Symmetric Products and Quaternion Cycle Spaces

Jacob Mostovoy

A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy
to the
University of Edinburgh
1997

Abstract

Symmetric products play an important role both in algebraic topology and algebraic geometry. In topology they provide models for Eilenberg-MacLane spaces; in the context of algebraic geometry they are the basic examples of spaces of algebraic cycles. As shown by the works of H. B. Lawson, P. Lima-Filho and E. Friedlander the interplay between these two incarnations of symmetric products leads to a homotopy-theoretic definition of both classical and new algebraic invariants of algebraic varieties.

The objects of study in this thesis are symmetric products and spaces of algebraic cycles. The first new result concerns symmetric products and it describes the geometry of truncated symmetric products (or, in other terminology, symmetric products modulo 2). We prove that if M is a closed compact connected manifold, a necessary and sufficient condition for its symmetric products modulo 2 to be manifolds is that M is a circle. We also show that the symmetric products of the circle modulo 2 are homeomorphic to real projective spaces and give an interpretation of this homeomorphism as a real topological analogue of Vieta's theorem.

The second result concerns the spaces of real algebraic cycles, first studied by T. K. Lam. We describe a method of calculating the homotopy groups of the spaces of real cycles with integral coefficients on projective spaces; we give an explicit formula for groups which lie in the "stable range".

The third result (or, rather, a group of results) is the construction of a quaternionic analogue of Lawson's theory of algebraic cycles. We define quaternionic cycles as those, which are invariant with respect to a free involution on $\mathbf{CP}^{2n+1}$, induced by the action of the quaternion j on $\mathbf{H}^n$. Basic properties of quaternionic algebraic cycles are studied; a rational "quaternionic suspension theorem" is proved and the spaces of quaternionic cycles with rational coefficients on $\mathbf{CP}^{2n+1}$ are described. We also present a method of calculating the Betti numbers of the spaces of quaternionic cycles of degree 2 and odd codimension on $\mathbf{CP}^{\infty}$.

Some other results that are included in the thesis are a twisted version of the Dold-Thom theorem and an interpretation of the Kuiper-Massey theorem via symmetric products.

After the main results on quaternionic cycles were proved, the author learned that similar results were obtained by Lawson, Lima-Filho and Michelsohn. Their version of the quaternionic suspension theorem is stronger and requires more sophisticated machinery for the proof.

Acknowledgements

First of all I would like to express my deep gratitude to Elmer Rees, whose advice and help have been invaluable. I would also like to thank Viktor Buchstaber and Ben Mann who provided much of the inspiration for my work; Andrew Ranicki, Michael Singer and other members of the department who patiently answered my questions; H. Blaine Lawson and Paulo Lima-Filho who clarified many things for me; Simon Willerton with whom I discussed almost everything in the world; Paul Moseley, Adam Hughes, Robert Pefferly and other postgraduate students for being wonderful company; and, finally, Mr. D. Usanov and my other friends for things too numerous to mention.

Dedicated to the memory of Gen. Sarmiento -
the brave mouse who lived behind the fridge.

Contents

Introduction

It is hard to say who first introduced the notion of a symmetric product of a topological space. Inequivalent definitions can be found in the literature of the first half of this century; one of them successfully survived. Nowadays the n-th symmetric product $SP^n(X)$ of a space X is defined as a quotient of the n-fold Cartesian product of X by the action of the symmetric group on n letters. It can be loosely described as a configuration space of n indistinguishable particles on X, which are allowed to collide. For spaces with basepoints one can also define infinite symmetric products [1].

This notion is very natural; symmetric products occur, for example, as spaces of divisors on Riemann surfaces. The importance of symmetric products in topology was revealed when Jean-Pierre Serre conjectured that the infinite symmetric product of an n-sphere has the homotopy type of the Eilenberg-MacLane space $K(\mathbf{Z}, n)$. This, for example, meant that rather clumsy and artificial (but very useful) monsters - Eilenberg-MacLane spaces - had nice and natural geometric models. Albrecht Dold, René Thom and Ioan James set out to prove the Serre's conjecture; Dold and Thom came up with a proof first (see the article of I. James [**27**]). In [**17**] they announced their celebrated theorem

For a connected locally finite CW-complex X

$$\pi_m(SP^\infty(X)) \simeq \widetilde{H}_m(X, \mathbf{Z}) \text{ for any } m \geq 0.$$

This theorem made symmetric products into a popular object of study. Homology groups of finite symmetric products were studied. The first strong result about them was obtained by Norman Steenrod; the final blow - the complete description - was dealt by R. James Milgram [**44**]. The concept of "symmetric products with coefficients in a group" was introduced already in [**16**]; M. C. McCord in [**43**] generalised this idea to

[1] A different definition was given by Borsuk and Ulam in [**7**]. The symmetric products and potencies of Borsuk and Ulam were first studied in [**6**],[**8**],[**24**],[**45**].

a wide class of groups and exhibited a connection between Eilenberg-MacLane spaces and classifying spaces of groups.

An important generalisation was suggested by Herbert Federer. Symmetric products of a manifold X can be considered as spaces of polyhedral 0-chains; what if we topologise the spaces of polyhedral cycles of higher dimensions? This question was answered by Frederick Justin Almgren in **[2]**; it turned out that the results of Dold and Thom extend quite naturally to this situation:

Let $C_i(X)$ denote the space of closed integral currents of dimension i on a manifold X. Then

$$\pi_m(C_i(X)) = \widetilde{H}_{m+i}(X, \mathbf{Z}) \quad \text{for any } i, m \geq 0.$$

When $i = 0$ Almgren's theorem specialises to the Dold-Thom theorem.

About a quarter of a century after Almgren's work H. Blaine Lawson realised that one can take the same approach to the algebraic cycles on a projective variety; namely, study the homotopy groups of Chow varieties with the Dold-Thom Theorem in mind. In his paper **[31]** he proved the Complex Suspension Theorem:

The stabilised space of algebraic cycles of dimension p on a complex projective variety X is homotopy equivalent to the the stabilised space of the algebraic cycles of dimension $p+1$ on the projective cone of X.

An immediate corollary of this theorem is that the space of algebraic p-cycles on $\mathbf{CP}^n$ is homotopy equivalent to the infinite symmetric product $SP^\infty(\mathbf{CP}^{n-p})$. This gives a complete topological description of the stabilised spaces of algebraic cycles on $\mathbf{CP}^n$.

In **[38]** Paulo Lima-Filho related Lawson's results to Almgren's theorem. The Complex Suspension Theorem turned out to be the algebraic analogue of the Thom Isomorphism Theorem (which can be realised geometrically as a weak homotopy equivalence between spaces of closed integral currents).

Lawson's paper **[31]** was the first in a sequence of works by Lawson, Marie-Louise Michelsohn, Paulo Lima-Filho, Eric Friedlander and others. New invariants of algebraic varieties - "Lawson homology groups" were defined, their properties were investigated. The subject is still being intensively developed. Of particular interest for us here are the results of Tsz Kin Lam **[30]**, who proved a real analogue of Lawson's suspension theorem.

The main motivation behind the work presented in our thesis is a desire to understand how the relations between reals, complex numbers and quaternions are reflected in the geometric and topological structure of symmetric products and cycle spaces.

It is a famous classical fact that Vieta's theorem, which essentially claims that the correspondence between monic complex polynomials in one variable and their root systems is one-to-one, can be expressed as a geometric statement about symmetric products:

$$SP^n(\mathbf{CP}^1) = \mathbf{CP}^n.$$

(Strictly speaking, the statement above describes the correspondence between *homogeneous* polynomials and their roots.) A quaternionic analogue of this theorem is well-known too:

$$SP^n(\mathbf{RP}^2) = \mathbf{RP}^{2n}.$$

If it is not immediately clear why this statement has something to do with quaternions, notice that the antipodal map on $\mathbf{CP}^1$, which is used to define $\mathbf{RP}^2$, can be represented as the action of the quaternion j on the complex projectivisation of $\mathbf{H}$.

The real analogue of these geometric theorems turns out to be quite nontrivial. If we want to characterise real polynomials by the systems of their real roots, the correspondence we obtain, blindly following the examples above, is neither continuous nor one-to-one. The solution is to use the space of root systems, where the multiplicities of the roots are taken to be not integers, but elements of $\mathbf{Z}/2$. Geometrically this corresponds to using so-called symmetric products mod 2. We obtain a continuous map

$$\mathbf{RP}^n \to SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$$

from the space of homogeneous real polynomials modulo multiplication by a constant to the space of their root systems, reduced modulo 2. This map is still not one-to-one, but, as it turns out, it is homotopic to a homeomorphism. We also prove that this is the only case when a symmetric product modulo 2 of a manifold is again a manifold. (These results are to appear separately in [**47**].)

This is the main result of Chapter 1, which essentially is a collection of various examples of symmetric products. Not all of these examples are vital for the next chapters, but they seem curious enough to deserve a description. Some of them are new, such as "twisted" symmetric products for which we obtain a Dold-Thom-type theorem. Symmetric products with rational coefficients, which give nice models for rational Eilenberg-MacLane spaces, also appear to be a new object, though the construction we use is probably older than the author of this thesis.

Most of the Chapter 2 is a brief account of the works of Lawson, Michelsohn and Lam on complex, real and group-invariant algebraic cycles. The new result here is the calculation of the homotopy groups of the spaces of real cycles with coefficients in $\mathbf{Z}$. To avoid the boredom of lengthy computations we placed some of them in the Appendix.

(In fact, the technical results in the Appendix might be of some independent interest: the object of study there, which is the quotient of $\mathbf{CP}^n$ by complex conjugation, is very well-known and has attracted a considerable amount of attention before. It is very probable that these auxiliary results are not new; however, we have not been able to find them in the literature).

In Chapter 3 we introduce quaternionic varieties and cycles, prove the (rational) suspension theorem and describe (rationally) the spaces of quaternionic cycles on projective spaces, thus obtaining a quaternionic version of the theory. The word "quaternionic" here means "invariant with respect to the action of the quaternion j"; so, for instance, our quaternionic varieties have little to do with quaternionic Kähler manifolds. This definition of a quaternionic object is not very restrictive (i.e. there are sufficiently many quaternionic varieties) and rather reasonable in the sense that it does not seem to be easy to give a better one. (The first draft of the preprint, which contained the results of Chapter 3 had a cautious title "Quaternion-flavoured cycle spaces").

In Chapter 4 we study the spaces of cycles of degree 1 and 2. On a projective space, spaces of complex cycles of degree one are Grassmanians. As Lawson and Michelsohn showed in **[34]**, the inclusion of the cycles of degree 1 into the space of all cycles gives maps

$$BU(n) \to K(\mathbf{Z},2) \times \ldots \times K(\mathbf{Z},2n)$$

and

$$BU \to K(\mathbf{Z},2) \times K(\mathbf{Z},4) \times K(\mathbf{Z},6) \times \ldots$$

These maps have nice properties and are rather important for some questions of algebraic topology (see **[9]**). In particular, they classify the Chern classes of the universal bundle, which generate the cohomology rings of infinite Grassmanians.

In Section 1 we briefly describe the real and quaternionic analogues of these maps, which give Pontrjagin classes for $BO(n)$ and symplectic Pontrjagin classes for $BSp(n)$ respectively. (The real mod 2 case, which gives the Stiefel-Whitney classes of the universal bundle over $BO(n)$ was studied in **[30]**).

These analogues of Lawson-Michelsohn maps in the quaternionic case exist only in even codimension, since in odd codimension there are no cycles of degree 1. Section 2 is a discussion of the following question: how "good" are the spaces of quaternionic cycles of degree 2 and odd codimension? We show that they are not as good as we would like them to be, i.e. that the inclusions into the "spaces of all cycles" do not induce inclusions on homology.

The questions, discussed in this thesis, seem to be of interest today. When the first draft of the preprint on quaternionic cycles **[48]** was ready, it turned out that a stronger version of the quaternionic suspension theorem was obtained by Lawson, Lima-Filho and Michelsohn [**32**]. Their account of the theory of quaternionic cycles [**33**] is currently in preparation.

Another recent work, closely related to the matters in the scope of our investigations, is the paper by V.I.Arnold [**3**]. He makes use of the fact that $SP^n(\mathbf{RP}^2) = \mathbf{RP}^{2n}$ and obtains a generalisation of the Kuiper-Massey theorem with the help of symmetric products. The last section of the Appendix in this thesis contains a similar interpretation of the Kuiper-Massey theorem; we also use symmetric products.

There are other questions which might deserve a place in this thesis, but are not discussed here. Well, according to Koz'ma Prutkov, *"Никто не обнимет необъятного"*[2]

[2] a somewhat handicapped translation is "one cannot embrace the infinite".

CHAPTER 1

Symmetric Products

In this chapter we define various types of symmetric products, describe some of their properties and give numerous examples. Examples, in fact, take the central place; some of them provide an introduction to the theory of algebraic cycles, others are of independent interest and have only a loose connection with the following chapters.

About the notation: we write "=" for homeomorphisms, "$\simeq$" means a homotopy equivalence. Eilenberg-MacLane spaces are denoted by $K(\cdot, i)$. From the very beginning we assume all topological spaces in question to be finite polyhedra, unless clearly otherwise. This restriction is not necessary in many cases, but it is acceptable for our purposes.

1. Basic definitions and Dold-Thom theorem

DEFINITION 1.1. The n-th symmetric product of a space X (denoted by $SP^n(X)$) is the quotient of the n-fold Cartesian product X^n by the action of the symmetric group S_n, which permutes the factors in X^n. The topology on $SP^n(X)$ is the quotient topology.

We will write points in $SP^n(X)$ either as (unordered) n-tuples of the form $(x_1, \dots, x_n)$, where x_i are (possibly coinciding) points in X, or as sums $k_1x_1 + \dots + k_mx_m$, where k_i are positive integers - multiplicities of points.

If X is endowed with a basepoint $*$, there are inclusion maps

$$SP^n(X) \hookrightarrow SP^{n+1}(X),$$

defined as

$$(x_1, \dots, x_n) \to (x_1, \dots, x_n, *).$$

DEFINITION 1.2. The direct limit of the sequence of these inclusions as $n \to \infty$ is called the infinite symmetric product of X and denoted by $SP^\infty(X)$.

$SP^\infty(X)$ is an abelian topological monoid[1]: addition is defined as

$$(x_1, \dots, x_k) + (x_{k+1}, \dots, x_n) = (x_1, \dots, x_k, x_{k+1}, \dots, x_n),$$

[1] i.e. an abelian topological semigroup with a unit.

and $*$ is a unit.

Let X be a space with a basepoint $*$ and let $p \in \mathbf{N}$. There is a monoid map

$$p : SP^\infty(X) \hookrightarrow SP^\infty(X),$$

which is multiplication by the integer p.

DEFINITION 1.3. The infinite symmetric product of X with coefficients in $\mathbf{Z/p}$, or infinite symmetric product modulo p (notation: $SP^\infty(X, \mathbf{Z/p})$) is the quotient monoid of $SP^\infty(X)$ by $p \cdot SP^\infty(X)$. The n-th symmetric product of X with coefficients in $\mathbf{Z/p}$ (notation: $SP^n(X, \mathbf{Z/p})$) is the image of $SP^n(X)$ under the quotient map

$$SP^\infty(X) \to SP^\infty(X, \mathbf{Z/p}).$$

In other words, $SP^\infty(X, \mathbf{Z/p})$ is the topological vector space over $\mathbf{Z/p}$, generated by points of X modulo the subspace generated by $*$. Note, that while $SP^\infty(X)$ is a monoid, $SP^\infty(X, \mathbf{Z/p})$ is a topological group.

The last important object to define here is a symmetric product with integral coefficients. Let X again be a space with a basepoint $*$. On the space $X \vee X$ there is an involution τ, which exchanges points in the first and the second copies of X. It induces an involution on $SP^\infty(X \vee X)$, which we also call τ.

DEFINITION 1.4. The infinite symmetric product of X with integral coefficients (we denote it by $SP^\infty(X, \mathbf{Z})$) is the quotient of $SP^\infty(X \vee X)$ by the equivalence relation

$$x \sim x + x' + \tau(x'), \quad x, x' \in SP^\infty(X \vee X).$$

In fact, $SP^\infty(X, \mathbf{Z})$ is the Grothendieck group of the monoid $SP^\infty(X)$; points in $SP^\infty(X, \mathbf{Z})$ can be written as sums $k_1x_1 + \ldots + k_mx_m$, where x_i are distinct points of X and k_i are nonzero integers.

All types of symmetric products defined above are CW-complexes. Continuous maps between spaces induce continuous maps between their symmetric products of all types; homotopic maps between spaces induce homotopic maps between symmetric products. This means that the homotopy type of symmetric products of X depends only on the homotopy type of X (for details see [**16**]).

If Y is an arbitrary topological subspace of X it follows from the definition that $SP^n(Y)$ is a subspace of $SP^n(X)$ for any n (including $n = \infty$). A corresponding statement for symmetric products with coefficients in $\mathbf{Z}$ or $\mathbf{Z/p}$ is not obvious. It is true for subpolyhedra, but in general, however, the fact that Y is a subset of X does not imply that symmetric products of Y can be treated as subspaces of symmetric products of X. This is illustrated by the following example:

EXAMPLE 1.1. Let D be the unit disk in $\mathbf{C}$, provided with a basepoint:

$$D = \{z \in \mathbf{C} |\ z\bar{z} \leq 1\},\ * = 1$$

and D_0 denotes the punctured disk $D - 0$. (Notice that the D_0 is not a subpolyhedron of D; moreover, the inclusion $D_0 \hookrightarrow D$ is not a cofibration.) We define $\widehat{SP^\infty}(D_0, \mathbf{Z})$ as the subspace of $SP^\infty(D, \mathbf{Z})$ which consists of the points $\sum k_i x_i$, where x_i are distinct points in D_0.

Obviously, a punctured disk is homotopy equivalent to a circle. As we will see later,

$$SP^\infty(S^1, \mathbf{Z}) \simeq S^1 \simeq SP^\infty(D_0, \mathbf{Z}),$$

but $\widehat{SP^\infty}(D_0, \mathbf{Z})$ is contractible.

Indeed, as $* = 1$ is a unit, any point in $\widehat{SP^\infty}(D_0, \mathbf{Z})$ can be written as a sum $k_0 * + k_1 x_1 + \ldots + k_m x_m$, such that $\sum_{i=0}^m k_i = 0$. Define the deformation retraction

$$\phi_t : \widehat{SP^\infty}(D_0, \mathbf{Z}) \times [0,1] \to \widehat{SP^\infty}(D_0, \mathbf{Z})$$

by "linear contraction to zero":

$$\phi_t(k_0 * + k_1 x_1 + \ldots + k_m x_m) = k_0 \cdot (1-t) + k_1 \cdot (x_1(1-t)) + \ldots + k_m \cdot (x_m(1-t)),$$

when $t < 1$ and extend it to $t = 1$ by continuity. Then $\phi_1(\widehat{SP^\infty}(D_0, \mathbf{Z})) = 1$, so $* = 1$ is a deformation retract of $\widehat{SP^\infty}(D_0, \mathbf{Z})$.

REMARK. Terminology and notation for symmetric products varies in literature. Symmetric products with coefficients in $\mathbf{Z}/2$ are sometimes called truncated symmetric products (see, for instance [5]); alternative notation for $SP^\infty(X, \mathbf{Z}/2)$ is $TP^\infty(X)$, $XP^\infty(X)$ or $AG(X, \mathbf{Z}/2, *)$. Symmetric products with coefficients in $\mathbf{Z}$ are sometimes denoted by $AG(X, *)$ (after Dold and Thom).

The most fundamental result about symmetric products was proved by Dold and Thom; parts a), b) and c) are known as the Dold-Thom Theorem (**[17]**,**[16]**)[2]:

THEOREM 1.1. *For connected X the following is true:*

a)

$$SP^\infty(X) \simeq \prod_i K(\widetilde{H}_i(X), i);$$

b)

$$SP^\infty(X, \mathbf{Z}/\mathrm{p}) \simeq \prod_i K(\widetilde{H}_i(X, \mathbf{Z}/\mathrm{p}), i);$$

[2] In **[16]** the conditions on the topological space X were: in part a) X is a locally finite CW-complex; in parts b), c) and d) X is a locally countable polyhedron.

c) the natural inclusion

$$SP^\infty(X) \hookrightarrow SP^\infty(X, \mathbf{Z})$$

is a homotopy equivalence;
d) the map

$$\pi_i(X) \to \pi_i(SP^\infty(X)) = \widetilde{H}_i(X)$$

induced by the inclusion $X \hookrightarrow SP^\infty(X)$ *is the Hurewicz homomorphism.*

It is hard to overestimate the significance of the Dold-Thom theorem. It gives geometric models for Eilenberg-MacLane spaces and allows the construction of homology without singular chains or spectra. The most important thing for us here is that it is one of the main ingredients of Lawson's theory of algebraic cycles.

There exist several proofs ([**16**],[**50**],[**57**],[**37**], see also a discussion in [**21**]). The method of the original proof in [**16**] was to verify that the functors $\pi_*(SP^\infty(\cdot))$ and $\pi_*(SP^\infty(\cdot, G))$, $G = \mathbf{Z}, \mathbf{Z/p}$ satisfy Eilenberg-Steenrod axioms for singular homology. The reason they do so is that they carry cofibrations

$$A \hookrightarrow X \to X/A$$

into (quasi)fibrations[3]

$$SP^\infty(A) \to SP^\infty(X) \to SP^\infty(X/A),$$

$$SP^\infty(A, G) \to SP^\infty(X, G) \to SP^\infty(X/A, G).$$

The homotopy exact sequence of a (quasi)fibration plays the role of the exact homological sequence and so on.

REMARK. It is important that $\pi_* SP^\infty(X, \cdot)$ can be identified with singular homology *as a functor*, i.e. maps between polyhedra induce the same maps on $\pi_* SP^\infty(\cdot, \cdot)$ and $H_*(\cdot, \cdot)$. See [**14**].

Other important results on symmetric products include Steenrod's theorem about the splitting of the homology of symmetric products [**15**], Macdonald's results about their Poincaré polynomials [**39**] and Milgram's theorem [**44**], which completely describes the homology of finite symmetric products. However, we are not going to use them.

[3]Quasifibrations were introduced in [**16**] as a tool for studying symmetric products. For homotopy-theoretic applications they are almost as good as genuine fibrations; in particular, there exists a long exact sequence of homotopy groups of a quasifibration.

2. Comparative zoology of symmetric products

Explicit examples of symmetric products are relatively scarce; one of the reasons for this is that the following theorems are true:

THEOREM 1.2. *Let M be a closed manifold. Then $SP^n(M)$ is a closed manifold for $n > 1$ if and only if* $\dim M = 2$.

THEOREM 1.3. *Let M be a closed compact connected manifold. Then $SP^n(M, \mathbf{Z}/2)$ is a closed manifold for $n > 1$ if and only if* $\dim M = 1$.

The first theorem is well-known. To see why it is true, let us consider a particular case: a symmetric square of a smooth manifold. The action of $S_2 = \mathbf{Z}/2$ is free away from the diagonal. A tubular neighbourhood of the diagonal is diffeomorphic to its normal bundle. The $\mathbf{Z}/2$-action respects the projection in the normal bundle and acts on the fibres by the antipodal map; so a neighbourhood of a point of $SP^2(M)$ which is of the form $(x, x), x \in M$ (i.e. the neighbourhood of a diagonal point) is homeomorphic to $\mathbf{R}^d \times c(\mathbf{RP}^{d-1})$, where $d = \dim M$, $c(X)$ - cone on X. This can be homeomorphic to $\mathbf{R}^{2d}$ only if $d = 2$.

We defer the proof of the second theorem until section 4.

However, we still have quite a variety of examples. Three of them are of particular interest, as they represent the simplest cases of complex, real and quaternionic cycle spaces. Let us start with the most famous and basic one.

EXAMPLE 1.2. **Complex projective line.**

$$SP^n(\mathbf{CP}^1) = \mathbf{CP}^n.$$

$\mathbf{CP}^n$ can be thought of as the space of homogeneous polynomials of degree n in two variables modulo multiplication by a nonzero complex number. Every polynomial determines a point in $SP^n(\mathbf{CP}^1)$ by its root system; conversely, every point of $SP^n(\mathbf{CP}^1)$ determines a polynomial (modulo multiplication by a scalar). So we have a continuous bijection between $SP^n(\mathbf{CP}^1)$ and $\mathbf{CP}^n$, i.e. a homeomorphism.

If we want to find some kind of a real analogue to this example, the first choice seems to be the symmetric products of $\mathbf{RP}^1$. They were described by H. Morton in [**46**]: $SP^n(\mathbf{RP}^1)$ is a disk bundle over S^1, in fact that associated to the Whitney sum of $n - 1$ copies of the Möbius band vector bundle.

However, the symmetric products of $\mathbf{RP}^1$ do not describe the real root systems of real polynomials in a continuous fashion. The most appropriate "real analogue" of the example above turns out to be the following

EXAMPLE 1.3. **Real projective line.**

$$SP^n(\mathbf{RP}^1, \mathbf{Z}/2) = \mathbf{RP}^n.$$

Take $\mathbf{RP}^n$ to be the space of *real* homogeneous polynomials of degree n in two variables modulo multiplication by a scalar. There is a continuous map

$$P : \mathbf{RP}^n \to SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$$

which sends a polynomial to its roots, taken with multiplicities modulo 2. (The choice of basepoints here is not important.) This map is onto, but not one-to-one. It is not hard to see that the fibres of P are contractible; this implies that P is a homotopy equivalence. We will not go into details here, as in section 4 we will show that this map is actually homotopic to a homeomorphism.[4]

Our third example is of quaternionic nature.

EXAMPLE 1.4. **Real projective plane.**

$$SP^n(\mathbf{RP}^2) = \mathbf{RP}^{2n}.$$

On $\mathbf{CP}^1$ there is an antiholomorphic involution j - the antipodal map. In homogeneous coordinates it can be written as

$$j : (z_0, z_1) \to (-\overline{z_1}, \overline{z_0}).$$

(If we consider a pair (z_0, z_1) to represent a quaternion $z_0 + j \cdot z_1$, this involution becomes the left multiplication by j; this is where quaternions come in.) The real projective plane is the quotient $\mathbf{CP}^1/j$. The action of j induces an action on homogeneous polynomials; we call a polynomial $P(X, Y)$ j-invariant if

$$\overline{P(X, Y)} = P(-\bar{Y}, \bar{X}).$$

A polynomial $P = \sum_{i=0}^{n} a_i X^i Y^{n-i}$ is j-invariant if and only if

$$a_k = (-1)^{n-k} \bar{a}_{n-k}.$$

This condition is real linear, hence the space of j-invariant polynomials of degree $2n$ modulo multiplication by a nonzero real number is homeomorphic to $\mathbf{RP}^{2n}$. (Notice that this condition also implies that the degree of P must be even.)

The root system of a j-invariant polynomial is j-invariant; in fact, there is a continuous 1-1 correspondence between points in $SP^n(\mathbf{RP}^2)$ and j-invariant homogeneous

[4]The fact that $SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$ is *homotopy equivalent* to $\mathbf{RP}^n$, has been proved by B. Mann and R. J. Milgram. Their (unpublished) proof was based on different arguments.

polynomials of degree $2n$ (defined up to a constant real factor). This proves the statement.

REMARK. The fact that the symmetric products of $\mathbf{RP}^2$ are projective spaces $\mathbf{RP}^{2n}$ follows directly from J. C. Maxwell's theorem on spherical harmonics [**12**], (see also J. J. Sylvester's paper [**58**], where he proves it "more mathematically" than Maxwell). It is essentially a topological reformulation of Maxwell's theorem; see a forthcoming paper by V. Arnold [**3**].

There are other explicit examples of symmetric products: Riemann surfaces and non-orientable Riemann surfaces. For Riemann surfaces (i.e. complex algebraic curves) X_g of positive genus g there are Abel-Jacobi maps

$$SP^n(X_g) \to T^{2g}$$

from symmetric products of X_g into a $2g$-dimensional torus - the Jacobian of the curve. When $n > 2g - 2$ by virtue of the Riemann-Roch Theorem the fibres of this map are complex projective spaces. For detailed investigations of these spaces see [**41**] and [**40**]; a discussion of the case of small n can be found in [**49**]. For non-orientable Riemann surfaces the situation is pretty much the same, instead of the classical Riemann-Roch theorem one can use the Riemann-Roch Theorem of [**54**]. For n big enough symmetric products of non-orientable surfaces will be real projective bundles over the Jacobian (which in this case can be an odd-dimensional torus), see [**18**].

Symmetric products with coefficients other than $\mathbf{N}$ and $\mathbf{Z}$ were studied already by Dold and Thom. Now we will discuss some further generalisations.

EXAMPLE 1.5. **Rational coefficients.**

Consider the sequence of maps

$$SP^\infty(X) \xrightarrow{\times 2} SP^\infty(X) \xrightarrow{\times 3} SP^\infty(X) \xrightarrow{\times 4} \dots$$

If one thinks of elements of $SP^\infty(X)$ as sums of points of X taken with positive integral multiplicities, the second copy of $SP^\infty(X)$ in this sequence can be thought of as the space of sums of points with rational multiplicities with denominators at most 2. The third copy will yield denominators which divide 6 and so on; so we have a good reason to call the direct limit of this sequence "a symmetric product of X with rational coefficients" and denote it by $SP^\infty(X, \mathbf{Q})$. The multiplication maps

$$SP^\infty(X) \xrightarrow{\times n} SP^\infty(X)$$

induces maps

$$\pi_*(SP^\infty(X)) \xrightarrow{\times n} \pi_*(SP^\infty(X)).$$

Homotopy and homology commute with direct limits, so the Dold-Thom Theorem holds for $SP^\infty(X,\mathbf{Q})$:

THEOREM 1.4.

$$SP^\infty(X,\mathbf{Q}) \simeq \prod K(\widetilde{H}_i(X,\mathbf{Q}),i).$$

Thus we have a simple model for rational Eilenberg-MacLane spaces:

$$K(\mathbf{Q},n) \simeq SP^\infty(S^n,\mathbf{Q}).$$

REMARK. Our construction of rational spaces $K(\mathbf{Q},n)$ is, of course, equivalent to the standard construction via mapping telescopes, described, for example, in [**26**].

All the coefficient groups or monoids we encountered before - $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Z/p}$, $\mathbf{Q}$- were abelian. There is, however, one case when we can use non-abelian coefficients.

EXAMPLE 1.6. **Non-abelian symmetric products of a circle.**

Let S^1 be a circle, parametrised by the map of an interval $f : [0,1] \to S^1$ with $f(0) = f(1) = *$ being a basepoint. For a topological monoid or group G (not even necessarily discrete) define $SP^\infty(S^1,G)$ to be the set of finite formal sums

$$g_1 f(x_1) + \ldots + g_m f(x_m), \quad g_i \in G, \quad x_i \in [0,1] \text{ and } x_i \le x_{i+1}$$

with identifications

$$g_0 f(0) + \sum g_i f(x_i) + g_m f(1) \sim \sum g_i f(x_i)$$

and

$$g_i f(x_i) + g_{i+1} f(x_{i+1}) \sim g_i g_{i+1} f(x_i) \text{ if } x_i = x_{i+1}.$$

There exists a filtration on $SP^\infty(S^1,G)$ by the number of terms in these sums. (Notice that in case $G = \mathbf{Z/p}$ this filtration does *not* coincide with the filtration by $SP^k(S^1,\mathbf{Z/p})$.) We topologise the n-th member of this filtration $(SP^\infty(S^1,G))_n$ as follows: if Δ_n is an n-dimensional simplex $0 \le t_1 \le \ldots \le t_n \le 1$, there is a map

$$G^n \times \Delta_n \to (SP^\infty(S^1,G))_n,$$

which sends $(g_1,\ldots,g_n;t_1,\ldots,t_n)$ into a formal sum $g_1 f(t_1) + \ldots + g_n f(t_n)$. Then the topology we put on $(SP^\infty(S^1,G))_n$ is the quotient topology. Finally, $SP^\infty(S^1,G)$ has the topology of a direct limit.

Under some mild restrictions on G the following is true:

THEOREM 1.5. (M.C.McCord [**43**])

$$SP^\infty(S^1,G) \simeq BG,$$

where BG is the classifying space for G.

When G is discrete, this theorem looks rather familiar:

$$SP^{\infty}(S^1, G) \simeq K(G, 1),$$

as, for discrete groups, BG has the homotopy type of $K(G,1)$; in particular, if G is any of the abelian groups discussed above (or $\mathbf{N}$), we recover the statement of the Dold-Thom theorem.

McCord's paper [**43**] also describes a generalisation of symmetric products with coefficients to a wide class of commutative coefficient groups and monoids. As sets, symmetric products with coefficients are sets of finitely supported functions on polyhedra with values in an abelian monoid. Quite similarly one can define symmetric products with local coefficients as spaces of finitely supported sections of a local system of groups. The first example of the next section deals with such a case.

3. Zoology II: group actions

Group actions on topological spaces induce corresponding actions on their infinite symmetric products. We will not attempt to study the subject in full detail, limiting our attention to two examples. The first example is a Dold-Thom type theorem for twisted integral homology. This result will not be used further in this thesis, but it seems to be a nice illustration of how the interpretation of homology via symmetric products can be carried over to a group-equivariant situation. It is not unlikely that this theorem might be used to interpret Poincarè duality for non-orientable manifolds in the spirit of [**22**].

In the second example we construct the homology transfer map. It will be useful later for the study of the spaces of real and quaternionic cycles.

We will utilise the machinery of simplicial sets[5]. The main general reference is [**42**]; a study of symmetric products in the simplicial setting can be found in [**57**].

First let us recall briefly the simplicial description of symmetric products. Let K be a finite connected simplicial set and let $|\cdot|$ denote the geometric realisation functor. The n-th symmetric product of K (notation: $SP^n(K)$) is a simplicial set whose q-simplices are unordered n-tuples of q-simplices of K with obvious face and degeneracy operators. Assume that K has a basepoint $*$. Then one can define the infinite symmetric product $SP^{\infty}(K)$ in the same way as it is defined for topological spaces. The key fact here is that for finite K_1 and K_2 there is a natural homeomorphism

$$|K_1 \times K_2| = |K_1| \times |K_2|.$$

[5]or, in other terminology, semi-simplicial complexes.

It follows that

$$|SP^n(K)| = SP^n(|K|)$$

and, correspondingly,

$$|SP^\infty(K)| = SP^\infty(|K|).$$

Denote by $C(K)$ the simplicial abelian group generated by the simplices of K and by $SP^\infty(K, \mathbf{Z})$ the quotient of $C(K)$ by the subgroup, generated by the degeneracies of $*$. Then the inclusion map

$$SP^\infty(K) \hookrightarrow SP^\infty(K, \mathbf{Z})$$

is a homotopy equivalence. Any simplicial abelian group can be given the structure of a chain complex, the boundary being defined via face operators. The homology of $C(K)$ and $SP^\infty(K, \mathbf{Z})$ regarded as chain complexes is just the unreduced and reduced integral homology of K respectively. (See [**57**].)

The naturality of the geometric realisation together with the naturality of the homeomorphism $|K_1 \times K_2| = |K_1| \times |K_2|$ implies that there is a natural homeomorphism

$$|SP^\infty(K, \mathbf{Z})| = SP^\infty(|K|, \mathbf{Z}),$$

which is, in fact, a group isomorphism.

REMARK. One can treat symmetric products with other coefficients in a similar fashion. Define $SP^\infty(K, \mathbf{Z}/2)$ as the simplicial abelian group whose q-simplices are finite linear combinations of q-simplices of K with coefficients in $\mathbf{Z}/2$ and degeneracies of $*$ are units. Then there is a simplicial fibration

$$SP^\infty(K, \mathbf{Z}) \xrightarrow{\times 2} SP^\infty(K, \mathbf{Z}) \to SP^\infty(K, \mathbf{Z}/2),$$

which is carried by the geometric realisation functor to the fibration

$$SP^\infty(|K|) \xrightarrow{\times 2} SP^\infty(|K|) \to |SP^\infty(K, \mathbf{Z}/2)|.$$

This means that

$$|SP^\infty(K, \mathbf{Z}/2)| = SP^\infty(|K|, \mathbf{Z}/2).$$

A useful corollary of this is that a triangulation of a topological space X induces a triangulation of $SP^n(X, \mathbf{Z}/2)$.

EXAMPLE 1.7. **Twisted coefficients.**

Let K be as above and assume K has a basepoint. Let $\widetilde{K}$ be the orienting double cover of K and let T be the monodromy of $\widetilde{K}$. Then $\mathbf{Z}/2$ acts on $C(\widetilde{K})$; denote by $C^-(K)$ the subgroup of elements of $C(\widetilde{K})$ such that $T(c) = -c$. The homology of K with twisted integral coefficients $H_*(K, \mathbf{Z_T})$ is defined as the homology of $C^-(K)$.

We can also define *reduced* twisted homology groups $\widetilde{H}_*(K, \mathbf{Z_T})$ as follows: if

$$\widehat{K} = \widetilde{K}/\{*, T(*)\}$$

then the action of T descends to $\widehat{K}$. We denote by $SP^\infty(K, \mathbf{Z_T})$ the subgroup of such elements of $SP^\infty(\widehat{K}, \mathbf{Z})$ that change sign under T. The reduced twisted homology $\widetilde{H}_*(K, \mathbf{Z_T})$ is the homology of $SP^\infty(K, \mathbf{Z_T})$ regarded as a chain complex.

The homotopy groups of a simplicial abelian group coincide with the homology of the same group, regarded as a chain complex. This implies

LEMMA 1.6.

$$\pi_*(SP^\infty(K, \mathbf{Z_T})) = \widetilde{H}_*(K, \mathbf{Z_T}).$$

This is the simplicial version of the "twisted" Dold-Thom Theorem.

If $X = |K|$, $\mathbf{Z}/2$ acts on $\widehat{X} = |\widehat{K}|$ and, hence, on $SP^\infty(\widehat{X}, \mathbf{Z})$.

DEFINITION 1.5. The twisted infinite symmetric product of X is defined as the subspace of points $\sum k_i x_i$ in $SP^\infty(\widehat{X}, \mathbf{Z})$ such that

$$T(\sum k_i x_i) = -\sum k_i x_i.$$

Finally, define $H_*(X, \mathbf{Z_T})$ and $\widetilde{H}_*(X, \mathbf{Z_T})$ as $H_*(K, \mathbf{Z_T})$ and $\widetilde{H}_*(K, \mathbf{Z_T})$ respectively.

REMARK. The relation between reduced and unreduced twisted homology is slightly different from the case of ordinary homology. Namely, one can easily check that for connected X

$$H_i(X, \mathbf{Z_T}) = \widetilde{H}_i(X, \mathbf{Z_T}) \quad \text{for} \quad i \geq 2,$$
$$\widetilde{H}_0(X, \mathbf{Z_T}) = 0$$

and that there is a short exact sequence[6]

$$0 \to H_1(X, \mathbf{Z_T}) \to \widetilde{H}_1(X, \mathbf{Z_T}) \to \mathbf{Z} \to 0.$$

THEOREM 1.7.

$$SP^\infty(X, \mathbf{Z_T}) \simeq \prod_{i>0} K(\widetilde{H}_i(X, \mathbf{Z_T}), i)$$

PROOF. By Moore's Theorem [**16**] every abelian topological group is a product of Eilenberg-MacLane spaces, so it is enough to check that

$$\pi_*(SP^\infty(X, \mathbf{Z_T})) = \widetilde{H}_*(X, \mathbf{Z_T}).$$

[6] as this sequence shows, we could have called $\widetilde{H}_*(X, \mathbf{Z_T})$ *extended* rather than *reduced* twisted homology.

If $X = |K|$ this follows from Lemma 1.6, provided that

$$SP^{\infty}(|K|, \mathbf{Z_T}) \simeq |SP^{\infty}(K, \mathbf{Z_T})|.$$

To see this, recall that there is a natural group isomorphism

$$SP^{\infty}(|K|, \mathbf{Z}) = |SP^{\infty}(K, \mathbf{Z})|.$$

Its naturality implies that it commutes with the action T of $\mathbf{Z}/2$, so, in particular, the subgroups of elements which are sent to their inverses by T are isomorphic. □

EXAMPLE 1.8. **The homology transfer map.**

The homology transfer map can be described in the simplicial setting as follows. Let K be as above and let a finite group G act on it simplicially. We also assume for simplicity that the action of G fixes the basepoint in K. There is a simplicial quotient map

$$p : K \to K/G.$$

The map

$$\mathbf{Tr} : SP^{\infty}(K/G, \mathbf{Z}) \to SP^{\infty}(K, \mathbf{Z})$$

is defined by sending a simplex Δ of K/G to $\sum_{g \in G} g\Delta'$, where $p(\Delta') = \Delta$. It is easy to see that if $SP^{\infty}(K/G, \mathbf{Z})$ and $SP^{\infty}(K, \mathbf{Z})$ are regarded as chain complexes, $\mathbf{Tr}$ is a chain map, so it induces a map on reduced homology

$$Tr : \widetilde{H}_*(K/G) \to \widetilde{H}_*(K).$$

This is the homology transfer map. It is a standard result [**19**] that, when tensored with $\mathbf{Q}$, this map becomes an isomorphism onto the G-fixed subgroup of $\widetilde{H}_*(K)$.

Applying geometric realisation to the map $\mathbf{Tr}$ one obtains a map

$$|\mathbf{Tr}| : SP^{\infty}(|K|/G, \mathbf{Z}) \to SP^{\infty}(|K|, \mathbf{Z}).$$

The induced map of the homotopy groups coincides with the homology transfer. If we take rational coefficients instead of integers, the map we obtain

$$SP^{\infty}(|K|/G, \mathbf{Q}) \to SP^{\infty}(|K|, \mathbf{Q})$$

gives an isomorphism between the homotopy groups of $SP^{\infty}(|K|/G, \mathbf{Q})$ and the G-fixed subgroups of the homotopy of $SP^{\infty}(|K|, \mathbf{Q})$

Another way to represent the transfer map rationally is to consider the inclusion map

$$i : (SP^{\infty}(K, \mathbf{Z}))^G \hookrightarrow SP^{\infty}(K, \mathbf{Z})$$

of the G-fixed subgroup of the symmetric product of K into the symmetric product of K. There is a commutative triangle

$$\begin{array}{ccc} SP^\infty(K/G,\mathbf{Z}) & \xrightarrow{\mathbf{Tr}} & SP^\infty(K,\mathbf{Z}) \\ \mathbf{Tr}\searrow & & \nearrow i \\ & (SP^\infty(K,\mathbf{Z}))^G & \end{array}$$

Notice that after tensoring with rationals, the map

$$SP^\infty(K/G,\mathbf{Z}) \xrightarrow{\mathbf{Tr}} (SP^\infty(K,\mathbf{Z}))^G$$

becomes one-to-one. It follows that

$$i_* : \pi_*((SP^\infty(K,\mathbf{Z}))^G) \otimes \mathbf{Q} \to (\pi_*(SP^\infty(K,\mathbf{Z})) \otimes \mathbf{Q})^G$$

is an isomorphism. It can also be written as

$$i_* : \pi_*((SP^\infty(K,\mathbf{Q}))^G) \to \widetilde{H}(K,\mathbf{Q})^G.$$

Applying geometric realisations one can obtain corresponding statements for topological spaces.

REMARK. Here we considered symmetric products with coefficients in $\mathbf{Z}$. Everything said above applies to products with positive integral coefficients.

It seems to be possible to elaborate on the last example and establish yet another version of the Dold-Thom Theorem for Smith's "special homology" [**51**]. We will not attempt to do it, though, as the result wouldn't justify the effort.

There are other generalisations of the Dold-Thom Theorem. One can take a similar approach in K-theory. The role of infinite symmetric products for K-cohomology is played by "spaces of bundles" i.e. spaces of maps from a space to an infinite Grassmanian; for K-homology it is played by the spaces of elliptic operators on a space; see also [**55**] on this subject.

There is also a Dold-Thom theorem for intersection homology, established by Pawel Gajer [**23**]. It relates the intersection homology groups of a variety to the "intersection homotopy groups" of its infinite symmetric product.

Friedlander in [**20**] showed that the Dold-Thom Theorem also works in the context of étale homotopy and homology; this version is important for the theory of algebraic cycles over fields other than $\mathbf{C}$.

4. Geometry of symmetric products modulo 2

Here we will prove theorem 1.3. One part of it is covered by the following

PROPOSITION 1.8. *Let M be a closed compact connected manifold. If* $\dim M \geq 2$ *and $n > 1$ then $SP^n(M, \mathbf{Z}/2)$ is not a closed manifold.*

PROOF. Denote by Δ the "big diagonal" in $(M)^n$, i.e. the set of points with at least two coinciding coordinates. Consider the sequence of natural maps

$$(M)^n \xrightarrow{\rho_1} SP^n(M) \xrightarrow{\rho_2} SP^n(M, \mathbf{Z}/2).$$

The image of Δ under $\rho_2\rho_1$ is $SP^{n-2}(M, \mathbf{Z}/2) \subset SP^n(M, \mathbf{Z}/2)$. Also notice that the composite map $\rho_2\rho_1$, restricted to $(M)^n - \Delta$ is an $n!$-sheeted covering

$$\rho_2\rho_1 : (M)^n - \Delta \to SP^n(M, \mathbf{Z}/2) - SP^{n-2}(M, \mathbf{Z}/2).$$

Now suppose $SP^n(M, \mathbf{Z}/2)$ is a manifold, $\dim M \geq 2$. Clearly, $\dim SP^n(M, \mathbf{Z}/2) = n \dim M$.

Choose any point $y \in M$. Then the point $\rho_2(ny)$ (or, in other words, $\rho_2\rho_1(y, y, \dots, y)$) belongs to $SP^{n-2}(M, \mathbf{Z}/2) \subset SP^n(M, \mathbf{Z}/2)$. Take a small ball B around $\rho_2(ny)$. Then

$$\dim B \cap SP^{n-2}(M, \mathbf{Z}/2) = (n-2) \dim M$$

and, hence, $H^{n \dim M - 2}(B \cap SP^{n-2}(M, \mathbf{Z}/2)) = 0$. By Alexander duality this implies that the first homology group of

$$W = B - (B \cap SP^{n-2}(M, \mathbf{Z}/2))$$

is zero. Also, as $\dim M \geq 2$, it is clear that $\widetilde{W} = \rho_1^{-1}\rho_2^{-1}(W)$ is connected. But as

$$\rho_2\rho_1 : \widetilde{W} \to W$$

is an $n!$-sheeted covering with S_n being the monodromy group, $H_1(W)$ cannot be trivial; so we get a contradiction.

(Indeed, we have a surjection

$$\pi_1(W) \to S_n.$$

For any n there is a non-trivial map $S_n \to \mathbf{Z}/2$. As $H_1(W) = \pi_1(W)/[\pi_1(W), \pi_1(W)]$, the composite map $\pi_1(W) \to \mathbf{Z}/2$ must factor through a non-trivial map $H_1(W) \to \mathbf{Z}/2$, so $H_1(W) \neq 0$.) □

To prove the second part we have to show that $SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$ is a manifold, as $\mathbf{RP}^1$ is the only closed compact 1-dimensional manifold. We will actually prove more.

PROPOSITION 1.9. *The map $P : \mathbf{RP}^n \to SP^n(M, \mathbf{Z}/2)$, defined in Example 1.3 of section 2, is homotopic to a homeomorphism.*

First we state an auxiliary lemma about the symmetric products of a 2-disk[7]:

LEMMA 1.10. *$SP^k(D^2)$ is homeomorphic to a $2k$-dimensional closed ball D^{2k}.*

PROOF. Let $D^2 = \{z \mid z \in \mathbf{C}, z\bar{z} \leq 1\}$. Define a map $\psi_1 : SP^k(D^2) \to \mathbf{C}^k$ by sending a point $(a_1, \dots, a_k) \in SP^k(D^2)$ to the coefficients of the polynomial $(z - a_1) \cdot \dots \cdot (z - a_k)$; and a map $\psi_2 : \mathbf{C}^k \to \mathbf{C}^k = \mathbf{R}^{2k}$ by

$$(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, \dots, r_n e^{i\phi_k}) \to (r_1 e^{i\phi_1}, \sqrt{r_2} e^{i\phi_2}, \dots, \sqrt[n]{r_n} e^{i\phi_k}).$$

Then the composite map $\psi_2\psi_1 : SP^k(D^2) \to \mathbf{R}^{2k}$ is one-to-one and carries $SP^k(D^2)$ onto some neighbourhood of zero in $\mathbf{R}^{2k}$. Any ray starting at $0 \in \mathbf{R}^{2k}$ meets the boundary of $\psi_2\psi_1 SP^k(D^2)$ precisely in one point. This implies that $\psi_2\psi_1 SP^k(D^2)$ and, hence, $SP^k(D^2)$ is homeomorphic to a ball. □

Let us study the structure of the map P, defined in Example 1.3. For any polyhedron X with a basepoint $*$ and any $m \geq 1$ there are natural inclusions

$$SP^{m-1}(X, \mathbf{Z}/2) \hookrightarrow SP^m(X, \mathbf{Z}/2),$$

defined as

$$\sum k_i x_i \to \sum k_i x_i + *.$$

For $X = \mathbf{RP}^1$ the space $C_m = SP^m(\mathbf{RP}^1, \mathbf{Z}/2) - SP^{m-2}(\mathbf{RP}^1, \mathbf{Z}/2)$ can be identified with the configuration space of m unordered distinct points on the circle. Clearly C_m is an (open) manifold.

$\mathbf{RP}^n$ can be decomposed as the union of subspaces $\cup R_i$, where R_i is the space of polynomials, which have *precisely* i real roots of odd multiplicity. Here $i \in \{0, 2, \dots, n\}$ when n is even and $i \in \{1, 3, \dots, n\}$ when n is odd. Notice that $\cup \partial R_i$ is the hypersurface of polynomials with multiple real zeros; it is a subset of the zero set of the discriminant. The map P respects the decompositions into C_i and R_i we have introduced on $\mathbf{RP}^n$ and $SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$. Denote by p_i the restriction of P to R_i:

$$p_i : R_i \to C_i.$$

Every polynomial with i roots of odd multiplicity can be uniquely factorised as a product of a nonnegative polynomial of degree $n - i$ (normalised in such a way that the sum of squares of its coefficients is equal to 1) and a polynomial of the form

$$(a_1 x - b_1 y) \dots (a_i x - b_i y),$$

where (a_k, b_k) are distinct points of $\mathbf{RP}^1$.

[7]This lemma is, apparently, well-known; however, we haven't been able to find a reference for it.

The space of normalised nonnegative polynomials of degree $n-i$ is homeomorphic to $SP^{\frac{1}{2}(n-i)}(D^2)$, where D^2 is considered as the upper closed hemisphere of $\mathbf{CP}^1$. This means that

$$R_i = C_i \times SP^{\frac{1}{2}(n-i)}(D^2) = C_i \times D^{n-i}$$

and p_i is the projection on the first factor. The boundary of R_i can be described as a union of two components. The first one is $F_i \stackrel{def}{=} C_i \times \partial D^{n-i} = C_i \times S^{n-i-1}$, obviously $F_i \subset R_i$. The second one is $\partial R_i - F_i = \partial R_i \cap \overline{R_{i-2}}$. The following lemmas can be easily verified directly from the definition of R_i.

LEMMA 1.11. *Any $x \in F_i$ has a neighbourhood N_x such that $N_x \cap R_k$ is empty for $k < i$.*

LEMMA 1.12. *For each i $\overline{F}_i$ divides $\mathbf{RP}^n$ into two components; the closure of one of them is $\cup_{k \leq i} R_k$; the closure of the other is $\cup_{k>i}\overline{R_k}$.*

The map P can be described as a composition of collapsing maps

$$\mathbf{RP}^n = Q_0 \stackrel{q_0}{\to} Q_2 \stackrel{q_2}{\to} \dots \stackrel{q_{n-2}}{\to} Q_n = SP^n(\mathbf{RP}^1, \mathbf{Z}/2),\ n \text{ even},$$

or

$$\mathbf{RP}^n = Q_1 \stackrel{q_1}{\to} Q_3 \stackrel{q_3}{\to} \dots \stackrel{q_{n-2}}{\to} Q_n = SP^n(\mathbf{RP}^1, \mathbf{Z}/2),\ n \text{ odd},$$

where q_i are the maps, which collapse the fibres of p_i to single points, Q_i are the corresponding quotient spaces. We will see that these maps are actually homotopic to homeomorphisms.

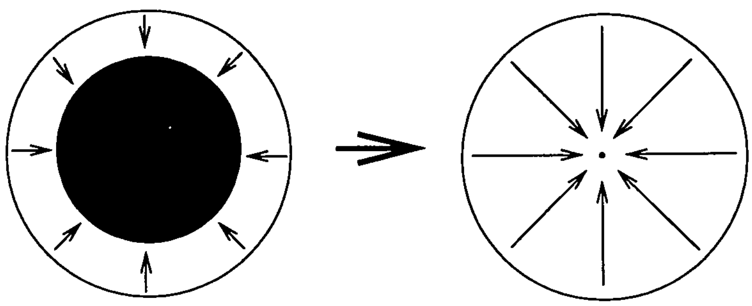

Figure 1.1. The deformation retraction ϕ_t

Let us first look at two examples: $n = 2$ and $n = 3$. For $n = 2$ the map P is 1-1 everywhere, apart from the disk, bounded by the conic

$$x_1^2 - 4x_0x_2 = 0,$$

i.e. the defined by the condition that the discriminant of the polynomial is 0; here x_i are the homogeneous coordinates on $\mathbf{RP}^2$. This disc, which describes polynomials with double roots or no real roots, maps into a single point $*$ in $SP^2(\mathbf{RP}^1, \mathbf{Z}/2)$.

Take a "one-sided tubular neighbourhood" of this disk and define the deformation retraction

$$\phi_t : \mathbf{RP}^2 \times [0,1] \to \mathbf{RP}^2$$

so that ϕ_t is the identity map outside the neighbourhood of the disk for all t and is the contraction of the disk to a single point when $t = 1$, see Fig. 1.1. Then $P(\phi_t^{-1})$ is a homotopy of P to a homeomorphism.

In the case of $n = 3$ the map P is 1-1 everywhere in $\mathbf{RP}^3$, apart from a full torus $R_1 = \mathbf{RP}^1 \times D^2$; as we said above, on this torus P is the projection onto $\mathbf{RP}^1$. If the boundary of R_1 happened to be smooth, we would be able to use the same method as for $n = 2$ "fibrewise". However, the surface, defined by the condition for the discriminant to be zero, is singular and we cannot use the normal bundle to construct a one-sided tubular neighbourhood.

This is the situation we are also going to encounter in higher dimensions, so we need some results concerning one-sided tubular neighbourhoods in such a situation.

DEFINITION 1.6. **[10]** Let X be a topological space and B a subset of X. Then B is *collared* in X if there is a homeomorphism h carrying $B \times [0,1)$ onto an open neighbourhood of B such that $h(b,0) = b$ for all $b \in B$. If B can be covered by a collection of open subsets (relative to B) each of which is collared in X, then B is *locally collared* in X. B is said to be *bi-collared* in X if there is a homeomorphism h carrying $B \times (-1,1)$ onto an open neighbourhood of B such that $h(b,0) = b$ for all $b \in B$. The notion of B being *locally bi-collared* is defined similarly.

An example of a locally collared (in fact, collared) subset is the boundary of a manifold. An example of a locally bi-collared subset is a smooth submanifold of codimension 1 in a smooth manifold.

THEOREM 1.13. **[10]** *A locally collared subset of a metric space is collared.*

THEOREM 1.14. (relevant references are **[52]** or **[10]**) *If $B \to Int(X)$ is a piecewise linear embedding of an $(n-1)$-dimensional PL-manifold into a PL n-manifold X, then B is locally bi-collared in X.*

Now we are in the position to prove the following

PROPOSITION 1.15. *The maps $q_i : Q_i \to Q_{i+2}$ are homotopic to homeomorphisms.*

PROOF. First of all notice that $\mathbf{RP}^n$ together with the hypersurface of polynomials with multiple real roots is a compact stratified set, hence by the main theorem of [**28**] it has a triangulation, compatible with the stratification. The subsets F_i are not closed, but they are locally triangulable. As we mentioned above, F_i is homeomorphic to $C_i \times S^{n-i-1}$, which is a manifold. Thus, by lemma 1.11, for any point $x \in F_i$ we can find such a triangulation of $\mathbf{RP}^n$, that there exists a triangulated neighbourhood N of x, homeomorphic to a ball D^n, $N \cap F_i$ is homeomorphic to a ball D^{n-1} and $N \cap \partial R_k$ is empty for $k < i$. Applying Theorem 1.14 we get that F_i is bi-collared in some neighbourhood of x inside $\mathbf{RP}^n$ and, hence, collared in some neighbourhood of x inside each of the components into which $\overline{F}_i$ divides $\mathbf{RP}^n$. In particular, it is collared in $\mathbf{RP}^n - Int(\bigcup_{k\leq i} R_i)$. As x is an arbitrary point of F_i, it follows that F_i is locally collared and, by Theorem 1.13, is collared in $\mathbf{RP}^n - Int(\bigcup_{k\leq i} R_i)$.

Now we construct a "spindle neighbourhood" of F_i as follows. Let λ be a continuous function on $\overline{F_i}$, such that $\lambda = 0$ on $\overline{F_i} - F_i$ and $0 < \lambda < 1$ on F_i. Then our spindle neighbourhood S_i is the subset of the collar $F_i \times [0, 1)$ defined as

$$S_i = \{(x, t) \mid x \in F_i; 0 \leq t \leq \lambda\}.$$

There is a deformation retraction

$$\Phi_t : (S_i \cup R_i) \times [0, 1] \to S_i \cup R_i,$$

which commutes with the projection p_i and in fibres looks like the retraction ϕ_t on Fig.1. The union of R_i and the spindle neighbourhood of F_i maps homeomorphically into Q_i under the composite collapsing map $\mathbf{RP}^n \to Q_i$. We extend Φ_t to Q_i by the identity map; it is immediately clear that this extension of Φ_t is continuous everywhere, possibly apart from the image of $\bigcup_{k<i} R_k$. The continuity there easily follows from the following fact:

LEMMA 1.16. *The decompositon of* $\mathbf{RP}^n$ *into fibres of* P *is upper semicontinuous, i.e. for any neighbourhood* U *of a disk* $P^{-1}(y)$, $y \in SP^n(\mathbf{RP}^1, \mathbf{Z}/2)$ *there exists a neighbourhood* V*, such that any fibre of* P *which intersects* V *lies in* U.

(This lemma follows directly from Proposition 1, Chapter 1 of [**13**], which says that if a map between topological spaces is closed, the decomposition of the source space into the fibres of the map is upper semicontionuous. It can be easily verified directly.)

Now $q_i\Phi_t^{-1}$ is a homotopy, carrying q_i into a homeomorphism; this proves proposition 1.15. □

If two maps are homotopic to homeomorphisms, their composite is also homotopic to a homeomorphism. This proves Theorem 1.3.

REMARK. There are powerful tools, such as Edwards cell-like approximation theorem [**13**], to treat situations like ours. However, Edwards theorem only works in dimensions ≥ 5 and requires some conditions, which are not totally trivial to verify in our case; so we chose the direct approach.

CHAPTER 2

Spaces of Complex and Real Algebraic Cycles

Here we give an overview of Lawson's results [**31**] on complex algebraic cycles and their real version, obtained by T.K.Lam [**30**]. We also calculate the homotopy groups of real cycles with coefficients in $\mathbf{Z}$ on projective spaces. Finally, we consider a theorem of Lawson and Michelsohn about group actions on the cycle spaces [**36**].

We are going to deal only with projective varieties. From now on we adopt the following notation: group (monoid) quotients are denoted by "$/\!\!/$" while the usual "/" denotes the topological quotient.

1. Complex cycle spaces

Definition 2.1. An algebraic cycle $c = \sum k_i V_i,\ i, k_i \in \mathbf{Z}$ in $\mathbf{CP}^n$ is a finite formal sum of irreducible subvarieties V_i of $\mathbf{CP}^n$ of the same dimension. The degree of a cycle c is defined as

$$\deg(c) = \sum k_i \deg V_i,$$

the dimension of c is defined as $\dim V_i$. A cycle is called effective if $k_i \geq 0$ for all i.

The set of effective cycles of the same degree d and dimension p in $\mathbf{CP}^n$ has the structure of an algebraic variety. This variety is called a Chow variety and denoted by $\mathcal{C}_{p,d}(\mathbf{CP}^n)$. Algebraic varieties carry a natural topology, so algebraic cycles form topological spaces. Alternatively, algebraic cycles define linear functionals on differential forms by integration; so one can put the topology of the dual space on the sets of cycles; these two topologies coincide ([**31**]). Similarly one defines spaces $\mathcal{C}_{p,d}(X)$ of cycles with support on a subvariety of $\mathbf{CP}^n$. (See [**53**], [**56**]).

In most cases Chow varieties are singular, reducible and hard to study. However, as discovered by H.B.Lawson in [**31**], when the degree of cycles tends to infinity, the homotopy type of $\mathcal{C}_{p,d}(\mathbf{CP}^n)$ becomes calculable. To state this rigorously we have to define the stabilised Chow varieties.

Let $X \subset \mathbf{CP}^n$ be a subvariety of $\mathbf{CP}^n$. Take a disjoint union

$$\mathcal{C}_{p,\cdot}(X) = \{0\} \coprod_d \mathcal{C}_{p,d}(X),$$

where $\{0\}$ denotes an "empty" cycle. This is an abelian monoid under unions of cycles.

There are several methods of obtaining a "homotopy group completion" of $C_{p,\cdot}(X)$.

The first one is provided by the mapping telescope construction. For each element $\alpha \in \pi_0(C_{p,\cdot}(X))$ select a representative c_α. Then there are maps

$$c_\alpha : C_{p,\cdot}(X) \xrightarrow{+c_\alpha} C_{p,\cdot}(X).$$

Let $\beta_1, \beta_2, \dots$ be a sequence of elements of $\pi_0(C_{p,\cdot}(X))$, such that every element of $\pi_0(C_{p,\cdot}(X))$ appears in it infinitely often. Then $C_p(X)$ is defined as the mapping telescope $\mathrm{Tel}(C_{p,\cdot}(X), c_{\beta_i})$ (see **[31]** and **[20]**). This definition relies on a chosen set of representatives for $\pi_0(C_{p,\cdot}(X))$, but the homotopy type of the space obtained this way does not depend on the choice.

The motivation for using the telescope construction comes from two basic examples: 0-cycles on a variety X and cycles on $\mathbf{CP}^n$. Recall that for a sequence of inclusions the mapping telescope is homotopy equivalent to the direct limit space.

For 0-cycles $C_{0,\cdot}(X)$ is the disjoint union $\coprod_d SP^d(X)$. Suppose that X is connected. If $y \in X$ then $\{ky \mid k \in \mathbf{Z}, k \geq 0\}$ forms a set of representatives for $\pi_0(C_{p,\cdot}(X))$. So $C_0(X)$ is homotopy equivalent to the direct limit of the sequence

$$\coprod_d SP^d(X) \hookrightarrow \coprod_d SP^d(X) \hookrightarrow \dots$$

where the inclusions are

$$SP^d(X) \xrightarrow{+y} SP^{d+1}(X).$$

So it is clear that the connected components of $C_0(X)$ are all homotopy equivalent to $SP^\infty(X)$; also $\pi_0(C_0(X)) = \mathbf{Z}$.

The situation with p-cycles on $\mathbf{CP}^n$ is similar: choose a p-dimensional linear subspace l and define inclusions

$$C_{p,d}(\mathbf{CP}^n) \hookrightarrow C_{p,d+1}(\mathbf{CP}^n)$$

by adding l to every cycle in $C_{p,d}(\mathbf{CP}^n)$:

$$c \to c + l.$$

Then the connected components of the stabilised Chow variety $C_p(\mathbf{CP}^n)$ are homotopy equivalent to the direct limit of the sequence of these inclusions as $d \to \infty$.

The second way to define a group completion of $C_{p,\cdot}(X)$ is to consider the space $\Omega B(C_{p,\cdot}(X))$, where $B(\cdot)$ is the classifying space of an abelian monoid, provided by the bar construction, and Ω denotes the loop space. The space $\Omega B(C_{p,\cdot}(X))$ is homotopy equivalent to $C_p(X)$; see Corollary 2.6 of **[20]**.

Yet another (and, probably, the most attractive) method of obtaining stabilised cycle spaces is taking the "naïve" group completion $\widetilde{C}_p(X)$, which is just the Grothendieck group of $C_{p,\cdot}(X)$, endowed with the quotient topology from the map

$$C_{p,\cdot}(X) \times C_{p,\cdot}(X) \xrightarrow{a-b} \widetilde{C}_p(X), \quad a, b \in C_{p,\cdot}(X).$$

One can consider $\widetilde{C}_p(X)$ as a space of cycles with coefficients in $\mathbf{Z}$. As $\widetilde{C}_p(X)$ is a group, $\pi_0(\widetilde{C}_p(X))$ also has a group structure. It is easy to see that the connected components of $\widetilde{C}_p(X)$ are homeomorphic. Indeed, if $\alpha, \beta \in \pi_0(\widetilde{C}_p(X))$, choose a cycle c in the component represented by $\beta - \alpha$. Then the map

$$(\widetilde{C}_p(X))_\alpha \xrightarrow{+c} (\widetilde{C}_p(X))_\beta$$

is the desired homeomorphism.

For connected X the connected components of $\widetilde{C}_0(X)$ are indexed by the degrees of 0-cycles, i.e. by the sums of coefficients; the same is true for $\widetilde{C}_p(\mathbf{CP}^n)$.

It follows from the theorem of Lima-Filho [**37**] that the spaces $C_p(X)$ and $\widetilde{C}_p(X)$ are homotopy equivalent. A particular case of this statement is part c) of the Dold-Thom theorem, see page 8. Indeed,

$$C_0(X) \simeq \mathbf{Z} \times SP^\infty(X)$$

and

$$\widetilde{C}_0(X) = \mathbf{Z} \times SP^\infty(X, \mathbf{Z}).$$

We will use the last definition of the stabilised cycle spaces, i.e. the abelian groups $\widetilde{C}_p(X)$.

There is a geometric construction, which Lawson called "complex suspension". $\mathbf{CP}^n$ can be embedded into $\mathbf{CP}^{n+1}$ as

$$(z_0, \dots, z_n) \to (z_0, \dots, z_n, 0).$$

If $X \subset \mathbf{CP}^n$ the complex suspension $\Sigma\!\!\!/ X \in \mathbf{CP}^{n+1}$ is defined as the union of all lines, joining X and the point $(0, \dots, 0, 1)$. In other words, $\Sigma\!\!\!/ X$ is the projective cone on X. If X is a projective variety, $\Sigma\!\!\!/ X$ is a projective variety of the same degree. Indeed, it is defined using exactly the same equations as define X. Obviously, $\Sigma\!\!\!/ \mathbf{CP}^n$ is $\mathbf{CP}^{n+1}$.

Complex suspension induces continuous maps on the cycle spaces

$$\Sigma\!\!\!/ : C_{p,d}(X) \to C_{p+1,d}(\Sigma\!\!\!/ X).$$

So there is a map

$$\Sigma\!\!\!/ : C_{p,\cdot}(X) \to C_{p+1,\cdot}(\Sigma\!\!\!/ X)$$

which gives rise to a map of group completions

$$\Sigma\!\!\!/ : \widetilde{C}_p(X) \to \widetilde{C}_{p+1}(\Sigma\!\!\!/ X).$$

The main result of [**31**] is the Complex Suspension Theorem:

THEOREM 2.1. *If X is a subvariety of $\mathbf{CP}^n$, the maps*

$$\Sigma\!\!\!/ : \widetilde{\mathcal{C}}_p(X) \to \widetilde{\mathcal{C}}_{p+1}(\Sigma\!\!\!/ X)$$

are homotopy equivalences for all $p \geq 0$.

An immediate corollary of this theorem is the following description of cycle spaces on $\mathbf{CP}^n$:

THEOREM 2.2.

$$\widetilde{\mathcal{C}}_p(\mathbf{CP}^n) \simeq \prod_{i=0}^{n-p} K(\mathbf{Z}, 2i).$$

Indeed, the connected components of the space of 0-cycles on $\mathbf{CP}^{n-p}$ are just the infinite symmetric products of $\mathbf{CP}^{n-p}$ with coefficients in $\mathbf{Z}$. So, according to the Dold-Thom Theorem

$$\widetilde{\mathcal{C}}_0(\mathbf{CP}^{n-p}) \simeq \mathbf{Z} \times K(\mathbf{Z}, 2) \times \ldots \times K(\mathbf{Z}, 2(n-p-1)) \times K(\mathbf{Z}, 2(n-p)).$$

Now apply the Complex Suspension Theorem p times; this immediately gives the result.

REMARK. It is not surprising that the cycle spaces turn out to be products of $K(\pi, n)$'s: according to the Moore's Theorem (see [**16**]) any abelian topological monoid is homotopy equivalent to a product of Eilenberg-MacLane spaces.

2. Real cycle spaces

Complex conjugation induces an anti-holomorphic involution on $\mathbf{CP}^n$:

$$t : (z_0, \ldots, z_n) \to (\bar{z}_0, \ldots, \bar{z}_n).$$

A variety is called *real* if its ideal is fixed by t; a *real* cycle is a formal sum of real subvarieties[1]. The complex suspension $\Sigma\!\!\!/$ of a real cycle is again a real cycle; this makes it possible to adapt the methods used for complex cycles, to study real cycle spaces. The definitions and results below can be found in [**30**].

For a real variety X denote by $RC_{p,\cdot}(X)$ the submonoid of real cycles in $C_{p,\cdot}(X)$. The submonoid of $RC_{p,\cdot}(X)$ which consists of the cycles of the form $c + t(c), c \in C_{p,\cdot}(X)$ is called the monoid of Galois sums and denoted by $DC_{p,\cdot}(X)$.

The stabilised spaces of real cycles, which we denote by $\mathcal{RC}_p(X)$ and $\mathcal{DC}_p(X)$ respectively, are defined in the same way as the stabilised space $\mathcal{C}_p(X)$ via the mapping telescope construction or as the loop spaces $\Omega B(RC_{p,\cdot}(X))$ and $\Omega B(DC_{p,\cdot}(X))$.

[1]so we distinguish a *real variety* from its *real locus*.

As in the case of complex cycles, one can consider spaces $\widetilde{\mathcal{RC}}_p(X)$ and $\widetilde{\mathcal{DC}}_p(X)$ of "real cycles with coefficients in $\mathbf{Z}$", defined as the group completions of $RC_{p,\cdot}(X)$ and $DC_{p,\cdot}(X)$ respectively. (Clearly $\widetilde{\mathcal{RC}}_p(X)$ and $\widetilde{\mathcal{DC}}_p(X)$ are subgroups of $\widetilde{\mathcal{C}}_p(X)$.) These spaces are homotopy equivalent to the spaces $\mathcal{RC}_p(X)$ and $\mathcal{DC}_p(X)$ by the theorem of Lima-Filho.

The quotient group of $\widetilde{\mathcal{RC}}_p(X)$ by $\widetilde{\mathcal{DC}}_p(X)$ is denoted by $\widetilde{\mathcal{E}}_p(X)$ and is referred to as the space of real cycles modulo 2 (which may be somewhat misleading).

The main result about real cycles is the Suspension Theorem:

THEOREM 2.3. **[30]** *If X is a real subvariety of $\mathbf{CP}^n$, the maps*

$$\not{\Sigma} : \widetilde{\mathcal{RC}}_p(X) \to \widetilde{\mathcal{RC}}_{p+1}(\not{\Sigma} X),$$
$$\not{\Sigma} : \widetilde{\mathcal{DC}}_p(X) \to \widetilde{\mathcal{DC}}_{p+1}(\not{\Sigma} X),$$
$$\not{\Sigma} : \widetilde{\mathcal{E}}_p(X) \to \widetilde{\mathcal{E}}_{p+1}(\not{\Sigma} X)$$

are homotopy equivalences for all $p \geq 0$.

The spaces of cycles of dimension 0 have a following description:

$$\widetilde{\mathcal{DC}}_0(X) = \mathbf{Z} \times SP^\infty(X/t, \mathbf{Z}),$$
$$\widetilde{\mathcal{E}}_0(X) = \mathbf{Z}/2 \times SP^\infty(X_{\mathbf{R}}, \mathbf{Z}/2),$$

where $X_{\mathbf{R}}$ is the real locus of X. The space $\widetilde{\mathcal{RC}}_0(X)$ does not have such a simple description in terms of symmetric products, however $\widetilde{\mathcal{RC}}_0(X) \otimes \mathbf{Z}/2$, i.e. the space of real 0-cycles with coefficients in $\mathbf{Z}/2$ is homotopy equivalent to the product

$$\mathbf{Z}/2 \times SP^\infty((X/t)/X_{\mathbf{R}}, \mathbf{Z}/2) \times SP^\infty(X_{\mathbf{R}}, \mathbf{Z}/2).$$

Combining Theorem 2.3 with the Dold-Thom Theorem, we can describe the spaces of real cycles modulo 2 on $\mathbf{CP}^n$:

THEOREM 2.4. **[30]**

$$\widetilde{\mathcal{E}}_p(\mathbf{CP}^n) \simeq \prod_{i=0}^{n-p} K(\mathbf{Z}/2, i).$$

Lam in his thesis did not calculate the homotopy groups of $\widetilde{\mathcal{RC}}_p(\mathbf{CP}^n)$; here we present a method of doing this. By virtue of the Suspension Theorem it is enough to consider 0-cycles.

First we describe the homotopy of $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$ rationally. As $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$ is just the t-invariant subspace of $\mathbf{Z} \times SP^\infty(\mathbf{CP}^n, \mathbf{Z})$ its rational homotopy groups coincide with the rational homology of $\mathbf{CP}^n/t$ (see the discussion of the homology transfer map in the previous chapter). The rational and mod 2 homology of $\mathbf{CP}^n/t$ are calculated in Appendix A; from Proposition A.3 directly follows

THEOREM 2.5.

$$\pi_i(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) \otimes \mathbf{Q} = \mathbf{Q} \text{ for } i = 4k \leq 4n \text{ and } 0 \text{ otherwise.}$$

For the corresponding spaces with coefficients in $\mathbf{Z}/2$ we have a direct product:

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2 = \mathbf{Z}/2 \times SP^\infty((\mathbf{CP}^n/t)/\mathbf{RP}^n, \mathbf{Z}/2) \times SP^\infty(\mathbf{RP}^n, \mathbf{Z}/2).$$

Hence, for $i > 0$

$$\pi_i(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2) = H_i((\mathbf{CP}^n/t)/\mathbf{RP}^n), \mathbf{Z}/2) \oplus H_i(\mathbf{RP}^n, \mathbf{Z}/2)$$

and $\pi_0(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2) = \mathbf{Z}/2$. From Proposition A.2 it follows that

$$rk(\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2)) = [\frac{1}{2}k] + 1 \text{ for } k \leq n,$$

$$rk(\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2)) = [\frac{1}{2}k] + 1 - (k-n) \text{ for } n < k \leq 2n,$$

where $[x]$ denotes the integral part of x.

Finally, we have a "coefficient fibration"

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \overset{\times 2}{\to} \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \to \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2.$$

If we know $\pi_*(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) \otimes \mathbf{Q}$ and $\pi_*(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \otimes \mathbf{Z}/2)$, then from the exact sequence of the coefficient fibration we can determine the homotopy groups of $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$, given that these groups contain no torsion apart from $\mathbf{Z}/2$ - torsion. This is established by the following lemma:

LEMMA 2.6. *Let X be a real projective variety. Then if $H_*(X)$ is torsion-free, $\pi_*(\widetilde{\mathcal{RC}}_0(X))$ does not contain any torsion apart from $\mathbf{Z}/2$ -torsion.*

PROOF. Let $S^-(X)$ be the subgroup of $SP^\infty(X, \mathbf{Z})$, consisting of all c such that $t(c) = -c$. There is a fibration

$$\widetilde{\mathcal{RC}}_0(X) \times S^-(X) \to \mathbf{Z} \times SP^\infty(X, \mathbf{Z}) \to \mathbf{Z}/2 \times SP^\infty((X/t)/X_{\mathbf{R}}, \mathbf{Z}/2).$$

As $H_*(X)$ is torsion-free and $H_*((X/t)/X_{\mathbf{R}}, \mathbf{Z}/2)$ contains only $\mathbf{Z}/2$-torsion, the statement of the lemma follows from the exact sequence of this fibration. □

So we can determine the homotopy groups of $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$ (and, hence, $\mathcal{RC}_0(\mathbf{CP}^n)$) completely. As an example, $\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n))$ for $n \leq 6$ are shown in Fig. 2.1. (Obviously, $\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) = 0$ for all $k > 2n$, so we do not show these groups in the table.)

The general formulae for the ranks of the torsion in $\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n))$ are not too pleasant. However, when $n \to \infty$ things become simpler. Indeed, from the exact

n	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
1	Z	Z/2	0										
2	Z	Z/2	Z/2	0	Z								
3	Z	Z/2	Z/2	Z/2	Z	Z/2	0						
4	Z	Z/2	Z/2	Z/2	Z ⊕ Z/2	Z/2	Z/2	0	Z				
5	Z	Z/2	Z/2	Z/2	Z ⊕ Z/2	Z/2 ⊕ Z/2	Z/2	Z/2	Z	Z/2	0		
6	Z	Z/2	Z/2	Z/2	Z ⊕ Z/2	Z/2 ⊕ Z/2	Z/2 ⊕ Z/2	Z/2	Z ⊕ Z/2	Z/2	Z/2	0	Z

Figure 2.1. Homotopy groups of $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$ for $n \le 6$.

sequence of the coefficient fibration it easily follows that if $k \le n$, the rank of torsion in $\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n))$ is equal to $[\frac{k-1}{4}] + 1 = [\frac{k+3}{4}]$. Notice that this number does not depend on n.

LEMMA 2.7. *The natural inclusion*

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \hookrightarrow \widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1})$$

induces isomorphisms in homotopy groups of dimensions $\le n$.

PROOF. There is a fibration

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \hookrightarrow \widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}) \to \widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}) /\!\!/ \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n).$$

The space $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}) /\!\!/ \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)$ can be considered as a space of "real" 0-cycles of degree zero on $\mathbf{CP}^{n+1}/\mathbf{CP}^n = S^{2n+2}$. In particular, we have a fibration

$$SP^\infty(S^{2n+2}/t, \mathbf{Z}) \to \widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}) /\!\!/ \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \to SP^\infty(S^{n+1}, \mathbf{Z}/2).$$

By Proposition A.4 $H_i(S^{2n+2}/t) = 0$ for $i \le n$ so $\pi_i(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}) /\!\!/ \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) = 0$ for $i \le n$ as well. This implies that

$$\pi_i(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) \to \pi_i(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}))$$

is an isomorphism for $i < n$ and an epimorphism for $i = n$. However, calculation shows that $\pi_n(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^n)) = \pi_n(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^{n+1}))$. □

If we define $\widetilde{\mathcal{RC}}_0(\mathbf{CP}^\infty)$ as the direct limit of the sequence of inclusions

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^1) \hookrightarrow \widetilde{\mathcal{RC}}_0(\mathbf{CP}^2) \hookrightarrow \ldots \hookrightarrow \widetilde{\mathcal{RC}}_0(\mathbf{CP}^n) \hookrightarrow \ldots$$

the arguments above give us

THEOREM 2.8. *The free part of* $\pi_k(\widetilde{\mathcal{RC}}_0(\mathbf{CP}^\infty))$ *is* $\mathbf{Z}$ *if* k *is divisible by* 4 *and* 0 *otherwise. All torsion is* $\mathbf{Z}/2$ *-torsion; the rank of the torsion part of* π_k *is* $[\frac{k+3}{4}]$.

3. Cycle spaces and group actions

Lam's results on real cycles led Lawson and Michelsohn to the investigation of the actions of arbitrary finite groups on the cycle spaces.

If a finite group G acts on $\mathbf{C}^{n+1}$ linearly, there is an induced action of G on $\mathbf{CP}^n$ by linear automorphisms and a corresponding action of G on the cycle spaces. Extend the action of G on $\mathbf{C}^{n+2}$ to be trivial on the last coordinate; there are corresponding extensions of the induced actions on $\mathbf{CP}^{n+1}$ and the cycle spaces. Let X be a G-invariant subvariety of $\mathbf{CP}^n$; denote by $\widetilde{\mathcal{C}}_p(X)^G$ the space of G-invariant cycles with integral coefficients on X and by $G \cdot \widetilde{\mathcal{C}}_p(X)$ the subgroup of cycles, which are of the form $\sum_{g\in G} g(c),\ c \in \widetilde{\mathcal{C}}_p(X)$. The following theorem can be found in [**36**]:

THEOREM 2.9. *For each* p *the complex suspension maps*

$$\Sigma\!\!\!/ : \widetilde{\mathcal{C}}_p(X)^G \to \widetilde{\mathcal{C}}_p(\Sigma\!\!\!/ X)^G$$

and

$$\Sigma\!\!\!/ : G \cdot \widetilde{\mathcal{C}}_p(X) \to G \cdot \widetilde{\mathcal{C}}_p(\Sigma\!\!\!/ X)$$

induce isomorphisms on all homotopy groups $\pi_i(\cdot) \otimes \Lambda$, $i \geq 0$, *tensored with any ring* Λ *in which the order of* G *is invertible.*

The proof of this theorem closely follows the proof of the Complex Suspension Theorem; one has to check only that all constructions used there can be made G-invariant.

In fact, the requirement of *linearity* of the action of G on $\mathbf{CP}^n$ can be slightly relaxed. If we require that G acts on $\mathbf{CP}^n$ by linear or antilinear (rather than just linear) automorphisms, the action of G can also be extended to $\mathbf{CP}^{n+1}$: trivially in the case of a linear automorphism and by complex conjugation in the case of an antilinear automorphism. Theorem 2.9 will still be true in this situation; the proof remains virtually unchanged.

In the same fashion as in the Example 1.5 we can introduce spaces of cycles with rational coefficients $\widetilde{\mathcal{C}}_p(X) \otimes \mathbf{Q}$ as direct limits of sequences

$$\widetilde{\mathcal{C}}_p(X) \overset{\times 2}{\to} \widetilde{\mathcal{C}}_p(X) \overset{\times 3}{\to} \dots$$

Then

$$\pi_*(\widetilde{\mathcal{C}}_p(X)) \otimes \mathbf{Q} = \pi_*(\widetilde{\mathcal{C}}_p(X) \otimes \mathbf{Q}).$$

Now we can re-state Theorem 2.9 as follows:

THEOREM 2.10. *Let G act on $\mathbf{CP}^n$ by linear or antilinear automorphisms. For each p the complex suspension maps*

$$\Sigma\!\!\!/ : \widetilde{\mathcal{C}}_p(X)^G \otimes \mathbf{Q} \to \widetilde{\mathcal{C}}_p(\Sigma\!\!\!/ X)^G \otimes \mathbf{Q}$$

and

$$\Sigma\!\!\!/ : G \cdot \widetilde{\mathcal{C}}_p(X) \otimes \mathbf{Q} \to G \cdot \widetilde{\mathcal{C}}_p(\Sigma\!\!\!/ X) \otimes \mathbf{Q}$$

are homotopy equivalences.

CHAPTER 3

Spaces of Quaternionic Cycles

In this chapter we define the quaternionic cycle spaces, describe their main properties and prove the "rational suspension theorem". As a corollary we obtain a full description of the spaces of quaternionic cycles with rational coefficients on projective spaces.

1. Some generalities

There is no quaternionic algebraic geometry as such because of the noncommutativity of quaternions. However, real objects in classical algebraic geometry often have a certain quaternionic flavour.

The usual way to define real objects is to consider varieties (functions, polynomials) invariant (equivariant) with respect to the complex conjugation on $\mathbf{CP}^n$. But there are other antiholomorphic involutions on complex manifolds, and they give rise to an alternative treatment of algebraic reality. N.Alling and N.Greenleaf in **[1]** defined a real algebraic curve as a complex curve with a given antiholomorphic involution; they provide examples which suggest that this is the right set-up for some problems of real algebraic geometry.

The simplest example of an antiholomorphic involution on a curve, apart from the complex conjugation on $\mathbf{CP}^1$, is the antipodal map on $\mathbf{CP}^1$, given in homogeneous coordinates as

$$(z_0, z_1) \to (-\bar{z}_1, \bar{z}_0).$$

It has a natural generalisation to $\mathbf{CP}^{2n+1}$:

$$j : (z_0, z_1, z_2, z_3, \dots) \to (-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2, \dots),$$

and it is natural to ask what kind of "reality" is associated with this involution.

The answer is that this "reality" can be described as "quasi-quaternionic". This is not very surprising, if we notice that when the homogeneous coordinates in $\mathbf{CP}^{2n+1}$ are written as a set of $n+1$ quaternions, j becomes left multiplication by the quaternion j. So, in the absence of better candidates for the title "quaternionic" (at least in this context), we will use this term to refer to j-invariant objects.

This concept of a quaternionic object is not artificial; for example, j-invariant manifolds play an important role in the study of symplectic cobordism, see [**11**]. As H.B.Lawson pointed out, quaternionic cycles are related to complex and real cycles in the same way as quaternionic representations of groups are related to complex and real representations. In any case, the results we will obtain seem to justify the definitions.

2. Action of j on $\mathbf{CP}^{2n+1}$

The involution $j : \mathbf{CP}^{2n+1} \to \mathbf{CP}^{2n+1}$ is fixed point-free. It reverses the orientation of $\mathbf{CP}^{2n+1}$, so the quotient space $\mathbf{CP}^{2n+1}/j$ is a non-orientable manifold. Notice that j induces involutions on the fibres of the map

$$\mathbf{CP}^{2n+1} \to \mathbf{HP}^n.$$

This means that there is a fibration

$$\mathbf{RP}^2 \to \mathbf{CP}^{2n+1}/j \to \mathbf{HP}^n.$$

The spectral sequence of this fibration collapses at the term E^2, and thus the homology groups of $\mathbf{CP}^{2n+1}/j$ are as follows:

$$H_i(\mathbf{CP}^{2n+1}/j) = \begin{cases} \mathbf{Z} & \text{when } i \equiv 0 \ (mod\ 4) \quad i \leq 4n+2, \\ \mathbf{Z}/2 & \text{when } i \equiv 1 \ (mod\ 4) \quad i \leq 4n+2, \\ 0 & \text{otherwise;} \end{cases}$$

Hence the rational homology groups are equal to $\mathbf{Q}$ in dimensions $4i$ and 0 otherwise. In some sense $\mathbf{CP}^{2n+1}/j$ is a hybrid between complex and quaternionic projective spaces, having the same rational homology as $\mathbf{HP}^n$, and the same higher homotopy groups as $\mathbf{CP}^{2n+1}$.

The sequence of natural inclusions

$$\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^3 \hookrightarrow \ldots \hookrightarrow \mathbf{CP}^\infty$$

gives rise to a sequence

$$\mathbf{CP}^1/j \hookrightarrow \mathbf{CP}^3/j \hookrightarrow \ldots \hookrightarrow \mathbf{CP}^\infty/j$$

It is easy to see that the induced sequence of maps in homology

$$H_*(\mathbf{CP}^1/j) \to H_*(\mathbf{CP}^3/j) \to \ldots \to H_*(\mathbf{CP}^\infty/j)$$

is also a sequence of inclusions.

REMARKS. As $\mathbf{CP}^\infty/j$ has $\mathbf{CP}^\infty$ as double cover, only two of its homotopy groups are non-trivial: $\pi_1 = \mathbf{Z}/2$ and $\pi_2 = \mathbf{Z}$. However, as homology groups tell us, it is not a product of Eilenberg-MacLane spaces. In fact, the homotopy type of $\mathbf{CP}^\infty/j$ is completely defined by its Postnikov invariant $k \in H^3(\mathbf{RP}^\infty, \mathbf{Z}) = \mathbf{Z}/2$. As $\mathbf{CP}^\infty/j$ does not split as a product, its Postnikov invariant is equal to 1.

Like $\mathbf{RP}^\infty$ and $\mathbf{CP}^\infty$, $\mathbf{CP}^\infty/j$ is also a classifying space. The group it classifies is the normaliser of the maximal torus $S^1 \subset Sp(1)$; see [**11**].

DEFINITION 3.1. A homogeneous polynomial P of $2n+2$ variables is called j-invariant if $P(z_i) = \overline{P(j(z_i))}$. (cf. Example 1.4).

We denote the graded ring of j-invariant polynomials by $\mathbf{C}[z_0, \dots, z_{2n+1}]^j$.

REMARK. We will use the letter j to denote the generator of the action of $\mathbf{Z}/4$ on the space of polynomials. It sends $P(z_i)$ to $\overline{P(j(z_i))}$; this should not lead to confusion.

PROPOSITION 3.1. *The real vector space* $\mathbf{C}[z_0, \dots, z_{2n+1}]^j_d$ *of j-invariant polynomials of degree d has dimension 0 if d is odd and* $\binom{2n+1+d}{d}$ *if d is even.*

PROOF. The map $j : P(z_i) \to \overline{P(j(z_i))}$ is real linear, so to describe it completely it is enough to consider the action of j on monomials.

Denote by m the monomial $z_0^{d_0} \cdot \ldots \cdot z_{2n+1}^{d_{2n+1}}$ and by m' the monomial $z_0^{d_1} \cdot z_1^{d_0} \cdot \ldots \cdot z_{2n}^{d_{2n+1}} \cdot z_{2n+1}^{d_{2n}}$. (We obtain m' from m interchanging the degrees of z_{2k} and z_{2k+1}.) Also set $D_0 = d_0 + d_2 + \ldots + d_{2n}$ and $D_1 = d_1 + \ldots + d_{2n+1}$. Then

$$j(m) = \overline{m(j(z))} = (-1)^{D_0} m'$$

and

$$j(m') = \overline{m'(j(z))} = (-1)^{D_1} m.$$

First suppose $m \neq m'$. The action of j sends $am + bm'$ to $(-1)^{D_0}\bar{b}m + (-1)^{D_1}\bar{a}m'$. So if $am + bm'$ is fixed by j, we have

$$(-1)^{D_0}\bar{b} = a,$$
$$(-1)^{D_1}\bar{a} = b.$$

If $d = D_0 + D_1$ is odd this implies $a = b = 0$. If d is even this means that there are 2 real linear conditions on the coefficients of the pair m, m' and therefore for a given monomial m, j-fixed expressions of the form $am + bm'$ form a 2-dimensional real vector space.

When $m = m'$ we have $D_0 = D_1$ so the degree of m is even. If in this case am is fixed by j, this implies that $(-1)^{D_0}\bar{a} = a$; this is a real linear condition on a. So a j-fixed monomial spans a 1-dimensional real vector space.

A polynomial is j-invariant if and only if it is a sum of terms which are either j-fixed expressions of the form $am + bm'$, where $m \neq m'$, or j-fixed monomials. From this we get that $\dim \mathbf{C}[z_0, \dots, z_{2n+1}]_d^j = 0$ for d odd. For d even the dimension of $\mathbf{C}[z_0, \dots, z_{2n+1}]_d^j$ is equal to the number of different monomials in $z_0, \dots, z_{2n+1}$, i.e. for d even

$$\dim \mathbf{C}[z_0, \dots, z_{2n+1}]_d^j = \binom{2n+1+d}{d}.$$

□

3. Quaternionic varieties and cycles

DEFINITION 3.2. X is a quaternionic (or j-invariant) subvariety of $\mathbf{CP}^{2n+1}$ if its ideal is fixed by the action of j on the space of polynomials.

If $X \subset \mathbf{CP}^{2n+1}$ is quaternionic, then j induces involutions on the cycle spaces:

$$j : \mathcal{C}_{p,d}(X) \to \mathcal{C}_{p,d}(X).$$

DEFINITION 3.3. The fixed subspaces are called the spaces of quaternionic (or j-invariant) cycles; we denote them by $\mathcal{QC}_{p,d}(X)$.

In a certain sense, quaternionic varieties and cycles are a particular case of real varieties and cycles. Indeed, define a map

$$v : \mathbf{CP}^{2n+1} \to \mathbf{CP}^{(n+1)(2n+3)-1}$$

in the following way: choose a basis P_i, $1 \le i \le (n+1)(2n+3)$ for the vector space $\mathbf{C}[z_0, \dots, z_{2n+1}]_2^j$ of j-invariant polynomials of degree 2; then v sends a point $z \in \mathbf{CP}^{2n+1}$ to the point $(P_1(z), \dots, P_{(n+1)(2n+3)}(z))$. It is clearly a regular embedding (in fact, it is essentially the Veronese embedding). The following diagram is commutative:

$$\begin{array}{ccc} \mathbf{CP}^{2n+1} & \hookrightarrow & \mathbf{CP}^{2n^2+5n+2} \\ \downarrow j & & \downarrow conj. \\ \mathbf{CP}^{2n+1} & \hookrightarrow & \mathbf{CP}^{2n^2+5n+2} \end{array}$$

This means that any j-invariant subset of $\mathbf{CP}^{2n+1}$ can be embedded into $\mathbf{CP}^{2n^2+5n+2}$ as a real subset.

The simplest examples of quaternionic subvarieties are j-invariant linear subspaces. If a j-invariant linear subspace satisfies $F(z) = 0$, where F is linear, then it also satisfies $F(j(z)) = 0$. These two equations can be written as a single linear equation in quaternions; hence we get

PROPOSITION 3.2. *All j-invariant linear subspaces of $\mathbf{CP}^{2n+1}$ are inverse images of linear subspaces in $\mathbf{HP}^n$ under the map $\mathbf{CP}^{2n+1} \to \mathbf{HP}^n$. In particular, j-invariant complex lines are just the fibres of this map.*

A corollary of this is that all j-invariant linear subspaces are odd-dimensional. More generally, the following is true:

PROPOSITION 3.3. *The degree of an even-dimensional quaternionic cycle is even.*

Before proving this, we will establish a technical lemma:

LEMMA 3.4. *Let X be a quaternionic cycle of codimension $2k$ or $2k+1$ in $\mathbf{CP}^{2n+1}$. Then there exists a j-invariant linear subspace of dimension $2k-1$, disjoint from X.*

PROOF. Induction on k. For $k=0$ there is nothing to prove.

Assume the statement of the lemma is true for all cycles of codimension $\leq 2k-1$. Notice that for any cycle of codimension ≥ 2 there exists a complex j-invariant line, disjoint from it. Indeed, a cycle which intersects all j-invariant lines in $\mathbf{CP}^{2n+1}$ maps onto $\mathbf{HP}^n$ under the map $\mathbf{CP}^{2n+1} \to \mathbf{HP}^n$. However, this map does not raise dimensions of algebraic subvarieties of $\mathbf{CP}^{2n+1}$, so such a cycle must have complex dimension $2n$ or $2n+1$.

For a cycle X of codimension $2k$ or $2k+1$ find a complex j-invariant line l, disjoint from it and a j-invariant linear subspace L of dimension $2n-1$, such that $L \cap l = \emptyset$. The projection P_l away from l onto L commutes with j and preserves the dimension of X. This means that the codimension of $P_l(X)$ in L is $\leq 2k-1$ and by the induction hypothesis, there is a j-invariant linear subspace $L' \subset L$ such that $\dim L' = 2k-3$ and $P_l(X) \cap L' = \emptyset$. The linear span of l and L' is j-invariant, has dimension $2k-1$ and does not intersect X. □

PROOF OF PROPOSITION 3.3. If a divisor is j-invariant, it is defined by a j-invariant polynomial. By Proposition 3.1 such a polynomial has even degree and, hence, the degree of the divisor is also even.

But this is enough, as the case of arbitrary codimension $2k$ can be reduced to the case of divisors with the help of a projection away from a suitably chosen j-invariant linear subspace of dimension $2k-1$. □

REMARK. This proposition has another proof, which is more transparent geometrically. For any even-dimensional quaternionic subvariety take a j-invariant linear subspace of complementary dimension, which intersects it transversally. The intersection is j-invariant and, as j is free, consists of an even number of points. This proves the proposition given that we can always find a transversal j-invariant linear subspace. It is not hard to show this, but we will not give a proof here.

There is yet another difference between the cycles of even and odd dimensions.

PROPOSITION 3.5. *The involution j preserves the orientation of the even-dimensional cycles and reverses the orientation of the odd-dimensional cycles.*

PROOF. We will show below (Lemma 3.6) that the spaces $\mathcal{QC}_{p,d}(\mathbf{CP}^{2n+1})$ are connected. This means that we need to check the statement of the proposition for only one cycle in each $\mathcal{QC}_{p,d}(\mathbf{CP}^{2n+1})$. For p odd we can look at the cycles of the form $d \cdot l$, where l is a j-invariant linear subspace; for p even we can check the cycles of the form $\frac{1}{2}d(l + j(l))$, where l is a linear subspace. In both cases the statement is trivially true. □

LEMMA 3.6. *The spaces $\mathcal{QC}_{p,d}(\mathbf{CP}^{2n+1})$ are connected.*

PROOF. The proof is essentially the same as for the complex and real cycles. First we consider the case of p odd, $p = 2k + 1$.

Inside $\mathcal{QC}_{2k+1,d}(\mathbf{CP}^{2n+1})$ there is a copy of the Grassmanian of k-dimensional subspaces of $\mathbf{HP}^n$, which represents the d-fold multiples of linear subspaces (see Proposition 3.2). We will show that for any cycle $c \in \mathcal{QC}_{2k+1,d}(\mathbf{CP}^{2n+1})$ there is a path, joining it to a point on this Grassmanian. As quaternionic Grassmanians are path connected, this will prove the lemma for p odd.

By Lemma 3.4 we can choose j-invariant linear subspaces l and L of dimensions $2(n - k) - 1$ and $2k + 1$ respectively, such that

$$c \cap l = l \cap L = \emptyset.$$

Then $\mathbf{CP}^{2n+1} - l$ can be thought of as a vector bundle of rank $2(n - k) - 1$ over L, the fibre over a point $x \in L$ being the linear span of x and l, with l taken out. Denote by ϕ_t the multiplication by a real number t in this bundle:

$$\phi_t : \mathbf{CP}^{2n+1} - l \to \mathbf{CP}^{2n+1} - l;$$

clearly, ϕ_t commutes with the action of j. The support of c does not intersect l, so the path $[0, 1] \to \phi_t(c)$ joins the cycle $d \cdot L$ with c.

The case of p even is completely analogous. Using the same method as above one can show that for any p-cycle there is a path in $\mathcal{QC}_{p,d}(\mathbf{CP}^{2n+1})$ which joins it to a p-cycle that is contained in a j-invariant linear subspace of dimension $p + 1$. The space of such cycles is a fibration over the quaternionic Grassmanian with the fibre being the space of quaternionic divisors on $\mathbf{CP}^{p+1}$. It is connected and this proves the lemma. □

There are two basic examples of quaternionic cycle spaces: 0-cycles and divisors on $\mathbf{CP}^{2n+1}$. For 0-cycles on a connected X we have (by definition)

$$\mathcal{QC}_{0,d}(X) = SP^d(X/j).$$

Spaces of divisors are described by Proposition 3.1:

$$\mathcal{QC}_{2n,d}(\mathbf{CP}^{2n+1}) = \mathbf{RP}^D, \text{ where } D = \binom{2n+1+d}{d}$$

for d even and

$$\mathcal{QC}_{2n,d}(\mathbf{CP}^{2n+1}) = \emptyset$$

for d odd. When $n = 0$, divisors and 0-cycles are the same thing, so, as we have seen in Example 1.4

$$SP^n(\mathbf{RP}^2) = \mathbf{RP}^{2n}.$$

For a j-invariant subvariety X the stabilised quaternionic cycle spaces $\mathcal{QC}_p(X)$ are defined in complete analogy with the complex and real case via the mapping telescope construction. The abelian group $\widetilde{\mathcal{QC}}_p(X)$ can be defined as the j-fixed subgroup of $\widetilde{\mathcal{C}}_p(X)$.

The fact that any j-invariant subvariety can be realised as a real subvariety immediately implies that $\mathcal{QC}_p(X)$ and $\widetilde{\mathcal{QC}}_p(X)$ are homotopy equivalent. Indeed, the j-equivariant Veronese map v sends quaternionic cycles on $X \subset \mathbf{CP}^{2n+1}$ to real cycles on $v(X)$. Moreover, it induces a homeomorphism between the corresponding spaces of quaternionic and real cycles, so

$$\mathcal{QC}_p(X) = \mathcal{RC}_p(v(X)) \simeq \widetilde{\mathcal{RC}}_p(v(X)) = \widetilde{\mathcal{QC}}_p(X).$$

As before, we will state all the results for "naïve" group completions, i.e. for the spaces $\widetilde{\mathcal{QC}}_p(X)$.

4. Rational suspension theorem and cycles on $\mathbf{CP}^{2n+1}$

The complex suspension $\Sigma\!\!\!/$, taken twice (we denote it by $\Sigma\!\!\!/\!\Sigma\!\!\!/$), carries quaternionic cycles on $X \subset \mathbf{CP}^{2n+1}$ to quaternionic cycles on $\Sigma\!\!\!/\!\Sigma\!\!\!/\mathbf{CP}^{2n+1} = \mathbf{CP}^{2n+3}$. This gives maps

$$\Sigma\!\!\!/\!\Sigma\!\!\!/ : \widetilde{\mathcal{QC}}_p(X) \to \widetilde{\mathcal{QC}}_{p+2}(\Sigma\!\!\!/\!\Sigma\!\!\!/ X).$$

To show that these maps are homotopy equivalences one could try to mimic Lawson's proof of the Complex Suspension Theorem. However, there are certain difficulties inherent in this approach. The reason is that some questions which have simple answers in codimension 1 (in the case of complex cycles) have to be answered in codimension 2; this turns out to be much harder.

But if we work with the spaces of cycles with rational coefficients (see Chapter 2, Section 3), Theorem 2.10 becomes applicable. Here we prove the following statement:

THEOREM 3.7. *If X is a quaternionic subvariety of $\mathbf{CP}^{2n+1}$, the maps*

$$\not{\Sigma} : \widetilde{\mathcal{QC}}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{QC}}_{p+2}(\not{\Sigma} X) \otimes \mathbf{Q}$$

are homotopy equivalences for all p.

PROOF. Extend the "quaternionic" action of $\mathbf{Z}/2$ by j on $\mathbf{CP}^{2n+1}$ to an action of $\mathbf{Z}/4$ on $\mathbf{CP}^{2n+2}$:

$$j_1 : (z, z_{2n+2}) \to (jz, \bar{z}_{2n+2}),$$

where $z = (z_0, z_1, \dots, z_{2n+1}) \in \mathbf{CP}^{2n+1}$, j_1 is the action of the generator of $\mathbf{Z}/4$. This action can be extended further to an action of $\mathbf{Z}/4$ on $\mathbf{CP}^{2n+3}$:

$$j_2 : (z, z_{2n+2}, z_{2n+3}) \to (jz, \bar{z}_{2n+2}, \bar{z}_{2n+3}).$$

Applying complex suspension to the space of j-invariant cycles and then to the space of j_1-invariant cycles, by Theorem 2.10 we get a homotopy equivalence

$$\widetilde{\mathcal{QC}}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{C}}_{p+2}(\not{\Sigma} X)^{\mathbf{Z}/4} \otimes \mathbf{Q}.$$

The cycle space on the right is invariant under j_2, which changes the last two coordinates only by conjugation. Now our purpose is to show that $\widetilde{\mathcal{C}}_{p+2}(\not{\Sigma} X)^{\mathbf{Z}/4} \otimes \mathbf{Q}$ is homotopy equivalent to $\widetilde{\mathcal{QC}}_{p+2}(\not{\Sigma} X) \otimes \mathbf{Q}$.

Define a map

$$T : \mathbf{CP}^{2n+3} \to \mathbf{CP}^{2n+3}$$

by

$$T : (z_0, \dots, z_{2n+1}, z_{2n+2}, z_{2n+3}) \to (z_0, \dots, z_{2n+1}, -z_{2n+3}, z_{2n+2}).$$

The automorphism T has order 4. It can be checked that

$$j_2 T = T j_2 = j \text{ and } j T^3 = T^3 j = j_2.$$

If c is a j-invariant cycle, then it is easy to see from the properties of T, that $c + Tc + T^2c + T^3c$ is a j_2-invariant cycle and if c is j_2-invariant then $c + Tc + T^2c + T^3c$ is j-invariant. This means that we have maps

$$P_1 : \widetilde{\mathcal{QC}}_{p+2}(\not{\Sigma} X) \otimes \mathbf{Q} \xrightarrow{1+T+T^2+T^3} \widetilde{\mathcal{C}}_{p+2}(\not{\Sigma} X)^{j_2} \otimes \mathbf{Q}$$

and

$$P_2 : \widetilde{\mathcal{C}}_{p+2}(\not{\Sigma} X)^{j_2} \otimes \mathbf{Q} \xrightarrow{1+T+T^2+T^3} \widetilde{\mathcal{QC}}_{p+2}(\not{\Sigma} X) \otimes \mathbf{Q}.$$

These maps establish the desired homotopy equivalence. To see this, notice that

$$P_1 P_2 = (1 + T + T^2 + T^3)^2$$

induces multiplication by 16 in $\pi_*(\widetilde{\mathcal{QC}}_{p+2}(\not\# X) \otimes \mathbf{Q})$, as T is homotopic to the identity through linear maps

$$T_t : (\dots, z_{2n+2}, z_{2n+3}) \to (\dots, z_{2n+2}\cos t - z_{2n+3}\sin t, z_{2n+2}\sin t + z_{2n+3}\cos t),$$

which commute with j. (Here $T_0 = \mathrm{Id}$, $T_{\pi/2} = T$.) But in a rational vector space multiplication by 16 is an isomorphism, hence P_1P_2 is a homotopy equivalence. The same is true for P_2P_1, so P_1 and P_2 are also homotopy equivalences.

Finally, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{QC}}_{p+2}(\not\# X) \otimes \mathbf{Q} & \xrightarrow{P_1} & \widetilde{\mathcal{C}}_{p+2}(\not\# X)^{j_2} \otimes \mathbf{Q} \\ & \nwarrow & \uparrow \\ & & \widetilde{\mathcal{QC}}_p(X) \end{array}$$

where the vertical arrow is $\not\# \times 4$ and the diagonal arrow is $\not\#$. P_1 is a homotopy equivalence; this means that

$$\not\# : \widetilde{\mathcal{QC}}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{QC}}_{p+2}(\not\# X) \otimes \mathbf{Q}$$

is a homotopy equivalence. □

The Rational Suspension Theorem for quaternionic cycles allows us to describe the spaces of quaternionic cycles on $\mathbf{CP}^{2n+1}$ rationally:

THEOREM 3.8.

$$\widetilde{\mathcal{QC}}_{2p}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \prod_{i=0}^{n-p} K(\mathbf{Q}, 4i)$$

and

$$\widetilde{\mathcal{QC}}_{2p+1}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \prod_{i=0}^{n-p} K(\mathbf{Q}, 4i).$$

PROOF. The statement for $\widetilde{\mathcal{QC}}_{2p}(\mathbf{CP}^{2n+1})$ is a mere corollary of the Dold-Thom theorem. It is less trivial for $\widetilde{\mathcal{QC}}_{2p+1}(\mathbf{CP}^{2n+1})$, as all we can get from the Suspension Theorem is

$$\widetilde{\mathcal{QC}}_{2p+1}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2(n-p)+1}) \otimes \mathbf{Q}.$$

To describe the space $\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2k+1}) \otimes \mathbf{Q}$ we will use the same trick as in the proof of the Suspension Theorem. Namely, the space $\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2k+1}) \otimes \mathbf{Q}$ is homotopy equivalent to the space $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$, where the generator of $\mathbf{Z}/4$ acts on $\mathbf{CP}^{2k}$ as follows:

$$j_1 : (z, z_{2k}) \to (jz, \bar{z}_{2k}),$$

where $z = (z_0, \dots, z_{2k-1})$.

To calculate the homotopy groups of $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$ notice that there is a transfer map

$$\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q} \hookrightarrow \widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k}) \otimes \mathbf{Q},$$

which induces an isomorphism from the homotopy groups of $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$ onto the $\mathbf{Z}/4$-fixed part of the rational homology of $\mathbf{CP}^{2k}$.

Choosing appropriate representatives for the homology classes, one can easily see that $4k$-dimensional classes in $H_*(\mathbf{CP}^{2k})$ are fixed by the action of $\mathbf{Z}/4$ and $4k+2$-dimensional classes change sign under the action of a generator. So

$$H_i(\mathbf{CP}^{2k}/(\mathbf{Z}/4), \mathbf{Q}) = \mathbf{Q} \quad \text{if } i = 4m, i \leq 4k$$

and 0 otherwise. This proves the theorem. □

REMARK. According to the Theorem 3.8, $\pi_0(\widetilde{\mathcal{QC}}_p(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q}) = \mathbf{Q}$. Actually, as the spaces $\mathcal{QC}_{p,d}(\mathbf{CP}^{2n+1})$ are connected, it is easy to see that $\pi_0(\widetilde{\mathcal{QC}}_p(\mathbf{CP}^{2n+1})) = \mathbf{Z}$; the components are indexed by the degree of cycles for odd p and by half the degree for even p.

5. More on quaternionic 1-cycles

As we have seen, there are two essentially different types of quaternionic cycles: cycles of even and odd dimension. The "basic" case of a space of even-dimensional cycles is the case of 0-cycles. Spaces of 0-cycles are just infinite symmetric products and they are well-understood. The "basic" case of odd-dimensional cycles seems to be more obscure: even though we have already calculated the rational homotopy type of the spaces of quaternionic 1-cycles, it is still desirable to get a better understanding of their structure. The following construction provides natural geometric representatives for the rational homotopy of $\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1})$.

Notice that

$$\mathbf{HP}^n = \mathcal{QC}_{1,1}(\mathbf{CP}^{2n+1}),$$

as every point of $\mathbf{HP}^n$ corresponds to a j-invariant line in $\mathbf{CP}^{2n+1}$. Similarly, for all m there are inclusions

$$i_m : SP^m(\mathbf{HP}^n) \hookrightarrow \mathcal{QC}_{1,m}(\mathbf{CP}^{2n+1}),$$

defined by sending a point in the symmetric product into the corresponding linear combination of j-invariant lines. Clearly these inclusions agree and one can pass to the limit.

THEOREM 3.9. *The map*

$$i_\infty : SP^\infty(\mathbf{HP}^n) \hookrightarrow \widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1})$$

induces isomorphisms for all $k > 0$ between $\pi_k(SP^\infty(\mathbf{HP}^n))$ and the quotient of $\pi_k(\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1}))$ by torsion.

PROOF. Consider the composite map

$$SP^\infty(\mathbf{HP}^n) \xrightarrow{i_\infty} \widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1}) \xrightarrow{i'} \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}),$$

where i' is the inclusion map of the space of quaternionic cycles into the complex cycle space.

By the Dold-Thom Theorem

$$\pi_m(SP^\infty(\mathbf{HP}^n)) = \widetilde{H}_m(\mathbf{HP}^n) = \mathbf{Z} \text{ for } 0 < m = 4k \leq 4n \text{ and } 0 \text{ otherwise.}$$

We also know from Theorem 3.8 that

$$\text{rk } \pi_m(\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1})) = 1 \text{ for } 0 \leq m = 4k \leq 4n \text{ and } 0 \text{ otherwise.}$$

Finally, by Theorem 2.2

$$\pi_m(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})) = \mathbf{Z} \text{ for } 0 \leq m = 2k \leq 4n \text{ and } 0 \text{ otherwise.}$$

This implies that it is sufficient to prove the following

LEMMA 3.10. *Under the composite map*

$$i'i_\infty : SP^\infty(\mathbf{HP}^n) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})$$

the generator of $\pi_{4k}(SP^\infty(\mathbf{HP}^n))$ maps into the generator of $\pi_{4k}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}))$.

We will use induction by n.

For $n = 0$ the statement is trivial. Suppose the result is true for some $n-1$, i.e. for all k there is an isomorphism

$$\pi_{4k}(SP^\infty(\mathbf{HP}^{n-1})) \to \pi_{4k}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n-1})).$$

We will use the following notation: by $\mathbf{CP}^{2n-1}$ we will mean the subspace of $\mathbf{CP}^{2n+1}$, defined by the equations $z_{2n} = z_{2n+1} = 0$; by L_1^{2n} and by L_2^{2n} we denote the hyperplanes, defined by $z_{2n+1} = 0$ and $z_{2n} = 0$ respectively. Finally, x_0 denotes the point $(0, \ldots, 0, 1) \in \mathbf{CP}^{2n+1}$. Notice that $\mathbf{CP}^{2n-1}$ is a j-invariant subspace and

$$\mathbf{CP}^{2n-1} = L_1^{2n} \cap L_2^{2n}.$$

This is illustrated by Fig. 3.1.

The cofibration

$$\mathbf{HP}^{n-1} \hookrightarrow \mathbf{HP}^n \to \mathbf{HP}^n/\mathbf{HP}^{n-1} = S^{4n}$$

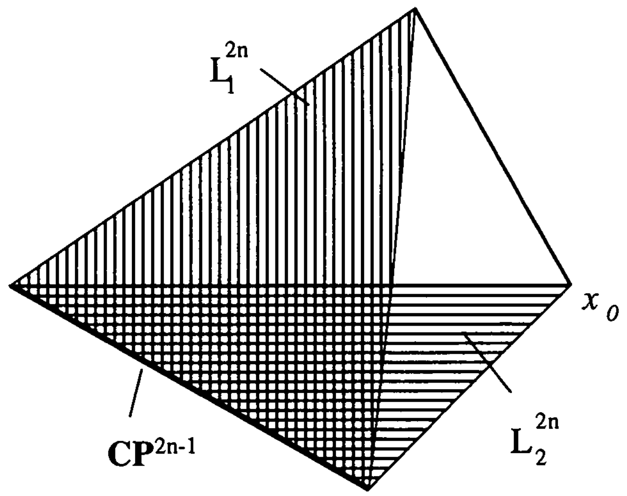

Figure 3.1.

gives rise to maps of the corresponding symmetric products; the inclusion map

$$SP^\infty(\mathbf{HP}^{n-1}) \hookrightarrow SP^\infty(\mathbf{HP}^n)$$

induces isomorphisms on the homotopy groups of dimensions $\leq 4n-4$. Similarly, the map

$$\mathbf{Z} = \pi_{4n}(SP^\infty(\mathbf{HP}^n)) \to \pi_{4n}(SP^\infty(S^{4n})) = \mathbf{Z}$$

is an isomorphism, so one can think of the map

$$S^{4n} = \mathbf{HP}^n/\mathbf{HP}^{n-1} \to SP^\infty(S^{4n})$$

as representing a homotopy class of $SP^\infty(\mathbf{HP}^n)$, namely the generator of the $4n$-th homotopy group.

By virtue of the Complex Suspension Theorem the situation is very much the same for complex cycle spaces (cf. [**34**]). For any linear embedding of $\mathbf{CP}^{2n}$ into $\mathbf{CP}^{2n+1}$ there is a fibration

$$\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n}) \hookrightarrow \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})/\!\!/\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n})$$

and the inclusion of 1-cycles on $\mathbf{CP}^{2n}$ into the space of 1-cycles on $\mathbf{CP}^{2n+1}$ induces isomorphisms of homotopy groups in dimensions $\leq 4n-2$. We also have an isomorphism

$$\mathbf{Z} = \pi_{4n}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})) \to \pi_{4n}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})/\!\!/\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n})) = \mathbf{Z}.$$

If $\not{\Sigma}$ denotes the complex suspension from L_1^{2n} to the point x_0, there are maps

$$p_1 : L_1^{2n} \hookrightarrow \widetilde{\mathcal{C}}_0(L_1^{2n}) \xrightarrow{\not{\Sigma}} \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})$$

and

$$p_2 : \mathbf{CP}^{2n-1} \hookrightarrow \widetilde{\mathcal{C}}_0(\mathbf{CP}^{2n-1}) \xrightarrow{\not\mathcal{L}} \widetilde{\mathcal{C}}_1(L_2^{2n}),$$

where the inclusions $L_1^{2n} \hookrightarrow \widetilde{\mathcal{C}}_0(L_1^{2n})$ and $\mathbf{CP}^{2n-1} \hookrightarrow \widetilde{\mathcal{C}}_0(\mathbf{CP}^{2n-1})$ are just the standard inclusions of the type $X \hookrightarrow SP^\infty(X, \mathbf{Z})$.

Notice that p_2 is the restriction of p_1 to $\mathbf{CP}^{2n-1}$. This implies that there is a map

$$p : S^{4n} = L_1^{2n}/\mathbf{CP}^{2n-1} \xrightarrow{\not\mathcal{L}} \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(L_2^{2n}),$$

which can be thought of as representing the generator of $\pi_{4n}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}))$.

So there is a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{HP}^{n-1} & \hookrightarrow & SP^\infty(\mathbf{HP}^{n-1}) & \hookrightarrow & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n-1}) & \hookrightarrow & \widetilde{\mathcal{C}}_1(L_2^{2n}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{HP}^{n} & \hookrightarrow & SP^\infty(\mathbf{HP}^{n}) & \hookrightarrow & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) & \overset{\text{id}}{=} & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{4n} & \hookrightarrow & SP^\infty(S^{4n}) & \hookrightarrow & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n-1}) & \to & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(L_2^{2n}) \\
 & & \| & & & & \| \\
 & & K(\mathbf{Z}, 4n) & & & & K(\mathbf{Z}, 4n)
\end{array}$$

From the commutativity of this diagram it follows that Lemma 3.10 is true for $k < n$. It is also obviously true for $k > n$, as all the groups in question are trivial. To see that it holds for $k = n$ one has to show that the map

$$p' : S^{4n} \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(L_2^{2n})$$

of the bottom row of the diagram represents the generator of $\pi_{4n}(\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(L_2^{2n}))$.

Recall that we consider $\mathbf{HP}^n$ to be the space of j-invariant lines in $\mathbf{CP}^{2n+1}$. Define a map

$$\mu : S^{4n} = \mathbf{HP}^n/\mathbf{HP}^{n-1} \to L_1^{2n}/\mathbf{CP}^{2n-1} = S^{4n}$$

by sending a j-invariant line into its point of intersection with L_1^{2n}. All j-invariant lines which are not transversal to L_1^{2n} (i.e. are contained in it) are also contained in $\mathbf{CP}^{2n-1}$; they are precisely those lines, which are parametrised by $\mathbf{HP}^{n-1}$, so, clearly, μ is continuous. Moreover, as any j-invariant line is defined uniquely by specifying a point on it, μ is a homeomorphism. It is also easy to see that if the orientations on S^{4n} are defined by the natural orientations of $\mathbf{HP}^n$ and L_1^{2n}, then μ is orientation-preserving.

Now we will find the homotopy of the map p' to the map $p \circ \mu$ and this will establish the lemma.

Define a map

$$\phi_t : \mathbf{CP}^{2n+1} \times [1, \infty) \to \mathbf{CP}^{2n+1}$$

as the multiplication of the last homogeneous coordinate by a real number t:

$$(z_0, \dots, z_{2n}, z_{2n+1}) \to (z_0, \dots, z_{2n}, t \cdot z_{2n+1}).$$

It is a linear transformation for any $t \in [1, \infty)$, so it carries lines into lines. All the points of L_1^{2n} are fixed by ϕ_t and L_2^{2n} is carried into itself. Hence we can consider a map

$$\phi_t \circ p' : \mathbf{HP}^n/\mathbf{HP}^{n-1} \times [1, \infty) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) /\!\!/ \widetilde{\mathcal{C}}_1(L_2^{2n}).$$

Clearly, one can extend this map by continuity to $t = \infty$; then

$$\phi_\infty \circ p' = p \circ \mu$$

so ϕ_t is the desired homotopy.

□

REMARK. There are two possible choices of the "natural" generators in the homotopy groups of $\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q}$: via the map

$$i_\infty : SP^\infty(\mathbf{HP}^n) \hookrightarrow \widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1})$$

and via the inclusion into the space of complex cycles

$$\widetilde{\mathcal{QC}}_1(\mathbf{CP}^{2n+1}) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}),$$

as both $SP^\infty(\mathbf{HP}^n)$ and the space of complex 1-cycles come with preferred generators in their homotopy groups. This, in turn, is due to the existence of natural orientations on complex and quaternionic projective spaces.

In the proof of Theorem 3.9 we have, in fact, shown that these choices agree.

CHAPTER 4

Spaces of Cycles of Low Degrees

Here we investigate the properties of quaternionic cycles of degrees 1 and 2. In particular, we show that the inclusion of the infinite quaternionic Grassmanian into the the space of all quaternionic cycles classifies the symplectic Pontrjagin classes. We also show that the inclusions of the spaces of quaternionic cycles of degree 2 and odd codimension into the space of all quaternionic cycles is not an injection on homology.

1. Cycles of degree one

Algebraic cycles of degree 1 in the projective space are just linear subspaces; the spaces $\mathcal{C}_{p,1}(\mathbf{CP}^n)$, $\mathcal{RC}_{p,1}(\mathbf{CP}^n)$ and $\mathcal{QC}_{2p+1,1}(\mathbf{CP}^{2n+1})$ are complex, real and quaternionic Grassmanians respectively.

It will be convenient for us to speak about cycles of a given codimension, so we introduce the following notation:

$$\widetilde{\mathcal{C}}^q(\mathbf{CP}^n) \stackrel{def}{=} \mathcal{C}_{n-q}(\mathbf{CP}^n),$$
$$\widetilde{\mathcal{E}}^q(\mathbf{CP}^n) \stackrel{def}{=} \widetilde{\mathcal{E}}_{n-q}(\mathbf{CP}^n),$$
$$\widetilde{\mathcal{RC}}^q(\mathbf{CP}^n) \stackrel{def}{=} \mathcal{RC}_{n-q}(\mathbf{CP}^n),$$
$$\widetilde{\mathcal{QC}}^q(\mathbf{CP}^{2n+1}) \stackrel{def}{=} \mathcal{QC}_{2n+1-q}(\mathbf{CP}^{2n+1}).$$

In this notation Theorems 2.2, 2.4, 2.5 and 3.8, describing the homotopy groups of the cycle spaces on $\mathbf{CP}^n$ will give us

$$\widetilde{\mathcal{C}}^q(\mathbf{CP}^n) \simeq \prod_{i=0}^{q} K(\mathbf{Z}, 2i),$$

$$\widetilde{\mathcal{E}}^q(\mathbf{CP}^n) \simeq \prod_{i=0}^{q} K(\mathbf{Z}/2, i),$$

$$\widetilde{\mathcal{RC}}^q(\mathbf{CP}^n) \otimes \mathbf{Q} \simeq \prod_{i=0}^{[\frac{1}{2}q]} K(\mathbf{Q}, 4i),$$

$$\widetilde{\mathcal{QC}}^q(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \prod_{i=0}^{[\frac{1}{2}q]} K(\mathbf{Q}, 4i).$$

Notice that in these expression the right-hand side does not depend on n. So we can define the space of the complex cycles of codimension q on $\mathbf{CP}^\infty$ as the direct limit of the sequence of homotopy equivalences:

$$\widetilde{\mathcal{C}}^q(\mathbf{CP}^q) \xrightarrow{\not{\Sigma}} \widetilde{\mathcal{C}}^q(\mathbf{CP}^{q+1}) \xrightarrow{\not{\Sigma}} \dots$$

The spaces of real and quaternionic cycles on $\mathbf{CP}^\infty$ are defined analogously.

Inclusions of the spaces of the linear cycles of fixed codimension into the spaces of all cycles on $\mathbf{CP}^\infty$ give us following maps:

$$BU(q) \hookrightarrow \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) = \prod_{i=0}^{q} K(\mathbf{Z}, 2i),$$
$$BO(q) \hookrightarrow \widetilde{\mathcal{E}}^q(\mathbf{CP}^\infty) = \prod_{i=0}^{q} K(\mathbf{Z}/2, i),$$
$$BO(q) \hookrightarrow \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) = \prod_{i=0}^{[\frac{1}{2}q]} K(\mathbf{Z}, 4i) \times \prod K(torsion, \cdot),$$
$$BSp(q) \hookrightarrow \widetilde{\mathcal{QC}}^q(\mathbf{CP}^\infty) = \prod_{i=0}^{q} K(\mathbf{Z}, 4i) \times \prod K(torsion, \cdot).$$

Homotopy classes of maps of a topological space X into the Eilenberg-MacLane space $K(\pi, n)$ are in natural 1-1 correspondence with $H^n(X, \pi)$. So it is reasonable to ask which cohomology classes of the infinite Grassmanians are represented by the maps above. For the complex cycles this question was raised and answered by Lawson and Michelsohn in [**34**]:

THEOREM 4.1. *The map*

$$BU(q) \hookrightarrow \prod_{i=1}^{q} K(\mathbf{Z}, 2i)$$

represents the total Chern class of the universal bundle over $BU(q)$.

REMARK. In general, there are certain choices involved in representing cohomology classes by maps into Eilenberg-MacLane spaces. For instance, one has to choose the fundamental class in the cohomology of a $K(\pi, n)$. An implicit claim in the theorem above is that there is a natural choice of the fundamental classes in the homotopy of the cycle spaces. Namely, there is an inclusion map

$$\mathbf{CP}^n \hookrightarrow \widetilde{\mathcal{C}}^n(\mathbf{CP}^\infty),$$

the natural orientation of $\mathbf{CP}^n$ determines the fundamental classes in the homotopy of the space of cycles.

The real modulo 2 case was tackled by Lam:

THEOREM 4.2. [**30**] *The map*

$$BO(q) \hookrightarrow \prod_{i=1}^{q} K(\mathbf{Z}/2, i)$$

represents the total Stiefel-Whitney class of the universal bundle over $BO(q)$.

So our next theorem comes as no surprise:

THEOREM 4.3. *The composite inclusion map*

$$BO(q) \hookrightarrow \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \longrightarrow \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} = \prod_{i=0}^{[\frac{1}{2}q]} K(\mathbf{Q}, 4i)$$

into the space of real cycles with rational coefficients represents (up to signs) the total rational Pontrjagin class of the universal bundle over $BO(q)$.

Our choice of fundamental classes for $\widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q}$ will become apparent from the proof.

PROOF. For every $q > 0$ there is a commutative diagram

$$\begin{array}{ccccc} BO(q) & \hookrightarrow & \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) & \hookrightarrow & \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \\ \downarrow & & \downarrow & & \downarrow \\ BU(q) & \hookrightarrow & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) & \hookrightarrow & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \end{array}$$

The map

$$H^*(BU(q), \mathbf{Q}) \to H^*(BO(q), \mathbf{Q})$$

induced by the complexification map $BO(q) \hookrightarrow BU(q)$, sends the odd Chern classes of the universal bundle over $BU(q)$ to 0, and even Chern classes to the Pontrjagin classes of the universal bundle over $BO(q)$, taken with appropriate signs. So the pullbacks of the fundamental classes of $\prod_{i=1}^{q} K(\mathbf{Z}, 2i)$ to $BO(q)$ are, up to sign, the Pontrjagin classes. So the result follows from Theorem 4.1, given that we can choose the fundamental classes in $\widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q}$ to be the pullbacks of the fundamental classes in $\widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q}$.

By the Suspension Theorems 2.1 and 2.3 it is enough to consider the natural inclusion

$$\widetilde{\mathcal{RC}}_0(\mathbf{CP}^q) \otimes \mathbf{Q} \hookrightarrow \widetilde{\mathcal{C}}_0(\mathbf{CP}^q) \otimes \mathbf{Q}.$$

This map can be viewed as the homology transfer map on the level of symmetric products, see Chapter 1, Section 3. So the induced map on the homotopy groups is one-to-one. Moreover, this map between products of Eilenberg-MacLane spaces splits up to homotopy as a product of maps between $K(\mathbf{Q}, m)$'s with the same m. This can be seen as follows:

Let

$$\Gamma_1 = \{\gamma_{4k} \mid \gamma_{4k} : S^{4k} \to \widetilde{\mathcal{R}\mathcal{C}}_0(\mathbf{CP}^q) \otimes \mathbf{Q};\;\; 0 < k \leq [\frac{1}{2}q]\}$$

be the set of representatives for the generators of the homotopy groups of the space of real 0-cycles and

$$\Gamma_2 = \{\gamma_{2k} \mid \gamma_{2k} : S^{2k} \to \widetilde{\mathcal{C}}_0(\mathbf{CP}^q) \otimes \mathbf{Q};\;\; 0 < k \leq q\}$$

be the set of representatives for the generators of the homotopy groups of the complex cycle space. As the inclusion of the space of real cycles into the space of complex cycles is one-to-one on homotopy, we can choose Γ_1 and Γ_2 in such a way that Γ_1 goes to a subset of Γ_2 under this inclusion. Then we have a commutative diagram

$$\begin{array}{ccc} \bigvee_{k \leq [\frac{1}{2}q]} S^{4k} & \overset{\vee\gamma_{4k}}{\to} & \widetilde{\mathcal{R}\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \\ \downarrow & & \downarrow \\ \bigvee_{k \leq q} S^{2k} & \overset{\vee\gamma_{2k}}{\to} & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \end{array}$$

which extends to a diagram of monoid homomorphisms

$$\begin{array}{ccc} SP^\infty(\bigvee_{k \leq [\frac{1}{2}q]} S^{4k}, \mathbf{Q}) & \to & \widetilde{\mathcal{R}\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \\ \downarrow & & \downarrow \\ SP^\infty(\bigvee_{k \leq q} S^{2k}, \mathbf{Q}) & \to & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} \end{array}$$

The horizontal maps in this diagram induce isomorphisms on homotopy groups and, hence, are homotopy equivalences. One can also easily see that for any CW-complexes X and Y

$$SP^\infty(X \vee Y) = SP^\infty(X) \times SP^\infty(Y)$$

and so the vertical map on the left-hand side splits as a product between rational symmetric products of spheres of the same dimension, i.e. Eilenberg-MacLane spaces $K(\mathbf{Q}, m)$ with the same m. □

The quaternionic case is very similar:

THEOREM 4.4. *The composite map*

$$BSp(q) \hookrightarrow \widetilde{\mathcal{Q}\mathcal{C}}^{2q}(\mathbf{CP}^\infty) \hookrightarrow \widetilde{\mathcal{Q}\mathcal{C}}^{2q}(\mathbf{CP}^\infty) \otimes \mathbf{Q} = \prod_{i=0}^{q} K(\mathbf{Q}, 4i)$$

represents the total rational symplectic Pontrjagin class of the universal bundle over $BSp(q)$.

PROOF. There is a commutative diagram

$$\begin{array}{ccccc} BSp(q) & \hookrightarrow & \widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^\infty) & \hookrightarrow & \widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^\infty)\otimes\mathbf{Q} \\ \downarrow & & \downarrow & & \downarrow \\ BU(2q) & \hookrightarrow & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty) & \hookrightarrow & \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty)\otimes\mathbf{Q} \end{array}$$

The map

$$H^*(BU(2q),\mathbf{Z}) \to H^*(BSp(q),\mathbf{Z})$$

induced by the inclusion $BSp(q) \hookrightarrow BU(2q)$, sends the odd Chern classes of the universal bundle over $BU(q)$ to 0, and even Chern classes to the symplectic Pontrjagin classes, i.e. the Chern classes of the universal bundle over $BSp(q)$ ([**35**]). In the proof of the Theorem 3.9 we have seen that the inclusion map of the space of quaternionic 1-cycles into the space of complex 1-cycles (coefficients are taken to be rational) is one-to-one on homotopy groups. An argument, completely analogous to the one we used in the proof of the previous theorem shows that this inclusion map splits as a product of maps between Eilenberg-MacLane spaces $K(\mathbf{Q},m)$ with the same m. Hence, we can choose the fundamental classes for $\widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^\infty)\otimes\mathbf{Q}$ as the pullbacks of the fundamental classes in the space of complex cycles $\widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty)\otimes\mathbf{Q}$. The result now follows from Theorem 4.1 and the commutativity of the diagram above. □

REMARK. The choice of the fundamental classes for the spaces of quaternionic cycles as pullbacks is natural in the sense that it agrees with the choice of the fundamental classes via the inclusion map

$$\mathbf{HP}^n \hookrightarrow \widetilde{\mathcal{QC}}^{2n}(\mathbf{CP}^\infty),$$

see the remark at the end of the previous chapter.

2. Quaternionic cycles of degree two and odd codimension

As we have seen above, there is an embedding of the infinite Grassmanian of the effective cycles of degree 1 into $\widetilde{\mathcal{C}}^q(\mathbf{CP}^n)$.

The map on homology

$$H_*(BU(q),\mathbf{Z}) \to H_*(\widetilde{\mathcal{C}}^q(\mathbf{CP}^n),\mathbf{Z})$$

induced by the inclusion is injective, as the corresponding map on cohomology is surjective by Theorem 4.1. This naturally leads to the following question: is the inclusion of the subspace of the effective cycles of degree 2 into the space of all cycles injective on homology? The spaces of the complex cycles of degree 2 were studied by Burt Totaro **[60]**,**[59]**. In particular, he obtained a negative answer to the question above.

There are no reasons to expect the situation to be different for the real cycle spaces or the spaces of quaternionic cycles, except for one case. Recall that the properties of the spaces of quaternionic cycles on $\mathbf{CP}^{2n+1}$ depend quite dramatically on the complex codimension: in the case of even codimension there exist quaternionic cycles of any degree; in the case of odd codimension the degree must be even by Proposition 3.3. This means that the smallest degree of an effective cycle in $\widetilde{\mathcal{QC}}^{2q+1}(\mathbf{CP}^n)$ is not 1, but 2. So it makes sense to ask:

Is the inclusion of the subspace of the quaternionic effective cycles of degree 2 into the space of all cycles of fixed odd complex codimension injective on homology ?

The most interesting (or, rather, the simplest) case here is the case of cycles on $\mathbf{CP}^\infty$. The answer in this case is generally "no". Below we describe the method of calculating the Betti numbers b_i of the space of effective cycles of degree 2 and odd codimension on $\mathbf{CP}^\infty$. It turns out that for codimension ≥ 3 these Betti numbers are bigger than the corresponding Betti numbers of the space of all cycles.

Denote by D_n the space of effective quaternionic cycles of degree 2 and complex codimension $2n+1$ on $\mathbf{CP}^\infty$. It is the union of 2 components, A_n and B_n, where

A_n is the closure of the space of *irreducible* j-invariant quadrics;

B_n is the space of *reducible* j-invariant quadrics, i.e. those, which are unions of linear subspaces of the form $l+j(l)$. (In odd codimension l and $j(l)$ are always distinct).

Our method is to use the Mayer-Vietoris sequence to express the Betti numbers of D_n via the Betti numbers of A_n, B_n and $A_n \cap B_n$. From now on all cohomology groups have rational coefficients.

LEMMA 4.5. *The rational cohomology ring of A_n is a free polynomial algebra:*

$$H^*(A_n) = \mathbf{Q}[a_1, \dots, a_n], \ \dim a_k = 4k.$$

PROOF. Every irreducible j-invariant quadric of codimension $2n+1$ is contained in a single quaternionic linear subspace of complex codimension $2n$. This gives us a fibration

$$A_n \to BSp(n)$$

whose fibre is the space of j-invariant quadratic divisors on $\mathbf{CP}^\infty$. By Proposition 3.1 this fibre is homeomorphic to $\mathbf{RP}^\infty$, hence, $H^*(A_n) = H^*(BSp(n))$. □

LEMMA 4.6. *The rational cohomology ring of B_n is isomorphic to the subalgebra $H^{4*}(BU(2n+1))$.*

PROOF. B_n is homeomorphic to the quotient of $BU(2n+1)$ by the action of j. So the rational cohomology of B_n is the subring of $H^*(BU(2n+1))$, fixed by the action of j. Now we verify that the action of j on $H^*(BU(2n+1))$ sends c_{2k} to c_{2k}, and c_{2k+1} to $-c_{2k+1}$.

Indeed, denote by γ_{2n+1} the universal bundle over $BU(2n+1)$. The map

$$j : BU(2n+1) \to BU(2n+1)$$

induces a bundle γ'_{2n+1} over $BU(2n+1)$. This bundle is equivalent to the conjugate universal bundle $\overline{\gamma_{2n+1}}$: the fibre of γ'_{2n+1} over the point l is the linear subspace $j(l)$, we have an antilinear isomorphism between l and $j(l)$; on the other hand we have antilinear isomorphisms between l and $\bar{l}$ - hence we have linear isomorphisms between fibres of γ'_{2n+1} and $\overline{\gamma_{2n+1}}$. These give us a map of total spaces, which establishes the equivalence.

The Chern classes of the conjugate bundle are $(-1)^k c_k$. So, by the naturality of the Chern classes, c_{2k} are invariant under the action of j, while c_{2k+1} change sign.

□

LEMMA 4.7. *The rational cohomology ring of $A_n \cap B_n$ is a free polynomial algebra:*

$$H^*(A_n \cap B_n) = \mathbf{Q}[d, a_1, \dots, a_n], \ \dim a_k = 4k, \ \dim d = 4.$$

PROOF. The space $A_n \cap B_n$ consists of all the pairs of linear subspaces $(l, j(l))$, which have complex codimension 1 in their quaternionic span. To find its cohomology we first study its double cover, which we denote by C_n. Points of C_n can be written as pairs (l, H), where l is a codimension-1 hyperplane in the quaternionic linear subspace H of complex codimension $2n$.

Denote by $\xi_{m,n}$ the tautological bundle over the quaternionic Grassmanian $\mathbf{H}G(m+n, m)$ of m-dimensional (quaternionic) subspaces in $\mathbf{H}^{m+n}$. Then a point in the complex projectivisation of the dual bundle $\xi^*_{m,n}$ represents a pair (l', H'), where H' is a quaternionic linear subspace of $\mathbf{H}^{m+n}$ of complex codimension $2n$ and l is a complex hyperplane in it.

So C_n is a direct limit over m of the sequence of (complex) projectivisations:

$$P_{\mathbf{C}}(\xi^*_{1,n}) \hookrightarrow \dots \hookrightarrow P_{\mathbf{C}}(\xi^*_{m,n}) \hookrightarrow P_{\mathbf{C}}(\xi^*_{m+1,n}) \hookrightarrow \dots$$

Multiplicatively, $H^*(P_{\mathbf{C}}(\xi^*_{m,n}))$ is equal to $H^*(\mathrm{HG}(m+n,m)) \otimes H^*(\mathbf{CP}^{2m-1})$ modulo a relation of degree $2m$. When $m \to \infty$ the degree of this relation tends to infinity. So the cohomology of C_n is the inverse limit of the corresponding cohomology rings $H^*(P_{\mathbf{C}}(\xi^*_{m,n}))$ which is the free polynomial algebra $H^*(BSp(n)) \otimes H^*(\mathbf{CP}^\infty)$. Notice also that the projection map $C_n \to BSp(n)$ induces an inclusion on cohomology: $x \in H^*(BSp(n))$ goes to $x \otimes 1 \in H^*(BSp(n)) \otimes H^*(\mathbf{CP}^\infty)$.

The monodromy of C_n acts in the fibres the map $C_n \to BSp(n)$; it fixes $H^*(BSp(n))$ and changes the sign of the generator of $H^*(\mathbf{CP}^\infty)$. Recall that $H^*(A_n \cap B_n)$ is the subring of $H^*(C_n)$, fixed by the monodromy; this establishes the lemma. □

LEMMA 4.8. *The map*

$$H^*(B_n) \to H^*(A_n \cap B_n)$$

induced by the inclusion $A_n \cap B_n \hookrightarrow B_n$ is onto. (Explicitly, if we identify the generators of $H^*(B_n)$ with the even Chern classes and pairwise products of odd Chern classes in $H^*(BU(2n+1))$, classes a_k are hit by c_{2k} and the class d is hit by c_1^2.)

PROOF. The proof is essentially a continuation of the argument of the proof of Lemma 4.7.

The map $A_n \cap B_n \hookrightarrow B_n$ can be lifted to an inclusion map of double covers

$$C_n \hookrightarrow BU(2n+1).$$

The monodromy of the double covers commutes with this inclusion, so the map in cohomology

$$H^*(B_n) \to H^*(A_n \cap B_n)$$

can be considered as the map between the subrings of $H^*(C_n)$ and $H^*(BU(2n+1))$ that are fixed by the monodromy. We have seen that $H^*(C_n)$ is freely generated by $4k$-dimensional classes a_k and a 2-dimensional class, which we denote by u. So in order to establish the lemma we have to show that the classes $a_k \in H^{4k}(C_n)$ are hit by $c_{2k} \in H^{4k}(BU(2n+1))$ and u is hit by c_1.

Denote by $\eta_{m,n}$ the tautological m-plane bundle over the complex Grassmanian $\mathbf{C}\mathrm{G}(m+n,m)$ of m-dimensional subspaces in $\mathbf{C}^{m+n}$. Then the projectivisation of its dual bundle $P_{\mathbf{C}}(\eta^*_{m,n})$ is the set of pairs of the form (l, H), where H is a subspace of codimension n (dimension m) in $\mathbf{C}^{m+n}$ and l is a hyperplane in H.

Denote by F_n the direct limit of the sequence

$$P_{\mathbf{C}}(\eta^*_{1,n}) \hookrightarrow \dots \hookrightarrow P_{\mathbf{C}}(\eta^*_{m,n}) \hookrightarrow P_{\mathbf{C}}(\eta^*_{m+1,n}) \hookrightarrow \dots$$

so that, roughly speaking, F_n is a projectivisation of the "infinite-dimensional bundle" $\eta_{\infty,n}$ over $BU(n)$. The cohomology of F_n can be found in the same fashion as the cohomology of C_n: it is a free polynomial algebra $H^*(BU(n)) \otimes H^*(\mathbf{CP}^\infty)$; the projection

$$p_n : F_n \to BU(n)$$

induces an inclusion in cohomology: $x \in H^*(BU(n))$ goes to $x \otimes 1 \in H^*(F_n) = H^*(BU(n)) \otimes H^*(\mathbf{CP}^\infty)$.

Notice that the identification of $\mathbf{H}^{m+n}$ with $\mathbf{C}^{2m+2n}$ gives us maps

$$P_{\mathbf{C}}(\xi^*_{m,n}) \hookrightarrow P_{\mathbf{C}}(\eta^*_{2m,2n}).$$

and, in the limit, we have a map

$$C_n \hookrightarrow F_{2n}.$$

It is not hard to see that the induced map on cohomology

$$H^*(BU(2n)) \otimes H^*(\mathbf{CP}^\infty) \to H^*(BSp(n)) \otimes H^*(\mathbf{CP}^\infty)$$

sends c_{2k} to a_k, c_{2k+1} to zero and the generator of $H^*(\mathbf{CP}^\infty)$, which we denote by u', to u.

The point in introducing the spaces F_n is that the inclusion $C_n \hookrightarrow BU(2n+1)$ factors through F_{2n}; the inclusion

$$i_{2n} : F_{2n} \hookrightarrow BU(2n+1)$$

sends a pair (l, H) into l. The induced map in cohomology can be calculated as follows.

As before, denote by γ_k the universal k-plane bundle over $BU(k)$. There are two pullback bundles over F_{2n}: a $2n$-plane bundle $p^*_{2n}(\gamma_{2n})$ and a $2n+1$-plane bundle $i^*_{2n}(\gamma_{2n+1})$.

The Chern classes of $p^*_{2n}(\gamma_{2n})$ are, by naturality, just the classes $c_k \otimes 1 \in H^*(F_{2n})$. To find the Chern classes of $i^*_{2n}(\gamma_{2n+1})$ notice that $p^*_{2n}(\gamma_{2n})$ is a subbundle of $i^*_{2n}(\gamma_{2n+1})$.

The difference of these bundles is exactly the tautological line bundle S over F_{2n}, considered as a projectivisation of some infinite-dimensional bundle (see above). The Chern class of such a line bundle over a projectivisation is exactly the generator of the cohomology of the fibre, so $c_1(S) = u'$.

Hence, by the Whitney sum formula

$$c_k(i^*_{2n}(\gamma_{2n+1})) = c_k \otimes 1 + c_{k-1} \otimes u'.$$

This, by the naturality of Chern classes, determines the map in cohomology

$$p^*_{2n} : H^*(BU(2n+1)) \to H^*(F_{2n}).$$

Now, combining this with the map $H^*(F_{2n}) \to H^*(C_n)$, which we have already described, we obtain the complete description of the map $H^*(BU(2n+1)) \to H^*(C_n)$: even Chern classes c_{2k} are sent to a_k; odd Chern classes c_{2k+1} are sent to $a_k \otimes u$. In particular, u is hit by c_1. But, as we said in the beginning of the proof, this is sufficient to verify the lemma. □

By Lemmas 4.5, 4.6 and 4.7 the cohomology of A_n, B_n and $A_n \cap B_n$ in dimensions other than $4k$ is zero, so the Mayer-Vietoris sequence consists of exact sequences of the form

$$0 \to H^{4q}(D_n) \to H^{4q}(A_n) \oplus H^{4q}(B_n) \to H^{4q}(A_n \cap B_n) \to H^{4q+1}(D_n) \to 0.$$

By Lemma 4.8 the map

$$H^{4q}(A_n) \oplus H^{4q}(B_n) \to H^{4q}(A_n \cap B_n)$$

is onto so

$$H^{4q+1}(D_n) = 0,$$

and the sequences

$$0 \to H^{4q}(D_n) \to H^{4q}(A_n) \oplus H^{4q}(B_n) \to H^{4q}(A_n \cap B_n) \to 0$$

are exact. This means that

$$b_{4q+1}(D_n) = b_{4q+2}(D_n) = b_{4q+3}(D_n) = 0,$$

$$b_{4q}(D_n) = b_{4q}(A_n) + b_{4q}(B_n) - b_{4q}(A_n \cap Bn).$$

As we know the cohomology of A_n, B_n and $A_n \cap B_n$ we have an effective method of calculating $b_i(D_n)$. In particular,

$$b_8(D_1) = 1 + 4 - 3 = 2$$

This is the case of cycles of complex codimension 3; the space of all cycles of codimension 3 is (rationally) $K(\mathbf{Q}, 4)$,

$$b_8(K(\mathbf{Q}, 4)) = 1,$$

so the inclusion of the cycles of degree 2 cannot induce an injection on homology.

For the cycles of complex codimension $2n + 1 > 3$ we have

$$b_8(D_n) = 2 + 5 - 4 = 3,$$

while b_8 of the corresponding product of the Eilenberg-MacLane spaces is equal to 2; hence the same conclusion.

APPENDIX A

On the quotient of $\mathbf{CP}^n$ by complex conjugation

Denote by t the involution, induced on $\mathbf{CP}^n$ by complex conjugation. Here we describe the rational and mod 2 homology of $\mathbf{CP}^n/t$ and give a proof of the (weak) Kuiper-Massey theorem via symmetric products.

1. Mod 2 and rational homology of $\mathbf{CP}^n$ modulo conjugation

First we describe a t-equivariant cell decomposition of $\mathbf{CP}^n$. It consists of cells $a^{\pm}_{i,j}$ $(0 < i \leq j)$ and $a_{0,j}$, defined as

$$a^{+}_{i,j} = \{(z_1, \dots, z_i, x_{i+1}, \dots, x_j, 1, 0, \dots) \mid z_k \in \mathbf{C},\ x_k \in \mathbf{R},\ Im(z_i) > 0\},$$

$$a^{-}_{i,j} = \{(z_1, \dots, z_i, x_{i+1}, \dots, x_j, 1, 0, \dots) \mid z_k \in \mathbf{C},\ x_k \in \mathbf{R},\ Im(z_i) < 0\},$$

$$a_{0,j} = \{x_1, \dots, x_j, 1, 0, \dots) \mid x_k \in \mathbf{R}\}.$$

Obviously, $\dim a^{\pm}_{i,j} = i + j$.

The boundary operators in this cell complex modulo 2 are as follows:

for $j > i > 1$

$$\partial a^{\pm}_{i,j} = a^{+}_{i-1,j} + a^{-}_{i-1,j} + a^{+}_{i,j-1} + a^{-}_{i,j-1},$$

for $i = j > 1$

$$\partial a^{\pm}_{i,i} = a^{+}_{i-1,i} + a^{-}_{i-1,i},$$

for $i = 1, j > 1$

$$\partial a^{\pm}_{1,j} = a_{0,j} + a^{+}_{1,j-1} + a^{-}_{1,j-1},$$

for $i = j = 1$

$$\partial a^{\pm}_{1,1} = a_{0,1},$$

for $i = 0$

$$\partial a_{0,j} = 0.$$

This cell decomposition induces a cell structure on $\mathbf{CP}^n/t$; cells $a^{\pm}_{i,j}$ map into cells we denote by $a_{i,j}$; cells $a_{0,j}$ are fixed by t, we denote their images by $a_{0,j}$ as well.

The boundaries modulo 2 are as follows:

for $i \neq 1$

$$\partial a_{i,j} = 0,$$

for $i = 1$

$$\partial a_{1,j} = a_{0,j}.$$

It is easy to calculate the dimensions of the vector spaces of cellular chains:
for $k \geq n$

$$dim\ C_k = [\frac{1}{2}k] + 1 - (k - n),$$

for $k \leq n$

$$dim\ C_k = [\frac{1}{2}k] + 1,$$

where $[x]$ denotes the integral part of x.

Given that the only non-trivial boundaries are $\partial a_{1,j} = a_{j+1}$, we get that mod 2 Betti numbers, i.e. ranks of $H_*(\mathbf{CP}^n/t, \mathbf{Z}/2)$ are as follows:

PROPOSITION A.1. *For* $n + 1 < k \leq 2n$
$rk(H_k(\mathbf{CP}^n/t, \mathbf{Z}/2)) = [\frac{1}{2}k] + 1 - (k - n),$
for $1 < k \leq n + 1$
$rk(H_k(\mathbf{CP}^n/t, \mathbf{Z}/2)) = [\frac{1}{2}k] - 1,$
for $k \leq 1$
$rk(\widetilde{H}_k(\mathbf{CP}^n/t, \mathbf{Z}/2)) = 0.$

The inclusion

$$\mathbf{RP}^n \hookrightarrow \mathbf{CP}^n/t$$

agrees with the standard cell structure on $\mathbf{RP}^n$. Each cell of $\mathbf{RP}^n$ is a mod 2 boundary in $\mathbf{CP}^n/t$; hence the natural inclusion $\mathbf{RP}^n \hookrightarrow \mathbf{CP}^n/t$ induces zero maps on homology. So from the exact homology sequence of the cofibration

$$\mathbf{RP}^n \hookrightarrow \mathbf{CP}^n/t \to (\mathbf{CP}^n/t)/\mathbf{RP}^n$$

we get the following

PROPOSITION A.2.
For $0 \leq k \leq n + 1$ $\quad rk(\widetilde{H}_k((\mathbf{CP}^n/t)/\mathbf{RP}^n, \mathbf{Z}/2)) = [\frac{1}{2}k],$
for $n + 1 \leq k \leq 2n$ $\quad rk(\widetilde{H}_k((\mathbf{CP}^n/t)/\mathbf{RP}^n, \mathbf{Z}/2)) = [\frac{1}{2}k] + 1 - (k - n).$

To calculate the rational homology of $\mathbf{CP}^n/t$, recall that the transfer homomorphism in homology

$$H_n(\mathbf{CP}^n/t, \mathbf{Q}) \to H_n(\mathbf{CP}^n, \mathbf{Q})$$

provides the isomorphism between $H_n(\mathbf{CP}^n/t, \mathbf{Q})$ and the t-fixed part of $H_n(\mathbf{CP}^n, \mathbf{Q})$, see [**19**]. It is straightforward to see that the action of t on the homology of $\mathbf{CP}^n$ is trivial on $4k$-dimensional classes and is multiplication by -1 on the classes of dimension $4k + 2$. Thus we have:

PROPOSITION A.3.

$$H_i(\mathbf{CP}^n/t, \mathbf{Q}) = \mathbf{Q} \text{ for } 2n \geq i = 4k, \text{ and } 0 \text{ otherwise.}$$

The action of t respects the inclusions $\mathbf{CP}^{n-1} \hookrightarrow \mathbf{CP}^n$. So there is an involution on the quotient $S^{2n} = \mathbf{CP}^n/\mathbf{CP}^{n-1}$. If we represent $S^{2n} = \mathbf{CP}^n/\mathbf{CP}^{n-1}$ as a 1-point compactification of $\mathbf{R}^{2n}$, we can write t as follows:

$$t : (x_1, \dots, x_n, y_1, \dots, y_n) \to (x_1, \dots, x_n, -y_1, \dots, -y_n).$$

From this it is clear that S^{2n}/t is just an n-fold suspension of $\mathbf{RP}^n$, hence we have

PROPOSITION A.4.

$$\widetilde{H}_i(S^{2n}/t) = 0 \quad \text{for } i < n.$$

REMARKS. In order to derive the formulae for the boundaries in the above cell complex on $\mathbf{CP}^n$ it is necessary to consider the "cells at infinity". This problem can be rigorously addressed as follows.

Let (x_k, y_k), $x_k + iy_k = z_k$, $0 \leq k \leq n$ be the coordinates in $\mathbf{C}^{n+1}$ and S^{2n+1} be the sphere, defined by the equation

$$\sum x_k^2 + y_k^2 = 1.$$

We define a cell decomposition of S^{2n+1} into cells $b_{i,j}^{\pm;\pm}$, $b_{i,0}^{\pm}$ and $b_{0,j}^{\pm}$ as

$$b_{i,j}^{\pm;\pm} = \{(x_1, y_1, \dots, x_{n+1}, y_{n+1}) \mid x_k = 0 \text{ for } k > i,\ x_i \gtrless 0,\ y_k = 0 \text{ for } k > j,\ y_j \gtrless 0, \},$$

where i and j range from 1 to $n+1$, and

$$b_{0,j}^{\pm} = \{(x_1, y_1, \dots, x_{n+1}, y_{n+1}) \mid x_k = 0 \text{ for all } k,\ y_k = 0 \text{ for } k > j,\ y_j \gtrless 0\},$$

$$b_{i,0}^{\pm} = \{(x_1, y_1, \dots, x_{n+1}, y_{n+1}) \mid x_k = 0 \text{ for } k > i,\ x_i \gtrless 0,\ y_k = 0 \text{ for all } k\}.$$

The boundaries in this cell complex can be calculated directly without any reference to "cells at infinity". Notice that under the standard map

$$p : S^{2n+1} \to \mathbf{CP}^n$$

cells $b_{i,j}^{+;+}$ and $b_{i,j}^{-;-}$ go to $a_{j,i-1}^{+}$ for $i > j$. Similarly, for $i > j$ $b_{i,j}^{-;+}$ and $b_{i,j}^{+;-}$ go to $a_{j,i-1}^{-}$. Cells $b_{i,i}^{\pm;\pm}$ are mapped to the cells of lower dimension.

This turns out to be sufficient to calculate the boundaries in the cell complex of $\mathbf{CP}^n$ via the boundaries of the cell complex of S^{2n+1}.

As Elmer Rees pointed out, there is an elegant way of calculating the homology groups of $(\mathbf{CP}^n/t)/\mathbf{RP}^n$. We will give a rough sketch of the argument.

Let $S^{2n+1} \subset \mathbf{C}^{n+1}$ be as above and $f : S^{2n+1} \to \mathbf{R}$ be the function, given by

$$f(z_0, \dots, z_n) = |z_0^2 + z_1^2 + \dots + z_n^2|.$$

Clearly, f is invariant under the action of S^1 on S^{2n+1}, which sends $(z_0, \dots, z_n)$ to $(z_0 e^{i\theta}, \dots, z_n e^{i\theta})$; it is also invariant under the complex conjugation in $\mathbf{C}^{n+1}$. This implies that f descends to a function on $\mathbf{CP}^n/t$. The value of this function on $\mathbf{RP}^n \subset \mathbf{CP}^n/t$ is constant (and equal to 1) so, in fact, f descends to a function $\tilde{f}$ on $(\mathbf{CP}^n/t)/\mathbf{RP}^n$. The minimal value of $\tilde{f}$ is 0, it is attained on the subset of real codimension 2, which is the image of a hypersurface

$$z_0^2 + z_1^2 + \dots + z_n^2 = 0$$

in $(\mathbf{CP}^n/t)/\mathbf{RP}^n$. It is not too hard to see that this subset is homeomorphic to a real Grassmanian $\mathrm{G}(n+1, 2)$. One can also check that $\tilde{f}$ has one maximum and no other critical points. This means that $(\mathbf{CP}^n/t)/\mathbf{RP}^n$ is homeomorphic to the Thom space of a real 2-plane bundle over $\mathrm{G}(n+1, 2)$ and thus the homology of $(\mathbf{CP}^n/t)/\mathbf{RP}^n$ can be expressed via the homology of $\mathrm{G}(n+1, 2)$ with the help of the Thom isomorphism.

2. Kuiper-Massey theorem via symmetric products

One case when we can explicitly identify the quotient $\mathbf{CP}^n/t$ is when $n = 2$. (Case $n = 1$ is trivial).

THEOREM A.5. *$\mathbf{CP}^2/t$ is homeomorphic to a 4-sphere.*

There are many different proofs of this famous statement, known as Kuiper-Massey theorem[1]. Here we offer an interpretation of it via symmetric products.

Recall that $\mathbf{CP}^2$ can be regarded as the symmetric square of a Riemann sphere. The map

$$\mathbf{CP}^1 \to \mathbf{CP}^1/t = D^2$$

induces a map

$$\mathbf{CP}^2 = SP^2(\mathbf{CP}^1) \to SP^2(D^2) = D^4.$$

This map factors through $\mathbf{CP}^2/t$; it is the map

$$\mathbf{CP}^2/t \to D^4,$$

[1] Actually, this statement, which is, apparently, rather classical (according to [25], footnote to p.149, Pontrjagin was aware of it) can be refined so that it becomes less trivial. Namely, $\mathbf{CP}^2/t$ with its natural smooth structure turns out to be *diffeomorphic* to S^4. [4],[29].

which is of interest for us. Over $Int(D^4)$ it is an unbranched double cover and it is 1-1 over $\partial D^4 = S^3$. From this it is immediately clear that $\mathbf{CP}^2/t$ is homeomorphic to a 4-sphere.

Bibliography

1. N.L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture notes in Math., vol. 219, Springer-Verlag, 1971.
2. F.J. Almgren Jr., *Homotopy groups of the integral cycle groups*, Topology **1** (1962), 257–299.
3. V. I. Arnold, *Topological content of the Maxwell theorem on the multipole representation of spherical functions*, to appear.
4. Арнольд, В. И., *Разветвлённое накрытие* $\mathbf{CP}^2 \to S^4$, *гиперболичность и проективная топология*, Сиб. Мат. Ж. **29** (1988), no. 5, 36–47.
5. C.-F. Bödigheimer, F.R. Cohen, and R.J. Milgram, *Truncated symmetric products and configuration spaces*, Math. Z. **214** (1993), 179–216.
6. K. Borsuk, *On the third symmetric potency of the circumference*, Fund. Math. **36** (1949), 236–244.
7. K. Borsuk and S. Ulam, *On symmetric products of topological spaces*, Bull. Am. Math. Soc. **37** (1931), 875–882.
8. R. Bott, *On the third symmetric potency of* S_1, Fund. Math. **39** (1952), 264–268.
9. C. P. Boyer, H. B. Lawson Jr., P. Lima-Filho, B. M. Mann, and M.-L. Michelsohn, *Algebraic cycles and infinite loop spaces*, Invent. Math. **113** (1993), 373–388.
10. M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. **75** (1962), no. 2, 331–341.
11. V. M. Buchshtaber, *Characteristic classes in cobordism and topological applications of the theory of single-valued and two-valued formal groups*, J.Soviet Mathematics **11** (1979), no. 6.
12. J. Clerk Maxwell, A treatise on electricity and magnetism, vol. 1, ch. Spherical harmonics, Clarendon Press, Oxford, 1873.
13. R. Daverman, *Decompositions of manifolds*, Pure and Applied Mathematics, vol. 124, Academic Press, 1986.
14. A. Deleanu, *The relative Dold-Thom theorem*, Arch. Math. (Basel) **17** (1966), 234–238.
15. A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. **68** (1958), 54–80.
16. A. Dold and Thom R., *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. **67** (1958), no. 2, 239–281.
17. A. Dold and R. Thom, *Une généralisation de la notion d'espace fibré. Application aux produits symétriques infinis*, C. R. Acad. Sci Paris **242** (1956), 1680–1682.
18. J. L. Dupont and G. Lustig, *On manifolds satisfying* $w_1^2 = 0$, Topology **10** (1971), no. 2, 81–92.
19. E. E. Floyd, *Periodic maps via Smith theory*, Borel's seminar on transformation groups, Annals of Mathematical Studies, no. 46, Princeton, 1960, pp. 35–48.
20. E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. **77** (1991), 55–93.

21. E. Friedlander and B. Mazur, *Filtrations on homology of algebraic varieties*, Memoirs of the AMS **110** (1994), no. 529.
22. P. Gajer, *Poincarè duality and integral cycles*, Compositio Math. **98** (1995), no. 2, 193–203.
23. ———, *The intersection Dold-Thom theorem*, Topology **35** (1996), no. 4, 939–967.
24. T. Ganea, *Symmetrische Potenzen topologischer Räume*, Math. Nachr. **11** (1954), 305–316.
25. Гийу, Л. и Марен, А. (ред.), *В поисках утраченной топологии*, М.,Мир, 1989.
26. P.A. Griffiths and J.W. Morgan, *Rational homotopy theory and differential forms*, Birkhäuser, 1981.
27. I. James, *Reminiscences of a topologist*, The Mathematical Intelligencer **12** (1990), no. 1, 50–55.
28. F. E. A. Johnson, *On the triangulation of stratified sets and singular varieties*, Trans. Am. Math. Soc. **275** (1983), no. 1, 333–343.
29. N. H. Kuiper, *The quotient space of* $\mathbf{CP}^2$ *by complex conjugation is the 4-sphere*, Math. Ann. **208** (1974), 175–177.
30. T.K. Lam, *Spaces of real algebraic cycles and homotopy theory*, Ph.D. thesis, SUNY, 1990.
31. H.B. Lawson, *Algebraic cycles and homotopy theory*, Ann. of Math. **129** (1989), 253–291.
32. H.B. Lawson, P. Lima-Filho, and M.-L. Michelsohn, *On equivariant algebraic suspension*, Preprint, 1996.
33. ———, *Quaternionic algebraic cycles*, to appear, 1996.
34. H.B. Lawson and M.-L. Michelsohn, *Algebraic cycles, Bott periodicity and the Chern characteristic map*, Proc. Symp. Pure Math **48** (1988), 241–263.
35. ———, Spin Geometry, ch. Appendix B. Characteristic Classes, Princeton University Press, Princeton, New Jersey, 1989.
36. ———, *Algebraic cycles and group actions*, Differential Geometry, Longman, 1993.
37. P. Lima-Filho, *Completions and fibrations for topological monoids*, Trans. Am. Math. Soc. **340** (1993), no. 1, 127–147.
38. ———, *On the generalized cycle map*, J. Diff. Geom. **38** (1993), 105–130.
39. I.G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Camb. Phil. Soc. **58** (1962), 563–568.
40. ———, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.
41. A. Mattuck, *Symmetric products and Jacobians*, Am. Journal of Math. **83** (1961), 189–206.
42. J.P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies, vol. 11, Princeton, 1967.
43. M.C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Am. Math. Soc. **146** (1969), 273–298.
44. R.J. Milgram, *Homology of symmetric products*, Trans. Am. Math. Soc. **138** (1969), 251–265.
45. R. Molski, *On symmetric products*, Fund. Math. **44** (1957), 165–170.
46. H.R. Morton, *Symmetric products of the circle*, Proc. Camb. Phil. Soc **63** (1967), 349–352.
47. J. Mostovoy, *Geometry of truncated symmetric products and real roots of real polynomials*, to appear in the Bulletin of the LMS.
48. ———, *Quaternion-flavoured cycle spaces*, Preprint, May 1996.
49. D. Mumford, *Curves and their Jacobians*, The University of Michigan Press, Ann Arbor, 1975.
50. D. Puppe, *Homotopie und Homologie in abelschen Gruppen- und Monoidkomplexen*, Math. Z. **68** (1958), 367–421.

51. M. Richardson and P. A. Smith, *Periodic transformations of complexes*, Ann. of Math **39** (1938), 611–633.
52. T. B. Rushing, *Topological embeddings*, Pure and Applied Mathematics, vol. 52, Academic Press, 1973.
53. P. Samuel, *Methodes d'algebre abstraite en géometrie algébrique*, Springer-Verlag, 1955 (revised ed. 1967).
54. M. Schiffer and D. Spencer, *Functionals of finite Riemann surfaces*, Princeton, 1954.
55. G. Segal, *K-homology and algebraic K-theory*, K-theory and operator algebras, proceedings of the conference in Athens, Georgia, 1975, Lecture notes in math., vol. 575, Springer-Verlag.
56. I. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, New York, 1974.
57. E. Spanier, *Infinite symmetric products, function spaces and duality*, Ann. of Math **69** (1959), 142–198.
58. J. J. Sylvester, *Note on spherical harmonics*, Phil. Mag. (1876), 291–307,400.
59. B. Totaro, *The Chow variety of cycles of degree 2*, Preprint, Oct. 1990.
60. ———, *The map from the Chow variety of cycles of degree 2 to the space of all cycles*, Preprint, Feb. 1990.