)$$

The Lagrangian eq. (1.33) is invariant under a $\text{SU}(3)$ gauge transformation, i.e. invariant under a transformation by a spacetime-dependent special unitary matrix $U(x)$. Under this transformation, the fields transform as

$$\psi(x) \rightarrow U(x) \psi(x), \quad (1.40a)$$

$$A_\mu(x) \rightarrow U(x) A_\mu(x) U(x)^{-1} + \frac{i}{g_s} (\partial_\mu U(x)) U(x)^{-1}. \quad (1.40b)$$

The gauge transformation can be written $U(x) = \exp(i\alpha^a(x) T^a)$, with $\alpha^a(x)$ a

set of real-valued fields. For small α^a we can write eqs. (1.40a) and (1.40b) as

$$\psi(x) \rightarrow \psi(x) + i\alpha^a T^a \psi(x), \quad (1.41a)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{g_s}(\partial_\mu \alpha^a(x))T^a + i[\alpha^a(x)T^a, A_\mu(x)]. \quad (1.41b)$$

The underlying motivation of this thesis is to test whether the theory defined by eq. (1.33) is sufficient to describe nature in the high energy collisions that occur at the Large Hadron Collider. In order to test QCD in this context, we need to be able to make physical predictions from eq. (1.33), i.e. we need to be able to go from a Lagrangian to the partonic scattering amplitudes of eq. (1.31).

As an intermediate step, correlation functions of n fields can be elegantly obtained from a Lagrangian using the path integral formalism, by taking functional derivatives of the partition function. Before considering QCD, let us review this for a scalar field theory with a single field ϕ with mass m . Then a correlator of n fields can be calculated by introducing a source term $J(x)\phi(x)$ to the Lagrangian, and taking n functional derivatives with respect to J , i.e

$$G^{(n)}(x_1, \dots, x_n) = \frac{1}{\mathcal{Z}_{\text{scalar}}} \left(\frac{1}{i} \frac{\delta}{\delta J(x_1)} \right) \cdots \left(\frac{1}{i} \frac{\delta}{\delta J(x_n)} \right) \mathcal{Z}_{\text{scalar}}[J] \Big|_{J=0}, \quad (1.42)$$

where $\mathcal{Z}_{\text{scalar}}$ is the partition function,

$$\mathcal{Z}_{\text{scalar}}[J] = \int \mathcal{D}\phi e^{iS_{\text{scalar}}[\phi, J]}, \quad (1.43)$$

and S_{scalar} is the action

$$S_{\text{scalar}}[\phi, J] = \int d^4x \mathcal{L}_{\text{scalar}}(x) + J(x)\phi(x). \quad (1.44)$$

An n -point correlation function has (at least) n poles, each corresponding to one of the momenta approaching the physical mass of the associated particle, $p_i^2 \rightarrow m^2$. The LSZ formula equates the residue of the correlation function at this

multiple pole to the S -matrix element,

$$\begin{aligned} \langle p'_1, \dots, p'_{m'} | S | p_1, \dots, p_m \rangle &= \lim_{p_i^2, p_j'^2 \rightarrow m^2} \prod_{j=1}^{m'} (p_j'^2 - m^2 + i0) \prod_{k=1}^m (p_k^2 - m^2 + i0) \\ &\times \int \prod_{i=1}^{m'} [d^4 x_i e^{ip'_i \cdot x_i}] \prod_{i=1}^m [d^4 y_i e^{-ip_i \cdot y_i}] G(x_i, \dots, x_{m'}, y_i, \dots, y_m). \end{aligned} \quad (1.45)$$

Equation (1.45) allows us to complete the connection between the Lagrangian, and a scattering amplitude, which enables us to make predictions for collider experiments. If the coupling of the theory is sufficiently small, we can use the machinery of perturbative QFT, and obtain scattering amplitudes order-by-order in the coupling by drawing all possible amputated, connected Feynman diagrams of a given number of loops, and performing the integral over all loop momenta.

For gauge theories such as QCD, however, there is an issue we must address before using the techniques of perturbative QFT. Let us write the correlator of some gauge invariant operator $\mathcal{O}$ as

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{O} e^{iS_{\text{QCD}}[A, \psi, \bar{\psi}]}}{\int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_{\text{QCD}}[A, \psi, \bar{\psi}]}}. \quad (1.46)$$

In addition to the normal difficulty of QFT in having an infinite-dimensional measure, both the numerator and denominator of eq. (1.46) suffer from another difficulty we need to address, namely that the integration is over all configurations of A , including those related by gauge transformations, eq. (1.41b). To remove this redundancy, we would like specify a gauge by imposing a gauge fixing condition $G(A^a) = 0$.

We follow the Faddeev-Popov procedure in order to factor out the integral over gauge orbits. This involves writing the identity as a functional integral

$$1 = \int \mathcal{D}a \left(\det \frac{\delta G(a)}{\delta a} \right) \delta^{(n)}(G(a)). \quad (1.47)$$

We would like to insert this identity into the numerator and denominator of eq. (1.46). We wish the delta function to impose the gauge fixing condition, so let us parametrise the gauge transformations of the gluon field by a^a ,

$$A_\mu^a(a) = A_\mu^a + \frac{1}{g_s} D_\mu a^a. \quad (1.48)$$

Here we have used the fact that the gauge transformation eq. (1.41b) can be expressed in terms of the covariant derivative eq. (1.36). Then due to the gauge invariance of the action and the operator $\mathcal{O}$, we can factor out the integral over a , i.e. we can factor out the divergent integral over gauge-equivalent fields, leaving us with

$$\frac{\int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{O} \delta(G(A^a)) \det\left(\frac{\delta G(A^a(\alpha))}{\delta \alpha}\right) e^{iS_{\text{QCD}}[A, \psi, \bar{\psi}]}}{\int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_{\text{QCD}}[A, \psi, \bar{\psi}]}}. \quad (1.49)$$

We have now fixed a gauge, but due to presence of the delta function and the functional determinant, eq. (1.49) is not yet in a form that we can use the standard tools of perturbative QFT. To put eq. (1.49) in this form we wish to write both of these terms as path integrals over unphysical, auxiliary fields, of some exponentiated action.

To deal with the delta function, we can write

$$\delta(G(A)) = \int \mathcal{D}h e^{iS_{\text{NL}}}, \quad (1.50)$$

where h is a scalar field in the adjoint representation, sometimes called the Nakanishi–Lautrup (NL) field. In the NL action,

$$S_{\text{NL}} = \int d^4x h^a(x) G^a(A(x)) \quad (1.51)$$

the field h simply acts as a Lagrange multiplier, so the path integral imposes the gauge-fixing condition as desired.

To deal with the determinant, we note that the Gaussian integral over complex Grassman numbers θ_i and θ_i^* involving a Hermitian matrix M is equal to the determinant of the matrix,

$$\left(\prod_i \int d\theta_i d\theta_i^* \right) e^{-\theta_i^* M_{ij} \theta_j} = \det M. \quad (1.52)$$

By analogy, we introduce the complex, Grassman valued fields in the adjoint representation c^a , $\bar{c}^a$, in order to write

$$\det\left(\frac{\delta G(A^a(\alpha))}{\delta \alpha}\right) = \int \mathcal{D}c \mathcal{D}\bar{c} e^{iS_{\text{FP}}} \quad (1.53)$$

with

$$S_{\text{FP}} = \int d^4x d^4y \bar{c}^a(x) \frac{\delta G(A^a(\alpha, x))}{\delta \alpha(y)} c^b(y). \quad (1.54)$$

In this thesis we will use Lorentz gauge,

$$G^a(A) = \partial_\mu A^{a\mu}, \quad (1.55)$$

which leads to

$$\frac{\delta G(A^a(\alpha, x))}{\delta \alpha^b(y)} = \delta^{ab} \partial^\mu D_\mu \delta^d(x - y). \quad (1.56)$$

This gives us the ghost action

$$S_{FP} = - \int d^4x (\partial^\mu \bar{c}^a)(D_\mu c^a). \quad (1.57)$$

On the other hand, the gauge-fixing action becomes

$$S_{NL} = \int d^4x h^a(x) \partial_\mu A^{a\mu}. \quad (1.58)$$

This is still difficult to handle, but if we give the NL field an arbitrary mass term,

$$S_{NL} = \int d^4x h^a(x) \partial_\mu A^{a\mu} + \frac{\xi}{2} h^a(x) h^a(x), \quad (1.59)$$

we can perform the path integral eq. (1.50) by ‘completing the square’, obtaining

$$\int \mathcal{D}h e^{iS_{NL}} = e^{i \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu^a)(\partial^\nu A_\nu^a)}. \quad (1.60)$$

We will set $\xi = 1$, known as Feynman gauge.

$$\begin{aligned} \mathcal{L}_{\text{QCD, gauge-fixed}} &= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2\xi} (\partial^\mu A_\mu^a)(\partial^\nu A_\nu^a) + (\partial^\mu \bar{c}^a)(D_\mu c^a) \\ &+ \sum_f \bar{\psi}^f [i\not{D} - m_f] \psi^f \\ &= A_\mu^a [\delta^{ab} \partial^2 \eta^{\mu\nu}] A_\nu^b - \bar{c}^a [\delta^{ab} \partial^2] c^b + \sum_f \bar{\psi}^f [i\not{D} - m_f] \psi^f \\ &+ g_s f^{abc} (\partial^\mu \bar{c}^a) A_\mu^b c^c - g_s f^{abc} (\partial_\mu A_\nu^a) A^{b\mu} A^{c\nu} + \sum_f g_s A_\mu^a \bar{\psi}^f \gamma^\mu T^a \psi^f \\ &- \frac{1}{4} g_s^2 f^{abc} f^{cde} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu} \end{aligned} \quad (1.61)$$

In the final three lines of eq. (1.61) we have collected terms with exactly two, three, or four powers of the fields. This form is amenable to the extraction of the Feynman rules which are listed in eqs. (1.62a)–(1.62g), where we take all momenta to be outgoing, and we label vertices anti-clockwise. A discussion of

external states, namely spinors and gluon polarisation vectors, will be given in section 1.6.

$$a_1 \mu_1 \text{---} \text{---} \text{---} \text{---} a_2 \mu_2 \quad \frac{-i\delta^{a_1 a_2} \eta^{\mu_1 \mu_2}}{p^2 + i0} \quad (1.62a)$$

$$v_2 \xrightarrow[p]{\quad} v_1 \quad \frac{i\delta_{i_1 v_2}(\not{p} + m_f)}{p^2 - m_f^2 + i0} \quad (1.62b)$$

$$a_1 \cdots \cdots \rangle \cdots \cdots a_2 \quad \frac{i\eta_G \delta^{ab}}{p^2 + i0} \quad (1.62c)$$

The fermion-gauge and ghost-gauge interactions are given by

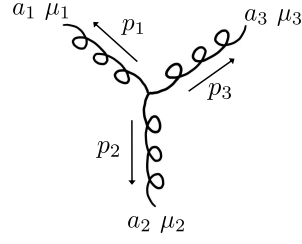


$$ig_s \gamma^\mu T_{i_1 i_2}^a \quad (1.62d)$$

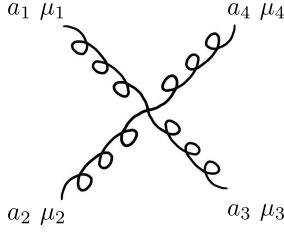


$$g_s f^{a_1 a_2 a_3} p_1^{\mu_2} \quad (1.62e)$$

While the gluon self-interactions are



$$\begin{aligned}
& -g_s f^{a_1 a_2 a_3} \left[\eta^{\mu_1 \mu_2} (p_1 - p_2)^{\mu_3} \right. \\
& \quad \left. + \eta^{\mu_2 \mu_3} (p_2 - p_3)^{\mu_1} \right. \\
& \quad \left. + \eta^{\mu_3 \mu_1} (p_3 - p_1)^{\mu_2} \right]
\end{aligned} \tag{1.62f}$$



$$\begin{aligned}
& -ig_s^2 \left[f^{a_x a_1 a_2} f^{a_x a_3 a_4} (\eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} - \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3}) \right. \\
& \quad + f^{a_x a_1 a_3} f^{a_x a_4 a_2} (\eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} - \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4}) \\
& \quad \left. + f^{a_x a_1 a_4} f^{a_x a_2 a_3} (\eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} - \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4}) \right]
\end{aligned} \tag{1.62g}$$

The fields appearing in eq. (1.61) are so called bare fields, and starting at one-loop, amplitudes computed using the above Feynman rules will suffer from two forms of infinities: infra-red (IR) divergences arise from regions of soft loop-integration momentum while ultraviolet (UV) divergences arise from regions of infinite loop-integration momentum. The convergence of loop integrals depends on the dimension of spacetime: if we modify the spacetime dimension to be $d = 4 - 2\epsilon$, the IR and UV divergences will appear as poles in the Laurent expansion of the amplitude in ϵ . This is known as dimensional regularisation. To keep the coupling, g_s , dimensionless in d dimensions, we re-scale it by an arbitrary parameter μ

$$g_s^2 \rightarrow \mu^{4-d} g_s^2. \tag{1.63}$$

Due to the KLN [27, 28] theorem, the IR divergences from the virtual loop integration must cancel with the integration over ‘unobserved’ (i.e. soft and/or collinear) real radiation. On the other hand, UV divergences can be dealt with via perturbative renormalisation, which we now briefly review. We perform a

rescaling of the fields, conventionally

$$\begin{aligned}
\psi_R &= \sqrt{Z_2} \psi, \\
A_R^\mu &= \sqrt{Z_3} A^\mu, \\
c_R &= \sqrt{Z_2^c} c, \\
m_R &= Z_m m, \\
g_{SR} &= Z_g \mu^\epsilon g_S.
\end{aligned} \tag{1.64}$$

The factors are then expanded as $Z_i = 1 + \delta_i$, where the δ_i are known as counterterms. The counterterms are chosen order-by-order in the renormalised coupling to remove the UV divergences that appear in calculations. There are an infinite number of ways to fix the counterterms to remove the UV divergence: the prescription used to fix the finite part of the counterterm is called the renormalisation scheme. Observables in a renormalised theory should be finite and independent of the renormalisation scheme.

The simplest renormalisation scheme is the so-called minimal-subtraction (MS) scheme, which is simply the prescription that the counterterms have no finite parts. In dimensional regularisation, the counterterms therefore depend on μ , so that in effect, MS is a one-parameter family of renormalisation schemes. We identify the dimensional regularisation scale with the renormalisation scale, $\mu = \mu_R$, that we saw earlier in eq. (1.31). That physical observables are independent of μ is a powerful constraint. Renormalised Green's functions depend explicitly on μ_R , and implicitly depend on μ_R via the renormalised masses and coupling. The requirement that Green's functions should not depend on the value of μ_R is expressed through the Callan-Symanzik equation[29, 30]

$$\left[\mu_R \frac{\partial}{\partial \mu_R} + \beta(g_{SR}) \frac{\partial}{\partial g_{SR}} + n_\psi \gamma_2 + n_A \gamma_3 + \sum_f m_f \gamma_{m_f R} \right] G_R = 0, \tag{1.65}$$

where

$$\beta(g_{SR}) = \mu_R \frac{\partial g_{SR}}{\partial \mu_R}, \quad \gamma_i = -\frac{\mu_R}{2Z_i} \frac{\partial Z_i}{\partial \mu_R}, \quad \gamma_{m_f R} = \frac{\mu_R}{m_f} \frac{\partial m_{fR}}{\partial \mu_R}, \tag{1.66}$$

which are termed the beta function, field anomalous dimensions and mass anomalous dimensions respectively. The beta function of QCD is of crucial importance. To the lowest order in the coupling, the beta function is scheme

independent and given by

$$\beta(g_{SR}) = -\frac{g_{SR}^3}{(4\pi)^2}\beta_0, \quad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f, \quad (1.67)$$

for gauge group $SU(N_c)$ and n_f flavours of quarks. This lets us relate the coupling measured at some physical scale μ_0 , to the coupling at some other scale μ_R ,

$$\alpha_s(\mu_R^2) = \frac{\alpha_s(\mu_0^2)}{1 + \frac{\alpha_s(\mu_0^2)}{4\pi}\beta_0 \log\left(\frac{\mu_R^2}{\mu_0^2}\right)}, \quad (1.68)$$

where $\alpha_s = g_s/(4\pi)$. Typically, the coupling constant is measured at the scale of the mass of the Z boson, M_Z , with a world average value of $\alpha_s(M_Z^2) = 0.1179 \pm 0.0009$ [31]. As $N_c = 3$, and $n_f < 33/2$, we see that $\beta_0 < 0$, and therefore that the QCD coupling enjoys the famous property of asymptotic freedom [32, 33]. On the other hand, eq. (1.68) also tells us that the coupling diverges at some scale $\Lambda_{\text{QCD}} \sim 300$ MeV, at which point perturbation theory is inapplicable.

1.4 Jets

The partonic cross section, eq. (1.31), describes the transition of two incoming partons into several outgoing partons. We have seen above that the running of the coupling, eq. (1.68), allows us to treat the incoming partons as free particles, but it is also responsible for our inability to detect free partons in a detector.

A key feature of partonic amplitudes is their collinear enhancement: as partons move away from the initial collision point they are likely to split into two partons co-moving in the direction of the original parton. The iterated splitting is typically modelled within Monte Carlo event generators using a parton shower model. As the parton shower evolves, the partons lose energy, until their energy is of order Λ_{QCD} . At this stage the partons combine into hadrons. This is an inherently nonperturbative process, and in Monte Carlo event generators this physics is described using qualitative models such as the string model [34] or independent fragmentation [35].

In order to compare the mess of hadrons observed in a hadron calorimeter with the simulated partons or hadrons produced by an event generator, we need to introduce the concept of a jet algorithm. A pedagogical introduction to jet

algorithms is given in ref. [36]. There is no one optimal way to define a jet, and the choice of jet definition depends on the process in question. Throughout this thesis, we will use the anti-kt algorithm [37], which is commonly used in experimental analyses at the LHC. The anti-kt algorithm uses the following distance measures:

$$d_{ij} = \min(p_{i\perp}^{-2}, p_{j\perp}^{-2}) \frac{\Delta R_{ij}^2}{R^2}, \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2, \quad (1.69a)$$

$$d_{iB} = p_{i\perp}^{-2}. \quad (1.69b)$$

We refer to the parameter R as the jet radius. Throughout this thesis we will always choose this parameter to be $R = 0.4$. The procedure to go from a list of particle momenta to a list of jet momenta is given by the following algorithm:

1. Work out all the d_{ij} and d_{iB} according to eq. (1.69).
2. Find the minimum of d_{ij} or d_{iB} .
3. If it is a d_{ij} , replace i and j with their combined momenta. If it is a d_{iB} , declare particle i to be a jet and remove it from the list of particles, provided $d_{iB} > |p_{\text{cut}\perp}|^2$, else discard this particle.
4. Return to step 1, until no particles remain.

A scheme to combine momenta is required for step 3. We use the common choice of the E -scheme or 4-vector recombination scheme, as advocated by ref. [38].

Unlike earlier jet algorithms, the anti-kt algorithm is IR-safe, i.e. the definitions of jets are not altered by the emission of a gluon of zero energy, or by the collinear branching of a parton. The latter property is particularly easy to see, as any collinear pair will have small ΔR_{ij} , and will be immediately recombined.

One disadvantage of the anti-kt algorithm is that unlike the related kt algorithm, the substructure of the jet cannot be usefully related to the pattern of QCD branching, and instead the anti-kt algorithm will tend to result in roughly circular jets in the y - ϕ plane [36]. This will not be of much importance for us, as in this thesis we are primarily interested in describing the physics of partons that are widely separated in rapidity, and in chapter 4 and chapter 5 we will only encounter one or two hard partons in a single jet.

1.5 Colour management

The high energy limit of QCD amplitudes depends heavily on the interplay between colour and kinematics. Organisation of the colour structure of amplitudes also leads to great computational simplifications, and will allow us to obtain compact results in chapter 3 and chapter 4 which would otherwise be intractable.

In the following section I will present some useful identities for manipulating the colour structures of Feynman diagrams. In section 1.5.2 we will list projection operators that will allow us to decompose a colour structure into a basis of the exchange of irreducible representations. Such projection operators will be used frequently in chapter 4. Finally in section 1.5.3 we will discuss several colour bases which will be of great benefit in organising amplitudes throughout this thesis.

1.5.1 Identities for $SU(N_c)$ colour algebra

Only fields that transform in the adjoint and fundamental representation appear in eq. (1.33), so we will limit our discussion to these representations. We will however leave the dimension of the fundamental representation as N_c . For completeness let us first collect the identities mentioned in section 1.3, specialising eq. (1.37) to the fundamental and adjoint representations:

$$[T^a, T^b]_{ij} = if^{abc}T_{ij}^c, \quad (1.70a)$$

$$f^{aeb}f^{edc} - f^{aec}f^{edb} = f^{ade}f^{ecb}, \quad (1.70b)$$

$$T_{ij}^{a\dagger} = T_{ij}^a, \quad (1.70c)$$

$$\text{tr}(T^a) = 0, \quad (1.70d)$$

$$\text{tr}(T^a T^b) = T_F \delta^{ab}. \quad (1.70e)$$

For any representation, $t_R^a t_R^a$ commutes with all generators (which follows from eq. (1.37)), so by Schur's lemma must be proportional to the $(\dim(R) \times \dim(R))$ identity. For the fundamental and adjoint representation,

$$(T^a T^a)_{ij} = C_F \delta_{ij}, \quad i^2 f^{acd} f^{dcb} = C_A \delta^{ab}, \quad (1.71)$$

where C_F and C_A are known as quadratic Casimir invariants. Using $\delta_{ii} = N_c$, $\delta^{aa} = N^2 - 1$ and eq. (1.70e) we find

$$C_F = T_F \frac{N_c^2 - 1}{N_c}. \quad (1.72)$$

Pragmatically, eq. (1.71) and eq. (1.70e) lets us reduce ‘bubble’ diagrams of generators to the identity matrix in the representation of the external legs. To determine C_A we will need to do more work, but we will derive many useful identities along the way.

We note that any $N \times N$ complex matrix M can be written as a complex linear combination of the fundamental generators and the identity,

$$M_{ij} = M_0 \delta_{ij} + M_a T_{ij}^a. \quad (1.73)$$

We can project the M_0 coefficient by taking the trace of eq. (1.73) and we can project the M_a coefficients by first multiplying eq. (1.73) by another generator T^b and then taking the trace. Substituting these results back into eq. (1.73), we obtain

$$M_{ij} = \frac{1}{N_c} \text{tr}(M) \delta_{ij} + \frac{1}{T_F} \text{tr}(MT^a) T_{ij}^a. \quad (1.74)$$

Rewriting the traces in index notation, and rearranging, we get

$$\left[T_{ij}^a T_{kl}^a - T_F \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \right] M_{lk} = 0. \quad (1.75)$$

As this is true for any M , we obtain the identity

$$T_{ij}^a T_{kl}^a = T_F \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right). \quad (1.76)$$

Many useful results follow from eq. (1.76), including

$$T_{ij}^a T_{jk}^b T_{kl}^a = -\frac{T_F}{N_c} T_{il}^b, \quad (1.77)$$

which is the first example of several ‘triangle’ identities that we will obtain, relating the contraction of three generators to a single generator. Consequently,

$$\text{tr}(T^a T^b T^a T^c) = -\frac{T_F^2}{N_c} \delta^{bc}. \quad (1.78)$$

The completeness relation eq. (1.73) lets us express the anticommutator of two

fundamental generators as

$$\{T^a, T^b\}_{ij} = \frac{2T_F}{N_c} \delta^{ab} \delta_{ij} + d^{abc} T_{ij}^c, \quad (1.79)$$

where d^{abc} is symmetric under $a \leftrightarrow b$. Equation (1.79) furthermore implies

$$d^{abc} = \frac{1}{T_F} \text{tr}(\{T^a, T^b\} T^c), \quad (1.80)$$

so we see that d^{abc} is in fact totally symmetric, and can be viewed as the symmetric analogue of the structure constants

$$if^{abc} = \frac{1}{T_F} \text{tr}([T^a, T^b] T^c). \quad (1.81)$$

Equation (1.70a) and eq. (1.79) can be combined to give

$$(T^a T^b)_{ij} = \frac{1}{2} \{T^a, T^b\} + \frac{1}{2} [T^a, T^b] = \frac{T_F}{N_c} \delta_{ij} + \frac{1}{2} (d^{abc} + if^{abc}) T_{ij}^c. \quad (1.82)$$

From here we can easily obtain another two triangle identities

$$\text{tr}(T^a T^b T^c) = \frac{T_F}{2} (d^{abc} + if^{abc}), \quad (1.83)$$

$$if^{abc} (T^a T^b)_{ij} = -\frac{C_A}{2} T_{ij}^c. \quad (1.84)$$

Let us return to the task of determining C_A . Using eq. (1.83) and eq. (1.77) we can obtain the useful identities

$$if^{acd} \text{tr}(T^d T^c T^b) = T_F^2 \left(\frac{C_F}{T_F} + \frac{1}{N_c} \right) \delta^{ab} = T_F^2 N_c \delta^{ab}, \quad (1.85)$$

$$d^{acd} \text{tr}(T^d T^c T^b) = T_F^2 \left(\frac{C_F}{T_F} - \frac{3}{N_c} \right) \delta^{ab} = T_F^2 \left(\frac{N_c^2 - 4}{N_c} \right) \delta^{ab}. \quad (1.86)$$

Finally we can determine that

$$C_A = 2T_F N_c, \quad (1.87)$$

and for completeness we list two final ‘bubble identities’

$$d^{acd} d^{dcb} = 2T_F \left(\frac{N_c^2 - 4}{N_c} \right) \delta^{ab}, \quad if^{acd} d^{dcb} = 0. \quad (1.88)$$

When considering loop-level colour structures, it is useful to introduce a generator

notation for f and d

$$(F^c)_{ab} = i\sqrt{2T_F}f^{acb}, \quad (D^c)_{ab} = \sqrt{2T_F}d^{acb}. \quad (1.89)$$

This allows us to write trace identities and (anti)commutation relations in a form analogous to the fundamental identities above. For example, eq. (1.70b) becomes

$$[F^b, F^c]_{ad} = i\sqrt{2T_F}F_{ad}^e f^{ecb}, \quad (1.90)$$

and we collect here a number of useful ‘triangle’ identities:

$$\text{tr}(F^a F^b F^c) = i(2T_F)^{\frac{3}{2}} N_c T_F f^{abc}, \quad (1.91a)$$

$$\text{tr}(F^a F^b D^c) = (2T_F)^{\frac{3}{2}} N_c T_F d^{abc}, \quad (1.91b)$$

$$\text{tr}(F^a D^b D^c) = i(2T_F)^{\frac{3}{2}} \left(\frac{N_c^2 - 4}{N_c} \right) T_F f^{abc}, \quad (1.91c)$$

$$\text{tr}(D^a D^b D^c) = (2T_F)^{\frac{3}{2}} \left(\frac{N_c^2 - 4}{N_c} \right) T_F d^{abc}. \quad (1.91d)$$

Lastly, in chapter 3 we encounter a totally symmetric tensor which can be expressed as a sum over all permutations

$$d_A^{a_1 a_2 a_3 a_4} = \frac{1}{4!} \sum_{\sigma \in S_4} \text{tr}(F^{a_{\sigma_1}} F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}}). \quad (1.92)$$

where the subscript A indicates the generators in the associated trace are in the adjoint representation. In this more general notation we can express

$$d_F^{a_1 a_2 a_3} = \frac{T_F}{3!} d^{a_1 a_2 a_3}. \quad (1.93)$$

1.5.2 Projection operators

In this thesis we will use the standard boldface notation for representations of $\text{SU}(N_c)$, where we label the representations by the dimension of the representation for $N_c = 3$. We use a bar to indicate a conjugate representation. For example, the fundamental, antifundamental and adjoint representations are labeled by $\mathbf{3}$, $\bar{\mathbf{3}}$ and $\mathbf{8}$ respectively. Although we label the representations by their dimension for $N_c = 3$, the following projection operators are valid for arbitrary N_c [39].

Given a colour structure with an external fundamental and antifundamental

index, we can decompose the colour structure into a sum over exchanges of irreducible representations that appear in the decomposition

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \quad (1.94)$$

We can construct explicit projection operators to project the contribution from either representation [40, 41]

$$\hat{P}_{\mathbf{1}_{i_1 \bar{i}_1 i_2 \bar{i}_2}} = \frac{1}{N_c} \delta_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2}, \quad (1.95)$$

$$\hat{P}_{\mathbf{8}_{i_1 \bar{i}_1 i_2 \bar{i}_2}} = \frac{1}{T_F} T_{i_1 \bar{i}_1}^{a_t} T_{i_2 \bar{i}_2}^{a_t}, \quad (1.96)$$

which we can check are orthogonal and idempotent. Projecting the octet exchange of QCD processes will be of central importance to the studies carried out in chapter 4. Considering the other possible direct products of $\mathbf{3}$, $\bar{\mathbf{3}}$ and $\mathbf{8}$, we see that only $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\mathbf{8} \otimes \mathbf{8}$ yield an octet exchange:

$$\mathbf{3} \otimes \mathbf{8} = \mathbf{3} \oplus \bar{\mathbf{6}} \oplus \mathbf{15}, \quad (1.97)$$

$$\mathbf{8} \otimes \mathbf{8} = \{\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{8}_s \oplus \mathbf{27}\} \oplus [\mathbf{8}_a \oplus \mathbf{10} \oplus \overline{\mathbf{10}}], \quad (1.98)$$

where for future reference in eq. (1.98) we have placed the symmetric (antisymmetric) representations in curly (square) brackets. In eq. (1.98) we have included the “null” representation, $\mathbf{0}$, which vanishes for SU(3), but its analogue exists for general N_c .

For external gluons we list only the projection operators which are used in this thesis, namely the singlet and antisymmetric-octet projection operators:

$$\hat{P}_{\mathbf{1}}^{a_1 a'_1 a_2 a'_2} = \frac{1}{N_c^2 - 1} \delta^{a_1 a'_1} \delta^{a_2 a'_2} \quad (1.99)$$

$$\hat{P}_{\mathbf{8}_a}^{a_1 a'_1 a_2 a'_2} = \frac{1}{2T_F N_c} f^{a_1 a'_1 a_t} f^{a_t a_2 a'_2}. \quad (1.100)$$

The remaining projection operators for eq. (1.98) are listed in ref. [39].

By contrast, the decomposition of $\mathbf{3} \otimes \mathbf{8}$ does not contain the adjoint representation and such channels will be of little interest to us in this thesis. The exchange of a fundamental representation will only be of tangential interest to us in chapter 4, where we will require

$$\hat{P}_{\mathbf{3}_{i\bar{i}}}^{aa'} = \frac{1}{C_F} (T^a T^{a'})_{i\bar{i}}. \quad (1.101)$$

The remaining projection operators associated with eq. (1.97) can be found in ref. [42] (where $T_F = 1/2$).

1.5.3 Colour ordering

From the QCD Feynman rules, eqs. (1.62a)–(1.62g), we see that at any loop order, amplitudes can only contain factors of T^a or f^{abc} . To reveal the general colour structure of amplitudes, we can eliminate all instances of the structure constants in favour of fundamental generators using eq. (1.81). We can reduce the number of traces using the identity eq. (1.76). In particular, we can always reduce the colour structure of any n -gluon tree-level Feynman diagram to a single trace. This leads us to the colour-decomposition of the n -gluon tree-level amplitude,

$$\begin{aligned} \mathcal{M}^{(0)}(\{p_1, \lambda_1, a_1\}, \dots, \{p_n, \lambda_n, a_n\}) \\ = g_s^{n-2} \sum_{\sigma \in S_n / \mathbb{Z}_n} \text{tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) M_{ng}^{(0)}(p_{\sigma_1}^{\lambda_{\sigma_1}}, \dots, p_{\sigma_n}^{\lambda_{\sigma_n}}). \end{aligned} \quad (1.102)$$

The coefficients $M_n^{(0)}$ are called colour-ordered amplitudes. The sum is over S_n , the set of all permutations of n objects, modulo $\mathbb{Z}_n$, the set of cyclic permutations (thereby accounting for the cyclicity of the trace). Crucially, the colour-ordered amplitudes are individually gauge invariant [43]. Because the amplitudes $M_n^{(0)}$ depend on a cyclic ordering of particles, they have a simpler analytic structure than $\mathcal{M}_n^{(0)}$. For example, the five gluon colour-ordered amplitude, $M_{5g}^{(0)}(1, 2, 3, 4, 5)$, can only have poles in the cyclically ordered two-particle Mandelstam invariants, $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$, not in, e.g. s_{13} .

The colour-ordered amplitudes are not all independent. For example, helicity configurations related by parity can be obtained by complex conjugation

$$(M^{(0)}(p_1^{\lambda_1}, \dots, p_n^{\lambda_n}))^* = M^{(0)}(p_1^{-\lambda_1}, \dots, p_n^{-\lambda_n}), \quad (1.103)$$

and the cyclicity of the trace means the amplitudes are invariant under a cyclic permutation of the gluons

$$M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, \dots, p_n^{\lambda_n}) = M^{(0)}(p_n^{\lambda_n}, p_1^{\lambda_1}, \dots, p_{n-1}^{\lambda_{n-1}}). \quad (1.104)$$

Another relationship among the colour-ordered amplitudes can be obtained by noting that the colour decomposition eq. (1.102) is valid for any gauge group,

not just the $SU(N_c)$ of QCD. Let us take the gauge group to be $U(N_c)$. The Lie algebra is isomorphic to that of $SU(N_c) \times U(1)$. We can therefore use the basis $\{T^a\} \cup \mathbb{1}_{N_c \times N_c}$. We will refer to the field associated with the identity generator as the ‘photon’. Let us now consider eq. (1.102) for an amplitude consisting of one photon and three gluons. Due to the identity appearing within the colour traces, we get

$$\begin{aligned} \mathcal{M}_{\gamma g g g}(p_1, p_2, p_3, p_4) \\ = g_s^2 \text{tr}(T^{a_2} T^{a_3} T^{a_4}) (M(p_1, p_2, p_3, p_4) + M(p_2, p_1, p_3, p_4) + M(p_2, p_3, p_1, p_4)) \\ + g_s^2 \text{tr}(T^{a_2} T^{a_4} T^{a_3}) (M(p_1, p_2, p_4, p_3) + M(p_2, p_1, p_4, p_3) + M(p_2, p_4, p_1, p_3)), \end{aligned} \quad (1.105)$$

where we have suppressed helicity dependence, as we will do so in the following whenever the helicities are irrelevant. The left hand side of eq. (1.105) must be zero, as the interaction term between the photon and a gluon is proportional to the commutator of their generators (see eq. (1.61)), and is therefore zero. The photon and gluons can only couple indirectly via, e.g. a fermion, and this requires a closed loop. It follows that the kinematic term multiplying each of the colour factors in eq. (1.105) must be zero, e.g.

$$M(p_1, p_2, p_3, p_4) + M(p_2, p_1, p_3, p_4) + M(p_2, p_3, p_1, p_4) = 0. \quad (1.106)$$

More generally we have

$$\begin{aligned} 0 = M(p_1, p_2, p_3, \dots, p_n) + M(p_2, p_1, p_3, \dots, p_n) + M(p_2, p_3, p_1, \dots, p_n) \\ + \dots + M(p_2, p_3, \dots, p_1, p_n) \end{aligned} \quad (1.107)$$

which will refer to as the ‘photon decoupling identity’.

In addition to the photon-decoupling identity, there are more intricate relations between the colour-ordered amplitudes, such as the Kleis-Kuijf relations [44]. Let

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_m), \quad \vec{\beta} = (\beta_1, \dots, \beta_{n-m-2}) \quad (1.108)$$

be two ordered sequences of numbers such that

$$\{1\} \cup \{\alpha_1, \dots, \alpha_m\} \cup \{\beta_1, \dots, \beta_{n-m-2}\} \cup \{n\} = \{1, 2, \dots, n-1, n\}. \quad (1.109)$$

The Kleis-Kuijf relations state

$$M_n^{(0)}(1, \alpha_1, \dots, \alpha_m, n, \beta_1, \dots, \beta_{n-m-2}) \quad (1.110)$$

$$= (-1)^{n-2-j} \sum_{\sigma \in \vec{\alpha} \amalg \vec{\beta}^\tau} M^{(0)}(1, \sigma_1, \dots, \sigma_{n-2}, n) \quad (1.111)$$

where $\vec{\beta}^\tau$ denotes the reversal of $\vec{\beta}$, $(\beta_1, \dots, \beta_{n-m-2})$ and $\vec{\alpha} \amalg \vec{\beta}^\tau$ denotes the shuffle product of $\vec{\alpha}$ and $\vec{\beta}^\tau$, i.e. the set of all permutations of the elements of $\vec{\alpha}$ and $\vec{\beta}^\tau$ which preserves the relative order of the elements of $\vec{\alpha}$ and the relative order of the elements of $\vec{\beta}^\tau$. The relations eq. (1.111) can be shown for any theory that consists of only antisymmetric trivalent vertices. A pedagogical explanation of how Yang-Mills theory can be cast into such a form can be found in ref. [45].

Instead of the trace basis, eq. (1.102), it would be desirable to have a basis of colour structures which accounted for the Kleiss-Kuijf relations. Such a basis was introduced in ref. [46] and was in fact inspired by the high-energy limit of amplitudes, which is the object of study of this thesis. In this basis, which we refer to as the DDF basis,

$$\begin{aligned} \mathcal{M}^{(0)}(\{p_1, \lambda_1, a_1\}, \dots, \{p_n, \lambda_n, a_n\}) = \\ g_s^{n-2} \sum_{\sigma \in S_{n-2}} f^{a_2 a_{\sigma_3} c_1} f^{c_1 a_{\sigma_4} c_2} \dots f^{c_{n-3} a_{\sigma_n} a_1} M_n^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_{\sigma_3}^{\lambda_{\sigma_3}} \dots, p_{\sigma_n}^{\lambda_{\sigma_n}}). \end{aligned} \quad (1.112)$$

Note that the sum is over $(n-2)!$ permutations, as opposed to the $(n-1)!$ entries in eq. (1.102). For completeness we mention that there are further relations among tree-level gluon amplitudes known as BCJ relations [47] which means there are in fact only $(n-3)!$ independent amplitudes, but this is not relevant for our purposes, as we will study at most six-gluon amplitudes in detail in this thesis.

Analogous decompositions to eq. (1.102) can be made for amplitudes involving quark-antiquark pairs. In particular, for a single quark-antiquark pair we have the decomposition

$$\begin{aligned} \mathcal{M}^{(0)}(\{p_{\bar{q}}, -\lambda_q, \bar{\iota}\}, \{p_q, \lambda_q, \iota\}, \{p_{g_1}, \lambda_{g_1}, a_1\}, \dots, \{p_{g_n}, \lambda_{g_n}, a_{g_n}\}) \\ = g_s^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{g\sigma_1}} \dots T^{a_{g\sigma_n}})_{\bar{\iota}\iota} M^{(0)}(p_{\bar{q}}^{-\lambda_q}, p_q^{\lambda_1}, p_{g\sigma_1}^{\lambda_{g\sigma_1}}, \dots, p_{g\sigma_n}^{\lambda_{g\sigma_n}}). \end{aligned} \quad (1.113)$$

At one-loop level, the only amplitudes we study in this thesis are pure-gluon

amplitudes. The analogue of eq. (1.112) is [48]

$$\begin{aligned} \mathcal{M}^{(1)} = g_s^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} & \left[\text{tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_n}}) M^{(1)\textcircled{\mathbf{8}}}(p_{\sigma_1}^{\lambda_{\sigma_1}}, \dots, p_{\sigma_n}^{\lambda_{\sigma_n}}) \right. \\ & \left. + 2n_f \text{tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) M^{(1)\textcircled{\mathbf{3}}}(p_{\sigma_1}^{\lambda_{\sigma_1}}, \dots, p_{\sigma_n}^{\lambda_{\sigma_n}}) \right] \end{aligned} \quad (1.114)$$

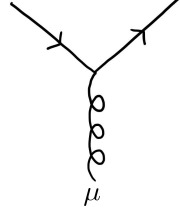
which we will refer to as the DDM basis. Here $S_{n-1} = S_n/\mathbb{Z}_n$ is the group of non-cyclic permutations and $\mathcal{R}$ is the total reflection, e.g. $\mathcal{R}(1, 2, \dots, n) = (n, \dots, 2, 1)$. The superscript $\textcircled{\mathbf{R}}$ on colour-ordered amplitudes denotes the colour representation of the particle circulating in the loop.

Working in a colour ordered basis means we now only need to consider colour ordered Feynman diagrams. We therefore list the colour ordered Feynman rules:

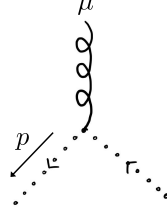
$$\begin{array}{c} \mu_1 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \mu_2 \\ \text{---} p \text{---} \end{array} \quad \frac{-i\eta^{\mu_1\mu_2}}{p^2 + i0} \quad (1.115a)$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} p \text{---} \end{array} \quad \frac{i\not{p}}{p^2 + i0} \quad (1.115b)$$

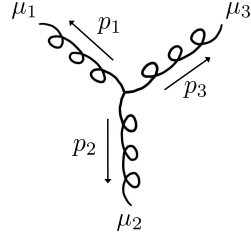
$$\begin{array}{c} \dots \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} p \text{---} \end{array} \quad \frac{i}{p^2 + i0} \quad (1.115c)$$



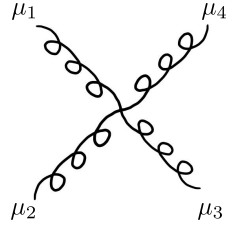
$$ig_s \gamma^\mu \quad (1.115d)$$



$$g_s p^\mu \quad (1.115e)$$



$$ig_s v_{3g}^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) \quad (1.115f)$$



$$ig_s^2 v_{4g}^{\mu_1 \mu_2 \mu_3 \mu_4} \quad (1.115g)$$

where we have introduced the following notation for the three- and four-gluon vertices:

$$v_{3g}^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) = \eta^{\mu_1 \mu_2}(p_1 - p_2)^{\mu_3} + \eta^{\mu_2 \mu_3}(p_2 - p_3)^{\mu_1} + \eta^{\mu_3 \mu_1}(p_3 - p_2)^{\mu_2}, \quad (1.116)$$

$$v_{4g}^{\mu_1 \mu_2 \mu_3 \mu_4} = 2\eta^{\mu_1 \mu_3}\eta^{\mu_2 \mu_4} - \eta^{\mu_1 \mu_2}\eta^{\mu_3 \mu_4} - \eta^{\mu_1 \mu_4}\eta^{\mu_3 \mu_2}. \quad (1.117)$$

1.6 Spinor-helicity formalism

In the previous section we saw how there was great benefit in organising amplitudes by the external colour information. In this section we see that further

benefits are obtained from organising amplitudes by another external quantum number, namely that of helicity.

A review of the spinor-helicity formalism and BCFW recursion relations contributed towards my masters degree, and in accordance with the declaration on page *iii*, I will simply list here the expressions needed throughout this thesis.

First let us summarise our conventions for Dirac spinors. We work in a chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{ab} \\ \bar{\sigma}^\mu_{\dot{a}b} & 0 \end{pmatrix}, \quad (1.118)$$

with $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ where σ^i are the usual Pauli-matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.119)$$

We can expand solutions of the Dirac equation in terms of plane waves of positive and negative frequency,

$$u_s(p)e^{-ip \cdot x} + v_s(p)e^{ip \cdot x},$$

where

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta_s \\ -\sqrt{p \cdot \bar{\sigma}} \eta_s \end{pmatrix}, \quad (1.120)$$

and where ξ_s and η_s are two component spinors. Let us work in a basis where $\xi_1 = \eta_1 = (1, 0)^T$ and $\xi_2 = \eta_2 = (0, 1)^T$. The helicity of a state is given by its eigenvalue of the helicity operator

$$\hat{h} = \frac{1}{2} \hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad (1.121)$$

which is the projection of angular momentum along the direction of motion. We say a field with $h = 1/2$ is ‘right-handed’ and a field with $h = -1/2$ is ‘left-handed’.

In a chiral basis, Lorentz transformations are block-diagonal. This indicates that the Dirac spinor representation of the Lorentz group is reducible. It decomposes into two irreducible representations, which we label by the chirality. We can project these using the Lorentz invariant projection operators

$$\mathcal{P}^\oplus = \frac{1}{2} (1 + \gamma^5), \quad \mathcal{P}^\ominus = \frac{1}{2} (1 - \gamma^5), \quad (1.122)$$

$$u^{\oplus} = \mathcal{P}^{\oplus} u, \quad u^{\ominus} = \mathcal{P}^{\ominus} u, \quad (1.123)$$

We have encircled the $+$ and $-$ chirality labels to distinguish them from positive and negative lightcone coordinates, introduced in section 1.6.2.

At the LHC, we can typically neglect the masses of the quarks. In dropping the mass term from the Lagrangian eq. (1.33), the chiral components of Dirac spinors decouple. Let us therefore work directly with two-component Weyl spinors, rather than four-component Dirac spinors:

$$u^{\oplus}(p) = v^{\ominus}(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \quad u^{\ominus}(p) = v^{\oplus}(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}. \quad (1.124)$$

Furthermore, in the massless limit, the helicity of a particle is a Lorentz-invariant property. Our choice of basis of spinors discussed above lets us identify chirality $\oplus$ ($\ominus$) with helicity $+1/2$ ($-1/2$).

Left- and right- facing brackets are related by the Levi-Civita tensor

$$|p|^a = \varepsilon^{ab} |p]_b, \quad |p\rangle^{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} \langle p|^{\dot{b}}. \quad (1.125)$$

The Levi-Civita tensor is antisymmetric

$$\varepsilon^{\dot{a}\dot{b}} = \varepsilon^{ab} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{ab}, \quad (1.126)$$

and obeys the Schouten identity

$$\varepsilon^{ab} \varepsilon^{cd} + \varepsilon^{ac} \varepsilon^{db} + \varepsilon^{ad} \varepsilon^{bc} = 0. \quad (1.127)$$

We use the following compact notation for Lorentz invariant inner products of spinors

$$[p_i|^a |p_j]_a = [i j], \quad \langle p_i |_{\dot{a}} |p_j\rangle^{\dot{a}} = \langle i j \rangle. \quad (1.128)$$

The spinor products inherit antisymmetry from the Levi-Civita tensor,

$$\langle i j \rangle = -\langle j i \rangle, \quad [i j] = -[j i]. \quad (1.129)$$

Square and angle products are related by complex conjugation,

$$[i j] = \text{sign}(p_i^0 p_j^0) \langle i j \rangle^*. \quad (1.130)$$

Similarly, spinor currents can be formed by contracting with the spin indices of σ^μ or $\bar{\sigma}^\mu$:

$$[p_i|^a \sigma_{ab}^\mu |p_j\rangle^{\dot{b}} = [i|\mu|j\rangle, \quad \langle p_i|_{\dot{a}} \bar{\sigma}^{\mu \dot{a}b} |p_j]_b = \langle i|\mu|j]. \quad (1.131)$$

Spinor currents are related by charge conjugation,

$$[p|\gamma^\mu|k\rangle = \langle k|\gamma^\mu|p], \quad (1.132)$$

and complex conjugation,

$$[p|\gamma^\mu|k\rangle^* = \text{sign}(k^0 p^0) [k|\gamma^\mu|p]. \quad (1.133)$$

Any massless 4-momentum can be written in bi-spinor form

$$p_{ab} = p_\mu (\sigma^\mu)_{ab}, \quad p^{\dot{a}b} = p_\mu (\bar{\sigma}^\mu)^{\dot{a}b}. \quad (1.134)$$

As the determinant of the bi-spinor is zero, we can write it as the outer product of two 2-vectors, i.e.

$$p_{ab} = |p]_a \langle p|_b, \quad p^{\dot{a}b} = |p\rangle^{\dot{a}} [p|^b. \quad (1.135)$$

Correspondingly, we use the notation

$$p_k^\mu [p_i|^a \sigma_{\mu ab} |p_j\rangle^{\dot{b}} = [i|k|j\rangle = [i|k\rangle \langle k|j], \quad p_k^\mu \langle p_i|_{\dot{a}} \bar{\sigma}_\mu^{\dot{a}b} |p_j]_b = \langle i|k|j] = \langle i|k\rangle [k|j]. \quad (1.136)$$

The Lorentz product of two spinor currents is given by the Fierz identity,

$$\langle i|\mu|j\rangle [k|\mu|l] = 2\langle i|l\rangle [k|j]. \quad (1.137)$$

The Gordon identity for massless particles is

$$[i|\mu|i\rangle = \langle i|\mu|i] = 2p_i^\mu. \quad (1.138)$$

We can hence express Mandelstam invariants in terms of spinors:

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \frac{1}{2} \langle i|\mu|i\rangle [j|\mu|j] = \langle i|j\rangle [j|i]. \quad (1.139)$$

We now wish to treat the polarisation vectors of a gluon with momentum p_g

with the spinor formalism. To start with, let us choose this vector to be an arbitrary light-like vector, $\varepsilon^\mu(p_g)$ which we can write as a bi-spinor, $\varepsilon_{\alpha\dot{\alpha}} = |r\rangle_\alpha [r]_{\dot{\alpha}}$. Currently ε has four degrees of freedom. Let us impose that the polarisation of the gluon is transverse, i.e.

$$\varepsilon(p_g) \cdot p_g = \frac{1}{2} \langle gr \rangle [gr] = 0. \quad (1.140)$$

This has two solutions, corresponding to the two polarisation states of the gluon. We could take these solutions to be $\varepsilon_{\alpha\dot{\alpha}} = |g\rangle_\alpha [r]_{\dot{\alpha}}$ and $\varepsilon_{\alpha\dot{\alpha}} = |r\rangle_\alpha [g]_{\dot{\alpha}}$. However, gauge invariance of the amplitude requires that the amplitude is not changed by a shift $\varepsilon(p_g)^\mu \rightarrow \varepsilon(p_g)^\mu + \alpha p_g^\mu$. We must therefore take our polarisation vectors to be proportional to

$$\varepsilon_{\alpha\dot{\alpha}}(p_g^\ominus, p_r) = \sqrt{2} \frac{|g\rangle_\alpha [r]_{\dot{\alpha}}}{[r \ g]}, \quad \varepsilon_{\alpha\dot{\alpha}}(p_g^\oplus, p_r) = \sqrt{2} \frac{|r\rangle_\alpha [g]_{\dot{\alpha}}}{\langle g \ r \rangle}. \quad (1.141)$$

As the spinors are two-dimensional, we can write an arbitrary shift of $[r]$ as $[r] \rightarrow [r'] = a[r] + b[g]$, or likewise $|r\rangle \rightarrow |r'\rangle = a|r\rangle + b|g\rangle$. Under these arbitrary shifts we see that $\varepsilon_{\alpha\dot{\alpha}}$ or $\varepsilon_{\alpha\dot{\alpha}}$ change by at most a term proportional to $|g\rangle[g]$. The normalisation in eq. (1.141) is chosen to satisfy $\varepsilon(p_g^\lambda, p_r)^* \cdot \varepsilon(p_g^\lambda, p_r) = -1$. For our purposes, it will be more convenient to work with the polarisation vectors in 4-vector form, i.e.

$$\varepsilon^\mu(p_g^\ominus, p_r) = \frac{1}{\sqrt{2}} \frac{\langle g|\mu|r\rangle}{[g \ r]}, \quad \varepsilon^\mu(p_g^\oplus, p_r) = -\frac{1}{\sqrt{2}} \frac{[g|\mu|r\rangle}{\langle g \ r \rangle}. \quad (1.142)$$

We note that the bi-spinor representation of momenta, eq. (1.135) is invariant under a change of phase

$$|p\rangle \rightarrow e^{i\phi} |p\rangle, \quad [p] \rightarrow e^{-i\phi} [p], \quad (1.143)$$

known as little-group scaling. Each propagator and vertex in eq. (1.115g) is invariant under this scaling, so the scaling of an amplitude is determined by the scaling of the external states. For gluons this is

$$\epsilon(p_g^\ominus, p_r) \rightarrow e^{2i\phi}, \quad \epsilon(p_g^\oplus, p_r) \rightarrow e^{-2i\phi}. \quad (1.144)$$

Little-group scaling places powerful constraints on the form of amplitudes.

1.6.1 MHV classification

The spinor-helicity formalism introduced in the previous section makes transparent many results which are obscured by the traditional approach outlined in section 1.3. For example, it is straightforward to show that the ‘all- $\oplus$ ’ and ‘all-but-one- $\oplus$ ’ amplitudes vanish, i.e., [49]

$$M_{ng}(p_1^\oplus, p_2^\oplus, \dots, p_n^\oplus) = 0, \quad (1.145a)$$

$$M_{ng}(p_1^\ominus, p_2^\oplus, \dots, p_n^\oplus) = 0. \quad (1.145b)$$

This motivates the MHV classification of amplitudes, where the simplest nonvanishing amplitudes, which have exactly two negative helicity gluons, are termed maximally-helicity-violating (MHV) amplitudes. The next simplest amplitudes, which have exactly three negative helicity gluons are termed next-to-maximally-helicity-violating (NMHV) amplitudes, etc.

The spinor formalism allows amplitudes to be written in a compact form. For example the celebrated Parke-Taylor formula for MHV tree-level gluon amplitudes reads [50]

$$M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, \dots, p_{n-1}^{\lambda_{n-1}}, p_n^{\lambda_n}) = 2^{\frac{n-2}{2}} \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \dots \langle n-1, n \rangle \langle n 1 \rangle}. \quad (1.146)$$

Similarly compact forms can be found for amplitudes involving quark-antiquark pairs [51]. For a single quark-antiquark pair and n gluons the MHV amplitude is given by

$$M(p_q^\oplus, p_{g_1}^\oplus, \dots, p_{g_i}^\ominus, \dots, p_{g_n}^\oplus, p_{\bar{q}}^\ominus) = -2^{\frac{n}{2}} \frac{\langle q g_i \rangle \langle \bar{q} g_i \rangle^3}{\langle q g_1 \rangle \langle g_1 g_2 \rangle \dots \langle g_n \bar{q} \rangle \langle \bar{q} q \rangle}, \quad (1.147a)$$

$$M(p_q^\ominus, p_{g_1}^\oplus, \dots, p_{g_i}^\ominus, \dots, p_{g_n}^\oplus, p_{\bar{q}}^\oplus) = 2^{\frac{n}{2}} \frac{\langle q g_i \rangle^3 \langle \bar{q} g_i \rangle}{\langle q g_1 \rangle \langle g_1 g_2 \rangle \dots \langle g_n \bar{q} \rangle \langle \bar{q} q \rangle}. \quad (1.147b)$$

For two quark-antiquark pairs and n gluons the MHV amplitude is given by [51]

$$\begin{aligned} & M(p_{q_1}^{\lambda_{q_1}}, p_{g_1}^\oplus, \dots, p_{g_m}^\oplus, p_{\bar{q}_2}^{-\lambda_{q_2}}, p_{q_2}^{\lambda_{q_2}}, p_{g_{m+1}}^\oplus, \dots, p_{g_n}^\oplus, p_{\bar{q}_1}^{-\lambda_{q_1}}) \\ &= 2^{\frac{n+2}{2}} A_0(p_{q_1}^{\lambda_{q_1}}, p_{\bar{q}_2}^{-\lambda_{q_2}}, p_{q_2}^{\lambda_{q_2}}, p_{\bar{q}_1}^{-\lambda_{q_1}}) \times \frac{\langle q_1 \bar{q}_2 \rangle}{\langle q_1 g_1 \rangle \langle g_1 g_2 \rangle \dots \langle g_m \bar{q}_2 \rangle} \times \frac{\langle q_2 \bar{q}_1 \rangle}{\langle q_2 g_{m+1} \rangle \dots \langle g_n \bar{q}_1 \rangle}, \end{aligned} \quad (1.148)$$

where all the dependence on the helicities of the quarks is given by

$$\begin{aligned} A_0(p_{q_1}^{\oplus}, p_{\bar{q}_2}^{\ominus}, p_{q_2}^{\oplus}, p_{\bar{q}_1}^{\ominus}) &= \frac{\langle \bar{q}_1 \bar{q}_2 \rangle^2}{\langle q_1 \bar{q}_1 \rangle \langle q_2 \bar{q}_2 \rangle}, & A_0(p_{q_1}^{\oplus}, p_{\bar{q}_2}^{\oplus}, p_{q_2}^{\ominus}, p_{\bar{q}_1}^{\ominus}) &= -\frac{\langle \bar{q}_1 q_2 \rangle^2}{\langle q_1 \bar{q}_1 \rangle \langle q_2 \bar{q}_2 \rangle}, \\ A_0(p_{q_1}^{\ominus}, p_{\bar{q}_2}^{\ominus}, p_{q_2}^{\oplus}, p_{\bar{q}_1}^{\oplus}) &= -\frac{\langle q_1 \bar{q}_2 \rangle^2}{\langle q_1 \bar{q}_1 \rangle \langle q_2 \bar{q}_2 \rangle}, & A_0(p_{q_1}^{\ominus}, p_{\bar{q}_2}^{\oplus}, p_{q_2}^{\ominus}, p_{\bar{q}_1}^{\oplus}) &= \frac{\langle q_1 q_2 \rangle^2}{\langle q_1 \bar{q}_1 \rangle \langle q_2 \bar{q}_2 \rangle}. \end{aligned} \quad (1.149)$$

NMHV amplitudes are more complicated, even at tree-level. However, they can be obtained from MHV amplitudes through recursive methods such as the BCFW recursion relations [52, 53]. For our purposes, we will just need the six-gluon NMHV amplitude, which is listed in appendix A.2.

1.6.2 Explicit representations of spinors in light-cone coordinates

In this section we discuss the notation and conventions we will use for multi-particle scattering, and we provide the explicit representation for spinors which we will use in chapter 3. Throughout this thesis we will utilise light-cone coordinates. In terms of the standard Cartesian coordinates

$$p^\mu = (p^0, p^x, p^y, p^z), \quad (1.150)$$

where $p^0 > 0$ for physical momenta, we define the light-cone coordinates

$$p^\pm = p^0 \pm p^z. \quad (1.151)$$

The rapidity, y , of a particle is then simply expressed in terms of the ratio of light-cone components,

$$y = \frac{1}{2} \log \left(\frac{p^+}{p^-} \right). \quad (1.152)$$

We concurrently use a complex representation for transverse momenta

$$p_\perp = p_x + ip_y = |p_\perp| e^{i\phi}, \quad (1.153)$$

where ϕ is the azimuthal angle of the momentum. In terms of light-cone components, the scalar product between two momenta p_i and p_j is

$$2p_i \cdot p_j = p_i^+ p_j^- + p_i^- p_j^+ - p_{i\perp} p_{j\perp}^* - p_{i\perp}^* p_{j\perp}. \quad (1.154)$$

For massless partons, the on-shell condition is

$$|p_\perp|^2 = p^+ p^- . \quad (1.155)$$

We will sometimes adopt a boldface notation for 2-dimensional transverse momenta, $\mathbf{p}_\perp = (p_x, p_y)$. Then we will use a Euclidean metric to form inner products, such that $\mathbf{p}_\perp^2 = |p_\perp|^2$.

We will frequently be interested in the scattering of $2 \rightarrow n - 2$ massless partons. Consider the scattering of partons with momenta $p_1 p_2 \rightarrow p_3 \cdots p_n$, in the partonic centre-of-mass frame. We take p_1 and p_2 to be along the $-$ and $+$ light-cone directions respectively. The momenta in this frame are

$$\begin{aligned} p_1 &= \left(\frac{p_1^-}{2}, 0, 0, \frac{-p_1^-}{2} \right) = (0, p_1^-; 0) , \\ p_2 &= \left(\frac{p_2^+}{2}, 0, 0, \frac{p_2^+}{2} \right) = (p_2^+, 0; 0) , \\ p_i &= \left(\frac{p_i^+ + p_i^-}{2}, \text{Re}[p_{i\perp}], \text{Im}[p_{i\perp}], \frac{p_i^+ - p_i^-}{2} \right) = (|p_{i\perp}|e^{y_i}, |p_{i\perp}|e^{-y_i}; |p_{i\perp}|e^{i\phi}) , \end{aligned} \quad (1.156)$$

with $i \in \{1, \dots, n - 1\}$. In the first equality in section 1.6.2 we list the standard Cartesian representation of the 4-momenta, eq. (1.150), and in the second equality we list the light-cone coordinates in the form $(p^+, p^-; p_\perp)$.

It will often simplify calculations if we adopt the convention of taking all momenta to be outgoing. We therefore consider the scattering in the unphysical region where all momenta are taken as outgoing, and then we analytically continue to the physical region where $p_1^0 < 0$ and $p_2^0 < 0$ (i.e. $p_1^- < 0$ and $p_2^+ < 0$). Thus partons are ‘incoming’ or ‘outgoing’ depending on the sign of their energy. Since the helicity of a positive-energy (negative-energy) massless spinor has the same (opposite) sign as its chirality, the helicities assigned to the partons depend on whether they are incoming or outgoing. We label outgoing (positive-energy) particles with their helicity; if they are incoming the actual helicity and charge are reversed, e.g. an incoming left-handed parton is labelled as an outgoing right-handed anti-parton.

The all-outgoing convention leads to the trivial statement of momentum conser-

vation, $\sum_i p_i = 0$. In terms of light-cone coordinates, this condition reads

$$\begin{aligned} 0 &= \sum_{i=3}^n p_{i\perp}, \\ p_2^+ &= - \sum_{i=3}^n p_i^+, \\ p_1^- &= - \sum_{i=3}^n p_i^-, \end{aligned} \tag{1.157}$$

and using the scalar product eq. (1.154), any two-parton Mandelstam invariant may be written as one of,

$$\begin{aligned} s &= 2p_1 \cdot p_2 = \sum_{i,j=3}^n p_i^+ p_j^-, \\ s_{2i} &= 2p_2 \cdot p_i = - \sum_{j=3}^n p_i^- p_j^+, \\ s_{1i} &= 2p_1 \cdot p_i = - \sum_{j=3}^n p_i^+ p_j^-, \\ s_{ik} &= 2p_i \cdot p_k = p_i^+ p_k^- + p_i^- p_k^+ - p_{i\perp} p_{k\perp}^* - p_{i\perp}^* p_{k\perp}, \end{aligned} \tag{1.158}$$

with $3 \leq i, k \leq n$.

It will sometimes be convenient to use an explicit representation for spinors. For partons along the beam axis (i.e. with $p_\perp = 0$), we use the representation [46]

$$u^\oplus(p_1) = \begin{pmatrix} 0 \\ -i\sqrt{p_1^-} \\ 0 \\ 0 \end{pmatrix}, \quad u^\ominus(p_1) = \begin{pmatrix} 0 \\ 0 \\ -i\sqrt{p_1^-} \\ 0 \end{pmatrix}, \tag{1.159}$$

$$u^\oplus(p_2) = \begin{pmatrix} \sqrt{p_2^+} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^\ominus(p_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{p_2^+} \end{pmatrix}, \tag{1.160}$$

while for partons $3 \leq i \leq n$ we use the representation

$$u^\oplus(p_i) = \begin{pmatrix} \sqrt{p_i^+} \\ \sqrt{p_i^-} e^{i\phi_i} \\ 0 \\ 0 \end{pmatrix}, \quad u^\ominus(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_i^-} e^{-i\phi_i} \\ -\sqrt{p_i^+} \end{pmatrix}. \quad (1.161)$$

Using this representation, the spinor products of eq. (1.129) are given by

$$\begin{aligned} \langle i j \rangle &= p_{i\perp} \sqrt{\frac{p_j^+}{p_i^+}} - p_{j\perp} \sqrt{\frac{p_i^+}{p_j^+}}, \\ \langle 2 i \rangle &= -i \sqrt{\frac{-p_2^+}{p_i^+}} p_{i\perp}, \\ \langle i 1 \rangle &= i \sqrt{-p_1^- p_i^+}, \\ \langle 2 1 \rangle &= -\sqrt{p_2^+ p_1^-}. \end{aligned} \quad (1.162)$$

1.7 Supersymmetric decomposition of one-loop QCD amplitudes

At tree level we have seen that remarkable simplifications occur when we organise amplitudes by the quantum numbers of their external states, namely colour in section 1.5.3 and helicity in section 1.6. In the one-loop colour decomposition of pure-gluon amplitudes, eq. (1.114), we have seen that it is of practical benefit to consider also the management of ‘internal quantum numbers’. In this section we take this idea one step further, and consider writing the gluon-loop and quark-loop components of these amplitudes as a linear combination of supersymmetric amplitudes.[54–56]

A $\mathcal{N} = 4$ SYM multiplet contains a gluon g , four gluinos f , and three complex (i.e. six real) scalar fields s . A $\mathcal{N} = 1$ chiral (which we write as $\mathcal{N} = 1\chi$) doublet consists of a fermion and a complex scalar. We can relate the contribution of a single gluon circulating in the loop with these supersymmetric multiplets

$$g = (g + 4f + 3s) - 4(f + s) + s = M^{\mathcal{N}=4} - 4M^{\mathcal{N}=1\chi} + M^{\text{scalar}} \quad (1.163a)$$

$$f = (f + s) - s = M^{\mathcal{N}=1\chi} - M^{\text{scalar}} \quad (1.163b)$$

In the notation of eq. (1.114), we can therefore write the gluon-loop and quark-loop contributions to QCD amplitudes as the following linear combinations of one-loop gluon amplitudes

$$M_{\text{QCD}}^{(1)\textcircled{8}} = M_{\mathcal{N}=4}^{(1)} - 4M_{\mathcal{N}=1\chi}^{(1)} + M_{\text{scalar}}^{(1)}, \quad (1.164a)$$

$$M_{\text{QCD}}^{(1)\textcircled{3}} = M_{\mathcal{N}=1\chi}^{(1)} - M_{\text{scalar}}^{(1)}. \quad (1.164b)$$

The supersymmetric amplitudes are much simpler than the QCD amplitudes. This will be seen explicitly in section 2.2. These amplitudes are cut-constructible, that is the full amplitude can be calculated solely from the knowledge of their cuts[56, 57].

This is not the case for the M^{scalar} piece, which has rational terms that cannot be directly determined from unitarity cuts. This piece is therefore the most difficult ingredient to compute, yet it is nevertheless simpler than the full QCD amplitude as the scalar does not circulate any spin information around the loop.

To bring this supersymmetric decomposition to the level of colour-dressed QCD amplitudes we define the colour-dressed contribution from the $\mathcal{N} = 4$ multiplet,

$$\mathcal{M}_{\mathcal{N}=4}^{(1)} = g_S^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \text{tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_n}}) M_{\mathcal{N}=4}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_n}^{\nu_{\sigma_n}}). \quad (1.165)$$

For the colour-dressed contributions from the $\mathcal{N} = 1$ multiplet and the scalar circulating in the loop, we separately consider the cases where these are in the adjoint or fundamental representations,

$$\mathcal{M}_{\mathcal{N}=1\chi}^{(1)\textcircled{8}} = g_S^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \text{tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_n}}) M_{\mathcal{N}=1\chi}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_n}^{\nu_{\sigma_n}}), \quad (1.166a)$$

$$\mathcal{M}_{\mathcal{N}=1\chi}^{(1)\textcircled{3}} = g_S^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \text{tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) M_{\mathcal{N}=1\chi}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_n}^{\nu_{\sigma_n}}), \quad (1.166b)$$

$$\mathcal{M}_{\text{scalar}}^{(1)\textcircled{8}} = g_S^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \text{tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_n}}) M_{\text{scalar}}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_n}^{\nu_{\sigma_n}}), \quad (1.166c)$$

$$\mathcal{M}_{\text{scalar}}^{(1)\textcircled{3}} = g_S^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \text{tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) M_{\text{scalar}}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_n}^{\nu_{\sigma_n}}), \quad (1.166d)$$

such that we can write the QCD one-loop colour-dressed amplitude as

$$\mathcal{M}_{\text{QCD}}^{(1)} = \mathcal{M}_{\text{QCD}}^{(1)\textcircled{8}} + N_f \mathcal{M}_{\text{QCD}}^{(1)\textcircled{3}} \quad (1.167)$$

$$= \left(\mathcal{M}_{\mathcal{N}=4}^{(1)} - 4\mathcal{M}_{\mathcal{N}=1\chi}^{(1)\textcircled{8}} + \mathcal{M}_{\text{scalar}}^{(1)\textcircled{8}} \right) + N_f \left(\mathcal{M}_{\mathcal{N}=1\chi}^{(1)\textcircled{3}} - \mathcal{M}_{\text{scalar}}^{(1)\textcircled{3}} \right). \quad (1.168)$$

This decomposition will be a useful organisational principle in section 2.2 and an invaluable one in chapter 3.

Chapter 2

QCD at high energy: The asymptotic limit

In section 1.1 we sketched the ideas of Regge theory. We found that under the assumption that the partial wave amplitude $m_l(t)$ had only simple poles in the complex l -plane, the $2 \rightarrow 2$ amplitude scaled as eq. (1.25) in the Regge limit $s \gg |t|$. These results were theory-independent, and instead relied upon the assumptions of the S -matrix formalism outlined in section 1.1.

In contrast to the S -matrix approach, quantum field theory gives us a concrete method to obtain scattering amplitudes given a Lagrangian which specifies a set of fields and their interactions. Gell-Mann and Goldberger first asked the question of whether the high-energy limit of amplitudes obtained in quantum field theory reproduced the results of Regge theory [58]. This was followed by several general investigations into the compatibility of field theory and Regge theory [59–63]. In quantum electrodynamics (QED) it was found that to leading logarithmic (LL) accuracy in $\log(s/|t|)$, the electron is indeed described by a Regge pole [64–69], but that the photon-exchange amplitudes are instead described by a cut singularity, eq. (1.29) [70, 71].

With the discovery of asymptotic freedom, eq. (1.68), QCD became a viable candidate for a theory of the strong interaction. Subsequently, the high energy limit of QCD amplitudes were investigated. Surprisingly, it was found that, to LL accuracy, QCD amplitudes are described by a simple Regge pole associated with a t -channel exchange of a gluon [72–76]. This key feature of QCD amplitudes, known as the Reggeisation of the gluon, will be explored further in the following

chapter. The Reggeisation of the gluon led to the derivation of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation, which resums the radiative corrections to parton-parton scattering to all orders at LL accuracy [73, 75, 77, 78].

At NLL, QCD amplitudes have both a pole and cut contribution, but remarkably, these contributions can be disentangled by the concept of signature, which as we saw in section 1.1 was introduced for the entirely different reason of ensuring convergence of partial amplitudes in the large l limit. This allowed the BFKL equation to be extended to NLL accuracy [79–82].

Whether the BFKL approach can be extended to next-to-next-to leading logarithmic (NNLL) accuracy is an open question. The study of QCD amplitudes at this logarithmic accuracy has been a subject of recent interest [83–86], and in chapter 3 we obtain a factorised expression that is necessary for the extension of the BFKL framework to this accuracy.

A comment on the title of this chapter is perhaps in order. In this chapter we take concrete mathematical limits of scattering amplitudes, and our interest is only in the behaviour of QCD amplitudes in these asymptotic limits. This is in contrast to the approach of chapter 4 where we find that the approximations made in this chapter are too severe to give a useful description of scattering amplitudes within the phase-space cuts of the LHC. Nevertheless, the asymptotic approach presented in this chapter has, and continues to, provide invaluable insights into the structure of amplitudes, and knowledge of amplitudes in this limit has been used to constrain amplitudes in general kinematics [87–92].

The purpose of this chapter is two-fold. Firstly, we introduce the necessary background for the investigation detailed in chapter 3. Secondly, the introduction to the BFKL approach provided in this chapter will allow us to compare and contrast each piece of the analogous High Energy Jets approach to high energy scattering reviewed in chapter 4.

We have found the following pedagogical references particularly useful, and have used them as a guide in many places throughout the following chapter: refs. [19, 93, 94].

2.1 LL contributions to partonic cross sections

2.1.1 The Regge limit of the LO dijet cross section

In this section we wish to investigate the high-energy limit of the LO partonic dijet cross section. To do so, we will first study LO $2 \rightarrow 2$ QCD amplitudes in the Regge limit. We will take the process $gg \rightarrow gg$ as a prototypical example. The necessary tree-level gluon amplitudes are given by the compact Parke-Taylor formula, eq. (1.146), and due to Bose symmetry, the amplitude is symmetric under the interchange of any two gluons.

We consider the amplitude for the production of two gluons with momenta p_3 and p_4 in the scattering of two gluons with momenta p_1 and p_2 . We adopt the all-outgoing convention discussed in section 1.6.2, i.e. momentum conservation is given by $\sum_{i=1}^4 p_i = 0$, and correspondingly, the momenta associated with the scattering partons have negative energy components $p_1^0, p_2^0 < 0$. The four-gluon amplitude depends on all quantum numbers of the initial and final state partons,

$$\mathcal{M}_{gg \rightarrow gg}^{(0)}(\{p_1, \lambda_1, a_1\}, \{p_2, \lambda_2, a_2\}, \{p_3, \lambda_3, a_3\}, \{p_4, \lambda_4, a_4\})$$

but we will leave the colour dependence implicit and write the amplitude more compactly as

$$\mathcal{M}_{gg \rightarrow gg}^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}).$$

Using the DDM basis for tree-level gluon amplitudes, eq. (1.112), this LO amplitude is given by the sum over two colour orderings

$$\mathcal{M}_{gg \rightarrow gg}^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) = (ig_s)^2 \sum_{\sigma \in S_2} f^{a_2 a_t a_{\sigma_3}} f^{a_{\sigma_4} a_t a_1} M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}}). \quad (2.1)$$

We wish to find the Regge limit of eq. (2.1), i.e. the limit $s \gg |t|$ for fixed $|t|$, where $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$. To find the limit of the colour dressed amplitude, eq. (2.1) we first take the limit of the colour-ordered amplitudes.

Let us take the physical helicity configuration $g_1^\oplus g_2^\oplus \rightarrow g_3^\oplus g_4^\oplus$ as an example. In our all-outgoing convention this corresponds to the MHV helicity configuration

$\emptyset \rightarrow g_1^\ominus g_2^\ominus g_3^\oplus g_4^\oplus$. The Parke-Taylor formula, eq. (1.146), for this amplitude is

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) = 2 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \quad (2.2)$$

In the light-cone notation of section 1.6.2, the Regge limit can be restated as the limit

$$p_3^+ \gg p_4^+, \quad p_4^- \gg p_3^-, \quad (2.3)$$

with $|p_{3\perp}| \sim |p_{4\perp}|$ finite.

The leading behaviour of the spinor products eq. (1.162) in this limit is

$$\begin{aligned} \langle 12 \rangle &\rightarrow \sqrt{p_3^+ p_4^-} \\ \langle 23 \rangle &\rightarrow -ip_{3\perp} \\ \langle 34 \rangle &\rightarrow -\frac{p_{4\perp}}{|p_{4\perp}|} \sqrt{p_3^+ p_4^-} \\ \langle 41 \rangle &\rightarrow i|p_{4\perp}|. \end{aligned} \quad (2.4)$$

From this list, it is straightforward to find the leading power behaviour of the colour-ordered amplitude in p_4^+/p_3^+ , which we write suggestively as

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) \xrightarrow{\text{Regge}} 1 \times \frac{s}{-|q_\perp|^2} \times \frac{p_{4\perp}^*}{|p_{4\perp}|}. \quad (2.5)$$

In this chapter we use q_i to denote t -channel momenta. Here $q = -p_2 - p_3$, such that $t = q^2$. In the Regge limit, only the transverse degrees of freedom of the t -channel invariant are relevant,

$$t = q^2 = (p_2 + p_3)^2 = p_2^+ p_3^- \xrightarrow{\text{Regge}} -p_3^+ p_3^- = -|p_{3\perp}|^2 = -|q_\perp|^2. \quad (2.6)$$

Thus, the amplitude eq. (2.5) scales like s and has only a single t -channel pole.

For the colour ordering $g_1 g_2 g_3 g_4$ we can similarly check that all colour-ordered amplitudes with $\lambda_1 = -\lambda_4$ or $\lambda_2 = -\lambda_3$ are equal to eq. (2.5), up to a phase, while all colour-ordered amplitudes with $\lambda_1 = \lambda_4$ or $\lambda_2 = \lambda_3$ are power-suppressed compared to eq. (2.5). In addition, we find that the amplitudes with colour ordering $g_1 g_2 g_4 g_3$ are also power-suppressed compared to eq. (2.5) for all helicity configurations [95]. The leading behaviour of the colour-dressed amplitudes

eq. (2.1) can be summarised by [73, 96]

$$\begin{aligned} \mathcal{M}_{gg \rightarrow gg}^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}; p_3^{\lambda_3}, p_4^{\lambda_4}) &\xrightarrow{\text{Regge}} 2s [ig_s f^{a_2 a_t a_3} C^{g \rightarrow g(0)}(p_2^{\lambda_2}; p_3^{\lambda_3})] \\ &\times \left(\frac{1}{-|q_\perp|^2} \right) \\ &\times [ig_s f^{a_4 a_t a_1} C^{g \rightarrow g(0)}(p_1^{\lambda_1}; p_4^{\lambda_4})] . \end{aligned} \quad (2.7)$$

which is depicted in fig. 2.1. This amplitude is written in a factorised form, consisting of so-called impact factors¹, $C^{g \rightarrow g(0)}$, times a t -channel pole. The helicity-conserving impact factors are simply equal to phases at LO [97],

$$C^{g \rightarrow g(0)}(p_2^\ominus, p_3^\oplus) = 1, \quad C^{g \rightarrow g(0)}(p_1^\ominus; p_4^\oplus) = \frac{p_{4\perp}^*}{p_{4\perp}}, \quad (2.8)$$

consistent with eq. (2.5), while the helicity-flip impact factors are power suppressed [98–100]. These factors transform under parity into their complex conjugates,

$$[C^{g \rightarrow g(0)}(p_i^{\lambda_i}; p_j^{\lambda_j})]^* = C^{g \rightarrow g(0)}(p_i^{-\lambda_i}; p_j^{-\lambda_j}). \quad (2.9)$$

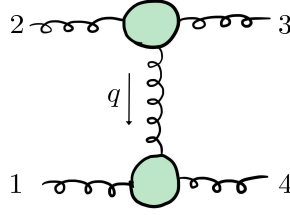


Figure 2.1 *Depiction of the factorised form of the LO $gg \rightarrow gg$ amplitude in the Regge limit, given in eq. (2.7). The green blobs represent the tree level impact factors, $C^{g \rightarrow g(0)}$, given in eq. (2.8).*

Note that eq. (2.7) consists of only a single colour structure, and that only the antisymmetric octet representation $\mathbf{8}_a$ is exchanged in the t channel, which can be checked using the projection operator eq. (1.100). We have therefore chosen to label the adjoint index corresponding to this t -channel exchange a_t .

Performing an analogous study of the MHV amplitudes for the other partonic channels, eqs. (1.147b) and (1.148), we find that any $2 \rightarrow 2$ scattering in QCD which permits an $\mathbf{8}$ exchange in the t channel at leading order is

¹In the high-energy literature, these factors are sometimes called ‘parton-parton-Reggeon’ vertices, with the term ‘impact factor’ reserved for the interference of such vertices. We call the amplitude-level vertex an ‘impact factor’, and will call the interference of two such vertices a ‘squared impact factor’ to avoid potential confusion.

described by [73, 77]

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4} \xrightarrow{\text{Regge}} [g_s T_{R_2 R_3}^{a_t} C^{f_2 \rightarrow f_3(0)}(p_2^{\lambda_2}, p_3^{\lambda_3})] \frac{s}{-|q_\perp|^2} [g_s T_{R_4 R_3}^{a_t} C^{f_1 \rightarrow f_4(0)}(p_1^{\lambda_1}, p_4^{\lambda_4})], \quad (2.10)$$

with impact factors given by eq. (2.8) or [97]

$$\begin{aligned} C^{q \rightarrow q(0)}(p_2^\ominus, p_3^\oplus) &= 1, & C^{q \rightarrow q(0)}(p_1^\ominus, p_4^\oplus) &= \left(\frac{p_{4\perp}^*}{p_{4\perp}} \right)^{\frac{1}{2}} \\ C^{\bar{q} \rightarrow \bar{q}(0)}(p_2^\ominus, p_3^\oplus) &= -1, & C^{\bar{q} \rightarrow \bar{q}(0)}(p_1^\ominus, p_4^\oplus) &= - \left(\frac{p_{4\perp}^*}{p_{4\perp}} \right)^{\frac{1}{2}}. \end{aligned} \quad (2.11)$$

The colour factors $T_{R_i R_j}^{a_t}$ are generators in a representation determined by the representation R of partons i and j , namely

$$T_{\mathbf{8}\mathbf{8}}^{a_t} = i f^{a_i a_t a_j}, \quad T_{\mathbf{\bar{3}}\mathbf{3}}^{a_t} = T_{i_j \bar{i}_i}^{a_t}, \quad T_{\mathbf{3}\mathbf{\bar{3}}}^{a_t} = T_{i_i \bar{j}_j}^{a_t}.$$

We note that the LO QCD amplitudes, eq. (2.10), obey the Regge-pole scaling of eq. (1.25), with $\alpha(t) = 1$. We therefore interpret eq. (2.10) as describing the t -channel exchange of a particle with spin 1, i.e. a gluon, an interpretation that is consistent with the t -channel $\mathbf{8}$ colour exchange. Equation (2.10) leads to a cross section that is independent of s at high energy, a feature of QCD first noted in refs. [101, 102]. In the next section we will look at the ingredients required to obtain the high energy behaviour of this cross section at NLO.

2.1.2 The Regge limit of the NLO dijet cross section

Let us now consider the one-loop $2 \rightarrow 2$ amplitudes in QCD, beginning as before with the four-gluon amplitude. Using the one-loop DDM colour basis, eq. (1.114), this amplitude is given by a sum over three colour orderings,

$$\begin{aligned} \mathcal{M}_{gg \rightarrow gg}^{(1)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) &= (ig_s)^4 \sum_{\sigma \in S_3/\mathcal{R}} [\text{tr}(F^{a_{\sigma_1}} F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}}) M^{(1)\textcircled{\mathbf{8}}}(p_{\sigma_1}^{\lambda_{\sigma_1}}, p_{\sigma_2}^{\lambda_{\sigma_2}}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}}) \\ &\quad + 2n_f \text{tr}(T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}) M^{(1)\textcircled{\mathbf{3}}}(p_{\sigma_1}^{\lambda_{\sigma_1}}, p_{\sigma_2}^{\lambda_{\sigma_2}}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}})]. \end{aligned} \quad (2.12)$$

Utilising the supersymmetric decomposition introduced in section 1.7, we can obtain the QCD amplitudes $M^{(1)\textcircled{\mathbf{8}}}$ and $M^{(1)\textcircled{\mathbf{3}}}$ from the one-loop four-gluon

amplitudes in $\mathcal{N} = 4$ and $\mathcal{N} = 1\chi$ SYM, and where a complex scalar is circulating in the loop.

The $\mathcal{N} = 4$ amplitude is simply given by the tree-level amplitude multiplied by a helicity-independent transcendental function,

$$M_{\mathcal{N}=4}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_4}^{\nu_{\sigma_4}}) = \frac{\kappa_\Gamma}{(4\pi)^2} M^{(0)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_4}^{\nu_{\sigma_4}}) V_{g4}(\sigma_1, \dots, \sigma_4), \quad (2.13)$$

with

$$V_{g4}(1, 2, 3, 4) = \frac{-2}{\epsilon^2} \left[\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \right] + \log^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2, \quad (2.14)$$

and where we use

$$\kappa_\Gamma \equiv (4\pi)^\epsilon \frac{\Gamma(1+\epsilon) \Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \quad (2.15)$$

to collect prefactors which are ubiquitous in one-loop amplitudes in the dimensional regularisation scheme. This is $16\pi^2 c_\Gamma$ where c_Γ is the more common factor in the literature. Our normalisation will be convenient when we later expand quantities in $\alpha_s/(4\pi)$.

The remaining $\mathcal{N} = 1\chi$ and scalar one-loop amplitudes are simplest for the ‘split helicity’ configuration, e.g. $M(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus)$, where they are also given by the tree-level amplitude times a transcendental function of the momenta,

$$M_{\mathcal{N}=1\chi}^{(1)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) = \frac{\kappa_\Gamma}{(4\pi)^2} M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) V_{f4}(1, 2, 3, 4), \quad (2.16)$$

$$M_{\text{scalar}}^{(1)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) = \frac{\kappa_\Gamma}{(4\pi)^2} M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\oplus) V_{s4}(1, 2, 3, 4). \quad (2.17)$$

with

$$V_{f4}(1, 2, 3, 4) = \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon + 2, \quad (2.18)$$

$$V_{s4}(1, 2, 3, 4) = \frac{1}{3} V_{f4}(1, 2, 3, 4) + \frac{2}{9}. \quad (2.19)$$

Before using eqs. (2.14), (2.18) and (2.19), which are defined for the Euclidean region where all invariants are negative, we must analytically continue to our physical region of $s_{12} > 0$ and $s_{23} < 0$. We give all Mandelstam invariants a small positive imaginary part, $t_i^{[m]} \rightarrow t_i^{[m]} + i0$. For logarithms of ratios of invariants,

this prescription yields

$$\log \left(\frac{-s_i}{-s_j} \right) = \log \left(\left| \frac{s_i}{s_j} \right| \right) - i\pi (\Theta(s_i) - \Theta(s_j)), \quad (2.20)$$

where the Heaviside function $\Theta(x)$ is defined as $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ otherwise.

The first thing to note is that, being multiplied by the tree-level gluon amplitude, only the helicity configurations that give rise to leading behaviour at tree-level will give rise to leading behaviour at NLO. Although a new colour ordering appears at one-loop, the tree-level amplitudes are related via a photon decoupling identity (see section 1.5.3):

$$M^{(0)}(p_1^{\lambda_1}, p_3^{\lambda_3}, p_2^{\lambda_2}, p_4^{\lambda_4}) = -M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) - M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_4^{\lambda_4}, p_3^{\lambda_3}). \quad (2.21)$$

In the high-energy limit, we know the last factor is suppressed giving us the relation

$$M^{(0)}(p_1^{\lambda_1}, p_3^{\lambda_3}, p_2^{\lambda_2}, p_4^{\lambda_4}) \underset{\text{Regge}}{=} -M^{(0)}(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}). \quad (2.22)$$

In other words, in the Regge limit, an interchange of $p_2^{\lambda_2} \leftrightarrow p_3^{\lambda_3}$ (or $p_1^{\lambda_1} \leftrightarrow p_4^{\lambda_4}$) leads only to an overall sign-change of the tree-level amplitude.

We then note that for these leading tree-level configurations, only eq. (2.14) depends on the large invariants s_{12} or s_{13} and therefore the LL behaviour of the 1-loop gluon amplitude is given purely by the $\mathcal{N} = 4$ component. The $\mathcal{N} = 4$ and scalar one-loop amplitudes for the ‘alternating’ helicity configurations (e.g. $M(p_1^\ominus, p_2^\oplus, p_3^\ominus, p_4^\oplus)$) are more complicated, where the $\mathcal{N} = 1\chi$ and scalar pieces include rational terms that are not simply the tree-level amplitude. These amplitudes are listed in e.g. ref. [100]. However, for these helicity configurations, the leading logarithmic behaviour is again given by the (helicity independent) $\mathcal{N} = 4$ component alone and we can simply state

$$\mathcal{M}_{gg \rightarrow gg}^{(1)} = \kappa_\Gamma N_c \frac{\alpha_s}{4\pi} \mathcal{M}_{gg \rightarrow gg}^{(0)} \alpha^{(1)}(|q_\perp|^2) \log \left(\frac{s}{|q_\perp|^2} \right), \quad (2.23)$$

with

$$\alpha^{(1)}(|q_\perp|^2) = \frac{2}{\epsilon} \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon, \quad (2.24)$$

which is known as the one-loop Regge trajectory of the gluon [73, 77]. This name will be justified later. Note that the one-loop ring colour-structure of eq. (2.12)

has reduced to N_c times the tree-level colour structure.

The amplitude eq. (2.23) is depicted schematically in fig. 2.2.

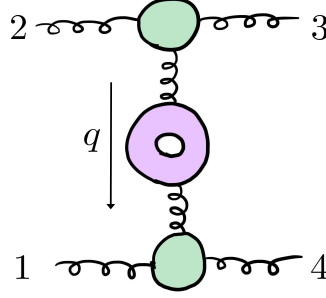


Figure 2.2 *Depiction of the factorised form of the NLO $gg \rightarrow gg$ amplitude in the Regge limit, given in eq. (2.23). The magenta blob represents the one-loop Regge trajectory, eq. (2.24).*

Remarkably, the LL behaviour of all partonic channels is given by [73, 77]

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(1)} = \frac{\alpha_s}{4\pi} \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(0)} \alpha^{(1)}(|q_\perp|^2) \log \left(\frac{s}{|q_\perp|^2} \right). \quad (2.25)$$

The IR pole of eq. (2.24) must be cancelled by the integration over phase space of the $2 \rightarrow 3$ tree-level amplitude. To see how this phase space integral can yield a factor of $\log \left(\frac{s}{-t} \right)$, let us clarify the relationship between this large logarithm and the rapidity difference between the outgoing partons. In general kinematics, in $2 \rightarrow 2$ scattering, the invariant mass of partons 3 and 4 is

$$s_{34} = p_3^+ p_4^- + p_3^- p_4^+ - p_{3\perp} p_{4\perp}^* - p_{3\perp}^* p_{4\perp}, \quad (2.26)$$

while from eq. (1.152), the rapidity difference between these partons is

$$\Delta y_{34} = y_3 - y_4 = \frac{1}{2} \log \left(\frac{p_3^+ p_4^-}{p_3^- p_4^+} \right) = \log \left(\frac{p_3^+ p_4^-}{|p_{3\perp}| |p_{4\perp}|} \right). \quad (2.27)$$

In the Regge limit we obtain $s_{34} \sim p_3^+ p_4^-$, and consequently we can identify

$$\Delta y_{34} \sim \log \left(\frac{s_{34}}{|q_\perp|^2} \right) \sim \log \left(\frac{s}{-t} \right).$$

When studying the high-energy limit of the dijet cross section, it is useful to

rewrite the n -parton contribution to eq. (1.30) as

$$\begin{aligned}
\sigma_{AB \rightarrow 2j}^n &= \sum_{f_1, f_2} \prod_{i=3}^n \left(\int_{\mathbf{p}_{i\perp}=0}^{p_{i\perp}=\infty} \frac{d^2 p_{i\perp}}{(2\pi)^3} \int \frac{dy_i}{2} \right) \frac{|\mathcal{M}^{f_1 f_2 \rightarrow f_3 f_4 \dots f_n}(\{p_j\})|^2}{\hat{s}^2} \\
&\times x_{f_1}(\{p_j\}) f_{f_1/A}(x_{f_1}(\{p_j\}), \mu_F) x_{f_2}(\{p_j\}) f_{f_2/B}(x_{f_2}(\{p_j\}), \mu_F) \\
&\times (2\pi)^4 \delta^2 \left(\sum_{k=3}^n \mathbf{p}_{k\perp} \right) \mathcal{O}_{2j}(\{p_j\}),
\end{aligned} \tag{2.28}$$

where we have performed the integrals over momentum fractions x_a and x_b . The momentum fractions that appear in eq. (2.28) are therefore functions of the final state momenta, $(\{p_j\})$. The function $\mathcal{O}_{2j}(\{p_j\})$ imposes the requirement of two jets in the final state via a jet algorithm, as discussed in section 1.4. In eq. (2.28) we have written the partonic centre-of-mass as $\hat{s}$, but we drop the hat in the remainder of this thesis unless specified otherwise, as we will typically be considering scattering at the parton level.

We now consider the $n = 5$ contribution to the Regge limit of eq. (2.28), i.e. we require the two jets are well separated in rapidity. Without loss of generality, we consider the production of three partons with momenta p_3 , p_4 and p_5 , in the limit $p_3^+ \sim -p_2^+$ and $p_5^- \sim -p_1^-$, and where $y_3 \geq y_4 \geq y_5$. From eq. (2.28), the phase space integral for the central parton is

$$\int_0^\infty \frac{d^2 \mathbf{p}_{4\perp}}{(2\pi)^3} \int_{y_5}^{y_3} \frac{dy_4}{2} \frac{|\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4 f_5}|^2}{s^2}, \tag{2.29}$$

and we see that if the amplitude obeys the Regge pole scaling of eq. (1.27), the rapidity integral of parton 4 yields a factor of $\Delta y_{35} \sim \log(s/\tau)$, where τ is a finite scale on the order of $|p_{3\perp}|^2 \sim |p_{5\perp}|^2$. In other words, the LL contribution to the cross section from the NLO real radiative correction is given by the phase space integral over the Multi-Regge Kinematic (MRK) region

$$p_3^+ \gg p_4^+ \gg p_5^+, \quad p_5^- \gg p_4^- \gg p_3^-, \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}|. \tag{2.30}$$

We now need to demonstrate, firstly, that the tree-level $2 \rightarrow 3$ QCD amplitudes in the MRK limit do indeed obey Regge pole scaling, and secondly, that the integral over the further soft limit $|p_{4\perp}|^2 \rightarrow 0$ of eq. (2.29) regulates the IR divergence of the virtual correction, eq. (2.25).

Fortunately, the only nonzero helicity configurations for $2 \rightarrow 3$ scattering at tree-level are (anti-)MHV configurations, so all the necessary colour-ordered amplitudes are given by eqs. (1.146), (1.147b) and (1.148). In the MRK limit, these amplitudes are described by

$$\begin{aligned} \mathcal{M}_{f_1 f_2 \rightarrow f_3 g f_5}^{(0)} &\xrightarrow{\text{Regge}} 2s [g_s T_{R_2 R_3}^{a_t} C^{f_2 \rightarrow f_3(0)}(p_2^{\lambda_2}, p_3^{\lambda_3})] \\ &\times \frac{1}{-|q_{1\perp}|^2} [i g_s f^{a_{t_1} a_{t_2} a_4} V^{g(0)}(q_1, p_4^{\lambda_4}, q_2)] \frac{1}{-|q_{2\perp}|^2} \\ &\times [g_s T_{R_5 R_3}^{a_t} C^{f_1 \rightarrow f_5(0)}(p_1^{\lambda_1}, p_5^{\lambda_5})], \end{aligned} \quad (2.31)$$

with $q_1 = -p_2 - p_3$ and $q_2 = -p_2 - p_3 - p_4$. This amplitude is depicted in fig. 2.3 for the case of incoming gluons. We see that the only real radiative correction that contributes leading behaviour in the MRK limit is where a gluon is emitted with a rapidity between that of the other partons. This means the amplitude consists of the exchange of **8** in both the t_1 - and t_2 - channels.

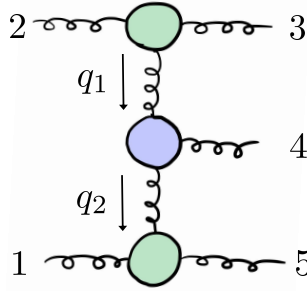


Figure 2.3 *Depiction of the factorised form of the LO $gg \rightarrow ggg$ amplitude in the MRK limit, given in eq. (2.31). The blue blob represents the tree-level Lipatov vertex, eq. (2.32).*

The emission of the central on-shell gluon is described by the so called Lipatov vertex [73], or central-emission vertex (CEV) ²

$$V^{g(0)}(q_1, p_4^\oplus, q_2) = \frac{q_{1\perp}^* q_{2\perp}}{p_{4\perp}}, \quad (2.32)$$

which, like the impact factors, transforms under parity into its complex conjugate,

$$[V^{g(0)}(q_1, p^\nu, q_2)]^* = V^{g(0)}(q_1, p^{-\nu}, q_2). \quad (2.33)$$

In the MRK limit we have $s \sim s_{35} \sim s_{34}s_{45}/|p_{4\perp}|^2$, so eq. (2.31) is consistent with

²Although in eq. (2.32) we list three 4-momentum arguments for $V^{g(0)}$, we note that this vertex conserves transverse momentum so only depends on 4 real degrees of freedom.

the Regge pole scaling of eq. (1.27), with a spin-1 particle exchanged in both t -channels. Crucially, the integrand of eq. (2.29), $|\mathcal{M}|^2/s^2$, is therefore independent of s as desired, and the integration over y_4 yields a leading logarithmic correction, $\mathcal{O}(\alpha_s \Delta y_{35})$.

As mentioned above, the integral over transverse momentum will generate an IR divergence which must cancel that of the virtual correction to the cross section,

$$2\text{Re} \left[\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{*(0)} \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{*(1)} \right] = 2N_c \kappa_\Gamma \frac{\alpha_s}{4\pi} |\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(0)}|^2 \alpha_s (|q_\perp|^2) \Delta y_{34} \quad (2.34)$$

such that the NLO cross section is finite. We find that in the limit of $|p_{4\perp}|^2 \rightarrow 0$, eq. (2.31) factorises further into a $2 \rightarrow 2$ amplitude times (the MRK limit of) an Eikonal function, so that the squared amplitude can be written as

$$|\mathcal{M}_{f_1 f_2 \rightarrow f_3 g f_5}^{(0)}|^2 \xrightarrow{|p_{4\perp}|^2 \rightarrow 0} \sum_{\lambda_4 \in \{\oplus, \ominus\}} |V^{g(0)}(q_1, p_4^{\lambda_4}, q_1)|^2 |\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_5}|^2. \quad (2.35)$$

The relevant integral over the soft region, i.e. where the transverse momenta of the central parton is less than some cut-off, $|p_{4\perp}|^2 < \lambda^2$, in dimensional regularisation with $d = 4 - 2\epsilon$ is

$$g_s^2 \mu^{2\epsilon} \int_{y_5}^{y_3} \frac{dy_3}{4\pi} \int_0^\lambda \frac{d^{2-2\epsilon} \mathbf{p}_{4\perp}}{(2\pi)^{2-2\epsilon}} \left(\frac{4N_c}{|p_{4\perp}|^2} \right) = -4N_c \frac{\alpha_s}{4\pi} \Delta y_{35} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{1}{\epsilon} \left(\frac{\mu^2}{\lambda^2} \right)^\epsilon. \quad (2.36)$$

Expanding both real and virtual contributions in ϵ , we see that the NLO partonic cross section is IR finite as expected.

2.1.3 MRK limit of QCD amplitudes to LL accuracy

That the universality of the one-loop corrections to QCD amplitudes in the Regge limit, eq. (2.25), follows from the universality of the tree-level amplitudes, eq. (2.10), can be seen as a consequence of s -channel unitarity, eq. (1.7). In Feynman gauge, the LL terms of eq. (2.23) arise from box diagrams, as these are the only diagrams which depend on both s and t . The Cutkosky rules [103], derived from eq. (1.7), allow us to obtain the imaginary part of a one-loop diagram from a phase-space integral of the interference of two on-shell tree-level amplitudes. Therefore, the imaginary part of the Regge limit of the box integrals can be determined from eq. (2.10). It is found that only the transverse degrees

of freedom contribute to this integral,

$$\alpha^{(1)}(|q_\perp|^2) = -|q_\perp|^2 \int \frac{d^{d-2}\mathbf{k}_\perp}{(2\pi)^{d-2}} \frac{1}{|k_\perp|^2 |q_\perp - k_\perp|^2} \quad (2.37)$$

which yields eq. (2.24) in dimensional regularisation with $d = 4 - 2\epsilon$. The entire one-loop diagram can then be obtained from its imaginary part via a (twice-subtracted) dispersion relation, eq. (1.11). This was the technique employed by ref. [73] in the first determination of the Regge trajectory.

The same dispersive technique was used in ref. [73] to find also the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ virtual corrections from the interference of on-shell $2 \rightarrow 3$ and $2 \rightarrow 4$ tree-level diagrams. Due to similar arguments to the above discussion of the cross section, the LL contribution to the Cutkosky phase-space integral is captured by the MRK region of phase space, further motivating an investigation of the MRK limit of tree-level amplitudes, for an arbitrary number of final state partons.

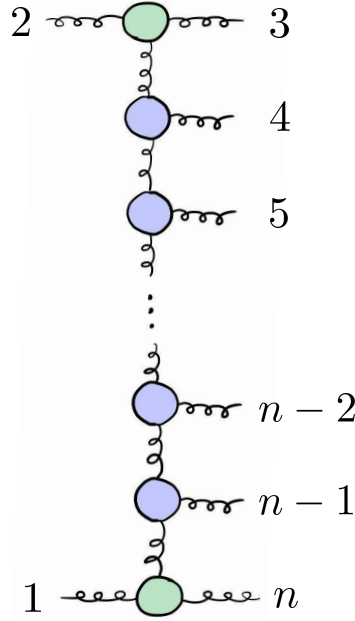


Figure 2.4 *Depiction of the factorised LO $gg \rightarrow (n-2)g$ amplitude in MRK, given by eq. (2.39).*

It was found that for $n \geq 4$, QCD amplitudes in the MRK limit,

$$p_3^+ \gg p_4^+ \gg \cdots p_n^+, \quad p_n^- \gg p_4^- \gg \cdots p_3^-, \quad |p_{3\perp}| \simeq |p_{4\perp}| \cdots \simeq |p_{n\perp}|, \quad (2.38)$$

are given by the simple generalisation of eq. (2.31), i.e. [22, 93]

$$\begin{aligned} \mathcal{M}_{f_1 f_2 \rightarrow f_3 g \cdot g f_n} \xrightarrow{\text{MRK}} & 2s \left[g_s T_{R_2 R_3}^{a_{t_1}} C^{f_2 \rightarrow f_3(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}) \right] \\ & \times \frac{1}{-|q_{1\perp}|^2} \prod_{i=4}^{n-1} \left(\left[i g_s f^{a_{t_{i-3}} a_{t_{i-2}} a_i} V^{g(0)}(q_{i-3}, p_i^{\lambda_i}, q_{i-2}) \right] \frac{1}{-|q_{i-2\perp}|^2} \right) \\ & \times \left[g_s T_{R_n R_3}^{a_{t_{n-3}}} C^{f_1 \rightarrow f_n(0)}(p_1^{\lambda_1}, p_n^{\lambda_n}) \right], \end{aligned} \quad (2.39)$$

depicted in fig. 2.4. In ref. [73] the $n = 5$ and $n = 6$ cases were used to obtain the $\mathcal{O}(\alpha_s^2 \log^2(s/-t))$ and $\mathcal{O}(\alpha_s^3 \log^3(s/-t))$ virtual corrections,

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(2)} = \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(0)} \frac{1}{2} \left(N_c \kappa_\Gamma \frac{\alpha_s}{4\pi} \alpha^{(1)}(|q_\perp|^2) \log \left(\frac{s}{|q_\perp|^2} \right) \right)^2, \quad (2.40)$$

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(3)} = \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(0)} \frac{1}{6} \left(N_c \kappa_\Gamma \frac{\alpha_s}{4\pi} \alpha^{(1)}(|q_\perp|^2) \log \left(\frac{s}{|q_\perp|^2} \right) \right)^3. \quad (2.41)$$

This provided strong motivation for the conjecture that virtual corrections exponentiate to all orders in α_s , which we will refer to as the Lipatov ansatz:

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4} \xrightarrow{\text{LL}} 2s \left[g_s T_{R_1 R_2}^{a_t} C_{f_1 f_2}^{(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}) \right] \left(\frac{s}{|q_\perp|^2} \right)^{\alpha(|q_\perp|^2)} \left[g_s T_{R_3 R_4}^{a_t} C_{f_3 f_4}^{(0)}(p_1^{\lambda_1}, p_4^{\lambda_4}) \right], \quad (2.42)$$

where eq. (2.24) is interpreted as the first term in the all orders expression for the Regge trajectory,

$$\alpha(|q_\perp|^2) = \frac{\alpha_s}{4\pi} N_c \kappa_\Gamma \alpha^{(1)}(|q_\perp|^2) + \mathcal{O}(\alpha_s^2). \quad (2.43)$$

We note that eq. (2.42) is purely real.

To all orders in α_s , the $f_1 f_2 \rightarrow f_3 f_4$ amplitude consists in principle of the t -channel exchange of any irreducible representation that appears in both the decomposition of $R_1 \otimes R_2$ and $R_3 \otimes R_4$. As an example, for $gg \rightarrow gg$, all the representations that appear in the decomposition eq. (1.98) can be exchanged in the t channel. However, we emphasise that the only colour structure that contributes to LL accuracy is **8** exchange in the t -channel.

To LL accuracy, we see that QCD amplitudes are indeed governed by a Regge pole, and this Reggeon, with effective ‘spin’ $(1 + \alpha(t))$, and which transforms in the adjoint representation, is called the Reggeised gluon. The exponentiated virtual corrections of eq. (2.42) are interpreted as the ‘Reggeisation’ of the gluon

propagator,

$$\frac{1}{|q_\perp|^2} \rightarrow \frac{1}{|q_\perp|^2} \left(\frac{s}{\tau} \right)^{\alpha(|q_\perp|^2)}, \quad (2.44)$$

where $\tau > 0$ is a Regge factorisation scale, which is of order of $|q_\perp|^2$, and much smaller than s . In $2 \rightarrow 2$ scattering, the natural choice is of course to set this scale equal to the only physical finite scale, $|q_\perp|^2$, but it will be useful to keep track of the dependence on the Regge factorisation scale in the following.

The amplitude eq. (2.42) is depicted in fig. 2.5 for the case of incoming gluons, where we use a zig-zag line to represent the dressed, or Reggeised, t -channel gluon propagator eq. (2.44).

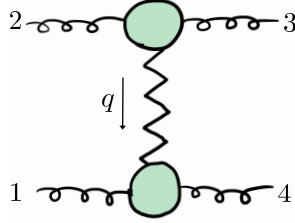


Figure 2.5 *Depiction of the LL $gg \rightarrow gg$ amplitude given by eq. (2.42). The zig-zag line represents the Reggeized gluon propagator, eq. (2.44).*

Finally, to describe $2 \rightarrow (n-2)$ amplitudes in the MRK limit to LL accuracy, to all orders in α_s , in refs. [73, 77] the Lipatov ansatz was made for all t channel gluon exchanges,

$$\begin{aligned} \mathcal{M}_{f_1 f_2 \rightarrow f_3 g \dots g f_n} \xrightarrow{\text{LL}} & 2s \left[g_s T_{R_2 R_3}^{a_{t_1}} C^{f_2 \rightarrow f_3(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}) \right] \\ & \times \prod_{j=1}^{n-3} \frac{1}{-|q_{j\perp}|^2} \left(\frac{s_{i+1, i+3}}{\tau} \right)^{\alpha(|q_{j\perp}|^2)} \\ & \times \prod_{i=4}^n \left[i g_s f^{a_{t_{i-3}} a_{t_{i-2}} a_i} V^{g(0)}(q_{i-3}, p_i^{\lambda_i}, q_{i-2}) \right] \\ & \times \left[g_s T_{R_n R_3}^{a_{t_{n-3}}} C^{f_1 \rightarrow f_n(0)}(p_1^{\lambda_1}, p_n^{\lambda_n}) \right], \end{aligned} \quad (2.45)$$

which is depicted in fig. 2.6.

These amplitudes, along with the s -channel unitarity condition, eq. (1.7), were the starting point of the derivation of the LL BFKL equation [73, 75, 77, 78]. A thorough derivation of this equation is beyond the scope of this introduction. However we will sketch the main steps in order to draw a connection between the factorised components of the amplitude, and the kernel of this evolution equation.

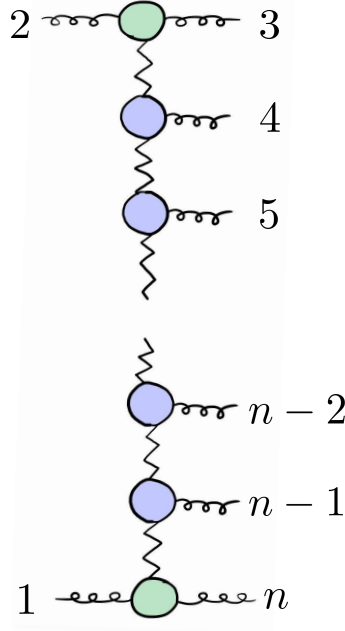


Figure 2.6 *Depiction of the factorised LL $gg \rightarrow (n-2)g$ amplitude in MRK, given by eq. (2.45).*

In the case of forward scattering, the optical theorem, eq. (1.9), relates the imaginary part of the amplitude to the total cross section. For the inclusive partonic cross section for incoming partons of flavour f_1 and f_2 , the optical theorem reads

$$\sigma_{f_1 f_2}(s) = \frac{1}{s} \text{Im}[\overline{\mathcal{M}}_{f_1 f_2 \rightarrow f_1 f_2}(s, t=0)] = \frac{1}{2s} \sum_{n=4}^{\infty} \sum_{f_i, a_i, \lambda_i} \int d\Phi_{n-2} |\overline{\mathcal{M}}_{f_1 f_2 \rightarrow f_3 \dots f_n}|^2, \quad (2.46)$$

where the sum runs over all possible states permitted in the s -channel. The notation $\overline{\mathcal{M}}$ denotes the amplitude has been summed (averaged) over final (initial) colours and helicities. Conservation of colour of partons f_1 and f_2 requires that a colour singlet is exchanged in the t -channel. As argued above, the LL behaviour of the integral on the right hand side will arise from the MRK region of phase space, so we can substitute the factorised amplitudes eq. (2.45), and modify the phase space measure $d\Phi_n$ such that we only integrate over MRK phase space. The resulting nested rapidity integrals can be disentangled by a Laplace transform,

$$\int_0^{\infty} d\Delta e^{-(\omega+1)\Delta} \sigma_{f_1 f_2}(s(\Delta)). \quad (2.47)$$

where $\Delta = \log(s/\tau)$ and τ is some finite scale that is immaterial at LL accuracy.

To deal with the remaining transverse integration, it is convenient to change variables from the real-emission momenta $p_{i\perp}$ (with $i = 3, \dots, n$) to the t -channel momenta $q_{i\perp}$ (with $i = 1, \dots, n-3$). The colour-summed self-interference of the impact factors can be factored out of the process into the ‘squared impact factors’,

$$\Phi_{f_1}(|q_{1\perp}|^2) = \frac{c_R g_S^2}{|q_{1\perp}|^2}, \quad \Phi_{f_2}(|q_{(n-3)\perp}|^2) = \frac{c_R g_S^2}{|q_{(n-3)\perp}|^2}, \quad (2.48)$$

where c_R is equal to c_F for c_A depending on whether f_i is a quark or gluon. We emphasise that the building blocks of the squared impact factors can be obtained from the Regge limit of LO $2 \rightarrow 2$ amplitudes, as depicted in fig. 2.7. For brevity, let us re-label $q_{1\perp} \rightarrow q_{a\perp}$, $q_{(n-3)\perp} \rightarrow q_{b\perp}$, and correspondingly $p_{a\perp} = q_{a\perp}$ and $p_{b\perp} = -q_{b\perp}$ are the transverse momenta of the extremal dijets.

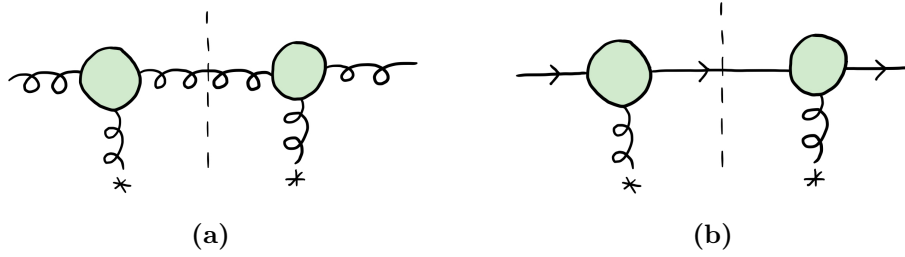


Figure 2.7 *Partial interference diagrams which contain the building blocks required for constructing the BFKL impact factors at LL accuracy: (a) $g \rightarrow g$ impact factor at tree level; (b) $q \rightarrow q$ impact factor at tree level. The * indicates an off-shell gluon.*

The Laplace transform of the total cross section is then written in the factorised form

$$\int_0^\infty d\Delta e^{-(\omega+1)\Delta} \sigma_{f_1 f_2} = \frac{1}{N_c^2 - 1} \int d^2 \mathbf{q}_{a\perp} d^2 \mathbf{q}_{b\perp} \Phi_{f_1}(|q_{a\perp}|^2) f_\omega(\mathbf{q}_{a\perp}, \mathbf{q}_{b\perp}) \Phi_{f_2}(|q_{b\perp}|^2), \quad (2.49)$$

where f_ω captures the integration over the ‘central’ physics of the t -channel gluon ladder. This function can be expressed as the solution to the iterated equation

$$\omega f_\omega(\mathbf{q}_{a\perp}, \mathbf{q}_{b\perp}) = \delta^{(d-2)}(\mathbf{q}_{a\perp} - \mathbf{q}_{b\perp}) + \int d^{d-2} \mathbf{q}_\perp \mathcal{K}(\mathbf{q}_{a\perp}, \mathbf{q}_\perp) f_\omega(\mathbf{q}_\perp, \mathbf{q}_{b\perp}) \quad (2.50)$$

known as the (singlet) BFKL equation. The kernel of this equation is given by a virtual component, which is proportional to the Regge trajectory and conserves

momentum in the t -channel, plus a real emission component $\mathcal{K}_r$,

$$\mathcal{K}(\mathbf{q}_{i\perp}, \mathbf{q}_{j\perp}) = g_s^2 N_c \alpha^{(1)}(|q_{i\perp}|^2) \delta^{(d-2)}(\mathbf{q}_{i\perp} - \mathbf{q}_{j\perp}) + \mathcal{K}_r(\mathbf{q}_{i\perp}, \mathbf{q}_{j\perp}), \quad (2.51)$$

where the latter is proportional to the square of the Lipatov vertex, summed over helicity and colour, and where a singlet projection operator, eq. (1.99), has been inserted at each ‘rung’ of the ladder, yielding a factor of N_c ,

$$\mathcal{K}_r(\mathbf{q}_{i\perp}, \mathbf{q}_{j\perp}) = \frac{g_s^2 N_c}{2(2\pi)^{d-1}} \sum_{\lambda} |V^{g(0)}(q_{i\perp}, (q_{i\perp} - q_{j\perp})^\lambda, q_{j\perp})|^2 = \frac{g_s^2 N_c}{(2\pi)^{d-1}} \frac{2}{|q_{i\perp} - q_{j\perp}|^2}. \quad (2.52)$$

We emphasise that the building blocks of the LL BFKL kernel, namely the Regge trajectory and the Lipatov vertex can be obtained from one loop $2 \rightarrow 2$ amplitudes and tree-level $2 \rightarrow 3$ amplitudes, as depicted in fig. 2.8. We also emphasise that the Kernel is IR finite, due to the cancellation of the IR-poles as discussed in section 2.1.2.

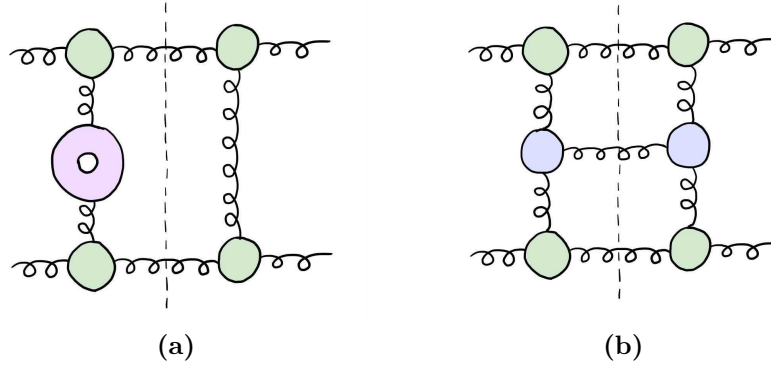


Figure 2.8 *Interference diagrams which contain the building blocks required for constructing the BFKL kernel at LL accuracy: (a) gluon Regge trajectory at one loop; (b) one-gluon central emission vertex at tree level. Tree-level impact factors (shown in green) are required for the computation of cross sections, but are not required to determine the evolution of the t -channel Reggeon.*

Writing the inverse Laplace transform of f_ω as

$$f(\mathbf{q}_{i\perp}, \mathbf{q}_{j\perp}, \Delta) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\omega e^{\omega\Delta} f_\omega(\mathbf{q}_{i\perp}, \mathbf{q}_{j\perp}), \quad (2.53)$$

we can write the total $f_1 f_2$ -initiated partonic cross section as

$$\sigma_{f_1 f_2}(s) = \frac{1}{N_c^2 - 1} \int d^2 \mathbf{q}_{a\perp} d^2 \mathbf{q}_{b\perp} \Phi_{f_1}(|q_{a\perp}|^2) f(\mathbf{q}_{a\perp}, \mathbf{q}_{b\perp}, \Delta) \Phi_{f_2}(|q_{b\perp}|^2). \quad (2.54)$$

Thus the LL behaviour of the total cross section is determined by the solution of eq. (2.50). Guides to solving the BFKL equation can be found in refs. [11, 19, 94]. For the case where $|q_{a\perp}|^2 \sim |q_{b\perp}|^2$ the solution can be obtained analytically, and the resulting total partonic cross section is

$$\sigma_{f_1 f_2}(s) = k s^{\frac{N_c \alpha_S}{\pi} 4 \log(2)}. \quad (2.55)$$

There are a number of issues with the LL BFKL approach [11, 100]:

1. The growth in s of the cross section is in violation of the Froissart bound eq. (1.21), and therefore in violation of unitarity, eq. (1.7). This has been the subject of much discussion in the literature, see e.g. ref. [104].
2. It can be shown [11] that the transverse momentum distribution of a t -channel momentum within the BFKL gluon ladder is a Gaussian whose width is a maximum at the centre of the ladder, $\Delta/2$, and grows with Δ . This means that even under the constraint $|q_{1\perp}|^2 \sim |q_{(n-3)\perp}|^2$, for sufficiently large Δ , the transverse momenta far from the ends of the ladder will violate the MRK condition, eq. (2.38), and eventually will fall below Λ_{QCD} , where perturbation theory is not justified [105].
3. There is no scale variation of the coupling constant at LL accuracy.
4. As real emissions are only produced in the MRK limit, jets only contain a single parton.
5. The form of the amplitudes in MRK given in eq. (2.45) are gauge invariant in the MRK limit, but are not gauge invariant outwith this limit. This leads to essentially arbitrary predictions in much of the phase space measured in current experiments.
6. Energy and longitudinal momentum are not conserved. One consequence of this fact is that the momentum fractions,

$$x_{f_1} = \frac{1}{\sqrt{s_{\text{had}}}} \left(\sum_{i=3}^n |p_{i\perp}| e^{y_i} \right), \quad x_{f_2} = \frac{1}{\sqrt{s_{\text{had}}}} \left(\sum_{i=3}^n |p_{i\perp}| e^{-y_i} \right), \quad (2.56)$$

of the incoming partons are approximated as

$$x_{f_1} = \frac{|p_{3\perp}|}{\sqrt{s_{\text{had}}}} e^{y_3}, \quad x_{f_2} = \frac{|p_{n\perp}|}{\sqrt{s_{\text{had}}}} e^{-y_n}, \quad (2.57)$$

where s_{had} is the hadronic centre of mass. This generally leads to a poor approximation of the PDF values in phenomenological applications, which are highly sensitive to x_{f_i} in regions of experimental accessibility. We emphasise that a kinematic configuration with two well-separated jets corresponds to large momentum fractions.

The first two points are inherent to the BFKL approach, and means we cannot trust its results to arbitrarily high energies. If we are to verify our understanding of QCD, we must therefore test the all-orders results of the BFKL formalism at high-but-not-asymptotically high energies. This compounds issues 3–6 in the above list. One way to improve on points 3–6 are to move to NLL accuracy, which is the subject of the next section.

2.2 Building blocks of the NLL BFKL kernel

In the previous section we saw that to LL accuracy, the $2 \rightarrow 2$ amplitude, eq. (2.42), was described by a Regge pole, eq. (1.25). At NLL accuracy this is no longer the case. The situation can be clarified by recalling the concept of signature, eq. (1.20). In terms of signature, the results of the previous section can be summarised as

$$\mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(+)} \Big|_{\text{LL}} = 0, \quad \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(-)} \Big|_{\text{LL}} = \mathcal{M}_{f_1 f_2 \rightarrow f_3 f_4}^{(0)} \left(\frac{s}{\tau} \right)^{\alpha(|q_\perp|^2)}. \quad (2.58)$$

We interpreted this as the exchange of a single Reggeon. It can be shown that signature odd (even) amplitudes only receive contributions from an odd (even) number of Reggeons [106]. The signature-even $2 \rightarrow 2$ amplitude at NLL receives a contribution from the exchange of two Reggeons in the t -channel which leads to a Regge cut behaviour, eq. (1.29). However, the signature-odd component of the amplitude is still governed by the exchange of a single Reggeon in the t channel, and the results of the previous section can be generalised, as we will see below.

Beyond LL accuracy, it is beneficial to introduce the signature-even logarithm [106],

$$L \equiv \frac{1}{2} \left[\log \left(\frac{-s - i0}{-t} \right) + \log \left(\frac{-u - i0}{-t} \right) \right] \xrightarrow{s \gg |t|} \log \left| \frac{s}{t} \right| - \frac{i\pi}{2}. \quad (2.59)$$

Expanding the $2 \rightarrow 2$ amplitude in powers of L ,

$$\mathcal{M}^{(\pm)} = g_s^2 \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n \sum_{m=0}^n L^m \mathcal{M}^{(\pm, n, m)}, \quad (2.60)$$

one can show that the coefficients $\mathcal{M}^{(+, n, m)}$ and $\mathcal{M}^{(-, n, m)}$ are imaginary and real respectively (although we emphasise that L itself has an imaginary part). These reality properties derive from the reality of s and u channel discontinuities, e.g. in eq. (1.11), having the interpretation of a spectral density of positive energy states. We note that this implies the signature-even amplitude at NLL accuracy can only contribute to a squared amplitude, and thus to a cross section, at NNLL accuracy. We therefore consider only the signature-odd amplitude in the remainder of this section.

At this point we will restrict the discussion to gluon scattering for clarity, where due to Bose symmetry, the signature-odd amplitude must necessarily have an antisymmetric representation exchanged in the t -channel. In this case eq. (2.42) generalises to NLL accuracy in a straightforward way [99]:

$$\begin{aligned} \mathcal{M}_{gg \rightarrow gg}^{(-)} \xrightarrow{\text{NLL}} & 2s \left[ig_s f^{a_2 a_t a_3} C^{g \rightarrow g}(p_2^{\lambda_2}, p_3^{\lambda_3}) \right] \\ & \times \frac{1}{|q_\perp|^2} \frac{1}{2} \left[\left(\frac{s}{\tau} \right)^{\alpha(t)} + \left(\frac{-s}{\tau} \right)^{\alpha(t)} \right] \\ & \times \left[ig_s f^{a_4 a_t a_1} C^{g \rightarrow g}(p_1^{\lambda_1}, p_4^{\lambda_4}) \right]. \end{aligned} \quad (2.61)$$

Note that while in principle any of the antisymmetric components of eq. (1.98) could contribute, we once again see that only the octet contributes to the odd part of the amplitude to NLL accuracy. Note that to describe the $2 \rightarrow 2$ amplitude to NLL we need to obtain the impact factor and Regge trajectory to one higher power in α_s compared to eq. (2.42).

We expand the gluon Regge trajectory in α_s as

$$\alpha(t) = \frac{\alpha_s}{4\pi} N_c \kappa_\Gamma \alpha^{(1)}(t) + \left(\frac{\alpha_s}{4\pi} N_c \kappa_\Gamma \right)^2 \alpha^{(2)}(|q_\perp|^2) + \mathcal{O}(\alpha_s^3), \quad (2.62)$$

with the one-loop coefficient given in eq. (2.24), and the two-loop coefficient given by [107–111]

$$\alpha^{(2)}(t) = \left(\frac{\mu^2}{|q_\perp|^2} \right)^{2\epsilon} \left(\frac{\beta_0}{\epsilon^2} + \frac{\gamma_K^{(2)}}{8\epsilon} + \frac{\gamma_\lambda^{(2)}}{2} + \zeta_2 \beta_0 \right) + \mathcal{O}(\epsilon) \quad (2.63)$$

which is neatly expressed in terms of the beta function eq. (1.67), and the cusp [112, 113] and wedge [114, 115] anomalous dimensions

$$\begin{aligned}\gamma_K^{(2)} &= \frac{11}{3} - \frac{2}{3} \frac{N_f}{N_c}, \\ \gamma_\wedge^{(2)} &= 8 \left(\frac{64}{9} + \frac{\delta_R}{3} - 2\zeta_2 \right) - \frac{80}{9} \frac{N_f}{N_c}.\end{aligned}\tag{2.64}$$

In eq. (2.64), δ_R is a regularisation parameter which equals 1 in the conventional dimensional regularisation (CDR) or 't Hooft-Veltman (HV) scheme, and which equals 0 in the dimensional reduction (DRED) or four-dimensional helicity (FDH) schemes.

Analogous to eq. (2.63), we expand the helicity-conserving impact factor,

$$C^{g \rightarrow g}(p_i^\oplus, p_j^\ominus) = C^{g \rightarrow g(0)}(p_i^\oplus, p_j^\ominus) + \frac{\alpha_s}{4\pi} C^{g \rightarrow g(1)}(p_i^\oplus, p_j^\ominus) + \mathcal{O}(\alpha_s^2), \tag{2.65}$$

with one-loop corrections,

$$C^{g \rightarrow g(1)}(p_i^\oplus, p_j^\ominus) = \kappa_\Gamma C^{g \rightarrow g(0)}(p_i^\oplus, p_j^\ominus) c_{\text{QCD}}^{g \rightarrow g(1)}(|q_\perp|^2). \tag{2.66}$$

Note that the one-loop coefficient $c_{\text{QCD}}^{g \rightarrow g(1)}$ is real and independent of the helicity configuration. The coefficient $C^{g \rightarrow g(1)}$, and therefore also $C^{g \rightarrow g}$, depends on the renormalisation and Regge-factorisation scales μ^2 and τ , but we omit this dependence for brevity.

Rather than simply quoting the one-loop QCD impact factor, let us see how this expression can be obtained via the supersymmetric decomposition of section 1.7. This will firstly provide a simple example of the practical benefit of such a decomposition, and secondly, will allow us to collect the one-loop expressions that will be required for chapter 3.

Recall the one-loop four-gluon amplitudes listed in eq. (2.13). Taking the Regge limit of the real part of eq. (2.14), subtracting the LL contribution of $\alpha(|q_\perp|^2) \log(s/\tau)$, and then dividing by two (i.e. taking the natural convention of splitting the NLL remainder equally between the two impact factors), we obtain the coefficient

$$c_{\mathcal{N}=4}^{g \rightarrow g(1)}(|q_\perp|^2) = \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon \left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{\tau}{|q_\perp|^2} \right) + \frac{\pi^2}{2} - \frac{\delta_R}{6} \right). \tag{2.67}$$

Note the dependence on τ , which enters due to the subtraction of the τ -dependent

LL piece. By contrast, neither eq. (2.16) or eq. (2.17) contributed at LL accuracy, so we can simply take the $\mathcal{N} = 1\chi$ and scalar impact factors to be half the contribution of eq. (2.18) and eq. (2.19) in the Regge limit:

$$c_{\mathcal{N}=1}^{g \rightarrow g(1)}(|q_\perp|^2) = \frac{1}{2\epsilon} \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon + 1, \quad (2.68)$$

$$c_{\text{scalar}}^{g \rightarrow g(1)}(|q_\perp|^2) = \frac{1}{6\epsilon} \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon + \frac{4}{9}, \quad (2.69)$$

which are of course independent of τ . Now we can assemble the linear combinations, eqs. (2.67)–(2.69), required for the gluon-loop contribution to the QCD impact factor,

$$\begin{aligned} c_{\text{QCD}}^{g \rightarrow g(1)\textcircled{8}}(|q_\perp|^2) &= c_{\mathcal{N}=4}^{g \rightarrow g(1)}(|q_\perp|^2) - 4c_{\mathcal{N}=1}^{g \rightarrow g(1)}(|q_\perp|^2) + c_{\text{scalar}}^{g \rightarrow g(1)}(|q_\perp|^2) \\ &= \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon \left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \log \left(\frac{\tau}{|q_\perp|^2} \right) + \frac{\pi^2}{2} - \frac{\delta_R}{6} - \frac{11}{6\epsilon} - \frac{32}{9} \right) - \frac{11}{6\epsilon}, \end{aligned} \quad (2.70)$$

and the quark-loop contribution to the QCD impact factor,

$$\begin{aligned} c_{\text{QCD}}^{g \rightarrow g(1)\textcircled{3}}(|q_\perp|^2) &= c_{\mathcal{N}=1}^{g \rightarrow g(1)}(|q_\perp|^2) - c_{\text{scalar}}^{g \rightarrow g(1)}(|q_\perp|^2) \\ &= \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon \left(\frac{1}{3\epsilon} + \frac{5}{9} \right) + \frac{2}{6\epsilon}. \end{aligned} \quad (2.71)$$

Finally adding the gluon- and quark-loop contributions we obtain the full QCD $g \rightarrow g$ impact factor, $c_{\text{QCD}}^{g \rightarrow g(1)} = N_c c_{\text{QCD}}^{g \rightarrow g(1)\textcircled{8}} + N_f c_{\text{QCD}}^{g \rightarrow g(1)\textcircled{3}}$, which is in agreement with [99, 100].

For completeness we give the analogous one-loop correction to the QCD quark impact factor:

$$\begin{aligned} c_{\text{QCD}}^{q \rightarrow q(1)}(|q_\perp|^2) &= N_c \left(\frac{\mu^2}{|q_\perp|^2} \right)^\epsilon \left(-\frac{1}{\epsilon^2} - \frac{11}{3\epsilon} + \frac{1}{\epsilon} \log \left(\frac{\tau}{|q_\perp|^2} \right) + \frac{\pi^2}{2} - \frac{\delta_R}{3} + \frac{19}{18} \right) \\ &\quad - N_f \left(\frac{1}{3\epsilon} + \frac{5}{9} \right) - \frac{11}{6\epsilon}. \end{aligned} \quad (2.72)$$

The $2 \rightarrow 3$ scattering amplitudes, $g_1 g_2 \rightarrow g_3 g_4 g_5$, can be likewise split into odd and even components under signature symmetry,

$$\mathcal{M}_{5g} = \mathcal{M}_{5g}^{(+,+)} + \mathcal{M}_{5g}^{(+,-)} + \mathcal{M}_{5g}^{(-,+)} + \mathcal{M}_{5g}^{(-,-)}, \quad (2.73)$$

with

$$\begin{aligned}
& \mathcal{M}_{5g}^{(\tau_a, \tau_b)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}) \\
&= \frac{1}{4} \mathcal{M}_{5g}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}) + \frac{\tau_a}{4} \mathcal{M}_{5g}(p_1^{\nu_1}, p_3^{\nu_3}, p_2^{\nu_2}, p_4^{\nu_4}, p_5^{\nu_5}) \\
&+ \frac{\tau_b}{4} \mathcal{M}_{5g}(p_5^{\nu_5}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_1^{\nu_1}) + \frac{\tau_a \tau_b}{4} \mathcal{M}_{5g}(p_5^{\nu_5}, p_3^{\nu_3}, p_2^{\nu_2}, p_4^{\nu_4}, p_1^{\nu_1}),
\end{aligned} \tag{2.74}$$

where the two signature indices τ_a and τ_b can separately be $+1$ or -1 . By Bose symmetry, the upper and lower vertices (where we have in mind a MRK factorisation such as the one in fig. 2.3) then acquire a corresponding colour symmetry: the upper vertex is symmetric/antisymmetric upon interchanging the colour indices a_2 and a_3 for $\tau_a = +$ and $\tau_a = -$, respectively. Similarly, the lower one is symmetric/antisymmetric upon interchanging a_1 and a_5 for $\tau_b = +$ and $\tau_b = -$, respectively. Specifically, the $\mathcal{M}^{(-, -)}$ amplitude, which we discuss below, is antisymmetric in colour in both the upper and lower vertices.

In MRK, eq. (2.74) is understood to involve the exchanges $s_{34} \leftrightarrow s_{24} = -s_{34}$ and $s_{45} \leftrightarrow s_{41} = -s_{45}$ along with the associated change in sign of $s = s_{12} = -s_{13} = -s_{52} = s_{35}$. At NLL accuracy, the signature odd-odd component of the amplitude, $\mathcal{M}^{(-, -)}$, is governed by an antisymmetric octet exchange, $\mathbf{8}_a$, in the t_1 and t_2 channels, corresponding to a single Reggeon. One can then generalise eq. (2.31) to NLL accuracy,

$$\begin{aligned}
\mathcal{M}_{5g}^{(-, -)[\mathbf{8}_a]} &\xrightarrow{\text{MRK}} \frac{1}{4} s \left[i g_s f^{a_2 a_{t_1} a_3} C^{g \rightarrow g}(p_2^{\lambda_2}, p_3^{\lambda_3}) \right] \frac{1}{t_1} \left[\left(\frac{s_{34}}{\tau} \right)^{\alpha(t_1)} + \left(\frac{-s_{34}}{\tau} \right)^{\alpha(t_1)} \right] \\
&\times [i g_s f^{a_{t_1} a_{t_2} a_4} V^g(q_1, p_4^{\nu_4}, q_2)] \frac{1}{t_2} \left[\left(\frac{s_{45}}{\tau} \right)^{\alpha(t_2)} + \left(\frac{-s_{45}}{\tau} \right)^{\alpha(t_2)} \right] \\
&\times [i g_s f^{a_5 a_{t_2} a_1} C^{g \rightarrow g}(p_1^{\lambda_1}, p_5^{\lambda_5})] ,
\end{aligned} \tag{2.75}$$

where the CEV now must include a one-loop correction in the expansion

$$V^g(q_1, p_4^{\nu_4}, q_2) = V^{g(0)}(q_1, p_4^{\nu_4}, q_2) + \frac{\alpha_s}{4\pi} V^{g(1)}(q_1, p_4^{\nu_4}, q_2) + \mathcal{O}(\alpha_s^2) . \tag{2.76}$$

Analogous to eq. (2.66) we write the one-loop vertex as

$$V^{g(1)}(q_1, p_4^{\nu_4}, q_2) = V^{g(0)}(q_1, p_4^{\nu_4}, q_2) v^{g(1)}(q_{1\perp}, q_{2\perp}) . \tag{2.77}$$

where $v^{g(1)}$ is real and helicity-independent. Similarly to $c^{g(1)}$, we omit the

dependence on the scales μ^2 and τ in the arguments of $v^{g(1)}$ and V^g .

As for the discussion of the impact factors, let us introduce the one-loop correction to the Lipatov vertex in the context of the supersymmetric decomposition of section 1.7.

The helicity independent $\mathcal{N} = 4$ contribution to the one-loop five-gluon amplitude is [54]

$$V_{g5}(1, 2, 3, 4, 5) = \frac{-1}{\epsilon^2} \sum_{j=1}^5 \left(\frac{\mu^2}{-s_{j,j+1}} \right)^\epsilon + \sum_{j=1}^5 \log \left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}} \right) + \frac{5}{6}\pi^2 - \frac{\delta_R}{3} \quad (2.78)$$

which we must analytically continue to our physical region, i.e. $s_{12}, s_{34}, s_{45} > 0$ and $s_{23}, s_{51} < 0$.

Expanding the $\mathcal{N} = 4$ amplitude in the MRK region, and subtracting the LL pieces of the $\mathcal{N} = 4$ amplitude (given by the sum of $\alpha(|q_{1\perp}|^2) \log(s_{34}/\tau)$ and $\alpha(|q_{2\perp}|^2) \log(s_{45}/\tau)$), as well as subtracting the NLL parts which are captured by the impact factors (i.e. $c_{\mathcal{N}=1}^{g(1)}(|q_{1\perp}|^2)$ and $c_{\mathcal{N}=1}^{g(1)}(|q_{2\perp}|^2)$), we obtain the coefficient

$$\begin{aligned} v_{\mathcal{N}=4}^{g(1)}(|q_{1\perp}|^2, |p_{4\perp}|^2, |q_{2\perp}|^2) = & -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{|p_{4\perp}|^2} \right)^\epsilon + \frac{\pi^2}{3} - \frac{1}{2} \log^2 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right) \\ & + \frac{1}{\epsilon} \left[\left(\frac{\mu^2}{|q_{1\perp}|^2} \right)^\epsilon + \left(\frac{\mu^2}{|q_{2\perp}|^2} \right)^\epsilon \right] \log \left(\frac{\tau}{|p_{4\perp}|^2} \right). \end{aligned} \quad (2.79)$$

The one-loop five-gluon amplitudes in the $\mathcal{N} = 1\chi$ and scalar cases functions are more complicated, and helicity dependent, yet in the MRK limit it is found[116] that they yield the helicity-independent, IR finite, one-loop CEVs

$$\begin{aligned} v_{\mathcal{N}=1}^{g(1)}(|q_{1\perp}|^2, |p_{4\perp}|^2, |q_{2\perp}|^2) = & -\frac{1}{2} |p_{4\perp}|^2 \frac{L_0 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{|q_{2\perp}|^2}, \\ v_{\text{scalar}}^{g(1)}(q_{1\perp}, q_{2\perp}) = & \frac{1}{6} |p_{4\perp}|^2 \left(\frac{q_{1\perp} q_{2\perp}^*}{|q_{1\perp}|^2 |q_{2\perp}|^2} + \frac{L_0 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{|q_{2\perp}|^2} \right. \\ & \left. + 2q_{1\perp} q_{2\perp}^* (2q_{1\perp} q_{2\perp}^* - |q_{1\perp}|^2 - |q_{2\perp}|^2) \frac{L_2 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{(|q_{2\perp}|^2)^3} \right), \end{aligned} \quad (2.80)$$

where

$$L_0(x) = \frac{1}{1-x} \log(x), \quad L_2 = \frac{1}{(1-x)^3} \left(\log(x) - \frac{x}{2} + \frac{1}{2x} \right). \quad (2.81)$$

As indicated by the arguments of these functions, the scalar CEVs depend on four real degrees of freedom, one more than the supersymmetric CEVs. Also note that the $\mathcal{N} = 1\chi$ and scalar pieces have no dependence on τ , due to the Regge trajectory of QCD coinciding with that of $\mathcal{N} = 4$ SYM at one loop. Following the same procedure as for the impact factor above, we can assemble the full QCD Lipatov vertex $v_{\text{QCD}}^{g(1)}(q_{1\perp}, q_{2\perp})$, from the gluon- and quark-loop contributions,

$$\begin{aligned} v_{\text{QCD}}^{g(1)\textcircled{8}}(|q_\perp|^2) &= v_{\mathcal{N}=4}^{g(1)}(|q_\perp|^2) - 4v_{\mathcal{N}=1}^{g(1)}(|q_\perp|^2) + v_{\text{scalar}}^{g(1)}(|q_\perp|^2) \\ &= -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{|p_{4\perp}|^2} \right)^\epsilon + \frac{1}{\epsilon} \left[\left(\frac{\mu^2}{|q_{1\perp}|^2} \right)^\epsilon + \left(\frac{\mu^2}{|q_{2\perp}|^2} \right)^\epsilon \right] \log \left(\frac{\tau}{|p_{4\perp}|^2} \right) \\ &\quad + \frac{\pi^2}{3} - \frac{1}{2} \log^2 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right) + \frac{11}{3} \frac{1}{2} |p_{4\perp}|^2 \frac{L_0 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{|q_{2\perp}|^2} \\ &\quad - \frac{1}{3} |p_{4\perp}|^2 q_{1\perp} q_{2\perp}^* \left(\frac{1}{|q_{1\perp}|^2 |q_{2\perp}|^2} - (2q_{1\perp} q_{2\perp}^* - |q_{1\perp}|^2 - |q_{2\perp}|^2) \frac{L_2 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{(|q_{2\perp}|^2)^3} \right), \end{aligned} \quad (2.82)$$

$$\begin{aligned} v_{\text{QCD}}^{g(1)\textcircled{3}}(|q_\perp|^2) &= v_{\mathcal{N}=1}^{g(1)}(|q_\perp|^2) - v_{\text{scalar}}^{g(1)}(|q_\perp|^2) \\ &= \frac{1}{3} |p_{4\perp}|^2 q_{1\perp} q_{2\perp}^* \left(\frac{1}{|q_{1\perp}|^2 |q_{2\perp}|^2} - (2q_{1\perp} q_{2\perp}^* - |q_{1\perp}|^2 - |q_{2\perp}|^2) \frac{L_2 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{(|q_{2\perp}|^2)^3} \right) \\ &\quad - \frac{2}{3} \frac{1}{2} |p_{4\perp}|^2 \frac{L_0 \left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2} \right)}{|q_{2\perp}|^2} \end{aligned} \quad (2.83)$$

in agreement with (the unrenormalised version of) the full QCD vertex given in [116]. In eq. (2.82) and eq. (2.83), we have written the coefficient of L_0 to clarify that this coefficient is proportional to β_0 in the full QCD vertex.

Let us note in passing that the highest transcendental weight functions of eq. (2.70) and eq. (2.82) (counting ϵ^{-1} as weight one), coincide exactly with the $\mathcal{N} = 4$ contribution to these pieces. Preliminary results suggest that this is not the case for the one-loop two-gluon CEV, where the $\mathcal{N} = 1\chi$ piece yields a highest-weight term.

In obtaining the NLL behaviour of cross sections, it is no longer sufficient to consider the contribution of real emissions integrated over just the MRK phase space. Consider relaxing the strict rapidity ordering requirement between two partons. Integrating the two partons over their common rapidity yields a factor of $\mathcal{O}(\alpha_s^2 \Delta y_{3n})$, i.e. a NLL contribution.

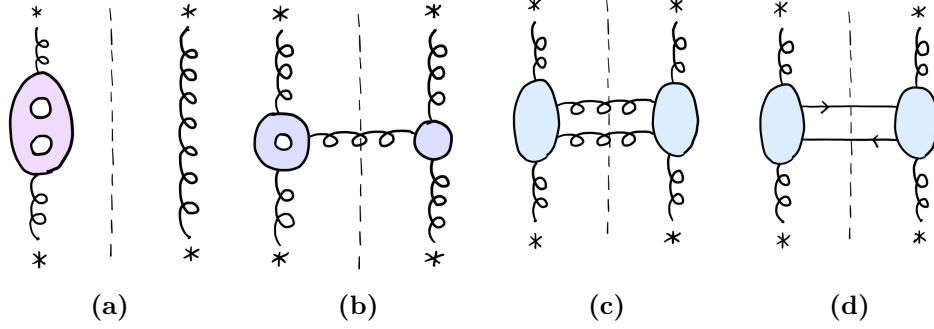


Figure 2.9 *Partial interference diagrams which contain the building blocks required for constructing the BFKL equation at NLL accuracy: (a) gluon Regge trajectory at two loops; (b) one-gluon central-emission vertex at one loop; (c) two-gluon central emission vertex at tree level; (d) central emission vertex for a quark-antiquark pair at tree-level.*

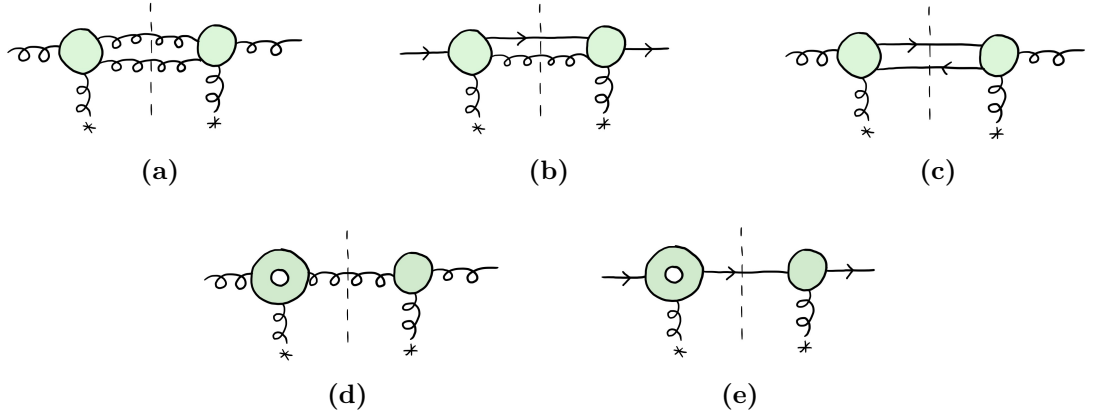


Figure 2.10 *Partial interference diagrams which contain the building blocks required for constructing the BFKL impact factors at NLL accuracy: (a) $g \rightarrow gg$ impact factor at tree level; (b) $q \rightarrow qg$ impact factor at tree level; (c) $g \rightarrow q\bar{q}$ impact factor at tree level; (d) $g \rightarrow g$ impact factor at one loop; (e) $q \rightarrow q$ impact factor at one loop;*

To obtain the real corrections to the impact factors, we can take the NMRK limit of five-parton amplitudes. These amplitudes are necessarily (anti-)MHV amplitudes, so can be obtained from the expressions listed in section 1.6.1. We

consider the production of three partons of momenta p_i , with $i = 3, 4, 5$ in the scattering of two partons of momenta p_1 and p_2 , with the momenta parametrised in terms of lightcone coordinates as in section 1.6.2. We refer to the process in question by $f_1 f_2 \rightarrow f_3 f_4 f_5$, where f_i are the physical flavours of the partons involved in the scattering. In the following we will consider the ‘forward’ NMRK configuration where the rapidities and transverse momenta of the outgoing gluons satisfy

$$y_3 \simeq y_4 \gg y_5; \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}|. \quad (2.84)$$

Let us begin with the pure-gluon case. The forward NMRK limit of the $gg \rightarrow ggg$ amplitude yields [97, 117]

$$\begin{aligned} M_{gg \rightarrow ggg} \xrightarrow{\text{NMRK}} & 2s \left[(ig_s)^2 \sum_{\sigma \in S_2} f^{a_2 a_x a_{\sigma_3}} f^{a_x a_{t_2} a_{\sigma_4}} C^{g \rightarrow gg(0)}(p_2^{\lambda_2}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}}) \right] \\ & \times \left(\frac{1}{-|q_{2\perp}|^2} \right) \\ & \times [ig_s f^{a_5 a_{t_2} a_1} C^{g \rightarrow g(0)}(p_1^{\lambda_1}; p_5^{\lambda_5})], \end{aligned} \quad (2.85)$$

which has a structure analogous to eq. (2.7). The expression in square brackets defines the LO $g \rightarrow gg$ impact factor. Here $C^{g \rightarrow gg(0)}$ itself can be expressed as a product of a factor which depends on the individual helicities of the gluons, $F^{g \rightarrow gg(0)}$, times a factor which depends on only the sum of helicities of the gluons, $A^{g \rightarrow gg(0)}$,

$$C^{g \rightarrow gg(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) = F^{g \rightarrow gg(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) A_{\Lambda_{234}}(p_2, p_3, p_4), \quad (2.86)$$

where $\Lambda_{234} = \lambda_2 + \lambda_3 + \lambda_4$. The impact factor can hence be constructed from

$$\begin{aligned} F^{g \rightarrow gg(0)}(p_2^\ominus, p_3^\oplus, p_4^\oplus) &= 1, \\ F^{g \rightarrow gg(0)}(p_2^\ominus, p_3^\oplus, p_4^\ominus) &= \left(1 + \frac{p_3^+}{p_4^+} \right)^{-2}, \\ F^{g \rightarrow gg(0)}(p_2^\ominus, p_3^\ominus, p_4^\oplus) &= \left(1 + \frac{p_4^+}{p_3^+} \right)^{-2}, \end{aligned} \quad (2.87)$$

and

$$A_\oplus(p_2, p_3, p_4) = 2 \frac{q_{2\perp}}{p_{3\perp}} \frac{p_3^+}{(p_3^+ p_{4\perp} - p_4^+ p_{3\perp})}, \quad (2.88)$$

where as usual, the parity-conjugate expressions are obtained from eq. (2.87) and eq. (2.88) by complex conjugation. As expected, the amplitude for $gq \rightarrow ggg$ is described by eq. (2.85) but with the lower impact factor replaced by a $q \rightarrow q$

impact factor.

Similarly, the LO $g \rightarrow q\bar{q}$ impact factor can be obtained by taking the (forward) NMRK limit of the $gg \rightarrow q\bar{q}g$ or $gq_1 \rightarrow q_2\bar{q}_2q_1$ amplitude. To be concrete we state the result for the former [97, 118],

$$\begin{aligned}
M_{gg \rightarrow q\bar{q}g} \xrightarrow{\text{NMRK}} & 2s \left[g_S^2 (T^{a_2} T^{a_{t_2}})_{i_3 i_4} C^{g \rightarrow q\bar{q}(0)}(p_2^{\lambda_2}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}}) \right. \\
& \left. + g_S^2 (T^{a_{t_2}} T^{a_2})_{i_3 i_4} C^{g \rightarrow q\bar{q}(0)}(p_2^{\lambda_2}, p_{\sigma_4}^{\lambda_{\sigma_4}}, p_{\sigma_3}^{\lambda_{\sigma_3}}) \right] \\
& \times \left(\frac{1}{-|q_{2\perp}|^2} \right) [i g_S f^{a_5 a_{t_2} a_1} C^{g \rightarrow g(0)}(p_1^{\lambda_1}; p_5^{\lambda_5})] .
\end{aligned} \tag{2.89}$$

The factors $C^{g \rightarrow q\bar{q}(0)}$ have an analogous structure to eq. (2.86), i.e.

$$C^{g \rightarrow q\bar{q}(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) = F^{g \rightarrow q\bar{q}(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) A_{\Lambda_{234}}(p_2, p_3, p_4), \tag{2.90}$$

where A_λ is the same process independent factor defined in eq. (2.88). The new ingredients are

$$\begin{aligned}
F^{g \rightarrow q\bar{q}(0)}(p_2^\oplus, p_3^\oplus, p_4^\ominus) &= \frac{1}{\sqrt{2}} \sqrt{\frac{p_3^+}{p_4^+}} \left(1 + \frac{p_3^+}{p_4^+} \right)^{-2}, \\
F^{g \rightarrow q\bar{q}(0)}(p_2^\oplus, p_3^\ominus, p_4^\oplus) &= \frac{1}{\sqrt{2}} \sqrt{\frac{p_4^+}{p_3^+}} \left(1 + \frac{p_4^+}{p_3^+} \right)^{-2}.
\end{aligned} \tag{2.91}$$

Finally, we can take the forward NMRK limit of, e.g. $qg \rightarrow qgg$ scattering to obtain the LO $q \rightarrow qg$ impact factor [97]

$$\begin{aligned}
M_{qg \rightarrow qgg} \xrightarrow{\text{NMRK}} & 2s \left[g_S^2 F^{q \rightarrow qg\bar{q}(0)}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) \left((T^{a_3} T^{a_{t_2}})_{i_3 i_2} A_{\lambda_4}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) \right. \right. \\
& \left. \left. + (T^{a_{t_2}} T^{a_3})_{i_3 i_2} B_{\lambda_4}(p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}) \right) \right] \\
& \times \left(\frac{1}{-|q_{2\perp}|^2} \right) [i g_S f^{a_4 a_{t_2} a_1} C^{g \rightarrow g(0)}(p_1^{\lambda_1}; p_4^{\lambda_4})] .
\end{aligned} \tag{2.92}$$

where the overall factor is given by

$$\begin{aligned} F^{q \rightarrow qg\bar{q}(0)}(p_2^\ominus, p_3^\oplus, p_4^\oplus) &= \frac{1}{\sqrt{2}} \left(1 + \frac{p_4^+}{p_3^+}\right)^{-\frac{1}{2}}, \\ F^{q \rightarrow qg\bar{q}(0)}(p_2^\oplus, p_3^\ominus, p_4^\oplus) &= \frac{1}{\sqrt{2}} \left(1 + \frac{p_4^+}{p_3^+}\right)^{-\frac{3}{2}}, \end{aligned} \quad (2.93)$$

and where the function B_λ is related to A_λ

$$A_\lambda(p_2, p_3, p_4) + A_\lambda(p_2, p_4, p_3) + B_\lambda(p_2, p_3, p_4) = 0, \quad (2.94)$$

which is the NMRK limit of a photon decoupling identity eq. (1.107).

A key property of the NLL impact factors in eqs. (2.85), (2.89) and (2.92) is their further factorisation in the MRK limit, where we further require $y_3 \gg y_4$. We find that in this limit, the $g \rightarrow gg$ impact factor factorises into a $g \rightarrow g$ impact factor times a Lipatov vertex,

$$\begin{aligned} &\sum_{\sigma \in S_2} f^{a_2 a_x a_{\sigma_3}} f^{a_x a_{t_2} a_{\sigma_4}} C^{g \rightarrow gg(0)}(p_2^{\lambda_2}, p_{\sigma_3}^{\lambda_{\sigma_3}}, p_{\sigma_4}^{\lambda_{\sigma_4}}) \\ &\xrightarrow{\text{MRK}} [ig_s f^{a_2 a_x a_3} C^{g \rightarrow g(0)}(p_2^{\lambda_2}, p_3^{\lambda_3})] \frac{1}{|q_{1\perp}|^2} [ig_s f^{a_x a_{t_2} a_4} V^{g(0)}(q_1, p_4^{\lambda_4}, q_2)], \end{aligned} \quad (2.95)$$

which reproduces the $n = 5$ case of eq. (2.39) as expected. Similarly, the MRK limit of the $q \rightarrow qg$ yields a $q \rightarrow q$ impact factor times a Lipatov vertex. By contrast, the $g \rightarrow q\bar{q}$ impact factor is power suppressed in MRK, which is also consistent with eq. (2.39).

To complete the catalogue of expressions required to obtain cross sections at NLL accuracy, we need expressions for the emission of two gluons or a quark-antiquark pair in a central NMRK limit. To obtain these expressions we can take the central NMRK limit of certain six-parton amplitudes. Our labelling follows the natural generalisation of the five-parton case above, i.e. we consider the production of four gluons of momenta p_i , with $i = 3, 4, 5, 6$ in the scattering of two partons of momenta p_1 and p_2 . We are now interested in the central NMRK region, detailed further in appendix A.1, where the rapidities and transverse momenta of the outgoing partons satisfy

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|. \quad (2.96)$$

In this limit, the relevant t -channel momenta are $q_1 = -(p_2 + p_3)$ and $q_3 = p_1 + p_6$,

with $s_{23} = t_1 = q_1^2 \simeq -|q_{1\perp}|^2$, $s_{61} = t_3 = q_3^2 \simeq -|q_{3\perp}|^2$.

The tree level $gg \rightarrow gggg$ amplitude factorises as expected in the central NMRK limit [97, 117–119],

$$\begin{aligned} \mathcal{M}_{gg \rightarrow gggg} \xrightarrow{\text{NMRK}} & 2s \left[ig_s f^{a_2 a_t a_3} C^{g \rightarrow g(0)}(p_2^{\lambda_2}; p_3^{\lambda_3}) \right] \left(\frac{1}{-|q_{1\perp}|^2} \right) \\ & \times \left[(ig_s)^2 \sum_{\sigma \in S_2} f^{a_{t_1} a_x a_{\sigma_4}} f^{a_x a_{t_3} a_{\sigma_5}} A^{gg(0)}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) \right] \\ & \times \left(\frac{1}{-|q_{3\perp}|^2} \right) \left[ig_s f^{a_6 a_{t_3} a_1} C^{g \rightarrow g(0)}(p_1^{\lambda_1}; p_6^{\lambda_6}) \right], \end{aligned} \quad (2.97)$$

where the sum is over the two permutations of the gluon labels 4 and 5. Here $A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3)$ is the tree-level two-gluon central-emission vertex (CEV). The determination of this quantity at one loop is the motivation of chapter 3.

There are four helicity configurations associated to the central-emission vertex for the emission of two gluons, two of which read [97, 117–119]

$$\begin{aligned} A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) &= \frac{q_{1\perp}^* q_{3\perp}}{p_{4\perp}} \sqrt{\frac{x_4}{x_5}} \frac{1}{\langle 45 \rangle}, \quad (2.98) \\ A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_1 &= -\frac{p_{4\perp}^*}{p_{4\perp}} \left\{ -\frac{1}{s_{45}} \left[\frac{y_5 p_{5\perp} |q_{1\perp}|^2}{p_{5\perp}^*} + \frac{x_4 p_{4\perp} |q_{3\perp}|^2}{p_{4\perp}^*} + \frac{s_{234} p_{4\perp} p_{5\perp}}{p_4^- p_5^+} \right] \right. \\ & \quad \left. + \frac{(q_{3\perp} + p_{5\perp})^2}{s_{234}} - \frac{q_{3\perp} + p_{5\perp}}{s_{45}} \left[\frac{p_{4\perp}}{y_4} - \frac{p_{5\perp}}{x_5} \right] \right\}, \quad (2.99) \end{aligned}$$

where the kinematic invariants are given in eqs. (A.3), (A.5) and (A.8), and where we use the momentum fractions,

$$x_i = \frac{p_i^+}{p_4^+ + p_5^+}, \quad y_i = \frac{p_i^-}{p_4^- + p_5^-}, \quad i = 4, 5. \quad (2.100)$$

In eqs. (2.98) and (2.99), similarly to eq. (2.8), we have encircled the helicity index and we have appended the index $|_1$ to the opposite-helicity vertex (eq. (2.99)) in order to stress that this is the first amongst several equivalent representations we are going to consider for this quantity. The CEVs for the remaining two helicity configurations, $\eta_4 = \eta_5 = \ominus$ and $\eta_4 = -\eta_5 = \ominus$, can be obtained from eqs. (2.98) and (2.99) respectively by complex conjugation.

Note that the impact factors, eq. (2.9), and the central-emission vertex, eqs. (2.98)

and (2.99), transform under parity into their complex conjugates,

$$[A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3)]^* = A^{gg(0)}(q_1, p_4^{-\nu_4}, p_5^{-\nu_5}, q_3), \quad (2.101)$$

If the further MRK limit is taken, i.e. further requiring $y_4 \gg y_5$, the two-gluon CEV factorises into a product of two Lipatov vertices times an intermediate t -channel pole. The MRK limit is discussed in appendix A.4.1.

In ref. [120] an alternative form of the opposite-helicity two-gluon central-emission vertex was found by using the CSW rules [121],

$$\begin{aligned} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_2 = & \sqrt{\frac{x_4}{x_5}} \frac{|q_{1\perp}|^2 x_4 p_{5\perp}^3}{\langle 45 \rangle (x_5 |p_{4\perp}|^2 + x_4 |p_{5\perp}|^2) p_{4\perp} (p_{4\perp} + p_{5\perp})} \\ & + \sqrt{\frac{x_4}{x_5}} \frac{q_{1\perp}^* x_4 (q_{3\perp} + p_{5\perp})^3}{\left(q_{3\perp} \sqrt{\frac{x_4}{x_5}} - \langle 45 \rangle\right) (x_5 |p_{4\perp}|^2 + x_4 |q_{2\perp}|^2) p_{4\perp}} \\ & + \frac{q_{1\perp} q_{3\perp} \sqrt{x_4 x_5} x_5}{\langle 45 \rangle (x_5 p_{4\perp} - x_4 q_{2\perp})} - \sqrt{\frac{x_4}{x_5}} \frac{q_{1\perp}^* q_{3\perp} x_4}{(p_{4\perp} + p_{5\perp}) [45]}. \end{aligned} \quad (2.102)$$

We have appended the index $|_2$ to emphasise that this is the second representation we have encountered for this expression. Equation (2.102) is a compact form of the vertex, consisting of only four rational terms, but unlike eq. (2.99), this representation appears to have unphysical poles, e.g. at $p_{4\perp} + p_{5\perp} = 0$. This is a feature which is shared with the new representations we will obtain from the BCFW recursion relations in section 3.1.1.

In the central NMRK limit, the colour-ordered amplitude of the ordering $\{1, 4, 2, 3, 5, 6\}$ is leading, and factorises as

$$M_{6g}^{(0)}(p_1^{\nu_1}, p_4^{\nu_4}, p_2^{\nu_2}, p_3^{\nu_3}, p_5^{\nu_5}, p_6^{\nu_6}) = s C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3}) \frac{1}{t_1} B^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{\nu_6}). \quad (2.103)$$

However, analogous to eq. (2.94), the tree-level CEV for this colour ordering is not independent to that of the orderings $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 5, 4, 6\}$ due to the relationship [119],

$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) + A^{gg(0)}(q_1, p_5^{\nu_5}, p_4^{\nu_4}, q_3) + B^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) = 0. \quad (2.104)$$

This relation similarly follows from a photon decoupling identity [46]. In chapter 3

we will see that these three colour orderings are no longer independent, and the implications of this fact will be a central focus of that chapter. As such, it will be useful to introduce the shorthand notation

$$\sigma_A = \{1, 2, 3, 4, 5, 6\}, \quad \sigma_{A'} = \{1, 2, 3, 5, 4, 6\}, \quad \sigma_B = \{1, 4, 2, 3, 5, 6\}. \quad (2.105)$$

Using eqs. (2.98) and (2.104) we find

$$B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) = \frac{q_{1\perp}^* q_{3\perp}}{p_{4\perp} p_{5\perp}}. \quad (2.106)$$

We will derive a new compact form for $B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)$ in section 3.1.1.

Finally, we need the analogous quark-antiquark CEV, [97, 118]

$$\left[g_s^2 (T^{a_{t_3}} T^{a_{t_1}})_{i_4 i_5} A^{\bar{q}q(0)}(q_1, p_4^{\lambda_4}, p_5^{\lambda_5}, q_3) + g_s^2 (T^{a_{t_1}} T^{a_{t_3}})_{i_4 i_5} A^{\bar{q}q(0)}(q_1, p_5^{\lambda_5}, p_4^{\lambda_4}, q_3) \right]. \quad (2.107)$$

Unlike the two-gluon vertex, charge conservation means we have just one distinct helicity configuration for the quark-antiquark CEV,

$$\begin{aligned} A^{\bar{q}q(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) = & -\sqrt{\frac{p_4^+}{p_5^+}} \left(\frac{x_5 |q_{3\perp}|^2}{s_{45}} + \frac{p_{5\perp}}{p_{4\perp}} \frac{y_5 |q_{3\perp}|^2}{s_{45}} + \frac{p_5^+}{p_4^+} \frac{p_{4\perp}^*}{q_{2\perp}^*} \right. \\ & \left. + \frac{q_{2\perp} (p_4^- p_5^+ - p_{4\perp}^* p_{5\perp} - q_{2\perp}^* p_{5\perp})}{p_{4\perp} s_{45}} - \frac{|p_{5\perp}|^2}{s_{45}} \right). \end{aligned} \quad (2.108)$$

We will now summarise the many factorised expressions introduced in this section. While high-energy factorisation at amplitude-level is of interest in its own right, we will list the factorised pieces by how they enter into the α_s correction to the BFKL kernel, eq. (2.51):

- The two-loop correction to the Regge trajectory, eqs. (2.62) and (2.63),
- The one-loop correction to the CEV of one gluon, eqs. (2.76), (2.77), (2.82) and (2.83),
- The tree-level CEV of two partons eqs. (2.97)–(2.99), (2.107) and (2.108),

or the α_s correction to the BFKL impact factors, eq. (2.48):

- The one-loop correction to the one-parton impact factors, eqs. (2.65), (2.66) and (2.70)–(2.72),

- The tree-level impact factors of two partons, eqs. (2.85)–(2.94).

Unfortunately the NLL corrections to the dijet cross section were found to be large [79]. It is a highly interesting question whether the extension of the framework to NNLL accuracy is possible, and, if so, whether the NNLL predictions stabilise the expansion in $\log(s/|t|)$. An overview of the current status of the programme is the subject of the next section. Before moving on to NNLL accuracy, we will briefly review a remarkable feature of gluon amplitudes in $\mathcal{N} = 4$ SYM, namely that they can be described exactly as a sum of the one-loop factorised expressions we derived in this section. We will make use of this feature in chapter 3.

2.3 The exact factorisation of one-loop amplitudes in $\mathcal{N} = 4$ SYM

In section 2.1.2 we found that the Regge limit of the dispersive part of the one-loop four-gluon amplitude in $\mathcal{N} = 4$ SYM was given by

$$\text{Re}[V_{g4}(1, 2, 3, 4)] \xrightarrow{\text{Regge}} c_{\mathcal{N}=4}^{g \rightarrow g(1)}(|q_\perp|^2) + \alpha^{(1)}(|q_\perp|^2) \log\left(\frac{p_3^+ p_4^-}{\tau}\right) + c_{\mathcal{N}=4}^{g \rightarrow g(1)}(|q_\perp|^2). \quad (2.109)$$

Using exact Mandelstam variables in the RHS of eq. (2.109) (i.e. using $s = s_{34}$ and $-t = -s_{23}$) rather than the MRK limit of these variables ($p_3^+ p_4^-$ and $|q_\perp|^2$ respectively), we are able to write $\text{Re}[V_{g4}]$ in general kinematics *exactly*:

$$\text{Re}[V_{g4}(1, 2, 3, 4)] = c_{\mathcal{N}=4}^{g \rightarrow g(1)}(-s_{23}) + \alpha^{(1)}(-s_{23}) \log\left(\frac{s_{34}}{\tau}\right) + c_{\mathcal{N}=4}^{g \rightarrow g(1)}(-s_{23}). \quad (2.110)$$

An analogous decomposition holds in the Euclidean region for the entire amplitude (not just its real part) [122], but we have specified to the real part of the physical Minkowski region in preparation for the investigation we will perform in chapter 3. As an aside, we note that this means V_{g4} can be written exactly in a ‘dual’ form with $s_{23} \leftrightarrow s_{34}$.

Similar to eq. (2.110), we can write the one-loop five gluon amplitude in $\mathcal{N} = 4$ SYM in terms of the one-loop factorised functions derived in the MRK limit, by

taking their arguments as follows [122]:

$$\begin{aligned}
\text{Re}[V_{g5}(1, 2, 3, 4, 5)] &= c_{\mathcal{N}=4}^{g \rightarrow g(1)}(-s_{23}) + \alpha^{(1)}(-s_{23}) \log\left(\frac{s_{34}}{\tau}\right) \\
&\quad + v_{\mathcal{N}=4}^{g(1)}(-s_{23}, \frac{s_{34}s_{45}}{s_{345}}, -s_{51}) \\
&\quad + \alpha^{(1)}(-s_{51}) \log\left(\frac{s_{45}}{\tau}\right) + c_{\mathcal{N}=4}^{g \rightarrow g(1)}(-s_{51}).
\end{aligned} \tag{2.111}$$

Given that the one-loop amplitudes obey this exact factorisation, it is natural to wonder whether any tree level amplitudes admit an exact factorisation. In chapter 4, we will see that this is indeed the case, and this observation will lead to many of the advantages of the HEJ framework.

Equation (2.110) and eq. (2.111) are also highly suggestive of the form gluonic scattering amplitudes may take at higher multiplicity and higher loop level. In chapter 3 will see that an analogous result holds for one-loop six-gluon amplitudes in $\mathcal{N} = 4$ SYM [122], a fact that will greatly aid us in the organisation of the calculations performed in that chapter.

2.4 Building blocks of the NNLL BFKL kernel

Extending the BFKL equation to next-to-next-to-leading logarithmic (NNLL) accuracy faces many challenges³. The most dramatic issue is the fact that, starting at this logarithmic order, the signature-odd component of $2 \rightarrow 2$ amplitudes is not governed by a single Reggeized gluon. Instead, it also involves contributions from three Reggeon exchange [92, 126–130] and the mixing of the latter with a single Reggeon. These give rise to a Regge cut as well as a Regge pole in the complex angular momentum plane. This makes the determination of the three-loop Regge trajectory, characterizing the Regge-pole term, a subtle problem, as it requires the disentanglement of the Regge pole and cut contributions, see e.g. [127, 129, 131]. Recently this problem has been resolved [85, 92] and the three-loop Regge trajectory in QCD has been determined [84–86]. The universality of the three-loop Regge trajectory has yet to be verified against the full three-loop $qg \rightarrow qg$ amplitude, but if this highly non-trivial check is passed, there is hope that the analytic structure of amplitudes in the Regge limit can be understood beyond NLL accuracy also for the multi-

³The BFKL equation has been extended to NNLL [123, 124] and to N³LL accuracy [125] in planar $\mathcal{N} = 4$ SYM theory.

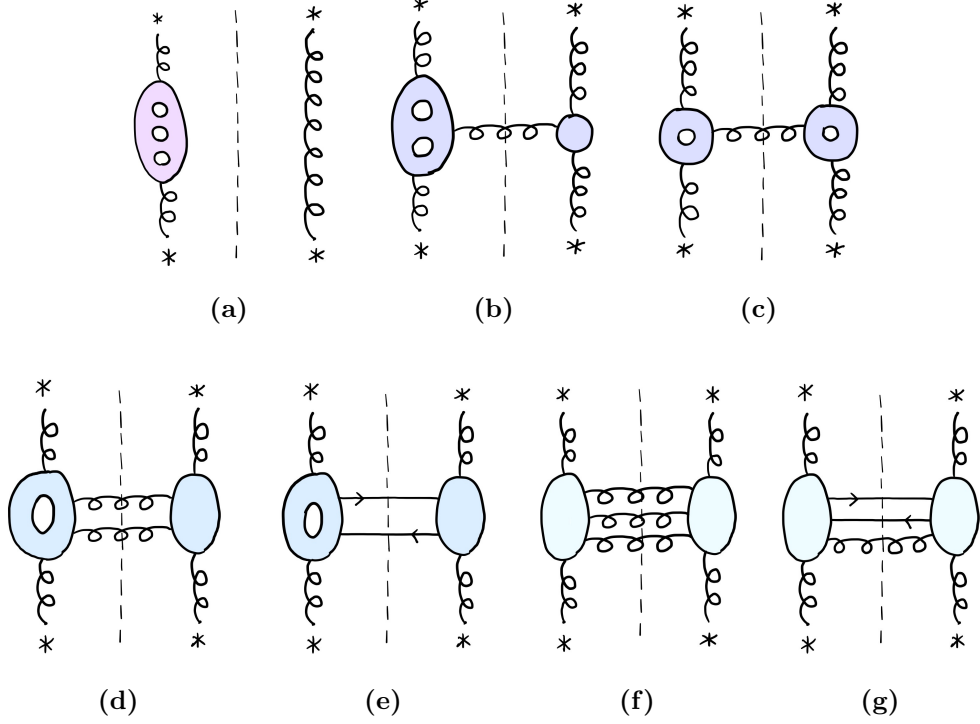


Figure 2.11 *Partial interference diagrams which contain the building blocks required for constructing the BFKL kernel at NNLL accuracy: (a) gluon Regge trajectory at three loops; (b) one-gluon CEV at two loops; (c) self-interference of one-gluon CEV at one loop; (d)–(e) two-parton CEVs at one loop; (f)–(g) three-parton CEVs at tree level.*

parton case, and indeed that the BFKL programme could eventually be extended to NNLL.

If the BFKL programme can be extended to NNLL accuracy, the kernel requires many ingredients. We list the current status of these building blocks, which are illustrated in fig. 2.11:

- The Regge trajectory at three loops, fig. 2.11a: As mentioned above, this has recently been determined from the three-loop $gg \rightarrow gg$ [84–86] amplitude, but the universality of this quantity must be checked against the three-loop $gq \rightarrow gq$ or $q_1 q_2 \rightarrow q_1 q_2$ amplitude.
- The two-loop correction to the CEV of one gluon, fig. 2.11b: This quantity is still to be computed, and may be obtained from the forthcoming two-loop five-parton amplitudes in QCD. The two-loop correction to the single-gluon CEV is known in planar $\mathcal{N} = 4$ SYM theory [132].

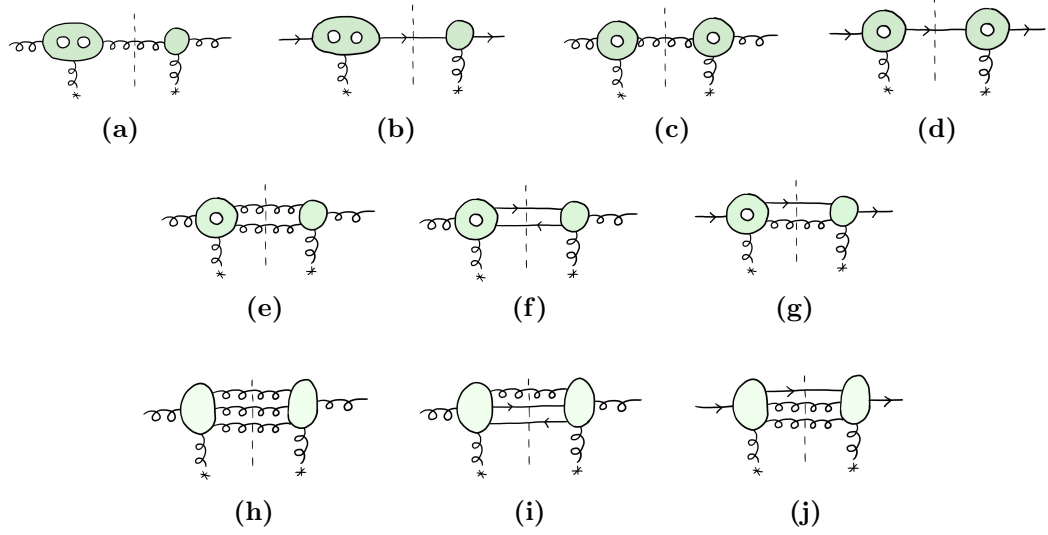


Figure 2.12 *Partial interference diagrams which contain the building blocks required for constructing the BFKL squared impact factors at NLL accuracy: (a)–(b) single-parton impact factors at two loops; self-interference of the one-loop single-parton impact factors; (e)–(g) two-parton impact factors at two loops; (h)–(j) three-parton impact factors at tree level.*

- The square of the one-loop correction to the CEV of one gluon, fig. 2.11c: This is known in planar $\mathcal{N} = 4$ SYM theory to all orders in ϵ [132]. Beyond $\mathcal{O}(\epsilon^0)$, the one-loop single-gluon CEV requires the one-loop pentagon in $6 - 2\epsilon$ dimensions [133–136]
- The one-loop correction to the CEV of two gluons, fig. 2.11d: This expression is not yet known. The extraction of the $\mathcal{N} = 4$ contribution to this expression is the object of chapter 3 and work is in progress to obtain this vertex in full QCD.
- The one-loop correction to the CEV of a quark-antiquark pair, fig. 2.11e: This expression is not yet known.
- The tree-level CEV of three gluons, fig. 2.11f: This piece has been known for a long time [46, 120, 137].
- The tree-level CEV of a quark-antiquark pair and one gluon, fig. 2.11g: This object is not known, but could be easily obtained from the known seven-parton tree-level amplitudes [138].

Let us also note that in order to compute jet cross sections at NNLL accuracy

through the BFKL framework, α_s^2 corrections to the squared-impact factors, eq. (2.48), will be needed. This requires the following building blocks:

- The two-loop correction to the single-parton impact factors, fig. 2.12a and fig. 2.12b: These are known [127].
- The square of the one-loop impact factors, fig. 2.12c and fig. 2.12d: As for the kernel above, the finite terms are known, but higher orders in ϵ are required. Also, the square of the one-loop helicity-violating impact factor [100] contributes to this logarithmic accuracy.
- The one-loop correction to the two-parton impact factors, figs. 2.12e–2.12g: The two-gluon impact factor has been obtained [83] but an analogous exercise is required for the $g \rightarrow q\bar{q}$ and $q \rightarrow qg$ impact factors.
- The tree-level three-parton impact factors, figs. 2.12h–2.12j: These are known [46].

The above list represents some great theoretical achievements, but there is clearly much work still to be done. In the following chapter we present the determination of the one-loop two-gluon CEV in $\mathcal{N} = 4$ SYM [4], which is a component of the same quantity in QCD.

Chapter 3

One-loop CEV for two gluons in $\mathcal{N} = 4$ SYM

In chapter 2 we reviewed the BFKL framework for resumming high-energy logarithms at LL and NLL accuracy. In section 2.4 we gave an overview of the status of this program at NNLL accuracy. One of the missing ingredients is the one-loop correction to the central-emission vertex of two gluons. In this chapter, we compute this CEV in $\mathcal{N} = 4$ SYM. Utilizing the supersymmetric decomposition of QCD loop amplitudes reviewed in section 1.7 [54, 56, 57], the $\mathcal{N} = 4$ result is a first step toward the determination of the same vertex in QCD. To this end we use the one-loop six-gluon amplitudes in $\mathcal{N} = 4$ SYM based on refs. [56, 57]. By considering the appropriate NMRK limit of these amplitudes and comparing the result with a Regge-factorized form, we extract the two-gluon central-emission vertex in both the same helicity and opposite helicity configurations, i.e. the one loop corrections to eq. (2.99) and eq. (3.7).

An important motivation for our investigation, besides the extension of the BFKL program, stems from the potential of the NMRK limit to unravel the analytic structure (as well as the colour structure) of multi-leg amplitudes. Understanding the behaviour of amplitudes in various limits, where drastic simplifications occur, is a key strategy towards understanding them in general kinematics. In this section we therefore emphasise aspects of the structure of amplitudes in the NMRK limit which might find utility outwith this limit.

The structure of this chapter is as follows. In section 3.1 we revisit the tree-level two-gluon central-emission vertex and we obtain new representations of this

vertex. In section 3.2 we take the central NMRK limit of the one-loop six-gluon amplitude in $\mathcal{N} = 4$ SYM. We find that the dispersive part of the amplitude factorises in terms of a one-loop two-gluon CEV. We obtain this CEV for the case of same-helicity gluons in section 3.2.1 and for the case of opposite-helicity gluons in section 3.2.2. In section 3.2.3 we consider the NLO $gg \rightarrow gggg$ squared matrix element and show that several simplifications occur in this context. We discuss the implications of these results for the NNLL BFKL kernel. In section 3.3 we speculate how the behaviour we have observed for six-gluon one-loop amplitudes may be extended to higher multiplicity and loop order. Finally, we conclude in section 3.4.

To avoid interrupting the flow of the main derivation, several sections have been included as appendices. In appendix A.1 we provide details on the NMRK region of interest; in appendix A.2 we list the tree-level and one-loop six-gluon NMHV amplitudes in $\mathcal{N} = 4$ SYM in general kinematics; in appendix A.3 we display the absorptive part of the one-loop six-gluon amplitude in $\mathcal{N} = 4$ SYM in the NMRK limit; in appendix A.4 we compute the MRK, soft and collinear limits of the one-loop two-gluon CEV, demonstrating consistency with known results in these further limits.

3.1 Revisiting the two-gluon CEV at tree level

The purpose of this section is to prepare the ground for obtaining the two-gluon CEV at one loop. In section 3.1.1 we obtain new representations for the opposite-helicity CEV from amplitudes that were derived via Britto-Cachazo-Feng-Witten (BCFW) recursion relations [52, 53]. Not only do these representations make clear the collinear structure of the tree-level CEV, the individual rational terms that comprise the BCFW amplitudes appear also in the one-loop six-gluon NMHV amplitudes of $\mathcal{N} = 4$ SYM. The study of these rational terms therefore also lays the groundwork for section 3.2, where we obtain the one-loop two-gluon CEV in $\mathcal{N} = 4$ SYM. In section 3.1.2 we introduce a new set of kinematic variables which, in particular, greatly simplifies the task of showing equivalence between the different representations of the CEV. Finally in section 3.1.3 we discuss some symmetries of the tree-level amplitude which will inform our organisation of the one-loop calculation.

3.1.1 BCFW representations

In ref. [97], the opposite-helicity vertex of eq. (2.99) was obtained from the tree-level six-gluon amplitude given in ref. [139]. Compact expressions for NMHV six-gluon amplitudes were later derived using the BCFW on-shell recursion relations [52, 53]. These BCFW-based representations for the tree amplitude conveniently relate to the rational functions appearing at one loop, giving us an extra incentive to explore them here. We present one of the amplitudes obtained in ref. [52] here, which we will use as an example in the following¹:

$$\begin{aligned}
M_{(\text{eq. (3.1)})}^{(0)}(p_1^\oplus, p_2^\oplus, p_3^\ominus, p_4^\oplus, p_5^\ominus, p_6^\ominus) = & - \frac{\langle 3|1+2|4]^4}{\langle 1\,2\rangle\langle 2\,3\rangle[4\,5][5\,6]s_{123}\langle 1|2+3|4\rangle\langle 3|4+5|6\rangle} \\
& - \frac{\langle 5\,6\rangle^4[2\,4]^4}{\langle 5\,6\rangle\langle 6\,1\rangle[2\,3][3\,4]s_{234}\langle 5|6+1|2\rangle\langle 1|2+3|4\rangle} \\
& - \frac{\langle 3\,5\rangle^4[1\,2]^4}{[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle s_{345}\langle 3|4+5|6\rangle\langle 5|6+1|2\rangle}.
\end{aligned} \tag{3.1}$$

From eq. (3.1), we see that the BCFW-based representations have the additional advantage of making the collinear singularity structure of the tree amplitudes manifest. For example, eq. (3.1) has separate poles at $\langle 4\,5\rangle = 0$ and $[45] = 0$, while eq. (2.99) only displays a pole at $s_{45} = 0$. The form in eq. (2.99) entangles the two types of contributions, from different intermediate helicity states; also, individual terms are more singular than the actual amplitude. The BCFW-based representations are therefore more amenable to the study of collinear limits, a property which is utilised in appendix A.4.3. In addition to physical singularities (spinor products of two cyclically adjacent momenta, and Mandelstam invariants of three cyclically adjacent momenta), the individual terms in these BCFW-based representations also contain unphysical singularities (which cancel in the sum of terms appearing in the amplitude). The unphysical singularities of eq. (3.1) arise

¹This amplitude is equal to the complex conjugate of eq. (A.10) and we have expanded the D_i functions to display the physical and spurious poles of the amplitude.

from the following spinor strings,

$$\begin{aligned}
\langle 1|2+3|4\rangle &\xrightarrow{\text{NMRK}} -i|q_{3\perp}| \sqrt{\frac{p_4^+}{p_6^+}} (q_{2\perp}^* + p_{4\perp}^* \frac{p_5^+}{p_4^+}), \\
\langle 3|4+5|6\rangle &\xrightarrow{\text{NMRK}} -q_{3\perp}^* \sqrt{\frac{p_3^+}{p_6^+}} (p_{4\perp} + p_{5\perp}), \\
\langle 5|6+1|2\rangle &\xrightarrow{\text{NMRK}} -ip_{5\perp} \sqrt{\frac{p_3^+}{p_5^+}} (q_{2\perp}^* + p_{5\perp}^* \frac{p_4^-}{p_5^-}).
\end{aligned} \tag{3.2}$$

On the right hand side, we see that these terms lead to unphysical singularities that survive into the central NMRK limit. This is in contrast to the further MRK limit of the same spinor strings, which no longer give rise to unphysical singularities.

For six-particle kinematics there is a simple relationship between the square of such spinor strings and two- and three-particle Mandelstam invariants, e.g.

$$\begin{aligned}
\langle 1|2+3|4\rangle \langle 4|2+3|1\rangle &= s_{123}s_{345} \left(1 - \frac{s_{12}s_{45}}{s_{123}s_{345}} \right), \\
\langle 3|4+5|6\rangle \langle 6|4+5|3\rangle &= s_{123}s_{345} \left(1 - \frac{s_{12}s_{45}}{s_{123}s_{345}} \right), \\
\langle 5|6+1|2\rangle \langle 2|6+1|5\rangle &= s_{123}s_{345} \left(1 - \frac{s_{12}s_{45}}{s_{123}s_{345}} \right).
\end{aligned} \tag{3.3}$$

The ratios of Mandelstam invariants that appear on the right-hand side are conformally-invariant cross ratios [140], which we denote

$$\begin{aligned}
u_A &= \frac{s_{12}s_{45}}{s_{123}s_{345}} \xrightarrow{\text{NMRK}} \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)}, \\
v_A &= \frac{s_{23}s_{56}}{s_{234}s_{123}} \xrightarrow{\text{NMRK}} \frac{(-t_1)p_5^+}{(|q_{2\perp}|^2 + p_4^- p_5^+)(p_4^+ + p_5^+)}, \\
w_A &= \frac{s_{34}s_{61}}{s_{345}s_{234}} \xrightarrow{\text{NMRK}} \frac{(-t_3)p_4^-}{(p_4^- + p_5^-)(|q_{2\perp}|^2 + p_4^- p_5^+)},
\end{aligned} \tag{3.4}$$

where the A subscript refers to the ordering σ_A of eq. (2.105). Via the relations listed in eq. (3.3) we see that each of the unphysical singularity surfaces of the rational terms in eq. (3.1) correspond to one of the cross ratios equalling unity. We therefore choose to label the unphysical singularities of the BCFW representations by the corresponding cross-ratio that vanishes at that unphysical

singularity surface,

$$\begin{aligned}
P_v^A &= q_{2\perp}^* - p_{4\perp} \frac{p_5^+}{p_4^+}, \\
P_u^A &= p_{4\perp} + p_{5\perp}, \\
P_w^A &= q_{2\perp}^* + p_{5\perp}^* \frac{p_4^-}{p_5^-}.
\end{aligned} \tag{3.5}$$

such that

$$P_x^A = 0 \quad \Rightarrow \quad x_A = 1, \quad \forall x \in \{u, v, w\}. \tag{3.6}$$

Now we can return to eq. (3.1) to obtain the following representation of the tree-level opposite-helicity central-emission vertex,

$$A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_3 = R_{uv}^A + R_{vw}^A + R_{wu}^A, \tag{3.7}$$

with

$$\begin{aligned}
R_{uv}^A &= \frac{-|q_{3\perp}|^2 q_{1\perp}^* (p_4^+)^2}{(p_4^+ + p_5^+) \langle 4 \ 5 \rangle^* P_u^A P_v^A}, \\
R_{vw}^A &= \frac{(p_{4\perp}^*)^3 q_{3\perp} q_{1\perp} p_5^+}{(|q_{2\perp}|^2 + p_4^- p_5^+) P_v^A P_w^A}, \\
R_{wu}^A &= \frac{-(p_{5\perp})^4 q_{3\perp}^* |q_{1\perp}|^2 p_{4+}}{p_{4\perp} (p_5^+)^2 (p_4^- + p_5^-) \langle 4 \ 5 \rangle P_w^A P_u^A}.
\end{aligned} \tag{3.8}$$

We have appended a $|_3$ to eq. (3.7) to emphasise that this is our third representation for the opposite helicity vertex. We will show that this representation is equivalent to eq. (2.99) below. In addition to the physical singularities of eq. (A.3) and eq. (A.5), individual terms in eq. (3.7) feature the unphysical singularities of eq. (3.2). We denote the rational terms in eq. (3.8) by their singularities, where R_{xy}^A features unphysical poles at $P_x^A = 0$ and $P_y^A = 0$.

We can obtain another equivalent representation for the opposite-helicity vertex from eq. (3.1) by taking the central NMRK limit of

$$M_{(\text{eq. (3.1)})}^{(0)}(p_3^\oplus, p_4^\ominus, p_5^\oplus, p_6^\oplus, p_1^\ominus, p_2^\ominus)^*.$$

We note that here the helicities of gluons 2 and 3 differ to the helicity configuration of eq. (3.1), but this does not matter for the extraction of the CEV, as we divide by the gluon impact factors eq. (2.8). We are left with our fourth representation

of the CEV,

$$A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_4 = R_{\bar{u}\bar{v}}^A + R_{\bar{v}\bar{w}}^A + R_{\bar{w}\bar{u}}^A, \quad (3.9)$$

with

$$\begin{aligned} R_{\bar{u}\bar{v}}^A &= \frac{-|q_{3\perp}|^2 q_{1\perp} (p_5^+)^2}{(p_4^+ + p_5^+) \langle 4 \ 5 \rangle \bar{P}_u^A \bar{P}_v^A}, \\ R_{\bar{v}\bar{w}}^A &= \frac{(p_{5\perp} + q_{3\perp})^4 q_{3\perp}^* q_{1\perp}^* (p_4^+)^2}{p_{4\perp} p_5^+ (|q_{2\perp}|^2 + p_4^- p_5^+) \bar{P}_v^A \bar{P}_w^A}, \\ R_{\bar{w}\bar{u}}^A &= \frac{-(p_{4\perp}^*)^3 q_{3\perp} |q_{1\perp}|^2}{p_4^+ (p_4^- + p_5^-) \langle 4 \ 5 \rangle^* \bar{P}_w^A \bar{P}_u^A}. \end{aligned} \quad (3.10)$$

The notation $R_{\bar{x}\bar{y}}^A$ indicates these rational terms have unphysical singularities at $\bar{P}_x^A = 0$ and $\bar{P}_y^A = 0$, where we use a bar to denote complex conjugation.

Of course, the six rational terms in eq. (3.8) and eq. (3.10) are not independent due to the relation,

$$R_{\bar{u}\bar{v}}^A + R_{\bar{v}\bar{w}}^A + R_{\bar{w}\bar{u}}^A = R_{\bar{u}\bar{w}}^A + R_{\bar{v}\bar{u}}^A + R_{\bar{w}\bar{v}}^A. \quad (3.11)$$

These rational terms, when divided by the appropriate MHV central-emission vertex, correspond to the NMRK limit of the dual-conformal-invariant R -invariants of $\mathcal{N} = 4$ SYM [141]. In this thesis however, we do not normalise by the MHV vertex as this will avoid the introduction of the MHV phase, and will lead to simpler expressions when we take further kinematic limits of these rational terms.

In order to check that the above results are indeed independent of the helicities of the non-central gluons 1, 2, 3 and 6, we need a complete set of NMHV amplitudes. In addition to eq. (3.1), we need a BCFW representation of the NMHV amplitude for the ‘split’ helicity configuration, $(\ominus, \ominus, \ominus, \oplus, \oplus, \oplus)$, and the ‘alternating’ helicity configuration, $(\ominus, \oplus, \ominus, \oplus, \ominus, \oplus)$. Such a complete set of amplitudes was derived in ref. [52]. We list this set of representations using a compact notation in appendix A.2, in eqs. (A.9)–(A.11), where our illustrative example eq. (3.1) is equal to the complex conjugate of eq. (A.10). A second complete set of representations, eqs. (A.14)–(A.15), can be obtained from the one-loop NMHV amplitudes in $\mathcal{N} = 4$ SYM, as discussed further in appendix A.2. These two sets of representations are complementary in the sense that if an amplitude from one set of representations yields the vertex eq. (3.7) in the central NMRK limit, the

amplitude from the other set of representations yields the vertex eq. (3.9).

As discussed in section 2.2, the CEV for the $\sigma_{A'}$ ordering can be obtained from the CEV for the σ_A ordering by interchange of $p_4 \leftrightarrow p_5$, followed by complex conjugation. Following an analogous procedure to the above, we obtain the representations

$$A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) = R_{uv}^{A'} + R_{vw}^{A'} + R_{wu}^{A'}, \quad (3.12)$$

and

$$A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) = R_{\bar{u}\bar{v}}^{A'} + R_{\bar{v}\bar{w}}^{A'} + R_{\bar{w}\bar{u}}^{A'}, \quad (3.13)$$

where the rational terms of the σ_A and $\sigma_{A'}$ ordering are related by

$$R_{xy}^A \xleftrightarrow[p_4 \leftrightarrow p_5]{} \left(R_{\bar{x}\bar{y}}^{A'}\right)^*. \quad (3.14)$$

For the σ_B ordering, the BCFW representations of the amplitudes in ref. [52] have unphysical singularities which arise from the spinor strings

$$\begin{aligned} \langle 1|4 + 2|3 \rangle &\xrightarrow{\text{NMRK}} -i|q_{3\perp}| \sqrt{\frac{p_3^+}{p_6^+}} q_{2\perp}^*, \\ \langle 2|3 + 5|6 \rangle &\xrightarrow{\text{NMRK}} i q_{3\perp}^* \sqrt{\frac{p_3^+}{p_6^+}} q'_{2\perp}, \\ \langle 5|6 + 1|4 \rangle &\xrightarrow{\text{NMRK}} -p_{5\perp} q_{2\perp}^* \sqrt{\frac{p_4^+}{p_5^+}} - p_{4\perp}^* q'_{2\perp} \sqrt{\frac{p_5^+}{p_4^+}}. \end{aligned} \quad (3.15)$$

With reference to the conformally-invariant cross ratios for this colour-ordering,

$$\begin{aligned} u_B &= \frac{s_{14}s_{35}}{s_{142}s_{235}} \xrightarrow{\text{NMRK}} \frac{p_4^+ p_5^-}{|q'_{2\perp}|^2 + p_4^+ p_5^-}, \\ v_B &= \frac{s_{42}s_{56}}{s_{423}s_{142}} \xrightarrow{\text{NMRK}} \frac{p_4^- p_5^+}{|q_{2\perp}|^2 + p_4^- p_5^+}, \\ w_B &= \frac{s_{23}s_{61}}{s_{235}s_{423}} \xrightarrow{\text{NMRK}} \frac{(-t_1)(-t_3)}{(|q_{2\perp}|^2 + p_4^- p_5^+)(|q'_{2\perp}|^2 + p_4^+ p_5^-)}, \end{aligned} \quad (3.16)$$

we identify the following unphysical singularity surfaces of the BCFW amplitudes,

in analogy to eq. (3.5):

$$\begin{aligned}
P_v^B &= p_{5\perp}^* + q_{3\perp}^* , \\
P_u^B &= p_{4\perp} + q_{3\perp} , \\
P_w^B &= \frac{p_{5\perp} q_{2\perp}^*}{p_5^+} + \frac{p_{4\perp} q_{2\perp}'}{p_4^+} .
\end{aligned} \tag{3.17}$$

Taking the NMRK limit of the amplitudes in appendix A.2.1 we obtain the representation

$$B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) = R_{uv}^B + R_{vw}^B + R_{wu}^B , \tag{3.18}$$

with rational terms

$$\begin{aligned}
R_{uv}^B &= \frac{|q_{1\perp}|^2 |q_{3\perp}|^2}{p_{4\perp} p_{5\perp}^* P_u^B P_v^B} , \\
R_{vw}^B &= \frac{(p_{4\perp}^*)^3 q_{1\perp} q_{3\perp} p_5^+}{p_4^+ (|q_{2\perp}|^2 p_4^+ + |p_{4\perp}|^2 p_5^+) P_v^B P_w^B} , \\
R_{wu}^B &= \frac{(p_{5\perp})^3 q_{1\perp}^* q_{3\perp}^* p_4^+}{p_5^+ (|q_{2\perp}'|^2 p_5^+ + |p_{5\perp}|^2 p_4^+) P_w^B P_u^B} ,
\end{aligned} \tag{3.19}$$

and the representation

$$B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) = R_{\bar{u}\bar{v}}^B + R_{\bar{v}\bar{w}}^B + R_{\bar{w}\bar{u}}^B , \tag{3.20}$$

with rational terms

$$\begin{aligned}
R_{\bar{u}\bar{v}}^B &= 0 , \\
R_{\bar{v}\bar{w}}^B &= \frac{(\bar{P}_v^B)^3 q_{1\perp}^* q_{3\perp}^*}{p_{4\perp} p_5^+ (|q_{2\perp}|^2 + p_4^- p_5^+) \bar{P}_u^B} , \\
R_{\bar{w}\bar{u}}^B &= \frac{(\bar{P}_u^B)^3 q_{1\perp} q_{3\perp}}{p_{5\perp} p_4^+ (|q_{2\perp}'|^2 + p_5^- p_4^+) \bar{P}_w^B} .
\end{aligned} \tag{3.21}$$

We note that under the transformation $p_4 \leftrightarrow p_5$ one obtains

$$u_B \xleftrightarrow[p_4 \leftrightarrow p_5]{} v_B , \quad w_B \xleftrightarrow[p_4 \leftrightarrow p_5]{} w_B , \tag{3.22}$$

and

$$\begin{aligned}
R_{uv}^B \xleftrightarrow{p_4 \leftrightarrow p_5} (R_{uv}^B)^*, & \quad R_{vw}^B \xleftrightarrow{p_4 \leftrightarrow p_5} (R_{wu}^B)^*, \\
R_{\bar{u}\bar{v}}^B \xleftrightarrow{p_4 \leftrightarrow p_5} (R_{\bar{u}\bar{v}}^B)^* = 0, & \quad R_{\bar{v}\bar{w}}^B \xleftrightarrow{p_4 \leftrightarrow p_5} (R_{\bar{w}\bar{u}}^B)^*.
\end{aligned} \tag{3.23}$$

These properties will be useful for organising the one-loop opposite-helicity amplitude in section 3.2.2.

Although each rational term in the above representations of the CEVs have unphysical poles, the amplitudes, and therefore the CEVs must be free from unphysical poles. In order to demonstrate that this is the case, it will be useful to first introduce a minimal set of kinematic variables, which in particular encode transverse momentum conservation. This is the goal of the following section.

3.1.2 A minimal set of variables

In MRK the central-emission vertex for the emission of one gluon can be expressed in terms of one independent transverse momentum, i.e. in terms of two real parameters. We expect that the emission of one more gluon out of the central-emission vertex will require the introduction of the variables of one more on-shell massless particle, i.e. three more real parameters, such that the central-emission vertex for the emission of n gluons will be expressed in terms of a set of $(3n - 1)$ independent real parameters. In particular, the central-emission vertex for the emission of two gluons will be given in terms of five real parameters.

We write $p_{4\perp}$ and $p_{5\perp}$ in terms of MRK t -channel transverse momenta,

$$p_{4\perp} = q_{1\perp} - q_{2\perp}, \quad p_{5\perp} = q_{2\perp} - q_{3\perp}. \tag{3.24}$$

In the sum of the equations above, $q_{2\perp}$ cancels such that transverse momentum conservation in NMRK is fulfilled. We parametrise the ratios of transverse momenta in eq. (3.24) in terms of two complex variables, w and z ,

$$\begin{aligned}
\frac{p_{4\perp}}{q_{1\perp}} &= \frac{1}{1 - z}, & \frac{q_{2\perp}}{q_{1\perp}} &= \frac{-z}{1 - z}, \\
\frac{p_{5\perp}}{q_{3\perp}} &= \frac{-1}{1 - w}, & \frac{q_{2\perp}}{q_{3\perp}} &= \frac{-w}{1 - w}.
\end{aligned} \tag{3.25}$$

Inverting these relations gives us

$$z = -\frac{q_{2\perp}}{p_{4\perp}}, \quad w = \frac{q_{2\perp}}{p_{5\perp}}. \quad (3.26)$$

Due to our complex parametrization of the transverse momenta (see section 1.6.2), z and w are also complex. We will use $\bar{z}$ and $\bar{w}$ to denote their complex conjugates.

We need one further real variable which depends on the rapidity difference between p_4 and p_5 . For this purpose we define the positive variable

$$X \equiv \frac{p_4^+}{p_5^+} = \frac{x_4}{x_5}. \quad (3.27)$$

In summary, we may parametrise the central-emission vertex for the emission of two gluons, involving five real degrees of freedom, in terms of the set $\{w, z, X\}$.

The MRK limit is reached by taking $X \rightarrow \infty$. When we use the acronym MRK, we will always mean this canonical MRK limit with the strict rapidity ordering $y_3 \gg y_4 \gg y_5 \gg y_6$. The multi-Regge regime with strict rapidity ordering $y_3 \gg y_5 \gg y_4 \gg y_6$ is also contained within the central NMRK limit eq. (A.1), and we will refer to this kinematic condition as MRK'. The MRK' limit is reached by taking $X \rightarrow 0$.

Before obtaining expressions for the tree-level CEV in terms of the minimal variables, let us list some relevant Lorenz invariant quantities in terms of these variables. Of particular importance to the central NMRK limit is the spinor product,

$$\langle 45 \rangle = -q_{1\perp} \frac{w + Xz}{w\sqrt{X}(z-1)}, \quad (3.28)$$

which is treated exactly in this limit, and from eq. (1.139) we obtain

$$s_{45} = |q_{1\perp}|^2 \frac{|w + Xz|^2}{X|w|^2|z-1|^2}. \quad (3.29)$$

The three-particle invariants of eq. (A.5) become

$$\begin{aligned}
s_{123} &= |q_{1\perp}|^2 \frac{p_4^+ (1+X) |w-1|^2 |z|^2}{p_6^+ X |z-1|^2 |w|^2}, \\
s_{234} &= -|q_{1\perp}|^2 \frac{(1+X|z|^2)}{X|z-1|^2}, \\
s_{235} &= -|q_{1\perp}|^2 \frac{(|w+z-wz|^2 + X|z|^2)}{|z-1|^2 |w|^2}, \\
s_{345} &= |q_{1\perp}|^2 \frac{p_3^+ (|w|^2 + X|z|^2)}{p_4^+ |z-1|^2 |w|^2}.
\end{aligned} \tag{3.30}$$

We note that these quantities contain lightcone-momentum dependence: These factors will cancel when considering physical quantities, such as amplitudes, in the central NMRK limit. For example, rewriting eq. (2.99) and eq. (2.98) in terms of the minimal set, we obtain

$$A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) = \frac{q_{1\perp}^*}{q_{1\perp}} \frac{z(w-1)(z-1)X}{(w+Xz)}, \tag{3.31}$$

$$\begin{aligned}
A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_1 &= \frac{\bar{w}^2 z^2 |z-1|^2 X^2}{|w+Xz|^2 (|w|^2 + X|z|^2)} + \frac{z^2 X}{(1+X|z|^2)} \\
&+ \frac{Xz}{|w+Xz|^2} \left(\frac{|w-1|^2 X \bar{z}}{(1+X)} + \bar{w}(1+X|z|^2) \right. \\
&\quad \left. - (|w|^2 + X|z|^2) - \bar{w}z(1+X) \right).
\end{aligned} \tag{3.32}$$

Recalling eq. (3.28), we see that the quantity $(w+Xz)$ that appears in the denominator of eq. (3.31) originates from the central NMRK behaviour of the spinor product $\langle 45 \rangle$. In a similar way, with reference to eq. (3.30), we can associate the remaining terms in the denominators of eq. (3.32) with the central NMRK behaviour of Mandelstam invariants,

$$\begin{aligned}
s_{123} = 0 &\leftrightarrow 1+X = 0, \\
s_{234} = 0 &\leftrightarrow 1+X|z|^2 = 0, \\
s_{345} = 0 &\leftrightarrow |w|^2 + X|z|^2 = 0.
\end{aligned} \tag{3.33}$$

However, we note that because $X > 0$, these singular surfaces cannot be reached in NMRK for finite transverse momenta. This makes intuitive sense: in the central NMRK limit we require, e.g. gluons 3 and 4 to be widely separated in rapidity, and therefore these gluons cannot become collinear.

To express the remaining representations of the opposite-helicity vertex, eqs. (2.102), (3.7) and (3.9), in terms of the minimal set, it will be useful to first list the spinor strings eq. (3.2) in terms of these variables:

$$\begin{aligned}
\langle 1|2+3|4\rangle &= -iq_{1\perp}^*|q_{1\perp}|\sqrt{\frac{p_4^+}{p_6^+}}\frac{|w-1||z|(1+X\bar{z})}{|z-1||w|X(\bar{z}-1)}, \\
\langle 3|4+5|6\rangle &= |q_{1\perp}|^2\sqrt{\frac{p_3^+}{p_6^+}}\frac{\bar{z}(\bar{w}-1)(w-z)}{|w|^2|z-1|^2}, \\
\langle 5|6+1|2\rangle &= -i|q_{1\perp}|^2\sqrt{\frac{p_3^+}{p_5^+}}\frac{(w+X|z|^2)}{wX|z-1|^2}.
\end{aligned} \tag{3.34}$$

The rational terms eq. (3.8) become

$$\begin{aligned}
R_{uv}^A &= \frac{X^3|w-1|^2(\bar{z}-1)|z|^2}{(X+1)(\bar{w}+X\bar{z})(w-z)(1+X\bar{z})}, \\
R_{vw}^A &= \frac{zX(w-1)(z-1)}{(1+X|z|^2)(1+X\bar{z})(w+X|z|^2)}, \\
R_{wu}^A &= -\frac{z^3X^3(\bar{w}-1)|z-1|^2|z|^2}{(w+Xz)(|w|^2+X|z|^2)(w+X|z|^2)(w-z)},
\end{aligned} \tag{3.35}$$

while the rational terms eq. (3.10) become

$$\begin{aligned}
R_{\bar{u}\bar{v}}^A &= \frac{X|w-1|^2(z-1)|z|^2}{(X+1)(w+Xz)(\bar{w}-\bar{z})(1+Xz)}, \\
R_{\bar{v}\bar{w}}^A &= \frac{X^3(\bar{w}-1)(\bar{z}-1)z^4\bar{z}}{(1+X|z|^2)(1+Xz)(\bar{w}+X|z|^2)}, \\
R_{\bar{w}\bar{u}}^A &= -\frac{Xz(w-1)\bar{w}^4|z-1|^2}{(\bar{w}+X\bar{z})(|w|^2+X|z|^2)(\bar{w}+X|z|^2)(\bar{w}-\bar{z})}.
\end{aligned} \tag{3.36}$$

In addition to the physical singularities listed in eq. (3.33), we see that, as expected, the rational terms in eq. (3.35) and eq. (3.36) have unphysical singularities arising from the central NMRK behaviour of the spinor strings. With reference to eq. (3.5) we associate

$$\begin{aligned}
P_v^A = 0 &\leftrightarrow 1+X\bar{z} = 0, \\
P_u^A = 0 &\leftrightarrow w-z = 0, \\
P_w^A = 0 &\leftrightarrow w+X|z|^2 = 0.
\end{aligned} \tag{3.37}$$

Returning to eq. (2.102), we find that in terms of the minimal variables, the

CSW-derived representation of the opposite helicity amplitude becomes

$$\begin{aligned}
A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_4 = & -\frac{\bar{w}z^3|1-z|^2X^2}{(w-z)(w+Xz)(|w|^2+X|z|^2)} + \frac{X^2z^3(1-\bar{z})}{(1+Xz)(1+X|z|^2)} \\
& + \frac{(1-w)(1-z)zX}{(1+X)(1+Xz)(w+Xz)} + \frac{(1-w)(1-\bar{z})\bar{w}zX^2}{(1+X)(w-z)(\bar{w}+X\bar{z})},
\end{aligned} \tag{3.38}$$

which has the expected physical singularities of eqs. (3.28) and (3.33), and also spurious poles at $P_{u_A} = 0$ and $\bar{P}_{v_A} = 0$.

Now that we have expressions for all four representations of the opposite-helicity vertex (eqs. (3.7), (3.9), (3.32) and (3.38)) in terms of the minimal set $\{w, z, X\}$, it is a relatively straightforward exercise to analytically verify the equivalence of these representations.

For the $\sigma_{A'}$ ordering we can apply the map $p_4 \rightarrow p_5$ to obtain the same-helicity vertex for this colour ordering. In terms of the minimal variables this map is given by

$$\begin{aligned}
z & \xrightarrow[p_4 \leftrightarrow p_5]{} 1 + w \frac{1-z}{z}, \\
w & \xrightarrow[p_4 \leftrightarrow p_5]{} 1 + z \frac{1-w}{w}, \\
X & \xrightarrow[p_4 \leftrightarrow p_5]{} \frac{1}{X},
\end{aligned} \tag{3.39}$$

whereby from eq. (3.32) we obtain

$$A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) = \frac{q_{1\perp}^*}{q_{1\perp}} \frac{w(w-1)(z-1)}{(w+Xz)}. \tag{3.40}$$

For the opposite-helicity vertex we must perform the map $p_4 \leftrightarrow p_5$ followed by complex conjugation, whereby from eq. (3.35) and eq. (3.36) we obtain

$$\begin{aligned}
R_{uv}^{A'} &= \frac{|w-1|^2\bar{w}X^2|z|^2(\bar{z}-1)}{(\bar{w}+X\bar{z})(X+1)(w-z)(\bar{w}+\bar{z}-\bar{w}\bar{z}+X\bar{z})}, \\
R_{vw}^{A'} &= \frac{wz(w-1)(z-1)(\bar{w}+\bar{z}-\bar{w}\bar{z})^4}{\bar{z}(|w+z-wz|^2+X|z|^2)(\bar{w}+\bar{z}-\bar{w}\bar{z}+X\bar{z})(w(\bar{w}+\bar{z}-\bar{w}\bar{z})+X|z|^2)}, \\
R_{wu}^{A'} &= -\frac{(\bar{w}-1)|w|^2X^2|z-1|^2z^2|z|^2}{(w+Xz)(|w|^2+X|z|^2)(w(\bar{w}+\bar{z}-\bar{w}\bar{z})+X|z|^2)(w-z)}.
\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}
R_{\bar{u}\bar{v}}^{A'} &= \frac{|w-1|^2 w(z-1)|z|^2}{(X+1)(\bar{w}-\bar{z})(w+Xz)(w+z-wz+Xz)}, \\
R_{\bar{v}\bar{w}}^{A'} &= \frac{\bar{w}(\bar{w}-1)(\bar{z}-1)z^2|z|^2 X^2}{(|w+z-wz|^2 + X|z|^2)(w+z-wz+Xz)(\bar{w}(w+z-wz) + X|z|^2)}, \\
R_{\bar{w}\bar{u}}^{A'} &= -\frac{(w-1)|w|^2 \bar{w}^4 |z-1|^2 z}{\bar{z}(|w|^2 + X|z|^2)(\bar{w}+X\bar{z})(\bar{w}(w+z-wz) + X|z|^2)(\bar{w}-\bar{z})}.
\end{aligned} \tag{3.42}$$

respectively.

For the σ_B ordering we find that the same-helicity CEV becomes

$$B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) = -\frac{q_{1\perp}^*}{q_{1\perp}}(w-1)(z-1), \tag{3.43}$$

while the rational terms which comprise the opposite-helicity vertex representations eq. (3.19) and eq. (3.21) become

$$\begin{aligned}
R_{uv}^B &= \frac{z|w-1|^2|z-1|^2}{\bar{z}(w+z-wz)}, \\
R_{vw}^B &= -\frac{z(w-1)(z-1)}{\bar{z}(1+X|z|^2)(w+z-wz+X|z|^2)}, \\
R_{wu}^B &= -\frac{(\bar{w}-1)X^2 z^3(\bar{z}-1)\bar{z}}{(w+z-wz)(w+z-wz+X|z|^2)(|w+z-wz|^2 + X|z|^2)},
\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}
R_{\bar{u}\bar{v}}^B &= 0, \\
R_{\bar{v}\bar{w}}^B &= -\frac{(\bar{w}-1)(\bar{z}-1)|z|^2 z^2 X^2}{(1+X|z|^2)(\bar{w}+\bar{z}-\bar{w}\bar{z}+X|z|^2)}, \\
R_{\bar{w}\bar{u}}^B &= -\frac{z(w-1)(z-1)(\bar{w}+\bar{z}-\bar{w}\bar{z})^3}{\bar{z}(\bar{w}+\bar{z}-\bar{w}\bar{z}+X|z|^2)(|w+z-wz|^2 + X|z|^2)},
\end{aligned} \tag{3.45}$$

respectively. We can straightforwardly associate two of the factors that appear in denominators of these rational terms with the unphysical singularity surfaces of eq. (3.17),

$$\begin{aligned}
P_u^B = 0 &\leftrightarrow w+z-wz=0, \\
P_w^B = 0 &\leftrightarrow w+z-wz+X|z|^2=0.
\end{aligned} \tag{3.46}$$

However, there is a subtlety in expressing the condition $P_v^B = q_{2\perp}^* = 0$ in terms

of the minimal variables. From eq. (3.26) we see that this condition is equivalent to taking $\bar{w}$ and $\bar{z}$ to zero at the same rate. This means the ratio $\bar{w}/\bar{z}$ is well defined in this limit.

Before leaving this section, we note that in the central NMRK limit, the dual-conformally invariant cross ratios depend on the minimal set, $\{X, w, z\}$, alone. For the σ_A ordering, the cross ratios eq. (3.4) become:

$$\begin{aligned} u_A &\xrightarrow{\text{NMRK}} \frac{|w + Xz|^2}{(1 + X)(|w|^2 + X|z|^2)}, \\ v_A &\xrightarrow{\text{NMRK}} \frac{X|1 - z|^2}{(1 + X|z|^2)(1 + X)}, \\ w_A &\xrightarrow{\text{NMRK}} \frac{X|1 - w|^2|z|^2}{(|w|^2 + X|z|^2)(1 + X|z|^2)}, \end{aligned} \quad (3.47)$$

while for the σ_B ordering, the cross ratios are:

$$\begin{aligned} u_B &\xrightarrow{\text{NMRK}} \frac{X|z|^2}{|w + z - wz|^2 + X|z|^2}, \\ v_B &\xrightarrow{\text{NMRK}} \frac{1}{1 + X|z|^2}, \\ w_B &\xrightarrow{\text{NMRK}} \frac{|w - 1|^2|z - 1|^2 X|z|^2}{(1 + X|z|^2)(|w + z - wz|^2 + X|z|^2)}. \end{aligned} \quad (3.48)$$

3.1.3 The target-projectile exchange

Due to the reversal symmetry of gluon amplitudes, each colour ordering of the amplitude

$$\begin{aligned} \mathcal{M}_{6g}^{(0)} \Big|_{\text{NMRK}} &= g_s^4 (F^{a_3})_{a_2 c_1} (F^{a_6})_{a_1 c_3} \\ &\times \sum_{\sigma \in S_2} (F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3} M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, p_6^{\nu_6}) \Big|_{\text{NMRK}}, \end{aligned} \quad (3.49)$$

is manifestly symmetric under the ‘target-projectile’ exchange

$$\{p_1, \lambda_1, a_1\} \leftrightarrow \{p_2, \lambda_2, a_2\}, \{p_3, \lambda_3, a_3\} \leftrightarrow \{p_6, \lambda_6, a_6\}, \{p_4, \lambda_4, a_4\} \leftrightarrow \{p_5, \lambda_5, a_5\}. \quad (3.50)$$

The manifest symmetry is broken on the right-hand side of eq. (2.97) by the choice of impact factors in eq. (2.8), which in turn is dictated by the momentum representation, eq. (1.162), of the spinor products. However, in transverse-

momentum space the impact factors in eq. (2.8) are just phases, and hence the central-emission vertex for the emission of two gluons is invariant under target-projectile exchange up to an overall phase. We may consider then the (kinematic part of the) target-projectile map F , which interchanges

$$p_1 \xleftrightarrow{F} p_2, \quad p_3 \xleftrightarrow{F} p_6, \quad p_4 \xleftrightarrow{F} p_5. \quad (3.51)$$

In NMRK, in light-cone coordinates, this target-projectile map acts as

$$q_{1\perp} \xleftrightarrow{F} -q_{3\perp}, \quad p_{4\perp} \xleftrightarrow{F} p_{5\perp}, \quad p_4^\pm \xleftrightarrow{F} p_5^\mp. \quad (3.52)$$

In terms of the minimal set $\{w, z, X\}$ these flip relations can be expressed as

$$w \xleftrightarrow{F} z, \quad X \xleftrightarrow{F} \frac{|z|^2}{|w|^2} X. \quad (3.53)$$

Under these flips, the cross ratios for the σ_A ordering, eq. (3.4), change as

$$u_A \xleftrightarrow{F} u_A, \quad v_A \xleftrightarrow{F} w_A, \quad (3.54)$$

while the spurious poles change as

$$w + X|z|^2 \xrightarrow{F} z(1 + X\bar{z}), \quad w - z \xrightarrow{F} -(w - z), \quad 1 + X\bar{z} \xrightarrow{F} \frac{w + X|z|^2}{w}, \quad (3.55)$$

i.e. the first and third spurious-pole surfaces in eq. (3.37) are swapped. Furthermore, the rational coefficients eqs. (3.35) and (3.36) change as

$$\begin{aligned} R_{uv}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{wu}^A, & R_{\bar{u}\bar{v}}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{\bar{w}\bar{u}}^A, \\ R_{vw}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{vw}^A, & R_{\bar{v}\bar{w}}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{\bar{v}\bar{w}}^A, \\ R_{wu}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{uv}^A, & R_{\bar{w}\bar{u}}^A &\xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{\bar{u}\bar{v}}^A, \end{aligned} \quad (3.56)$$

from which we see that both eqs. (3.7) and (3.9) are invariant under the target-projectile map F , up to an overall phase. By contrast, for the σ_B ordering we find that each cross ratio (eq. (3.16)) is invariant under this map,

$$u_B \xleftrightarrow{F} u_B, \quad v_B \xleftrightarrow{F} v_B, \quad w_B \xleftrightarrow{F} w_B, \quad (3.57)$$

the spurious singularity surfaces (3.46) are likewise invariant, while each rational term in eqs. (3.18) and (3.20) is invariant up to an overall phase,

$$R_{xy}^B \xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{xy}^B, \quad R_{\bar{x}\bar{y}}^B \xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} R_{\bar{x}\bar{y}}^B, \quad x, y \in \{u, v, w\}, x \neq y.$$

For the purposes of chapter 3, we introduce the map

$$\{p_2, \lambda_2, a_2\} \leftrightarrow \{p_3, \lambda_3, a_3\}, \{p_1, \lambda_1, a_1\} \leftrightarrow \{p_6, \lambda_6, a_6\}, \{p_4, \lambda_4, a_4\} \leftrightarrow \{p_5, \lambda_5, a_5\}. \quad (3.58)$$

We refer to the kinematic interchanges of this map as G ,

$$p_2 \xleftrightarrow{G} p_3, \quad p_1 \xleftrightarrow{G} p_6, \quad p_4 \xleftrightarrow{G} p_5. \quad (3.59)$$

Of particular interest will be the action of this map on the cross ratios in the NMRK limit:

$$u_A \xleftrightarrow{G} u_A, \quad v_A \xleftrightarrow{G} v_{A'}, \quad w_A \xleftrightarrow{G} w_{A'}, \quad (3.60a)$$

$$u_B \xleftrightarrow{G} v_B, \quad w_B \xleftrightarrow{G} w_B. \quad (3.60b)$$

Having investigated the tree-level CEV, and in particular the rational terms R_{xy}^I which comprise the BCFW representations of this CEV, we proceed to the study of this vertex at one loop.

3.2 The two-gluon central-emission vertex at one loop

In this section we will extract the two-gluon central-emission vertex at one loop in $\mathcal{N} = 4$ SYM. The plan is analogous to the extraction of the tree-level vertex in section 2.2. We will begin with the one-loop amplitude in general kinematics, and find the leading behaviour in the central NMRK limit defined in appendix A.1. We expect that in this limit, the amplitude will factorise into the known one-loop impact factors and Regge trajectories associated with the two large rapidity spans, and a new piece which depends only on the central degrees of freedom, i.e. the gluons 4 and 5 and the transverse t -channel momenta. We will identify this new central piece with the one-loop CEV.

The study of the two-gluon CEV in $\mathcal{N} = 4$ SYM is not only of interest in its own right; via the supersymmetric decomposition of QCD amplitudes at

one loop [54, 56, 57], it is also an ingredient for the two-gluon CEV in QCD. In sections 2.1.2 and 2.2 we extended the relationship eq. (1.168) to the level of one-loop Regge-factorised expressions, clarifying the connection between the factorised expressions within QCD and $\mathcal{N} = 4$ SYM:

$$\begin{aligned} c_{\text{QCD}}^{g \rightarrow g(1)} &= N_c \left(c_{\mathcal{N}=4}^{g \rightarrow g(1)} - 4c_{\mathcal{N}=1}^{g \rightarrow g(1)} + c_{\text{scalar}}^{g \rightarrow g(1)} \right) + N_f \left(c_{\mathcal{N}=1}^{g \rightarrow g(1)} - c_{\text{scalar}}^{g \rightarrow g(1)} \right) \\ v_{\text{QCD}}^{g \rightarrow g(1)} &= N_c \left(v_{\mathcal{N}=4}^{g(1)} - 4v_{\mathcal{N}=1}^{g(1)} + v_{\text{scalar}}^{g(1)} \right) + N_f \left(v_{\mathcal{N}=1}^{g(1)} - v_{\text{scalar}}^{g(1)} \right). \end{aligned} \quad (3.61)$$

For the convenience of the reader, we repeat eq. (1.165), specifying to the six-gluon case, which will be the master equation for all that follows,

$$\mathcal{M}_{\mathcal{N}=4}^{(1)} = g_s^6 \sum_{\sigma \in S_5/\mathcal{R}} \text{tr}(F^{a_{\sigma_1}} \dots F^{a_{\sigma_6}}) M_{\mathcal{N}=4}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_6}^{\nu_{\sigma_6}}). \quad (3.62)$$

Let us begin by discussing the relevant helicity configurations. As at tree level, there are two distinct helicity configurations for the one-loop two-gluon central-emission vertex. In order to derive the two-gluon CEV we take the central NMRK limit, as defined in eq. (A.1), of $\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{-\nu_2}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{-\nu_1})$. Thus, to obtain the same-helicity vertex, i.e. $\nu_4 = \nu_5$, the six-gluon amplitude we begin with must necessarily be an (anti-)MHV amplitude. In section 3.2.1 we take the central NMRK limit of (anti-)MHV one-loop six-gluon amplitudes in $\mathcal{N} = 4$ SYM to obtain the one-loop same-helicity two-gluon CEV of this theory. In turn, in order to extract the one-loop opposite-helicity vertex we must instead take the NMRK limit of a helicity configuration with $\nu_4 \neq \nu_5$, which is necessarily an NMHV amplitude. This study is performed in section 3.2.2.

In both cases we can simplify the procedure by recalling some results from section 3.1. Although eq. (3.62) consists of a sum of 60 colour orderings, only 12 of these are not power suppressed by the rational terms in the central NMRK limit. This is straightforward for the MHV amplitudes, which are proportional to the tree-level result, up to logarithmically varying functions, as we will see in eq. (3.70). For the NMHV amplitudes, eqs. (A.17) and (A.19), this is not the case. However, each of the pairs of individual rational terms in these amplitudes are power suppressed in the NMRK limit for colour orderings that do not have both p_2 and p_3 adjacent, and p_6 and p_1 adjacent. This means we need only study the 12 colour orderings that are leading at tree-level also for the one-loop NMHV amplitudes. It will be useful to consider the representative colour orderings, eq. (2.105), that we discussed in section 3.1. We further introduce the notation

$\overset{\leftrightarrow}{\sigma}$ to indicate a kinematic (momentum and spin) interchange of $2 \leftrightarrow 3$, and $\overset{\leftrightarrow}{\sigma}$ to indicate a kinematic interchange of $1 \leftrightarrow 6$. Heuristically, this notation is intended to suggest the interchange of particles on the top or bottom of ‘half-ladder’ diagrams such as fig. 2.6. As an example of this notation,

$$\overset{\leftrightarrow}{\sigma}_A = \{6, 3, 2, 4, 5, 1\}. \quad (3.63)$$

Following eq. (2.73), we decompose eq. (3.62) into symmetric (+) and antisymmetric (−) components,

$$\mathcal{M}^{(1)} = \mathcal{M}^{(1)(+,+)} + \mathcal{M}^{(1)(+,-)} + \mathcal{M}^{(1)(-,+)} + \mathcal{M}^{(1)(-,-)}. \quad (3.64)$$

As discussed in section 2.2, due to the Bose symmetry of gluon amplitudes, these kinematic interchanges translate directly into interchanges of the corresponding colour indices, so for example $\mathcal{M}^{(1)(-,-)}$ corresponds to the component of the amplitude in which an antisymmetric colour representation is exchanged in both the t_1 and t_3 channels, $\mathcal{M}^{(1)(+,-)}$ corresponds to one in which a symmetric representation is exchanged in the t_1 channel, while an antisymmetric one in the t_3 channel, etc. The notation in eq. (3.63) allows for the compact representation of singature components in terms of colour-ordered amplitudes. For example, when there is an adjoint representation circulating in the loop we obtain

$$\begin{aligned} \mathcal{M}^{(1)(+,+)} &= \frac{g_s^6}{4} \\ &\times \left\{ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) \left(M^{(1)}(\sigma_A) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) \right) \right. \\ &+ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) \left(M^{(1)}(\sigma_{A'}) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \\ &\left. + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) \left(M^{(1)}(\sigma_B) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) \right) \right\}. \end{aligned} \quad (3.65)$$

$$\begin{aligned}
\mathcal{M}^{(1)(+,-)} &= \frac{g_s^6}{4} \\
&\times \left\{ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} [F^{a_6}, F^{a_1}]) \left(M^{(1)}(\sigma_A) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_A) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right. \\
&+ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} [F^{a_6}, F^{a_1}]) \left(M^{(1)}(\sigma_{A'}) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_{A'}) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) \right) \\
&\left. + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} [F^{a_6}, F^{a_1}] F^{a_5}) \left(M^{(1)}(\sigma_B) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_B) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right\}.
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
\mathcal{M}^{(1)(-,+)} &= \frac{g_s^6}{4} \\
&\times \left\{ \text{tr}([F^{a_2}, F^{a_3}] F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) \left(M^{(1)}(\sigma_A) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) + M^{(1)}(\underset{\leftrightarrow}{\sigma}_A) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right. \\
&+ \text{tr}([F^{a_2}, F^{a_3}] F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) \left(M^{(1)}(\sigma_{A'}) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) + M^{(1)}(\underset{\leftrightarrow}{\sigma}_{A'}) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) \right) \\
&\left. + \text{tr}([F^{a_2}, F^{a_3}] F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) \left(M^{(1)}(\sigma_B) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) + M^{(1)}(\underset{\leftrightarrow}{\sigma}_B) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right\}.
\end{aligned} \tag{3.67}$$

$$\begin{aligned}
\mathcal{M}^{(1)(-,-)} &= g_s^6 F^{a_3 a_2 c_1} F^{a_6 a_1 c_3} \\
&\times \frac{1}{4} \left\{ \text{tr}(F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \times \left(M^{(1)}(\sigma_A) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_A) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right. \\
&+ \text{tr}(F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \times \left(M^{(1)}(\sigma_{A'}) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_{A'}) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) \right) \\
&\left. + \text{tr}(F^{c_1} F^{a_4} F^{c_3} F^{a_5}) \times \left(M^{(1)}(\sigma_B) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) - M^{(1)}(\underset{\leftrightarrow}{\sigma}_B) + M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right\},
\end{aligned} \tag{3.68}$$

which in particular are valid for one-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, eq. (3.62), but eqs. (3.65) and (3.68) also apply to $\mathcal{M}_{\mathcal{N}=1\chi}^{(1)\textcircled{8}}$, $\mathcal{M}_{\text{scalar}}^{(1)\textcircled{8}}$ and $\mathcal{M}_{\text{QCD}}^{(1)\textcircled{8}}$.

Having introduced some general aspects of our organisation the one-loop amplitudes we now look at the particulars of the same-helicity and opposite-helicity configurations in turn.

3.2.1 Same-helicity vertex

In this section we take the NMRK limit of the MHV amplitude,

$$\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{-\nu_2}, p_4^\oplus, p_5^\oplus, p_6^{-\nu_1}). \quad (3.69)$$

After the CEV for $\nu_4 = \nu_5 = \oplus$ has been obtained, the CEV for $\nu_4 = \nu_5 = \ominus$ may be obtained by complex conjugation, just as for the tree-level vertex, eq. (2.101). The colour-ordered amplitudes for one-loop six-gluon MHV amplitudes in $\mathcal{N} = 4$ SYM are [56]

$$M_{\mathcal{N}=4}^{(1)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_6}^{\nu_{\sigma_6}}) = \frac{\kappa_\Gamma}{(4\pi)^2} M^{(0)}(p_{\sigma_1}^{\nu_{\sigma_1}}, \dots, p_{\sigma_6}^{\nu_{\sigma_6}}) V_{g6}(\sigma_1, \dots, \sigma_6), \quad (3.70)$$

where the transcendental terms are given by the function,

$$\begin{aligned} V_{g6}(\sigma_1, \dots, \sigma_6) = & \sum_{i=1}^6 -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-t_i^{[2]}} \right)^\epsilon - \sum_{i=1}^6 \log \left(\frac{-t_i^{[2]}}{-t_i^{[3]}} \right) \log \left(\frac{-t_{i+1}^{[2]}}{-t_i^{[3]}} \right) \\ & + \frac{1}{2} \sum_{i=1}^3 \log^2 \left(\frac{-t_i^{[3]}}{-t_{i+1}^{[3]}} \right) - \sum_{i=1}^3 \text{Li}_2 \left[1 - \frac{(-t_i^{[2]})(-t_{i+3}^{[2]})}{(-t_i^{[3]})(-t_{i-1}^{[3]})} \right] + \pi^2, \end{aligned} \quad (3.71)$$

which does not depend on the helicities of the gluons. Here we have used the compact notation,

$$t_i^{[r]} = (p_{\sigma_i} + \dots + p_{\sigma_{i+r-1}})^2. \quad (3.72)$$

Similar to the discussion in section 2.1.2, we need to analytically continue eq. (3.71), where all Mandelstam invariants are negative, to our physical region described in appendix A.1. Particular care must be taken when performing the analytic continuation of the arguments of the dilogarithms in eq. (3.71) to the physical region, which is discussed in more detail in appendix appendix A.3.

From eq. (3.70) we see that the one-loop amplitudes in $\mathcal{N} = 4$ SYM in the same-helicity configuration are given by the tree-level gluon amplitude times the transcendental function V_{g6} . As a step towards taking the full NMRK limit of eq. (3.69), let us keep only the leading power terms of the rational functions. This means we neglect all but the 12 leading colour orderings in eq. (3.69), and for these 12 leading colour orderings, we take the NMRK limit of the tree-level colour-ordered amplitudes that appear in eq. (3.70). We recall that in the NMRK limit, an interchange of $p_2^{\nu_2} \leftrightarrow p_3^{\nu_3}$ or $p_6^{\nu_6} \leftrightarrow p_1^{\nu_1}$ leads only to a sign change of

the leading tree-level amplitude. This lets us write, for example, the $(-, -)$ component of the colour-dressed amplitude in the NMRK limit as

$$\begin{aligned}
\mathcal{M}^{(1)(-, -)}_{\mathcal{N}=4}(p_4^\nu, p_5^\nu) &\xrightarrow{\text{NMRK}} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} F^{a_3 a_2 c_1} F^{a_6 a_1 c_3} \\
&\times \frac{1}{4} \left\{ \text{tr}(F^{c_1} F^{a_4} F^{a_5} F^{c_3}) M^{(0)}(\sigma_A) \Big|_{\text{NMRK}} \left(V_{g6}(\sigma_A) + V_{g6}(\overset{\leftrightarrow}{\sigma}_A) + V_{g6}(\underset{\leftrightarrow}{\sigma}_A) + V_{g6}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right. \\
&\quad + \text{tr}(F^{c_1} F^{a_5} F^{a_4} F^{c_3}) M^{(0)}(\sigma_{A'}) \Big|_{\text{NMRK}} \left(V_{g6}(\sigma_{A'}) + V_{g6}(\overset{\leftrightarrow}{\sigma}_{A'}) + V_{g6}(\underset{\leftrightarrow}{\sigma}_{A'}) + V_{g6}(\overset{\leftrightarrow}{\sigma}_B) \right) \\
&\quad \left. + \text{tr}(F^{c_1} F^{a_4} F^{c_3} F^{a_5}) M^{(0)}(\sigma_B) \Big|_{\text{NMRK}} \left(V_{g6}(\sigma_B) + V_{g6}(\overset{\leftrightarrow}{\sigma}_B) + V_{g6}(\underset{\leftrightarrow}{\sigma}_B) + V_{g6}(\overset{\leftrightarrow}{\sigma}_{A'}) \right) \right\}, \tag{3.73}
\end{aligned}$$

where we factorised the tree amplitude for each colour ordering. The positive relative signs in this equation, in contrast to eq. (3.68), are due to the sign flips in the tree amplitude $M^{(0)}(\sigma)$. As in eq. (2.94), these sign flips can be viewed as the NMRK limit of photon decoupling identities. Here, and in the following, we suppress the momentum and helicity arguments of $\mathcal{M}_{\mathcal{N}=4}^{(1)}$, apart from the two central gluons 4 and 5. Next, we write the tree-level colour-ordered amplitudes in the central NMRK in a factorized form, e.g. for the A colour ordering we use

$$M^{(0)}(\sigma_A) \Big|_{\text{NMRK}} = C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \frac{1}{|q_{3\perp}|^2} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \tag{3.74}$$

and similarly for the A' and B orderings.

Turning our attention to the transcendental functions, we observe a major simplification in the central NMRK limit upon considering *the dispersive part of the amplitude*, defined by taking the real part of the transcendental functions. We find that the latter is given solely by the $(-, -)$ component, eq. (3.73), as the dispersive parts of the remaining components of eq. (3.64), quoted in eqs. (3.65) and (3.67), vanish. This simplification is due to the fact that, for the 12 leading configurations (shown in e.g. eq. (3.73)), the real part of the transcendental functions V_{g6} in the central NMRK limit is unchanged by an interchange of $p_2 \leftrightarrow p_3$ or $p_6 \leftrightarrow p_1$. We will prove this fact below. In conclusion, the dispersive

part of the colour-dressed amplitude in the central NMRK limit is given by

$$\begin{aligned}
& \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}} \\
& g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} \frac{1}{|q_{3\perp}|^2} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \times \\
& \left\{ \text{tr} (F^{c_1} F^{a_4} F^{a_5} F^{c_3}) A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \text{Re} [V_{g6}(\sigma_A)|_{\text{NMRK}}] \right. \\
& \quad + \text{tr} (F^{c_1} F^{a_5} F^{a_4} F^{c_3}) A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) \text{Re} [V_{g6}(\sigma_{A'})|_{\text{NMRK}}] \\
& \quad \left. + \text{tr} (F^{c_1} F^{a_5} F^{c_3} F^{a_4}) B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \text{Re} [V_{g6}(\sigma_B)|_{\text{NMRK}}] \right\}.
\end{aligned} \tag{3.75}$$

By contrast, the absorptive part of the amplitude, defined by taking the imaginary part of the transcendental functions, in NMRK receives contribution from all terms in the decomposition of eq. (3.64). The absorptive part of the amplitude is discussed further in appendix A.3, and the four signature components are listed in eqs. (A.25) and (A.28). In section 3.2.3 we show that this absorptive part does not contribute to the colour-summed NLO $gg \rightarrow gggg$ squared matrix element in the central NMRK limit. This means that, in particular, only the dispersive part of the one-loop amplitude contributes to the dijet cross section at NNLL accuracy. We now move to the study of the real part of V_{g6} for the relevant colour orderings, leaving the analogous discussion of the imaginary parts of V_{g6} to appendix A.3.

In section 2.3 we saw that the real parts of the one-loop four- and five-gluon MHV transcendental functions, V_{g4} and V_{g5} , could be written *without any approximation* as a sum of one-loop Regge-factorised expressions. As noted in ref. [122], this decomposition can be partly generalised to the six-gluon case, namely the real part of V_{g6} can be written, without any approximation, as a combination of the one-loop Regge trajectory of eq. (2.24), the impact factor of eq. (2.67) and the Lipatov vertex of eq. (2.79) – plus a remainder term that vanishes in the MRK limit. We will refer to this term as the NMRK remainder function (ΔV below). The arguments of the one-loop MRK functions can be chosen such that the remainder function is free from IR divergences, and is a function of conformally-invariant cross ratios alone. For future convenience we choose to group these one-loop MRK functions into two components. One component is a function which collects the ‘central’ behaviour that will be relevant for our central NMRK

limit,

$$U(\tilde{t}_1, \tilde{\eta}_{12}, \tilde{t}_2, \tilde{s}_2, \tilde{\eta}_{23}, \tilde{t}_3) = v_{\mathcal{N}=4}^{g(1)}(\tilde{t}_1, \tilde{\eta}_{12}, \tilde{t}_2) + \alpha^{(1)}(\tilde{t}_2) \log\left(\frac{\tilde{s}_2}{\tau}\right) + v_{\mathcal{N}=4}^{g(1)}(\tilde{t}_2, \tilde{\eta}_{23}, \tilde{t}_3) . \quad (3.76)$$

The other component collects the remaining ‘non-central’ behaviour,

$$E(\tilde{t}_1, \tilde{s}_1; \tilde{t}_3, \tilde{s}_3) = c_{\mathcal{N}=4}^{g(1)}(\tilde{t}_1; \tau) + \alpha^{(1)}(\tilde{t}_1) \log\left(\frac{\tilde{s}_1}{\tau}\right) + \alpha^{(1)}(\tilde{t}_3) \log\left(\frac{\tilde{s}_3}{\tau}\right) + c_{\mathcal{N}=4}^{g(1)}(\tilde{t}_3; \tau) . \quad (3.77)$$

The arguments decorated with tildes in eq. (3.76) and eq. (3.77) are placeholders: the arguments assigned will differ for each colour ordering in eq. (3.75). However, we will always make the natural choice of assigning $\tilde{t}_1 = -t_1 = -s_{23}$ and $\tilde{t}_3 = -t_3 = -s_{61}$, and we will assign $\tilde{t}_2 = -t_2 = -s_{234}$ when considering the MRK limit, and $\tilde{t}_2 = -t'_2 = -s_{235}$ when considering the MRK' limit. Based on the structure of V_{g6} in eq. (3.71), for a given colour ordering we will also choose the remaining arguments to be constructed from colour-ordered two- and three-particle Mandelstam invariants. This choice will be informed by our knowledge of the MRK and MRK' limit of this amplitude. However, these criteria do not lead to a unique assignment of the arguments of eqs. (3.76) and (3.77): the remaining freedom will be discussed separately for each colour ordering.

As a first example, let us see how this re-writing can be performed for the real part of $V_{g6}(\sigma_A)$. We first separate the non-central function E , making the natural choice of Mandelstam variables for its arguments,

$$\text{Re}[V_{g6}(\sigma_A)] = E(-s_{23}, s_{34}; -s_{61}, s_{56}) + V_A . \quad (3.78)$$

This choice of arguments for E is the simplest choice which is compatible with the known MRK limit²,

$$E(-s_{23}, s_{34}; -s_{61}, s_{56}) \xrightarrow{(\text{N})\text{MRK}} E(|q_{1\perp}|^2, p_3^+ p_4^-; |q_{3\perp}|^2, p_5^+ p_6^-) . \quad (3.79)$$

The function V_A in eq. (3.78) collects the remaining ‘central’ behaviour and it can be identified as the transcendental function associated with the two-gluon CEV for the A colour ordering. As anticipated, this function itself can be neatly

²As this function does not depend on the central degrees of freedom in the NMRK, no further simplification is obtained by taking the further MRK limit.

written as

$$V_A = U_A + \Delta V_A(u_A, v_A, w_A), \quad (3.80)$$

where, again guided by the MRK limit, we fix U_A to equal the function U , eq. (3.76), with arguments assigned as follows

$$U_A = U \left(-s_{23}, \frac{s_{34}s_{45}}{s_{345}\sqrt{u_A}}, -s_{234}, s_{45}, \frac{s_{45}s_{56}}{s_{456}\sqrt{u_A}}, -s_{61} \right). \quad (3.81)$$

This choice of kinematic arguments leads to a simple NMRK remainder function ΔV_A , which is IR-finite, vanishes in the MRK limit and depends only on the cross ratios, eq. (3.4),

$$\begin{aligned} \Delta V_A(u_A, v_A, w_A) = & \frac{\pi^2}{3} - \frac{1}{4} \log^2(u_A) - \frac{1}{2} \log(u_A) \log(v_A w_A) \\ & - \text{Li}_2(1 - u_A) - \text{Li}_2(1 - v_A) - \text{Li}_2(1 - w_A). \end{aligned} \quad (3.82)$$

The MRK limit of the cross ratios is $u_A \rightarrow 1$, $v_A \rightarrow 0$, $w_A \rightarrow 0$ (see eq. (A.39)). It is easy to see that eq. (3.82) vanishes in this limit.

As mentioned above, eq. (3.81) is not the unique choice which leads to an IR-finite NMRK remainder. In eq. (3.81) we have made the further choice of requiring U_A to be invariant under target-projectile exchange, eq. (3.51). Yet even this physically-motivated requirement does not uniquely specify arguments for U_A : a rescaling of the $\tilde{t}_2$ variable in eq. (3.81) by an arbitrary power of u_A , i.e. $s_{234} \rightarrow s_{234} \times (u_A)^p$ for real p , yields a NMRK remainder function which is an IR-finite function of cross ratios and is invariant under target-projectile exchange. Our ultimate choice of arguments in eq. (3.81) was guided by the intuition that the trajectory should be evaluated exactly at the t -channel invariant $t_2 = s_{234}$ also away from the strict MRK limit. It would be interesting to explore this same procedure at higher orders in ϵ , or at two loops, where, if such an organisation proves possible, there may be additional constraints on the kinematic arguments.

We stress that eq. (3.78) holds in general kinematics if no approximation is made to the kinematic invariants. However, we will proceed to make the kinematic approximations valid in the central NMRK limit, in order to exploit the invariance of the real part under the interchanges $p_2 \leftrightarrow p_3$ or $p_6 \leftrightarrow p_1$ in this limit, as in eq. (3.75). Thus, we specify eq. (3.80) to the NMRK limit, by making the

approximation,

$$U_A \xrightarrow{\text{NMRK}} U \left(|q_{1\perp}|^2, \frac{p_4^- s_{45}}{p_4^- + p_5^-}, |q_{2\perp}|^2 + p_5^+ p_4^-, s_{45}, \frac{s_{45} p_5^+}{p_4^+ + p_5^+}, |q_{3\perp}|^2 \right), \quad (3.83)$$

and also taking the NMRK approximation of the cross ratios, as listed in eq. (3.4), for the arguments of eq. (3.82). This NMRK approximation of V_A will later form one of the building blocks of the colour-dressed two-gluon CEV. The structure of the NMRK limit of the dispersive part of $M_{\mathcal{N}=4}^{(1)}$ is depicted in fig. 3.1.

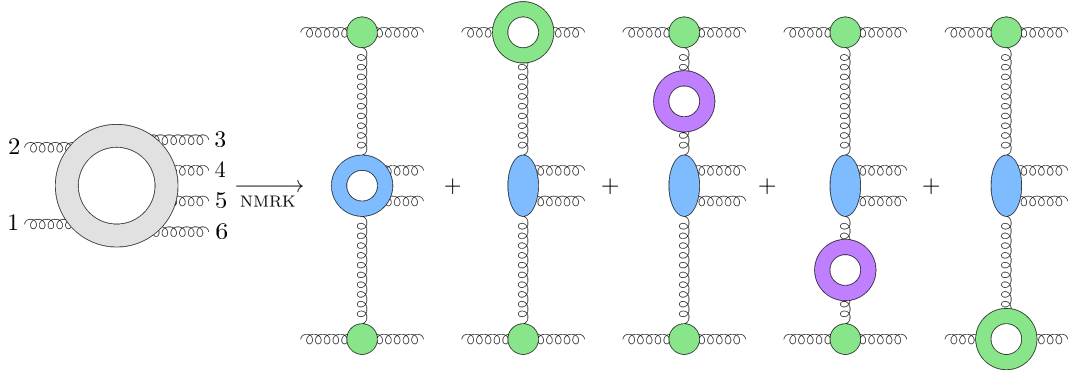


Figure 3.1 *Schematic illustration of the behaviour of the dispersive part of the one-loop partial amplitude $M_{\mathcal{N}=4}^{(1)}(\sigma_A)$ in the central NMRK limit. The first diagram on the right-hand side represents the transcendental functions that depend on the central degrees of freedom, V_A , which we identify as the one-loop correction to the two-gluon CEV for this colour ordering. The remaining four diagrams on the right-hand side are given by the known one-loop impact factors and gluon Regge trajectory contributions, and are collected in the function E of eq. (3.77).*

The function $V_{g6}(\sigma_{A'})$ is simply related to $V_{g6}(\sigma_A)$ by an interchange of $p_4 \leftrightarrow p_5$. We now use the known MRK' limit of the amplitude as our guide, writing

$$\text{Re} [V_{g6}(\sigma_{A'})|_{\text{NMRK}}] = E(s_{23}, s_{35}; s_{61}, s_{46}) + V_{A'}, \quad (3.84)$$

$$V_{A'} = U_{A'} + \Delta V_A(u_{A'}, w_{A'}, v_{A'}), \quad (3.85)$$

where $U_{A'}$ is related to eq. (3.81) by the interchange of $p_4 \leftrightarrow p_5$. As was true for the A ordering, in order to use the compact form of eq. (3.75), we will need to make the NMRK approximations to the arguments of ΔV_A in eq. (3.85), and

similarly we make the approximation,

$$U_{A'} \xrightarrow{\text{NMRK}} U \left(|q_{1\perp}|^2, \frac{p_5^- s_{45}}{p_4^- + p_5^-}, |q'_{2\perp}|^2 + p_4^+ p_5^-, s_{45}, \frac{s_{45} p_4^+}{p_4^+ + p_5^+}, |q_{3\perp}|^2 \right). \quad (3.86)$$

Note that the NMHV remainder function for the A' ordering is the same as for the A ordering, because $\Delta V_A(u, w, v) = \Delta V_A(u, v, w)$ based on eq. (3.82).

We now turn to study of the real part of $V_{g6}(\sigma_B)$. The rational coefficient of this transcendental function, $M^{(0)}(\sigma_B)$, has a leading contribution in both the MRK and MRK' limits. It will be useful therefore to re-write this transcendental function in two ways: one guided by the known MRK limit of the amplitude, and the other by the known MRK' limit. As usual, we first separate the central and non-central functions,

$$\text{Re}[V_{g6}(\sigma_B)] = E(-s_{23}, -s_{42}; -s_{61}, s_{56}) + V_B = E(-s_{23}, s_{35}; -s_{61}, -s_{14}) + V_{B'}. \quad (3.87)$$

The forms of the central functions V_B and $V_{B'}$ are guided, respectively, by the known MRK and MRK' limits of this function – note that both refer to the same colour ordering. These central functions are given by

$$V_B = U_B + \Delta V_B(u_B, v_B, w_B), \quad V_{B'} = U_{B'} + \Delta V_B(v_B, u_B, w_B), \quad (3.88)$$

where we have chosen to use the kinematic arguments,

$$\begin{aligned} U_B &= U \left(-s_{23}, \frac{s_{14}s_{42}}{s_{142}u_B}, -s_{234}, -s_{235}, \frac{s_{35}s_{56}}{s_{356}u_B}, -s_{61} \right), \\ U_{B'} &= U \left(-s_{23}, \frac{s_{35}s_{56}}{s_{356}v_B}, -s_{235}, -s_{234}, \frac{s_{14}s_{42}}{s_{142}v_B}, -s_{61} \right), \end{aligned} \quad (3.89)$$

for the one-loop MRK functions.

Unlike the A colour ordering, target-projectile symmetry does not constrain the arguments of eq. (3.89): from eq. (3.57) we see that in NMRK, each cross ratio is invariant under target-projectile exchange. Our choice of kinematic arguments in eq. (3.89) were instead motivated by the requirement that U_B and $U_{B'}$ are related by the map G given in eq. (3.59). This map is a symmetry of the real part of $V_{g6}(\sigma_B)$. For the natural assignment of variables in eq. (3.87), the real parts of E are related by this map.³ Choosing U_B and $U_{B'}$ to be related by G therefore

³Note that we always choose the signs of the arguments of E such that it is in fact a real function.

means the respective NMRK remainder functions are also related by this map. The NMRK remainder function that follows from the choices made in eq. (3.89) is

$$\begin{aligned}\Delta V_B(u_B, v_B, w_B) &= \Delta V_B(v_B, u_B, w_B) \\ &= \frac{\pi^2}{3} - \log(u_B) \log(v_B) - \text{Li}_2(1 - u_B) - \text{Li}_2(1 - v_B) - \text{Li}_2(1 - w_B) .\end{aligned}\tag{3.90}$$

Recalling eq. (3.60b) we see that eq. (3.90) is indeed invariant under G . We note, however, that by considering suitable rescaling of the arguments of U_B and $U_{B'}$ by powers of u_B or v_B respectively (the quantities which tend to unity in the MRK or MRK' limits respectively), there is one alternative set of kinematic assignments which lead to an NMRK remainder function that possesses the same desirable properties as eq. (3.90): as for the σ_A ordering, our final choice of arguments in eq. (3.89) was determined by requiring that $\tilde{t}_2$ is identified with the relevant t -channel momentum even in general kinematics, namely $-t_2$ and $-t'_2$ in the MRK and MRK' limits, respectively.

In order to use eq. (3.87) within the compact form of eq. (3.75), we must make the NMRK approximations

$$U_B \xrightarrow{\text{NMRK}} U\left(|q_{1\perp}|^2, -\frac{p_4^-}{p_5^-} t'_2, -t_2, -t'_2, -\frac{p_5^+}{p_4^+} t_2, |q_{3\perp}|^2\right),\tag{3.91}$$

$$U_{B'} \xrightarrow{\text{NMRK}} U\left(|q_{1\perp}|^2, -\frac{p_5^-}{p_4^-} t_2, -t'_2, -t_2, -\frac{p_4^+}{p_5^+} t'_2, |q_{3\perp}|^2\right),\tag{3.92}$$

where it is understood that on the right hand side we use the NMRK limits of

$$-t_2 = -s_{234} \xrightarrow{\text{NMRK}} |q_{2\perp}|^2 + p_5^+ p_4^-, \quad -t'_2 = -s_{235} \xrightarrow{\text{NMRK}} |q'_{2\perp}|^2 + p_4^+ p_5^- .\tag{3.93}$$

Similarly, we apply the NMRK approximations listed in eq. (3.16) for the cross-ratio arguments of eq. (3.90). We see that in the NMRK limit, U_B and $U_{B'}$ (and therefore V_B and $V_{B'}$) are simply related by an interchange of $p_4 \leftrightarrow p_5$.

Having studied the three colour orderings of eq. (3.75), we will now move to a basis of colour structures that more closely resembles the colour structure of the tree-level amplitude, eq. (3.49). This basis will let us identify one-loop corrections associated with each of the tree-level colour structures, as well as highlight the new colour structure that starts contributing at one-loop level. Denoting the

three traces by

$$T_A = \text{tr}(F^{c_1} F^{a_4} F^{a_5} F^{c_3}), \quad T_{A'} = \text{tr}(F^{c_1} F^{a_5} F^{a_4} F^{c_3}), \quad T_B = \text{tr}(F^{c_1} F^{a_5} F^{c_3} F^{a_4}), \quad (3.94)$$

we consider the following rotation of the sum inside the curly brackets in eq. (3.75)

$$\begin{aligned} T_A M_A + T_{A'} M_{A'} + T_B M_B &= (T_A - T_B) \times \frac{1}{3} (2M_A - M_{A'} - M_B) \\ &+ (T_{A'} - T_B) \times \frac{1}{3} (2M_{A'} - M_A - M_B) \\ &+ \frac{1}{3} (T_A + T_{A'} + T_B) \times (M_A + M_{A'} + M_B). \end{aligned} \quad (3.95)$$

In this chapter we set $T_F = 1$. Using the identities eqs. (1.90) and (1.91a) we relate the first two combinations of colour structure on the right-hand side of eq. (3.95) to the tree-level structures $(F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3}$ of eq. (3.49), and define the third colour factor in terms of the fully-symmetric trace eq. (1.92). Making use of the additional relations that exist within NMRK, namely

$$\begin{aligned} E(|q_{1\perp}|^2, -s_{42}; |q_{3\perp}|^2, s_{56})|_{\text{NMRK}} &= E(|q_{1\perp}|^2, s_{34}; |q_{3\perp}|^2, s_{56})|_{\text{NMRK}}, \\ E(|q_{1\perp}|^2, s_{35}; |q_{3\perp}|^2, -s_{14})|_{\text{NMRK}} &= E(|q_{1\perp}|^2, s_{35}; |q_{3\perp}|^2, s_{46})|_{\text{NMRK}}, \end{aligned} \quad (3.96)$$

we now express eq. (3.75) as

$$\begin{aligned} \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\oplus) \right] &\xrightarrow{\text{NMRK}} \frac{g_s^6}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} \times \\ &\sum_{\sigma \in S_2} \left\{ (F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3} \left(A_{\mathbb{F}}^{gg(1)}(q_1, p_{\sigma_4}^\oplus, p_{\sigma_5}^\oplus, q_3) \right. \right. \\ &\quad \left. \left. + N_c \kappa_\Gamma A^{gg(0)}(q_1, p_{\sigma_4}^\oplus, p_{\sigma_5}^\oplus, q_3) E(|q_{1\perp}|^2, s_{3\sigma_4}; |q_{3\perp}|^2, s_{\sigma_5 6}) \right) \right. \\ &\quad \left. + d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3} A_d^{gg(1)}(q_1, p_{\sigma_4}^\oplus, p_{\sigma_5}^\oplus, q_3) \right\} \times \frac{1}{|q_{3\perp}|^2} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}), \end{aligned} \quad (3.97)$$

where the totally symmetric colour structure d_A starts contributing at one-loop level, with

$$A_d^{gg(1)}(q_1, p_4^\oplus, p_5^\oplus, q_3) = \kappa_\Gamma A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) (V_A - V_B). \quad (3.98)$$

The symmetry of the d_A colour structure under the permutation S_2 of eq. (3.97)

means that after the sum over the σ_4, σ_5 permutations is performed, there are only three independent colour structures in this equation, as in eq. (3.75).

For the kinematic coefficient of the tree-level colour structures we find, using eqs. (2.104) and (3.95),

$$\begin{aligned} A_{\mathbb{F}}^{gg(1)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \\ = \frac{N_c \kappa_\Gamma}{3} \left(A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) (2V_A + V_B) - A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) (V_{A'} - V_{B'}) \right). \end{aligned} \quad (3.99)$$

We interpret this term as the one-loop coefficient of the all-orders expansion of this colour structure,

$$A_{\mathbb{F}}^{gg}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) = A_{\mathbb{F}}^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) + \frac{\alpha_S}{4\pi} A_{\mathbb{F}}^{gg(1)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) + \mathcal{O}(\alpha_S^2), \quad (3.100)$$

with

$$A_{\mathbb{F}}^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) = A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3). \quad (3.101)$$

As the structure d_A first appears at one-loop level we write

$$A_d^{gg}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) = \frac{\alpha_S}{4\pi} A_d^{gg(1)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) + \mathcal{O}(\alpha_S^2), \quad (3.102)$$

in anticipation of higher-order corrections to this colour structure. We note that $A_d^{gg(1)}$ and $A_{\mathbb{F}}^{gg(1)}$ both depend on μ^2 and τ , but as for the one-loop coefficients of the impact factor and Lipatov vertex, we omit this dependence for brevity.

The notation introduced in this section makes it straightforward to verify that the eqs. (3.98) and (3.99) yield the expected MRK limit. We first note that by construction, eqs. (3.80) and (3.88) both tend to the same MRK limit,

$$U_A|_{\text{MRK}} = U_B|_{\text{MRK}} = U_{\text{MRK}} = U(|q_{1\perp}|^2, |p_{4\perp}|^2, |q_{2\perp}|^2, p_4^+ p_5^-, |p_{5\perp}|^2, |q_{3\perp}|^2). \quad (3.103)$$

Using the MRK limits of the relevant cross ratios listed in eq. (A.39) together with the values of the dilogarithm,

$$\text{Li}_2(0) = 0, \quad \text{Li}_2(1) = \frac{\pi^2}{6}, \quad (3.104)$$

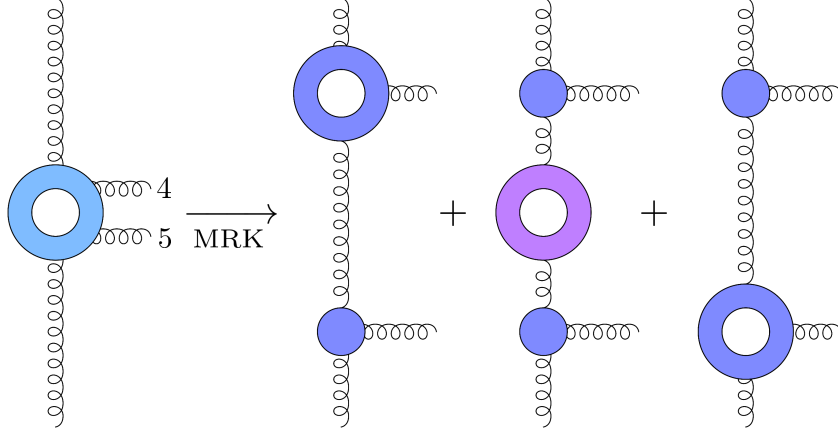


Figure 3.2 *Schematic illustration of the further factorisation of the one-loop two-gluon CEV for the ordering σ_A in the MRK limit.*

we see that both $\Delta V_A(u_A, v_A, w_A)$ and $\Delta V_B(u_B, v_B, w_B)$ vanish in the MRK limit as desired. This leads to the simple results,

$$A_{\mathbb{F}}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \xrightarrow{\text{MRK}} V^{g(0)}(q_1, p_4^{\oplus}, q_2) \left(\frac{-1}{|q_{2\perp}|^2} \right) V^{g(0)}(q_2, p_5^{\oplus}, q_3) U_{\text{MRK}}, \quad (3.105)$$

$$A_d^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \xrightarrow{\text{MRK}} 0. \quad (3.106)$$

Equation (3.105) is depicted schematically in fig. fig. 3.2. The function U_{MRK} generates all three of the loop corrections on the right-hand side of the figure, thanks to the definition eq. (3.76). A similar analysis shows that

$$A_{\mathbb{F}}^{gg(1)}(q_1, p_5^{\oplus}, p_4^{\oplus}, q_3) \xrightarrow{\text{MRK}'} V^{g(0)}(q_1, p_5^{\oplus}, q'_2) \left(\frac{-1}{|q'_{2\perp}|^2} \right) V^{g(0)}(q'_2, p_4^{\oplus}, q_3) U_{\text{MRK}'}, \quad (3.107)$$

$$A_d^{gg(1)}(q_1, p_5^{\oplus}, p_4^{\oplus}, q_3) \xrightarrow{\text{MRK}'} 0, \quad (3.108)$$

where

$$U_{A'}|_{\text{MRK}'} = U_{B'}|_{\text{MRK}'} = U_{\text{MRK}'} = U(|q_{1\perp}|^2, |p_{5\perp}|^2, |q'_{2\perp}|^2, p_5^+ p_4^-, |p_{4\perp}|^2, |q_{3\perp}|^2). \quad (3.109)$$

In appendix A.4 we investigate the soft and collinear limits of eq. (3.97). In section 3.3 we conjecture how the one-loop results of this section may be extended to all orders. In the next subsection we obtain expressions analogous to eqs. (3.98)

and (3.99) but for the case where gluons 4 and 5 have opposite helicities.

3.2.2 Opposite-helicity vertex

We now perform an analogous study to section 3.2.1 for the opposite-helicity case, where we must necessarily start from an NMHV helicity configuration. We present the procedure for a configuration of the form

$$\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{-\nu_2}, p_4^\oplus, p_5^\ominus, p_6^{-\nu_1}).$$

As for the tree-level vertex, after the CEV for $\nu_4 = -\nu_5 = \oplus$ has been obtained, the CEV for $\nu_4 = -\nu_5 = \ominus$ may be obtained by complex conjugation. The colour-ordered one-loop six-gluon NMHV amplitudes in $\mathcal{N} = 4$ SYM are given in eqs. (A.17) and (A.19). Unlike eq. (3.70), these NMHV amplitudes are not proportional to the tree-level amplitude. Instead, the transcendental functions of the NMHV amplitude, W_i , are multiplied by pairs of the rational functions that appear in the BCFW representations of the tree-level amplitude listed in appendix A.2.1. In the NMRK, after dividing by the appropriate impact factors and t -channel propagators, these rational coefficients are simply given by sums of pairs of the rational functions which we have studied in section 3.1.1,

$$S_{xy}^I = R_{xy}^I + R_{\bar{x}\bar{y}}^I \quad x, y \in \{u, v, w\}, \quad I \in \{A, A', B\}. \quad (3.110)$$

Like the tree-level amplitudes, these rational coefficients of the transcendental functions W_i change only by a sign under exchange of $p_2^{\nu_2} \leftrightarrow p_3^{\nu_3}$ or $p_6^{\nu_6} \leftrightarrow p_1^{\nu_1}$. For the 12 leading colour orderings, we further find that an exchange of $p_2 \leftrightarrow p_3$ or $p_6 \leftrightarrow p_1$ does not alter the real part of the functions W_i . These properties collectively allow the dispersive part of the amplitude to be written as

$$\begin{aligned} & \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} \\ & g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} \frac{1}{|q_{3\perp}|^2} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\ & \times \left\{ \text{tr} (F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \left[S_{uv}^A \text{Re} [W_1(\sigma_A)] + S_{vw}^A \text{Re} [W_2(\sigma_A)] + S_{wu}^A \text{Re} [W_3(\sigma_A)] \right] \right. \\ & \quad + \text{tr} (F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \left[S_{uv}^{A'} \text{Re} [W_1(\sigma_{A'})] + S_{vw}^{A'} \text{Re} [W_2(\sigma_{A'})] + S_{wu}^{A'} \text{Re} [W_3(\sigma_{A'})] \right] \\ & \quad \left. + \text{tr} (F^{c_1} F^{a_5} F^{c_3} F^{a_4}) \left[S_{uv}^B \text{Re} [W_1(\sigma_B)] + S_{vw}^B \text{Re} [W_2(\sigma_B)] + S_{wu}^B \text{Re} [W_3(\sigma_B)] \right] \right\}, \end{aligned} \quad (3.111)$$

analogous to eq. (3.75). On the right-hand side of eq. (3.111), the transcendental functions are understood to be taken in the NMRK limit, but for brevity we do not indicate this explicitly. As in section 3.2.1, we find that the dispersive part of the amplitude is given solely by $(-, -)$ exchange. This is not true for the absorptive part, which is studied in appendix A.3. We will see in section 3.2.3 that the absorptive part does not contribute to the helicity- and colour-summed NLO squared matrix element for $gg \rightarrow gggg$, or to the dijet cross section at NNLL. The absorptive part will only contribute at N³LL. For the remainder of this section we focus on the dispersive part of the amplitude, eq. (3.111).

Unlike the tree-level expressions eq. (3.7), the sums of two rational terms, eq. (3.110), do contain genuine unphysical poles. To demonstrate that the dispersive part of the NMRK amplitude, eq. (3.111), is nonetheless free from unphysical singularities it is useful to use the fact that the sum of the three S_{xy}^I is twice the tree-level CEV to rewrite eq. (3.111) as

$$\begin{aligned}
& \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} \\
& \times \left\{ \text{tr} (F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \left[2A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \text{Re} [W_2(\sigma_A)] \right. \right. \\
& \quad \left. \left. + S_{uv}^A \text{Re} [W_1(\sigma_A) - W_2(\sigma_A)] + S_{wu}^A \text{Re} [W_3(\sigma_A) - W_2(\sigma_A)] \right] \right. \\
& + \text{tr} (F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \left[2A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \text{Re} [W_2(\sigma_{A'})] \right. \\
& \quad \left. + S_{uv}^{A'} \text{Re} [W_1(\sigma_{A'}) - W_2(\sigma_{A'})] + S_{wu}^{A'} \text{Re} [W_3(\sigma_{A'}) - W_2(\sigma_{A'})] \right] \\
& + \text{tr} (F^{c_1} F^{a_5} F^{c_3} F^{a_4}) \left[2B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \text{Re} [W_1(\sigma_B)] \right. \\
& \quad \left. + S_{vw}^B \text{Re} [W_2(\sigma_B) - W_1(\sigma_B)] + S_{wu}^B \text{Re} [W_3(\sigma_B) - W_1(\sigma_B)] \right] \left. \right\} \\
& \times \frac{1}{|q_{3\perp}|^2} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}).
\end{aligned} \tag{3.112}$$

This rewriting simplifies our task for two reasons:

- From section 3.1 we know the functions $A^{gg(0)}$ and $B^{gg(0)}$ are free from unphysical singularities.
- The difference between two cyclic permutations of W is much simpler than a given permutation of W .

To expand on the second point, the difference between two cyclic permutations of

W can be expressed as a single product of logarithms whose arguments are simple functions of the cross ratios. This property holds even in general kinematics, although in the following we implicitly consider the cross ratios to be taken in the NMRK limit. Let us first consider these terms in the σ_A ordering,

$$\begin{aligned} W_1(\sigma_A) - W_2(\sigma_A) &= \log(v_A) \log\left(\frac{w_A}{u_A}\right), \\ W_3(\sigma_A) - W_2(\sigma_A) &= \log(w_A) \log\left(\frac{v_A}{u_A}\right). \end{aligned} \quad (3.113)$$

We note in passing that these functions are related by the interchange of $u_A \leftrightarrow v_A$, which has the physical interpretation of target-projectile exchange, eq. (3.54). This provides our motivation for arranging eq. (3.112) as we have done, where $W_2(\sigma_A)$ multiplies the corresponding tree-level CEV, while the remaining terms are related by

$$S_{uv}^A \text{Re}[W_1(\sigma_A) - W_2(\sigma_A)] \xrightarrow{F} \frac{w\bar{z}}{\bar{w}z} S_{wu}^A \text{Re}[W_3(\sigma_A) - W_2(\sigma_A)]. \quad (3.114)$$

Table 3.1 lists the behaviour of the cross ratios at each unphysical-singularity surface. From this table we see that at $P_v^A = 0$ or $P_w^A = 0$, the transcendental terms which multiply the divergent rational terms in eq. (3.112) vanish. For the remaining unphysical singularity at $P_u^A = 0$, we find

$$\begin{aligned} S_{uv}^A (W_1(\sigma_A) - W_2(\sigma_A)) + S_{wu}^A (W_3(\sigma_A) - W_2(\sigma_A)) \\ \xrightarrow{P_u^A \rightarrow 0} (S_{uv}^A + S_{wu}^A) \log^2 \left(\frac{|z-1|^2 X}{(1+X)(1+|z|^2 X)} \right), \end{aligned} \quad (3.115)$$

which is finite at $P_u^A = 0$, because in the sum $S_{uv}^A + S_{wu}^A = 2A^{gg(0)} - S_{vw}^A$, the apparent pole at $P_u^A = 0$ is a removable singularity.

	$P_u^A = 0$	$P_v^A = 0$	$P_w^A = 0$
u_A	1	$\frac{ w-1 ^2 X}{(1+X)(1+ w ^2 X)}$	$\frac{ z-1 ^2 X}{(1+X)(1+ z ^2 X)}$
v_A	$\frac{ z-1 ^2 X}{(1+X)(1+ z ^2 X)}$	1	$\frac{ z-1 ^2 X}{(1+X)(1+ z ^2 X)}$
w_A	$\frac{ z-1 ^2 X}{(1+X)(1+ z ^2 X)}$	$\frac{ w-1 ^2 X}{(1+X)(1+ w ^2 X)}$	1

Table 3.1 *Behaviour of the cross ratios in the NMRK limit at the three unphysical-singularity surfaces of the rational terms of the ordering σ_A , listed in eq. (3.37).*

The $\sigma_{A'}$ ordering is exactly analogous. For the σ_B ordering we have singled out in eq. (3.112) the transcendental function $W_1(\sigma_B)$ to multiply the tree-level amplitude. The other transcendental functions that appear in this colour structure are then

$$\begin{aligned} W_2(\sigma_B) - W_1(\sigma_B) &= \log(v_B) \log\left(\frac{u_B}{w_B}\right), \\ W_3(\sigma_B) - W_1(\sigma_B) &= \log(u_B) \log\left(\frac{v_B}{w_B}\right), \end{aligned} \quad (3.116)$$

which are related to each other by the map G introduced in section 3.2.1. For the NMHV tree-level rational functions in the NMRK limit, this map is equivalent to the composition of interchanging $p_4 \leftrightarrow p_5$ with complex conjugation, i.e. the remaining terms in eq. (3.112) have the property,

$$S_{vw}^B (W_2(\sigma_B) - W_1(\sigma_B)) \xleftrightarrow{p_4 \leftrightarrow p_5} (S_{wu}^B)^* (W_3(\sigma_B) - W_1(\sigma_B)), \quad (3.117)$$

where we have used eq. (3.23). Using the properties summarised in table 3.2, we see that this ordering is also free from unphysical singularities. At $P_u^B = 0$ or $P_v^B = 0$, the rational terms in eq. (3.112) which diverge are multiplied by transcendental terms (eq. (3.116)) that vanish. At $P_w^B = 0$ we see that the transcendental terms in eq. (3.116) become equal, allowing us to write

$$S_{vw}^B (W_2(\sigma_B) - W_1(\sigma_B)) + S_{wu}^B (W_3(\sigma_B) - W_1(\sigma_B)) \xrightarrow{P_w^B \rightarrow 0} (S_{vw}^B + S_{wu}^B) \log^2 \left(\frac{1}{1 + |z|^2 X} \right). \quad (3.118)$$

The sum of rational terms $S_{vw}^B + S_{wu}^B$ is finite at $P_w^B = 0$, so we have shown that the dispersive part of the amplitude eq. (3.112) is free from unphysical singularities, as expected.

	$P_u^B = 0$	$P_v^B = 0$	$P_w^B = 0$
u_B	1	$\frac{1}{1 + 1 + \frac{w}{z} ^2 \frac{1}{X}}$	$\frac{1}{1 + z ^2 X}$
v_B	$\frac{1}{1 + z ^2 X}$	1	$\frac{1}{1 + z ^2 X}$
w_B	$\frac{1}{1 + z ^2 X}$	$\frac{1}{1 + 1 + \frac{w}{z} ^2 \frac{1}{X}}$	1

Table 3.2 *Behaviour of the cross ratios in the NMRK at the three unphysical-singularity surfaces of the rational terms of the ordering σ_B , listed in eq. (3.46).*

As in section 3.2.1, we can write the real part of the transcendental functions exactly in terms of the one-loop trajectory, impact factors and Lipatov vertex,

plus a NMRK remainder term. We first separate the transcendental function into ‘central’ and ‘non-central’ pieces,

$$2\text{Re}[W_2(\sigma_A)] = E(-t_1, s_{34}; -t_3, s_{56}) + 2W_2^A, \quad (3.119)$$

with E given in eq. (3.77), and we further separate the central function into

$$2W_2^A = U_A + \Delta W_A(u_A, v_A, w_A). \quad (3.120)$$

The function U_A is given by eq. (3.81). In other words, this is a helicity-independent term. In contrast, for the opposite-helicity case the remainder function is given by

$$\Delta W_A(u_A, v_A, w_A) = -\frac{1}{4} \log^2(u_A) + \frac{1}{2} \log(u_A) \log(v_A w_A) - \log(v_A) \log(w_A). \quad (3.121)$$

Note that, as for the same-helicity case, eq. (3.82), the variable assignment in eq. (3.81) leads to a NMRK remainder function which is IR-finite and symmetric under $v_A \leftrightarrow w_A$. However, we note that unlike eq. (3.82), eq. (3.121) does not vanish in the MRK limit. We will return to this point later. When working in NMRK, in order to exploit the equality of the real parts of the transcendental functions under an exchange of $p_1 \leftrightarrow p_6$ or $p_2 \leftrightarrow p_3$, we make the approximation eq. (3.83), and we make the NMRK approximation of the cross ratios in eq. (3.121). For the A' ordering we similarly write

$$\begin{aligned} 2\text{Re}[W_2(\sigma_{A'})] &= E(t_1, s_{35}; t_3, s_{46}) + 2W_2^{A'}, \\ 2W_2^{A'} &= U_{A'} + \Delta W_A(u_{A'}, w_{A'}, v_{A'}). \end{aligned} \quad (3.122)$$

where $U_{A'}$ is related to U_A by exchange of $p_4 \leftrightarrow p_5$.

As in section 3.2.1, we can write the transcendental function of the σ_B ordering in terms of the one-loop factorised pieces in two different ways,

$$\begin{aligned} 2\text{Re}[W_1(\sigma_B)] &= E(-t_1, -s_{24}; -t_3, s_{56}) + 2W_1^B \\ &= E(-t_1, s_{35}; -t_3, -s_{14}) + 2W_1^{B'}, \end{aligned} \quad (3.123)$$

with

$$2W_1^B = U_B + \Delta W_B(u_B, v_B, w_B), \quad 2W_1^{B'} = U_{B'} + \Delta W_B(v_B, u_B, w_B). \quad (3.124)$$

The helicity-independent functions U_B and $U_{B'}$ are given in eq. (3.89), and the NMRK remainder is

$$\Delta W_B(u_B, v_B, w_B) = \log(v_B) \log\left(\frac{w_B}{u_B}\right) + \log(u_B) \log\left(\frac{w_B}{v_B}\right). \quad (3.125)$$

We note that, similar to eq. (3.90), eq. (3.125) is IR finite, symmetric under $u_B \leftrightarrow v_B$, but does not vanish in either MRK or MRK' limits. We emphasise that eq. (3.123) is exact, but in the following we make the NMRK approximation (eq. (3.92)) for U_B or $U_{B'}$, after which the functions U_B and $U_{B'}$ are related by $p_4 \leftrightarrow p_5$. We also make the NMRK approximation to the cross ratios in eq. (3.125), after which this function is symmetric under $p_4 \leftrightarrow p_5$.

Following the procedure of section 3.2.1, we now wish to express the opposite-helicity amplitude using the colour-structure basis of eq. (3.97). To this end, we apply the colour rotation of eq. (3.95) to the sum inside the curly brackets in eq. (3.112). We find that the structure of the kinematic coefficients is similar to the same-helicity case, motivating the following expression for general helicities, generalising eq. (3.97),

$$\begin{aligned} \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^{\nu_4}, p_5^{\nu_5}) \right] &\xrightarrow{\text{NMRK}} \\ &\frac{g_s^6}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{|q_{1\perp}|^2} \times \\ &\sum_{\sigma \in S_2} \left\{ (F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3} \left(A_{\mathbb{F}}^{gg(1)}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) \right. \right. \\ &\quad \left. \left. + N_c \kappa_\Gamma A^{gg(0)}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) E(|q_{1\perp}|^2, s_{3\sigma_4}; |q_{3\perp}|^2, s_{\sigma_5 6}) \right) \right. \\ &\quad \left. + d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3} A_d^{gg(1)}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) \right\} \times \frac{1}{|q_{3\perp}|^2} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}). \end{aligned} \quad (3.126)$$

For the opposite-helicity case we find that the kinematic coefficient of the tree-

level colour structure is

$$\begin{aligned}
& A_{\mathbb{F}}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \\
&= \frac{N_c \kappa_{\Gamma}}{3} \left[2A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \left(2W_2^A + W_1^B \right) - 2A^{gg(0)}(q_1, p_5^{\ominus}, p_4^{\oplus}, q_3) \left(W_2^{A'} - W_1^{B'} \right) \right. \\
&\quad + 2 \left(S_{uv}^A(W_1^A - W_2^A) + S_{wu}^A(W_3^A - W_2^A) \right) - \left(S_{uv}^{A'}(W_1^{A'} - W_2^{A'}) + S_{wu}^{A'}(W_3^{A'} - W_2^{A'}) \right) \\
&\quad \left. - \left(S_{vw}^B(W_2^B - W_1^B) + S_{wu}^B(W_3^B - W_1^B) \right) \right].
\end{aligned} \tag{3.127}$$

There is some freedom in how to partition the kinematic coefficient of d_A as a sum over colour orderings. Before discussing our choice of how we organise this term, let us investigate the MRK limit of eq. (3.127). We first drop the rational terms that are power suppressed in this limit,

$$\begin{aligned}
& A_{\mathbb{F}}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \\
& \xrightarrow{\text{MRK}} \frac{N_c \kappa_{\Gamma}}{3} \left[A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \left(3U_{\text{MRK}} + 2\Delta W_A(u_A, v_A, w_A) + \Delta W_B(u_B, v_B, w_B) \right) \right. \\
& \quad \left. + 2 \left(S_{uv}^A(W_1^A - W_2^A) + S_{wu}^A(W_3^A - W_2^A) \right) - S_{vw}^B(W_2^B - W_1^B) \right],
\end{aligned} \tag{3.128}$$

where U_{MRK} is the helicity-independent MRK limit of the central transcendental functions, introduced in eq. (3.103). We now note that the nonvanishing terms of ΔW_A and ΔW_B cancel against the MRK limits of the terms proportional to S^A and S^B respectively, i.e.

$$\begin{aligned}
& A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \Delta W_A(u_A, v_A, w_A) \underset{\text{MRK}}{=} -S_{uv}^A(W_1^A - W_2^A) - S_{wu}^A(W_3^A - W_2^A), \\
& A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \Delta W_B(u_B, v_B, w_B) \underset{\text{MRK}}{=} S_{vw}^B(W_2^B - W_1^B).
\end{aligned} \tag{3.129}$$

With this we see that the MRK limit of eq. (3.128),

$$A_{\mathbb{F}}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \xrightarrow{\text{MRK}} V^{g(0)}(q_1, p_4^{\oplus}, q_2) \left(\frac{-1}{|q_{2\perp}|^2} \right) V^{g(0)}(q_2, p_5^{\ominus}, q_3) U_{\text{MRK}}, \tag{3.130}$$

coincides with the MRK limit of the same-helicity amplitude, eq. (3.105),

manifesting the helicity independence of the MRK limit. A similar analysis shows that the MRK' limit of eq. (3.128),

$$A_{\mathbb{F}}^{gg(1)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \xrightarrow{\text{MRK}'} V^{g(0)}(q_1, p_5^\ominus, q'_2) \left(\frac{-1}{|q'_{2\perp}|^2} \right) V^{g(0)}(q'_2, p_4^\oplus, q_3) U_{\text{MRK}'} , \quad (3.131)$$

coincides with the MRK' limit of the same-helicity vertex, eq. (3.107).

Equation (3.129) also provides our motivation for writing the kinematic coefficient of the totally symmetric colour structure as

$$\begin{aligned} A_d^{gg(1)}(q_1, p_4^\oplus, p_5^\ominus, q_3) = & \kappa_\Gamma \left[2A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) (W_2^A - W_1^B) \right. \\ & \left. + S_{uv}^A(W_1^A - W_2^A) + S_{wu}^A(W_3^A - W_2^A) + S_{vw}^B(W_2^B - W_1^B) \right], \end{aligned} \quad (3.132)$$

with the aforementioned property eq. (3.117) leading to

$$\begin{aligned} A_d^{gg(1)}(q_1, p_5^\ominus, p_4^\oplus, q_3) = & \kappa_\Gamma \left[2A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) (W_2^{A'} - W_1^{B'}) \right. \\ & \left. + S_{uv}^{A'}(W_1^{A'} - W_2^{A'}) + S_{wu}^{A'}(W_3^{A'} - W_2^{A'}) + S_{vw}^B(W_3^B - W_1^B) \right]. \end{aligned} \quad (3.133)$$

This choice gives us the simple MRK and MRK' limits,

$$\begin{aligned} A_d^{gg(1)}(q_1, p_4^\oplus, p_5^\ominus, q_3) & \xrightarrow{\text{MRK}} 0, \\ A_d^{gg(1)}(q_1, p_5^\ominus, p_4^\oplus, q_3) & \xrightarrow{\text{MRK}'} 0. \end{aligned} \quad (3.134)$$

similar to eqs. (3.106) and (3.108).

3.2.3 NLO $gg \rightarrow gggg$ squared matrix element in the central NMRK limit

Having obtained the central NMRK limit of the six-gluon one-loop amplitude in $\mathcal{N} = 4$ SYM for all leading helicity configurations, we now consider the contribution of this amplitude to the squared matrix element for $gg \rightarrow gggg$ at NLO, i.e. the interference of this amplitude with the tree-level amplitude given

in eq. (2.97), summed over all colour indices:

$$2\text{Re} \left[\mathcal{M}_{6g}^{(0)*} \mathcal{M}_{6g}^{(1)} \right] = 2\text{Re} \left[\mathcal{M}_{6g}^{(0)*} \text{Disp} \left[\mathcal{M}_{6g}^{(1)} \right] \right] - 2\text{Im} \left[\mathcal{M}_{6g}^{(0)*} \text{Absorp} \left[\mathcal{M}_{6g}^{(1)} \right] \right]. \quad (3.135)$$

We consider these two contributions in turn.

Dispersive part

We first consider the contribution of the dispersive part of the one-loop amplitude, eq. (3.126), to eq. (3.135). The colour structures contributing to $\mathcal{M}_{6g}^{(0)*} \text{Disp} [\mathcal{M}_{6g}^{(1)}]$ are depicted as interference diagrams in fig. 3.3. For each of them, after summing over the colour indices a_2, a_3 and a_1, a_6 , the factors associated with the impact factors are proportional to the identity in the adjoint representation, i.e.

$$\text{tr}(F^{c_1} F^{\tilde{c}_1}) = 2N_c \delta^{c_1 \tilde{c}_1}, \quad \text{tr}(F^{c_3} F^{\tilde{c}_3}) = 2N_c \delta^{c_3 \tilde{c}_3}. \quad (3.136)$$

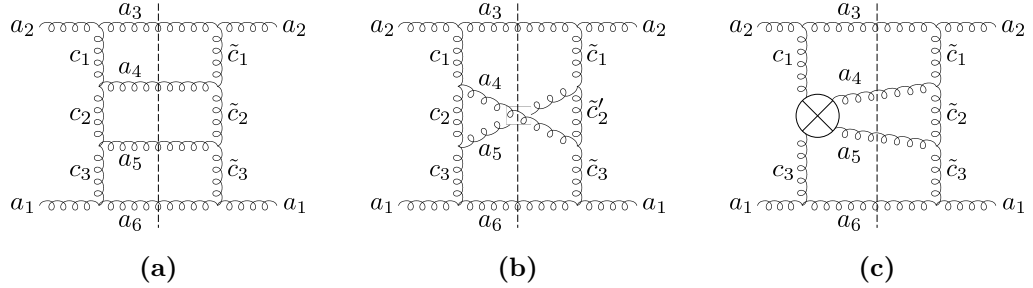


Figure 3.3 Three of the colour structures contributing to $\mathcal{M}_{6g}^{(0)*} \text{Disp} [\mathcal{M}_{6g}^{(1)}]$ using the basis of one-loop colour structures in eq. (3.126). The remaining three colour structures can be obtained by interchanging $a_4 \leftrightarrow a_5$. A three-gluon vertex represents the adjoint generator F , while the $\otimes$ symbol represents the fully symmetric trace d_A of eq. (1.92).

We then see that the colour structures involving the totally symmetric tensor d_A vanish after contraction with the partially antisymmetric tree-level colour-structure:

$$4N_c^2 \delta^{c_1 \tilde{c}_1} \delta^{c_3 \tilde{c}_3} d_A^{c_1 a_4 a_5 c_3} (F^{a_4} F^{a_5})_{\tilde{c}_1 \tilde{c}_3} = -8N_c^2 d_A^{c_1 a_4 a_5 c_3} f^{c_1 a_4 \tilde{c}_2} f^{\tilde{c}_2 a_5 c_3} = 0. \quad (3.137)$$

The coefficient $A_d^{gg(1)}$ of the two-gluon CEV in eq. (3.126) therefore does not

contribute to the NLO $gg \rightarrow 4 \text{ jet}$ cross section, or by extension, to the N³LO dijet cross section after phase-space integration. We emphasise that this conclusion applies to any helicity configuration of the one-loop two-gluon CEV.

By a similar argument, the symmetric colour structure of the two-gluon CEV does not contribute to the NNLL BFKL kernel. To see this we note that at this logarithmic accuracy, the relevant interference, or ‘ladder’ diagrams describing an arbitrary number of real emissions along the ladder, including precisely one instance of the one-loop two-gluon CEV, have all other ‘rungs’ generated by the interference of tree-level impact factors and one-gluon CEVs (those appearing in fig. 2.8b). The colour structure containing d_A of such a ladder diagram is depicted in fig. 3.4. Given that the colour structure of the tree-level impact factors are either a fundamental generator or adjoint structure constant, and that the colour-structure of any one-gluon CEV is simply a structure constant, it follows that upon summing over the colour indices, the entire ladder of rungs above and below the one-loop two-gluon CEV collapse with the result proportional the identity in the adjoint representation (with each rung yielding an extra factor of $2N_c$). At the final stage, the contraction between d_A and the structure constant yields a vanishing result, just as in eq. (3.137).

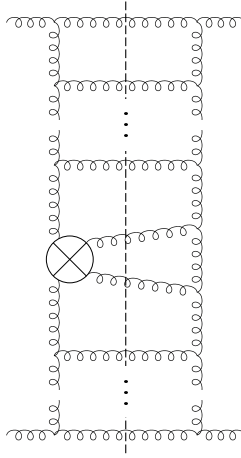


Figure 3.4 *A colour structure of an interference, or ‘ladder’ diagram at NNLL accuracy, containing one instance of the one-loop two-gluon CEV, and all other real emissions given by tree-level one-gluon CEVs. The $\otimes$ symbol represents the fully symmetric trace d_A of the two-gluon CEV. This colour structure vanishes upon summing over all colour indices, as discussed in the main text.*

We further note that the symmetric colour structure of the two-gluon CEV is expected to contribute to the cross section and to the BFKL kernel at N³LL

accuracy: at this order one must consider the square of the one-loop two-gluon CEV, where the interference involving two d_A terms survives, and yields a quartic Casimir invariant.

Having seen that the symmetric trace of four generators, d_A , which appears in the two-gluon CEV, does not contribute to the cross section or the BFKL kernel at NNLL, it is interesting to ask whether such symmetric colour structures contribute at this logarithmic accuracy to *any* of the interference diagrams making up the BFKL kernel in fig. 2.11. The answer is negative: upon summing over colour indices, the impact factors in any of these interference diagrams would be proportional to the identity in the adjoint representation. This argument holds for both gluon or quark impact factors. Having contracted the impact factors, the virtual loops and the central gluons only involve six colour vertices in total, which cannot form a quartic Casimir in any of the contributions in fig. 2.11. This may be contrasted with contributions to the $2 \rightarrow 2$ amplitude from triple Reggeon exchange, which do involve quartic Casimirs at NNLL accuracy [85, 92, 130].

Absorptive part

We now consider the contribution of the absorptive part of the one-loop amplitude to eq. (3.135). As mentioned above, for the absorptive part of the one-loop amplitude, all signature components contribute at leading power in the NMRK limit, although they do not have logarithmic dependence on the large invariants s_{34} or s_{56} . The four signature components are listed in eqs. (A.25) and (A.28) for $\nu_4 = \nu_5$ and in eqs. (A.29) and (A.32) for $\nu_4 = -\nu_5$. However, the leading-power behaviour of the tree-level amplitude is given by the $(-, -)$ component alone, and this projects out the $(-, -)$ component of the absorptive one-loop amplitude in $\mathcal{M}_{6g}^{(0)*} \text{Absorp}[\mathcal{M}_{6g}^{(1)}]$ upon summing over the external colour indices. For the $(-, -)$ component of the absorptive part of the amplitude (eq. (A.25) for the $\nu_4 = \nu_5$ case and eq. (A.29) for the $\nu_4 = -\nu_5$ case), we can factor out an adjoint generator associated to the tree-level impact factors. Upon considering the interference between these absorptive parts and the tree-level amplitude, we therefore find that the colour factors associated with the impact factors are identical to the dispersive case above, eq. (3.136).

To proceed and evaluate the imaginary part of $\mathcal{M}_{6g}^{(0)*} \text{Absorp}[\mathcal{M}_{6g}^{(1)}]$ we now distinguish between the same-helicity case, where the relevant absorptive part is given in eq. (A.25), and the opposite-helicity case, where the relevant absorptive

part is given in eq. (A.29). For the former case, the colour factors which contain $\text{tr}(F^{c_1} F^{a_4} F^{a_5} F^{c_3})$ or $\text{tr}(F^{c_1} F^{a_5} F^{a_4} F^{c_3})$ do not contribute to the absorptive interference term, due to either the vanishing of the colour factor or the reality of its kinematic coefficient. The remaining contribution is proportional to

$$\begin{aligned} & \text{tr}(F^{c_1} F^{a_5} F^{c_3} F^{a_4})(F^{a_4} F^{a_5})_{c_1 c_3} \text{Im} [B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) A^{gg(0)*}(q_1, p_4^\oplus, p_5^\oplus, q_3)] \\ & + \text{tr}(F^{c_1} F^{a_5} F^{c_3} F^{a_4})(F^{a_5} F^{a_4})_{c_1 c_3} \text{Im} [B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) A^{gg(0)*}(q_1, p_5^\oplus, p_4^\oplus, q_3)] = 0. \end{aligned} \quad (3.138)$$

To show that this quantity is zero we first note that the two colour factors are equal, which can be shown using the reversal-invariance property of the trace of four adjoint generators, along with a relabelling of the indices a_4 and a_5 . After factoring out the common colour factor one is left with the imaginary part of a kinematic factor which can be shown to be real using eq. (2.104). We conclude that the absorptive part of the amplitude does not contribute to the squared matrix element for $gg \rightarrow gggg$ at NLO in the central NMRK limit for the helicity configurations with $\nu_4 = \nu_5$.

Repeating the above analysis for the opposite-helicity case we find a non-vanishing result. Specifically, for $\nu_4 = \oplus$ and $\nu_5 = \ominus$ we find that

$$\begin{aligned} & 2\text{Im} \left[\mathcal{M}_{6g}^{(0)*}(p_4^\oplus, p_5^\ominus) \text{Absorp} \left[\mathcal{M}_{6g}^{(1)}(p_4^\oplus, p_5^\ominus) \right] \right] \\ & \xrightarrow{\text{NMRK}} 2\kappa_\Gamma g_s^8 \frac{\alpha_s}{4\pi} \frac{\pi}{2} \frac{s^2}{|q_{1\perp}|^2 |q_{3\perp}|^2} |C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})|^2 \\ & \times 8 N_c^5 (N_c^2 - 1) \left\{ \text{Im} \left[\left(S_{uv}^A \log(v_A) + S_{wu}^A \log(w_A) \right) A^{gg(0)*}(q_1, p_4^\oplus, p_5^\ominus, q_3) \right] \right. \\ & \quad + \text{Im} \left[\left(S_{uv}^{A'} \log(v_{A'}) + S_{wu}^{A'} \log(w_{A'}) \right) A^{gg(0)*}(q_1, p_5^\ominus, p_4^\oplus, q_3) \right] \\ & \quad \left. - \text{Im} \left[S_{uv}^B B^{gg(0)*}(q_1, p_4^\oplus, p_5^\ominus, q_3) \right] \log \left(\frac{w_B}{u_B v_B} \right) \right\} \times |C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1})|^2, \end{aligned} \quad (3.139)$$

which is generically non-zero.

Next we observe that the absorptive interference term for $\nu_4 = \ominus$ and $\nu_5 = \oplus$ is equal to the complex conjugate of eq. (3.139). Therefore, the absorptive part of the one-loop amplitude in NMRK does not contribute to the helicity-summed squared matrix element for $gg \rightarrow gggg$ at NLO. Thus, the contribution of the one-loop six-gluon amplitude in the central NMRK limit to the cross section at this logarithmic accuracy is entirely driven by the dispersive part.

3.3 All-order conjectures for amplitudes at NLL in a central NMRK limit

In section 3.2 we discussed the six-gluon $\mathcal{N} = 4$ SYM amplitude to one-loop accuracy. However, the results of section 3.2.1 and section 3.2.2 are suggestive of the factorised form that amplitudes may take in the central NMRK limit at higher orders, and potentially at higher multiplicity. In this section we therefore make conjectures for the form that the dispersive part of amplitudes may take at NLL accuracy beyond one loop and beyond the six-gluon case.

The separation of the non-central E functions in eq. (3.126), multiplying the tree-level two-gluon CEV, strongly suggests a factorised picture, analogous to eq. (2.75), for the production of two gluons in the central NMRK limit. The one-loop two-gluon CEV extracted this way is a component of BFKL at NNLL accuracy. As noted in ref. [83], one can place limits on the pattern of Reggeization by requiring that the MRK limit of said CEV agrees with the known behaviour of amplitudes in MRK. The following Reggeization ansatz for the production of two gluons in the central NMRK limit is compatible with both the known MRK limit and with the structure of the dispersive part of the one-loop amplitude eq. (3.126),

$$\begin{aligned}
\text{Disp} \left[\mathcal{M}_{6g}^{(-,-)} \right] &\xrightarrow{\text{NMRK}} s [g_s(F^{a_3})_{a_2 c_1} C^g(p_2^{\nu_2}, p_3^{\nu_3})] \\
&\times \sum_{\sigma \in S_2} \left\{ \frac{1}{|q_{1\perp}|^2} \frac{1}{2} \left[\left(\frac{s_{3\sigma_4}}{\tau} \right)^{\alpha(t_1)} + \left(\frac{-s_{3\sigma_4}}{\tau} \right)^{\alpha(t_1)} \right] \right. \\
&\times [g_s^2(F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3} A_{\mathbb{F}}^{gg}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) + d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3} A_d^{gg}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3)] \\
&\times \frac{1}{|q_{3\perp}|^2} \frac{1}{2} \left[\left(\frac{s_{\sigma_5 6}}{\tau} \right)^{\alpha(t_3)} + \left(\frac{-s_{\sigma_5 6}}{\tau} \right)^{\alpha(t_3)} \right] \Big\} \\
&\times [g_s(F^{a_6})_{a_1 c_3} C^g(p_1^{\nu_1}, p_6^{\nu_6})] .
\end{aligned} \tag{3.140}$$

This factorization is illustrated schematically in fig. 3.5 (a). The first thing to note is that, unlike amplitudes at LL, or signature-odd amplitudes at NLL, the amplitude is not fully factorised. Rather, eq. (3.140) is a sum over two colour orderings, each of which is factorised in the familiar form of impact factors, Reggeized gluons and a two-gluon CEV.

Further to this point are the assumptions we have made regarding the colour

structure d_A beyond one-loop accuracy. In eq. (3.140) we have assumed that this totally-symmetric colour factor couples to Reggeized gluons at higher orders in α_s at NLL accuracy. Specifically, eq. (3.140) assumes a particular form for this Reggeization, based on the partitioning of the kinematic coefficient of d_A , sections 3.2.1 and 3.2.2. While this partitioning seems natural from the point of view of requiring compatibility with the known MRK limit, the nature of Reggeization of this term of course cannot be determined from the one-loop amplitude studied in this thesis. It may be constrained by taking the central NMRK limit of the two-loop six-gluon amplitude. A similar structure is accessible through the non-central NMRK limit of the two-loop five-gluon amplitude.

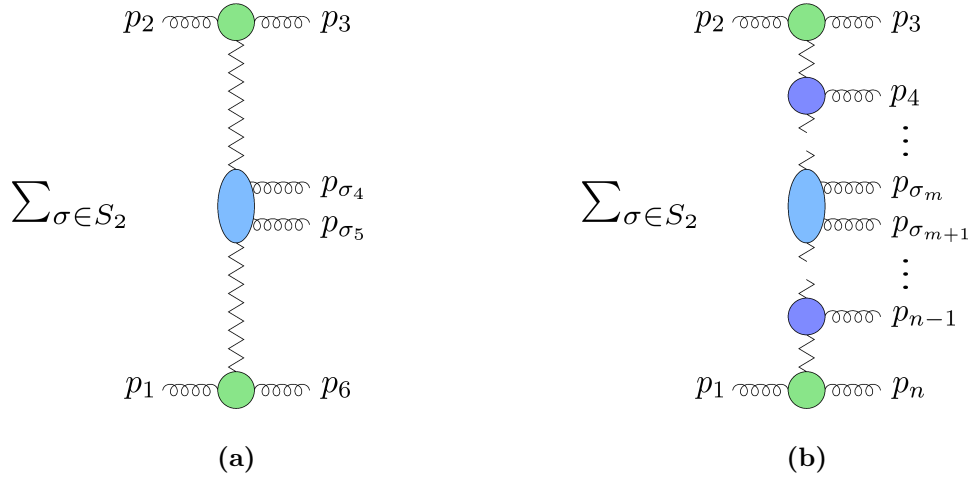


Figure 3.5 (a) The expected factorised picture of the dispersive part of a six-gluon amplitude in the central NMRK limit. (b) The expected factorization of an n -gluon amplitude in a central NMRK limit. The coloured blobs represent impact factors, and one- and two gluon CEVs, defined to all orders in perturbation theory, similarly to fig. 2.6.

With these assumptions in mind, we may consider how eq. (3.140) may be generalised to the case of $2 \rightarrow n$ scattering in any of the central NMRK limits where the rapidity separation between a pair of consecutive central gluons denoted by m and $m + 1$ (with $4 \leq m \leq n - 2$) is not necessarily large, while all other rapidity separations are large. We propose the following ansatz for an arbitrary

number of legs:

$$\begin{aligned}
& \text{Disp} [\mathcal{M}_{ng}^{[8_a]}] \xrightarrow{\text{NMRK}} s [g_S(F^{a_3})_{a_2 c_1} C^g(p_2^{\nu_2}, p_3^{\nu_3})] \\
& \times \prod_{i=4}^{m-1} \left(\frac{1}{t_{i-3}} \frac{1}{2} \left[\left(\frac{s_{i-1,i}}{\tau} \right)^{\alpha(t_{i-3})} + \left(\frac{-s_{i-1,i}}{\tau} \right)^{\alpha(t_{i-3})} \right] [g_S(F^{a_i})_{c_{i-3} c_{i-2}} V^g(q_{i-3}, p_i^{\nu_i}, q_{i-2})] \right) \\
& \times \sum_{\sigma \in S_2} \left\{ \frac{1}{t_{m-3}} \frac{1}{2} \left[\left(\frac{s_{m-1, \sigma_m}}{\tau} \right)^{\alpha(t_{m-3})} + \left(\frac{-s_{m-1, \sigma_m}}{\tau} \right)^{\alpha(t_{m-3})} \right] \right. \\
& \times \left[g_S^2(F^{a_{\sigma_m}} F^{a_{\sigma_{m+1}}})_{c_{m-3} c_{m-1}} A_{\text{F}}^{gg}(q_{m-3}, p_{\sigma_m}^{\nu_{\sigma_m}}, p_{\sigma_{m+1}}^{\nu_{\sigma_{m+1}}}, q_{m-1}) \right. \\
& \quad \left. \left. + d_A^{c_{m-3} a_{\sigma_m} a_{\sigma_{m+1}} c_{m-1}} A_d^{gg}(q_{m-3}, p_{\sigma_m}^{\nu_{\sigma_m}}, p_{\sigma_{m+1}}^{\nu_{\sigma_{m+1}}}, q_{m-1}) \right] \right\} \\
& \times \frac{1}{t_{m-1}} \frac{1}{2} \left[\left(\frac{s_{\sigma_{m+1}, m+2}}{\tau} \right)^{\alpha(t_{m-1})} + \left(\frac{-s_{\sigma_{m+1}, m+2}}{\tau} \right)^{\alpha(t_{m-1})} \right] \Bigg\} \\
& \times \prod_{i=m+2}^{n-1} \left([g_S(F^{a_i})_{c_{i-3} c_{i-2}} V^g(q_{i-3}, p_i^{\nu_i}, q_{i-2})] \frac{1}{t_{i-2}} \frac{1}{2} \left[\left(\frac{s_{i, i+1}}{\tau} \right)^{\alpha(t_{i-2})} + \left(\frac{-s_{i, i+1}}{\tau} \right)^{\alpha(t_{i-2})} \right] \right) \\
& \times [g_S(F^{a_n})_{a_1 c_{n-3}} C^g(p_1^{\nu_1}, p_n^{\nu_n})] ,
\end{aligned} \tag{3.141}$$

which is written entirely in terms of the components extracted from $n \leq 6$ amplitudes. This factorised structure is depicted schematically in fig. 3.5b, where real emissions that are strongly ordered in rapidity are described in a similar way to the MRK structure depicted in fig. 2.6.

3.4 Conclusions

In this chapter we have obtained the one-loop corrections to the two-gluon central-emission vertex (CEV) in $\mathcal{N} = 4$ SYM theory. This is a first step toward the determination of the same quantity in QCD. In contrast with the one-gluon CEV, we find that the two-gluon CEV, eq. (3.97), receives one-loop corrections involving a new colour structure consisting of a fully symmetric trace of four adjoint generators, which is not present at tree level. We find, however, that this new structure does not contribute to the squared matrix element for $gg \rightarrow gggg$ at NLO, so it does not enter the NNLL BFKL kernel.

Compared to the factorised expressions reviewed in chapter 2, the kinematic dependence of the two-gluon CEV is rather complex. In the same-helicity two-

gluon CEV dilogarithms survive in the NMRK limit. In the opposite-helicity CEV, there is an interplay between spurious poles of the rational coefficients and zeroes of the transcendental functions multiplying them, see eqs. (3.112)–(3.116) and tables 3.1 and 3.2.

In ref. [122], it was noted that the transcendental functions of the one-loop six-gluon MHV amplitude can be written exactly in terms of the known one-loop Regge components, eq. (3.78), consisting of impact factors, Reggeized gluons and Lipatov vertices, plus a remainder term (3.82) written in terms of the conformally-invariant cross ratios (3.4). In eqs. (3.119) and (3.121), we have shown that this holds also for the NMHV amplitude, and thus for the whole one-loop six-gluon amplitude in $\mathcal{N} = 4$ SYM. It would be interesting to extend this analysis to the two-loop six-gluon amplitude in full $\mathcal{N} = 4$ SYM.

Our one-loop analysis of the two-gluon CEV provides first hints of the factorization and exponentiation properties of the six-gluon amplitude. An interesting question raised by this chapter is whether the totally symmetric colour structure Reggeises at two loops. This provides strong motivation to study the five- and six- gluon amplitudes in the NMRK limit.

In the NNLL BFKL kernel, the one-loop two-gluon CEV is multiplied by the corresponding tree-level CEV with the gluons p_4 and p_5 emitted along the ladder, which are integrated over their phase space, see fig. 2.11d. In the phase-space integration, both gluons may become soft, as well as collinear to each other, yielding poles of up to $\mathcal{O}(1/\epsilon^3)$. Accordingly, the one-loop two-gluon CEV must be evaluated to $\mathcal{O}(\epsilon^3)$ in the limit that gluons p_4 and p_5 are soft, and/or collinear to each other. To higher order in ϵ , the one-loop splitting functions are known [142, 143], but the one-loop double-soft functions [144] are yet to be determined. We leave the analysis of the higher-order terms of the one-loop two-gluon CEV to future work.

We note that the computational tools developed to aid the manipulations performed in this chapter, as well as the minimal kinematic variables introduced in section 3.1.2, can be straightforwardly adapted to other high-energy limits. These would greatly aid in the extraction of the remaining tree-level and one-loop building blocks, depicted in figs. 2.11e, 2.11g, 2.12f, 2.12g and 2.12i.

Chapter 4

QCD at high energy: The HEJ framework

In section 1.1 we saw how, under a range of general assumptions, amplitudes in the high-energy limit could be described by the exchange of Reggeons (see e.g. eq. (1.25)). In chapter 2 we saw that to LL accuracy, QCD amplitudes do indeed have this simple asymptotic behaviour. We also saw that cross sections could be described at NLL accuracy by amplitudes which are again governed by simple Regge poles. In section 2.1.3 we sketched the BFKL approach to obtain the LL behaviour of partonic cross sections. We summarised numerous issues with this approach at the end of section 2.1.3. Moving to NLL and eventually to NNLL will ameliorate some of these issues, and increase the applicability of the BFKL framework to current collider experiments. However, in this section we pursue another direction towards the generation of stable, precise and reliable predictions for the LHC which include the sizeable high-energy logarithmic corrections currently omitted by most physics searches at the LHC.

In this chapter, we will see that by embracing Monte Carlo (MC) event generation to perform phase space integration, we can address many of the problems of the BFKL approach, even at LL accuracy. This is the approach of the High Energy Jets (HEJ) framework [145].

In chapter 2 we took full advantage of the approximations permitted in the (N)MRK limit in order to obtain factorised approximations of amplitudes that were so simple that, in some cases, cross sections could be obtained by analytic phase space integration. By adopting MC event generation for phase

space integration we are enabled to take more complicated expressions for our factorised building blocks of amplitudes. We can therefore make much less severe approximations of QCD amplitudes, which, as we shall see, results in factorised approximations of amplitudes with many advantages over the analogous expressions in chapter 2. In particular, the minimal approximations of the HEJ approach lead to factorised amplitudes which:

- obey full 4-momentum conservation, not just momentum conservation in the transverse plane,
- are gauge invariant in all of phase-space, not just in the relevant (N)MRK limit,
- preserve crossing relations between incoming and outgoing partons which are broken in the strict (N)MRK limit,
- do not rely on adopting an explicit representation for spinors or polarisation vectors and therefore preserve target-projectile symmetry and little group scaling.

Furthermore, unlike eq. (2.57), momentum fractions can be constructed exactly, so our approximation to the partonic amplitudes does not alter the sampling of PDFs. MC phase space generation also allows for the imposition of arbitrary phase space cuts, allowing us to generate predictions for specific experimental cuts. This is greatly facilitated by the ability to interface with modern analysis tools such as Rivet [146].

Finally, the adoption of MC event generation allows for the matching of the HEJ amplitudes to fixed-order predictions where they are available. This will be the subject of chapter 5. Where fixed-order results are not yet available, or prohibitively computationally expensive, the factorised all-multiplicity HEJ amplitudes can be truncated to a given order in α_s to provide an approximation to the required amplitudes [147].

In this chapter we focus on the HEJ approximation of QCD scattering amplitudes. In particular we derive two new factorised expressions which are required to bring the HEJ framework to NLL accuracy. In chapter 3 we saw how Regge-factorised expressions grew in complexity with increasing logarithmic accuracy, like a microcosm of amplitudes in general kinematics. This study of a Regge-factorised expression therefore demanded the adoption of modern techniques

to the organisation of scattering amplitudes. We similarly found that the techniques originally used to derive the LL HEJ building blocks led to intractable complexity when applied to the new NLL cases. Similar to chapter 3, modern developments in the organisation of amplitudes were necessary to make the calculations manageable.

In this chapter we therefore not only present the derivation of two new factorised expressions (in section 4.2.2 and section 4.3.2), we also re-derive all of the pure-QCD factorised expressions of the HEJ framework, embracing the developments reviewed in section 1.5.3 and section 1.6. This allows us to give a unified presentation of the HEJ building blocks, while also allowing us introduce new techniques and notation in the simpler context of lower-multiplicity amplitudes.

4.1 Factorisation of $2 \rightarrow 2$ amplitudes at LO

In this section we revisit $2 \rightarrow 2$ scattering from the point of view of the HEJ framework. A key observation in ref. [148] was that the amplitude for the scattering of distinguishable quarks, $q_1 q_2 \rightarrow q_1 q_2$, already has the factorised form characteristic of all QCD amplitudes in the Regge limit:

$$\begin{aligned} \mathcal{M}_{q_1 q_2 \rightarrow q_1 q_2}(p_a, p_b; p_1, p_2) &= [ig_s T_{i_1 \bar{i}_a}^{a_A} \bar{u}^{\lambda_1}(p_1) \gamma^{\mu_A} u^{\lambda_a}(p_a)] \\ &\times \left[\frac{-ig_{\mu_A \mu_B} \delta^{a_A a_B}}{t} \right] \\ &\times [ig_s T_{i_2 \bar{i}_b}^{a_B} \bar{u}^{\lambda_2}(p_2) \gamma^{\mu_B} u^{\lambda_b}(p_b)] , \end{aligned} \quad (4.1)$$

where $p_a + p_b = p_1 + p_2$ and $t = (p_a - p_1)^2$. This motivates us to take the 4-vector function [149]

$$\mathcal{J}_{q \rightarrow q}^{\mu a}(\{p_a, \lambda_a, \bar{i}_a\}; \{p_1, \lambda_1, i_1\}) = ig_s T_{i_1 \bar{i}_a}^a J_{q \rightarrow q}^\mu(p_a^{\lambda_a}; p_1^{\lambda_1}), \quad (4.2)$$

$$J_{q \rightarrow q}^\mu(p_a^{\lambda_a}; p_1^{\lambda_1}) = \bar{u}^{\lambda_1}(p_1) \gamma^\mu u^{\lambda_a}(p_a), \quad (4.3)$$

to be our $q \rightarrow q$ impact factor, rather than the simpler scalar function given in eq. (2.11). Of course, we may approximate eq. (4.2) further in the high energy limit. Writing $p_1 = p_a - p_t$, and noting that all components of p_t are small, we can rewrite

$$\bar{u}^{\lambda_1}(p_1) \gamma^{\mu_A} u^{\lambda_a}(p_a) \xrightarrow{s \gg |t|} \delta^{\lambda_1 \lambda_a} e^{i\phi_A} 2p_a^{\mu_A},$$

up to some phase ϕ_A . We can make a similar approximation to the lower quark line, whereby eq. (4.1) becomes

$$\begin{aligned}\mathcal{M}_{q_1 q_2 \rightarrow q_1 q_2}(p_a, p_b; p_1, p_2) = & 2s \left[ig_s T_{i_1 \bar{i}_a}^{a_A} e^{i\phi_A} \delta^{\lambda_1 \lambda_a} \right] \\ & \times \left[\frac{-i\delta^{a_A a_B}}{t} \right] \\ & \times \left[ig_s T_{i_2 \bar{i}_b}^{a_B} e^{i\phi_B} \delta^{\lambda_2 \lambda_b} \right],\end{aligned}\tag{4.4}$$

in agreement with eq. (2.10).

For the same reason, we take the Reggeised gluon propagator, truncated to LO, to be

$$\frac{-i\eta_{\mu_i \mu_j} \delta^{a_i a_j}}{t},\tag{4.5}$$

rather than the simpler

$$\frac{i\delta^{a_i a_j}}{|p_{t\perp}|^2},\tag{4.6}$$

where we need to include the metric tensor in eq. (4.5) in order to contract the 4-vector impact factors of eq. (4.3). In the Regge limit of $2 \rightarrow 2$ scattering, where $p_a^+ \sim p_1^+$, we have the relation

$$t = p_t^2 = (p_a - p_1)^2 = -p_a^+ p_1^- \xrightarrow{\text{MRK}} -|p_{1\perp}|^2 = -|p_{t\perp}|^2,$$

so the correct asymptotic behaviour will be obtained by taking the LO factorised gluon propagator to be eq. (4.5), i.e. we gain LO accuracy without losing LL accuracy. By taking eq. (4.2) to be our $q \rightarrow q$ impact factor and eq. (4.5) to be our LO factorised gluon propagator, we will be able to obtain a factorised description of QCD amplitudes which are not only accurate in a given Regge kinematic region, but which also describe the LO $q_1 q_2 \rightarrow q_1 q_2$ amplitude exactly:

$$\mathcal{M}_{q_1 q_2 \rightarrow q_1 q_2}(p_a, p_b; p_1, p_2) = \mathcal{J}_{q \rightarrow q}^{\mu_t a_t}(p_a; p_1) \left(\frac{-i}{t} \right) \mathcal{J}_{q \rightarrow q}^{a_t \mu_t}(p_b; p_2).\tag{4.7}$$

In ref. [150] it was observed that this exact factorisation applies not only to $q_1 q_2 \rightarrow q_1 q_2$ scattering, but also for certain helicity configurations of $qg \rightarrow qg$ scattering. We will explore this in section 4.1.2. In the following section we will investigate $q_1 q_2 \rightarrow q_1 q_2$ scattering more thoroughly, and introduce techniques and notation that will be used throughout this chapter.

4.1.1 $\bar{q}_1 q_1 \bar{q}_2 q_2$

In this section we investigate the Regge limit of the LO amplitude of two distinguishable quark-antiquark pairs, introducing the techniques and notation that we will use throughout this chapter. We take all momenta to be outgoing, i.e. we consider the amplitude $\emptyset \rightarrow \bar{q}_1 q_1 \bar{q}_2 q_2$. We correspondingly use the notation

$$\mathcal{M}_{\bar{q}_1 q_1 \bar{q}_2 q_2}(\{p_{\bar{q}_1}, \lambda_{\bar{q}_1}, \bar{\iota}_{\bar{q}_1}\}, \{p_{q_1}, \lambda_{q_1}, \iota_{q_1}\}, \{p_{\bar{q}_2}, \lambda_{\bar{q}_2}, \bar{\iota}_{\bar{q}_2}\}, \{p_{q_2}, \lambda_{q_2}, \iota_{q_2}\}), \quad (4.8)$$

which by crossing relations can describe the physical processes

$$\begin{aligned} & \mathcal{M}_{q_1 q_2 \rightarrow q_1 q_2} \\ &= \mathcal{M}_{\bar{q}_1 q_1 \bar{q}_2 q_2}(\{-p_{\bar{q}_1}, -\lambda_{\bar{q}_1}, \bar{\iota}_{\bar{q}_1}\}, \{p_{q_1}, \lambda_{q_1}, \iota_{q_1}\}, \{-p_{\bar{q}_2}, -\lambda_{\bar{q}_2}, \bar{\iota}_{\bar{q}_2}\}, \{p_{q_2}, \lambda_{q_2}, \iota_{q_2}\}), \end{aligned} \quad (4.9)$$

with $p_{\bar{q}_1}^0, p_{\bar{q}_2}^0 < 0$ and $p_{q_1}^0, p_{q_2}^0 > 0$, or

$$\begin{aligned} & \mathcal{M}_{q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2} \\ &= \mathcal{M}_{\bar{q}_1 q_1 \bar{q}_2 q_2}(\{-p_{\bar{q}_1}, -\lambda_{\bar{q}_1}, \bar{\iota}_{\bar{q}_1}\}, \{-p_{q_1}, -\lambda_{q_1}, \iota_{q_1}\}, \{p_{\bar{q}_2}, \lambda_{\bar{q}_2}, \bar{\iota}_{\bar{q}_2}\}, \{p_{q_2}, \lambda_{q_2}, \iota_{q_2}\}), \end{aligned} \quad (4.10)$$

with $p_{\bar{q}_1}^0, p_{q_1}^0 < 0$ and $p_{\bar{q}_2}^0, p_{q_2}^0 > 0$ etc. As the notation of eq. (4.8) is lengthy, in this section we abbreviate the amplitude to

$$\mathcal{M}(\bar{q}_1, q_1, \bar{q}_2, q_2), \quad (4.11)$$

where we use the parton flavour label as shorthand for all the quantum numbers of that parton. In particular, taking all momenta as outgoing makes the MHV classification of a given helicity configuration clear, and will allow us to efficiently define ‘all-outgoing’ factorised approximations to amplitudes.

While 4-point amplitudes are sufficiently simple to work with colour-dressed amplitudes and arbitrary helicities (as we have already done in e.g. eq. (4.1)), in this section we will nevertheless make use of several techniques introduced in chapter 1. Namely, we will work with helicity amplitudes, which will enable us to use the spinor-helicity techniques introduced in section 1.6, and we will work with colour-ordered amplitudes, introduced in section 1.5.3. Although these techniques will yield no simplifications for the process in question, adopting them now will allow us to illustrate the usage of these techniques in a simple context, and will allow us to introduce the notation we will use for higher multiplicity amplitudes, where traditional techniques would lead to untractable expressions.

Let us begin by writing the colour decomposition of this amplitude as,

$$\mathcal{M}(\bar{q}_1, q_1, \bar{q}_2, q_2) = T_{i_{q_1} \bar{i}_{\bar{q}_1}}^{a_t} T_{i_{q_2} \bar{i}_{\bar{q}_2}}^{a_t} M(\bar{q}_1, q_1, \bar{q}_2, q_2), \quad (4.12)$$

which is depicted in fig. 4.1. In such illustrations of the colour decomposition of amplitudes, we use the vertex of eq. (1.62d) to depict a fundamental generator and we will likewise use the vertex of eq. (1.62f) to depict an adjoint generator. The labeling of legs by colour indices in such illustrations should prevent confusion with Feynman diagrams.

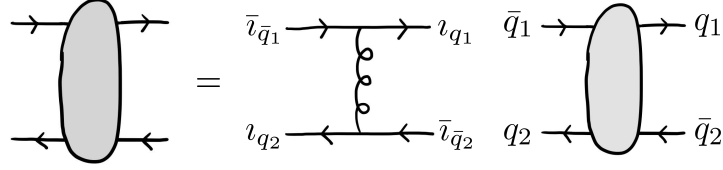


Figure 4.1 *Illustration of the colour decomposition of the amplitude $\emptyset \rightarrow q_1 \bar{q}_1 q_1 \bar{q}_1$, given in eq. (4.12).*

As mentioned, this does not yield any simplification over eq. (4.1), as there is only a single colour structure for this amplitude. Equation (4.12) does however give us an opportunity to apply the colour ordered Feynman rules of eq. (1.115g). The colour-ordered amplitude in question consists of a single Feynman diagram, as illustrated in fig. 4.2.

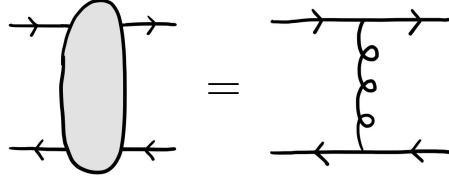


Figure 4.2 *The single colour ordered Feynman diagram of the single partial amplitude in eq. (4.12).*

Let us write down this single colour-ordered Feynman diagram for the helicity configuration $\emptyset \rightarrow \bar{q}_1^\ominus q_1^\oplus \bar{q}_2^\oplus q_2^\ominus$:

$$iM(\bar{q}_1^\ominus, q_1^\oplus, \bar{q}_2^\oplus, q_2^\ominus) = [ig_s[q_1|\mu|\bar{q}_1\rangle]] \left(\frac{-i}{t} \right) [ig_s\langle q_2|\mu|\bar{q}_2\rangle], \quad (4.13)$$

where we define $p_t = -p_{q_1} - p_{\bar{q}_1}$ and $t = p_t^2$. Upon application of the Fierz identity, eq. (1.137), and expressing the Mandelstam invariant t in terms of

spinors, according to eq. (1.139), the amplitude simplifies further to

$$M(\bar{q}_1^\ominus, q_1^\oplus, \bar{q}_2^\oplus, q_2^\ominus) = g_s^2 \frac{[q_1 \bar{q}_2] \langle \bar{q}_1 q_2 \rangle}{\langle q_1 \bar{q}_1 \rangle [q_1 \bar{q}_1]} = g_s^2 \frac{-\langle \bar{q}_1 q_2 \rangle^2}{\langle q_1 \bar{q}_1 \rangle \langle q_2 \bar{q}_2 \rangle}. \quad (4.14)$$

In the last equality we have made use of momentum conservation to bring the amplitude into the compact form given in eq. (1.148). While this form has obvious advantages, it disguises the factorised form made manifest in eq. (4.13). In contrast to the majority of contemporary research on amplitudes, the factorised form of eq. (4.13) gives us motivation to work with individual Feynman diagrams in the following sections. From this factorised form of the amplitude we read off the colour-ordered impact factors

$$J_{\bar{q}qg^*}^\mu(p_{\bar{q}}^\oplus, p_q^\ominus, -(p_{\bar{q}} + p_q)) = ig_s [q|\mu|\bar{q}], \quad (4.15a)$$

$$J_{\bar{q}qg^*}^\mu(p_{\bar{q}}^\ominus, p_q^\oplus, -(p_{\bar{q}} + p_q)) = ig_s \langle q|\mu|\bar{q} \rangle, \quad (4.15b)$$

which are related by complex conjugation. We have explicitly listed the outgoing momentum of the off-shell gluon, g^* . As the colour structure in eq. (4.12) has the desired factorised form, we can define the colour-dressed impact factor

$$\mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}, \{p_q, \lambda_q, \iota_q\}, p_t) = T_{i_{q_1} \bar{\iota}_{\bar{q}_1}}^{a_t} J_{\bar{q}qg^*}^{\mu_t}(p_{\bar{q}}^{\lambda_{\bar{q}}}, p_q^{\lambda_q}, p_t). \quad (4.16)$$

Following the convention that we use for full amplitudes, e.g. eq. (4.12), we will use caligraphic letters for colour dressed objects, and Roman letters for colour-ordered objects. We will also label the momentum p , Lorentz index μ and colour index a associated to a given t -channel by the subscript t , unless this becomes prohibitively unwieldy.

The colour dressed amplitude eq. (4.8) can be written without any approximation as

$$\mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}(\{p_{\bar{q}_1}, \lambda_{\bar{q}_1}, \bar{\iota}_{\bar{q}_1}\}, \{p_{q_1}, \lambda_{q_1}, \iota_{q_1}\}, p_t) \left(\frac{-i}{t} \right) \mathcal{J}_{\bar{q}qg^* \mu_t a_t}^{a_t}(\{p_{\bar{q}_2}, \lambda_{\bar{q}_2}, \bar{\iota}_{\bar{q}_2}\}, \{p_{q_2}, \lambda_{q_2}, \iota_{q_2}\}, -p_t),$$

or, adopting the analogous abbreviation, eq. (4.12) can be written as

$$\mathcal{M}(\bar{q}_1, q_1, \bar{q}_2, q_2) = \mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}(\bar{q}_1, q_1, p_t) \left(\frac{-i}{t} \right) \mathcal{J}_{\bar{q}qg^* \mu_t a_t}(\bar{q}_2, q_2, -p_t). \quad (4.17)$$

Just as for full amplitudes, we may utilise crossing relations to obtain impact

factors for physical processes, for example

$$\mathcal{J}_{q \rightarrow qg}^{\mu_t a_t}(\{-p_{\bar{q}}, -\lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}; \{p_q, \lambda_q, \iota_q\}, p_t) = \mathcal{J}_{\bar{q}qg}^{\mu_t a_t}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}, \{p_q, \lambda_q, \iota_q\}, p_t), \quad (4.18)$$

with $p_q^0 > 0$, $p_{\bar{q}}^0 < 0$ or

$$\mathcal{J}_{\bar{q} \rightarrow \bar{q}g}^{\mu_t a_t}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}; \{-p_q, -\lambda_q, \iota_q\}, p_t) = \mathcal{J}_{q\bar{q}g}^{\mu_t a_t}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}, \{p_q, \lambda_q, \iota_q\}, p_t), \quad (4.19)$$

with $p_q^0 < 0$, $p_{\bar{q}}^0 > 0$. This is permitted because, unlike the approach of chapter 2, we have not yet had to specify which is the ‘incoming’ parton with $p_{\perp} = 0$. At this stage such crossing relations may seem like an overindulgence in formalism. However, there will be a great advantage of being able to perform such crossings when we study the factorisation of higher multiplicity amplitudes.

As mentioned above, another advantage of not specifying a specific representation for spinors in eq. (4.15) is that target-projectile symmetry remains unbroken. This is unlike the analogous expressions in eq. (2.11) where target-projectile duality is explicitly broken by a phase. Finally, the HEJ impact factors eq. (4.15) obey the expected little-group scaling, which is further information that is lost in the approach of chapter 2.

4.1.2 $\bar{q}qgg$

We now perform the analogous study of the process $\emptyset \rightarrow \bar{q}qgg$, whose amplitude we denote in full as

$$\mathcal{M}_{\bar{q}qgg}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\iota}_{\bar{q}}\}, \{p_q, \lambda_q, \iota_q\}, \{p_{g_1}, \lambda_{g_1}, a_{g_1}\}, \{p_{g_2}, \lambda_{g_2}, a_{g_2}\}), \quad (4.20)$$

which in this section we abbreviate to $\mathcal{M}(\bar{q}, q, g_1, g_2)$. Let us begin by writing the colour decomposition of the amplitude,

$$\mathcal{M}(\bar{q}, q, g_1, g_2) = (T^{a_{g_1}} T^{a_{g_2}})_{\iota_q \bar{\iota}_{\bar{q}}} M(\bar{q}, q, g_1, g_2) + (T^{a_{g_2}} T^{a_{g_1}})_{\iota_q \bar{\iota}_{\bar{q}}} M(\bar{q}, q, g_2, g_1). \quad (4.21)$$

which is depicted in fig. 4.3.

We note that unlike eq. (4.1), neither of these colour factors contribute solely to the exchange of the **8** representation in the t -channel. We can project the **8** contribution from each colour factor by applying the projection operator

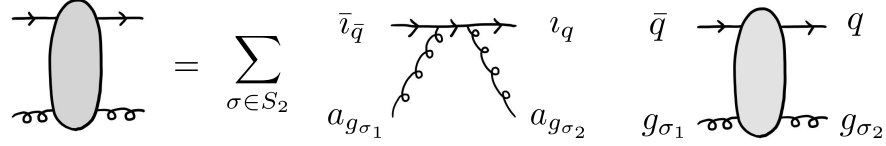


Figure 4.3 *Illustration of the colour decomposition of the amplitude $\emptyset \rightarrow q\bar{q}gg$, given in eq. (4.21).*

eq. (1.96):

$$\begin{aligned} \mathcal{M}^{[8]}(\bar{q}, q, g_1, g_2) &= \frac{1}{T_F} T_{i_q \bar{i}_q}^{a_t} \text{tr}(T^{a_t} T^{a_{g_1}} T^{a_{g_2}}) M(\bar{q}, q, g_1, g_2) \\ &+ \frac{1}{T_F} T_{i_q \bar{i}_q}^{a_t} \text{tr}(T^{a_t} T^{a_{g_2}} T^{a_{g_1}}) M(\bar{q}, q, g_2, g_1). \end{aligned} \quad (4.22)$$

Here we write the t -channel representation in square brackets, $[\mathbf{R}]$ to avoid confusion with the ‘loop’ representation $(\mathbf{R})$ of section 1.7. This projection has allowed us to factor out $T_{i_q \bar{i}_q}^{a_t}$ from each colour structure. We next apply the projection operator eq. (1.100) to the external adjoint colour indices to obtain

$$\mathcal{M}^{[8_a]}(\bar{q}, q, g_1, g_2) = \frac{i}{2} T_{i_q \bar{i}_q}^{a_t} f^{a_t a_{g_1} a_{g_2}} (M(\bar{q}, q, g_1, g_2) - M(\bar{q}, q, g_2, g_1)). \quad (4.23)$$

As in section 2.1.1 we associate the factorised adjoint generator with the impact factor of the gluons.

Our aim is to factor out an overall $\mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}$ from this colour-dressed amplitude, so our remaining task is to factor out $J_{\bar{q}qg^*}^{\mu_t}$ from each colour ordered amplitude on the right-hand side of eq. (4.23). As mentioned above, for some helicity configurations, it is possible to write the colour-ordered amplitudes in this way without making any kinematic approximations. To see how this arises, it is simplest to start by writing the colour-ordered amplitudes as a sum of Feynman diagrams.

The Feynman diagrams of the colour-ordered amplitude $M(\bar{q}, q, g_1, g_2)$ are shown in fig. 4.4. In section 1.6.1 we saw that the only non-zero amplitudes are those where the gluons have opposite helicity. Let us analyse the first diagram in fig. 4.4. The three gluon vertex in this diagram, eq. (1.116), contracted with the gluon polarisation vectors, is equal to

$$ig_s (\eta^{\mu_{g_1} \mu_{g_2}} (p_{g_1} - p_{g_2})^{\mu_t} + \eta^{\mu_t \mu_{g_2}} 2p_{g_2}^{\mu_{g_1}} - \eta^{\mu_t \mu_{g_1}} 2p_{g_1}^{\mu_{g_2}}) \epsilon_{\mu_{g_1}}(p_{g_1}^{\lambda_{g_1}}, p_{r_1}) \epsilon_{\mu_{g_2}}(p_{g_2}^{\lambda_{g_2}}, p_{r_2}) \quad (4.24)$$

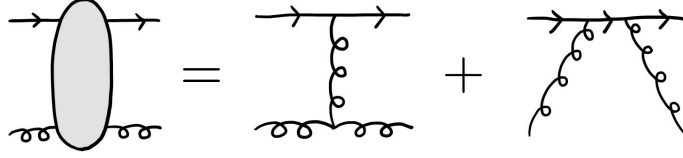


Figure 4.4 The Feynman diagrams of the partial amplitudes in eq. (4.23).

If $\lambda_{g_1} = -\lambda_{g_2}$, then setting $p_{r_1} = p_{g_2}$ and $p_{r_2} = p_{g_1}$ leads to this diagram vanishing. This can be verified using the representations of the polarisation vectors given in eq. (1.142). In particular, we note that for this choice of reference momenta, we obtain

$$\epsilon(p_{g_1}^\lambda, p_{g_2}) \cdot \epsilon(p_{g_2}^{-\lambda}, p_{g_1}) = 0.$$

This is what ref. [151] refers to as a ‘minimal gauge’, i.e. one that minimises the number of non-zero inner products of polarisation vectors.

Let us now consider the remaining diagram in fig. 4.4 for the helicity configuration $\emptyset \rightarrow \bar{q}^\ominus, q^\oplus, g_1^\ominus, g_2^\oplus$:

$$\begin{aligned} iM(\bar{q}^\ominus, q^\oplus, g_1^\ominus, g_2^\oplus) &= -g_s^2 \frac{[q|\mu_{g_1}(p_q + p_{g_1})\mu_{g_2}|\bar{q}]}{s_{qg_1}} \left(\frac{-i}{t} \right) \frac{\langle g_1|\mu_{g_1}|g_2 \rangle [g_2|\mu_{g_2}|g_1]}{\sqrt{2}[g_1g_2]} \frac{1}{\sqrt{2}\langle g_2g_1 \rangle} \\ &= (i[q|\mu_t|\bar{q}]) \left(\frac{-i}{t} \right) \left(i \frac{[qg_2]}{[qg_1]} [g_2|\mu_t|g_1] \right). \end{aligned} \quad (4.25)$$

This colour-ordered amplitude has the desired factorisation property, which motivates us to take the colour-ordered ggg^* current to be

$$J_{ggg^*}^{\mu_t}(p_{g_1}^\ominus, p_{g_2}^\oplus, -p_t) = ig_s \frac{[qg_2]}{[qg_1]} \langle g_1|\mu_t|g_2 \rangle. \quad (4.26)$$

It may be concerning that this current has an apparent dependence on p_q , in contrast to eq. (4.15) which depends only on the momenta which are ‘local in rapidity’. We emphasise that in obtaining eq. (4.26), we have not had to specify that any partons lie on the beam axis. Via momentum conservation we obtain the equality

$$\frac{[qg_2]}{[qg_1]} = -\frac{\langle \bar{q}g_1 \rangle}{\langle \bar{q}g_2 \rangle}, \quad (4.27)$$

which, after using the explicit representation of inner products listed in section 1.6.2, makes it clear that this factor does not depend on the quark momenta

upon choosing either p_q or $p_{\bar{q}}$ to lie on the beam axis.

When we move to higher multiplicity amplitudes, the relationship eq. (4.27) does not hold, so there is an ambiguity in how to generalise eq. (4.26) for higher multiplicity processes, without first specifying a parton to lie on the beam axis. Currently in HEJ we use the implementation given in ref. [150], which uses an explicit representation of physical spinors and therefore avoids this ambiguity. However, for the reasons discussed beneath eq. (4.19) it would be desirable to adopt a representation-independent $g \rightarrow g$ impact in the future, where the ambiguity discussed above may be fixed by analysing five-parton amplitudes.

For the other colour ordering in eq. (4.21) we similarly obtain

$$\begin{aligned} M(\bar{q}^\ominus, q^\oplus, g_2^\oplus, g_1^\ominus) &= \frac{[q|\mu_{g_2}(p_q + p_{g_2})\mu_{g_1}|\bar{q}]}{s_{qg_2}} \left(\frac{-i}{t} \right) \frac{\langle g_1|\mu_{g_1}|g_2\rangle}{\sqrt{2}[g_1g_2]} \frac{[g_2|\mu_{g_2}|g_1\rangle}{\sqrt{2}\langle g_2g_1\rangle} \\ &= (i[q|\mu_t|\bar{q}]) \left(\frac{-i}{t} \right) \left(-i \frac{\langle qg_1\rangle}{\langle qg_2\rangle} \langle g_1|\mu_t|g_2\rangle \right). \end{aligned} \quad (4.28)$$

Returning to the colour-dressed amplitude, we conclude that we can write the octet component of the amplitude exactly as

$$\mathcal{M}^{[8_a]}(\bar{q}, q, g_1, g_2) = \mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}(\bar{q}, q, p_t) \left(\frac{-i}{t} \right) \mathcal{J}_{ggg^* \mu_t}^{a_t}(g_1, g_2, -p_t) \quad (4.29)$$

where we take the colour dressed ggg^* impact factor to be

$$\begin{aligned} \mathcal{J}_{ggg^*}^{\mu_t a_t}(\{p_{g_1}, \lambda_{g_1}, a_{g_1}\}, \{p_{g_2}, \lambda_{g_2}, a_{g_2}\}, -p_t) &= \\ \frac{1}{T_F} f^{a_{g_1} a_{g_2} a_t} \left(J_{ggg^*}^\mu(p_{g_1}^{\lambda_{g_1}}, p_{g_2}^{\lambda_{g_2}}, -p_t) - J_{ggg^*}^\mu(p_{g_2}^{\lambda_{g_2}}, p_{g_1}^{\lambda_{g_1}}, -p_t) \right). \end{aligned} \quad (4.30)$$

We note in passing a further observation of ref. [150]. We have seen above that both colour-ordered amplitudes of eq. (4.21) can be written exactly as a contraction of two spinor currents over a t -channel pole. Returning to eq. (4.21), we see that the full $\emptyset \rightarrow q\bar{q}gg$ amplitude can be expressed as a contraction of currents over a t -channel pole, not just the octet-exchange component. However, in order to build up amplitudes of higher multiplicities, or to apply these factorised ggg^* impact factors to, e.g. $gg \rightarrow gg$ scattering, we must also factorise the colour factors of the amplitude, necessitating the projection operations that lead to eq. (4.23).

Before moving on to the study of five-parton amplitudes, we will briefly make contact with an observation made in section 2.3. In that section we saw that one-

loop gluon amplitudes in $\mathcal{N} = 4$ SYM obeyed an ‘exact factorisation’, analogous to the ‘exact factorisation’ of tree level amplitudes we have seen above. In section 2.3 we also saw that the one-loop $2 \rightarrow 2$ amplitude had an exactly-factorised dual form, corresponding to a different kinematic limit. This motivates us to ask whether tree level amplitudes also have an exact dual form. We have seen above that the colour ordered amplitudes for the process $\emptyset \rightarrow q\bar{q}gg$ can be written without any approximation as a contraction of two currents over a t -channel propagator. This is the form of amplitudes characteristic of the Regge limit $s \gg -t$. However, this process also contributes leading behaviour in the backwards scattering limit $s \gg -u$ which is described to all orders by a Reggeized quark exchange in the u channel [152]. We see that we can indeed write these amplitudes in a ‘dual’ form suggestive of such a quark exchange exchange, e.g.

$$M(\bar{q}^\ominus, q^\oplus, g_1^\ominus, g_2^\oplus) = \left[[q|_i \sigma_{ij}^{\mu_{g_1}} \frac{\langle g_1 | \mu_{g_1} | g_2 \rangle}{\sqrt{2}[g_1 g_2]} \right] \times \frac{i\delta_{jk} (\not{p}_q + \not{p}_{g_1})}{s_{qg_1}} \times \left[\frac{[g_2 | \mu_{g_2} | g_1 \rangle}{\sqrt{2}\langle g_2 g_1 \rangle} \sigma_{kl}^{\mu_{g_2}} | \bar{q} \rangle_l \right] \quad (4.31)$$

where we emphasise that we have made no approximation to the amplitude. Note that in contrast to the gluon propagator eq. (4.5) we factorised above, we have instead factorised a u -channel quark propagator. The corresponding colour-ordered impact factor is [152]

$$\mathcal{J}_{qq^*}^j(p_q^\oplus, p_g^\ominus, p_t) = [q|_i \sigma_{ij}^{\mu_{g_1}} \frac{\langle g_1 | \mu_{g_1} | g_2 \rangle}{\sqrt{2}[g_1 g_2]} \quad (4.32)$$

which has a free spin index rather than the Lorentz index of eq. (4.16) and eq. (4.30). For the remainder of the thesis we will be concerned exclusively with gluon Reggeisation, but we note that the HEJ framework could be straightforwardly adapted to include quark-Reggeisation effects. We leave such investigations for future work.

4.2 Factorisation of $2 \rightarrow 3$ amplitudes at LO

We now perform an analogous study to section 4.1 for five-parton amplitudes. In chapter 2 we first derived all necessary LL building blocks from MRK limits of amplitudes, and subsequently derived all NLL real-emission building blocks from NMRK limits of amplitudes. In this section we will instead begin by taking a

NMRK limit of an amplitude, obtaining a HEJ NLL building block and then show that the resulting factorised amplitude further factorises into HEJ LL building blocks when we move to the MRK limit. This approach avoids repetition of material, and also places an emphasis on the nested factorisation properties of Regge-factorised expressions.

4.2.1 $\bar{q}_1 q_1 g \bar{q}_2 q_2$

In this section we consider the process $\emptyset \rightarrow \bar{q}_1 q_1 g \bar{q}_2 q_2$, with all momenta taken to be outgoing,

$$\mathcal{M}_{\bar{q}_1 q_1 g \bar{q}_2 q_2}(\{p_{\bar{q}_1}, \lambda_{\bar{q}_1}, \bar{\iota}_{\bar{q}_1}\}, \{p_{q_1}, \lambda_{q_1}, \iota_{q_1}\}, \{p_g, \lambda_g, a_g\}, \{p_{q_2}, \lambda_{q_2}, \iota_{q_2}\}, \{p_{\bar{q}_2}, \lambda_{\bar{q}_2}, \bar{\iota}_{\bar{q}_2}\}), \quad (4.33)$$

which, as usual, we abbreviate to

$$\mathcal{M}(\bar{q}_1, q_1, g, \bar{q}_2, q_2). \quad (4.34)$$

We likewise use the abbreviation

$$M_{\bar{q}_1 q_1 g \bar{q}_2 q_2}(p_{\bar{q}_1}^{\lambda_{\bar{q}_1}}, p_{q_1}^{\lambda_{q_1}}, p_g^{\lambda_g}, p_{q_2}^{\lambda_{q_2}}, p_{\bar{q}_2}^{\lambda_{\bar{q}_2}}) = M(\bar{q}_1, q_1, g, \bar{q}_2, q_2), \quad (4.35)$$

for colour-ordered amplitudes, where we suppress helicity dependence unless we are discussing a specific helicity configuration.

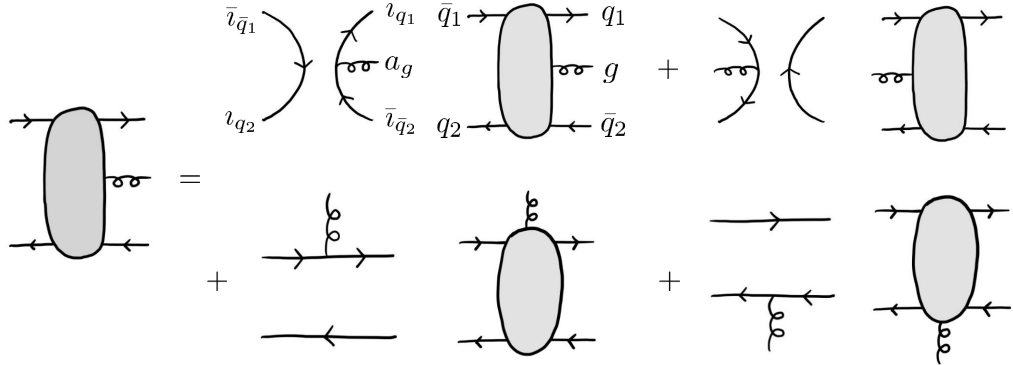


Figure 4.5 *Illustration of the colour decomposition of the amplitude eq. (4.36).*

The colour decomposition for this amplitude can be written

$$\begin{aligned}\mathcal{M}(\bar{q}_1, q_1, g, \bar{q}_2, q_2) = & \delta_{i_{q_2} i_{\bar{q}_1}} T_{i_{q_1} i_{\bar{q}_2}}^{a_g} M(\bar{q}_1, q_1, g, \bar{q}_2, q_2) + \delta_{i_{q_1} i_{\bar{q}_2}} T_{i_{q_2} i_{\bar{q}_1}}^{a_g} M(\bar{q}_1, q_1, \bar{q}_2, q_2, g) \\ & + \delta_{i_{q_2} i_{\bar{q}_2}} T_{i_{q_1} i_{\bar{q}_1}}^{a_g} M(\bar{q}_1, g, q_1, \bar{q}_2, q_2) + \delta_{i_{q_1} i_{\bar{q}_1}} T_{i_{q_2} i_{\bar{q}_2}}^{a_g} M(\bar{q}_1, q_1, \bar{q}_2, g, q_2),\end{aligned}\quad (4.36)$$

which is depicted in fig. 4.5. We now wish to consider this amplitude in the extremal NMRK limit where $y_{q_1} \sim y_{\bar{q}_1} \gg y_g \sim y_{q_2} \sim y_{\bar{q}_2}$. In this limit, we expect to be able to factorise the quark current $\mathcal{J}_{q\bar{q}g}^{\mu_{t_1} a_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1})$ from each term in eq. (4.36), where $p_{t_1} = -p_{q_1} - p_{\bar{q}_1}$. From section 2.1.2 we know that colour-octet exchange in the t_1 channel will dominate in the NMRK limit. We therefore use the colour-octet projection operator eq. (1.96) on each of the colour structures in eq. (4.36), which allows a colour factor of $T_{i_{q_1} i_{\bar{q}_1}}^{a_{t_1}}$ to be factorised from each partial amplitude:

$$\begin{aligned}\mathcal{M}^{[8]}(\bar{q}_1, q_1, g, \bar{q}_2, q_2) = & \frac{1}{T_F} T_{i_{q_1} i_{\bar{q}_1}}^{a_{t_1}} (T^{a_{t_1}} T^{a_g})_{i_{q_2} i_{\bar{q}_2}} M(\bar{q}_1, q_1, g, \bar{q}_2, q_2) \\ & + \frac{1}{T_F} T_{i_{q_1} i_{\bar{q}_1}}^{a_{t_1}} (T^{a_g} T^{a_{t_1}})_{i_{q_2} i_{\bar{q}_2}} M(\bar{q}_1, q_1, \bar{q}_2, q_2, g) \\ & + T_{i_{q_1} i_{\bar{q}_1}}^{a_{t_1}} \delta_{i_{q_2} i_{\bar{q}_2}} \delta^{a_{t_1} a_g} M(\bar{q}_1, g, q_1, \bar{q}_2, q_2).\end{aligned}\quad (4.37)$$

The final colour structure in eq. (4.36) vanishes upon contracting with the projection operator due to eq. (1.70d). The final colour structure in eq. (4.37) was unaltered by the octet projection, but we have rewritten it to clarify that we can factorise $T_{i_{q_1} i_{\bar{q}_1}}^{a_{t_1}}$ from each colour structure. We have illustrated eq. (4.37) in fig. 4.6. Having dealt with the colour factors, we now move to

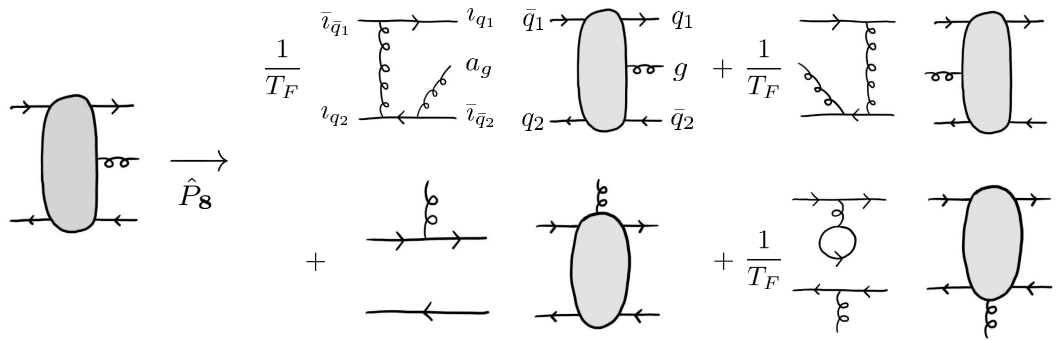


Figure 4.6 *Illustration of the colour decomposition of the amplitude in eq. (4.37), which is obtained from eq. (4.36) upon projecting the octet exchange in the t_1 channel. The final colour factor is zero.*

the kinematic approximations of the colour-ordered amplitudes. Our goal is to

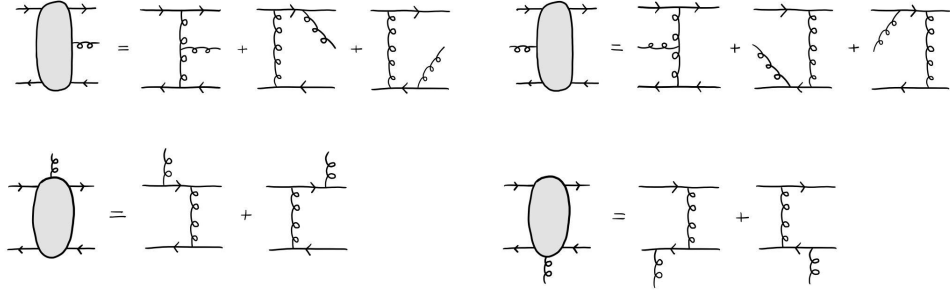


Figure 4.7 *The colour-ordered Feynman diagrams which comprise the partial amplitudes of eq. (4.36) and eq. (4.37).*

factorise $J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1})$ from the three surviving colour-ordered amplitudes in eq. (4.37). The colour-ordered amplitudes of eq. (4.37) are expressed in terms of Feynman diagrams in fig. 4.7. While the final colour-ordered amplitude does not have a colour structure that contributes to $\mathbf{8}$ exchange in the t_1 channel, it is interesting to note in passing that it does have the desired kinematic factorisation property. From fig. 4.7 we see that the only diagrams which are not already of the desired factorised form are where the on-shell gluon is emitted from the $q_1-\bar{q}_1$ line. Such diagrams will be ubiquitous in future sections, so we will spend some time discussing these on-shell gluon emissions. We consider the partial diagrams of fig. 4.8.

Applying the colour-ordered Feynman rules, the diagram fig. 4.8a is

$$\begin{aligned}
& -i \frac{[q_1 | \mu_g (q_1 + g) \mu_{t_2} | \bar{q}_1 \rangle}{s_{q_1 g}} \epsilon^{\mu_g}(p_g^\ominus, p_r) \\
&= \frac{-i}{s_{q_1 g}} ([q_1 | \mu_g | q_1 \rangle [q_1 | \mu_{t_2} | \bar{q}_1 \rangle + [q_1 | \mu_g | g \rangle [g | \mu_{t_2} | \bar{q}_1 \rangle]) \frac{\langle g | \mu_g | r \rangle}{\sqrt{2} [gr]} \\
&= -i \frac{[q_1 r]}{\sqrt{2} [q_1 g] [gr]} [q_1 | \mu_{t_2} | \bar{q}_1 \rangle \\
&= -\frac{2p_{q_1}^{\mu_g}}{s_{q_1 g}} \epsilon_{\mu_g}(p_g^\ominus, p_r) J_{\bar{q}qg^*}^{\mu_{t_2}}(p_{\bar{q}_1}^\ominus, p_{q_1}^\oplus, p_{t_1}),
\end{aligned} \tag{4.38}$$

where we see the factorisation-breaking term vanishes when we contract it with the polarisation vector $\epsilon^{\mu_g}(p_g^\ominus, p_r)$ for any choice of p_r . We opt to leave the polarisation vector explicit in the final line, as this is a convenient way to implement such terms in **FORM** [153], for use with **HEJ**, and simplifies the process of performing crossing relations.

The partial diagram in fig. 4.8d is likewise of the desired factorised form, and can

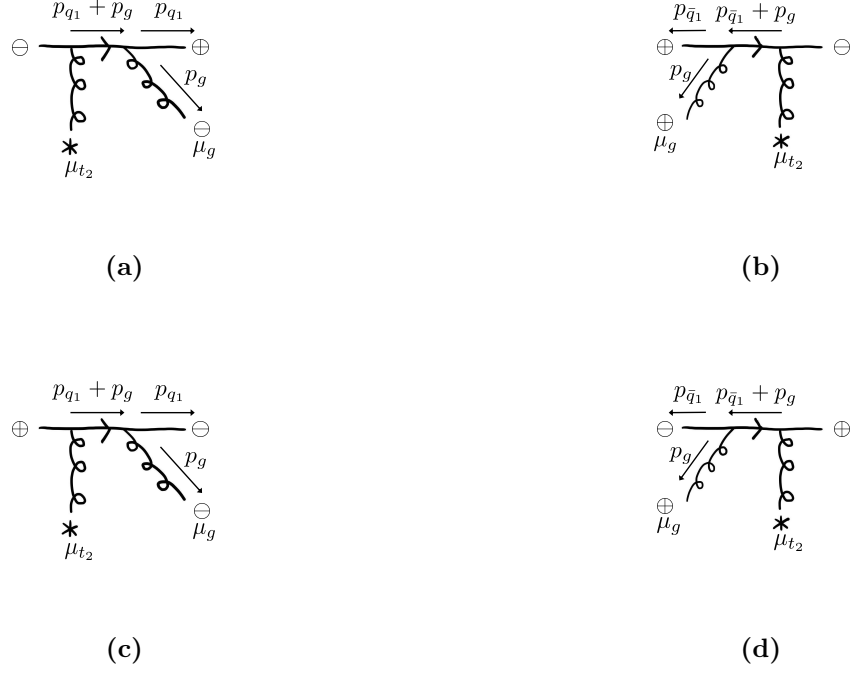


Figure 4.8 *Examples of partial Feynman diagrams where a single on-shell gluon is emitted from a quark line. The kinematic approximations we must make for such diagrams are discussed in the main text.*

be written

$$i \frac{[q_1 | \mu_{t_2} (\bar{q}_1 + g) \mu_g | \bar{q}_1 \rangle}{s_{q_1 g}} \epsilon^{\mu_g}(p_g^\oplus, p_r) = i \frac{2p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1 g}} \epsilon^{\mu_g}(p_g^\oplus, p_r) J_{\bar{q}qg}^{\mu_{t_2}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}). \quad (4.39)$$

without any kinematic approximation. We summarise this simplification by stating a single gluon emission from a quark (or antiquark) line, is described exactly by an Eikonal factor if the gluon and quark (or antiquark) have opposite helicity. This means that, for example, $M(\bar{q}_1^\ominus, q_1^\oplus, g^\ominus, \bar{q}_2^\oplus, q_2^\ominus)$ is already in the desired factorised form, and we need make no kinematic approximations to this colour-ordered amplitude.

What about the case where the gluon and quark (or antiquark) have the same

helicity, e.g. figs. 4.8b and 4.8c? Let us consider fig. 4.8c as an example:

$$\begin{aligned}
& -i \frac{\langle q_1 | \mu_g (q_1 + g) \mu_{t_2} | \bar{q}_1 \rangle}{s_{q_1 g}} \epsilon^{\mu_g} (p_g^\ominus, p_r) \\
&= \frac{-i}{s_{q_1 g}} (\langle q_1 | \mu_g | q_1 \rangle \langle q_1 | \mu_{t_2} | \bar{q}_1 \rangle + \langle q_1 | \mu_g | g \rangle \langle g | \mu_{t_2} | \bar{q}_1 \rangle) \frac{\langle g | \mu_g | r \rangle}{\sqrt{2} [gr]} \\
&= \frac{-i}{\sqrt{2} [q_1 g]} \left(\frac{[q_1 r]}{[gr]} \langle q_1 | \mu_{t_2} | \bar{q}_1 \rangle + \langle g | \mu_{t_2} | \bar{q}_1 \rangle \right). \tag{4.40}
\end{aligned}$$

Here we see that the factorisation-breaking piece, highlighted in red, does not vanish when we contract with the polarisation vector, for any choice of p_r . Furthermore, the factorised term does depend on p_r , and even vanishes for $p_r = p_{q_1}$. To show that the factorisation-breaking piece is suppressed compared to the factorised part of the full amplitude, we must contract it with the rest of the diagram in question, and analyse the scaling of the resulting Lorentz invariant terms in the NMRK limit. The same is true for fig. 4.8b.

Let us now consider the three-gluon vertex diagram that appears in fig. 4.7. This diagram has the desired factorisation properties, so our aim is to write this term as compactly as possible. The relevant t -channel momenta are

$$p_{t_1} = -p_{q_1} - p_{\bar{q}_1}, \quad p_{t_2} = p_{q_2} + p_{\bar{q}_2}, \tag{4.41}$$

with $p_g = p_{t_1} - p_{t_2}$. The three-gluon vertex, eq. (1.116), yields a kinematic factor

$$\eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - \eta^{\mu_{t_1} \mu_g} (2p_g + p_{t_2})^{\mu_{t_2}} + \eta^{\mu_{t_2} \mu_g} (2p_g - p_{t_1})^{\mu_{t_1}}. \tag{4.42}$$

Contracting this three-gluon vertex with quark currents lets us utilise the Dirac equation to make further simplifications. For example, the full Feynman diagram for $\lambda_{q_1} = \lambda_{q_2} = \oplus$ becomes

$$i g_s^3 \frac{[q_1 | \mu_{t_1} | \bar{q}_1] [q_2 | \mu_{t_2} | \bar{q}_2]}{t_1 t_2} (\eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1} \mu_g} + 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2} \mu_g}). \tag{4.43}$$

The kinematic factor in this expression will be a useful building block for the following sections so we define

$$\tilde{v}_{3g}^{\mu_{t_1} \mu_{t_2} \mu_g} (-p_{t_1}, p_{t_2}, p_g) = \eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1} \mu_g} + 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2} \mu_g}, \tag{4.44}$$

where the tilde indicates that this is a modified version of the three-gluon Feynman rule.

We now return to our statement that for certain helicity configurations, the colour ordered amplitudes can be written in the desired factorised form without any kinematic approximation. For the helicity configuration $\emptyset \rightarrow \bar{q}_1^\ominus q_1^\oplus g^\ominus \bar{q}_2^\ominus q_2^\oplus$, we can combine the result of eq. (4.38) with the tidied form of the three-gluon vertex, eq. (4.43), to write

$$M(\bar{q}_1^\ominus, q_1^\oplus, g^\ominus, \bar{q}_2^\ominus, q_2^\oplus) = J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}_1}^\ominus, p_{q_1}^\oplus, p_{t_1}) \left(\frac{-i}{t_1} \right) J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\ominus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1}), \quad (4.45)$$

with

$$\begin{aligned} J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\ominus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1}) &= ig_s^2 \epsilon_{\mu_g}(p_g^\ominus, p_r) \left\{ \frac{1}{s_{\bar{q}_2 g}} [q_2 | \mu_1(\bar{q}_2 + g) \mu_g | \bar{q}_2 \rangle \right. \\ &+ \left. \frac{1}{t_2} \left(\eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1} \mu_g} + 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2} \mu_g} - t_1 \eta^{\mu_{t_1} \mu_{t_2}} \frac{2p_{q_1}^{\mu_g}}{s_{q_1 g}} \right) [q_2 | \mu_{t_2} | \bar{q}_2 \rangle \right\}. \end{aligned} \quad (4.46)$$

Note that, for the helicity configuration $\emptyset \rightarrow \bar{q}_1^\ominus q_1^\oplus g^\oplus \bar{q}_2^\ominus q_2^\oplus$, the resulting $g\bar{q}qg^*$ current is simpler than eq. (4.46). In this case, both Feynman diagrams which have an on-shell gluon emission from a quark line can be written exactly as a quark current times an eikonal factor. This means that the quark current $[q_2 | \mu_{t_2} | \bar{q}_2 \rangle$ can also be factored out from $J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\oplus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1})$. We will return to this observation below. Although $J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\oplus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1})$ is simpler than $J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\ominus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1})$, here we opt to write this impact factor for an arbitrary gluon helicity:

$$\begin{aligned} J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^{\lambda_g}, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1}) &= ig_s^2 \epsilon_{\mu_g}(p_g^{\lambda_g}, p_r) \left\{ \frac{1}{s_{\bar{q}_2 g}} [q_2 | \mu_{t_1}(\bar{q}_2 + g) \mu_g | \bar{q}_2 \rangle \right. \\ &+ \left. \frac{1}{t_2} \left(\eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1} \mu_g} + 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2} \mu_g} - t_1 \frac{2p_{q_1}^{\mu_g}}{s_{q_1 g}} \right) [q_2 | \mu_{t_2} | \bar{q}_2 \rangle \right\}. \end{aligned} \quad (4.47)$$

We can similarly write

$$M(\bar{q}_1^\ominus, q_1^\oplus, \bar{q}_2^\ominus, q_2^\oplus, g^{\lambda_g}) = J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}_1}^\ominus, p_{q_1}^\oplus, p_{t_1}) \left(\frac{-i}{t_1} \right) J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, p_g^{\lambda_g}, -p_{t_1}), \quad (4.48)$$

with

$$J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_2}^{\ominus}, p_{q_2}^{\oplus}, p_g^{\lambda_g}, -p_{t_1}) = -ig_s^2 \epsilon_{\mu_g}(p_g^{\lambda_g}, p_r) \left\{ \frac{1}{s_{q_2g}} [q_2 | \mu_g(q_2 + g) \mu_{t_1} | \bar{q}_2] \right. \\ \left. + \frac{1}{t_2} \left(\eta^{\mu_{t_1} \mu_{t_2}} (p_{t_1} + p_{t_2})^{\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1} \mu_g} + 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2} \mu_g} - t_1 \frac{2p_{q_1}^{\mu_g}}{s_{\bar{q}_1g}} \right) [q_2 | \mu_{t_2} | \bar{q}_2] \right\}. \quad (4.49)$$

We can demonstrate that the factorised expressions eq. (4.47) and eq. (4.49) are gauge invariant by replacing $\epsilon^{\mu_g}(p_g^{\lambda_g}, p_r)$ with $p_r^{\mu_g}$. In both cases we find (up to an overall sign)

$$ig_s^2 \left\{ [q_2 | \mu_{t_2} | \bar{q}_2] + \frac{1}{t_2} ((p_{t_1} - p_{t_2})_{\mu_g} (p_{t_1} + p_{t_2})^{\mu_g} - t_1) [q_2 | \mu_{t_2} | \bar{q}_2] \right\} = 0, \quad (4.50)$$

as desired.

The factorised expressions eq. (4.47) and eq. (4.49) were obtained from colour-ordered amplitudes that are ‘exactly factorised’. We claim that these expressions capture the leading NMRK behaviour of the colour-ordered amplitudes that are not exactly factorised, and that these expressions can therefore be used to provide a factorised approximation for any helicity configuration in the NMRK. Let us demonstrate this is true for the helicity configuration $\emptyset \rightarrow \bar{q}_1^{\oplus} q_1^{\ominus} g^{\ominus} \bar{q}_2^{\ominus} q_2^{\oplus}$. We claim that the factorised approximation

$$M(\bar{q}_1^{\oplus}, q_1^{\ominus}, g^{\ominus}, \bar{q}_2^{\ominus}, q_2^{\oplus}) \xrightarrow{\text{NMRK}} J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_1}^{\oplus}, p_{q_1}^{\ominus}) \left(\frac{-i}{t_1} \right) J_{g\bar{q}qg^*}^{\mu_{t_2}}(p_g^{\ominus}, p_{\bar{q}_2}^{\ominus}, p_{q_2}^{\oplus}), \quad (4.51)$$

is valid in NMRK. This approximation is equivalent to neglecting the red term in eq. (4.40). We first note that this approximation is gauge invariant, as the red term does not depend on the reference momentum p_r . That the factorisation breaking term had to be separately gauge invariant follows from the gauge invariance of the colour-ordered amplitude and the gauge invariance of $J_{\bar{q}qg^*}^{\mu}$ which was demonstrated in eq. (4.50). We can write the relevant Feynman diagram (which is eq. (4.40) contracted with $[q_2 | \mu_{t_2} | \bar{q}_2] / t_2$) as

$$\frac{-i}{\sqrt{2}[q_1g]} \left(\frac{[q_1r]}{[gr]} \langle q_1 \bar{q}_2 \rangle [\bar{q}_1 q_2] + \langle g \bar{q}_2 \rangle [\bar{q}_1 q_2] \right). \quad (4.52)$$

We need to demonstrate that the red factorisation breaking term does not contribute to the NMRK limit of the full amplitude. To find the scaling of the full amplitude, it is simplest to analyse the compact form of eq. (1.148). Taking

the large rapidity difference of this process to be Δ , we see that the full amplitude scales as $\sim e^\Delta$, while the factorisation breaking term in eq. (4.52) scales as ~ 1 . We are therefore justified in dropping the factorisation breaking term. As an aside, we note that the factorised term in eq. (4.52) scales as 0, ~ 1 or $\sim e^\Delta$ depending on the choice of reference momenta, which serves as a reminder of the subtleties of analysis at the level of Feynman diagrams. A similar gauge invariant scaling analysis for the remaining helicity configurations shows that all partial amplitudes with the ordering $q_1 \bar{q}_1 g q_2 \bar{q}_2$ or $q_1 \bar{q}_1 q_2 \bar{q}_2 g$ can be approximated in NMRK with the factorised expressions eq. (4.47) and eq. (4.49).

It remains to discuss the colour-ordered amplitude $M(\bar{q}_1, g, q_1, \bar{q}_2, q_2)$. From fig. 4.7 we see that this amplitude consists of two Feynman diagrams that are naively non-factorised. As helicity conservation requires $\lambda_{q_1} = -\lambda_{\bar{q}_1}$, there are no helicity configurations for which this amplitude is exactly factorised. We might nevertheless hope that the ‘eikonal approximation’ of dropping the non-factorised terms will be valid in NMRK, and that we can therefore factorise an overall $J_{\bar{q}qg^*}$ from this colour-ordered amplitude. Let us pursue this idea, taking the example

$$\begin{aligned}
M(\bar{q}_1^\ominus, g^\ominus, q_1^\oplus, \bar{q}_2^\ominus, q_2^\oplus) &= ig_s^3 \left([q_1 | \mu | \bar{q}_1] \left(\frac{[\bar{q}_1 r]}{[\bar{q}_1 g][gr]} - \frac{[r q_1]}{[rg][gq_1]} \right) - \frac{[q_1 | \mu | g]}{[q_1 g]} \right) \frac{[q_2 | \mu | q_2]}{\sqrt{2} s_{q_1 \bar{q}_2}} \\
&= ig_s^3 \left([q_1 | \mu | \bar{q}_1] \frac{[\bar{q}_1 q_1]}{[\bar{q}_1 g][gq_1]} - \frac{[q_1 | \mu | g]}{[q_1 g]} \right) \frac{[q_2 | \mu | q_2]}{\sqrt{2} s_{q_1 \bar{q}_2}} \\
&= ig_s^3 \frac{[q_1 q_2]}{\sqrt{2} [q_1 g] s_{q_1 \bar{q}_2}} \left(\frac{\langle \bar{q}_2 \bar{q}_1 \rangle [\bar{q}_1 q_1]}{[\bar{q}_1 g]} + \langle g \bar{q}_2 \rangle \right).
\end{aligned} \tag{4.53}$$

In the second equality we have made use of the Schouten identity, eq. (1.127). This makes it explicit that the amplitude does not depend on p_r . Following a similar scaling analysis as before, we find that both the factorised, and non-factorised terms scale like ~ 1 . This means it is not valid in NMRK to neglect the non-factorised term, but moreover, this partial amplitude can be neglected compared to the flavour orderings $q_i \bar{q}_i g q_j \bar{q}_j$ we studied above. We therefore drop the colour-ordered amplitudes with flavour-orderings $q_1 g \bar{q}_1 q_2 \bar{q}_2$, even though they contribute to t_1 -channel **8** exchange.

Recalling eq. (4.37), we can now combine eq. (4.47) and eq. (4.49) to form the

colour-dressed current

$$\begin{aligned}
& \mathcal{J}_{\bar{q}qg^*}^{\mu_t a_t}(\{p_{\bar{q}}, \lambda_{\bar{q}}, \bar{\imath}_{\bar{q}}\}, \{p_q, \lambda_q, \imath_q\}, \{p_g, \lambda_g, a_g\}, p_t) \\
&= \frac{1}{T_F} \left((T^{a_t} T^{a_g})_{i_q i_{\bar{q}}} J_{g\bar{q}qg^*}^{\mu_t}(p_g^{\lambda_g}, p_{\bar{q}}^{\lambda_{\bar{q}}}, p_q^{\lambda_q}, p_t) + (T^{a_g} T^{a_t})_{i_q i_{\bar{q}}} J_{\bar{q}qg^*}^{\mu_t}(p_{\bar{q}}^{\lambda_{\bar{q}}}, p_q^{\lambda_q}, p_g^{\lambda_g}, p_t) \right),
\end{aligned} \tag{4.54}$$

in agreement with the ‘unordered’ current derived in ref. [154]. Although in obtaining eq. (4.54) we have had to specify the rapidity hierarchy $y_{q_1} \sim y_{\bar{q}_1} \gg y_g \sim y_{q_2} \sim y_{\bar{q}_2}$, we have not had to specify a parton to be on the beam axis. This allows us to utilise crossing to obtain three physical currents: $\mathcal{J}_{q \rightarrow qg^*}^{\mu_t a_t}$, $\mathcal{J}_{\bar{q} \rightarrow \bar{q}g^*}^{\mu_t a_t}$ and $\mathcal{J}_{g \rightarrow q\bar{q}}^{\mu_t a_t}$. This is in contrast to the approach taken in ref. [97], where these quantities had to be obtained separately, see eq. (2.91) and eq. (2.93). We also note in passing that eq. (4.47) and eq. (4.49) also possess the desired little-group scaling, which is lost in eqs. (2.89) and (2.92).

By studying the amplitude $\emptyset \rightarrow \bar{q}_1 q_1 g \bar{q}_2 q_2$ in the NMRK limit $y_{q_1} \sim y_{\bar{q}_1} \gg y_g \sim y_{q_2} \sim y_{\bar{q}_2}$, we have obtained a factorised expression for an on-shell gluon and $q\bar{q}$ pair, $\mathcal{J}_{\bar{q}qg^*}^{a_t \mu_t}$, where the rapidities of all these partons are well separated from all other partons in the scattering process. We now wish to consider the MRK limit $y_{q_1} \sim y_{\bar{q}_1} \gg y_g \gg y_{q_2} \sim y_{\bar{q}_2}$ of the same process, and show that the LO amplitude further factorises into two quark impact factors, $\mathcal{J}_{\bar{q}_1 q_1}^{a_{t_1} \mu_{t_1}}$ and $\mathcal{J}_{\bar{q}_2 q_2}^{a_{t_2} \mu_{t_2}}$ contracted with a central factor which we will interpret as the 2-tensor analogue of the Lipatov vertex, eq. (2.32). This is equivalent to showing that $\mathcal{J}_{\bar{q}qg^*}^{a_t \mu_t}$ itself factorises into an impact factor contracted with the 2-tensor Lipatov vertex. The procedure we will perform to obtain this emission vertex is exactly analogous to the study performed above. Firstly, we will project an antisymmetric octet in the t_2 -channel in order to factor a fundamental generator from the colour factors. Secondly, we will observe that there are certain helicity configurations which already have the desired factorised structure. We will use these exactly-factorised helicity configurations to define the 2-tensor Lipatov vertex. We will finally check that this vertex describes all helicity configurations in the MRK limit, and that we therefore have captured significantly more of the LO behaviour of the amplitude without losing LL accuracy when we integrate this amplitude.

Let us begin by applying the projection operator eq. (1.96) to the colour structures

of eq. (4.54):

$$\mathcal{J}_{\bar{q}qgg^*}^{\mu_{t_1} a_{t_1}}(\bar{q}, q, g, p_t) \xrightarrow{\hat{p}_8} \frac{i}{2T_F} f^{a_{t_1} a_{t_2} a_g} T_{i_q i_{\bar{q}}}^{a_{t_2}} (J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^{\lambda_g}, p_{\bar{q}}^{\lambda_{\bar{q}}}, p_q^{\lambda_q}, p_t) - J_{\bar{q}qgg^*}^{\mu_{t_1}}(p_{\bar{q}}^{\lambda_{\bar{q}}}, p_q^{\lambda_q}, p_g^{\lambda_g}, p_t)). \quad (4.55)$$

Beneath eq. (4.46) we mentioned that for the helicity configuration $\emptyset \rightarrow \bar{q}_1^\ominus q_1^\oplus g^\oplus \bar{q}_2^\ominus q_2^\oplus$ (more generally, where a gluon has opposite helicity to both cyclically-adjacent quarks or antiquarks) the quark current $J_{\bar{q}qg^*}^{\mu_{t_2}}$ could be further factored out of $J_{g\bar{q}qg^*}^{\mu_{t_1}}$, making no kinematic approximations. Let us now make this remark more explicit. We can write

$$J_{g\bar{q}qg^*}^{\mu_{t_1}}(p_g^\oplus, p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_1}) = V_{g^*g^*g^*}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_g^\oplus, p_{t_2}) \left(\frac{-i}{t_2} \right) J_{\bar{q}qg^* \mu_{t_2}}(p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_2}) \quad (4.56)$$

without any approximation by taking

$$V_{g^*g^*g^*}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_g^\oplus, p_{t_2}) = ig_s \epsilon_{\mu_g}(p_g^\oplus, p_r) \left\{ 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2}\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1}\mu_g} + \eta^{\mu_{t_1}\mu_{t_2}} \left((p_{t_1} + p_{t_2})^{\mu_g} - t_1 \frac{2p_{q_1}^{\mu_g}}{s_{q_1g}} + t_2 \frac{2p_{\bar{q}_2}^{\mu_g}}{s_{\bar{q}_2g}} \right) \right\} \quad (4.57)$$

to be our 2-tensor colour-ordered Lipatov vertex. For the colour ordering $\bar{q}_1 q_1 \bar{q}_2 q_2 g$, we have the exact factorisation

$$J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, p_g^\ominus, -p_{t_1}) = V_{g^*g^*g^*}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_{t_2}, p_g^\ominus) \left(\frac{-i}{t_2} \right) J_{\bar{q}qg^* \mu_{t_2}}(p_{\bar{q}_2}^\ominus, p_{q_2}^\oplus, -p_{t_2}) \quad (4.58)$$

with

$$V_{g^*g^*g^*}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_{t_2}, p_g^\ominus) = -ig_s \epsilon_{\mu_g}(p_g^\ominus, p_r) \left\{ 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2}\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1}\mu_g} + \eta^{\mu_{t_1}\mu_{t_2}} \left((p_{t_1} + p_{t_2})^{\mu_g} - t_1 \frac{2p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1g}} + t_2 \frac{2p_{q_2}^{\mu_g}}{s_{q_2g}} \right) \right\}. \quad (4.59)$$

That these vertices not only describe the helicity configurations in eq. (4.56) and eq. (4.58) exactly, but also capture the MRK behaviour of all helicity configurations can be demonstrated using similar scaling arguments to the above, taking both $\Delta_1 = y_{q_1} - y_g$ and $\Delta_2 = y_g - y_{q_2}$ to be infinite.

Recalling eq. (4.55), we can write the colour dressed 2-tensor Lipatov vertex as

$$\begin{aligned}
\mathcal{V}_{g^*gg^*}^{\mu_{t_1}\mu_{t_2}a_{t_1}a_{t_2}}(-p_{t_1}, \{p_g, \lambda_g, a_g\}, p_{t_2}) \\
&= \frac{i}{2T_F} f^{a_{t_1}a_{t_2}a_g} (V_{g^*gg^*}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_g^{\lambda_g}, p_{t_2}) - V_{g^*g^*g}^{\mu_{t_1}\mu_{t_2}}(-p_{t_1}, p_{t_2}, p_g^{\lambda_g})) \\
&= \frac{ig_s}{2T_F} f^{a_{t_1}a_{t_2}a_g} \epsilon_{\mu_g}(p_g^{\lambda_g}, p_r) \left\{ 2p_g^{\mu_{t_1}} \eta^{\mu_{t_2}\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1}\mu_g} \right. \\
&\quad \left. + \eta^{\mu_{t_1}\mu_{t_2}} \left((p_{t_1} + p_{t_2})^{\mu_g} - \frac{t_1}{2} \left(\frac{2p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1g}} + \frac{2p_{q_1}^{\mu_g}}{s_{q_1g}} \right) + \frac{t_2}{2} \left(\frac{2p_{\bar{q}_2}^{\mu_g}}{s_{\bar{q}_2g}} + \frac{2p_{q_2}^{\mu_g}}{s_{q_2g}} \right) \right) \right\}.
\end{aligned} \tag{4.60}$$

The procedure to check this is gauge invariant is similar to eq. (4.50), and we obtain

$$\frac{ig_s}{2T_F} f^{a_{t_1}a_{t_2}a_g} \left\{ \eta^{\mu_{t_1}\mu_{t_2}} ((p_{t_1} - p_{t_2})_{\mu_g} (p_{t_1} + p_{t_2})^{\mu_g} - t_1 + t_2) \right\} = 0 \tag{4.61}$$

upon substituting $\epsilon_{\mu_g}(p_g^{\lambda_g}, p_r) \rightarrow (p_g)_{\mu_g}$ in eq. (4.60).

Equation (4.60) is compact, and makes minimal approximations to the LO amplitude. However, the non-trivial Lorentz structure of this expression makes it computationally difficult to describe multiple real emissions with this vertex, which requires contracting several instances of eq. (4.60). It would therefore be desirable to make minimal further approximations to eq. (4.60) in order to factor out an overall $\eta^{\mu_{t_1}\mu_{t_2}}$. This would mean that several real gluon emissions would be described by several vertices which are simply multiplied together, as in e.g. eq. (2.39).

To this end, let us consider the two non-trivial Lorentz structures of eq. (4.60), contracted with arbitrary quark currents:

$$J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1})(2p_g^{\mu_{t_1}} \eta^{\mu_{t_2}\mu_g} - 2p_g^{\mu_{t_2}} \eta^{\mu_{t_1}\mu_g}) J_{\bar{q}qg^* \mu_{t_2}}(p_{\bar{q}_2}, p_{q_2}, -p_{t_2}). \tag{4.62}$$

As discussed beneath eq. (4.3), we can approximate

$$J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) J_{\bar{q}qg^*}^{\mu_{t_2}}(p_{\bar{q}_2}, p_{q_2}, -p_{t_2}) \xrightarrow{\text{MRK}} -4g_s^2 e^{i(\phi_{q_1} + \phi_{q_2})} p_{q_1}^{\mu_{t_1}} p_{q_2}^{\mu_{t_2}} \tag{4.63}$$

in the MRK limit, where $\phi_{q_1} + \phi_{q_2}$ is a phase which depends on the spinor representation that we use. Making this approximation, we can write eq. (4.62) as

$$-4g_s e^{i(\phi_{q_1} + \phi_{q_2})} (s_{q_1g} p_{q_2}^{\mu_g} - s_{q_2g} p_{q_1}^{\mu_g}). \tag{4.64}$$

Contracting eq. (4.63) with $\eta^{\mu_{t_1}\mu_{t_2}}$ we can write the prefactor as

$$-4g_s^2 e^{i(\phi_{q_1} + \phi_{q_2})} \underset{\text{MRK}}{=} 2\eta^{\mu_{t_1}\mu_{t_2}} \frac{2}{s_{q_1 q_2}} J_{\bar{q}qg^*}^{\mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) J_{\bar{q}qg^*}^{\mu_{t_2}}(p_{\bar{q}_2}, p_{q_2}, -p_{t_2}), \quad (4.65)$$

meaning we can approximate the term eq. (4.62) in the MRK as

$$J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) \underset{\text{MRK}}{=} \eta^{\mu_{t_1}\mu_{t_2}} \frac{1}{s_{q_1 q_2}} (s_{q_1 g} p_{q_2}^{\mu_g} - s_{q_2 g} p_{q_1}^{\mu_g}) J_{\bar{q}qg^* \mu_{t_2}}(p_{\bar{q}_2}, p_{q_2}, -p_{t_2}). \quad (4.66)$$

We would arrive at an analogous expression if we had instead approximated the quark currents as $e^{i\phi_{\bar{q}_1}} 2p_{\bar{q}_1}^{\mu_{t_1}}$ or $e^{i\phi_{\bar{q}_2}} 2p_{\bar{q}_2}^{\mu_{t_2}}$. We therefore symmetrise over these equally valid choices. Recalling eq. (4.60), we now have made approximations such that each term in the effective vertex has a common factor of $\eta_{\mu_{t_1}\mu_{t_2}}$. Factoring out the metric tensor, we are left with the scalar Lipatov vertex

$$\begin{aligned} \mathcal{V}_{g^*gg^*}^{a_{t_1}a_{t_2}}(-p_{t_1}, \{p_g, \lambda_g, a_g\}, p_{t_2}) &= \frac{ig_s}{2T_F} f^{a_{t_1}a_{t_2}a_g} \epsilon_{\mu_g}(p_g^{\lambda_g}, p_r) \\ &\times \left[(p_{t_1} + p_{t_2})^{\mu_g} - \frac{t_1}{2} \left(\frac{2p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1 g}} + \frac{2p_{q_1}^{\mu_g}}{s_{q_1 g}} \right) + \frac{t_2}{2} \left(\frac{2p_{\bar{q}_2}^{\mu_g}}{s_{\bar{q}_2 g}} + \frac{2p_{q_2}^{\mu_g}}{s_{q_2 g}} \right) + \right. \\ &+ \frac{1}{2} \left\{ \frac{s_{q_1 g} p_{q_2}^{\mu_g} - s_{q_2 g} p_{q_1}^{\mu_g}}{s_{q_1 q_2}} + \frac{s_{\bar{q}_1 g} p_{q_2}^{\mu_g} - s_{q_2 g} p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1 q_2}} \right. \\ &\quad \left. \left. + \frac{s_{q_1 g} p_{\bar{q}_2}^{\mu_g} - s_{\bar{q}_2 g} p_{q_1}^{\mu_g}}{s_{q_1 \bar{q}_2}} + \frac{s_{\bar{q}_1 g} p_{\bar{q}_2}^{\mu_g} - s_{\bar{q}_2 g} p_{\bar{q}_1}^{\mu_g}}{s_{\bar{q}_1 \bar{q}_2}} \right\} \right]. \end{aligned} \quad (4.67)$$

as derived in ref. [148].

4.2.2 $\bar{q}qggg$

In the previous section we analysed the LO amplitude for $\emptyset \rightarrow \bar{q}_1 q_1 g \bar{q}_2 q_2$ in the NMRK limit $y_{q_1} \sim y_{\bar{q}_1} \gg y_g \sim y_{q_2} \sim y_{\bar{q}_2}$ in order to obtain the factorised expression $\mathcal{J}_{\bar{q}qgg^*}^{\mu_t a_t}$. In this section we perform an analogous study of the LO amplitude for $\emptyset \rightarrow \bar{q}qg_1 g_2 g_3$ in the NMRK limit $y_{q_1} \sim y_{\bar{q}_1} \gg y_{g_1} \sim y_{g_2} \sim y_{g_3}$ in order to obtain the factorised expression $\mathcal{J}_{ggg^*}^{\mu_t a_t}$. This is the first of two new factorised expressions that we will derive in this thesis. The method we use will be completely analogous. We will first project the colour structures which have octet exchange in the t_1 channel. We will then analyse the corresponding kinematic factors, where we will find that certain helicity configurations can be expressed in the desired factorised form without any approximation. We will finally check that these factorised expressions can be used as an approximation for the remaining

helicity configurations in the above NMRK limit.

We start as usual by writing the colour ordered amplitude

$$\mathcal{M}(\bar{q}, q, g_1, g_2, g_3) = \sum_{\sigma \in S_3} (T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}})_{i\bar{i}} M(\bar{q}, q, g_{\sigma_1}, g_{\sigma_2}, g_{\sigma_3}), \quad (4.68)$$

which is depicted in fig. 4.9, and we use the projection operator eq. (1.96) to obtain

$$\mathcal{M}^{[8]}(\bar{q}, q, g_1, g_2, g_3) = \frac{1}{T_F} T_{i\bar{i}}^{a_{t_1}} \sum_{\sigma \in S_3} \text{tr}(T^{a_{t_1}} T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}}) M(\bar{q}, q, g_{\sigma_1}, g_{\sigma_2}, g_{\sigma_3}), \quad (4.69)$$

which is depicted in fig. 4.10.

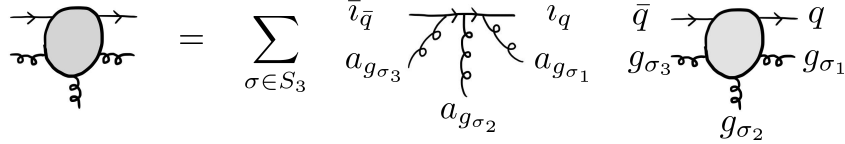


Figure 4.9 *Illustration of the colour decomposition of the amplitude in eq. (4.68).*

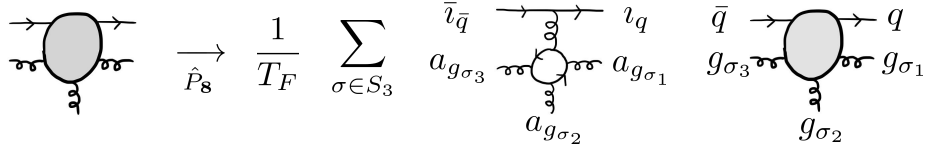


Figure 4.10 *Illustration of the colour decomposition of the amplitude in eq. (4.69).*

Due to the Bose symmetry of the gluon legs there is only one distinct colour ordered amplitude in eq. (4.69), which is depicted in fig. 4.11. We see that there are three diagrams which are not of the desired factorised form. Consider the final diagram in fig. 4.11, which consists of three gluon emissions from the quark line. We recall the observation made in section 4.2.1, that such a diagram factorises where the quark and antiquark have opposite helicities to the cyclically adjacent gluons. For example, for the helicity configuration $\emptyset \rightarrow \bar{q}^{\ominus} q^{\oplus} g_1^{\ominus} g_2^{\oplus} g_3^{\oplus}$,

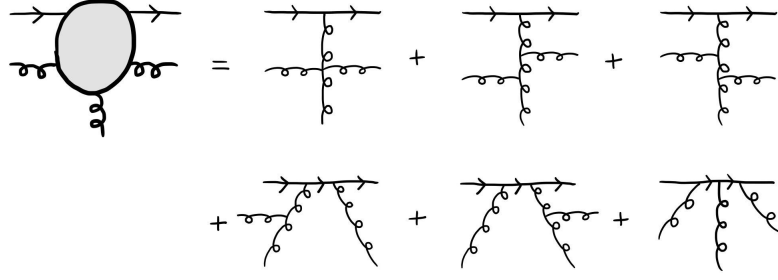


Figure 4.11 *The single colour-ordered amplitude that appears in eq. (4.68), expressed as a sum of colour-ordered Feynman diagrams.*

this diagram is simply

$$g_s^3 \epsilon_{\mu_{g_1}}(p_{g_1}^\ominus; r_{g_1}) \epsilon_{\mu_{g_2}}(p_{g_2}^\oplus; r_{g_2}) \epsilon_{\mu_{g_3}}(p_{g_3}^\oplus; r_{g_3}) (i[q|\mu|\bar{q}]) \left(\frac{-i}{t_1} \right) \left(\frac{2p_q^{\mu_{g_1}}}{s_{qg_1}} \right) \left(\frac{2p_{\bar{q}}^{\mu_{g_3}}}{s_{\bar{q}g_3}} \right), \quad (4.70)$$

which is true for any choice of reference momenta r_{g_i} .

The next observation we recall from section 4.2.1, is that for three-gluon vertices containing two on-shell gluons of opposite helicity, the vertex vanishes upon setting the reference momenta to be that of the other on-shell gluon. We therefore specify $r_{g_1} = p_{g_2}$ and $r_{g_2} = p_{g_1}$. In order to choose a ‘minimal gauge’ [151], we look at the inner products,

$$\epsilon(p_{g_3}^\oplus; r_{g_3}) \cdot \epsilon(p_{g_1}^\ominus; p_{g_2}) = \frac{[g_3 g_2] \langle r_3 g_1 \rangle}{\langle g_3 r_3 \rangle [g_1 g_2]}, \quad (4.71a)$$

$$\epsilon(p_{g_3}^\oplus; r_{g_3}) \cdot \epsilon(p_{g_2}^\oplus; p_{g_1}) = \frac{[g_3 g_2] \langle r_3 g_1 \rangle}{\langle g_3 r_3 \rangle \langle g_2 g_1 \rangle}, \quad (4.71b)$$

which motivates us to choose $p_{r_3} = p_{g_1}$ as the final reference momentum. In particular, this choice makes the four-gluon vertex vanish. More importantly, the fourth diagram in fig. 4.11 can be written, without approximation, as

$$g_s^3 \epsilon_{\mu_{g_1}}(p_{g_1}^\ominus; r_{g_1}) \epsilon_{\mu_{g_2}}(p_{g_2}^\oplus; r_{g_2}) \epsilon_{\mu_{g_3}}(p_{g_3}^\oplus; r_{g_3}) (i[q|\mu_{t_1}|\bar{q}]) \left(\frac{-i}{t_1} \right) \times \left(\frac{2p_q^{\mu_{g_1}}}{s_{qg_1}} \right) \frac{1}{s_{g_2 g_3}} (2p_{g_3}^{\mu_{g_2}} \eta^{\mu_{t_1} \mu_{g_3}} - 2p_{g_2}^{\mu_{g_3}} \eta^{\mu_{t_1} \mu_{g_2}}). \quad (4.72)$$

This means that the full colour-ordered amplitude can be written, without

approximation, as

$$M(\bar{q}^{\ominus}, q^{\oplus}, g_1^{\ominus}, g_2^{\oplus}, g_3^{\oplus}) = J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}}^{\ominus}, p_q^{\oplus}, p_{t_1}) \left(\frac{-i}{t_1} \right) J_{gggg^*}^{\mu_{t_1}}(p_{g_1}^{\ominus}, p_{g_2}^{\oplus}, p_{g_3}^{\oplus}, -p_{t_1}), \quad (4.73)$$

with

$$\begin{aligned} J_{gggg^*}^{\mu_{t_1}}(p_{g_1}^{\ominus}, p_{g_2}^{\oplus}, p_{g_3}^{\oplus}, -p_{t_1}) &= g_s \epsilon^{\mu_{g_1}}(p_{g_1}^{\ominus}; p_{g_2}) \epsilon^{\mu_{g_2}}(p_{g_2}^{\oplus}; p_{g_1}) \epsilon^{\mu_{g_3}}(p_{g_3}^{\oplus}; p_{g_1}) \\ &\times \left\{ s_{q\bar{q}} \frac{2p_q^{\mu_{g_1}}}{s_{qg_1}} \left(\eta^{\mu_{t_1}\mu_{g_2}} \frac{2p_{\bar{q}}^{\mu_{g_3}}}{s_{\bar{q}g_3}} + \eta^{\mu_{t_1}\mu_{g_3}} \frac{2p_{g_3}^{\mu_{g_2}}}{s_{g_2g_3}} - \eta^{\mu_{t_1}\mu_{g_2}} \frac{2p_{g_2}^{\mu_{g_3}}}{s_{g_2g_3}} \right) \right. \\ &\quad \left. + \frac{2p_{g_3}^{\mu_{g_1}}}{s_{g_2g_3}} (2p_{g_2}^{\mu_{g_3}} \eta^{\mu_{t_1}\mu_{g_2}} - 2p_{g_3}^{\mu_{g_2}} \eta^{\mu_{t_1}\mu_{g_3}}) \right\}. \end{aligned} \quad (4.74)$$

It remains to check that this factorised expression correctly describes the helicity configuration $\emptyset \rightarrow \bar{q}^{\oplus} q^{\ominus} g_1^{\ominus} g_2^{\oplus} g_3^{\oplus}$ in the NMRK limit. This may be checked by similar scaling arguments to the ones in section 4.2.1. We demonstrate this property numerically in fig. 4.12, where we see that the approximation to the LO amplitude does agree with the LO amplitude as we approach the NMRK limit.

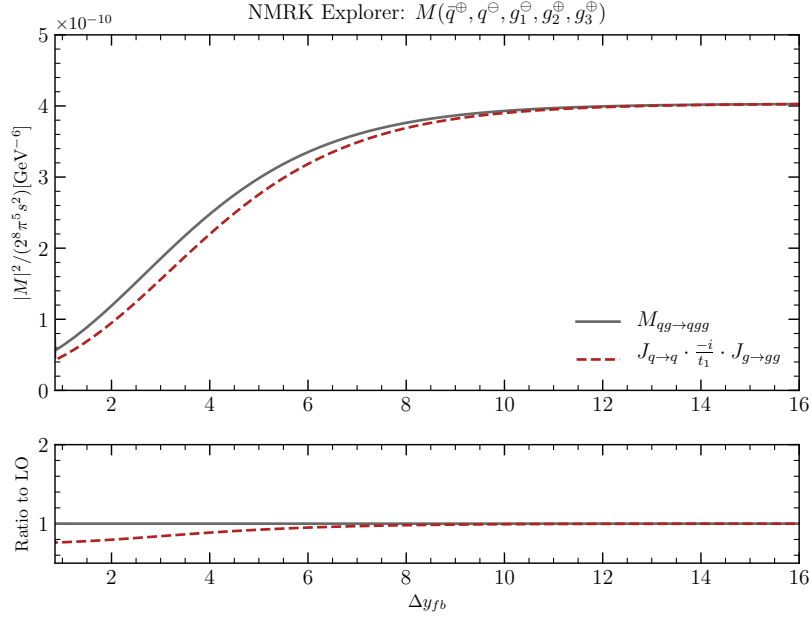


Figure 4.12 A slice through phase space comparing the exact colour-ordered amplitude (black, solid) to the factorised approximation (red, dashed) for the helicity configuration $\emptyset \rightarrow \bar{q}^{\oplus} q^{\ominus} g_1^{\ominus} g_2^{\oplus} g_3^{\oplus}$, where the latter uses eq. (4.74). The descriptions converge towards the right hand side of the plot, where the phase-space point approaches the NMRK limit. The details of the phase space slice are given in eq. (A.90).

Charge conservation requires $\lambda_q = -\lambda_{\bar{q}}$, and in section 1.6.1 we saw that the all- $\oplus$ or all-but-one- $\oplus$ diagrams (and their complex conjugates) vanish. It therefore remains to consider the helicity configurations where $\lambda_{g_1} = \lambda_{g_3} = -\lambda_{g_2}$. Unlike eq. (4.73), no colour-ordered amplitudes are exactly factorised for this class of helicity configurations. We nevertheless follow the same steps, choosing $\emptyset \rightarrow \bar{q}^\ominus q^\oplus g_1^\ominus g_2^\oplus g_3^\ominus$ as a representative helicity configuration. We choose the minimal gauge $p_{r_1} = p_{g_2}$, $p_{r_2} = p_{g_1}$, $p_{r_3} = p_{g_2}$. With this choice of reference momenta, we see that we need to make only a single kinematic approximation to the entire colour-ordered amplitude, namely, we must make the Eikonal approximation for g_3 in the final diagram of fig. 4.11. Doing so, we obtain the factorised expression

$$\begin{aligned}
J_{ggg^*}^{\mu_{t_1}}(p_{g_1}^\ominus, p_{g_2}^\oplus, p_{g_3}^\ominus) &= g_s \epsilon_\ominus^{\mu_{g_1}}(p_{g_1}; p_{g_2}) \epsilon_\oplus^{\mu_{g_2}}(p_{g_2}; p_{g_1}) \epsilon_\ominus^{\mu_{g_3}}(p_{g_3}; p_{g_2}) \\
&\times \left\{ s_{q\bar{q}} \frac{2p_q^{\mu_{g_1}}}{s_{qg_1}} \left(\eta^{\mu_{t_1}\mu_{g_2}} \frac{2p_{\bar{q}}^{\mu_{g_3}}}{s_{\bar{q}g_3}} + \eta^{\mu_{t_1}\mu_{g_3}} \frac{2p_{g_3}^{\mu_{g_2}}}{s_{g_2g_3}} \right) \right. \\
&\quad \left. + \frac{2p_{g_3}^{\mu_{g_2}}}{s_{g_2g_3}} (2p_{g_1}^{\mu_{g_3}} \eta^{\mu_{t_1}\mu_{g_1}} - 2p_{g_3}^{\mu_{g_1}} \eta^{\mu_{t_1}\mu_{g_3}}) \right\}. \tag{4.75}
\end{aligned}$$

It remains to check that this factorised expression correctly describes the helicity configurations $\emptyset \rightarrow \bar{q}^\ominus q^\oplus g_1^\ominus g_2^\oplus g_3^\ominus$ and $\emptyset \rightarrow \bar{q}^\oplus q^\ominus g_1^\ominus g_2^\oplus g_3^\ominus$ in the NMRK limit. This is indeed the case, which can be most simply checked by comparing the NMRK scaling of the factorisation-breaking term (i.e. scaling in the limit of large $\Delta = y_{q_1} - y_{g_1}$) to the scaling of the contracted form of the relevant helicity amplitude in section 1.6.1. We can also check this property numerically, which we demonstrate for the former helicity configuration in fig. 4.13.

We can now return to the colour dressed amplitude, eq. (4.69), which in the NMRK limit we can approximate with the factorised expression

$$\mathcal{M}(\bar{q}, q, g_1, g_2, g_3) \xrightarrow{\text{NMRK}} \mathcal{J}_{\bar{q}qg^*}^{\mu_{t_1} a_{t_1}}(p_{\bar{q}}, p_q, p_{t_1}) \left(\frac{-i}{t_1} \right) \mathcal{J}_{ggg^* \mu_{t_1}}^{a_{t_1}}(p_{g_1}, p_{g_2}, p_{g_3}, -p_{t_1}), \tag{4.76}$$

using the colour-dressed current

$$\mathcal{J}_{ggg^*}^{\mu_{t_1} a_t}(p_{g_1}, p_{g_2}, p_{g_3}, -p_{t_1}) = \frac{1}{T_F} \sum_{\sigma \in S_3} \text{tr}(T^{a_t} T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}}) J_{ggg^*}^{\mu_t}(p_{g_{\sigma_1}}, p_{g_{\sigma_2}}, p_{g_{\sigma_3}}, -p_{t_1}). \tag{4.77}$$

which is the HEJ analogue of the impact factor in eq. (2.85).

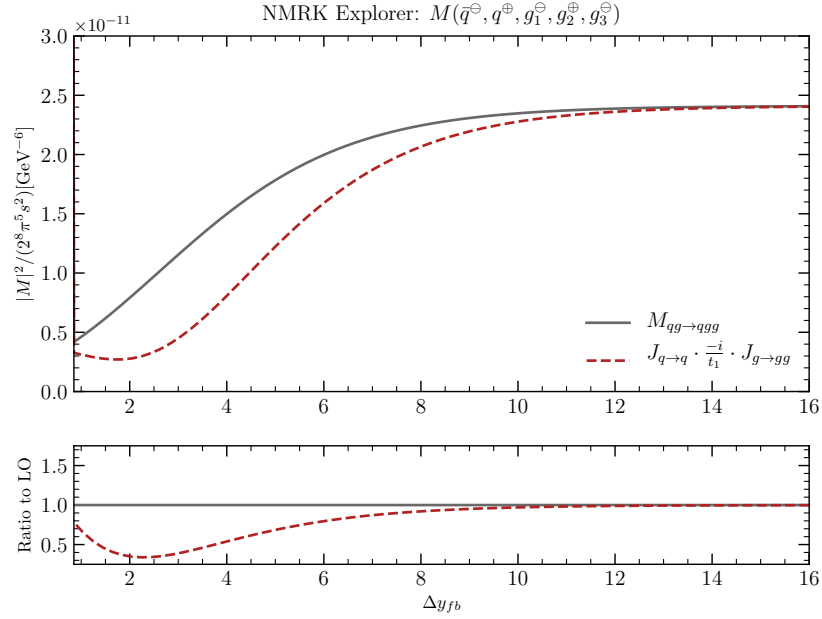


Figure 4.13 *A slice through phase space comparing the exact colour-ordered amplitude (black, solid) to the factorised approximation (red, dashed) for the helicity configuration $\emptyset \rightarrow \bar{q}^{\ominus} q^{\oplus} g_1^{\ominus} g_2^{\oplus} g_3^{\ominus}$, where the latter uses eq. (4.74). The descriptions converge towards the right hand side of the plot, where the phase-space point approaches the NMRK limit. The details of the phase space slice are given in eq. (A.90).*

Figures 4.12 and 4.13 provide crucial checks on the behaviour of the colour-ordered amplitudes. However, our ultimate interest is in comparison to experiment. The next step is therefore to present analogous plots for the squared, colour-dressed amplitude, eq. (4.76), averaged (summed) over initial (final) state helicities and colours. To perform the sum over helicities, we exploited the Bose symmetry of the gluon legs, and the charge conjugation properties of the colour-ordered amplitudes, so that only three independent squared matrix elements needed to be hard-coded. To compute the squared colour structures we made use of the Mathematica package `ColorMath` [155]. The results are displayed in fig. 4.14, which compares the approximation of eq. (4.76) with the approximation using the Lipatov vertex eq. (4.67).

In fig. 4.14a we see that both of these approximations converge to the LO colour-summed matrix-element squared in the MRK limit. By contrast, in fig. 4.14b we see that only the approximation of eq. (4.76) converges with the LO result in the central NMRK limit. This factorised approximation is also seen to give a good description of the LO matrix-element squared throughout phase space, and not just in the kinematic limits. However, fig. 4.14 only plots one-dimensional

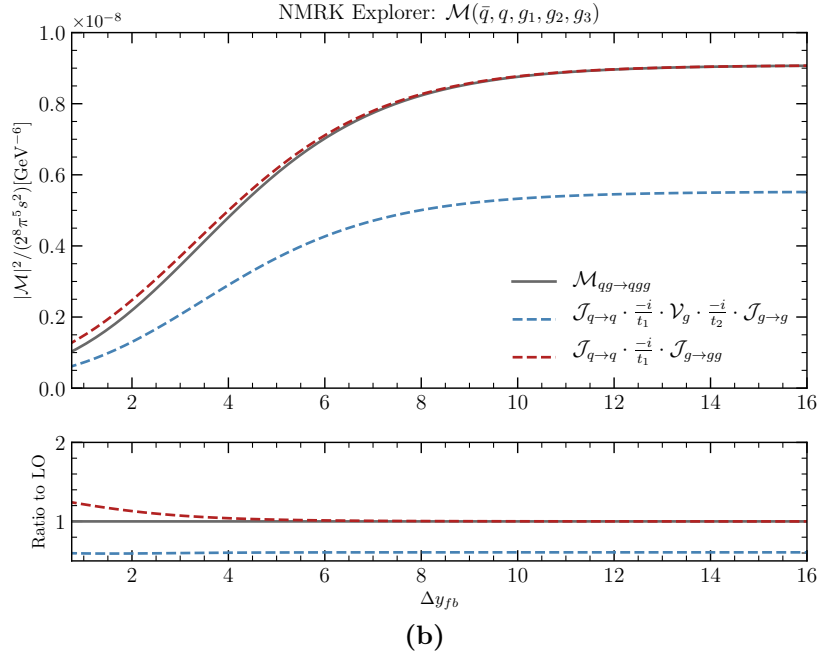
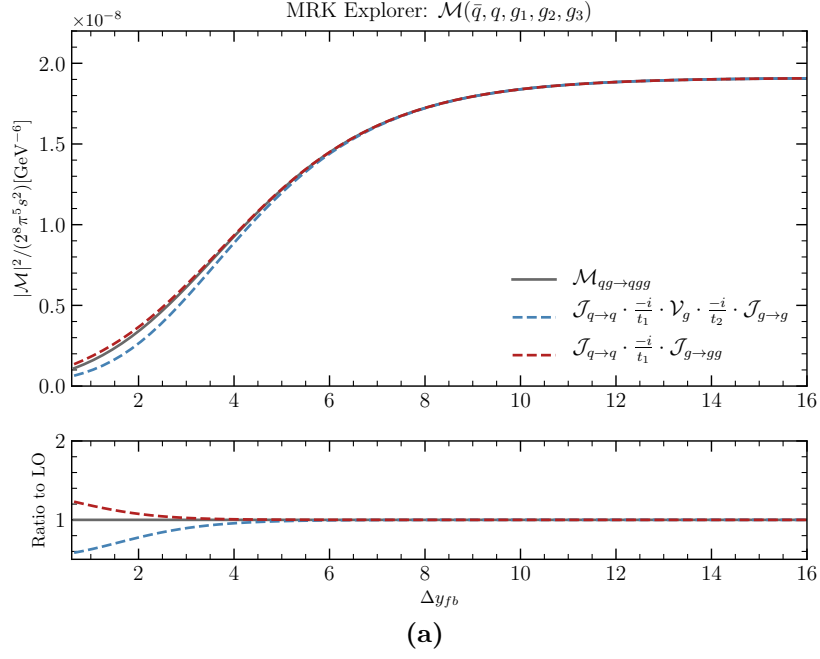


Figure 4.14 *A comparison of the exact LO amplitude for $qg \rightarrow qgg$, squared, and summed over external helicity and colour configurations (black, solid) to the factorised approximation given in eq. (4.76) (red, dashed), and an approximation where the $g \rightarrow gg$ impact factor of the latter is further factorised into the Lipatov vertex eq. (4.67) times a $g \rightarrow g$ impact factor, eq. (4.30). Panel (a) plots the phase space slice parametrised in eq. (A.89) which tends towards an MRK phase space point. Panel (b) plots the phase space slice parametrised in eq. (A.90) which tends towards an NMRK phase space point.*

slices through phase space. To assess how the factorised approximation performs within the phase space of the LHC we must integrate over phase space, imposing appropriate fiducial and jet cuts, and convolute with PDFs (all plots in this thesis were produced using the **NNPDF** 3.1 NLO PDF set [156]).

The resulting differential cross sections for several observables are plotted in fig. 4.15. For clarity, we present differential cross sections for a single partonic channel, namely $ug \rightarrow ugg$, requiring $y_u \geq y_g$. Although jet rapidities are typically measured at the LHC in the range $|y_j| < 4.5$, we have extended this cut to $|y_j| < 6$ to better demonstrate convergence in theoretically controlled limits.

In fig. 4.15a we see that, as expected, all three descriptions agree when we require a large rapidity separation between all jets. The observable $\min(\Delta y_{jj})$ is useful to check convergence in the MRK limit. This is a vital theoretical check, but this observable is unfortunately not typically measured in experimental analyses. More commonly measured observables are the rapidity separation between the hardest two jets, Δy_{12} , or the invariant mass of the hardest two jets, m_{12} . These are correlated with the MRK limit, but even the tails of these distributions are inclusive over all rapidities of the softest jet. This provides strong motivation to bring the HEJ framework to NLL accuracy. In figs. 4.15c and 4.15d we see that the new NLL approximation (red) provides a much better description of the LO amplitude (black) across the entire distributions, compared to the original LL HEJ approximation (blue).

In fig. 4.15e we have chosen to investigate an observable which will be described exactly by the red line but not the blue line. In the limit of large Δy_{qg} , i.e. the rapidity separation between the outgoing quark and the central gluon, the cross section will only receive a contribution from events which have the two gluons close in rapidity, but where the rapidity of the gluons are well separated from that of the quark. We expect this region of phase space to be well described by the NLL description, but not by the LL description, which also requires a large separation between the two gluons. In the plot we see these expectations are satisfied, and we see that the red line quickly converges to LO while the blue line diverges significantly. In contrast, in fig. 4.15f we have chosen to plot an observable where we expect the new NLL approximation to give a poor description of the LO amplitude. At large Δy_{gg} , the outgoing quark is necessarily close in rapidity to the forward gluon. This violates the NMRK limit in which the NLL approximation was derived, and indeed we see that the red line diverges in the tail of fig. 4.15f. In this region of phase space we should instead use the NLL

$\mathcal{J}_{q \rightarrow qg}$ current (eq. (4.54)) contracted with a $\mathcal{J}_{g \rightarrow g}$ current (eq. (4.30)).

Finally, in fig. 4.15b we plot an observable which is uncorrelated with (N)MRK phase space, namely the scalar sum of the squared transverse momenta of all jets, H_T . Here we see that the NLL approximation gives a much better description of the LO amplitude than the LL approximation. This is to be expected, because of the milder approximations we made to the amplitude in obtaining the NLL factorised expression.

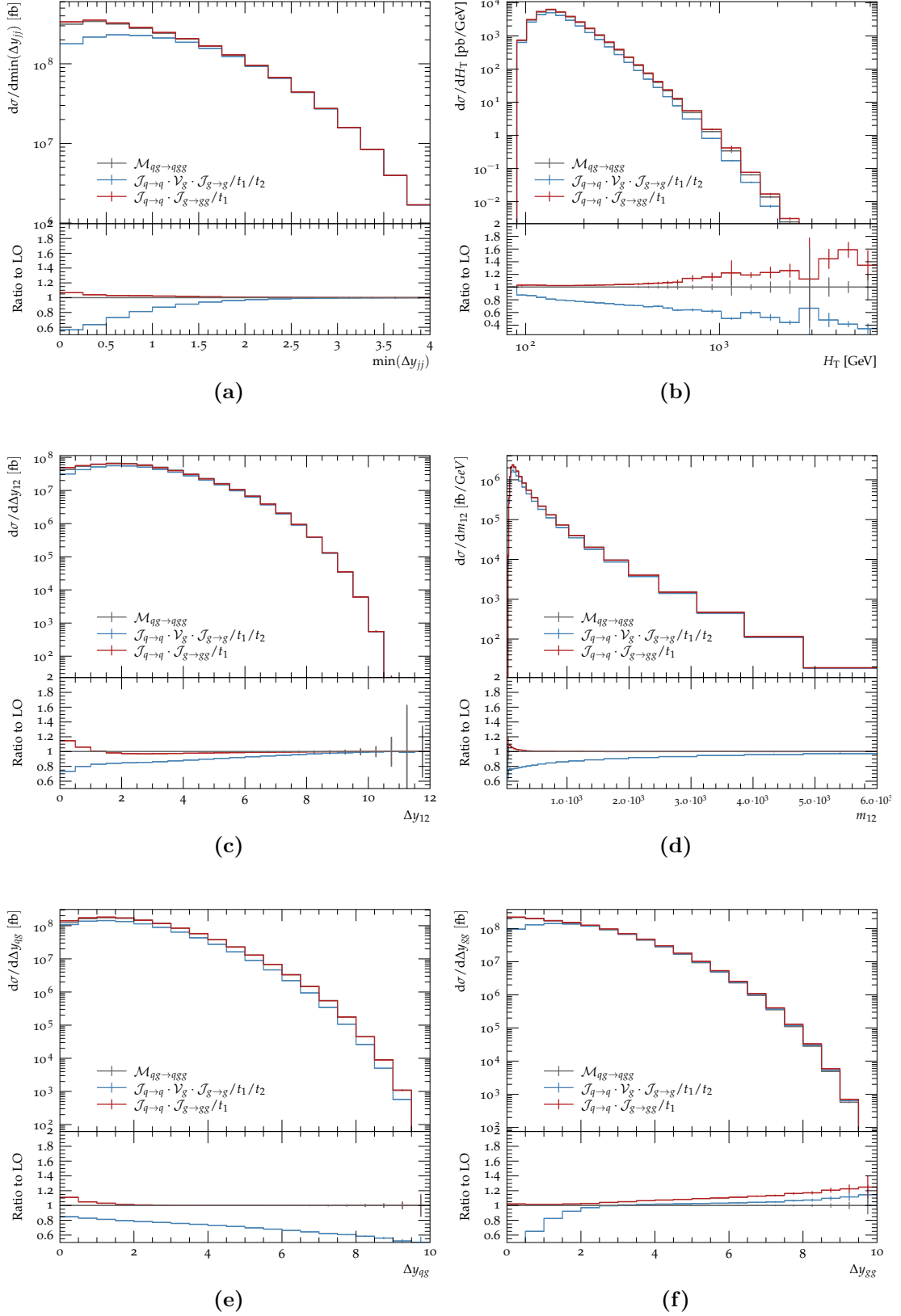


Figure 4.15 *Differential cross sections for the process $ud \rightarrow ugg$ at a 13 TeV collider, with $|p_{j\perp}| > 30$ GeV, $|y_j| < 6$, jet radius $R=0.4$ anti-kt, scale $H_T/2$.*

4.3 Factorisation of $2 \rightarrow 4$ amplitudes at LO

We now come to the derivation of NLL building blocks of the HEJ framework which must necessarily be extracted from six-parton amplitudes, namely the two-parton CEVs depicted in figs. 2.9c and 2.9d. The central quark-antiquark vertex was derived in ref. [1] and is presented in section 4.3.1. The central two-gluon vertex is a new result, and is presented in section 4.3.2.

4.3.1 $\bar{q}_1 q_1 \bar{q}_2 q_2 \bar{q}_3 q_3$

In this section, we obtain a CEV for the emission of a quark-antiquark pair, by taking the central NMRK limit $y_{\bar{q}_1} \sim y_{q_1} \gg y_{\bar{q}_2} \sim y_{q_2} \gg y_{\bar{q}_3} \sim y_{q_3}$ of the LO amplitude for $\emptyset \rightarrow \bar{q}_1 q_1 \bar{q}_2 q_2 \bar{q}_3 q_3$. In stark contrast to the following section, for this process it is more straightforward to work with the traditional (colour-dressed) Feynman rules of eqs. (1.62a)–(1.62g) rather than working with a colour decomposition. In Feynman gauge, there are just seven Feynman diagrams for this process, depicted in fig. 4.16. In the colour-flow basis of [157] there are six colour structures, one of which receives contribution from seven colour-ordered diagrams. Nevertheless, we will see that following our usual procedure of projecting octet t -channel exchanges leads to a factorised expression which retains more of the singularity structure of the LO amplitude compared to the original derivation of this expression [1]. Whether this minor improvement leads to a better approximation of the LO amplitude throughout the phase space of the LHC will be left to a future study.

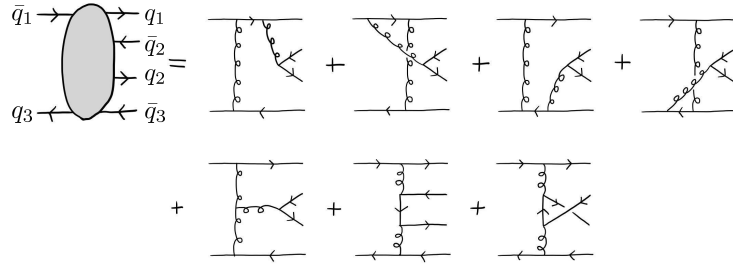


Figure 4.16 *Illustration of the Feynman diagrams of the amplitude $\mathcal{M}_{\bar{q}_1 q_1 \bar{q}_2 q_2 \bar{q}_3 q_3}$.*

From fig. 4.16, we see that the final three diagrams are of the desired factorised form, so we only need to approximate the first four diagrams. Let us

take the first Feynman diagram as an example, for the helicity configuration $\emptyset \rightarrow \bar{q}_1^\oplus q_1^\ominus \bar{q}_2^\oplus q_2^\ominus \bar{q}_3^\oplus q_3^\ominus$:

$$\begin{aligned}
& -ig_s^3 (T^{a_x} T^{a_{t_3}})_{i_1 \bar{i}_1} T_{i_2 \bar{i}_2}^{a_x} \frac{\langle q_1 | \mu_x (q_1 + \bar{q}_2 + q_2) \mu_{t_3} | \bar{q}_1 \rangle \langle q_2 | \mu_x | \bar{q}_2 \rangle}{s_{q_1 \bar{q}_2 q_2}} \frac{1}{s_{\bar{q}_2 q_2}} \\
& \times \left(\frac{-i}{t_3} \right) (ig_s T_{i_3 \bar{i}_3}^{a_{t_3}} \langle q_3 | \mu_{t_3} | \bar{q}_3 \rangle) .
\end{aligned} \tag{4.78}$$

We only need to apply the projection operator eq. (1.96) to the $i_1 \bar{i}_1$ colour structure, which results in

$$\begin{aligned}
& -ig_s^3 T_{i_1 \bar{i}_1}^{a_{t_1}} \frac{1}{T_F} \text{tr} (T^{a_{t_1}} T^{a_x} T^{a_{t_3}}) T_{i_2 \bar{i}_2}^{a_x} \frac{\langle q_1 | \mu_x (q_1 + \bar{q}_2 + q_2) \mu_{t_3} | \bar{q}_1 \rangle \langle q_2 | \mu_x | \bar{q}_2 \rangle}{s_{q_1 \bar{q}_2 q_2}} \frac{1}{s_{\bar{q}_2 q_2}} \\
& \times \left(\frac{-i}{t_3} \right) (ig_s T_{i_3 \bar{i}_3}^{a_{t_3}} \langle q_3 | \mu_{t_3} | \bar{q}_3 \rangle) .
\end{aligned} \tag{4.79}$$

Following the argument presented in ref [1], the $\not{q}_2 + \not{q}_2$ term may be dropped, as it is subleading compared to the $\not{q}_1$ term in the central NMRK limit. This leads to the desired factorised form for this diagram:

$$\begin{aligned}
& (ig_s T_{i_1 \bar{i}_1}^{a_{t_1}} \langle q_1 | \mu_{t_1} | \bar{q}_1 \rangle) \left(\frac{-i}{t_1} \right) \\
& \times -ig_s^2 \frac{1}{T_F} \text{tr} (T^{a_{t_1}} T^{a_x} T^{a_{t_3}}) T_{i_2 \bar{i}_2}^{a_x} \frac{s_{\bar{q}_1 q_1}}{s_{\bar{q}_2 q_2}} \frac{2p_{q_1}^{\mu_x} \eta^{\mu_{t_1} \mu_{t_3}}}{s_{q_1 \bar{q}_2 q_2}} \langle q_2 | \mu_x | \bar{q}_2 \rangle \\
& \times \left(\frac{-i}{t_3} \right) (ig_s T_{i_3 \bar{i}_3}^{a_{t_3}} \langle q_3 | \mu_{t_3} | \bar{q}_3 \rangle) .
\end{aligned} \tag{4.80}$$

We then perform the same procedure for the remaining three non-factorised diagrams in fig. 4.16. Upon application of eq. (1.83), we see that only the antisymmetric colour structure contributes in the NMRK limit. The first four diagrams of fig. 4.16 can hence be written

$$\begin{aligned}
& (ig_s T_{i_1 \bar{i}_1}^{a_{t_1}} \langle q_1 | \mu_{t_1} | \bar{q}_1 \rangle) \left(\frac{-i}{t_1} \right) \\
& \times g_s^2 f^{a_{t_1} a_{t_3} a_x} T_{i_2 \bar{i}_2}^{a_x} \eta^{\mu_{t_1} \mu_{t_3}} \left[\frac{s_{\bar{q}_1 q_1}}{s_{\bar{q}_2 q_2}} \left(\frac{2p_{q_1}^{\mu_x}}{s_{q_1 \bar{q}_2 q_2}} + \frac{2p_{\bar{q}_1}^{\mu_x}}{s_{\bar{q}_1 \bar{q}_2 q_2}} \right) - \frac{s_{\bar{q}_3 q_3}}{s_{\bar{q}_2 q_2}} \left(\frac{2p_{q_3}^{\mu_x}}{s_{q_3 \bar{q}_2 q_2}} + \frac{2p_{\bar{q}_3}^{\mu_x}}{s_{\bar{q}_3 \bar{q}_2 q_2}} \right) \right] \langle q_2 | \mu_x | \bar{q}_2 \rangle \\
& \times \left(\frac{-i}{t_3} \right) (ig_s T_{i_3 \bar{i}_3}^{a_{t_3}} \langle q_3 | \mu_{t_3} | \bar{q}_3 \rangle) .
\end{aligned} \tag{4.81}$$

The remaining three diagrams in fig. 4.16 can be included without any approxi-

mation. This leads to the factorised approximation of the amplitude

$$\begin{aligned}
\mathcal{M}(\bar{q}_1, q_1, \bar{q}_2, q_2, \bar{q}_3, q_3) &\xrightarrow{\text{NMRK}} \mathcal{J}_{\bar{q}qg^* \mu_{t_1}}^{a_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) \left(\frac{-i}{t_1} \right) \\
&\times \mathcal{V}_{g^* \bar{q}qg^*}^{\mu_{t_1} \mu_{t_3} a_{t_1} a_{t_3}}(-p_{t_1}, p_{\bar{q}_2}, p_{q_2}, p_{t_3}) \\
&\times \left(\frac{-i}{t_3} \right) \mathcal{J}_{\bar{q}qg^* \mu_{t_3}}^{a_{t_3}}(p_{\bar{q}_3}, p_{q_3}, -p_{t_3}),
\end{aligned} \tag{4.82}$$

with

$$\begin{aligned}
&\mathcal{V}_{g^* \bar{q}qg^*}^{\mu_{t_1} \mu_{t_3} a_{t_1} a_{t_3}}(-p_{t_1}, p_{\bar{q}_2}^{\oplus}, p_{q_2}^{\ominus}, p_{t_3}) \\
&= g_s^2 \left\{ f^{a_{t_1} a_{t_3} a_x} T_{i_2 \bar{i}_2}^{a_x} \frac{\langle q_2 | \mu_x | \bar{q}_2 \rangle}{s_{\bar{q}_2 q_2}} \left[v_{3g}^{\mu_{t_1} \mu_{t_3} \mu_x}(-p_{t_1}, p_{t_3}, p_{\bar{q}_2} + p_{q_2}) \right. \right. \\
&\quad \left. \left. + \eta^{\mu_{t_1} \mu_{t_3}} \left(s_{\bar{q}_1 q_1} \left(\frac{2p_{q_1}^{\mu_x}}{s_{q_1 \bar{q}_2 q_2}} + \frac{2p_{\bar{q}_1}^{\mu_x}}{s_{\bar{q}_1 \bar{q}_2 q_2}} \right) - s_{\bar{q}_3 q_3} \left(\frac{2p_{q_3}^{\mu_x}}{s_{q_3 \bar{q}_2 q_2}} + \frac{2p_{\bar{q}_3}^{\mu_x}}{s_{\bar{q}_3 \bar{q}_2 q_2}} \right) \right) \right] \right. \\
&\quad \left. - i (T^{a_{t_3}} T^{a_{t_1}})_{i_2 \bar{i}_2} \left[\frac{\langle q_2 | \mu_{t_3} (q_3 + \bar{q}_3 + q_2) \mu_{t_1} | \bar{q}_2 \rangle}{s_{\bar{q}_3 q_3 q_2}} \right] \right. \\
&\quad \left. - i (T^{a_{t_1}} T^{a_{t_3}})_{i_2 \bar{i}_2} \left[\frac{\langle q_2 | \mu_{t_1} (q_1 + \bar{q}_1 + q_2) \mu_{t_3} | \bar{q}_2 \rangle}{s_{\bar{q}_1 q_1 q_2}} \right] \right\}.
\end{aligned} \tag{4.83}$$

in agreement with the expression derived in ref [1], up to a modification of the three-particle Mandelstam invariants in the third line, which is subleading in the NMRK limit.

4.3.2 $\bar{q}_1 q_1 g g \bar{q}_2 q_2$

The final LO factorised expression we need to bring HEJ to NLL accuracy is the 2-tensor analogue of the two-gluon CEV in eq. (2.97). The latter was obtained by taking the central NMRK limit of $\emptyset \rightarrow gggggg$. In this section we instead start from the amplitude for $\emptyset \rightarrow \bar{q}_1 q_1 g_1 g_2 \bar{q}_2 q_2$ in the central NMRK limit $y_{\bar{q}_1} \sim y_{q_1} \gg y_{g_1} \sim y_{g_2} \gg y_{\bar{q}_2} \sim y_{q_2}$. We consider this process as it is easier to factorise the simple quark currents $\mathcal{J}_{\bar{q}_1 q_1 g^*}^{\mu_{t_1}}$ and $\mathcal{J}_{\bar{q}_2 q_2 g^*}^{\mu_{t_2}}$ from this process.

As always, we begin with the colour dressed amplitude

$$\begin{aligned}
\mathcal{M}(\bar{q}_1, q_1, g_1, g_2, \bar{q}_2, q_2) = & \sum_{\sigma \in S_2} (T^{a_{\sigma_1}} T^{a_{\sigma_2}})_{i_2 \bar{i}_1} \delta_{i_1 \bar{i}_2} M(\bar{q}_1, q_1, \bar{q}_2, q_2, g_{\sigma_1}, g_{\sigma_2}) \\
& + \delta_{i_2 \bar{i}_1} (T^{a_{\sigma_1}} T^{a_{\sigma_2}})_{i_1 \bar{i}_2} M(\bar{q}_1, q_1, g_{\sigma_1}, g_{\sigma_2}, \bar{q}_2, q_2) \\
& + T_{i_2 \bar{i}_1}^{a_{\sigma_1}} T_{i_1 \bar{i}_2}^{a_{\sigma_2}} M(\bar{q}_1, q_1, g_{\sigma_2}, \bar{q}_2, g_{\sigma_1}, q_2) \\
& + \delta_{i_1 \bar{i}_1} (T^{a_{\sigma_1}} T^{a_{\sigma_2}})_{i_2 \bar{i}_2} M(\bar{q}_1, q_1, \bar{q}_2, g_{\sigma_1}, g_{\sigma_2}, q_2) \\
& + (T^{a_{\sigma_1}} T^{a_{\sigma_2}})_{i_1 \bar{i}_1} \delta_{i_2 \bar{i}_2} M(\bar{q}_1, g_{\sigma_1}, g_{\sigma_2}, q_1, \bar{q}_2, q_2) \\
& + T_{i_1 \bar{i}_1}^{a_{\sigma_1}} T_{i_2 \bar{i}_2}^{a_{\sigma_2}} M(\bar{q}_1, g_{\sigma_1}, q_1, \bar{q}_2, g_{\sigma_2}, q_2)
\end{aligned} \tag{4.84}$$

which is depicted in fig. 4.17, and we apply the projection operator eq. (1.96) to both $\bar{q}q$ pairs in order to extract the component of the amplitude where an octet is exchanged in both t_1 and t_3 channels. In doing so we obtain

$$\begin{aligned}
\mathcal{M}^{[8]}(\bar{q}_1, q_1, g_1, g_2, \bar{q}_2, q_2) = & T_{i_1 \bar{i}_1}^{a_{t_1}} T_{i_2 \bar{i}_2}^{a_{t_3}} \frac{1}{T_F^2} \sum_{\sigma \in S_2} \text{tr} (T^{a_{t_1}} T^{a_{t_3}} T^{a_{\sigma_1}} T^{a_{\sigma_2}}) M(\bar{q}_1, q_1, \bar{q}_2, q_2, g_{\sigma_1}, g_{\sigma_2}) \\
& + \text{tr} (T^{a_{t_1}} T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{t_3}}) M(\bar{q}_1, q_1, g_{\sigma_1}, g_{\sigma_2}, \bar{q}_2, q_2) \\
& + \text{tr} (T^{a_{t_1}} T^{a_{\sigma_2}} T^{a_{t_3}} T^{a_{\sigma_1}}) M(\bar{q}_1, q_1, g_{\sigma_2}, \bar{q}_2, g_{\sigma_1}, q_2) \\
& + T_F^2 \delta^{a_{t_1} a_{\sigma_1}} \delta^{a_{t_3} a_{\sigma_2}} M(\bar{q}_1, g_{\sigma_1}, q_1, \bar{q}_2, g_{\sigma_2}, q_2),
\end{aligned} \tag{4.85}$$

which is depicted in fig. 4.18.

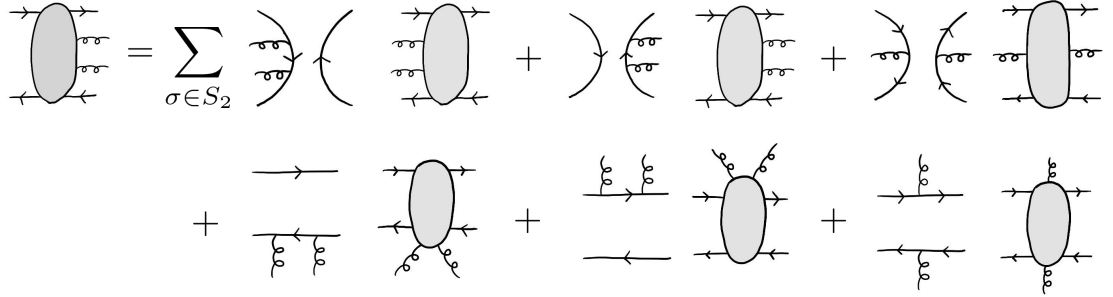


Figure 4.17 *Illustration of the colour decomposition of the amplitude in eq. (4.84). The sum is over permutations of the gluons.*

We begin our analysis of the final entry of eq. (4.85). The Feynman diagrams of this colour-ordered amplitude are depicted in fig. 4.19. The behaviour of this partial amplitude in the central NMRK is analogous to the amplitude eq. (4.53) we studied previously in the MRK limit. The factorisable and non-factorisable parts of this amplitude scale similarly in the central NMRK limit, so the use of

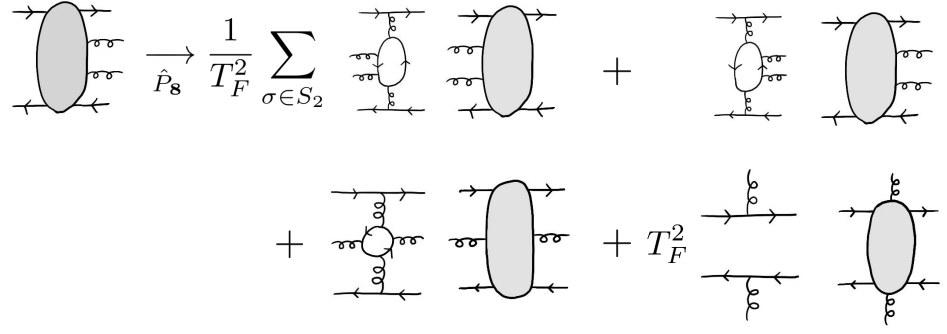


Figure 4.18 *Illustration of the colour decomposition of the amplitude in eq. (4.85). The sum is over permutations of the gluons.*

the Eikonal approximation for these gluon emissions is not valid. Furthermore, this colour-ordered amplitude is suppressed in the central NMRK limit compared to the other colour-ordered amplitudes in eq. (4.85), so we therefore drop this colour-ordered amplitude completely.

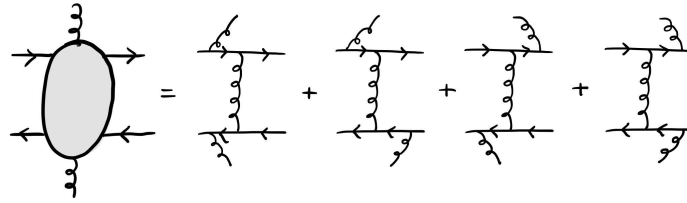


Figure 4.19 *Feynman diagram expansion of a colour ordered amplitude which is suppressed in the central NMRK limit.*

The next colour-ordered amplitude we will study is $M(\bar{q}_1, q_1, g_2, \bar{q}_2, q_2, g_1)$, which is depicted in fig. 4.20. With the exception of the first diagram, which contains a four-gluon vertex, the diagrammatic elements are all similar to diagrams we encountered in section 4.2.1. In order to obtain a factorised approximation of this amplitude we adopt the following procedure:

- We make the eikonal approximation to all instances of a gluon emission from a quark line.
- We use the approximated form of the three-gluon vertex, $\tilde{v}_{3g}$, defined in eq. (4.44), for all gluon emissions from a t -channel gluon.
- We require the central vertex to be gauge invariant.

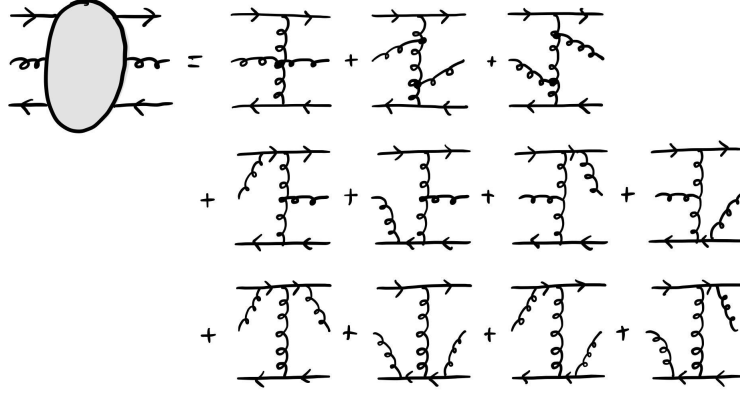


Figure 4.20 *Feynman diagrams of the colour-ordered amplitude $M(\bar{q}_1, q_1, g_2, \bar{q}_2, q_2, g_1)$.*

The first point will allow us to write the amplitude in the desired factorised form:

$$\begin{aligned}
 M(\bar{q}_1, q_1, g_2, \bar{q}_2, q_2, g_1) &\xrightarrow{\text{NMRK}} J_{\bar{q}qg^* \mu_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) \left(\frac{-i}{t_1} \right) \\
 &\times V_{g^*gg^*g}^{\mu_{t_1}\mu_{t_3}}(-p_{t_1}, p_{g_2}, p_{t_3}, p_{g_1}) \\
 &\times \left(\frac{-i}{t_3} \right) J_{\bar{q}qg^* \mu_{t_3}}(p_{\bar{q}_2}, p_{q_1}).
 \end{aligned} \tag{4.86}$$

The final point is the requirement that the vertex $V_{g^*gg^*g}^{\mu_{t_1}\mu_{t_3}}(p_{t_1}, p_{g_2}, p_{t_3}, p_{g_1})$ vanishes under replacement of either $\epsilon^{\mu_{g_1}}(p_{g_1}) \rightarrow p_{g_1}^{\mu_{g_1}}$ or $\epsilon^{\mu_{g_2}}(p_{g_2}) \rightarrow p_{g_2}^{\mu_{g_2}}$. Having specified a method of approximating all diagrams apart from the four-gluon vertex, this final point fixes the approximation we must make to this diagram. Prior to any kinematic approximation, this vertex is given by eq. (1.117), i.e.

$$v_{4g}^{\mu_{t_1}\mu_{g_1}\mu_{g_2}\mu_{t_3}} = 2\eta^{\mu_{t_1}\mu_{t_3}}\eta^{\mu_{g_1}\mu_{g_2}} - \eta^{\mu_{t_1}\mu_{g_2}}\eta^{\mu_{t_3}\mu_{g_1}} - \eta^{\mu_{t_1}\mu_{g_1}}\eta^{\mu_{t_3}\mu_{g_2}}. \tag{4.87}$$

However, to maintain gauge invariance of the two-gluon CEV, we must make the replacement

$$v_{4g}^{\mu_{t_1}\mu_{g_1}\mu_{g_2}\mu_{t_3}} \rightarrow 2\eta^{\mu_{t_1}\mu_{t_3}}\eta^{\mu_{g_1}\mu_{g_2}}. \tag{4.88}$$

While this final step appears ad-hoc, this method of approximation nevertheless maintains accuracy of the partial amplitude in the central NMRK limit for all helicity configurations. This property will be demonstrated in fig. 4.21.

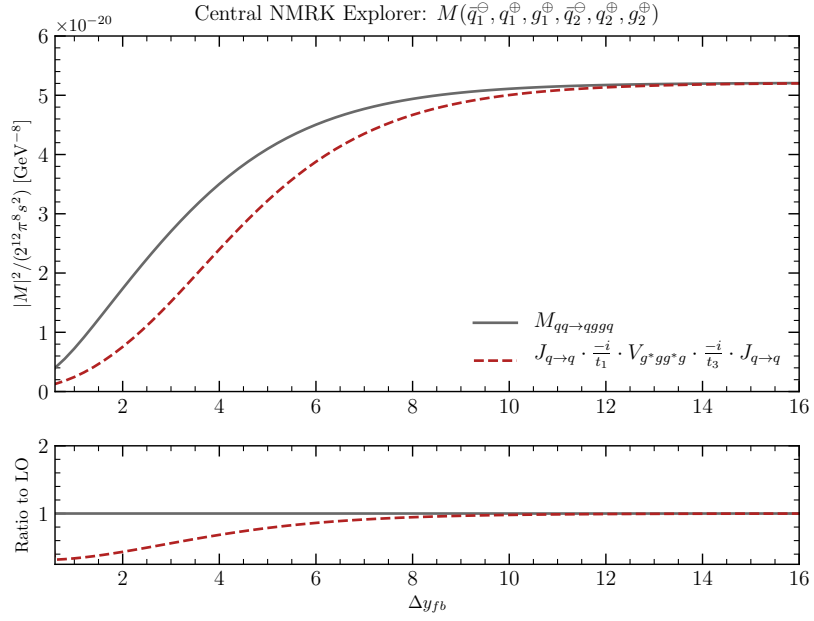


Figure 4.21 A slice through phase space comparing the exact colour-ordered amplitude $M(\bar{q}_1^-, q_1^+, g_1^+, \bar{q}_2^-, q_2^+, g_2^+)$ (black, solid) to the factorised approximation given in eq. (4.86) (red, dashed). The descriptions converge towards the right hand side of the plot, where the phase-space point approaches the NMRK limit. The details of the phase space slice are given in eq. (A.92).

The full vertex obtained in this manner is

$$\begin{aligned}
V_{g^* g g^* g}^{\mu_{t_1} \mu_{t_3}}(-p_{t_1}, p_{g_1}^{\lambda_{g_1}}, p_{t_2}, p_{g_2}^{\lambda_{g_2}}) &= i g_s \epsilon_{\mu_{g_1}}(p_{g_1}^{\lambda_{g_1}}; r_{g_1}) \epsilon_{\mu_{g_2}}(p_{g_2}^{\lambda_{g_2}}; r_{g_2}) \\
&\times \left\{ -\frac{1}{t_2} \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_2} \mu_{g_1}}(-p_{t_1}, p_{t_2}, p_{g_1}) \tilde{v}_{3g}^{\mu_{t_2} \mu_{t_3} \mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) \right. \\
&- \frac{1}{t_2'} \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_2'} \mu_{g_2}}(-p_{t_1}, p_{t_2'}, p_{g_2}) \tilde{v}_{3g}^{\mu_{t_2'} \mu_{t_3} \mu_{g_1}}(-p_{t_2'}, p_{t_3}, p_{g_1}) \\
&+ \frac{t_1}{t_2} \left(\frac{2p_{q_1}^{\mu_{g_1}}}{s_{\bar{q}_1 g_1}} \right) \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) - \frac{t_3}{t_2'} \left(\frac{2p_{q_2}^{\mu_{g_1}}}{s_{q_2 g_1}} \right) \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_1}, p_{t_2'}, p_{g_2}) \\
&+ \frac{t_1}{t_2'} \left(\frac{2p_{q_1}^{\mu_{g_2}}}{s_{q_1 g_2}} \right) \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_3} \mu_{g_1}}(-p_{t_2'}, p_{t_3}, p_{g_1}) - \frac{t_3}{t_2} \left(\frac{2p_{q_2}^{\mu_{g_2}}}{s_{\bar{q}_2 g_2}} \right) \tilde{v}_{3g}^{\mu_{t_1} \mu_{t_3} \mu_{g_1}}(-p_{t_1}, p_{t_2}, p_{g_1}) \\
&- \eta^{\mu_{t_1} \mu_{t_3}} \left(t_1 \left(\frac{2p_{\bar{q}_1}^{\mu_{g_1}}}{s_{\bar{q}_1 g_1}} \right) \left(\frac{2p_{q_1}^{\mu_{g_2}}}{s_{q_1 g_2}} \right) + t_3 \left(\frac{2p_{q_2}^{\mu_{g_1}}}{s_{q_2 g_1}} \right) \left(\frac{2p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2 g_2}} \right) - 2\eta^{\mu_{g_1} \mu_{g_2}} \right) \\
&\left. + t_1 t_3 \eta^{\mu_{t_1} \mu_{t_3}} \left(\frac{1}{t_2} \left(\frac{2p_{\bar{q}_1}^{\mu_{g_1}}}{s_{\bar{q}_1 g_1}} \right) \left(\frac{2p_{q_2}^{\mu_{g_2}}}{s_{\bar{q}_2 g_2}} \right) + \frac{1}{t_2'} \left(\frac{2p_{q_2}^{\mu_{g_1}}}{s_{q_2 g_1}} \right) \left(\frac{2p_{\bar{q}_1}^{\mu_{g_2}}}{s_{q_1 g_2}} \right) \right) \right\}.
\end{aligned} \tag{4.89}$$

We demonstrate the gauge invariance of this expression by replacing

$$\epsilon_{\mu_{g_1}}(p_{g_1}^{\lambda_{g_1}}; r_{g_1}) \rightarrow p_{g_1 \mu_{g_1}}:$$

$$\begin{aligned} V_{g^* g g^* g}^{\mu_{t_1} \mu_{t_3}}(-p_{t_1}, p_{g_1}^{\lambda_{g_1}}, p_{t_2}, p_{g_2}^{\lambda_{g_2}}) &\rightarrow i g_s \epsilon_{\mu_{g_2}}(p_{g_2}^{\lambda_{g_2}}; r_{g_2}) \left\{ 2 \eta^{\mu_{t_1} \mu_{t_3}} p_{g_1}^{\mu_{g_2}} \right. \\ &- \frac{1}{t_2} (t_1 - t_2) V^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) - \frac{1}{t'_2} V^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_1}, p_{t'_2}, p_{g_2}) (t'_2 - t_3) \\ &+ \frac{t_1}{t_2} V^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) - \frac{t_3}{t'_2} V^{\mu_{t_1} \mu_{t_3} \mu_{g_2}}(-p_{t_1}, p_{t'_2}, p_{g_2}) \\ &+ \frac{t_1}{t'_2} \left(\frac{2 p_{q_1}^{\mu_{g_2}}}{s_{q_1 g_2}} \right) (t'_2 - t_3) - \frac{t_3}{t_2} \left(\frac{2 p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2 g_2}} \right) (t_1 - t_2) \\ &\left. - \eta^{\mu_{t_1} \mu_{t_3}} \left(t_1 \left(1 - \frac{t_3}{t'_2} \right) \left(\frac{2 p_{q_1}^{\mu_{g_2}}}{s_{q_1 g_2}} \right) + t_3 \left(1 - \frac{t_1}{t_2} \right) \left(\frac{2 p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2 g_2}} \right) \right) \right\} = 0. \end{aligned} \quad (4.90)$$

We likewise obtain zero if we replace the polarisation vector of g_2 with its momentum vector.

We now consider the partial amplitude $M(\bar{q}_1, q_1, g_1, g_2, \bar{q}_2, q_2)$ where the gluons are now cyclically adjacent, as depicted in fig. 4.22.

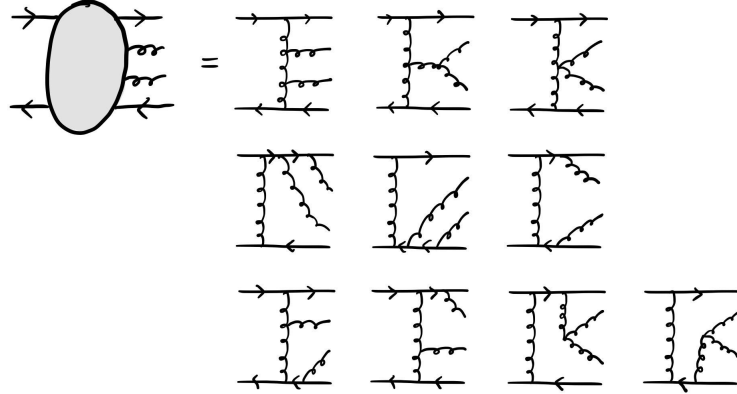


Figure 4.22 *Feynman diagrams of the colour-ordered amplitude $M(\bar{q}_1, q_1, g_1, g_2, \bar{q}_2, q_2)$.*

We follow the same procedure as before, but this only sets the approximations used for three of the ten Feynman diagrams. Additional care must be taken where both gluons are emitted from the same quark leg, as these diagrams contain three-parton Mandelstam invariants, which is a new feature compared to the diagrams in fig. 4.20. We must preserve the intricate web of cancellations that are necessary for the gauge invariance of the full amplitude.

Consider one such Feynman diagram,

$$ig_s^4 \frac{[q_1 | \mu_{g_1} (q_1 + g_1) \mu_{g_2} (q_1 + g_1 + g_2) \mu_{t_3} | \bar{q}] \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle}{s_{q_1 g_1} s_{q_1 g_1 g_2}} \quad (4.91)$$

stripped of polarisation vectors. We choose to approximate this diagram as

$$ig_s^4 \frac{2q_1^{\mu_{g_1}}}{s_{qg_1}} \frac{2q_1^{\mu_{g_2}}}{(s_{qg_1} + s_{qg_2})} [q_1 | \mu_{t_3} | \bar{q}] \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle. \quad (4.92)$$

Motivated by the results of section 4.2.1, we expect that the omitted $\not{g}_1$ and $\not{g}_2$ terms will lead to subleading terms for any choice of r_{g_1} or r_{g_2} . This will be verified numerically below. The modification of the three-particle invariant can be easily justified, because in the NMRK region of interest, $s_{g_1 g_2} \ll s_{qg_1} + s_{qg_2}$.

Contracting eq. (4.92) with p_{g_1} yields

$$ig_s^4 \frac{2q_1^{\mu_{g_2}}}{(s_{qg_1} + s_{qg_2})} [q_1 | \mu_{t_3} | \bar{q}] \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle. \quad (4.93)$$

Now consider the Feynman diagram where a three-gluon vertex is coupled to the same quark leg:

$$ig_s^4 \frac{[q_1 | \mu_\sigma (q + g_1 + g_2) \mu_{t_3} | \bar{q}]}{s_{qg_1 g_2}} \frac{1}{s_{g_1 g_2}} V_{3g}^{\sigma \mu_{g_2} \mu_{g_1}} (-p_{g_1} - p_{g_2}, p_{g_2}, p_{g_1}) \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle, \quad (4.94)$$

If we make an analogous approximation to this diagram,

$$ig_s^4 \frac{2q_1^{\mu_\sigma}}{(s_{q_1 g_1} + s_{q_1 g_2})} [q_1 | \mu_{t_3} | \bar{q}] \frac{1}{s_{g_1 g_2}} V_{3g}^{\sigma \mu_{g_2} \mu_{g_1}} (-p_{g_1} - p_{g_2}, p_{g_2}, p_{g_1}) \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle, \quad (4.95)$$

and contract with $p_{g_1}^{\mu_{g_1}}$ we obtain

$$ig_s^4 \left\{ \frac{2q_1^{\mu_{g_2}}}{(s_{q_1 g_1} + s_{q_1 g_2})} + \frac{2g_1^{\mu_{g_2}}}{s_{g_1 g_2}} \right\} [q_1 | \mu_{t_3} | \bar{q}] \left(\frac{1}{t_3} \right) \langle q_2 | \mu_{t_3} | \bar{q}_2 \rangle, \quad (4.96)$$

where the first term in the braces cancels with eq. (4.93). The approximations made in eq. (4.92) and eq. (4.95) were chosen to imitate the cancellation of gauge variation terms between the exact diagrams. At this point we have approximated eight of the ten Feynman diagrams in fig. 4.22. The total variation of the sum of

these eight diagrams under the replacement $\epsilon_{\mu_{g_1}} \rightarrow p_{g_1}$ is

$$\eta^{\mu_{t_1}\mu_{t_3}} \left\{ \frac{1}{t_1 t_3} (q_1 + q_3)^{\mu_{g_2}} + \left(\frac{1}{t_1} - \frac{1}{t_3} \right) \frac{p_{g_1}^{\mu_{g_2}}}{s_{g_1 g_2}} \right\} + \frac{1}{t_1 t_3} (2\eta^{\mu_{t_1}\mu_{g_2}} p_{g_2}^{\mu_{t_3}} - 2\eta^{\mu_{t_3}\mu_{g_2}} p_{g_2}^{\mu_{t_1}}) \quad (4.97)$$

Prior to any approximation, it can be shown (after much algebra) that these same eight diagrams lead to a total variation of

$$\eta^{\mu_{t_1}\mu_{t_3}} \left\{ \frac{1}{t_1 t_3} (q_1 + q_3)^{\mu_{g_2}} + \left(\frac{1}{t_1} - \frac{1}{t_3} \right) \frac{p_{g_1}^{\mu_{g_2}}}{s_{g_1 g_2}} \right\} + \frac{1}{t_1 t_3} (2\eta^{\mu_{t_1}\mu_{g_2}} p_{g_2}^{\mu_{t_3}} + \eta^{\mu_{t_1}\mu_{g_2}} p_{g_1}^{\mu_{t_3}} - 2\eta^{\mu_{t_3}\mu_{g_2}} p_{g_2}^{\mu_{t_1}}), \quad (4.98)$$

which, by virtue of gauge invariance of the full amplitude, cancels with the total variation of the remaining two Feynman diagrams

$$v_{4g}^{\mu_{t_1}\mu_{g_1}\mu_{g_2}\mu_{t_3}} + \frac{1}{s_{g_1 g_2}} v_{3g}^{\mu_{\sigma}\mu_{g_2}\mu_{g_1}} (-p_{g_1} - p_{g_2}, p_{g_2}, p_{g_1}) v_{3g}^{\mu_{t_1}\mu_{t_2}\mu_{\sigma}} (-p_{t_1}, p_{t_3}, p_{g_1} + p_{g_2}). \quad (4.99)$$

We can therefore include these latter two diagrams exactly, and include an additional single term proportional to $\eta^{\mu_{t_1}\mu_{g_1}} \eta^{\mu_{t_3}\mu_{g_2}}$. This term generates a term proportional to $p_{g_1}^{\mu_{t_1}} \eta^{\mu_{t_3}\mu_{g_2}}$ under the replacement $\epsilon_{\mu_{g_1}} \rightarrow p_{g_1}$ and generates a term proportional to $\eta^{\mu_{t_1}\mu_{g_1}} p_{g_2}^{\mu_{t_3}}$ under the replacement $\epsilon_{\mu_{g_2}} \rightarrow p_{g_2}$. Thus, we introduce a term that restores gauge invariance whilst maintaining the target-projectile symmetry of the amplitude.

In keeping with eq. (4.88), we can interpret this correction factor as a modification of the four-gluon vertex. The exact vertex is

$$v_{4g}^{\mu_{t_1}\mu_{g_1}\mu_{g_2}\mu_{t_3}} = 2\eta^{\mu_{t_1}\mu_{g_2}} \eta^{\mu_{t_3}\mu_{g_1}} - \eta^{\mu_{t_1}\mu_{t_3}} \eta^{\mu_{g_1}\mu_{g_2}} - \eta^{\mu_{t_1}\mu_{g_1}} \eta^{\mu_{t_3}\mu_{g_2}}. \quad (4.100)$$

but we must make the replacement

$$v_{4g}^{\mu_{t_1}\mu_{g_1}\mu_{g_2}\mu_{t_3}} \rightarrow 2\eta^{\mu_{t_1}\mu_{g_2}} \eta^{\mu_{t_3}\mu_{g_1}} - \eta^{\mu_{t_1}\mu_{t_3}} \eta^{\mu_{g_1}\mu_{g_2}} - 2\eta^{\mu_{t_1}\mu_{g_1}} \eta^{\mu_{t_3}\mu_{g_2}}. \quad (4.101)$$

The full factorised expression is then

$$\begin{aligned}
V_{g^*ggg^*}^{\mu_{t_1}\mu_{t_3}}(-p_{t_1}, p_{g_1}^{\lambda_{g_1}}, p_{g_2}^{\lambda_{g_2}}, p_{t_2}) &= ig_s \epsilon_{\mu_{g_1}}(p_{g_1}^{\lambda_{g_1}}; r_{g_1}) \epsilon_{\mu_{g_2}}(p_{g_2}^{\lambda_{g_2}}; r_{g_2}) \\
&\times \left\{ \eta^{\mu_{t_1}\mu_{t_3}} \left(t_1 \left(\frac{2p_{q_1}^{\mu_{g_1}}}{s_{q_1g_1}} \right) \left(\frac{2p_{q_1}^{\mu_{g_2}}}{(s_{q_1g_1} + s_{q_1g_2})} \right) + t_3 \left(\frac{2p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2g_2}} \right) \left(\frac{2p_{\bar{q}_2}^{\mu_{g_1}}}{(s_{\bar{q}_2g_1} + s_{\bar{q}_2g_2})} \right) \right. \right. \\
&\quad \left. \left. - \frac{t_1 t_3}{t_2} \left(\frac{2p_{q_1}^{\mu_{g_1}}}{s_{q_1g_1}} \right) \left(\frac{2p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2g_2}} \right) \right) \right. \\
&\quad + \frac{1}{t_2} \tilde{v}_{3g}^{\mu_{t_1}\mu_{t_2}\mu_{g_1}}(-p_{t_1}, p_{t_2}, p_{g_1}) \tilde{v}_{3g}^{\mu_{t_2}\mu_{t_3}\mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) \\
&\quad - \frac{t_1}{t_2} \left(\frac{2p_{q_1}^{\mu_{g_1}}}{s_{q_1g_1}} \right) \tilde{v}_{3g}^{\mu_{t_1}\mu_{t_3}\mu_{g_2}}(-p_{t_2}, p_{t_3}, p_{g_2}) + \frac{t_3}{t_2} \tilde{v}_{3g}^{\mu_{t_1}\mu_{t_3}\mu_{g_1}}(-p_{t_1}, p_{t_2}, p_{g_1}) \left(\frac{2p_{\bar{q}_2}^{\mu_{g_2}}}{s_{\bar{q}_2g_2}} \right) \\
&\quad + \frac{1}{s_{g_1g_2}} v_{3g}^{\mu_\sigma\mu_{g_2}\mu_{g_1}}(-p_{g_1} - p_{g_2}, p_{g_2}, p_{g_1}) \left(v_{3g}^{\mu_{t_1}\mu_{t_2}\mu_\sigma}(-p_{t_1}, p_{t_3}, p_{g_1} + p_{g_2}) \right. \\
&\quad \left. - \frac{2q_1^{\mu_\sigma}}{(s_{q_1g_1} + s_{q_1g_2})} + \frac{2\bar{q}_2^{\mu_\sigma}}{(s_{\bar{q}_2g_1} + s_{\bar{q}_2g_2})} \right) \\
&\quad \left. + 2\eta^{\mu_{t_1}\mu_{g_2}} \eta^{\mu_{t_3}\mu_{g_1}} - \eta^{\mu_{t_1}\mu_{t_3}} \eta^{\mu_{g_1}\mu_{g_2}} - 2\eta^{\mu_{t_1}\mu_{g_1}} \eta^{\mu_{t_3}\mu_{g_2}} \right\}.
\end{aligned} \tag{4.102}$$

The vertex eq. (4.102) is gauge invariant by construction, but we still need to show that the approximations we have made in obtaining this expression are valid in the central NMRK limit. We verify this numerically in fig. 4.23, where, just as in fig. 4.23, we find the approximation converges to the full LO colour-ordered amplitude as we approach the NMRK limit.

The partial amplitude for $V_{g^*g^*gg}^{\mu_{t_1}\mu_{t_3}}(-p_{t_1}, p_{t_3}, p_{g_2}, p_{g_1})$ is obtained in a similar manner and is related to eq. (4.102) by the interchanges $q_1 \leftrightarrow q_2$ and $\bar{q}_1 \leftrightarrow \bar{q}_2$.

We are now in a position to finally return to the colour dressed amplitude eq. (4.85), which we can approximate as

$$\begin{aligned}
\mathcal{M}(\bar{q}_1, q_1, g_1, g_2, \bar{q}_2, q_2) &\xrightarrow{\text{NMRK}} \mathcal{J}_{\bar{q}qg^*\mu_{t_1}}^{a_{t_1}}(p_{\bar{q}_1}, p_{q_1}, p_{t_1}) \left(\frac{-i}{t_1} \right) \\
&\times \mathcal{V}_{g^*ggg^*}^{\mu_{t_1}\mu_{t_3}a_{t_1}a_{t_3}}(-p_{t_1}, p_{g_1}, p_{g_2}, p_{t_3}) \\
&\times \left(\frac{-i}{t_3} \right) \mathcal{J}_{\bar{q}qg^*\mu_{t_3}}(p_{\bar{q}_2}, p_{q_2}, -p_{t_3}),
\end{aligned} \tag{4.103}$$

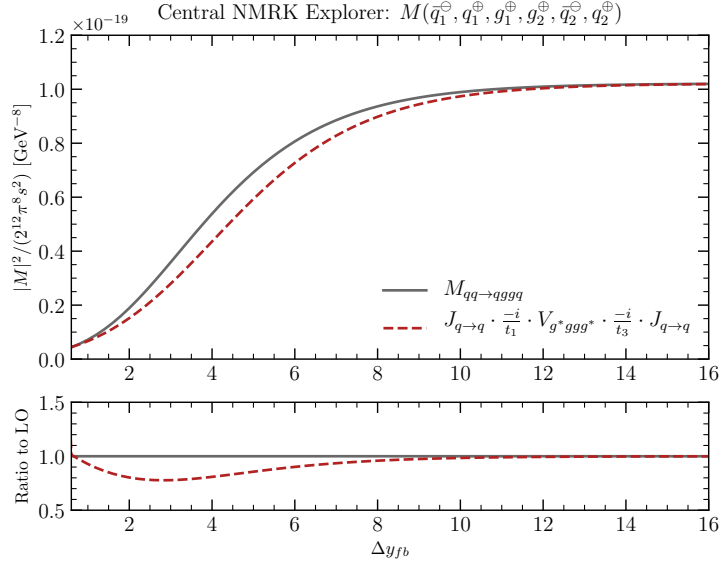


Figure 4.23 *A slice through phase space comparing the exact colour-ordered amplitude $M(\bar{q}_1^\ominus, q_1^\oplus, g_1^\oplus, g_2^\oplus, \bar{q}_2^\ominus, q_2^\oplus)$ (black, solid) to the factorised approximation which uses eq. (4.102) (red, dashed). The descriptions converge towards the right hand side of the plot, where the phase-space point approaches the NMRK limit. The details of the phase space slice are given in eq. (A.92).*

taking the colour dressed CEV to be

$$\begin{aligned}
 \mathcal{V}_{g^* g g g^*}^{\mu_{t_1} \mu_{t_3} a_{t_1} a_{t_3}}(-p_{t_1}, p_{g_1}, p_{g_2}, p_{t_3}) &= \frac{1}{T_F^2} \sum_{\sigma \in S_2} \text{tr}(T^{a_{t_1}} T^{a_{t_3}} T^{a_{\sigma_1}} T^{a_{\sigma_2}}) V_{g^* g^* g g}^{\mu_{t_1} \mu_{t_3}}(-p_{t_1}, p_{t_3}, p_{g_{\sigma_1}}, p_{g_{\sigma_2}}) \\
 &\quad + \text{tr}(T^{a_{t_1}} T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{t_3}}) V_{g^* g g g^*}^{\mu_{t_1} \mu_{t_3}}(-p_{t_1}, p_{g_{\sigma_1}}, p_{g_{\sigma_2}}, p_{t_3}) \\
 &\quad + \text{tr}(T^{a_{t_1}} T^{a_{\sigma_2}} T^{a_{t_3}} T^{a_{\sigma_1}}) V_{g^* g g^* g}^{\mu_{t_1} \mu_{t_3}}(-p_{t_1}, p_{g_{\sigma_2}}, p_{t_3}, p_{g_{\sigma_1}}).
 \end{aligned}
 \tag{4.104}$$

In fig. 4.24b we see that the factorised approximation of eq. (4.103) (the red, dashed line) converges with the exact LO amplitude (the black, solid line) as we approach a central NMRK phase space point, verifying that the many approximations we have made in obtaining eq. (4.103) are indeed valid in the NMRK limit. Furthermore, we see that the approximation of eq. (4.103) converges to the LO result far faster than they both converge to the asymptotic NMRK value. This demonstrates that the mild approximations made to obtain eq. (4.103) have retained much of the LO physics of the amplitude. The red and black lines converge at $\Delta y_{fb} \sim 3$, well within the range of the hadronic calorimeters at the LHC. In fig. 4.24b we also see that, as expected, a factorised

description consisting of only LL HEJ building blocks (the blue, dashed line) does not correctly describe a central NMRK phase space point. These last two points highlight the phenomenological importance of bringing HEJ to full NLL accuracy, and in particular the adoption of the CEV given in eq. (4.104). As a test of our implementation of both factorised approximations, in fig. 4.24a we have verified that all three lines converge in the MRK limit, as expected.

The plots in fig. 4.24 are crucial tests of the formalism, but they only explore a single one-dimensional slice through phase space. To more properly demonstrate the applicability of eq. (4.104) at the LHC, we need to integrate over the final-state phase space, and convolute with PDFs, according to eq. (1.30). The results are shown in fig. 4.25, which depicts several differential cross sections, discussed below. For clarity, we have specified the single partonic channel $ud \rightarrow uggd$, and we have required $y_u \geq y_g \geq y_d$. We have required $|p_{j\perp}| > 30$ GeV. Although jet rapidities are typically measured in the range $|y_j| < 4.5$, we have extended this cut to $|y_j| < 6$ to better demonstrate convergence in theoretically controlled limits.

Figure 4.25a can be considered the integrated analogue of fig. 4.24a, and demonstrates that both factorised approximations converge to the LO result in the MRK limit. Regarding figs. 4.25b–4.25d, we recall the comment made in section 4.2.2, that the tails of commonly measured observables such as Δy_{fb} , Δy_{12} and m_{12} , receive contribution from outwith the MRK limit. For $n > 2$ jets, these observables are inclusive over N^{n-2} MRK regions of phase space. In fig. 4.25b we see that the region of large y_{fb} is very poorly described by the blue line. In fact, this observation provided us with the initial motivation to derive eq. (4.104): in fig. 4.25b we see that the approximation of the red line, which is exact in the MRK and central NMRK limits, is in good agreement with the LO result across this distribution.

To test our understanding of eq. (4.104), we looked not only at observables which we knew should be well described by the red line; we also looked at regions of phase space where this approximation should fail. One example is given in fig. 4.25e. The extreme Δy_{gg} limit requires $y_u \sim y_{g_1} \gg y_{g_2} \sim y_d$. This is a NNMRK limit which explicitly breaks the assumption under which eq. (4.104) was derived. Instead, this region should be described by a contraction of two $q \rightarrow qg$ impact factors (given in eq. (4.54)). This NNLL description is depicted in fig. 4.25e by the green line. As an aside, this plot illustrates the broader point that a building block derived for a NLL approximation can be used to

describe an event at higher logarithmic accuracy, provided that IR divergences are appropriately regulated. In this case, the IR divergences of the $q \rightarrow qg$ impact factors are regulated by the requirement of 4 jets in the final state.

Finally, in fig. 4.25f we show an observable uncorrelated with the high energy limit. Here we see that the NLL (red) approximation gives an improved description over the LL (blue) approximation across this distribution, owing to the former description being derived under milder approximations of the LO amplitude.

We now have derived all the necessary real-emission building blocks of the HEJ formalism required to give a LL or NLL accurate description of cross sections. In the next section we discuss the inclusion of virtual corrections within the framework.

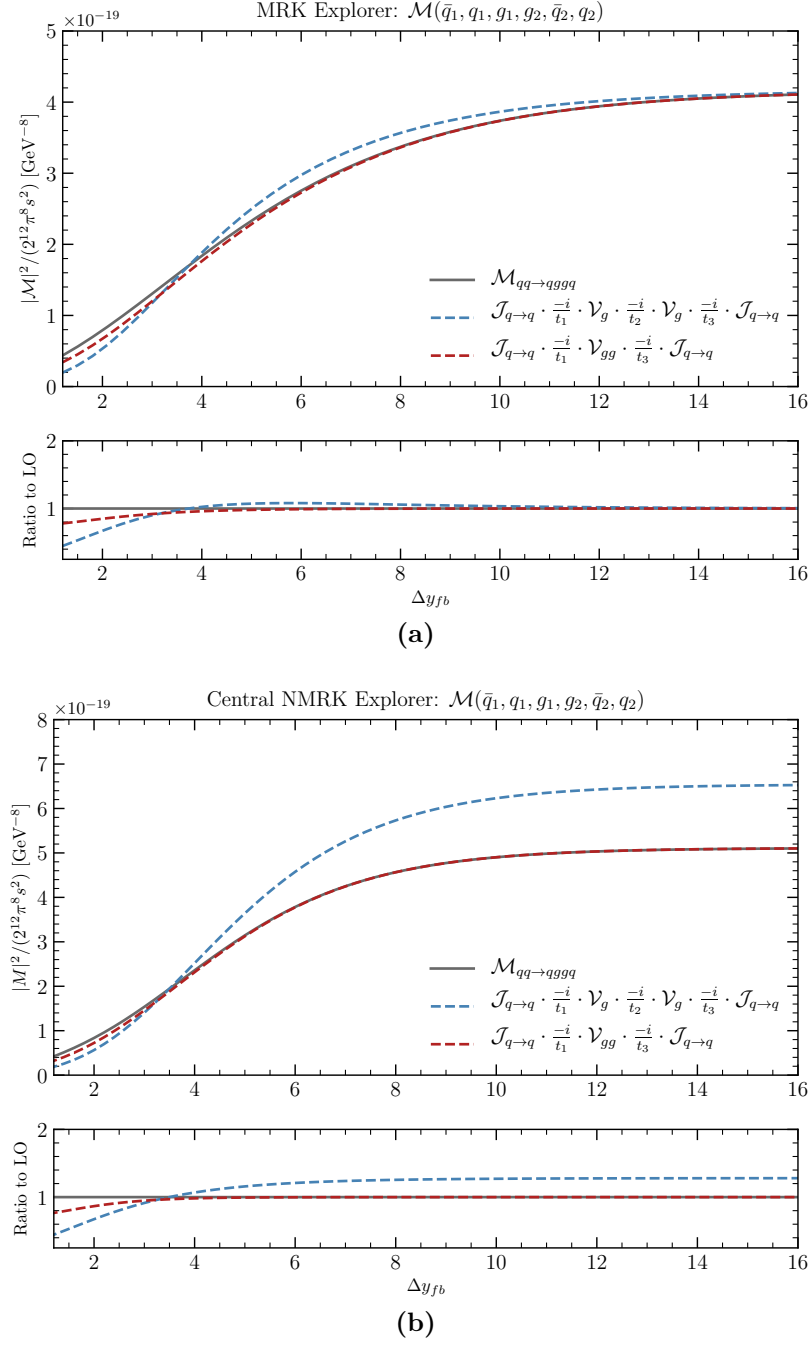


Figure 4.24 A comparison of the exact LO amplitude for $q_1 q_2 \rightarrow q_1 g g q_2$, squared, and summed over external helicity and colour configurations (black, solid), to the factorised approximation given in eq. (4.103) (red, dashed), and an approximation where the central gg CEV of the latter is further factorised into a product of two Lipatov vertices given in eq. (4.67). Panel (a) plots the phase space slice parametrised in eq. (A.91) which tends towards an MRK phase space point. Panel (b) plots the phase space slice parametrised in eq. (A.92) which tends towards a central NMRK phase space point.

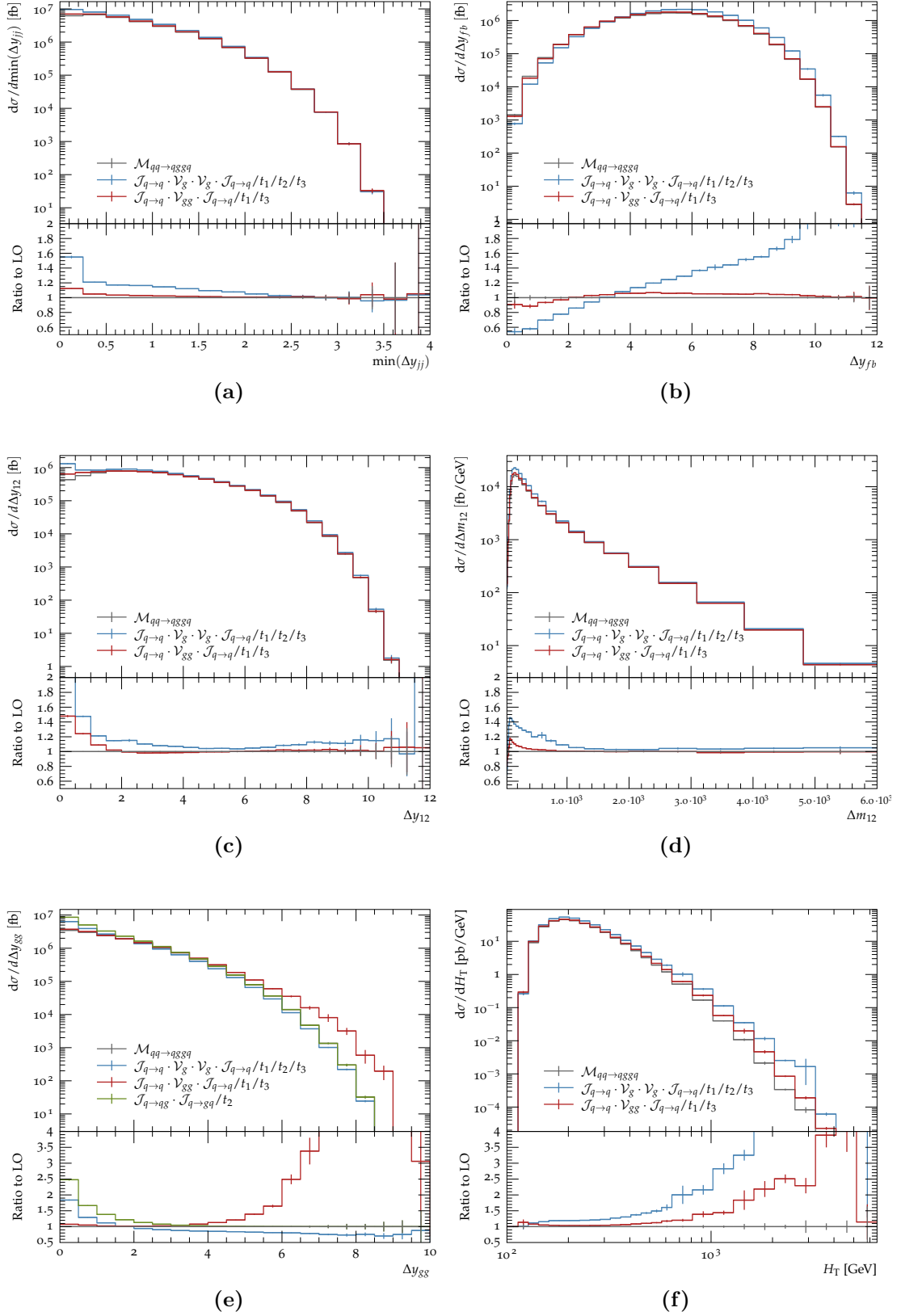


Figure 4.25 *Differential cross sections for the process $ud \rightarrow ugdd$ at a 13 TeV collider, with $|p_{j\perp}| > 30$ GeV, $|y_j| < 6$, jet radius $R=0.4$ anti-kt, scale $H_T/2$.*

4.4 HEJ at LL and NLL accuracy

In the previous section we adopted the all-outgoing convention for helicities and momenta, and we labelled the factorised building blocks of amplitudes by the outgoing parton flavour of each building block. In this section we will only use physical momenta, i.e. we consider the $2 \rightarrow n$ scattering of $f_a f_b \rightarrow f_1 \cdots f_n$, where momentum conservation reads

$$p_a + p_b = \sum_{i=1}^n p_i, \quad (4.105)$$

with $p_a^0, p_b^0, p_i^0 > 0$. Correspondingly, we label the factorised components by the physical parton flavours, as in eq. (4.18) for example.

At LL accuracy, truncated to LO, HEJ amplitudes are constructed in analogy to eq. (2.39):

$$\begin{aligned} \mathcal{M}_{f_a f_b \rightarrow f_1 g \cdots g f_n}^{\text{LL HEJ}}|_{\text{LO}} &= \mathcal{J}_{f_a \rightarrow f_1 g^*}^{\mu_{t_1} a_{t_1}}(p_a^{\lambda_a}; p_1^{\lambda_1}, p_{t_1}) \\ &\times \prod_{i=1}^{n-1} \left(\frac{-i}{t_i} \right) \\ &\times \prod_{i=2}^{n-1} \mathcal{V}_{g^* \rightarrow g g^*}^{\mu_{t_i} a_{t_{i-1}} a_{t_i}}(p_{t_{i-1}}; p_i^{\lambda_i}, p_{t_i}) \\ &\times \mathcal{J}_{g^* f_n \rightarrow f_2 \mu_{t_{n-1}}}^{a_{t_{n-1}}}(p_{t_{n-1}}, p_b^{\lambda_b}; p_n^{\lambda_n}), \end{aligned} \quad (4.106)$$

where $p_{t_i} = p_a - \sum_{j=1}^i p_j$ and $t_i = p_i^2$. In eq. (4.106) we have used the version of the Lipatov vertex given in eq. (4.60). As discussed in section 4.2.1, we can approximate the Lipatov vertex further to eq. (4.67). Then eq. (4.106) reduces to

$$\begin{aligned} \mathcal{M}_{f_a f_b \rightarrow f_1 g \cdots g f_n}^{\text{LL HEJ}}|_{\text{LO}} &= \mathcal{J}_{f_a \rightarrow f_1 g^*}^{\mu a_{t_1}}(p_a^{\lambda_a}; p_1^{\lambda_1}, p_{t_1}) \\ &\times \prod_{i=1}^{n-1} \left(\frac{-i}{t_i} \right) \\ &\times \prod_{i=2}^{n-1} \mathcal{V}_{g^* \rightarrow g g^*}^{a_{t_{i-1}} a_{t_i}}(p_{t_{i-1}}; p_i^{\lambda_i}, p_{t_i}) \\ &\times \mathcal{J}_{g^* f_n \rightarrow f_2 \mu}^{a_{t_{n-1}}}(p_{t_{n-1}}, p_b^{\lambda_b}; p_n^{\lambda_n}). \end{aligned} \quad (4.107)$$

At LL accuracy, to all orders in α_s , HEJ amplitudes are constructed in analogy

to eq. (2.45):

$$\begin{aligned}
\mathcal{M}_{f_a f_b \rightarrow f_1 g \dots g f_n}^{\text{LL HEJ}} &= \mathcal{J}_{f_a \rightarrow f_1 g^*}^{\mu a_{t_1}}(p_a^{\lambda_a}; p_1^{\lambda_1}, p_{t_1}) \\
&\times \prod_{i=1}^{n-1} \left(\frac{-i}{t_i} \right) e^{\alpha(|q_{i\perp}|^2) \Delta y_{i,i+1}} \\
&\times \prod_{i=2}^{n-1} \mathcal{V}_{g^* \rightarrow g g^*}^{a_{t_{i-1}} a_{t_i}}(p_{t_{i-1}}; p_i^{\lambda_i}, p_{t_i}) \\
&\times \mathcal{J}_{g^* f_n \rightarrow f_2 \mu}^{a_{t_{n-1}}}(p_{t_{n-1}}, p_b^{\lambda_b}, p_n^{\lambda_n}).
\end{aligned} \tag{4.108}$$

This corresponds to making the Lipatov ansatz, eq. (2.44), to capture virtual corrections to all orders, fixing the Reggeisation scale to be $\tau_i = s_{i,i+1} \sqrt{p_i^- p_{i+1}^+ / (p_i^+ p_{i+1}^-)}$.

Given that the soft limit of the HEJ version of the Lipatov vertex, eq. (4.67) has the same soft limit as the version of the Lipatov vertex given in eq. (2.32), the cancellation of the real and virtual IR poles of the cross section occurs as in section 2.1.2, which generalises to all orders. In the HEJ framework, the cancellation is achieved via phase space slicing, where the IR regularised amplitude is given by eq. (4.108) with the replacement [148]

$$\alpha(|q_{i\perp}|^2) \rightarrow \tilde{\alpha}(|q_{i\perp}|^2; \lambda) = -\frac{\alpha_s N_c}{2\pi} \log \left(\frac{|q_{i\perp}|^2}{\lambda} \right), \tag{4.109}$$

and where the regularised amplitude should only be evaluated for $|p_{i\perp}|^2 > \lambda$. A further refinement of this phase-space slicing method is given in ref. [158].

Moving to NLL requires the inclusion of the two-loop Regge trajectory, the one-loop impact factors and the one-loop Lipatov vertex within the HEJ framework. The IR divergences of these pieces should collectively cancel with the IR divergences that arise from the phase-space integration of the NLL real-emission building blocks derived in this chapter (eqs. (4.54), (4.77), (4.83) and (4.104)). This work is ongoing. Nevertheless, we can already utilise the NLL real-emission building blocks to include a well-defined subset of NLL corrections within the HEJ framework. By requiring that the on-shell partons emitted from the real-emission building blocks form their own jets, the soft and collinear regions are avoided.

Thus, we still can apply the (regularised) Lipatov ansatz to each t -channel propagator of the factorised amplitude. NLL corrections of this type were first introduced in ref. [154], where the use of eq. (4.54) allowed the resummation of LL corrections to the process $qg \rightarrow ggg$, where the final states are ordered in

rapidity.

Chapter 5

Fixed-order matching for HEJ

In the previous chapter, we studied the construction of HEJ amplitudes at LL and NLL accuracy. These amplitudes include high-energy effects to all orders in α_s which are omitted by the standard fixed-order approach to perturbative QCD. Conversely, the HEJ amplitudes make approximations to a given fixed order in α_s in order to facilitate the all-orders resummation. An ideal prediction for LHC phenomenology would combine fixed-order and logarithmic accuracy. This chapter discusses progress towards this goal.

Since its inception, the predictions of HEJ could be matched to LO results for low jet multiplicities [148]. This LO matching procedure was greatly improved with the release of HEJ 2 [147, 159]. We will briefly review the improved LO matching procedure in section 5.1. In section 5.2 we will then discuss how the results of HEJ may be further matched to NLO accuracy. We will provide details of our recent study of W production in association with at least two jets at the LHC, where for the first time we matched the results of HEJ to a NLO calculation.

5.1 LO matching

In section 2.1.2 we encountered the inclusive dijet cross section, eq. (2.28). In this section we will review the implementation of the dijet cross section in HEJ 2, which takes as its input LO events generated by any fixed-order Monte Carlo event generator. LL resummation is applied through the method discussed in

section 4.4. The resummed, and LO-matched cross section is

$$\begin{aligned}
\sigma_{2j}^{\text{resum,match}} = & \sum_{f_1, f_2} \sum_m \prod_{j=1}^m \left(\int_{p_{j\perp}^B=0}^{p_{j\perp}^B=\infty} \frac{d^2 \mathbf{p}_{j\perp}^B}{(2\pi)^3} \int \frac{dy_j^B}{2} \right) (2\pi)^4 \delta^{(2)} \left(\sum_{k=1}^m \mathbf{p}_{k\perp}^B \right) \\
& \times x_a^B f_{a,f_1}(x_a^B, \mu_F) x_b^B f_{b,f_2}(x_b^B, \mu_F) \frac{\left| \mathcal{M}_{f_1 f_2 \rightarrow f_1 g \dots g f_2}^{(0)}(\{p_j^B\}) \right|^2}{(\hat{s}^B)^2} \\
& \times \left(\left| \mathcal{M}_{f_1 f_2 \rightarrow f_1 g \dots g f_2}^{\text{HEJ LL}} \Big|_{\text{LO}} \right|^2 \right)^{-1} \times (2\pi)^{-4+3m} 2^m \\
& \times \sum_{n=2}^{\infty} \int_{p_{1\perp}=p_{\perp,\min}}^{p_{1\perp}=\infty} \frac{d^2 \mathbf{p}_{1\perp}}{(2\pi)^3} \int_{p_{n\perp}=p_{\perp,\min}}^{p_{n\perp}=\infty} \frac{d^2 \mathbf{p}_{n\perp}}{(2\pi)^3} \\
& \times \prod_{i=2}^{n-1} \int_{p_{i\perp}=\lambda}^{p_{i\perp}=\infty} \frac{d^2 \mathbf{p}_{i\perp}}{(2\pi)^3} (2\pi)^4 \delta^{(2)} \left(\sum_{k=1}^n \mathbf{p}_{k\perp} \right) \\
& \times \mathbf{T}_y \prod_{i=1}^n \left(\int \frac{dy_i}{2} \right) \mathcal{O}_{mj}^e(\{p_i\}) \left(\prod_{l=1}^{m-1} \delta^{(2)}(\mathbf{p}_{\mathcal{J}_l\perp}^B - \mathbf{j}_{l\perp}) \right) \left(\prod_{l=1}^m \delta(y_{\mathcal{J}_l}^B - y_{\mathcal{J}_l}) \right) \\
& \times x_a f_{a,f_1}(x_a, \mu_F) x_b f_{b,f_2}(x_b, \mu_F) \frac{\left| \mathcal{M}_{f_1 f_2 \rightarrow f_1 g \dots g f_2}^{\text{HEJ LL}}(\{p_i\}) \right|^2}{\hat{s}^2} \mathcal{O}_{2j}(\{p_i\}) \\
& \times \frac{(\hat{s}^B)^2}{x_a^B f_{a,f_1}(x_a^B, \mu_F) x_b^B f_{b,f_2}(x_b^B, \mu_F)}.
\end{aligned} \tag{5.1}$$

Let us digest this equation line-by-line. The first two lines are the phase space integration over the LO matrix element. The momenta of the LO matrix element are the Born-level jet momenta p_i^B . The momentum fractions x_i^B are constructed from these momenta, as in eq. (2.56). The third line divides the the LO matrix element by the HEJ matrix element truncated to LO, see eq. (4.107). Lines 4–7 are the phase space integration over all of the resummation phase space. The factor $\mathbf{T}_y$ imposes rapidity ordering. The factor $\mathcal{O}_{mj}^e(\{p_i\})$ imposes the requirement that the resummed amplitude is clustered into exactly m jets, according to a jet algorithm (see section 1.4). The δ -functionals on line 6 connect the kinematics of the resummed jets with the Born-level jets. The Born-level jet momenta are by definition equal to the Born-level parton momenta, $p_i^B = p_{\mathcal{J}_i}^B$. In order to re-impose momentum conservation, the resummed jet momenta are mapped onto Born-level kinematics through the algorithm devised in ref. [158],

$$\mathbf{j}_{l\perp} = \mathbf{p}_{\mathcal{J}_l\perp} + \mathbf{q}_{\perp} \frac{|\mathbf{p}_{\mathcal{J}_l\perp}|}{P_{\perp}}, \tag{5.2}$$

where $\mathbf{q}_{\perp}$ is the sum of the transverse momenta of partons outside jets after resummation, and $P_{\perp}$ is the scalar sum of the jet transverse momenta after

resummation. Hence, eq. (5.2) distributes the unobserved transverse momenta to the resummed jet momenta in proportion to the transverse momenta of each jet. This minimises the impact to the jet transverse momenta. For a given Born-level phase space point, all resummation momenta that yield $\mathbf{j}_{l\perp} = \mathbf{p}_{\mathcal{J}_l}^B$ need to be sampled. The momenta fractions x_i in line 7 are reconstructed from the resummed momenta. The final line removes the factors in line 2, which allows both the LO and resummed phase space integrals to be expressed in the form of standard phase space integrals such as eq. (2.28). In particular, this means the first two lines can be obtained from an independent LO Monte Carlo event generator.

With eq. (5.1), the output of HEJ is automatically matched to the LO input. Where LO results are not yet available, or where they are prohibitively computationally expensive, the HEJ fixed order generator [147] may be used to generate LO events for any jet multiplicity, which can then be used as the LO input to eq. (5.1).

In the next section, we discuss a method by which the results of HEJ may be further matched to NLO accuracy.

5.2 NLO matching

In order to discuss the matching of fixed-order and all-orders results beyond LO, we introduce some convenient notation in section 5.2.1. With this notation, we can then describe a method by which the results of HEJ can be matched to that of a full NLO calculation. In section 5.2.2 we then look at an example of the implementation of this method.

5.2.1 Comparison of series expansions

In this discussion we will consider only pure-QCD processes, but the discussion can be straightforwardly adapted to include electroweak corrections. For an n -jet process, we will write the leading order approximation to the cross section as

$$\sigma_{nj}^{\text{LO}} = f_{nj}^{(n)} \alpha_s^n, \quad (5.3)$$

where the superscript (n) indicates the power of the coupling associated to the coefficient f , and the subscript indicates the exclusive number of jets. We write the NLO approximation to the n -jet cross section as

$$\sigma_{nj}^{\text{NLO}} = f_{nj}^{(n)} \alpha_s^n + \left(f_{nj}^{(n+1)} + f_{(n+1)j}^{(n+1)} \right) \alpha_s^{n+1}. \quad (5.4)$$

This requires as theoretical input both the one-loop virtual corrections to the n -parton amplitude and the leading order amplitudes for $(n+1)$ -parton production. The coefficient $f_{(n+1)j}^{(n+1)}$, being simply the LO $(n+1)$ -jet cross section, is IR finite. The coefficient $f_{nj}^{(n+1)}$ is also IR finite due to the cancellation of IR singularities between the virtual corrections and unresolved real emissions.

We can further define the exclusive n -jet cross section, which consists only of n -jet configurations:

$$\sigma_{nj\text{-E}}^{\text{NLO}} = f_{nj}^{(n)} \alpha_s^n + f_{nj}^{(n+1)} \alpha_s^{(n+1)}. \quad (5.5)$$

This depends on our choice of jet algorithm at $\mathcal{O}(\alpha_s^{(n+1)})$.

As developments continue within the amplitudes community, we can successively build up an increasingly accurate prediction for the cross section, e.g.

$$\sigma_{2j}^{\text{N}^n\text{LO}} = \sum_{m=2}^{n+2} \left(\sum_{k=2}^m f_{kj}^{(m)} \right) \alpha_s^m, \quad (5.6)$$

In contrast to the fixed-order approach, HEJ can make predictions for an arbitrary number of jets. We write these exclusive predictions as

$$\sigma_{nj\text{-E}}^{\text{HEJ}} = \sum_{m=n}^{\infty} h_{nj}^{(m)} \alpha_s^m. \quad (5.7)$$

We can write the all-orders prediction to the (inclusive) 2-jet cross section as

$$\sigma_{2j}^{\text{HEJ}} = \sum_{n=2}^{\infty} \left(\sum_{m=2}^n h_{mj}^{(n)} \right) \alpha_s^n, \quad (5.8)$$

to be compared with eq. (5.6). We may truncate the exclusive jet rates, eq. (5.7), to any order in α_s . Truncating to the leading power of α_s gives what we might call HEJ@LO:

$$\sigma_{nj\text{-E}}^{\text{HEJ@LO}} = h_{nj}^{(n)} \alpha_s^n. \quad (5.9)$$

Similarly, we may truncate the exclusive n -jet rates to the first two powers of α_s . This we will refer to as exclusive HEJ@NLO:

$$\sigma_{nj-E}^{\text{HEJ@NLO}} = h_{nj}^{(n)} \alpha_s^n + h_{nj}^{(n+1)} \alpha_s^{n+1}. \quad (5.10)$$

In order to compare to fixed order predictions, we may supplement the above exclusive n -jet prediction with the HEJ@LO $(n+1)$ -jet predictions to give the inclusive HEJ@NLO result

$$\sigma_{nj}^{\text{HEJ@NLO}} = h_{nj}^{(n)} \alpha_s^n + \left(h_{nj}^{(n+1)} + h_{(n+1)j}^{(n+1)} \right) \alpha_s^{n+1}, \quad (5.11)$$

cf. eq. (5.4). As discussed in section 5.1, with the release of HEJ 2 [147], $h_{nj}^{(n)}$ is matched by default to $f_{nj}^{(n)}$, where this is available. We implicitly assume this LO-matching for the remainder of this section.

We now wish to introduce a method of modifying eq. (5.8) so that it agrees with eq. (5.6) up to α_s^3 , but without damaging the logarithmic accuracy of HEJ. Perhaps the simplest approach is to multiply σ_{2j}^{HEJ} by the ratio $\sigma_{2j}^{\text{NLO}}/\sigma_{2j}^{\text{HEJ@NLO}}$. Expanding this ratio in α_s we obtain

$$\begin{aligned} \frac{\sigma_{2j}^{\text{NLO}}}{\sigma_{2j}^{\text{HEJ@NLO}}} &= 1 + \alpha_s \frac{(f_{2j}^{(3)} - h_{2j}^{(3)})}{f_{2j}^{(2)}} \left[\prod_0^\infty (-1)^n \left(\alpha_s \frac{(f_{3j}^{(3)} + h_{2j}^{(3)})}{f_{2j}^{(2)}} \right)^n \right] \\ \Rightarrow \frac{\sigma_{2j}^{\text{NLO}}}{\sigma_{2j}^{\text{HEJ@NLO}}} &\xrightarrow[|t| \rightarrow \infty]{s} 1 \end{aligned} \quad (5.12)$$

In the final line we have taken the Regge limit, $s/|t| \rightarrow \infty$. This is the limit where the approximations of HEJ become exact (to a given logarithmic accuracy), in particular, $f_{2j}^{(3)} \rightarrow h_{2j}^{(3)}$. If we apply this reweighting factor to eq. (5.8), we obtain

$$\sigma_{2j}^{\text{HEJ}} \left(\frac{\sigma_{2j}^{\text{NLO}}}{\sigma_{2j}^{\text{HEJ@NLO}}} \right) = f_{2j}^{(2)} \alpha_s^2 + \left(f_{2j}^{(3)} + f_{3j}^{(3)} \right) \alpha_s^3 + \left(h_{2j}^{(4)} + h_{3j}^{(4)} + f_{4j}^{(4)} \right) \alpha_s^4 + \mathcal{O}(\alpha_s^5), \quad (5.13)$$

which is indeed NLO accurate in all of phase space.

While this section has proceeded via a discussion of cross-sections, the same argument holds for any particular bin in a distribution of any observable in which we might be interested. Such a bin-by-bin reweighting procedure will be performed in the next section.

5.2.2 NLO matching for W plus dijets at the LHC

In this thesis we have so far been concerned with QCD processes. The HEJ framework may be straightforwardly adapted to deal with electroweak corrections, as the production of a colour-singlet particle does not affect the resummation of higher-order QCD corrections [148]. To LL accuracy, the production of a W boson in association with at least two jets may be accounted for by the inclusion of the impact factor

$$\mathcal{J}_{f_a \rightarrow f_1 W g^*}^{\mu_t a_t}(p_a^{\lambda_a}; p_1^{\lambda_1}, p_\ell^\ominus, p_{\bar{\ell}}^\oplus, p_{t_1}) = ig_s T_{i_1 i_a}^{at} \frac{g_W^2}{2} \frac{\bar{u}^\ominus(p_\ell) \gamma_{\mu_W} v^\ominus(p_{\bar{\ell}})}{p_W^2 - M_W^2 + i\Gamma_W M_W} \quad (5.14)$$

$$\times \left(\frac{\bar{u}^{\lambda_1}(p_1) \gamma^{\mu_W} (\not{p}_1 + \not{p}_W) \gamma^{\mu_t} u^{\lambda_a}(p_a)}{(p_1 + p_W)^2} + \frac{\bar{u}^{\lambda_1}(p_1) \gamma^{\mu_t} (\not{p}_a - \not{p}_W) \gamma^{\mu_W} u^{\lambda_a}(p_a)}{(p_a - p_W)^2} \right),$$

where f_a and f_1 are (anti)quark flavours related by a diagonal CKM matrix element, which we approximate as unity.

In the study of ref. [1], a class of NLL corrections were also included by deriving the analogues of eq. (4.54) and eq. (4.83) but where a W boson is emitted from these building blocks. As discussed in section 4.4, these building blocks allow for LL resummation to be controlled for processes that first contribute to the cross section at NLL accuracy.

In this section we compare the results of HEJ, a full NLO calculation (produced using Sherpa [160] with the extension of OpenLoops [161]) and HEJ matched to NLO, to experimental data of W +dijet production at the LHC, at energies of 7 TeV [5] and 8 TeV [6].

The data was collected using slightly different cuts for the electron and muon channel, and then extrapolated to a “combined” selection contrasted with predictions generated with the following cuts listed in table 5.1. The rapidity selection criteria for jets, accepting jets of rapidities in the full detector, not just the central part, is particularly important for the study of high-energy logarithms.

The matching of logarithmic and NLO accuracy for the distributions of interest was obtained using the bin-by-bin matching procedure described in section 5.2.1. This required calculating all distributions of $W + 2$ -jets at both fixed-order NLO and for the NLO expansion of the leading and sub-leading logarithmic contributions to the cross section. For the inclusive two-jet rate,

Lepton $p_{\perp}$	$p_{\perp} > 25 \text{ GeV}$
Lepton rapidity	$ \eta < 2.5$
Missing transverse momentum	$E_{\perp}^{\text{miss}} > 25 \text{ GeV}$
Reconstructed transverse mass of boson	$m_{\perp} > 40 \text{ GeV}$
Jet $p_{\perp}$	$p_{\perp} > 30 \text{ GeV}$
Jet rapidity	$ y < 4.4$
Jet isolation	$\Delta R(l, \text{jet}) > 0.5$ (7 TeV [5]), 0.4 (8 TeV [6]) jet is removed

Table 5.1 *The selection cuts used for the comparisons to data.*

HEJ@NLO is obtained by expanding the virtual corrections from the reggeized propagator in the matrix elements for two-parton production,

$$e^{\alpha(|q_{\perp}|^2)\Delta y_{1,2}} \rightarrow 1 + \alpha(|q_{\perp}|^2)\Delta y_{1,2}, \quad (5.15)$$

and truncating the real emissions to a single additional gluon. The organisation of the cancellation of the poles in ε is performed with the same subtraction method as used in the all-order calculation. For the full HEJ calculation, LO $W + n$ -jet input samples were produced using Sherpa [160]. The aforementioned LL corrections, and class of NLL corrections were then resummed with HEJ 2. In practice, it was found that multiplicities beyond $W + 6$ -jets could be omitted.

Predictions can then be obtained at full NLO and resummed (leading and sub-leading) accuracy for each distribution by the following procedure:

1. obtain histograms for the process at (a) full NLO, (b) HEJ@NLO and (c) full HEJ (including the fixed-order 2-jet and 3-jet subprocesses not subject to resummation which enter (b), but not including these for ≥ 4 -jets).
2. scale each bin in the all-order resummed distribution 1(c) by the ratio of the same bin in the histograms of 1(a) and 1(b).

The final weight in each bin, w_{HEJ2NLO} , is then given by

$$w_{\text{HEJ2NLO}} = w_{\text{HEJ2}} \frac{w_{\text{NLO}}}{w_{\text{HEJ@NLO}}} + w_{\text{FO } W+\geq 4j}, \quad (5.16)$$

where w_{NLO} is the weight from full NLO (1(a) above), $w_{\text{HEJ@NLO}}$ is the full HEJ prediction truncated at α_s^3 (1(b) above) and w_{HEJ2} is the inclusive $W + 2$ -jet prediction from HEJ at all orders plus the 2-jet and 3-jet processes we must add at fixed order (1(c) above). The remaining term $w_{\text{FO } W+\geq 4j}$ represents the

fixed-order processes at 4-jets and above which do not enter the NLO matching.

In practice, the matching is performed for W^+ and W^- separately, but we find that the matching corrections are similar. This matching of kinematic distributions to full NLO accuracy corrects distributions in the sub-asymptotic kinematic regions where the leading and sub-leading logarithmic approximation is not expected to be a good approximation to the full perturbative result, e.g. in high-transverse momentum regions.

We show the matching corrections in figs. 5.1 and 5.2 for several example distributions. Fig. 5.1 shows the corrections for $W + 2$ -jet production at 7 TeV for distributions which appear in ref. [5] and fig. 5.2 shows examples for $W + 2$ -jet production at 8 TeV for distributions which appear in ref. [6]. We chose a central scale of $\mu_R = \mu_F = H_T/2$. This scale choice is guided purely by the convention for fixed-order predictions — it has not been tuned in order to obtain a good fit of data to the predictions of HEJ. We varied μ_R and μ_F independently by a factor of two around this central scale choice, keeping their ratio between 0.5 and 2. The scale variation bands are obtained from the envelope of these variations. The jet parameters and cuts are given in the figures. We see that for $p_{j,\perp}$ distributions (figs. 5.1a, 5.1b, 5.2a and 5.2b), the reweighting factors, given by $w_{\text{NLO}}/w_{\text{HEJ@NLO}}$, all have a similar shape; the ratios start around 1.5 for low $p_{j,\perp}$ values and then fall until reaching a stable value around 0.7-0.8 for $p_{j,\perp}$ values above about 300 GeV. A similar drop is also seen in the matching corrections as a function of $p_{W,\perp}$ in fig. 5.2c. On the other hand, the impact of matching is independent of the azimuthal separation of the jets, and hence the matching factor in fig. 5.1c is relatively flat. None of the figures show any significant difference between $W^+ + 2j$ production and $W^- + 2j$ production.

The improvement obtained in the description of data from applying these NLO matching corrections (as well as the NLL improvements to HEJ described in ref. [1]) is illustrated in figs. 5.3 and 5.4 (labelled “HEJ2 NLO”, shown in green). We also show the previously available predictions based on just the leading logarithmic description [162] combined with matching to high-multiplicity Born-level matrix elements (labelled “HEJ1”, shown in blue). The HEJ1 predictions failed systematically in the regions of large ($> 100\text{GeV}$) transverse momenta of the jets — this specifically breaks the multi-Regge kinematic conditions that the leading logarithmic approximation is based upon.

Figs. 5.3a-5.3b compare the predictions for the inclusive and exclusive jet rates

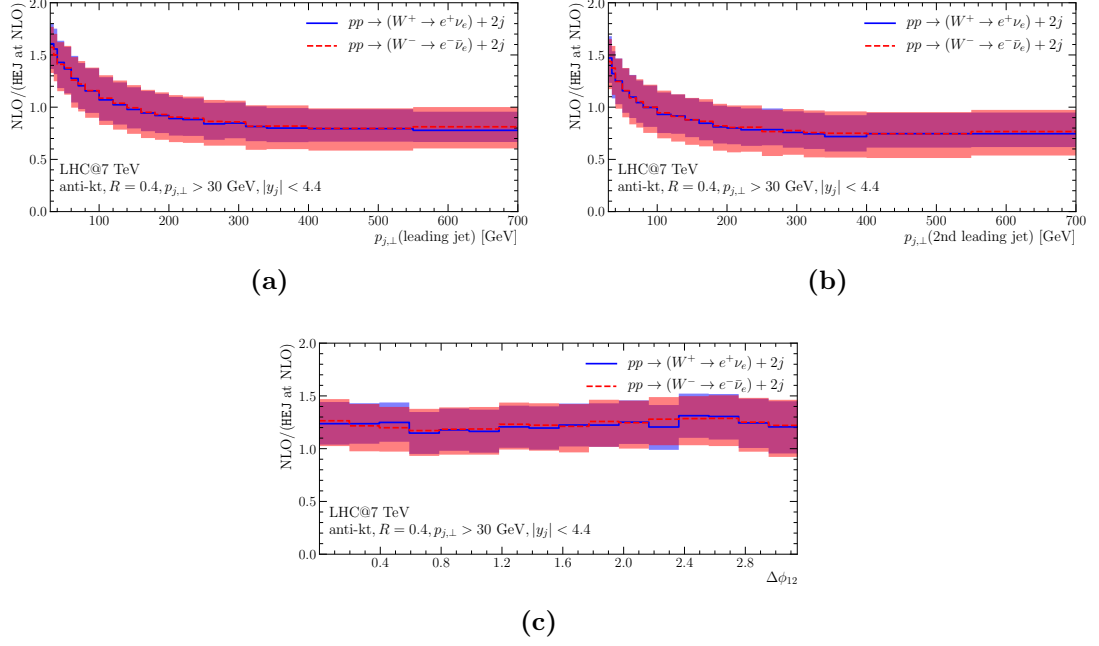


Figure 5.1 *The NLO matching corrections obtained for the following distributions presented in ref. [5] for inclusive $W^+ + 2j$ production (blue, solid) and $W^- + 2j$ production (red, dashed): (a) transverse momentum of the leading jet, (b) transverse momentum of the second jet and (c) the difference in azimuthal angle between the leading two jets.*

respectively to data. The inclusive 2-jet rate is matched to NLO accuracy, which as expected decreases the scale dependence compared to the predictions obtained with HEJ1 reported in [5]. The central predictions for both the inclusive and exclusive rates now agree more closely with the data. The scale variation of the 2-jet bin is significantly reduced due to the NLO matching in that bin. It is worth noting that while the scale variation in the inclusive 2-jet bin from pure NLO is smaller than that of the NLO-matched HEJ 2, the scale variation in the *exclusive* 2-jet bin of fig. 5.3b are similar for the NLO-calculation and the NLO-matched HEJ2.

Figs. 5.3c-5.3d demonstrates, as expected, that the impact of including the sub-leading processes in the resummation is significant for the description of scatterings at large transverse momenta. Indeed, all the problems identified with the description in HEJ1 are solved in HEJ2. The description of the spectrum of the leading and sub-leading jet is better still than that of pure W +dijets at NLO. The scale variation is slightly larger which is due to the corrections at α_s^4 and above included in the HEJ2 predictions.

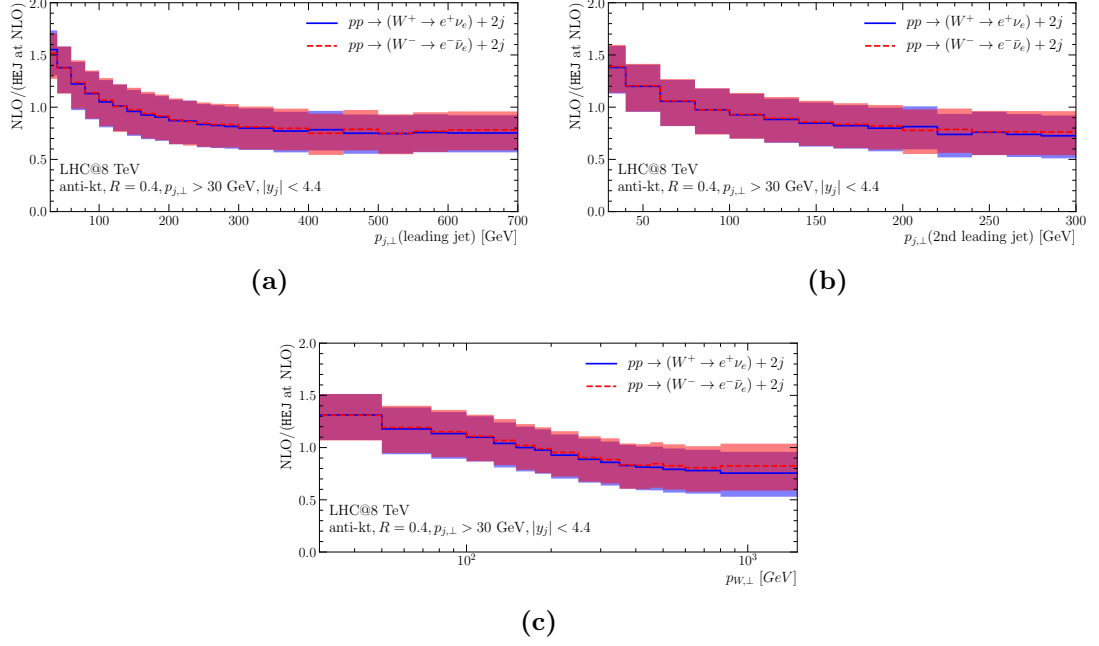


Figure 5.2 *The NLO matching corrections for the following distributions presented in ref. [6] for inclusive $W^+ + 2j$ production (blue, solid) and $W^- + 2j$ production (red, dashed): (a) transverse momentum of the leading jet, (b) transverse momentum of the second jet and (c) the transverse momentum of the W boson.*

Finally, fig. 5.3e shows the differential distribution for the azimuthal angle between the two hardest jets, ϕ_{12} . Again, the already decent description of HEJ1 is slightly improved in HEJ2, and the scale variation is reduced. Overall, the agreement with data for this observable is similar to that reached at pure NLO.

The comparison to the relevant results from the updated analysis for 8 TeV collisions and similar cuts is presented in fig. 5.4. Of course very little changes in the calculations, and the conclusions are similar to those arrived at for the 7 TeV analysis. The inclusive rates arrived at with HEJ2 show an extremely good agreement with data across all the matched multiplicities 2–6 inclusive. The transverse momentum is described extremely well for data available for the first and second hardest jet, and for the W -boson.

The comparison with data clearly demonstrates that the combined effect of matching to W +dijets at NLO and the inclusion of sub-leading channels in the all-order treatment has repaired the short-comings of the previous approach based solely on Born level matching and leading-logarithmic resummation. The scale variation is reduced in line with the NLO input, and the distributions associated with sub-leading regions of phase space, such as large transverse momenta, are

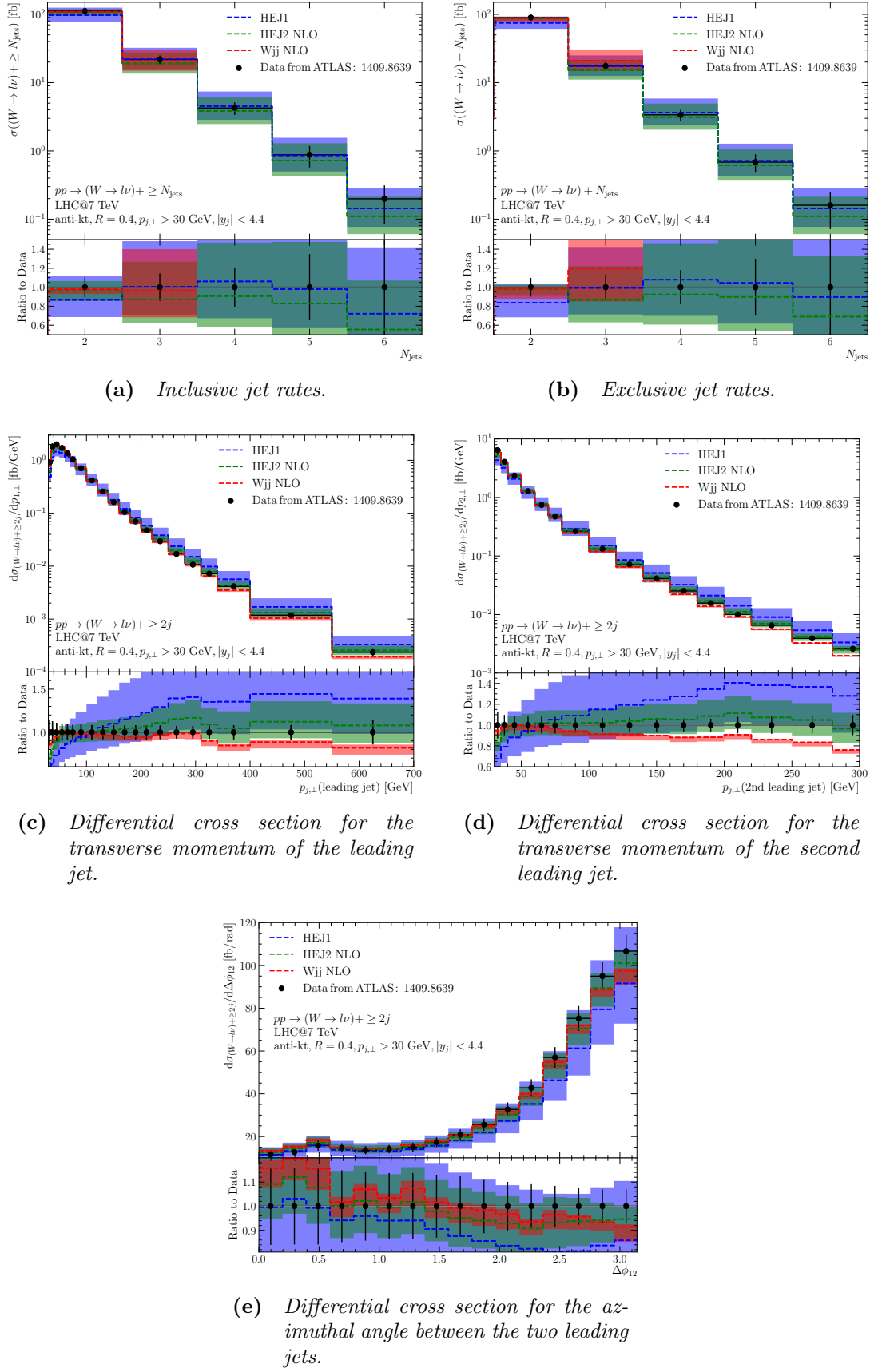
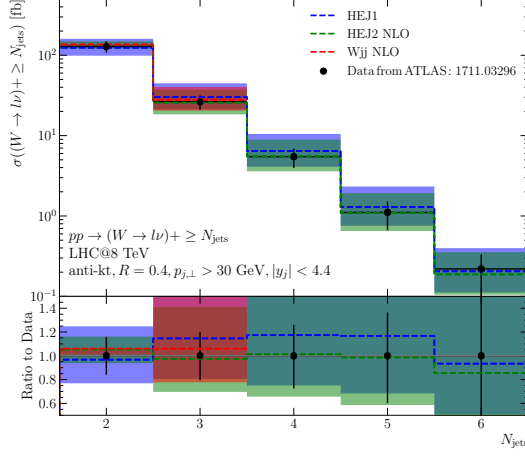
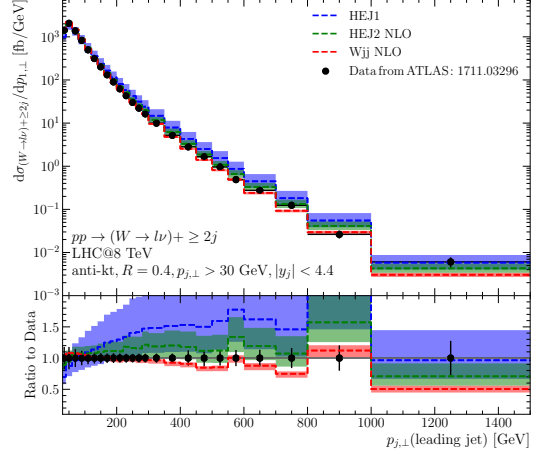


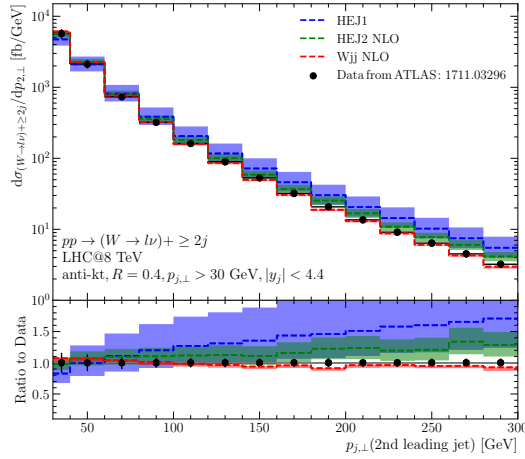
Figure 5.3 Predictions for $pp \rightarrow (W \rightarrow l\nu) + \geq 2j$ for LHC@7TeV, compared to ATLAS data from [5]. The original HEJ1 predictions are shown in blue, the pure NLO predictions are shown in red and the improved, NLO-matched HEJ2 predictions are shown in green.



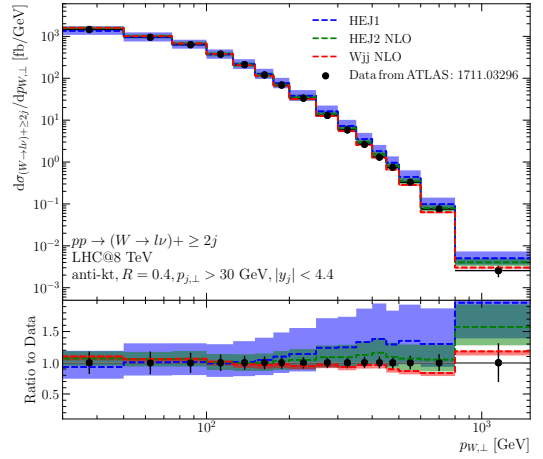
(a) Inclusive jet rates.



(b) Differential cross section for the transverse momentum of the leading jet.



(c) Differential cross section for the transverse momentum of the second leading jet.



(d) Differential cross section for the transverse momentum of the W boson.

Figure 5.4 Predictions for $pp \rightarrow (W \rightarrow l\nu) + \geq 2j$ for LHC@8TeV, compared to ATLAS data from [6]. The original HEJ1 predictions are shown in blue, the pure NLO predictions are shown in red and the improved, NLO-matched HEJ2 predictions are shown in green.

now described well.

In this section, we have focused on transverse observables rather than observables correlated with the high energy limit. One complicating factor for studying such distributions in the present W plus dijet case is due to the experimental rapidity cuts detailed in table 5.1. The smaller acceptance range of lepton rapidity compared to jet rapidity enhances the region of phase space where the W boson is far in rapidity from the extremal jets. This does not formally violate the assumptions under which the factorised W production building blocks were derived, but in practice this region of phase space is poorly described. This issue motivates the study of bin-by-bin NLO matching in the simpler scenario of inclusive dijet production.

A key priority for further improvement in HEJ is to move to full NLL accuracy. This necessitates alterations to the treatment of soft and collinear gluon radiation, and the NLL virtual corrections will capture more of the fixed-order NLO physics, thereby reducing the NLO matching corrections. The move to full NLL will therefore not only increase the logarithmic accuracy of the HEJ framework, but will also facilitate improvements to the matching of HEJ to fixed-order calculations.

Chapter 6

Conclusions

In this thesis we have studied several aspects of QCD scattering amplitudes in the high-energy limit, from the frontier of analytic investigations to precision phenomenology at the LHC.

In chapter 3 we obtained the one-loop correction to the two-gluon CEV in $\mathcal{N} = 4$ SYM, which is one of the missing building blocks required to bring the BFKL equation to NNLL accuracy. Our organisation of kinematic and colour terms should guide the organisation of this vertex in full QCD, and the techniques we used could also be straightforwardly adapted to obtain the remaining tree-level and one-loop building blocks of the NNLL BFKL kernel.

Determining the NNLL behaviour of QCD cross-sections will answer a decades-old question about the convergence of expanding in high-energy logarithms [79]. However, in chapter 3 we encountered several interesting aspects of the NMRK limit of the six-gluon amplitudes, which vanished or cancelled at the level of cross sections. We found that the two-gluon CEV that arose from the dispersive part of the amplitude had, in addition to the familiar tree-level colour structures, a new, totally symmetric colour structure. As argued in section 3.2.3, this colour structure vanishes at cross-section level, upon summing over external colour configurations. It will be an interesting future direction to determine whether this colour structure receives large logarithmic corrections at two loops. We also gave a full treatment of the absorptive part of the amplitude in appendix A.3. This part of the amplitude vanishes when considering the colour- and helicity-summed cross section at NNLL accuracy, but the structure of these terms and their relation to Regge-cut contributions warrants a deeper understanding.

Finally, we found that the structure of the one-loop amplitudes in the MRK limit provided a useful scheme to organise the amplitudes even in general kinematics. This motivates an investigation of whether this organisational principle is useful at higher loop order, higher parton multiplicities, and at higher powers in ϵ .

Separate to the formal investigation of the structure of amplitudes at high logarithmic accuracy are the questions of whether these high-energy effects can and should be accounted for at the LHC. In chapter 4 we saw that the answer to both questions was affirmative. In particular, for regions of phase space with large rapidity separations between jets, the inclusion of high energy logarithms to all orders in α_s is essential to restore stability to the standard fixed-order approach. The High Energy Jets (HEJ) framework is one well-developed tool to incorporate such high-energy effects. In chapter 4, we saw that there were several phenomenologically motivated reasons to bring HEJ to full NLL accuracy. In sections 4.2.2 and 4.3.2 we derived the two remaining real-emission building blocks required for a NLL description of cross sections. We saw that these building blocks were essential to give a good description of rapidity-based observables commonly measured at the LHC, which are inclusive over NMRK as well as MRK regions of phase space. The remaining task is to regulate the IR divergence of these real-emission building blocks by inclusion of corresponding virtual corrections. This ongoing work will not only increase the formal accuracy of HEJ to NLL, but will also extend the applicability of the new building blocks to a greater region of phase space.

The study of high-energy resummation should supplement our understanding of fixed-order QCD rather than supplant it. As such, it is important that the predictions of HEJ should be able to be readily matched to fixed-order calculations, where they are available. In section 5.1 we reviewed the method by which HEJ is by default matched to fixed-order predictions, and in section 5.2 we gave details of the first bin-by-bin matching of HEJ predictions to a full NLO calculation. The favourable comparison to LHC data is a promising indication that the combination of fixed-order calculations and all-orders high-energy effects is a viable route to stable and precise QCD predictions for the LHC era.

There is still much to learn about the all-orders structure of QCD scattering amplitudes, and there is much work to be done to utilise this knowledge in precise phenomenological studies.

Appendix A

Appendices

Here we collect several appendices containing supplementary material, or material which would interrupt the development of ideas in the main body of the thesis. In appendix A.1 we give details of the kinematic region which is central to the derivation of the two-gluon CEV in chapter 3. In appendix A.2 we list the NMHV amplitudes from which the opposite-helicity two-gluon CEV was obtained. In chapter 3 we focused on the dispersive part of the one-loop six gluon amplitude, which is the part of the amplitude that admits a Regge-pole factorisation, and the part which enters in the NLL BFKL kernel. In appendix A.3 we study the absorptive part of the one-loop amplitudes in NMRK. In appendix A.4 we take several kinematic limits of the CEV derived in chapter 3, showing that these limits are compatible with previously known results.

A.1 Central next-to-multi-Regge kinematics

We consider the production of four partons of momenta p_i , with $3 \leq i \leq 6$, with the new constraint that partons 4 and 5 are in the central region along the gluon ladder,

$$\begin{aligned} p_3^+ \gg p_4^+ \simeq p_5^+ \gg p_6^+, \quad p_3^- \ll p_4^- \simeq p_5^- \ll p_6^-, \\ |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|, \end{aligned} \tag{A.1}$$

and hence,

$$y_3 \gg y_4 \simeq y_5 \gg y_6. \tag{A.2}$$

The leading behaviour of the Mandelstam invariants in eq. (1.158) becomes

$$\begin{aligned}
s &\simeq p_3^+ p_6^- , \\
s_{2i} &\simeq -p_3^+ p_i^- , \\
s_{1i} &\simeq -p_i^+ p_6^- , \\
s_{jk} &\simeq p_j^+ p_k^- , \quad j \neq 4 \text{ and } k \neq 5 , \\
s_{45} &= p_4^+ p_5^- + p_4^- p_5^+ - p_{4\perp}^* p_{5\perp} - p_{5\perp}^* p_{4\perp} ,
\end{aligned} \tag{A.3}$$

with $3 \leq i, j, k \leq 6$, with $p_j^+ > p_k^+$ for $j < k$. Note that the mass-shell condition in eq. (1.155) and the Mandelstam invariants in eq. (A.3) imply that

$$\begin{aligned}
s &\simeq \frac{s_{34} s_{56}}{p_4^- p_5^+} \\
&= \frac{s_{34} s_{56} p_4^+ p_5^-}{|p_{4\perp}|^2 |p_{5\perp}|^2} .
\end{aligned} \tag{A.4}$$

All three-particle Mandelstam invariants either simplify to a two-particle Mandelstam invariant, or are equal to one of the following, up to an overall sign,

$$\begin{aligned}
s_{123} &\simeq (p_4^+ + p_5^+) p_6^- , \\
s_{234} &\simeq -|p_{3\perp} + p_{4\perp}|^2 - p_4^- p_5^+ , \\
s_{235} &\simeq -|p_{3\perp} + p_{5\perp}|^2 - p_4^+ p_5^- , \\
s_{345} &\simeq p_3^+ (p_4^- + p_5^-) .
\end{aligned} \tag{A.5}$$

In the Euclidean region, where the Mandelstam invariants are taken as all negative, the ordering in eq. (A.1) becomes

$$-s_{12} \gg -s_{34}, -s_{56}, -s_{123}, -s_{345} \gg -s_{23}, -s_{45}, -s_{61}, -s_{234}, -s_{235} . \tag{A.6}$$

Introducing a parameter $\lambda \ll 1$, the hierarchy in eq. (A.1) above is equivalent to the rescaling,

$$\{s_{34}, s_{56}, s_{123}, s_{345}\} = \mathcal{O}(\lambda) , \quad \{s_{23}, s_{45}, s_{61}, s_{234}, s_{235}\} = \mathcal{O}(\lambda^2) . \tag{A.7}$$

The spinor products in eq. (1.162) become

$$\begin{aligned}
\langle p_2 p_1 \rangle &\simeq -\sqrt{p_3^+ p_6^-}, \\
\langle p_2 p_k \rangle &= -i\sqrt{\frac{-p_2^+}{p_k^+}} p_{k\perp} \simeq -i\sqrt{\frac{p_3^+}{p_k^+}} p_{k\perp}, \\
\langle p_k p_1 \rangle &= i\sqrt{-p_1^- p_k^+} \simeq i\sqrt{p_k^+ p_6^-}, \\
\langle p_j p_k \rangle &= p_{j\perp} \sqrt{\frac{p_k^+}{p_j^+}} - p_{k\perp} \sqrt{\frac{p_j^+}{p_k^+}} \simeq -p_{k\perp} \sqrt{\frac{p_j^+}{p_k^+}}, \quad j \neq 4 \text{ and } k \neq 5 \\
\langle p_4 p_5 \rangle &= p_{4\perp} \sqrt{\frac{p_5^+}{p_4^+}} - p_{5\perp} \sqrt{\frac{p_4^+}{p_5^+}},
\end{aligned} \tag{A.8}$$

with $3 \leq j, k \leq 6$, with $p_j^+ > p_k^+$.

A.2 Six-gluon NMHV amplitudes in general kinematics

In this appendix we list the tree-level and one-loop six-gluon amplitudes in $\mathcal{N} = 4$ SYM which are used in the main text to obtain the two-gluon opposite-helicity CEV. For these amplitudes we use the shorthand notation for spinors listed in section 1.6.

A.2.1 Tree-level six-gluon NMHV amplitudes

Here we list a representation of the tree-level six-gluon NMHV colour-ordered amplitudes obtained through the BCFW on-shell recursion relations [52, 53],

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\ominus, p_4^\oplus, p_5^\oplus, p_6^\oplus) = \frac{\langle 1|2+3|4]^4}{D_2^*} + \frac{\langle 3|4+5|6]^4}{D_3}, \tag{A.9}$$

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\ominus, p_5^\oplus, p_6^\oplus) = \frac{\langle 4|1+2|3]^4}{D_1^*} + \frac{\langle 2|4]^4[5|6]^4}{D_2} + \frac{\langle 1|2]^4[3|5]^4}{D_3^*}, \tag{A.10}$$

$$M^{(0)}(p_1^\ominus, p_2^\oplus, p_3^\ominus, p_4^\oplus, p_5^\ominus, p_6^\oplus) = \frac{\langle 1|3]^4[4|6]^4}{D_1} + \frac{[2|4]^4\langle 1|5]^4}{D_2^*} + \frac{[2|6]^4\langle 3|5]^4}{D_3}. \tag{A.11}$$

The functions in the denominators are

$$D_1(1, 2, 3, 4, 5, 6) = -\langle 1\,2\rangle\langle 2\,3\rangle[4\,5][5\,6]s_{123}\langle 1|2+3|4\rangle\langle 3|4+5|6\rangle, \quad (\text{A.12})$$

and cyclic permutations thereof. We use the shorthand notation

$$D_i(\sigma) = D_i(\sigma_i, \sigma_{i+1}, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}, \sigma_{i+5}), \quad (\text{A.13})$$

where for the amplitudes in this section it is assumed – and left implicit – that the ordering σ is always taken to be that of the parent $M^{(0)}$, i.e. $\{1, 2, 3, 4, 5, 6\}$. $D_{i+1}(\sigma)$ is obtained from $D_i(\sigma)$ by replacing $\sigma_i \rightarrow \sigma_{i+1}$ in the latter, with all indices taken modulo 6. However, there are only three independent cyclic orderings due to the relation $D_{i+3}(\sigma) = D_i^*(\sigma)$.

We will see that the individual rational terms that occur in eqs. (A.9–A.11) will appear also in the one-loop amplitudes listed in the following subsection. The coefficient of the ϵ^{-2} pole of these one-loop amplitudes must be proportional to the tree-level amplitude, thus, using the one-loop amplitudes, we obtain an alternative set of representations for the tree-level amplitudes:

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\ominus, p_4^\oplus, p_5^\oplus, p_6^\oplus) = \frac{(s_{123})^4}{D_1^*} + \frac{\langle 2\,3\rangle^4[5\,6]^4}{D_2} + \frac{\langle 1\,2\rangle^4[4\,5]^4}{D_3^*}, \quad (\text{A.14})$$

$$M^{(0)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\ominus, p_5^\oplus, p_6^\oplus) = \frac{\langle 1\,2\rangle^4[5\,6]^4}{D_1} + \frac{\langle 1|2+4|3\rangle^4}{D_2^*} + \frac{\langle 4|1+2|6\rangle^4}{D_3}, \quad (\text{A.15})$$

$$M^{(0)}(p_1^\ominus, p_2^\oplus, p_3^\ominus, p_4^\oplus, p_5^\ominus, p_6^\oplus) = \frac{\langle 5|1+3|2\rangle^4}{D_1^*} + \frac{\langle 3|2+4|6\rangle^4}{D_2} + \frac{\langle 1|2+6|4\rangle^4}{D_3^*}. \quad (\text{A.16})$$

The representations eqs. (A.9–A.11) are complementary to the representations eqs. (A.14–A.16) in the sense that where the NMRK limit of the rational terms of the former give rise directly to R_{xy}^I (defined in eqs. (3.8) and (3.19)), the rational terms of the latter give rise to $R_{\bar{x}\bar{y}}^I$ (defined in eqs. (3.10) and (3.21)), and vice versa.

A.2.2 One-loop six-gluon NMHV amplitudes in $\mathcal{N} = 4$ SYM

We will need to study the NMRK limit of the colour-ordered amplitudes derived in ref. [57], namely:

$$\begin{aligned} \frac{(4\pi)^2}{\kappa_\Gamma} M_{\mathcal{N}=4}^{(1)}(p_1^\ominus, p_2^\ominus, p_3^\ominus, p_4^\oplus, p_5^\oplus, p_6^\oplus) &= \left(\frac{(s_{123})^4}{D_1^*} \right) W_1 \\ &+ \left(\frac{\langle 1|2+3|4]^4}{D_2^*} + \frac{\langle 23\rangle^4[56]^4}{D_2} \right) W_2 \\ &+ \left(\frac{\langle 3|4+5|6]^4}{D_3} + \frac{\langle 12\rangle^4[45]^4}{D_3^*} \right) W_3, \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} \frac{(4\pi)^2}{\kappa_\Gamma} M_{\mathcal{N}=4}^{(1)}(p_1^\ominus, p_2^\ominus, p_3^\oplus, p_4^\ominus, p_5^\oplus, p_6^\oplus) &= \left(\frac{\langle 4|1+2|3]^4}{D_1^*} + \frac{\langle 12\rangle^4[56]^4}{D_1} \right) W_1 \\ &+ \left(\frac{\langle 1|2+4|3]^4}{D_2^*} + \frac{\langle 24\rangle^4[56]^4}{D_2} \right) W_2 \\ &+ \left(\frac{\langle 4|1+2|6]^4}{D_3} + \frac{\langle 12\rangle^4[35]^4}{D_3^*} \right) W_3, \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} \frac{(4\pi)^2}{\kappa_\Gamma} M_{\mathcal{N}=4}^{(1)}(p_1^\ominus, p_2^\oplus, p_3^\ominus, p_4^\oplus, p_5^\ominus, p_6^\oplus) &= \left(\frac{\langle 13\rangle^4[46]^4}{D_1} + \frac{\langle 5|1+3|2]^4}{D_1^*} \right) W_1 \\ &+ \left(\frac{[24]^4\langle 15\rangle^4}{D_2^*} + \frac{\langle 3|2+4|6]^4}{D_2} \right) W_2 \\ &+ \left(\frac{[26]^4\langle 35\rangle^4}{D_3} + \frac{\langle 1|2+6|4]^4}{D_3^*} \right) W_3. \end{aligned} \quad (\text{A.19})$$

The transcendental functions are given by

$$\begin{aligned} W_i(\sigma) &= -\frac{1}{2\epsilon^2} \sum_{j=1}^6 \left(\frac{\mu^2}{-t_j^{[2]}} \right)^\epsilon - \log \left(\frac{-t_i^{[3]}}{-t_i^{[2]}} \right) \log \left(\frac{-t_i^{[3]}}{-t_{i+1}^{[2]}} \right) - \log \left(\frac{-t_i^{[3]}}{-t_{i+3}^{[2]}} \right) \log \left(\frac{-t_i^{[3]}}{-t_{i+4}^{[2]}} \right) \\ &+ \log \left(\frac{-t_i^{[3]}}{-t_{i+2}^{[2]}} \right) \log \left(\frac{-t_i^{[3]}}{-t_{i+5}^{[2]}} \right) + \frac{1}{2} \log \left(\frac{-t_i^{[2]}}{-t_{i+3}^{[2]}} \right) \log \left(\frac{-t_{i+1}^{[2]}}{-t_{i+4}^{[2]}} \right) \\ &+ \frac{1}{2} \log \left(\frac{-t_{i+5}^{[2]}}{-t_i^{[2]}} \right) \log \left(\frac{-t_{i+1}^{[2]}}{-t_{i+2}^{[2]}} \right) + \frac{1}{2} \log \left(\frac{-t_{i+2}^{[2]}}{-t_{i+3}^{[2]}} \right) \log \left(\frac{-t_{i+4}^{[2]}}{-t_{i+5}^{[2]}} \right) + \frac{\pi^2}{3}, \end{aligned} \quad (\text{A.20})$$

where we use the same notation for the arguments and cyclic permutations as for the function D_i , discussed below eq. (A.12). While in eqs. (A.17–A.19) the

ordering argument is left implicit, it is useful to display the ordering explicitly when the transcendental functions of different colour orderings appear in the same expression, e.g. in eq. (3.111). We note that the function W_i has cyclic symmetry in the form $W_{i+3} = W_i$ so there are only three independent cyclic permutations. In eq. (A.20) we use the compact notation for Mandelstam invariants introduced in eq. (3.72), and we use the prescription of eq. (2.20). We emphasise that the function W_i does not depend on the individual helicities of the particles.

A.3 Absorptive part of the one-loop six-gluon amplitude in NMRK

In section 3.2 we found that in the central NMRK, the dispersive amplitude received a contribution from the $(-, -)$ component alone. The same is not true for the absorptive part of the amplitude, which is the object of study of this appendix. To this end we list the NMRK limit of the imaginary parts of the transcendental functions eq. (3.71) and eq. (A.20), for the 12 colour orderings that appear in eqs. (3.65)–(3.67). We begin by giving the results for the σ_A orderings:

$$\begin{aligned}
\text{Im}\left[V_6(\sigma_A)\right] &= 2\text{Im}\left[W_2(\sigma_A)\right] \xrightarrow{\text{NMRK}} -\frac{4\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- s_{45} p_5^+ t_3}{(\mu^2)^4}\right), \\
\text{Im}\left[V_6(\overset{\leftrightarrow}{\sigma}_A)\right] &= 2\text{Im}\left[W_2(\overset{\leftrightarrow}{\sigma}_A)\right] \xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{p_4^- s_{45} p_5^+ t_3}{t_1 (\mu^2)^2}\right), \\
\text{Im}\left[V_6(\underset{\leftrightarrow}{\sigma}_A)\right] &= 2\text{Im}\left[W_2(\underset{\leftrightarrow}{\sigma}_A)\right] \xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- s_{45} p_5^+}{t_3 (\mu^2)^2}\right), \\
\text{Im}\left[V_6(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_A)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- p_5^+ t_3}{s_{45} (\mu^2)^2}\right) + 2\pi \log\left(\frac{u_A}{1 - u_A}\right), \\
2\text{Im}\left[W_2(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_A)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- p_5^+ t_3}{s_{45} (\mu^2)^2}\right) - 2\pi \log\left(\frac{v_A w_A}{u_A}\right).
\end{aligned} \tag{A.21}$$

We provide the results for W_1 and W_3 as differences from those for W_2 . Interestingly, $\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_A$ is the only ordering which leads to differing imaginary parts

for different cyclic permutations of W_i :

$$\begin{aligned}
\text{Im} \left[W_1(\sigma_A) - W_2(\sigma_A) \right] &= 0, & \text{Im} \left[W_3(\sigma_A) - W_2(\sigma_A) \right] &= 0, \\
\text{Im} \left[W_1(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 0, & \text{Im} \left[W_3(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 0, \\
\text{Im} \left[W_1(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 0, & \text{Im} \left[W_3(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 0, \\
\text{Im} \left[W_1(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 2\pi \log(v_A), & \text{Im} \left[W_3(\overset{\leftrightarrow}{\sigma}_A) - W_2(\overset{\leftrightarrow}{\sigma}_A) \right] &= 2\pi \log(w_A).
\end{aligned}
\tag{A.22}$$

We further note that only for the ordering $\overset{\leftrightarrow}{\sigma}_A$ does the analytic continuation of the dilogarithms in eq. (3.71) to our physical region produce an imaginary part. For this colour ordering, the cross ratio

$$\frac{s_{63}s_{45}}{s_{632}s_{245}},$$

which is related to u_A by interchange of $p_1 \leftrightarrow p_6$ and $p_2 \leftrightarrow p_3$, has two positive invariants in the numerator and two negative invariants in the denominator. The analytic continuation of the dilogarithm that depends on this cross ratio leads to factor of $i\pi \log(1 - u_A)$ which is related to the factorisation breaking term discovered in ref. [122]. The interpretation of this quantity is different in our present study; ref. [122] considers the single planar colour ordering of the amplitude, analytically continued to the physical region where $s_{12}, s_{45} > 0$, $s_{23}, s_{34}, s_{56}, s_{61}, s_{234}, s_{345} < 0$, while we consider a certain colour-ordered amplitude in the physical region where $s_{12}, s_{34}, s_{45}, s_{56}, s_{345} > 0$, $s_{23}, s_{61}, s_{234} < 0$. However, the consequence is the same, namely that this term is not compatible with Regge-pole factorisation of the amplitude, which only applies to its dispersive part. Instead it contributes to a Regge cut.

Expressions for the $\sigma_{A'}$ orderings are related by an interchange of $p_4 \leftrightarrow p_5$.

For the σ_B orderings we obtain

$$\begin{aligned}
\text{Im}\left[V_6(\sigma_B)\right] &= 2\text{Im}\left[W_1(\sigma_B)\right] \xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_5^+ p_5^- t_3}{p_4^+ p_4^- (\mu^2)^2}\right), \\
\text{Im}\left[V_6(\overset{\leftrightarrow}{\sigma}_B)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^+ p_5^- t_3}{p_4^- p_5^+ (\mu^2)^2}\right) + 2\pi \log\left(\frac{v_B}{1-v_B}\right), \\
2\text{Im}\left[W_1(\overset{\leftrightarrow}{\sigma}_B)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^+ p_5^- t_3}{p_4^- p_5^+ (\mu^2)^2}\right) + 2\pi \log\left(\frac{u_B v_B}{w_B}\right), \\
\text{Im}\left[V_6(\underset{\leftrightarrow}{\sigma}_B)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- p_5^+ t_3}{p_4^+ p_5^- (\mu^2)^2}\right) + 2\pi \log\left(\frac{u_B}{1-u_B}\right), \\
2\text{Im}\left[W_1(\underset{\leftrightarrow}{\sigma}_B)\right] &\xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^- p_5^+ t_3}{p_4^+ p_5^- (\mu^2)^2}\right) + 2\pi \log\left(\frac{u_B v_B}{w_B}\right), \\
\text{Im}\left[V_6(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B)\right] &= 2\text{Im}\left[W_1(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B)\right] \xrightarrow{\text{NMRK}} -\frac{2\pi}{\epsilon} + \pi \log\left(\frac{t_1 p_4^+ p_4^- t_3}{p_5^+ p_5^- (\mu^2)^2}\right).
\end{aligned} \tag{A.23}$$

Again the orderings that lead to imaginary contributions from the dilogarithms (namely $\overset{\leftrightarrow}{\sigma}_B$, $\underset{\leftrightarrow}{\sigma}_B$) are the same orderings for which the imaginary parts of the distinct cyclic permutations of W_i differ:

$$\begin{aligned}
\text{Im}\left[W_2(\sigma_B) - W_1(\sigma_B)\right] &= 0, & \text{Im}\left[W_3(\sigma_B) - W_1(\sigma_B)\right] &= 0, \\
\text{Im}\left[W_2(\overset{\leftrightarrow}{\sigma}_B) - W_1(\overset{\leftrightarrow}{\sigma}_B)\right] &= 2\pi \log\left(\frac{w_B}{u_B}\right), & \text{Im}\left[W_3(\overset{\leftrightarrow}{\sigma}_B) - W_1(\overset{\leftrightarrow}{\sigma}_B)\right] &= -2\pi \log(u_B), \\
\text{Im}\left[W_2(\underset{\leftrightarrow}{\sigma}_B) - W_1(\underset{\leftrightarrow}{\sigma}_B)\right] &= -2\pi \log(v_B), & \text{Im}\left[W_3(\underset{\leftrightarrow}{\sigma}_B) - W_1(\underset{\leftrightarrow}{\sigma}_B)\right] &= 2\pi \log\left(\frac{w_B}{v_B}\right), \\
\text{Im}\left[W_2(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B) - W_1(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B)\right] &= 0, & \text{Im}\left[W_3(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B) - W_1(\overset{\leftrightarrow}{\underset{\leftrightarrow}{\sigma}}_B)\right] &= 0.
\end{aligned} \tag{A.24}$$

Absorptive part of the same-helicity amplitude

From the above lists of imaginary parts we can now assemble the absorptive parts of the colour-dressed amplitude for the case where $\nu_4 = \nu_5 = \oplus$:

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(-,-)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \\
& \times \frac{\pi}{2} \left\{ \text{tr} (F^{c_1} F^{a_4} F^{a_5} F^{c_3}) A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \left(\frac{-5}{\epsilon} + \log \left(\frac{t_1 s_{45} (p_4^- p_5^+)^2 t_3}{(\mu^2)^5} \frac{u_A}{1 - u_A} \right) \right) \right. \\
& \quad + \text{tr} (F^{c_1} F^{a_5} F^{a_4} F^{c_3}) A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) \left(\frac{-5}{\epsilon} + \log \left(\frac{t_1 s_{45} (p_4^+ p_5^-)^2 t_3}{(\mu^2)^5} \frac{u_{A'}}{1 - u_{A'}} \right) \right) \\
& \quad \left. + \text{tr} (F^{c_1} F^{a_5} F^{c_3} F^{a_4}) B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \left(\frac{-4}{\epsilon} + \log \left(\frac{t_1^2 t_3^2}{(\mu^2)^4} \frac{v_B}{1 - v_B} \frac{u_B}{1 - u_B} \right) \right) \right\} \\
& \times \frac{1}{t_3} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}),
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(+,-)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr} (\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} [F^{a_6}, F^{a_1}]) A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_3 s_{45}}{t_1 \mu^2} \frac{1 - u_A}{u_A} \right) \right) \right. \\
& \quad + \text{tr} (\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} [F^{a_6}, F^{a_1}]) A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_3 s_{45}}{t_1 \mu^2} \frac{1 - u_{A'}}{u_{A'}} \right) \right) \\
& \quad \left. + \text{tr} (\{F^{a_2}, F^{a_3}\} F^{a_4} [F^{a_6}, F^{a_1}] F^{a_5}) B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \log \left(\left(\frac{p_5^-}{p_4^+} \right)^2 \frac{1 - u_B}{u_B} \frac{v_B}{1 - v_B} \right) \right\},
\end{aligned} \tag{A.26}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(-,+)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr} ([F^{a_2}, F^{a_3}] F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 s_{45}}{t_3 \mu^2} \frac{1 - u_A}{u_A} \right) \right) \right. \\
& \quad + \text{tr} ([F^{a_2}, F^{a_3}] F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 s_{45}}{t_3 \mu^2} \frac{1 - u_{A'}}{u_{A'}} \right) \right) \\
& \quad \left. + \text{tr} ([F^{a_2}, F^{a_3}] F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \log \left(\left(\frac{p_5^+}{p_4^-} \right)^2 \frac{u_B}{1 - u_B} \frac{1 - v_B}{v_B} \right) \right\},
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(+,+)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 t_3}{s_{45} \mu^2} \frac{u_A}{1 - u_A} \right) \right) \right. \\
& \quad + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) A^{gg(0)}(q_1, p_5^\oplus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 t_3}{s_{45} \mu^2} \frac{u_{A'}}{1 - u_{A'}} \right) \right) \\
& \quad \left. + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) B^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) \log \left(\frac{1 - u_B}{u_B} \frac{1 - v_B}{v_B} \right) \right\}.
\end{aligned} \tag{A.28}$$

In section 3.2.3 we show that the absorptive parts listed in eqs. (A.25–A.28) do not contribute to the NLO $gg \rightarrow gggg$ squared matrix element. These terms will only contribute to the squared matrix element at higher orders in the coupling.

Absorptive part of the opposite-helicity amplitude

Here we similarly list the absorptive parts of the colour-dressed amplitude for the case where $\nu_4 = -\nu_5 = \oplus$:

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(-,-)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^\ominus, p_3^\oplus) \frac{1}{t_1} \\
& \times \frac{\pi}{2} \left\{ \text{tr}(F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \left[A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \left(\frac{-5}{\epsilon} + \log \left(\frac{t_1 s_{45} (p_4^+ p_5^-)^2 t_3}{(\mu^2)^5} \frac{u_A}{v_A w_A} \right) \right) \right. \right. \\
& \quad \left. \left. + S_{uv}^A \log(v_A) + S_{wu}^A \log(w_A) \right] \right. \\
& \quad + \text{tr}(F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \left[A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \left(\frac{-5}{\epsilon} + \log \left(\frac{t_1 s_{45} (p_4^+ p_5^-)^2 t_3}{(\mu^2)^5} \frac{u_{A'}}{v_{A'} w_{A'}} \right) \right) \right. \\
& \quad \left. \left. + S_{uv}^{A'} \log(v_{A'}) + S_{wu}^{A'} \log(w_{A'}) \right] \right. \\
& \quad \left. + \text{tr}(F^{c_1} F^{a_5} F^{c_3} F^{a_4}) \left[B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \left(\frac{-4}{\epsilon} + \log \left(\frac{t_1^2 t_3^2}{(\mu^2)^4} \right) \right) \right. \right. \\
& \quad \left. \left. - S_{uv}^B \log \left(\frac{w_B}{u_B v_B} \right) \right] \right\} \times \frac{1}{t_3} F^{a_6 a_1 c_3} C^{g(0)}(p_6^\oplus, p_1^\ominus),
\end{aligned} \tag{A.29}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(+,-)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} [F^{a_6}, F^{a_1}]) \right. \\
& \quad \times \left[A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_3 s_{45}}{t_1 \mu^2} \frac{v_A w_A}{u_A} \right) \right) - S_{uv}^A \log(v_A) - S_{wu}^A \log(w_A) \right] \\
& + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} [F^{a_6}, F^{a_1}]) \\
& \quad \times \left[A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_3 s_{45}}{t_1 \mu^2} \frac{v_{A'} w_{A'}}{u_{A'}} \right) \right) - S_{uv}^{A'} \log(v_{A'}) - S_{wu}^{A'} \log(w_{A'}) \right] \\
& + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} [F^{a_6}, F^{a_1}] F^{a_5}) \\
& \quad \times \left[2B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \log \left(\frac{p_5^-}{p_4^+} \right) + S_{vw}^B \log \left(\frac{u_B}{v_B w_B} \right) - S_{wu}^B \log \left(\frac{v_B}{w_B u_B} \right) \right] \Bigg\}, \tag{A.30}
\end{aligned}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(-,+)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr}([F^{a_2}, F^{a_3}] F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) \right. \\
& \quad \times \left[A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 s_{45}}{t_3 \mu^2} \frac{v_A w_A}{u_A} \right) \right) - S_{uv}^A \log(v_A) - S_{wu}^A \log(w_A) \right] \\
& + \text{tr}([F^{a_2}, F^{a_3}] F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) \\
& \quad \times \left[A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 s_{45}}{t_3 \mu^2} \frac{v_{A'} w_{A'}}{u_{A'}} \right) \right) - S_{uv}^{A'} \log(v_{A'}) - S_{wu}^{A'} \log(w_{A'}) \right] \\
& + \text{tr}([F^{a_2}, F^{a_3}] F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) \\
& \quad \times \left[2B^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \log \left(\frac{p_5^-}{p_4^+} \right) - S_{vw}^B \log \left(\frac{u_B}{v_B w_B} \right) + S_{wu}^B \log \left(\frac{v_B}{w_B u_B} \right) \right] \Bigg\}, \tag{A.31}
\end{aligned}$$

$$\begin{aligned}
& \text{Absorp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)(+,+)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}} \frac{\pi}{2} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
& \times \left\{ \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} F^{a_5} \{F^{a_6}, F^{a_1}\}) \right. \\
& \quad \times \left[A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 t_3}{s_{45} \mu^2} \frac{u_A}{v_A w_A} \right) \right) + S_{uv}^A \log(v_A) + S_{wu}^A \log(w_A) \right] \\
& \quad + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_5} F^{a_4} \{F^{a_6}, F^{a_1}\}) \\
& \quad \times \left[A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3) \left(\frac{-1}{\epsilon} + \log \left(\frac{t_1 t_3}{s_{45} \mu^2} \frac{u_{A'}}{v_{A'} w_{A'}} \right) \right) + S_{uv}^{A'} \log(v_{A'}) + S_{wu}^{A'} \log(w_{A'}) \right] \\
& \quad \left. + \text{tr}(\{F^{a_2}, F^{a_3}\} F^{a_4} \{F^{a_6}, F^{a_1}\} F^{a_5}) \times \left[S_{uv}^B \log \left(\frac{w_B}{u_B v_B} \right) \right] \right\}.
\end{aligned} \tag{A.32}$$

In section 3.2.3 we show that at the level of the NLO $gg \rightarrow gggg$ squared matrix element, eq. (A.29) is the only absorptive component of the amplitude which can contribute for a specific helicity configuration. This contribution vanishes in the sum over helicities ν_4 and ν_5 . All components eq. (A.25)–(A.32) will contribute at higher orders in the coupling.

A.4 Kinematic limits of the two-gluon central-emission vertex

Starting from the central NMRK region, it is possible to take further kinematic limits to regions which are contained within NMRK. In these more restricted kinematic regions further simplifications occur. A key example are the regions where we additionally require $p_4^+ \gg p_5^+$ or $p_5^+ \gg p_4^+$ which we refer to as the MRK and MRK' limits respectively. These limits of the two-gluon CEV are studied in appendix A.4.1. From the central NMRK region we may alternatively take the limit where one of the central gluons becomes soft, which is the subject of appendix A.4.2, or the limit where the two central gluons become collinear to each other, which is the subject of appendix A.4.3. These further limits act as checks on the results presented in chapter 3, and also provide intuition for the kinematic dependence of the cross ratios and the two-gluon CEV. One may also investigate the limits where both of the central gluons become soft, but this is not pursued in this appendix.

A.4.1 MRK limit

Recall that the MRK (MRK') limit is obtained by taking $X \rightarrow \infty$ ($X \rightarrow 0$). In the MRK or MRK' limit, the tree-level two-gluon CEV factorises into a product of Lipatov vertices times a t -channel pole. This factorization property applies for any helicity configuration, for example, for opposite helicities

$$\begin{aligned} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} &= V^{g(0)}(q_1, p_4^\oplus, q_2) \frac{1}{t_2} V^{g(0)}(q_2, p_5^\ominus, q_3) \\ &= \frac{z}{\bar{z}} (\bar{w} - 1)(\bar{z} - 1), \end{aligned} \quad (\text{A.33})$$

$$\begin{aligned} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} &= V^{g(0)}(q_1, p_5^\ominus, q'_2) \frac{1}{t'_2} V^{g(0)}(q'_2, p_4^\oplus, q_3) \\ &= \frac{z}{\bar{z}} \frac{(\bar{w}(\bar{z} - 1) - \bar{z})}{(w(z - 1) - z)} (w - 1)(z - 1), \end{aligned} \quad (\text{A.34})$$

with Lipatov vertex $V^{g(0)}$ in eq. (2.32). Considering the opposite helicity configuration at one loop, we saw in section 3.2.2 that different transcendental terms multiply rational terms which are not simply the tree amplitudes. This prompts an analysis of these individual rational terms in the MRK and MRK' limits. As a consistency check on the MRK and MRK' limits of these individual rational terms, the relevant sums of these rational terms given in eqs. (3.7), (3.9), (3.12) and (3.13) should reproduce the limits eq. (A.33) and eq. (A.34).

For the σ_A ordering in the MRK, from eqs. (3.8) and (3.10) we obtain

$$\begin{aligned} R_{uv}^A &\xrightarrow{\text{MRK}} \frac{w - 1}{w - z} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \\ R_{v\bar{w}}^A &\xrightarrow{\text{MRK}} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \\ R_{wu}^A &\xrightarrow{\text{MRK}} -\frac{z - 1}{w - z} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \end{aligned} \quad (\text{A.35})$$

while $R_{\bar{u}\bar{v}}^A$, R_{vw}^A and $R_{\bar{w}\bar{u}}^A$ vanish in this limit. All six of these rational terms vanish in the MRK' limit. The limits listed in eq. (A.35) can be used to show eq. (A.33), starting from either of the representations eq. (3.7) or eq. (3.9). On the other hand, for the $\sigma_{A'}$ permutation, all rational terms vanish in MRK. In MRK', $R_{uv}^{A'}$,

$R_{\bar{v}\bar{w}}^{A'}$ and $R_{wu}^{A'}$ vanish, while the remaining terms tend to

$$\begin{aligned}
R_{\bar{v}\bar{w}}^{A'} &\xrightarrow{\text{MRK}'} -\frac{(\bar{w}-1)\bar{z}^2}{(\bar{w}-\bar{z})(\bar{w}(\bar{z}-1)-\bar{z})} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} \\
R_{vw}^{A'} &\xrightarrow{\text{MRK}'} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} \\
R_{\bar{w}\bar{u}}^{A'} &\xrightarrow{\text{MRK}'} \frac{\bar{w}^2(\bar{z}-1)}{(\bar{w}-\bar{z})(\bar{w}(\bar{z}-1)-\bar{z})} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} .
\end{aligned} \tag{A.36}$$

Using the representation eq. (3.12) or eq. (3.13) we see that these limits are consistent with the limit of the tree-level amplitude eq. (A.34). Finally for the B configurations, we get terms that survive in the MRK limit,

$$\begin{aligned}
R_{uv}^B &\xrightarrow{\text{MRK}} -\frac{(w-1)(z-1)}{w(z-1)-z} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \\
R_{\bar{u}\bar{v}}^B &\xrightarrow{\text{MRK}} 0 \\
R_{vw}^B &\xrightarrow{\text{MRK}} 0 \\
R_{\bar{v}\bar{w}}^B &\xrightarrow{\text{MRK}} -A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \\
R_{wu}^B &\xrightarrow{\text{MRK}} \frac{1}{w(z-1)-z} A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3)|_{\text{MRK}} \\
R_{\bar{w}\bar{u}}^B &\xrightarrow{\text{MRK}} 0,
\end{aligned} \tag{A.37}$$

and also terms that survive in the MRK' limit,

$$\begin{aligned}
R_{uv}^B &\xrightarrow{\text{MRK}'} -\frac{(\bar{w}-1)(\bar{z}-1)}{\bar{w}(\bar{z}-1)-\bar{z}} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} \\
R_{\bar{u}\bar{v}}^B &\xrightarrow{\text{MRK}'} 0 \\
R_{vw}^B &\xrightarrow{\text{MRK}'} \frac{1}{\bar{w}(\bar{z}-1)-\bar{z}} A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} \\
R_{\bar{v}\bar{w}}^B &\xrightarrow{\text{MRK}'} 0 \\
R_{wu}^B &\xrightarrow{\text{MRK}'} 0 \\
R_{\bar{w}\bar{u}}^B &\xrightarrow{\text{MRK}'} -A^{gg(0)}(q_1, p_5^\ominus, p_4^\oplus, q_3)|_{\text{MRK}'} .
\end{aligned} \tag{A.38}$$

The MRK and MRK' limits of the one-loop two gluon CEV in $\mathcal{N} = 4$ SYM are discussed in section 3.2. The notation introduced in that section makes it trivial to obtain the MRK and MRK' limits of the transcendental terms, with the exception of terms that can be compactly described in terms of the cross ratios of eqs. (3.4), and (3.16). To facilitate the investigation of these latter transcendental

functions, we list the cross ratios in the MRK and MRK' limits:

$$\begin{aligned}
u_A &\xrightarrow{\text{MRK}} 1, & v_A &\xrightarrow{\text{MRK}} \frac{(-t_1)p_5^+}{(-t_2)p_4^+}, & w_A &\xrightarrow{\text{MRK}} \frac{(-t_3)p_4^-}{(-t_2)p_5^-}, \\
u_{A'} &\xrightarrow{\text{MRK}'} 1, & v_{A'} &\xrightarrow{\text{MRK}'} \frac{(-t_1)p_4^+}{(-t'_2)p_5^+}, & w_{A'} &\xrightarrow{\text{MRK}'} \frac{(-t_1)p_5^-}{(-t'_2)p_4^-}, \\
u_B &\xrightarrow{\text{MRK}} 1, & v_B &\xrightarrow{\text{MRK}} \frac{p_4^+ p_5^-}{(-t_2)}, & w_B &\xrightarrow{\text{MRK}} \frac{(-t_1)(-t_3)}{(-t_2)p_4^+ p_5^-}, \\
u_B &\xrightarrow{\text{MRK}'} \frac{p_4^+ p_5^-}{(-t'_2)}, & v_B &\xrightarrow{\text{MRK}'} 1, & w_B &\xrightarrow{\text{MRK}'} \frac{(-t_1)(-t_3)}{(-t'_2)p_4^- p_5^+}.
\end{aligned} \tag{A.39}$$

A.4.2 Soft limits

In the limit that either of the gluons of the tree two-gluon CEV become soft, $p_4 \rightarrow 0$ or $p_5 \rightarrow 0$, the amplitude given in eq. (2.97) is singular. It factorises into an eikonal factor, which behaves like $1/p_4$ or $1/p_5$, and a five-gluon amplitude in MRK. For example, taking the limit where all components of p_4 tend to zero at the same rate, eq. (2.97) factorises as

$$\begin{aligned}
&M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}) \Big|_{\text{NMRK}} \\
&\xrightarrow{p_4 \rightarrow 0} \left[\text{Soft}^{(0)}(p_3, p_4^{\nu_4}, p_5) \right]_{\text{NMRK}} \left[M_{5g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_5^{\nu_5}, p_6^{\nu_6}) \right]_{\text{MRK}}, \tag{A.40}
\end{aligned}$$

where the five-gluon amplitude in MRK is

$$M_{5g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_5^{\nu_5}, p_6^{\nu_6}) \Big|_{\text{MRK}} = s C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3}) \frac{1}{t_1} V^{g(0)}(q_1, p_5^{\nu_5}, q_3) \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{\nu_6}), \tag{A.41}$$

with $q_3 = q_1 - p_5$ and $V^{g(0)}$ given in eq. (2.32). In eq. (A.40), the positive-helicity soft factor, or eikonal factor is

$$\text{Soft}^{(0)}(p_3, p_4^\oplus, p_5) = \frac{\langle 3 5 \rangle}{\langle 3 4 \rangle \langle 4 5 \rangle}. \tag{A.42}$$

Note that this eikonal factor does not depend on the helicities of particles 3 and 5, and in fact not even on their parton flavour. The negative-helicity eikonal factor is obtained by complex conjugation. In the NMRK limit the positive-helicity eikonal factor becomes

$$\text{Soft}^{(0)}(p_3, p_4^\oplus, p_5) \Big|_{\text{NMRK}} = \text{Soft}_A^{(0)}(p_4^\oplus, p_5) = \frac{p_{5\perp}}{p_{4\perp}} \sqrt{\frac{p_4^+}{p_5^+}} \frac{1}{\langle 4 5 \rangle}, \tag{A.43}$$

where in the first equality we have introduced a notation which makes it clear that this function no longer depends on p_3 once we have taken the NMRK limit. Using this notation, eq. (A.40) implies that the two-gluon CEV factorises as

$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \xrightarrow{p_4 \rightarrow 0} \text{Soft}_A^{(0)}(p_4^{\nu_4}, p_5) V^{g(0)}(q_1, p_5^{\nu_5}, q_3). \quad (\text{A.44})$$

For the equal-helicity vertex eq. (2.98), which comes from a MHV amplitude, this limit is straightforward to verify. In fact, for such an MHV amplitude, the relation at the amplitude level, eq. (A.40), holds exactly in general kinematics without even requiring that $p_4 \rightarrow 0$. For the opposite-helicity vertex, in order to check eq. (A.44) we have found it most convenient to use the representation in eq. (3.7).

As a step towards obtaining the soft limit of the one-loop amplitude, we first consider the rational coefficients. Specifically, for the limit where all components of p_4 tend to zero at the same rate, the minimal variables of eq. (3.26) and eq. (3.27) satisfy

$$X \rightarrow 0, \quad z \rightarrow -\frac{q_{1\perp}}{p_{4\perp}}, \quad w \rightarrow \frac{q_{1\perp}}{p_{5\perp}}, \quad Xz \rightarrow \text{finite}. \quad (\text{A.45})$$

In this limit R_{uv}^A , R_{vw}^A and $R_{\bar{w}\bar{u}}^A$ tend to zero. The others are given by

$$\begin{aligned} R_{wu}^A &\rightarrow \frac{Xz^2(\bar{w} - 1)}{w + Xz}, \\ R_{\bar{u}\bar{v}}^A &\rightarrow -\frac{Xz^2|w - 1|^2}{(1 + Xz)(w + Xz)}, \\ R_{\bar{v}\bar{w}}^A &\rightarrow \frac{Xz^2(\bar{w} - 1)}{1 + Xz}. \end{aligned} \quad (\text{A.46})$$

Therefore $R_{uv}^A + R_{vw}^A + R_{wu}^A$ and $R_{\bar{u}\bar{v}}^A + R_{\bar{v}\bar{w}}^A + R_{\bar{w}\bar{u}}^A$ tend to the same limit (that of eq. (A.44)). The results for the A' ordering are no longer related to these by $p_4 \leftrightarrow p_5$ because the specific choice of taking $p_4 \rightarrow 0$ breaks the symmetry. We therefore list them here. The rational coefficients $R_{uv}^{A'}$, $R_{\bar{v}\bar{w}}^{A'}$ and $R_{\bar{w}\bar{u}}^{A'}$ tend to zero. The remaining non-zero coefficients are

$$\begin{aligned} R_{\bar{u}\bar{v}}^{A'} &\rightarrow -\frac{wz(1 - \bar{w})}{w + Xz}, \\ R_{vw}^{A'} &\rightarrow \frac{wz(1 - \bar{w})^2}{w\bar{w} - w - Xz}, \\ R_{wu}^{A'} &\rightarrow \frac{w\bar{w}Xz^2(1 - \bar{w})}{(w + Xz)(w\bar{w} - w - Xz)}, \end{aligned} \quad (\text{A.47})$$

such that again $R_{uv}^{A'} + R_{vw}^{A'} + R_{wu}^{A'}$ and $R_{\bar{u}\bar{v}}^{A'} + R_{\bar{v}\bar{w}}^{A'} + R_{\bar{w}\bar{u}}^{A'}$ give the expected tree-level limit:

$$-\frac{wz(1-\bar{w})}{w+Xz} = \text{Soft}_{A'}^{(0)}(p_4^{\nu_4}, p_5) V^{g(0)}(q_1, p_5^\ominus, q_3), \quad (\text{A.48})$$

where, analogous to eq. (A.43), we have introduced the notation

$$\text{Soft}^{(0)}(p_5, p_4^\oplus, p_6) \Big|_{\text{NMRK}} = \text{Soft}_{A'}^{(0)}(p_4^{\nu_4}, p_5) = -\sqrt{\frac{p_5^+}{p_4^+}} \frac{1}{\langle 45 \rangle}. \quad (\text{A.49})$$

Finally, for the B colour ordering $R_{\bar{u}\bar{v}}^B$, R_{vw}^B and R_{wu}^B tend to zero, leaving

$$\begin{aligned} R_{uv}^B &\rightarrow -(\bar{w}-1)z, \\ R_{\bar{v}\bar{w}}^B &\rightarrow -\frac{Xz^2(\bar{w}-1)}{1+Xz-\bar{w}}, \\ R_{\bar{w}\bar{u}}^B &\rightarrow -\frac{z(\bar{w}-1)^2}{\bar{w}-Xz-1}. \end{aligned} \quad (\text{A.50})$$

These then lead to the expected tree-level result where $R_{uv}^B + R_{vw}^B + R_{wu}^B$ and $R_{\bar{u}\bar{v}}^B + R_{\bar{v}\bar{w}}^B + R_{\bar{w}\bar{u}}^B$ both sum to

$$-(\bar{w}-1)z = \text{Soft}_B^{(0)}(p_4) V^{g(0)}(q_1, p_5^\ominus, q_3), \quad (\text{A.51})$$

where as for the other colour orderings we have introduced

$$\text{Soft}^{(0)}(p_1, p_4^\oplus, p_2) \Big|_{\text{NMRK}} = \text{Soft}_B^{(0)}(p_4) = \frac{1}{p_{4\perp}}, \quad (\text{A.52})$$

which does not even depend on the helicity of the soft particle. The three relevant eikonal functions in the NMRK obey

$$\text{Soft}_A^{(0)}(p_4^{\nu_4}, p_5) + \text{Soft}_{A'}^{(0)}(p_4^{\nu_4}, p_5) + \text{Soft}_B^{(0)}(p_4) = 0, \quad (\text{A.53})$$

consistent with the more general eq. (2.104), and the eikonal identity.

We now turn to the colour-ordered amplitudes at one-loop. Beginning with the same-helicity vertex, taking the limit $p_4 \rightarrow 0$ in eqs. (3.78), (3.84) and (3.87), we

find

$$\begin{aligned}
\text{Re}[V_6(\sigma_A)|_{\text{NMRK}}] &\xrightarrow{p_4 \rightarrow 0} E(t_1, s_{35}; t_3, s_{56}) + v^{g(1)}(t_1, |p_{5\perp}|^2, t_3) + s_A^{(1)}(p_4^{\nu_4}, p_5) \\
\text{Re}[V_6(\sigma_{A'})|_{\text{NMRK}}] &\xrightarrow{p_4 \rightarrow 0} E(t_1, s_{35}; t_3, s_{56}) + v^{g(1)}(t_1, |p_{5\perp}|^2, t_3) + s_{A'}^{(1)}(p_4, p_5), \\
\text{Re}[V_6(\sigma_B)|_{\text{NMRK}}] &\xrightarrow{p_4 \rightarrow 0} E(t_1, s_{35}; t_3, s_{56}) + v^{g(1)}(t_1, |p_{5\perp}|^2, t_3) + s_B^{(1)}(p_4),
\end{aligned} \tag{A.54}$$

where $E(t_1, s_{35}; t_3, s_{56})$ is the natural collection of impact factors and Regge trajectories in the absence of p_4 and $v^{g(1)}$ is the one-loop Lipatov vertex correction given in eq. (2.79). The one-loop soft factor [163] is

$$s^{(1)}(p_3, p_4^{\nu_4}, p_5) = - \left(\frac{\mu^2(-s_{35})}{(-s_{34})(-s_{45})} \right)^\epsilon \left(\frac{1}{\epsilon^2} + \frac{\pi^2}{6} \right) + \mathcal{O}(\epsilon^2), \tag{A.55}$$

which must first be analytically continued to our physical region, discussed in appendix A.1, and then taken in the NMRK limit. In eq. (A.54) we have used the notation

$$s_A^{(1)}(p_4, p_5) = \text{Re} \left[s^{(1)}(p_3, p_4, p_5) \Big|_{\text{NMRK}} \right], \tag{A.56}$$

$$s_{A'}^{(1)}(p_4, p_5) = \text{Re} \left[s^{(1)}(p_4, p_5, p_6) \Big|_{\text{NMRK}} \right], \tag{A.57}$$

$$s_B^{(1)}(p_4) = \text{Re} \left[s^{(1)}(p_1, p_4, p_2) \Big|_{\text{NMRK}} \right], \tag{A.58}$$

which, as for the analogous tree-level functions, more clearly indicate the simpler kinematic dependence of these one-loop soft functions in the NMRK limit.

Eq. (A.54) exactly matches the expected sum of the one-loop corrections to the non-central pieces, the Lipatov vertex for the single emission of p_5 and the one-loop soft function. From eq. (3.75), these are multiplied by the soft limit of the corresponding tree-level amplitude for that ordering.

Turning now to the opposite-helicity case in eq. (3.112), and starting with the A colour ordering, we find

$$\begin{aligned}
S_{wu}^A &\xrightarrow{p_4 \rightarrow 0} \text{Soft}_A^{(0)}(p_4^{\nu_4}, p_5) V^{g(0)}(q_1, p_5^\ominus, q_3), \\
\text{Re}[W_1(\sigma_A) - W_2(\sigma_A)] &\xrightarrow{p_4 \rightarrow 0} 0, \\
\text{Re}[W_3(\sigma_A) - W_2(\sigma_A)] &\xrightarrow{p_4 \rightarrow 0} \log \left(\frac{s_{34} t_3}{s_{35} t_1} \right) \log \left(\frac{|p_{5\perp}|^2}{s_{45}} \right).
\end{aligned} \tag{A.59}$$

This allows us to extract a common factor of the tree amplitude multiplying

$$\text{Re} [2W_2(\sigma_A) + (W_3(\sigma_A) - W_2(\sigma_A))].$$

Using eqs. (3.119) and (3.120), we find

$$\begin{aligned} & \text{Re} [2W_2(\sigma_A) + (W_3(\sigma_A) - W_2(\sigma_A))]_{\text{NMRK}} \\ &= \text{Re} [V_6(\sigma_A) - \Delta V_A(u_A, v_A, w_A) + \Delta W_A(u_A, v_A, w_A) + (W_3(\sigma_A) - W_2(\sigma_A))]_{\text{NMRK}} \\ &\xrightarrow{p_4 \rightarrow 0} \text{Re} [V_6(\sigma_A)]_{\text{NMRK}, p_4 \rightarrow 0}, \end{aligned} \tag{A.60}$$

where the cancellation in the last line arises from the soft limit of eqs. (3.82), (3.121) and (A.59). Eq. (A.54) then immediately gives the expected one-loop structure which equals that found for the same-helicity vertex. The helicity-dependence in the soft limit is contained only in the tree-level factors.

Similarly for the A' -ordering we find

$$\begin{aligned} & S_{uv}^{A'} \xrightarrow{p_4 \rightarrow 0} \text{Soft}_{A'}^{(0)}(p_4^{\nu_4}, p_5) V^{g(0)}(q_1, p_5^\ominus, q_3), \\ & \text{Re} [W_1(\sigma_{A'}) - W_2(\sigma_{A'})]_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} \log \left(\frac{s_{46} t_1}{s_{56} t_3} \right) \log \left(\frac{|p_{5\perp}|^2}{s_{45}} \right), \\ & \text{Re} [W_3(\sigma_{A'}) - W_2(\sigma_{A'})]_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} 0. \end{aligned} \tag{A.61}$$

Extracting a common coefficient of the tree amplitude leaves

$$\text{Re} [2W_2(\sigma_{A'}) + (W_1(\sigma_{A'}) - W_2(\sigma_{A'}))]_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} \text{Re} [V_6(\sigma_{A'})]_{\text{NMRK}, p_4 \rightarrow 0}, \tag{A.62}$$

as expected. Finally for the B ordering,

$$\begin{aligned} & S_{vw}^B + S_{wu}^B \xrightarrow{p_4 \rightarrow 0} \text{Soft}_B^{(0)}(p_4) V^{g(0)}(q_1, p_5^\ominus, q_3), \\ & \text{Re} [W_2(\sigma_B) - W_1(\sigma_B)]_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} \log \left(\frac{s_{24} s_{56}}{s_{12} t_1} \right) \log \left(\frac{s_{14} s_{35}}{s_{12} t_3} \right), \\ & \text{Re} [W_3(\sigma_B) - W_1(\sigma_B)]_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} \log \left(\frac{s_{24} s_{56}}{s_{12} t_1} \right) \log \left(\frac{s_{14} s_{35}}{s_{12} t_3} \right). \end{aligned} \tag{A.63}$$

As the final two limits are equal, one again factors out the tree amplitude leaving

$$\text{Re} [2W_1(\sigma_B) + (W_2(\sigma_B) - W_1(\sigma_B))] |_{\text{NMRK}} \xrightarrow{p_4 \rightarrow 0} \text{Re} [V_6(\sigma_B)] |_{\text{NMRK}, p_4 \rightarrow 0} . \quad (\text{A.64})$$

Finally, performing the change of basis of colour structures in eq. (3.95), we collect the results of this appendix in the following expression for general helicities,

$$\begin{aligned} & \text{Disp} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^{\nu_4}, p_5^{\nu_5}) \right] \xrightarrow{\text{NMRK}, p_4 \rightarrow 0} \\ & \frac{g_s^6}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} V^{g(0)}(q_1, p_5^{\nu_5}, q_3) \times \\ & \left\{ (F^{a_4} F^{a_5})_{c_1 c_3} \left[\text{Soft}_A(p_4^{\nu_4}, p_5) \left(3 v^{g(1)}(t_1, |p_{5\perp}|^2, t_3) + N_c \left(2s_A^{(1)}(p_4, p_5) + s_B^{(1)}(p_4) \right) \right. \right. \right. \\ & \quad \left. \left. + 3 E(t_1, s_{35}; t_3, s_{56}) \right) + N_c \text{Soft}_{A'}(p_4^{\nu_4}, p_5) \left(s_{A'}^{(1)}(p_4, p_5) - s_B^{(1)}(p_4) \right) \right] \\ & \quad + (F^{a_5} F^{a_4})_{c_1 c_3} \left[\text{Soft}_{A'}(p_4^{\nu_4}, p_5) \left(3 v^{g(1)}(t_1, |p_{5\perp}|^2, t_3) + N_c \left(2s_{A'}^{(1)}(p_4, p_5) + s_B^{(1)}(p_4) \right) \right. \right. \\ & \quad \left. \left. + 3 E(t_1, s_{35}; t_3, s_{56}) \right) + N_c \text{Soft}_A(p_4^{\nu_4}, p_5) \left(s_A^{(1)}(p_4, p_5) - s_B^{(1)}(p_4) \right) \right] \\ & \quad + d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3} \left[\text{Soft}_A(p_4^{\nu_4}, p_5) \left(s_A^{(1)}(p_4, p_5) - s_B^{(1)}(p_4) \right) \right. \\ & \quad \left. \left. + \text{Soft}_{A'}(p_4^{\nu_4}, p_5) \left(s_{A'}^{(1)}(p_4, p_5) - s_B^{(1)}(p_4) \right) \right] \right\} \times \frac{1}{t_3} F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) . \end{aligned} \quad (\text{A.65})$$

A.4.3 Collinear limits

In this appendix we investigate the behaviour of the one-loop amplitude eq. (3.126) in the collinear limit,

$$p_4 \rightarrow z_4 P, \quad p_5 \rightarrow z_5 P, \quad z_4 + z_5 = 1. \quad (\text{A.66})$$

We begin by recalling the collinear limit of the colour ordered six-gluon tree-level amplitude in general kinematics. It factorises into a divergent splitting amplitude,

which behaves like $1/\langle 45 \rangle$ or $1/[45]$, and a five-gluon amplitude,

$$M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}) \xrightarrow{p_4 \parallel p_5} \sum_{\nu=\pm} \text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}), \quad (\text{A.67})$$

where we sum over the helicities of the intermediate state P^ν . For an equal-helicity pair, the tree-level splitting amplitudes are

$$\text{Split}_{\ominus}^{(0)}(p_4^{\oplus}, p_5^{\oplus}) = \frac{1}{\sqrt{z_4 z_5} \langle 45 \rangle}, \quad \text{Split}_{\oplus}^{(0)}(p_4^{\oplus}, p_5^{\oplus}) = 0, \quad (\text{A.68})$$

while for an opposite-helicity pair, they are

$$\text{Split}_{\oplus}^{(0)}(p_4^{\oplus}, p_5^{\ominus}) = \frac{z_5^2}{\sqrt{z_4 z_5} \langle 45 \rangle}, \quad \text{Split}_{\ominus}^{(0)}(p_4^{\oplus}, p_5^{\ominus}) = -\frac{z_4^2}{\sqrt{z_4 z_5} [45]}. \quad (\text{A.69})$$

In the central NMRK limit, eq. (A.67) becomes

$$M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}) \Big|_{\text{NMRK}} \xrightarrow{p_4 \parallel p_5} \sum_{\nu=\pm} \text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}) \Big|_{\text{MRK}}, \quad (\text{A.70})$$

where the five-gluon amplitude in MRK is given in eq. (A.41), with $q_3 = q_1 - P$. In particular, the two-gluon CEV factorises as

$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \stackrel{p_4 \parallel p_5}{=} \sum_{\nu=\pm} \text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) V^{g(0)}(q_1, P^\nu, q_3). \quad (\text{A.71})$$

For the equal-helicity vertex eq. (2.98), it is straightforward to verify eq. (A.71). For the opposite-helicity vertex, we have found it more convenient to check eq. (A.71) by starting from eq. (3.7) or eq. (3.9) rather than eq. (2.99), where as we have mentioned in section 3.1.1, the BCFW representations make clear the collinear singularity structure of the amplitudes. Working in the minimal variables, the collinear limit can be expressed as

$$\begin{aligned} w &\xrightarrow{p_4 \parallel p_5} -\frac{z_4}{z_5} z, \\ X &\xrightarrow{p_4 \parallel p_5} \frac{z_4}{z_5}. \end{aligned} \quad (\text{A.72})$$

For the σ_A ordering we find that four of the six rational terms factorise into a splitting function and Lipatov vertex, while two of the rational terms do not diverge in this limit (unless we move to the further soft-collinear limits of $z_4 \rightarrow 0$

or $z_4 \rightarrow 1$),

$$\begin{aligned}
R_{uv}^A &\xrightarrow{p_4||p_5} V^{g(0)}(q_1, P^\oplus, q_3) \text{Split}_{\ominus}^{(0)}(p_4^\oplus, p_5^\ominus), \\
R_{\bar{u}\bar{v}}^A &\xrightarrow{p_4||p_5} V^{g(0)}(q_1, P^\ominus, q_3) \text{Split}_{\oplus}^{(0)}(p_4^\oplus, p_5^\ominus), \\
R_{vw}^A &\xrightarrow{p_4||p_5} -\frac{(z-1)(zz_4+z_5)z_5}{(\bar{z}-1)(\bar{z}z_4+z_5)(|z|^2z_4+z_5)}, \\
R_{\bar{v}\bar{w}}^A &\xrightarrow{p_4||p_5} -\frac{z^4z_4^2(\bar{z}-1)(\bar{z}z_4+z_5)}{z_5(z-1)(zz_4+z_5)(|z|^2z_4+z_5)}, \\
R_{wu}^A &\xrightarrow{p_4||p_5} V^{g(0)}(q_1, P^\ominus, q_3) \text{Split}_{\oplus}^{(0)}(p_4^\oplus, p_5^\ominus), \\
R_{\bar{w}\bar{u}}^A &\xrightarrow{p_4||p_5} V^{g(0)}(q_1, P^\oplus, q_3) \text{Split}_{\ominus}^{(0)}(p_4^\oplus, p_5^\ominus).
\end{aligned} \tag{A.73}$$

Analogous results for the $\sigma_{A'}$ ordering can be found by exchanging $p_4 \leftrightarrow p_5$ and taking the complex conjugate, as in eq. (3.14). For the σ_B ordering, as expected, none of the rational terms diverge in this limit,

$$\begin{aligned}
R_{uv}^B &\xrightarrow{p_4||p_5} \frac{|z-1|^2|zz_4+z_5|^2}{z_5((z-1)z_4+z_5)\bar{z}}, \\
R_{\bar{u}\bar{v}}^B &\xrightarrow{p_4||p_5} 0, \\
R_{vw}^B &\xrightarrow{p_4||p_5} \frac{z_5(z-1)(zz_4+z_5)}{\bar{z}(|z|^2z_4+z_5)((z+\bar{z}-1)z_4+z_5)}, \\
R_{\bar{v}\bar{w}}^B &\xrightarrow{p_4||p_5} \frac{z^3z_4^2(\bar{z}-1)(\bar{z}z_4+z_5)}{z_5(|z|^2z_4+z_5)((z+\bar{z}-1)z_4+z_5)}, \\
R_{wu}^B &\xrightarrow{p_4||p_5} \frac{z_5z_4^2(\bar{z}-1)(\bar{z}z_4+z_5)}{((z-1)z_4+z_5)((z+\bar{z}-1)z_4+z_5)(|z_5+(z-1)z_4|^2+z_4z_5)}, \\
R_{\bar{w}\bar{u}}^B &\xrightarrow{p_4||p_5} \frac{(z-1)(zz_4+z_5)((\bar{z}-1)z_4+z_5)^3}{z_5((z+\bar{z}-1)z_4+z_5)(|z_5+(z-1)z_4|^2+z_4z_5)}.
\end{aligned} \tag{A.74}$$

From these lists of rational terms we can immediately verify that the divergent pieces satisfy the condition in eq. (A.71) from the representations eq. (3.7) or eq. (3.9). We note that in this limit the divergent pieces also satisfy

$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \Big|_{p_4||p_5} = -A^{gg(0)}(q_1, p_5^{\nu_5}, p_4^{\nu_4}, q_3) \Big|_{p_4||p_5}. \tag{A.75}$$

In addition to eq. (A.66), the amplitude in eq. (2.97) is divergent in the collinear regions $p_{3\perp} = -q_{1\perp} \rightarrow 0$ or $p_{6\perp} = q_{3\perp} \rightarrow 0$. In those regions, eq. (2.97) must not diverge more rapidly than $1/|q_{i\perp}|$, with $i = 1, 3$, in order for the related cross

section not to diverge more than logarithmically. Since eq. (2.97) displays poles as $|q_{1\perp}|^2$ and $|q_{3\perp}|^2$ go to zero, the central-emission vertex must be at least linear in $|q_{i\perp}|$ as $|q_{i\perp}| \rightarrow 0$,

$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \stackrel{|q_{i\perp}| \rightarrow 0}{=} \mathcal{O}(|q_{i\perp}|), \quad (\text{A.76})$$

with $i = 1, 3$, which is indeed fulfilled by eq. (2.99).

Let us now consider the collinear limit of the one-loop amplitudes in $\mathcal{N} = 4$ SYM in the NMRK limit. It is most convenient if we work at the level of colour-ordered amplitudes, where in general kinematics the colour-ordered one-loop six gluon amplitude has the collinear limit [56],

$$M_{6g}^{(1)}(p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}, p_1^{\nu_1}) \xrightarrow{p_4 \parallel p_5} \sum_{\nu=\pm} \left(\text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(1)}(p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}, p_1^{\nu_1}) \right. \\ \left. + \text{Split}_{-\nu}^{(1)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(0)}(p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}, p_1^{\nu_1}) \right), \quad (\text{A.77})$$

where the one-loop splitting functions were obtained in [56]. In $\mathcal{N} = 4$ SYM, these splitting functions are proportional to the tree-level splitting function,

$$\text{Split}_{-\nu}^{(1)}(p_4^{\nu_4}, p_5^{\nu_5}) = \kappa_\Gamma \text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) r_S^{\text{SUSY}}(z_4, s_{45}), \quad (\text{A.78})$$

where the loop factor,

$$r_S^{\text{SUSY}}(z_4, s_{45}) = -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{z_4(1-z_4)(-s_{45})} \right)^\epsilon + 2 \log(z_4) \log(1-z_4) - \frac{\pi^2}{6}, \quad (\text{A.79})$$

is independent of the helicities. In the central NMRK limit we expect to obtain

$$M_{6g}^{(1)}(p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}, p_1^{\nu_1}) \Big|_{\text{NMRK}} \xrightarrow{p_4 \parallel p_5} \sum_{\nu=\pm} \left(\text{Split}_{-\nu}^{(0)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(1)}(p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}, p_1^{\nu_1}) \Big|_{\text{MRK}} \right. \\ \left. + \text{Split}_{-\nu}^{(1)}(p_4^{\nu_4}, p_5^{\nu_5}) M_{5g}^{(0)}(p_2^{\nu_2}, p_3^{\nu_3}, P^\nu, p_6^{\nu_6}, p_1^{\nu_1}) \Big|_{\text{MRK}} \right). \quad (\text{A.80})$$

Let us check this first for the same-helicity case. As we are working at the level of colour-ordered amplitudes, we consider the three representative orderings of

eq. (3.75). For the σ_A colour ordering, we find

$$s_{234} \xrightarrow{p_4 || p_5} z_5 t_1 + z_4 t_3, \quad (\text{A.81})$$

so that in the collinear limit,

$$v_A + w_A \xrightarrow{p_4 || p_5} 1. \quad (\text{A.82})$$

In particular, this allows us to use the identity,

$$\text{Li}_2(1-x) + \text{Li}_2(x) = \frac{\pi^2}{6} - \log(1-x) \log(x). \quad (\text{A.83})$$

to write

$$\begin{aligned} \Delta V_A(u_A, v_A, w_A) \xrightarrow{p_4 || p_5} & -\frac{1}{4} \log^2 \left(\frac{s_{45}}{|P_\perp|^2} \right) - \frac{1}{2} \log \left(\frac{s_{45}}{|P_\perp|^2} \right) \log \left(\frac{t_1 t_3 z_4 z_5}{s_{234}^2} \right) \\ & + \log \left(\frac{z_4 t_3}{s_{234}} \right) \log \left(\frac{z_5 t_1}{s_{234}} \right). \end{aligned} \quad (\text{A.84})$$

For the full transcendental function of this colour ordering we find

$$\begin{aligned} \text{Re} [V_6(\sigma_A)|_{\text{NMRK}}] \xrightarrow{p_4 || p_5} & E(t_1, p_3^+ P^-; t_3, P^+ p_6^-) + v_{\mathcal{N}=4}^{g(1)}(t_1, |P_\perp|^2, t_3) \\ & + \text{Re} [r^{\text{SUSY}}(z_4, s_{45})], \end{aligned} \quad (\text{A.85})$$

which is invariant under $p_4 \leftrightarrow p_5$. After manipulating the colour factors to obtain the natural association of one adjoint generator with each three-point tree-level factor, we obtain

$$\begin{aligned} \text{Re} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\oplus) \right] \xrightarrow{\text{NMRK}, p_4 || p_5} & \frac{g_s^6}{(4\pi)^2} \kappa_\Gamma s_{12} \left[F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \right] \frac{1}{t_1} \\ & \times \left[F^{c_1 c_3 c_2} V^{g(0)}(q_1, P^\oplus, q_3) \right] \left[F^{c_2 a_5 a_4} \text{Split}_\ominus^{(0)}(p_4^\oplus, p_5^\oplus) \right] \\ & \times \frac{1}{t_3} \left[F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \right] \times \left\{ N_c \text{Re} [V_6(\sigma_A)]|_{\text{NMRK}, p_4 || p_5} \right\}. \end{aligned} \quad (\text{A.86})$$

The opposite-helicity case proceeds similarly. From eq. (A.73) we see that in the collinear limit, both S_{uv}^A and S_{wu}^A are equal to the tree-level amplitude, and these

terms combine with the ΔW_A term to play the same role as ΔV_A in this limit,

$$\begin{aligned}
& A^{gg(0)}(q_1, p_4^\oplus, p_4^\ominus, q_3) \Delta W_A(u_A, v_A, w_A) + S_{uv}^A \log(v_A) \left(\frac{w_A}{u_A} \right) + S_{wu}^A \log(w_A) \left(\frac{v_A}{u_A} \right) \\
& \xrightarrow{p_4 || p_5} \left(\sum_{\nu=\pm} V^{g(0)}(q_1, P^\nu, q_3) \text{Split}_{-\nu}^{(0)}(p_4^\oplus, p_5^\ominus) \right) \Delta V_A(u_A, v_A, w_A)|_{p_4 || p_5}
\end{aligned} \tag{A.87}$$

This means we can straightforwardly write

$$\begin{aligned}
& \text{Re} \left[\mathcal{M}_{\mathcal{N}=4}^{(1)}(p_4^\oplus, p_5^\ominus) \right] \xrightarrow{\text{NMRK}, p_4 || p_5} \\
& \frac{g_s^6}{(4\pi)^2} \kappa_{\Gamma S_{12}} \left[F^{a_3 a_2 c_1} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \right] \frac{1}{t_1} \\
& \times \sum_{\nu=\pm} \left(\left[F^{c_1 c_3 c_2} V^{g(0)}(q_1, P^\nu, q_3) \right] \left[F^{c_2 a_5 a_4} \text{Split}_{-\nu}^{(0)}(p_4^\oplus, p_5^\ominus) \right] \right) \\
& \times \frac{1}{t_3} \left[F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \right] \times \left\{ N_c \text{Re} [V_6(\sigma_A)]|_{\text{NMRK}, p_4 || p_5} \right\}.
\end{aligned} \tag{A.88}$$

A.5 Explorer phase space slices

The phase space parameterisation used in the plots of chapter 4, which we term ‘explorer plots’, are listed here in order to allow the plots to be reproduced, but we stress that the level of accuracy does not depend on the specific choices of these parameters. The purpose of these plots are to verify agreement with fixed order calculations in multi-Regge kinematic regions, and to provide us with an intuition for the rapidity separations required for the convergence of the approximated LO amplitudes. The more stringent test of the phenomenological applicability of the approximations require us to convolute with PDFs and integrate over phase space, as performed in chapter 4.

We generated the phase space points in cartesian coordinates, and we parameterise these coordinates in terms of hardness $|p_{i\perp}|$, rapidity y_i and azimuthal angle ϕ_i :

$$p_i = (p_i^0, p_i^x, p_i^y, p_i^z) = |p_{i\perp}| (\cosh y_i, \cos \phi_i, \sin \phi_i, \sinh y_i)$$

The $2 \rightarrow 3$ MRK phase space slice used in fig. 4.14a is:

$$\begin{aligned}
|p_{1\perp}| &= |p_{2\perp}| = 40 \text{ GeV} \\
y_1 &= \Delta, \quad y_2 = 0, \quad y_3 = -\Delta, \\
\phi_1 &= 0, \quad \phi_2 = \frac{2\pi}{3}, \\
p_{3\perp} &= -p_{1\perp} - p_{2\perp}
\end{aligned} \tag{A.89}$$

The $2 \rightarrow 3$ NMRK phase space slice used in figs. 4.12, 4.13 and 4.14b is:

$$\begin{aligned}
|p_{1\perp}| &= |p_{2\perp}| = 40 \text{ GeV} \\
y_1 &= \frac{\Delta}{2}, \quad y_2 = -\frac{\Delta}{2} + 0.25, \quad y_3 = -\frac{\Delta}{2} - 0.25, \\
\phi_1 &= 0, \quad \phi_2 = \frac{2\pi}{3}, \\
p_{3\perp} &= -p_{1\perp} - p_{2\perp}
\end{aligned} \tag{A.90}$$

The $2 \rightarrow 4$ MRK phase space slice used in fig. 4.24a is:

$$\begin{aligned}
|p_{1\perp}| &= |p_{2\perp}| = |p_{3\perp}| = 40 \text{ GeV} \\
y_1 &= \frac{3\Delta}{2}, \quad y_2 = \frac{\Delta}{2}, \quad y_3 = -\frac{\Delta}{2}, \quad y_4 = -\frac{3\Delta}{2} \\
\phi_1 &= 0, \quad \phi_2 = \frac{\pi}{4}, \quad \phi_3 = -\frac{3\pi}{2}, \\
p_{4\perp} &= -p_{1\perp} - p_{2\perp} - p_{3\perp}
\end{aligned} \tag{A.91}$$

The $2 \rightarrow 4$ central NMRK phase space slice used in figs. 4.21, 4.23 and 4.24a is:

$$\begin{aligned}
|p_{1\perp}| &= |p_{2\perp}| = |p_{3\perp}| = 40 \text{ GeV} \\
y_1 &= \Delta, \quad y_2 = 0.25, \quad y_3 = -0.25, \quad y_4 = -\Delta \\
\phi_1 &= 0, \quad \phi_2 = \frac{\pi}{4}, \quad \phi_3 = -\frac{3\pi}{2}, \\
p_{4\perp} &= -p_{1\perp} - p_{2\perp} - p_{3\perp}
\end{aligned} \tag{A.92}$$

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