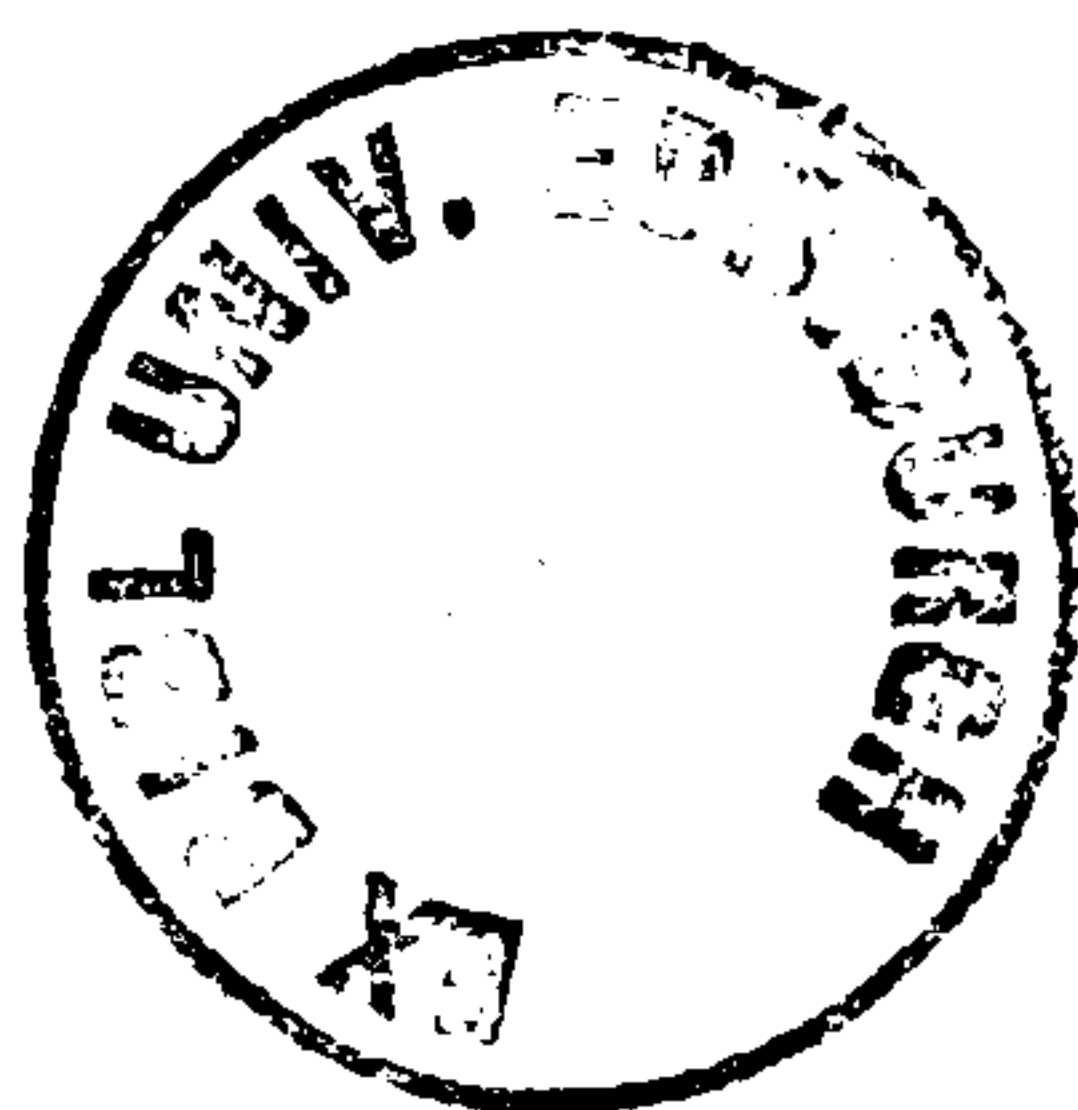


Matricial Norm
Structures and Inequalities
for the
Schatten p -Ideals

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Declaration

This thesis was composed entirely by the author. The work contained herein was carried out under the supervision of Dr. T. A. Gillespie at the University of Edinburgh between October 1992 and January 1996. No part of this thesis has been submitted for any other degree and nor have any of the results been submitted for publication. Any results not due to the author are acknowledged by explicit reference.

Abstract

We construct ideals in an arbitrary C^* -algebra (which correspond to those of the Schatten p -ideals in $\mathcal{B}(\mathcal{H})$) using the idea of a single element of an algebra and an isometric $*$ -representation due to J. A. Erdos. We show that many of the familiar results on the Schatten p -ideals have a counterpart on these new structures. In particular, these ideals may be thought of as a generalisation to C^* -algebras of the unitary matrix spaces in $\mathcal{B}(\mathcal{H})$ and we show that interpolation results due to Jonathan Arazy extend to these new classes. The representation also enables us to view the concept of Volterra, Riesz and Fredholm elements of a C^* -algebra in a new light by representing these classes on a Hilbert space.

Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, assumed to be separable if infinite dimensional. Fix $1 \leq p \leq \infty$ and let $s, t \in \mathbb{N}$. For each $1 \leq i \leq s$ and $1 \leq j \leq t$, let $n_i, m_j \in \mathbb{N}$ and set $n = \sum_{i=1}^s n_i$ and $m = \sum_{j=1}^t m_j$. Suppose that, for each i, j , $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$. Then $(A_{ij}) \in \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$ and is a block $s \times t$ matrix. We can now consider the quantity:

$$\left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)}.$$

We use this construction to equip the Schatten p -ideals with new quasi-norms (in some cases, they are genuine norms) and we prove a number of results which relate the value of $\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)}$ to $\left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)}$.

We characterise those matricially normed spaces which are completely isometric to subspaces of $\mathcal{C}_2(\mathcal{H})$ using a condition on the matricial norm. We also formulate results concerning the interpolation of matricially normed spaces using the complex, real and approximation methods. In particular, we show that, under the appropriate compatibility criteria, the complex interpolation method C_θ applied to an L^p - and an L^q -matricially normed space gives an L^r -matricially normed space where $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ and that the real interpolation method $K_{\theta,1}$ applied to two L^1 -matricially normed spaces yields another L^1 -matricially normed space.

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Thanks also to Dr. J. A. Erdos for bringing [14] to my attention, to Prof. Gilles Pisier for providing me with a preprint of [19] and to Dr. C. J. Smith for a useful discussion concerning the Tarry-Escott problem.

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“To be or not to be? *Not* to be.”

Jack Slater as Hamlet, *Last Action Hero*.

“And worrying is less work than doing something to fix the worry. This is especially true if we’re careful to pick the biggest possible problems to worry about. Everybody wants to save the earth; nobody wants to help Mom do the dishes.”

P. J. O’Rourke, *All The Trouble In The World: The Lighter Side Of Famine, Pestilence, Destruction And Death*.

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Preface

For the reader's convenience, chapter one contains some results from established theory and can be divided into three main parts: the Schatten p -ideals of operators, matricially normed spaces and interpolation theory. All of the results are stated without proof; standard references are given.

C^* -algebras were characterised by Gelfand and Naimark in the following manner: if A is a C^* -algebra, then there is an isometric $*$ -representation of A onto a norm-closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since the Schatten classes of operators are ideals in $\mathcal{B}(\mathcal{H})$, it is reasonable to ask if 'abstract' C^* -algebras support the notion of ' $\mathcal{C}_p$ elements'. This question may be answered in the affirmative and the development of such a class of elements is the subject of chapter two.

Two ideas are crucial. The first is the notion of a single element of an algebra which takes the place of an operator of rank one on a Hilbert space. The second is an isometric $*$ -representation of a C^* -algebra due to J. A. Erdos which has the property that the image of a single element under this representation is an operator of rank at most one. (We also point out why the representation due to Gelfand and Naimark is inappropriate for the job at hand.)

Using these two ingredients, we construct ideals in an arbitrary C^* -algebra by mimicking the usual construction of the Schatten p -ideals of operators. To be precise, we formulate a notion of the s -numbers of an element of the C^* -algebra and form ideals according to which symmetric sequence spaces these s -numbers lie in.

We show that a great many of the standard results can be proven for these ideals. In particular, analogues of the interpolation results due to Jonathan Arazy for the Schatten p -ideals may be proven for these new structures.

Erdos' representation also allows us to consider the notion of Volterra, Fredholm and Riesz elements of a C^* -algebra in a new light by representing such elements as operators on a Hilbert space.

Chapter three is given over to the study of norm inequalities for the Schatten p -ideals. However, an essential question we need to consider initially is what we call the *pointwise ordering property* for scalar matrices, which concerns the values of $\|(\alpha_{ij})\|_p$ and $\|(\beta_{ij})\|_p$ where $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$. We show that $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$

for $p = 2m$ ($m \in \mathbb{N}$) or infinity and give bounds for all the other cases in which $p > 2$. We also give a counter-example which shows that the pointwise ordering property does not hold for any $1 \leq p < 2$.

For the norm inequalities themselves, we are primarily concerned with the following idea. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, assumed to be separable if infinite dimensional. Fix $1 \leq p \leq \infty$ and let $s, t \in \mathbb{N}$. For each $1 \leq i \leq s$ and $1 \leq j \leq t$, let $n_i, m_j \in \mathbb{N}$ and set $n = \sum_{i=1}^s n_i$ and $m = \sum_{j=1}^t m_j$. Suppose that, for each i, j , $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$. Then $(A_{ij}) \in \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$ and is a block $s \times t$ matrix. We can now consider the following quantity:

$$\left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)}.$$

We use this construction to equip the Schatten p -ideals with new quasi-norms (in some cases, they are genuine norms) and we prove a number of results which relate the value of $\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)}$ to that of $\left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)}$.

Specifically, we use $(*, p)$ to denote that, whenever $n = \sum_{i=1}^s n_i$, $m = \sum_{j=1}^t m_j$ and $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ ($1 \leq i \leq s$, $1 \leq j \leq t$), we have:

- (i) $\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)} \geq \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)}$ if $1 \leq p \leq 2$;
- (ii) $\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)} \leq \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)}$ if $2 \leq p \leq \infty$.

We show that $(*, p)$ holds if p is an even integer or infinity but that it fails for $1 \leq p < 2$ and $2 < p < 4$ in general; we give bounds in all the other cases. However, 2×2 matrices exhibit the $(*, p)$ property for all $1 \leq p \leq \infty$.

Finally, we give l^p -type estimates for the Schatten p -norms applied to scalar matrices and we demonstrate that the bounds given are the best possible.

It was proved by Zhong-Jin Ruan that every L^∞ -matricially normed space is completely isometric to a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This begs an obvious question: is every L^p -matricially normed space completely isometric to a subspace of $\mathcal{C}_p(\mathcal{H})$ for some Hilbert space $\mathcal{H}$?

It is reasonably clear that a matricially normed space $(V, \{\|\cdot\|_n\})$ is completely isometric to a subspace of $\mathcal{C}_2(\mathcal{H})$ if, for all $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on $M_n(V)$ is an inner product norm. What is not so clear is that there is a very simple property for the matricial norm which is equivalent to the norm being an inner product norm. This property is the following:

$$\|(v_{ij})\|_n = \sqrt{\sum_{i,j=1}^n \|v_{ij}\|_1^2},$$

where $(v_{ij}) \in M_n(V)$.

Given two ‘compatible’ spaces, the theory of interpolation allows us to derive information about a space which is, in some sense, between the two spaces we began with.

Since matricially normed spaces can have a suitably nice structure, it is natural to ask questions about the interpolation of such spaces.

We consider the three main methods of interpolation, namely the real method $K_{\theta,q}$ and the complex methods C^θ and C_θ as well as giving some attention at the end to the approximation method $E_{\alpha,q}$.

Our first task is to establish that, with the appropriate conditions, the interpolation of two matricially normed spaces yields another matricially normed space (an intuitively obvious and relatively easy fact).

Next, we consider the more complicated question of interpolating an L^p - and an L^q -matricially normed space. We are able to show that the complex method C_θ yields an L^r -matricially normed space where $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. We demonstrate that applying the real method $K_{\theta,1}$ to two L^1 -matricially normed spaces gives another L^1 -matricially normed space. The approximation method produces results of interest but little practical use. This concludes chapter four.

The Maple¹ and Mathematica² packages were used to do many of the calculations in chapter three. Appendix A contains some of the programs which were used whilst the graphs which were produced have been collected together in appendix B.

This thesis was written under the active direction of Dr. T. A. Gillespie and was prepared using L^AT_EX.

¹ *Maple* is a registered trademark of Waterloo Maple Software

² *Mathematica* is a registered trademark of Wolfram Research, Inc.

Chapter 1

Background Theory and Notation

1.1 Introduction

In this section, we introduce the notation and definitions which will play a prominent role in the rest of the thesis and list (without proof) key results of established theory that will be needed later on.

We use $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ to denote the natural, real and complex numbers respectively. If $z \in \mathbf{C}$, then $\Re z$ will denote the real part of z . $\{E_{ij}\}_{i,j=1}^n$ will denote the matrix units in $M_n(\mathbf{C})$ and $x \otimes E_{ij}$ will denote the element in $M_n(X) = X \otimes M_n(\mathbf{C})$ which has x in the $(i, j)^{\text{th}}$ position and zeroes elsewhere. The symbol $\blacksquare$ will be used to denote the end of a proof.

1.2 Symmetric Norms

Definition 1.1 *If $(a_n) \in c_0$, we use (a_n^*) to denote the non-increasing re-arrangement of (a_n) , that is to say, the elements of (a_n^*) are the moduli of the entries of (a_n) arranged in decreasing order of magnitude and repeated according to multiplicity. Let $\hat{c}$ denote the set of those sequences in c_0 which have only finitely many non-zero terms. A norm $\|\cdot\|_{\Phi}$ on $\hat{c}$ is said to be symmetric if:*

(i) $\|(1, 0, 0, \dots)\|_{\Phi} = 1$;

(ii) $\|\cdot\|_{\Phi}$ is invariant under all permutations and all maps $(a_n) \mapsto (e^{i\theta_n} a_n)$, that is to say, $\|(a_n)\|_{\Phi} = \|(a_n^*)\|_{\Phi}$ for all $(a_n) \in \hat{c}$.

We set

$$s_{\Phi} = \left\{ (a_n) \in c_0 : \|(a_n)\|_{\Phi} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \|(a_1, \dots, a_n, 0, 0, \dots)\|_{\Phi} < \infty \right\},$$

and $\|\cdot\|_{\Phi}$ is a norm on s_{Φ} . We use $s_{\Phi}^{(0)}$ to denote the closure of $\hat{c}$ with respect to $\|\cdot\|_{\Phi}$. If $\|\cdot\|_{\Phi}$ is a symmetric norm on $\hat{c}$, define the conjugate norm $\|\cdot\|_{\Phi'}$ on $\hat{c}$ by

$$\|(b_n)\|_{\Phi'} \stackrel{\text{def}}{=} \sup \left\{ \left| \sum_n a_n b_n \right| : (a_n) \in \hat{c}, \|(a_n)\|_{\Phi} \leq 1 \right\}.$$

Then $\|\cdot\|_{\Phi}$ is a symmetric norm. Finally, a symmetric norm $\|\cdot\|_{\Phi}$ is mononormalising if

$$\lim_{n \rightarrow \infty} \|(a_{n+1}, a_{n+2}, \dots)\|_{\Phi} = 0,$$

for all $(a_n) \in s_{\Phi}$.

Remark 1.2 Note that, if $\|\cdot\|_{\Phi}$ is a symmetric norm, (ii) implies that

$$\|(a_1, a_2, \dots, a_n, 0, \dots)\|_{\Phi} \leq \|(a_1, a_2, \dots, a_n, a_{n+1}, 0, \dots)\|_{\Phi},$$

for all $a_i \in \mathbb{C}$.

Remark 1.3 Two symmetric norms which we shall be particularly interested in are:

(i) l^p : $\|(a_n)\|_p = (\sum_n |a_n|^p)^{\frac{1}{p}}$ for $1 \leq p \leq \infty$;

(ii) $l^{p,q}$: $\|(a_n)\|_{p,q} = \left(\sum_n a_n^{*q} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}}$ for $1 \leq p, q \leq \infty$.

1.3 The Schatten p -Ideals

Let $\mathcal{H}$ denote a Hilbert space, assumed to be separable if infinite dimensional. We use $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ (or sometimes $\mathcal{C}_{\infty}(\mathcal{H})$) and $\mathcal{FR}(\mathcal{H})$ to denote, respectively, the sets of all bounded, compact and finite rank linear operators on $\mathcal{H}$.

Remark 1.4 Note that, in Chapter two, we shall use the symbol $\mathcal{FR}(\mathcal{B}(\mathcal{H}))$ to denote the set of finite rank elements of the C^* -algebra $\mathcal{B}(\mathcal{H})$ and this should be distinguished from $\mathcal{FR}(\mathcal{H})$; similar remarks apply to $\mathcal{K}(\mathcal{B}(\mathcal{H}))$ and so on.

We shall use $\mathcal{H}^n$ to denote the direct sum of n copies of the Hilbert space, allowing us to identify $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$ in the usual manner.

If $A \in \mathcal{K}(\mathcal{H})$, then the operator $(A^*A)^{\frac{1}{2}}$ is unique, compact and positive and we use $s(A) = (s_n(A))$ to denote the sequence of (necessarily non-negative) eigenvalues of the operator $(A^*A)^{\frac{1}{2}}$, arranged in decreasing order of magnitude and repeated according to multiplicity. We shall refer to the $s_n(A)$ as the **s-numbers** of the operator A .

Definition 1.5 Let $\|\cdot\|_{\Phi}$ be a symmetric norm. We define the spaces

$$\mathcal{C}_{\Phi}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : s(T) \in s_{\Phi}\},$$

and

$$\mathcal{C}_{\Phi}^{(0)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : s(T) \in s_{\Phi}^{(0)}\}.$$

Set

$$\|T\|_{\Phi} = \|s(T)\|_{\Phi} = \lim_{n \rightarrow \infty} \min \left(\|T\|, \|T - F_1\|, \dots, \|T - F_n\|, 0, 0, \dots \right)_{\Phi},$$

where the minimum is taken over all sequences (F_n) where F_n has rank at most n and $(\|T\|, \|T - F_1\|, \|T - F_2\|, \dots) \in s_\Phi$ or $s_\Phi^{(0)}$ as appropriate. Such a sequence will be referred to in the sequel as a Φ -associated sequence. A Φ -associated sequence (F_n) for T will be said to be minimal if $s_{n+1}(T) = \|T - F_n\|$ for all $n \geq 0$. The spaces $C_\Phi(\mathcal{H})$ and $C_\Phi^{(0)}(\mathcal{H})$ will be referred to as unitary matrix spaces. In the cases that the s -numbers lie in l^p or $l^{p,q}$, we use the notation $C_p(\mathcal{H})$ and $C_{p,q}(\mathcal{H})$ and call these the spaces of C_p and Lorentz operators respectively.

Theorem 1.6 Let $\|\cdot\|_\Phi$ be a symmetric norm. Then:

(i) $\|\cdot\|_\Phi$ is a norm on both $C_\Phi(\mathcal{H})$ and $C_\Phi^{(0)}(\mathcal{H})$. Moreover, with respect to this norm, both $C_\Phi(\mathcal{H})$ and $C_\Phi^{(0)}(\mathcal{H})$ are Banach spaces. Also, $C_2(\mathcal{H})$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle A, B \rangle = \tau(B^* A),$$

where τ denotes the trace on $C_1(\mathcal{H})$ and the norm derived from this inner product is $\|\cdot\|_2$;
(ii) $C_\Phi^{(0)}(\mathcal{H})$ is the closure of $\mathcal{FR}(\mathcal{H})$ with respect to $\|\cdot\|_\Phi$. Also, $C_\Phi(\mathcal{H}) = C_\Phi^{(0)}(\mathcal{H})$ if and only if $\|\cdot\|_\Phi$ is mononormalising;
(iii) $C_\Phi(\mathcal{H})$ and $C_\Phi^{(0)}(\mathcal{H})$ are 2-sided, $*$ -ideals. In both cases, the involution is isometric and we have the estimate

$$\|RST\|_\Phi \leq \|R\| \|S\|_\Phi \|T\|,$$

whenever $R, T \in B(\mathcal{H})$ and $S \in C_\Phi(\mathcal{H})$ or $C_\Phi^{(0)}(\mathcal{H})$;

(iv) for $1 \leq p \leq q < \infty$, $C_p(\mathcal{H}) \subseteq C_q(\mathcal{H})$ and $\|A\|_q \leq \|A\|_p$ for all $A \in C_p(\mathcal{H})$;

(v) if $T \in K(\mathcal{H})$, then for all $n \geq 0$,

$$s_{n+1}(A) = \min_{\text{rank } F \leq n} \|A - F\|,$$

where $\text{rank } F = \dim \{Fh : h \in \mathcal{H}\}$ and $\dim\{X\} = 0$ is interpreted as $X = \{0\}$;

(vi) $T \in C_\Phi(\mathcal{H}) \Leftrightarrow (T^*T)^{\frac{1}{2}} \in C_\Phi(\mathcal{H})$.

Proposition 1.7 (Hölder's Inequality) Choose symmetric norms $\|\cdot\|_{\Phi_1}$, $\|\cdot\|_{\Phi_2}$ and $\|\cdot\|_{\Phi_3}$ such that, if $(x_n) \in s_{\Phi_2}$ and $(y_n) \in s_{\Phi_3}$, then $(x_n y_n) \in s_{\Phi_1}$ and

$$\|(x_n y_n)\|_{\Phi_1} \leq \|(x_n)\|_{\Phi_2} \|(y_n)\|_{\Phi_3}.$$

Then:

(i) if $A \in C_{\Phi_2}(\mathcal{H})$ and $B \in C_{\Phi_3}(\mathcal{H})$, then $AB \in C_{\Phi_1}(\mathcal{H})$ and $\|AB\|_{\Phi_1} \leq \|A\|_{\Phi_2} \|B\|_{\Phi_3}$;

(ii) if $A \in C_{\Phi_2}^{(0)}(\mathcal{H})$ and $B \in C_{\Phi_3}^{(0)}(\mathcal{H})$, then $AB \in C_{\Phi_1}^{(0)}(\mathcal{H})$ and $\|AB\|_{\Phi_1} \leq \|a\|_{\Phi_2} \|b\|_{\Phi_3}$.

In particular, suppose that $1 \leq p, q, r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $A \in C_p(\mathcal{H})$ and $B \in C_q(\mathcal{H})$, then $AB \in C_r(\mathcal{H})$ and we have the estimate $\|AB\|_r \leq \|A\|_p \|B\|_q$.

Proposition 1.8 *Let $\|\cdot\|_{\Phi}$ be a symmetric norm which is not equivalent to $\|\cdot\|_1$. Then, for $B \in \mathcal{C}_{\Phi'}(\mathcal{H})$,*

$$\|B\|_{\Phi'} = \sup \left\{ |\tau(BC)| : C \in \mathcal{C}_{\Phi}^{(0)}(\mathcal{H}), \|C\|_{\Phi} \leq 1 \right\}.$$

Theorem 1.9 (Duality) *Let $\|\cdot\|_{\Phi}$ be a symmetric norm which is not equivalent to $\|\cdot\|_1$. Then there is an isometric isomorphism from $\mathcal{C}_{\Phi'}(A)$ onto $\left(\mathcal{C}_{\Phi}^{(0)}(\mathcal{H})\right)^*$, implemented by the map*

$$T \in \mathcal{C}_{\Phi'}(A) \mapsto f_T \in \left(\mathcal{C}_{\Phi}^{(0)}(\mathcal{H})\right)^*, \quad (1.1)$$

where $f_T(S) = \tau(ST)$ for all $S \in \mathcal{C}_{\Phi}^{(0)}(\mathcal{H})$.

In particular:

- (i) $(\mathcal{K}(\mathcal{H}))^* \cong \mathcal{C}_1(\mathcal{H})$;
- (ii) $(\mathcal{C}_p(\mathcal{H}))^* \cong \mathcal{C}_q(\mathcal{H})$ where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;
- (iii) $(\mathcal{C}_1(\mathcal{H}))^* \cong \mathcal{B}(\mathcal{H})$.

In each case, the duality is implemented by the map defined in (1.1).

Finally, $\mathcal{C}_{\Phi}(\mathcal{H})$ is reflexive if and only if both $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\Phi'}$ are mononormalising.

Theorem 1.10 (Riesz-Thorin Convexity Theorem) *Let $\mathcal{D}$ be a dense subspace of $\mathcal{C}_1(\mathcal{H})$ and $\psi : \mathcal{D} \rightarrow \mathcal{B}(\mathcal{H})$ be a linear map. Let*

$$\|\psi\|_{p,q} = \sup \left\{ \|\psi(A)\|_q : A \in \mathcal{D}, \|A\|_p \leq 1 \right\}.$$

Then as a function of (p^{-1}, q^{-1}) , $\log \|\psi\|_{p,q}$ is convex on the unit square $[0, 1] \times [0, 1]$.

Let T_R and T_I denote the real and imaginary parts respectively of an operator $T \in \mathcal{B}(\mathcal{H})$, that is to say, $T_R = \frac{1}{2}(T + T^*)$ and $T_I = \frac{1}{2i}(T - T^*)$.

Definition 1.11 *A Volterra operator T on a Hilbert space is a completely continuous quasi-nilpotent operator, that is to say, it is a compact operator for which the following three equivalent conditions hold:*

- (i) *the spectral radius of T is zero;*
- (ii) *the spectrum of T consists only of 0;*
- (iii) $\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = 0$.

We use $\mathcal{V}(\mathcal{H})$ to denote the set of all Volterra operators on $\mathcal{H}$.

Theorem 1.12 (Macaev) *Let $T \in \mathcal{V}(\mathcal{H})$ such that $T_I \in \mathcal{C}_p(\mathcal{H})$ for some $1 < p < \infty$. Then $T_R \in \mathcal{C}_p(\mathcal{H})$ and we have the estimate*

$$\|T_R\|_p \leq \gamma_p \|T_I\|_p,$$

where the constant γ_p depends only on p .

Remark 1.13 Proofs of the above results will be found in [3], [13], [15], [20] and [23].

1.4 Matricially Normed Spaces

Let V be a vector space over $\mathbb{C}$. If $(v_{ij}) \in M_n(V)$ and $(w_{ij}) \in M_m(V)$, we write

$$(v_{ij}) \oplus (w_{ij}) = \begin{bmatrix} (v_{ij}) & \theta \\ \theta & (w_{ij}) \end{bmatrix} \in M_{n+m}(V),$$

where θ denotes a rectangular matrix (of the appropriate size) of zero elements over V .

Definition 1.14 *If, for each $n \in \mathbb{N}$, there is a norm $\|\cdot\|_n$ on $M_n(V)$, the family of norms $\{\|\cdot\|_n\}$ is called a matrix norm. V is called a space with matrix norm and we use the notation $(V, \{\|\cdot\|_n\})$. We say that a space V with a matrix norm is a matricially normed space if, for all $(\alpha_{ij}) \in M_n(\mathbb{C})$, $(v_{ij}) \in M_n(V)$, $\theta \in M_m(V)$ and $n, m \in \mathbb{N}$:*

- (i) $\|(v_{ij}) \oplus \theta\|_{n+m} = \|(v_{ij})\|_n$;
- (ii) $\|(\alpha_{ij})(v_{ij})\|_n, \|(v_{ij})(\alpha_{ij})\|_n \leq \|(\alpha_{ij})\| \|(v_{ij})\|_n$.

Remark 1.15 (i) Not all authors make the above distinction between a matricially normed space and a space with a matrix norm;

(ii) Some authors take their definition of a matricially normed space to be that there are norms defined on all $m \times n$ matrices over V . If we only have norms on square matrices, we can introduce a norm $\|\cdot\|_{m,n}$ on the space $M_{m,n}(V)$ of $m \times n$ matrices over V by

$$\|(v_{ij})\|_{m,n} = \begin{cases} \|[(v_{ij}), \theta]\|_m & \text{if } m > n \\ \|(v_{ij})\|_m & \text{if } m = n \\ \left\| \begin{bmatrix} (v_{ij}) \\ \theta \end{bmatrix} \right\|_n & \text{if } m < n. \end{cases}$$

If the matricial norms $\|\cdot\|_{m,n}$ are already defined and we introduce a new norm $\|\cdot\|'_{m,n}$ by the above procedure, then it is well-known that $\|\cdot\|'_{m,n} = \|\cdot\|_{m,n}$.

Definition 1.16 *For $1 \leq p < \infty$, we say that a matricially normed space $(V, \{\|\cdot\|_n\})$ is an L^p -matricially normed space if*

$$\|(v_{ij}) \oplus (w_{ij})\|_{n+m} = (\|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p)^{\frac{1}{p}},$$

for all $(v_{ij}) \in M_n(V)$ and $(w_{ij}) \in M_m(V)$ and is an L^∞ -matricially normed space if

$$\|(v_{ij}) \oplus (w_{ij})\|_{n+m} = \max \{ \|(v_{ij})\|_n, \|(w_{ij})\|_m \},$$

for all $(v_{ij}) \in M_n(V)$ and $(w_{ij}) \in M_m(V)$.

In particular, $\mathcal{C}_p(\mathcal{H})$ is an L^p -matricially normed space ($1 \leq p < \infty$) whilst $\mathcal{K}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are both L^∞ -matricially normed spaces. In the sequel, we shall use $(X, \|\cdot\|_p)$ to denote the space $X \subseteq \mathcal{C}_p(\mathcal{H})$ equipped with the matricial Schatten p -norm.

Definition 1.17 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. We identify $M_n(V^*)$ with $M_n(V)^*$ by

$$\langle (v_{ij}), (f_{ij}) \rangle = \sum_{i,j=1}^n \langle v_{ij}, f_{ij} \rangle.$$

We use the dual norm on $M_n(V)^*$ to induce a matrix norm $\{\|\cdot\|_n^*\}$ on $M_n(V^*)$ and we say that $(V^*, \{\|\cdot\|_n^*\})$ is the matricial dual space of V .

The following results are well-known (see [21], for example).

Proposition 1.18 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Then:

(i) for all $(\alpha_{ij}) \in M_{m,n}(\mathbb{C})$, $(v_{ij}) \in M_n(V)$ and $(\beta_{ij}) \in M_{n,m}(\mathbb{C})$,

$$\|(\alpha_{ij})(v_{ij})(\beta_{ij})\|_m \leq \|(\alpha_{ij})\| \| (v_{ij}) \|_n \|(\beta_{ij})\|;$$

(ii) for all $(v_{ij}) \in M_n(V)$ and all $1 \leq k, l \leq n$,

$$\|v_{kl}\|_1 \leq \| (v_{ij}) \|_n \leq \sum_{k,l=1}^n \|v_{kl}\|_1.$$

Hence V is complete under the $\|\cdot\|_1$ norm if and only if $M_n(V)$ is complete under the $\|\cdot\|_n$ norm for $n = 2, 3, \dots$

(iii) Let $a \in M_n(V)$, $b \in M_{n,m}(V)$, $c \in M_{m,n}(V)$ and $d \in M_m(V)$ for some $n, m \in \mathbb{N}$. Then

$$\left\| \begin{pmatrix} a & \theta \\ \theta & d \end{pmatrix} \right\|_{n+m} \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{n+m}.$$

(iv) The matricial dual space $(V^*, \{\|\cdot\|_n^*\})$ is a matricially normed space. Moreover, if V is an L^p -matricially normed space for some $1 \leq p \leq \infty$, then $(V^*, \{\|\cdot\|_n^*\})$ is an L^q -matricially normed space where q is the conjugate index to p .

Definition 1.19 Let V and W be matricially normed spaces and $T : V \rightarrow W$ be a linear map. For all $n \geq 2$, define a map

$$T_n : (v_{ij}) \in M_n(V) \mapsto (Tv_{ij}) \in M_n(W).$$

Then

$$T \text{ is } \begin{cases} \text{completely bounded if } \|T\|_{cb} \stackrel{\text{def}}{=} \sup \{\|T_n\| : n \in \mathbb{N}\} < \infty, \\ \text{a complete contraction if } \|T\|_{cb} \leq 1, \\ \text{a complete isometry if } T_n \text{ is an isometry } \forall n \in \mathbb{N}. \end{cases}$$

Definition 1.20 A matricially normed space $(V, \{\|\cdot\|_n\})$ is an (abstract) operator space if it is completely isometric to a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 1.21 (Ruan) Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Then V is an operator space if and only if it is an L^∞ -matricially normed space.

Proof. See [9] or [21, Theorem 3.1]. ■

Theorem 1.22 (Shin, Lee, Byun, Kim) *A matricially normed space $(V, \{\|\cdot\|_n\})$ is an operator space if and only if*

$$\|(x_{ij})\|_{s,t} \leq \left\| \left[\|x_{ij}\|_{k_i, l_j} \right] \right\|,$$

for each $(x_{ij}) \in M_{s,t}(V)$ where $s = k_1 + \dots + k_m$ and $t = l_1 + \dots + l_n$.

Proof. See [22, Theorem 2.6]. ■

1.5 Interpolation Theory

Here, we give a (very) brief sketch of the real, complex and approximation methods of interpolation (which we shall require for Chapters two and three). Full details and auxiliary results are contained in [4, Chapters 2, 3, 4 and 7].

Definition 1.23 *A category $\mathcal{C}$ consists of objects $A, B, \dots$ and morphisms $S, T, \dots$. Between objects and morphisms, a three place relation is defined, namely $T : A \rightsquigarrow B$. If $T : A \rightsquigarrow B$ and $S : B \rightsquigarrow C$, then there is a morphism ST , the product of S and T , such that $ST : A \rightsquigarrow C$. The product of morphisms satisfies the associative law*

$$T(SR) = (TS)R.$$

Moreover, for any object A in $\mathcal{C}$, there is a morphism $I = I_A$ such that for all morphisms $T : A \rightsquigarrow A$, we have $TI = IT = T$.

A **functor** is a map F between two categories $\mathcal{C}$ and $\mathcal{D}$ which sends an object A and a morphism T of $\mathcal{C}$ to an object $F(A)$ and a morphism $F(T)$ of $\mathcal{D}$. If $T : A \rightsquigarrow B$, then $F(T) : F(A) \rightsquigarrow F(B)$ and

$$F(ST) = F(S)F(T), \quad F(I_A) = I_{F(A)}.$$

If $\mathcal{C}$ and $\mathcal{D}$ are the categories of all topological vector spaces and all finite dimensional Euclidean spaces respectively, for example, the morphisms are the continuous linear operators. If we define $F(A) = A$ and $F(T) = T$, then F is a functor from $\mathcal{D}$ into $\mathcal{C}$.

We use $\mathcal{N}$ and $\mathcal{B}$ to denote the categories of normed vector spaces and Banach spaces respectively; in both categories, the morphisms are the bounded linear operators.

Definition 1.24 (a) *Let A_0 and A_1 be normed vector spaces. We say that they are compatible if there is a Hausdorff topological space $\mathcal{U}$ such that A_0 and A_1 are subspaces of $\mathcal{U}$. Then we can form the spaces $A_0 \cap A_1$ and $A_0 + A_1$. These are normed vector spaces with respect to the norms $\|\cdot\|_{A_0 \cap A_1}$ and $\|\cdot\|_{A_0 + A_1}$ respectively where*

$$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1}) \quad \text{and} \quad \|a\|_{A_0 + A_1} = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}).$$

Moreover, if A_0 and A_1 are complete, then $A_0 \cap A_1$ and $A_0 + A_1$ are also complete.

(b) Let $\mathcal{C}$ denote a sub-category of $\mathcal{N}$. We let $\mathcal{C}_1$ denote the category of compatible couples $\overline{A} = (A_0, A_1)$, that is to say, such that A_0 and A_1 are compatible and $A_0 \cap A_1$ and $A_0 + A_1$ are spaces in $\mathcal{C}$;

(c) two basic functors from $\mathcal{C}_1$ to $\mathcal{C}$ which we can define are

$$\Delta(\overline{A}) = A_0 \cap A_1 \text{ and } \Sigma(\overline{A}) = A_0 + A_1.$$

(d) Let $\overline{A} = (A_0, A_1)$ be a couple in $\mathcal{C}_1$. A space A in $\mathcal{C}$ is an intermediate space between A_0 and A_1 (or with respect to $\overline{A}$) if

$$\Delta(\overline{A}) \subseteq A \subseteq \Sigma(\overline{A}),$$

with continuous inclusions.

(e) Suppose that (A_1, A_2) and (B_1, B_2) are couples in some category $\mathcal{C}$ and that A and B are intermediate spaces for (A_1, A_2) and (B_1, B_2) respectively. We say that $\langle A, B \rangle$ is an interpolation pair for $\langle (A_1, A_2), (B_1, B_2) \rangle$ (or that A and B are interpolation spaces with respect to (A_1, A_2) and (B_1, B_2)) if every operator T which is bounded from (A_1, A_2) into (B_1, B_2) —that is to say, $T : A_1 \rightarrow B_1$ and $T : A_2 \rightarrow B_2$ boundedly—is also bounded from A into B . In the case that $\langle A, B \rangle$ is an interpolation pair for $\langle (A_1, A_2), (B_1, B_2) \rangle$, there is a constant M such that

$$\|T\|_{\mathcal{B}(A, B)} \leq M \|T\|_{1,2},$$

for every bounded operator $T : (A_1, A_2) \rightarrow (B_1, B_2)$ where

$$\|T\|_{1,2} = \max \left\{ \|T\|_{\mathcal{B}(A_1, B_1)}, \|T\|_{\mathcal{B}(A_2, B_2)} \right\}.$$

The smallest such M is called the interpolation constant of $\langle A, B \rangle$ and is denoted by

$$M \{ \langle A, B \rangle ; \langle (A_1, A_2), (B_1, B_2) \rangle \};$$

(f) A and B are exact interpolation spaces if

$$\|T\|_{A, B} \leq \max \left\{ \|T\|_{A_0, B_0}, \|T\|_{A_1, B_1} \right\},$$

and are exact of exponent θ if

$$\|T\|_{A, B} \leq \|T\|_{A_0, B_0}^{1-\theta} \|T\|_{A_1, B_1}^{\theta};$$

(g) a functor F from $\mathcal{C}_1$ into $\mathcal{C}$ is an interpolation functor if, whenever $\overline{A}$ and $\overline{B}$ are couples in $\mathcal{C}_1$, then $F(\overline{A})$ and $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. F is exact of exponent θ if $F(\overline{A})$ and $F(\overline{B})$ are exact of exponent θ and F is exact if $F(\overline{A})$ and $F(\overline{B})$ are exact.

1.5.1 The Real Method $K_{\theta,q}$

In this subsection, we consider the category $\mathcal{N}$. For a couple $\overline{A} = (A_0, A_1)$, define

$$K(t, a) = \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1} \right),$$

for all $a \in \Sigma(\overline{A})$.

Fix θ and q in the ranges $0 < \theta < 1, 1 \leq q \leq \infty$ or $0 \leq \theta \leq 1, q = \infty$. Define $\Phi_{\theta,q}$ by

$$\Phi_{\theta,q}(\varphi(t)) \stackrel{\text{def}}{=} \begin{cases} \left(\int_0^\infty \left[t^{-\theta} \varphi(t) \right]^q dt/t \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t>0} t^{-\theta} \varphi(t) & \text{if } q = \infty, \end{cases}$$

where $\varphi(t)$ is a non-negative function.

Definition 1.25 Define

$$\overline{A}_{\theta,q;K} = K_{\theta,q}(\overline{A}) = \left\{ a \in \Sigma(\overline{A}) : \Phi_{\theta,q}(K(t, a)) < \infty \right\},$$

and define a norm $\|\cdot\|_{\theta,q;K}$ by

$$\|a\|_{\theta,q;K} = \Phi_{\theta,q}(K(t, a)).$$

Theorem 1.26 $K_{\theta,q}$ is an exact interpolation functor of exponent θ on the category $\mathcal{N}$.

1.5.2 The Complex Method C_θ

Now let us change to the category $\mathcal{B}_1$ of compatible Banach couples. Given a couple $\overline{A} = (A_0, A_1)$, define $\mathcal{F}(\overline{A})$ to be the space of functions f with values in $A_0 + A_1$ which:

- (i) are bounded and continuous on the strip $S = \{z : 0 \leq \Re z \leq 1\}$;
- (ii) are analytic on the open strip $S_0 = \{z : 0 < \Re z < 1\}$;
- (iii) for $j = 0$ and 1 , the functions $t \mapsto f(j + it)$ are continuous from the real line into A_j and tend to zero as $|t| \rightarrow 0$.

Then $\mathcal{F}(\overline{A})$ is a vector space. Moreover, it is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{F}}$ given by

$$\|f\|_{\mathcal{F}} = \max \left(\sup_{t_1} \|f(it_1)\|_{A_0}, \sup_{t_2} \|f(1 + it_2)\|_{A_1} \right).$$

Definition 1.27 Suppose $0 < \theta < 1$. We use $C_\theta(\overline{A}) = \overline{A}_{[\theta]}$ to denote the space of all $a \in \Sigma(\overline{A})$ such that $a = f(\theta)$ for some $f \in \mathcal{F}(\overline{A})$. We define a norm $\|\cdot\|_{[\theta]}$ on $\overline{A}_{[\theta]}$ by

$$\|a\|_{[\theta]} = \inf \left\{ \|f\|_{\mathcal{F}} : f(\theta) = a, f \in \mathcal{F}(\overline{A}) \right\}.$$

Theorem 1.28 The space $\overline{A}_{[\theta]}$ is a Banach space and an intermediate space with respect to $\overline{A}$. The interpolation functor C_θ is an exact interpolation functor of exponent θ .

1.5.3 The Complex Method C^θ

We remain in $\mathcal{B}_1$. Let $\mathcal{G}(\overline{A})$ consist of those functions which are defined on the strip S , take values in $\Sigma(\overline{A})$ and which satisfy the following:

- (i) there is a constant c such that $\|g(z)\|_{\Sigma(\overline{A})} \leq c(1 + |z|)$ for all z ;
- (ii) g is continuous on S and analytic on S_0 ;
- (iii) $g(j + it_1) - g(j + it_2)$ has values in A_j for all $t_1, t_2 \in \mathbb{R}$ and for $j = 0, 1$ and

$$\|g\|_{\mathcal{G}} = \max \left(\sup_{t_1, t_2} \left\| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right\|_{A_0}, \sup_{t_3, t_4} \left\| \frac{g(1 + it_3) - g(1 + it_4)}{t_3 - t_4} \right\|_{A_1} \right),$$

is finite.

The space $\mathcal{G}(\overline{A})$, reduced modulo constant functions and equipped with the norm $\|\cdot\|_{\mathcal{G}}$, is a Banach space.

Definition 1.29 For $0 < \theta < 1$, we define the space $C^\theta(\overline{A}) = \overline{A}^{[\theta]}$ as the space which consists of all $a \in \Sigma(\overline{A})$ such that $a = g'(\theta)$ for some $g \in \mathcal{G}(\overline{A})$ and define a norm on $\overline{A}^{[\theta]}$ by

$$\|a\|^{[\theta]} = \inf \{ \|g\|_{\mathcal{G}} : g \in \mathcal{G}, a = g'(\theta) \}.$$

Theorem 1.30 The space $\overline{A}^{[\theta]}$ is a Banach space and an intermediate space with respect to $\overline{A}$. The interpolation functor C^θ is an exact interpolation functor of exponent θ .

1.5.4 The Approximation Method $E_{\alpha, q}$

Definition 1.31 Let G be an abelian group whose operation is denoted by $+$, the inverse of $g \in G$ is $-g$ and the identity is 0 . An $\mathbb{R}$ -valued function $\|\cdot\|_G$ is a quasi-norm if, for all $g, h \in G$:

- (a) $\|g\|_G \geq 0$ and $\|g\|_G = 0 \Leftrightarrow g = 0$;
- (b) $\|-g\|_G = \|g\|_G$;
- (c) $\|g + h\|_G \leq c(\|g\|_G + \|h\|_G)$ where $c \geq 1$.

Definition 1.32 If G and H are quasi-normed abelian groups, a mapping $T : G \rightarrow H$ is called a homomorphism if $T(f + g) = T(f) + T(g)$ for all $f, g \in G$.

Here we consider the category $\mathcal{G}$ of all quasi-normed abelian groups and we take the morphisms to be the bounded homomorphisms. The notions of compatible couples, intermediate spaces, interpolation spaces and interpolation functors retain their same meanings.

Given a couple $\overline{A} = (V, W)$ in $\mathcal{G}_1$ and an element $a \in V + W$, define

$$E(t, a) = E(t, a, V, W) = \inf_{\|v\|_V \leq t} \|a - v\|_W,$$

where $0 < t < \infty$.

Definition 1.33 Let $\overline{A} = (V, W)$ be a compatible couple of quasi-normed abelian groups. We define the approximation space

$$E_{\alpha,q}(V, W) = \left\{ a \in \Sigma(\overline{A}) : \|a\|_{\alpha,q;E} \stackrel{\text{def}}{=} \Phi_{-\alpha,q}(E(t, a)) < \infty \right\},$$

where $0 < \alpha < \infty$, $0 < q \leq \infty$ or $\alpha = 0$, $q = 0$ and $\Phi_{\theta,r}$ is as defined in Section 1.5.1.

Theorem 1.34 (i) Let $\overline{A}$ be a compatible quasi-normed couple and set $\theta = \frac{1}{\alpha+1}$ and $r = \theta q$. Then

$$\left(E_{\alpha,r}(\overline{A}) \right)^\theta = K_{\theta,q}(\overline{A}),$$

with equivalent norms where we use B^ρ to denote the space B when equipped with the quasi-norm $\|\cdot\|_B^\rho$.

(ii) If $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$ and $\alpha_0 \neq \alpha_1$, then

$$K_{\theta,q} \left(E_{\alpha_0,r_0}(\overline{A}), E_{\alpha_1,r_1}(\overline{A}) \right) = E_{\alpha,q}(\overline{A}),$$

and

$$\left[E_{\beta,r} \left(E_{\alpha_0,r_0}(\overline{A}), E_{\alpha_1,r_1}(\overline{A}) \right) \right]^\theta = E_{\alpha,q}(\overline{A}),$$

with equivalent norms where $r = \theta q$ and $\beta = \frac{\alpha_1 - \alpha}{\alpha - \alpha_0}$ or, equivalently, $\theta = \frac{1}{\beta+1}$.

Chapter 2

Generalised Unitary Matrix Spaces

2.1 Introduction

In [10], Erdos constructed an isometric $*$ -representation of a C^* -algebra on a Hilbert space and gave a necessary and sufficient condition for an element of the algebra to have an operator of rank one as its image under this representation. Subsequently, with collaborators Giotopoulos and Lambrou, Erdos extended these results to a semi-simple Banach algebra (see [11]).

In this chapter, we extend Erdos' results on a C^* -algebra to give a characterisation of elements of the algebra whose images are finite rank, Schatten, Volterra, Fredholm or Riesz operators under this representation.

2.2 Erdos' $*$ -Representation

For the reader's convenience, we begin with the construction of Erdos' representation. We sketch the important steps (which we shall need to refer to later), proving only (part of) the main result (Theorem 2.9). Full details can be found in [10].

Definition 2.1 *An element a of a C^* -algebra A is single if, whenever $bac = 0$ for some $b, c \in A$, then at least one of ba, ac is zero. We use $\sigma(A)$ to denote the set of single elements of A .*

Remark 2.2 Obviously, this definition is valid for all algebras. It is clear that if $a \in \sigma(A)$ and $x \in A$, then a^*, xa and ax are all in $\sigma(A)$. For a Hilbert space $\mathcal{G}$, the single elements of $\mathcal{B}(\mathcal{G})$ are just the operators of rank at most one on $\mathcal{G}$.

Lemma 2.3 ([10], Lemma 2.2) *If $s \in \sigma(A)$ is normal, then there exists a complex number λ such that:*

- (i) $s^2 = \lambda s$;
- (ii) $s = \lambda e$ for some single self-adjoint idempotent e .

If the C^* -algebra A contains a non-zero single element s , then s^*s is non-zero, single, self-adjoint and hence normal. So there will exist a non-zero, single, self-adjoint idempotent e by Lemma 2.3.

Lemma 2.4 ([10], Theorem 2.5) *If $s, t \in \sigma(A)$, then the set $sAt = \{sat : a \in A\}$ is a zero or one dimensional linear subspace of A . Moreover, $s \in sAs$.*

From now on, we assume that the C^* -algebra A contains a non-zero, single, self-adjoint idempotent e . Consider the left ideal Ae . If $x, y \in Ae$, then $x = xe$ and $y = ye$. The proof of Lemma 2.4 tells us that there exists a complex number $\lambda(x, y)$ such that

$$y^*x = ey^*xe = \lambda(x, y)e.$$

Define

$$\langle x, y \rangle = \lambda(x, y).$$

Theorem 2.5 ([10], Theorem 3.2) *The left ideal Ae , with the C^* -algebra norm and the inner product $\langle \cdot, \cdot \rangle$ defined above, is a Hilbert space.*

For any non-zero, self-adjoint, single idempotent e , we shall use $\mathcal{H}_e$ to denote the left ideal Ae considered as a Hilbert space. Define an adjoint-preserving representation ρ_e of A into $\mathcal{B}(\mathcal{H}_e)$ by

$$\rho_e(a)x = ax,$$

for all $x \in \mathcal{H}_e$.

Proposition 2.6 ([10], Theorem 3.3) *If $s \in \sigma(A)$, then $\rho_e(s)$ has rank one or rank zero. Every rank one operator on $\mathcal{H}_e$ is the image under ρ_e of some element of AeA . Moreover, if $s \in AeA$, then*

$$\|\rho_e(s)\| = \|s\|.$$

Definition 2.7 *A non-zero representation ρ of A on a Hilbert space $\mathcal{H}$ is said to be (topologically) irreducible if the set $\{\rho(a)h : a \in A\}$ is dense in $\mathcal{H}$ for all non-zero $h \in \mathcal{H}$.*

By Proposition 2.6, $\rho_e(A)$ contains every rank one operator on $\mathcal{H}_e$ and hence ρ_e is irreducible.

Lemma 2.8 ([10], Theorem 3.5 and Corollary 3.6) *(i) If ρ is a non-zero, continuous, irreducible representation of A on a Hilbert space $\mathcal{H}$ for which there is a single element s in A such that $\rho(s) \neq 0$, then ρ is similar to ρ_e where e is the single, self-adjoint idempotent such that $s = se$. In the case that ρ is also adjoint-preserving, we*

can replace similarity with unitary equivalence;

(ii) For any non-zero single element $s \in A$, there is one and only one unitary equivalence class $[\rho]$ of irreducible, adjoint-preserving representations such that $\rho(s) \neq 0$ for all $\rho \in [\rho]$. Also, if $\rho \in [\rho]$, then $\rho(s)$ has rank one and $\|\rho(s)\| = \|s\|$.

We are now in a position to state the main result ([10, Theorem 3.7]).

Theorem 2.9 (Erdos) *There exists an isometric $*$ -representation of the C^* -algebra A such that the image of each non-zero single element has rank one.*

Proof. Consider the set of all continuous, irreducible, adjoint-preserving representations ρ of A for which there exist a single element $s \in A$ such that $\rho(s) \neq 0$. (This is the condition of Lemma 2.8(i).) Divide this collection into its unitary equivalence classes. Choose one ρ from each equivalence class, find a single element s such that $\rho(s) \neq 0$ and a single, self-adjoint idempotent e such that $s = se$. Let ξ denote the set of all such e 's.

Let $\{[\rho_\gamma] : \gamma \in \Gamma\}$ be the set of all unitary equivalence classes of irreducible, adjoint-preserving representations of A where Γ is some indexing set. Choose one representative from each equivalence class to form a set $\{\rho_\gamma : \gamma \in \Gamma\}$ and note that we can think of Γ as being the union of the sets ξ and $\Gamma \setminus \xi$.

For each $\gamma \in \Gamma$, let $\mathcal{H}_\gamma$ be the Hilbert space of ρ_γ , that is to say,

$$\mathcal{H}_\gamma = \begin{cases} A\gamma \text{ considered as a Hilbert space} & \text{if } \gamma \in \xi \\ \text{GNS construction} & \text{if } \gamma \in \Gamma \setminus \xi. \end{cases}$$

Define a Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \{\oplus \mathcal{H}_e : e \in \xi\} \oplus \{\oplus \mathcal{H}_\gamma : \gamma \in \Gamma \setminus \xi\},$$

and a $*$ -representation π of A on $\mathcal{H}$ by

$$\pi(a) = \{\oplus \rho_e(a) : e \in \xi\} \oplus \{\oplus \rho_\gamma(a) : \gamma \in \Gamma \setminus \xi\},$$

for all $a \in A$.

If s is a non-zero single element of A , Lemma 2.8(ii) tells us that $\rho_e(s) \neq 0$ for exactly one $e \in \xi$ and that $\rho_\gamma(s) = 0$ for every $\gamma \in \Gamma \setminus \xi$. Moreover, for this particular $e \in \xi$, $\rho_e(s)$ has rank one. The proof that π is an isometry is omitted. ■

Throughout the rest of this chapter, π and $\mathcal{H}$ will be the $*$ -representation and the Hilbert space of the preceding Theorem.

2.3 Single Elements of Algebras

In this section, we consider some properties of single elements of an arbitrary algebra A that we shall require later on. A key consideration is the relationship between $M_n(\sigma(A))$ and $\sigma(M_n(A))$. (Note that, in Proposition 3.24, we give a partial mechanism for classifying single scalar matrices.)

Lemma 2.10 *Let A be an algebra. For all $n \in \mathbb{N}$, $\sigma(M_n(A)) \subseteq M_n(\sigma(A))$.*

Proof. Let $(a_{ij}) \in \sigma(M_n(A))$ and fix $1 \leq k, l \leq n$ where $n \geq 2$. Suppose there exist $b, c \in A$ such that $ba_{kl}c = 0$. Then $(b \otimes E_{kk})(a_{ij})(c \otimes E_{ll}) = 0$ where $\{E_{ij}\}_{i,j=1}^n$ are the matrix units. Since (a_{ij}) is single, we have that $(b \otimes E_{kk})(a_{ij}) = 0$ or $(a_{ij})(c \otimes E_{ll}) = 0$. Thus, either $ba_{kl} = 0$ or $a_{kl}c = 0$, and hence a_{ij} is single. ■

Lemma 2.11 *Let $(a_{ij}) \in \sigma(M_n(A))$ and $(\lambda_{ij}), (\mu_{ij}) \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Then $(\lambda_{ij})(a_{ij})(\mu_{ij}) \in \sigma(M_n(A))$.*

Proof. This is clear since $B = M_n(A)$ is an algebra in its own right. ■

Proposition 2.12 *Let A be a C^* -algebra and $(\alpha_{ij}) \in M_n(A)$. Then (α_{ij}) is single if and only if $(a_{ij}) = (\lambda_{ij}a)$ for some $a \in \sigma(A)$ and $(\lambda_{ij}) \in \sigma(M_n(\mathbb{C}))$.*

Proof. Suppose that (a_{ij}) is single. If (a_{ij}) is zero, there is nothing to prove so suppose that (a_{ij}) is non-zero with, without loss of generality, $a_{11} \neq 0$. Then, for each (i, j) , there exist single, self-adjoint idempotents e_{ij} and f_{ij} such that $a_{ij} = f_{ij}a_{ij}e_{ij}$.

Now, the $(1, 1)$ entry of the product $(a_{ij})(E_{11} + E_{k1})$ is $a_{11} + a_{1k}$, which is single (though possibly zero) and so $\pi(a_{11} + a_{1k}) = \pi(a_{11}) + \pi(a_{1k})$ is an operator of rank at most one. Hence, if a_{1k} is non-zero, the operators $\pi(a_{11})$ and $\pi(a_{1k})$ must act on the same $\mathcal{H}_e$ for some $e \in \xi$ and so $e_{11} = e_{1k}$.

Similarly, the $(1, 1)$ entry of the product $(E_{11} + E_{1k})(a_{ij})$ is the single element $a_{11} + a_{k1}$ and, by a similar argument, we have that $e_{11} = e_{k1}$.

Also, the $(1, 1)$ entry of the product $(E_{11} + E_{1k})(a_{ij})(E_{11} + E_{l1})$ is the single element $a_{11} + a_{k1} + a_{1l} + a_{kl}$. Now $\pi(a_{11} + a_{k1} + a_{1l} + a_{kl}) = \pi(a_{11}) + \pi(a_{k1}) + \pi(a_{1l}) + \pi(a_{kl})$ and the first three operators on the right hand side all act on the same $\mathcal{H}_e$ by the above arguments (unless they are zero). Hence, if a_{kl} is non-zero, we must have that $e_{11} = e_{kl}$.

So there exists a single, self-adjoint idempotent e such that $e_{ij} = e$ for all i, j . If we consider the single element $(a_{ij})^* = (a_{ji}^*)$ of $M_n(A)$, then $a_{ij}^* = e_{ij}a_{ij}^*f_{ij}$ and the above arguments tell us that there exists a single, self-adjoint idempotent f such that $f_{ij} = f$ for all i, j .

Then, for each (i, j) , $a_{ij} = fa_{ij}e \in fAe$, which is a linear subspace of dimension one (since a_{11} is non-zero and belongs to fAe). So there exists a single element $a \in fAe$ and scalars λ_{ij} such that $a_{ij} = \lambda_{ij}a$.

It remains to show that (λ_{ij}) is single. Let $(\mu_{ij}), (\gamma_{ij}) \in M_n(\mathbb{C})$ and suppose that $(\mu_{ij})(\lambda_{ij})(\gamma_{ij}) = 0$. Then $(\mu_{ij}a^*)(\lambda_{ij}a)(\gamma_{ij}a^*) = 0$ and hence $(\mu_{ij})(\lambda_{ij})a^*a = 0$ or $(\lambda_{ij})(\gamma_{ij})aa^* = 0$ since $(\lambda_{ij}a)$ is single. So $(\mu_{ij})(\lambda_{ij}) = 0$ or $(\lambda_{ij})(\gamma_{ij}) = 0$ since $a^*a, aa^* \neq 0$. Thus (λ_{ij}) is single.

The other implication is obvious. ■

Finally, we make a few observations (none of which will be used in the rest of this chapter) concerning the unitisation of a C^* -algebra. Given a C^* -algebra A , we can

adjoin a unit by the usual method to get A_e . What relationships are there between the single elements of the algebra A and the single elements of A_e ?

Lemma 2.13 *Suppose that the C^* -algebra A contains a non-zero single element. Then $(0, \lambda)$ is not single in A_e for any $\lambda \neq 0$.*

Proof. If A has a non-zero single element, then by the remarks following Lemma 2.3, A contains a non-zero, self-adjoint, single idempotent, a , say. Then, for any non-zero $\lambda \in \mathbb{C}$, we have that

$$(a, -1)(0, \lambda)(a, 0) = (\lambda a^2 - \lambda a, 0) = (0, 0),$$

since a is an idempotent. However

$$(a, -1)(0, \lambda) = (\lambda a, -\lambda) \neq (0, 0) \text{ and } (0, \lambda)(a, 0) = (\lambda a, 0) \neq (0, 0).$$

Hence $(0, \lambda)$ is not single for $\lambda \neq 0$. ■

Lemma 2.14 *Suppose that $(a, 0) \in \sigma(A_e)$. Then $a \in \sigma(A)$.*

Proof. Suppose that $(a, 0) \in A_e$ is single and that there exist $b, c \in A$ such that $bac = 0$. Then $(b, 0)(a, 0)(c, 0) = (0, 0)$ and hence $(b, 0)(a, 0) = (0, 0)$ or $(a, 0)(c, 0) = (0, 0)$ since $(a, 0)$ is single in A_e . So $(ba, 0) = (0, 0)$ or $(ac, 0) = (0, 0)$ and thus $ba = 0$ or $ac = 0$. Hence a is single. ■

Corollary 2.15 *Let A be a non-zero C^* -algebra. Then $(0, \lambda) \in \sigma(A_e)$ if and only if $\lambda = 0$.*

Proof. The implication $\Leftarrow$ is trivial. Suppose that $(0, \lambda) \in \sigma(A_e)$ for some $\lambda \neq 0$. Then, for any non-zero $a \in A$, we have $(a, 0) = (\frac{1}{\lambda}a, 0)(0, \lambda)$ is single and hence a is single by Lemma 2.14. But then $(0, \lambda)$ is not single by Lemma 2.13, a contradiction. ■

2.4 Finite Rank Elements

From now on, unless stated otherwise, we shall assume that A is a C^* -algebra which contains a non-zero single element and which may or may not have an identity.

Definition 2.16 *We say that an element of a C^* -algebra A is **n-single** if it can be written as a sum of n single elements of the algebra.*

Definition 2.17 *We say that an element of a C^* -algebra A has **rank n** if n is the least $m \in \mathbb{N}$ for which it is m -single. An element of the algebra has **finite rank** if it has rank n for some $n \in \mathbb{N}$ and we shall use $\mathcal{FR}(A)$ to denote the set of finite rank elements of A .*

There already is a notion of a finite rank element of a C^* -algebra in the literature, namely the following (see [2], for example).

Definition 2.18 *An element x in a C^* -algebra A is said to have ω -finite rank if the wedge operator*

$$x \wedge x : a \in A \mapsto xax \in A,$$

has finite rank. We denote the set of ω -finite rank elements of A by $\mathcal{FR}_\omega(A)$.

Remark 2.19 By the remarks at the start of Chapter four of [10], we know that $\mathcal{FR}(A)$ is identical to the socle of A . Moreover, $\mathcal{FR}_\omega(A) = \text{soc}(A)$ by [2, Theorem $C^*1.2$]. Hence

Proposition 2.20 *For any C^* -algebra A , $\mathcal{FR}(A) = \mathcal{FR}_\omega(A) = \text{soc}(A)$.*

Proposition 2.21 *Let $a \in A$ be an element of rank n for some $n \in \mathbb{N}$. Then $\pi(a)$ is an operator of rank at most n on $\mathcal{H}$.*

Proof. Suppose $a = a_1 + \dots + a_n$ has rank n . Then $\pi(a) = \sum_{i=1}^n \pi(a_i)$ is the sum of n operators, each of which has rank at most one and hence $\pi(a)$ is an operator of rank at most n . ■

Proposition 2.22 *$\mathcal{FR}(A)$ is a 2-sided, $*$ -ideal in A .*

Proof. Immediate from the definition of a finite rank element and the properties of single elements which are listed in Remark 2.2. ■

Theorem 2.23 *For a Hilbert space $\mathcal{G}$, $\mathcal{FR}(\mathcal{G}) = \mathcal{FR}(\mathcal{B}(\mathcal{G}))$.*

Proof. Since the single elements of $\mathcal{B}(\mathcal{G})$ are those operators of rank at most one on $\mathcal{G}$, it follows that an operator F has rank at most n ($n \in \mathbb{N}$) as an operator on $\mathcal{G}$ if and only if F has rank at most n as an element of the C^* -algebra $\mathcal{B}(\mathcal{G})$. Hence

$$\begin{aligned} F \in \mathcal{FR}(\mathcal{G}) &\Leftrightarrow F \text{ can be expressed as a finite linear combination of} \\ &\quad \text{rank one operators on } \mathcal{G}, \\ &\Leftrightarrow F \text{ can be expressed as a finite linear combination of} \\ &\quad \text{single elements of } \mathcal{B}(\mathcal{G}), \\ &\Leftrightarrow F \in \mathcal{FR}(\mathcal{B}(\mathcal{G})). \end{aligned}$$

■

Recall that the Hilbert space $\mathcal{H}$ is defined in the proof of Theorem 2.9.

Definition 2.24 *In the sequel, we shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ satisfies $(\dagger)$ if:*
(i) $T\mathcal{H}_e \subseteq \mathcal{H}_e$ for all $e \in \xi$;
(ii) $T\mathcal{H}_\gamma \subseteq 0$ for all $\gamma \in \Gamma \setminus \xi$.

By virtue of its construction, $\pi(a)$ is a rank one operator satisfying $(\dagger)$ for all single elements $a \in A$. Hence $\pi(b)$ satisfies $(\dagger)$ for all $b \in \mathcal{FR}(A)$.

Proposition 2.25 *If $F \in \mathcal{FR}(\mathcal{H})$ satisfies $(\dagger)$, then F is the image under π of some $f \in \mathcal{FR}(A)$.*

Proof. By Proposition 2.6, we know that $\rho_e(AeA)$ contains every rank one operator on $\mathcal{H}_e$ for all $e \in \xi$. $AeA \subseteq Ae$ and every element of Ae is necessarily single. Since Ae is a vector space, it follows that $\rho_e(Ae)$ contains every operator of finite rank on $\mathcal{H}_e$ for all $e \in \xi$. Hence, if $F \in \mathcal{B}(\mathcal{H})$ is an operator of finite rank which satisfies $(\dagger)$, then there exists an $f \in \mathcal{FR}(A)$ such that $\pi(f) = F$. ■

2.5 Four Lemmas

Here we gather together four results which we shall require special cases of later on. Throughout this section, Δ will denote an indexing set. For each $\alpha \in \Delta$, $\mathcal{G}_\alpha$ is a Hilbert space and $\mathcal{G} = \oplus_{\alpha \in \Delta} \mathcal{G}_\alpha$.

Lemma 2.26 *Suppose that $R \in \mathcal{K}(\mathcal{G})$ such that $R(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$. Then, among the best approximations with rank at most n to R (with respect to the operator norm), there is at least one, K_n say, such that $K_n(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$.*

Proof. $R = \oplus_{\alpha \in \Delta} R_\alpha$ where $R_\alpha \in \mathcal{K}(\mathcal{G}_\alpha)$ for each $\alpha \in \Delta$. Now, for each $\alpha \in \Delta$, we have a Schmidt representation $R_\alpha = \sum_{j \in J_\alpha} s_j(R_\alpha) \langle \cdot, \phi_j^{(\alpha)} \rangle \psi_j^{(\alpha)}$ where J_α is some indexing set.

Since R is compact, the set $\{s_j(R_\alpha)\}_{j \in J_\alpha, \alpha \in \Delta}$ is a set of positive numbers which is either finite or is countable and can be arranged in decreasing order so that the entries in the sequence converge to zero.

Suppose that $s_{j_1}(R_{\alpha_1}), \dots, s_{j_n}(R_{\alpha_n})$ are the first n entries in such a re-arrangement where $\alpha_1, \dots, \alpha_n \in \Delta$.

Identify each of the operators

$$s_{j_1}(R_{\alpha_1}) \langle \cdot, \phi_{j_1}^{(\alpha_1)} \rangle \psi_{j_1}^{(\alpha_1)}, \dots, s_{j_n}(R_{\alpha_n}) \langle \cdot, \phi_{j_n}^{(\alpha_n)} \rangle \psi_{j_n}^{(\alpha_n)},$$

with their analogous counterpart which act on the whole of $\mathcal{G}$ and let K_n be the sum of these n operators.

The K_n is an operator of rank at most n and $K_n(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$. Moreover, $\|R - K_n\| = s_{j_{n+1}}(R_{\alpha_{n+1}})$, the $(n+1)^{\text{st}}$ largest of the s -numbers of R . ■

Remark 2.27 Note that there could well be other best approximations which do not satisfy $(\dagger)$: in the case of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are equally good approximations from the set of operators which have rank at most one.

Lemma 2.28 *Let $(T_{ij}) \in \mathcal{K}(\mathcal{G}^n)$ such that $T_{ij}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$ and $1 \leq i, j \leq n$. Then, among the best approximations with rank at most n to (T_{ij}) (with respect to the operator norm), there is at least one, (F_{ij}) say, such that $F_{ij}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$ and $1 \leq i, j \leq n$.*

Proof. For each $\alpha \in \Delta$, let $\mathcal{G}_\alpha^{(1)}, \dots, \mathcal{G}_\alpha^{(n)}$ be n copies of $\mathcal{G}_\alpha$. For $1 \leq i \leq n$, let

$$\mathcal{G}_i = \left\{ \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha^{(i)} \right\},$$

so that the $\mathcal{G}_i$'s are all copies of $\mathcal{G}$.

Now, each T_{ij} ($1 \leq i, j \leq n$) is block diagonal and so we can write

$$T_{ij} = \bigoplus_{\alpha \in \Delta} T_{ij}^{(\alpha)}.$$

Now

$$\begin{aligned} (T_{ij}) &= \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} T_{11}^{(\alpha_1)} & & \\ & T_{11}^{(\alpha_2)} & \\ & & \ddots \end{bmatrix} & \cdots & \begin{bmatrix} T_{1n}^{(\alpha_1)} & & \\ & T_{1n}^{(\alpha_2)} & \\ & & \ddots \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} T_{n1}^{(\alpha_1)} & & \\ & T_{n1}^{(\alpha_2)} & \\ & & \ddots \end{bmatrix} & \cdots & \begin{bmatrix} T_{nn}^{(\alpha_1)} & & \\ & T_{nn}^{(\alpha_2)} & \\ & & \ddots \end{bmatrix} \end{bmatrix}, \end{aligned}$$

with respect to the decomposition $\mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_n$.

However,

$$(T_{ij}) = \begin{bmatrix} \begin{bmatrix} T_{11}^{(\alpha_1)} & \cdots & T_{1n}^{(\alpha_1)} \\ \vdots & & \vdots \\ T_{n1}^{(\alpha_1)} & \cdots & T_{nn}^{(\alpha_1)} \end{bmatrix} & & \\ & \begin{bmatrix} T_{11}^{(\alpha_2)} & \cdots & T_{1n}^{(\alpha_2)} \\ \vdots & & \vdots \\ T_{n1}^{(\alpha_2)} & \cdots & T_{nn}^{(\alpha_2)} \end{bmatrix} & \\ & & \ddots \end{bmatrix},$$

with respect to the decomposition $\bigoplus_{\alpha \in \Delta} \{\mathcal{G}_\alpha^{(1)} \oplus \dots \oplus \mathcal{G}_\alpha^{(n)}\}$.

(T_{ij}) is still a compact operator and, with respect to the new decomposition, it is a block diagonal operator. Hence, by Lemma 2.26, the best approximation (F_{ij}) to (T_{ij}) from the operators of rank at most m with respect to the new decomposition may be assumed to be block diagonal also.

However, with respect to the original decomposition, (F_{ij}) is still an operator of rank at most m and each $F_{ij}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ for all $\alpha \in \Delta$. The result now follows since the quantity $\|(T_{ij}) - (F_{ij})\|$ is independent of the chosen decomposition for the underlying Hilbert space. $\blacksquare$

Definition 2.29 Define a map $\theta : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{G})$ by $\theta(T_{ij})_{i,j \in \Delta} = (S_{ij})_{i,j \in \Delta}$ where

$$S_{ij} = \begin{cases} T_{ii} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.30 Let $S, T \in \mathcal{B}(\mathcal{G})$ such that $S = \theta(S)$. Then $\tau(ST) = \tau(S\theta(T))$.

$$\begin{aligned} \tau(ST) &= \sum_{\alpha \in \Delta} \tau(ST)_{\alpha\alpha} = \sum_{\alpha \in \Delta} \tau \left(\sum_{\beta \in \Delta} S_{\alpha\beta} T_{\beta\alpha} \right) \\ &= \sum_{\alpha \in \Delta} \tau(S_{\alpha\alpha} T_{\alpha\alpha}) \text{ by construction of } S \\ &= \tau(S\theta(T)). \end{aligned} \quad \blacksquare$$

Lemma 2.31 Let $\|\cdot\|_\Phi$ be a symmetric norm. Then $\theta : \mathcal{C}_\Phi^{(0)}(\mathcal{G}) \rightarrow \mathcal{C}_\Phi^{(0)}(\mathcal{G})$ and $\theta : \mathcal{C}_{\Phi'}(\mathcal{G}) \rightarrow \mathcal{C}_{\Phi'}(\mathcal{G})$. In both cases, the map is contractive.

Proof. Clearly, $\theta : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{G})$: suppose that $(T_{ij}) \in \mathcal{B}(\mathcal{G})$ and $(S_{ij}) = \theta(T_{ij})$. Then

$$\|(S_{ij})\|_{\mathcal{B}(\mathcal{G})} = \sup_{i \in \Delta} \|S_{ii}\|_{\mathcal{B}(\mathcal{G}_i)} = \sup_{i \in \Delta} \|T_{ii}\|_{\mathcal{B}(\mathcal{G}_i)} \leq \sup_{i,j \in \Delta} \|T_{ij}\|_{\mathcal{B}(\mathcal{G}_j, \mathcal{G}_i)} \leq \|(T_{ij})\|_{\mathcal{B}(\mathcal{G})}.$$

Now, $\theta : \mathcal{K}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{G})$ contractively (by the above). We show that $\theta(\mathcal{K}(\mathcal{G})) \subseteq \mathcal{K}(\mathcal{G})$. Suppose that $(T_{ij}) \in \mathcal{K}(\mathcal{G})$. Then the sequence

$$\left(\left\| \begin{bmatrix} T_{11} & T_{12} & \cdots \\ T_{21} & T_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & T_{22} & T_{23} & \cdots \\ 0 & T_{32} & T_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right\|, \dots \right) \in c_0.$$

Now

$$\|T_{ii}\| \leq \left\| \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & T_{ii} & T_{ii+1} & \cdots \\ 0 & T_{i+1i} & T_{i+1i+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right\|,$$

and so the sequence

$$(\|T_{11}\|, \|T_{22}\|, \dots) \in c_0,$$

also. Since $T_{ii} \in \mathcal{K}(\mathcal{G}_i)$ for each $i \in \Delta$, it follows that $\theta(T)$ is compact. So $\theta : \mathcal{K}(\mathcal{G}) \rightarrow \mathcal{K}(\mathcal{G})$ contractively.

Let $\psi : \mathcal{C}_1(\mathcal{G}) \rightarrow \mathcal{C}_1(\mathcal{G})$ be the conjugate map of $\theta : \mathcal{K}(\mathcal{G}) \rightarrow \mathcal{K}(\mathcal{G})$. Now

$$\langle \theta(A_{ij}), (B_{ij}) \rangle = \langle (A_{ij}), \psi(B_{ij}) \rangle,$$

and set $(C_{ij}) = \psi(B_{ij})$. The “duality map” is

$$(S_{ij}) \in \mathcal{C}_1(\mathcal{G}) \mapsto f_{(S_{ij})} \in (\mathcal{K}(\mathcal{G}))^*,$$

where $f_{(S_{ij})}(T_{ij}) = \tau((T_{ij})(S_{ij}))$. So

$$\tau((A_{ij})(C_{ij})) = \tau(\theta(A_{ij})(B_{ij})) = \sum_{\alpha \in \Delta} \tau(A_{\alpha\alpha}B_{\alpha\alpha}),$$

by the above and hence we require that

$$\psi(B_{ij}) = (C_{ij}) = \begin{bmatrix} B_{11} & & \\ & B_{22} & \\ & & \ddots \end{bmatrix} = \theta(B_{ij}).$$

Thus $\psi : \mathcal{C}_1(\mathcal{G}) \rightarrow \mathcal{C}_1(\mathcal{G})$ and $\|\psi\|_{\mathcal{C}_1(\mathcal{G}) \rightarrow \mathcal{C}_1(\mathcal{G})} = \|\theta\|_{\mathcal{K}(\mathcal{G}) \rightarrow \mathcal{K}(\mathcal{G})}$. Hence $\theta : \mathcal{C}_1(\mathcal{G}) \rightarrow \mathcal{C}_1(\mathcal{G})$ is a contraction.

The spaces $s_{\Phi}^{(0)}$ are, in the parlance of [1], symmetric sequence spaces, and hence $s_{\Phi}^{(0)}$ is an interpolation space for (l^1, c_0) for all symmetric norms $\|\cdot\|_{\Phi}$ (see Corollary 2.73). Hence $\mathcal{C}_{\Phi}^{(0)}(\mathcal{G})$ is an interpolation space for $(\mathcal{C}_1(\mathcal{G}), \mathcal{K}(\mathcal{G}))$. Then, by interpolation, $\theta : \mathcal{C}_{\Phi}^{(0)}(\mathcal{G}) \rightarrow \mathcal{C}_{\Phi}^{(0)}(\mathcal{G})$ is a contraction.

By dualising, we have that $\theta : \mathcal{C}_{\Phi'}(\mathcal{G}) \rightarrow \mathcal{C}_{\Phi'}(\mathcal{G})$ is a contraction. ■

2.6 Generalised Unitary Matrix Spaces

Lemma 2.32 *Suppose that $R \in \mathcal{K}(\mathcal{H})$ satisfies $(\dagger)$. Then, among the best approximations with rank at most n to R (with respect to the operator norm), there is at least one, K_n say, such that K_n satisfies $(\dagger)$.*

Proof. This is the analogue of Lemma 2.26 applied to the property $(\dagger)$. ■

Lemma 2.33 *Let $(T_{ij}) \in \mathcal{K}(\mathcal{H}^n)$ such that T_{ij} satisfies $(\dagger)$ for all $1 \leq i, j \leq n$. Then, among the best approximations with rank at most n to (T_{ij}) (with respect to the operator norm), there is at least one, (F_{ij}) say, such that F_{ij} satisfies $(\dagger)$ for all $1 \leq i, j \leq n$.*

Proof. This is the analogue of Lemma 2.28 applied to the property $(\dagger)$. ■

Proposition 2.34 *Let A be a C^* -algebra and $a \in A$. Then, for all $n \in \mathbb{N}$, there exists an element $f_n \in A$ of rank at most n such that*

$$\|a - f_n\| = \min \{ \|a - f\| : f \text{ has rank at most } n \}.$$

Proof. By construction, $\pi(a)$ has ‘block diagonal form’, that is to say, $\pi(a) : \mathcal{H}_\gamma \rightarrow \mathcal{H}_\gamma$ for all $\gamma \in \Gamma$. We know that the operator $\{\oplus_{e \in \xi} \rho_e(a)\} \oplus 0 \in \mathcal{B}(\mathcal{H})$ and has a best approximation from the operators of rank at most n , F_n say, which satisfies $(\dagger)$ by Lemma 2.32. Hence, there exists $f_n \in \mathcal{FR}(A)$ with rank at most n such that $\pi(f_n) = F_n$ by Proposition 2.25. So the quantity

$$\min \{ \|\pi(a) - F_n\| : F_n \text{ satisfies } (\dagger), F_n \text{ has rank at most } n \},$$

exists. Hence

$$\min \{ \|\pi(a) - \pi(f_n)\| : f_n \text{ has rank at most } n \},$$

also exists and, finally,

$$\min \{ \|a - f_n\| : f_n \text{ has rank at most } n \}$$

exists, as required. ■

Hence, we can introduce the notion of s -numbers for elements of a C^* -algebra.

Definition 2.35 *Let A be a C^* -algebra. For any $a \in A$, define the s -numbers $(s_n(a))$ of a by*

$$s_{n+1}(a) = \min \{ \|a - f_n\| : f_n \in A \text{ has rank at most } n \},$$

for all $n \geq 0$. We shall use $s(a)$ to denote the sequence $(s_n(a))_{n=1}^\infty$.

Definition 2.36 *Let $\|\cdot\|_\Phi$ be a symmetric norm. We define the spaces*

$$\mathcal{C}_\Phi(A) = \{a \in A : s(a) \in s_\Phi\},$$

and

$$\mathcal{C}_\Phi^{(0)}(A) = \{a \in A : s(a) \in s_\Phi^{(0)}\}.$$

Set

$$\|a\|_\Phi = \|s(a)\|_\Phi = \lim_{n \rightarrow \infty} \min \|(\|a\|, \|a - f_1\|, \dots, \|a - f_n\|, 0, 0, \dots)\|_\Phi,$$

where the minimum is taken over all sequences (f_n) where f_n has rank at most n and $(\|a\|, \|a - f_1\|, \|a - f_2\|, \dots) \in s_\Phi$ or $s_\Phi^{(0)}$ as appropriate. Such a sequence will be referred to in the sequel as a Φ -associated sequence. A Φ -associated sequence (f_n) for a will be said to be minimal if $s_{n+1}(a) = \|a - f_n\|$ for all $n \geq 0$. The spaces $\mathcal{C}_\Phi(A)$ and $\mathcal{C}_\Phi^{(0)}(A)$ will be referred to as generalised unitary matrix spaces. In the cases that the s -numbers lie in c_0 , l^p or $l^{p,q}$, we use the notation $\mathcal{K}(A)$, $\mathcal{C}_p(A)$ and $\mathcal{C}_{p,q}(A)$ and call these the spaces of compact, $\mathcal{C}_p$ and Lorentz elements respectively.

Remark 2.37 Let $\mathcal{G}$ be an infinite-dimensional Hilbert space and I be the identity operator. Then $A = \{\lambda I : \lambda \in \mathbb{C}\}$ is a C^* -algebra. Clearly, every element of A is single and hence compact as an element of the algebra. However, the non-zero elements of A are not compact as operators on $\mathcal{G}$.

Proposition 2.38 *Let $\|\cdot\|_\Phi$ be a symmetric norm which is not equivalent to $\|\cdot\|_\infty$. Then $\mathcal{C}_1(A) \subseteq \mathcal{C}_\Phi^{(0)}(A) \subseteq \mathcal{C}_\Phi(A) \subseteq \mathcal{K}(A)$ and $\|\cdot\|_\infty \leq \|\cdot\|_\Phi \leq \|\cdot\|_1$.*

Proof. This follows immediately from the properties of the underlying symmetric sequence spaces. ■

The following easy lemma will prove to be useful.

Lemma 2.39 *Let $\|\cdot\|_\Phi$ be a symmetric norm and let (a_n) and (b_n) be sequences such that $0 \leq b_n \leq a_n$ for each n . Then $\|(b_n)\|_\Phi \leq \|(a_n)\|_\Phi$. Also, for any $m \in \mathbb{N}$, the map*

$$T_m : (a_n) \mapsto (\underbrace{a_1, \dots, a_1}_{m \text{ copies}}, \underbrace{a_2, \dots, a_2}_{m \text{ copies}}, \dots),$$

is a bounded linear operator on both s_Φ and $s_\Phi^{(0)}$ with norm at most m .

Proposition 2.40 *Both $\mathcal{C}_\Phi(A)$ and $\mathcal{C}_\Phi^{(0)}(A)$ are 2-sided, $*$ -ideals in A for each symmetric norm $\|\cdot\|_\Phi$. Moreover, the involution is isometric with respect to $\|\cdot\|_\Phi$ and we have the estimate $\|xay\|_\Phi \leq \|x\| \|a\|_\Phi \|y\|$.*

Proof. Let $a \in \mathcal{C}_\Phi(A)$ and let (a_n) be a minimal Φ -associated sequence to a . For each $n \in \mathbb{N}$, a_n can be written as the sum of at most n single elements, $a_n = s_1 + \dots + s_k$, say, where $1 \leq k \leq n$. Then $a_n^* = s_1^* + \dots + s_k^*$ is also the sum of at most n single elements and

$$\|(\|a^*\|, \|a^* - a_1^*\|, \dots)\|_\Phi = \|(\|a\|, \|a - a_1\|, \dots)\|_\Phi < \infty,$$

since the involution is isometric with respect to the C^* -norm and so $a^* \in \mathcal{C}_\Phi(A)$. Since the norm is the minimum taken over all Φ -associated sequences, this implies that $\|a^*\|_\Phi \leq \|a\|_\Phi$. By applying the argument again, we see that, in fact, $\|a^*\|_\Phi = \|a\|_\Phi$.

Let $x, y \in A$. Then $xa_n y = xs_1 y + \dots + xs_k y$ is the sum of at most n single elements for each n . Now

$$\begin{aligned} \|(\|xay\|, \|xay - xa_1 y\|, \dots)\|_\Phi &\leq \|(\|x\| \|a\| \|y\|, \|x\| \|a - a_1\| \|y\|, \dots)\|_\Phi \\ &= \|x\| \|(\|a\|, \|a - a_1\|, \dots)\|_\Phi \|y\| = \|x\| \|a\|_\Phi \|y\|, \end{aligned}$$

and so $xay \in \mathcal{C}_\Phi(A)$ with $\|xay\|_\Phi \leq \|x\| \|a\|_\Phi \|y\|$.

Let $a, b \in \mathcal{C}_\Phi(A)$ with minimal Φ -associated sequences $(a_n), (b_n)$ respectively and let α, β be scalars. Define a sequence (c_n) by

$$c_1 = 0 \text{ and } c_{2n} = c_{2n+1} = \alpha a_n + \beta b_n,$$

for all $n \in \mathbb{N}$. Then each c_n has rank at most n since it can be written as a sum of, at most, n single elements. Also

$$\|(\|\alpha a + \beta b\|, \|\alpha a + \beta b - c_1\|, \|\alpha a + \beta b - c_2\|, \dots)\|_\Phi$$

$$\begin{aligned}
&\leq 2\|(\|\alpha a + \beta b\|, \|\alpha a + \beta b - \alpha a_1 - \beta b_1\|, \|\alpha a + \beta b - \alpha a_2 - \beta b_2\|, \dots)\|_{\Phi} \\
&\quad \text{since successive } c_n \text{'s are equal,} \\
&\leq 2\|(\|\alpha a\|, \|\alpha a - \alpha a_1\|, \|\alpha a - \alpha a_2\|, \dots)\|_{\Phi} \\
&\quad + 2\|(\|\beta b\|, \|\beta b - \beta b_1\|, \|\beta b - \beta b_2\|, \dots)\|_{\Phi},
\end{aligned}$$

since $\|\cdot\|_{\Phi}$ is a norm and hence $\alpha a + \beta b \in \mathcal{C}_{\Phi}(A)$. ■

Theorem 2.41 *Let $\|\cdot\|_{\Phi}$ be a symmetric norm. Then:*

- (i) $\pi(a)$ satisfies $(\dagger)$ for all $a \in \mathcal{K}(A)$;
- (ii) for all $a \in \mathcal{K}(A)$, $s(a) = s(\pi(a))$. In particular,

$$\|a\|_{\Phi} = \|\pi(a)\|_{\mathcal{C}_{\Phi}(\mathcal{H})},$$

for all $a \in \mathcal{C}_{\Phi}(A)$ and hence $\|\cdot\|_{\Phi}$ is a norm on both $\mathcal{C}_{\Phi}(A)$ and $\mathcal{C}_{\Phi}^{(0)}(A)$;

(iii) both $\mathcal{C}_{\Phi}(A)$ and $\mathcal{C}_{\Phi}^{(0)}(A)$ are Banach spaces with respect to $\|\cdot\|_{\Phi}$;

(iv) if $B \in \mathcal{C}_{\Phi}(\mathcal{H})$ (resp $\mathcal{C}_{\Phi}^{(0)}(\mathcal{H})$) and satisfies $(\dagger)$, then there exists $b \in \mathcal{C}_{\Phi}(A)$ (resp $\mathcal{C}_{\Phi}^{(0)}(A)$) such that $\pi(b) = B$.

Proof. (i) If $a \in \mathcal{K}(A)$, it has an Φ -associated sequence (a_n) , say, such that $\|a - a_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus $\|\pi(a) - \pi(a_n)\| \rightarrow 0$ since π is an isometry and hence $\pi(a)$ satisfies $(\dagger)$ since each $\pi(a_n)$ does.

(ii) Let $n \geq 0$. Then

$$\begin{aligned}
s_{n+1}(\pi(a)) &= \min \{ \|\pi(a) - F_n\| : F_n \text{ has rank at most } n \} \\
&= \min \{ \|\pi(a) - F_n\| : F_n \text{ satisfies } (\dagger) \text{ and has rank at most } n \} \\
&\quad \text{by (i) and Lemma 2.32} \\
&= \min \{ \|\pi(a) - \pi(f_n)\| : f_n \text{ has rank at most } n \} \\
&= \min \{ \|a - f_n\| : f_n \text{ has rank at most } n \} \\
&= s_{n+1}(a),
\end{aligned}$$

and so

$$\|a\|_{\Phi} = \|s(a)\|_{\Phi} = \|s(\pi(a))\|_{\Phi} = \|\pi(a)\|_{\mathcal{C}_{\Phi}(\mathcal{H})},$$

as required.

(iii) Let (a_n) be a Cauchy sequence in $\mathcal{C}_{\Phi}(A)$. Then the sequences (a_n) and $(\pi(a_n))$ are Cauchy in A (since $\|\cdot\| \leq \|\cdot\|_{\Phi}$) and $\mathcal{C}_{\Phi}(\mathcal{H})$ respectively and so converge to some $a \in A$ and $T \in \mathcal{C}_{\Phi}(\mathcal{H})$ respectively. Now

$$\|\pi(a) - T\| = \|\pi(a) - \pi(a_n) + \pi(a_n) - T\| \leq \|a - a_n\| + \|\pi(a_n) - T\| \rightarrow 0,$$

as $n \rightarrow \infty$ and so $\pi(a) = T$. Since each of the $\pi(a_n)$'s satisfy $(\dagger)$, so must T . Letting (T_n) be a Φ -associated sequence for T , we may assume without loss that each T_n satisfies

(†) and hence there is a sequence (t_n) in A such that $\pi(t_n) = T_n$ and each t_n has rank at most n . Finally,

$$\|(\|a\|, \|a - t_1\|, \dots)\|_{\Phi} = \|(\|T\|, \|T - T_1\|, \dots)\|_{\Phi} < \infty,$$

and hence (t_n) is an Φ -associated sequence for a . So $\mathcal{C}_{\Phi}(A)$ is complete. The proof that $\mathcal{C}_{\Phi}^{(0)}(A)$ is complete is analogous.

(iv) First of all, we consider the compact operators. Suppose that $B \in \mathcal{K}(\mathcal{H})$ satisfies (†) and let (B_n) be its compacting sequence. Then each B_n has rank at most n and may be assumed to satisfy (†). So there is a sequence (k_n) in $\mathcal{FR}(A)$ such that each k_n has rank at most n and $\pi(k_n) = B_n$ for all $n \in \mathbb{N}$. Since $\|B - \pi(k_n)\| \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\|k_n - k_m\| = \|\pi(k_n) - \pi(k_m)\| \leq \|\pi(k_n) - B\| + \|B - \pi(k_m)\| \rightarrow 0 \text{ as } m, n \rightarrow \infty,$$

and so (k_n) is a Cauchy sequence in $\mathcal{K}(A)$. So $k_n \rightarrow k$ for some $k \in \mathcal{K}(A)$ since $\mathcal{K}(A)$ is a Banach space. Then $B = \pi(k)$, as required.

Suppose that $B \in \mathcal{C}_{\Phi}(\mathcal{H})$ satisfies (†). Then, in particular, B is compact and hence there exists a $b \in \mathcal{K}(A)$ such that $\pi(b) = B$. Finally, we have that $s(b) = s(\pi(b)) = s(B)$ and so $\|s(b)\|_{\Phi} = \|s(B)\|_{\Phi} < \infty$. So $b \in \mathcal{C}_{\Phi}(A)$. ■

Theorem 2.42 *Let $\|\cdot\|_{\Phi}$ be a symmetric norm. Then $\mathcal{C}_{\Phi}^{(0)}(A)$ is the closure of $\mathcal{FR}(A)$ with respect to $\|\cdot\|_{\Phi}$. Moreover, $\mathcal{C}_{\Phi}^{(0)}(A) = \mathcal{C}_{\Phi}(A)$ if and only if $\|\cdot\|_{\Phi}$ is mononormalising.*

Proof. Let (a_n) be a sequence in $\mathcal{FR}(A)$ such that $\|a - a_n\|_{\Phi} \rightarrow 0$ as $n \rightarrow \infty$. Then $\|\pi(a) - \pi(a_n)\|_{\Phi} \rightarrow 0$ and hence $\pi(a) \in \mathcal{C}_{\Phi}^{(0)}(\mathcal{H})$. Now $\pi(a)$ satisfies (†) and $s(a) = s(\pi(a)) \in s_{\Phi}^{(0)}$ and so $a \in \mathcal{C}_{\Phi}^{(0)}(A)$, as required. Finally, $\mathcal{C}_{\Phi}^{(0)}(A) = \mathcal{C}_{\Phi}(A)$ if and only if $s_{\Phi}^{(0)} = s_{\Phi}$, which occurs precisely when $\|\cdot\|_{\Phi}$ is mononormalising. ■

As we have seen, the maps

$$\pi : \mathcal{C}_{\Phi}(A) \text{ (resp, } \mathcal{C}_{\Phi}^{(0)}(A)) \rightarrow \{T \in \mathcal{C}_{\Phi}(\mathcal{H}) \text{ (resp, } \mathcal{C}_{\Phi}^{(0)}(\mathcal{H})) : T \text{ satisfies (†)}\},$$

are isometric $*$ -isomorphisms. We introduce a natural norm on $M_n(\mathcal{C}_{\Phi}(A))$ and show that

$$\pi : \mathcal{C}_{\Phi}(A) \rightarrow \mathcal{C}_{\Phi}(\mathcal{H}) \text{ and } \pi : \mathcal{C}_{\Phi}^{(0)}(A) \rightarrow \mathcal{C}_{\Phi}^{(0)}(\mathcal{H}),$$

are complete isometries for all symmetric norms $\|\cdot\|_{\Phi}$. We begin with a well-known result.

Proposition 2.43 *Let A be a C^* -algebra and $\mathcal{G}$ be a Hilbert space. Then a faithful $*$ -representation $\varphi : A \rightarrow \mathcal{B}(\mathcal{G})$ is a complete isometry.*

Proof. By [6, Definition 1.1.7, Propositions 1.3.7 and 1.8.1], φ is an isometry. That φ is a complete isometry follows immediately from the fact that

$$\varphi_n : (a_{ij}) \in M_n(A) \mapsto (\pi(a_{ij})) \in B(\mathcal{G}^n),$$

is also a faithful $*$ -representation and hence an isometry for all $n \geq 2$. ■

Since $\pi : A \rightarrow B(\mathcal{H})$ is (by Theorem 2.9) a faithful $*$ -representation, we know that this map is a complete isometry by the above proposition. So we must have that

$$\|(a_{ij})\| = \|\pi_n(a_{ij})\| = \|(\pi(a_{ij}))\|,$$

for all $n \in \mathbb{N}$, $(a_{ij}) \in M_n(A)$ where $\|\cdot\|$ is the norm on the C^* -algebra $M_n(A)$.

Lemma 2.44 *Suppose that $(a_{ij}) \in M_n(A)$ has rank k for some $k \in \mathbb{N}$. Then each a_{ij} has rank at most k ($1 \leq i, j \leq n$).*

Proof. Fix $1 \leq i, j \leq n$ and write out (a_{ij}) in terms of k single elements of $M_n(A)$. By Lemma 2.10, the $(i, j)^{\text{th}}$ entry of each of these summands is a single element of A and hence a_{ij} is the sum of at most k single elements. ■

Proposition 2.45 *Let $\|\cdot\|_\Phi$ be a symmetric norm. Then $\mathcal{C}_\Phi(M_n(A)) = M_n(\mathcal{C}_\Phi(A))$ for all $n \geq 2$.*

Proof. Suppose that $(a_{ij}) \in \mathcal{C}_\Phi(M_n(A))$. Then (a_{ij}) has a Φ -associated sequence $(a_{ij}^{(k)})$, say. By the lemma, $a_{ij}^{(k)}$ has rank at most k for all $1 \leq i, j \leq n$ and

$$\|a_{ij} - a_{ij}^{(k)}\| \leq \|(a_{ij}) - (a_{ij}^{(k)})\|.$$

Hence

$$\left\| \left(\|a_{ij}\|, \|a_{ij} - a_{ij}^{(1)}\|, \dots \right) \right\|_\Phi \leq \left\| \left(\|(a_{ij})\|, \|(a_{ij}) - (a_{ij}^{(1)})\|, \dots \right) \right\|_\Phi < \infty.$$

Thus $(a_{ij}^{(1)}, a_{ij}^{(2)}, \dots)$ is a Φ -associated sequence for a_{ij} and so $a_{ij} \in \mathcal{C}_\Phi(A)$. So we have proved $\subseteq$.

Suppose now that each $a_{ij} \in \mathcal{C}_\Phi(A)$ with Φ -associated sequence $(a_{ij}^{(1)}, a_{ij}^{(2)}, \dots)$. Define a sequence $(b_{ij}^{(k)})$ in $M_n(A)$ as follows:

$$\begin{aligned} (b_{ij}^{(1)}) &= \dots = (b_{ij}^{(n^2-1)}) = 0, \\ (b_{ij}^{(n^2)}) &= \dots = (b_{ij}^{(2n^2-1)}) = \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1n}^{(1)} \\ \vdots & & \vdots \\ a_{n1}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix}, \end{aligned}$$

$$\begin{pmatrix} b_{ij}^{(2n^2)} \end{pmatrix} = \dots = \begin{pmatrix} b_{ij}^{(3n^2-1)} \end{pmatrix} = \begin{bmatrix} a_{11}^{(2)} & \dots & a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{n1}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix},$$

and so on. So each $\begin{pmatrix} b_{ij}^{(k)} \end{pmatrix}$ has rank at most k . Now, for each $m \in \mathbb{N}$, we have

$$\|(a_{ij}) - \begin{pmatrix} b_{ij}^{(m \cdot n^2)} \end{pmatrix}\| = \dots = \|(a_{ij}) - \begin{pmatrix} b_{ij}^{(m+1)n^2-1} \end{pmatrix}\| \leq \sum_{i,j=1}^n \|a_{ij} - a_{ij}^{(m)}\|.$$

Then

$$\begin{aligned} & \left\| \left(\underbrace{\|(a_{ij})\|, \dots, \|(a_{ij})\|}_{n^2 \text{ copies}}, \underbrace{\|(a_{ij}) - \begin{pmatrix} b_{ij}^{(1)} \end{pmatrix}\|, \dots, \|(a_{ij}) - \begin{pmatrix} b_{ij}^{(1)} \end{pmatrix}\|}_{n^2 \text{ copies}}, \dots \right) \right\|_{\Phi} \\ & \leq n^2 \left\| \left(\sum_{i,j=1}^n \|a_{ij}\|, \sum_{i,j=1}^n \|a_{ij} - a_{ij}^{(1)}\|, \sum_{i,j=1}^n \|a_{ij} - a_{ij}^{(2)}\|, \dots \right) \right\|_{\Phi} \\ & \leq n^2 \sum_{i,j=1}^n \left\| \left(\|a_{ij}\|, \|a_{ij} - a_{ij}^{(1)}\|, \|a_{ij} - a_{ij}^{(2)}\|, \dots \right) \right\|_{\Phi} < \infty, \end{aligned}$$

since $(a_{ij}^{(1)}, a_{ij}^{(2)}, \dots)$ is an Φ -associated sequence for each a_{ij} . Hence $(a_{ij}) \in \mathcal{C}_{\Phi}(M_n(A))$ and we have the inclusion $\supseteq$. $\blacksquare$

This allows us to define a natural norm—the $\mathcal{C}_{\Phi}(M_n(A))$ norm which we continue to denote by $\|\cdot\|_{\Phi}$ on $M_n(\mathcal{C}_{\Phi}(A))$ —by

$$\|(a_{ij})\|_{\Phi} = \lim_{n \rightarrow \infty} \min \left\| \left(\|(a_{ij})\|, \|(a_{ij}) - \begin{pmatrix} a_{ij}^{(1)} \end{pmatrix}\|, \dots, \|(a_{ij}) - \begin{pmatrix} a_{ij}^{(n)} \end{pmatrix}\|, 0, \dots \right) \right\|_{\Phi},$$

where the minimum is taken over all Φ -associated sequences $(a_{ij}^{(k)})$ of (a_{ij}) in $M_n(A)$.

Lemma 2.46 *Suppose that $(T_{ij}) \in \mathcal{B}(\mathcal{H}^n)$ is single and that each $T_{ij} \in \mathcal{B}(\mathcal{H})$ satisfies $(\dagger)$. Then there exists a matrix $(t_{ij}) \in \sigma(M_n(A))$ such that $\pi_n(t_{ij}) = (T_{ij})$.*

Proof. By Lemma 2.10, each $T_{ij} \in \mathcal{B}(\mathcal{H})$ is single and satisfies $(\dagger)$ by hypothesis. So, for each $1 \leq i, j \leq n$, there exists a $t_{ij} \in A$ such that $\pi(t_{ij}) = T_{ij}$. So $\pi_n(t_{ij}) = (T_{ij})$.

It remains to show that $(t_{ij}) \in \sigma(M_n(A))$. Suppose that $(a_{ij}), (b_{ij}) \in M_n(A)$ and $(a_{ij})(t_{ij})(b_{ij}) = 0$. Then $\|(a_{ij})(t_{ij})(b_{ij})\| = 0$ and hence $\pi_n(a_{ij})\pi_n(t_{ij})\pi_n(b_{ij}) = 0$ since π is a complete isometry. So $\pi_n(a_{ij})(T_{ij}) = 0$ or $(T_{ij})\pi_n(b_{ij}) = 0$ since (T_{ij}) is single and hence $(a_{ij})(t_{ij}) = 0$ or $(t_{ij})(b_{ij}) = 0$. Thus (t_{ij}) is single. $\blacksquare$

Theorem 2.47 *For all symmetric norms $\|\cdot\|_{\Phi}$, $\pi : \mathcal{C}_{\Phi}(A) \rightarrow \mathcal{C}_{\Phi}(\mathcal{H})$ is a complete isometry.*

Proof. Let $(a_{ij}) \in \mathcal{C}_\Phi(M_n(A)) = M_n(\mathcal{C}_\Phi(A))$. Then $\pi_n(a_{ij}) \in \mathcal{C}_\Phi(\mathcal{H}^n)$ and so it has a minimal Φ -associated sequence (F_k) , say. If $F_k = (F_{ij}^{(k)})$, then each $F_{ij}^{(k)}$ can be assumed to satisfy $(\dagger)$ since each $\pi(a_{ij})$ does (by Lemma 2.33). Now

$$(F_{ij}^{(k)}) = (G_{ij}^{(1)}) + \dots + (G_{ij}^{(l)}),$$

for some $1 \leq l \leq k$ where each $(G_{ij}^{(m)})$ is single and each $G_{ij}^{(m)}$ satisfies $(\dagger)$.

Hence, by Lemma 2.46, there exist matrices $(g_{ij}^{(1)}), \dots, (g_{ij}^{(l)}) \in \sigma(M_n(A))$ such that $\pi_n(g_{ij}^{(m)}) = (G_{ij}^{(m)})$ for all $1 \leq m \leq l$. Then

$$(f_{ij}^{(k)}) \stackrel{\text{def}}{=} (g_{ij}^{(1)}) + \dots + (g_{ij}^{(l)}),$$

has rank at most k in $M_n(A)$ and $\pi_n(f_{ij}^{(k)}) = (F_{ij}^{(k)}) = F_k$.

Since $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is a complete isometry,

$$\|(a_{ij}) - (f_{ij}^{(k)})\| = \|\pi_n((a_{ij})) - F_k\|,$$

and so

$$\begin{aligned} & \|\pi_n((a_{ij}))\|_{\mathcal{C}_\Phi(\mathcal{H}^n)} \\ &= \lim_{n \rightarrow \infty} \min \left\| \left(\|(a_{ij})\|, \|(a_{ij}) - (f_{ij}^{(1)})\|, \dots, \|(a_{ij}) - (a - f_{ij}^{(n)})\|, 0, \dots \right) \right\|_\Phi, \end{aligned}$$

which must, in turn, be equal to $\|(a_{ij})\|_\Phi$ because of the minimality of the sequence (F_k) : if, for any k , we can do better than $(f_{ij}^{(k)})$, then we can do better than F_k , a contradiction. ■

Another notion of compactness which has been studied by many authors is the following (see [2]).

Definition 2.48 *An element x in a C^* -algebra A is said to be ω -compactly acting if the wedge operator*

$$x \wedge x : a \in A \mapsto xax \in A,$$

is compact. We denote the set of ω -compactly acting elements of A by $\mathcal{K}_\omega(A)$.

These two notions of compactness are equivalent.

Theorem 2.49 $\mathcal{K}(A) = \mathcal{K}_\omega(A)$.

Proof. $\mathcal{K}(A) = \overline{\mathcal{FR}(A)} = \overline{\text{soc}(A)} = \mathcal{K}_\omega(A)$ by [2, Theorem $C^*1.3$]. ■

Corollary 2.50 *For any C^* -algebra A , $x \in \mathcal{FR}(A)$ (respectively, $\mathcal{K}(A)$) if and only if $x^*x \in \mathcal{FR}(A)$ (respectively, $\mathcal{K}(A)$).*

Proof. By [2, Theorem BA4.4], Proposition 2.20 and Theorem 2.49. ■

Lemma 2.51 Suppose $a \in A$. Then $\pi \left[(a^*a)^{\frac{1}{2}} \right] = [\pi(a^*a)]^{\frac{1}{2}}$.

Proof. Since $(a^*a)^{\frac{1}{2}}$ is positive, there exists an element $b \in A$ such that $b^*b = (a^*a)^{\frac{1}{2}}$. Then $\pi \left[(a^*a)^{\frac{1}{2}} \right] = \pi(b)^*\pi(b) \geq 0$. Finally, $\pi \left[(a^*a)^{\frac{1}{2}} \right] \pi \left[(a^*a)^{\frac{1}{2}} \right] = \pi[a^*a]$ and hence $\pi \left[(a^*a)^{\frac{1}{2}} \right] = [\pi(a^*a)]^{\frac{1}{2}}$. ■

Theorem 2.52 Let $\|\cdot\|_{\Phi}$ be a symmetric norm. Then $a \in \mathcal{C}_{\Phi}(A)$ (respectively, $\mathcal{C}_{\Phi}^{(0)}(A)$) if and only if $(a^*a)^{\frac{1}{2}} \in \mathcal{C}_{\Phi}(A)$ (respectively, $\mathcal{C}_{\Phi}^{(0)}(A)$).

Proof. The result is true, with equality of norms, for $\pi(a)$ in $\mathcal{C}_{\Phi}(\mathcal{H})$ and $\mathcal{C}_{\Phi}^{(0)}(\mathcal{H})$, by polar decomposition: if $T = VH$ is the polar decomposition of some operator T , then $H = V^*T$ where H is, of course, $(T^*T)^{\frac{1}{2}}$ and V is a partial isometry. Now

$$\begin{aligned} a \in \mathcal{C}_{\Phi}(A) &\Rightarrow \pi(a) \in \mathcal{C}_{\Phi}(\mathcal{H}) \text{ satisfies } (\dagger) \\ &\Rightarrow \pi \left[(a^*a)^{\frac{1}{2}} \right] = [\pi(a^*a)]^{\frac{1}{2}} \in \mathcal{C}_{\Phi}(\mathcal{H}) \text{ satisfies } (\dagger) \\ &\Rightarrow (a^*a)^{\frac{1}{2}} \in \mathcal{C}_{\Phi}(A). \end{aligned}$$

Conversely, if $(a^*a)^{\frac{1}{2}} \in \mathcal{C}_{\Phi}(A)$, then $a^*a \in \mathcal{C}_{\Phi}(A) \subseteq \mathcal{K}(A)$ and so $a \in \mathcal{K}(A)$. Then

$$\|\pi(a)\|_{\Phi} = \left\| [\pi(a^*a)]^{\frac{1}{2}} \right\|_{\Phi} = \left\| \pi \left[(a^*a)^{\frac{1}{2}} \right] \right\|_{\Phi} < \infty.$$

Hence $\pi(a) \in \mathcal{C}_{\Phi}(\mathcal{H})$ and satisfies $(\dagger)$ as $a \in \mathcal{K}(A)$. So $a \in \mathcal{C}_{\Phi}(A)$ as required. The case of $\mathcal{C}_{\Phi}^{(0)}(A)$ is exactly the same. ■

Proposition 2.53 (Hölder's Inequality) Choose symmetric norms $\|\cdot\|_{\Phi_1}$, $\|\cdot\|_{\Phi_2}$ and $\|\cdot\|_{\Phi_3}$ such that, if $(x_n) \in s_{\Phi_2}$ and $(y_n) \in s_{\Phi_3}$, then $(x_n y_n) \in s_{\Phi_1}$ and

$$\|(x_n y_n)\|_{\Phi_1} \leq \|(x_n)\|_{\Phi_2} \|(y_n)\|_{\Phi_3}.$$

- (i) If $a \in \mathcal{C}_{\Phi_2}(A)$ and $b \in \mathcal{C}_{\Phi_3}(A)$, then $ab \in \mathcal{C}_{\Phi_1}(A)$ and $\|ab\|_{\Phi_1} \leq \|a\|_{\Phi_2} \|b\|_{\Phi_3}$;
- (ii) if $a \in \mathcal{C}_{\Phi_2}^{(0)}(A)$ and $b \in \mathcal{C}_{\Phi_3}^{(0)}(A)$, then $ab \in \mathcal{C}_{\Phi_1}^{(0)}(A)$ and $\|ab\|_{\Phi_1} \leq \|a\|_{\Phi_2} \|b\|_{\Phi_3}$.

In particular, if $a \in \mathcal{C}_p(A)$ and $b \in \mathcal{C}_q(A)$ where $1 \leq p, q < \infty$, then $ab \in \mathcal{C}_r(A)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $\|ab\|_r \leq \|a\|_p \|b\|_q$.

Theorem 2.54 $\mathcal{C}_2(A)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle a, b \rangle = \tau(\pi(b^*a)),$$

for all $a, b \in \mathcal{C}_2(A)$ where τ is the usual trace on $\mathcal{C}_1(\mathcal{H})$.

Proof. Recall that the inner product $\langle S, T \rangle = \tau(T^*S)$ gives rise to the $\|\cdot\|_2$ norm on $\mathcal{C}_2(\mathcal{H})$. Now

$$\begin{aligned} \langle a + \lambda b, c \rangle &= \tau(\pi(c^*(a + \lambda b))) = \tau(\pi(c^*a + \lambda c^*b)) \\ &= \tau(\pi(c^*a)) + \lambda \tau(\pi(c^*b)) \\ &= \langle a, c \rangle + \lambda \langle b, c \rangle, \end{aligned}$$

and

$$\begin{aligned}\overline{\langle a, b \rangle} &= \overline{\tau(\pi(b^*a))} = \tau((\pi(b^*a))^*) \\ &= \tau(\pi(a^*b)) = \langle b, a \rangle,\end{aligned}$$

for all $a, b, c \in \mathcal{C}_2(A)$ and $\lambda \in \mathbb{C}$. Finally,

$$\langle a, a \rangle = \tau(\pi(a^*a)) = \tau(\pi(a)^*\pi(a)) = \|\pi(a)\|_2^2 = \|a\|_2^2,$$

for all $a \in \mathcal{C}_2(A)$. Since $\|\cdot\|_2$ is a norm under which $\mathcal{C}_2(A)$ is complete, it follows that $\mathcal{C}_2(A)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$. ■

Definition 2.55 Define a map $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by $\phi(T_{ij}) = (S_{ij})$ where

$$S_{ij} = \begin{cases} T_{ii} & \text{if } i \in \xi \text{ and } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.56 Let $S, T \in \mathcal{B}(\mathcal{H})$ such that S satisfies $(\dagger)$. Then $\tau(ST) = \tau(S\phi(T))$.

Proof. This is the analogue of Lemma 2.30 applied to the property $(\dagger)$. ■

Lemma 2.57 Let $\|\cdot\|_\Phi$ be a symmetric norm. Then $\phi : \mathcal{C}_\Phi^{(0)}(A) \rightarrow \mathcal{C}_\Phi^{(0)}(A)$ and $\phi : \mathcal{C}_{\Phi'}(A) \rightarrow \mathcal{C}_{\Phi'}(A)$. In both cases, the map is contractive.

Proof. This is the analogue of Lemma 2.31 applied to the property $(\dagger)$. ■

Lemma 2.58 Let $\|\cdot\|_\Phi$ be a symmetric norm which is not equivalent to $\|\cdot\|_1$. Then, for $b \in \mathcal{C}_{\Phi'}(A)$:

$$\|b\|_{\Phi'} = \sup \left\{ |\tau(\pi(bc))| : c \in \mathcal{C}_\Phi^{(0)}(A), \|c\|_\Phi \leq 1 \right\}.$$

Proof.

$$\begin{aligned}\|b\|_{\Phi'} &= \|\pi(b)\|_{\Phi'} \\ &= \sup \left\{ |\tau(\pi(b)C)| : C \in \mathcal{C}_\Phi^{(0)}(\mathcal{H}), \|C\|_\Phi \leq 1 \right\} \\ &= \sup \left\{ |\tau(\pi(b)C)| : C \in \mathcal{C}_\Phi^{(0)}(\mathcal{H}) \text{ satisfies } (\dagger), \|C\|_{\mathcal{C}_\Phi(\mathcal{H})} \leq 1 \right\} \\ &\quad \text{by Lemmas 2.56 and 2.57} \\ &= \sup \left\{ |\tau(\pi(bc))| : c \in \mathcal{C}_\Phi^{(0)}(A), \|c\|_\Phi \leq 1 \right\}. \quad \blacksquare\end{aligned}$$

Theorem 2.59 Let $\|\cdot\|_\Phi$ be a symmetric norm which is not equivalent to $\|\cdot\|_1$. Then the following pairs are isometrically isomorphic:

- (i) $\mathcal{C}_{\Phi'}(A) \cong \left(\mathcal{C}_\Phi^{(0)}(A)\right)^*$;
- (ii) $\{T \in \mathcal{B}(\mathcal{H}) : T \text{ satisfies } (\dagger)\} \cong \left(\mathcal{C}_1^{(0)}(A)\right)^*$.

Proof. We prove (i) in detail and (ii) follows in an analogous fashion. For $b \in \mathcal{C}_{\Phi'}(A)$, define a functional f_b on $\mathcal{C}_{\Phi}^{(0)}(A)$ by

$$f_b(a) = \tau(\pi(ab)),$$

for all $a \in \mathcal{C}_{\Phi}^{(0)}(A)$.

The map $b \mapsto f_b$ is clearly a linear homomorphism and is bounded since

$$|f_b(a)| = |\tau(\pi(ab))| \leq \|a\|_{\Phi} \|b\|_{\Phi'},$$

by Hölder's inequality and so $\|f_b\| \leq \|b\|_{\Phi'}$.

The map $b \mapsto f_b$ is an isometry since

$$\begin{aligned} \|b\|_{\Phi'} &= \sup \left\{ |\tau(\pi(bc))| : c \in \mathcal{C}_{\Phi}^{(0)}(A), \|c\|_{\Phi} \leq 1 \right\} \text{ by Lemma 2.58} \\ &= \|f_b\|. \end{aligned}$$

Only surjectivity remains. Let $f \in (\mathcal{C}_{\Phi}^{(0)}(A))^*$. Set

$$X_{\Phi}^{(0)}(\mathcal{H}) \stackrel{\text{def}}{=} \left\{ T \in \mathcal{C}_{\Phi}^{(0)}(\mathcal{H}) : T \text{ satisfies } (\dagger) \right\},$$

and define a functional g on $X_{\Phi}^{(0)}(\mathcal{H})$ by $g(\pi(a)) = f(a)$. Extend g to a continuous functional h on the whole of $\mathcal{C}_{\Phi}^{(0)}(\mathcal{H})$ by setting $h = g \circ \phi$. Then there exists an operator $B \in \mathcal{C}_{\Phi'}(\mathcal{H})$ such that $h = h_B$. Let $C = \phi(B)$. Then $C \in \mathcal{C}_{\Phi'}(\mathcal{H})$ by Lemma 2.57 and $\exists c \in \mathcal{C}_{\Phi'}(A)$ such that $\pi(c) = C$. Thus

$$\begin{aligned} f(a) &= g(\pi(a)) = h_B(\pi(a)) = \tau(\pi(a)B) \\ &= \tau(\pi(a)C) \text{ by Lemma 2.56} \\ &= \tau(\pi(ac)) = f_c(a), \end{aligned}$$

and hence the map $b \mapsto f_b$ is a surjection. ■

Corollary 2.60 $\mathcal{C}_{\Phi}(A)$ is reflexive if and only if both $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\Phi'}$ are mononormalising.

Proof. This is merely the analogue of [15, Theorem 3.12.2]. ■

Theorem 2.61 (Riesz-Thorin Convexity Theorem) Let $\mathcal{D}$ be a dense subspace of $\mathcal{C}_1(A)$ and $\psi : \mathcal{D} \rightarrow A$ be a linear map. Let $\|\psi\|_{p,q}$ denote the norm of ψ with respect to $\|\cdot\|_p$ and $\|\cdot\|_q$, that is to say,

$$\|\psi\|_{p,q} = \sup \left\{ \|\psi(a)\|_q : a \in \mathcal{D}, \|a\|_p \leq 1 \right\}.$$

As a function of (p^{-1}, q^{-1}) , $\log \|\psi\|_{p,q}$ is convex on the unit square $[0, 1] \times [0, 1]$.

Proof. Suppose $T \in \mathcal{C}_1(\mathcal{H})$. Then T can be written as a sum $T_1 + T_2$ where T_1 satisfies $(\dagger)$ and $\phi(T_2) = 0$. The spaces of such operators T_1 and T_2 will be denoted by $X_1(\mathcal{H})$ and $Y_1(\mathcal{H})$ respectively. Since $\pi(\mathcal{D})$ is dense in $X_1(\mathcal{H})$, $\pi(\mathcal{D}) \oplus Y_1(\mathcal{H})$ is dense in $\mathcal{C}_1(\mathcal{H})$.

Define a map $\varphi : \pi(\mathcal{D}) \oplus Y_1(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by

$$\varphi(\pi(a) \oplus B) = \pi(\psi(a)),$$

for all $a \in \mathcal{D}$ and $B \in Y_1(\mathcal{H})$. Clearly, φ is linear since both ψ and π are and φ is well-defined. Then, by the Riesz-Thorin Convexity Theorem for $\mathcal{C}_p(\mathcal{H})$ spaces (Theorem 1.10), we have that $\log \|\varphi\|_{p,q}$ is convex as a function of (p^{-1}, q^{-1}) on $[0, 1] \times [0, 1]$. Then

$$\begin{aligned} \|\varphi\|_{p,q} &= \sup \left\{ \|\varphi(\pi(a) \oplus B)\|_q : \pi(a) \oplus B \in \pi(\mathcal{D}) \oplus Y_1(\mathcal{H}), \|\pi(a) \oplus B\|_p \leq 1 \right\} \\ &= \sup \left\{ \|\pi(\psi(a))\|_q : \pi(a) \oplus B \in \pi(\mathcal{D}) \oplus Y_1(\mathcal{H}), \|\pi(a) \oplus B\|_p \leq 1 \right\} \\ &= \sup \left\{ \|\pi(\psi(a))\|_q : \pi(a) \in \pi(\mathcal{D}), \|\pi(a)\|_p \leq 1 \right\} \\ &= \sup \left\{ \|\psi(a)\|_q : a \in \mathcal{D}, \|a\|_p \leq 1 \right\} \\ &= \|\psi\|_{p,q}. \end{aligned}$$

Hence, as a function of (p^{-1}, q^{-1}) , $\log \|\psi\|_{p,q}$ is convex on the unit square. $\blacksquare$

We end this section with an alternative view of s -numbers and then give two examples.

Lemma 2.62 *Let A be a C^* -algebra with identity e . Suppose $a \in A$ is invertible. Then $\pi(a) \in \mathcal{B}(\mathcal{H})$ is invertible.*

Proof. Suppose that $a \in A$ is invertible. Since each ρ_γ ($\gamma \in \Gamma$) is an irreducible $*$ -representation, $\rho_\gamma(e)$ is the identity operator on $\mathcal{H}_\gamma$ and hence $\pi(e)$ is the identity on $\mathcal{H}$. Then $\pi(a)^{-1} = \pi(a^{-1})$ since

$$I = \pi(e) = \pi(aa^{-1}) = \pi(a)\pi(a^{-1}),$$

as required. $\blacksquare$

Proposition 2.63 *Let A be a C^* -algebra with identity e . Suppose $a \in \mathcal{K}(A)$ and that μ is an entry of the sequence $s(a)$. Then $(a^*a)^{\frac{1}{2}} - \mu e$ is not invertible.*

Proof. Suppose there exists an $n \in \mathbb{N}$ such that $\mu = s_n(a) = s_n(\pi(a))$. Then $\mu = \min \{\|\pi(a) - F_n\| : F_n \text{ has rank at most } n\}$ and hence μ is an eigenvalue of $[\pi(a)^*\pi(a)]^{\frac{1}{2}}$. So $[\pi(a)^*\pi(a)]^{\frac{1}{2}} - \mu I$ is not invertible and hence, by Lemma 2.62, $(a^*a)^{\frac{1}{2}} - \mu e$ is not invertible. $\blacksquare$

Remark 2.64 We shall return to this view of the definition of the s -numbers in Section 2.12.

If the proof of the pudding is in the eating, then the justification for our approach in this chapter lies in the following two results.

Theorem 2.65 *Let $\mathcal{G}$ be a Hilbert space. Then*

$$\mathcal{C}_{\Phi}(\mathcal{B}(\mathcal{G})) = \mathcal{C}_{\Phi}(\mathcal{G}) \text{ and } \mathcal{C}_{\Phi}^{(0)}(\mathcal{B}(\mathcal{G})) = \mathcal{C}_{\Phi}^{(0)}(\mathcal{G}),$$

with equality of norms for all symmetric norms $\|\cdot\|_{\Phi}$.

Proof. Recalling the argument used in Proposition 2.23,

$$\begin{aligned} K \in \mathcal{C}_{\Phi}(\mathcal{B}(\mathcal{G})) &\Leftrightarrow \exists \text{ a sequence } (K_n) \text{ such that } K_n \text{ is an element of } \mathcal{B}(\mathcal{G}) \text{ of} \\ &\quad \text{rank at most } n \text{ and } \|(\|K\|, \|K - K_1\|, \dots)\|_{\Phi} \in s_{\Phi} \\ &\Leftrightarrow \exists \text{ a sequence } (K_n) \text{ such that } K_n \text{ is an operator of rank} \\ &\quad \text{at most } n \text{ and } \|(\|K\|, \|K - K_1\|, \dots)\|_{\Phi} \in s_{\Phi} \text{ (by Remark 2.2)} \\ &\Leftrightarrow K \in \mathcal{C}_{\Phi}(\mathcal{G}), \end{aligned}$$

and the second part follows by replacing s_{Φ} with $s_{\Phi}^{(0)}$. ■

Theorem 2.66 *Let $\|\cdot\|_{\Phi}$ be a symmetric norm. Then*

$$\mathcal{C}_{\Phi}(l^{\infty}) = s_{\Phi} \text{ and } \mathcal{C}_{\Phi}^{(0)}(l^{\infty}) = s_{\Phi}^{(0)},$$

with equality of norms.

Proof. For a sequence (x_n) in c_0 , let (x_n^*) denote the non-increasing re-arrangement of $(|x_n|)$. Then $\|(x_n)\|_{\Phi} = \|(x_n^*)\|_{\Phi}$, for example, $\|(x_n)\|_{p,q} = \left(\sum_{n=1}^{\infty} x_n^{*q} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}}$, etc.

Clearly, the single elements of l^{∞} are those sequences which have at most one non-zero entry. So if $(x_n) \in l^{\infty}$, then $s_1((x_n)) = \|(x_n)\| = x_1^*$.

Now, suppose that x_{m_1} is an entry in (x_n) which gives rise to x_1^* , that is to say, it is the largest (or joint largest) entry in modulus. Let

$$(y_1) = (0, \dots, 0, x_{m_1}, 0, \dots) \quad (\text{entry in the } m_1^{\text{th}} \text{ place}).$$

Then $s_2((x_n)) = \|(x_n) - (y_1)\| = x_2^*$. We proceed in the obvious manner: let x_{m_2} be an entry which gives rise to x_2^* and set

$$(y_2) = (0, \dots, 0, x_{m_1}, 0, \dots, 0, x_{m_2}, 0, \dots) \quad (\text{entries in the } m_1^{\text{th}} \text{ and } m_2^{\text{th}} \text{ places}),$$

and so $s_2((x_n)) = \|(x_n) - (y_2)\| = x_3^*$. Continuing in this way, we get

$$s_m((x_n)) = x_m^*,$$

for all $m \in \mathbb{N}$ and so

$$(x_n) \in s_{\Phi} \Leftrightarrow (x_n^*) \in s_{\Phi} \Leftrightarrow (s_n((x_n))) \in s_{\Phi} \Leftrightarrow (x_n) \in \mathcal{C}_{\Phi}(l^{\infty}),$$

and the equality of norms comes by $s_m((x_n)) = x_m^*$. The second part follows by replacing s_{Φ} with $s_{\Phi}^{(0)}$. ■

2.7 Interpolation of Generalised Unitary Matrix Spaces

As we mentioned earlier, the spaces $s_{\Phi}^{(0)}$ are the symmetric sequence spaces of [1], where the author uses E and $\|\cdot\|_E$ for $s_{\Phi}^{(0)}$ and $\|\cdot\|_{\Phi}$. We now adopt this notation to help simplify the expressions that our theorems yield. (By [1, page 458], the following results all hold for $\tilde{E}$, which is s_{Φ} .) Primarily, we generalise the interpolation results due to Arazy (see [1]) from $\mathcal{C}_E(\mathcal{H})$ to $\mathcal{C}_E(A)$ and show that the interpolation of generalised unitary matrix spaces reduces to interpolation of symmetric sequence spaces. This is achieved by mimicking Arazy's approach.

Recall from Definition 1.24 the usual norms on $A_0 \cap A_1$ and $A_0 + A_1$ for a compatible couple (A_0, A_1) . We begin with a lemma.

Lemma 2.67 *Let A be a C^* -algebra and E_1, E_2 be symmetric sequence spaces. Then*

$$\mathcal{C}_{E_1+E_2}(A) = \mathcal{C}_{E_1}(A) + \mathcal{C}_{E_2}(A),$$

with equality of norms.

Proof. It is proved in [1, Remark 2.3(c)] that $E_1 + E_2$ is a symmetric sequence space and also that the lemma is true for $A = \mathcal{B}(\mathcal{G})$ (see [1, Lemma 2.5]).

Now, suppose that $R \in \mathcal{C}_{E_1+E_2}(\mathcal{H})$ satisfies $(\dagger)$. Then $R \in \mathcal{C}_{E_1}(\mathcal{H}) + \mathcal{C}_{E_2}(\mathcal{H})$ and

$$\begin{aligned} \|R\|_{\mathcal{C}_{E_1+E_2}(\mathcal{H})} &= \|R\|_{\mathcal{C}_{E_1}(\mathcal{H})+\mathcal{C}_{E_2}(\mathcal{H})} \\ &= \inf \left\{ \|S\|_{\mathcal{C}_{E_1}(\mathcal{H})} + \|T\|_{\mathcal{C}_{E_2}(\mathcal{H})} : R = S + T \right\} \\ &= \inf \left\{ \|S\|_{\mathcal{C}_{E_1}(\mathcal{H})} + \|T\|_{\mathcal{C}_{E_2}(\mathcal{H})} : R = S + T, S \text{ and } T \text{ satisfy } (\dagger) \right\}, \end{aligned}$$

since, by construction of ϕ , $\phi(S)$ and $\phi(T)$ satisfy $(\dagger)$ with

$$R = \phi(R) = \phi(S + T) = \phi(S) + \phi(T),$$

and

$$\|\phi(S)\|_{\mathcal{C}_{E_1}(\mathcal{H})} \leq \|S\|_{\mathcal{C}_{E_1}(\mathcal{H})}, \quad \|\phi(T)\|_{\mathcal{C}_{E_2}(\mathcal{H})} \leq \|T\|_{\mathcal{C}_{E_2}(\mathcal{H})}.$$

Now, given $S \in \mathcal{C}_{E_1}(\mathcal{H})$ and $T \in \mathcal{C}_{E_2}(\mathcal{H})$ such that S and T both satisfy $(\dagger)$, there exist $s \in \mathcal{C}_{E_1}(A)$, $t \in \mathcal{C}_{E_2}(A)$ such that $\pi(s) = S$ and $\pi(t) = T$. So, if $r \in \mathcal{C}_{E_1+E_2}(A)$, we have

$$\begin{aligned} \|r\|_{\mathcal{C}_{E_1+E_2}(A)} &= \|\pi(r)\|_{\mathcal{C}_{E_1+E_2}(\mathcal{H})} \\ &= \inf \left\{ \|s\|_{\mathcal{C}_{E_1}(A)} + \|t\|_{\mathcal{C}_{E_2}(A)} : r = s + t, s \in \mathcal{C}_{E_1}(A), t \in \mathcal{C}_{E_2}(A) \right\} \\ &= \|r\|_{\mathcal{C}_{E_1}(A)+\mathcal{C}_{E_2}(A)}, \end{aligned}$$

and hence $r \in \mathcal{C}_{E_1}(A) + \mathcal{C}_{E_2}(A)$. The reverse inclusion is immediate and hence the lemma is proved. ■

Theorem 2.68 *Let A be a C^* -algebra and E_1, E_2 be symmetric sequence spaces. Then*

$$\mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)) = \mathcal{C}_{\mathcal{F}(E_1, E_2)}(A),$$

for all exact interpolation functors $\mathcal{F}$ with equality of norms.

Proof. It is proved in [1, Theorem 2.4] that the result is true for $A = B(\mathcal{G})$ and that $\mathcal{F}(E_1, E_2)$ is a symmetric sequence space E , say.

Now π maps the spaces $\mathcal{C}_{E_1}(A)$, $\mathcal{C}_{E_2}(A)$ and $\mathcal{C}_{E_1+E_2}(A) = \mathcal{C}_{E_1}(A) + \mathcal{C}_{E_2}(A)$ isometrically into $\mathcal{C}_{E_1}(\mathcal{H})$, $\mathcal{C}_{E_2}(\mathcal{H})$ and $\mathcal{C}_{E_1+E_2}(\mathcal{H}) = \mathcal{C}_{E_1}(\mathcal{H}) + \mathcal{C}_{E_2}(\mathcal{H})$ respectively. Hence π is a bounded operator from the couple $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$ into the couple $(\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H}))$ and

$$\|\pi\|_{1,2} \stackrel{\text{def}}{=} \max \left\{ \|\pi\|_{B(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_1}(\mathcal{H}))}, \|\pi\|_{B(\mathcal{C}_{E_2}(A), \mathcal{C}_{E_2}(\mathcal{H}))} \right\} = 1.$$

Since $\mathcal{F}$ is an exact interpolation functor,

$$\pi : \mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)) \rightarrow \mathcal{F}(\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})) = \mathcal{C}_E(\mathcal{H}),$$

and

$$\|\pi\|_{B(\mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), \mathcal{C}_E(\mathcal{H}))} \leq \|\mathcal{F}\| \|\pi\|_{1,2} = 1.$$

Hence

$$a \in \mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)) \Rightarrow \pi(a) \in \mathcal{C}_E(\mathcal{H}) \Rightarrow a \in \mathcal{C}_E(A),$$

and

$$\|a\|_{\mathcal{C}_E(A)} = \|\pi(a)\|_{\mathcal{C}_E(\mathcal{H})} \leq \|a\|_{\mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))}. \quad (2.1)$$

Conversely, $\pi^{-1} \circ \phi$ contractively maps the spaces $\mathcal{C}_{E_1}(\mathcal{H})$, $\mathcal{C}_{E_2}(\mathcal{H})$ and $\mathcal{C}_{E_1+E_2}(\mathcal{H}) = \mathcal{C}_{E_1}(\mathcal{H}) + \mathcal{C}_{E_2}(\mathcal{H})$ onto $\mathcal{C}_{E_1}(A)$, $\mathcal{C}_{E_2}(A)$ and $\mathcal{C}_{E_1+E_2}(A) = \mathcal{C}_{E_1}(A) + \mathcal{C}_{E_2}(A)$ respectively and hence $\pi^{-1} \circ \phi$ is a bounded operator from the couple $(\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H}))$ onto the couple $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$ and $\|\pi^{-1} \circ \phi\|_{1,2} = 1$. Since $\mathcal{F}$ is an exact interpolation functor,

$$\pi^{-1} \circ \phi : \mathcal{C}_E(\mathcal{H}) = \mathcal{F}(\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})) \rightarrow \mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)),$$

and

$$\|\pi^{-1} \circ \phi\|_{B(\mathcal{C}_E(\mathcal{H}), \mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)))} \leq \|\mathcal{F}\| \|\pi^{-1} \circ \phi\|_{1,2} = 1.$$

Hence

$$a \in \mathcal{C}_E(A) \Rightarrow \pi(a) \in \mathcal{C}_E(\mathcal{H}) \Rightarrow a = (\pi^{-1} \circ \phi)(\pi(a)) \in \mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)),$$

and

$$\begin{aligned} \|a\|_{\mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))} &= \|(\pi^{-1} \circ \phi)(\pi(a))\|_{\mathcal{F}(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))} \\ &\leq \|\pi(a)\|_{\mathcal{C}_E(\mathcal{H})} = \|a\|_{\mathcal{C}_E(A)}. \end{aligned} \quad (2.2)$$

The equality of norms follows by (2.1) and (2.2). ■

In particular, the interpolation functor Δ is exact, so the previous theorem tells us that

$$\mathcal{C}_{E_1 \cap E_2}(A) = \mathcal{C}_{E_1}(A) \cap \mathcal{C}_{E_2}(A),$$

with equal norms. This enables us to prove the following.

Lemma 2.69 *Suppose that E , E_1 and E_2 are symmetric sequence spaces and let $\mathcal{G}$ be a Hilbert space. Then the following are equivalent:*

- (i) E is an intermediate space for (E_1, E_2) ;
- (ii) $\mathcal{C}_E(\mathcal{G})$ is an intermediate space for $(\mathcal{C}_{E_1}(\mathcal{G}), \mathcal{C}_{E_2}(\mathcal{G}))$;
- (iii) $\mathcal{C}_E(A)$ is an intermediate space for $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$.

Proof. (i) $\Leftrightarrow$ (ii): this was demonstrated in [1].

(ii) $\Rightarrow$ (iii): if we take the Hilbert space to be $\mathcal{H}$, the result is immediate since π isometrically embeds $\mathcal{C}_F(A)$ in $\mathcal{C}_F(\mathcal{H})$ for all symmetric sequence spaces F .

(iii) $\Rightarrow$ (ii): choose one $e \in \xi$ and let ψ be the embedding of $\mathcal{H}_e$ in $\mathcal{H}$. Then $\pi^{-1} \circ \psi$ isometrically embeds $\mathcal{C}_F(\mathcal{H}_e)$ in $\mathcal{C}_F(A)$ for all symmetric sequence spaces F . ■

Theorem 2.70 *Let A be a C^* -algebra and E , E_1 , E_2 , F , F_1 and F_2 be symmetric sequence spaces. Then $\langle E, F \rangle$ is an interpolation pair for $\langle (E_1, E_2), (F_1, F_2) \rangle$ if and only if $\langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle$ is an interpolation pair for $\langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle$. Moreover, in that case, the interpolation constants $M \{ \langle E, F \rangle; \langle (E_1, E_2), (F_1, F_2) \rangle \}$ and $M \{ \langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle; \langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle \}$ are equal.*

Proof. We know that $\langle E, F \rangle$ is an interpolation pair for $\langle (E_1, E_2), (F_1, F_2) \rangle$ if and only if $\langle \mathcal{C}_E(\mathcal{H}), \mathcal{C}_F(\mathcal{H}) \rangle$ is an interpolation pair for $\langle (\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})), (\mathcal{C}_{F_1}(\mathcal{H}), \mathcal{C}_{F_2}(\mathcal{H})) \rangle$ and, in that case, the interpolation constants are equal (see [1, Corollary 2.8]).

Now, E is an intermediate space for (E_1, E_2) if and only if $\mathcal{C}_E(A)$ is an intermediate space for $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$ by Lemma 2.69. Similar remarks apply to the F 's. To proceed, let us assume that $\langle \mathcal{C}_E(\mathcal{H}), \mathcal{C}_F(\mathcal{H}) \rangle$ is an interpolation pair for the couple $\langle (\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})), (\mathcal{C}_{F_1}(\mathcal{H}), \mathcal{C}_{F_2}(\mathcal{H})) \rangle$ and set

$$M_1 \stackrel{\text{def}}{=} M \{ \langle \mathcal{C}_E(\mathcal{H}), \mathcal{C}_F(\mathcal{H}) \rangle; \langle (\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})), (\mathcal{C}_{F_1}(\mathcal{H}), \mathcal{C}_{F_2}(\mathcal{H})) \rangle \} < \infty,$$

by hypothesis.

Let S be a bounded operator from $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$ into $(\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A))$. Then

$$T \stackrel{\text{def}}{=} \pi \circ S \circ \pi^{-1} \circ \phi,$$

is a bounded operator from $(\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H}))$ into $(\mathcal{C}_{F_1}(\mathcal{H}), \mathcal{C}_{F_2}(\mathcal{H}))$ since

$$\|TX\|_{\mathcal{C}_{F_1}(\mathcal{H})} = \left\| \left(\pi \circ S \circ \pi^{-1} \circ \phi \right) (X) \right\|_{\mathcal{C}_{F_1}(\mathcal{H})}$$

$$\begin{aligned}
&= \left\| \left(S \circ \pi^{-1} \circ \phi \right) (X) \right\|_{\mathcal{C}_{F_i}(A)} \\
&\leq \|S\|_{\mathcal{B}(\mathcal{C}_{E_i}(A), \mathcal{C}_{F_i}(A))} \left\| \left(\pi^{-1} \circ \phi \right) (X) \right\|_{\mathcal{C}_{E_i}(A)} \\
&= \|S\|_{\mathcal{B}(\mathcal{C}_{E_i}(A), \mathcal{C}_{F_i}(A))} \|\phi(X)\|_{\mathcal{C}_{E_i}(\mathcal{H})} \\
&\leq \|S\|_{\mathcal{B}(\mathcal{C}_{E_i}(A), \mathcal{C}_{F_i}(A))} \|X\|_{\mathcal{C}_{E_i}(\mathcal{H})},
\end{aligned}$$

for $i = 1, 2$ and hence $\|T\|_{1,2} \leq \|S\|_{1,2}$.

Fix $a \in \mathcal{C}_E(A)$ and $b \stackrel{\text{def}}{=} Sa$. Then $\pi(a) \in \mathcal{C}_E(\mathcal{H})$ and $T(\pi(a)) \in \mathcal{C}_F(\mathcal{H})$ by hypothesis. But $T(\pi(a)) = \pi(b)$ and

$$\begin{aligned}
\|b\|_{\mathcal{C}_F(A)} &= \|\pi(b)\|_{\mathcal{C}_F(\mathcal{H})} = \|T(\pi(a))\|_{\mathcal{C}_F(\mathcal{H})} \\
&\leq \|T\|_{\mathcal{B}(\mathcal{C}_E(\mathcal{H}), \mathcal{C}_F(\mathcal{H}))} \|\pi(a)\|_{\mathcal{C}_E(\mathcal{H})} \\
&\leq M_1 \|T\|_{1,2} \|a\|_{\mathcal{C}_E(A)} \text{ by the definition of } M_1 \\
&\leq M_1 \|S\|_{1,2} \|a\|_{\mathcal{C}_E(A)}.
\end{aligned}$$

Hence $S : \mathcal{C}_E(A) \rightarrow \mathcal{C}_F(A)$ is a bounded operator and so $\langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle$ is an interpolation pair for $\langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle$. Finally,

$$M \{ \langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle; \langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle \} \leq M_1.$$

Conversely, let us assume that $\langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle$ is an interpolation pair for the couple $\langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle$ and set

$$M_2 \stackrel{\text{def}}{=} M \{ \langle \mathcal{C}_E(A), \mathcal{C}_F(A) \rangle; \langle (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)), (\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A)) \rangle \} < \infty.$$

Let ξ be the set constructed in Theorem 2.9 and $e \in \xi$ be the first element under some ordering of this set. Recall that the Hilbert space $\mathcal{H}_e$ is the left ideal Ae . Suppose that V is a bounded operator from $(\mathcal{C}_{E_1}(\mathcal{H}_e), \mathcal{C}_{E_2}(\mathcal{H}_e))$ into $(\mathcal{C}_{F_1}(\mathcal{H}_e), \mathcal{C}_{F_2}(\mathcal{H}_e))$ and let $V \oplus 0$ denote the map

$$V \oplus (\oplus_{\Gamma \setminus \{e\}} 0) : \mathcal{H} \rightarrow \mathcal{H}.$$

Set

$$W \stackrel{\text{def}}{=} \pi^{-1} \circ (V \oplus 0) \circ \pi.$$

Then W is a bounded operator from $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$ into $(\mathcal{C}_{F_1}(A), \mathcal{C}_{F_2}(A))$ and observe that $\|W\|_{1,2} \leq \|V\|_{1,2}$ since

$$\begin{aligned}
\|Wa\|_{\mathcal{C}_{F_i}(A)} &= \left\| \left(\pi^{-1} \circ (V \oplus 0) \circ \pi \right) (a) \right\|_{\mathcal{C}_{F_i}(A)} \\
&= \|(V \oplus 0)(\pi(a))\|_{\mathcal{C}_{F_i}(\mathcal{H})} \\
&\leq \|V \oplus 0\|_{\mathcal{B}(\mathcal{C}_{E_i}(\mathcal{H}), \mathcal{C}_{F_i}(\mathcal{H}))} \|\pi(a)\|_{\mathcal{C}_{E_i}(\mathcal{H})} \\
&= \|V\|_{\mathcal{B}(\mathcal{C}_{E_i}(\mathcal{H}_e), \mathcal{C}_{F_i}(\mathcal{H}_e))} \|a\|_{\mathcal{C}_{E_i}(A)},
\end{aligned}$$

for $i = 1, 2$.

Now, fix $X \in \mathcal{C}_E(\mathcal{H}_e)$ and $Y \stackrel{\text{def}}{=} VX$. Then $X \oplus 0 \in \mathcal{C}_E(\mathcal{H})$ and satisfies $(\dagger)$ so there exists an $x \in \mathcal{C}_E(A)$ such that $\pi(x) = X \oplus 0$. Since $W : \mathcal{C}_E(A) \rightarrow \mathcal{C}_F(A)$ by hypothesis, set $y \stackrel{\text{def}}{=} Wx \in \mathcal{C}_F(A)$. Then

$$\pi(y) = \pi(Wx) = (V \oplus 0)(\pi(x)) = (V \oplus 0)(X \oplus 0) = VX \oplus 0 = Y \oplus 0.$$

So

$$\begin{aligned} \|Y\|_{\mathcal{C}_F(\mathcal{H}_e)} &= \|Y \oplus 0\|_{\mathcal{C}_F(\mathcal{H})} = \|\pi(y)\|_{\mathcal{C}_F(\mathcal{H})} = \|y\|_{\mathcal{C}_F(A)} \\ &= \|Wx\|_{\mathcal{C}_F(A)} \leq \|W\|_{\mathcal{B}(\mathcal{C}_E(A), \mathcal{C}_F(A))} \|x\|_{\mathcal{C}_E(A)} \\ &\leq M_2 \|W\|_{1,2} \|\pi(x)\|_{\mathcal{C}_E(\mathcal{H})} \text{ by definition of } M_2 \\ &\leq M_2 \|V\|_{1,2} \|X \oplus 0\|_{\mathcal{C}_E(\mathcal{H})} \\ &= M_2 \|V\|_{1,2} \|X\|_{\mathcal{C}_E(\mathcal{H}_e)}. \end{aligned}$$

Then $V : \mathcal{C}_E(\mathcal{H}_e) \rightarrow \mathcal{C}_F(\mathcal{H}_e)$ is a bounded operator and hence $\langle \mathcal{C}_E(\mathcal{H}_e), \mathcal{C}_F(\mathcal{H}_e) \rangle$ is an interpolation pair for $\langle (\mathcal{C}_{E_1}(\mathcal{H}_e), \mathcal{C}_{E_2}(\mathcal{H}_e)), (\mathcal{C}_{F_1}(\mathcal{H}_e), \mathcal{C}_{F_2}(\mathcal{H}_e)) \rangle$ and

$$M \{ \langle \mathcal{C}_E(\mathcal{H}_e), \mathcal{C}_F(\mathcal{H}_e) \rangle; \langle (\mathcal{C}_{E_1}(\mathcal{H}_e), \mathcal{C}_{E_2}(\mathcal{H}_e)), (\mathcal{C}_{F_1}(\mathcal{H}_e), \mathcal{C}_{F_2}(\mathcal{H}_e)) \rangle \} \leq M_2.$$

This completes the proof since

$$\begin{aligned} &M \{ \langle \mathcal{C}_E(\mathcal{H}_e), \mathcal{C}_F(\mathcal{H}_e) \rangle; \langle (\mathcal{C}_{E_1}(\mathcal{H}_e), \mathcal{C}_{E_2}(\mathcal{H}_e)), (\mathcal{C}_{F_1}(\mathcal{H}_e), \mathcal{C}_{F_2}(\mathcal{H}_e)) \rangle \} \\ &= M \{ \langle \mathcal{C}_E(\mathcal{H}), \mathcal{C}_F(\mathcal{H}) \rangle; \langle (\mathcal{C}_{E_1}(\mathcal{H}), \mathcal{C}_{E_2}(\mathcal{H})), (\mathcal{C}_{F_1}(\mathcal{H}), \mathcal{C}_{F_2}(\mathcal{H})) \rangle \}, \end{aligned}$$

the interpolation constants being independent of the underlying Hilbert space. $\blacksquare$

Hence, just as for unitary matrix spaces, the theory of interpolation of generalised unitary matrix spaces reduces to the theory of interpolation of symmetric sequence spaces and, just as in [1], we now have some easy corollaries.

Corollary 2.71 *Let A be a C^* -algebra and E , E_1 and E_2 be symmetric sequence spaces. Then E is an interpolation space for (E_1, E_2) if and only if $\mathcal{C}_E(A)$ is an interpolation space for $(\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A))$. Moreover, in that case,*

$$M \{ E; (E_1, E_2) \} = M \{ \mathcal{C}_E(A); (\mathcal{C}_{E_1}(A), \mathcal{C}_{E_2}(A)) \}.$$

Proof. By setting $F_1 = E_1$, $F_2 = E_2$ and $F = E$ in the previous theorem. $\blacksquare$

In the next corollary, we consider the Lorentz elements (see Definition 2.36) of a C^* -algebra and let $[\cdot, \cdot]_{\theta, q; K}$ be the generalised K -method (see [1, Section 2.3.f]).

Corollary 2.72 *Let A be a C^* -algebra, $1 \leq p_1, p_2 \leq \infty$ where $p_1 \neq p_2$, $0 < \theta < 1$ and $1 \leq q \leq \infty$. Define p by $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then, for every $1 \leq q_1, q_2 \leq \infty$, we have*

$$[\mathcal{C}_{p_1, q_1}(A), \mathcal{C}_{p_2, q_2}(A)]_{\theta, q; K} = \mathcal{C}_{p, q}(A).$$

In particular,

$$[\mathcal{C}_{p_1}(A), \mathcal{C}_{p_2}(A)]_{\theta, q; K} = \mathcal{C}_{p, q}(A) \text{ and } [\mathcal{C}_{p_1}(A), \mathcal{C}_{p_2}(A)]_{\theta, p; K} = \mathcal{C}_p(A).$$

Proof. The interpolation functor $[\cdot, \cdot]_{\theta, q; K}$ is exact, $\mathcal{C}_{r, r}(A) = \mathcal{C}_r(A)$ for all r and $[l^{p_1, q_1}, l^{p_2, q_2}]_{\theta, q; K} = l^{p, q}$ with equivalent (not exact) norms by [4]. The result is immediate by applying Theorem 2.70. ■

Corollary 2.73 *Let A be a C^* -algebra and E be a symmetric sequence space. Then $\mathcal{C}_E(A)$ is an interpolation space for $(\mathcal{C}_1(A), \mathcal{K}(A))$. Moreover,*

$$M\{\mathcal{C}_E(A); (\mathcal{C}_1(A), \mathcal{K}(A))\} = 1.$$

Proof. Immediate since every symmetric sequence space E is an interpolation space for (l^1, c_0) and $M\{E; (l^1, c_0)\} = 1$ (see [5, Section 1]). ■

Corollary 2.74 *Let A be a C^* -algebra and $1 < q \leq p < \infty$. Then $\mathcal{C}_{p, q}(A)$ is uniformly convex.*

Proof. If A_0 and A_1 are Banach spaces such that at least one is uniformly convex, then $[A_0, A_1]_{\theta, q; K}$ is uniformly convex (see [1, page 471]). Since $\mathcal{C}_2(A)$ is a Hilbert space, it is uniformly convex and we fill in the other cases using $\mathcal{C}_{p, q}(A) = [\mathcal{C}_2(A), \mathcal{C}_\infty(A)]_{\theta, q; K}$ where $2 < p < \infty$ and $\mathcal{C}_{p, q}(A) = [\mathcal{C}_1(A), \mathcal{C}_2(A)]_{\theta, q; K}$ where $1 < p < 2$. Alternatively, this can be proved directly since $\pi : \mathcal{C}_{p, q}(A) \rightarrow \mathcal{C}_{p, q}(\mathcal{H})$ is an isometry and $\mathcal{C}_{p, q}(\mathcal{H})$ is uniformly convex. ■

To identify the classes of elements obtained when using the approximation method, the notion of ‘Lebesgue elements’ of a C^* -algebra is required.

Definition 2.75 *Let A be a C^* -algebra and fix $p, q > 0$. Define the quasi-normed spaces*

$$\mathcal{L}_{p, q}(A) = \left\{ a \in A : \|a\|_{\mathcal{L}_{p, q}} \stackrel{\text{def}}{=} \left(\int_0^\infty \left[t^{\frac{1}{p}} s_{[t]+1}(a) \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} < \infty \right\},$$

and use $\mathcal{L}_p(A)$ for $\mathcal{L}_{p, p}(A)$.

Let A be a C^* -algebra and equip $\mathcal{FR}(A)$ with the quasi-norm $\|\cdot\|_{\mathcal{FR}}$ which is defined by $\|a\|_{\mathcal{FR}} = \text{rank } a$.

Lemma 2.76 *For $a \in A$, $E(t, a, \mathcal{FR}(A), A) = s_{[t]+1}(a)$.*

Proof. $E(t, a, \mathcal{FR}(A), A) = \inf_{\text{rank } b \leq t} \|a - b\| = \inf_{\text{rank } b \leq [t]} \|a - b\| = s_{[t]+1}(a)$. ■

Theorem 2.77 *Fix $p, q > 0$.*

(i) $E_{\frac{1}{p}, q}(\mathcal{FR}(A), A) = \mathcal{L}_{p, q}(A)$ with equality of quasi-norms;

(ii) if $\theta = \frac{p}{p+1}$ and $r = \theta q$, then $K_{\theta,q}(\mathcal{FR}(A), A) = (\mathcal{L}_{p,q}(A))^\theta$ with equivalent quasi-norms.

(iii) Let $p_0, p_1, q > 0$, $\theta = \frac{1}{\beta+1}$ and $r = \theta q$. Set $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then

$$E_{\beta,r}(\mathcal{L}_{p_0}(A), \mathcal{L}_{p_1}(A)) = (\mathcal{L}_{p,q}(A))^\frac{1}{\beta},$$

with equivalent quasi-norms.

Proof. (i)

$$\begin{aligned} \|a\|_{\frac{1}{p},q;E} &= \Phi_{\frac{1}{p},q}(E(t,a)) = \left(\int_0^\infty \left[t^{\frac{1}{p}} E(t,a) \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left[t^{\frac{1}{p}} s_{[t]+1}(a) \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} = \|a\|_{\mathcal{L}_{p,q}}. \end{aligned}$$

(ii)

$$\begin{aligned} (\mathcal{L}_{p,r}(A))^\theta &= \left(E_{\frac{1}{p},r}(\mathcal{FR}(A), A) \right)^\theta \text{ by (i)} \\ &= K_{\theta,q}(\mathcal{FR}(A), A) \text{ by [4, Theorem 7.1.7]}. \end{aligned}$$

(iii) This follows from (ii) by applying the reiteration theorem for the real method. ■

2.8 Volterra Elements

Next, we shall introduce a class of elements of the C^* -algebra such that their images under π are Volterra operators on the Hilbert space $\mathcal{H}$: see Definition 1.11.

Definition 2.78 An element a of a C^* -algebra A is said to be a **quasi-nilpotent element** of the algebra if $\lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$. An element a is said to be a **Volterra element** if it is compact and quasi-nilpotent. We use $\mathcal{Q}(A)$ and $\mathcal{V}(A)$ to denote the sets of quasi-nilpotent and Volterra elements of A respectively.

There are alternate definitions in terms of the wedge operator.

Definition 2.79 Let A be a C^* -algebra. We say that $a \in A$ is a ω -quasi-nilpotent element if the operator

$$a \wedge a : b \in A \mapsto aba \in A,$$

is quasi-nilpotent and we use $\mathcal{Q}_\omega(A)$ to denote the set of ω -quasi-nilpotent elements of A . Moreover, we say that $a \in A$ is a ω -Volterra element if a is both ω -compact and ω -quasi-nilpotent. We use $\mathcal{V}_\omega(A)$ to denote the set of all ω -Volterra elements of A .

Proposition 2.80 Let A be a C^* -algebra. Then $\mathcal{V}(A) = \mathcal{V}_\omega(A)$.

Proof. Since $\mathcal{K}(A) = \mathcal{K}_\omega(A)$, it suffices to show that $\mathcal{Q}(A) = \mathcal{Q}_\omega(A)$. For all $n \in \mathbb{N}$, $(a \wedge a)^n = a^n \wedge a^n$ and, upon showing that $\|a \wedge a\| = \|a\|^2$, we have that

$$\|(a \wedge a)^n\|^{\frac{1}{n}} = \|a^n\|^{\frac{2}{n}},$$

and hence $a \in \mathcal{Q}_\omega(A)$ if and only if $a \in \mathcal{Q}(A)$, as required.

Now, since A is a C^* -algebra, we have

$$\|x\| = \sup \{\|xy\| : y \in A, \|y\| \leq 1\},$$

and so, for $a \neq 0$, we have

$$\|aa^*a\| \geq \left\| aa^*a \frac{a^*}{\|a\|} \right\| = \frac{1}{\|a\|} \|aa^*\|^2 = \|a\|^3.$$

Thus $\|a \wedge a\| \geq \|a\|^2$. But clearly $\|a \wedge a\| \leq \|a\|^2$ and the result follows. ■

Proposition 2.81 *If $a \in \mathcal{V}(A)$, then $\pi(a)$ is a Volterra operator on $\mathcal{H}$.*

Proof. $\pi(a)$ is a compact operator on $\mathcal{H}$ and it is quasi-nilpotent since

$$\begin{aligned} \|\pi(a)^n\|^{\frac{1}{n}} &= \|\pi(a^n)\|^{\frac{1}{n}} \text{ since } \pi \text{ is a homomorphism} \\ &= \|a^n\|^{\frac{1}{n}} \text{ since } \pi \text{ is an isometry} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$
■

Proposition 2.82 *Let $\mathcal{G}$ be a Hilbert space. Then $\mathcal{V}(\mathcal{B}(\mathcal{G})) = \mathcal{V}(\mathcal{G})$.*

Proof. Immediate from the facts that $\mathcal{K}(\mathcal{B}(\mathcal{G})) = \mathcal{K}(\mathcal{G})$ and that T is a quasi-nilpotent operator on the Hilbert space $\mathcal{G}$ if and only if T is quasi-nilpotent as an element of the C^* -algebra $\mathcal{K}(\mathcal{G})$. ■

Theorem 2.83 *The limit in norm of a sequence of Volterra elements is a Volterra element.*

Proof. Suppose that (a_n) is a sequence of Volterra elements and that $\|a_n - a\| \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\|\pi(a_n) - \pi(a)\| = \|\pi(a_n - a)\| = \|a_n - a\| \rightarrow 0,$$

as $n \rightarrow \infty$. Since each $\pi(a_n)$ is a Volterra operator, we have that $\pi(a)$ is also a Volterra operator since the limit in norm of a sequence of Volterra operators is, again, a Volterra operator (see [15, Theorem 1.4.1]).

Now $\pi(a)$ is a compact operator and satisfies (†) so we have that $a \in \mathcal{K}(\mathcal{A})$. Moreover, the proof of the Proposition 2.81 shows that a must be a quasi-nilpotent element and hence $a \in \mathcal{V}(\mathcal{A})$ as required. ■

Proposition 2.84 *If $a \in \mathcal{V}(\mathcal{A})$, then $a^* \in \mathcal{V}(\mathcal{A})$.*

Proof. If $a \in \mathcal{V}(\mathcal{A})$, then $a \in \mathcal{K}(\mathcal{A})$ and $\lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$. By Proposition 2.40, we have that $a^* \in \mathcal{K}(\mathcal{A})$. Finally, $\|(a^*)^n\|^{\frac{1}{n}} = \|a^n\|^{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$. ■

Before we state and prove our next two results, we give a definition and make a few observations.

Definition 2.85 *We define the order of an element $a \in \mathcal{V}(\mathcal{A})$ to be the infimum $p(a)$ of the numbers p such that $a \in \mathcal{C}_p(A)$. In the case that $a \in \mathcal{K}(A)$ is not a member of $\mathcal{C}_p(A)$ for any $1 \leq p < \infty$, we say that a has infinite order and set $p(a) = \infty$.*

It is obvious that $p(a) = p(a^*)$ and that the order of a in the algebra is the same as the order of $\pi(a)$ as an operator on $\mathcal{H}$.

For $a \in A$, write $a_R = \frac{1}{2}(a + a^*)$ and $a_I = \frac{1}{2i}(a - a^*)$. Then we have that

$$\pi(a_R) = \pi(a)_R \text{ and } \pi(a_I) = \pi(a)_I,$$

for all $a \in \mathcal{V}(\mathcal{A})$.

Proposition 2.86 *Let $a \in \mathcal{V}(\mathcal{A})$. If $p(a_I) \geq 1$, then $p(a_R) \leq p(a) = p(a_I)$.*

Proof.

$$\begin{aligned} p(a_R) &= p(\pi(a_R)) = p(\pi(a)_R) \\ &\leq p(\pi(a)) \text{ by [16, Theorem 3.6.1]} \\ &= p(\pi(a)_I) = p(\pi(a_I)) = p(a_I). \end{aligned}$$

■

Theorem 2.87 (Macaev) *Let $a \in \mathcal{V}(A)$. Fix $1 < p < \infty$ and suppose that $a_I \in \mathcal{C}_p(A)$. Then $a_R \in \mathcal{C}_p(A)$ also and we have the estimate*

$$\|a_R\|_p \leq \gamma_p \|a_I\|_p,$$

where the constant γ_p depends only on p .

Proof. Fix $a \in \mathcal{V}(\mathcal{A})$ and assume that $a_I \in \mathcal{C}_p(A)$. Then, by the above, we have that $\pi(a)_I = \pi(a_I) \in \mathcal{C}_p(\mathcal{H}) \cap \mathcal{V}(\mathcal{A})$. Hence, by the Macaev theorem for operators (see Theorem 1.12), we have that $\pi(a)_R \in \mathcal{C}_p(\mathcal{H})$ and that

$$\|\pi(a)_R\|_{\mathcal{C}_p(\mathcal{H})} \leq \gamma_p \|\pi(a)_I\|_{\mathcal{C}_p(\mathcal{H})},$$

where γ_p is some constant depending only on p . So $\pi(a_R) \in \mathcal{C}_p(\mathcal{H})$ and obviously satisfies (†) since $\pi(a)$ does. Thus $a_R \in \mathcal{C}_p(A)$ and we have

$$\begin{aligned} \|a_R\|_p &= \|\pi(a_R)\|_{\mathcal{C}_p(\mathcal{H})} = \|\pi(a)_R\|_{\mathcal{C}_p(\mathcal{H})} \leq \gamma_p \|\pi(a)_I\|_{\mathcal{C}_p(\mathcal{H})} \\ &= \gamma_p \|\pi(a_I)\|_{\mathcal{C}_p(\mathcal{H})} = \gamma_p \|a_I\|_p, \end{aligned}$$

as required since $\pi(a_I)$ also satisfies (†). ■

In the case $p = 2$, we can be more precise by adapting a theorem due to Sahnovič.

Proposition 2.88 (Sahnovič) *Let $a \in \mathcal{V}(A)$. If $a_I \in \mathcal{C}_2(A)$, then $a_R \in \mathcal{C}_2(A)$ and*

$$\|a_R\|_2^2 = \|a_I\|_2^2 = \frac{1}{2}\|a\|_2^2.$$

Proof. Suppose $a_I \in \mathcal{C}_2(A)$. Then $\pi(a)_I \in \mathcal{C}_2(\mathcal{H})$ by previous work. Since $\pi(a) \in \mathcal{V}(\mathcal{H})$, it follows from Sahnovič's theorem for operators on Hilbert space (see [16, Theorem 1.10.2]) that $\pi(a)_R \in \mathcal{C}_2(\mathcal{H})$ and

$$\|\pi(a)_R\|_{\mathcal{C}_2(\mathcal{H})}^2 = \|\pi(a)_I\|_{\mathcal{C}_2(\mathcal{H})}^2 = \frac{1}{2}\|\pi(a)\|_{\mathcal{C}_2(\mathcal{H})}^2. \quad (2.3)$$

Now $\pi(a_R) = \frac{1}{2}(\pi(a) + \pi(a)^*)$ satisfies $(\dagger)$ since $\pi(a)$ does. So $a_R \in \mathcal{C}_2(A)$ as required. The norm estimates follow immediately from (2.3) since π is an isometry. ■

2.9 Fredholm Elements

For an arbitrary C^* -algebra A , the set $\mathcal{K}(A)$ of compact elements is a closed, 2-sided, $*$ -ideal of A . Then the quotient space $A/\mathcal{K}(A)$ is a C^* -algebra with respect to the quotient norm given by

$$\|a + \mathcal{K}(A)\| = \inf \{\|a + b\| : b \in \mathcal{K}(A)\}.$$

Definition 2.89 *For a C^* -algebra A , we shall refer to the quotient algebra $A/\mathcal{K}(A)$ as the Calkin algebra of A .*

If $\mathcal{G}$ is a Hilbert space, it is clear from our previous work that the Calkin algebra of $\mathcal{B}(\mathcal{G})$ is just the Calkin algebra of $\mathcal{G}$, $\mathcal{B}(\mathcal{G})/\mathcal{K}(\mathcal{G})$.

We begin by proving a (limited) analogue of Atkinson's theorem. An account of Fredholm theory, detailing all of the results which follow (and more besides!) for compact operators can be found in [18, Section 3.3].

Theorem 2.90 (Atkinson) *Let A be a C^* -algebra with identity e . For $a \in A$, the following are equivalent:*

- (i) *the image of a is invertible in the Calkin algebra $A/\mathcal{K}(A)$ of A ;*
- (ii) *there exists an element $b \in A$ such that both $ab - e$ and $ba - e$ are in $\mathcal{K}(A)$;*
- (iii) *the image of a is invertible in $A/\mathcal{FR}(A)$;*
- (iv) *there exists an element $b \in A$ such that both $ab - e$ and $ba - e$ are in $\mathcal{FR}(A)$.*

Proof. (i) $\Leftrightarrow$ (ii):

$$\begin{aligned} & \text{The image of } a \text{ is invertible in } A/\mathcal{K}(A) \\ \Leftrightarrow & \exists b + \mathcal{K}(A) \in A/\mathcal{K}(A) \text{ such that} \\ & (a + \mathcal{K}(A))(b + \mathcal{K}(A)) = (e + \mathcal{K}(A)) = (b + \mathcal{K}(A))(a + \mathcal{K}(A)) \\ \Leftrightarrow & ab + \mathcal{K}(A) = e + \mathcal{K}(A) = ba + \mathcal{K}(A) \\ \Leftrightarrow & ab - e, ba - e \in \mathcal{K}(A). \end{aligned}$$

(iii) $\Leftrightarrow$ (iv): Similarly.

(i) $\Leftrightarrow$ (iii): by BA2.4 and the remarks on page 36 of [2], we know that invertibility modulo $\text{soc}(A)$ is equivalent to invertibility modulo $I(A)$, the kernel of the hull of the presocle of A , referred to as the set of *inessential elements* of A [2, Definition F3.1]. Moreover, [2, C*2.4] tells us that $I(A) = \overline{\text{soc}(A)}$ and hence invertibility modulo $\mathcal{FR}(A)$ is equivalent to invertibility modulo $\mathcal{K}(A)$ by Proposition 2.20 and Theorem 2.49. ■

Definition 2.91 *We shall say that an element a of a C^* -algebra A with identity e is a Fredholm element if it satisfies the conditions of Atkinson's theorem and we shall use $\mathcal{F}(A)$ to denote the set of Fredholm elements of A . Also, $\mathcal{F}(\mathcal{G})$ will denote the set of Fredholm operators on the Hilbert space $\mathcal{G}$.*

Proposition 2.92 *Let $\mathcal{G}$ be a Hilbert space. Then $\mathcal{F}(\mathcal{G}) = \mathcal{F}(\mathcal{B}(\mathcal{G}))$.*

Throughout the rest of this section, A will denote a C^* -algebra A with identity e .

Proposition 2.93 *Let $a \in \mathcal{F}(A)$. Then $\pi(a)$ is a Fredholm operator on $\mathcal{H}$.*

Proof. Since $\pi(e)$ is the identity on $\mathcal{H}$, the result follows by Atkinson's theorem. ■

Proposition 2.94 *Let $a, b \in \mathcal{F}(A)$. Then $a^*, ab \in \mathcal{F}(A)$.*

Proof. If $a \in \mathcal{F}(A)$, then there exists $b \in A$ such that $ab - e, ba - e \in \mathcal{K}(A)$ and hence $(ab - e)^*, (ba - e)^* \in \mathcal{K}(A)$. So $b^*a^* - e, a^*b^* - e \in \mathcal{K}(A)$ and thus $a^* \in \mathcal{F}(A)$.

Finally, if $a, b \in \mathcal{F}(A)$, then $a + \mathcal{K}(A), b + \mathcal{K}(A)$ are invertible in $A/\mathcal{K}(A)$ and hence $ab + \mathcal{K}(A)$ is invertible in $A/\mathcal{K}(A)$. So $ab \in \mathcal{F}(A)$. ■

2.10 Riesz Elements

For a Banach space X , the following conditions on an operator $T \in \mathcal{B}(X)$ are well-known (see [8], for example) to be equivalent:

- (i) T is a Riesz operator;
- (ii) T is essentially quasi-compact;
- (iii) T is essentially quasi-finite rank.

In the rest of this section, let A be a C^* -algebra with identity e . Define

$$\Phi : a \in A \mapsto a + \mathcal{K}(A) \in A/\mathcal{K}(A) \text{ and } \Psi : a \in A \mapsto a + \mathcal{FR}(A) \in A/\mathcal{FR}(A).$$

Definition 2.95 *An element $a \in A$ is said to be a Riesz element if the spectral radius of $\Phi(a)$ in $A/\mathcal{K}(A)$ is zero; a is said to be essentially quasi-compact if*

$$\lim_{n \rightarrow \infty} \kappa(a^n)^{\frac{1}{n}} = 0,$$

where

$$\kappa(a) \stackrel{\text{def}}{=} \inf \{\|a - k\| : k \in \mathcal{K}(A)\};$$

a is said to be **essentially quasi-finite rank** if

$$\lim_{n \rightarrow \infty} \iota(a^n)^{\frac{1}{n}} = 0,$$

where

$$\iota(a) \stackrel{\text{def}}{=} \inf \{ \|a - f\| : f \in \mathcal{FR}(A) \}.$$

We denote the sets of Riesz, essentially quasi-compact and essentially quasi-finite-rank elements by $\mathcal{R}(A)$, $\mathcal{EQK}(A)$ and $\mathcal{EQFR}(A)$ respectively and use $\mathcal{R}(\mathcal{G})$ to denote the set of Riesz operators on the Hilbert space $\mathcal{G}$.

Remark 2.96 $A/\mathcal{K}(A)$ is a normed algebra but we only have a semi-norm on $A/\mathcal{FR}(A)$.

We have the following trivial result.

Proposition 2.97 $\mathcal{EQFR}(A) = \mathcal{EQK}(A) = \mathcal{R}(A)$.

Proof. Since the closure of $\mathcal{FR}(A)$ with respect to the C^* -norm is $\mathcal{K}(A)$, the first equality is immediate. Finally,

$$\begin{aligned} a \in \mathcal{R}(A) &\Leftrightarrow \text{the spectral radius of } \Phi(a) = a + \mathcal{K}(A) \text{ in } A/\mathcal{K}(A) \text{ is zero} \\ &\Leftrightarrow \lambda\Phi(e) - \Phi(a) \text{ is invertible in } A/\mathcal{K}(A) \text{ for all } \lambda \neq 0 \\ &\Leftrightarrow \Phi(a) \text{ is quasi-nilpotent in } A/\mathcal{K}(A) \\ &\Leftrightarrow a \in \mathcal{EQK}(A), \end{aligned}$$

by definition of the norm in the quotient algebra $A/\mathcal{K}(A)$. ■

Theorem 2.98 $a \in \mathcal{R}(A) \Leftrightarrow \lambda e - a \in \mathcal{F}(A) \text{ for all } \lambda \neq 0$.

Proof. Now, $\lambda e - a \in \mathcal{F}(A)$ for all $\lambda \neq 0$ if and only if $\Phi(\lambda e - a)$ is invertible in $A/\mathcal{K}(A)$ for all $\lambda \neq 0$ (by Atkinson's theorem) and this occurs precisely when $a \in \mathcal{R}(A)$, by the proof of the previous proposition. ■

Proposition 2.99 If $a \in \mathcal{R}(A)$, then $\pi(a)$ is a Riesz operator on $\mathcal{H}$.

Proof. If $a \in \mathcal{R}(A)$, then $\lambda e - a \in \mathcal{F}(A)$ for all $\lambda \neq 0$ and hence $\lambda\pi(e) - \pi(a) \in \mathcal{F}(\mathcal{H})$ for all $\lambda \neq 0$. Thus $\pi(a) \in \mathcal{R}(\mathcal{H})$, as required. ■

Proposition 2.100 Let $\mathcal{G}$ be a Hilbert space. Then $\mathcal{R}(\mathcal{G}) = \mathcal{R}(\mathcal{B}(\mathcal{G}))$.

2.11 The Gelfand-Naimark Representation Theorems

We now make a few comments on the comparison between the Gelfand-Naimark theorems and the representation due to Erdos which we have used extensively.

Let $C_0(X)$ denote the algebra of all continuous, complex-valued functions which vanish at infinity on some locally compact Hausdorff space X . Let A be a commutative Banach algebra and let $\hat{A}$ denote the set of all non-zero multiplicative linear functionals on A . Then $\hat{A}$, equipped with the Gelfand topology, is a locally compact Hausdorff space.

Theorem 2.101 (Commutative Gelfand-Naimark) *Suppose that A is a commutative C^* -algebra. Then the Gelfand representation $x \mapsto \hat{x}$ is an isometric $*$ -representation of A onto $C_0(\hat{A})$.*

Proposition 2.102 *Let A be a commutative C^* -algebra and E be a symmetric sequence space. Then the Gelfand representation implements an isometric $*$ -representation of $\mathcal{C}_E(A)$ onto $\mathcal{C}_E(C_0(\hat{A}))$.*

Proof. First, it is clear that $a \in \sigma(A) \Leftrightarrow \hat{a} \in \sigma(C_0(\hat{A}))$. The rest now follows trivially: (a_n) is an E -associated sequence for a if and only if $(\hat{a}_n)$ is an E -associated sequence for $\hat{a}$ and, since the map $a \in A \mapsto \hat{a} \in C_0(\hat{A})$ is an isometry, we have that $\|a\|_E = \|\hat{a}\|_E$. ■

Let A be a C^* -algebra with identity e . For each $z \in A$, we construct a positive functional p such $p(e) = 1$ and $p(z^*z) = \|z\|^2$ and let

$$\phi_z : A \rightarrow \mathcal{B}(\mathcal{H}_z),$$

be the Gelfand-Naimark-Segal $*$ -representation of A on the Hilbert space $\mathcal{H}_z$ which is constructed from p . (Note that ϕ_z is norm decreasing and that $\|\phi_z(z)\| = \|z\|$.)

Theorem 2.103 (Gelfand-Naimark) *Let A be a C^* -algebra. Then*

$$\phi : a \in A \mapsto \bigoplus_{z \in A} \phi_z(a) \in \mathcal{B}(\mathcal{H}),$$

is an isometric $*$ -representation of A onto a norm-closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ where

$$\mathcal{H} = \bigoplus_{z \in A} \mathcal{H}_z.$$

Suppose that $a \in A$ is non-zero. Then $\phi(a) \in \mathcal{B}(\mathcal{H})$ is *not* single since the direct sum $S \oplus T$ of two non-zero operators is not single. Hence, the s -numbers of a in A and the s -numbers of $\phi(a)$ in $\mathcal{B}(\mathcal{H})$ will not, in general, be the same. (In the case that $a \in \sigma(A)$, the failure will be quite spectacular.) What is true is the following result.

Proposition 2.104 *An element $a \in A$ is single if and only if $\phi(a)$ is a single element of $\phi(A)$.*

Remark 2.105 When considering the s -numbers, therefore, the representation due to Erdos comes in to its own; the theorems to Gelfand and Naimark cannot move from an ‘abstract’ to a ‘concrete’ C^* -algebra without losing such information.

2.12 Related Results

The notion of generalised unitary matrix spaces is not, as we thought when we began the work, new. About six months after this work was completed, the papers [14] and [24] came to our attention. Both authors consider the following idea: suppose that a is non-zero. Then a^*a is positive and let (λ_n) denote its eigenvalues, arranged in decreasing order of magnitude and repeated according to multiplicity. There is a maximal orthogonal family of hermitian minimal idempotents $\{e_\alpha\}$ of A such that $e_\alpha a^*a = a^*a e_\alpha$. So $a^*a = \sum_n \lambda_n e_n$ where $e_n \in \{e_\alpha\}$ and we set $s_n(a) = \sqrt{\lambda_n}$. Giotopoulos also has the idea of using a representation but contents himself with stating that results can be transferred from $\mathcal{C}_p(\mathcal{H})$ to A_p (his symbol for what we call $\mathcal{C}_p(A)$).

Nevertheless, there are new results here. In particular, the interpolation of such generalised unitary matrix spaces and the representation of these classes on Hilbert space are new. The ideas of Volterra, Riesz and Fredholm elements are well-known but representing these classes on Hilbert space using Erdos’ construction is a new development.

Chapter 3

New Norms and Quasi-Norms on the Schatten p -Ideals

3.1 Introduction

The main thrust of this chapter is an idea based on a result in [22, Theorem 2.6]—see Theorem 1.22—where, in the context of L^∞ -matricially normed spaces, the authors consider a “norm of norms”. We generalise the study of such a structure to the Schatten p -ideals and consider some of the properties that these structures have. In order to study these structures, we need to make extensive use of the Maple and Mathematica computer packages.

However, we need to begin by making some observations about the Schatten p -norms when applied to scalar matrices.

3.2 The Pointwise Ordering Property

We begin by introducing a notion that will be of great importance in what follows.

Definition 3.1 *We shall say that a norm $\|\cdot\|$ on $M_n(\mathbb{C})$ satisfies the **pointwise ordering property**, denoted by (PO) in the sequel, if, whenever $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$, we have that $\|(\alpha_{ij})\| \leq \|(\beta_{ij})\|$.*

We commence our investigation with a negative result.

Proposition 3.2 *Fix $1 \leq q < 2$. Then $\exists \epsilon > 0$ such that*

$$\left\| \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right\|_q < \left\| \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\|_q.$$

Hence $\|\cdot\|_q$ does not satisfy (PO) on $M_2(\mathbb{C})$.

Proof. The details will be given later in appendix A.3. In passing, we remark that this result will be seen to be a consequence of Theorem 3.31. ■

Remark 3.3 It is clear that there is not one ϵ which will give the inequality for all $1 \leq q < 2$.

Remark 3.4 By Proposition 1.18(iii), we know that

$$\left\| \begin{pmatrix} a & \theta \\ \theta & d \end{pmatrix} \right\|_{n+m} \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{n+m},$$

where $(V, \{\|\cdot\|_n\})$ is a matricially normed space and $a \in M_n(V)$, $b \in M_{n,m}(V)$, $c \in M_{m,n}(V)$ and $d \in M_m(V)$ for some $n, m \in \mathbb{N}$. However, if we have one zero and one non-zero block on a diagonal, Proposition 3.2 shows that we cannot replace the zero block with a non-zero block and always get an increase in the norm. A result, however, in this direction is the following.

Proposition 3.5 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Fix $n, m \in \mathbb{N}$ and let $a \in M_n(V)$, $b \in M_{n,m}(V)$, $c \in M_{m,n}(V)$ and $d \in M_m(V)$. Then

$$\left\| \begin{pmatrix} a & b \\ c & \theta \end{pmatrix} \right\|_{n+m} \leq \left\| \begin{pmatrix} a & (1 \pm i)b \\ (1 \pm i)c & d \end{pmatrix} \right\|_{n+m},$$

where θ is the $m \times m$ matrix of zero elements of V .

Proof. Note that $\begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix} = \begin{pmatrix} a & \pm ib \\ \pm ic & -d \end{pmatrix}$ and so

$$\begin{aligned} \left\| \begin{pmatrix} 2a & (1 \pm i)b \\ (1 \pm i)c & \theta \end{pmatrix} \right\|_{n+m} &= \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & \pm ib \\ \pm ic & -d \end{pmatrix} \right\|_{n+m} \\ &\leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{n+m} + \left\| \begin{pmatrix} a & \pm ib \\ \pm ic & -d \end{pmatrix} \right\|_{n+m} \\ &= 2 \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{n+m}. \end{aligned}$$

The result now follows by dividing through by 2 and replacing b and c with $(1 \mp i)b'$ and $(1 \mp i)c'$ respectively. $\blacksquare$

Now let us return to the (PO) property for the Schatten p -norms on $M_n(\mathbb{C})$. We begin with an important (for our purposes, at least) lemma which shows how the Schatten p -norm of a matrix is controlled by the size of the matrix.

Lemma 3.6 (i) Let $A \in \mathcal{C}_p(\mathcal{H}, \mathcal{K})$ for some $1 \leq p \leq \infty$ and Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Then

$$\|A\|_{\mathcal{C}_p(\mathcal{H}, \mathcal{K})} \leq \min(\dim \mathcal{H}, \dim \mathcal{K})^{\frac{1}{p}} \|A\|_{\mathcal{C}_q(\mathcal{H}, \mathcal{K})},$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$;

(ii) in particular, if $(\alpha_{ij}) \in M_n(\mathbb{C})$, then $n^{\frac{-1}{p}} \|(\alpha_{ij})\|_p \leq n^{\frac{-1}{q}} \|(\alpha_{ij})\|_q$ for all $1 \leq p \leq q \leq \infty$.

Proof. By applying Hölder's inequality to $\|A\|_p = \|I_{\mathcal{K}}A\|_p = \|AI_{\mathcal{H}}\|_p$ where $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ are the identity operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. ■

Theorem 3.7 Suppose that $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$.

(i) If $p = 2m$ for some $m \in \mathbb{N}$ or $p = \infty$, then $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$;

(ii) if $1 \leq p < 2$, then

$$\|(\alpha_{ij})\|_p \leq n^{B_p} \|(\beta_{ij})\|_p,$$

where $B_p = \frac{2-p}{2p}$;

(iii) if $2m < p < 2m+1$ for some $m \in \mathbb{N}$, then

$$\|(\alpha_{ij})\|_p \leq n^{C_p} \|(\beta_{ij})\|_p,$$

where $C_p = \frac{p-2m}{2mp}$;

(iv) if $2m+1 \leq p < 2(m+1)$ for some $m \in \mathbb{N}$, then

$$\|(\alpha_{ij})\|_p \leq n^{D_p} \|(\beta_{ij})\|_p,$$

where $D_p = \frac{2(m+1)-p}{2(m+1)p}$.

Proof. (i) For each $p = 2m$ where $m \in \mathbb{N}$, we observe that $[(\alpha_{ij})^* (\alpha_{ij})]^m$ is positive and that

$$\|\alpha\|_p = \|(\alpha^* \alpha)^m\|_1^{\frac{1}{p}} = [\tau [(\alpha^* \alpha)^m]]^{\frac{1}{p}}.$$

Now

$$\begin{aligned} \tau [(\alpha^* \alpha)^m] &= \sum_{i=1}^n \sum_{k_1=1}^n \cdots \sum_{k_{m-1}=1}^n (\alpha^* \alpha)_{ik_1} \cdots (\alpha^* \alpha)_{k_{m-1}i} \\ &= \sum_{i=1}^n \sum_{k_1=1}^n \cdots \sum_{k_{m-1}=1}^n \sum_{j_1=1}^n \cdots \sum_{j_m=1}^n (\overline{\alpha_{j_1 i}} \alpha_{j_1 k_1}) \cdots (\overline{\alpha_{j_m k_{m-1}}} \alpha_{j_m i}). \end{aligned} \quad (3.1)$$

If $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$, then replacing each α by the corresponding β in (3.1) will certainly not decrease the total value. So

$$\tau [(\alpha^* \alpha)^m] \leq \tau [(\beta^* \beta)^m] \text{ and hence } \|(\alpha_{ij})\|_{2m} \leq \|(\beta_{ij})\|_{2m}.$$

Finally, we prove the case $p = \infty$. Let $e = (e_1, \dots, e_n)$, $f = (f_1, \dots, f_n) \in \mathbb{C}^n$ be unit vectors. Then $e' = (|e_1|, \dots, |e_n|)$ and $f' = (|f_1|, \dots, |f_n|)$ are also unit vectors and

$$|\langle (\alpha_{ij}) e, f \rangle| = \left| \sum_{i,j=1}^n \alpha_{ij} e_j f_i \right| \leq \sum_{i,j=1}^n \alpha_{ij} |e_j| |f_i| \leq \sum_{i,j=1}^n \beta_{ij} |e_j| |f_i| = \langle (\beta_{ij}) e', f' \rangle,$$

and hence $\|(\alpha_{ij})\|_{\infty} \leq \|(\beta_{ij})\|_{\infty}$.

(ii) Suppose that $1 \leq p < 2$. Then

$$\begin{aligned} n^{\frac{-1}{p}} \|(\alpha_{ij})\|_p &\leq n^{\frac{-1}{2}} \|(\alpha_{ij})\|_2 \text{ by Lemma 3.6} \\ &\leq n^{\frac{-1}{2}} \|(\beta_{ij})\|_2 \text{ since } \|(\alpha_{ij})\|_2 \leq \|(\beta_{ij})\|_2 \\ &\leq n^{\frac{-1}{2}} \|(\beta_{ij})\|_p \text{ since } \|(\beta_{ij})\|_2 \leq \|(\beta_{ij})\|_p, \end{aligned}$$

and so

$$\|(\alpha_{ij})\|_p \leq n^{\frac{1}{p}} n^{\frac{-1}{2}} \|(\beta_{ij})\|_p.$$

(iii) Suppose that $2m < p < 2m + 1$ for some $m \in \mathbb{N}$. Then

$$\|(\alpha_{ij})\|_p \leq \|(\alpha_{ij})\|_{2m} \leq \|(\beta_{ij})\|_{2m} \leq n^{\frac{1}{2m}} n^{\frac{-1}{p}} \|(\beta_{ij})\|_p,$$

as required.

(iv) Suppose that $2m + 1 \leq p < 2(m + 1)$ for some $m \in \mathbb{N}$. Then

$$n^{\frac{-1}{p}} \|(\alpha_{ij})\|_p \leq n^{\frac{-1}{2(m+1)}} \|(\alpha_{ij})\|_{2(m+1)} \leq n^{\frac{-1}{2(m+1)}} \|(\beta_{ij})\|_{2(m+1)} \leq n^{\frac{-1}{2(m+1)}} \|(\beta_{ij})\|_p,$$

as required. ■

Remark 3.8 (a) In the sequel, we shall use A_p to denote the appropriate constant B_p , C_p or D_p ;

(b) at each stage of the above calculations, the inequalities are best possible;

(c) we have already seen that $\|\cdot\|_p$ does not satisfy the pointwise ordering property when $1 \leq p < 2$ but we can now bound the behaviour of the norm using (ii) of the theorem;

(d) there does not seem to be any profit in using $\|\alpha^* \alpha\|_{\frac{p}{2}}^{\frac{1}{2}}$ instead of $\|\alpha\|_p$ in the hope of getting a better estimate.

Example 3.9 $\|(\alpha_{ij})\|_7 \leq n^{\frac{1}{38}} \|(\beta_{ij})\|_7$ and so we can bound the values of $\|(\alpha_{ij})\|_7$ and $\|(\beta_{ij})\|_7$ using a constant ≤ 2 for matrices which are less than 7×10^{16} square. In the more modest case of 3×3 matrices, we can say that $\|(\alpha_{ij})\|_7 \leq 1.0198 \|(\beta_{ij})\|_7$.

In specific cases, we can say more.

Proposition 3.10 Fix $p \in \mathbb{N}$. Suppose that $(\alpha_{ij}), (\beta_{ij}) \in M_n(\mathbb{C})$ are such that $(\alpha_{ij}) \geq 0$ (in the usual operator sense) and that $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$. Then $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$.

Proof.

$$\begin{aligned} \|(\alpha_{ij})\|_p &= \|(\alpha_{ij})^p\|_1^{\frac{1}{p}} = [\tau \{(\alpha_{ij})^p\}]^{\frac{1}{p}} \text{ since } (\alpha_{ij}) \geq 0 \\ &= \left[\sum_{i=1}^n \sum_{k_1=1}^n \cdots \sum_{k_{p-1}=1}^n \alpha_{ik_1} \alpha_{k_1 k_2} \cdots \alpha_{k_{p-1} i} \right]^{\frac{1}{p}} \\ &\leq \left[\sum_{i=1}^n \sum_{k_1=1}^n \cdots \sum_{k_{p-1}=1}^n \beta_{ik_1} \beta_{k_1 k_2} \cdots \beta_{k_{p-1} i} \right]^{\frac{1}{p}} \\ &= [\tau \{(\beta_{ij})^p\}]^{\frac{1}{p}} \leq \|(\beta_{ij})^p\|_1^{\frac{1}{p}} \\ &\leq \|(\beta_{ij})\|_p \text{ using Hölder's inequality.} \end{aligned}$$

■



Remark 3.11 This result is not valid if we replace the positivity of (α_{ij}) with mere self-adjointness: for each $1 \leq q < 2$, we can find $n \in \mathbb{N}$ such that

$$\left\| \begin{pmatrix} 1 & n \\ n & n^2 \end{pmatrix} \right\|_q < \left\| \begin{pmatrix} 0 & n \\ n & n^2 \end{pmatrix} \right\|_q.$$

We illustrate with the case $n = 100$ (see figure B.1 on page 105) where the horizontal axis denotes the quantity p and the vertical denotes the quantity

$$\left\| \begin{pmatrix} 1 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p - \left\| \begin{pmatrix} 0 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p.$$

(The norms of the two matrices are equal when, to four decimal places, $p = 1.9287$.)

Proposition 3.12 Suppose that $(\alpha_{ij}), (\beta_{ij}) \in M_n(\mathbb{C})$ are such that $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$. If:

- (i) (α_{ij}) has rank at most one, then $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$ for all $p \in [1, \infty]$;
- (ii) (β_{ij}) has rank at most one, then $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$ for all $p \in [2, \infty]$.

Proof. (i) Suppose that (α_{ij}) has rank at most one. Choose $p \in [1, \infty)$ and $m \in \mathbb{N}$ such that $2m \geq p$. Then

$$\|(\alpha_{ij})\|_p = \|(\alpha_{ij})\|_{2m} \leq \|(\beta_{ij})\|_{2m} \leq \|(\beta_{ij})\|_p.$$

(ii) Suppose that (β_{ij}) has rank at most one and fix $p \in [2, \infty]$. Then

$$\|(\alpha_{ij})\|_p \leq \|(\alpha_{ij})\|_2 \leq \|(\beta_{ij})\|_2 = \|(\beta_{ij})\|_p,$$

as required. ■

Remark 3.13 (ii) of the above result cannot be extended to $[1, \infty]$: the example in Remark 3.11 demonstrates this.

We can now use (i) of Theorem 3.7 to prove a result for 2×2 matrices.

Proposition 3.14 The norm $\|\cdot\|_p$ on $M_2(\mathbb{C})$ satisfies (PO) for all $2 \leq p \leq \infty$.

Proof. Suppose that $0 \leq \alpha_{ij} \leq \beta_{ij}$ for all $1 \leq i, j \leq n$ and suppose that $\mu_1 \geq \mu_2 \geq 0$ and $\lambda_1 \geq \lambda_2 \geq 0$ are the eigenvalues of the matrices $[(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$ and $[(\beta_{ij})^* (\beta_{ij})]^{\frac{1}{2}}$ respectively. Then $\mu_1^2 + \mu_2^2 \leq \lambda_1^2 + \lambda_2^2$ and $\mu_1 \leq \lambda_1$ by (i) of Theorem 3.7.

Fix $2 < p < \infty$ and refer to figure 3.1 on page 53. By re-scaling, we may assume that $\lambda_1^2 + \lambda_2^2 = 1$ and set $c = \lambda_1^p + \lambda_2^p$. Let (P) denote the curve $x^p + y^p = c$. So (P) passes through both (λ_1, λ_2) and (λ_2, λ_1) .

Let (x_p, y_p) lie on (P). For $0 \leq x_p \leq \lambda_2$, the curve (P) is sandwiched between the curve $x^2 + y^2 = 1$ and the line $y = \lambda_1$; a similar situation is in effect for $x_p \geq \lambda_1$. In the region $\lambda_2 \leq x_p \leq \lambda_1$, the curve (P) lies to the right of the curve $x^2 + y^2 = 1$.

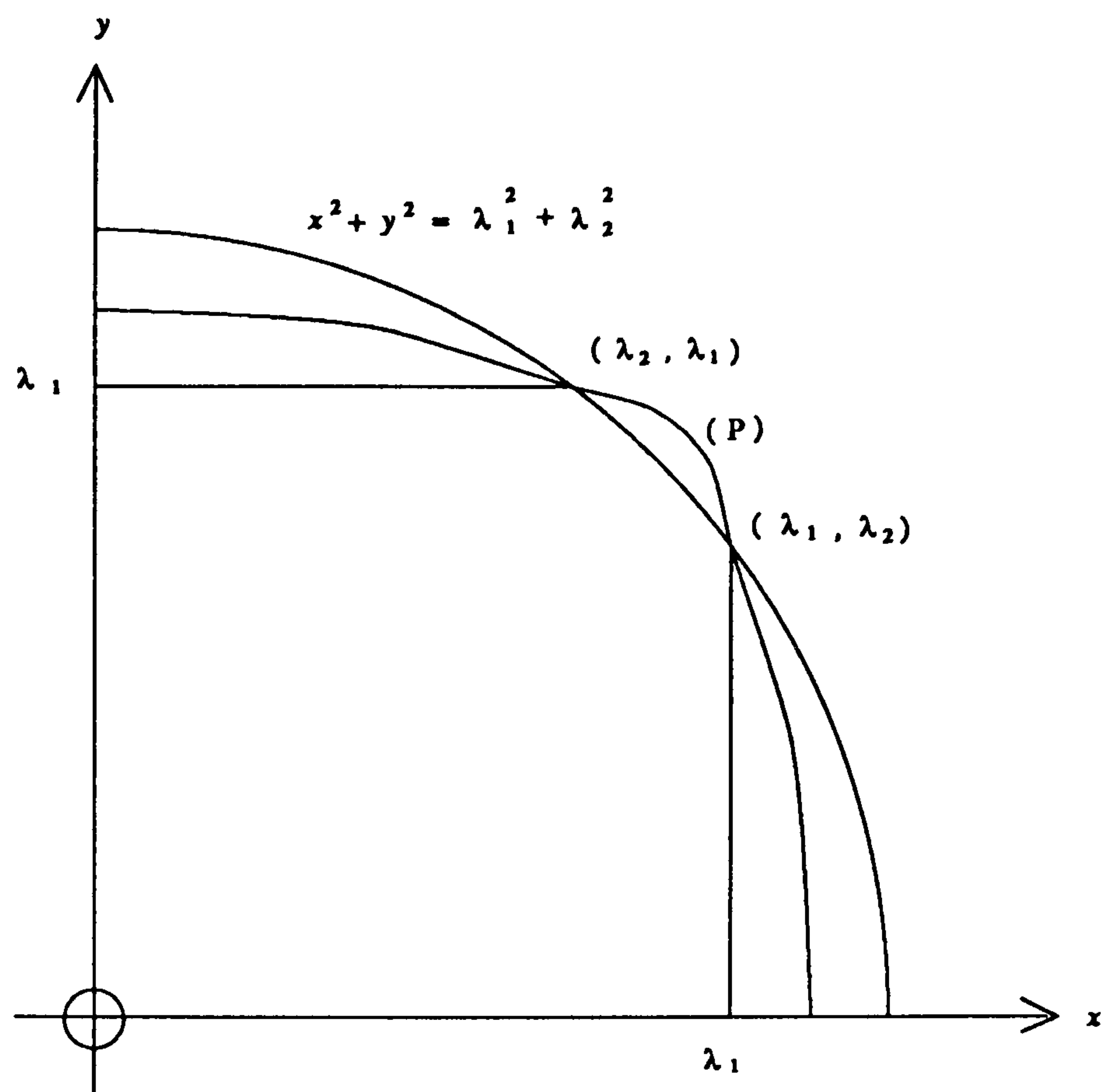


Figure 3.1: Pointwise ordering in two dimensions

Since (μ_1, μ_2) and (μ_2, μ_1) lie on or inside the curve $x^2 + y^2 = 1$, on or to the left of the line $x = \lambda_1$ and on or below the line $y = \lambda_1$, both points must be either on, or to the left of, (P) and hence $\mu_1^p + \mu_2^p \leq \lambda_1^p + \lambda_2^p$, that is to say, $\|(\alpha_{ij})\|_p \leq \|(\beta_{ij})\|_p$. ■

Remark 3.15 Is there an analytic proof of Proposition 3.14?

Remark 3.16 Trying to prove a similar result for $M_n(\mathbb{C})$ where $n \geq 3$ by the above method does not seem to work. However, we believe the following to be true:

Conjecture 3.17 For $2 \leq p \leq \infty$, $\|\cdot\|_p$ satisfies the pointwise ordering property on $M_n(\mathbb{C})$ for all $n \in \mathbb{N}$.

Remark 3.18 (i) Suppose that $(\alpha_{ij}), (\beta_{ij}) \in M_n(\mathbb{C})$ for some $n \geq 2$, that $0 \leq \alpha_{ij} \leq \beta_{ij}$ and that $(h_{ij}) = [(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$ and $(k_{ij}) = [(\beta_{ij})^* (\beta_{ij})]^{\frac{1}{2}}$. Then it need not follow that $h_{ij} \leq k_{ij}$ for all i, j or even that $0 \leq h_{ij}$ for all i, j . (Of course, in the case of 2×2 matrices, $\alpha_{ij} \geq 0$ for all i, j implies that $h_{ij} \geq 0$ for all i, j .)

Let $A = \begin{bmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 0 & 5 & 4 \end{bmatrix}$ and so $A^*A = \begin{bmatrix} 5 & 12 & 3 \\ 12 & 70 & 29 \\ 3 & 29 & 18 \end{bmatrix}$. Using Maple, we see that S , the square root of A^*A , is

$$\begin{bmatrix} 1.855 & 1.247 & -.0443 \\ 1.247 & 7.857 & 2.590 \\ -.0443 & 2.590 & 3.360 \end{bmatrix},$$

to three decimal places.

Let $B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$. Then

$$[B^*B]^{\frac{1}{2}} = \begin{pmatrix} 3.818 & 1.556 \\ 1.556 & 3.253 \end{pmatrix} \text{ and } [C^*C]^{\frac{1}{2}} = \begin{pmatrix} 4.384 & 2.404 \\ 2.404 & 2.687 \end{pmatrix}.$$

(ii) Also, it need not follow that the entries in $[(\alpha_{ij})^*(\alpha_{ij})]^{\frac{1}{2}}$ are smaller than those in $[(|\alpha_{ij}|)^*(|\alpha_{ij}|)]^{\frac{1}{2}}$.

Let $D = \begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix}$. Then $[D^*D]^{\frac{1}{2}} = \begin{pmatrix} 4.070 & 0.657 \\ 0.657 & 3.545 \end{pmatrix}$, which should be compared with the entries of $[B^*B]^{\frac{1}{2}}$.

3.3 Expressions for Trace Class Norms

The only result of this section which will be used in the sequel is Lemma 3.22(i). However, we need some additional facts in order to prove the other assertions of said lemma.

Definition 3.19 Let $A \in M_n(\mathbb{C})$. We use $A^{(i_1, \dots, i_{n-r})}$, $A_{(j_1, \dots, j_{n-r})}$ and $A_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})}$ to denote, respectively, the $r \times n$ matrix obtained by deleting rows $i_1, \dots, i_{n-r}$ of A , the $n \times r$ matrix obtained by deleting columns $j_1, \dots, j_{n-r}$ of A and the $r \times r$ matrix obtained by deleting rows $i_1, \dots, i_{n-r}$ and columns $j_1, \dots, j_{n-r}$ of A .

Theorem 3.20 (Cauchy-Binet Theorem) Suppose $C = AB$ where $A \in M_{r,n}(\mathbb{C})$ and $B \in M_{n,r}(\mathbb{C})$ where $r < n$. Then

$$\det C = \sum_{j_1 < \dots < j_{n-r}} [\det A_{(j_1, \dots, j_{n-r})}] [\det B^{(j_1, \dots, j_{n-r})}].$$

Proof. See [12, page 9] for details. ■

Proposition 3.21 Let $\alpha \in M_n(\mathbb{C})$ and let $\beta = (\alpha^* \alpha)^{\frac{1}{2}}$. Suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of β . Then, for $1 \leq r \leq n-1$,

$$\begin{aligned} & \lambda_1^2 \lambda_2^2 \cdots \lambda_r^2 + \lambda_1^2 \cdots \lambda_{r-1}^2 \lambda_{r+1}^2 + \cdots + \lambda_{n-r+1}^2 \lambda_{n-r+2}^2 \cdots \lambda_n^2 \\ &= \sum_{i_1 < \dots < i_{n-r}} \det (\alpha^* \alpha)_{(i_1, \dots, i_{n-r})}^{(i_1, \dots, i_{n-r})} \\ &= \sum_{i_1 < \dots < i_{n-r}} \sum_{j_1 < \dots < j_{n-r}} \left| \det \alpha_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right|^2 \\ &= \sum_{i_1 < \dots < i_{n-r}} \sum_{j_1 < \dots < j_{n-r}} \left| \det \beta_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right|^2, \end{aligned}$$

where the first line is the sums of squares of the $\binom{n}{r}$ r -tuples $\lambda_{k_1} \cdots \lambda_{k_r}$.

Proof. That the first two quantities are equal is a well-known fact (for example, see [17, Theorem 7.1.2]). Now

$$(\alpha^* \alpha)_{(i_1, \dots, i_{n-r})}^{(i_1, \dots, i_{n-r})} = (\alpha^*)_{(i_1, \dots, i_{n-r})}^{(i_1, \dots, i_{n-r})} \alpha_{(i_1, \dots, i_{n-r})},$$

is the product of an $r \times n$ and an $n \times r$ matrix where $r < n$. Hence (by the Cauchy-Binet theorem),

$$\begin{aligned} \det(\alpha^* \alpha)_{(i_1, \dots, i_{n-r})}^{(i_1, \dots, i_{n-r})} &= \sum_{j_1 < \dots < j_{n-r}} \left[\det(\alpha^*)_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right] \left[\det \alpha_{(i_1, \dots, i_{n-r})}^{(j_1, \dots, j_{n-r})} \right] \\ &= \sum_{j_1 < \dots < j_{n-r}} \left[\det(\alpha^*)_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right] \overline{\left[\det(\alpha^*)_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right]} \\ &= \sum_{j_1 < \dots < j_{n-r}} \left| \det(\alpha^*)_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right|^2 \\ &= \sum_{j_1 < \dots < j_{n-r}} \left| \det \alpha_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right|^2. \end{aligned}$$

Thus

$$\sum_{i_1 < \dots < i_{n-r}} \det(\alpha^* \alpha)_{(i_1, \dots, i_{n-r})}^{(i_1, \dots, i_{n-r})} = \sum_{i_1 < \dots < i_{n-r}} \sum_{j_1 < \dots < j_{n-r}} \left| \det \alpha_{(j_1, \dots, j_{n-r})}^{(i_1, \dots, i_{n-r})} \right|^2,$$

as required. The result involving the matrix β now follows immediately. ■

Lemma 3.22 (i) Let $(\alpha_{ij}) \in M_2(\mathbb{C})$. Then

$$\|(\alpha_{ij})\|_1 = \sqrt{\|(\alpha_{ij})\|_2^2 + 2 |\det(\alpha_{ij})|}.$$

(ii) Let $\alpha = (\alpha_{ij}) \in M_3(\mathbb{C})$. Then

$$\|(\alpha_{ij})\|_1^2 = \|(\alpha_{ij})\|_2^2 + 2 \sqrt{\sum_{i,j=1}^3 \left| \det \alpha_{(j)}^{(i)} \right|^2 + 2 |\det(\alpha_{ij})| \|(\alpha_{ij})\|_1}.$$

(iii) Let $(\alpha_{ij}) \in M_4(\mathbb{C})$. Then

$$\|(\alpha_{ij})\|_1^2 = \|(\alpha_{ij})\|_2^2 + 2\beta,$$

where β satisfies the relation

$$\beta^2 = \sum_{i < j} \sum_{k < l} \left| \det \alpha_{(k,l)}^{(i,j)} \right|^2 - 2 |\det \alpha| + 2 \|(\alpha_{ij})\|_1 \sqrt{\sum_{i,k=1}^4 \left| \det \alpha_{(k)}^{(i)} \right|^2 + 2 |\det \alpha| \beta}.$$

Proof. (i) is easy to prove directly and it also follows from (ii) (which will be proved) by embedding the matrix in $M_3(\mathbb{C})$. (iii) is more complicated and the proof is omitted.

Let $(\alpha_{ij}) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and λ_1, λ_2 and λ_3 be the eigenvalues of $(\alpha^* \alpha)^{\frac{1}{2}}$. Then

$$\alpha^* \alpha = \begin{pmatrix} |a|^2 + |d|^2 + |g|^2 & \bar{a}b + \bar{d}e + \bar{g}h & \bar{a}c + \bar{d}f + \bar{g}j \\ a\bar{b} + d\bar{e} + g\bar{h} & |b|^2 + |e|^2 + |h|^2 & \bar{b}c + \bar{e}f + \bar{h}j \\ a\bar{c} + d\bar{f} + g\bar{j} & b\bar{c} + e\bar{f} + h\bar{j} & |c|^2 + |f|^2 + |j|^2 \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix},$$

say. Hence

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 = \det \alpha^* \alpha \Rightarrow \lambda_1 \lambda_2 \lambda_3 = |\det \alpha|,$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = A + E + J = \|(\alpha_{ij})\|_2^2$$

$$\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 = AE + AJ + EJ - FH - BD - CG = \sum_{i,j=1}^3 \left| \det \alpha_{(j)}^{(i)} \right|^2.$$

Now

$$\begin{aligned} (\lambda_1 + \lambda_2 + \lambda_3)^2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \\ &= \|(\alpha_{ij})\|_2^2 + 2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3), \end{aligned}$$

and

$$\begin{aligned} (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)^2 &= \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 + 2\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3) \\ &= \sum_{i,j=1}^3 \left| \det \alpha_{(j)}^{(i)} \right|^2 + 2|\det \alpha| \|(\alpha_{ij})\|_1. \end{aligned}$$

Hence

$$\|(\alpha_{ij})\|_1^2 = \|(\alpha_{ij})\|_2^2 + 2\sqrt{\sum_{i,j=1}^3 \left| \det \alpha_{(j)}^{(i)} \right|^2 + 2|\det(\alpha_{ij})| \|(\alpha_{ij})\|_1},$$

as required. ■

Remark 3.23 A similar result for $M_5(\mathbb{C})$ is even more complicated (and even less informative). It seems unlikely that there is a general formula for $M_n(\mathbb{C})$.

We now give (as promised) a characterisation of single matrices. (Note that there are many more equivalent statements.)

Proposition 3.24 Suppose that $(\alpha_{ij}) \in M_n(\mathbb{C})$. Then the following are equivalent:

- (i) (α_{ij}) is single;
- (ii) $\sum_{i_1 < \dots < i_{n-2}} \det(\alpha^* \alpha)_{(i_1, \dots, i_{n-2})}^{(i_1, \dots, i_{n-2})} = 0$;
- (iii) $\sum_{i,j=1}^n |\alpha_{ij}|^2 = \left[\sum_{i,j=1}^n \left| \sum_{k=1}^n \alpha_{ki}^* \alpha_{kj} \right|^2 \right]^{\frac{1}{2}}$.

Proof. (iii) merely asserts that $\|(\alpha_{ij})\|_2 = \|(\alpha_{ij})\|_4$ whilst (ii) shows that (via Proposition 3.21) $[(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$ has at most one non-zero eigenvalue. ■

3.4 The Norm Inequality

Throughout this section, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, assumed to be separable if infinite dimensional.

Definition 3.25 We say that $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ is a numerical partition if $\sum_{i=1}^s n_i = n$ and $\sum_{j=1}^t m_j = m$ where $s, t, n_i, m_j \in \mathbb{N}$. Given a numerical partition $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we form the direct sums

$$\mathcal{K}^m = \mathcal{K}^{m_1} \oplus \dots \oplus \mathcal{K}^{m_t} \text{ and } \mathcal{H}^n = \mathcal{H}^{n_1} \oplus \dots \oplus \mathcal{H}^{n_s}.$$

Now fix $1 \leq p \leq \infty$ and suppose that $(A_{ij}) \in \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$ where $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ for all i, j ; in other words, we think of (A_{ij}) as a block $s \times t$ matrix. This enables us to consider the quantity

$$\|(A_{ij})\|_{\mathcal{D}_p(n, m; n_1, \dots, n_s, m_1, \dots, m_t)} \stackrel{\text{def}}{=} \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)}. \quad (3.2)$$

Definition 3.26 Fix $n, m \in \mathbb{N}$. We say that $\|\cdot\|_p$ on $\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$ satisfies $(*, p)$ if:

(i) in the case that $1 \leq p \leq 2$, we have

$$\|(A_{ij})\|_p \geq \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)},$$

for all numerical partitions $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and all $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$;

(ii) in the case that $2 \leq p \leq \infty$, we have

$$\|(A_{ij})\|_p \leq \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^t, \mathbb{C}^s)},$$

for all numerical partitions $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and all $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$.

Remark 3.27 In the sequel (and with the exception of the statement of a result), we shall generally suppress the underlying spaces on the norms, writing

$$\|(A_{ij})\|_p \leq \left\| \left[\|A_{ij}\|_p \right] \right\|_p$$

as shorthand for (ii) and using $\|\cdot\|_{\mathcal{D}_p}$ for $\|\cdot\|_{\mathcal{D}_p(n, m; n_1, \dots, n_s; m_1, \dots, m_t)}$. Also, we shall refer to the constructions given in (3.2) as $\mathcal{D}_p$ structures.

Theorem 3.28 $(M_2(\mathbb{C}), \|\cdot\|_p)$ satisfies $(*, p)$ for all $1 \leq p \leq \infty$.

Proof. There are four numerical partitions that can be made.

$$(a) \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_p} = \left\| \left[\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}^2)} \right] \right\|_{\mathcal{C}_p(\mathbb{C})} : \text{this is just the usual } \|\cdot\|_p \text{ norm;}$$

$$(b) \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_p} = \left\| \left[\left\| \begin{pmatrix} a \\ c \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}, \mathbb{C}^2)} \left\| \begin{pmatrix} b \\ d \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}, \mathbb{C}^2)} \right] \right\|_{\mathcal{C}_p(\mathbb{C}^2, \mathbb{C})} : \text{this is easily seen to}$$

be just the usual $\|\cdot\|_2$ norm on $M_2(\mathbb{C})$ and hence $\|\cdot\|_{\mathcal{D}_p} \leq \|\cdot\|_p$ if $1 \leq p \leq 2$ with the inequality reversed if $2 \leq p \leq \infty$;

$$(c) \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_p} = \left\| \begin{bmatrix} \left\| \begin{pmatrix} a & b \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}^2, \mathbb{C})} \\ \left\| \begin{pmatrix} c & d \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}^2, \mathbb{C})} \end{bmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}, \mathbb{C}^2)} : \text{this is, again, just the usual } \|\cdot\|_2$$

norm on $M_2(\mathbb{C})$;

$$(d) \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_p} = \left\| \begin{pmatrix} |a| & |b| \\ |c| & |d| \end{pmatrix} \right\|_{\mathcal{C}_p(\mathbb{C}^2)}$$

Now

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_1} \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_1 \text{ by Lemma 3.22 as } |ad - bc| \geq ||a||d| - |b||c|,$$

it is clear that we have equality when $p = 2$ and that

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{D}_\infty} \geq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_\infty,$$

by Theorem 1.22. The intermediate cases can now be filled in using an argument along the lines of Proposition 3.14.

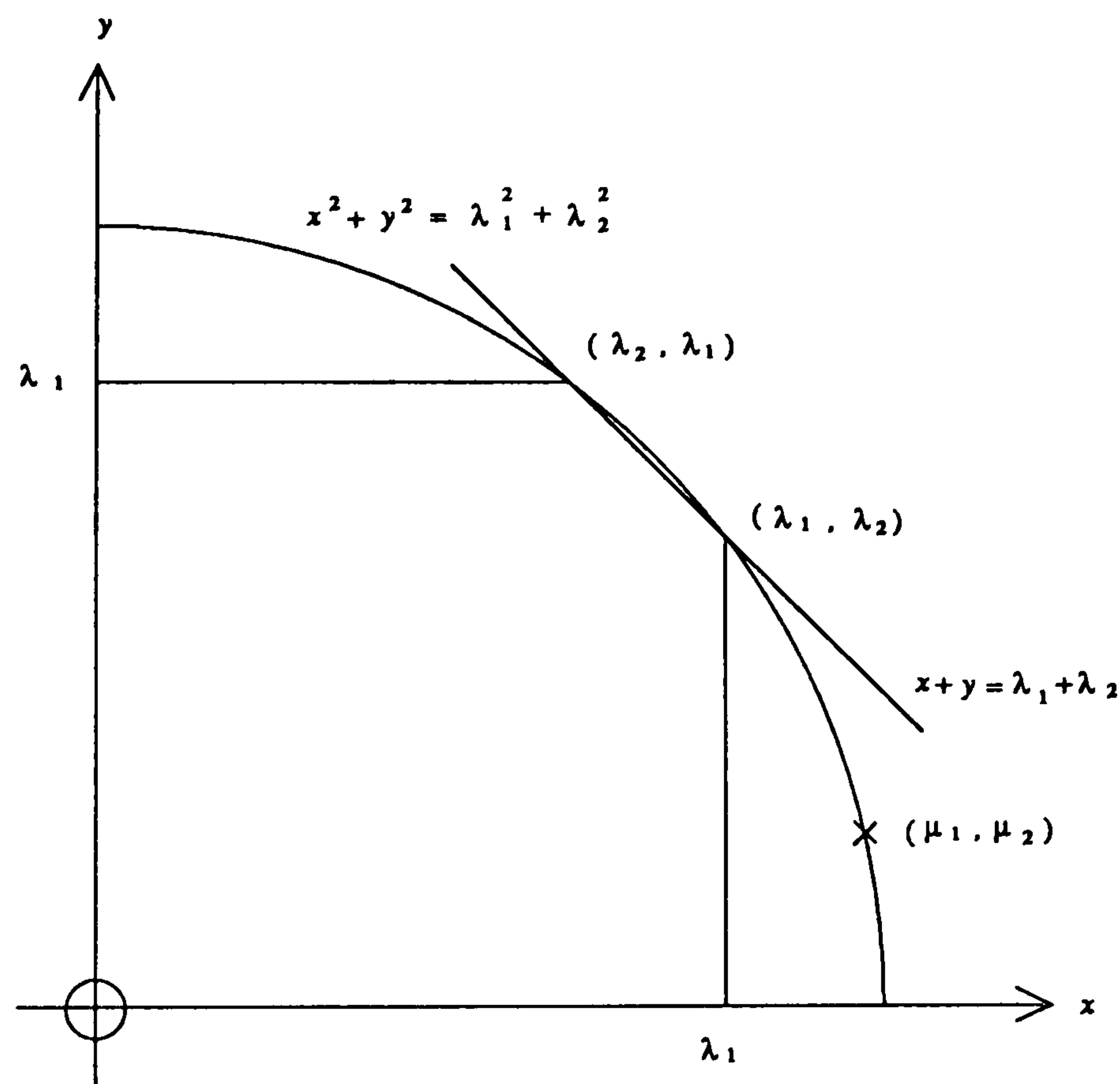


Figure 3.2: The norm inequality in two dimensions

Let $\lambda_1 \geq \lambda_2 \geq 0$ and $\mu_1 \geq \mu_2 \geq 0$ be the eigenvalues of the matrices $[(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$ and $[(|\alpha_{ij}|)^* (|\alpha_{ij}|)]^{\frac{1}{2}}$ respectively.

Then $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$ and $\mu_1 \geq \lambda_1$. Clearly, $\mu_2 \leq \lambda_2 \leq \lambda_1 \leq \mu_1$.

Fix $1 < p < 2$ and refer to figure 3.2 on page 58. By re-scaling, we may assume that $\lambda_1^2 + \lambda_2^2 = \mu_1^2 + \mu_2^2 = 1$ and set $c = \lambda_1^p + \lambda_2^p$. Let (P) denote the curve $x^p + y^p = c$. So (P) passes through both (λ_1, λ_2) and (λ_2, λ_1) .

Suppose that (x_p, y_p) lies on (P). For $0 \leq x_p \leq \lambda_2$, the curve (P) lies above the circle but below the straight line. For $x \geq \lambda_1$, (P) again lies below the straight line and above the circle. Finally, for $\lambda_2 \leq x_p \leq \lambda_1$, (P) lies above the straight line but below the circle. Hence, the points (μ_1, μ_2) and (μ_2, μ_1) must lie below (P). Since (P) passes through (λ_1, λ_2) , we have that $\mu_1^p + \mu_2^p \leq \lambda_1^p + \lambda_2^p$, that is to say,

$$\left\| \begin{pmatrix} |a| & |b| \\ |c| & |d| \end{pmatrix} \right\|_p \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_p.$$

A similar argument to the above gives the reverse inequality for $2 < p < \infty$. ■

Unfortunately, construction (d) is not a norm.

Proposition 3.29 Fix $1 \leq q < 2$ and let $\|\cdot\|_{\mathcal{D}_q}$ on $M_2(\mathbb{C})$ be given by

$$\|(\alpha_{ij})\|_{\mathcal{D}_q} = \|(|\alpha_{ij}|)\|_q.$$

Then there exists $\epsilon > 0$ such that

$$\left\| \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right\|_{\mathcal{D}_q} + \left\| \begin{pmatrix} 1 & 1 \\ 1 & -\epsilon \end{pmatrix} \right\|_{\mathcal{D}_q} < \left\| \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right\|_{\mathcal{D}_q}.$$

Hence $\|\cdot\|_{\mathcal{D}_q}$ is not a norm on $M_2(\mathbb{C})$.

Proof. The proof (which is quite tedious and not very instructive) involves evaluating the Taylor series of

$$f(q, \epsilon) = \left\| \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right\|_q - 2 \left\| \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right\|_q,$$

where $1 \leq q < 2$ and $\epsilon > 0$. Some of the details, as well as more detail concerning the next example, may be found in Section A.3 ■

Example 3.30 If $1 \leq q \leq 1.999$, then it suffices to take $\epsilon = 0.0001$:

$$\left\| \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right\|_{1.999} - 2 \left\| \begin{pmatrix} 1 & 1 \\ 1 & 0.0001 \end{pmatrix} \right\|_{1.999} = 0.439 \times 10^{-7}.$$

Sometimes, though, all the $\mathcal{D}_p$ structures are norms.

Theorem 3.31 (i) If $\|\cdot\|_p$ satisfies (PO) on $M_n(\mathbb{C})$, then $\|\cdot\|_{\mathcal{D}_p}$ is a norm;
(ii) if $p \neq 2q$ ($q \in \mathbb{N}$) or ∞ , then $\|\cdot\|_{\mathcal{D}_p}$ is a $\min\{s, t\}^{A_p}$ -quasi-norm, where A_p is the constant introduced in Proposition 3.7.

Proof. (i)

$$\begin{aligned}
\|(A_{ij}) + (B_{ij})\|_{\mathcal{D}_p} &= \left\| \left[\|A_{ij} + B_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)} \\
&\leq \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} + \|B_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)} \text{ by (PO)} \\
&\leq \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)} + \left\| \left[\|B_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)} \\
&= \|(A_{ij})\|_{\mathcal{D}_p} + \|(B_{ij})\|_{\mathcal{D}_p},
\end{aligned}$$

as required.

(ii) Similar to (i), except that we need to introduce the constant $\min\{s, t\}^{A_p}$ in the second line instead of using the pointwise ordering property. ■

Remark 3.32 The contrapositive of (i) says that if $\|\cdot\|_{\mathcal{D}_p}$ is not a norm, then $\|\cdot\|_p$ does not obey the (PO) property, confirming Proposition 3.2.

Whilst $M_2(\mathbb{C})$ has the property $(*, p)$ for all $1 \leq p \leq \infty$, $M_3(\mathbb{C})$ does not.

Example 3.33 Consider the matrix $A = \begin{bmatrix} n & -n & 0 \\ -n & \frac{1}{n} & \frac{1}{n} - n \\ 0 & \frac{1}{n} - n & \frac{1}{n} - n \end{bmatrix}$ and let $X = (|A_{ij}|)$. (We note in passing that A is self-adjoint.) Then we can find $n \in \mathbb{N}$ such that, for all $1 \leq p < 2$, we have

$$\|A\|_p < \|X\|_p,$$

which shows that $(*, p)$ does not hold for any $1 \leq p < 2$. However, we also have that, for any $2 < p < 4$, we can find $n \in \mathbb{N}$ such that

$$\|A\|_p > \|X\|_p,$$

which shows that $(*, p)$ does not hold in general for $2 < p < 4$.

We illustrate in the case that $n = 100$. Figure B.2 on page 106 shows the general behaviour whilst figure B.3 on page 107 gives a more detailed picture in the neighbourhood of $p = 4$. In both figures, the horizontal axis denotes p whilst the vertical denotes $\|(A_{ij})\|_p - \|(|A_{ij}|)\|_p$. So $M_3(\mathbb{C})$ does not possess the $(*, p)$ property for $1 \leq p < 2$ or $2 < p < 4$.

Remark 3.34 This example contradicts the all too plausible notion that

$$\|A\|_p \leq \|B\|_p \text{ for } p = 2m \text{ } (m \in \mathbb{N}), \infty \Rightarrow \|A\|_p \leq \|B\|_p \text{ for all } p \in [2, \infty].$$

At the level of sequences in l^p , this is clearly false (for example, see some of the sequences given in [7]). However, it is far from clear that such sequences can arise as the eigenvalues of two matrices in our highly specialised case.

Let us now consider some situations in which the property $(*, p)$ does hold.

Theorem 3.35 Fix $n, m \in \mathbb{N}$ and a numerical partition $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$. Let $p = 2q$ for some $q \in \mathbb{N}$ or ∞ . Then, if $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ ($1 \leq i \leq s, 1 \leq j \leq t$):

$$\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)} \leq \left\| \left[\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)}.$$

Proof. Clearly, there is equality when $p = 2$ and the case $p = \infty$ has been established in [22]: see Theorem 1.22. It remains only to consider the case $p = 2q$ for some $q \geq 2$. Now

$$\begin{aligned} \tau \left([(A_{ij})^* (A_{ij})]^q \right) &= \left| \tau \left([(A_{ij})^* (A_{ij})]^q \right) \right| = \left| \sum_{i=1}^t \tau \left\{ [(A_{ij})^* (A_{ij})]^q \right\}_{ii} \right| \\ &= \left| \sum_{i=1}^t \tau \left(\sum_{k_1=1}^t \cdots \sum_{k_{q-1}=1}^t [(A_{ij})^* (A_{ij})]_{ik_1} \cdots [(A_{ij})^* (A_{ij})]_{k_{q-1}i} \right) \right| \\ &= \left| \sum_{i=1}^t \tau \left(\sum_{k_1=1}^t \cdots \sum_{k_{q-1}=1}^t \sum_{j_1=1}^s \cdots \sum_{j_q=1}^s A_{j_1 i}^* A_{j_1 k_1} \cdots A_{j_q k_{q-1}}^* A_{j_q i} \right) \right| \\ &\leq \sum_{i=1}^t \sum_{k_1=1}^t \cdots \sum_{k_{q-1}=1}^t \sum_{j_1=1}^s \cdots \sum_{j_q=1}^s \left| \tau \left(A_{j_1 i}^* A_{j_1 k_1} \cdots A_{j_q k_{q-1}}^* A_{j_q i} \right) \right| \\ &\leq \sum_{i=1}^t \sum_{k_1=1}^t \cdots \sum_{k_{q-1}=1}^t \sum_{j_1=1}^s \cdots \sum_{j_q=1}^s \|A_{j_1 i}^*\|_{2q} \|A_{j_1 k_1}\|_{2q} \cdots \|A_{j_q k_{q-1}}^*\|_{2q} \|A_{j_q i}\|_{2q} \\ &= \tau \left(\left[(\|A_{ij}\|_{2q})^* (\|A_{ij}\|_{2q}) \right]^q \right). \end{aligned}$$

Then

$$\begin{aligned} \|(A_{ij})\|_{2q} &= \left\| [(A_{ij})^* (A_{ij})]^q \right\|_1^{\frac{1}{2q}} = \left[\tau \left\{ [(A_{ij})^* (A_{ij})]^q \right\} \right]^{\frac{1}{2q}} \\ &\leq \left[\tau \left\{ \left[(\|A_{ij}\|_{2q})^* (\|A_{ij}\|_{2q}) \right]^q \right\} \right]^{\frac{1}{2q}} = \left\| [\|A_{ij}\|_{2q}] \right\|_{2q}. \quad \blacksquare \end{aligned}$$

Though the $(*, p)$ property does not hold for $M_3(\mathbb{C})$ for $1 \leq p < 2$ or $2 < p < 4$, we can say something about cases.

Theorem 3.36 (i) $M_3(\mathbb{C})$ has the $(*, p)$ property for all $p \geq 4$;
(ii) for $M_4(\mathbb{C})$ and $M_5(\mathbb{C})$, there is at most one region $[2m, 2(m+1)]$ ($m \in \mathbb{N}$) in $[4, \infty)$ such that $(*, p)$ fails.

Proof. Fix $(\alpha_{ij}) \in M_n(\mathbb{C})$ and let $f(q) = \|(\alpha_{ij})\|_q^q - \|(|\alpha_{ij}|)\|_q^q$. Then $f(q) = (\lambda_1^q + \cdots + \lambda_n^q) - (\mu_1^q + \cdots + \mu_n^q)$ where the λ_i 's and the μ_j 's are the eigenvalues of $[(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$ and $[(|\alpha_{ij}|)^* (|\alpha_{ij}|)]^{\frac{1}{2}}$ respectively. If $f \equiv 0$, then there is nothing to prove. So assume $f \not\equiv 0$.

(i) If $n = 3$, then f has at most two roots, one of which is at 2. If there is only one root or if the second root is less than or equal to 4, we are done since $f(2m) \leq 0$ for all $m \in \mathbb{N}$. So suppose there is a second root $r > 4$. Then there exists an $m \in \mathbb{N}$ such that

$r \in [2m, 2(m+1)]$. Now, there does not exist an $s \in [2m, 2(m+1)]$ such that $f(s) > 0$ or else there would have to be two roots of f in $[2m, 2(m+1)]$, a contradiction. So the graph of f touches (but does not cross) the x -axis at r and hence $f(p) \leq 0$ for all $p \geq 4$. (ii) For $M_4(\mathbb{C})$, there are at most two roots— r and s , say—which are greater than or equal to 4. If there exists an $m \in \mathbb{N}$ such that $r < 2m < s$, then we have that r and s are local maxima with $f(r) = f(s) = 0$. If $r, s \in [2m, 2(m+1)]$, then we possibly get failure in this region. The case for $M_5(\mathbb{C})$ is similar. ■

Remark 3.37 (i) In the same way, if $n \geq 2$, then for $M_{2n}(\mathbb{C})$ and $M_{2n+1}(\mathbb{C})$, there are at most $(n-1)$ regions $[2m_1, 2(m_1+1)], \dots, [2m_{n-1}, 2(m_{n-1}+1)]$ in $[4, \infty)$ such that $(*, p)$ fails.

(ii) We can replace $\|(|\alpha_{ij}|)\|_q$ with $\left\| \left[\|\alpha_{ij}\|_q \right] \right\|_q$. The only difference this might make is that we have fewer than n eigenvalues; in that case, just add in the appropriate number of zeros.

Remark 3.38 (i) $0 \leq \alpha_{ij} \leq \beta_{ij} \not\Rightarrow \tau[(\alpha^* \alpha)^{\frac{1}{2}}] \leq \tau[(\beta^* \beta)^{\frac{1}{2}}]$: let $(\alpha_{ij}) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ and $(\beta_{ij}) = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}$. Then $(\alpha^* \alpha)^{\frac{1}{2}}$ has eigenvalues $\sqrt{15 + 5\sqrt{5}}$ and $\sqrt{15 - 5\sqrt{5}}$ and hence the value of the trace is $\sqrt{50}$. On the other hand, $(\beta^* \beta)^{\frac{1}{2}}$ has eigenvalues $\frac{7+\sqrt{17}}{2}$ and $\frac{7-\sqrt{17}}{2}$ which gives the matrix a trace of 7.

(ii) Similarly, it is not true in general that $\tau\{(\alpha^* \alpha)^{\frac{1}{2}}\} \leq \tau\{[(|\alpha_{ij}|)^*(|\alpha_{ij}|)]^{\frac{1}{2}}\}$: suppose that $(\gamma_{ij}) = \begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix}$. Then $(\gamma^* \gamma)^{\frac{1}{2}}$ has eigenvalues $\sqrt{15 + \sqrt{29}}$ and $\sqrt{15 - \sqrt{29}}$ and hence the value of the trace is $\sqrt{58}$.

In the context of infinite dimensional Hilbert spaces, we have one more result.

Proposition 3.39 For $p \neq 2q$ ($q \in \mathbb{N}$) or ∞ , we have

$$\|(A_{ij})\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \leq st \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{C_p(\mathbb{C}^t, \mathbb{C}^s)}.$$

Proof. $\|(A_{ij})\|_p \leq \sum_{i=1}^s \sum_{j=1}^t \|A_{ij}\|_p \leq st \max_{i,j} \|A_{ij}\|_p \leq st \left\| \left[\|A_{ij}\|_p \right] \right\|_p$. ■

The remainder of the results in this section will be concerned with the situation in which $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional, that is to say, are $\mathbb{C}^r$ for some r . For these, we need the constants A_p which were established in Proposition 3.7.

Theorem 3.40 (i) For $1 \leq p < 2$ or $2q+1 \leq p < 2(q+1)$ ($q \in \mathbb{N}$),

$$\|(A_{ij})\|_{C_p(\mathbb{C}^m, \mathbb{C}^n)} \leq h^{A_p} \left\| \left[\|A_{ij}\|_{C_p(\mathbb{C}^{m_j}, \mathbb{C}^{n_i})} \right] \right\|_{C_p(\mathbb{C}^t, \mathbb{C}^s)},$$

where $h = \min \{m, n\}$;

(ii) for $2q < p < 2q + 1$ ($q \in \mathbb{N}$),

$$\|(A_{ij})\|_{C_p(\mathbb{C}^m, \mathbb{C}^n)} \leq (kl)^{A_p} \left\| \left[\|A_{ij}\|_{C_p(\mathbb{C}^{m_j}, \mathbb{C}^{n_i})} \right] \right\|_{C_p(\mathbb{C}^t, \mathbb{C}^s)},$$

where, for each pair (i, j) , $g_{ij} = \min \{m_j, n_i\}$, $k = \max \{g_{ij} : 1 \leq i \leq s, 1 \leq j \leq t\}$ and $l = \min \{s, t\}$.

Proof. (i) For example, suppose that $2q + 1 \leq p < 2(q + 1)$. Recall that we found A_p in this ranges by considering the $\|\cdot\|_{2(q+1)}$ norm. Now

$$\begin{aligned} \|(A_{ij})\|_p &\leq h^{A_p} \|(A_{ij})\|_{2(q+1)} \leq h^{A_p} \left\| \left[\|A_{ij}\|_{2(q+1)} \right] \right\|_{2(q+1)} \\ &\leq h^{A_p} \left\| \left[\|A_{ij}\|_p \right] \right\|_{2(q+1)} \leq h^{A_p} \left\| \left[\|A_{ij}\|_p \right] \right\|_p, \end{aligned}$$

since $\|\cdot\|_{2(q+1)} \leq \|\cdot\|_p$. The case $1 \leq p < 2$ is analogous.

(ii) Suppose that $2q < p < 2q + 1$. Then

$$\begin{aligned} \|(A_{ij})\|_p &\leq \|(A_{ij})\|_{2q} \leq \left\| \left[\|A_{ij}\|_{2q} \right] \right\|_{2q} \leq \left\| \left[g_{ij}^{A_p} \|A_{ij}\|_p \right] \right\|_{2q} \\ &\leq \left\| \left[k^{A_p} \|A_{ij}\|_p \right] \right\|_{2q} = k^{A_p} \left\| \left[\|A_{ij}\|_p \right] \right\|_{2q} \\ &\leq k^{A_p} l^{A_p} \left\| \left[\|A_{ij}\|_p \right] \right\|_p = (kl)^{A_p} \left\| \left[\|A_{ij}\|_p \right] \right\|_p. \quad \blacksquare \end{aligned}$$

Corollary 3.41 In the ranges of Theorem 3.40, $\|(\alpha_{ij})\|_{C_p(\mathbb{C}^n)} \leq n^{A_p} \|(|\alpha_{ij}|)\|_{C_p(\mathbb{C}^n)}$.

Proof. This result follows by the theorem as $h = n$ or $k = 1$ and $l = n$. ■

Remark 3.42 In the case $1 \leq p < 2$ and $n_i = m_j = 1$ for all i and j , these bounds are the best possible. Fix $n \in \mathbb{N}$ such that $n \geq 2$ and let ω ($\neq 1$) be a n^{th} root of unity. Set

$$(\alpha_{ij}) = \left[\omega^{(i-1)(j-1)} \right]_{ij}.$$

Then $\alpha^* \alpha = nI_n$ and hence

$$\|(\alpha_{ij})\|_p = \sqrt{n} n^{\frac{1}{p}}.$$

However, each of the n^2 entries of $(|\alpha_{ij}|)$ are all 1 and hence $\|(|\alpha_{ij}|)\|_p = n$.

Example 3.43 From the corollary, we have that

$$\|(\alpha_{ij})\|_{C_5(\mathbb{C}^3)} \leq 3^{\frac{1}{20}} \|(|\alpha_{ij}|)\|_{C_5(\mathbb{C}^3)} \text{ where } 3^{\frac{1}{20}} \simeq 1.05647,$$

and

$$\|(\alpha_{ij})\|_{C_5(\mathbb{C}^3)} \leq 4^{\frac{1}{20}} \left\| \left[\frac{\left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\|_{C_5(\mathbb{C}^2)}}{\sqrt{|a_{31}|^2 + |a_{32}|^2}} \quad \frac{\sqrt{|a_{13}|^2 + |a_{23}|^2}}{|a_{33}|} \right] \right\|_{C_5(\mathbb{C}^2)}.$$

Remark 3.44 (i) We know that the property $(*, p)$ does not hold for $2 < p < 4$ and so this gives us bounds for comparing $\|\cdot\|_p$ and $\left\|\left[\|\cdot\|_p\right]\right\|_p$;
(ii) $(*, p)$ also fails for $1 \leq p < 2$ and we can use similar arguments to get the following bounds.

Proposition 3.45 For $1 \leq p < 2$ and k, l as in Theorem 3.40,

$$(kl)^{A_p} \|(A_{ij})\|_{C_p(\mathbf{C}^m, \mathbf{C}^n)} \geq \left\| \left[\|A_{ij}\|_{C_p(\mathbf{C}^m, \mathbf{C}^n)} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)}.$$

Proof.

$$\begin{aligned} \left\| \left[\|(A_{ij})\|_p \right] \right\|_p &\leq l^{A_p} \left\| \left[\|(A_{ij})\|_p \right] \right\|_2 \leq l^{A_p} k^{A_p} \left\| \left[\|(A_{ij})\|_2 \right] \right\|_2 \\ &= (kl)^{A_p} \|(A_{ij})\|_2 \leq (kl)^{A_p} \|(A_{ij})\|_p. \end{aligned} \quad \blacksquare$$

Proposition 3.46 (i) If (A_{ij}) is an operator of rank at most one, then

$$\|(A_{ij})\|_p \leq \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)},$$

for all $p \in [1, \infty]$;

(ii) if $\left[\|A_{ij}\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \right]$ has rank at most one, then

$$\|(A_{ij})\|_p \leq \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)},$$

for all $p \in [2, \infty]$.

Proof. The proof is analogous to that of Proposition 3.12. Again, (ii) cannot be extended to $[1, 2)$; consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$. Then $\left\| \left[\|A_{ij}\| \right] \right\|_p = 3$ for all p and, since (A_{ij}) has rank greater than one, $\|(A_{ij})\|_p > 3$ for all $1 \leq p < 2$. $\blacksquare$

We conclude the study of this matricial norm inequality with what we believe the true position to be.

Conjecture 3.47 For $4 \leq p \leq \infty$ and $\mathcal{H}, \mathcal{K}$ possibly infinite dimensional,

$$\|(A_{ij})\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \leq \left\| \left[\|A_{ij}\|_{C_p(\mathcal{K}^m, \mathcal{H}^n)} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)}.$$

We end this section with a brief look at two other matricial norm inequalities.

Proposition 3.48 Let $p \in \mathbb{N}$ and $(H_{ij}) = [(A_{ij})^* (A_{ij})]^{\frac{1}{2}}$. Then

$$\|(A_{ij})\|_p \leq \left\| \left[\|H_{ij}\|_p \right] \right\|_p.$$

Proof.

$$\begin{aligned}
\|(A_{ij})\|_p &= \|(H_{ij})\|_p = \|(H_{ij})^p\|_1^{\frac{1}{p}} = [\tau\{(H_{ij})^p\}]^{\frac{1}{p}} \\
&= \left[\sum_{i=1}^t \tau \left(\sum_{k_1=1}^t \cdots \sum_{k_{p-1}=1}^t H_{ik_1} \cdots H_{k_{p-1}i} \right) \right]^{\frac{1}{p}} \\
&\leq \left[\sum_{i=1}^t \sum_{k_1=1}^t \cdots \sum_{k_{p-1}=1}^t |\tau(H_{ik_1} \cdots H_{k_{p-1}i})| \right]^{\frac{1}{p}} \\
&\leq \left[\sum_{i=1}^t \sum_{k_1=1}^t \cdots \sum_{k_{p-1}=1}^t \|H_{ik_1}\|_p \cdots \|H_{k_{p-1}i}\|_p \right]^{\frac{1}{p}} \\
&= [\tau\{(\|H_{ij}\|_p)^p\}]^{\frac{1}{p}} \leq \|(\|H_{ij}\|_p)^p\|_1^{\frac{1}{p}} \leq \|(\|H_{ij}\|_p)\|_p. \quad \blacksquare
\end{aligned}$$

Corollary 3.49 *Let $p \in \mathbb{N}$ and (A_{ij}) be positive. Then*

$$\|(A_{ij})\|_p \leq \left\| \left[\|A_{ij}\|_p \right] \right\|_p.$$

Remark 3.50 This does not extend to self-adjoint operators—see Example 3.33. However, we can use this to get a (somewhat crude) estimate for self-adjoint matrices.

Proposition 3.51 *Let $(\alpha_{ij}) \in M_n(\mathbb{C})$ be self-adjoint and fix $p \in \mathbb{N}$. Then*

$$\|(\alpha_{ij})\|_p \leq n^{\frac{1}{p}} \|(\alpha_{ij})\| + \left(\sum_{i=1}^n |\alpha_{ii}|^p \right)^{\frac{1}{p}} + \|(|\alpha_{ij}|)\|_p.$$

Proof. $(\alpha_{ij}) = \frac{1}{2} (\|(\alpha_{ij})\| I + (\alpha_{ij})) - \frac{1}{2} (\|(\alpha_{ij})\| I - (\alpha_{ij}))$ is a decomposition of (α_{ij}) into two positive matrices. Set

$$\beta_{kl} = \|[\|(\alpha_{ij})\| I + (\alpha_{ij})]_{kl}\| \text{ and } \gamma_{kl} = \|[\|(\alpha_{ij})\| I - (\alpha_{ij})]_{kl}\|.$$

Then

$$\beta_{ij} = \begin{cases} \|(\alpha_{ij})\| + \alpha_{ii} & \text{if } i = j \\ |\alpha_{ij}| & \text{if } i \neq j \end{cases} \text{ and } \gamma_{ij} = \begin{cases} \|(\alpha_{ij})\| - \alpha_{ii} & \text{if } i = j \\ |\alpha_{ij}| & \text{if } i \neq j \end{cases}$$

$$\text{So } (\beta_{ij}) = \|(\alpha_{ij})\| I + (|\alpha_{ij}|) \text{ and } (\gamma_{ij}) = \|(\alpha_{ij})\| I + (|\alpha_{ij}|) - 2 \begin{pmatrix} \alpha_{11} & & \\ & \ddots & \\ & & \alpha_{nn} \end{pmatrix}.$$

Fix $p \in \mathbb{N}$. Then

$$\begin{aligned}
\|(\alpha_{ij})\|_p &\leq \frac{1}{2} \| \|(\alpha_{ij})\| I + (\alpha_{ij}) \|_p + \frac{1}{2} \| \|(\alpha_{ij})\| I - (\alpha_{ij}) \|_p \\
&\leq \frac{1}{2} \|(\beta_{ij})\|_p + \frac{1}{2} \|(\gamma_{ij})\|_p \text{ by Corollary 3.49}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \| \|(\alpha_{ij})\| I + (|\alpha_{ij}|)\|_p + \frac{1}{2} \left\| \|(\alpha_{ij})\| I + (|\alpha_{ij}|) - 2 \begin{pmatrix} \alpha_{11} & & \\ & \ddots & \\ & & \alpha_{nn} \end{pmatrix} \right\|_p \\
&\leq \|(\alpha_{ij})\| \|I\|_p + \|(|\alpha_{ij}|)\|_p + \left\| \begin{pmatrix} \alpha_{11} & & \\ & \ddots & \\ & & \alpha_{nn} \end{pmatrix} \right\|_p \\
&= n^{\frac{1}{p}} \|(\alpha_{ij})\| + \left(\sum_{i=1}^n |\alpha_{ii}|^p \right)^{\frac{1}{p}} + \|(|\alpha_{ij}|)\|_p. \quad \blacksquare
\end{aligned}$$

Conjecture 3.52 Suppose that $2 \leq p \leq \infty$. Then

$$\|(A_{ij})\|_p \leq \left\| \left[\|H_{ij}\|_p \right] \right\|_p,$$

where $(H_{ij}) = [(A_{ij})^* (A_{ij})]^{\frac{1}{2}}$.

We can say a little more in the case of scalar matrices.

Proposition 3.53 Let $(h_{ij}) = [(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$. Then

$$\|(\alpha_{ij})\|_1 \leq \| [|h_{ij}|] \|_1,$$

with equality if and only if $[|h_{ij}|]$ is positive.

Proof.

$$\begin{aligned}
\|(\alpha_{ij})\|_1 &= \|(h_{ij})\|_1 = \tau(h_{ij}) = \sum_{i=1}^n h_{ii} \\
&= \sum_{i=1}^n |h_{ii}| = \tau [|h_{ij}|] \leq \| [|h_{ij}|] \|_1,
\end{aligned}$$

with equality between the last two quantities if and only if $[|h_{ij}|]$ is positive. ■

In fact:

Proposition 3.54 Suppose that $(\alpha_{ij}) \in M_3(\mathbb{C})$ and that $(h_{ij}) = [(\alpha_{ij})^* (\alpha_{ij})]^{\frac{1}{2}}$. Then $\|(\alpha_{ij})\|_1 = \| [|h_{ij}|] \|_1$.

Proof. Suppose that $(h_{ij}) = \begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{pmatrix}$ and set $(k_{ij}) = \begin{pmatrix} a & |b| & |c| \\ |b| & d & |e| \\ |c| & |e| & f \end{pmatrix}$.

Now $a \geq 0$ and $\det \begin{pmatrix} a & |b| \\ |b| & d \end{pmatrix} = \det \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \geq 0$. Finally,

$$\begin{aligned}
\det(k_{ij}) &= a(df - |e|^2) - |b|(|b|f - |c||e|) + |c|(|b||e| - |c|d) \\
&= adf - a|e|^2 - |b|^2f + 2|b||c||e| - |c|^2d \\
&\geq adf - a|e|^2 - |b|^2f + 2\Re(b\bar{c}e) - |c|^2d \\
&= adf - a|e|^2 - |b|^2f + b\bar{c}e + \bar{b}c\bar{e} - |c|^2d \\
&= \det(h_{ij}) \geq 0.
\end{aligned}$$

So $\|(\alpha_{ij})\|_1 = \|(h_{ij})\|_1 = \|(k_{ij})\|_1$ by the previous result. ■

Suppose that $(H_{ij}) = [(A_{ij})^* (A_{ij})]^{\frac{1}{2}}$. Is there a relationship between $\left\| \left[\|A_{ij}\|_p \right] \right\|_p$ and $\left\| \left[\|H_{ij}\|_p \right] \right\|_p$ for all $p \in [1, \infty]$? Up to the use solely of $\leq$ and $\geq$, this question can be answered, for the cases in which $p \neq 2$, in the negative. Consider the following examples:

$$(i) A_1 = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}; (ii) A_2 = \begin{pmatrix} -3 & 1 & 3 \\ -3 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}; (iii) A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix};$$

$$(iv) A_4 = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 0 & 2 \\ -1 & 5 & -1 \end{pmatrix}.$$

In the four graphs in appendix B (figures B.4 to B.7 on pages 108 to 111), the horizontal axis denotes p whilst the vertical axis shows $\left\| \left[\|A_{ij}\|_p \right] \right\|_p - \left\| \left[\|H_{ij}\|_p \right] \right\|_p$.

3.5 The $\mathcal{D}_p$ Spaces

Let us now consider some properties of the matricial norm structure considered in the last section.

Definition 3.55 Fix $1 \leq p, q \leq \infty$ and suppose that $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, assumed to be separable if infinite dimensional. Fix $n, m \in \mathbb{N}$ and a numerical partition $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$. Define $\mathcal{D}_{p,q}(\mathcal{K}^m, \mathcal{H}^n)$ to be the set

$$\left\{ (T_{ij}) \in \mathcal{B}(\mathcal{K}^m, \mathcal{H}^n) : \|(T_{ij})\|_{\mathcal{D}_{p,q}} \stackrel{\text{def}}{=} \left\| \left[\|T_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_q(\mathbf{C}^t, \mathbf{C}^s)} < \infty \right\}.$$

In the sequel, we shall write $\mathcal{D}_p$ for $\mathcal{D}_{p,p}$.

Proposition 3.56 (i) $\mathcal{D}_{p,q}(\mathcal{K}^m, \mathcal{H}^n) = \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$;

(ii) $\|\cdot\|_{\mathcal{D}_{p,q}}$ is a quasi-norm on $\mathcal{D}_{p,q}$ for all $1 \leq p, q \leq \infty$;

(iii) for $1 \leq p \leq \infty$, $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$ if and only if $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$. Moreover, $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$ is complete;

(iv) for $1 \leq p \leq \infty$, $\mathcal{F}(\mathcal{K}^m, \mathcal{H}^n)$ is dense in $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$ with respect to $\|\cdot\|_{\mathcal{D}_p}$.

Proof. (i) Obvious by definition of the quantity $\|\cdot\|_{\mathcal{D}_{p,q}}$.

(ii) By previous work.

(iii) Suppose that $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$. Then

$$\left\| (T_{ij}^{(r)}) - (T_{ij}^{(q)}) \right\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)} \rightarrow 0,$$

as $r, q \rightarrow \infty$. Hence, for all i, j , we have that

$$\left\| T_{ij}^{(r)} - T_{ij}^{(q)} \right\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r, q \rightarrow \infty$ and so

$$\left\| \left[\|T_{ij}^{(r)} - T_{ij}^{(q)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)} \leq \sum_{i=1}^s \sum_{j=1}^t \|T_{ij}^{(r)} - T_{ij}^{(q)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r, q \rightarrow \infty$ and so $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$.

Conversely, if $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$, then

$$\|T_{ij}^{(r)} - T_{ij}^{(q)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r, q \rightarrow \infty$ for all i, j and hence there exists a $T_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ such that

$$\|T_{ij}^{(r)} - T_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r \rightarrow \infty$ since the Schatten p -ideals are Banach spaces. Then $(T_{ij}) \in \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n) = \mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$ and

$$\left\| \left[\|T_{ij}^{(r)} - T_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)} \leq \sum_{i=1}^s \sum_{j=1}^t \|T_{ij}^{(r)} - T_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r \rightarrow \infty$, showing that $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$ is complete. Finally,

$$\left\| \left[\|T_{ij}^{(r)} - T_{ij}^{(q)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)} \leq \sum_{i=1}^s \sum_{j=1}^t \|T_{ij}^{(r)} - T_{ij}^{(q)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r, q \rightarrow \infty$ and so $(T_{ij}^{(r)})$ is a Cauchy sequence in $\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$.

(iv) Suppose $(G_{ij}) \in \mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$. Then each $G_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ and so there exists a sequence $(F_{ij}^{(r)})$ such that $F_{ij}^{(r)}$ has rank at most r and $\|G_{ij} - F_{ij}^{(r)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0$ as $r \rightarrow \infty$. Then

$$\|(G_{ij}) - (F_{ij}^{(r)})\|_{\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)} \leq \sum_{i=1}^s \sum_{j=1}^t \|G_{ij} - F_{ij}^{(r)}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \rightarrow 0,$$

as $r \rightarrow \infty$ as required. ■

Lemma 3.57 (Duality) Fix $1 \leq q \leq 2 \leq p \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Fix $n, m \in \mathbb{N}$ and a numerical partition $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$. For $1 \leq i, l \leq s$ and $1 \leq j, k \leq t$, let $A_{ij} \in \mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ and $B_{kl} \in \mathcal{C}_q(\mathcal{H}^{n_l}, \mathcal{K}^{m_k})$. Then $(A_{ij}) \in \mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)$ and $(B_{kl}) \in \mathcal{C}_q(\mathcal{H}^n, \mathcal{K}^m)$. So $(A_{ij})(B_{kl}) \in \mathcal{C}_1(\mathcal{H}^n)$. Then:

- (i) $|\tau[(A_{ij})(B_{kl})]| \leq \|(A_{ij})\|_{\mathcal{D}_p} \|(B_{kl})\|_{\mathcal{D}_q}$;
- (ii) if $(*, q)$ holds, then $(*, p)$ holds.

Proof. $(A_{ij})(B_{kl}) \in \mathcal{C}_1(\mathcal{H}^n)$ is a block $s \times s$ matrix and the $(i, j)^{th}$ of the matrix is $\sum_{c=1}^t A_{ic} B_{cj} : \mathcal{H}^{n_j} \rightarrow \mathcal{H}^{n_i}$. Now

$$|\tau[(A_{ij})(B_{kl})]| = \left| \sum_{d=1}^s \tau \left(\sum_{c=1}^t A_{dc} B_{cd} \right) \right|$$

$$\begin{aligned}
&\leq \sum_{d=1}^s \sum_{c=1}^t |\tau(A_{dc}B_{cd})| \\
&\leq \sum_{d=1}^s \sum_{c=1}^t \|A_{dc}\|_{\mathcal{C}_p(\mathcal{K}^{m_c}, \mathcal{H}^{n_d})} \|B_{cd}\|_{\mathcal{C}_q(\mathcal{H}^{n_d}, \mathcal{K}^{m_c})} \\
&= \tau \left[\left(\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right) \left(\|B_{kl}\|_{\mathcal{C}_q(\mathcal{H}^{n_l}, \mathcal{K}^{m_k})} \right) \right] \\
&\leq \left\| \left(\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right) \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)} \left\| \left(\|B_{kl}\|_{\mathcal{C}_q(\mathcal{H}^{n_l}, \mathcal{K}^{m_k})} \right) \right\|_{\mathcal{C}_q(\mathbf{C}^s, \mathbf{C}^t)} \\
&\quad \text{(which proves (i) of the lemma),} \\
&\leq \left\| \left(\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right) \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)} \| (B_{kl}) \|_{\mathcal{C}_q(\mathcal{H}^n, \mathcal{K}^m)} \text{ (if } (*, q) \text{ holds),}
\end{aligned}$$

and hence $\|(A_{ij})\|_{\mathcal{C}_p(\mathcal{K}^m, \mathcal{H}^n)} \leq \left\| \left(\|A_{ij}\|_{\mathcal{C}_p(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})} \right) \right\|_{\mathcal{C}_p(\mathbf{C}^t, \mathbf{C}^s)}$ as required. $\blacksquare$

Remark 3.58 The converse of (ii) is, as we have seen, false.

Proposition 3.59 (The Dual Of $\mathcal{D}_p$) Let $1 < p < \infty$ and let q be the conjugate index to p . For each $(T_{ij}) \in \mathcal{D}_q(\mathcal{H}^n, \mathcal{K}^m)$, the equation

$$f_{(T_{ij})}((S_{ij})) = \tau((S_{ij})(T_{ij})),$$

defines a continuous linear functional on $\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)$. Moreover, the mapping

$$(T_{ij}) \mapsto f_{(T_{ij})},$$

is a contractive isomorphism from $\mathcal{D}_q(\mathcal{H}^n, \mathcal{K}^m)$ onto $(\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n))^*$.

Proof. By the lemma, we have that

$$|\tau((S_{ij})(T_{ij}))| \leq \|(S_{ij})\|_{\mathcal{D}_p} \|(T_{ij})\|_{\mathcal{D}_q},$$

and so we see that $f_{(T_{ij})}$ is a bounded functional with $\|f_{(T_{ij})}\| \leq \|(T_{ij})\|_{\mathcal{D}_q}$.

Now, suppose that f is a bounded linear functional on $\mathcal{D}_p$. Then f is a bounded linear functional on $\mathcal{C}_p$ and hence there exists an operator $(T_{ij}) \in \mathcal{C}_q(\mathcal{H}^n, \mathcal{K}^m)$ such that $f = f_{(T_{ij})}$. But then $(T_{ij}) \in \mathcal{D}_q(\mathcal{H}^n, \mathcal{K}^m)$, as required. $\blacksquare$

Theorem 3.60 (Hölder's Inequality) Let $1 \leq p \leq \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Fix $n, m, l \in \mathbb{N}$ and numerical partitions $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and $(m, l : m_1, \dots, m_t; l_1, \dots, l_u)$. For $1 \leq i \leq s$, $1 \leq j \leq t$, let $A_{ij} \in \mathcal{C}_q(\mathcal{K}^{m_j}, \mathcal{H}^{n_i})$ and for $1 \leq v \leq t$, $1 \leq w \leq u$, let $B_{vw} \in \mathcal{C}_r(\mathcal{L}^{l_w}, \mathcal{K}^{m_v})$. Hence $(A_{ij}) \in \mathcal{C}_q(\mathcal{K}^m, \mathcal{H}^n)$ and $(B_{ij}) \in \mathcal{C}_r(\mathcal{L}^l, \mathcal{K}^m)$. Then $(A_{ij})(B_{ij}) \in \mathcal{C}_p(\mathcal{L}^l, \mathcal{H}^n)$ and we have the following estimates:

(i) if p is an even integer or infinity, then

$$\|(A_{ij})(B_{ij})\|_{\mathcal{D}_p(\mathcal{L}^l, \mathcal{H}^n)} \leq \|(A_{ij})\|_{\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)} \|(B_{ij})\|_{\mathcal{D}_r(\mathcal{L}^l, \mathcal{K}^m)};$$

(ii) otherwise,

$$\|(A_{ij})(B_{ij})\|_{\mathcal{D}_p(\mathcal{L}^l, \mathcal{H}^n)} \leq \min\{s, u\}^{A_p} \|(A_{ij})\|_{\mathcal{D}_p(\mathcal{K}^m, \mathcal{H}^n)} \|(B_{ij})\|_{\mathcal{D}_r(\mathcal{L}^l, \mathcal{K}^m)}.$$

Proof. (i) $(A_{ij})(B_{ij})$ is a block $s \times u$ matrix whose $(a, b)^{\text{th}}$ entry is

$$\sum_{k=1}^t A_{ak} B_{kb} : \mathcal{L}^{lb} \rightarrow \mathcal{H}^{na},$$

and

$$\begin{aligned} \left\| \sum_{k=1}^t A_{ak} B_{kb} \right\|_{\mathcal{C}_p(\mathcal{L}^{lb}, \mathcal{H}^{na})} &\leq \sum_{k=1}^t \|A_{ak} B_{kb}\|_{\mathcal{C}_p(\mathcal{L}^{lb}, \mathcal{H}^{na})} \\ &\leq \sum_{k=1}^t \|A_{ak}\|_{\mathcal{C}_q(\mathcal{K}^{mk}, \mathcal{H}^{na})} \|B_{kb}\|_{\mathcal{C}_r(\mathcal{L}^{lb}, \mathcal{K}^{mk})}, \end{aligned}$$

and this term is the $(a, b)^{\text{th}}$ entry of the matrix product

$$\left[\|A_{ij}\|_{\mathcal{C}_q(\mathcal{K}^{mj}, \mathcal{H}^{ni})} \right] \left[\|B_{vw}\|_{\mathcal{C}_r(\mathcal{L}^{lw}, \mathcal{H}^{mv})} \right].$$

So

$$\begin{aligned} \|(A_{ij})(B_{ij})\|_{\mathcal{D}_p(\mathcal{L}^l, \mathcal{K}^m)} &= \left\| \left[\left\| \sum_{k=1}^t A_{ak} B_{kb} \right\|_{\mathcal{C}_p(\mathcal{L}^{lb}, \mathcal{H}^{na})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^s, \mathbf{C}^s)} \\ &\leq \left\| \left[\|A_{ij}\|_{\mathcal{C}_q(\mathcal{K}^{mj}, \mathcal{H}^{ni})} \right] \left[\|B_{vw}\|_{\mathcal{C}_r(\mathcal{L}^{lw}, \mathcal{K}^{mv})} \right] \right\|_{\mathcal{C}_p(\mathbf{C}^s, \mathbf{C}^s)} \text{ by (PO)} \\ &\leq \left\| \left[\|A_{ij}\|_{\mathcal{C}_q(\mathcal{K}^{mj}, \mathcal{H}^{ni})} \right] \right\|_{\mathcal{C}_q(\mathbf{C}^t, \mathbf{C}^s)} \left\| \left[\|B_{vw}\|_{\mathcal{C}_r(\mathcal{L}^{lw}, \mathcal{K}^{mv})} \right] \right\|_{\mathcal{C}_r(\mathbf{C}^s, \mathbf{C}^t)} \\ &= \|(A_{ij})\|_{\mathcal{D}_q(\mathcal{K}^m, \mathcal{H}^n)} \|(B_{ij})\|_{\mathcal{D}_r(\mathcal{L}^l, \mathcal{K}^m)}, \end{aligned}$$

as required.

(ii) Similar, except we use the constants A_p (from Proposition 3.7) instead of the point-wise ordering property. ■

Since $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ is a Banach space for $p = 2q$ ($q \in \mathbb{N}$) or ∞ , we are in a position to interpolate.

Proposition 3.61 (Interpolation) Choose $p \neq q$ from the set $\{2m (m \in \mathbb{N}), \infty\}$ and fix $0 < \theta < 1$. Then $\|(A_{ij})\|_{[\theta]} \geq \|(A_{ij})\|_r$ where $\|\cdot\|_{[\theta]}$ is the interpolated norm on $(\mathcal{D}_p, \mathcal{D}_q)_{[\theta]}$ and $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$.

Proof. See Section 1.5 for a brief reminder of the complex interpolation method C_θ . It is clear that $\mathcal{F}(\mathcal{C}_p, \mathcal{C}_q) = \mathcal{F}(\mathcal{D}_p, \mathcal{D}_q)$. So

$$\begin{aligned} \|f\|_{\mathcal{F}(\mathcal{D}_p, \mathcal{D}_q)} &= \max \left(\sup_{t_1} \|f(it_1)\|_{\mathcal{D}_p}, \sup_{t_2} \|f(1+it_2)\|_{\mathcal{D}_q} \right) \\ &\geq \max \left(\sup_{t_1} \|f(it_1)\|_p, \sup_{t_2} \|f(1+it_2)\|_q \right) \\ &= \|f\|_{\mathcal{F}(\mathcal{C}_p, \mathcal{C}_q)}, \end{aligned}$$

since p and q were chosen to lie in the ranges where the $(*, p)$ property holds.

Now, suppose that $(A_{ij}) \in (\mathcal{D}_p, \mathcal{D}_q)_{[\theta]}$. Then

$$\begin{aligned} \|(A_{ij})\|_{[\theta]} &= \inf \left\{ \|f\|_{\mathcal{F}(\mathcal{D}_p, \mathcal{D}_q)} : f(\theta) = (A_{ij}) \right\} \\ &\geq \inf \left\{ \|f\|_{\mathcal{F}(\mathcal{C}_p, \mathcal{C}_q)} : f(\theta) = (A_{ij}) \right\} \\ &= \|(A_{ij})\|_r. \end{aligned}$$

Note, however, that $\|\cdot\|_{[\theta]} \neq \|\cdot\|_{\mathcal{D}_r}$ in general—otherwise we would have that $\|\cdot\|_{\mathcal{D}_3} \geq \|\cdot\|_3$, which is a contradiction. There are some obvious exceptions—for example, if both the $\|\cdot\|_{\mathcal{D}_p}$ and $\|\cdot\|_{\mathcal{D}_q}$ coincide with the usual $\|\cdot\|_2$ norm, then the interpolated norm will be, again, $\|\cdot\|_2$. ■

3.6 l^p -Type Estimates for the Schatten p -Norms

It is well-known that we can give bounds for the $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ norms on $M_n(\mathbb{C})$ by the following

$$\|(\alpha_{ij})\|_1 \leq \sum_{i,j=1}^n \|\alpha_{ij}\|_1, \quad \|(\alpha_{ij})\|_2 = \sqrt{\sum_{i,j=1}^n \|\alpha_{ij}\|_2^2} \quad \text{and} \quad \max_{i,j} \|\alpha_{ij}\| \leq \|(\alpha_{ij})\|_\infty,$$

and recall that the estimates for $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are best possible; the identity matrix suffices for both norms. In this section, we give similar estimates for the other Schatten p -norms.

Proposition 3.62 *Fix a numerical partition $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and $1 \leq p \leq \infty$. Suppose that $\alpha_{ij} : \mathbb{C}^{m_j} \rightarrow \mathbb{C}^{n_i}$ for each $1 \leq i \leq s$, $1 \leq j \leq t$. For each pair (i, j) , let $l_{ij} = \min \{m_j, n_i\}$ and set $k = \max \{l_{ij} : 1 \leq i \leq s, 1 \leq j \leq t\}$. Then, for $p \geq 2$:*

$$\|(\alpha_{ij})\|_{\mathcal{C}_p(\mathbb{C}^m, \mathbb{C}^n)} \leq (kst)^{\frac{p-2}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_{\mathcal{C}_p(\mathbb{C}^{m_j}, \mathbb{C}^{n_i})}^p \right]^{\frac{1}{p}}.$$

Moreover, in the case that $n_i = m_j = 1$ for all i and j , this upper bound is the best possible.

Proof.

$$\begin{aligned} \|(\alpha_{ij})\|_p &\leq \|(\alpha_{ij})\|_2 = \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_2^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum_{i=1}^s \sum_{j=1}^t \left\{ l_{ij}^{\frac{1}{2} - \frac{1}{p}} \|\alpha_{ij}\|_p \right\}^2 \right]^{\frac{1}{2}} \quad \text{by Lemma 3.6} \\ &= \left[\sum_{i=1}^s \sum_{j=1}^t l_{ij}^{\frac{p-2}{p}} \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \left[\sum_{i=1}^s \sum_{j=1}^t k^{\frac{p-2}{p}} \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}} = k^{\frac{p-2}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}} \\
&\leq k^{\frac{p-2}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t 1^q \right]^{\frac{1}{q}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^p \right]^{\frac{1}{p}} \quad \text{where } \frac{1}{2} = \frac{1}{p} + \frac{1}{q} \\
&= (kst)^{\frac{p-2}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^p \right]^{\frac{1}{p}},
\end{aligned}$$

as required. It remains only to show that this is best possible: consider the $n \times n$ matrix (α_{ij}) which has 1 as each of its entries. Then $\|(\alpha_{ij})\|_p = n$ and the rhs is $n^{\frac{p-2}{p}} n^{\frac{2}{p}}$ since $s = t = n$ and $k = 1$. ■

Proposition 3.63 *Let (α_{ij}) be as in the statement of the previous result. Then, for $1 < p < 2$, we have*

$$\|(\alpha_{ij})\|_p \leq n^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^p \right]^{\frac{1}{p}}.$$

Proof.

$$\begin{aligned}
\|(\alpha_{ij})\|_p &\leq n^{\frac{1}{p}} n^{\frac{-1}{2}} \|(\alpha_{ij})\|_2 = n^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_2^2 \right]^{\frac{1}{2}} \\
&\leq n^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}} \leq n^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^p \right]^{\frac{1}{p}}. \quad \blacksquare
\end{aligned}$$

However, given the tendency of inequalities to flip at 2, it is more sensible to try another approach to getting estimates.

Proposition 3.64 *For $1 \leq p < 2$,*

$$\|(\alpha_{ij})\|_p \geq h^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}},$$

where $h = \min \{m_j, n_i : 1 \leq i \leq s, 1 \leq j \leq t\}$. Again, in the case that $n_i = m_j = 1$ for all i and j , this lower bound is best possible.

Proof. For each i, j , $\|\alpha_{ij}\|_2 \geq \min \{m_j, n_i\}^{\frac{2-p}{2p}} \|\alpha_{ij}\|_p \geq h^{\frac{2-p}{2p}} \|\alpha_{ij}\|_p$ using Lemma 3.6. Then

$$\begin{aligned}
\|(\alpha_{ij})\|_p &\geq \|(\alpha_{ij})\|_2 = \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_2^2 \right]^{\frac{1}{2}} \\
&\geq \left[\sum_{i=1}^s \sum_{j=1}^t h^{\frac{2-p}{p}} \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}} = h^{\frac{2-p}{2p}} \left[\sum_{i=1}^s \sum_{j=1}^t \|\alpha_{ij}\|_p^2 \right]^{\frac{1}{2}},
\end{aligned}$$

as required. Finally, to see that this lower bound is the best possible, consider the $n \times n$ matrix which has 1's everywhere and take $h = 1$. ■

3.7 The Inequality applied to Matricially Normed Spaces

Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space and recall from Chapter one that we may assume that we have norms defined on $M_{m,n}(V)$ for all $m, n \in \mathbb{N}$.

Definition 3.65 Let $1 \leq p \leq \infty$. We shall say that V satisfies $(*, p)$ if:

(i) in the case that $1 \leq p \leq 2$, we have

$$\|(v_{ij})\|_{n,m} \geq \left\| \left[\|v_{ij}\|_{n_i, m_j} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)},$$

for all numerical partitions $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and all $v_{ij} \in M_{n_i, m_j}(V)$;

(ii) in the case that $2 \leq p \leq \infty$, we have

$$\|(v_{ij})\|_{n,m} \leq \left\| \left[\|v_{ij}\|_{n_i, m_j} \right] \right\|_{C_p(\mathbf{C}^t, \mathbf{C}^s)},$$

for all numerical partitions $(n, m : n_1, \dots, n_s; m_1, \dots, m_t)$ and all $v_{ij} \in M_{n_i, m_j}(V)$.

It has been shown (see Theorem 1.22) that $(*, \infty)$ is equivalent to V being an (abstract) operator space. $(*, 2)$ will be seen to be equivalent to the property $(*)$ of Chapter four and, hence, is equivalent to V being a Hilbert-Schmidt space (Theorem 4.5). What can we say about the condition $(*, p)$ for other values of p ?

Proposition 3.66 Suppose that $(V, \{\|\cdot\|_n\})$ is a matricially normed space which satisfies $(*, 1)$. Then V is an L^1 -matricially normed space.

Proof. For any $v \in M_n(V)$, $w \in M_m(V)$, we have that $\|v \oplus w\|_{n+m} \leq \|v\|_n + \|w\|_m$ by the triangle inequality. Suppose that V is not L^1 : then there exists $n, m \in \mathbb{N}$ and $x \in M_n(V)$, $y \in M_m(V)$ such that $\|x \oplus y\|_{n+m} < \|x\|_n + \|y\|_m$. But

$$\|x \oplus y\|_{n+m} = \left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} \geq \left\| \begin{pmatrix} \|x\|_n & 0 \\ 0 & \|y\|_m \end{pmatrix} \right\|_{C_1(\mathbf{C}^2)} \text{ by } (*, 1) = \|x\|_n + \|y\|_m,$$

which is a contradiction. Hence V is L^1 . ■

Definition 3.67 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space and let $1 \leq p \leq \infty$. We say that V is a sub- L^p -matricially normed space if

$$\|(v_{ij}) \oplus (w_{ij})\|_{n+m} \leq (\|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p)^{\frac{1}{p}},$$

for all $n, m \in \mathbb{N}$ and $(v_{ij}) \in M_n(V)$, $(w_{ij}) \in M_m(V)$. We define the term sup- L^p -matricially normed space analogously.

Corollary 3.68 In a similar fashion:

- (i) $1 < p < 2$, $(*, p) \Rightarrow$ 'sup- L^p ', that is to say, $\|v \oplus w\|_{n+m} \geq (\|v\|_n^p + \|w\|_m^p)^{\frac{1}{p}}$;
- (ii) $2 < q < \infty$, $(*, q) \Rightarrow$ 'sub- L^p ', that is to say, $\|v \oplus w\|_{n+m} \leq (\|v\|_n^q + \|w\|_m^q)^{\frac{1}{q}}$.

Remark 3.69 It is clear that, for a matricially normed space V to be completely isometric to a subspace of $C_p(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then V must (where appropriate) satisfy the norm inequalities given earlier in this chapter.

Chapter 4

Interpolation of Matricially Normed Spaces

4.1 Introduction

This chapter can essentially be divided into two parts. In the first, we give an easy characterisation of those matricially normed spaces which are completely isometric to subspaces of $\mathcal{C}_2(\mathcal{H})$. In the latter, we investigate the interpolation of matricially normed spaces using the real, complex and approximation methods.

4.2 (Abstract) Hilbert-Schmidt Spaces

Ruan (and subsequently with Effros—see [21] and [9]) has given a characterisation of L^∞ -matricially normed spaces in terms of being completely isometric to a subspace of $\mathcal{B}(\mathcal{H})$. (The reader is referred to Section 1.4 for some basic results in the theory of matricially normed spaces.) In this first section, we give a similar characterisation (Theorem 4.5) of L^2 -matricially normed spaces. The characterisation itself is trivial; the interest centres upon those conditions which are equivalent to the existence of the complete isometry.

Our idea is based upon the following well-known result.

Proposition 4.1 *Let $\mathcal{H}$ be a Hilbert space. Then*

$$\|(T_{ij})\|_{\mathcal{C}_2(\mathcal{H}^n)} = \sqrt{\sum_{i,j=1}^n \|T_{ij}\|_{\mathcal{C}_2(\mathcal{H})}^2},$$

for all $n \in \mathbb{N}$ and $(T_{ij}) \in \mathcal{C}_2(\mathcal{H}^n)$.

Just as we have the notion of an (abstract) operator space in [21], so we introduce the following terminology.

Definition 4.2 Let $1 \leq p < \infty$. A space is said to be an (abstract) $\mathcal{C}_p$ space if it is completely isometric to a subspace of $\mathcal{C}_p(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In the case that $p = 2$, we use the term (abstract) Hilbert-Schmidt space.

Definition 4.3 A matricially normed space $(V, \{\|\cdot\|_n\})$ will be said to satisfy $(*)$ if

$$\|(v_{ij})\|_n = \sqrt{\sum_{i,j=1}^n \|v_{ij}\|_1^2},$$

for all $n \in \mathbb{N}$ and $(v_{ij}) \in M_n(V)$.

So, in particular, $(\mathcal{C}_2(\mathcal{H}), \|\cdot\|_{\mathcal{C}_2(\mathcal{H})})$ satisfies $(*)$.

Definition 4.4 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. We say that $\{\|\cdot\|_n\}$ is re-arrangement invariant if, for all $n \in \mathbb{N}$, $\|(v_{ij})\|_n$ is insensitive to the ordering of the elements in $(v_{ij}) \in M_n(V)$.

Theorem 4.5 Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Then the following are equivalent:

- (i) V is an (abstract) Hilbert-Schmidt space;
- (ii) V satisfies $(*)$;
- (iii) $\|\cdot\|_n$ is an inner product norm for all $n \in \mathbb{N}$;
- (iv) V is an L^2 -matricially normed space and $\{\|\cdot\|_n\}$ is re-arrangement invariant.

Proof. (ii) $\Rightarrow$ (iv): This is obvious by the definition of $(*)$.

(iv) $\Rightarrow$ (ii): Fix $n \geq 2$ and $(v_{ij}) \in M_n(V)$. Then

$$\begin{aligned} \|(v_{ij})\|_n &= \|(v_{ij}) \oplus O_{n^2-n}\|_{n^2} \\ &= \left\| \begin{pmatrix} v_{11} & & \\ & \ddots & \\ & & v_{nn} \end{pmatrix} \right\|_{n^2} = \left(\sum_{i,j=1}^n \|v_{ij}\|_1^2 \right)^{\frac{1}{2}}, \end{aligned}$$

since $(V, \{\|\cdot\|_n\})$ is an L^2 -matricially normed space.

(ii) $\Rightarrow$ (iii): Case $n = 1$: [The proof was suggested by T. A. Gillespie] Let $a, b \in V$. Then

$$\begin{aligned} \|a + b\|_1^2 + \|a - b\|_1^2 &= \left\| \begin{pmatrix} a + b & 0 \\ a - b & 0 \end{pmatrix} \right\|_2^2 \text{ by } (*) \\ &= 2 \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \right\|_2^2 \\ &= 2 \left\| \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \right\|_2^2 \text{ since the scalar matrix is unitary} \\ &= 2 (\|a\|_1^2 + \|b\|_1^2) \text{ by } (*). \end{aligned}$$

So $\|\cdot\|_1$ satisfies the parallelogram law and is, therefore, an inner product norm.

Case $n \geq 2$: Fix $n \geq 2$ and let $(v_{ij}), (w_{ij}) \in M_n(V)$. Then

$$\begin{aligned} & \| (v_{ij}) + (w_{ij}) \|_n^2 + \| (v_{ij}) - (w_{ij}) \|_n^2 \\ &= \sum_{i,j=1}^n \|v_{ij} + w_{ij}\|_1^2 + \sum_{i,j=1}^n \|v_{ij} - w_{ij}\|_1^2 \text{ by } (*) \\ &= 2 \sum_{i,j=1}^n (\|v_{ij}\|_1^2 + \|w_{ij}\|_1^2) \text{ since } \|\cdot\|_1 \text{ satisfies the parallelogram law} \\ &= 2 (\| (v_{ij}) \|_n^2 + \| (w_{ij}) \|_n^2) \text{ by } (*), \end{aligned}$$

and hence $\|\cdot\|_n$ is an inner product norm for all $n \geq 2$.

(iii) $\Rightarrow$ (ii): For all $n \in \mathbb{N}$, let $\langle \cdot, \cdot \rangle_n$ be the inner product which gives rise to $\|\cdot\|_n$. Let $v \in V$ and $1 \leq i, j \leq n$. Then

$$\langle v \otimes E_{ij}, v \otimes E_{ij} \rangle_n = \|v \otimes E_{ij}\|_n^2 = \|v\|_1^2 = \langle v, v \rangle_1,$$

where $\{E_{ij}\}_{i,j=1}^n$ are the usual matrix units. Now, fix $n \geq 2$ and let $(v_{ij}) \in M_n(V)$. Then

$$\begin{aligned} \| (v_{ij}) \|_n^2 &= \langle (v_{ij}), (v_{ij}) \rangle_n = \left\langle \sum_{i,j=1}^n v_{ij} \otimes E_{ij}, \sum_{k,l=1}^n v_{kl} \otimes E_{kl} \right\rangle_n \\ &= \sum_{i,j=1}^n \langle v_{ij} \otimes E_{ij}, v_{ij} \otimes E_{ij} \rangle_n + \sum_{i,j=1}^n \sum_{(k,l) \neq (i,j)} \langle v_{ij} \otimes E_{ij}, v_{kl} \otimes E_{kl} \rangle_n \\ &= \sum_{i,j=1}^n \|v_{ij}\|_1^2 + \sum_{i,j=1}^n \sum_{(k,l) \neq (i,j)} \langle v_{ij} \otimes E_{ij}, v_{kl} \otimes E_{kl} \rangle_n. \end{aligned} \quad (4.1)$$

We now show that the second term of (4.1) is zero. Let $v, w \in V$ and $(k, l) \neq (i, j)$. Then we claim that

$$\|v \otimes E_{ij} + w \otimes E_{kl}\|_n = \|v \otimes E_{ij} - w \otimes E_{kl}\|_n. \quad (4.2)$$

Without loss, we may assume that both v and w are non-zero and we split into two cases.

Case $j \neq l$: $v \otimes E_{ij} + w \otimes E_{kl}$ for $j \neq l$. Let A be an $n \times n$ matrix which has 1's all down the diagonal except in the $(l, l)^{\text{th}}$ place where it has a -1 and zeroes everywhere else. Then A is unitary and the product $(v \otimes E_{ij} + w \otimes E_{kl})A$ has only two non-zero entries: in the $(i, j)^{\text{th}}$ place we have v and in the $(k, l)^{\text{th}}$ place we have $-w$. So

$$\|v \otimes E_{ij} + w \otimes E_{kl}\|_n = \|v \otimes E_{ij} - w \otimes E_{kl}\|_n.$$

Case $i \neq k$: Similar, except we pre-multiply by a unitary matrix B which has 1's all down the diagonal except in the $(k, k)^{\text{th}}$ place where it has a -1 and zeroes everywhere else. So (4.2) holds.

Now, for $(k, l) \neq (i, j)$, the polarisation identity yields

$$\langle v_{ij} \otimes E_{ij}, v_{kl} \otimes E_{kl} \rangle_n = \frac{1}{4} \sum_{s=0}^3 i^s \|v_{ij} \otimes E_{ij} + i^s v_{kl} \otimes E_{kl}\|_n^2,$$

and, by (4.2), the first and third terms and the second and fourth terms cancel out. Thus

$$\|(v_{ij})\|_n = \left(\sum_{i,j=1}^n \|v_{ij}\|_1^2 \right)^{\frac{1}{2}},$$

as required.

(i) $\Rightarrow$ (ii): Suppose that $\theta : V \rightarrow \mathcal{C}_2(\mathcal{H})$ is a complete isometry. Then V must satisfy (*) since $\mathcal{C}_2(\mathcal{H})$ does.

(ii) $\Rightarrow$ (i): Let $\tilde{V}$ denote the Banach space completion of V , that is to say, the closure of V in V^{**} . Since the norm on V is an inner product norm (by (ii) $\Rightarrow$ (iii)), $\tilde{V}$ is a Hilbert space, $\mathcal{H}$ say, and we have an isometric embedding $\phi : V \rightarrow \mathcal{H}$.

Let $\mathcal{G}$ be the direct sum of $(\dim \mathcal{H})$ copies of $\mathcal{H}$. Then clearly the map

$$\psi : h \in \mathcal{H} \mapsto h \oplus 0 \oplus 0 \oplus \dots \in \mathcal{G},$$

is an isometry. Also, $\dim(\mathcal{C}_2(\mathcal{H})) = (\dim \mathcal{H})^2$ since $\{h_i \otimes h_j : (i, j) \in I^2\}$ is a basis for $\mathcal{C}_2(\mathcal{H})$ if $\{h_i : i \in I\}$ is a basis for $\mathcal{H}$ (see [18, Proposition 3.5.14]). So $\dim \mathcal{G} = \dim \mathcal{C}_2(\mathcal{H})$ and hence there is an isometric surjection $\varphi : \mathcal{G} \rightarrow \mathcal{C}_2(\mathcal{H})$. Hence the map

$$\theta \stackrel{\text{def}}{=} \varphi \circ \psi \circ \phi : V \rightarrow \mathcal{C}_2(\mathcal{H}),$$

is an isometry. Finally, let $n \in \mathbb{N}$ and $(v_{ij}) \in M_n(V)$. Then

$$\begin{aligned} \|\theta_n(v_{ij})\|_{\mathcal{C}_2(\mathcal{H}^n)}^2 &= \|[\theta(v_{ij})]\|_{\mathcal{C}_2(\mathcal{H}^n)}^2 = \sum_{i,j=1}^n \|\theta(v_{ij})\|_{\mathcal{C}_2(\mathcal{H})}^2 \\ &= \sum_{i,j=1}^n \|v_{ij}\|_1^2 \text{ since } \theta \text{ is an isometry} \\ &= \|(v_{ij})\|_n^2 \text{ by } (*), \end{aligned}$$

and hence $\theta : V \rightarrow \mathcal{C}_2(\mathcal{H})$ is a complete isometry, as required. ■

Proposition 4.6 *Let $(V, \{\|\cdot\|_n\})$ be a Hilbert-Schmidt space. Then $(V^*, \{\|\cdot\|_n^*\})$ is also a Hilbert-Schmidt space.*

Proof. By Proposition 1.18(iv), the matricial dual of an L^2 -matricially normed space is again an L^2 -matricially normed space. Fix $n \in \mathbb{N}$ and $(f_{ij}) \in M_n(V^*)$. Let (g_{ij}) be any re-arrangement of (f_{ij}) . Then

$$\|(f_{ij})\|_n^* = \sup \{ |\langle (v_{ij}), (f_{ij}) \rangle| : \|(v_{ij})\|_n = 1 \}$$

$$\begin{aligned}
&= \sup \left\{ \left| \sum_{i,j=1}^n f_{ij}(v_{ij}) \right| : \|(v_{ij})\|_n = 1 \right\} \\
&= \sup \left\{ \left| \sum_{i,j=1}^n g_{ij}(w_{ij}) \right| : \|(w_{ij})\|_n = 1 \right\},
\end{aligned}$$

where (w_{ij}) is the re-arrangement of (v_{ij}) corresponding to that of (g_{ij}) with respect to (f_{ij}) and so

$$\|(f_{ij})\|_n^* = \sup \{ |\langle (w_{ij}), (g_{ij}) \rangle| : \|(w_{ij})\|_n = 1 \} = \|(g_{ij})\|_n^*.$$

Hence $\{\|\cdot\|_n^*\}$ is re-arrangement invariant and hence V^* is a Hilbert-Schmidt space, as required. $\blacksquare$

Remark 4.7 Does the condition $(*)$ hold for all L^2 -matricially normed spaces? At present, we are unable to answer this question. However, we can prove a result (Theorem 4.10) in this direction concerning the re-norming of an L^2 -matricially normed space.

We shall use $(v_{ij}) * (\alpha_{ij})$ to denote the Schur (or Hadamard) product, that is to say, $(v_{ij}) * (\alpha_{ij}) = (\alpha_{ij} v_{ij})$ where $(v_{ij}) \in M_n(V)$ and $(\alpha_{ij}) \in M_n(\mathbb{C})$.

Lemma 4.8 *Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space, $n \in \mathbb{N}$, $(v_{ij}) \in M_n(V)$ and $(\alpha_{ij}) \in M_n(\mathbb{C})$. Then*

$$\|(v_{ij}) * (\alpha_{ij})\|_n \leq n^{\frac{2}{q}} \|(v_{ij})\|_n \left(\sum_{i,j=1}^n |\alpha_{ij}|^p \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In particular,

$$\|(v_{ij}) * (\alpha_{ij})\|_n \leq \|(v_{ij})\|_n \sum_{i,j=1}^n |\alpha_{ij}|.$$

Proof. Fix $n \in \mathbb{N}$, $(v_{ij}) \in M_n(V)$ and $(\alpha_{ij}) \in M_n(\mathbb{C})$. Then

$$\begin{aligned}
\|(v_{ij}) * (\alpha_{ij})\|_n &= \|(\alpha_{ij} v_{ij})\|_n \\
&\leq \sum_{i,j=1}^n \|\alpha_{ij} v_{ij}\|_1 = \sum_{i,j=1}^n |\alpha_{ij}| \|v_{ij}\|_1 \\
&\leq \left(\sum_{i,j=1}^n |\alpha_{ij}|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^n \|v_{ij}\|_1^q \right)^{\frac{1}{q}} \text{ by Hölder's inequality} \\
&\leq \left(n^2 \max_{1 \leq i,j \leq n} \|v_{ij}\|_1^q \right)^{\frac{1}{q}} \left(\sum_{i,j=1}^n |\alpha_{ij}|^p \right)^{\frac{1}{p}} \\
&\leq n^{\frac{2}{q}} \|(v_{ij})\|_n \left(\sum_{i,j=1}^n |\alpha_{ij}|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Definition 4.9 Let V be a vector space and $\{|\cdot|_n\}$, $\{\|\cdot\|_n\}$ be two matrix norms on V . We say that $\{|\cdot|_n\}$ and $\{\|\cdot\|_n\}$ are equivalent matrix norms if $|\cdot|_n$ and $\|\cdot\|_n$ are equivalent norms for all $n \in \mathbb{N}$.

Theorem 4.10 Let $(V, \{\|\cdot\|_n\})$ be an L^2 -matricially normed space. Then there is a matrix norm $\{|\cdot|_n\}$ on V which is equivalent to $\{\|\cdot\|_n\}$ and which satisfies (*).

Proof. First, we construct $|\cdot|_n$ by setting

$$|v|_1 = \|v\|_1 \text{ for all } v \in V \text{ and } |(v_{ij})|_n = \left(\sum_{i,j=1}^n |v_{ij}|_1^2 \right)^{\frac{1}{2}},$$

for all $n \geq 2$ and $(v_{ij}) \in M_n(V)$. Clearly, $|\cdot|_n$ is a norm for all n and it satisfies (*) by construction. Since $|\cdot|_1$ and $\|\cdot\|_1$ are identical, we fix $n \geq 2$.

Let $(v_{ij}) \in M_n(V)$. We write (v_{ij}) as a sum of $2n - 2$ matrices, each of which has entries only on some diagonal. The decomposition is achieved in the following manner: the first of the summands consists only of the leading diagonal of (v_{ij}) . Then we take diagonals running in the opposite direction (of which there are $2n - 3$), at each stage omitting entries which lie on the leading diagonal. So

$$(v_{ij}) = V_1 + \dots + V_{2n-2},$$

where

$$V_1 = \begin{pmatrix} v_{11} & & & \\ & \ddots & & \\ & & v_{nn} & \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & v_{12} & \cdots & 0 \\ v_{21} & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & 0 & v_{13} & \cdots & 0 \\ 0 & 0 & & & \vdots \\ v_{31} & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \dots, \quad V_{2n-2} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & v_{n,n-1} & 0 \end{pmatrix}.$$

Let I_n denote the $n \times n$ identity scalar matrix. Then

$$\begin{aligned} \|(v_{ij})\|_n &= \left\| \begin{pmatrix} I_n & \cdots & I_n \end{pmatrix} \begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_{2n-2} \end{pmatrix} \begin{pmatrix} I_n \\ \vdots \\ I_n \end{pmatrix} \right\|_n \\ &\leq \left\| \begin{pmatrix} I_n & \cdots & I_n \end{pmatrix} \right\| \left\| \begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_{2n-2} \end{pmatrix} \right\|_{n(2n-2)} \left\| \begin{pmatrix} I_n \\ \vdots \\ I_n \end{pmatrix} \right\| \end{aligned}$$

$$\begin{aligned}
&= (2n-2) \left\{ \|V_1\|_n^2 + \dots + \|V_{2n-2}\|_n^2 \right\}^{\frac{1}{2}} \\
&\quad \text{because } \left\| \begin{pmatrix} I_n & \cdots & I_n \end{pmatrix} \right\| = \sqrt{2n-2} \|I_n\| = \sqrt{2n-2}, \\
&= 2(n-1) \left(\sum_{i,j=1}^n \|v_{ij}\|_1^2 \right)^{\frac{1}{2}} \\
&\quad \text{because each of the } V_i \text{'s have non-zero entries only on their diagonal,} \\
&= 2(n-1) |(v_{ij})|_n,
\end{aligned}$$

by definition of $\{|\cdot|_n\}$. Also,

$$\begin{aligned}
| (v_{ij}) |_n &= \left\| \begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_{2n-2} \end{pmatrix} \right\|_{n(2n-2)} \\
&= \left\| \underbrace{\begin{pmatrix} (v_{ij}) & & \\ & (v_{ij}) & \\ & & \ddots \\ & & & (v_{ij}) \end{pmatrix}}_{(2n-2) \text{ copies of } (v_{ij})} * \underbrace{\begin{pmatrix} I_n & & \\ & (E_{12} + E_{21}) & \\ & & \ddots \\ & & & (E_{n,n-1} + E_{n-1,n}) \end{pmatrix}}_{=T} \right\|_{n(2n-2)} \\
&\leq \left\| \begin{pmatrix} (v_{ij}) & & \\ & \ddots & \\ & & (v_{ij}) \end{pmatrix} \right\|_{n(2n-2)} \sum_{i,j=1}^{n(2n-2)} |T_{ij}| \quad (\text{by Lemma 4.8}) \\
&= n^2 \sqrt{2(n-1)} \|(v_{ij})\|_n,
\end{aligned}$$

since T contains n^2 non-zero entries and they are all 1's. Hence

$$\frac{1}{n^2 \sqrt{2(n-1)}} |(v_{ij})|_n \leq \|(v_{ij})\|_n \leq 2(n-1) |(v_{ij})|_n,$$

for all $n \geq 2$, $(v_{ij}) \in M_n(V)$. ■

We may generalise the property (*) as follows.

Definition 4.11 Let $1 \leq p \leq \infty$. A matrix norm $\{\|\cdot\|_n\}$ on a vector space V will be said to be a K^p -matrix norm if

$$\|(v_{ij})\|_n = \left(\sum_{i,j=1}^n \|v_{ij}\|_1^p \right)^{\frac{1}{p}},$$

for all $n \in \mathbb{N}$ and $(v_{ij}) \in M_n(V)$ where the rhs has the usual supremum interpretation when $p = \infty$.

It is known that, for $p \neq 2$, the natural norm on the Schatten p -ideal $\mathcal{C}_p(\mathcal{H})$ is not a K^p -matrix norm. Are there, however, other matricially normed spaces which support such a matrix norm? This question can be answered in the negative and, to show this, we begin with a lemma (cf Proposition 2.12).

Lemma 4.12 *Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space. Then*

$$\left\| \begin{pmatrix} v & \cdots & v \\ \vdots & & \vdots \\ v & \cdots & v \end{pmatrix} \right\|_n = n\|v\|_1,$$

for all $n \in \mathbb{N}$, $v \in V$.

Proof. Fix $n \in \mathbb{N}$ and $v \in V$. Then

$$\begin{aligned} n^2\|v\|_1 &= \left\| \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} v & \cdots & v \\ \vdots & & \vdots \\ v & \cdots & v \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_n \\ &\leq n \left\| \begin{pmatrix} v & \cdots & v \\ \vdots & & \vdots \\ v & \cdots & v \end{pmatrix} \right\|_n \\ &= n \left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (v) \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \right\|_n \\ &\leq n^2\|v\|_1. \end{aligned}$$

■

Corollary 4.13 *Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space such that $\{\|\cdot\|_n\}$ is a K^p -matrix norm for some $p \neq 2$. Then $V = \{0\}$.*

Proof. Choose $v \in V$ and $n \geq 2$. Then

$$\left\| \begin{pmatrix} v & \cdots & v \\ \vdots & & \vdots \\ v & \cdots & v \end{pmatrix} \right\|_n = \begin{cases} n^{\frac{2}{p}}\|v\|_1 & \text{if } p \text{ is finite} \\ \|v\|_1 & \text{if } p = \infty, \end{cases}$$

by the definition of a K^p -matrix norm whereas

$$\left\| \begin{pmatrix} v & \cdots & v \\ \vdots & & \vdots \\ v & \cdots & v \end{pmatrix} \right\|_n = n\|v\|_1,$$

by Lemma 4.12. So $p \neq 2$ implies that $\|v\|_1 = 0$.

■

4.3 The McCarthy-Clarkson Inequalities

Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space, $1 \leq p < \infty$ and q be the conjugate index to p . Consider the following four norm inequalities where $n \in \mathbb{N}$, $(v_{ij}), (w_{ij}) \in M_n(V)$:

$$\begin{aligned} 2^{p-1} (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p) &\leq \|(v_{ij}) + (w_{ij})\|_n^p + \|(v_{ij}) - (w_{ij})\|_n^p & (1 \leq p \leq 2) \\ &\leq 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p) \\ \|(v_{ij}) + (w_{ij})\|_n^q + \|(v_{ij}) - (w_{ij})\|_n^q &\leq 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p)^{\frac{q}{p}} & (1 < p \leq 2) \\ 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p) &\leq \|(v_{ij}) + (w_{ij})\|_n^p + \|(v_{ij}) - (w_{ij})\|_n^p & (2 \leq p < \infty) \\ &\leq 2^{p-1} (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p) \\ 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_n^p)^{\frac{q}{p}} &\leq \|(v_{ij}) + (w_{ij})\|_n^q + \|(v_{ij}) - (w_{ij})\|_n^q & (2 \leq p < \infty) \end{aligned}$$

Definition 4.14 *These four inequalities will, henceforth, be referred to as the (abstract) McCarthy-Clarkson inequalities and will be denoted by $MC_1(p)$, $MC_2(p)$, $MC_3(p)$ and $MC_4(p)$ respectively.*

Recall the definitions of sub- and sup- L^p (see definition 3.67).

Proposition 4.15 *Let $(V, \{\|\cdot\|_n\})$ be a matricially normed space and let $1 \leq p < \infty$.*

- (i) If V satisfies $MC_1(p)$ or $MC_2(p)$, then V is sub- L^p ;*
- (ii) if V satisfies $MC_3(p)$ or $MC_4(p)$, then V is sup- L^p .*

Proof. (i) Let $n, m \in \mathbb{N}$ and $(v_{ij}) \in M_n(V)$, $(w_{ij}) \in M_n(W)$. If V satisfies $MC_1(p)$, then

$$\begin{aligned} 2\|(v_{ij}) \oplus (w_{ij})\|_{n+m}^p &= \|(v_{ij}) \oplus (w_{ij})\|_{n+m}^p + \|(v_{ij}) \oplus (- (w_{ij}))\|_{n+m}^p \\ &\leq 2 (\|(v_{ij}) \oplus 0\|_{n+m}^p + \|0 \oplus (w_{ij})\|_{n+m}^p) \text{ by } MC_1(p) \\ &= 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p), \end{aligned}$$

and so

$$\|(v_{ij}) \oplus (w_{ij})\|_{n+m}^p \leq \|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p.$$

If, on the other hand, V satisfies $MC_2(p)$, then

$$\begin{aligned} 2\|(v_{ij}) \oplus (w_{ij})\|_{n+m}^q &= \|(v_{ij}) \oplus (w_{ij})\|_{n+m}^q + \|(v_{ij}) \oplus (- (w_{ij}))\|_{n+m}^q \\ &\leq 2 (\|(v_{ij}) \oplus 0\|_{n+m}^p + \|0 \oplus (w_{ij})\|_{n+m}^p)^{\frac{q}{p}} \text{ by } MC_2(p) \\ &= 2 (\|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p)^{\frac{q}{p}}, \end{aligned}$$

and hence

$$\|(v_{ij}) \oplus (w_{ij})\|_{n+m} \leq (\|(v_{ij})\|_n^p + \|(w_{ij})\|_m^p)^{\frac{1}{p}}.$$

(ii) From (i) since the inequalities $MC_3(p)$ and $MC_4(p)$ are the reverse of those in $MC_1(p)$ and $MC_2(p)$ respectively. ■

4.4 Interpolation of Matricially Normed Spaces

The reader is referred to Section 1.5 for details of the complex, real and approximation methods of interpolation.

4.4.1 The Complex Methods

We begin this section by proving the intuitively obvious fact that complex interpolation of matricially normed spaces yields another matricially normed space.

Theorem 4.16 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be matricially normed spaces such that (V, W) is a compatible Banach couple. Then, for each $0 < \theta < 1$, $((V, W)_{[\theta]}, \{\|\cdot\|_{[\theta]}\})$ is a matricially normed space where $\|\cdot\|_{[\theta]}$ is the norm obtained by interpolating the norms $|\cdot|_n$ and $|||\cdot|||_n$ for each $n \in \mathbb{N}$.*

Proof. Clearly, it makes sense to establish that, for $0 < \theta < 1$ and $n \geq 2$,

$$(M_n(V), M_n(W))_{[\theta]} = M_n((V, W)_{[\theta]}). \quad (4.3)$$

However, we first address the question of compatibility. Since V and W are Banach spaces with respect to the norms $|\cdot|_1$ and $|||\cdot|||_1$ respectively, $M_n(V)$ and $M_n(W)$ are Banach with respect to $|\cdot|_n$ and $|||\cdot|||_n$ respectively for all $n \geq 2$. Since V and W are compatible, there is a Hausdorff topological vector space $\mathcal{U}$ such that V and W are subspaces of $\mathcal{U}$. Then, by considering the identification of $M_n(\mathcal{U})$ with $\oplus_{i=1}^{n^2} \mathcal{U}$, we have that $M_n(\mathcal{U})$ is a Hausdorff topological vector space and $M_n(V)$ and $M_n(W)$ are subspaces of $M_n(\mathcal{U})$. So, for all $n \geq 2$, $(M_n(V), M_n(W))$ is a compatible Banach couple.

The proof is divided into four parts:

- (a) $M_n((V, W)_{[\theta]}) \subseteq (M_n(V), M_n(W))_{[\theta]}$;
- (b) $(M_n(V), M_n(W))_{[\theta]} \subseteq M_n((V, W)_{[\theta]})$;
- (c) $\|(x_{ij}) \oplus O_m\|_{[\theta]} = \|(x_{ij})\|_{[\theta]}$;
- (d) $\|(\alpha_{ij})(x_{ij})(\beta_{ij})\|_{[\theta]} \leq \|(\alpha_{ij})\| \|(x_{ij})\|_{[\theta]} \|(\beta_{ij})\|$.

Some notation will help simplify matters. Set

$$\mathcal{F} = \mathcal{F}((V, W)) \text{ and } \mathcal{F}_n = \mathcal{F}((M_n(V), M_n(W))) \text{ for } n \geq 2.$$

(a) Suppose that $(a_{ij}) \in M_n((V, W)_{[\theta]})$. Then, for $1 \leq i, j \leq n$, $a_{ij} \in (V, W)_{[\theta]}$. Hence, there exists an $f_{ij} \in \mathcal{F}$ such that $f_{ij}(\theta) = a_{ij}$. Define a map

$$G : z \in \mathbb{C} \mapsto (f_{ij}(z)) \in M_n(V) + M_n(W) = M_n(V + W).$$

Then $G(\theta) = (f_{ij}(\theta)) = (a_{ij})$. It remains to show that $G \in \mathcal{F}_n$ by verifying the conditions (i), (ii) and (iii) of Section 1.5.2:

(i), (ii): Since each of the f_{ij} are bounded and continuous on S and analytic on S_0 , it

follows that G has the same properties.

(iii): the maps $t \mapsto G(it)$ and $t \mapsto G(1 + it)$ are continuous from the real line into $M_n(V)$ and $M_n(W)$ respectively and tend to zero as $|t| \rightarrow \infty$ since each of the f_{kl} has the analogous property in $\mathcal{F}$ ($1 \leq k, l \leq n$).

So $(a_{ij}) \in (M_n(V), M_n(W))_{[\theta]}$, as required.

(b) This inclusion hinges upon the following observation, namely that, for a vector space X , the subspace $X \oplus O_{n-1} \subseteq M_n(X)$ can be canonically identified (by ϕ , say) with X .

Suppose $(b_{ij}) \in (M_n(V), M_n(W))_{[\theta]}$. Then $(b_{ij}) = G(\theta)$ for some $G \in \mathcal{F}_n$. Since $(b_{ij}) \in M_n(V) + M_n(W) = M_n(V + W)$, each $b_{ij} \in V + W$ for all $1 \leq i, j \leq n$. We show that there exist $f_{ij} \in \mathcal{F}$ such that $f_{ij}(\theta) = b_{ij}$. Then each $b_{ij} \in (V, W)_{[\theta]}$ and hence $(b_{ij}) \in M_n((V, W)_{[\theta]})$, completing the inclusion and establishing (4.3).

Fix $1 \leq i, j \leq n$. Let F_{ij} be the linear map which annihilates all but the $(i, j)^{\text{th}}$ entry of (b_{ij}) and permutes b_{ij} to the $(1, 1)$ position. Define

$$f_{ij} : z \in \mathbb{C} \mapsto (\phi \circ F_{ij} \circ G)(z) \in V + W,$$

where ϕ is the canonical identification of $X \oplus O_{n-1}$ with X . Then

$$f_{ij}(\theta) = (\phi \circ F_{ij})(b_{ij}) = \phi(b_{ij} \oplus O_{n-1}) = b_{ij},$$

as required. We now show that $f_{ij} \in \mathcal{F}$:

(i), (ii): again, the boundedness and continuity on S , as well as the analyticity on S_0 , of f_{ij} follow from the same properties of G ;

(iii): since the functions $t \mapsto G(it)$ and $t \mapsto G(1 + it)$ are continuous from $\mathbb{R}$ into $M_n(V)$ and $M_n(W)$ respectively and tend to zero as $|t| \rightarrow \infty$, the maps $t \in \mathbb{R} \mapsto f_{kl}(it) \in V$ and $t \in \mathbb{R} \mapsto f_{kl}(1 + it) \in W$ are continuous and tend to zero as $|t| \rightarrow \infty$.

So each $f_{kl} \in \mathcal{F}$ and (4.3) follows.

Now, we prove that the two conditions for a matricially normed space hold.

(c) Fix $n, m \in \mathbb{N}$ and let $g \in \mathcal{F}_n$. Then

$$g \oplus 0 : z \in \mathbb{C} \mapsto g(z) \oplus 0 \in M_{n+m}(V + W),$$

is in $\mathcal{F}_{n+m}$. Now, since $\{|\cdot|_n\}$ and $\{|||\cdot|||_n\}$ are matricial norms, it follows immediately that $\{||\cdot||_{\mathcal{F}_n}\}$ is also a matricial norm. So, in particular, we have that

$$\|a\|_{\mathcal{F}_n} \leq \|a \oplus d\|_{\mathcal{F}_{n+m}} \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{F}_{n+m}},$$

via arguments such as

$$\begin{aligned} \|g \oplus 0\|_{\mathcal{F}_{n+m}} &= \max \left(\sup_s |(g \oplus 0)(is)|_{n+m}, \sup_t |||(g \oplus 0)(1 + it)|||_{n+m} \right) \\ &= \max \left(\sup_s |g(is)|_n, \sup_t |||g(1 + it)|||_n \right) \\ &= \|g\|_{\mathcal{F}_n}. \end{aligned}$$

(In the sequel, this fact—as well as the analogous fact for the matricial norms $\{\|\cdot\|_{\mathcal{C}_n}\}$ and $\{\|\cdot\|_{\theta,q;K}\}$ —will be used extensively, usually without explicit reference.)

Consider $(x_{ij}) \in M_n((V, W)_{[\theta]})$. Then $(x_{ij}) \oplus O_m \in M_{n+m}((V, W)_{[\theta]})$ and

$$\begin{aligned} \|(x_{ij}) \oplus O_m\|_{[\theta]} &= \inf \left\{ \|f\|_{\mathcal{F}_{n+m}} : f \in \mathcal{F}_{n+m}, f(\theta) = (x_{ij}) \oplus O_m \right\} \\ &= \inf \left\{ \|g \oplus 0\|_{\mathcal{F}_{n+m}} : g \in \mathcal{F}_n, g(\theta) = (x_{ij}) \right\} \\ &= \inf \left\{ \|g\|_{\mathcal{F}_n} : g \in \mathcal{F}_n, g(\theta) = (x_{ij}) \right\} \\ &= \|(x_{ij})\|_{[\theta]}, \end{aligned}$$

as required.

(d) Defining an interpolation functor C_θ by

$$C_\theta((V, W)) = (V, W)_{[\theta]},$$

we have that C_θ is an exact interpolation functor of exponent θ . Fix $n \in \mathbb{N}$ and let $(\alpha_{ij}), (\beta_{ij}) \in M_n(\mathbb{C})$. Define

$$T : (v_{ij}) \in M_n(V) \mapsto (\alpha_{ij})(v_{ij})(\beta_{ij}) \in M_n(V),$$

and

$$T : (w_{ij}) \in M_n(W) \mapsto (\alpha_{ij})(w_{ij})(\beta_{ij}) \in M_n(W).$$

Then

$$T : (x_{ij}) \in M_n((V, W)_{[\theta]}) \mapsto (\alpha_{ij})(x_{ij})(\beta_{ij}) \in M_n((V, W)_{[\theta]}),$$

and we have the estimate

$$\|T\|_{M_n((V, W)_{[\theta]}) \rightarrow M_n((V, W)_{[\theta]})} \leq \|T\|_{V, V}^{1-\theta} \|T\|_{W, W}^\theta \leq \|(\alpha_{ij})\| \|(\beta_{ij})\|,$$

since $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ are matricially normed spaces. Hence

$$\|(\alpha_{ij})(x_{ij})(\beta_{ij})\|_{[\theta]} \leq \|(\alpha_{ij})\| \|(x_{ij})\|_{[\theta]} \|(\beta_{ij})\|.$$

This completes the proof. ■

Theorem 4.17 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be matricially normed spaces such that (V, W) is a compatible Banach couple. Then, for each $0 < \theta < 1$, $((V, W)^{[\theta]}, \{\|\cdot\|^{[\theta]}\})$ is a matricially normed space, where $\|\cdot\|^{[\theta]}$ is the norm obtained by interpolating the norms $|\cdot|_n$ and $|||\cdot|||_n$ for all $n \in \mathbb{N}$.*

Proof. Again, the proof is divided into four parts:

- (a) $M_n((V, W)^{[\theta]}) \subseteq (M_n(V), M_n(W))^{[\theta]}$;
- (b) $(M_n(V), M_n(W))^{[\theta]} \subseteq M_n((V, W)^{[\theta]})$;

- (c) $\|(x_{ij}) \oplus O_m\|^{[\theta]} = \|(x_{ij})\|^{[\theta]}$;
 (d) $\|(\alpha_{ij})(x_{ij})(\beta_{ij})\|^{[\theta]} \leq \|(\alpha_{ij})\| \|(x_{ij})\|^{[\theta]} \|(\beta_{ij})\|$.

Set

$$\mathcal{G} = \mathcal{G}((V, W)) \text{ and } \mathcal{G}_n = \mathcal{G}(M_n(V), M_n(W)) \text{ for } n \geq 2.$$

(a) Suppose that $(a_{ij}) \in M_n((V, W)^{[\theta]})$. Then, for $1 \leq i, j \leq n$, $a_{ij} \in (V, W)^{[\theta]}$. Hence there exists $g_{ij} \in \mathcal{G}$ such that $g'_{ij}(\theta) = a_{ij}$ for each $1 \leq i, j \leq n$. Define a map

$$G : z \in \mathbb{C} \mapsto (g_{ij}(z)) \in M_n(V) + M_n(W) = M_n(V + W).$$

Then $G'(\theta) = (g'_{ij}(\theta)) = (a_{ij})$. It remains to show that $G \in \mathcal{G}_n$ by verifying conditions (i), (ii) and (iii) of Section 1.5.3:

(i)

$$\|G(z)\|_{M_n(V+W)} \leq \sum_{i,j=1}^n \|g_{ij}(z)\|_{V+W} \leq n^2 c (1 + |z|),$$

for some constant c and all z .

(ii) clearly, G is continuous on S and analytic on S_0 since all the g_{ij} 's are.

(iii) $G(it_1) - G(it_2)$ and $G(1+it_1) - G(1+it_2)$ have values in $M_n(V)$ and $M_n(W)$ respectively for all $t_1, t_2 \in \mathbb{R}$ since the analogous property holds for each of the g_{ij} 's. Finally,

$$\begin{aligned} & \|G\|_{\mathcal{G}_n} \\ &= \max \left(\sup_{t_1, t_2} \left\| \frac{G(it_1) - G(it_2)}{t_1 - t_2} \right\|_n, \sup_{t_3, t_4} \left\| \left\| \frac{G(1+it_3) - G(1+it_4)}{t_3 - t_4} \right\| \right\|_n \right) \\ &\leq \max \left(\sup_{t_1, t_2} \sum_{k,l=1}^n \left\| \frac{g_{kl}(it_1) - g_{kl}(it_2)}{t_1 - t_2} \right\|_1, \sup_{t_3, t_4} \sum_{k,l=1}^n \left\| \left\| \frac{g_{kl}(1+it_3) - g_{kl}(1+it_4)}{t_3 - t_4} \right\| \right\|_1 \right) \\ &\leq \sum_{k,l=1}^n \left[\max \left(\sup_{t_1, t_2} \left\| \frac{g_{kl}(it_1) - g_{kl}(it_2)}{t_1 - t_2} \right\|_1, \sup_{t_3, t_4} \left\| \left\| \frac{g_{kl}(1+it_3) - g_{kl}(1+it_4)}{t_3 - t_4} \right\| \right\|_1 \right) \right] \\ &= \sum_{k,l=1}^n \|g_{kl}\|_{\mathcal{G}} < \infty, \end{aligned}$$

and so $G \in \mathcal{G}_n$, as required.

(b) Let $(b_{ij}) \in (M_n(V), M_n(W))^{[\theta]}$. So $(b_{ij}) = G'(\theta)$ for some $G \in \mathcal{G}_n$. Since $(b_{ij}) \in M_n(V) + M_n(W) = M_n(V + W)$, $b_{ij} \in V + W$ for all $1 \leq i, j \leq n$. We show that, for all i, j , there exists a $g_{ij} \in \mathcal{G}$ such that $b_{ij} = g'_{ij}(\theta)$. Then $b_{ij} \in (V, W)^{[\theta]}$ and hence $(b_{ij}) \in M_n((V, W)^{[\theta]})$.

Fix $1 \leq i, j \leq n$ and let F_{ij} be the map which annihilates all but the $(i, j)^{\text{th}}$ entry of (b_{ij}) and permutes b_{ij} to the $(1, 1)$ position. Define

$$g_{ij} : z \in \mathbb{C} \mapsto (\phi \circ F_{ij} \circ G)(z) \in V + W,$$

where ϕ is the canonical identification of $X \oplus O_{n-1}$ with X .

Then $g'_{ij}(\theta) = (\phi \circ F_{ij} \circ G')(\theta) = b_{ij}$. We now show that $g_{ij} \in \mathcal{G}$:

(i) $\|g_{ij}(z)\|_{V+W} \leq \|G(z)\|_{M_n(V+W)} \leq c(1 + |z|)$ for all z .

(ii) clearly, each g_{ij} is continuous on S and analytic on S_0 .

(iii) immediate as we are dealing with matricial norms $\{|\cdot|_n\}$ and $\{|||\cdot|||_n\}$. Hence $g_{ij} \in \mathcal{G}$, as required.

(c) Fix $n, m \in \mathbb{N}$ and let $g \in \mathcal{G}_n$. Then $g \oplus 0 \in \mathcal{G}_{n+m}$ and

$$\begin{aligned} \|g \oplus 0\|_{\mathcal{G}_{n+m}} &= \max \left(\sup_{t_1, t_2} \left| \frac{(g \oplus 0)(it_1) - (g \oplus 0)(it_2)}{t_1 - t_2} \right|_{n+m}, \right. \\ &\quad \left. \sup_{t_3, t_4} \left\| \frac{(g \oplus 0)(1 + it_3) - (g \oplus 0)(1 + it_4)}{t_3 - t_4} \right\|_{n+m} \right) \\ &= \max \left(\sup_{t_1, t_2} \left| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right|_n, \sup_{t_3, t_4} \left\| \frac{g(1 + it_3) - g(1 + it_4)}{t_3 - t_4} \right\|_n \right) \\ &= \|g\|_{\mathcal{G}_n}. \end{aligned}$$

Now, let $(x_{ij}) \in M_n((V, W)^{[\theta]})$. Then

$$\begin{aligned} \|(x_{ij}) \oplus O_m\|^{[\theta]} &= \inf \left\{ \|f\|_{\mathcal{G}_{n+m}} : f \in \mathcal{G}_{n+m}, f'(\theta) = (x_{ij}) \oplus O_m \right\} \\ &= \inf \left\{ \|g \oplus 0\|_{\mathcal{G}_{n+m}} : g \in \mathcal{G}_n, g'(\theta) = (x_{ij}) \right\} \\ &= \inf \left\{ \|g\|_{\mathcal{G}_n} : g \in \mathcal{G}_n, g'(\theta) = (x_{ij}) \right\} \\ &= \|(x_{ij})\|^{[\theta]}, \end{aligned}$$

as required.

(d) the scalar multiplication is proved exactly as in Theorem 4.16. ■

Next, we prove several results concerning the interpolation of L^p - and L^q -matricially normed spaces. Our first result (Theorem 4.18) is known (see [19], for example) but we give a proof which uses only the basic definitions.

Theorem 4.18 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two L^∞ -matricially normed spaces such that (V, W) is a compatible Banach couple. Then, for $0 < \theta < 1$, the following matricially normed spaces are L^∞ -matricially normed spaces:*

- (i) $((V, W)_{[\theta]}, \{|\cdot|_{[\theta]}\})$;
- (ii) $((V, W)^{[\theta]}, \{|||\cdot|||^{[\theta]}\})$.

Proof. (i) Let $a \oplus b \in M_{n+m}((V, W)_{[\theta]})$. Then

$$\begin{aligned} \|a \oplus b\|_{[\theta]} &= \inf \left\{ \|f\|_{\mathcal{F}_{n+m}} : f(\theta) = a \oplus b \right\} \\ &= \inf \left\{ \|g \oplus h\|_{\mathcal{F}_{n+m}} : g \in \mathcal{F}_n, g(\theta) = a, h \in \mathcal{F}_m, h(\theta) = b \right\}. \end{aligned}$$

Now

$$\|g \oplus h\|_{\mathcal{F}_{n+m}} = \max \left(\sup_{t_1} |(g \oplus h)(it_1)|_{n+m}, \sup_{t_2} |||(g \oplus h)(1 + it_2)|||_{n+m} \right)$$

$$\begin{aligned}
&= \max \left(\sup_{t_1} \{ \max [|g(it_1)|_n, |h(it_1)|_m] \}, \right. \\
&\quad \left. \sup_{t_2} \{ \max [\|g(1+it_2)\|_n, \|h(1+it_2)\|_m] \} \right) \\
&= \sup_t (|g(it)|_n, |h(it)|_m, \|g(1+it)\|_n, \|h(1+it)\|_m),
\end{aligned}$$

and so

$$\begin{aligned}
&\|a \oplus b\|_{[\theta]} \\
&= \inf \left\{ \sup_t (|g(it)|_n, |h(it)|_m, \|g(1+it)\|_n, \|h(1+it)\|_m) : g(\theta) = a, h(\theta) = b \right\}.
\end{aligned}$$

Now

$$\begin{aligned}
&\max (\|a\|_{[\theta]}, \|b\|_{[\theta]}) \\
&= \max \left(\inf \left\{ \sup_{t_1} |g(it_1)|_n, \sup_{t_2} \|g(1+it_2)\|_n : g \in \mathcal{F}_n, g(\theta) = a \right\}, \right. \\
&\quad \left. \inf \left\{ \sup_{t_3} |h(it_3)|_m, \sup_{t_4} \|h(1+it_4)\|_m : h \in \mathcal{F}_m, h(\theta) = b \right\} \right),
\end{aligned}$$

and hence

$$\|a \oplus b\|_{[\theta]} \leq \max (\|a\|_{[\theta]}, \|b\|_{[\theta]}).$$

But

$$\|a\|_{[\theta]}, \|b\|_{[\theta]} \leq \|a \oplus b\|_{[\theta]},$$

since $((V, W)_{[\theta]}, \{\|\cdot\|_{[\theta]}\})$ is a matricially normed space, giving us equality.

(ii) Let $a \oplus b \in M_{n+m}((V, W)^{[\theta]})$. Then

$$\begin{aligned}
\|a \oplus b\|^{[\theta]} &= \inf \{ \|f\|_{\mathcal{G}_{n+m}} : f'(\theta) = a \oplus b \} \\
&= \inf \{ \|g \oplus h\|_{\mathcal{G}_{n+m}} : g \in \mathcal{G}_n, g'(\theta) = a, h \in \mathcal{G}_m, h'(\theta) = b \}.
\end{aligned}$$

Now

$$\begin{aligned}
&\|g \oplus h\|_{\mathcal{G}_{n+m}} \\
&= \max \left(\sup_{t_1, t_2} \left| \frac{(g \oplus h)(it_1) - (g \oplus h)(it_2)}{t_1 - t_2} \right|_{n+m}, \right. \\
&\quad \left. \sup_{t_3, t_4} \left\| \left\| \frac{(g \oplus h)(1+it_3) - (g \oplus h)(1+it_4)}{t_3 - t_4} \right\| \right\|_{n+m} \right) \\
&= \max \left(\sup_{t_1, t_2} \left[\max \left\{ \left\| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right\|_n, \left\| \frac{h(it_1) - h(it_2)}{t_1 - t_2} \right\|_m \right\} \right], \right. \\
&\quad \left. \sup_{t_3, t_4} \left[\max \left\{ \left\| \left\| \frac{g(1+it_3) - g(1+it_4)}{t_3 - t_4} \right\| \right\|_n, \left\| \left\| \frac{h(1+it_3) - h(1+it_4)}{t_3 - t_4} \right\| \right\|_m \right\} \right] \right) \\
&= \sup_{t_1, t_2} \left\{ \left\| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right\|_n, \left\| \frac{h(it_1) - h(it_2)}{t_1 - t_2} \right\|_m, \right. \\
&\quad \left. \left\| \left\| \frac{g(1+it_1) - g(1+it_2)}{t_1 - t_2} \right\| \right\|_n, \left\| \left\| \frac{h(1+it_1) - h(1+it_2)}{t_1 - t_2} \right\| \right\|_m \right\},
\end{aligned}$$

and so

$$\begin{aligned} & \|a \oplus b\|^{[\theta]} \\ &= \inf \left\{ \sup_{t_1, t_2} \left(\left| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right|_n, \left| \frac{h(it_1) - h(it_2)}{t_1 - t_2} \right|_m, \right. \right. \\ & \quad \left. \left. \left\| \frac{g(1 + it_1) - g(1 + it_2)}{t_1 - t_2} \right\|_n, \left\| \frac{h(1 + it_1) - h(1 + it_2)}{t_1 - t_2} \right\|_m \right) \right\}, \end{aligned}$$

where the infimum is taken over those functions g and h such that $g'(\theta) = a$ and $h'(\theta) = b$. Now

$$\begin{aligned} & \max \left(\|a\|^{[\theta]}, \|b\|^{[\theta]} \right) \\ &= \max \left(\inf \left\{ \sup_{t_1, t_2} \left| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right|_n, \sup_{t_3, t_4} \left\| \frac{g(1 + it_3) - g(1 + it_4)}{t_3 - t_4} \right\|_n \right\}, \right. \\ & \quad \left. \inf \left\{ \sup_{t_5, t_6} \left| \frac{h(it_5) - h(it_6)}{t_5 - t_6} \right|_n, \sup_{t_7, t_8} \left\| \frac{h(1 + it_7) - h(1 + it_8)}{t_7 - t_8} \right\|_n \right\} \right), \end{aligned}$$

and hence

$$\|a \oplus b\|^{[\theta]} \leq \max \left(\|a\|^{[\theta]}, \|b\|^{[\theta]} \right).$$

The result now follows since the space is once again matricially normed. $\blacksquare$

To handle more general cases, we introduce some notation and use Lemma 4.20.

Definition 4.19 Let (A_n) be a sequence of Banach spaces. For $1 \leq p \leq \infty$, let

$$l^p((A_n)) \stackrel{\text{def}}{=} \left\{ (a_n) : a_n \in A_n, \|(a_n)\|_{l^p((A_n))} \stackrel{\text{def}}{=} \left(\sum_{n=1}^{\infty} \|a_n\|_{A_n}^p \right)^{\frac{1}{p}} < \infty \right\},$$

where the case $p = \infty$ has the usual ‘sup’ interpretation.

Lemma 4.20 Let $1 \leq p, q < \infty$ and (A_n) and (B_n) be two sequences of Banach spaces such that (A_n, B_n) is a compatible Banach couple for all $n \in \mathbb{N}$. Let $0 < \theta < 1$. Then

$$(l^p((A_n)), l^q((B_n)))_{[\theta]} = l^r((A_n, B_n)_{[\theta]}),$$

with equal norms where $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ and

$$(l^p((A_n)), l^\infty((B_n)))_{[\theta]} = l^r((A_n, B_n)_{[\theta]}),$$

with equal norms where $\frac{1}{r} = \frac{1-\theta}{p}$.

Proof. This is an analogue of [4, Theorem 5.6.2], applied to the sequences of the form $l^p((A_n))$ rather than $l^p(A)$. The proof requires only the obvious modifications. $\blacksquare$

We are now in a position to prove our interpolation theorem for matricially normed spaces.

Theorem 4.21 Let $1 \leq p, q \leq \infty$. Let $(V, \{|\cdot|_n\})$ be an L^p -matricially normed space and $(W, \{|||\cdot|||_n\})$ be an L^q -matricially normed space such that (V, W) is a compatible Banach couple. Then, for $0 < \theta < 1$, $((V, W)_{[\theta]}, \{\|\cdot\|_{[\theta]}\})$ is an L^r -matricially normed space where $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$.

Proof. Theorem 4.18 dealt with the case $p = q = \infty$ and so we assume that at least one of p and q is finite. By Lemma 4.20, the spaces

$$(l^p(M_{n_1}(V), M_{n_2}(V), \dots), l^q(M_{n_1}(W), M_{n_2}(W), \dots))_{[\theta]},$$

and

$$l^r(M_{n_1}((V, W)_{[\theta]}), M_{n_2}((V, W)_{[\theta]}), \dots),$$

coincide with equality of norms for all choices of $n_i \in \mathbb{N}$ since we have shown that $(M_n(V), M_n(W))_{[\theta]} = M_n((V, W)_{[\theta]})$ as spaces.

Let $x \in M_n((V, W)_{[\theta]})$ and $y \in M_m((V, W)_{[\theta]})$ for some $n, m \in \mathbb{N}$. Then

$$\begin{aligned} & \|x \oplus y\|_{[\theta]} \\ &= \inf \{ \|f\|_{\mathcal{F}} : f(\theta) = x \oplus y \} \\ &= \inf \{ \|f_1 \oplus f_2\|_{\mathcal{F}} : f_1(\theta) = x, f_2(\theta) = y \} \\ &= \inf \left\{ \max \left(\sup_{t_1} |(f_1 \oplus f_2)(it_1)|_{n+m}, \right. \right. \\ & \quad \left. \left. \sup_{t_2} |||(f_1 \oplus f_2)(1 + it_2)|||_{n+m} \right) : f_1(\theta) = x, f_2(\theta) = y \right\} \\ &= \inf \left\{ \max \left(\sup_{t_1} \|(f_1(it_1), f_2(it_1))\|_{l^r(M_n(V), M_m(V))}, \right. \right. \\ & \quad \left. \left. \sup_{t_2} \|(f_1(1 + it_2), f_2(1 + it_2))\|_{l^q(M_n(W), M_m(W))} \right) : f_1(\theta) = x, f_2(\theta) = y \right\} \\ &= \|(f_1(\theta), f_2(\theta))\|_{l^r(M_n((V, W)_{[\theta]}), M_m((V, W)_{[\theta]}))} \\ &= \left(\|x\|_{[\theta]}^r + \|y\|_{[\theta]}^r \right)^{\frac{1}{r}}. \quad \blacksquare \end{aligned}$$

Remark 4.22 Is there a direct proof of Theorem 4.21 along the lines of proof of Theorem 4.18?

Theorem 4.23 Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two matricially normed spaces whose matrix norms are both re-arrangement invariant. Then, for $0 < \theta < 1$, the interpolated matrix norms $\{\|\cdot\|_{[\theta]}\}$ and $\{\|\cdot\|^{[\theta]}\}$ are both re-arrangement invariant.

Proof. Fix $(a_{kl}) \in M_n((V, W)_{\theta})$ and (b_{kl}) be any re-arrangement of (a_{kl}) . Then

$$\begin{aligned} \|(a_{kl})\|_{[\theta]} &= \inf \{ \|f\|_{\mathcal{F}_n} : f(\theta) = (a_{kl}) \} \\ &= \inf \{ \|(f_{kl})\|_{\mathcal{F}_n} : f \in \mathcal{F}, f_{kl}(\theta) = a_{kl} \text{ for } 1 \leq k, l \leq n \}. \end{aligned}$$

Now let (g_{kl}) be the re-arrangement of (f_{kl}) which corresponds to that of (b_{kl}) with respect to (a_{kl}) . Then

$$\begin{aligned}\|(g_{kl})\|_{\mathcal{F}_n} &= \max \left(\sup_s |(g_{kl})(is)|_n, \sup_t |||(g_{kl})(1+it)|||_n \right) \\ &= \max \left(\sup_s |(g_{kl}(is))|_n, \sup_t |||(g_{kl}(1+it))|||_n \right) \\ &= \max \left(\sup_s |(f_{kl}(is))|_n, \sup_t |||(f_{kl}(1+it))|||_n \right) \\ &= \|(f_{kl})\|_{\mathcal{F}_n}.\end{aligned}$$

Now, if $f_{kl}(\theta) = a_{kl}$, then $g_{kl}(\theta) = b_{kl}$ for all $1 \leq k, l \leq n$. Hence

$$\begin{aligned}\|(a_{kl})\|_{[\theta]} &= \inf \left\{ \|(f_{kl})\|_{\mathcal{F}_n} : f_{kl}(\theta) = (a_{kl}) \right\} \\ &= \inf \left\{ \|(g_{kl})\|_{\mathcal{F}_n} : g_{kl}(\theta) = (b_{kl}) \right\} \\ &= \|(b_{kl})\|_{[\theta]},\end{aligned}$$

and thus $\{\|\cdot\|_{[\theta]}\}$ is re-arrangement invariant, as required. The proof that $\{\|\cdot\|^{[\theta]}\}$ is re-arrangement invariant is analogous. ■

Corollary 4.24 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two Hilbert-Schmidt spaces. Then, for $0 < \theta < 1$, $((V, W)_{[\theta]}, \{\|\cdot\|_{[\theta]}\})$ is a Hilbert-Schmidt space.*

Proof. Immediate by Theorems 4.5, 4.21 and 4.23. ■

4.4.2 The Real Method

Throughout this subsection (unless explicitly stated to the contrary), we fix $1 \leq q \leq \infty$ and θ in the range $\begin{cases} 0 < \theta < 1 & \text{if } q \neq \infty \\ 0 \leq \theta \leq 1 & \text{if } q = \infty. \end{cases}$

Theorem 4.25 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be matricially normed spaces such that (V, W) is a compatible couple of normed linear spaces. Then $(K_{\theta,q}(V, W), \{\|\cdot\|_{\theta,q;K}\})$ is a matricially normed space where $\|\cdot\|_{\theta,q;K}$ is the norm obtained by interpolating the norms $|\cdot|_n$ and $|||\cdot|||_n$ for all $n \in \mathbb{N}$.*

Proof. Clearly, $(M_n(V), M_n(W))$ is a compatible couple of normed linear spaces for all $n \geq 2$ and, once again, the proof is in four parts:

- (a) $K_{\theta,q}(M_n(V), M_n(W)) \subseteq M_n(K_{\theta,q}(V, W));$
- (b) $M_n(K_{\theta,q}(V, W)) \subseteq K_{\theta,q}(M_n(V), M_n(W));$
- (c) $\|(x_{ij}) \oplus O_m\|_{\theta,q;K} = \|(x_{ij})\|_{\theta,q;K};$
- (d) $\|(\alpha_{ij})(x_{ij})(\beta_{ij})\|_{\theta,q;K} \leq \|(\alpha_{ij})\| \| (x_{ij}) \|_{\theta,q;K} \|(\beta_{ij})\|.$

For $n \in \mathbb{N}$, $t > 0$ and $a \in M_n(V) + M_n(W)$, let

$$K_n(t, a) = \inf_{a=a_0+a_1} \{|a_0|_n + t|||a_1|||_n\}.$$

(a) Fix $(a_{ij}) \in K_{\theta,q}(M_n(V), M_n(W))$. Therefore $(a_{ij}) \in M_n(V) + M_n(W) = M_n(V+W)$ and $\Phi_{\theta,q}(K_n(t, (a_{ij}))) < \infty$. So, for each $1 \leq i, j \leq n$, $a_{ij} \in V + W$. Then

$$\begin{aligned} K_1(t, a_{ij}) &= \inf_{a_{ij}=a_{ij}^0+a_{ij}^1} \left\{ \|a_{ij}^0\|_1 + t \|a_{ij}^1\|_1 \right\} \\ &\leq \inf_{(a_{ij})=(a_{ij})^0+(a_{ij})^1} \left\{ \|(a_{ij})^0\|_n + t \|(a_{ij})^1\|_n \right\} \\ &= K_n(t, (a_{ij})), \end{aligned}$$

and so

$$\begin{aligned} \Phi_{\theta,q}(K_1(t, a_{ij})) &= \left(\int_0^\infty \left(t^{-\theta} K_1(t, a_{ij}) \right)^q \frac{1}{t} dt \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^\infty \left(t^{-\theta} K_n(t, (a_{ij})) \right)^q \frac{1}{t} dt \right)^{\frac{1}{q}} \\ &= \Phi_{\theta,q}(K_n(t, (a_{ij}))) < \infty, \end{aligned}$$

by hypothesis and hence each $a_{ij} \in K_{\theta,q}(V, W)$, as required.

(b) Let $(b_{ij}) \in M_n(K_{\theta,q}(V, W))$. Then, for each $1 \leq i, j \leq n$, each $b_{ij} \in K_{\theta,q}(V, W)$ and so $K_1(t, b_{ij}) < \infty$. Now each $b_{ij} \in V + W$ and so $(b_{ij}) \in M_n(V) + M_n(W)$. Then

$$\begin{aligned} K_n(t, (b_{ij})) &= \inf_{(b_{ij})=(b_{ij})^0+(b_{ij})^1} \left\{ \|(b_{ij})^0\|_n + t \|(b_{ij})^1\|_n \right\} \\ &\leq \sum_{i,j=1}^n K_1(t, b_{ij}), \end{aligned}$$

and so

$$\Phi_{\theta,q}(K_n(t, (b_{ij}))) \leq \sum_{i,j=1}^n \Phi_{\theta,q}(K_1(t, b_{ij})) < \infty.$$

Hence $(b_{ij}) \in K_{\theta,q}(M_n(V), M_n(W))$.

(c) Now

$$\begin{aligned} K_{n+m}(t, (x_{ij} \oplus O_m)) &= \inf_{(x_{ij} \oplus O_m)=(y_{ij})+(z_{ij})} \left\{ \|(y_{ij})\|_{n+m} + t \|(z_{ij})\|_{n+m} \right\} \\ &= \inf_{(x_{ij})=(v_{ij})+(w_{ij})} \left\{ \|(v_{ij})\|_n + t \|(w_{ij})\|_n \right\} \\ &= K_n(t, (x_{ij})), \end{aligned}$$

and hence

$$\begin{aligned} \|(x_{ij}) \oplus O_m\|_{\theta,q;K} &= \Phi_{\theta,q}(K_{n+m}(t, (x_{ij} \oplus O_m))) \\ &= \Phi_{\theta,q}(K_n(t, (x_{ij}))) \\ &= \|(x_{ij})\|_{\theta,q;K}, \end{aligned}$$

as required.

(d) this is proved exactly as in the case of complex interpolation since the real interpolation functor $K_{\theta,q}$ is also exact of exponent θ . ■

We now consider the real interpolation of L^p -matricially normed spaces and we do this in three cases: Propositions 4.26, 4.28 and 4.31.

Proposition 4.26 *Let θ, q be in the usual ranges and let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two L^1 -matricially normed spaces such that (V, W) is a compatible couple of normed linear spaces. Then:*

- (i) *the interpolation space $(K_{\theta,1}(V, W), \{\|\cdot\|_{\theta,1;K}\})$ is an L^1 -matricially normed space;*
- (ii) *for $q > 1$, the interpolation space $(K_{\theta,q}(V, W), \{\|\cdot\|_{\theta,q;K}\})$ satisfies*

$$\|(a_{ij}) \oplus (b_{ij})\|_{\theta,q;K} \leq \|(a_{ij})\|_{\theta,q;K} + \|(b_{ij})\|_{\theta,q;K},$$

for all $m, n \in \mathbb{N}$, $(a_{ij}) \in M_n(K_{\theta,q}(V, W))$, $(b_{ij}) \in M_m(K_{\theta,q}(V, W))$.

Proof.

$$\begin{aligned} & K_{n+m}(t, (a_{ij}) \oplus (b_{ij})) \\ &= \inf_{(a_{ij}) \oplus (b_{ij}) = (c_{ij}) + (d_{ij})} \left\{ |(c_{ij})|_{n+m} + t |||(d_{ij})|||_{n+m} \right\} \\ &= \inf_{(a_{ij}) \oplus (b_{ij}) = (s_{ij}) \oplus (t_{ij}) + (v_{ij}) \oplus (w_{ij})} \left\{ |(s_{ij}) \oplus (t_{ij})|_{n+m} + t |||(v_{ij}) \oplus (w_{ij})|||_{n+m} \right\} \quad (4.4) \\ &= \inf \left\{ |(s_{ij})|_n + |(t_{ij})|_m + t [|||(v_{ij})|||_n + |||(w_{ij})|||_m] \right\} \\ &= \inf \left\{ [| (s_{ij}) |_n + t |||(v_{ij}) |||_n] + [| (t_{ij}) |_m + t |||(w_{ij}) |||_m] \right\} \\ &= K_n(t, (a_{ij})) + K_m(t, (b_{ij})), \end{aligned}$$

and so

$$\begin{aligned} & \|(a_{ij}) \oplus (b_{ij})\|_{\theta,q;K} = \Phi_{\theta,q}(K_{n+m}(t, (a_{ij}) \oplus (b_{ij}))) \\ &= \left(\int_0^\infty \left[t^{-\theta} \{K_n(t, (a_{ij})) + K_m(t, (b_{ij}))\} \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} \quad (4.5) \end{aligned}$$

$$\begin{aligned} & \leq \left(\int_0^\infty \left[t^{-\theta} K_n(t, (a_{ij})) \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} + \left(\int_0^\infty \left[t^{-\theta} K_m(t, (b_{ij})) \right]^q \frac{1}{t} dt \right)^{\frac{1}{q}} \quad (4.6) \\ &= \|(a_{ij})\|_{\theta,q;K} + \|(b_{ij})\|_{\theta,q;K}, \end{aligned}$$

for a general q and note that we get equality between (4.5) and (4.6) in the case that $q = 1$, making the interpolation space L^1 -matricially normed. ■

We next make an easy observation.

Lemma 4.27 *Let $1 \leq p < \infty$, $z > 0$ and $s, t, v, w \geq 0$. Then*

$$\left\{ (s^p + t^p)^{\frac{1}{p}} + z(v^p + w^p)^{\frac{1}{p}} \right\}^p \geq (s + zv)^p + (t + zw)^p.$$

Proof.

$$\begin{aligned} \{(s + zv)^p + (t + zw)^p\}^{\frac{1}{p}} &= \left\| \begin{pmatrix} s + zv & 0 \\ 0 & t + zw \end{pmatrix} \right\|_p \\ &\leq \left\| \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \right\|_p + z \left\| \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \right\|_p \\ &= (s^p + t^p)^{\frac{1}{p}} + z(v^p + w^p)^{\frac{1}{p}}. \quad \blacksquare \end{aligned}$$

Proposition 4.28 *Let $1 < p < \infty$, $0 < \theta < 1$ and $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two L^p -matricially normed spaces such that (V, W) is a compatible couple of normed linear spaces. Then the interpolation space $(K_{\theta,p}(V, W), \{\|\cdot\|_{\theta,p;K}\})$ is sup- L^p , that is to say, it satisfies*

$$\|(a_{ij}) \oplus (b_{ij})\|_{\theta,p;K}^p \geq \|(a_{ij})\|_{\theta,p;K}^p + \|(b_{ij})\|_{\theta,p;K}^p,$$

for all $m, n \in \mathbb{N}$, $(a_{ij}) \in M_n(K_{\theta,p}(V, W))$, $(b_{ij}) \in M_m(K_{\theta,p}(V, W))$.

Proof.

$$\begin{aligned} & K_{n+m}(t, (a_{ij}) \oplus (b_{ij})) \\ &= \inf \left\{ |(s_{ij}) \oplus (t_{ij})|_{n+m} + t |||(v_{ij}) \oplus (w_{ij})|||_{n+m} \right\} \text{ by (4.4)} \\ &= \inf \left\{ [| (s_{ij})|_n^p + |(t_{ij})|_m^p]^{\frac{1}{p}} + t [|||(v_{ij})|||_n^p + |||(w_{ij})|||_m^p]^{\frac{1}{p}} \right\} \\ &\geq \inf \left\{ [| (s_{ij})|_n + t |||(v_{ij})|||_n]^p + [| (t_{ij})|_m + t |||(w_{ij})|||_m]^p \right\}^{\frac{1}{p}} \text{ by Lemma 4.27} \\ &= [K_n(t, (a_{ij}))^p + K_m(t, (b_{ij}))^p]^{\frac{1}{p}}, \end{aligned}$$

and so

$$\begin{aligned} \|(a_{ij}) \oplus (b_{ij})\|_{\theta,p;K}^p &= \int_0^\infty \left[t^{-\theta} K_{n+m}(t, (a_{ij}) \oplus (b_{ij})) \right]^p \frac{1}{t} dt \\ &\geq \int_0^\infty t^{-p\theta} [K_n(t, (a_{ij}))^p + K_m(t, (b_{ij}))^p] \frac{1}{t} dt \\ &= \int_0^\infty \left[t^{-\theta} K_n(t, (a_{ij})) \right]^p \frac{1}{t} dt + \int_0^\infty \left[t^{-\theta} K_m(t, (b_{ij})) \right]^p \frac{1}{t} dt \\ &= \|(a_{ij})\|_{\theta,p;K}^p + \|(b_{ij})\|_{\theta,p;K}^p. \quad \blacksquare \end{aligned}$$

Remark 4.29 This immediately suggests that we dualise and use Lemma 4.30 with the following idea: since the interpolation of two L^p -matricially normed spaces is sup- L^p , the dual of the interpolated space will be sub- L^q . Dualising our matricially normed spaces and *then* interpolating means we get sup- L^q . If we have equality of norms with this duality, the space interpolated from the duals is, in fact, L^q and hence the interpolation space from the L^p -matricially normed spaces must be L^p . However, the duality theorem for the real method holds, in general, only with equivalent (and not exact) norms.

Lemma 4.30 *Fix $1 < p < \infty$ and let $(V, \{|||\cdot|||_n\})$ be a matricially normed space such that*

$$\|v \oplus w\|_{n+m}^p \leq \|v\|_n^p + \|w\|_m^p,$$

for all $v \in M_n(V)$, $w \in M_m(V)$. Then

$$\|f \oplus g\|_{n+m}^{*q} \geq \|f\|_n^{*q} + \|g\|_m^{*q},$$

for all $f \in M_n(V)^*$, $g \in M_m(V)^*$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. (This is essentially a watered down version of the proof of [21, Theorem 5.1].)

$$\begin{aligned}
\|f \oplus g\|_{n+m}^* &= \sup \left\{ |\langle (x_{ij}), f \oplus g \rangle| : \|(x_{ij})\|_{n+m} \leq 1 \right\} \\
&= \sup \left\{ |\langle v \oplus w, f \oplus g \rangle| : \|v \oplus w\|_{n+m} \leq 1 \right\} \\
&\geq \sup \left\{ |\langle v \oplus w, f \oplus g \rangle| : \|v\|_n^p + \|w\|_n^p \leq 1 \right\} \\
&= \sup \left\{ |\langle v, f \rangle + \langle w, g \rangle| : \|v\|_n^p + \|w\|_n^p \leq 1 \right\}.
\end{aligned}$$

Now, fix $v \in M_n(V)$ and $w \in M_m(W)$, both with norm one, such that

$$\langle v, f \rangle = \|f\|_n^* \text{ and } \langle w, g \rangle = \|g\|_m^*.$$

Set $v_0 \stackrel{\text{def}}{=} \|f\|_n^{*q-1} v$ and $w_0 \stackrel{\text{def}}{=} \|g\|_m^{*q-1} w$. Then

$$(\|v_0\|_n^p + \|w_0\|_m^p)^{\frac{1}{p}} = \left(\|f\|_n^{*q} + \|g\|_m^{*q} \right)^{\frac{1}{p}},$$

since $p(q-1) = q$ and so

$$\begin{aligned}
|\langle v_0, f \rangle + \langle w_0, g \rangle| &= \|f\|_n^{*q} + \|g\|_m^{*q} \\
&= \left(\|f\|_n^{*q} + \|g\|_m^{*q} \right)^{\frac{1}{q}} (\|v_0\|_n^p + \|w_0\|_m^p)^{\frac{1}{p}} \\
&\geq \left(\|f\|_n^{*q} + \|g\|_m^{*q} \right)^{\frac{1}{q}}.
\end{aligned}$$

■

Proposition 4.31 *Let θ, q be in the usual ranges and let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two L^∞ -matricially normed spaces such that (V, W) is a compatible couple of normed linear spaces. Then the interpolation space $(K_{\theta,q}(V, W), \{\|\cdot\|_{\theta,q,K}\})$ satisfies*

$$\|(a_{ij}) \oplus (b_{ij})\|_{\theta,q,K} \geq \max \left\{ \|(a_{ij})\|_{\theta,q,K}, \|(b_{ij})\|_{\theta,q,K} \right\},$$

for all $m, n \in \mathbb{N}$, $(a_{ij}) \in M_n(K_{\theta,q}(V, W))$, $(b_{ij}) \in M_m(K_{\theta,q}(V, W))$.

Proof. Suppose that $(a_{ij}) \oplus (b_{ij}) = (s_{ij}) \oplus (t_{ij}) + (v_{ij}) \oplus (w_{ij})$. Then

$$\begin{aligned}
&|(s_{ij}) \oplus (t_{ij})|_{n+m} + t |||(v_{ij}) \oplus (w_{ij})|||_{n+m} \\
&= \max [|(s_{ij})|_n, |(t_{ij})|_m] + t \max [|||(v_{ij})|||_n, |||(w_{ij})|||_m] \\
&\geq \max [|(s_{ij})|_n + t |||(v_{ij})|||_n, |(t_{ij})|_m + t |||(w_{ij})|||_m] \\
&\geq \max \{K_n(t, (a_{ij})), K_m(t, (b_{ij}))\},
\end{aligned}$$

since $|(s_{ij})|_n + t |||(v_{ij})|||_n \geq K_n(t, (a_{ij}))$ and similarly for the second term. Hence

$$K_{n+m}(t, (a_{ij}) \oplus (b_{ij})) \geq \max \{K_n(t, (a_{ij})), K_m(t, (b_{ij}))\}.$$

Thus

$$\begin{aligned}
\|(a_{ij}) \oplus (b_{ij})\|_{\theta,q,K} &= \Phi_{\theta,q}(K_{n+m}(t, (a_{ij}) \oplus (b_{ij}))) \\
&\geq \Phi_{\theta,q}(\max \{K_n(t, (a_{ij})), K_m(t, (b_{ij}))\}) \\
&\geq \max \{\Phi_{\theta,q}(K_n(t, (a_{ij}))), \Phi_{\theta,q}(K_m(t, (b_{ij})))\} \\
&= \max \left\{ \|(a_{ij})\|_{\theta,q,K}, \|(b_{ij})\|_{\theta,q,K} \right\}.
\end{aligned}$$

■

Proposition 4.32 *Let $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ be two matricially normed spaces such that (V, W) is a compatible couple of normed linear spaces and whose matrix norms are re-arrangement invariant. Then $\{\|\cdot\|_{\theta, q; K}\}$ is re-arrangement invariant.*

Proof. Fix $(a_{ij}) \in M_n(K_{\theta, q}(V, W))$ and (b_{ij}) be any re-arrangement of (a_{ij}) . For any decomposition $(a_{ij}) = (s_{ij}) + (t_{ij})$, let (v_{ij}) and (w_{ij}) be the analogous re-arrangements of (s_{ij}) and (t_{ij}) respectively. Then

$$\begin{aligned} K_n(t, (a_{ij})) &= \inf_{(a_{ij})=(s_{ij})+(t_{ij})} \{ |(s_{ij})|_n + t |||(t_{ij})|||_n \} \\ &= \inf_{(b_{ij})=(v_{ij})+(w_{ij})} \{ |(v_{ij})|_n + t |||(w_{ij})|||_n \} \\ &= K_n(t, (b_{ij})). \end{aligned}$$

Thus $\|(a_{ij})\|_{\theta, q; K} = \|(b_{ij})\|_{\theta, q; K}$ and hence $\{\|\cdot\|_{\theta, q; K}\}$ is re-arrangement invariant. $\blacksquare$

4.4.3 The Approximation Method

Theorem 4.33 *Suppose that $(V, \{|\cdot|_n\})$ and $(W, \{|||\cdot|||_n\})$ are two matricially quasi-normed abelian groups such that (V, W) is a compatible couple. Then, for α, q in the usual ranges and $(\beta_{ij}) \in M_n(\mathbb{C})$, $(x_{ij}) \in M_n(E_{\alpha, q}(V, W))$:*

- (i) $E_{\alpha, q}(M_n(V), M_n(W)) = M_n(E_{\alpha, q}(V, W))$;
- (ii) $\|(x_{ij}) \oplus O_m\|_{\alpha, q; E} = \|(x_{ij})\|_{\alpha, q; E}$ for all $m \in \mathbb{N}$;
- (iii) $\|(\beta_{ij})(x_{ij})\|_{\alpha, q; E} \leq \|(\beta_{ij})\|^{1+\alpha} \|(x_{ij})\|_{\alpha, q; E}$.

Proof. (i) follows by the same arguments as used to prove the analogous claim for the real interpolation method.

(ii) Suppose that $(x_{ij}) \in M_n(E_{\alpha, q}(V, W))$. Then there is a sequence $(y_{ij}^{(k)})$ in $M_n(V)$ such that $\left| \left(y_{ij}^{(k)} \right) \right|_n \leq t$ for all k and

$$\left| \left| (x_{ij}) - \left(y_{ij}^{(k)} \right) \right| \right|_n \rightarrow E(t, (x_{ij})),$$

as $k \rightarrow \infty$. Then

$$\begin{aligned} E(t, (x_{ij}) \oplus O_m) &= \inf_{(z_{ij}) \in M_{n+m}(V), |(z_{ij})|_{n+m} \leq t} |||(x_{ij}) \oplus O_m - (z_{ij})|||_{n+m} \\ &\leq \left| \left| (x_{ij}) \oplus O_m - \left(y_{ij}^{(k)} \right) \oplus O_m \right| \right|_{n+m} \\ &= \left| \left| (x_{ij}) - \left(y_{ij}^{(k)} \right) \right| \right|_n, \end{aligned}$$

for all $k \in \mathbb{N}$ and hence $E(t, (x_{ij}) \oplus O_m) \leq E(t, (x_{ij}))$.

Conversely, there exists a sequence $(z_{ij}^{(k)})$ in $M_{n+m}(V)$ such that $\left| \left(z_{ij}^{(k)} \right) \right|_{n+m} \leq t$ for all k and

$$\left| \left| (x_{ij}) - \left(z_{ij}^{(k)} \right) \right| \right|_{n+m} \rightarrow E(t, (x_{ij}) \oplus O_m),$$

as $k \rightarrow \infty$. For each k , let $(y_{ij}^{(k)})$ be the top left hand $n \times n$ sub-matrix of $(z_{ij}^{(k)})$. Then $\|(y_{ij}^{(k)})\|_n \leq t$ and

$$E(t, (x_{ij})) \leq \left\| \left\| (x_{ij}) - (y_{ij}^{(k)}) \right\| \right\|_n \leq \left\| \left\| (x_{ij}) \oplus O_m - (z_{ij}^{(k)}) \right\| \right\|_{n+m},$$

for all $k \in \mathbb{N}$ and hence $E(t, (x_{ij})) \leq E(t, (x_{ij}) \oplus O_m)$, giving us equality for all t .

(iii) For $x \in M_n(E_{\alpha,q}(V, W))$, let (y_k) be a sequence in $M_n(V)$ such that $\|y_k\| \leq t$ and $\|x - y_k\|_n \rightarrow E(t, x)$ as $k \rightarrow \infty$. Let $\beta \in M_n(\mathbb{C})$. Then

$$\begin{aligned} E(\|\beta\| t, \beta x) &= \inf_{z \in M_n(V), \|z\|_n \leq \|\beta\| t} \|\|\beta x - z\|\|_n \\ &\leq \|\|\beta x - \beta y_k\|\|_n \leq \|\beta\| \|x - y_k\|_n, \end{aligned}$$

for all k and so

$$E(\|\beta\| t, \beta x) \leq \|\beta\| E(t, x),$$

for all t . Hence

$$\begin{aligned} \|\beta x\|_{\alpha,q,E} &= \left(\int_0^\infty [t^\alpha E(t, \beta x)]^q \frac{1}{t} dt \right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty [\|\beta\|^\alpha s^\alpha E(\|\beta\| s, \beta x)]^q \frac{1}{\|\beta\| s} \|\beta\| ds \right)^{\frac{1}{q}} \text{ where } t = \|\beta\| s \\ &= \|\beta\|^\alpha \left(\int_0^\infty [s^\alpha E(\|\beta\| s, \beta x)]^q \frac{1}{s} ds \right)^{\frac{1}{q}} \\ &\leq \|\beta\|^\alpha \left(\int_0^\infty [s^\alpha \|\beta\| E(s, x)]^q \frac{1}{s} ds \right)^{\frac{1}{q}} \\ &= \|\beta\|^{1+\alpha} \left(\int_0^\infty [s^\alpha E(s, x)]^q \frac{1}{s} ds \right)^{\frac{1}{q}} \\ &= \|\beta\|^{1+\alpha} \|x\|_{\alpha,q,E}. \end{aligned}$$

■

Appendix A

Maple and Mathematica Computer Programs

A.1 Basic Calculations

Below is the program that was used in conjunction with the Maple package to evaluate the Schatten p -norms of complex scalar matrices and that of the matrix obtained by taking the modulus of all the entries. Using `:` instead of `;` at the end of a line means that the result is not echoed to the screen.

```
with(linalg):Digits:=20:
A:=matrix(3,3,[-2,-1,6,4,3,1,0,-1,2]);
B:=htranspose(A):
C:=evalm(B&*A):
K:=evalf(eigenvals(C)):
k:=coeff(sqrt(K[1]),I,0);
l:=coeff(sqrt(K[2]),I,0);
m:=coeff(sqrt(K[3]),I,0);
one_norm, k+l+m;
f:=(i,j) -> abs(A[i,j]):
X:=matrix(3,3,f);
Y:=transpose(X):
Z:=evalm(Y&*X):
L:=evalf(eigenvals(Z)):
x:=coeff(sqrt(L[1]),I,0);
y:=coeff(sqrt(L[2]),I,0);
z:=coeff(sqrt(L[3]),I,0);
one_norm, x+y+z;
```

More information can be obtained from a graph using

```
plot((k^p + l^p + m^p)^(1/p) - (x^p + y^p + z^p)^(1/p), p=1..15);
```

which plots the value of $\|A\|_p - \|X\|_p$ between $p = 1$ and $p = 15$.

A.2 The Square Root of a Matrix

The following program evaluates the square root of A^*A directly. The calculation is set for 750 iterations.

```
with(linalg):Digits:=20:
A:=matrix(3,3,[1,6,1,2,3,1,0,5,4]);
B:=htranspose(A):
C:=evalm(B&*A);
X:=evalf(norm(A,2)):
f:=(i,j) -> (1/(X*X))*C[i,j]:
Id:=array(1..3,1..3,identity):
P:=matrix(3,3,f):
R:=evalm(Id-P):
Q:=evalm(.5*R):
for i from 1 to 750 do
Q:=evalm(.5*(R+Q&*Q))
od:
S:=evalm(X*(Id-Q));
evalm(S&*S);
det(S);
```

$$A := \begin{bmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 0 & 5 & 4 \end{bmatrix}$$

$$C := \begin{bmatrix} 5 & 12 & 3 \\ 12 & 70 & 29 \\ 3 & 29 & 18 \end{bmatrix}$$

$$S :=$$

$$\begin{bmatrix} 1.8553107989329613495 & 1.2473412158250937189 & -.044291428817702782215 \\ 1.2473412158250937189 & 7.8571302328271856991 & 2.5902981287290062849 \\ -.044291428817702782215 & 2.5902981287290062849 & 3.3598205121161301035 \end{bmatrix}$$

$$\begin{bmatrix} 4.9999999999999999984 & 11.999999999999999999 & 3.0000000000000000000 \\ 11.999999999999999999 & 69.99999999999999996 & 28.99999999999999998 \\ 3.0000000000000000000 & 28.99999999999999998 & 17.99999999999999999 \end{bmatrix}$$

$$30.999999999999999991$$

It's 'Mathematica equivalent' is

```

a={{1,6,1},{2,3,1},{0,5,4}}
b=Transpose[a]
c=b.a
f=N[MatrixPower[c,.5],10]

```

The Mathematica programme is particularly good on 2×2 matrices but varies between minutes and weeks for 3×3 matrices. Also, rounding errors mean that the square root occasionally has some entries with a (very small) complex part. The Maple program is far better, in general, as it always gets the answer (though a lot of iterations are usually required since convergence can be painfully slow) and never it returns a matrix with complex entries.

A.3 The Working behind Propositions 3.2 and 3.29

We begin with the details behind Example 3.30.

```

q:=1.999;
A:=matrix(2,2,[1,1,1,.0001]);
B:=htranspose(A):
C:=evalm(B&*A):
K:=evalf(eigenvals(C)):
k:=coeff(sqrt(K[1]),I,0);
l:=coeff(sqrt(K[2]),I,0);
c:=evalf((k^q +l^q)^(1/q));
X:=matrix(2,2,[2,2,2,0]);
Y:=htranspose(X):
Z:=evalm(Y&*X):
L:=evalf(eigenvals(Z)):
x:=coeff(sqrt(L[1]),I,0);
y:=coeff(sqrt(L[2]),I,0);
d:=evalf((x^q +y^q)^(1/q));
difference, d-2*c;

```

```

q := 1.999
A :=  $\begin{bmatrix} 1 & 1 \\ 1 & .0001 \end{bmatrix}$ 
k := .61796162896456492989
l := 1.6180616289645649298
c := 1.7322160128380125104

```

$$X := \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$x := 3.2360679774997896964$$

$$y := 1.2360679774997896964$$

$$d := 3.4644320696245358212$$

$$\text{difference, } .439485108004 \cdot 10^{-7}$$

As mentioned previously, a genuine proof involves evaluating the Taylor series of

$$f(q, \epsilon) = \left\| \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right\|_q - 2 \left\| \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right\|_q,$$

where $1 \leq q < 2$ and $\epsilon > 0$.

The matrix

$$\left[\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}^* \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right]^{\frac{1}{2}},$$

has eigenvalues

$$x = \sqrt{5} + 1 \text{ and } y = \sqrt{5} - 1,$$

whereas the matrix

$$\left[\begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix}^* \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right]^{\frac{1}{2}},$$

has eigenvalues

$$x_\epsilon = \sqrt{\frac{3 + \epsilon^2 + \sqrt{5 + 8\epsilon + 2\epsilon^2 + \epsilon^4}}{2}} \text{ and } y_\epsilon = \sqrt{\frac{3 + \epsilon^2 - \sqrt{5 + 8\epsilon + 2\epsilon^2 + \epsilon^4}}{2}},$$

for $\epsilon > 0$.

In principle, therefore, we can express f as the difference of two sequences in l^q and we can use Maple to evaluate the Taylor series of f . It begins

$$\begin{aligned} & -2 \frac{(\%2 + \%1)^{(\frac{1}{q})} \%3}{q(\%2 + \%1)} \epsilon \\ & -2(\%2 + \%1)^{(\frac{1}{q})} \left(\frac{\%4}{q(\%2 + \%1)} + \frac{1}{2} \frac{\%3^2 \left(\frac{1}{q} - 1\right)}{q(\%2 + \%1)^2} \right) \epsilon^2 \\ & -2(\%2 + \%1)^{(\frac{1}{q})} \left(\frac{\%3 \left(\frac{1}{q} - 1\right) \%4}{q(\%2 + \%1)^2} + \frac{\%5}{q(\%2 + \%1)} + \frac{1}{6} \frac{\%3^3 \left(\frac{1}{q} - 1\right) \left(\frac{1}{q} - 2\right)}{q(\%2 + \%1)^3} \right) \epsilon^3, \end{aligned}$$

where the variables %1, %2 and %3 are

$$\%1 = .61803398874989484820^q,$$

$$\%2 = 1.6180339887498948482^q,$$

$$\%3 = .17082039324993690892 \%2^q - 1.1708203932499369089 \%1^q.$$

(The full expansion in general requires over thirty variables which depend explicitly on q and so we have truncated after ϵ^3 and the third undetermined factor to give merely a flavour of the Taylor series.)

The calculation in specific cases is much simpler. For example, in the case $q = 1.99$, the Taylor series is

$$\begin{aligned} &.0049925790344222239890 \epsilon - .57723828102591409422 \epsilon^2 \\ &- .0018468337845400606093 \epsilon^3 + .047775128293225434562 \epsilon^4 \\ &+ .00045529305494018654584 \epsilon^5 - .0079069396706575731332 \epsilon^6 \\ &- .00013067444793482650669 \epsilon^7 + .0016259392452467697463 \epsilon^8 \\ &+ .000029718615973609454360 \epsilon^9 + O(\epsilon^{10}), \end{aligned}$$

up to the ninth power.

Since $\%3 < 0$ for $1 \leq q < 2$ (see figure B.8 on page 112), we have that, for ϵ sufficiently small, $f(q, \epsilon) > 0$, that is to say

$$\left\| \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \right\|_q - 2 \left\| \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix} \right\|_q > 0,$$

and hence $\|\cdot\|_{\mathcal{D}_q}$ is not a norm.

Appendix B

Graphical Output

This appendix contains the graphical output of chapter three.

Figure B.1 on page 105 is the graph of

$$\left\| \begin{pmatrix} 1 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p - \left\| \begin{pmatrix} 0 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p,$$

plotted for $1 \leq p \leq 3$. This demonstrates the failure of the pointwise ordering property for $\|\cdot\|_p$ when $1 \leq p < 2$ and shows that we cannot replace positivity with mere self-adjointness in Proposition 3.10.

Figures B.2 and B.3 demonstrate the failure of the $(*, p)$ property for $1 \leq p < 2$ and $2 < p < 4$. Figure B.2 on page 106 shows the general behaviour whilst figure B.3 on page 107 gives a more detailed picture in the neighbourhood of $p = 4$. In both figures, the horizontal axis denotes p whilst the vertical denotes

$$\left\| \begin{bmatrix} 100 & -100 & 0 \\ -100 & 0.01 & -99.99 \\ 0 & -99.99 & -99.99 \end{bmatrix} \right\|_p - \left\| \begin{bmatrix} 100 & 100 & 0 \\ 100 & 0.01 & 99.99 \\ 0 & 99.99 & 99.99 \end{bmatrix} \right\|_p.$$

Figures B.4 to B.7 on pages 108 to 111 show the lack of a relationship (up to the use solely of $\leq$ and $\geq$) between $\| [A_{ij}] \|_p$ and $\| [H_{ij}] \|_p$ where $H = (A^* A)^{\frac{1}{2}}$. The four examples we consider are:

- (i) $A_1 = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix};$
- (ii) $A_2 = \begin{pmatrix} -3 & 1 & 3 \\ -3 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix};$
- (iii) $A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix};$
- (iv) $A_4 = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 0 & 2 \\ -1 & 5 & -1 \end{pmatrix}.$

In each of the four figures, the horizontal axis denotes p whilst the vertical axis shows

$$\| |A_{ij}| \|_p - \| |H_{ij}| \|_p.$$

Figure B.8 on page 112 shows the value of the variable $\%3$ which was obtained in section A.3; in particular, this justifies our earlier assertion that $\%3 < 0$ for $1 \leq q < 2$.

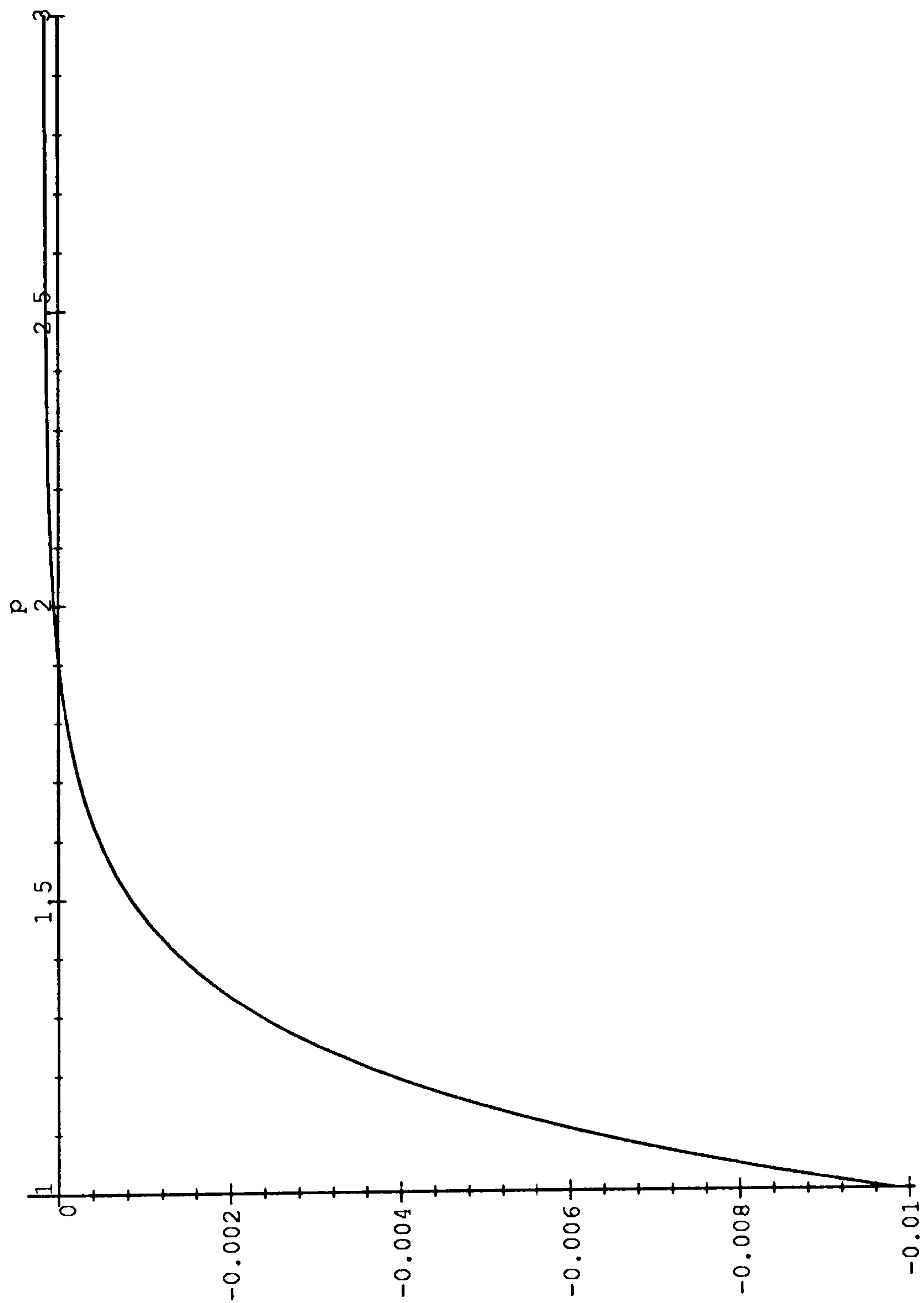


Figure B.1: $\left\| \begin{pmatrix} 1 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p - \left\| \begin{pmatrix} 0 & 100 \\ 100 & 100^2 \end{pmatrix} \right\|_p$

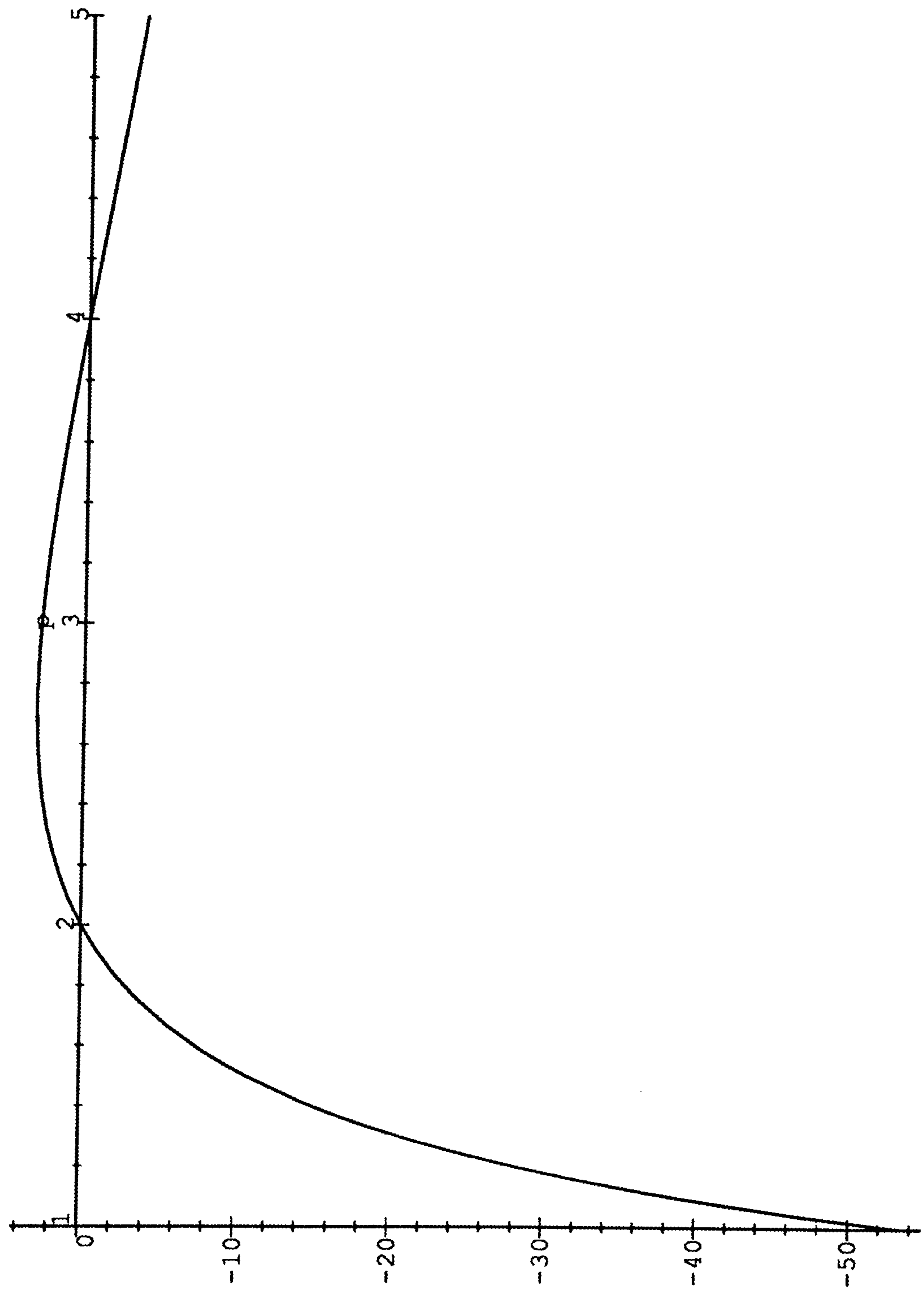


Figure B.2: $\left\| \begin{bmatrix} 100 & -100 & 0 \\ -100 & 0.01 & -99.99 \\ 0 & -99.99 & -99.99 \end{bmatrix} \right\|_p - \left\| \begin{bmatrix} 100 & 100 & 0 \\ 100 & 0.01 & 99.99 \\ 0 & 99.99 & 99.99 \end{bmatrix} \right\|_p$

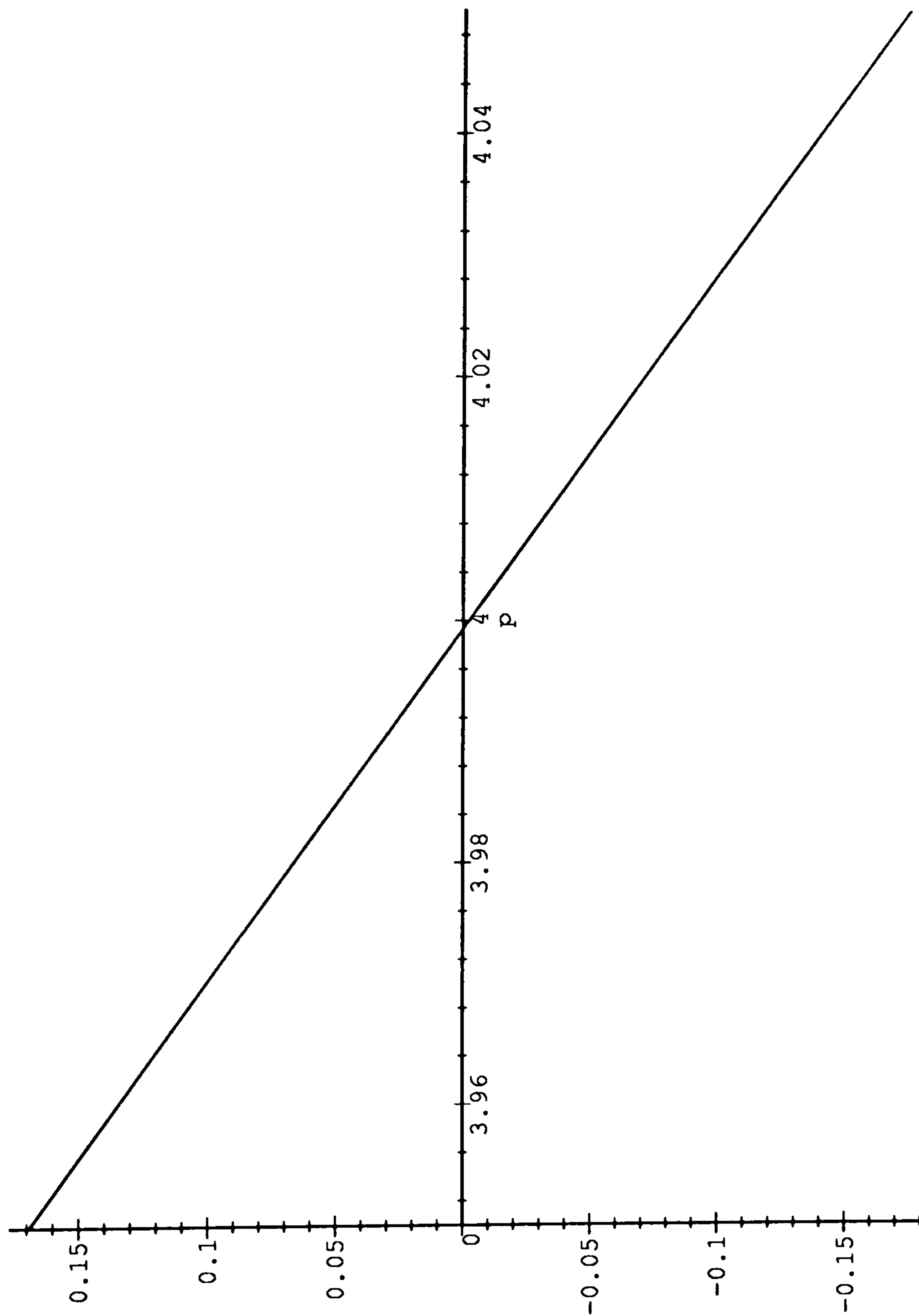


Figure B.3: $\left\| \begin{bmatrix} 100 & -100 & 0 \\ -100 & 0.01 & -99.99 \\ 0 & -99.99 & -99.99 \end{bmatrix} \right\|_p - \left\| \begin{bmatrix} 100 & 100 & 0 \\ 100 & 0.01 & 99.99 \\ 0 & 99.99 & 99.99 \end{bmatrix} \right\|_p$

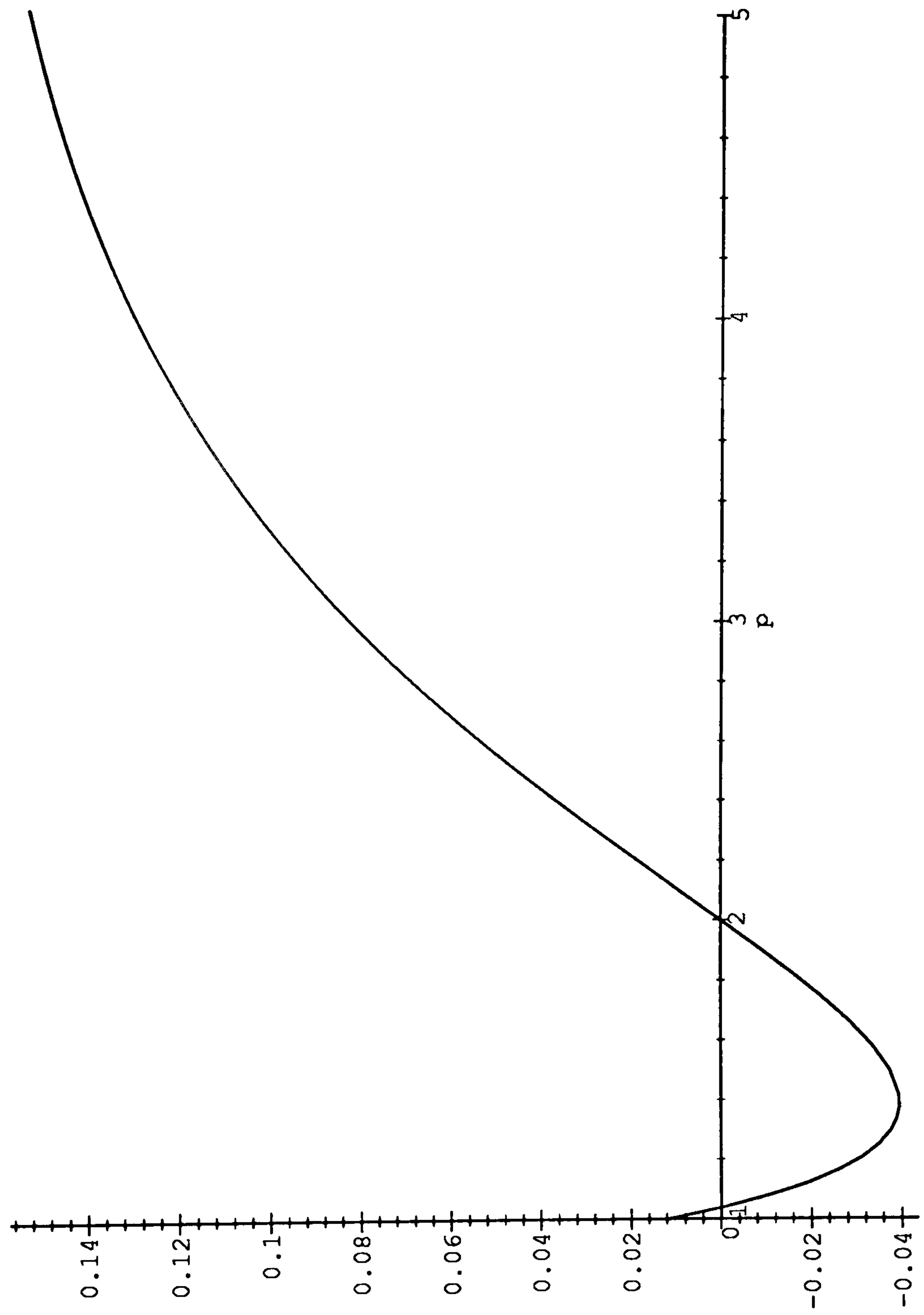


Figure B.4: $\| [|A_{ij}|] \|_p - \| [|H_{ij}|] \|_p$ for $(A_{ij}) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

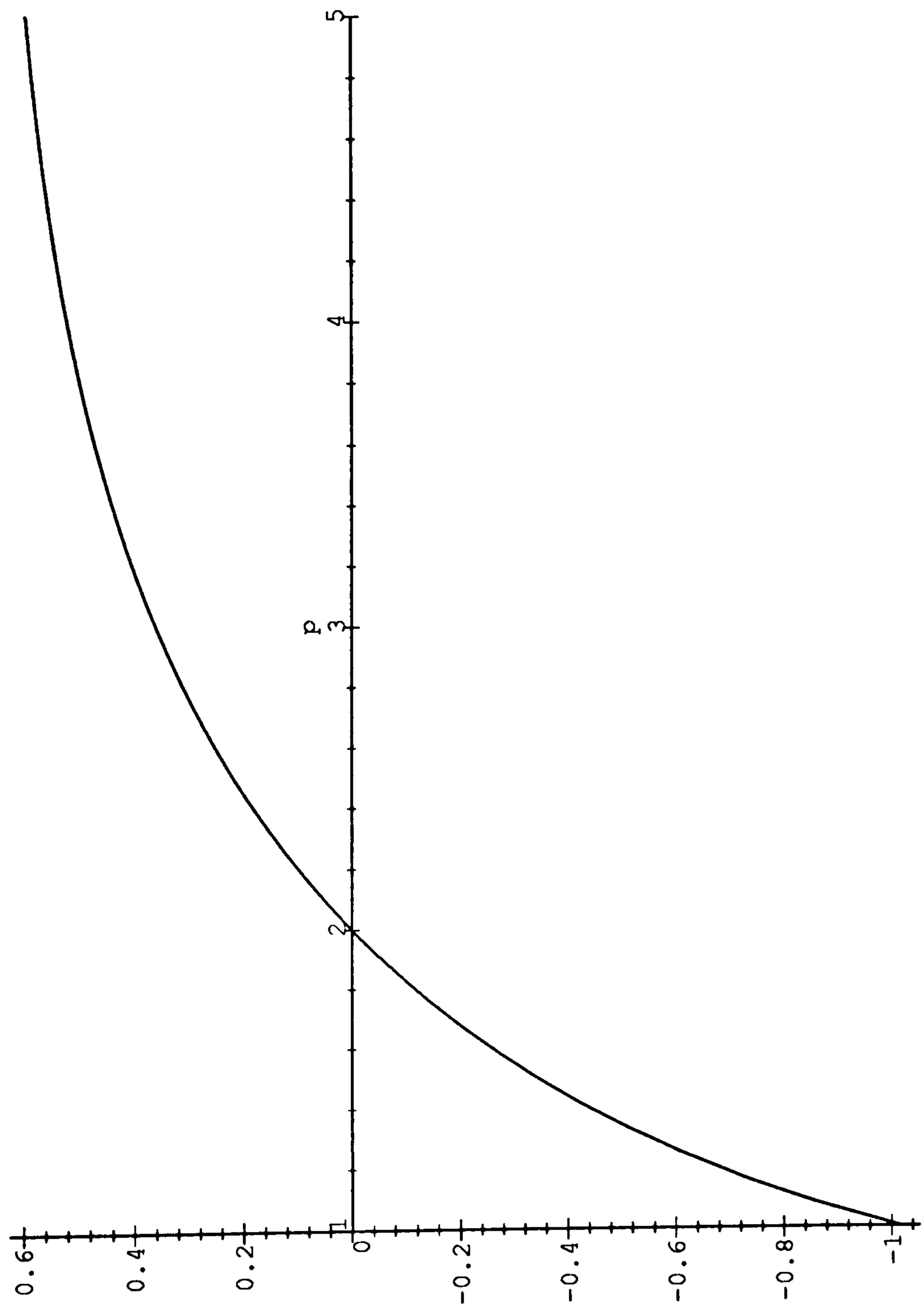


Figure B.5: $\| [A_{ij}] \|_p - \| [H_{ij}] \|_p$ for $(A_{ij}) = \begin{pmatrix} -3 & 1 & 3 \\ -3 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$

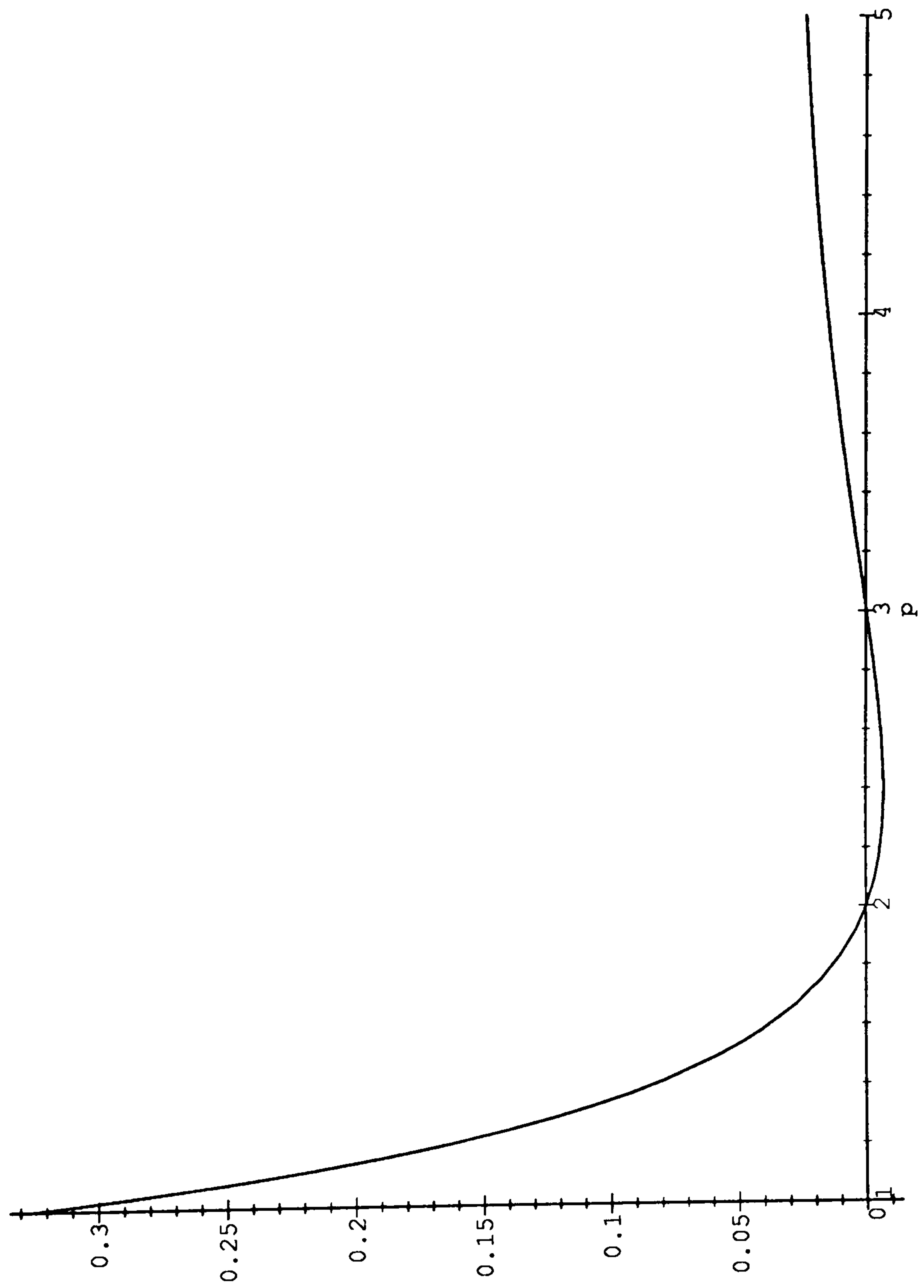


Figure B.6: $\| [|A_{ij}|] \|_p - \| [|H_{ij}|] \|_p$ for $(A_{ij}) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

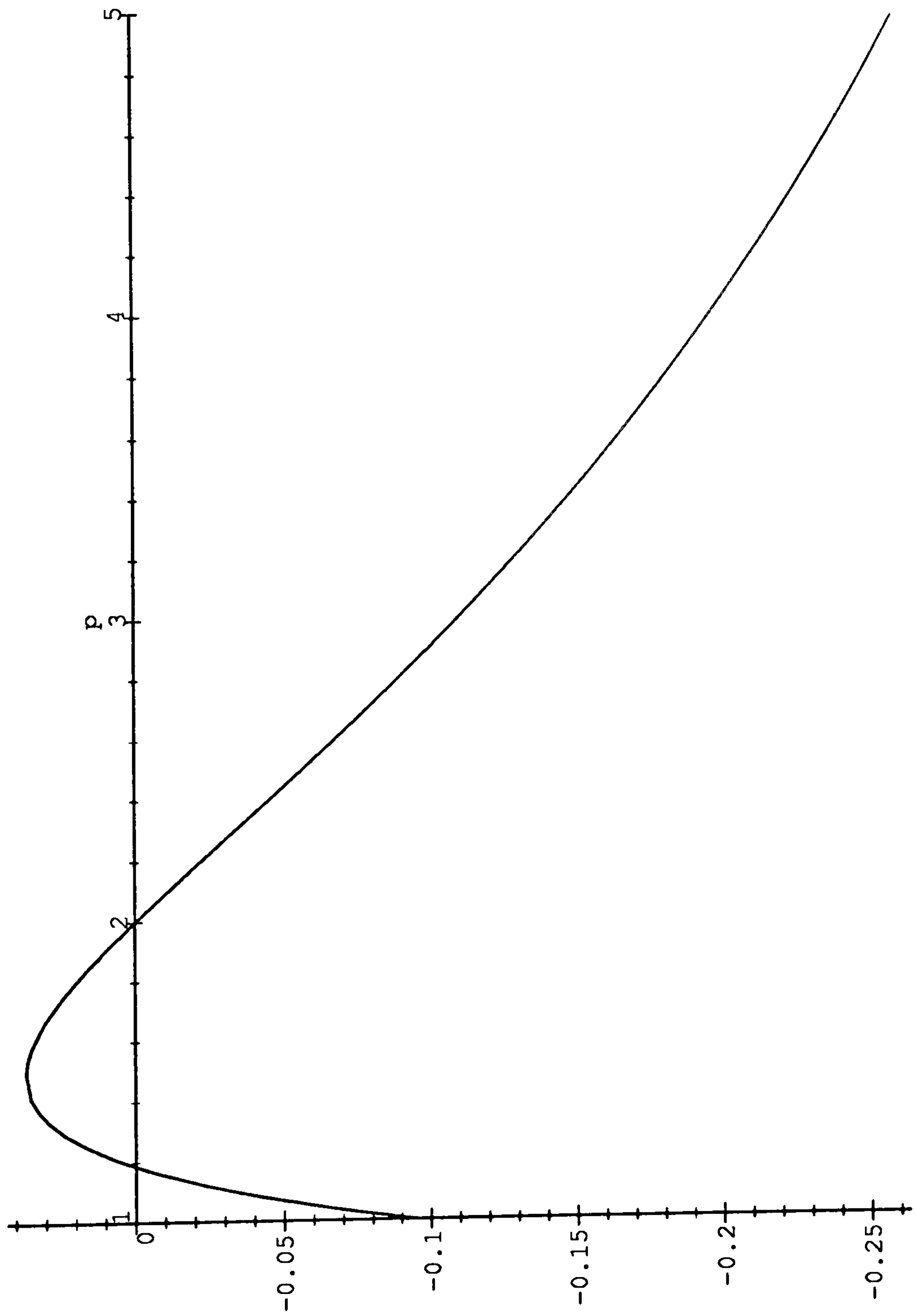


Figure B.7: $\| [|A_{ij}|] \|_p - \| [|H_{ij}|] \|_p$ for $(A_{ij}) = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 0 & 2 \\ -1 & 5 & -1 \end{pmatrix}$

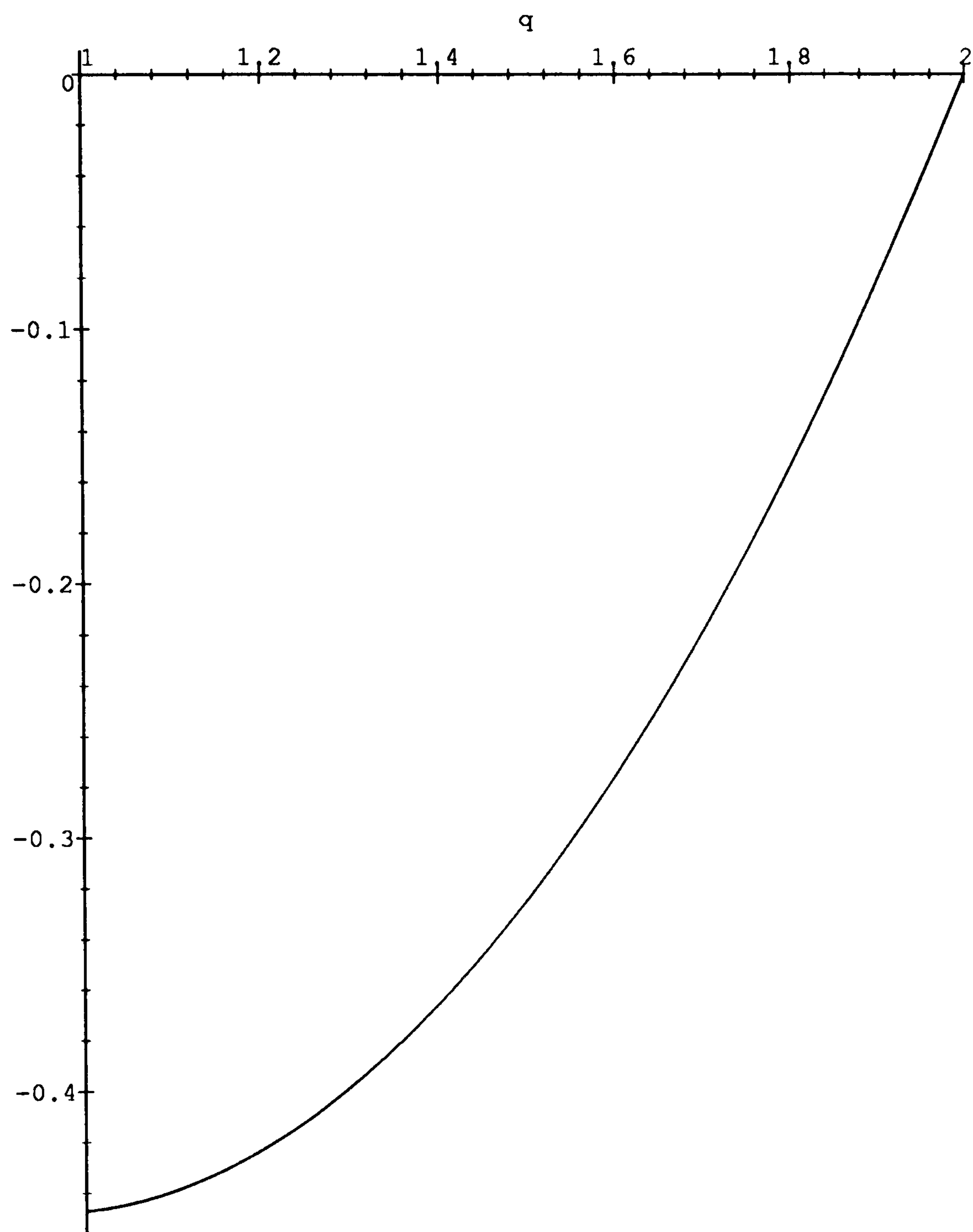


Figure B.8: Graph for %3

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