THESIS

FOR THE PH.D. DEGREE

ON

MATRIX REPRESENTATIONS OF SYMMETRIC GROUPS.

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PREFACE.

The Thesis consists of two parts. In part I, a concentrated summary of the symmetric group, its matrix representations and characters, is given. In part II, the irreducible matrix representations of the symmetric groups of degrees 4 and 5, are obtained in a rational form.

Thanks are expressed to Professor Aitken for his kind supervision.

PART I.

THE SYMMETRIC GROUP AND MATRIX REPRESENTATIONS

The Symmetric Group.

The symmetric group of order n! (or degree n) is the group whose elements are the n! permutations on n symbols (1, 2, ..., n). A typical element s of the group is indicated thus :

$$s = \begin{pmatrix} 1, 2, \ldots, n \\ s_1, s_2, \ldots, s_n \end{pmatrix}$$

the notation implying that the operation s replaces 1 by s_1, 2 by s_2, ..., n by s_n, where each of the symbols s_1, s_2, ..., s_n assumes one of the values (1,2, ..., n) no two of them assuming the same value.

A permutation which sends t_j into t_{j+1}, $j = 1,2, \ldots, n-1$ and t_n into t_1 (i.e. where each of the letters t_1, t_2, ..., t_n supposed arranged along a circle, is sent into its successor) is termed a cycle on n letters and is denoted by the symbol (t_1, t_2, ..., t_n). Thus:

$$(t_1, t_2, \ldots, t_n) = \begin{pmatrix} t_1, t_2, \ldots, t_n \\ t_n, t_1, \ldots, t_{n-1} \end{pmatrix}.$$

It is immaterial with which of its letters a cycle starts off. Thus :

$$(t_1, t_2, \ldots, t_n) = (t_2, t_3, \ldots, t_n, t_1) = \ldots = (t_n, t_1, t_2, \ldots, t_{n-1}).$$

The inverse of the cycle t is the cycle t with the sense reversed. Thus:

$$t^{-1} = (t_n, t_{n-1}, \cdots, t_2, t_1).$$

Any permutation can be written in a unique manner as the product of cycles; the order in which the factor cycles are written being immaterial (since no two of the cycles have a common letter). e.g. when n = 5

$$\begin{pmatrix} 1, 2, 3, 4, 5 \\ 3, 5, 1, 2, 4 \end{pmatrix} = (2,4,5)(1,3) = (1,3)(2,4,5)$$

$$\begin{pmatrix} 1, 2, 3, 4, 5 \\ 2, 3, 1, 4, 5 \end{pmatrix} = (1,3,2)(4)(5) = (1,3,2);$$

the unchanged symbols may or may not be mentioned.

If s and t are two permutations, the product st is defined to be the permutation obtained by operating first with the permutation t and then with the permutation s, so that e.g.,

$(1, 2)(1, 2, 3) = (1, 3)$

$(1, 2, 3)(1, 2) = (2, 3)$

The rule for finding the product of two permutations expressed in this terminology is explained as follows. In the first bracket (in the first example) 1 is followed by 2, in the second bracket 2 is followed by 3. Hence in the product 1 is followed by 3. In the first bracket 2 is followed by 1, in the second bracket 1 is followed by 2. Hence in the product 2 is unchanged. Lastly 3 is unchanged in (1, 2) and leads to 1 in (1, 2, 3).

The number of unary cycles (i.e. cycles on one letter, or fixed letters) in a permutation is denoted by α_1, the number of binary cycles (i.e. cycles on two letters, transpositions or interchanges) by α_2, and so on. Since the sum of the orders of the cycles, including unary cycles is equal to the number of symbols n we have the basic relation:

(1) $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$.

A permutation s which factors into α_1 unary cycles, α_2 binary cycles, etc., is said to be of the cycle structure (α) and the collection of permutations each of which has a stated common cycle structure (α) is referred to as the class (α) of the symmetric group on n letters.

The number of distinct classes is the number of distinct solutions in integers (positive or zero) of the equation (1). This is precisely the number of partitions of n into integers positive or zero; for

if $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = \lambda_1$

(2) $\alpha_2 + \alpha_3 + \cdots + \alpha_n = \lambda_2$

$\cdots\cdots\cdots\cdots$

$\alpha_n = \lambda_n$

then

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n; \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0;$

and conversely given any such partition of n we can associate with it a solution (α) of (1) defined by

(3) $\alpha_1 = \lambda_1 - \lambda_2, \quad \alpha_2 = \lambda_2 - \lambda_3, \quad \cdots, \quad \alpha_n = \lambda_n$.

The zeros which may occur at the end of a partition of n are omitted; thus if $\lambda_k > 0$, $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$, the partition $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is denoted by $(\lambda_1, \lambda_2, \cdots, \lambda_k)$, using an exponential notation when two or more adjacent λ's are equal.

As an example, the symmetric group of order 6 has three classes. The first consists of the identity which has three cycles of order 1 and the corresponding partition is (3). The second class contains the three substitutions (1,2), (1,3), (2,3) where there is one cycle of order 1 and one cycle of order 2 and the partition is (2,1). The third class comprises the substitutions (1,2,3), (1,3,2) corresponding to the partition (1^3).

The number of permutations s which belong to a class (α) is denoted by $n_{(\alpha)}$ and is found (by Cauchy) to be

$$(4) \qquad n_{(\alpha)} = \frac{n!}{1^{\alpha_1}\alpha_1! \cdot 2^{\alpha_2}\alpha_2! \cdots n^{\alpha_n}\alpha_n!}.$$

If s_1 and t are two permutations of the symmetric group on n letters, $s_2 = t s_1 t^{-1}$ is called the transform of s_1 by the permutation t, s_1 being the transform of s_2 by t^{-1}. Such two permutations as s_1 and s_2 which are the transforms of one another are called conjugate elements of the symmetric group. Two permutations s_2 and s_3 which are conjugate to the same permutation s_1 are conjugate to one another. The complete set of permutations s_1, s_2, s_3, ... which are conjugate to each other belongs to the same class (α). Conversely all the permutations which belong to the same class (α) are conjugate to each other, for the permutation

$$s = (s_{11}, s_{12}, \ldots, s_{1p})(s_{21}, s_{22}, \ldots, s_{2q})(s_{31}, s_{32}, \ldots)\ldots$$

is transferred into the permutation

$$t = (t_{11}, t_{12}, \ldots, t_{1p})(t_{21}, t_{22}, \ldots, t_{2q})(t_{31}, t_{32}, \ldots)\ldots$$

by the operation which replaces the symbols

$$\{s_{11}, s_{12}, \ldots, s_{1p}, s_{21}, s_{22}, \ldots, s_{2q}, s_{31}, s_{32}, \ldots\}$$

by the symbols

$$\{t_{11}, t_{12}, \ldots, t_{1p}, t_{21}, t_{22}, \ldots, t_{2q}, t_{31}, t_{32}, \ldots\}$$

A class (α), also, contains the reciprocal of each of the permutations contained in it, since the reciprocal of any permutation s is simply obtained by analysing s into a product of cycles and then reversing the sense of these cycles.

If $(x_1, x_2, \ldots, x_n)$ are n indeterminates, the difference product

$$\prod_{i<k} (x_i - x_k) = (x_1 - x_2)(x_1 - x_3) \cdots (x_{n-1} - x_n)$$

is denoted by $\Delta(x)$, and evidently every interchange on two consecutive symbols changes the sign of $\Delta(x)$. An interchange on two non-consecutive symbols can be expressed as the product of an odd number of interchanges on consecutive symbols, e.g.,

$$(3,6) = (3,4)(4,5)(5,6)(4,5)(3,4).$$

Since a permutation p which contains one cycle on m $\leqslant$

n letters can be analysed into the product of m-1 binary cycles, thus :

$$p = (t_1, t_2, \dots, t_m) = (t_1, t_2)(t_1, t_3) \cdots (t_1, t_m) \text{ ,}$$

a general permutation, of cycle structure (α), will leave $\Delta(x)$ unaltered if $\alpha_2 + \alpha_4 + \cdots$ is even and will change its sign if $\alpha_2 + \alpha_4 + \cdots$ is odd. In the first case the class (α) is said to be a class of even permutations or simply an even class; in the second case (α) is an odd class.

If t is an odd permutation, ts is an odd permutation for every even permutation s; so simply there are exactly as many even permutations as odd ones in the symmetric group on n letters.

Matrix Representations & Group Characters.

If to each element s_i of the symmetric group on n letters there corresponds a matrix A_i such that $A_i A_j = A_k$ whenever $s_i s_j = s_k$ the matrices A_i are said to form a matrix representation A of the symmetric group. e.g., the group of the six permutation matrices,

$$\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}$$

provides a matrix representation of the symmetric group on 3 letters, namely

1, (1,2), (1,3), (2,3), (1,2,3), (1,3,2).

The matrices A_i need not all be distinct. To several elements of the group may correspond identical matrices, so that the representation is not simply, but multiply isomorphic with the group. e.g., A_i may be the one-rowed unit matrix for every element of the symmetric group.

If A is a matrix representation of the symmetric group, and H is any fixed matrix of the same dimension as A, then HAH^{-1} is also a matrix representation of the symmetric group, since

$$H A_i H^{-1} \cdot H A_j H^{-1} = H A_i A_j H^{-1}.$$

The matrix representations A and HAH^{-1} are said to be equivalent or identical.

If there exists a matrix H such that

$$H A_i H^{-1} = \left[\begin{array}{c:c} B_i' & A_i' \\ \hdashline 0 & C_i' \end{array}\right]$$

for all i, where B_i' and C_i' are square matrices, then A is said to be reducible.

If there exists a matrix K such that

$$K A_i K^{-1} = \left[\begin{array}{c:c} B_i & 0 \\ \hdashline 0 & C_i \end{array}\right]$$

for all i, then A is said to be completely reducible.

The reducibility was shown by Schur to imply complete reducibility, and the matrix representation A is said to be reducible to the two matrix representations B and C, or equivalent to their direct sum, thus :

$$A = B \dotplus C$$

Each of B and C may be reducible, and so on, until ultimately

$$A = \begin{bmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_k \end{bmatrix} = \Gamma_1 \dotplus \Gamma_2 \dotplus \cdots \dotplus \Gamma_k,$$

where $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are irreducible.

Frobenius has shown that for the symmetric group of order $n!$ there are exactly p irreducible matrix representations, $\Gamma_1, \Gamma_2, \cdots, \Gamma_p$ corresponding to the p partitions of n, and any matrix representation is equivalent to a direct sum of these irreducible matrix representations, each being repeated any number of times, or omitted.

As an example, with

$$H = \frac{1}{12}\begin{bmatrix} 4 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and hence,

$$H^{-1} = \begin{bmatrix} 1 & 4 & 4 \\ 1 & -4 & 0 \\ 1 & 0 & -4 \end{bmatrix}$$

the six permutation matrices of order 3 x 3 are equivalent to

$$\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & -1 \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & -1 \\ \cdot & 1 & \cdot \end{bmatrix}$$

and the matrix representation of the symmetric group of order 3! by the permutation matrices of order 3 x 3 is reducible to the unit representation

1, 1, 1, 1, 1, 1

and another irreducible representation of order 2 x 2 namely

$$\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & \cdot \end{bmatrix}.$$

The determinants of these matrices give a scalar representation, for if $A_i A_j = A_k$, $|A_i||A_j| = |A_k|$. These determinants are

1, -1, -1, -1, 1, 1

and this is the third irreducible representation of the symmetric group on 3 letters, but it is not contained in the representation by the permutation matrices of order 3 x 3.

The spur (i.e., the sum of the diagonal elements) of the matrix corresponding to the element s_i, in the irreducible representation Γ_j is called the character of s_i and is written $\chi^{(j)}(s_i)$. The set of characters of the n! elements of the symmetric group on n letters corresponding to Γ_j is called a group character, and is written $\chi^{(j)}$. The $n_{(\alpha)}$ elements of the class (α) being transforms of one another, the corresponding matrices have the same spur, and the characters of the $n_{(\alpha)}$ elements are equal. This value is referred to as the character of the class (α), in the representation Γ_j and is written $\chi^{(j)}_{(\alpha)}$. $\chi^{(j)}_{(1^n)}$ is called the degree of the character and is written $f^{(j)}$.

There are p^2 distinct numbers which are the characters of the p classes corresponding to the p partitions. These numbers are usually arranged in a square table which is called the table of characters. e.g., in the case of the symmetric group on 3 letters the table of characters is

$n(\alpha) =$	1	3	2
	1	1	1
	2	0	-1
	1	-1	1

The group characters are all real numbers and satisfy orthogonal relations. These relations are :

$$\sum_{(\alpha)} n_{(\alpha)} \left\{ \chi^{(i)}_{(\alpha)} \right\}^2 = n!$$

$$\sum_{(\alpha)} n_{(\alpha)} \chi^{i}_{(\alpha)} \chi^{j}_{(\alpha)} = 0 \qquad j \neq i$$

$$\sum_i \left\{ \chi^{(i)}_{(\alpha)} \right\}^2 = n! / n_{(\alpha)}$$

$$\sum_i \chi^{(i)}_{(\alpha)} \chi^{(i)}_{(\beta)} = 0 \quad \beta \neq \alpha .$$

These are sometimes written in the form:

$$G D G' = n!\, I \quad , \quad G' G = n!\, D^{-1}$$

where G is the matrix of group characters, G' is its transposition, and D is the diagonal matrix with elements $n_{(\alpha)}$.

There is also the property that the sum of the squares of the degrees of the characters is equal to n! the order of the symmetric group.

Any linear function of the characters with positive integral coefficients is called a compound character. The spurs of the matrices in any matrix representation of the symmetric group is a compound character of the group, the coefficient of any simple character being the number of times the corresponding representation is repeated in the equivalent direct sum of irreducible representations.

The number of times a representation is repeated in the equivalent direct sum of irreducible representations is given explicitly and uniquely by the orthogonal relations of the group characters. For if the character vector of a given representation Γ is χ, and $\chi^{(j)}$

is the character vector of the irreducible representation Γ_j, then

$$\chi^{(j)} D \chi' = m \times n!$$

implies that Γ_j is contained m times in Γ.

e.g., the permutation matrices of order 3 x 3 have the character vector $[3, 1, 0]$,

$$D = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix}$$

and we find

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & \cdot & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ \cdot \end{bmatrix} = 3! \begin{bmatrix} 1 \\ 1 \\ \cdot \end{bmatrix} .$$

The interpretation is that the representation of the symmetric group on 3 letters by the permutation matrices of order 3 x 3, contains the unit representation once, the irreducible representation of order 2 x 2 once but does not contain the alternating scal ar representation.

The elements of the symmetric group on 3 letters satisfy the multiplication table :

	s_1	s_2	s_3	s_4	s_5	s_6
$s_1^{-1} =$	s_1	s_2	s_3	s_4	s_5	s_6
$s_2^{-1} =$	s_2	s_1	s_5	s_6	s_3	s_4
$s_3^{-1} =$	s_3	s_6	s_1	s_5	s_4	s_2
$s_4^{-1} =$	s_4	s_5	s_6	s_1	s_2	s_3
$s_5^{-1} =$	s_6	s_3	s_4	s_2	s_1	s_5
$s_6^{-1} =$	s_5	s_4	s_2	s_3	s_6	s_1

By the way of construction of the multiplication table no element can appear twice in any row or in any column, and all the diagonal elements are the unit element $s_\nu = I$. Introducing indeterminates $x_1, x_2, \ldots, x_6$ in place of the elements $s_1, s_2, \ldots, s_6$ we get the matrix

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_5 & x_6 & x_3 & x_4 \\ x_3 & x_6 & x_1 & x_5 & x_4 & x_2 \\ x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_6 & x_3 & x_4 & x_2 & x_1 & x_5 \\ x_5 & x_4 & x_2 & x_3 & x_6 & x_1 \end{bmatrix}$$

$$= x_1 M_1 + x_2 M_2 + \ldots + x_6 M_6 ,$$

where $M_1, M_2, \ldots, M_6$ are permutation matrices of order 6 x 6. These permutation matrices M provide a matrix representation of the symmetric group on 3 letters. This representation is called (by Frobenius) the regular matrix representation of the symmetric group on 3 letters and x is called the regular group matrix of that group. The regular matrix representation is reducible and in the general case is equivalent to the direct sum of the p irreducible matrix representations of the symmetric group, each being repeated $f^{(j)}$ times, the degree of the character of the representation.

Formulae For The Characters.

There are two theorems, each of which gives explicitly the character of any class of the symmetric group on n letters, corresponding to any partition of n. These theorems are due to Frobenius and Littlewood and Richardson respectively.

Frobenius gives the remarkable formula which exhibits the characters of the symmetric group as coefficients in a series of expansions one for each class of the group. Littlewood and Richardson give a graphical method for evaluating the character of any class corresponding to any partition.

Let $z_1, z_2, \ldots, z_n$ be n indeterminates, and let

$$\Delta(z) = \prod_{r<s} (z_r - z_s).$$

Let S_r be the sum of the r^{th} powers of $z_1, z_2, \ldots, z_n$ thus:

$$S_r = \sum_{i=1}^{n} z_i^{\,r},$$

and corresponding to the class $(\alpha) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the summetric group on n letters, let

$$S^{(\alpha)} = S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_n^{\alpha_n}.$$

Then if $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is a partition of n, Frobenius' theorem states that:

$$S^{(\alpha)} \Delta(z) = \sum \pm \chi^{(\lambda)}_{(\alpha)} z_1^{\lambda_1+n-1} z_2^{\lambda_2+n-2} \cdots z_n^{\lambda_n},$$

the summation being with respect to all permutations of the suffixes, and also with respect to all partitions (λ) of n; of the alternative signs, the positive sign is taken for a positive permutation and the negative sign for a negative permutation

A partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ is represented by a diagram of horizontal rows of nodes (all beginning on the same vertical line) the first row containing λ_1 nodes, the second λ_2 nodes and the last λ_κ nodes. Thus the partition $(3, 2^2, 1^3)$ of 10 is represented by the diagram

By simply interchanging the rows and columns a second diagram is obtained which is termed the associate of the original diagram and the partition which defines this new diagram is the partition (μ) which is termed the associate of the partition (λ). Thus the associate of $(3, 2^2, 1^3)$ is $(6, 3, 1)$. In general the associate of $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_\kappa)$ is

$$\left(\kappa^{\lambda_\kappa}, (\kappa-1)^{\lambda_{\kappa-1}-\lambda_\kappa}, (\kappa-2)^{\lambda_{\kappa-2}-\lambda_{\kappa-1}}, \ldots, 1^{\lambda_1-\lambda_2}\right).$$

when the diagram (or its defining partition) coincides with its associate it is termed self-associated. e.g., (3,2,1) is a self-associated partition of n = 6.

A diagram consisting of nodes arranged in rows and columns is said to be regular if in any row which contains say r nodes, these occur in the first r columns, and in any column which contains say s nodes, these occur in the first s rows. The diagram of a partition is simply a regular diagram, and conversely every regular diagram is the diagram of a partition.

If to a regular diagram one node is added so that the resulting diagram is also regular, this is called a regular application of a node. The addition of r nodes to a regular diagram is called a regular application of r nodes if the nodes are added to any row until they are exhausted, or until the number of nodes in this row exceeds the number in the preceding row by one, the nodes being then added to the preceding row according to the same rule, and so on, until their nodes are exhausted, provided that the final diagram obtained is regular. If the number of rows involved is even it is called a negative application, if odd, a positive application.

Littlewood and Richardson, making use of Frobenius' formula, have given the following theorem:
If $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ is a partition of n, and (a) denotes the class of the symmetric group, with cycles of orders $a_1, a_2, a_3, \cdots$, then $\chi^{(\lambda)}_{(a)}$ is obtained from the number of methods of building the diagram of the partition (λ) by consecutive regular applications of $a_1, a_2, a_3, \cdots$, nodes, by subtracting the number of methods which contain an odd number of negative applications from the number of methods which contain an even number of negative applications.

As an example of the practical use of the theorem, Littlewood and Richardson serve the character $\chi^{(16, 2^2)}$

for the class of the symmetric group on 20 letters, which contains one cycle of order 15, two cycles of order 2 and one cycle of order 1. The possible diagrams, using numbers in place of nodes to indicate at which step the nodes are added, are the following :

$$\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&4\\
2&2\\
3&3
\end{array}$$

$$\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&4\\
2&3\\
2&3
\end{array}$$

$$\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&2&2&4\\
1&3\\
1&3
\end{array}$$

$$\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&3&3&4\\
1&2\\
1&2
\end{array}$$

The first and second diagrams contain respectively o and 2 negative applications and each contributes $+1$ to the character. The last two diagrams each contain one negative appliaation, and these each contribute -1. Hence for this c lass

$$\chi^{(16,2^2)} = 1+1-1-1=0.$$

As an immediate corollary of the theorem the degree of the character of the representation corresponding to the partition (λ), is equal to the number of ways of building the diagram of the partition (λ) by n regular applications of nodes.

The theorem gives, also, many results concerning the characters of the symmetric group, e.g., in the case $(\lambda) = (n)$, there is one row and one method of building the diagram, so that $\chi_{(\alpha)}^{(n)} = 1$ for every class (α).

In the case $(\lambda) = (1^n)$ there is one column and one method of building the diagram, each cycle of even order contributing -1, so that

$$\chi_{(\alpha)}^{(1^n)} = 1 \quad \text{for a positive class}$$
$$= -1 \quad \text{for a negative class}$$

In the case of the associate partition (μ)

$$\chi_{(\alpha)}^{(\mu)} = \chi_{(\alpha)}^{(\lambda)} \chi_{(\alpha)}^{(1^n)}$$

so that the characters of the odd classes corresponding to a self-associated partition are all zero.

Characteristics and Schur Functions.

The expression

$$\phi^{(\lambda)}(S) = \frac{1}{n!} \sum_{(\alpha)} n_{(\alpha)} \chi_{(\alpha)}^{(\lambda)} S^{(\alpha)}$$
$$= \sum_{(\alpha)} \frac{1}{\alpha_1!\, \alpha_2! \dots \alpha_n!} \chi_{(\alpha)}^{(\lambda)} \left(\frac{S_1}{1}\right)^{\alpha_1} \left(\frac{S_2}{2}\right)^{\alpha_2} \cdots \left(\frac{S_n}{n}\right)^{\alpha_n}$$

is called the simple characteristic of the symmetric group of order n corresponding to the partition (λ) of n, and the characters $\chi_{(\alpha)}^{(\lambda)}$ are called the components of that characteristic.

The simple characteristic corresponding to the partition $(\lambda) = (n)$ is called the principal characteristic and is written

$$\vartheta_n(S) = \frac{1}{n!}\sum_{(\alpha)} n_{(\alpha)} S^{(\alpha)} .$$

It furnishes at a glance the structure of the corresponding symmetric group. The simple characteristic corresponding to the partition $(\lambda) = (1^n)$ is called the alternating characteristic and is written

$$\pi_n(S) = \frac{1}{n!}\sum_{(\alpha)} (-1)^{\alpha_2+\alpha_4+\cdots} n_{(\alpha)} S^{(\alpha)} .$$

Schur gives the simple characteristics in the form of determinants whose elements are special types of symmetric functions in the indeterminates $z_1, z_2, \cdots, z_n$ and they are called after him (by Littlewood and Richardson) Schur functions or S-functions.

Given the indeterminates $z_1, z_2, \cdots, z_n$ there are

(i) The elementary symmetric functions of degree r denoted by σ_r,

σ_r = sum of r-ary products of the z's without repetition of any z_i.

The expression

$$f(t) = \prod_{i=1}^{n} (1 - z_i t)$$

$$= 1 - \sigma_1 t + \sigma_2 t^2 - \cdots + (-1)^n \sigma_n t^n$$

is a generating function of the σ_r.

(ii) The complete homogeneous symmetric functions of degree r denoted by p_r,

p_r = sum of r-ary powers and products of the z's allowing all possible repetitions of the z_i.

The expression

$$g(t) = \prod_{i=1}^{n} (1 - z_i t)^{-1}$$
$$= \prod_{i=1}^{n} (1 + z_i t + z_i^2 t^2 + \cdots)$$
$$= 1 + p_1 t + p_2 t^2 + \cdots$$

is a generating function of the p_r.

The generating functions of σ_r and p_r are reciprocal series and we have the relations: $\sigma_0 p_0 = 1$,

$$\sigma_r - \sigma_{r-1} p_1 + \sigma_{r-2} p_2 - \cdots + (-1)^r p_r = 0 \qquad r = 1, 2, 3, \ldots$$

These relations are expressed by the statement that the two matrices

$$P_K = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_{K-1} \\ & p_0 & p_1 & \cdots & p_{K-2} \\ & & & & \\ & & & & p_0 \end{bmatrix}$$

$$\Sigma_K = \begin{bmatrix} \sigma_0 & -\sigma_1 & \sigma_2 & \cdots (-1)^{K-1} \sigma_{K-1} \\ & \sigma_0 & -\sigma_1 & \cdots (-1)^{K-2} \sigma_{K-2} \\ & & & \\ & & & \sigma_0 \end{bmatrix}$$

are reciprocal for k = 1,2,3 ..., (the elements below the main diagonal in each matrix being zero).

The substitution for $S^{(\alpha)}$ from Frobenius formula into the simple characteristic $\phi^{(\lambda)}(S)$ simply gives

$$\phi^{(\lambda)}(S) = \frac{\sum \pm \prod_i z_i^{\lambda_i + n - i}}{\sum \pm \prod_i z_i^{n-i}} ,$$

in each case the summation is taken with respect to all permutations of the suffixes, the negative sign being taken for a negative permutation. This is usually written in the form:

$$\phi^{(\lambda)}(S) = |z_i^{\lambda_j+n-j}| / |z_i^{n-j}|,$$

i indicating rows and j columns.

Such a quotient of two alternants have been studied by Jacobi, Trudi and Naegelsbach long before group characters were discovered by Frobenius, and Muir calls these functions bi-alternants. Jacobi, and independently Trudi, express the quotient as a determinant in which the elements are the symmetric functions p_r. Naegelsbach expresses the same quotient as a determinant in which the elements are the symmetric functions σ_r. Schur is the first to define these functions with any reference to group characters, and hence they are named after him.

Jacobi and Trudi express the above quotient as an nth order determinant whose diagonal elements are p_{λ_i}, the other elements in any row being obtained by methodically increasing (decreasing) the suffix carried by p as we move from any column to its neighbour on the right (left). If k is such that $\lambda_k > 0$ whilst $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$ the last n-k rows of that determinant have unity in the diagonal and zeros preceding the diagonal, so that $\phi^{(\lambda)}(S)$ is expressed as a determinant of order k of the type described. This determinant is called the Schur function or the S-function corresponding to the partition $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and is written.

(5) $\{\lambda\} = |p_{\lambda_i - i + j}|$,

each p with a negative suffix being zero.

As a special case the principal characteristic $q_m(S)$ is

(6) $\{n\} = p_n$.

Naegelsbach, using the reciprocity of the matrices P and Σ expresses the above quotient in the form

(7) $\{\lambda\} = |\sigma_{\mu_i - i + j}|$

where (μ) is the associate partition of (λ).

As a special case the alternating characteristic $\pi_n(S)$ is

(8) $\{1^n\} = \sigma_n$.

In virtue of (5) and (6) the simple characteristic $\phi^{(\lambda)}(S)$ is written (by Murnaghan) in the form

(9) $\phi^{(\lambda)}(S) = |q_{\lambda_i - i + j}(S)|$.

Setting $S_1 = 1$, $S_2 = S_3 = \cdots = S_n = 0$ in (2) and (9) we easily get

(10) $\chi^{(\lambda)}_{(1^n)} = \dfrac{n!\ \Delta(l)}{l_1!\ l_2! \cdots l_k!}$

where

$$l_1 = \lambda_1 + k - 1,\ l_2 = \lambda_2 + k - 2,\ \ldots,\ l_k = \lambda_k\ ;\ \Delta(l) = \prod_{p<q} (l_p - l_q)$$

which is known as Frobenius' formula for the degree of the character of the representation corresponding to the partition $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$.

In the above definition of the S-functions the parts of the partition (λ) are expressed in descending order, i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. When this inequality does not hold, $\{\lambda\}$ is defined not by the simple rearrangement of the parts in descending order, but by the equation,

$$\{\lambda\} = \{\lambda_1, \lambda_2, \cdots, \lambda_k\} = |p_{\lambda_i - i + j}|;$$

the parts λ_i must be integral, but need not be positive.

Every S-function so defined is either zero or equal to an S-function expressed with the parts in descending order, with a possible change of sign. To reduce such an S-function to this form, there are three rules.

(i) In any S-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part increased by unity, the S-function being thereby changed in sign, i.e.

$$\{\lambda_1, \cdots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \lambda_{i+2}, \cdots, \lambda_k\}$$
$$= -\{\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}-1, \lambda_i+1, \lambda_{i+2}, \cdots, \lambda_k\}$$

(ii) In any S-function if any part exceed by unity the preceding part the value of the S-function is zero, i.e.

$$\text{if} \quad \lambda_{i+1} = \lambda_i + 1, \ \{\lambda\} = 0.$$

(iii) The value of any S-function is zero if the last part is a negative number.

As simple illustrative examples

$$\{2,-1,5\} = -\{2,4,0\} = \{3^2\}$$

$$\{4,3,6\} = -\{4,5,4\} = 0$$

$$\{6,-2,1\} = -\{6,0,-1\} = 0$$

The S-functions $\{\lambda\}$ of the indeterminates $z_1, z_2, \dots z_n$ have been associated with the two series,

$$f(t) = \pi(1-z_i t) = 1 - \{1\}t + \{1^2\}t^2 - \dots + (-1)^n\{1^n\}t^n$$

$$g(t) = \pi(1-z_i t)^{-1} = 1 + \{1\}t + \{2\}t^2 + \dots .$$

The second series is chosen as the basic series and the concept of S-functions is generalised (by Littlewood and Richardson) to any basic series of the form

$$g(t) = 1 + \Sigma p_r t^r ,$$

irrespective of convergence or divergence. S_r is defined as the coefficient of t^{r-1} in the formal quotient,

(11) $$\frac{g'(t)}{g(t)} = \Sigma S_r t^{r-1} ,$$

as is the case with the basic series $g(t) = \pi(1-z_i t)^{-1}$, the dash denoting differentiation. With those values for the S_r's the S-functions are defined by the formula

(12) $$\{\lambda\} = \frac{1}{n!} \sum_{(\alpha)} n_{(\alpha)} \chi^{(\lambda)}_{(\alpha)} S^{(\alpha)} .$$

The S-functions of some series are of special interest. The most important series is the q-series.

$$\phi(q, zt) \,/\, \phi(q, t)$$

where

(13) $$\phi(q,t) = (1-t)(1-qt)(1-q^2 t) \cdots \text{to } \infty .$$

Littlewood and Richardson obtain the S-functions of the q-series as :

(14) $$\{\lambda_1, \lambda_2, \cdots, \lambda_k\} = \frac{\Pi(1-q^{\lambda_i-\lambda_j-i+j})}{\Pi\,[\lambda_i+k-i]!}\,(R_\lambda^z)$$

where

$$[r]! = (1-q)(1-q^2)\cdots(1-q^r),$$

(R_λ^z) is the product of the first λ_i terms from each i-th row of the set

$$\begin{array}{llll} 1-z, & 1-qz, & 1-q^2z, & \cdots \\ q-z, & q-qz, & q-q^2z, & \cdots \\ q^2-z, & q^2-qz, & q^2-q^2z, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

and $1 \leqslant i < j \leqslant k$.

Calculation of the Characters.

Frobenius' formula (p. 14) is theoretically sufficient for all cases, and indeed, for theory, it has a very remarkable simplicity. For numerical computation, however, the large number of terms involved make its use unwidely except when n is small; and much quicker methods are available. However the formula readily gives formulae for the characters corresponding to partitions into two parts. Thus $\chi_{(\alpha)}^{(n-p,\,p)}$ is the coefficient of $z_1^{\,n-p+1}\,z_2^{\,p}$ in the product

$$(z_1+z_2)^{\alpha_1}(z_1^2+z_2^2)^{\alpha_2}(z_1^3+z_2^3)^{\alpha_3}\cdots(z_1-z_2)$$

which is

$$\chi_{(\alpha)}^{(n-p,\,p)} = \sum\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\binom{\alpha_3}{\beta_3}\cdots - \sum\binom{\alpha_1}{\beta_1'}\binom{\alpha_2}{\beta_2'}\binom{\alpha_3}{\beta_3'}\cdots$$

The summation being with respect to all solutions of

$$\beta_1 + 2\beta_2 + 3\beta_3 + \cdots = p \;,\; \beta_1' + 2\beta_2' + 3\beta_3' + \cdots = p-1$$

This simply gives

$$\chi_{(\alpha)}^{(n-1,1)} = \alpha_1 - 1$$

$$\chi_{(\alpha)}^{(n-2,2)} = \tfrac{1}{2}\alpha_1(\alpha_1 - 3) + \alpha_2$$

$$\chi_{(\alpha)}^{(n-3,3)} = \tfrac{1}{6}\alpha_1(\alpha_1 - 1)(\alpha_1 - 5) + (\alpha_1 - 1)\alpha_2 + \alpha_3$$

$$\chi_{(\alpha)}^{(n-4,4)} = \tfrac{1}{24}\alpha_1(\alpha_1 - 1)(\alpha_1 - 2)(\alpha_1 - 7) + \tfrac{1}{2}\alpha_1(\alpha_1 - 3)\alpha_2$$

$$+ (\alpha_1 - 1)\alpha_3 + \tfrac{1}{2}\alpha_2(\alpha_2 - 1) + \alpha_4$$

. . . .

The S-functions of the series $\phi(q,zt)/\phi(q,t)$ (p. 24) yield many relations between the characters of the symmetric groups. The formula (12) (p. 24), together with the orthogonal properties of the group characters, gives

$$S^{(\alpha)} = \sum_{(\lambda)} \chi_{(\alpha)}^{(\lambda)} \{\lambda\}.$$

For the series $\phi(q,zt)/\phi(q,t)$

$$S_r = (1 - z^r)/(1 - q^r)$$

so that

$$\Pi \frac{(1 - z^r)^{\alpha_r}}{(1 - q^r)^{\alpha_r}} = \sum_{(\lambda)} \chi_{(\alpha)}^{(\lambda)} \frac{\Pi(1 - q^{\lambda_i - \lambda_j - i + j})}{\Pi[\lambda_i + n - i]!}(R_\lambda^2).$$

This equation is sufficient to determine all the characters of the symmetric groups. The value of any character may be expressed in terms of the cycles in the class, in the following way. Corresponding to any partition (λ), in the expression (R_λ^2) the least power of q that occurs has index $\lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots$, so that picking out the coefficient of q^r, only those characters occur for which the partition (λ) satisfies

$$\lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots \leqslant r.$$

The coefficients of $1, z, z^2, z^3, \ldots$ give

$$\begin{aligned}
\chi^{(n)}_{(\alpha)} &= 1\\
\chi^{(n)}_{(\alpha)} + \chi^{(n-1,1)}_{(\alpha)} &= \alpha_1\\
\chi^{(n-1,1)}_{(\alpha)} + \chi^{(n-2,1^2)}_{(\alpha)} &= \tfrac{1}{2}\alpha_1(\alpha_1-1) - \alpha_2\\
\chi^{(n-2,1^2)}_{(\alpha)} + \chi^{(n-3,1^3)}_{(\alpha)} &= \tfrac{1}{6}\alpha(\alpha_1-1)(\alpha_1-2) - \alpha_1\alpha_2 + \alpha_3\\
\chi^{(n-3,1^3)}_{(\alpha)} + \chi^{(n-4,1^4)}_{(\alpha)} &= \tfrac{1}{24}\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-3) - \tfrac{1}{2}\alpha_1(\alpha_1-1)\alpha_2\\
&\quad + \alpha_1\alpha_3 - \alpha_4 + \tfrac{1}{2}\alpha_2(\alpha_2-1)
\end{aligned}$$

. . . .

The solution of these equations gives the characters

$$\begin{aligned}
\chi^{(n-1,1)}_{(\alpha)} &= \alpha_1 - 1\\
\chi^{(n-2,1^2)}_{(\alpha)} &= \tfrac{1}{2}(\alpha_1-1)(\alpha_1-2) - \alpha_2\\
\chi^{(n-3,1^3)}_{(\alpha)} &= \tfrac{1}{6}(\alpha_1-1)(\alpha_1-2)(\alpha_1-3) - (\alpha_1-1)\alpha_2 + \alpha_3\\
\chi^{(n-4,1^4)}_{(\alpha)} &= \tfrac{1}{24}(\alpha_1-1)(\alpha_1-2)(\alpha_1-3)(\alpha_1-4) - \tfrac{1}{2}(\alpha_1-1)(\alpha_1-2)\alpha_2\\
&\quad + (\alpha_1-1)\alpha_3 - \alpha_4 + \tfrac{1}{2}\alpha_2(\alpha_2-1)
\end{aligned}$$

. . . .

In a similar manner the coefficients of qz, qz^2, qz^3, ... give the characters

$$\begin{aligned}
\chi^{(n-2,2)}_{(\alpha)} &= \tfrac{1}{2}\alpha_1(\alpha_1-3) + \alpha_2\\
\chi^{(n-3,2,1)}_{(\alpha)} &= \tfrac{1}{3}\alpha_1(\alpha_1-2)(\alpha_1-4) - \alpha_3\\
\chi^{(n-4,2,1^2)}_{(\alpha)} &= \tfrac{1}{8}\alpha_1(\alpha_1-2)(\alpha_1-3)(\alpha_1-5) - \tfrac{1}{2}\alpha_1\alpha_2(\alpha_1-3)\\
&\quad - \tfrac{1}{2}\alpha_2(\alpha_2-1) + \alpha_4
\end{aligned}$$

. . . .

and so on.

For the actual calculation of the tables of characters of the symmetric groups, by far the quickest method, is by the use of recurrence relations. Also it is only necessary to compute one half of a table of the characters, and from each character the value of the character corresponding to the associate partition may be written down by merely changing the sign for an odd class.

If (α) is a class of the symmetric group on n letters which contains a cycle on r letters, and $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_\kappa)$ is a partition of n, then the most important recurrence formula is stated thus :

$$\chi^{(\lambda)}_{(\alpha)} = \sum \pm \chi^{(\mu)}_{(\alpha')}$$

where (α') is the class of the symmetric group on n-r letters which has the same cycle structure as (α) but with one less cycle on r letters. The summation is over the characters $\chi^{(\mu)}$ which correspond to the S-functions

$$\{\lambda_1 - r, \lambda_2, \cdots, \lambda_\kappa\},\ \{\lambda_1, \lambda_2 - r, \cdots, \lambda_\kappa\},\ \cdots \{\lambda_1, \lambda_2, \cdots, \lambda_\kappa - r\},$$

the minus sign being taken when the S-function with the parts reduced to descending order becomes negative. This recurrence formula is sometimes written in the form, $\{\lambda_1, \lambda_2, \cdots, \lambda_\kappa\}_{(\alpha)} = \sum_{j=1}^{\kappa} \{\lambda_1, \lambda_2, \cdots, \lambda_j - r, \cdots, \lambda_\kappa\}_{(\alpha')}$. It was first given by Littlewood and Richardson in the special case r = 1, and then given by Murnaghan for any r. It enables us to write down at once those characters of the symmetric group on n letters which correspond to a class containing at least one cycle on r letters when the characters of that class of the symmetric group on n-r letters which contain one less cycle on r letters are known (r = 1,2,3, ..., n-1). Indeed it allows the whole table of characters of any symmetric group to be computed from tables of characters of lower degrees.

E.g., the character $\chi^{(4,3^2,1^2)}$ of the symmetric group of order 12! for those classes which contain a cycle of order 5 are obtained thus:

The corresponding S-functions are

$\{-1,3,3,1,1\}, \{4,-2,3,1,1\}, \{4,3,-2,1,1\},$
$\{4,3,3,-4,1\}, \{4,3,3,1,-4\},$

which reduce to

$\{2,2,1,1,1\}, \{4,2,0,0,-1\}, \{4,3,0,0,0\},$
$-\{4,3,3,0,-3\}, \{4,3,3,1,-4\}.$

The second, fourth and fifth are zero, so that

$$\chi_{(\alpha)}^{(4,3^2,1^2)} = \chi_{(\alpha')}^{(4,3)} + \chi_{(\alpha')}^{(2^2,1^3)}$$

For the classes

$1^7, 1^5 2, 1^4 3, 1^3 4, 1^3 2^2, 1^2 2 3, 1^2 5, 16, 124, 12^3,$
$13^2, 25, 2^2 3, 34, 7$

the characters $\chi^{(4,3)}$ and $\chi^{(2^2,1^3)}$ take the values

14, 4, -1, -2, 2, 1, -1, 0, 0, 0,
2, -1, -1, 1, 0,
14, -6, 2, 0, 2, 0, -1, 1, 0, -2,
-1, 1, 2, 0, 0,

so that $\chi^{(4,3^2,1^2)}$ for the given classes, takes the values

$1^7 5, 1^5 2 5, 1^4 3 5, 1^3 4 5, 1^3 2^2 5, 1^2 235, 1^2 5^2, 156, 1245,$
28, -2, 1, -2, 4, 1, -2, 1, 0

$1 2^3 5, \quad 1 3^2 5, \quad 2 5^2, \quad 2^2 3 5, \quad 3 4 5, \quad 5 7,$

$-2, \quad 1, \quad 0, \quad 1, \quad 1, \quad 0.$

The recurrence formula yields directly the characters of the classes containing but one cycle on n letters. If $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_K)$ then $\lambda_1 + K - 1 < n$ unless $\lambda_2 = \lambda_3 = \cdots = \lambda_K = 1$ in which case $\lambda_1 + K - 1 = n$. Also $\lambda_2 - n + K - 2 < \lambda_1 - n + K - 1$, and so on. So in both cases $\{\lambda_1, \lambda_2 - n, \lambda_3, \cdots \lambda_K\} = \{\lambda_1, \lambda_2, \lambda_3 - n, \cdots, \lambda_K\} = \cdots$
$= \{\lambda_1, \lambda_2, \cdots, \lambda_K - n\} = 0$, since the last term, when it is rearranged in non-increasing order, is negative. In the first case $\{\lambda_1 - n, \lambda_2, \lambda_3, \cdots, \lambda_K\}$ is also zero whilst in the second case $\{\lambda_1 - n, \lambda_2, \lambda_3, \ldots, \lambda_K\} = (-1)^{K-1}\{0^K\} = (-1)^{K-1}$. Thus

$$\chi^{(n-K+1,\, 1^{K-1})}_{\alpha_n = 1} = (-1)^{K-1}$$

otherwise $\chi^{(\lambda)}_{\alpha_n = 1} = 0$

In a similar manner the recurrence formula yields directly the characters of the class containing one unary cycle and one cycle on n - 1 letters, thus

$$\chi^{(n-K,\, 2,\, 1^{K-2})}_{\alpha_1 = 1,\ \alpha_{n-1} = 1} = (-1)^{K-1}$$

$$\chi^{(n-2,\, 2)}_{\alpha_1 = 1,\ \alpha_{n-1} = 1} = -1$$

$$\chi^{(n)}_{\alpha_1 = 1,\ \alpha_{n-1} = 1} = 1$$

$$\begin{aligned} \chi^{(1^n)}_{\alpha_1 = 1,\ \alpha_{n-1} = 1} &= 1 \qquad n \text{ even} \\ &= -1 \qquad n \text{ odd} \end{aligned}$$

$$\chi^{(\lambda)}_{\alpha_1 = 1,\ \alpha_{n-1} = 1} = 0 \qquad \text{otherwise.}$$

The recurrence formula may be combined with Frobenius' formula (p.22) giving the dimension of the representation or equivalently the character attached to the unit class, to determine directly characters of classes containing one or more unary cycles. e.g., the character of the class $\alpha_1 = 12$, $\alpha_8 = 1$ of the symmetric group on 20 letters, corresponding to the partition (9, 6, 3, 2) is easily calculated as follows.

The recurrence formula applied with r = 8 gives

$$\{9,6,3,2\}_{(\alpha)} = \{1,6,3,2\}_{(\alpha')} + \{9,-2,3,2\}_{(\alpha')}$$
$$+ \{9,6,-5,2\}_{(\alpha')} + \{9,6,3,-6\}_{(\alpha')} .$$

The first, third and fourth of the terms on the right vanish and there remains $\{9,-2,3,2\} = \{9,2,1\}$. Since (α') is the unit class the dimension formula of Frobenius yields, since

$$(l_1, l_2, l_3) = (11,3,1), \frac{12!}{11!\,3!\,1!}\, 8 . 10 . 2 . = 320 .$$

The characters of a class containing only cycles of the same length are easily calculated by formula (9) (p.22). For example if n = 2 m the characters of the class containing m binary cycles are simply found by setting $S_2 = 1$, $S_1 = S_3 = \dots = S_n = 0$. Evidently $q_j(S) = 0$ if j is odd whilst $q_{2p}(S) = \frac{1}{2^p\, p!}$ Thus for n = 12, the character of the class $\alpha_2 = 6$ corresponding to the partition (5, 4, 2, 1) is

$$2^6\, 6! \begin{vmatrix} 0 & (2^3 3!)^{-1} & 0 & (2^4 4!)^{-1} \\ 0 & (2^2 2!)^{-1} & 0 & (2^3 3!)^{-1} \\ 1 & 0 & 2^{-1} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -5$$

This also may be combined with the recurrence formula. e.g. the character of the class $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$ of the symmetric group on 12 letters, corresponding to the partition $(7, 1^5)$ is easily calculated in the following way.

The recurrence formula applied twice, first with r = 1 and then with r = 2 yields

$$\{7, 1^5\}_{(\alpha)} = \{4, 1^5\}_{(\alpha'')} - \{6, 1^3\}_{(\alpha'')} + \{5, 1^4\}_{(\alpha'')} - \{7, 1^2\}_{(\alpha'')},$$

where (α'') is the class of the symmetric group on 9 letters consisting of permutations each of which has three ternary cycles. Since this class is positive $\{4, 1^5\}_{(\alpha'')} = \{6, 1^3\}_{(\alpha'')}$. Also

$$\{5, 1^4\}_{(\alpha'')} = 3^3 \cdot 3! \begin{vmatrix} 0 & (3^2 \cdot 2!)^{-1} & 0 & 0 & (3^3 \cdot 3!)^{-1} \\ 1 & 0 & 0 & 3^{-1} & 0 \\ 0 & 1 & 0 & 0 & 3^{-1} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -2$$

$$\{7, 1^2\}_{(\alpha'')} = 3^3 \cdot 3! \begin{vmatrix} 0 & 0 & (3^3 \cdot 3!)^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

so that the desired character is - 3.

Explicit formulae for the characters of those classes of the symmetric group on n letters which consist of $\alpha_1 =$ n - p unary cycles and $\alpha_p = 1$ cycle on p letters; (p = 2, 3, 4, ...) are of importance in the physical applications. These formulae are given by Frobenius for p = 2, 3, 4, and by Murnaghan, using his recurrence formula, for any p.

A partition (λ) of n is conveniently specified as follows:

Let the principal diagonal of the diagram of the partition (i.e., the diagonal starting at the upper left-hand corner) strike s columns. Let $a_1 > a_2 > \dots > a_s \geqslant 0$ be the number of nodes to the right of the diagonal in the rows, 1, 2, ..., s respectively, and $b_1 > b_2 > \dots > b_s \geqslant 0$ be the number of nodes below the diagonal in the columns, 1, 2, ..., s respectively. Then the partition is described by

$$\begin{pmatrix} a_1, a_2, \cdots, a_s \\ b_1, b_2, \cdots, b_s \end{pmatrix}$$

and s is called its rank.

The associate of the partition (λ) is evidently described by

$$\begin{pmatrix} b_1, b_2, \cdots, b_s \\ a_1, a_2, \cdots, a_s \end{pmatrix};$$

the partition is self-associated if and only if

$$(b_1, b_2, \cdots, b_s) = (a_1, a_2, \cdots, a_s)$$

The number of nodes in the first row and column together $= a_1 + b_1 + 1$; when these are deleted the number of nodes in the new first row and column $= a_2 + b_2 + 1$; and so on, so that

$$\sum_{j=1}^{s} (a_j + b_j + 1) = n.$$

As an example the diagram of the partition $(6, 4, 3, 1^2)$ of 15 is

the nodes in the leading diagonal being in heavier type. In this notation of Frobenius the partition is denoted by

$$\begin{pmatrix} 5, & 2, & 0 \\ 4, & 1, & 0 \end{pmatrix}.$$

This is of rank 3 and differs from

$$\begin{pmatrix} 5, & 2 \\ 4, & 1 \end{pmatrix}$$

which is the partition $(6, 4, 2, 1^2)$ of 14 and whose rank is 2.

Murnaghan denotes by $\chi^{(\lambda)}(p)$ the characters of the class $\alpha_1 = n-p$, $\alpha_p = 1$, so that for instance $\chi^{(\lambda)}(2)$ are the characters of the transposition class, whilst $\chi^{(\lambda)}(1)$ are the dimensions of the various irreducible representations. He then gives the theorem:

$\chi^{(\lambda)}(p) \div \chi^{(\lambda)}(1)$ is the coefficient of y^{-2} in

$$-\frac{1}{p}\,\frac{y(y-1)(y-2)\cdots(y-p+1)}{n(n-1)(n-2)\cdots(n-p+1)}\left\{1-\frac{p}{1!}\,\frac{F'(y)}{F(y)}+\frac{p^2}{2!}\,\frac{F''(y)}{F(y)}-\frac{p^3}{3!}\,\frac{F'''(y)}{F(y)}+\cdots\right\},$$

$$F(y) = \prod_{j=1}^{s}\left\{\frac{y-a_j}{y+b_j+1}\right\}.$$

In the cases p = 2, 3, 4 the theorem yields Frobenius' formulae

$$\chi^{(\lambda)}(2) \div \chi^{(\lambda)}(1) = \left[\sum_{j=1}^{s}\left\{a_j(a_j+1) - b_j(b_j+1)\right\}\right] \div n(n-1).$$

$$\chi^{(\lambda)}(3) \div \chi^{(\lambda)}(1) = \Big[\sum_{j=1}^{s}\left\{a_j(a_j+1)(2a_j+1) + b_j(b_j+1)(2b_j+1)\right\}$$

$$- 3n(n-1)\Big] \div 2n(n-1)(n-2).$$

$$\chi^{(\lambda)}(4) \div \chi^{(\lambda)}(1) = \Big[\sum_{j=1}^{s}\left\{a_j^2(a_j+1)^2 - b_j^2(b_j+1)^2\right\}$$

$$- 2(2n-3)\left\{a_j(a_j+1) - b_j(b_j+1)\right\}\Big] \div$$

$$n(n-1)(n-2)(n-3).$$

The two notations of a partition are connected by the relations

(i) $a_j = \lambda_j - j \quad , j = 1, 2, \ldots, s$

(ii) $j - \lambda_j - 1, \quad j = s+1, s+2, \ldots, k \;\&\; b_j, j = 1, 2, \ldots, s$

are two complementary sets of the set (o, 1, 2, ..., k - 1). These relations transform Frobenius formula for $\chi^{(\lambda)}(2) \div \chi^{(\lambda)}(1)$ into the equivalent form,

$$\chi^{(\lambda)}(2) \div \chi^{(\lambda)}(1) = \sum_{j=1}^{k} \lambda_j(\lambda_j - 2j + 1) \div n(n-1),$$

which is known as Hund's formula.

Frobenius' formulae (p.26), for the characters of the symmetric group on n letters, corresponding to a partition of n into two parts are a type of formulae giving the characters of a given representation the class varying. The formulae derived by Littlewood and Richardson (p.27), from the q-series are a similar type. Murnaghan following a very simple method based on the formula (9) (p.22), for the characteristics of the symmetric groups, establishes formulae giving the characters of representations corresponding to partitions into not more than four parts, these being of extreme importance in the applications to nuclear physics.

These formulae are :

$$\chi_{(\alpha)}^{(\lambda_1,\lambda_2)} = \sum_{(\gamma)} \frac{\alpha_1 - 2\gamma_1 - 1}{\gamma_1 + 1} \binom{\alpha_1}{\gamma_1}\binom{\alpha_2}{\gamma_2}\cdots\binom{\alpha_n}{\gamma_n}$$

$$+ \sum_{(\beta)} \binom{\alpha_2}{\beta_2}\binom{\alpha_3}{\beta_3}\cdots\binom{\alpha_n}{\beta_n},$$

the summation in the first term being over all classes (γ) of the symmetric group on $\lambda_2 - 1$ letters, and in the second over all those classes (β) of the symmetric group on λ_2 letters which do not contain any unary cycle.

This formula is indeed an alternative form of Frobenius' formula (p. 25) for $\chi_{(\alpha)}^{(n-p,\,p)}$.

$$\chi_{(\alpha)}^{(\lambda_1,\lambda_2,\lambda_3)} = \sum_{(\beta)} \frac{\alpha_1 - 2\beta_1 - 1}{\beta_1 + 1} \chi_{(\beta')}^{(\lambda_2,\lambda_3)} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}$$

$$+ \sum_{(\delta)} \chi_{(\delta)}^{(\lambda_2,\lambda_3)} \binom{\alpha_2}{\delta_2}\binom{\alpha_3}{\delta_3}\cdots\binom{\alpha_n}{\delta_n}$$

$$+ \sum_{(\gamma)} \chi_{(\gamma)}^{(\lambda_2-1,\lambda_3+1)} \binom{\alpha_1}{\gamma_1}\binom{\alpha_2}{\gamma_2}\cdots\binom{\alpha_n}{\gamma_n}$$

The summation in the first term is over all classes (β) of the symmetric group on $\lambda_2 + \lambda_3 - 1$ letters, (β') being the class $(\beta_1+1, \beta_2, \cdots, \beta_n)$, in the second term over all classes (δ) of the symmetric group on $\lambda_2+\lambda_3$ letters which do not contain unary cycles, and in the third term over all classes (γ) of the symmetric group on $\lambda_2+\lambda_3-2$ letters.

$$\begin{aligned}\chi_{(\alpha)}^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} &= \sum_{(\beta)} \frac{\alpha_1-2\beta_1-1}{\beta_1+1}\chi_{(\beta')}^{(\lambda_2,\lambda_3,\lambda_4)} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n} \\ &+ \sum_{(\epsilon)} \chi_{(\epsilon)}^{(\lambda_2,\lambda_3,\lambda_4)} \binom{\alpha_2}{\epsilon_2}\binom{\alpha_3}{\epsilon_3}\cdots\binom{\alpha_n}{\epsilon_n} \\ &+ \tfrac{1}{2}\sum_{(\gamma)} \left\{\chi_{(\gamma')}^{(\lambda_2,\lambda_3,\lambda_4)} - \chi_{(\gamma'')}^{(\lambda_2,\lambda_3,\lambda_4)}\right\}\binom{\alpha_1}{\gamma_1}\binom{\alpha_2}{\gamma_2}\cdots\binom{\alpha_n}{\gamma_n} \\ &- \sum_{(\delta)} \chi_{(\delta)}^{(\lambda_2-1,\lambda_3-1,\lambda_4-1)} \binom{\alpha_1}{\delta_1}\binom{\alpha_2}{\delta_2}\cdots\binom{\alpha_n}{\delta_n}.\end{aligned}$$

The first summation is over all classes (β) of the symmetric group on $\lambda_2+\lambda_3+\lambda_4-1$ letters and $(\beta') = (\beta_1+1, \beta_2, \cdots, \beta_n)$. The second summation is over all classes (ϵ) of the symmetric group on $\lambda_2+\lambda_3+\lambda_4$ which do not contain unary cycles. The third summation is over all classes (γ) of the symmetric group on $\lambda_2+\lambda_3+\lambda_4-2$ letters and $(\gamma') = (\gamma_1+2, \gamma_2, \gamma_3, \cdots, \gamma_n)$, $(\gamma'') = (\gamma_1, \gamma_2+1, \gamma_3, \cdots, \gamma_n)$. The fourth summation is over all classes (δ) of the symmetric group on $\lambda_2+\lambda_3+\lambda_4-3$ letters.

As an illustrative example of the application of these formulae, if $\lambda_1 = n-5$, $\lambda_2 = 5$ is a partition of n, then there are five classes (γ) namely (1^4), $(1^2 2)$,

(2^2), $(1,3)$, (4) and two classes (β) namely $(2,3)$, (5), so that

$$\chi_{(\alpha)}^{(n-5,5)} = \frac{\alpha_1-9}{5}\binom{\alpha_1}{4}\binom{\alpha_2}{0}\cdots\binom{\alpha_n}{0} + \frac{\alpha_1-5}{3}\binom{\alpha_1}{2}\binom{\alpha_2}{1}\binom{\alpha_3}{0}\cdots\binom{\alpha_n}{0}$$
$$+ \frac{\alpha_1-1}{1}\binom{\alpha_1}{0}\binom{\alpha_2}{2}\binom{\alpha_3}{0}\cdots\binom{\alpha_n}{0} + \frac{\alpha_1-3}{2}\binom{\alpha_1}{1}\binom{\alpha_2}{0}\binom{\alpha_3}{1}\cdots\binom{\alpha_n}{0}$$
$$+ \frac{\alpha_1-1}{1}\binom{\alpha_1}{0}\binom{\alpha_2}{0}\binom{\alpha_3}{0}\binom{\alpha_4}{1}\cdots\binom{\alpha_n}{0}$$
$$+ \binom{\alpha_2}{1}\binom{\alpha_3}{1}\binom{\alpha_4}{0}\cdots\binom{\alpha_n}{0} + \binom{\alpha_2}{0}\binom{\alpha_3}{0}\binom{\alpha_4}{0}\binom{\alpha_5}{1}\cdots\binom{\alpha_n}{0}$$
$$= \frac{1}{120}\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-3)(\alpha_1-9)$$
$$+ \frac{1}{6}\alpha_1(\alpha_1-1)(\alpha_1-5)\alpha_2 + \frac{1}{2}(\alpha_1-1)\alpha_2(\alpha_2-1)$$
$$+ \frac{1}{2}\alpha_1(\alpha_1-3)\alpha_3 + (\alpha_1-1)\alpha_4 + \alpha_2\alpha_3 + \alpha_5 .$$

Applying the first formula Murnaghan establishes explicit formulae for the characters $\chi_{(\alpha)}^{(n-\lambda,\lambda)}$, $\lambda = 1, 2, \ldots, 8$; applying the second he establishes explicit formulae for the characters $\chi_{(\alpha)}^{(n-\lambda,\lambda_1,\lambda_2)}$, $\lambda_1+\lambda_2 = \lambda = 2, 3, \ldots, 8$; and applying the third he establishes explicit formulae for the characters $\chi_{(\alpha)}^{(n-\lambda,\lambda_1,\lambda_2,\lambda_3)}$, $\lambda_1+\lambda_2+\lambda_3 = \lambda = 3, 4, \cdots, 8$. These explicit formulae have the advantage that they are the same for all values of n, being expressed in terms of the cycles of the class (α)

The character tables for the various symmetric groups from n = 2 to n = 10, inclusive are given by

Littlewood. The character tables for n = 11, 12, 13 are given by M. Zia - Ud - Din.

The Analysis of the Direct Product Of Irreducible Representations of The Symmetric Groups.

If G_1 is the symmetric group with elements s_1, t_1, ... and G_2 the symmetric group with elements s_2, t_2, ... then the pairs (s_1, s_2) one element s_1 from G_1 and the other s_2 from G_2 constitute a group G (as s_1 runs over G_1 and s_2 over G_2) under the law of combination

$$(t_1, t_2)(s_1, s_2) = (t_1 s_1, t_2 s_2).$$

G is called the direct product of the two symmetric groups G_1 and G_2, its order g is the product of g_1 and g_2 the orders of G_1 and G_2, and the number of classes it contains is the product of the numbers of classes in G_1 and G_2.

If M_1 is an n_1 - dimensional representation of G_1, and M_2 an n_2 - dimensional representation of G_2, the direct product $M_1 \times M_2$ is an $n_1 n_2$ - dimensional representation of G. If ${}_1\chi(s_1)$ are the characters of M_1 and ${}_2\chi(s_2)$ are the characters of M_2, the characters of the representation $M_1 \times M_2$ of G are ${}_1\chi(s_1) \cdot {}_2\chi(s_2)$.

If ${}_1\chi_j(s_1)$, $j = 1, 2, \dots, p_1$

and ${}_2\chi_k(s_2)$, $k = 1, 2, \dots, p_2$

are the characters of the irreducible representations

Γ_j, $j = 1, 2, \dots p_1$, Γ_k, $k = 1, 2, \dots, p_2$

of G_1 and G_2 respectively, and $\chi_{j,k}(s_1, s_2)$ are the characters of the representation $\Gamma_j \times \Gamma_\kappa$ of G, then simply

$$\frac{1}{g} \sum_{s_1, s_2} \chi_{j_1, \kappa_1}(s_1, s_2) \cdot \chi_{j_2, \kappa_2}(s_1, s_2) = \left\{ \frac{1}{g_1} \sum_{s_1} {}_1\chi_{j_1}(s_1) \cdot {}_1\chi_{j_2}(s_1) \right\} \times$$

$$\left\{ \frac{1}{g_2} \sum_{s_2} {}_2\chi_{\kappa_1}(s_2) \, {}_2\chi_{\kappa_2}(s_2) \right\}$$

$$= 1 \ , \ j_2 = j_1 \ \& \ \kappa_2 = \kappa_1$$

$$= 0 \quad \text{otherwise},$$

so that there are exactly $p_1 p_2$ irreducible representations of G, namely the direct products of the irreducible representations of G_1 and G_2 taken in pairs. Thus the product of two simple characteristics of G_1 and G_2 is a simple characteristic of G.

If G_1 is the symmetric group on n' letters, G_2 the symmetric group on n'' letters and

$$(\lambda') = (\lambda_1', \lambda_2', \cdots, \lambda_{n'}') \ , \ \lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_{n'}'$$

is a partition of n',

$$(\lambda'') = (\lambda_1'', \lambda_2'', \ldots, \lambda_{n''}'') \ , \ \lambda_1'' \geq \lambda_2'' \geq \cdots \geq \lambda_{n''}''$$

is a partition of n'', then the characteristic which is the product of the two simple characteristics $\{\lambda'\}$ and $\{\lambda''\}$ is

$$\frac{1}{n'! \, n''!} \sum_{(\alpha')(\alpha'')} n_{(\alpha')}' \, n_{(\alpha'')}'' \, \chi_{(\alpha')}^{(\lambda')} \, \chi_{(\alpha'')}^{(\lambda'')} \, S'^{(\alpha')} \, S''^{(\alpha'')} \ ,$$

the summation being over all classes

$$(\alpha') = (\alpha_1', \alpha_2', \cdots, \alpha_{n'}') \ , (\alpha'') = (\alpha_1'', \alpha_2'', \cdots, \alpha_{n''}'')$$

of the symmetric groups G_1 and G_2 respectively, and

$$S'^{(\alpha')} = s_1'^{\alpha_1'} \, s_2'^{\alpha_2'} \cdots s_{n'}'^{\alpha_{n'}'}$$

$$S''^{(\alpha'')} = s_1''^{\alpha_1''} \, s_2''^{\alpha_2''} \cdots s_{n''}''^{\alpha_{n''}''} .$$

The group G is a subgroup of the symmetric group on $n = n' + n''$ letters, namely the subgroup which consists of the permutations on n letters which permute the letters of each of the two sets, one consisting of n' and the other of n'' letters, amongst themselves. Also $(\alpha) = (\alpha') + (\alpha'')$ is a class of the symmetric group on n letters; the equation $(\alpha) = (\alpha') + (\alpha'')$ meaning $\alpha_j = \alpha'_j + \alpha''_j$, $j = 1, 2, \dots, n$, where $\alpha'_j = 0$ if $j > n'$ and $\alpha''_j = 0$ if $j > n''$. The identification of S' and S'' gives the characteristic as

$$\frac{1}{n'!\,n''!} \sum_{(\alpha) = (\alpha') + (\alpha'')} n'_{(\alpha')}\, n''_{(\alpha'')}\, \chi^{(\lambda')}_{(\alpha')}\, \chi^{(\lambda'')}_{(\alpha'')}\, S^{(\alpha)}$$

where

$$\begin{aligned} S^{(\alpha)} &= s_1^{\alpha_1}\, s_2^{\alpha_2} \dots s_n^{\alpha_n}, \\ s_\kappa &= z_1^{\kappa} + z_2^{\kappa} + \dots + z_n^{\kappa} \qquad \kappa = 1, 2, \dots, n \end{aligned}$$

it being understood that the indeterminates S in the simple characteristics $\{\lambda'\}$ and $\{\lambda''\}$ are power sums of n variables z. This is simply a compound characteristic of the symmetric group on n letters.

The problem of analysing the representation of the symmetric group on n letters which is obtained in this way, into its irreducible components, is not merely a very interesting mathematical problem but also is of extreme importance in the applications to nuclear physics. It was treated extensively by two different methods.

One method is a recurrence method due to Murnaghan.

Murnaghan first gives the general result that those and only those $\{\lambda\}$ will occur in the product $\{\lambda'\}\{\lambda''\}$, for which the partition (λ) is obtained from (λ'), supposed written with n elements by the addition of n'' zeros, by adding (μ'') in all possible arrangements to each set of n'' of the n elements of (λ'), (μ'') running over those partitions of n'' which appear when $\{\lambda''\}$ is written as a linear combination of the various symmetric functions

$$T^{(\mu'')}(z) = \sum z_1^{\mu_1''} z_2^{\mu_2''} \cdots z_{n''}^{\mu_{n''}''},$$

of degree n'' in the n variables (z). In adding (μ'') to (λ') in this way some disordered partitions of n may be obtained and these are arranged in the usual way.

In the case $n'' = 1, \qquad (\lambda'') = (1)$

and $\qquad \phi^{(1)}(S) = T^{(1)}(z),$

so that $\qquad (\mu'') = (1)$.

(λ') written with n elements is $(\lambda_1', \lambda_2', \ldots, \lambda_{n'}'; 0)$

and then

$$\{\lambda'\}\{1\} = \{\lambda_1'+1, \lambda_2', \cdots, \lambda_{n'}'\} + \{\lambda_1', \lambda_2'+1, \cdots, \lambda_{n'}'\} + \cdots$$
$$+ \{\lambda_1', \lambda_2', \cdots, \lambda_{n'}'+1\} + \{\lambda_1', \lambda_2', \cdots, \lambda_{n'}', 1\}.$$

If (λ') is of k parts only, $k < n'$, every term on the right in which 1 is preceded by 0, is dropped and we get the formula:

$$\{\lambda_1, \lambda_2, \cdots, \lambda_k\}\{1\} = \{\lambda_1+1, \lambda_2, \cdots, \lambda_k\} + \{\lambda_1, \lambda_2+1, \cdots, \lambda_k\}$$
$$+ \cdots + \{\lambda_1, \lambda_2, \cdots, \lambda_k+1\} + \{\lambda_1, \lambda_2, \cdots, \lambda_k, 1\}.$$

In the case $n'' = 2$, there are similar formulae thus:

(i) $\phi^{(1^2)}(S) = T^{(1^2)}(z)$

and $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}\{1^2\}$ is obtained by writing $\{\lambda_1, \lambda_2, \ldots, \lambda_K\}$ in the form $\{\lambda_1, \lambda_2, \cdots \lambda_K, 0, 0\}$ and adding the pair (1, 1) in all possible ways.

(ii) $\phi^{(2)}(S) = T^{(2)}(z) + T^{(1^2)}(z)$

and $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}\{2\}$ is obtained by writing $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}$ in the form $\{\lambda_1, \lambda_2, \cdots, \lambda_K, 0\}$ and adding 2 and (1, 1) in all possible ways.

From this general result Murnaghan deduces his recurrence method involving two essential steps for the solution:

(i) The expression of each simple characteristic $\phi^{(\lambda'')}(S)$ of the symmetric group on n'' letters as a linear combination of the various symmetric functions,

$$T^{(\mu'')}(z) = \Sigma z_1^{\mu_1''} z_2^{\mu_2''} \ldots z_{n''}^{\mu_{n''}''}$$

of degree n'' in the variables (z);

(ii) The expression of the symmetric function of degree θ in m variables z,

$$T^{(\theta)}(z) = \Sigma z_1^{\theta_1} z_2^{\theta_2} \ldots z_m^{\theta_m}, \theta_1 + \theta_2 + \cdots + \theta_m = m$$

in terms of the simple characteristics $\phi^{(\nu)}(S)$ of the symmetric group on m letters.

These two steps have been done in tables due to Kostka up to n'', m = 11. Murnaghan's recurrence method for analysing $\{\lambda'\}\{\lambda''\}$ is then formalized as follows:

A. Precede by λ_1' each partition occurring in the, supposed known, analysis of the product

$$\{\lambda_2', \lambda_3', \dots, \lambda_{n'}'\}\{\lambda''\}.$$

B. Precede by $\lambda_1'+1$ each partition of the, supposed known, analysis of the product of $\{\lambda_2', \lambda_3', \dots, \lambda_{n'}'\}$ by a linear combination of simple characteristics of the symmetric group on $n'' - 1$ letters. This linear combination is obtained by first expressing $\{\lambda''\}$ as a linear combination of the symmetric functions $T^{(\mu'')}(z)$, of degree n'', of n letters (z) and then expressing the coefficient of z_1, which is a symmetric function, of degree $n'' - 1$, of $n - 1$ letters, in terms of the simple characteristics of the symmetric group on $n'' - 1$ letters.

C. Precede by $\lambda_1'+2$ each partition of the, supposed known, analysis of the product $\{\lambda_2', \lambda_3', \dots, \lambda_{n'}'\}$ by a linear combination of simple characteristics of the symmetric group on $n'' - 2$ letters. This linear combination is obtained from the coefficient of z_1^2 in the expression for $\{\lambda''\}$ in terms of the $T^{(\mu'')}(z)$.

D. Same as in B save that $\lambda_1'+1$ is replaced by $\lambda_1'+3$ and $n''-1$ by $n''-3$

E. Same as in B save that $\lambda_1'+1$ is replaced by $\lambda_1'+4$ and $n''-1$ by $n''-4$; and so on.

As an illustrative example, the analysis of $\{3,2\} \times \{2^2\}$ is obtained as follows :

A. $\{2^2\}\{2\} = \{4,2\} + \{2,4\} + \{2,2,2\} + \{3,3\}$

$+ \{3,2,1\} + \{2,3,1\}$,

Preceding with 3 and arranging parts we get

$$\{3,2^3\} + \{3^2,2,1\}$$

B. $\{2^2\} = T^{(2^2)}(z) + T^{(2,1^2)}(z) + 2T^{(1^4)}(z)$

The coefficient of z_1, is $T^{(2,1)}(z) + 2T^{(1^3)}(z) = \{2,1\}$.

$$\{2,1\}\{2\} = \{4,1\} + \{2,3\} + \{2,1,2\} + \{3,2\} + \{3,1,1\} + \{2,2,1\}$$

Preceding by 4 we get

$$\{4^2,1\} + \{4,3,2\} + \{4,3,1^2\} + \{4,2^2,1\}.$$

C. The coefficient of z_1^2 is $T^{(2)}(z) + T^{(1^2)}(z) = \{2\}$

$$\{2\}\{2\} = \{4\} + \{2,2\} + \{3,1\}.$$

Preceding by 5 we get

$$\{5,4\} + \{5,2^2\} + \{5,3,1\}.$$

There are no further terms and therefore

$$\{3,2\}\{2^2\} = \{3^2,2,1\} + \{3,2^3\} + \{4^2,1\} + \{4,3,2\} + \{4,3,1^2\} + \{4,2^2,1\} + \{5,4\} + \{5,3,1\} + \{5,2^2\}.$$

Murnaghan has constructed tables for the analysis of $\{\lambda'\}\{\lambda''\}$ for which $n' + n'' \leqslant 10$. He cuts the tables short by using the fact that in the analysis of $\{\mu'\} \times \{\mu''\}$, where $\{\mu'\}$ is the associate of $\{\lambda'\}$ and $\{\mu''\}$ the associate of $\{\lambda''\}$, the $\{\mu\}$ which will occur are the associates of $\{\lambda\}$ appearing in $\{\lambda'\}\{\lambda''\}$.

The second method gives the analysis of $\{\lambda'\}\{\lambda''\}$ without referring to any tables. It was mainly suggested by Littlewood and Richardson but completed by Robinson.

If $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$

is a partition of n and in a permutation of the n factors of the product

$$z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_k^{\lambda_k}$$

amongst the first r terms the number of times z_1 occurs $\geqslant$ the number of times z_2 occurs $\geqslant$ the number of times z_3 occurs, etc. for all values of r, then this permutation is called a lattice permutation.

Thus the lattice permutations of $z_1^3 z_2 z_3$ are

$$z_1^3 z_2 z_3 \qquad z_1^2 z_2 z_1 z_3 \qquad z_1^2 z_2 z_3 z_1$$
$$z_1 z_2 z_1^2 z_3 \qquad z_1 z_2 z_1 z_3 z_1 \qquad z_1 z_2 z_3 z_1^2 .$$

Simply the number of lattice permutations of $z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_k^{\lambda_k}$ is equal to the number of ways of building the diagram of the partition (λ) by n regular applications of nodes, i.e., the dimension of the corresponding irreducible representation.

If in the diagram of a partition (λ) of n, the n nodes are replaced by symbols $\alpha, \beta, \gamma, \ldots$ taken in any order the result is called a Young tableau.

Littlewood and Richardson first give the special case that the S-functions obtained in the product $\{\lambda\} \times p_r$ are those which correspond to Young tableaux that can be built by the addition of r identical symbols to a tableau corresponding to the S-function $\{\lambda\}$, no two identical symbols appearing in the same column.

They have also stated the general case but did not give its proof. Later Robinson has given a satisfactory proof. This general theorem is stated as follows.

The S-functions appearing in the product of $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \times \{\mu_1, \mu_2, \ldots, \mu_q\}$ are those which correspond to Young tableaux that can be built by adding to a Young tableaux corresponding to $\{\lambda\}$, μ_1 identical symbols α, μ_2 identical symbols β, μ_3 identical symbols γ, etc., subject to two conditions:

Firstly, after the addition of each set of identical symbols we must have a regular Young tableau with no two identical symbols in the same column.

Secondly, if the total set of added symbols is read from right to left in the consecutive rows of the final tableau, we obtain a lattice permutation of

$$\alpha^{\mu_1} \beta^{\mu_2} \gamma^{\mu_3} \ldots$$

The following illustrative example was given by Littlewood and Richardson.

To form the product $\{4,3,1\}\{2^2,1\}$, it is easier to build on $\{4,3,1\}$ and since the tableau remains unaltered its elements are represented by dots. Thus

$$[4,3,1]: \quad \begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & & & \end{array} \qquad\qquad [2^2,1]: \quad \begin{array}{ll} \alpha & \alpha \\ \beta & \beta \\ \gamma & \end{array}$$

and we get:

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta & \beta & \\ \cdot & \gamma & & & & \end{array} \qquad \begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta & \beta & \\ \cdot & & & & & \\ \gamma & & & & & \end{array} \qquad \begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta & & \\ \cdot & \beta & \gamma & & & \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \beta \\ \gamma \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \gamma \\ \beta \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot \\ \beta \\ \gamma \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot \\ \cdot & \beta & \beta \\ \gamma \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot \\ \cdot & \beta \\ \beta \\ \gamma \end{array}$$

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \alpha & \alpha \\ \cdot & \cdot & \cdot \\ \cdot & \beta \\ \beta & \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha & \beta \\ \cdot & \beta & \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha & \beta \\ \cdot & \beta \\ \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha & \beta \\ \cdot & \gamma \\ \beta \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha & \beta \\ \cdot \\ \beta \\ \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha \\ \cdot & \beta & \beta \\ \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha \\ \cdot & \beta \\ \beta & \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \alpha \\ \cdot & \beta \\ \beta \\ \gamma \end{array}$$

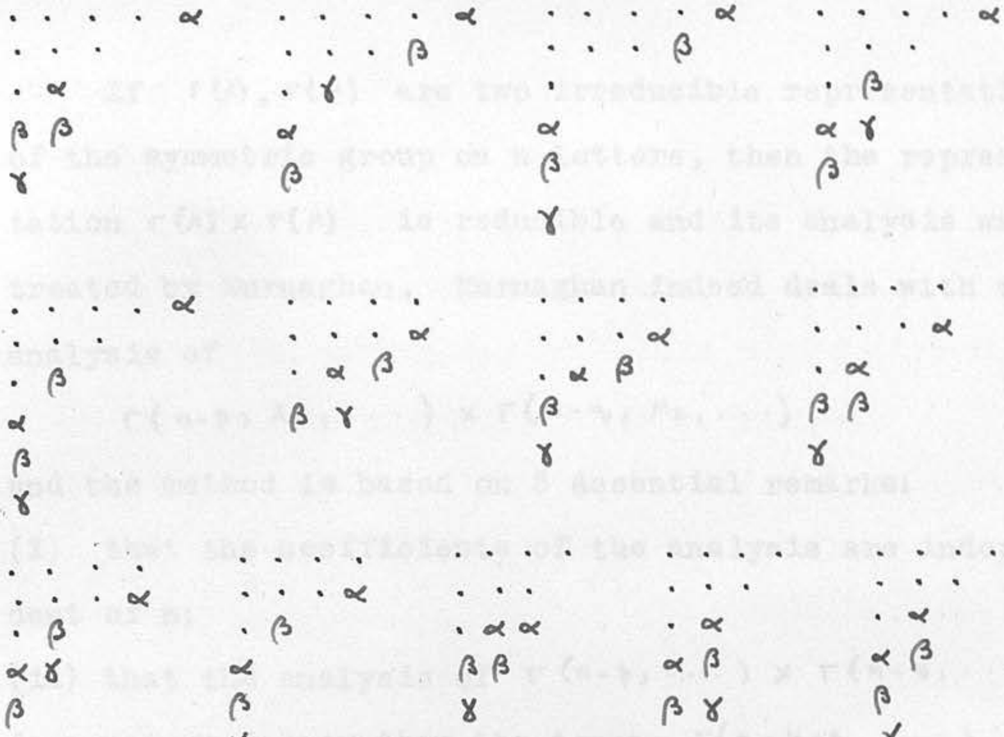
$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \alpha \\ \beta & \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \alpha & \gamma \\ \beta \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta \\ \cdot & \alpha \\ \beta \\ \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot \\ \cdot & \alpha & \beta \\ \beta & \gamma \end{array}$$

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot \\ \cdot & \alpha & \beta \\ \beta \\ \gamma \end{array}$$



$$\{4,3,1\}\{2^2,1\} = \{6,5,2\} + \{6,5,1^2\} + \{6,4,3\} + 2\{6,4,2,1\}$$
$$+ \{6,4,1^3\} + \{6,3,2^2\} + \{6,3,2,1^2\} + \{6,3^2,1\} + \{5^2,3\}$$
$$+ 2\{5^2,2,1\} + \{5^2,1^3\} + 2\{5,4,3,1\} + 2\{5,4,2^2\}$$
$$+ 3\{5,4,2,1^2\} + \{5,4,1^4\} + \{5,3^2,2\} + \{5,3^2,1^2\}$$
$$+ 2\{5,3,2^2,1\} + \{5,3,2,1^3\} + \{4,3^2,2,1\} + \{4^2,3,2\} + \{4^2,3,1^2\}$$
$$+ 2\{4^2,2^2,1\} + \{4,3,2^3\} + \{4,3,2^2,1^2\} + \{4^2,2,1^3\}.$$

The Analysis of The Kronecker Product Of The Irreducible Representations of The Symmetric Groups.

If A and B are two matrix representations of the symmetric group on n letters, then A X B is also a matrix representation of that symmetric group. For if

$$A_i A_j = A_k \quad , \quad B_i B_j = B_k \quad , \quad M_i = A_i \times B_i$$

then

$$M_i M_j = A_i \times B_i \cdot A_j \times B_j = A_i A_j \times B_i B_j = A_k \times B_k = M_k$$

If $\Gamma(\lambda), \Gamma(\mu)$ are two irreducible representations of the symmetric group on n letters, then the representation $\Gamma(\lambda) \times \Gamma(\mu)$ is reducible and its analysis was treated by Murnaghan. Murnaghan indeed deals with the analysis of

$$\Gamma(n-p, \lambda_2, \cdots) \times \Gamma(n-q, \mu_2, \cdots)$$

and the method is based on 3 essential remarks:

(i) that the coefficients of the analysis are independent of n;

(ii) that the analysis of $\Gamma(n-p, \cdots) \times \Gamma(n-q, \cdots)$ does not go deeper than the terms $\Gamma(n-p-q, \cdots)$;

(iii) that the coefficient of $\Gamma(\nu)$ in the analysis of $\Gamma(\lambda) \times \Gamma(\mu)$ is the same as the coefficient of $\Gamma(\mu)$ in the analysis of $\Gamma(\lambda) \times \Gamma(\nu)$.

The method is then completely described by the two illustrative examples :

(i) Since $\Gamma(\lambda) \times \Gamma(n) = \Gamma(\lambda)$, $\Gamma(n)$ does not appear in $\Gamma(\lambda) \times \Gamma(\mu)$ unless $(\mu) = (\lambda)$, and in this case it occurs exactly once.

Therefore writing (λ) for $\Gamma(\lambda)$

$$(n-1, 1) \times (n-1, 1) = (n) + c_1 (n-1, 1) + c_2 (n-2, 2) + q_2 (n-2, 1^2).$$

setting n = o,

$$(-1; 1) \times (-1, 1) = 0 + c_1 (-1, 1) + c_2 (-2, 2) + q_2 (-2, 1^2)$$

$$-1 \times -1 = 1 + c_1 \times -1 + c_2 \times 0 + q^2 \times 1$$

i.e., $c_1 = q_2$

setting n = 1, $c_2 = 1$

setting n = 2, $c_1 = c_2$

Hence $c_1 = c_2 = c_{1^2} = 1$ so that

$(n-1, 1) \times (n-1, 1) = (n) + (n-1, 1) + (n-2, 2) + (n-2, 1^2)$.

(ii) To obtain the analysis of $(n-1, 1) \times (n-2, 2)$ we remark that the coefficient of (n) is zero whilst that of $(n - 1, 1)$ is 1 (by example (i)), so that

$$(n-1,1) \times (n-2,2) = (n-1,1) + c_2 (n-2,2) + c_{1^2} (n-2,1^2) + c_3 (n-3,3) + c_{21} (n-3,2,1) + c_{1^3} (n-3,1^3).$$

Setting n = o, $c_{1^2} - c_{1^3} = 1$

Setting n = 1, $c_2 = c_{21}$

Setting n = 2, $c_2 = 1,\ c_3 = 1$

Setting n = 3, $c_3 = 1,\ c_{21} = c_{1^2}$

Hence $c_2 = 1,\ c_{1^2} = 1,\ c_3 = 1,\ c_{21} = 1,\ c_{1^3} = 0$

so that

$$(n - 1, 1) \times (n - 2, 2) = (n - 1, 1) + (n - 2, 2) + (n - 2, 1^2) + (n - 3, 3) + (n-3, 2,1).$$

In this way Murnaghan gives the analysis of $(n-p, \lambda_2, \ldots) \times (n-q, \mu_2, \ldots)$ for the following values of p and q,

p = 1, q = 1, 2, 3, 4, 5 there being 18 of these,

p = 2, q = 2, 3, 4 there being 19 of these,

p = 3, q = 3, 4 there being 21 of these.

From these he constructs tables giving the analysis of $r(\lambda) \times r(\mu)$ where (λ) and (μ) are partitions of n, n = 3, 4, 5, 6, 7, 8.

The tables are cut short by using the two facts

(i) If $(\tilde{\lambda})$ and $(\tilde{\mu})$ are the associate partitions of (λ) and (μ) respectively, then

$\Gamma(\tilde{\lambda}) \times \Gamma(\mu) = \Gamma(\lambda) \times \Gamma(\tilde{\mu})$ = the associates of the representations in the analysis of $\Gamma(\lambda) \times \Gamma(\mu)$.

(ii) $\Gamma(\tilde{\lambda}) \times \Gamma(\tilde{\mu}) = \Gamma(\lambda) \times \Gamma(\mu)$.

PART II.

THE IRREDUCIBLE MATRIX REPRESENTATIONS OF THE SYMMETRIC GROUPS.

It was indicated in part I p.8 that there are exactly p irreducible matrix representations of the symmetric group on n letters, where p is the number of partitions of n. The traces of the matrices in an irreducible matrix representation $\Gamma(\lambda)$, namely the characters of the symmetric group on n letters corresponding to the partition (λ) of n, have been invesitgated extensively, and tables with these traces have been constructed for n = 2, 3, ..., 13. Yet the matrices themselves have not been given much care, except in simple cases, namely n= 2, 3, and the particular partition (2^2) of n = 4. That is due to the fact that no method appeared easy or practical to obtain these matrices. But very recently Professor Aitken, while investigating the characters of the representations by the compound permutation matrices, has given a simple and direct method by which we can obtain the matrices in an irreducible matrix representation $\Gamma(\lambda)$, where (λ) is a partition of a certain type. This method, when combined with the other methods makes it practical to obtain all the irreducible matrix representations of the symmetric groups on 4 and 5 letters. It is our purpose here to obtain these matrix representations after describing the best method to be applied to each particular representation. In order that we shall have a complete list with all representations of the symmetric groups on 3, 4 and 5 letters (the case n = 2 is trivial) we shall give those of the symmetric group on 3 letters as illustrative examples of the methods described, and shall describe the method, due to Frobenius, applied to the representation $\Gamma(2^2)$ of the symmetric group on 4 letters.

First of all, the permutation matrices of order n x n provide a reducible matrix representation A of the symmetric group on n letters. By mere inspection of any permutation matrix A_i we note that cycles on one letter in the permutation s_i correspond to unit elements in the diagonal, while cycles on more than one letter correspond to chains of unit elements all off the diagonal. Thus the sum of the diagonal elements of A_i, or the trace, is equal to the number of cycles on one letter in the permutation s_i, namely α_1, in the notation of p. 3. Also we have seen p. 27 that :

$$\chi^{(n)}_{(\alpha)} = 1$$

$$\chi^{(n-1,1)}_{(\alpha)} = \alpha_1 - 1 \quad \text{for all } n.$$

We then conclude that the representation of the symmetric group on n letters, by the permutation matrices of order $n \times n$, is equivalent to the direct sum of the two irreducible representations $\Gamma(n)$ of order 1×1, and $\Gamma(n-1,1)$ of order $n-1 \times n-1$.

The operation $\text{row}_1 + \text{row}_2 + \text{row}_3 + \ldots + \text{row}_n$, applied to a permutation matrix A_i gives a matrix B_i differing from A_i, merely by, that all the elements in the first row of B_i are unity. The operations $\text{col}_n - \text{col}_1$, $\text{col}_{n-1} - \text{col}_1$, …, $\text{col}_2 - \text{col}_1$, applied to B_i give a matrix C_i in which the elements in the first row are $1, 0, 0, \ldots, 0$. Thus the operation $HA_i H^{-1}$ where,

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \cdot & 1 & \cdot & \cdots & \cdot \\ \cdot & \cdot & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \end{bmatrix} \text{(n rows)}$$

and

$$H^{-1} = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ \cdot & 1 & \cdot & \cdots & \cdot \\ \cdot & \cdot & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \text{(n rows)}$$

reduces each permutation matrix of order $n \times n$ to the unit matrix 1 and a matrix of order $n-1 \times n-1$. Thus the operation $H\,A\,H^{-1}$ reduces the representation A to its two irreducible components $\Gamma(n)$ and $\Gamma(n-1,1)$. This is certainly the easiest and most direct method to obtain the matrices in the irreducible representation $\Gamma(n-1,1)$ of the symmetric group on n letters.

The simplest example to illustrate the method is the representation $\Gamma(2,1)$ of the symmetric group on 3 letters. The elements of the group are

I , (12) , (13), (23), (123), (132)

or

123, 213, 321, 132, 312, 231 ,

where, in the notation of p. 1,

$$s_1\, s_2\, s_3 = \begin{pmatrix} 1, & 2, & 3 \\ s_1, & s_2, & s_3 \end{pmatrix}$$

The permutation matrices of order 3 x 3 corresponding to the group elements are :

$$\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}$$

The operation $H\,(\;)\,H^{-1}$, where

$$H = \begin{bmatrix} 1 & 1 & 1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} , \quad H^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}$$

applied to these permutation matrices, gives

$$\begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ 1 & -1 & -1 \\ . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ . & 1 & . \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ . & . & 1 \\ . & 1 & . \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . \\ . & . & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ 1 & -1 & -1 \\ . & 1 & . \end{bmatrix}$$

From these we have :

<u>the representation corresponding to the partition (3)</u>

$$1, 1, 1, 1, 1, 1,$$

<u>and the representation corresponding to the partition(2,1)</u>

$$\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ . & 1 \end{bmatrix} \begin{bmatrix} 1 & . \\ -1 & -1 \end{bmatrix} \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} . & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & . \end{bmatrix} .$$

We shall apply this method to obtain the matrix representation $\Gamma(3,1)$ of the symmetric group on 4 letters and the matric representation $\Gamma(4,1)$ of the symmetric group on 5 letters.

We note that the representation $\Gamma(2,1^{n-2})$ is obtained from the representation $\Gamma(n-1,1)$ by multiplying each matrix in $\Gamma(n-1,1)$, corresponding to a negative permutation of the symmetric group, by -1. We can apply that to obtain the representations $\Gamma(2,1^2)$ and $\Gamma(2,1^3)$ of the symmetric groups on 4 and 5 letters respectively.

Secondly, Frobenius has given the following method based on Young tableaux p. 46. A tableau in which the letters in each row and in each column are in ascending order, is called a standard tableau. The number of standard tableaux corresponding to a partition (λ) is simply the number of ways of building the graph of the partition (λ) by n regular applications of nodes (p.16) and hence equal to $f^{(\lambda)}$. These tableaux are ordered in the following manner. A and B representing two standard tableaux and the letters in the ith row and jth column being a_{ij} and b_{ij} respectively, we have the two sequences :

$$a_{11}, a_{12}, \dots, a_{1\lambda_1}, a_{21}, \dots, a_{2\lambda_2}, a_{31}, \dots, a_{p\lambda_p}$$
$$b_{11}, b_{12}, \dots, b_{1\lambda_1}, b_{21}, \dots, b_{2\lambda_2}, b_{31}, \dots, b_{p\lambda_p}.$$

If a_{ij}, the first letter in the first sequence such that $a_{ij} \neq b_{ij}$, is smaller than b_{ij}, the tableau A is said to precede B.

The term symmetric group on r letters is used to denote the sum of the group elements of the symmetric group which permutes these letters, and the term negative symmetric group is used to denote the same thing with a minus sign attached to each negative permutation.

Frobenius' theorem is then stated thus :
If for every pair of standard tableaux, it happens that there is at least one pair of letters in the same column in A_i and in the same row in A_j, we write, P_i = the product of symmetric groups on the elements in

rows of A_i, N_i = the product of -ve s.g's on elements in columns,

$$e_{ii} = P_i N_i$$

and $$e_{ij} = P_i \sigma_{ji} N_j$$

where σ_{ji} is the permutation which transforms the tableau A_j into the tableau A_i, then the matrix of order $f^{(\lambda)} \times f^{(\lambda)}$

$$\begin{bmatrix} e_{11} & e_{21} & e_{31} & \cdots & e_{f^{(\lambda)}1} \\ e_{12} & e_{22} & e_{32} & \cdots & e_{f^{(\lambda)}2} \\ e_{13} & e_{23} & e_{33} & \cdots & e_{f^{(\lambda)}3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{1f^{(\lambda)}} & e_{2f^{(\lambda)}} & e_{3f^{(\lambda)}} & \cdots & e_{f^{(\lambda)}f^{(\lambda)}} \end{bmatrix}$$

is a group matrix for the irreducible matrix representation $\Gamma(\lambda)$. Indeed this is of the form

$$\sum_i M_i x_i$$

where $x_i = s_i^{-1}$ and M_i is the matrix in $\Gamma(\lambda)$ representing the element s_i.

Frobenius has applied the method to the representation $\Gamma(2^2)$ of the symmetric group on 4 letters, and indeed this is the simplest method in that particular case.

If it does not happen that for every pair of standard tableau there are two elements in the same row in A_i and in the same column in A_j, the representation $\Gamma(\lambda)$ can be obtained by a similar method, but the elements e_{ij} in this case, are too complicated so that the method is not practical for use, and we shall not give it here.

Thirdly comes a method based on invariant matrices. If A is a given matrix, T (A) is an invariant matrix of A if

$$T(A) \cdot T(B) = T(A.B).$$

T (A) is said to be reducible if there exists a matrix H of the same order as T (A) such that

$$H\,T(A)\,H^{-1} = \begin{bmatrix} T_1(A) & U(A) \\ 0 & T_2(A) \end{bmatrix}$$

clearly T_1 (A) and T_2 (A) are also invariant matrices of A and T (A) is said to be equivalent to their direct sum.

Schur has shown that there are exactly p irreducible invariant matrices T (A), whose elements are functions (homogeneous) of degree n in the elements of A, p being as usual the number of partitions of n.

The invariant matrix corresponding to the partition (λ) of n is written $A^{[\lambda]}$ and its trace is the S-function $\{\lambda\}$ (p. 22) in the latent roots of A.

To any relation between the S-functions of the different degrees corresponds a relation between the invariant matrices of A, whose traces are the S-functions of the respective degrees in the latent roots of A, a sum or a product in the first relation being replaced by a direct sum or a direct product in the second, and an equality in the first being replaced by an equivalence.

Thus to the relation

$$\{2\}\{1\} = \{2,1\} + \{3\}$$

corresponds the relation

$$A^{[2]} \times A = A^{[2,1]} \dotplus A^{[3]}.$$

To the relation

$$\{1^2\}\{1\} = \{2,1\} + \{1^3\}$$

corresponds the relation

$$A^{[1^2]} \times A = A^{[2,1]} \dotplus A^{[1^3]}$$

or

$$A^{(2)} \times A = A^{[2,1]} \dotplus A^{(3)}.$$

$A^{[r]}$ being the rth induced power of A, and has trace $\{r\} = p_r$ and $A^{[1^r]} = A^{(r)}$, the rth compound of A and has trace $\{1^r\} = \sigma_r$.

Now if A is taken to be the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & \cdots & h_1 \\ a_2 & b_2 & c_2 & \cdots & h_2 \\ a_3 & b_3 & c_3 & \cdots & h_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & b_n & c_n & \cdots & h_n \end{bmatrix}$$

and we construct the invariant matrix $A^{[\lambda]}$ corresponding to the partition (λ) of n, there is a submatrix of $A^{[\lambda]}$ each element of which contains all the symbols a, b, c, ..., h and all the suffixes 1, 2, 3, ..., n permuted in some way. This part is called the most polarized part of $A^{[\lambda]}$ or its central core.

The main thing is that this central core is a group matrix for the irreducible matrix representation $\Gamma(\lambda)$ of the symmetric group on n letters. Indeed the central core is equal to

$$\sum_i M_i x_i$$

where

$$x_i = s_i \, (a_1 b_2 c_3 \cdots h_n)$$

and M_i is the matrix in $\Gamma(\lambda)$, representing the group element s_i.

The most simple example to illustrate the method is the representation $\Gamma(1^3)$ of the symmetric group on 3 letters. In this case A is the matrix,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and $A^{[1^3]} = A^{(3)}$, the determinant of the matrix. The central core is $A^{[1^3]}$ itself, namely

$a_1 b_2 c_3 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_3 b_1 c_2 + a_2 b_3 c_1$, and we get :

<u>the alternating scalar representation $\Gamma(1^3) \equiv$:</u>

1, -1, -1, -1, 1, 1.

We shall apply this method to obtain the representation $\Gamma(2^2,1)$ of the symmetric group on 5 letters. We shall obtain the central core of $A^{[2^2,1]}$ by reducing the central core of $A^{(3)} \times A^{(2)}$ to two components, the central core of $A^{[2^2,1]}$ and the central core of $A \times A^{(4)}$.

The representation $\Gamma(3,2)$ is obtained from the representation $\Gamma(2^2,1)$ by multiplying each matrix in

$\Gamma(2^2, 1)$, corresponding to a negative permutation by -1.

Lastly, Professor Aitken, while studying the characters of the compounds of permutation matrices of order n x n, has got the result that the traces of the 2nd, 3rd, ..., n-1 th compounds of the irreducible matrix representation $\Gamma(n-1, 1)$ are the characters

$$\chi^{(n-2, 1^2)}, \chi^{(n-3, 1^3)}, \ldots, \chi^{(2, 1^{n-2})}, \chi^{(1^n)}.$$

Thus the 2nd, 3rd, ..., (n - 1) th compounds of the irreducible representation $\Gamma(n-1, 1)$ are the irreducible representations

$$\Gamma(n-2, 1^2), \Gamma(n-3, 1^3), \ldots, \Gamma(2, 1^{n-2}), \Gamma(1^n).$$

This is undoubtedly the simplest method to obtain the irreducible matrix representations corresponding to the so-called, unicursal partitions.

The method is very simply illustrated by noting that the 2nd compound of the irreducible representation $\Gamma(2, 1)$ of the symmetric group on 3 letters (p. 57) is the alternating scalar representation

$$\Gamma(1^3) \equiv 1, -1, -1, -1, 1, 1.$$

We shall apply this method to obtain the matrix representation $\Gamma(3, 1^2)$ of the symmetric group on 5 letters.

Group of order 4!

The elements of the group are :

Class (1^4) : 1

1

1 2 3 4

Class $(1^2, 2)$: 2 3 4 5 6 7

(12) (13) (14) (23) (24) (34)

2134 3214 4231 1324 1432 1243

Class (2^2) : 8 9 10

(12)(34) (13)(24) (14)(23)

2143 3412 4321

Class (1,3): 11 12 13 14 15

(123) (124) (132) (134) (142)

3124 4132 2314 4213 2431

16 17 18

(143) (234) (243)

3241 1423 1342

Class (4) : 19 20 21 22

(1234) (1243) (1324) (1342)

4123 3142 4312 2413

23	24
(1423)	(1432)
3421	2341

Representation corresponding to partition (4) :

The unit representation 1, 1, 1, ..., 1.

Representation corresponding to partition (3, 1) :

$$\overset{1}{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}}$$

$$\overset{2}{\begin{bmatrix} -1 & -1 & -1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}} \overset{3}{\begin{bmatrix} 1 & \cdot & \cdot \\ -1 & -1 & -1 \\ \cdot & \cdot & 1 \end{bmatrix}} \overset{4}{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ -1 & -1 & -1 \end{bmatrix}} \overset{5}{\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}}$$

$$\overset{6}{\begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}} \overset{7}{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}}$$

$$\overset{8}{\begin{bmatrix} -1 & -1 & -1 \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}} \overset{9}{\begin{bmatrix} \cdot & \cdot & 1 \\ -1 & -1 & -1 \\ 1 & \cdot & \cdot \end{bmatrix}} \overset{10}{\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{11}{\begin{bmatrix} \cdot & 1 & \cdot \\ -1 & -1 & -1 \\ \cdot & \cdot & 1 \end{bmatrix}} \overset{12}{\begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ -1 & -1 & -1 \end{bmatrix}} \overset{13}{\begin{bmatrix} -1 & -1 & -1 \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}} \overset{14}{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{15}{\begin{bmatrix} -1 & -1 & -1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}} \overset{16}{\begin{bmatrix} 1 & \cdot & \cdot \\ -1 & -1 & -1 \\ \cdot & 1 & \cdot \end{bmatrix}} \overset{17}{\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix}} \overset{18}{\begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}}$$

$$\overset{19}{\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ -1 & -1 & -1 \end{bmatrix}} \overset{20}{\begin{bmatrix} \cdot & \cdot & 1 \\ -1 & -1 & -1 \\ \cdot & 1 & \cdot \end{bmatrix}} \overset{21}{\begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ -1 & -1 & -1 \end{bmatrix}} \overset{22}{\begin{bmatrix} -1 & -1 & -1 \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix}}$$

$$\overset{23}{\begin{bmatrix} \cdot & 1 & \cdot \\ -1 & -1 & -1 \\ 1 & \cdot & \cdot \end{bmatrix}} \overset{24}{\begin{bmatrix} -1 & -1 & -1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}} .$$

Representation corresponding to partition (2^2) :

The standard Young Tableaux are

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

so that

$$P_1 = I + (12) + (34) + (12)\ (34)$$

$$N_1 = I - (13) - (24) + (13)\ (24)$$

$$P_2 = I + (13) + (24) + (13)\ (24)$$

$$N_2 = I - (12) - (34) + (12)(34)$$

$$\sigma_{12} = \sigma_{21} = (23) .$$

Therefore

$$e_{11} = I + (12) - (13) - (24) + (34) + (12)(34) + (13)(24) + (14)(23) - (123) - (134) - (142) - (243) - (1234) + (1324) + (1423) - (1432).$$

$$e_{12} = -(13) + (14) + (23) - (24) - (123) + (124) + (132) - (134) - (142) + (143) + (234) - (243) - (1234) + (1243) + (1342) - (1432) .$$

$$e_{21} = -(12) + (14) + (23) - (34) + (123) - (124) - (132) + (134) + (142) - (143) - (234) + (243) + (1243) - (1324) + (1342) - (1423) .$$

$$e_{22} = I - (12) + (13) + (24) - (34) + (12)(34) + (13)(24) + (14)(23) - (124) - (132) - (143) - (234) + (1234) - (1324) - (1423) + (1432) .$$

and we get :

$$\overset{1}{\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}}$$

$$\overset{2}{\begin{bmatrix} 1 & -1 \\ \cdot & -1 \end{bmatrix}} \overset{3}{\begin{bmatrix} -1 & \cdot \\ -1 & 1 \end{bmatrix}} \overset{4}{\begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix}} \overset{5}{\begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix}} \overset{6}{\begin{bmatrix} -1 & \cdot \\ -1 & 1 \end{bmatrix}} \overset{7}{\begin{bmatrix} 1 & -1 \\ \cdot & -1 \end{bmatrix}}$$

$$\overset{8}{\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}} \overset{9}{\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}} \overset{10}{\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}}$$

$$\begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ \begin{bmatrix} \cdot & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & \cdot \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} \cdot & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{cc} 17 & 18 \\ \begin{bmatrix} -1 & 1 \\ -1 & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & -1 \\ 1 & -1 \end{bmatrix} \end{array}$$

$$\begin{array}{cccccc} 19 & 20 & 21 & 22 & 23 & 24 \\ \begin{bmatrix} -1 & \cdot \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ \cdot & -1 \end{bmatrix} & \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ \cdot & -1 \end{bmatrix} & \begin{bmatrix} -1 & \cdot \\ -1 & 1 \end{bmatrix} \end{array} .$$

This representation is multiply isomorphic with the group.

Representation corresponding to partition $(2, 1^2)$:

In the representation corresponding to the partition (3, 1), the matrices representing the elements in the classes $(1^2, 2)$ and (4), are multiplied by the scalar - 1.

Representation corresponding to partition (1^4) :

The alternating scalar representation.

Group of order 5!

The elements of the group are :

class (1^5) :	1			
	I			
	12345			
class $(1^3,2)$:	2	3	4	5
	(12)	(13)	(14)	(15)
	21345	32145	42315	52341
	6	7	8	9
	(23)	(24)	(25)	(34)
	13245	14325	15342	12435
	10	11		
	(35)	(45)		
	12543	12354		
class $(1,2^2)$:	12	13	14	15
	(12)(34)	(12)(35)	(12)(45)	(13)(24)
	21435	21543	21354	34125
	16	17	18	19
	(13)(25)	(13)(45)	(14)(23)	(14)(25)
	35142	32154	43215	45312
	20	21	22	23
	(14)(35)	(15)(23)	(15)(24)	(15)(34)
	42513	53241	54321	52431

24 25 26

(23)(45) (24)(35) (25)(34)

13254 14523 15432

class $(1^2,3)$: 27 28 29 30

(123) (124) (125) (132)

31245 41325 51342 23145

31 32 33 34

(134) (135) (142) (143)

42135 52143 24315 32415

35 36 37 38

(145) (152) (153) (154)

52314 25341 32541 42351

39 40 41 42

(234) (235) (243) (245)

14235 15243 13425 15324

43 44 45 46

(253) (254) (345) (354)

13542 14352 12534 12453

class (23) : 47 48 49 50

(123)(45)/(124)(35)/(125)(34)/(132)(45)

31254 41523 51432 23154

51 52 53 54

(134)(25)/(135)(24)/(142)(35)/(143)(25)

45132 54123 24513 35412

	55	56	57	58
	(145)(23)	(152)(34)	(153)(24)	(154)(23)
	53214	25431	34521	43251
	59	60	61	62
	(234)(15)	(235)(14)	(243)(15)	(245)(13)
	54231	45213	53421	35124
	63	64	65	66
	(253)(14)	(254)(13)	(345)(12)	(354)(12)
	43512	34152	21534	21453
class (14) :	67	68	69	70
	(1234)	(1235)	(1243)	(1245)
	41235	51243	31425	51324
	71	72	73	74
	(1253)	(1254)	(1324)	(1325)
	31542	41352	43125	53142
	75	76	77	78
	(1342)	(1345)	(1352)	(1354)
	24135	52134	25143	42153
	79	80	81	82
	(1423)	(1425)	(1432)	(1435)
	34215	54312	23415	52413
	83	84	85	86
	(1452)	(1453)	(1523)	(1524)
	25314	32514	35241	45321

	87	88	89	90
	(1532)	(1534)	(1542)	(1543)
	23541	42531	24351	32451
	91	92	93	94
	(2345)	(2354)	(2435)	(2453)
	15234	14253	15423	13524
	95	96		
	(2534)	(2543)		
	14532	13452		
class (5) :	97	98	99	100
	(12345)	(12354)	(12435)	(12453)
	51234	41253	51423	31524
	101	102	103	104
	(12534)	(12543)	(13245)	(13254)
	41532	31452	53124	43152
	105	106	107	108
	(13425)	(13452)	(13524)	(13542)
	54132	25134	45123	24153
	109	110	111	112
	(14235)	(14253)	(14325)	(14352)
	54213	34512	53412	25413
	113	114	115	116
	(14523)	(14532)	(15234)	(15243)
	35214	23514	45231	35421

117	118	119	120
(15324)	(15342)	(15423)	(15432)
43521	24531	34251	23451.

The operation $H\;(\;)\;H^{-1}$, where

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

and

$$H^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

applied to the permutation matrices of order 5 x 5 gives the representation corresponding to the partition (5) $\dotplus$ the representation corresponding to the partition (4,1). The representation corresponding to the partitions (1^5) and $(2, 1^3)$ are their associated representations.

The representation corresponding to the partition $(3, 1^2)$ is the 2nd compound of the representation corresponding to the partition (4, 1).

To obtain the representation corresponding to the partition $(2^2,1)$ and hence the representation corresponding to the partition $(3,2)$, we make use of the identity

$$\{2^2, 1\} = \begin{vmatrix} \sigma_3 & \sigma_4 \\ \sigma_1 & \sigma_2 \end{vmatrix}$$

i.e. $\sigma_3 \sigma_2 = \{2^2, 1\} + \sigma_1 \sigma_4$,

from which

$$A^{(3)} \times A^{(2)} = A^{[2^2,1]} \dot{+} A \times A^{(4)}.$$

We take

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{bmatrix}$$

so that $A^{(2)} =$

$$\begin{bmatrix} (a_1\ b_2) & (a_1\ c_2) & (a_1\ d_2) & (a_1\ e_2) & (b_1\ c_2) \\ (a_1\ b_3) & (a_1\ c_3) & (a_1\ d_3) & (a_1\ e_3) & (b_1\ c_3) \\ (a_1\ b_4) & (a_1\ c_4) & (a_1\ d_4) & (a_1\ e_4) & (b_1\ c_4) \\ (a_1\ b_5) & (a_1\ c_5) & (a_1\ d_5) & (a_1\ e_5) & (b_1\ c_5) \\ (a_2\ b_3) & (a_2\ c_3) & (a_2\ d_3) & (a_2\ e_3) & (b_2\ c_3) \\ (a_2\ b_4) & (a_2\ c_4) & (a_2\ d_4) & (a_2\ e_4) & (b_2\ c_4) \\ (a_2\ b_5) & (a_2\ c_5) & (a_2\ d_5) & (a_2\ e_5) & (b_2\ c_5) \\ (a_3\ b_4) & (a_3\ c_4) & (a_3\ d_4) & (a_3\ e_4) & (b_3\ c_4) \\ (a_3\ b_5) & (a_3\ c_5) & (a_3\ d_5) & (a_3\ e_5) & (b_3\ c_5) \\ (a_4\ b_5) & (a_4\ c_5) & (a_4\ d_5) & (a_4\ e_5) & (b_4\ c_5) \end{bmatrix}$$

$$\begin{matrix} (b_1\ d_2) & (b_1\ e_2) & (c_1\ d_2) & (c_1\ e_2) & (d_1\ e_2) \\ (b_1\ d_3) & (b_1\ e_3) & (c_1\ d_3) & (c_1\ e_3) & (d_1\ e_3) \end{matrix}\Bigg]$$

$$\left.\begin{array}{lllll}
(b_1\ d_4) & (b_1\ e_4) & (c_1\ d_4) & (c_1\ e_4) & (d_1\ e_4) \\
(b_1\ d_5) & (b_1\ e_5) & (c_1\ d_5) & (c_1\ e_5) & (d_1\ e_5) \\
(b_2\ d_3) & (b_2\ e_3) & (c_2\ d_3) & (c_2\ e_3) & (d_2\ e_3) \\
(b_2\ d_4) & (b_2\ e_4) & (c_2\ d_4) & (c_2\ e_4) & (d_2\ e_4) \\
(b_2\ d_5) & (b_2\ e_5) & (c_2\ d_5) & (c_2\ e_5) & (d_2\ e_5) \\
(b_3\ d_4) & (b_3\ e_4) & (c_3\ d_4) & (c_3\ e_4) & (d_3\ e_4) \\
(b_3\ d_5) & (b_3\ e_5) & (c_3\ d_5) & (c_3\ e_5) & (d_3\ e_5) \\
(b_4\ d_5) & (b_4\ e_5) & (c_4\ d_5) & (c_4\ e_5) & (d_4\ e_5)
\end{array}\right]$$

and $A^{(3)} =$

$$\left[\begin{array}{lllll}
(a_1\ b_2\ c_3) & (a_1\ b_2\ d_3) & (a_1\ b_2\ e_3) & (a_1\ c_2\ d_3) & (a_1\ c_2\ e_3) \\
(a_1\ b_2\ c_4) & (a_1\ b_2\ d_4) & (a_1\ b_2\ e_4) & (a_1\ c_2\ d_4) & (a_1\ c_2\ e_4) \\
(a_1\ b_2\ c_5) & (a_1\ b_2\ d_5) & (a_1\ b_2\ e_5) & (a_1\ c_2\ d_5) & (a_1\ c_2\ e_5) \\
(a_1\ b_3\ c_4) & (a_1\ b_3\ d_4) & (a_1\ b_3\ e_4) & (a_1\ c_3\ d_4) & (a_1\ c_3\ e_4) \\
(a_1\ b_3\ c_5) & (a_1\ b_3\ d_5) & (a_1\ b_3\ e_5) & (a_1\ c_3\ d_5) & (a_1\ c_3\ e_5) \\
(a_1\ b_4\ c_5) & (a_1\ b_4\ d_5) & (a_1\ b_4\ e_5) & (a_1\ c_4\ d_5) & (a_1\ c_4\ e_5) \\
(a_2\ b_3\ c_4) & (a_2\ b_3\ d_4) & (a_2\ b_3\ e_4) & (a_2\ c_3\ d_4) & (a_2\ c_3\ e_4) \\
(a_2\ b_3\ c_5) & (a_2\ b_3\ d_5) & (a_2\ b_3\ e_5) & (a_2\ c_3\ d_5) & (a_2\ c_3\ e_5) \\
(a_2\ b_4\ c_5) & (a_2\ b_4\ d_5) & (a_2\ b_4\ e_5) & (a_2\ c_4\ d_5) & (a_2\ c_4\ e_5) \\
(a_3\ b_4\ c_5) & (a_3\ b_4\ d_5) & (a_3\ b_4\ e_5) & (a_3\ c_4\ d_5) & (a_3\ c_4\ e_5)
\end{array}\right.$$

$$\left.\begin{array}{lllll}
(a_1\ d_2\ e_3) & (b_1\ c_2\ d_3) & (b_1\ c_2\ e_3) & (b_1\ d_2\ e_3) & (c_1\ d_2\ e_3) \\
(a_1\ d_2\ e_4) & (b_1\ c_2\ d_4) & (b_1\ c_2\ e_4) & (b_1\ d_2\ e_4) & (c_1\ d_2\ e_4) \\
(a_1\ d_2\ e_5) & (b_1\ c_2\ d_5) & (b_1\ c_2\ e_5) & (b_1\ d_2\ e_5) & (c_1\ d_2\ e_5) \\
(a_1\ d_3\ e_4) & (b_1\ c_3\ d_4) & (b_1\ c_3\ e_4) & (b_1\ d_3\ e_4) & (c_1\ d_3\ e_4) \\
(a_1\ d_3\ e_5) & (b_1\ c_3\ d_5) & (b_1\ c_3\ d_5) & (b_1\ d_3\ e_5) & (c_1\ d_3\ e_5) \\
(a_1\ d_4\ e_5) & (b_1\ c_4\ d_5) & (b_1\ c_4\ e_5) & (b_1\ d_4\ e_5) & (c_1\ d_4\ e_5) \\
(a_2\ d_3\ e_4) & (b_2\ c_3\ d_4) & (b_2\ c_3\ e_4) & (b_2\ d_3\ e_4) & (c_2\ d_3\ e_4)
\end{array}\right]$$

$$\left.\begin{matrix} (a_2\ d_3\ e_5) & (b_2\ c_3\ d_5) & (b_2\ c_3\ e_5) & (b_2\ d_3\ e_5) & (c_2\ d_3\ e_5) \\ (a_2\ d_4\ e_5) & (b_2\ c_4\ d_5) & (b_2\ c_4\ e_5) & (b_2\ d_4\ e_5) & (c_2\ d_4\ e_5) \\ (a_3\ d_4\ e_5) & (b_3\ c_4\ d_5) & (b_3\ c_4\ e_5) & (b_3\ d_4\ e_5) & (c_3\ d_4\ e_5) \end{matrix}\right]$$

The direct product $A^{(3)} \times A^{(2)}$ is a matrix of order 100 x 100. The most polarized part in this matrix, is a sub-matrix of order 10 x 10 and its elements lie in the 10th, 19th, 28th, 37th, 46th, 55th, 64th, 73rd, 82nd and 91st rows and columns. We write this sub-matrix as

$M = [m_1, m_2, m_3, \ldots, m_{10}]$

where each m is a column vector, namely

$$m_1 = \begin{bmatrix} (a_1\ b_2\ c_3)\ (d_4\ e_5) \\ (a_1\ b_2\ c_4)\ (d_3\ e_5) \\ (a_1\ b_2\ c_5)\ (d_3\ e_4) \\ (a_1\ b_3\ c_4)\ (d_2\ e_5) \\ (a_1\ b_3\ c_5)\ (d_2\ e_4) \\ (a_1\ b_4\ c_5)\ (d_2\ e_3) \\ (a_2\ b_3\ c_4)\ (d_1\ e_5) \\ (a_2\ b_3\ c_5)\ (d_1\ e_4) \\ (a_2\ b_4\ c_5)\ (d_1\ e_3) \\ (a_3\ b_4\ c_5)\ (d_1\ e_2) \end{bmatrix} \quad m_2 = \begin{bmatrix} (a_1\ b_2\ d_3)\ (c_4\ e_5) \\ (a_1\ b_2\ d_4)\ (c_3\ e_5) \\ (a_1\ b_2\ d_5)\ (c_3\ e_4) \\ (a_1\ b_3\ d_4)\ (c_2\ e_5) \\ (a_1\ b_3\ d_5)\ (c_2\ e_4) \\ (a_1\ b_4\ d_5)\ (c_2\ e_3) \\ (a_2\ b_3\ d_4)\ (c_1\ e_5) \\ (a_2\ b_3\ d_5)\ (c_1\ e_4) \\ (a_2\ b_4\ d_5)\ (c_1\ e_3) \\ (a_3\ b_4\ d_5)\ (c_1\ e_2) \end{bmatrix}$$

$$m_3 = \left[\begin{matrix} (a_1\ b_2\ e_3)\ (c_4\ d_5) \\ (a_1\ b_2\ e_4)\ (c_3\ d_5) \\ (a_1\ b_2\ e_5)\ (c_3\ d_4) \\ (a_1\ b_3\ e_4)\ (c_2\ d_5) \end{matrix}\right. \quad m_4 = \left[\begin{matrix} (a_1\ c_2\ d_3)\ (b_4\ e_5) \\ (a_1\ c_2\ d_4)\ (b_3\ e_5) \\ (a_1\ c_2\ d_5)\ (b_3\ e_4) \\ (a_1\ c_3\ d_4)\ (b_2\ e_5) \end{matrix}\right.$$

$$\begin{bmatrix} (a_1\ b_3\ e_5)\ (c_2\ d_4) \\ (a_1\ b_4\ e_5)\ (c_2\ d_3) \\ (a_2\ b_3\ e_4)\ (c_1\ d_5) \\ (a_2\ b_3\ e_5)\ (c_1\ d_4) \\ (a_2\ b_4\ e_5)\ (c_1\ d_3) \\ (a_3\ b_4\ e_5)\ (c_1\ d_2) \end{bmatrix} \qquad \begin{bmatrix} (a_1\ c_3\ d_5)\ (b_2\ e_4) \\ (a_1\ c_4\ d_5)\ (b_2\ e_3) \\ (a_2\ c_3\ d_4)\ (b_1\ e_5) \\ (a_2\ c_3\ d_5)\ (b_1\ e_4) \\ (a_2\ c_4\ d_5)\ (b_1\ e_3) \\ (a_3\ c_4\ d_5)\ (b_1\ e_2) \end{bmatrix}$$

$$m_5 = \begin{bmatrix} (a_1\ c_2\ e_3)\ (b_4\ d_5) \\ (a_1\ c_2\ e_4)\ (b_3\ d_5) \\ (a_1\ c_2\ e_5)\ (b_3\ d_4) \\ (a_1\ c_3\ e_4)\ (b_2\ d_5) \\ (a_1\ c_3\ e_5)\ (b_2\ d_4) \\ (a_1\ c_4\ e_5)\ (b_2\ d_3) \\ (a_2\ c_3\ e_4)\ (b_1\ d_5) \\ (a_2\ c_3\ e_5)\ (b_1\ d_4) \\ (a_2\ c_4\ e_5)\ (b_1\ d_3) \\ (a_3\ c_4\ e_5)\ (b_1\ d_2) \end{bmatrix} \quad m_6 = \begin{bmatrix} (a_1\ d_2\ e_3)\ (b_4\ c_5) \\ (a_1\ d_2\ e_4)\ (b_3\ c_5) \\ (a_1\ d_2\ e_5)\ (b_3\ c_4) \\ (a_1\ d_3\ e_4)\ (b_2\ c_5) \\ (a_1\ d_3\ e_5)\ (b_2\ c_4) \\ (a_1\ d_4\ e_5)\ (b_2\ c_3) \\ (a_2\ d_3\ e_4)\ (b_1\ c_5) \\ (a_2\ d_3\ e_5)\ (b_1\ c_4) \\ (a_2\ d_4\ e_5)\ (b_1\ c_3) \\ (a_3\ d_4\ e_5)\ (b_1\ c_2) \end{bmatrix}$$

$$m_7 = \begin{bmatrix} (b_1\ c_2\ d_3)\ (a_4\ e_5) \\ (b_1\ c_2\ d_4)\ (a_3\ e_5) \\ (b_1\ c_2\ d_5)\ (a_3\ e_4) \\ (b_1\ c_3\ d_4)\ (a_2\ e_5) \\ (b_1\ c_3\ d_5)\ (a_2\ e_4) \\ (b_1\ c_4\ d_5)\ (a_2\ e_3) \\ (b_2\ c_3\ d_4)\ (a_1\ e_5) \\ (b_2\ c_3\ d_5)\ (a_1\ e_4) \end{bmatrix} \quad m_8 = \begin{bmatrix} (b_1\ c_2\ e_3)\ (a_4\ d_5) \\ (b_1\ c_2\ e_4)\ (a_3\ d_5) \\ (b_1\ c_2\ e_5)\ (a_3\ d_4) \\ (b_1\ c_3\ e_4)\ (a_2\ d_5) \\ (b_1\ c_3\ e_5)\ (a_2\ d_4) \\ (b_1\ c_4\ e_5)\ (a_2\ d_3) \\ (b_2\ c_3\ e_4)\ (a_1\ d_5) \\ (b_2\ c_3\ e_5)\ (a_1\ d_4) \end{bmatrix}$$

$$\begin{bmatrix} (b_2\, c_4\, d_5)\,(a_1\, e_3) \\ (b_3\, c_4\, d_5)\,(a_1\, e_2) \end{bmatrix} \quad \begin{bmatrix} (b_2\, c_4\, e_5)\,(a_1\, d_3) \\ (b_3\, c_4\, e_5)\,(a_1\, d_2) \end{bmatrix}$$

$$m_9 = \begin{bmatrix} (b_1\, d_2\, e_3)\,(a_4\, c_5) \\ (b_1\, d_2\, e_4)\,(a_3\, c_5) \\ (b_1\, d_2\, e_5)\,(a_3\, c_4) \\ (b_1\, d_3\, e_4)\,(a_2\, c_5) \\ (b_1\, d_3\, e_5)\,(a_2\, c_4) \\ (b_1\, d_4\, e_5)\,(a_2\, c_3) \\ (b_2\, d_3\, e_4)\,(a_1\, c_5) \\ (b_2\, d_3\, e_5)\,(a_1\, c_4) \\ (b_2\, d_4\, e_5)\,(a_1\, c_3) \\ (b_3\, d_4\, e_5)\,(a_1\, c_2) \end{bmatrix} \quad m_{10} = \begin{bmatrix} (c_1\, d_2\, e_3)\,(a_4\, b_5) \\ (c_1\, d_2\, e_4)\,(a_3\, b_5) \\ (c_1\, d_2\, e_5)\,(a_3\, b_4) \\ (c_1\, d_3\, e_4)\,(a_2\, b_5) \\ (c_1\, d_3\, e_5)\,(a_2\, b_4) \\ (c_1\, d_4\, e_5)\,(a_2\, b_3) \\ (c_2\, d_3\, e_4)\,(a_1\, b_5) \\ (c_2\, d_3\, e_5)\,(a_1\, b_4) \\ (c_2\, d_4\, e_5)\,(a_1\, b_3) \\ (c_3\, d_4\, e_5)\,(a_1\, b_2) \end{bmatrix}$$

In the first column of the matrix $A^{(3)} \times A^{(2)}$, and the same rows as before, we have the column vector:

$$\begin{bmatrix} (a_1\, b_2\, c_3)\,(a_4\, b_5) \\ (a_1\, b_2\, c_4)\,(a_3\, b_5) \\ (a_1\, b_2\, c_5)\,(a_3\, b_4) \\ (a_1\, b_3\, c_4)\,(a_2\, b_5) \\ (a_1\, b_3\, c_5)\,(a_2\, b_4) \\ (a_1\, b_4\, c_5)\,(a_2\, b_3) \\ (a_2\, b_3\, c_4)\,(a_1\, b_5) \\ (a_2\, b_3\, c_5)\,(a_1\, b_4) \\ (a_2\, b_4\, c_5)\,(a_1\, b_3) \\ (a_3\, b_4\, c_5)\,(a_1\, b_2) \end{bmatrix}$$

The elements of that vector are not linearly independent. They are connected by five linearly independent identities. We easily see the following identities :

1) $(a_1 b_2 c_3)(a_4 b_5) - (a_1 b_2 c_4)(a_3 b_5) + (a_1 b_2 c_5)(a_3 b_4)$
$+ (a_1 b_3 c_4)(a_2 b_5) - (a_1 b_3 c_5)(a_2 b_4) + (a_1 b_4 c_5)(a_2 b_3)$
$- (a_2 b_3 c_4)(a_1 b_5) + (a_2 b_3 c_5)(a_1 b_4) - (a_2 b_4 c_5)(a_1 b_3)$
$+ (a_3 b_4 c_5)(a_1 b_2) = 0$, being the Laplacian expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix}$$

2) $(a_1 b_2 c_3)(a_4 b_5) - (a_1 b_2 c_4)(a_3 b_5) + (a_1 b_3 c_4)(a_2 b_5)$
$- (a_2 b_3 c_4)(a_1 b_5) = 0$, being the Laplacian expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{vmatrix}$$

3) $(a_1 b_2 c_3)(a_4 b_5) + (a_1 b_2 c_5)(a_3 b_4) - (a_1 b_3 c_5)(a_2 b_4)$
$+ (a_2 b_3 c_5)(a_1 b_4) = 0$, being the Laplacian expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ o & o & o & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix}$$

4) $-(a_1\,b_2\,c_4)(a_3\,b_5) + (a_1\,b_2\,c_5)(a_3\,b_4) + (a_1\,b_4\,c_5)(a_2\,b_3)$ $-(a_2\,b_4\,c_5)(a_1\,b_3) = o$, being the Laplacian expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ o & o & o & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix}$$

5) $(a_1\,b_3\,c_4)(a_2\,b_5) - (a_1\,b_3\,c_5)(a_2\,b_4) + (a_1\,b_4\,c_5)(a_2\,b_3)$ $+ (a_3\,b_4\,c_5)(a_1\,b_2) = o$, being the Laplacian expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ o & o & o & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix}$$

6) $-(a_2\,b_3\,c_4)(a_1\,b_5) + (a_2\,b_3\,c_5)(a_1\,b_4) - (a_2\,b_4\,c_5)(a_1\,b_3)$ $+ (a_3\,b_4\,c_5)(a_1\,b_2) = o$, being the Laplacian expansion

of the determinant

$$\begin{vmatrix} 0 & 0 & 0 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix}$$

Subtracting 6) from 1) we have

$$7) \quad (a_1 b_2 c_3)(a_4 b_5) - (a_1 b_2 c_4)(a_3 b_5) + (a_1 b_2 c_5)(a_3 b_4) + (a_1 b_3 c_4)(a_2 b_5) - (a_1 b_3 c_5)(a_2 b_4) + (a_1 b_4 c_5)(a_2 b_3) = 0$$

Subtracting 4) from 7) we have

$$8) \quad (a_1 b_2 c_3)(a_4 b_5) + (a_1 b_3 c_4)(a_2 b_5) - (a_1 b_3 c_5)(a_2 b_4) + (a_2 b_4 c_5)(a_1 b_3) = 0$$

Subtracting 7) from 5) we have

$$9) \quad -(a_1 b_2 c_3)(a_4 b_5) + (a_1 b_2 c_4)(a_3 b_5) - (a_1 b_2 c_5)(a_3 b_4) + (a_3 b_4 c_5)(a_1 b_2) = 0$$

From 7), 2), 3), 8) and 9) it follows that the operations

(1) $\text{row}_6 + \text{row}_1 - \text{row}_2 + \text{row}_3 + \text{row}_4 - \text{row}_5$

(2) $\text{row}_7 - \text{row}_1 + \text{row}_2 - \text{row}_4$

(3) $\text{row}_8 + \text{row}_1 + \text{row}_3 - \text{row}_5$

(4) $\text{row}_9 + \text{row}_1 + \text{row}_4 - \text{row}_5$

(5) $\text{row}_{10} - \text{row}_1 + \text{row}_2 - \text{row}_3$,

applied to the above vector, suppress its last five elements to zero.

Therefore, performing these operations, together with the operations

(6) $\text{Col}_5 + \text{col } 6$, $\text{col } 4 - \text{col } 6$, $\text{col}_3 - \text{col}_6$, $\text{col}_2 + \text{col}_6$,

col_1 - col_6

(7) $col_4 + col_7$, $col_2 - col_7$, $col_1 + col\ 7$.

(8) $col_5 + col_8$, $col_3 - col_8$, $col_1 - col\ 8$.

(9) $col_5 + col_9$, $col_4 - col_9$, $col_1 - col\ 9$.

(10) $col_3 + col_{10}$, $col_2 - col_{10}$, $col_1 + col\ 10$

on the matrix M, it reduces to $M_1 \dotplus M_2$ where M_1 is the most polarized part in $A^{[2^2 1]}$. These operations are equivalent to premultiplying M by

$$H = \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ 1 & -1 & 1 & 1 & -1 & 1 & . & . & . & . \\ -1 & 1 & . & -1 & . & . & 1 & . & . & . \\ 1 & . & 1 & . & -1 & . & . & 1 & . & . \\ 1 & . & . & 1 & -1 & . & . & . & 1 & . \\ -1 & 1 & -1 & . & . & . & . & . & . & 1 \end{bmatrix}$$

and postmultiplying by

$$H^{-1} = \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ -1 & 1 & -1 & -1 & 1 & 1 & . & . & . & . \\ 1 & -1 & . & 1 & . & . & 1 & . & . & . \\ -1 & . & -1 & . & 1 & . & . & 1 & . & . \\ -1 & . & . & -1 & 1 & . & . & . & 1 & . \\ 1 & -1 & 1 & . & . & . & . & . & . & 1 \end{bmatrix}$$

We write M_1 as $[M_{ij}]$, i, j = 1, 2, 3, 4, 5 and we then find :

$$M_{11} = (a_1\, b_2\, c_3)(d_4\, e_5) - (a_1\, d_2\, e_3)(b_4\, c_5) + (b_1\, c_2\, d_3)(a_4\, e_5)$$
$$- (b_1\, c_2\, e_3)(a_4\, d_5) - (b_1\, d_2\, e_3)(a_4\, c_5) + (c_1\, d_2\, e_3)(a_4\, b_5).$$

$$M_{21} = (a_1\, b_2\, c_4)(d_3\, e_5) - (a_1\, d_2\, e_4)(b_3\, c_5) + (b_1\, c_2\, d_4)(a_3\, e_5)$$
$$- (b_1\, c_2\, e_4)(a_3\, d_5) - (b_1\, d_2\, e_4)(a_3\, c_5) + (c_1\, d_2\, e_4)(a_3\, b_5).$$

$$M_{31} = (a_1\, b_2\, c_5)(d_3\, e_4) - (a_1\, d_2\, e_5)(b_3\, c_4) + (b_1\, c_2\, d_5)(a_3\, e_4)$$
$$- (b_1\, c_2\, e_5)(a_3\, d_4) - (b_1\, d_2\, e_5)(a_3\, c_4) + (c_1\, d_2\, e_5)(a_3\, b_4).$$

$$M_{41} = (a_1\, b_3\, c_4)(d_2\, e_5) - (a_1\, d_3\, e_4)(b_2\, c_5) + (b_1\, c_3\, d_4)(a_2\, e_5)$$
$$- (b_1\, c_3\, e_4)(a_2\, d_5) - (b_1\, d_3\, e_4)(a_2\, c_5) + (c_1\, d_3\, e_4)(a_2\, b_5).$$

$$M_{51} = (a_1\, b_3\, c_5)(d_2\, e_4) - (a_1\, d_3\, e_5)(b_2\, c_4) + (b_1\, c_3\, d_5)(a_2\, e_4)$$
$$- (b_1\, c_3\, e_5)(a_2\, d_4) - (b_1\, d_3\, e_5)(a_2\, c_4) + (c_1\, d_3\, e_5)(a_2\, b_4).$$

$$M_{12} = (a_1\, b_2\, d_3)(c_4\, e_5) + (a_1\, d_2\, e_3)(b_4\, c_5) - (b_1\, c_2\, d_3)(a_4\, e_5)$$
$$- (c_1\, d_2\, e_3)(a_4\, b_5).$$

$$M_{22} = (a_1\, b_2\, d_4)(c_3\, e_5) + (a_1\, d_2\, e_4)(b_3\, c_5) - (b_1\, c_2\, d_4)(a_3\, e_5)$$
$$- (c_1\, d_2\, e_4)(a_3\, b_5).$$

$$M_{32} = (a_1\, b_2\, d_5)(c_3\, e_4) + (a_1\, d_2\, e_5)(b_3\, c_4) - (b_1\, c_2\, d_5)(a_3\, e_4)$$
$$- (c_1\, d_2\, e_5)(a_3\, b_4).$$

$$M_{42}=(a_1\ b_3\ d_4)(c_2\ e_5)+(a_1\ d_3\ e_4)(b_2\ c_5)-(b_1\ c_3\ d_4)(a_2\ e_5)$$
$$-(c_1\ d_3\ e_4)(a_2\ b_5).$$

$$M_{52}=(a_1\ b_3\ d_5)(c_2\ e_4)+(a_1\ d_3\ e_5)(b_2\ c_4)-(b_1\ c_3\ d_5)(a_2\ e_4)$$
$$-(c_1\ d_3\ e_5)(a_2\ b_4).$$

$$M_{13}=(a_1\ b_2\ e_3)(c_4\ d_5)-(a_1\ d_2\ e_3)(b_4\ c_5)-(b_1\ c_2\ e_3)(a_4\ d_5)$$
$$+(c_1\ d_2\ e_3)(a_4\ b_5).$$

$$M_{23}=(a_1\ b_2\ e_4)(c_3\ d_5)-(a_1\ d_2\ e_4)(b_3\ c_5)-(b_1\ c_2\ e_4)(a_3\ d_5)$$
$$+(c_1\ d_2\ e_4)(a_3\ b_5).$$

$$M_{33}=(a_1\ b_2\ e_5)(c_3\ d_4)-(a_1\ d_2\ e_5)(b_3\ c_4)-(b_1\ c_2\ e_5)(a_3\ d_4)$$
$$+(c_1\ d_2\ e_5)(a_3\ b_4).$$

$$M_{43}=(a_1\ b_3\ e_4)(c_2\ d_5)-(a_1\ d_3\ e_4)(b_2\ c_5)-(b_1\ c_3\ e_4)(a_2\ d_5)$$
$$+(c_1\ d_3\ e_4)(a_2\ b_5)\ .$$

$$M_{53}=(a_1\ b_3\ e_5)(c_2\ d_4)-(a_1\ d_3\ e_5)(b_2\ c_4)-(b_1\ c_3\ e_5)(a_2\ d_4)$$
$$+(c_1\ d_3\ e_5)(a_2\ b_4)\ .$$

$$M_{14}=(a_1\ c_2\ d_3)(b_4\ e_5)-(a_1\ d_2\ e_3)(b_4\ c_5)+(b_1\ c_2\ d_3)(a_4\ e_5)$$
$$-(b_1\ d_2\ e_3)(a_4\ c_5)\ .$$

$$M_{24}=(a_1\ c_2\ d_4)(b_3\ e_5)-(a_1\ d_2\ e_4)(b_3\ c_5)+(b_1\ c_2\ d_4)(a_3\ e_5)$$
$$-(b_1\ d_2\ e_4)(a_3\ c_5).$$

$$M_{34} = (a_1\ c_2\ d_5)(b_3\ e_4) - (a_1\ d_2\ e_5)(b_3\ c_4) + (b_1\ c_2\ d_5)(a_3\ e_4)$$
$$-(b_1\ d_2\ e_5)(a_3\ c_4).$$

$$M_{44} = (a_1\ c_3\ d_4)(b_2\ e_5) - (a_1\ d_3\ e_4)(b_2\ c_5) + (b_1\ c_3\ d_4)(a_2\ e_5)$$
$$-(b_1\ d_3\ e_4)(a_2\ c_5).$$

$$M_{54} = (a_1\ c_3\ d_5)(b_2\ e_4) - (a_1\ d_3\ e_5)(b_2\ c_4) + (b_1\ c_3\ d_5)(a_2\ e_4)$$
$$-(b_1\ d_3\ e_5)(a_2\ c_4).$$

$$M_{15} = (a_1\ c_2\ e_3)(b_4\ d_5) + (a_1\ d_2\ e_3)(b_4\ c_5) + (b_1\ c_2\ e_3)(a_4\ d_5)$$
$$+(b_1\ d_2\ e_3)(a_4\ c_5)$$

$$M_{25} = (a_1\ c_2\ e_4)(b_3\ d_5) + (a_1\ d_2\ e_4)(b_3\ c_5) + (b_1\ c_2\ e_4)(a_3\ d_5)$$
$$+(b_1\ d_2\ e_4)(a_3\ c_5).$$

$$M_{35} = (a_1\ c_2\ e_5)(b_3\ d_4) + (a_1\ d_2\ e_5)(b_3\ c_4) + (b_1\ c_2\ e_5)(a_3\ d_4)$$
$$+(b_1\ d_2\ e_5)(a_3\ c_4).$$

$$M_{45} = (a_1\ c_3\ e_4)(b_2\ d_5) + (a_1\ d_3\ e_4)(b_2\ c_5) + (b_1\ c_3\ e_4)(a_2\ d_5)$$
$$+(b_1\ d_3\ e_4)(a_2\ c_5).$$

$$M_{55} = (a_1\ c_3\ e_5)(b_2\ d_4) + (a_1\ d_3\ e_5)(b_2\ c_4) + (b_1\ c_3\ e_5)(a_2\ d_4)$$
$$+(b_1\ d_3\ e_5)(a_2\ c_4).$$

Explicity in terms of a_1, a_2, ..., e_5, the suffixes arranged in accordance with the arrangement of the elements of the group, these are :

$$
\begin{aligned}
M_{11} = {} & a_1 b_2 c_3 d_4 e_5 - a_2 b_1 c_3 d_4 e_5 - a_3 b_2 c_1 d_4 e_5 \\
& + a_4 b_2 c_3 d_1 e_5 + a_5 b_2 c_3 d_4 e_1 - a_1 b_3 c_2 d_4 e_5 \\
& - a_1 b_2 c_3 d_5 e_4 + a_2 b_1 c_3 d_5 e_4 + a_3 b_2 c_1 d_5 e_4 \\
& - a_4 b_3 c_2 d_1 e_5 + a_4 b_5 c_3 d_1 e_2 + a_4 b_2 c_5 d_1 e_3 \\
& - a_5 b_3 c_2 d_4 e_1 + a_5 b_4 c_3 d_2 e_1 + a_5 b_2 c_4 d_3 e_1 \\
& + a_1 b_3 c_2 d_5 e_4 - a_1 b_4 c_5 d_2 e_3 - a_1 b_5 c_4 d_3 e_2 \\
& + a_3 b_1 c_2 d_4 e_5 - a_4 b_1 c_3 d_2 e_5 - a_5 b_1 c_3 d_4 e_2 \\
& + a_2 b_3 c_1 d_4 e_5 - a_4 b_2 c_1 d_3 e_5 - a_5 b_2 c_1 d_4 e_3 \\
& - a_5 b_2 c_3 d_1 e_4 - a_4 b_2 c_3 d_5 e_1 - a_3 b_1 c_2 d_5 e_4 \\
& - a_4 b_1 c_5 d_2 e_3 - a_5 b_1 c_4 d_3 e_2 - a_2 b_3 c_1 d_5 e_4 \\
& - a_4 b_5 c_1 d_3 e_2 - a_5 b_4 c_1 d_2 e_3 + a_2 b_4 c_5 d_1 e_3 \\
& + a_3 b_5 c_4 d_1 e_2 + a_5 b_3 c_2 d_1 e_4 + a_2 b_5 c_4 d_3 e_1 \\
& + a_3 b_4 c_5 d_2 e_1 + a_4 b_3 c_2 d_5 e_1 - a_5 b_4 c_2 d_3 e_1 \\
& - a_4 b_5 c_2 d_1 e_3 - a_5 b_3 c_4 d_2 e_1 - a_4 b_3 c_5 d_1 e_2 \\
& + a_4 b_1 c_2 d_3 e_5 + a_5 b_1 c_2 d_4 e_3 + a_5 b_1 c_3 d_2 e_4 \\
& + a_4 b_1 c_3 d_5 e_2 + a_4 b_3 c_1 d_2 e_5 + a_5 b_3 c_1 d_4 e_2 \\
& + a_5 b_2 c_1 d_3 e_4 + a_4 b_2 c_1 d_5 e_3 - a_5 b_4 c_3 d_1 e_2 \\
& - a_5 b_2 c_4 d_1 e_3 - a_4 b_5 c_3 d_2 e_1 - a_4 b_2 c_5 d_3 e_1 \\
& + a_1 b_5 c_4 d_2 e_3 + a_1 b_4 c_5 d_3 e_2 - a_5 b_1 c_2 d_3 e_4 \\
& - a_4 b_1 c_2 d_5 e_3 + a_5 b_1 c_4 d_2 e_3 + a_4 b_1 c_5 d_3 e_2 \\
& - a_5 b_3 c_1 d_2 e_4 - a_4 b_3 c_1 d_5 e_2 + a_5 b_4 c_1 d_3 e_2 \\
& + a_4 b_5 c_1 d_2 e_3 + a_5 b_4 c_2 d_1 e_3 - a_3 b_4 c_5 d_1 e_2 \\
& + a_5 b_3 c_4 d_1 e_2 - a_2 b_5 c_4 d_1 e_3 + a_4 b_5 c_2 d_3 e_1
\end{aligned}
$$

$$- a_3 b_5 c_4 d_2 e_1 + a_4 b_3 c_5 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 .$$

$$\begin{aligned}
M_{21} = & - a_3 b_2 c_1 d_4 e_5 + a_5 b_2 c_3 d_4 e_1 - a_1 b_5 c_3 d_4 e_2 \\
& + a_1 b_2 c_4 d_3 e_5 - a_2 b_1 c_4 d_3 e_5 + a_3 b_4 c_1 d_2 e_5 \\
& - a_3 b_5 c_1 d_4 e_2 + a_3 b_2 c_1 d_5 e_4 + a_4 b_5 c_3 d_1 e_2 \\
& - a_5 b_3 c_2 d_4 e_1 - a_5 b_4 c_3 d_2 e_1 + a_5 b_2 c_4 d_3 e_1 \\
& + a_3 b_1 c_2 d_4 e_5 - a_5 b_1 c_3 d_4 e_2 - a_4 b_2 c_1 d_3 e_5 \\
& + a_5 b_2 c_1 d_4 e_3 + a_3 b_2 c_4 d_1 e_5 - a_5 b_2 c_3 d_1 e_4 \\
& + a_2 b_5 c_3 d_4 e_1 - a_3 b_2 c_5 d_4 e_1 - a_1 b_4 c_2 d_3 e_5 \\
& + a_1 b_5 c_3 d_2 e_4 + a_1 b_3 c_5 d_4 e_2 - a_1 b_2 c_4 d_5 e_3 \\
& - a_3 b_1 c_2 d_5 e_4 - a_5 b_1 c_4 d_3 e_2 - a_5 b_4 c_1 d_2 e_3 \\
& + a_3 b_5 c_4 d_1 e_2 + a_5 b_3 c_2 d_1 e_4 + a_3 b_4 c_5 d_2 e_1 \\
& - a_5 b_4 c_2 d_3 e_1 + a_5 b_3 c_4 d_2 e_1 + a_3 b_5 c_1 d_2 e_4 \\
& - a_4 b_3 c_5 d_1 e_2 - a_3 b_4 c_1 d_5 e_2 + a_2 b_1 c_4 d_5 e_3 \\
& + a_4 b_1 c_2 d_3 e_5 - a_5 b_1 c_2 d_4 e_3 - a_3 b_1 c_4 d_2 e_5 \\
& + a_5 b_1 c_3 d_2 e_4 + a_3 b_1 c_5 d_4 e_2 + a_5 b_3 c_1 d_4 e_2 \\
& + a_2 b_4 c_1 d_3 e_5 - a_5 b_2 c_1 d_3 e_4 + a_4 b_2 c_1 d_5 e_3 \\
& - a_3 b_4 c_2 d_1 e_5 + a_5 b_4 c_3 d_1 e_2 - a_5 b_2 c_4 d_1 e_3 \\
& - a_2 b_5 c_3 d_1 e_4 + a_3 b_2 c_5 d_1 e_4 + a_3 b_5 c_2 d_4 e_1 \\
& - a_4 b_5 c_3 d_2 e_1 - a_2 b_3 c_5 d_4 e_1 - a_3 b_2 c_4 d_5 e_1 \\
& + a_1 b_4 c_2 d_5 e_3 - a_1 b_3 c_5 d_2 e_4 + a_5 b_1 c_2 d_3 e_4 \\
& - a_4 b_1 c_2 d_5 e_3 + a_5 b_1 c_4 d_2 e_3 - a_3 b_1 c_5 d_2 e_4 \\
& + a_3 b_1 c_4 d_5 e_2 - a_5 b_3 c_1 d_2 e_4 + a_5 b_4 c_1 d_3 e_2 \\
& - a_2 b_4 c_1 d_5 e_3 + a_5 b_4 c_2 d_1 e_3 - a_3 b_4 c_5 d_1 e_2 \\
& - a_5 b_3 c_4 d_1 e_2 - a_3 b_5 c_2 d_1 e_4 + a_2 b_3 c_5 d_1 e_4 \\
& - a_3 b_5 c_4 d_2 e_1 + a_4 b_3 c_5 d_2 e_1 + a_3 b_4 c_2 d_5 e_1 .
\end{aligned}$$

$$
\begin{aligned}
M_{31} = {} & a_3 b_2 c_1 d_4 e_5 - a_4 b_2 c_3 d_1 e_5 + a_1 b_4 c_3 d_2 e_5 \\
& - a_1 b_2 c_5 d_4 e_3 + a_2 b_1 c_5 d_4 e_3 + a_3 b_4 c_1 d_2 e_5 \\
& - a_3 b_5 c_1 d_4 e_2 - a_3 b_2 c_1 d_5 e_4 + a_4 b_3 c_2 d_1 e_5 \\
& + a_4 b_5 c_3 d_1 e_2 - a_4 b_2 c_5 d_1 e_3 - a_5 b_4 c_3 d_2 e_1 \\
& - a_3 b_1 c_2 d_4 e_5 + a_4 b_1 c_3 d_2 e_5 - a_4 b_2 c_1 d_3 e_5 \\
& + a_5 b_2 c_1 d_4 e_3 - a_2 b_4 c_3 d_1 e_5 + a_3 b_2 c_4 d_1 e_5 \\
& - a_3 b_2 c_5 d_4 e_1 + a_4 b_2 c_3 d_5 e_1 + a_1 b_5 c_2 d_4 e_3 \\
& - a_1 b_3 c_4 d_2 e_5 - a_1 b_4 c_3 d_5 e_2 + a_1 b_2 c_5 d_3 e_4 \\
& + a_3 b_1 c_2 d_5 e_4 + a_4 b_1 c_5 d_2 e_3 + a_4 b_5 c_1 d_3 e_2 \\
& - a_3 b_5 c_4 d_1 e_2 - a_3 b_4 c_5 d_2 e_1 - a_4 b_3 c_2 d_5 e_1 \\
& + a_4 b_5 c_2 d_1 e_3 + a_5 b_3 c_4 d_2 e_1 + a_3 b_5 c_1 d_2 e_4 \\
& - a_4 b_3 c_5 d_1 e_2 - a_3 b_4 c_1 d_5 e_2 - a_2 b_1 c_5 d_3 e_4 \\
& + a_4 b_1 c_2 d_3 e_5 - a_5 b_1 c_2 d_4 e_3 - a_3 b_1 c_4 d_2 e_5 \\
& + a_3 b_1 c_5 d_4 e_2 - a_4 b_1 c_3 d_5 e_2 - a_4 b_3 c_1 d_2 e_5 \\
& - a_5 b_2 c_1 d_3 e_4 - a_2 b_5 c_1 d_4 e_3 + a_4 b_2 c_1 d_5 e_3 \\
& - a_3 b_4 c_2 d_1 e_5 + a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5 \\
& + a_3 b_2 c_5 d_1 e_4 + a_3 b_5 c_2 d_4 e_1 - a_4 b_5 c_3 d_2 e_1 \\
& + a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1 - a_3 b_2 c_4 d_5 e_1 \\
& - a_1 b_5 c_2 d_3 e_4 + a_1 b_3 c_4 d_5 e_2 + a_5 b_1 c_2 d_3 e_4 \\
& - a_4 b_1 c_2 d_5 e_3 - a_3 b_1 c_5 d_2 e_4 - a_4 b_1 c_5 d_3 e_2 \\
& + a_3 b_1 c_4 d_5 e_2 + a_4 b_3 c_1 d_5 e_2 + a_2 b_5 c_1 d_3 e_4 \\
& - a_4 b_5 c_1 d_2 e_3 + a_3 b_4 c_5 d_1 e_2 - a_5 b_3 c_4 d_1 e_2 \\
& - a_3 b_5 c_2 d_1 e_4 - a_4 b_5 c_2 d_3 e_1 + a_3 b_5 c_4 d_2 e_1 \\
& + a_4 b_3 c_5 d_2 e_1 + a_3 b_4 c_2 d_5 e_1 - a_2 b_3 c_4 d_5 e_1 .
\end{aligned}
$$

$$M_{41} = a_2 b_1 c_3 d_4 e_5 - a_5 b_2 c_3 d_4 e_1 - a_1 b_4 c_3 d_2 e_5$$

$$+ a_1 b_2 c_5 d_4 e_3 - a_2 b_1 c_4 d_3 e_5 + a_2 b_1 c_5 d_4 e_3$$
$$- a_2 b_1 c_3 d_5 e_4 + a_3 b_4 c_1 d_2 e_5 - a_4 b_2 c_5 d_1 e_3$$
$$+ a_5 b_3 c_2 d_4 e_1 - a_5 b_4 c_3 d_2 e_1 + a_5 b_2 c_4 d_3 e_1$$
$$+ a_4 b_1 c_3 d_2 e_5 - a_5 b_1 c_3 d_4 e_2 - a_2 b_3 c_1 d_4 e_5$$
$$+ a_5 b_2 c_1 d_4 e_3 - a_2 b_4 c_3 d_1 e_5 + a_5 b_2 c_3 d_1 e_4$$
$$+ a_2 b_5 c_3 d_4 e_1 - a_3 b_2 c_5 d_4 e_1 - a_1 b_5 c_2 d_4 e_3$$
$$+ a_1 b_3 c_4 d_2 e_5 + a_1 b_4 c_3 d_5 e_2 - a_1 b_2 c_5 d_3 e_4$$
$$+ a_5 b_1 c_4 d_3 e_2 + a_2 b_3 c_1 d_5 e_4 + a_5 b_4 c_1 d_2 e_3$$
$$- a_2 b_4 c_5 d_1 e_3 - a_5 b_3 c_2 d_1 e_4 - a_2 b_5 c_4 d_3 e_1$$
$$- a_5 b_4 c_2 d_3 e_1 + a_4 b_5 c_2 d_1 e_3 + a_5 b_3 c_4 d_2 e_1$$
$$- a_3 b_4 c_1 d_5 e_2 - a_2 b_1 c_5 d_3 e_4 + a_2 b_1 c_4 d_5 e_3$$
$$- a_5 b_1 c_2 d_4 e_3 - a_3 b_1 c_4 d_2 e_5 + a_5 b_1 c_3 d_2 e_4$$
$$- a_4 b_1 c_3 d_5 e_2 - a_4 b_3 c_1 d_2 e_5 + a_5 b_3 c_1 d_4 e_2$$
$$+ a_2 b_4 c_1 d_3 e_5 - a_5 b_2 c_1 d_3 e_4 - a_2 b_5 c_1 d_4 e_3$$
$$+ a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5 - a_5 b_2 c_4 d_1 e_3$$
$$- a_2 b_5 c_3 d_1 e_4 + a_3 b_2 c_5 d_1 e_4 + a_3 b_5 c_2 d_4 e_1$$
$$- a_2 b_3 c_5 d_4 e_1 + a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1$$
$$+ a_1 b_5 c_2 d_3 e_4 - a_1 b_3 c_4 d_5 e_2 + a_5 b_1 c_2 d_3 e_4$$
$$- a_5 b_1 c_4 d_2 e_3 + a_3 b_1 c_4 d_5 e_2 - a_5 b_3 c_1 d_2 e_4$$
$$+ a_4 b_3 c_1 d_5 e_2 - a_5 b_4 c_1 d_3 e_2 + a_2 b_5 c_1 d_3 e_4$$
$$- a_2 b_4 c_1 d_5 e_3 + a_5 b_4 c_2 d_1 e_3 - a_5 b_3 c_4 d_1 e_2$$
$$+ a_2 b_5 c_4 d_1 e_3 - a_3 b_5 c_2 d_1 e_4 + a_2 b_3 c_5 d_1 e_4$$
$$- a_4 b_5 c_2 d_3 e_1 + a_2 b_4 c_5 d_3 e_1 - a_2 b_3 c_4 d_5 e_1 \; .$$

$$M_{51} = - a_2 b_1 c_3 d_4 e_5 + a_4 b_2 c_3 d_1 e_5 + a_1 b_5 c_3 d_4 e_2$$
$$- a_1 b_2 c_4 d_3 e_5 - a_2 b_1 c_4 d_3 e_5 + a_2 b_1 c_5 d_4 e_3$$
$$+ a_2 b_1 c_3 d_5 e_4 - a_3 b_5 c_1 d_4 e_2 - a_4 b_3 c_2 d_1 e_5$$

$$+ a_4 b_5 c_3 d_1 e_2 - a_4 b_2 c_5 d_1 e_3 + a_5 b_2 c_4 d_3 e_1$$
$$+ a_4 b_1 c_3 d_2 e_5 - a_5 b_1 c_3 d_4 e_2 + a_2 b_3 c_1 d_4 e_5$$
$$- a_4 b_2 c_1 d_3 e_5 - a_2 b_4 c_3 d_1 e_5 + a_3 b_2 c_4 d_1 e_5$$
$$+ a_2 b_5 c_3 d_4 e_1 - a_4 b_2 c_3 d_5 e_1 + a_1 b_4 c_2 d_3 e_5$$
$$- a_1 b_5 c_3 d_2 e_4 - a_1 b_3 c_5 d_4 e_2 + a_1 b_2 c_4 d_5 e_3$$
$$- a_4 b_1 c_5 d_2 e_3 - a_2 b_3 c_1 d_5 e_4 - a_4 b_5 c_1 d_3 e_2$$
$$+ a_2 b_4 c_5 d_1 e_3 + a_2 b_5 c_4 d_3 e_1 + a_4 b_3 c_2 d_5 e_1$$
$$- a_5 b_4 c_2 d_3 e_1 + a_4 b_5 c_2 d_1 e_3 + a_3 b_5 c_1 d_2 e_4$$
$$- a_4 b_3 c_5 d_1 e_2 - a_2 b_1 c_5 d_3 e_4 + a_2 b_1 c_4 d_5 e_3$$
$$+ a_4 b_1 c_2 d_3 e_5 + a_5 b_1 c_3 d_2 e_4 + a_3 b_1 c_5 d_4 e_2$$
$$- a_4 b_1 c_3 d_5 e_2 - a_4 b_3 c_1 d_2 e_5 + a_5 b_3 c_1 d_4 e_2$$
$$+ a_2 b_4 c_1 d_3 e_5 - a_2 b_5 c_1 d_4 e_3 + a_4 b_2 c_1 d_5 e_3$$
$$- a_3 b_4 c_2 d_1 e_5 + a_2 b_3 c_4 d_1 e_5 - a_5 b_2 c_4 d_1 e_3$$
$$- a_2 b_5 c_3 d_1 e_4 - a_4 b_5 c_3 d_2 e_1 - a_2 b_3 c_5 d_4 e_1$$
$$+ a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1 - a_3 b_2 c_4 d_5 e_1$$
$$- a_1 b_4 c_2 d_5 e_3 + a_1 b_3 c_5 d_2 e_4 - a_4 b_1 c_2 d_5 e_3$$
$$- a_3 b_1 c_5 d_2 e_4 + a_4 b_1 c_5 d_3 e_2 - a_5 b_3 c_1 d_2 e_4$$
$$+ a_4 b_3 c_1 d_5 e_2 + a_2 b_5 c_1 d_3 e_4 + a_4 b_5 c_1 d_2 e_3$$
$$- a_2 b_4 c_1 d_5 e_3 + a_5 b_4 c_2 d_1 e_3 - a_2 b_5 c_4 d_1 e_3$$
$$+ a_2 b_3 c_5 d_1 e_4 - a_4 b_5 c_2 d_3 e_1 + a_4 b_3 c_5 d_2 e_1$$
$$- a_2 b_4 c_5 d_3 e_1 + a_3 b_4 c_2 d_5 e_1 - a_2 b_3 c_4 d_5 e_1 \; .$$

$$M_{12} = - a_4 b_2 c_3 d_1 e_5 + a_1 b_2 c_4 d_3 e_5 - a_2 b_1 c_4 d_3 e_5$$
$$+ a_4 b_3 c_2 d_1 e_5 - a_4 b_5 c_3 d_1 e_2 - a_5 b_4 c_3 d_2 e_1$$
$$+ a_1 b_4 c_5 d_2 e_3 + a_1 b_5 c_4 d_3 e_2 + a_4 b_1 c_3 d_2 e_5$$
$$+ a_4 b_2 c_1 d_3 e_5 - a_3 b_2 c_4 d_1 e_5 + a_5 b_2 c_3 d_1 e_4$$

$$- a_1 b_3 c_4 d_2 e_5 - a_1 b_2 c_5 d_3 e_4 + a_4 b_5 c_1 d_3 e_2$$
$$+ a_5 b_4 c_1 d_2 e_3 - a_2 b_4 c_5 d_1 e_3 - a_3 b_5 c_4 d_1 e_2$$
$$- a_5 b_3 c_2 d_1 e_4 - a_2 b_5 c_4 d_3 e_1 - a_3 b_4 c_5 d_2 e_1$$
$$+ a_5 b_4 c_2 d_3 e_1 + a_4 b_5 c_2 d_1 e_3 + a_2 b_1 c_5 d_3 e_4$$
$$- a_4 b_1 c_2 d_3 e_5 + a_3 b_1 c_4 d_2 e_5 - a_5 b_1 c_3 d_2 e_4$$
$$- a_4 b_3 c_1 d_2 e_5 - a_5 b_2 c_1 d_3 e_4 + a_5 b_4 c_3 d_1 e_2$$
$$+ a_2 b_3 c_4 d_1 e_5 + a_3 b_2 c_5 d_1 e_4 + a_4 b_5 c_3 d_2 e_1$$
$$- a_1 b_5 c_4 d_2 e_3 + a_1 b_3 c_5 d_2 e_4 - a_1 b_4 c_5 d_3 e_2$$
$$+ a_5 b_1 c_2 d_3 e_4 - a_3 b_1 c_5 d_2 e_4 + a_5 b_3 c_1 d_2 e_4$$
$$- a_5 b_4 c_1 d_3 e_2 - a_4 b_5 c_1 d_2 e_3 - a_5 b_4 c_2 d_1 e_3$$
$$+ a_3 b_4 c_5 d_1 e_2 + a_2 b_5 c_4 d_1 e_3 - a_2 b_3 c_5 d_1 e_4$$
$$- a_4 b_5 c_2 d_3 e_1 + a_3 b_5 c_4 d_2 e_1 + a_2 b_4 c_5 d_3 e_1.$$

$$M_{22} = a_1 b_2 c_3 d_4 e_5 - a_2 b_1 c_3 d_4 e_5 + a_3 b_2 c_1 d_4 e_5$$
$$- a_4 b_2 c_3 d_1 e_5 - a_1 b_4 c_3 d_2 e_5 + a_1 b_5 c_3 d_4 e_2$$
$$- a_1 b_2 c_5 d_4 e_3 + a_2 b_1 c_5 d_4 e_3 - a_3 b_4 c_1 d_2 e_5$$
$$+ a_3 b_5 c_1 d_4 e_2 - a_4 b_5 c_3 d_1 e_2 + a_4 b_2 c_5 d_1 e_3$$
$$+ a_5 b_3 c_2 d_4 e_1 + a_1 b_4 c_5 d_2 e_3 - a_3 b_1 c_2 d_4 e_5$$
$$+ a_4 b_1 c_3 d_2 e_5 - a_5 b_2 c_1 d_4 e_3 + a_2 b_4 c_3 d_1 e_5$$
$$- a_3 b_2 c_4 d_1 e_5 - a_2 b_5 c_3 d_4 e_1 - a_1 b_5 c_3 d_2 e_4$$
$$- a_1 b_3 c_5 d_4 e_2 - a_4 b_1 c_5 d_2 e_3 + a_5 b_4 c_1 d_2 e_3$$
$$- a_2 b_4 c_5 d_1 e_3 - a_3 b_5 c_4 d_1 e_2 - a_5 b_3 c_2 d_1 e_4$$
$$- a_5 b_3 c_4 d_2 e_1 - a_3 b_5 c_1 d_2 e_4 + a_4 b_3 c_5 d_1 e_2$$
$$+ a_5 b_1 c_2 d_4 e_3 + a_3 b_1 c_4 d_2 e_5 - a_5 b_3 c_1 d_4 e_2$$
$$+ a_3 b_4 c_2 d_1 e_5 + a_5 b_2 c_4 d_1 e_3 + a_2 b_5 c_3 d_1 e_4$$
$$- a_3 b_5 c_2 d_4 e_1 + a_4 b_5 c_3 d_2 e_1 + a_2 b_3 c_5 d_4 e_1$$

$+ a_1 b_3 c_5 d_2 e_4 - a_5 b_1 c_4 d_2 e_3 + a_5 b_3 c_1 d_2 e_4$
$- a_5 b_4 c_2 d_1 e_3 + a_5 b_3 c_4 d_1 e_2 + a_3 b_5 c_2 d_1 e_4$
$- a_2 b_3 c_5 d_1 e_4 + a_3 b_5 c_4 d_2 e_1 - a_4 b_3 c_5 d_2 e_1.$

$M_{32} = - a_1 b_4 c_3 d_2 e_5 + a_1 b_2 c_3 d_5 e_4 - a_2 b_1 c_3 d_5 e_4$
$- a_3 b_4 c_1 d_2 e_5 + a_3 b_2 c_1 d_5 e_4 - a_4 b_3 c_2 d_1 e_5$
$+ a_4 b_2 c_5 d_1 e_3 + a_5 b_4 c_3 d_2 e_1 + a_2 b_4 c_3 d_1 e_5$
$- a_5 b_2 c_3 d_1 e_4 + a_1 b_3 c_4 d_2 e_5 - a_1 b_5 c_3 d_2 e_4$
$+ a_1 b_4 c_3 d_5 e_2 - a_1 b_2 c_4 d_5 e_3 - a_3 b_1 c_2 d_5 e_4$
$- a_4 b_1 c_5 d_2 e_3 + a_3 b_4 c_5 d_2 e_1 + a_4 b_3 c_2 d_5 e_1$
$- a_4 b_5 c_2 d_1 e_3 - a_5 b_3 c_4 d_2 e_1 - a_3 b_5 c_1 d_2 e_4$
$+ a_4 b_3 c_5 d_1 e_2 + a_3 b_4 c_1 d_5 e_2 + a_2 b_1 c_4 d_5 e_3$
$+ a_5 b_1 c_3 d_2 e_4 + a_4 b_3 c_1 d_2 e_5 - a_4 b_2 c_1 d_5 e_3$
$+ a_3 b_4 c_2 d_1 e_5 - a_5 b_4 c_3 d_1 e_2 - a_2 b_3 c_4 d_1 e_5$
$+ a_5 b_2 c_4 d_1 e_3 + a_2 b_5 c_3 d_1 e_4 - a_3 b_2 c_5 d_1 e_4$
$- a_2 b_4 c_3 d_5 e_1 + a_1 b_5 c_4 d_2 e_3 - a_1 b_3 c_4 d_5 e_2$
$+ a_4 b_1 c_2 d_5 e_3 - a_5 b_1 c_4 d_2 e_3 + a_3 b_1 c_5 d_2 e_4$
$- a_4 b_3 c_1 d_5 e_2 + a_4 b_5 c_1 d_2 e_3 - a_3 b_4 c_5 d_1 e_2$
$+ a_5 b_3 c_4 d_1 e_2 - a_2 b_5 c_4 d_1 e_3 + a_3 b_5 c_2 d_1 e_4$
$- a_4 b_3 c_5 d_2 e_1 - a_3 b_4 c_2 d_5 e_1 + a_2 b_3 c_4 d_5 e_1.$

$M_{42} = - a_2 b_1 c_3 d_4 e_5 + a_5 b_2 c_3 d_4 e_1 + a_1 b_3 c_2 d_4 e_5$
$- a_1 b_2 c_5 d_4 e_3 + a_2 b_1 c_4 d_3 e_5 - a_4 b_3 c_2 d_1 e_5$
$+ a_4 b_2 c_5 d_1 e_3 - a_5 b_2 c_4 d_3 e_1 - a_3 b_1 c_2 d_4 e_5$
$+ a_5 b_1 c_3 d_4 e_2 + a_2 b_3 c_1 d_4 e_5 - a_5 b_2 c_1 d_4 e_3$
$+ a_2 b_4 c_3 d_1 e_5 - a_5 b_2 c_3 d_1 e_4 - a_2 b_5 c_3 d_4 e_1$

$+ a_3 b_2 c_5 d_4 e_1 - a_1 b_4 c_2 d_3 e_5 + a_1 b_5 c_2 d_4 e_3$

$- a_1 b_3 c_5 d_4 e_2 + a_1 b_2 c_5 d_3 e_4 - a_5 b_1 c_4 d_3 e_2$

$+ a_2 b_5 c_4 d_3 e_1 - a_4 b_5 c_2 d_1 e_3 + a_4 b_3 c_5 d_1 e_2$

$+ a_4 b_1 c_2 d_3 e_5 + a_3 b_1 c_5 d_4 e_2 - a_5 b_3 c_1 d_4 e_2$

$- a_2 b_4 c_1 d_3 e_5 + a_5 b_2 c_1 d_3 e_4 + a_2 b_5 c_1 d_4 e_3$

$+ a_3 b_4 c_2 d_1 e_5 - a_5 b_4 c_3 d_1 e_2 - a_2 b_3 c_4 d_1 e_5$

$+ a_5 b_2 c_4 d_1 e_3 + a_2 b_5 c_3 d_1 e_4 - a_3 b_2 c_5 d_1 e_4$

$- a_3 b_5 c_2 d_4 e_1 - a_4 b_2 c_5 d_3 e_1 - a_1 b_5 c_2 d_3 e_4$

$+ a_1 b_4 c_5 d_3 e_2 - a_4 b_1 c_5 d_3 e_2 + a_5 b_4 c_1 d_3 e_2$

$- a_2 b_5 c_1 d_3 e_4 - a_3 b_4 c_5 d_1 e_2 + a_5 b_3 c_4 d_1 e_2$

$- a_2 b_5 c_4 d_1 e_3 + a_3 b_5 c_2 d_1 e_4 + a_4 b_5 c_2 d_3 e_1.$

$M_{52} = - a_4 b_2 c_3 d_1 e_5 + a_1 b_2 c_4 d_3 e_5 - a_2 b_1 c_3 d_5 e_4$

$- a_4 b_5 c_3 d_1 e_2 + a_4 b_2 c_5 d_1 e_3 - a_5 b_2 c_4 d_3 e_1$

$+ a_1 b_3 c_2 d_5 e_4 + a_1 b_5 c_4 d_3 e_2 + a_4 b_2 c_1 d_3 e_5$

$+ a_2 b_4 c_3 d_1 e_5 - a_3 b_2 c_4 d_1 e_5 + a_4 b_2 c_3 d_5 e_1$

$- a_1 b_4 c_2 d_3 e_5 - a_1 b_2 c_4 d_5 e_3 - a_3 b_1 c_2 d_5 e_4$

$- a_5 b_1 c_4 d_3 e_2 + a_2 b_3 c_1 d_5 e_4 + a_4 b_5 c_1 d_3 e_2$

$- a_2 b_4 c_5 d_1 e_3 - a_3 b_5 c_4 d_1 e_2 - a_5 b_3 c_2 d_1 e_4$

$+ a_5 b_4 c_2 d_3 e_1 + a_4 b_3 c_5 d_1 e_2 + a_2 b_1 c_5 d_3 e_4$

$+ a_4 b_1 c_3 d_5 e_2 - a_2 b_4 c_1 d_3 e_5 - a_4 b_2 c_1 d_5 e_3$

$+ a_3 b_4 c_2 d_1 e_5 + a_5 b_2 c_4 d_1 e_3 + a_2 b_5 c_3 d_1 e_4$

$- a_4 b_2 c_5 d_3 e_1 - a_2 b_4 c_3 d_5 e_1 + a_3 b_2 c_4 d_5 e_1$

$- a_1 b_5 c_2 d_3 e_4 + a_1 b_4 c_2 d_5 e_3 - a_1 b_3 c_4 d_5 e_2$

$+ a_5 b_1 c_2 d_3 e_4 - a_4 b_1 c_5 d_3 e_2 + a_3 b_1 c_4 d_5 e_2$

$- a_4 b_3 c_1 d_5 e_2 - a_2 b_5 c_1 d_3 e_4 + a_2 b_4 c_1 d_5 e_3$

$$\begin{aligned}
&- a_5 b_4 c_2 d_1 e_3 + a_5 b_3 c_4 d_1 e_2 + a_3 b_5 c_2 d_1 e_4\\
&- a_2 b_3 c_5 d_1 e_4 + a_2 b_4 c_5 d_3 e_1 - a_3 b_4 c_2 d_5 e_1 .
\end{aligned}$$

$$\begin{aligned}
M_{13} = {}& a_5 b_2 c_3 d_4 e_1 - a_1 b_2 c_5 d_4 e_3 + a_2 b_1 c_5 d_4 e_3\\
&+ a_4 b_5 c_3 d_1 e_2 - a_5 b_3 c_2 d_4 e_1 + a_5 b_4 c_3 d_2 e_1\\
&- a_1 b_4 c_5 d_2 e_3 - a_1 b_5 c_4 d_3 e_2 - a_5 b_1 c_3 d_4 e_2\\
&- a_5 b_2 c_1 d_4 e_3 + a_3 b_2 c_5 d_4 e_1 - a_4 b_2 c_3 d_5 e_1\\
&+ a_1 b_3 c_5 d_4 e_2 + a_1 b_2 c_4 d_5 e_3 - a_4 b_5 c_1 d_3 e_2\\
&- a_5 b_4 c_1 d_2 e_3 + a_2 b_4 c_5 d_1 e_3 + a_3 b_5 c_4 d_1 e_2\\
&+ a_2 b_5 c_4 d_3 e_1 + a_3 b_4 c_5 d_2 e_1 + a_4 b_3 c_2 d_5 e_1\\
&- a_5 b_4 c_2 d_3 e_1 - a_4 b_5 c_2 d_1 e_3 - a_2 b_1 c_4 d_5 e_3\\
&+ a_5 b_1 c_2 d_4 e_3 - a_3 b_1 c_5 d_4 e_2 + a_4 b_1 c_3 d_5 e_2\\
&+ a_5 b_3 c_1 d_4 e_2 + a_4 b_2 c_1 d_5 e_3 - a_5 b_4 c_3 d_1 e_2\\
&- a_4 b_5 c_3 d_2 e_1 - a_2 b_3 c_5 d_4 e_1 - a_3 b_2 c_4 d_5 e_1\\
&+ a_1 b_5 c_4 d_2 e_3 + a_1 b_4 c_5 d_3 e_2 - a_1 b_3 c_4 d_5 e_2\\
&- a_4 b_1 c_2 d_5 e_3 + a_3 b_1 c_4 d_5 e_2 - a_4 b_3 c_1 d_5 e_2\\
&+ a_5 b_4 c_1 d_3 e_2 + a_4 b_5 c_1 d_2 e_3 + a_5 b_4 c_2 d_1 e_3\\
&- a_3 b_4 c_5 d_1 e_2 - a_2 b_5 c_4 d_1 e_3 + a_4 b_5 c_2 d_3 e_1\\
&- a_3 b_5 c_4 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 + a_2 b_3 c_4 d_5 e_1 .
\end{aligned}$$

$$\begin{aligned}
M_{23} = {}& - a_1 b_5 c_3 d_4 e_2 + a_1 b_2 c_3 d_5 e_4 - a_2 b_1 c_3 d_5 e_4\\
&- a_3 b_5 c_1 d_4 e_2 + a_3 b_2 c_1 d_5 e_4 + a_4 b_5 c_3 d_1 e_2\\
&- a_5 b_3 c_2 d_4 e_1 + a_5 b_2 c_4 d_3 e_1 + a_2 b_5 c_3 d_4 e_1\\
&- a_4 b_2 c_3 d_5 e_1 + a_1 b_5 c_3 d_2 e_4 + a_1 b_3 c_5 d_4 e_2\\
&- a_1 b_4 c_3 d_5 e_2 - a_1 b_2 c_5 d_3 e_4 - a_3 b_1 c_2 d_5 e_4\\
&- a_5 b_1 c_4 d_3 e_2 + a_3 b_5 c_4 d_1 e_2 + a_5 b_3 c_2 d_1 e_4
\end{aligned}$$

$$
\begin{aligned}
&-a_5 b_4 c_2 d_3 e_1 + a_5 b_3 c_4 d_2 e_1 + a_3 b_5 c_1 d_2 e_4 \\
&-a_4 b_3 c_5 d_1 e_2 - a_3 b_4 c_1 d_5 e_2 + a_2 b_1 c_5 d_3 e_4 \\
&+a_4 b_1 c_3 d_5 e_2 + a_5 b_3 c_1 d_4 e_2 - a_5 b_2 c_1 d_3 e_4 \\
&-a_2 b_5 c_3 d_1 e_4 + a_3 b_5 c_2 d_4 e_1 - a_4 b_5 c_3 d_2 e_1 \\
&-a_2 b_3 c_5 d_4 e_1 + a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1 \\
&-a_3 b_2 c_4 d_5 e_1 - a_1 b_3 c_5 d_2 e_4 + a_1 b_4 c_5 d_3 e_2 \\
&+a_5 b_1 c_2 d_3 e_4 - a_4 b_1 c_5 d_3 e_2 + a_3 b_1 c_4 d_5 e_2 \\
&-a_5 b_3 c_1 d_2 e_4 + a_5 b_4 c_1 d_3 e_2 - a_5 b_3 c_4 d_1 e_2 \\
&-a_3 b_5 c_2 d_1 e_4 + a_2 b_3 c_5 d_1 e_4 - a_3 b_5 c_4 d_2 e_1 \\
&+a_4 b_3 c_5 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 + a_3 b_4 c_2 d_5 e_1 .
\end{aligned}
$$

$$
\begin{aligned}
M_{33} = &+ a_1 b_2 c_3 d_4 e_5 - a_2 b_1 c_3 d_4 e_5 + a_3 b_2 c_1 d_4 e_5 \\
&- a_5 b_2 c_3 d_4 e_1 + a_1 b_4 c_3 d_2 e_5 - a_1 b_5 c_3 d_4 e_2 \\
&- a_1 b_2 c_4 d_3 e_5 + a_2 b_1 c_4 d_3 e_5 + a_3 b_4 c_1 d_2 e_5 \\
&- a_3 b_5 c_1 d_4 e_2 + a_4 b_3 c_2 d_1 e_5 - a_5 b_4 c_3 d_2 e_1 \\
&+ a_5 b_2 c_4 d_3 e_1 + a_1 b_5 c_4 d_3 e_2 - a_3 b_1 c_2 d_4 e_5 \\
&+ a_5 b_1 c_3 d_4 e_2 - a_4 b_2 c_1 d_3 e_5 - a_2 b_4 c_3 d_1 e_5 \\
&+ a_2 b_5 c_3 d_4 e_1 - a_3 b_2 c_5 d_4 e_1 - a_1 b_3 c_4 d_2 e_5 \\
&- a_1 b_4 c_3 d_5 e_2 - a_5 b_1 c_4 d_3 e_2 + a_4 b_5 c_1 d_3 e_2 \\
&- a_2 b_5 c_4 d_3 e_1 - a_3 b_4 c_5 d_2 e_1 - a_4 b_3 c_2 d_5 e_1 \\
&+ a_5 b_3 c_4 d_2 e_1 - a_4 b_3 c_5 d_1 e_2 - a_3 b_4 c_1 d_5 e_2 \\
&+ a_4 b_1 c_2 d_3 e_5 + a_3 b_1 c_5 d_4 e_2 - a_4 b_3 c_1 d_2 e_5 \\
&- a_3 b_4 c_2 d_1 e_5 + a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5 \\
&+ a_3 b_5 c_2 d_4 e_1 + a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1 \\
&+ a_1 b_3 c_4 d_5 e_2 - a_4 b_1 c_5 d_3 e_2 + a_4 b_3 c_1 d_5 e_2 \\
&+ a_3 b_4 c_5 d_1 e_2 - a_5 b_3 c_4 d_1 e_2 - a_4 b_5 c_2 d_3 e_1 \\
&+ a_4 b_3 c_5 d_2 e_1 + a_3 b_4 c_2 d_5 e_1 - a_2 b_3 c_4 d_5 e_1 .
\end{aligned}
$$

$$
\begin{aligned}
M_{43} = {} & - a_5 b_2 c_3 d_4 e_1 + a_1 b_2 c_5 d_4 e_3 - a_2 b_1 c_3 d_5 e_4 \\
& - a_4 b_2 c_5 d_1 e_3 - a_5 b_4 c_3 d_2 e_1 + a_5 b_2 c_4 d_3 e_1 \\
& + a_1 b_3 c_2 d_5 e_4 + a_1 b_4 c_5 d_2 e_3 + a_5 b_2 c_1 d_4 e_3 \\
& + a_5 b_2 c_3 d_1 e_4 + a_2 b_5 c_3 d_4 e_1 - a_3 b_2 c_5 d_4 e_1 \\
& - a_1 b_5 c_2 d_4 e_3 - a_1 b_2 c_5 d_3 e_4 - a_3 b_1 c_2 d_5 e_4 \\
& - a_4 b_1 c_5 d_2 e_3 + a_2 b_3 c_1 d_5 e_4 + a_5 b_4 c_1 d_2 e_3 \\
& - a_2 b_5 c_4 d_3 e_1 - a_3 b_4 c_5 d_2 e_1 - a_4 b_3 c_2 d_5 e_1 \\
& + a_4 b_5 c_2 d_1 e_3 + a_5 b_3 c_4 d_2 e_1 + a_2 b_1 c_4 d_5 e_3 \\
& + a_5 b_1 c_3 d_2 e_4 - a_5 b_2 c_1 d_3 e_4 - a_2 b_5 c_1 d_4 e_3 \\
& - a_5 b_2 c_4 d_1 e_3 - a_2 b_5 c_3 d_1 e_4 + a_3 b_2 c_5 d_1 e_4 \\
& + a_3 b_5 c_2 d_4 e_1 + a_4 b_2 c_5 d_3 e_1 + a_2 b_4 c_3 d_5 e_1 \\
& + a_1 b_5 c_2 d_3 e_4 - a_1 b_4 c_2 d_5 e_3 - a_1 b_3 c_5 d_2 e_4 \\
& + a_4 b_1 c_2 d_5 e_3 - a_5 b_1 c_4 d_2 e_3 + a_3 b_1 c_5 d_2 e_4 \\
& - a_5 b_3 c_1 d_2 e_4 + a_2 b_5 c_1 d_3 e_4 - a_2 b_4 c_1 d_5 e_3 \\
& + a_2 b_5 c_4 d_1 e_3 - a_3 b_5 c_2 d_1 e_4 - a_4 b_5 c_2 d_3 e_1 \\
& + a_4 b_3 c_5 d_2 e_1 + a_3 b_4 c_2 d_5 e_1 - a_2 b_3 c_4 d_5 e_1 .
\end{aligned}
$$

$$
\begin{aligned}
M_{53} = {} & - a_2 b_1 c_3 d_4 e_5 + a_4 b_2 c_3 d_1 e_5 + a_1 b_3 c_2 d_4 e_5 \\
& - a_1 b_2 c_4 d_3 e_5 + a_2 b_1 c_5 d_4 e_3 - a_4 b_2 c_5 d_1 e_3 \\
& - a_5 b_3 c_2 d_4 e_1 + a_5 b_2 c_4 d_3 e_1 - a_3 b_1 c_2 d_4 e_5 \\
& + a_4 b_1 c_3 d_2 e_5 + a_2 b_3 c_1 d_4 e_5 - a_4 b_2 c_1 d_3 e_5 \\
& - a_2 b_4 c_3 d_1 e_5 + a_3 b_2 c_4 d_1 e_5 + a_2 b_5 c_3 d_4 e_1 \\
& - a_4 b_2 c_3 d_5 e_1 + a_1 b_4 c_2 d_3 e_5 - a_1 b_5 c_2 d_4 e_3 \\
& - a_1 b_3 c_4 d_2 e_5 + a_1 b_2 c_4 d_5 e_3 - a_4 b_1 c_5 d_2 e_3 \\
& + a_2 b_4 c_5 d_1 e_3 - a_5 b_4 c_2 d_3 e_1 + a_5 b_3 c_4 d_2 e_1 \\
& + a_5 b_1 c_2 d_4 e_3 + a_3 b_1 c_4 d_2 e_5 - a_4 b_3 c_1 d_2 e_5 \\
& + a_2 b_4 c_1 d_3 e_5 - a_2 b_5 c_1 d_4 e_3 + a_4 b_2 c_1 d_5 e_3 \\
& - a_3 b_4 c_2 d_1 e_5 - a_5 b_2 c_4 d_1 e_3 + a_3 b_5 c_2 d_4 e_1
\end{aligned}
$$

$$- a_4 b_5 c_3 d_2 e_1 - a_2 b_3 c_5 d_4 e_1 + a_4 b_2 c_5 d_3 e_1$$
$$+ a_2 b_4 c_3 d_5 e_1 - a_3 b_2 c_4 d_5 e_1 - a_1 b_4 c_2 d_5 e_3$$
$$+ a_1 b_5 c_4 d_2 e_3 - a_5 b_1 c_4 d_2 e_3 + a_4 b_5 c_1 d_2 e_3$$
$$- a_2 b_4 c_1 d_5 e_3 + a_5 b_4 c_2 d_1 e_3 - a_3 b_5 c_4 d_2 e_1$$
$$+ a_4 b_3 c_5 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 + a_3 b_4 c_2 d_5 e_1 \;.$$

$$M_{14} = a_4 b_2 c_3 d_1 e_5 - a_1 b_4 c_3 d_2 e_5 + a_3 b_4 c_1 d_2 e_5$$
$$- a_4 b_3 c_2 d_1 e_5 + a_4 b_2 c_5 d_1 e_3 + a_5 b_2 c_4 d_3 e_1$$
$$- a_1 b_4 c_5 d_2 e_3 - a_1 b_5 c_4 d_3 e_2 - a_4 b_1 c_3 d_2 e_5$$
$$- a_4 b_2 c_1 d_3 e_5 + a_2 b_4 c_3 d_1 e_5 - a_5 b_2 c_3 d_1 e_4$$
$$+ a_1 b_4 c_2 d_3 e_5 + a_1 b_5 c_3 d_2 e_4 - a_4 b_1 c_5 d_2 e_3$$
$$- a_5 b_1 c_4 d_3 e_2 + a_2 b_4 c_5 d_1 e_3 + a_3 b_5 c_4 d_1 e_2$$
$$+ a_5 b_3 c_2 d_1 e_4 + a_2 b_5 c_4 d_3 e_1 + a_3 b_4 c_5 d_2 e_1$$
$$- a_5 b_3 c_4 d_2 e_1 - a_3 b_5 c_1 d_2 e_4 - a_4 b_3 c_5 d_1 e_2$$
$$+ a_4 b_1 c_2 d_3 e_5 + a_5 b_1 c_3 d_2 e_4 + a_4 b_3 c_1 d_2 e_5$$
$$- a_2 b_4 c_1 d_3 e_5 + a_5 b_2 c_1 d_3 e_4 - a_3 b_4 c_2 d_1 e_5$$
$$- a_5 b_2 c_4 d_1 e_3 - a_2 b_5 c_3 d_1 e_4 - a_4 b_2 c_5 d_3 e_1$$
$$- a_1 b_5 c_2 d_3 e_4 + a_1 b_5 c_4 d_2 e_3 + a_1 b_4 c_5 d_3 e_2$$
$$- a_5 b_1 c_2 d_3 e_4 + a_5 b_1 c_4 d_2 e_3 + a_4 b_1 c_5 d_3 e_2$$
$$- a_5 b_3 c_1 d_2 e_4 + a_2 b_5 c_1 d_3 e_4 - a_3 b_4 c_5 d_1 e_2$$
$$+ a_5 b_3 c_4 d_1 e_2 - a_2 b_5 c_4 d_1 e_3 + a_3 b_5 c_2 d_1 e_4$$
$$- a_3 b_5 c_4 d_2 e_1 + a_4 b_3 c_5 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 \;.$$

$$M_{24} = - a_3 b_2 c_1 d_4 e_5 + a_5 b_2 c_3 d_4 e_1 + a_1 b_3 c_2 d_4 e_5$$
$$- a_1 b_5 c_3 d_4 e_2 + a_3 b_4 c_1 d_2 e_5 - a_4 b_3 c_2 d_1 e_5$$
$$+ a_4 b_5 c_3 d_1 e_2 - a_5 b_4 c_3 d_2 e_1 + a_3 b_1 c_2 d_4 e_5$$
$$- a_5 b_1 c_3 d_4 e_2 - a_2 b_3 c_1 d_4 e_5 + a_5 b_2 c_1 d_4 e_3$$

$+ a_3 b_2 c_4 d_1 e_5 - a_5 b_2 c_3 d_1 e_4 + a_2 b_5 c_3 d_4 e_1$

$- a_3 b_2 c_5 d_4 e_1 - a_1 b_5 c_2 d_4 e_3 - a_1 b_3 c_4 d_2 e_5$

$+ a_1 b_5 c_3 d_2 e_4 + a_1 b_3 c_5 d_4 e_2 - a_5 b_4 c_1 d_2 e_3$

$+ a_3 b_4 c_5 d_2 e_1 + a_4 b_5 c_2 d_1 e_3 - a_4 b_3 c_5 d_1 e_2$

$- a_5 b_1 c_2 d_4 e_3 - a_3 b_1 c_4 d_2 e_5 + a_5 b_1 c_3 d_2 e_4$

$+ a_3 b_1 c_5 d_4 e_2 + a_4 b_3 c_1 d_2 e_5 + a_2 b_5 c_1 d_4 e_3$

$- a_3 b_4 c_2 d_1 e_5 + a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5$

$- a_5 b_2 c_4 d_1 e_3 - a_2 b_5 c_3 d_1 e_4 + a_3 b_2 c_5 d_1 e_4$

$- a_4 b_5 c_3 d_2 e_1 - a_2 b_3 c_5 d_4 e_1 + a_1 b_5 c_4 d_2 e_3$

$- a_1 b_3 c_5 d_2 e_4 + a_5 b_1 c_4 d_2 e_3 - a_3 b_1 c_5 d_2 e_4$

$- a_4 b_5 c_1 d_2 e_3 + a_5 b_4 c_2 d_1 e_3 - a_3 b_4 c_5 d_1 e_2$

$- a_2 b_5 c_4 d_1 e_3 + a_2 b_3 c_5 d_1 e_4 + a_4 b_3 c_5 d_2 e_1 .$

$M_{34} = - a_4 b_2 c_3 d_1 e_5 + a_1 b_4 c_3 d_2 e_5 - a_3 b_2 c_1 d_5 e_4$

$+ a_4 b_5 c_3 d_1 e_2 - a_4 b_2 c_5 d_1 e_3 - a_5 b_4 c_3 d_2 e_1$

$+ a_1 b_3 c_2 d_5 e_4 + a_1 b_4 c_5 d_2 e_3 + a_4 b_1 c_3 d_2 e_5$

$- a_2 b_4 c_3 d_1 e_5 + a_3 b_2 c_4 d_1 e_5 + a_4 b_2 c_3 d_5 e_1$

$- a_1 b_3 c_4 d_2 e_5 - a_1 b_4 c_3 d_5 e_2 + a_3 b_1 c_2 d_5 e_4$

$+ a_4 b_1 c_5 d_2 e_3 - a_2 b_3 c_1 d_5 e_4 - a_5 b_4 c_1 d_2 e_3$

$- a_2 b_4 c_5 d_1 e_3 - a_3 b_5 c_4 d_1 e_2 - a_5 b_3 c_2 d_1 e_4$

$+ a_4 b_5 c_2 d_1 e_3 + a_5 b_3 c_4 d_2 e_1 + a_3 b_5 c_1 d_2 e_4$

$- a_3 b_1 c_4 d_2 e_5 - a_4 b_1 c_3 d_5 e_2 + a_4 b_2 c_1 d_5 e_3$

$+ a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5 + a_3 b_2 c_5 d_1 e_4$

$- a_4 b_5 c_3 d_2 e_1 + a_2 b_4 c_3 d_5 e_1 - a_3 b_2 c_4 d_5 e_1$

$- a_1 b_4 c_2 d_5 e_3 - a_1 b_3 c_5 d_2 e_4 + a_1 b_3 c_4 d_5 e_2$

$- a_4 b_1 c_2 d_5 e_3 - a_3 b_1 c_5 d_2 e_4 + a_3 b_1 c_4 d_5 e_2$

$+ a_5 b_3 c_1 d_2 e_4 - a_4 b_5 c_1 d_2 e_3 + a_2 b_4 c_1 d_5 e_3$

$+ a_5 b_4 c_2 d_1 e_3 - a_5 b_3 c_4 d_1 e_2 - a_3 b_5 c_2 d_1 e_4$

$$+a_2 b_3 c_5 d_1 e_4 + a_3 b_5 c_4 d_2 e_1 - a_2 b_3 c_4 d_5 e_1 .$$

$$
\begin{aligned}
M_{44} = {} & a_1 b_2 c_3 d_4 e_5 + a_2 b_1 c_3 d_4 e_5 - a_3 b_2 c_1 d_4 e_5 \\
& - a_4 b_2 c_3 d_1 e_5 - a_1 b_5 c_3 d_4 e_2 - a_1 b_2 c_4 d_3 e_5 \\
& + a_1 b_2 c_5 d_4 e_3 - a_2 b_1 c_4 d_3 e_5 + a_2 b_1 c_5 d_4 e_3 \\
& + a_3 b_5 c_1 d_4 e_2 + a_4 b_5 c_3 d_1 e_2 - a_4 b_2 c_5 d_1 e_3 \\
& + a_5 b_3 c_2 d_4 e_1 + a_1 b_5 c_4 d_3 e_2 - a_5 b_1 c_3 d_4 e_2 \\
& - a_2 b_3 c_1 d_4 e_5 + a_4 b_2 c_1 d_3 e_5 - a_2 b_4 c_3 d_1 e_5 \\
& + a_3 b_2 c_4 d_1 e_5 - a_3 b_2 c_5 d_4 e_1 - a_1 b_5 c_2 d_4 e_3 \\
& - a_1 b_2 c_5 d_3 e_4 + a_5 b_1 c_4 d_3 e_2 - a_4 b_5 c_1 d_3 e_2 \\
& - a_2 b_4 c_5 d_1 e_3 - a_3 b_5 c_4 d_1 e_2 - a_5 b_3 c_2 d_1 e_4 \\
& - a_5 b_4 c_2 d_3 e_1 + a_4 b_5 c_2 d_1 e_3 - a_2 b_1 c_5 d_3 e_4 \\
& - a_5 b_1 c_2 d_4 e_3 + a_5 b_3 c_1 d_4 e_2 + a_2 b_4 c_1 d_3 e_5 \\
& + a_5 b_4 c_3 d_1 e_2 + a_2 b_3 c_4 d_1 e_5 + a_3 b_2 c_5 d_1 e_4 \\
& + a_3 b_5 c_2 d_4 e_1 - a_2 b_3 c_5 d_4 e_1 + a_4 b_2 c_5 d_3 e_1 \\
& + a_1 b_5 c_2 d_3 e_4 + a_5 b_1 c_2 d_3 e_4 - a_5 b_4 c_1 d_3 e_2 \\
& + a_5 b_4 c_2 d_1 e_3 - a_5 b_3 c_4 d_1 e_2 - a_3 b_5 c_2 d_1 e_4 \\
& + a_2 b_3 c_5 d_1 e_4 - a_4 b_5 c_2 d_3 e_1 + a_2 b_4 c_5 d_3 e_1 .
\end{aligned}
$$

$$
\begin{aligned}
M_{54} = {} & - a_1 b_2 c_4 d_3 e_5 + a_1 b_2 c_3 d_5 e_4 - a_2 b_1 c_4 d_3 e_5 \\
& + a_2 b_1 c_3 d_5 e_4 - a_3 b_2 c_1 d_5 e_4 - a_4 b_3 c_2 d_1 e_5 \\
& + a_4 b_5 c_3 d_1 e_2 + a_5 b_2 c_4 d_3 e_1 + a_3 b_2 c_4 d_1 e_5 \\
& - a_5 b_2 c_3 d_1 e_4 + a_1 b_4 c_2 d_3 e_5 - a_1 b_4 c_3 d_5 e_2 \\
& - a_1 b_2 c_5 d_3 e_4 + a_1 b_2 c_4 d_5 e_3 - a_2 b_3 c_1 d_5 e_4 \\
& - a_4 b_5 c_1 d_3 e_2 + a_2 b_5 c_4 d_3 e_1 + a_4 b_3 c_2 d_5 e_1 \\
& - a_5 b_4 c_2 d_3 e_1 + a_4 b_5 c_2 d_1 e_3 - a_4 b_3 c_5 d_1 e_2 \\
& + a_3 b_4 c_1 d_5 e_2 - a_2 b_1 c_5 d_3 e_4 + a_2 b_1 c_4 d_5 e_3
\end{aligned}
$$

$$
\begin{aligned}
&+a_4b_1c_2d_3e_5-a_4b_1c_3d_5e_2+a_5b_2c_1d_3e_4\\
&-a_3b_4c_2d_1e_5+a_5b_4c_3d_1e_2+a_2b_3c_4d_1e_5\\
&-a_5b_2c_4d_1e_3-a_2b_5c_3d_1e_4+a_3b_2c_5d_1e_4\\
&-a_3b_2c_4d_5e_1-a_1b_4c_2d_5e_3+a_1b_4c_5d_3e_2\\
&-a_4b_1c_2d_5e_3+a_4b_1c_5d_3e_2+a_4b_3c_1d_5e_2\\
&-a_5b_4c_1d_3e_2+a_2b_5c_1d_3e_4+a_5b_4c_2d_1e_3\\
&-a_3b_4c_5d_1e_2-a_2b_5c_4d_1e_3+a_2b_3c_5d_1e_4\\
&-a_4b_5c_2d_3e_1+a_3b_4c_2d_5e_1-a_2b_3c_4d_5e_1\,.
\end{aligned}
$$

$$
\begin{aligned}
M_{15}=&-a_5b_2c_3d_4e_1+a_1b_5c_3d_4e_2-a_3b_5c_1d_4e_2\\
&-a_4b_2c_5d_1e_3+a_5b_3c_2d_4e_1-a_5b_2c_4d_3e_1\\
&+a_1b_4c_5d_2e_3+a_1b_5c_4d_3e_2+a_5b_1c_3d_4e_2\\
&+a_5b_2c_1d_4e_3-a_2b_5c_3d_4e_1+a_4b_2c_3d_5e_1\\
&-a_1b_5c_2d_4e_3-a_1b_4c_3d_5e_2+a_4b_1c_5d_2e_3\\
&+a_5b_1c_4d_3e_2-a_2b_4c_5d_1e_3-a_3b_5c_4d_1e_2\\
&-a_2b_5c_4d_3e_1-a_3b_4c_5d_2e_1-a_4b_3c_2d_5e_1\\
&+a_5b_3c_4d_2e_1+a_4b_3c_5d_1e_2+a_3b_4c_1d_5e_2\\
&-a_5b_1c_2d_4e_3-a_4b_1c_3d_5e_2-a_5b_3c_1d_4e_2\\
&+a_2b_5c_1d_4e_3-a_4b_2c_1d_5e_3+a_5b_2c_4d_1e_3\\
&+a_3b_5c_2d_4e_1+a_4b_2c_5d_3e_1+a_2b_4c_3d_5e_1\\
&+a_1b_4c_2d_5e_3-a_1b_5c_4d_2e_3-a_1b_4c_5d_3e_2\\
&+a_4b_1c_2d_5e_3-a_5b_1c_4d_2e_3-a_4b_1c_5d_3e_2\\
&+a_4b_3c_1d_5e_2-a_2b_4c_1d_5e_3+a_3b_4c_5d_1e_2\\
&-a_5b_3c_4d_1e_2+a_2b_5c_4d_1e_3+a_3b_5c_4d_2e_1\\
&-a_4b_3c_5d_2e_1+a_2b_4c_5d_3e_1-a_3b_4c_2d_5e_1\,.
\end{aligned}
$$

$$
\begin{aligned}
M_{25} = & - a_5 b_2 c_3 d_4 e_1 + a_1 b_5 c_3 d_4 e_2 - a_3 b_2 c_1 d_5 e_4 \\
& - a_4 b_5 c_3 d_1 e_2 + a_5 b_4 c_3 d_2 e_1 - a_5 b_2 c_4 d_3 e_1 \\
& + a_1 b_3 c_2 d_5 e_4 + a_1 b_5 c_4 d_3 e_2 + a_5 b_1 c_3 d_4 e_2 \\
& + a_5 b_2 c_3 d_1 e_4 - a_2 b_5 c_3 d_4 e_1 + a_3 b_2 c_5 d_4 e_1 \\
& - a_1 b_5 c_3 d_2 e_4 - a_1 b_3 c_5 d_4 e_2 + a_3 b_1 c_2 d_5 e_4 \\
& + a_5 b_1 c_4 d_3 e_2 - a_2 b_3 c_1 d_5 e_4 - a_4 b_5 c_1 d_3 e_2 \\
& - a_2 b_5 c_4 d_3 e_1 - a_3 b_4 c_5 d_2 e_1 - a_4 b_3 c_2 d_5 e_1 \\
& + a_5 b_4 c_2 d_3 e_1 + a_4 b_3 c_5 d_1 e_2 + a_3 b_4 c_1 d_5 e_2 \\
& - a_5 b_1 c_3 d_2 e_4 - a_3 b_1 c_5 d_4 e_2 + a_5 b_2 c_1 d_3 e_4 \\
& - a_5 b_4 c_3 d_1 e_2 + a_2 b_5 c_3 d_1 e_4 - a_3 b_2 c_5 d_1 e_4 \\
& + a_4 b_5 c_3 d_2 e_1 + a_2 b_3 c_5 d_4 e_1 + a_3 b_2 c_4 d_5 e_1 \\
& - a_1 b_5 c_2 d_3 e_4 + a_1 b_3 c_5 d_2 e_4 - a_1 b_3 c_4 d_5 e_2 \\
& - a_5 b_1 c_2 d_3 e_4 + a_3 b_1 c_5 d_2 e_4 - a_3 b_1 c_4 d_5 e_2 \\
& + a_4 b_3 c_1 d_5 e_2 - a_5 b_4 c_1 d_3 e_2 + a_2 b_5 c_1 d_3 e_4 \\
& + a_3 b_4 c_5 d_1 e_2 - a_2 b_3 c_5 d_1 e_4 + a_4 b_5 c_2 d_3 e_1 \\
& - a_4 b_3 c_5 d_2 e_1 - a_3 b_4 c_2 d_5 e_1 + a_2 b_3 c_4 d_5 e_1 \;.
\end{aligned}
$$

$$
\begin{aligned}
M_{35} = & - a_3 b_2 c_1 d_4 e_5 + a_4 b_2 c_3 d_1 e_5 + a_1 b_3 c_2 d_4 e_5 \\
& - a_1 b_4 c_3 d_2 e_5 + a_3 b_5 c_1 d_4 e_2 - a_4 b_5 c_3 d_1 e_2 \\
& - a_5 b_3 c_2 d_4 e_1 + a_5 b_4 c_3 d_2 e_1 + a_3 b_1 c_2 d_4 e_5 \\
& - a_4 b_1 c_3 d_2 e_5 - a_2 b_3 c_1 d_4 e_5 + a_4 b_2 c_1 d_3 e_5 \\
& + a_2 b_4 c_3 d_1 e_5 - a_3 b_2 c_4 d_1 e_5 + a_3 b_2 c_5 d_4 e_1 \\
& - a_4 b_2 c_3 d_5 e_1 - a_1 b_4 c_2 d_3 e_5 + a_1 b_3 c_4 d_2 e_5 \\
& - a_1 b_3 c_5 d_4 e_2 + a_1 b_4 c_3 d_5 e_2 - a_4 b_5 c_1 d_3 e_2 \\
& + a_3 b_5 c_4 d_1 e_2 + a_5 b_4 c_2 d_3 e_1 - a_5 b_3 c_4 d_2 e_1 \\
& - a_4 b_1 c_2 d_3 e_5 + a_3 b_1 c_4 d_2 e_5 - a_3 b_1 c_5 d_4 e_2
\end{aligned}
$$

$$\begin{aligned}
&+ a_4 b_1 c_3 d_5 e_2 + a_5 b_3 c_1 d_4 e_2 + a_2 b_4 c_1 d_3 e_5\\
&- a_5 b_4 c_3 d_1 e_2 - a_2 b_3 c_4 d_1 e_5 - a_3 b_5 c_2 d_4 e_1\\
&+ a_4 b_5 c_3 d_2 e_1 + a_2 b_3 c_5 d_4 e_1 - a_4 b_2 c_5 d_3 e_1\\
&- a_2 b_4 c_3 d_5 e_1 + a_3 b_2 c_4 d_5 e_1 + a_1 b_4 c_5 d_3 e_2\\
&- a_1 b_3 c_4 d_5 e_2 + a_4 b_1 c_5 d_3 e_2 - a_3 b_1 c_4 d_5 e_2\\
&- a_5 b_4 c_1 d_3 e_2 + a_5 b_3 c_4 d_1 e_2 + a_4 b_5 c_2 d_3 e_1\\
&- a_3 b_5 c_4 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 + a_2 b_3 c_4 d_5 e_1 .
\end{aligned}$$

$$\begin{aligned}
M_{45} = &- a_1 b_2 c_5 d_4 e_3 + a_1 b_2 c_3 d_5 e_4 - a_2 b_1 c_5 d_4 e_3\\
&+ a_2 b_1 c_3 d_5 e_4 - a_3 b_2 c_1 d_5 e_4 + a_4 b_2 c_5 d_1 e_3\\
&- a_5 b_3 c_2 d_4 e_1 + a_5 b_4 c_3 d_2 e_1 + a_3 b_2 c_5 d_4 e_1\\
&- a_4 b_2 c_3 d_5 e_1 + a_1 b_5 c_2 d_4 e_3 - a_1 b_5 c_3 d_2 e_4\\
&+ a_1 b_2 c_5 d_3 e_4 - a_1 b_2 c_4 d_5 e_3 - a_2 b_3 c_1 d_5 e_4\\
&- a_5 b_4 c_1 d_2 e_3 + a_2 b_4 c_5 d_1 e_3 + a_5 b_3 c_2 d_1 e_4\\
&+ a_5 b_4 c_2 d_3 e_1 - a_4 b_5 c_2 d_1 e_3 - a_5 b_3 c_4 d_2 e_1\\
&+ a_3 b_5 c_1 d_2 e_4 + a_2 b_1 c_5 d_3 e_4 - a_2 b_1 c_4 d_5 e_3\\
&+ a_5 b_1 c_2 d_4 e_3 - a_5 b_1 c_3 d_2 e_4 + a_4 b_2 c_1 d_5 e_3\\
&- a_3 b_2 c_5 d_1 e_4 - a_3 b_5 c_2 d_4 e_1 + a_4 b_5 c_3 d_2 e_1\\
&+ a_2 b_3 c_5 d_4 e_1 - a_4 b_2 c_5 d_3 e_1 - a_2 b_4 c_3 d_5 e_1\\
&+ a_3 b_2 c_4 d_5 e_1 - a_1 b_5 c_2 d_3 e_4 + a_1 b_5 c_4 d_2 e_3\\
&- a_5 b_1 c_2 d_3 e_4 + a_5 b_1 c_4 d_2 e_3 + a_5 b_3 c_1 d_2 e_4\\
&- a_4 b_5 c_1 d_2 e_3 + a_2 b_4 c_1 d_5 e_3 - a_5 b_4 c_2 d_1 e_3\\
&+ a_3 b_5 c_2 d_1 e_4 - a_2 b_3 c_5 d_1 e_4 + a_4 b_5 c_2 d_3 e_1\\
&- a_3 b_5 c_4 d_2 e_1 - a_2 b_4 c_5 d_3 e_1 + a_2 b_3 c_4 d_5 e_1 .
\end{aligned}$$

$$\begin{aligned}
M_{55} = &\ a_1 b_2 c_3 d_4 e_5 + a_2 b_1 c_3 d_4 e_5 - a_3 b_2 c_1 d_4 e_5\\
&- a_5 b_2 c_3 d_4 e_1 - a_1 b_4 c_3 d_2 e_5 + a_1 b_2 c_4 d_3 e_5
\end{aligned}$$

$$\begin{aligned}
&- a_1 b_2 c_5 d_4 e_3 + a_2 b_1 c_4 d_3 e_5 - a_2 b_1 c_5 d_4 e_3 \\
&+ a_3 b_4 c_1 d_2 e_5 + a_4 b_3 c_2 d_1 e_5 + a_5 b_4 c_3 d_2 e_1 \\
&- a_5 b_2 c_4 d_3 e_1 + a_1 b_4 c_5 d_2 e_3 - a_4 b_1 c_3 d_2 e_5 \\
&- a_2 b_3 c_1 d_4 e_5 + a_5 b_2 c_1 d_4 e_3 - a_3 b_2 c_4 d_1 e_5 \\
&- a_2 b_5 c_3 d_4 e_1 + a_3 b_2 c_5 d_4 e_1 - a_1 b_4 c_2 d_3 e_5 \\
&- a_1 b_2 c_4 d_5 e_3 + a_4 b_1 c_5 d_2 e_3 - a_5 b_4 c_1 d_2 e_3 \\
&- a_2 b_5 c_4 d_3 e_1 - a_3 b_4 c_5 d_2 e_1 - a_4 b_3 c_2 d_5 e_1 \\
&+ a_5 b_4 c_2 d_3 e_1 - a_4 b_5 c_2 d_1 e_3 - a_2 b_1 c_4 d_5 e_3 \\
&- a_4 b_1 c_2 d_3 e_5 + a_4 b_3 c_1 d_2 e_5 + a_2 b_5 c_1 d_4 e_3 \\
&+ a_3 b_4 c_2 d_1 e_5 - a_2 b_3 c_4 d_1 e_5 + a_5 b_2 c_4 d_1 e_3 \\
&+ a_4 b_5 c_3 d_2 e_1 + a_2 b_3 c_5 d_4 e_1 + a_3 b_2 c_4 d_5 e_1 \\
&+ a_1 b_4 c_2 d_5 e_3 + a_4 b_1 c_2 d_5 e_3 - a_4 b_5 c_1 d_2 e_3 \\
&- a_5 b_4 c_2 d_1 e_3 + a_2 b_5 c_4 d_1 e_3 + a_4 b_5 c_2 d_3 e_1 \\
&- a_4 b_3 c_5 d_2 e_1 - a_3 b_4 c_2 d_5 e_1 + a_2 b_3 c_4 d_5 e_1 .
\end{aligned}$$

Picking up the matrix coefficients of the terms $a_1 b_2 c_3 d_4 e_5$, $a_2 b_1 c_3 d_4 e_5$, ..., $a_2 b_3 c_4 d_5 e_1$, we get the representation corresponding to the partition $(2^2, 1)$; the conjugate representation is that corresponding to the partition (3, 2).

Representation corresponding to partition (5) :

The unit representation 1, 1, 1, ..., 1.

Representation corresponding to partition (4, 1) :

1

$$\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

2

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

3

$$\begin{bmatrix} 1 & . & . & . \\ -1 & -1 & -1 & -1 \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

4

$$\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ -1 & -1 & -1 & -1 \\ . & . & . & 1 \end{bmatrix}$$

5

$$\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

6

$$\begin{bmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

7

$$\begin{bmatrix} . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \end{bmatrix}$$

8

$$\begin{bmatrix} . & . & . & 1 \\ . & 1 & . & . \\ . & . & 1 & . \\ 1 & . & . & . \end{bmatrix}$$

9

$$\begin{bmatrix} 1 & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & 1 \end{bmatrix}$$

10

$$\begin{bmatrix} 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \end{bmatrix}$$

11

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

12

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

13

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

14

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

15

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

16

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

17

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

18

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

19

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

20

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

21

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

22

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

23

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$$\overset{24}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}} \overset{25}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{26}{\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}}$$

$$\overset{27}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{28}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{29}{\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{30}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{31}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{32}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{33}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{34}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}} \overset{35}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{36}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} \overset{37}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{38}{\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}}$$

$$\begin{array}{ccc} 39 & 40 & 41 \\ \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} & \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 42 & 43 & 44 \\ \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{cc} 45 & 46 \\ \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 47 & 48 & 49 \\ \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 50 & 51 & 52 \\ \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

$$\overset{53}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{54}{\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} \overset{55}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{56}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} \overset{57}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{58}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}}$$

$$\overset{59}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}} \overset{60}{\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} \overset{61}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}}$$

$$\overset{62}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} \overset{63}{\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{64}{\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}}$$

$$\overset{65}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}} \overset{66}{\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}}$$

67

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

68

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

69

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

70

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

71

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

72

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

73

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

74

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

75

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

76

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

77

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

78

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

79

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

80

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

81

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$\begin{array}{ccc} 82 & 83 & 84 \\ \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 85 & 86 & 87 \\ \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 88 & 89 & 90 \\ \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} & \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 91 & 92 & 93 \\ \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} 94 & 95 & 96 \\ \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \end{array}$$

97

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

98

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

99

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

100

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

101

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

102

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

103

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

104

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

105

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

106

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

107

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

108

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

109

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

110

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

111

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

112

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

113

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

114

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

115

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

116

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

117

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

118

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

119

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

120

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} .$$

Representation corresponding to partition (3, 2).

1

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

2

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

3

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

4

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

5

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & -1 & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

6

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

7

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

8

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & -1 & 1 & 1 & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

9

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

10

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

11

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

12

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

13

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

14

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

15

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

16

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

17

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

18

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

19

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

20

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

21

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

22

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

23

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

24

$$\begin{bmatrix} 1 & . & . & . & . \\ . & . & . & . & 1 \\ . & . & . & 1 & . \\ . & . & 1 & . & . \\ . & 1 & . & . & . \end{bmatrix}$$

25

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ . & 1 & . & . & . \\ . & . & . & 1 & . \\ . & . & 1 & . & . \\ . & . & . & . & 1 \end{bmatrix}$$

26

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ . & . & . & . & 1 \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & 1 & . & . & . \end{bmatrix}$$

27

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & 1 & . \\ -1 & . & -1 & . & 1 \\ . & -1 & . & . & . \\ . & . & -1 & . & . \end{bmatrix}$$

28

$$\begin{bmatrix} -1 & 1 & . & -1 & . \\ . & 1 & . & . & . \\ 1 & . & . & 1 & -1 \\ 1 & . & . & . & . \\ 1 & . & 1 & . & -1 \end{bmatrix}$$

29

$$\begin{bmatrix} -1 & . & -1 & . & 1 \\ -1 & . & . & -1 & 1 \\ . & . & 1 & . & . \\ -1 & 1 & . & -1 & . \\ -1 & . & . & . & . \end{bmatrix}$$

30

$$\begin{bmatrix} 1 & . & . & . & . \\ . & . & . & -1 & . \\ . & . & . & . & -1 \\ -1 & 1 & . & -1 & . \\ 1 & . & 1 & . & -1 \end{bmatrix}$$

31

$$\begin{bmatrix} -1 & 1 & . & -1 & . \\ -1 & . & . & . & . \\ -1 & . & -1 & . & 1 \\ . & . & . & 1 & . \\ -1 & 1 & -1 & . & . \end{bmatrix}$$

32

$$\begin{bmatrix} -1 & . & -1 & . & 1 \\ 1 & -1 & . & 1 & . \\ 1 & . & . & . & . \\ 1 & -1 & 1 & . & . \\ . & . & . & . & 1 \end{bmatrix}$$

33

$$\begin{bmatrix} . & . & . & 1 & . \\ . & 1 & . & . & . \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & . & -1 & . \\ -1 & 1 & -1 & . & . \end{bmatrix}$$

34

$$\begin{bmatrix} . & -1 & . & . & . \\ 1 & -1 & . & 1 & . \\ 1 & . & . & 1 & -1 \\ . & . & . & 1 & . \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

35

$$\begin{bmatrix} -1 & 1 & . & -1 & . \\ -1 & . & . & -1 & 1 \\ . & -1 & . & . & . \\ 1 & -1 & 1 & . & . \\ . & . & . & -1 & . \end{bmatrix}$$

36

$$\begin{bmatrix} . & . & . & . & -1 \\ 1 & -1 & 1 & 1 & -1 \\ . & . & 1 & . & . \\ 1 & -1 & 1 & . & . \\ 1 & . & 1 & . & -1 \end{bmatrix}$$

37

$$\begin{bmatrix} . & . & 1 & . & . \\ -1 & . & . & -1 & 1 \\ -1 & . & -1 & . & 1 \\ -1 & 1 & -1 & -1 & 1 \\ . & . & . & . & 1 \end{bmatrix}$$

38

$$\begin{bmatrix} -1 & . & -1 & . & 1 \\ . & . & -1 & . & . \\ 1 & . & . & 1 & -1 \\ . & . & . & . & -1 \\ -1 & 1 & -1 & . & . \end{bmatrix}$$

39

$$\begin{bmatrix} . & . & . & 1 & . \\ -1 & . & . & . & . \\ . & . & . & . & -1 \\ . & -1 & . & . & . \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

40

$$\begin{bmatrix} . & . & . & . & -1 \\ . & . & . & -1 & . \\ 1 & . & . & . & . \\ -1 & 1 & -1 & -1 & 1 \\ . & . & -1 & . & . \end{bmatrix}$$

41

$$\begin{bmatrix} . & -1 & . & . & . \\ . & . & . & -1 & . \\ -1 & 1 & -1 & -1 & 1 \\ 1 & . & . & . & . \\ . & . & -1 & . & . \end{bmatrix}$$

42

$$\begin{bmatrix} . & . & . & 1 & . \\ 1 & -1 & 1 & 1 & -1 \\ . & -1 & . & . & . \\ . & . & . & . & -1 \\ -1 & . & . & . & . \end{bmatrix}$$

43

$$\begin{bmatrix} . & . & 1 & . & . \\ 1 & -1 & 1 & 1 & -1 \\ . & . & . & . & -1 \\ . & -1 & . & . & . \\ -1 & . & . & . & . \end{bmatrix}$$

44

$$\begin{bmatrix} . & . & . & . & -1 \\ . & . & -1 & . & . \\ -1 & 1 & -1 & -1 & 1 \\ 1 & . & . & . & . \\ . & . & . & -1 & . \end{bmatrix}$$

45

$$\begin{bmatrix} . & -1 & . & . & . \\ . & . & -1 & . & . \\ 1 & . & . & . & . \\ -1 & 1 & -1 & -1 & 1 \\ . & . & . & -1 & . \end{bmatrix}$$

46

$$\begin{bmatrix} . & . & 1 & . & . \\ -1 & . & . & . & . \\ . & -1 & . & . & . \\ . & . & . & . & -1 \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

47

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

48

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

49

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

50

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

51

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

52

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

53

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

54

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

55

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

56

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

57

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & \cdot & -1 & 1 \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

58

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

59

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

60

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

61

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

62

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

63

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

64

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & 1 & \cdot & -1 \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

65

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

66

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

67

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

68

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ 1 & -1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

69

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

70

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

71

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

72

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

73

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

74

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

75

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

76

$$\begin{bmatrix} -1 & 1 & \cdot & -1 & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

77

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

78

$$\begin{bmatrix} -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

79

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

80

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

81

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

82

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

83

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

84

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

85

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

86

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

87

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

88

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

89

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

90

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

91

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

92

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

93

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

94

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

95

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ . & . & -1 & . & . \\ . & . & . & . & -1 \\ . & -1 & . & . & . \\ . & . & . & -1 & . \end{bmatrix}$$

96

$$\begin{bmatrix} . & . & 1 & . & . \\ . & . & . & . & 1 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & . & . & . & . \\ . & 1 & . & . & . \end{bmatrix}$$

97

$$\begin{bmatrix} -1 & 1 & . & -1 & . \\ 1 & . & 1 & . & -1 \\ 1 & . & . & . & . \\ 1 & . & . & 1 & -1 \\ . & 1 & . & . & . \end{bmatrix}$$

98

$$\begin{bmatrix} -1 & . & -1 & . & 1 \\ -1 & . & . & . & . \\ -1 & 1 & . & -1 & . \\ . & . & 1 & . & . \\ -1 & . & . & -1 & 1 \end{bmatrix}$$

99

$$\begin{bmatrix} 1 & . & . & 1 & -1 \\ 1 & -1 & . & 1 & . \\ . & -1 & . & . & . \\ -1 & . & -1 & . & 1 \\ . & . & -1 & . & . \end{bmatrix}$$

100

$$\begin{bmatrix} . & -1 & . & . & . \\ -1 & . & . & -1 & 1 \\ -1 & 1 & . & -1 & . \\ . & . & 1 & . & . \\ -1 & . & . & . & . \end{bmatrix}$$

101

$$\begin{bmatrix} 1 & . & . & 1 & -1 \\ . & . & -1 & . & . \\ -1 & . & -1 & . & 1 \\ . & -1 & . & . & . \\ 1 & -1 & . & 1 & . \end{bmatrix}$$

102

$$\begin{bmatrix} . & . & 1 & . & . \\ 1 & . & 1 & . & -1 \\ 1 & . & . & 1 & -1 \\ 1 & . & . & . & . \\ . & 1 & . & . & . \end{bmatrix}$$

103

$$\begin{bmatrix} -1 & 1 & . & -1 & . \\ -1 & 1 & -1 & . & . \\ . & . & . & 1 & . \\ -1 & . & -1 & . & 1 \\ -1 & . & . & . & . \end{bmatrix}$$

104

$$\begin{bmatrix} -1 & . & -1 & . & 1 \\ . & . & . & . & 1 \\ 1 & -1 & 1 & . & . \\ 1 & . & . & . & . \\ 1 & -1 & . & 1 & . \end{bmatrix}$$

105

$$\begin{bmatrix} 1 & -1 & 1 & . & . \\ 1 & . & 1 & . & -1 \\ . & . & . & . & -1 \\ -1 & 1 & . & -1 & . \\ . & . & . & -1 & . \end{bmatrix}$$

106

$$\begin{bmatrix} . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & . & . & . & . \\ 1 & -1 & 1 & . & . \\ 1 & -1 & . & 1 & . \end{bmatrix}$$

107

$$\begin{bmatrix} 1 & -1 & 1 & . & . \\ . & . & . & -1 & . \\ -1 & 1 & . & -1 & . \\ . & . & . & . & -1 \\ 1 & . & 1 & . & -1 \end{bmatrix}$$

108

$$\begin{bmatrix} . & . & . & . & -1 \\ -1 & . & . & . & . \\ . & . & . & 1 & . \\ -1 & . & -1 & . & 1 \\ -1 & 1 & -1 & . & . \end{bmatrix}$$

109

$$\begin{bmatrix} 1 & -1 & 1 & . & . \\ 1 & -1 & . & 1 & . \\ . & . & . & 1 & . \\ 1 & . & . & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

110

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ -1 & . & . & -1 & 1 \\ 1 & -1 & 1 & . & . \\ . & -1 & . & . & . \\ . & . & . & -1 & . \end{bmatrix}$$

111

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

112

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

113

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & 1 & -1 & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot \\ -1 & 1 & -1 & -1 & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

114

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & \cdot & 1 & \cdot \end{bmatrix}$$

115

$$\begin{bmatrix} 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

116

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

117

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

118

$$\begin{bmatrix} -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & 1 & -1 \\ -1 & 1 & -1 & \cdot & \cdot \end{bmatrix}$$

$$\overset{119}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & 1 & \cdot & -1 \\ 1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix}} \quad \overset{120}{\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & -1 & 1 \end{bmatrix}}$$

Representation corresponding to partition $(3, 1^2)$:

1

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

2

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

3

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

4

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \end{bmatrix}$$

5

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

6

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

7

$$\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

8

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

9

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

10

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

11

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

12

$$\begin{bmatrix} \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

13

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

14

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

15

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

16

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

17

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

18

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \end{bmatrix}$$

19

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

20

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

21

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

22

$$\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

23

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

24

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

25

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

26

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

27

$$\begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

28

$$\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \end{bmatrix}$$

29

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

30

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

31

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \end{bmatrix}$$

32

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \end{bmatrix}$$

33

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

34

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

35

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \end{bmatrix}$$

36

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

37

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

38

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \end{bmatrix}$$

39

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

40

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

41

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

42

$$\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

43

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

44

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

45

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

46

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

47

$$\begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

48

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \mathbf{1} \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

49

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

50

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

51

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

52

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

53

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

54

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

55

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \end{bmatrix}$$

56

$$\begin{bmatrix} \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

57

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

58

$$\begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \end{bmatrix}$$

59

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

60

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

61

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

62

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

63

$$\begin{bmatrix} . & . & -1 & . & . & . \\ . & . & 1 & . & 1 & 1 \\ . & . & . & . & -1 & . \\ -1 & -1 & -1 & . & . & . \\ 1 & . & . & . & . & . \\ -1 & . & . & 1 & 1 & . \end{bmatrix}$$

64

$$\begin{bmatrix} . & . & 1 & . & 1 & 1 \\ . & . & -1 & . & . & . \\ . & . & . & . & . & -1 \\ 1 & 1 & 1 & . & . & . \\ . & -1 & . & -1 & . & 1 \\ . & 1 & . & . & . & . \end{bmatrix}$$

65

$$\begin{bmatrix} . & -1 & . & -1 & . & 1 \\ . & . & -1 & . & -1 & -1 \\ -1 & . & . & 1 & 1 & . \\ . & . & . & . & . & 1 \\ . & . & . & -1 & . & . \\ . & . & . & . & -1 & . \end{bmatrix}$$

66

$$\begin{bmatrix} . & . & -1 & . & -1 & -1 \\ -1 & . & . & 1 & 1 & . \\ . & -1 & . & -1 & . & 1 \\ . & . & . & . & -1 & . \\ . & . & . & . & . & -1 \\ . & . & . & 1 & . & . \end{bmatrix}$$

67

$$\begin{bmatrix} . & . & . & 1 & . & . \\ 1 & . & . & -1 & -1 & . \\ . & . & . & . & 1 & . \\ . & 1 & . & 1 & . & -1 \\ . & . & . & . & . & 1 \\ . & . & -1 & . & -1 & -1 \end{bmatrix}$$

68

$$\begin{bmatrix} . & . & . & . & 1 & . \\ . & . & . & 1 & . & . \\ 1 & . & . & -1 & -1 & . \\ . & . & . & . & . & -1 \\ . & . & 1 & . & 1 & 1 \\ . & 1 & . & 1 & . & -1 \end{bmatrix}$$

69

$$\begin{bmatrix} . & 1 & . & 1 & . & -1 \\ . & . & . & -1 & . & . \\ . & . & . & . & . & 1 \\ -1 & . & . & 1 & 1 & . \\ . & . & -1 & . & -1 & -1 \\ . & . & . & . & 1 & . \end{bmatrix}$$

70

$$\begin{bmatrix} . & . & . & -1 & . & . \\ . & . & . & . & . & 1 \\ . & 1 & . & 1 & . & -1 \\ . & . & . & . & 1 & . \\ 1 & . & . & -1 & -1 & . \\ . & . & 1 & . & 1 & 1 \end{bmatrix}$$

71

$$\begin{bmatrix} . & . & 1 & . & 1 & 1 \\ . & . & . & . & . & -1 \\ . & . & . & . & -1 & . \\ . & -1 & . & -1 & . & 1 \\ -1 & . & . & 1 & 1 & . \\ . & . & . & -1 & . & . \end{bmatrix}$$

72

$$\begin{bmatrix} . & . & . & . & -1 & . \\ . & . & 1 & . & 1 & 1 \\ . & . & . & . & . & -1 \\ 1 & . & . & -1 & -1 & . \\ . & . & . & 1 & . & . \\ . & -1 & . & -1 & . & 1 \end{bmatrix}$$

73

$$\begin{bmatrix} . & -1 & . & . & . & . \\ . & 1 & . & 1 & . & -1 \\ . & . & . & . & . & 1 \\ -1 & -1 & -1 & . & . & . \\ . & . & 1 & . & . & . \\ . & . & -1 & . & -1 & -1 \end{bmatrix}$$

74

$$\begin{bmatrix} . & . & -1 & . & . & . \\ . & . & . & . & . & -1 \\ . & . & 1 & . & 1 & 1 \\ . & 1 & . & . & . & . \\ -1 & -1 & -1 & . & . & . \\ . & 1 & . & 1 & . & -1 \end{bmatrix}$$

75

$$\begin{bmatrix} . & -1 & . & -1 & . & 1 \\ 1 & 1 & 1 & . & . & . \\ . & . & -1 & . & -1 & -1 \\ . & -1 & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & 1 & . & . & . \end{bmatrix}$$

76

$$\begin{bmatrix} . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ -1 & -1 & -1 & . & . & . \\ . & . & . & . & . & 1 \\ . & 1 & . & 1 & . & -1 \\ . & . & 1 & . & 1 & 1 \end{bmatrix}$$

77

$$\begin{bmatrix} . & . & -1 & . & -1 & -1 \\ . & -1 & . & -1 & . & 1 \\ 1 & 1 & 1 & . & . & . \\ . & . & . & . & . & -1 \\ . & . & -1 & . & . & . \\ . & -1 & . & . & . & . \end{bmatrix}$$

78

$$\begin{bmatrix} . & . & 1 & . & . & . \\ -1 & -1 & -1 & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & 1 & 1 \\ . & . & . & . & . & -1 \\ . & -1 & . & -1 & . & 1 \end{bmatrix}$$

79

$$\begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

80

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

81

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

82

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

83

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

84

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

85

$$\begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

86

$$\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

87

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

88

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

89

$$\begin{bmatrix} -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

90

$$\begin{bmatrix} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

91

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

92

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

93

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

94

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

95

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

96

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

97

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \end{bmatrix}$$

98

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \end{bmatrix}$$

99

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

100

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

101

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

102

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

103

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \end{bmatrix}$$

104

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \end{bmatrix}$$

105

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

106

$$\begin{bmatrix} \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

107

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

108

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

109

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

110

$$\begin{bmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

111

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \end{bmatrix}$$

112

$$\begin{bmatrix} \cdot & \cdot & -1 & \cdot & -1 & -1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

113

$$\begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{bmatrix}$$

114

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & -1 \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

115

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

116

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

117

$$\begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

118

$$\begin{bmatrix} \cdot & -1 & \cdot & -1 & \cdot & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{array}{cc} 119 & 120 \\ \begin{bmatrix} 1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & -1 & \cdot & -1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix} \end{array}$$

Representation corresponding to partition $(2^2, 1)$:

In the representation corresponding to the partition (3, 2), the matrices representing the negative classes, namely the matrices, 2-11, 47-66, 67-96 are multiplied by -1.

Representation corresponding to partition $(2, 1^3)$:

In the representation corresponding to the partition (4, 1), the matrices 2-11, 47-96 are multiplied by -1.

Representation corresponding to partition (1^5) :

The alternating scalar representation.

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