

APPLICATIONS OF CURRENT ALGEBRAS AND CHIRAL

SYMMETRY BREAKING

Thesis

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DAVID JAMES WALLACE

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CHAPTER I

BACKGROUND CONCEPTS IN THE ALGEBRA OF CURRENTS

1. Current Commutators

This first chapter is intended to give a brief introduction to the aesthetic beauty and physical relevance of current algebras and the breaking of chiral symmetry. The particular approach taken is one which I had the pleasure of hearing cogently developed by Dr. B. Renner. It is hoped that the intuitive attractiveness of the approach compensates for the absence of technical details, which, along with references to the original papers, can be found in the early sections of refs. (1) and (2).

One of the most significant conceptual differences between the theories of classical and quantum mechanics lies in the fact that in the former the quantities which we measure, the position, velocity, angular momentum of a particle etc., are regarded as numerical quantities whereas in the latter they are associated with operators in a finite dimensional vector, or a Hilbert space. The relevant point for us here is the contrast between the commutative property of numbers and the non-commutative properties of operators, which are so intimately connected with the operations of measurement.

In quantum mechanics, one postulates commutation relationships for the hermitean operators representing physical observables. The basic equation is of course

$$\left[q^i, p^j \right] = i \delta^{ij} \quad (1.1)$$

where q^i, p^i are the generalised coordinates and canonical momenta of the system and we choose units such that $\hbar = 1$. The Lagrangian formulation of classical mechanics provides a very elegant basis on which to justify this equation, although the postulate stands or falls according to the comparison of its predictions with experiment. In a similar vein, one defines the angular momentum, L_i , commutation relationships

$$\left[L_i, L_j \right] = i \epsilon_{ijk} L_k \quad (1.2)$$

by analogy with the classical expression $L_i = \epsilon_{ijk} q_j p_k$ and expression (1.1).

In the study of elementary particles, other operators occur whose matrix elements are physically observable quantities, namely the currents which describe the weak and electromagnetic interactions. (A suitable review is given in ref. (3)). In view of the simplicity and considerable success of the above postulates in quantum mechanics, it is natural to believe that the operators representing the weak and electromagnetic currents also obey simple commutation relationships.

The clue to a possible form of these comes from a study of the strong interactions, which are known to be invariant under SU_2 and to be, in some sense, approximately invariant under the larger group SU_3 . The SU_2 invariance manifests itself in the appearance of multiplets of particles forming irreducible representations of SU_2 . The masses of the particles within a multiplet are equal up to what is believed to be the effect of

electromagnetic self-interactions. The approximate SU_2 invariance is realised by grouping together particles of approximately equal mass into irreducible representations of the SU_3 group.

In Lagrangian field theory, the one particle states and the field operators are given the group transformation properties appropriate to the multiplet of SU_2 or SU_3 in which the particle is classified. The formalism of infinitesimal local gauge transformations permits the definition from the Lagrangian of vector currents $V_\mu^a(x)$, whose integrated time components, $\int d^3x V_0^a(x)$ are just the generators Q^a of the group transformations and obey the appropriate commutation relationships. For SU_2 , these are exactly the same as the O_3 algebra in eq. (1.2), namely

$$[Q^a, Q^b] = i \epsilon^{abc} Q^c \quad (a, b = 1, 2, 3) \quad (1.3)$$

while for SU_3 , with (totally antisymmetric) structure constants f^{abc} one has

$$[Q^a, Q^b] = i f^{abc} Q^c \quad (a, b = 1, \dots, 8) \quad (1.4)$$

Now, it is an experimental fact that the vector currents which are related to the generators of the SU_3 transformations contain a subset which has exactly the correct isospin and hypercharge quantum numbers to represent the vector currents of the weak and electromagnetic interactions of the hadrons and thus one is led to postulate that the vector currents associated with the symmetry group of the strong interactions are the vector currents of the weak and electromagnetic interactions. Making a universality assumption about the weak interaction coupling constant (which is meaningful only because we have non-linear

relationships (1.3) and (1.4) amongst the currents) one can make predictions relating, for example, μ decay, neutron beta decay and charged pion beta decay, all of which are experimentally very well satisfied⁽³⁾.

For many applications of current algebra one must extend the postulates (1.3) and (1.4) for the weak vector current charges to more local equal-time commutation relationships, e.g. in SU_3 ,

$$\left[Q^a, V_\mu^b(y) \right] = i f^{abc} V_\mu^c(y) \quad (1.5a)$$

$$\delta(x^0 - y^0) \left[V_0^a(x), V_\mu^b(y) \right] = i f^{abc} V_\mu^c(x) \delta^4(x-y) \quad (1.5b)$$

Axial vector currents also appear in the theory of weak interactions and describe parity violating effects. The quantum numbers of the four currents are such that they can be contained in the octet representation of SU_3 . If we make the assumption that the axial currents belong to an octet representation, then they must obey

$$\left[Q^a, A_\mu^b(x) \right] = i f^{abc} A_\mu^c(x) \quad (1.6a)$$

and again one may further postulate a local form

$$\delta(x^0 - y^0) \left[V_0^a(x), A_\mu^b(y) \right] = i f^{abc} A_\mu^c(x) \delta^4(x-y) \quad (1.6b)$$

Finally, what are the commutators of the axial currents, or the axial charges $\bar{Q}^a(x^0) = \int d^3x A_0^a(x)$? In the limit of exact SU_3 symmetry, it is postulated that these currents belong to an octet representation. Their commutator is thus an antisymmetric combination of two octet operators and so belongs to the octet, decuplet or antidecuplet representations of SU_3 . Now, since aesthetically and practically we wish the algebra to be as simple

as possible, and since no decuplet operators have appeared in eqs. (1.5) and (1.6), one postulates

$$\left[\bar{Q}^a(x^0), \bar{Q}^b(x^0) \right] = i f^{abc} Q^c \quad (1.7a)$$

$$\left[\bar{Q}^a(x^0), A_\mu^b(x) \right] = i f^{abc} V_\mu^c(x) \quad (1.7b)$$

and $\delta(x^0-y^0) \left[A_0^a(x), A_\mu^b(y) \right] = i f^{abc} V_\mu^c(x) \delta^4(x-y) \quad (1.7c)$

Eqs. (1.5) to (1.7) are the basic current algebra relations which will be used in this work. By taking the combinations

$$Q^\pm = \frac{1}{2}(Q \pm \bar{Q}) \quad (1.8)$$

it is seen that the Q^\pm obey an $SU_3 \times SU_3$ "chiral" algebra.

It must be stressed that it requires stronger assumptions to justify the local forms of the commutation relationships. For example if the currents contain terms of the form $\partial^\mu T_{\mu\nu}$, where $T_{\mu\nu} = -T_{\nu\mu}$, such terms disappear when we go from eq. (1.7b) to (1.7c) by integrating over 3-space. Moreover it is known that the local commutators of time components with space components of currents must be modified by the introduction of Schwinger terms^(1,2). The form of these terms cannot be obtained from general principles, being dependent on the particular model used. However for most of the applications which we make, the once-integrated form of the local commutators is all that is required. Further comment on Schwinger terms is made in Chapter III.

It is also convenient to mention at this point that the full relevance of the SU_3 current commutators is obtained only when we know the combinations in which the currents V_μ and A_μ appear in the weak interactions. Here we rely on the Cabibbo theory⁽³⁾ of the leptonic and semi-leptonic weak interactions to normalise

the weak currents. Thus it is assumed that the charge-changing weak hadronic currents obey an SU_2 algebra, in parallel with the SU_2 algebra exhibited by the leptonic currents. This implies that the weak hadronic currents have the form

$$J_\mu = (V_\mu^{1+12} + A_\mu^{1+12}) \cos \theta + (V_\mu^{4+15} + A_\mu^{4+15}) \sin \theta \quad (1.9)$$

(Our SU_3 notation and conventions are given in Appendix B.)

2. Broken Chiral Symmetry

So far, the fact that SU_3 is only an approximate symmetry of the strong interactions has been completely glossed over. In a Lagrangian field theory, invariant under the transformations of a symmetry group, the Lagrangian must commute with the generators of the group. It is then immediate that the generators must be time-independent, and the associated currents must be conserved, $\partial_\mu V^\mu = 0$.

In a physical theory, the Lagrangian must not be exactly invariant under the SU_3 transformations, so that it must not commute with the generators of the group and hence these generators are necessarily time-dependent. Gell-Mann postulated that even though SU_3 is not an exact symmetry of the strong interactions, the commutators of the currents associated with the generators of the group should be unchanged by the symmetry-breaking, i.e. it is postulated that eqs. (1.5a) and (1.6a) need only be modified to equal-time commutators, so that they take a form similar to eqs. (1.7a) and (1.7b).

The possibility of adding symmetry breaking terms to an SU_3

symmetric Lagrangian without altering the current commutators is exhibited in many field-theory models, e.g. the triplet quark model, the σ model and the non-linear chiral Lagrangian model; in all these models, it is realised basically through having non-derivative symmetry breaking terms. In order to obtain quantitative predictions from such theories, it is in many cases necessary to make specific postulates about the form of the symmetry breaking. As a preliminary comment, it is natural to take the eighth component of an octet of operators to break SU_3 , in view of the success of the Gell-Mann-Okubo mass formula.

So far we have identified the integrated time components of the vector currents only, as the generators of a broken symmetry, SU_3 . Can the axial charges $\bar{Q}(t)$ also be regarded as the generators of some broken symmetry of the strong interactions, which then exhibit a broken chiral $SU_3 \times SU_3$ symmetry? Now the axial charge is a pseudoscalar quantity and can connect only states of different parity. Since particles do not appear in general as opposite-parity doublets, or even approximately so, then clearly the broken symmetry associated with axial currents is realised in a different fashion from broken SU_3 . The situation is discussed very clearly by Dashen⁽⁴⁾ for the case of chiral $SU_2 \times SU_2$.

Basically, it is assumed that the chiral symmetry is realised by the fact that the pion mass is very small compared with the other hadron masses. The relevance of this statement is seen by writing down the general form of the matrix element

$\langle N | A_{\mu}^a | N \rangle$, where $| N \rangle$ represents a nucleon state, and noting that $\partial^{\mu} A_{\mu}^a = 0$ is satisfied either by having the

nucleon mass, $m_N = 0$, or by having a pole at $q^2 = 0$ in matrix elements of ∂A , i.e. by the existence of a zero mass pseudo-scalar particle. Similarly, inspection of $\langle \sigma | A_\mu^a | \pi^c \rangle$ reveals that the axial current may be conserved if either $m_\sigma = m_\pi$ or $m_\pi = 0$. (Here σ represents a scalar, isosinglet meson.)

Further let us consider the matrix element of ∂A^a between nucleons

$$\langle N | \partial A^a | N \rangle = \bar{u} \tau^\beta \gamma_5 u d(q^2) \quad (1.10)$$

$$\text{where } d(q^2) = \frac{r}{q^2 - m_\pi^2} + \int \frac{\rho(m^2)}{q^2 - m^2} dm^2$$

Here r is the residue at the pion pole and $\rho(m^2)$ is the contribution from higher mass intermediate states. It is assumed that the strong interaction Lagrangian can be written in the form

$$\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_{\text{S.B.}} \quad (1.11)$$

where \mathcal{L}_0 is symmetric under $SU_2 \times SU_2$ and the symmetry limit is realised by $\epsilon \rightarrow 0$. Then ∂A^a is of order ϵ , since it may be related to the commutator $[A^0, \mathcal{L}] = \epsilon [A^0, \mathcal{L}_{\text{S.B.}}]$ and, if the symmetry limit is associated with the vanishing of the pion mass, then m_π^2 is also of order ϵ . It is then seen that for small q^2 , the first term is of order 1 whereas the second term is of order ϵ , i.e. for small q^2 , the pion pole will dominate matrix elements of the divergence of the axial current, regardless of the precise nature of the symmetry breaking.

In field theory terms, one can make this statement more precise, by ensuring that the symmetry breaking terms are chosen so as to give the **P.C.A.C. relation**

$$\partial A^a = f_\pi m_\pi^2 \phi^a \quad (1.12)$$

where ϕ^a is the pion field. However, this equation is, by itself, not meaningful. For example in Lagrangian field theory, we can redefine the fields by transformations of the form

$\bar{\phi} = \phi f(\phi)$ where $f(0) = 1$, without changing any measurable quantities. Under this transformation, ∂A will in general become a function of the new field $\bar{\phi}$, not simply $\bar{\phi}$ alone.

There is a parallel situation in axiomatic field theory where on-mass-shell predictions are independent of the particular interpolating field chosen, and the physical content lies in the statement that matrix elements of ∂A with the pion pole removed are smooth, slowly varying functions of q^2 , the square of momentum transfer. The equivalence of unsubtracted pion pole dominance and P.C.A.C. plus smoothness is illustrated in ref. 1, p. 42, in the proof of the Goldberger-Treiman relationship.

These remarks on the existence of a broken chiral $SU_2 \times SU_2$ symmetry of the strong interactions can, in principle, be extended to $SU_3 \times SU_3$, with the SU_3 of the vector charges realised by approximate mass multiplets and the axial SU_3 realised by the existence of an octet of almost zero-mass pseudoscalar mesons, π , K and η , although the quantitative validity of e.g. kaon pole dominance is much more open to question. Moreover, it will be shown in Chapter IV that within specific models of chiral symmetry breaking, it seems that ∂A^8 is not a smooth interpolating field for η .

3. The Form of Symmetry Breaking

The discussion of the previous section may be summarised as follows. The strong interactions are to be regarded as basically symmetric under chiral $SU_3 \times SU_3$ transformations, so that in a Lagrangian field theory, the basic part of the Lagrangian, \mathcal{L}_0 , should be invariant under $SU_3 \times SU_3$, commuting with the vector and axial charges generating the group transformations. To this one adds a "small" term, which ensures that

- (i) the current algebra is unchanged,
- (ii) the $SU_3 \times SU_3$ symmetry is broken by a term transforming as an SU_3 singlet, which gives a certain mass to the π , K and η and
- (iii) the SU_3 symmetry is broken by another term transforming as the eighth component of an octet, splitting the π , K and η .

The simplest way to satisfy these three criteria appears to be to take the symmetry breaking terms of the Lagrangian to belong to a specific representation of $SU_3 \times SU_3$, the simplest being the $(3, \bar{3}) + (\bar{3}, 3)$ representation. This proposal was made in 1962 by Gell-Mann, who envisaged that the pseudoscalar octet plus the pseudoscalar η' (960 MeV) might be placed in the $(3, \bar{3})$ representation of $SU_3 \times SU_3$, along with 9 scalar particles. Then u^a (scalar) and v^a (pseudoscalar) operators obey the equal-time algebra⁺,

⁺ The conventions for SU_3 tensors are given in Appendix B.

$$\begin{aligned}
 [Q^a, v^b] &= i f^{abc} v^c \\
 [Q^a, u^b] &= i f^{abc} u^c \\
 [\bar{Q}^a, v^b] &= i d^{abc} v^c \\
 [\bar{Q}^a, u^b] &= -i d^{abc} u^c
 \end{aligned}
 \tag{1.13}$$

u^0 and u^8 are the only members of this representation suitable for adding to the strong-interaction Lagrangian to break $SU_3 \times SU_3$, whilst still maintaining SU_2 , hypercharge conservation and parity invariance, so one writes

$$\mathcal{L} = \mathcal{L}_0 + \epsilon(u^0 + cu^8)
 \tag{1.14}$$

With the definition of the v^a as the pseudoscalar meson fields, this type of Lagrangian exhibits P.C.A.C.⁽⁶⁾, although, as discussed in the previous section, this is not in itself of any relevance, at least until we have motivation for choosing a specific off-mass-shell continuation, such as duality, perhaps. (Brief discussions of this point are given in Chapters IV and V.)

Eq. (1.14) is used in two approaches in this thesis.

Firstly, the specification of the symmetry breaking terms enables one to evaluate the σ -commutators $[A^a(x), \partial A^b(y)] \delta(x^0 - y^0)$, which may appear when one uses Ward identity techniques with many particles off-shell. It is then possible for example to attempt to estimate by the hard meson approach⁽⁷⁾ the corrections which these σ -commutators may have on soft-pion theorems, such as the Callan-Treiman relation. In Chapter III, the effects of σ -commutator terms on the prediction from $K \rightarrow 2\pi$ decays of the weak intermediate vector boson mass⁽⁸⁾ are studied, using the $(3, \bar{3})$ model, and found to be extremely significant.

Secondly, we use the chiral Lagrangian approach⁽⁹⁾ to study the η -X mixing problem and η and X decays. In this approach, \mathcal{L} is constructed explicitly from the fields of particles relevant to the problem under study, and amplitudes are then calculated in the tree graph approximation from the Lagrangian. More specifically, we shall use the non-linear chiral Lagrangian, in which \mathcal{L}_0 is realised on the eight fields of π , K and η , and u^0 and u^8 are functions of the π , K and η fields with the appropriate transformation properties (eq. (1.13)). Further comments on the chiral Lagrangian approach are made in Chapter V.

This section would be incomplete without commenting on other possible symmetry breaking terms with well-specified algebraic properties. It has been shown⁽⁶⁾ that if (a) the symmetry breaking is to transform as an SU_3 singlet together with the eighth component of an octet and (b) non-exotic quantum numbers (i.e. those which are not contained in octet and singlet representations, for mesons) are not allowed in the $SU_3 \times SU_3$ multiplet containing these terms, then only the $(1, 8) + (8, 1)$ and $(3, \bar{3}) + (\bar{3}, 3)$ representations are allowed. For, suppose the symmetry breaking terms belong to the representation (p, q) of the two commuting SU_3 subgroups SU_3^L and SU_3^R (with generators Q^\pm defined as in eq. (1.8)). Under the SU_3 subgroup, with generator $Q = Q^+ + Q^-$, the multiplet will transform as the tensor product of a p and q dimensional representation. In order to avoid exotic quantum numbers, p and q may only be 1, 3, $\bar{3}$, 6, $\bar{6}$ and 8 and it is readily seen that only $(1, 8) + (8, 1)$ and $(3, \bar{3})$ have octet quantum numbers alone. Possible loosening of the restrictions

(a) and (b) have been given by Renner and Sudbery⁽¹⁰⁾. They show that if (b) is replaced by the assumption that the chiral $SU_3 \times SU_3$ breaking must still preserve chiral $SU_2 \times SU_2$, then again only the $(3, \bar{3})$ and $(1, 8)$ representations are allowed. If the symmetry breaking term is allowed to belong to a four-dimensional representation of $SU_2 \times SU_2$, so that the π - π σ -commutator is an isoscalar⁺, then a larger number of representations is allowed including $(6, \bar{6}) + (\bar{6}, 6)$ and $(8, 8)$.

Of course, the approach employed in any particular problem may further limit the possible representations. For example, in non-linear pseudoscalar meson chiral Lagrangians, only terms belonging to the (n, \bar{n}) representations are possible, since these are the only linearly transforming representations which can be formed from the non-linearly transforming π , K and η fields⁽¹²⁾. This statement has a parallel in the soft meson calculations; contributions to amplitudes from the $(1, 8) + (8, 1)$ term are the same order of magnitude as corrections to soft meson limits⁽⁶⁾.

With these considerations, we shall adopt the view that the most suitable and simplest procedure is to assume $(3, \bar{3})$ symmetry breaking, since this is the one representation which is not

⁺ The existence of a large non-isoscalar σ -term would change the good prediction of pion mass difference⁽¹¹⁾ (using soft pion limits and spectral function sum rules) and the π - π scattering lengths discussed in Chapter IV.

a priori disfavoured for any reason⁺.

References to other previous work on $(3, \bar{3})$ symmetry breaking are given at the relevant points in the thesis.

⁺ It has been argued by Dashen⁽⁴⁾ that there is no very strong reason why one representation of $SU_3 \times SU_3$ should be enhanced over any other, in contrast to the case where SU_3 octet enhancement may take place through the dominance of some "low" mass scalar meson octet. However, we feel that this possible criticism of using a single $SU_3 \times SU_3$ representation to break the symmetry does not bear much weight since a convincing basic understanding of even octet dominance has still to be formulated; in both cases one appears to be simply putting in the quantum numbers which one wishes to get out of the theory.

CHAPTER II

ELECTROMAGNETIC EFFECTS IN $K \rightarrow 2\pi$ DECAYS

1. Introduction

The purpose of this chapter is to study the magnitude of electromagnetic effects in $K \rightarrow 2\pi$ decays. An accurate estimate of such effects is of considerable significance in weak interaction theory, since they give rise to $\Delta I = 3/2$ effects in the decays and it is only with knowledge of them that one may properly claim to know the extent to which the non-leptonic weak Hamiltonian obeys the $\Delta I = 1/2$ rule.

Experimentally, violation of the $\Delta I = 1/2$ rule in $K \rightarrow 2\pi$ decays appears in two ways. $K^+ \rightarrow \pi^+\pi^0$ is necessarily a $\Delta I = 3/2$ transition, and a small deviation from 2 of the ratio $\Gamma(K_1^0 \rightarrow \pi^+\pi^-)/\Gamma(K_1^0 \rightarrow \pi^0\pi^0)$ may also result from $\Delta I = 3/2$ effects.

Several previous studies of this problem have been made. These appear to fall into three categories. Firstly are the papers which attempt proper calculations of radiative corrections in second order electromagnetic perturbation theory^(13, 14). The results are divergent, and are expressed in terms of a cut-off parameter, which is usually taken to be of the order of 1 GeV. Quantitatively, the effects are small and, in $K \rightarrow 2\pi$ decays, of the wrong sign to account for the experimental $\Delta I = 3/2$ term. However such calculations neglect completely the strong interactions of the pions and kaon. Moreover they should be viewed very circumspectly in

view of our inability to use the standard electromagnetic Hamiltonian to describe quantitatively $\Delta I = 1$ effects, of which more will be said in the next section.

Hara and Nambu⁽¹⁵⁾ have attempted to obtain the decay amplitude for $K^+ \rightarrow \pi^+\pi^0$, by assuming that the weak Hamiltonian obeys the $\Delta I = 1/2$ rule and by applying low energy limits to a $K\pi\pi$ vertex which is quadratic in the meson four-momentum. They obtained $\mathcal{M}(K^+ \rightarrow \pi^+\pi^0) \propto (m_{\pi^+}^2 - m_{\pi^0}^2) = \Delta(m_{\pi}^2)$. Feynman⁽¹⁶⁾ however, has pointed out that while $\Delta(m_{\pi}^2)$ is generally attributed to electrodynamics, this derivation of $\mathcal{M}(K^+ \rightarrow \pi^+\pi^0)$ has not taken electrodynamics specifically into account. Moreover, the end result depends only on the $\Delta I = 2$ effect, $\Delta(m_{\pi}^2)$, with no contribution from what is generally a dominating $\Delta I = 1$ term.

Finally there have been some papers based on pole models for the decays. The original calculation seems to have been done by Riazuddin and Fayyazuddin⁽¹⁷⁾. They assume that the principal contribution to the decay $K^+ \rightarrow \pi^+\pi^0$ comes from the Feynman diagram Fig. 2.1.

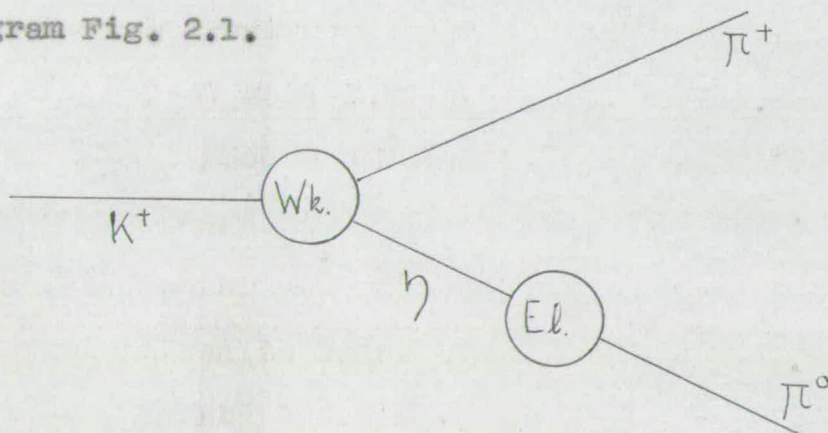


Fig. 2.1

The weak vertex $K^+ \rightarrow \pi^+\eta$ is a $\Delta I = 1/2$ transition, which is

estimated "by SU_3 " from $K^0 \rightarrow 2\pi$, and the $\eta \rightarrow \pi$ transition amplitude is estimated by assuming that the principal electromagnetic corrections to the $\pi^0 NN$ coupling constant arise from the similar diagram in Fig. 2.2.

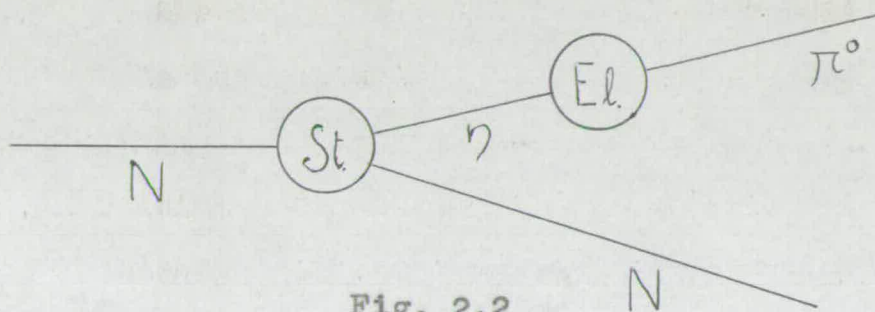


Fig. 2.2

This estimation has very large errors indeed since the magnitude of electromagnetic corrections to the $\pi^0 NN$ vertex is not known with any accuracy.

Fig. 2.1 was also studied by Faldt et al.⁽¹⁸⁾. The weak vertex is obtained in the same manner. A very large value for the $\eta \rightarrow \pi$ transition amplitude is obtained by using a current algebra calculation⁽¹⁹⁾ which enables it to be estimated from the

$\eta \rightarrow 3\pi$ decay width (more detailed comments on this type of calculation are given in Chapter V). The result is in good agreement with experiment. However in both refs. (17) and (18) the use of SU_3 for the weak vertex is not satisfactory, since the decay $K \rightarrow 2\pi$ is forbidden by the combination of SU_3 and CP invariance⁽²⁰⁾.

More recently, Greenberg⁽²¹⁾ has attempted to overcome this difficulty by assuming principally

(1) a pure octet non-leptonic weak Hamiltonian, with neutral currents

$$H_W = \sqrt{2} G d_{6\alpha\beta} J_\alpha^\mu J_{\mu\beta} \quad (2.1)$$

(ii) that the decay $K^+ \rightarrow \pi^+\pi^0$ proceeds mainly through Fig. 2.3.

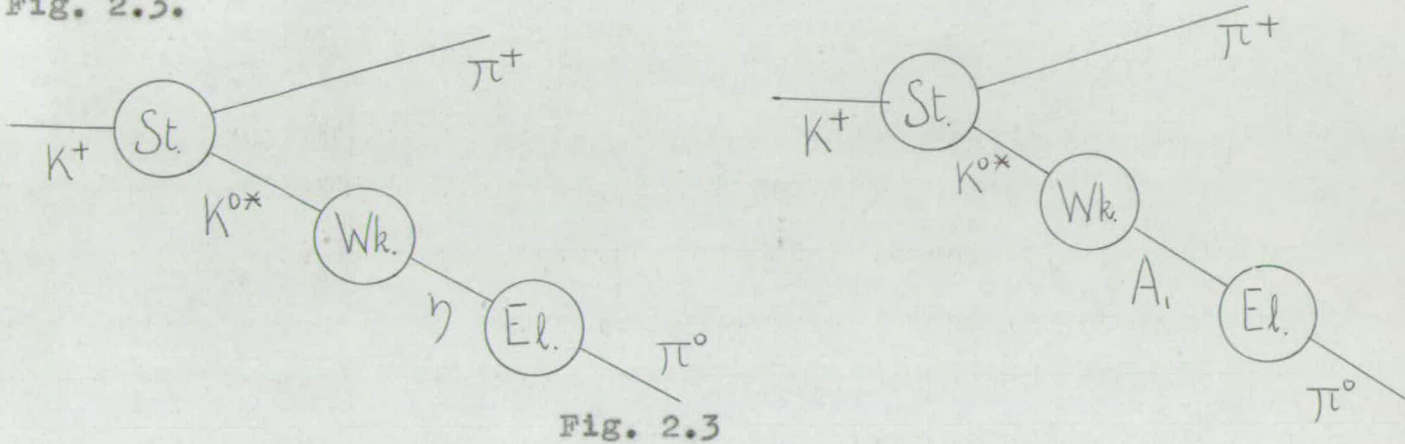


Fig. 2.3

(iii) that the weak matrix elements are to be evaluated by field current identity, or equivalently, by assuming that the vacuum alone need be retained in a sum over intermediate states between the currents in the Hamiltonian.

The electromagnetic transitions are assumed to be given by an effective Hamiltonian

$$H^{el} = B A_\mu^3 A_\mu^8$$

with matrix elements evaluated according to assumption (iii) above and normalised from the $\eta \rightarrow 3\pi$ decay amplitude. The result obtained is in good agreement with the experimental value of

$\mathcal{M}(K^+ \rightarrow \pi^+\pi^0) / \mathcal{M}(K_1^0 \rightarrow \pi^+\pi^-)$ and is of the correct sign to suppress the $\pi^0\pi^0$ relative to the $\pi^+\pi^-$ decay of K_1^0 . However, in view of the long chain of assumptions required to obtain it, there is considerable justification for studies of other models for the

decay⁺.

The main work in this chapter is a study⁽²²⁾ of the graph in Fig. 2.4.

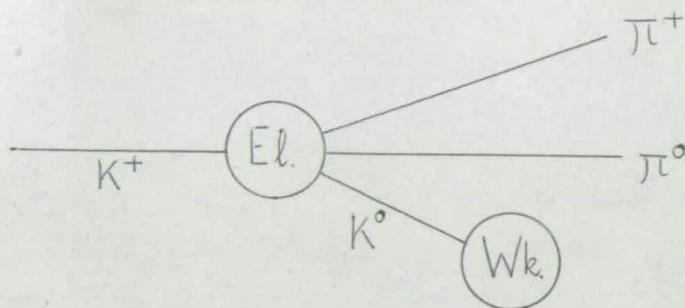


Fig. 2.4

Although the p-wave amplitude $K^+\bar{K}^0 \rightarrow \pi^+\pi^0$ does go by strong interactions, we are specifically interested in this as an electromagnetic interaction, since we wish the two pions to be emitted in an s-wave. This vertex is evaluated by performing two soft-pion reductions and relating the result to the kaon electromagnetic mass splitting. The weak vertex is related to the $K_1^0 \rightarrow \pi\pi$ amplitude, also by soft pion techniques. The details of these calculations are given in Section 3.

⁺ A recent calculation⁽¹¹⁴⁾ uses very similar assumptions to (i) and (iii) above. A slightly modified form of Oakes' model of symmetry breaking⁽³²⁾ is used to calculate the current divergences ∂A^{π^+} , ∂A^{π^0} etc. in terms of the pseudoscalar densities v^{π^+} and v^{π^0} and these relations are turned into field current identities for $A_\mu^{\pi^+}$ in terms of $\partial_\mu \pi^+$ etc., by assuming that v^{π^+} (v^{π^0}) is the correctly normalised π^+ (π^0) field. The ratio of $K^+ \rightarrow \pi^+\pi^0$ to $K_1^0 \rightarrow \pi^+\pi^-$ is in good agreement with experiment, although this is mainly because the coefficient of the $I = 1$ term in Oakes' model is so large (see comments in the next section). It is amusing to note the complete reversal of the conventional role of the Cabibbo angle from weak interactions to strong interactions.

2. The Electromagnetic Interactions of the Hadrons.

It has been appreciated for a long time that it is not possible to describe quantitatively the $\Delta I = 1$ electromagnetic interactions of hadrons by the standard second order Hamiltonian. The situation is typified by the wrong sign obtained for the $\Delta I = 1$ mass differences $n-p$ and $K^+ - K^0$ compared with the good result for the $\Delta I = 2$ pion mass difference⁽²⁴⁾. The problem is further emphasised when using current algebra techniques. The successful calculation of the pion mass difference, using soft meson techniques and spectral function sum rules, fails⁺ for the kaon mass difference⁽²⁵⁾ and current algebra also implies a suppression of the $\eta \rightarrow 3\pi$ decay width to orders of magnitude below the observed value⁽²⁶⁾.

The approach we take to this problem is essentially a development of the tadpole contribution to electromagnetic mass differences⁽²⁷⁾. In place of the tadpole contribution dominated by the $I = 1, I_3 = 0$

⁺ We do not regard as satisfactory the postulate of ref. (25) that the σ -terms in the soft kaon reduction should be equated to the tadpole term of Coleman et al.⁽²⁷⁾ since the actual magnitude of the σ -term contribution is critically dependent on the interpolating field chosen for the mesons. An example of this is the freedom in choosing v^8 or ∂A^8 as interpolating fields for a pure octet η ; the π, η σ -commutator occurring in $\langle \pi | H^{el} | \eta \rangle$ has coefficients $\sqrt{2} - c, \sqrt{2} + c$ respectively, in the ratio $\approx m_\eta^2 / m_\pi^2$. The postulate also fails if we combine simple pole models for $\langle K | H^{el} | K \rangle$ and $\langle \eta | H^{el} | \pi \rangle$ with current algebra. The numerator of the scalar meson pole turns out to be proportional to the relevant pseudoscalar mass, implying firstly a small effect and secondly large symmetry-breaking.

scalar meson, we introduce an effective $I = 1$, $I_3 = 0$ scalar Hamiltonian. Our further assumption is to specify the chiral properties of this operator by placing it in the same $(3, \bar{3})$ representation of $SU_3 \times SU_3$ as the symmetry breaking terms in eq. (1.14), so that the total effective Hamiltonian now takes the form

$$H^{\text{eff}} = H^{\text{el}} - \epsilon \cdot u_3 \quad (2.2)$$

where H^{el} is the standard second order electromagnetic Hamiltonian

$$H^{\text{el}} = \frac{(-1)}{2!} e^2 \int d^4y d^{\mu\nu}(y) \text{T} \{ V_\mu^{\text{el}}(y) V_\nu^{\text{el}}(0) \} \quad (2.3)$$

and $d^{\mu\nu}(y) = \langle \text{T} \{ a^\mu(y) a^\nu(0) \} \rangle_0$ is the photon propagator.

The scale of the u_3 contribution is fixed by equating it to the tadpole contribution in the kaon mass difference⁽⁹⁷⁾. Numerically⁽²⁷⁾, this is around 5 MeV. To be specific, we shall take the tadpole contribution as

$$\Delta_t(m_K^2) = \Delta(m_K^2) - \Delta(m_\pi^2) = m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2 \quad (2.4)$$

This equation is certainly true to lowest order in chiral symmetry breaking⁽⁴⁾, and when combined with SU_3 , is in good agreement with experimental data on mass differences^(27,28).

The u_3 effective Hamiltonian has been used by many previous authors. It was the basis of a current algebra calculation of $\eta \rightarrow 3\pi$ in ref. (19). Such a term has also been introduced within the theoretical framework of broken scale invariance⁽²⁹⁾, essentially as a renormalisation effect on the divergent Hamiltonian (2.3).

Alternatively, it may be induced by the need to avoid divergences in higher order weak interactions⁽³⁰⁾. In this approach the divergence properties of higher order weak interactions are studied by Bjorken limit techniques. The most divergent terms are related to matrix elements of weak current-divergence commutators, which are calculated by assuming a total symmetry-breaking Hamiltonian

$$H_{S.B.} = -\epsilon(u_0 + cu_8 + xu_3) \quad (2.5)$$

Demanding cancellation of these leading divergences within an SU_3 multiplet produces two relationships among c , x and the Cabibbo angle θ , namely

$$\tan^2 \theta = \frac{\rho}{3 + \rho} + \left[1 + \left(\frac{\rho}{3 + \rho} \right)^2 \right]^{1/2} - 1 \quad (2.6)$$

$$\sqrt{3} x/c = \frac{\rho^2}{3 + \rho + [\rho^2 + (3 + \rho)^2]^{1/2}} \quad (2.7)$$

where $\rho = -(1 + \sqrt{2}/c)$.

Now, applying SU_3 to suitable matrix elements of eq. (2.5) we obtain^(27,28)

$$\frac{x}{c} = \frac{-\Delta_t(m_K^2)}{m_K^2 - m_\pi^2} \cdot \frac{\sqrt{3}}{2} \approx .020 \quad (2.8)$$

Hence from eq. (2.7) $\rho \approx .45$ and hence $c \approx -.97$. This is not too discouragingly far away from the Gell-Mann, Oakes,

Renner value $(6)^+$, $c = -1.25$. Note that eq. (2.7) is very sensitive to the precise value of c ; $c = -1.25$ makes $x/c \approx .003$, so that if one does believe firmly in this value of c , then the work of Gatto et al. is not compatible with our approach to mass differences.

There has recently been an interesting suggestion by Oakes (32) , that c and x should be determined by a Cabibbo rotation of the chiral SU_3 symmetry breaking term which leaves chiral SU_2 unbroken, namely $u_0 - \sqrt{2} u_8$. Although this value of x is suitably large to provide an enhancement in $\eta \rightarrow 3\pi$ decay $(28,33)$, it disagrees with eq. (2.8) by a factor ≈ 2.5 and so does not give acceptable predictions of mass-splittings.

As a final remark, we note that the introduction of the $I = 1$ operator in eq. (2.2) is exactly paralleled in the dispersion relation approach (24) by the fact that the existence of the $I = 1$ A_2 trajectory implies Regge asymptotic behaviour in the $I = 1$ channel which automatically necessitates the existence of a subtraction constant in this channel (34) .

⁺ This value for c has also been obtained by using soft-pion extrapolations only (31) (and not soft K or η), but this derivation is not quite as water-tight as it appears at first sight because the extrapolation is in the small difference, $O(m_\pi^2)$ of the two large quantities $\epsilon \langle \pi | u^0 | \pi \rangle$ and $\epsilon c \langle \pi | u^8 | \pi \rangle$ which are both $O(m_K^2)$.

3. The Calculation ⁽²²⁾

We now return to the estimation of Fig. 2.4. The relevant electromagnetic transition is⁺

$$M^{el}(K^+ \rightarrow K^0 \pi^+ \pi^0) = (-i) \langle \pi^+ \pi^0 K^0 | H^{eff} | K^+ \rangle \quad (2.9)$$

with H^{eff} given by eqs. (2.2) and (2.3). We perform a double soft pion reduction on this matrix element (see the sections on low energy theorems in refs. (1) and (2)), averaging over the two orders of partial integration in order to project out the isospin symmetric π - π s-wave channel. With isoscalar σ -commutators, the possible $[A^0, \partial A^{\pi^+}]$ term vanishes and we have

$$\begin{aligned} M^{el}(K^+ \rightarrow K^0 \pi^+ \pi^0) &= \\ &= \frac{-e^2}{4/2 f_\pi^2} \int d^4y d^{\mu\nu}(y) \langle K^0 | T \{ v_\mu^{1-12}(y) v_\nu^{el}(0) + v_\mu^{el}(y) v_\nu^{1-12}(0) \} | K^+ \rangle \\ &\quad - \frac{ie\kappa}{4/2 f_\pi^2} \langle K^0 | u_{1-12} | K^+ \rangle \end{aligned} \quad (2.10)$$

The matrix elements in this equation are now related by SU_2 to the non-tadpole (nt) and tadpole (t) contributions to the kaon electromagnetic mass difference, with the Hamiltonian given in eqs. (2.2) and (2.3)

$$\Delta_{nt}(m_K^2) = \langle K^+ | H^{el} | K^+ \rangle - \langle K^0 | H^{el} | K^0 \rangle \quad (2.11)$$

$$\Delta_t(m_K^2) = -e\kappa (\langle K^+ | u_3 | K^+ \rangle - \langle K^0 | u_3 | K^0 \rangle) \quad (2.12)$$

⁺ Throughout this chapter we use the phase of the s-matrix for all amplitudes. The amplitudes are normalised so that only for the $\pi^0 \pi^0$ combination is there a statistical factor in the phase-space.

Only $\Delta I = 1$ effects are relevant in both eqs. (2.10) and (2.11) and the result is

$$\begin{aligned} \mathcal{M}^{el}(K^+ \rightarrow K^0 \pi^+ \pi^0) &= \frac{-i}{2\sqrt{2} f_\pi^2} (\Delta_{nt}(m_K^2) - \Delta_t(m_K^2)) \\ &= \frac{+i}{2\sqrt{2} f_\pi^2} (\Delta(m_K^2) - 2\Delta(m_\pi^2)) \quad (2.13) \end{aligned}$$

using eq. (2.4) for $\Delta_t(m_K^2)$. The chiral properties of the effective Hamiltonian are such that the non-tadpole and tadpole contributions now add constructively to enhance the electromagnetic effects.

By similar arguments we can relate the weak transition to the decay amplitude for $K^0 \rightarrow 2\pi$, neglecting electromagnetic effects and assuming a $V_\mu + A_\mu$ weak current. The result is

$$\langle 0 | \mathcal{H}^W | K^0 \rangle = -4 f_\pi^2 \langle \pi^0 \pi^0 | \mathcal{H}^W | K^0 \rangle \quad (2.14)$$

From eqs. (2.13) and (2.14), Fig. 2.4 gives

$$\begin{aligned} \mathcal{M}(K^+ \rightarrow \pi^+ \pi^0) &= \mathcal{M}^{el}(K^+ \rightarrow K^0 \pi^+ \pi^0) \cdot \frac{1}{0 - m_K^2} \langle 0 | -i \mathcal{H}^W | K^0 \rangle \\ &= -\sqrt{2} \frac{(\Delta(m_K^2) - 2\Delta(m_\pi^2))}{m_K^2} \mathcal{M}(K^0 \rightarrow \pi^0 \pi^0) \quad (2.15) \end{aligned}$$

Numerically, we obtain

$$\mathcal{M}(K^+ \rightarrow \pi^+ \pi^0) / \mathcal{M}(K^0 \rightarrow \pi^0 \pi^0) \simeq 3.6 \alpha \quad (2.16)$$

where α is the fine structure constant. This value is sufficiently close to the experimental ratio⁽³⁵⁾ for the modulus of the amplitudes in eq. (2.16), 6.6α , to suggest that electromagnetic

effects alone are sufficient to produce a large part of the $\Delta I \neq \frac{1}{2}$ transitions.

Moreover, we may also study the graphs corresponding to Fig. 2.4, which produce electromagnetic corrections to the K_1^0 decays. The terms containing isoscalar σ -commutators in the electromagnetic transitions are cancelled in the difference of the $K_1^0 \rightarrow \pi^0\pi^0$ and $\pi^+\pi^-$ decays, and again the electromagnetic amplitudes have $\Delta I = 1$ only in the soft pion limit. The three amplitudes for $K \rightarrow 2\pi$ decays then obey the sum rule obtained by allowing only $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ transitions⁽³⁶⁾,

$$\mathcal{M}(K_1^0 \rightarrow \pi^+\pi^-) - \mathcal{M}(K_1^0 \rightarrow \pi^0\pi^0) = 2 \mathcal{M}(K^+ \rightarrow \pi^+\pi^0) \quad (2.17)$$

and it is seen that the result (2.16) has the correct sign to produce the experimental enhancement⁽³⁵⁾ of the $\pi^+\pi^-$ mode over the $\pi^0\pi^0$ one. This result is to be contrasted with that of ref. (22), where the effect of a u_3 Hamiltonian was neglected.

4. Discussion

Let us first attempt to improve on the derivation of eq. (2.14). In performing low energy limits on the $K^0 \rightarrow 2\pi$ amplitude, we neglected the pion σ -commutator and neglected the error in extrapolation of the four-momentum, Δ_μ , associated with the Hamiltonian \mathcal{H}^W , from $\Delta^2 = 0$ in the physical amplitude to $\Delta^2 = m_K^2$ in the low energy limit. Now, in the absence of SU_3 symmetry, there must be a pole at $\Delta^2 = m_K^2$, due to the kaon coupling to \mathcal{H}^W . This is illustrated in Fig. 2.5.

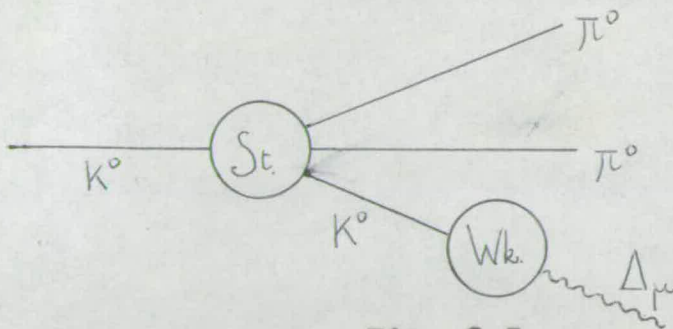


Fig. 2.5

The way to overcome this difficulty is to assume that the full $K^0 \rightarrow \pi^0 \pi^0$ amplitude is given as a "smooth" term (which, in practice, is taken to be a constant) together with the pole diagram Fig. (2.5). In the limit of the two q^0 's going simultaneously to zero, it can be shown^(22,37) that the divergence in the kaon pole as $\Delta^2 \rightarrow m_K^2$ is exactly cancelled by a divergence in the kaon pole of the σ -commutator term,

$\langle K^0 | T\{H^W(x)\sigma(0)\} | K^0 \rangle$. Moreover, when the same operations are performed for the reductions in the $\pi^+ \pi^-$ decay, using the Weinberg scattering amplitude⁽⁵⁴⁾ and retaining the σ term, i.e.

$$\langle K^0(\Delta) \pi^+(q_+) \pi^-(q_-) | K^0 \rangle = \frac{1}{f_\pi^2} (\langle K^0(\Delta) | \sigma | K^0 \rangle + \frac{1}{2} p_1 \cdot (q_+ - q_-)) \quad (2.18)$$

it is found that the p-wave term is just such as to compensate for the apparent dependence on the order of π^+ and π^- partial integrations. The equations are given in detail in ref. (22). It suffices to say that the smooth term C is determined unambiguously from the equation

$$\begin{aligned} C &= \lim_{q_+, q_- \rightarrow 0} (\langle \pi^+ \pi^- | H^W | K^0 \rangle - \text{fig 2.5}) \\ &= -\frac{1}{4f_\pi^2} \langle 0 | H^W | K^0 \rangle \end{aligned}$$

We now obtain for the on mass shell value of the decay $K^0 \rightarrow \pi^+ \pi^-$

$$\begin{aligned} \langle \pi^+ \pi^- | H^W | K^0 \rangle &= C + \text{on shell value of pole diagram} \\ &= - \frac{\langle 0 | H^W | K^0 \rangle}{4 f_\pi^2} \left\{ 1 - \frac{4 \langle K^0 | \sigma | K^0 \rangle}{m_K^2} \right\} \end{aligned} \quad (2.19)$$

where we have used eq. (2.18) for the scattering amplitude (the p-wave part does not contribute on-shell of course).

Thus we see that the original equation appears to be correct up to the effect of σ terms, which are $O(m_\pi^2)$. Indeed, if the isoscalar σ model is extended to the $(3, \bar{3})$ symmetry breaking model⁽⁶⁾, then in the soft-kaon limit, we have

$$\langle K^0 | \sigma | K^0 \rangle = - \frac{1}{2} m_\pi^2$$

which implies a 20% correction to the original estimate (2.14).

A criticism which may be applied to the above technique is that we do not use a $K\pi$ scattering amplitude which contains explicit extrapolations in the kaon four-momentum. If we attempt to rectify this by using the chiral symmetry breaking amplitude of Griffith⁽²³⁾, then ref. 22 shows that the same technique gives zero for the on shell $K \rightarrow 2\pi$ amplitude. The way out of this difficulty appears to lie in noticing that by taking the smooth function, C , as a constant we have demanded a form of kaon pole dominance of the weak Hamiltonian, which is incompatible with the use of kaon P.C.A.C. to obtain the $K\pi$ scattering amplitude.

Further comments on the use of the soft kaon limit in $K \rightarrow 2\pi$ decays are made in Chapter III.

Our second comments concern the amplitude $\mathcal{M}^{e\ell}(K^+ \rightarrow K^0 \pi^+ \pi^0)$.

This amplitude may be related by SU_3 to the experimentally known $\eta \rightarrow 3\pi$ amplitude. Many types of coupling are possible, but if we use symmetrised octet d-couplings for the pseudoscalar mesons and octet dominance for the electromagnetic Hamiltonian, i.e.

$$\mathcal{M}(\phi^a \rightarrow \phi^b \phi^c \phi^d) \propto d^{fg3}(d^{abf}d^{cdg} + d^{acf}d^{bdg} + d^{bcf}d^{adg}) \quad (2.20)$$

we obtain $\simeq 7.5\alpha$ for the modulus of the ratio in eq. (2.16). The difference between the two results may readily be attributed to the apparent enhancement of the $\eta \rightarrow 3\pi$ width over the value calculated using SU_3 assumptions.

Attempts, other than by " SU_3 " as in refs. (17) and (18), may also be made to estimate the contribution of Fig. 2.3. One way is to take the pion to zero four-momentum in $K \rightarrow \pi^+\eta$ and compare the result with $K^0 \rightarrow \pi^0\pi^0$ after one of the π^0 's has been taken to zero four-momentum. We assume octet dominance to write down the allowed coupling (f-coupling forbidden by CP invariance)

$$\langle \phi^a | J^C J^d | \phi^b \rangle \propto d^{abf}d^{cdf} \quad (2.21)$$

This approach enables us to avoid using SU_3 in an amplitude (i.e. $K \rightarrow \pi\pi$) which is non-zero only if SU_3 is broken⁽²⁰⁾.

The result is

$$\mathcal{M}(K^+ \rightarrow \pi^+\eta) = -\frac{1}{3} \mathcal{M}(K_1^0 \rightarrow \pi^0\pi^0) \quad (2.22)$$

This value is smaller than and of opposite sign to that obtained "by SU_3 " in ref. (18). To lowest order in chiral symmetry breaking $\langle \pi^0 | \mathcal{H}^{el} | \eta \rangle \simeq 0$ (here \mathcal{H}^{el} is the standard electromagnetic Hamiltonian, in eq. (2.3)). By SU_3 , with a pure octet η^+ ,

+ see footnote on next page.

$$\begin{aligned} \langle \eta | \epsilon x u_3 | \pi^0 \rangle &= \frac{\epsilon x}{\sqrt{3}} \left\{ \langle K^+ | u_3 | K^+ \rangle - \langle K^0 | u_3 | K^0 \rangle \right\} \\ &= -\Delta_t(m_K^2) / \sqrt{3} \end{aligned} \quad (2.23)$$

Hence, with the value of $\Delta_t(m_K^2)$ given by eq. (2.4), Fig. (2.3) is estimated to give

$$M(K^+ \rightarrow \pi^+ \pi^0) / M(K_1^0 \rightarrow \pi^0 \pi^0) = + \frac{(\Delta(m_K^2) - \Delta(m_\pi^2))}{3(m_\eta^2 - m_\pi^2)} \quad (2.24)$$

A comparison of eqs. (2.24) and (2.16) indicates that the η -pole graph may reduce the original $K^+ \rightarrow \pi^+ \pi^0$ amplitude by around 30%. The nett result still remains substantial however.

Finally we make the remark that all the amplitudes we obtain are real. Physically, one expects the K_1^0 and K^+ modes to have the phase of respectively the $I = 0$ (mainly) and $I = 2$ s-wave $\pi\pi$ phase shifts at centre of mass energy = m_K . The only equation where this might have significant effect is eq. (2.17). Provided the difference between the phase shifts, $\delta_2 - \delta_0$ is less than 90° , the signs are still such as to enhance $K_1^0 \rightarrow \pi^+ \pi^-$ over $K_1^0 \rightarrow \pi^0 \pi^0$. Experimentally⁽³⁸⁾, $\delta_2 - \delta_0 \simeq 40^\circ$, so the problem is not very vexing, empirically, at least.

We conclude from the work of this chapter that electromagnetic mechanisms, such as Fig. 2.4, may give substantial $\Delta I = 3/2$ corrections to $\Delta I = 1/2$ weak transitions. Moreover, if consistent

⁺ It is unlikely that the effect of η -X mixing will be large in this particular matrix element (see Chapter V).

use is made of the "tadpole" effective Hamiltonian, $-exu_3$, then these corrections have the correct sign to describe the enhancement of $K_1^0 \rightarrow \pi^+\pi^-$ over $K_1^0 \rightarrow \pi^0\pi^0$; this is in encouraging contrast to the wrong sign obtained from the naive use of the standard electromagnetic Hamiltonian⁽²²⁾.

CHAPTER III

K → 2π DECAYS IN THE INTERMEDIATE VECTOR BOSON MODEL

1. Introduction

The starting point for this chapter is a paper by Glashow, Schmitzer and Weinberg⁽⁸⁾. They study $K_1^0 \rightarrow 2\pi$ decays in the intermediate vector boson model (see ref. (3) Chapters 18 and 19), so that

$$\mathcal{H}_W = g(L_\mu + J_\mu)W^\mu + \text{h.c.} \quad (3.1)$$

Here L_μ is the lepton current, J_μ the usual hadron (Cabibbo) current and W_μ is the intermediate vector boson field. For a non-leptonic decay, mediated by a single vector boson line, the effective Hamiltonian is

$$\mathcal{H}^{\text{eff}} = -i g^2 \int d^4x d_W^{\mu\nu}(x) T \{ J_\mu(x) J_\nu^+(0) \} \quad (3.2)$$

where $d_W^{\mu\nu}(x) = \langle T \{ W^\mu(x) W^{\nu\dagger}(0) \} \rangle_0$ is the vector boson propagator. The coupling constant g is related to the usual Fermi constant G of leptonic weak interactions by taking the limit of large vector boson mass, $M_W \rightarrow \infty$,

$$G = \sqrt{2} g^2 / M_W^2 \quad (3.3)$$

In the $K \rightarrow 2\pi$ matrix element of (3.2), the two pions are reduced to zero four-momentum, averaging over the two orders of partial integration in the $\pi^+\pi^-$ case, and neglecting possible

σ -commutators. The result is as written previously in eq. (2.14). The kaon is now reduced to zero four momentum and the result written in the form of integrals over the spectral functions of vector and axial vector currents. The integrals are convergent if Weinberg's first and second spectral function sum rules (S.F.S.R.) hold in $SU_3 \times SU_3$ and a value for the mass M_W may be obtained from the $K_1^0 \rightarrow 2\pi$ width,

$$M_W \simeq 8 \text{ GeV.} \quad (3.4)$$

A similar calculation has been performed by Venturi⁽³⁹⁾. In this calculation, the weak current is assumed to have different vector and axial-vector Cabibbo angles, i.e.

$$J_\mu = V_\mu^{1+12} \cos \theta_V + A_\mu^{1+12} \cos \theta_A + V_\mu^{4+15} \sin \theta_V + A_\mu^{4+15} \sin \theta_A \quad (3.5)$$

First the kaon is reduced to zero four-momentum and then the two pions. The result is proportional to $\sin(\theta_A - \theta_V)$ and the integrals are again convergent if suitable combinations of S.F.S.R. hold. With $\theta_A - \theta_V \simeq .04$ a value of $M_W \simeq 5 \text{ GeV}$ is obtained. The smallness of $\theta_A - \theta_V$ is balanced by an increase in the spectral integral which does not vanish in the SU_3 symmetry limit⁽²⁰⁾ in contrast to the previous approach.

In both of these calculations, a special sequence of reductions is observed and σ -terms are ignored. The purpose of the work in this chapter is to test the reliability of the method by

- (i) investigating the dependence on the order of reduction,
- (ii) keeping σ terms and evaluating them by the $(3, \bar{3})$ symmetry breaking model.

As regards (i), we do not see any compelling reason why any order should be preferred over another; if the results do depend on the order of reduction, they are unreliable to that extent. One may then use the average, but this does not fully solve the problem. It is clear from refs. (8) and (39) that if σ terms are neglected, then a definite dependence on order is found.

As regards (ii), again there is no compelling reason a priori why σ -terms have to be kept or disregarded, as one can always define the off-shell reductions with or without them⁽⁴⁰⁾; again the results may be unreliable to the extent of the influence of the σ -terms. We find that if σ -terms are retained, the reductions have several attractive properties

- (a) the result is independent of the order of reduction
- (b) the $\Delta I = \frac{1}{2}$ rule is obeyed
- (c) the SU_3 properties of the matrix elements of \mathcal{H}_W are correctly exhibited.
- (d) There is now no problem of undesirable extrapolation in the momentum carried off by \mathcal{H}_W , acting as a spurion to conserve energy momentum.

However the results of Glashow et al. and Venturi are now substantially altered.

The lay-out of this chapter is as follows. In Section 2, we give a brief review of spectral function sum rules. In Section 3, the independence of the order of reductions is established in the general notation of Glashow and Weinberg⁽⁷⁾ and the specific calculations for $K \rightarrow 2\pi$ decays are performed in Section 4.

2. Spectral Function Sum Rules

We are interested in obtaining the spectral representation of the vacuum expectation value of the time-ordered product of two vector or axial-vector currents. The derivation of these representations for fields is given by Bjorken and Drell⁽⁴¹⁾, Chapter 16, and the derivation for currents follows from exactly the same principles of introducing a complete set of intermediate states and using Lorentz covariance, and the usual integral representation for the θ function. The result is

$$\begin{aligned}
 & \langle T \{ J_\mu^a(x) J_\nu^b(0) \} \rangle_0 \\
 &= i(2\pi)^{-4} \int \rho_1^{ab}(m^2) dm^2 \int d^4q e^{iq \cdot x} (-g_{\mu\nu} + q_\mu q_\nu / m^2) / (q^2 - m^2 + i\epsilon) \\
 &+ i(2\pi)^{-4} \int \rho_0^{ab}(m^2) dm^2 \int d^4q e^{iq \cdot x} q_\mu q_\nu / (q^2 - m^2 + i\epsilon) \quad (3.5)
 \end{aligned}$$

ρ_0^{ab} and ρ_1^{ab} are the spin 0 and spin 1 spectral functions, i.e.

$$q_\mu q_\nu \rho_0^{ab}(q^2) = (2\pi)^3 \sum_{J=0} \delta^4(p_n - q) \langle 0 | J_\mu^a(0) | n \rangle \langle n | J_\nu^b(0) | 0 \rangle \quad (3.6)$$

$$(-g_{\mu\nu} + q_\mu q_\nu / q^2) \rho_1^{ab}(q^2) = (2\pi)^3 \sum_{J=1} \delta^4(p_n - q) \langle 0 | J_\mu^a(0) | n \rangle \langle n | J_\nu^b(0) | 0 \rangle \quad (3.7)$$

In fact, it is necessary to modify eq. (3.5) by the introduction of Schwinger terms. To see this, take the ∂_μ of eq. (3.5). The derivative of the θ functions gives an equal-time commutator so that

$$\begin{aligned}
 \partial(x_0) \langle [J_0^a(x), J_\nu^b(0)] \rangle_0 &= +i(2\pi)^{-4} \int dm^2 \frac{\rho_1^{ab}(m^2)}{m^2} \cdot \int d^4q e^{iq \cdot x} i q_\nu \\
 &- (2\pi)^{-4} \int \rho_0^{ab}(m^2) dm^2 \int d^4q e^{iq \cdot x} \\
 &\cdot q_\nu \left\{ \frac{q^2}{q^2 - m^2} - \frac{m^2}{q^2 - m^2} \right\} \\
 &= i \partial_\nu \delta^4(x) \int dm^2 (\rho_1^{ab}(m^2) / m^2 + \rho_0^{ab}(m^2)) \quad (3.8)
 \end{aligned}$$

where in the first step we have used the spin-zero spectral representation, (3.6) for $\langle T \{ \partial J^a(x) J_\nu^b(0) \} \rangle_0$.

Eq. (3.8) gives a spectral representation for the vacuum expectation value of the equal time commutator. However, if we do this from first principles, by inserting intermediate states, it is easy to show that

$$\delta(x_0) \langle [J_0^a(x), J_\nu^b(0)] \rangle_0 = \begin{matrix} 0 & \nu=0 \\ i\delta_\nu^4(x) \int dm^2 (\rho_1^{ab}(m^2)/m^2 + \rho_0^{ab}(m^2)) & \nu=1,2,3. \end{matrix} \quad (3.9)$$

The non-zero term for $\nu = 1, 2, 3$ is the vacuum expectation value of the Schwinger term⁽⁴²⁾. Its existence shows that the local commutators postulated in eqs. (1.5b), (1.6b) and (1.7c) must be modified by the introduction of extra terms in the time-space component commutators.

A comparison of eqs. (3.8) and (3.9) shows that we must modify the $\mu = \nu = 0$ components of eq. (3.5), viz.

$$\begin{aligned} & \langle T \{ J_\mu^a(x) J_\nu^b(0) \} \rangle_0 \\ &= i(2\pi)^{-4} \int \rho_1^{ab}(m^2) dm^2 \int d^4q e^{iq \cdot x} (-g_{\mu\nu} + q_\mu q_\nu / m^2) / (q^2 - m^2 + i\epsilon) \\ &+ i(2\pi)^{-4} \int \rho_0^{ab}(m^2) dm^2 \int d^4q e^{iq \cdot x} q_\mu q_\nu / (q^2 - m^2 + i\epsilon) \\ &- i g_{\mu 0} g_{\nu 0} \delta^4(x) \int dm^2 (\rho_1^{ab}(m^2)/m^2 + \rho_0^{ab}(m^2)) \end{aligned} \quad (3.10)$$

The non-covariant nature of the time-ordered product of currents has been stressed by Johnson⁽¹¹⁵⁾.

Spectral function sum rules are an attempt to obtain information on chiral symmetry through relations among the spectral functions of the vector and axial vector currents. Weinberg's first and second sum rules (W1 and W2) are respectively

$$\int dm^2 (\rho_1^V(m^2)/m^2 + \rho_0^V(m^2)) = \int dm^2 (\rho_1^A(m^2)/m^2 + \rho_0^A(m^2)) \quad (3.11)$$

$$\int dm^2 \rho_1^V(m^2) = \int dm^2 \rho_1^A(m^2) \quad (3.12)$$

where ρ^V and ρ^A are the spectral functions of any of the vector and axial vector currents in the SU_3 octets.

Of these, W1, eq. (3.11), is on the sounder footing. Weinberg⁽⁴³⁾ proved it originally by assuming vector and axial current conservation, so that only a zero-mass pseudoscalar meson contributes to ρ_0^A . Using Ward identity techniques on the three-point function of two axial and one vector currents, he deduced the equality of the Schwinger terms in eq. (3.10). The assumption of current conservation may be relaxed, provided one stays within specific models of symmetry breaking (see the paper by Gerstein, Schnitzer and Weinberg in ref. 7). It is assumed in either case that the Schwinger terms in the $[J_0, J_i]$ commutators are c-numbers or at least commute with the currents; this is certainly a model-dependent assumption. The same assumption also enables one to prove equality of Schwinger terms by using the Jacobi identity for a charge and two currents⁽⁴⁴⁾. An alternative approach is to assume equality of the vector and axial current propagators (the Fourier transform of eq. (3.10)) in the asymptotic limit⁽⁴⁵⁾ $q \rightarrow \infty$.

The derivations of eq. (3.12) are not so convincing. Weinberg's proof⁽⁴³⁾ was based on free field behaviour of the currents A_ν and V_λ in the $p^2 \rightarrow \infty$ limit of the expression

$$\int d^4y \langle \pi^a | T \{ A_\nu^b(y) V_\lambda^c(0) \} | 0 \rangle e^{ip \cdot y} \quad .$$

Das et al.⁽⁴⁵⁾ derive W_2 by assuming a suitable superconvergence relation for the difference between vector and axial current propagators in the limit $q^2 \rightarrow \infty$. In both of these cases the derivations are strictly applied only for currents with the same SU_2 and hypercharge quantum numbers, though it is possible to make superconvergence assumptions to derive equality of the spectral integrals for all currents.

In practice, the spectral integrals are calculated by assuming saturation by suitable low mass intermediate states, π , ρ , A , etc. Clearly this is a stronger assumption for W_2 than W_1 , which is weighted by a factor $1/m^2$. The field algebra Lagrangian⁽⁴⁶⁾ is a model of pseudoscalar and spin-1 meson fields which explicitly exhibits the relations obtained by assuming single particle dominance of the S.F.S.R.

Successful predictions using S.F.S.R. include the following

- (i) $m_A^2/m_\rho^2 \simeq 2$, from the $I = 1$ SU_2 currents⁽⁴³⁾.
- (ii) from the vector currents, W_1 gives a good prediction of the K^* width⁽⁴⁵⁾ in terms of the ρ width (the contribution of a possible scalar χ is neglected since it is of second order in the symmetry-breaking).
- (iii) A convergent prediction of the pion electromagnetic mass difference⁽¹¹⁾.
- (iv) Successful use of the SU_3 form of W_1 in hard meson calculation on K_{l3} form factors (Chang and Leung⁽⁷⁾).
- (v) Reasonable agreement of W_1 with experimental values for ρ , ω and ϕ couplings to the electromagnetic current⁽⁴⁷⁾.

Notable failures of S.F.S.R., particularly W2 when extended to SU_3 , include

- (a) complete failure of W2 for the $I = 0$ and $I = 1$ vector currents⁽⁴⁸⁾,
- (b) a bad prediction of the K^* width⁽⁴⁵⁾ using W2,
- (c) strange results in hard meson calculations on K_{l3} form factors, where one tends to obtain f_K/f_π so close to 1 that the second order SU_3 symmetry breaking⁽⁴⁸⁾ of the $f_+(0)$ form factor is huge, $\approx 15-20\%$. (Glashow and Weinberg⁽⁷⁾.)

This brief summary is sufficient to show strong support for W1 in $SU_3 \times SU_3$, and W2 in $SU_2 \times SU_2$, but definite discrepancies if W2 is extended to $SU_3 \times SU_3$, at least in the single particle dominance approximation.

3. Independence of Order of Reduction

We now return to the general problem in which we are interested, namely the soft meson reductions in the matrix element

$$\langle \phi^i \phi^j | T \{ v_\mu^a(x) A_\nu^b(0) \} | \phi^k \rangle \quad (3.13)$$

When all mesons are taken off mass shell to zero four-momentum, with ∂A as interpolating fields, expression (3.13) is related directly to

$$\int d^4x_1 d^4x_2 d^4x_3 \langle T \{ \partial A^c(x_1) \partial A^d(x_2) \partial A^e(x_3) v_\mu^a(x) A_\nu^b(0) \} \rangle_0 \\ = \langle \partial A^c \partial A^d \partial A^e v_\mu^a A_\nu^b \rangle_0 \quad (3.14)$$

where this expression is introduced to simplify the notation.

We use the general notation of ref. (7) for chiral symmetry breaking. The Hamiltonian is assumed to have the form

$$\mathcal{H} = \mathcal{H}_0 - \epsilon_i \phi_i \quad (3.15)$$

where \mathcal{H}_0 is invariant under some group G , and the ϕ_i are hermitean local fields which form a basis for a real representation of the group G . The local commutation relations of the currents associated with the generators of G are assumed to be

$$\delta(x_0) [J_0^a(x), J_\nu^b(0)] = i C^{abc} J_\nu^c(x) \delta^4(x) + \text{S.T.} \quad (3.16)$$

where C^{abc} are the structure constants of the group and in general the non-covariant Schwinger terms (S.T.) will be disregarded since they would presumably be absent from a covariantly defined time-ordered product, T^* , in eqs. (3.13) and (3.14). The field transformations under G are defined by

$$\delta(x_0) \left[\int d^3x J_0^a(x), \phi_j(0) \right] = -i (T^a)_{jk} \phi_k(0) \quad (3.17)$$

The Euler-Lagrange equations imply

$$\partial^\mu J_\mu^a(x) = \epsilon_i (T^a)_{ij} \phi_j(x) \quad (3.18)$$

and the group properties of the T^a matrices are summed up by

$$[T^a, T^b] = C^{abc} T^c \quad (3.19)$$

$$(T^a)_{ij} = - (T^a)_{ji} \quad (3.20)$$

Eq. (3.20) is a direct consequence of the fact that the ϕ 's form a real representation.

We start with the relevant amplitude for the weak Hamiltonian

between single particle states. All equations are obtained by performing a partial integration on the first ∂J encountered in the product. Then

$$\begin{aligned} & \langle \partial J^a \partial J^b J^c J^d \rangle \\ &= - \epsilon_j T_{jl}^b (-i) T_{lk}^a \langle \phi_k J^c J^d \rangle - i \langle \partial J^b (C^{ace} J^e J^d + C^{ade} J^c J^e) \rangle \end{aligned} \quad (3.21)$$

Explicitly symmetrising the first term, which is the σ -commutator term, we have

$$T^b T^a = \frac{1}{2} \{ T^b, T^a \} + \frac{1}{2} C^{bae} T^e .$$

Substituting into eq. (3.21) and using eq. (3.18),

$$\begin{aligned} & \langle \partial J^a \partial J^b J^c J^a \rangle \\ &= + \frac{1}{2} \epsilon_j \{ T^b, T^a \}_{jl} \langle \phi_l J^c J^d \rangle + \frac{1}{2} C^{bae} \langle \partial J^e J^c J^d \rangle \\ & \quad - i \langle \partial J^b (C^{ace} J^e J^d + C^{ade} J^c J^e) \rangle \end{aligned} \quad (3.22)$$

If we now perform the remaining partial integrations in this equation and use the fact that the matrices - $(C^a)^{bc}$ themselves form representations of the group, i.e.

$$[C^a, C^b] = - C^{abc} C^c \quad (3.23)$$

then the result simplifies to

$$\begin{aligned} \langle \partial J^a \partial J^b J^c J^d \rangle &= \frac{1}{2} \epsilon_j \{ T^b, T^a \}_{jl} \langle \phi_l J^c J^d \rangle \\ & \quad + \frac{1}{2} \{ (C^{bce} C^{afe} + C^{ace} C^{bfe}) \langle J^f J^d \rangle + c \leftrightarrow d \} \\ & \quad - (C^{ace} C^{bdf} + C^{bce} C^{adf}) \langle J^f J^e \rangle \end{aligned} \quad (3.24)$$

The right hand side of this equation is explicitly symmetric in $a \Leftrightarrow b$, which is what we wished to show.

Exactly the same sort of manipulations are required to show that after partial integrations, expression (3.14) is completely symmetric in indices c, d and e , though of course the algebra is somewhat more complicated. Partially integrating on ∂J^c and symmetrising the c -commutator terms as in eq. (3.22), we have

$$\begin{aligned} & \langle \partial J^c \partial J^d \partial J^e J^a J^b \rangle \\ &= + \frac{1}{2} \left\{ \epsilon_j \left\{ T^c, T^d \right\}_{jl} \langle \phi_l \partial J^e J^a J^b \rangle + C^{dcf} \langle \partial J^f \partial J^e J^a J^b \rangle + d \Leftrightarrow e \right\} \\ & - i \langle \partial J^d \partial J^e (C^{cdf} J^f J^b + C^{cbf} J^a J^f) \rangle \end{aligned} \quad (3.25)$$

By partially integrating the ∂J^e in the first term on the right hand side, and using eq. (3.24) for the other terms we can immediately pick out one term which is completely symmetric in c, d , and e , namely

$$\begin{aligned} & \frac{1}{2} \epsilon_j \left\{ T^c, T^d \right\}_{jl} \langle \phi_l (C^{eaf} J^f J_b + C^{ebf} J^a J^f) \rangle + \text{cyclic perm} \\ & c \rightarrow d \rightarrow e \end{aligned} \quad (3.26)$$

In the remaining terms, the products of T matrices are forced into totally symmetric combinations and terms involving the commutator of two T 's. By patient use of eqs. (3.19) and (3.23) we can show that these remaining terms also are completely symmetric in c, d and e . The result is

$$\begin{aligned}
 & \langle \partial J^c \partial J^d \partial T^e J^a J^b \rangle \\
 & = + \frac{\epsilon_j}{12} \{ \{ T^c, T^d \}, T^e \}_{j\ell} \langle \phi_\ell J^a J^b \rangle + \frac{i}{4} (C^{caf} C^{dbg} C^{egh} + a \leftrightarrow b) \langle J^h J^f \rangle \\
 & - \frac{i}{6} (C^{chf} C^{dbg} C^{egf} \langle J^h J^a \rangle + a \leftrightarrow b) \\
 & + \frac{1}{4} \epsilon_j \{ T^c, T^d \}_{j\ell} \langle \phi_\ell (C^{eaf} J^f J^b + C^{ebf} J^a J^f) \rangle + \text{perm} \quad (3.27)
 \end{aligned}$$

where "perm" denotes the five similar expressions with $c \leftrightarrow d \leftrightarrow e$. Thus the result is totally symmetric in the meson indices, regardless of the order of partial integration.

4. Explicit Calculations for $K \rightarrow 2\pi$ Decays

We now proceed to give the explicit form of eq. (3.27) for the decays $K_1^0 \rightarrow \pi^0 \pi^0$, $K_1^0 \rightarrow \pi^+ \pi^-$ and $K^+ \rightarrow \pi^+ \pi^0$. We particularise to $SU_3 \times SU_3$, with the structure constants f^{abc} , and to the $3, \bar{3}$ symmetry-breaking model, so that we have (eq. (1.14))

$$\mathcal{H} = \mathcal{H}_0 - \epsilon(u_0 + cu_8) \quad (3.28)$$

For complete generality, we allow for different vector and axial-vector Cabibbo angles (eq. (3.5)).

The only terms which we cannot evaluate explicitly in eq. (3.27) are those of the type $\langle \phi_e J^a J^b \rangle$, where ϕ_e is the term arising from the π - π σ -commutator; the u^K arising from the π - K σ -commutator can be replaced by a ∂V^K and a further partial integration performed to reduce the term to a two point function. If we do this we obtain:

$$\langle \pi^+ \pi^- | \mathbb{H}^{\text{eff}} | K_1^0 \rangle = \frac{g^2}{f_\pi^2 f_K} \int d^4x d_W^{\mu\nu}(x) \left\{ \int d^4y \frac{1\epsilon(\sqrt{2}+c)}{3} \langle T \{ (\sqrt{2}u_0(y) + u_8(y)) \cdot (A_\mu^{4+15}(x) A_\nu^{4-15}(0) - V_\mu^{1+12} V_\nu^{1-12}) \cos\theta_V \sin\theta_A + V \leftrightarrow A \rangle \right. \quad (3.29)$$

$$+ \frac{2(\sqrt{2}+c)^2}{3c(\sqrt{2}-c/2)} \langle T \{ (A_\mu^4(x) A_\nu^4(0) - V_\mu^1(x) V_\nu^1(0)) \cos\theta_V \sin\theta_A + V \leftrightarrow A \rangle \rangle$$

$$+ \left. \langle T \{ (A_\mu^4 A_\nu^4 - V_\mu^1 V_\nu^1) + \frac{2(\sqrt{2}-2c)}{3} (V_\mu^1 V_\nu^1 - A_\mu^1 A_\nu^1) \} \rangle \sin(\theta_A - \theta_V) \right\}$$

$$\langle \pi^+ \pi^0 | \mathbb{H}^{\text{eff}} | K^+ \rangle = \sin(\theta_A - \theta_V) \frac{g^2}{f_\pi^2 f_K} \int d^4x d_W^{\mu\nu}(x) \cdot$$

$$\cdot \left\langle T \left\{ (A_\mu^4 A_\nu^4 - V_\mu^1 V_\nu^1) \frac{(\sqrt{2}+c)}{3c} + (V_\mu^1 V_\nu^1 - A_\mu^1 A_\nu^1) \frac{(\sqrt{2}-2c)}{3c} \right\} \right\rangle \quad (3.30)$$

In eq. (3.29), "V ↔ A" means replace A_μ by V_μ , $\cos \theta_A$ by $\cos \theta_V$, etc. No isoscalar σ term appears in eq. (3.30), of course. The equation for $K_1^0 \leftrightarrow \pi^0 \pi^0$ decay is redundant as eq. (2.17) is automatically respected (a useful check on the algebra!).

The first point to note is that $K^+ \leftrightarrow \pi^+ \pi^0$ has zero amplitude⁺ if $\theta_A = \theta_V$. The basic reason for this is that $[A^a, \mathbb{H}^{\text{eff}}] = [V^a, \mathbb{H}^{\text{eff}}]$ when $\theta_A = \theta_V$, so that, for example the isospin content of \mathbb{H}^{eff} is not changed by a soft pion reduction. Thus $\Delta I = 3/2$ terms appear only for $\theta_A \neq \theta_V$. (The same situation holds in Glashow, Schnitzer and Weinberg's treatment⁽⁸⁾ Using the same reduction technique as ref. (8) and suitable S.F.S.R.

⁺ The concept of describing $K^+ \rightarrow \pi^+ \pi^0$ by having $\theta_A \neq \theta_V$ has also been employed in a relativistic quark model⁽⁵⁰⁾, though not with great numerical success; the experimental widths quoted are wrong by a factor of 10.

one obtains

$$\mathcal{M}(K^+ \rightarrow \pi^+ \pi^0) / \mathcal{M}(K_1^0 \rightarrow \pi^0 \pi^0) \simeq - \frac{\sin(\theta_V - \theta_A)(2.6 \ln(M_W/m_\rho) - 2.0)}{\sin \theta \cos \theta (1.4 \ln(M_W/m_\rho) - 1.1)} .$$

This equation is very insensitive to the actual value of M_W . If one has faith in the soft meson reduction employed, an upper limit of

$$|\theta_V - \theta_A| \lesssim 5.10^{-3} \quad (3.31)$$

is obtained from the experimental values for the amplitudes, neglecting the electromagnetic corrections to the K^+ decay.)

As stated previously, the same situation does not hold in Venturi's reduction⁽³⁹⁾, where all amplitudes are proportional to $(\theta_A - \theta_V)$. Typically

$$K_1^0 \rightarrow \pi^0 \pi^0 \sim \sin(\theta_V - \theta_A) \langle T \{ A_{\mu\nu}^4 A_{\mu\nu}^4 - V_{\mu\nu}^4 V_{\mu\nu}^4 + 4A_{\mu\nu}^1 A_{\mu\nu}^1 - 4V_{\mu\nu}^1 V_{\mu\nu}^1 \} \rangle_0 \quad (3.32a)$$

$$K_1^0 \rightarrow \pi^+ \pi^- \sim \sin(\theta_V - \theta_A) \langle T \{ A_{\mu\nu}^4 A_{\mu\nu}^4 - V_{\mu\nu}^4 V_{\mu\nu}^4 + 2A_{\mu\nu}^1 A_{\mu\nu}^1 - 2V_{\mu\nu}^1 V_{\mu\nu}^1 \} \rangle_0 \quad (3.32b)$$

These two equations are unable to respect the near exactness of the $\Delta I = 1/2$ rule which is experimentally observed.

The second point is that the result of ref. (8) is reproduced by the second term of eq. (3.29) when $\theta_A = \theta_V$, except for an extra factor of $4(\sqrt{2} + c)^2 / (3c(\sqrt{2} - c/2))$. Since⁽⁶⁾ $c(\sqrt{2} + c) = O(m_\pi^2/m_K^2)$, it is clear that large cancellations have occurred between the terms retained in ref. (8) and the σ -commutator terms.

We now estimate the term in eq. (3.29) containing $\sqrt{2}u_0 + u_8$ by relating it by SU_3 to a term containing u^4 . For simplicity,

we set $\Theta_A = \Theta_V = \Theta$, this will only introduce an error $O(M(K^+ \rightarrow \pi^+\pi^0)/M(K_1^0 \rightarrow \pi\pi)) \simeq 5\%$, which is certainly negligible in this treatment. The coupling of the currents to the singlet u_0 gives zero; the coupling to u_8 can only be d-coupling, by C-invariance. The result is

$$\begin{aligned} & \langle T \{ (\sqrt{2}u_0 + u_8)(A_\mu^{4+15} A_\nu^{4-15} - V_\mu^{1+12} V_\nu^{1-12} + V \leftrightarrow A) \} \rangle \\ &= -\frac{\sqrt{3}}{2} \langle T \{ u^4 (V^{6-17} V^{1-12} + V^{1+12} V^{6+17} + V \leftrightarrow A) \} \rangle \end{aligned} \quad (3.33)$$

Finally, replacing u_4 by $-2/\sqrt{3}c \cdot \partial V^5 \tilde{\epsilon}^1$ and partially integrating one obtains

$$\begin{aligned} \langle \pi^+\pi^- | H^{\text{eff}} | K_1^0 \rangle &= \frac{\sqrt{2} + c}{\sqrt{2} - c/2} \cdot \frac{g^2 \cos\Theta \sin\Theta}{f_\pi^2 f_K} \cdot \int d^4x d^{\mu\nu}(x) \cdot \\ & \langle T \{ A_\mu^4(x) A_\nu^4(0) + V_\mu^4 V_\nu^4 - A_\mu^1 A_\nu^1 - V_\nu^1 V_\mu^1 \} \rangle_0 \end{aligned} \quad (3.34)$$

The spectral integral in this equation is exactly the same as that in ref. (8), but the amplitude now has an additional factor of $2(\sqrt{2} + c)/(\sqrt{2} - c/2) \simeq 2m_\pi^2/m_K^2$, if we use the value of c from Gell-Mann, Oakes and Renner⁽⁶⁾. Thus, regardless of whether one believes the S.F.S.R. or not, the σ -terms have provided substantial alterations to the result of Glashow, Schnitzer and Weinberg⁽⁸⁾.

If, notwithstanding the conclusions in Section 2, one uses S.F.S.R. to ensure convergence of the integrals in eqs. (3.30) and (3.34) and to help evaluate them, in the same manner as refs. (8) and (39), one obtains

$$M_W \simeq 1.4 \times 10^4 \text{ GeV} \quad (3.35)$$

$$\text{and } (\theta_V - \theta_A) \simeq 10^{-3} \quad (3.36)$$

The size of M_W is due to the fact that it appears inside a logarithm. The smallness of $(\theta_V - \theta_A)$ is very acceptable in view of the close connexions between⁽⁵¹⁾ $SU_3 \times SU_3$ algebra, universality of the weak interactions and Cabibbo current with $\theta_A = \theta_V$.

It is relevant to point out that the masses of the pions and kaons in the spectral integral in eqs. (3.30) and (3.34) are both set to zero, consistent with the soft meson techniques used to obtain these eqs. However, it is easily shown that if the spectral representations for physical mass pions and kaons are used, a quadratically divergent term appears with coefficient proportional to

$$f_\pi^2 m_\pi^2 - f_K^2 m_K^2 - f_s^2 m_s^2$$

where the subscript "s" denotes a possible χ meson intermediate state in $\langle V_\mu^\dagger V_\nu^\dagger \rangle_0$. From the experimental values f_π , f_K etc., it is clear that this expression cannot vanish. Thus there appears to be no equivalent of the calculation of finite pion mass difference by de Alwis⁽¹¹⁾, who uses physical pion spectral representation and a suitable alternative to W_2 to ensure convergence.

Finally we turn to the SU_3 properties of the reduction. The first point is that the very procedure of putting the pions and kaon on the same footing in the reduction may not be acceptable

since the decays are forbidden in the limit of SU_3 symmetry⁽²⁰⁾. Nevertheless, we note that the matrix element of eq. (3.34) does vanish in the SU_3 symmetry limit, so that our evaluation of the σ -term, by eq. (3.33) has been just right to eliminate the $1/c$ factors in eq. (3.29), which blow up in the SU_3 symmetry limit $c \rightarrow 0$. However, by the same token, eq. (3.33) must be criticised, because it is only true to the extent of SU_3 symmetry, and hence corrections to eq. (3.34) may be large.

The reduction with the σ -commutators retained also exhibits explicitly the non-vanishing of the $\langle \pi | J^a J^b | K \rangle$ weak vertex in the SU_3 limit (see eq. (2.21)). In this case the vanishing of the spectral integrals in the SU_3 symmetry limit is always cancelled by a $1/c$ factor, arising from the replacement of the π -K σ -commutator term, u_4 , by $-(2/\sqrt{3}c) \cdot \partial v^5$. This contrasts with the treatment of ref. (8), which implies the vanishing of the $\langle \pi | J^a J^b | K \rangle$ matrix element in the limit of SU_3 symmetry.

We may summarise our attitude in this chapter by saying that we have investigated a method of soft meson reduction in $K \rightarrow 2\pi$ decays which is in many ways more attractive than those employed previously. For this reason we regard the work as a move in the right direction. Although steps have been taken towards this end^(52,53),

⁺ It is amusing to note that if ρ and K^* intermediate states, neglected in the dispersive approach of ref. (52), are retained, and their contribution estimated using Greenberg's model⁽²¹⁾ for the weak Hamiltonian, then these pole terms cancel the subtraction term and the net result for $K \rightarrow \pi\pi$ is zero, in the approximation that $f_\pi = f_K$, and m_π^2 is negligible! Such difficulties appear to be symptomatic of more sophisticated current algebra approaches (see the comments in Section 4, Chapter II).

it seems unlikely that a proper hard-meson calculation will be very revealing, in view of the number of undetermined parameters which will be introduced.

CHAPTER IV

LINEAR EXPANSIONS OF PSEUDOSCALAR MESON SCATTERING

AMPLITUDES

1. Introduction

For this chapter we study the constraints imposed by current algebra, P.C.A.C. and chiral symmetry breaking which are collectively termed chirality, on pseudoscalar meson scattering amplitudes which are allowed to have at most a linear dependence on the Mandelstam invariants s , t and u and the q_1^2 associated with each particle off-mass-shell. The contents are designed partly as a review of the subject, in which the author has been involved, (although contributing little to its advance) and partly as an introduction to the chiral Lagrangian approach described in the next chapter. In the first section we discuss the physical ideas behind the linear expansion approach and the determination of scattering amplitudes for the π , K and η octet. In the second section a preliminary discussion is given of the possible relationship between chirality and duality, and the question of interpolating fields for η . In the third section the problem of $\eta - X$ mixing is discussed and finally we comment on the introduction of an SU_3 singlet axial current and compare the predictions with the Veneziano model terms.

The pioneering paper is due to Weinberg⁽⁵⁴⁾ who studied the problem of π - π scattering.⁺ The basic idea behind the approach

⁺ The scattering amplitudes for pseudoscalar mesons on massive targets are discussed in refs. (1) and (2).

is that the relevant amplitude is expanded in the most general form to first order in the Lorentz invariants s , t and q_1^2 , subject to the constraints of (a) crossing symmetry and (b) the identity $s + t + u = \sum_{i=1}^4 q_i^2$. One then uses as many tricks of low energy theorems as can be plausibly and acceptably mustered in order to determine the unknown coefficients appearing in the linear expansion.

Thus if one takes one particle off-mass-shell by an L.S.Z. reduction and uses the divergence of the appropriate current as interpolating field, then the amplitude must vanish when the particle is at zero four-momentum,

$$\begin{aligned} \langle \alpha | \partial A | \beta \rangle &= i q_\mu \langle \alpha | A_\mu | \beta \rangle \\ &\rightarrow 0 \text{ as } q_\mu \rightarrow 0 \end{aligned} \quad (4.1)$$

provided that there is no singularity in $\langle \alpha | A_\mu | \beta \rangle$ as $q_\mu \rightarrow 0$. Such singularities can arise from Born terms (see Fig. 4.1) in which the intermediate particle has the same mass as the other external particle. However, the absence of parity doublets for the pseudoscalar mesons ensures that there are no such diagrams

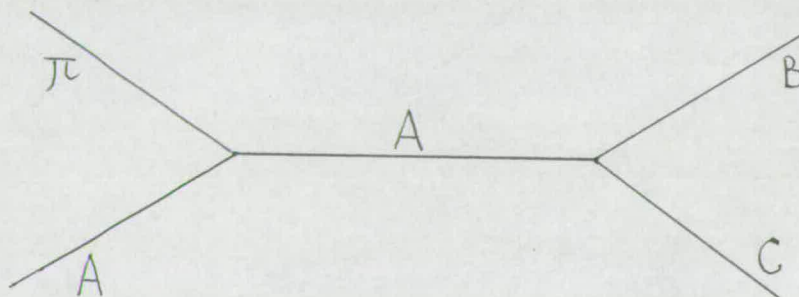


Fig. 4.1.

or singularities in amplitudes involving only pseudoscalar mesons. The condition (4.1) with all particles except one on mass shell is known as the Adler zero⁽⁵⁵⁾.

Further constraints are imposed when two or more particles are off mass shell and partial integrations are performed on the axial current divergences. The derivatives of the $\epsilon(x_0)$ functions (describing the time ordering) produce equal time commutators of the form $[A_0^i, \partial A^j]$ and $[A_0^i, A_\mu^j]$. Current algebra, and restrictions on chiral symmetry breaking determine these commutators and lead to further constraints on the constants in the linear expansion. For example, in the calculation of the $\pi\text{-}\pi$ scattering amplitude⁽⁵⁴⁾, there are initially three unknown constants in the linear expansion, which are determined by the conditions of (a) the Adler zero, (b) the $[A^i, \partial A^j]$ σ -commutator being an isoscalar (it may be $I = 0$ and 2 only, by $SU_2 \times SU_2$ algebra and vector current conservation) and (c) the $[A^i, A^j] = i \epsilon^{ijk} V^k$ commutator, which normalises the whole amplitude.

The linear expansion, determined thus by restrictions at points below threshold, is assumed to be a good approximation up to, and in some applications beyond the physical threshold. A necessary condition for the success of the scheme is that the imaginary part of the physical amplitude be small in the range of interest. In the low energy region, the s-wave phase shift δ_0 is given by the effective range formula

$$K \cot \delta = \frac{1}{a} + O(K^2) \quad (4.2)$$

where K is the magnitude of the three momentum of the particle



and a is the s-wave scattering length. Approximately, then,

$$\delta \sim a K$$

so that the imaginary part of the amplitude is largely determined by the scattering length. No internal inconsistencies appear in the approach, as the Weinberg scattering lengths are indeed small, $a_0 \simeq 0.20 m_\pi^{-1}$, $a_2 \simeq -0.06 m_\pi^{-1}$, and moreover in excellent agreement with present experimental analysis⁽⁵⁷⁾, $a_0 = (0.16 \pm 0.04) m_\pi^{-1}$ and $a_2 = (-0.05 \pm 0.01) m_\pi^{-1}$. The suffixes 0 and 2 refer to the appropriate isospin channel.

Attempts have been made to estimate the effects of quadratic terms in the Lorentz invariants. By taking all four pions simultaneously off mass shell and assuming that the $\langle \pi | \sigma | \pi \rangle$ vertex (where $\delta^{ij} \sigma = -i [A^i, \partial A^j]$) is very smooth, i.e. constant, off mass shell, one can obtain constraints which suggest that the quadratic terms have only a small effect⁽⁵⁸⁾. This treatment has been criticised by Sucher and Woo⁽⁵⁹⁾, however, who show that the approach can never determine uniquely an amplitude quadratic in Lorentz invariants. As an alternative, they modify the linear expansion by adding a term of the form $x \sqrt{4m_\pi^2 - s}$ and apply the further constraint of threshold unitarity. There is then a quadratic equation to solve for a_0 , with a small solution very close to Weinberg's and a large solution, $\simeq 2m_\pi^{-1}$. The conclusion from these papers appears to be that the small scattering length solution is stable to perturbations around the linear expansion, although the exact form of these additional terms is ambiguous.

⁺ Reference should be given at this point also to further attempts to include more structure in scattering amplitudes constrained by chirality. These include the hard pion methods⁽⁸⁶⁾, calculations imposing two particle unitarity⁽⁸⁷⁾, and renormalisation of the σ -model⁽⁸⁸⁾.

With these vindications of the validity of the linear expansion approach, it is interesting, in principle at least, to extend the work to other pseudoscalar meson processes. Griffith⁽²³⁾ calculated πK and KK scattering amplitudes, using the $(3, \bar{3})$ symmetry-breaking model to evaluate the σ -commutator terms. There are sufficient constraints to over-determine the amplitudes and the results $f_\pi = f_K$ and $c = +\sqrt{2}(m_\pi^2 - m_K^2)/(m_K^2 + \frac{1}{2}m_\pi^2)$, (in eq. (1.14)), appear as necessary consistency conditions. These are the same results as obtained by Gell-Mann, Oakes and Renner⁽⁶⁾. This should not be surprising, in view of the fact that the linear expansion for the scattering amplitude implies constancy of the σ -terms $\langle M^i | u^j | M^K \rangle$, which is just one of the starting assumptions of ref. 6. For this reason, one should regard these solutions as being correct only to first order of $SU_3 \times SU_3$ breaking effects, since P.C.A.C. correction terms $O(m_\pi^2, m_K^2)$ are neglected in this assumed constant behaviour of the three-point function.

These calculations again give satisfactorily small scattering lengths: for πK , $a_{\frac{1}{2}} = 0.22 m_\pi^{-1}$, $a_{\frac{3}{2}} = -0.11 m_\pi^{-1}$, and for KK , $a_0 = 0$ and $a_1 = -0.16 m_\pi^{-1}$. The threshold result for $K\bar{\pi}$ scattering is exactly the same as Weinberg's result for π scattering on the kaon, as a heavy target (see the reviews in refs. (1) and (2)). The $K\bar{K}$ scattering amplitude may be obtained from the KK amplitude by crossing, but clearly the threshold behaviour of the amplitude is totally inadequately represented by a linear expansion because of the ρ , ω and ϕ poles below or near threshold. Nevertheless it is interesting to note that the linear expansion method appears to try to reproduce the

result of these strong interactions by predicting the largest scattering length so far, $a_1 \simeq +0.50 m_\pi^{-1}$.

It is relevant to comment that this $K\pi$ scattering amplitude may also be used in $K_{\ell 4}$ calculations, in which the non-smooth behaviour of the kaon pole diagram (Fig. 4.2) is separated out from the full $K_{\ell 4}$ amplitude⁽⁶⁰⁾. The use of Griffith's off-

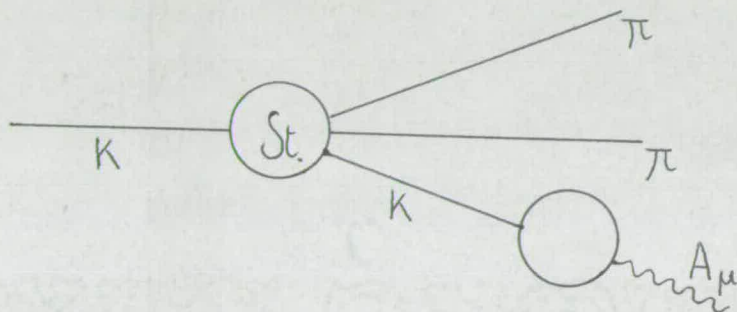


Fig. 4.2.

shell amplitude produces no contradictions with the simple Callan-Treiman equation⁽⁶¹⁾ in contrast to the results⁽⁶²⁾ of using a constant $K\pi$ amplitude. (This approach is the same as that employed in Chapter II, Section 4 (Fig. 2.5); the difficulties in that case do not arise here since the K meson is dominating the same operator A_{μ} , as in the calculation of the $K\pi$ scattering amplitude, at least up to the neglect of spin 1 intermediate states in A_{μ} .)

A very complete treatment of scattering amplitudes involving π , K and η has been given by Osborn⁽⁶³⁾. Similar work was done contemporaneously by this author⁽⁶⁴⁾, but with a somewhat different emphasis, and not in as great generality. The idea is simply to extend the previous results to include an η particle, using ∂A^8 as interpolating field for the η . There is a basic difference, however, between the cases when an η is, or is not present. The

amplitudes are determined uniquely from the demands of absence of exotic $I = 2$, $I = 3/2$ or $Y = 2$ σ -commutator terms (c.f. Weinberg's assumption of isoscalar π - π σ -commutator) whereas the former, which have no channels with such exotic quantum numbers, require knowledge of the actual form of the σ -commutators, as determined from the postulated chiral symmetry-breaking. Necessary consistency conditions again appear, namely $f_K = f_\pi$, the Gell-Mann - Okubo mass formula, $3m_\eta^2 = 4m_K^2 - m_\pi^2$ and, $c = \sqrt{2}(m_\pi^2 - m_K^2)/(m_K^2 + \frac{1}{2}m_\pi^2)$. However, Osborn shows that unless one performs further soft meson limits on the σ -commutator terms $\langle M^i | \sigma^j | M^K \rangle$ one cannot reproduce the final result $f_\eta = f_\pi$ of Gell-Mann, Oakes and Renner⁽⁶⁾. The results are exhibited for the sake of reference in Table I.

2. Duality and Interpolating Fields for η

The original motivation to this author for studying linear expansions of amplitudes involving η was to compare the results with the Veneziano model predictions. A general discussion of the concepts of duality, exchange degeneracy and the Veneziano model would be outwith the scope of this thesis, and excellent general reviews already exist in the literature⁽⁶⁵⁾. Instead, we confine ourselves to the papers of relevant interest.

The first of these is of course that by Lovelace⁽⁶⁶⁾. He obtains the well-satisfied constraint on the exchange degenerate ρ, f trajectory

$$a_\rho(m_\pi^2) = \frac{1}{2} \quad (4.3)$$

TABLE I : Scattering Amplitudes from Chirality, $\delta \propto \partial_A^8$

$\pi^a \pi^b \rightarrow \pi^c \pi^d$:	$\frac{1}{f^2} (\delta^{ab} \delta^{cd} (s - m_\pi^2) + \delta^{ac} \delta^{bd} (t - m_\pi^2) + \delta^{ad} \delta^{bc} (u - m_\pi^2))$
$\pi K \rightarrow \pi K$:	$\frac{1}{2f^2} (t + u - m_K^2 - m_\pi^2) \quad (I = 3/2 \text{ in s-channel})$
		$\frac{1}{4f^2} (3s - u + 2t - 2m_K^2 - 2m_\pi^2) \quad (I = 1/2 \quad " \quad " \quad)$
$KK \rightarrow KK$:	$0 \quad (I = 0 \quad " \quad " \quad)$
		$\frac{1}{f^2} (t + u - 2m_K^2) \quad (I = 1 \quad " \quad " \quad)$
$K\pi \rightarrow K\eta$:	$\frac{1}{4/3} f f_\eta (2(m_K^2 - m_\pi^2) - s - u + 2t - 2 \frac{(m_\eta^2 - m_\pi^2)}{m_\eta} \cdot q_\eta^2)$
$\eta\pi \rightarrow \eta\pi$:	$\frac{1}{3f_\eta^2} (s + t + u - 3m_\pi^2 - \frac{m_\eta^2 - m_\pi^2}{m_\eta} (q_{\eta_1}^2 + q_{\eta_2}^2))$
$K\eta \rightarrow K\eta$:	$\frac{1}{12f_\eta^2} (s + u + 10t + 2 \frac{(m_\eta^2 - m_\pi^2)}{m_\eta} (q_{\eta_1}^2 + q_{\eta_2}^2) - 6(3m_\eta^2 - m_K^2))$
$\eta\eta \rightarrow \eta\eta$:	$\frac{4m_\eta^2 - m_\pi^2}{3m_\eta^2} \cdot \frac{f^2}{f_\eta^4} (s + t + u - 3m_\eta^2)$

These amplitudes have the phase of the T matrix in $S = 1 + i(2\pi)^4 \delta^4(P_i - P_j)T$. Here $f = f_\pi = f_K$ is the meson decay constant; $f = f_\eta$ is not obtained unless more than two particles are taken off-shell⁽⁶³⁾. The Gell-Mann - Okubo mass formula, $4m_K^2 = 3m_\eta^2 + m_\pi^2$ appears as a necessary constraint in the $K\pi \rightarrow K\eta$ amplitude and has been used to simplify some of the other expressions.

by demanding that at the Adler point the denominator Γ function in the leading term Veneziano model for $\pi\pi$ scattering should have argument zero, in order to reproduce the Adler zero of current algebra. The approach has been extended, with successful results, to the case of πK and KK scattering and physically acceptable constraints on trajectories are predicted⁽⁶⁷⁾. The general case of pion emission has also been studied⁽⁶⁸⁾.

Moreover the amplitudes thus obtained have similar s , t and u structure to the amplitudes obtained by the linear expansion method, discussed in the previous section. A comparison of the following two equations, for $\pi^+\pi^+ \rightarrow \pi^+\pi^+$, will suffice to illustrate this.

$$M(\text{Ven.}) = \lambda \frac{\Gamma(1 - \alpha_\rho(t))\Gamma(1 - \alpha_\rho(u))}{\Gamma(1 - \alpha_\rho(t) - \alpha_\rho(u))} \quad (4.4)$$

$$M(\text{chiral}) = \frac{1}{f^2}(t + u - 2m_\pi^2) \quad (4.5)$$

Indeed, if eq. (4.4) is expanded for t and u close to their values at the Adler point ($t = u = m_\pi^2$), then eq. (4.5) is obtained, apart from the overall scale.

The remarkable feature of the results is that the πK Veneziano amplitude can accommodate both the π and K Adler conditions. An essential condition for this is, of course, that there is no necessary dependence on the particle momenta, q_1^2 , in **crossing alone eliminates such terms from the π - π amplitude** the chiral amplitude, crossing plus C-invariance does so for KK scattering, and the coefficient of possible q_1^2 terms in πK scattering turns out to be zero in the $3, \bar{3}$ symmetry breaking model.

One immediately runs into difficulties if one attempts to extend the Adler condition to the Veneziano model for processes involving η , or X particles. The simplest case of this is the $\pi\eta$ scattering amplitude, which involves the same trajectories as the $\pi\pi$ amplitude; if the Adler condition is applied for the pion, given the successful eq. (4.3), one obtains⁺ the highly unphysical condition^(64,65,78)

$$m_\eta^2 = m_\pi^2 \quad (4.6)$$

The condition (4.6) has an interesting parallel in the chiral amplitude for $\eta\pi$ scattering. As is evident from Table I, this amplitude does contain q_1^2 dependent terms, with coefficients proportional to $m_\eta^2 - m_\pi^2$, i.e. the chiral amplitude would be a function of s , t and u only if $m_\eta^2 = m_\pi^2$. The same situation occurs in all other amplitudes involving η , except $\eta\eta \rightarrow \eta\eta$, where crossing ensures that a term involving q_1^2 may always be

⁺ An alternative attitude is taken by Wong⁽⁷⁷⁾. He appeals to the splitting of the A_2 trajectory to assume that the A_2 exchanged in $\eta\pi$ scattering is not necessarily degenerate with the f trajectory, regardless of the exchange degeneracy exhibited in dual models of e.g. KK scattering. Although this allows us to escape from the unpleasant result (4.6), it is not an aesthetically satisfactory conclusion. A further study of incorporation of π and η Adler conditions by relaxation of trajectory conditions was made by this author; the conclusions are that if (a) parallel trajectories are maintained, (b) the maximum allowed f' coupling is added (consistent with the small $f' \rightarrow \pi\pi$ width) then the π and η Adler conditions imply $m_{A_2} - m_f \simeq 130$ MeV, a situation which is less satisfactory experimentally than mass degeneracy. Further if the π Adler condition alone is demanded, even with maximum f' coupling, it is still necessary to have the A_2 and f non-degenerate.

eliminated in favour of the combination $s + t + u$.

One possible understanding of the situation has been given by Osborn⁽⁶³⁾. Basically, the idea is that duality prescribes that amplitudes are constructed from non-exotic Mandelstam variables. Now if we prescribe a priori that the linear expansions should contain no dependence on exotic channel variables, then it can be seen explicitly⁽⁶³⁾ that the equations obtained in the $\pi\pi$, πK and KK amplitudes from (i) the Adler condition or (ii) the absence of exotic σ terms are equivalent. The implication is that, in dual amplitudes with no exotic channel, e.g. when there is an η , the application of the Adler condition is not justified. With this hypothesis, the undesirable results such as (4.6) are no longer inevitable.

An alternative viewpoint will be discussed in Section 4, where possible implications of η - X mixing are introduced. In view of the close connection between exchange degeneracy⁽⁶⁹⁾ (and in particular its realisation in the Veneziano model⁽⁶⁷⁾) and the quark nonet mixing scheme⁽⁷⁰⁾, it is not surprising that further insight may be gained by the study of this problem.

We finish off this section by discussing two points. Firstly we note the unsuccessful attempts to impose on the Veneziano model chirality constraints with more than one particle off mass shell. Initially the results of such calculations appeared very encouraging^(71,72), but further investigation⁽⁷³⁾ showed that there are a sufficient number of current algebra + $(3, \bar{3})$ symmetry-breaking conditions to prove inconsistencies with Veneziano expressions for off mass shell pseudoscalar scattering, of the "factorised" form,

$$M(s, t, u; q_a^2, a_b^2, m_c^2, m_d^2) = X_a(q_a^2)X_o(q_b^2)A(s, t, u) \quad (4.7)$$

where $A(s, t, u)$ is the relevant Veneziano amplitude. Specifically, from a study of the πK scattering amplitude with a pion and kaon off-mass-shell, one can show from current algebra alone that the form factor f_+ in $K\ell_3$ decays, defined by

$$\langle \pi | V_\mu | K \rangle = if_+(\Delta)(p_{\pi\mu} + p_{K\mu}) + if_-(\Delta)(p_{\pi\mu} - p_{K\mu}) \quad (4.8)$$

is of first order in SU_3 breaking, unless $m_\pi^2 = 0$. This is in direct contradiction with the Ademollo-Gatto theorem⁽²⁾, which is derived from the $SU_3 \times SU_3$ charge algebra. Moreover, even in the limit $m_\pi^2 = 0$, when all soft pion theorems are exact, it is possible to show that a considerable contribution from a $(1, 8) + (8, 1)$ symmetry-breaking term is required to avoid further inconsistencies. Indeed one may take the attitude that, in view of the generation in Lagrangian field theory of chiral algebras from the Adler condition alone⁽⁷⁴⁾, the observed coincidences of chiral and dual models may result largely from the application of the Adler condition to the latter, although this approach leaves the success of eq. (4.3) and its extensions in a somewhat unsatisfactory situation.

Secondly, we note that "smooth" off-shell amplitudes in the sense of no dependence on the mass variables q_1^2 are obtained by using the pseudoscalar density v_8 , instead of ∂A^8 , as interpolating field⁺ for γ . The amplitudes then obey the Adler condition for

⁺ In this work with current algebra manipulations, we normalise the u 's and v 's by taking the symmetry-breaking Lagrangian as

π and K still, but not for η . They are suitable candidates for comparison with dual theories, although they cannot be compared with the leading term Veneziano model which does not satisfy the π and K Adler conditions in amplitudes involving the η , unless unphysical constraints are imposed on the pseudoscalar meson masses. The smooth nature of the v_8 interpolating field was realised by this author shortly after writing paper⁽⁶⁴⁾. It has also been published in a chiral Lagrangian framework by Brooker and Taylor⁽⁷⁵⁾, and Turner⁽⁷⁶⁾. For reference, the relevant amplitudes are listed in Table II.

TABLE II : Scattering Amplitudes from Chirality, $\phi^{\eta} \propto v^8$

$$\begin{aligned}
 K\pi \rightarrow K\eta & : \frac{1}{4\sqrt{3}f^2} (2t - s - u + 2(m_K^2 - m_\eta^2)) \\
 \eta\pi \rightarrow \eta\pi & : \frac{1}{3f^2} (s + t + u - (2m_\eta^2 + m_\pi^2)) \\
 K\eta \rightarrow K\eta & : \frac{1}{12f^2} (s + u + 10t - 2m_\eta^2 - 10m_K^2) \\
 \eta\eta \rightarrow \eta\eta & : \frac{1}{3f^2} (3(s + t + u) - 8m_\eta^2 - m_\pi^2)
 \end{aligned}$$

In these amplitudes we have constrained $f_\eta = f$ (contrast Table I), since this appears as a necessary condition when more than two particles are taken off mass shell.

It is at first sight perhaps surprising that it is possible to determine uniquely these amplitudes involving η , since one no

⁺(contd.) as $u_0 + cu_8$. The ϵ in eq. (1.12) will be reintroduced for simplicity in the non-linear chiral Lagrangian calculations of the next chapter.

longer has direct use of the Adler condition for the η etc. The additional constraint is the fact that the pseudoscalar density v_0 does not have the η -pole

$$\langle 0 | v_0 | \eta \rangle = 0 \quad , \quad (4.9)$$

This equation arises as a necessary consistency condition from the study of the $\pi K \rightarrow \eta K$ amplitude, as does also the magnitude of the coupling of v_8 to η

$$\langle 0 | v_8 | \eta \rangle = \frac{\sqrt{3} m_\pi^2 f}{\sqrt{2+c}} \quad (4.10)$$

with

$$c = -\sqrt{2}(m_K^2 - m_\pi^2)/(m_K^2 + \frac{1}{2}m_\pi^2) \quad (4.11)$$

$$f_\pi = f_K = f_\eta \quad (4.12)$$

and

$$3m_\eta^2 = 4m_K^2 - m_\pi^2 \quad . \quad (4.13)$$

As an example of the use of eq. (4.9) we calculate the $\eta \eta \rightarrow \eta \eta$ scattering amplitude. The linear expansion has the form

$$M(\eta \eta \rightarrow \eta \eta) = a + b \sum q_i^2 \quad (4.14)$$

Taking one η off-shell by LSZ reduction, we have

$$M(\eta \eta \rightarrow \eta \eta) = \frac{\sqrt{2+c}}{\sqrt{3} m_\pi^2 f} (m_\eta^2 - q_1^2) \langle \eta \eta | v_8 | \eta \rangle$$

* Further confirmation that ∂A^8 is not a smooth interpolating field for a pure octet η , unless $m_\eta^2 = m_\pi^2$, is exhibited in a paper by Schulke⁽¹¹⁶⁾ who assumes that 3-point functions with pseudoscalar and scalar meson-poles removed, are constant off mass shell. We interpret the apparent inconsistency in eqs. (14) and (25) of this reference as telling us that the 3-point function is not a constant if we allow for scalar meson poles and use ∂A^8 as η interpolating field.

which can be rewritten with the use of eqs. (4.11) to (4.13),

$$M(\eta\eta \rightarrow \eta\eta) = \frac{m_\eta^2 - q_1^2}{m_\eta^2 f} \langle \eta\eta | \partial A_8 | \eta \rangle - \frac{(m_\eta^2 - q_1^2)}{m_\eta^2 f} \sqrt{\frac{2}{3}} c \langle \eta\eta | v_0 | \eta \rangle \quad (4.15)$$

Since $\langle \eta\eta | v_0 | \eta \rangle$ does not contain the η pole, there is no factor to cancel the $(m_\eta^2 - q_1^2)$ and hence by the linear expansion (4.14), $\langle \eta\eta | v_0 | \eta \rangle$ must be constant. Further applications of the same argument yield

$$\langle \eta\eta | v_0 | \eta \rangle = \sqrt{\frac{2}{3}} \frac{m_\eta^2}{(\sqrt{2} - c)f^2} \quad (4.16)$$

In the remaining term in eq. (4.15) let us take one particle off mass shell and take directly the limit $q_1 \rightarrow 0$. The result is

$$\lim_{\substack{q_1 \rightarrow 0 \\ q_3^2 \text{ arbitrary}}} M(\eta\eta \rightarrow \eta\eta) = - \frac{(m_\eta^2 - q_3^2)}{f^2} - \frac{\sqrt{2}c}{3f^2} \frac{m_\eta^2}{\sqrt{2} - c} \quad (4.17)$$

Comparing this equation with the corresponding limit of expression (4.14), we have the result

$$b = + \frac{1}{f^2} \quad (4.18)$$

$$a = \frac{-1}{3f^2} (8m_\eta^2 + m_\pi^2) \quad (4.19)$$

Finally we note that on-mass-shell the two amplitudes using v_8 and ∂A_8 are the same. We thus have explicitly an example of the independence of on-shell results on the particular interpolating field chosen; indeed the manipulations used to obtain eq. (4.16) and (4.17) are themselves simple examples of this same theorem.

+ (contd.) The same equations are also given by Auvil and Deshpande (83).

4.3 Mixing with an SU₃ Singlet Field

In this section, we turn to the question of describing η -X mixing in the $(3, \bar{3}) + (\bar{3}, 3)$ symmetry-breaking model. Here the η and X are suitable combinations of SU₃ singlet and octet pseudoscalar fields and the X is taken to be the X(960). The spin of this resonance is not unambiguously determined, but we assume it to be 0, as is slightly favoured, since applications of pole dominance to the other candidate for mixing with the η , the E(1420), would raise an even greater credibility gap!

Within the framework of the $(3, \bar{3})$ symmetry-breaking model the natural way to introduce η -X mixing is to take suitable combinations of the pseudoscalar densities v^8 and v^0 as interpolating fields for η and X. Thus if we define

$$\begin{aligned} \langle 0 | v^8 | \eta \rangle &= g_\eta^8 & \langle 0 | v^8 | X \rangle &= -g_X^8 \\ \langle 0 | v^0 | \eta \rangle &= g_\eta^0 & \langle 0 | v^0 | X \rangle &= g_X^0 \end{aligned} \quad (4.20)$$

and take ϕ^η and ϕ^X such that

$$\begin{aligned} \langle 0 | \phi^\eta | \eta \rangle &= 1 & \langle 0 | \phi^\eta | X \rangle &= 0 \\ \langle 0 | \phi^X | X \rangle &= 1 & \langle 0 | \phi^X | \eta \rangle &= 0 \end{aligned} \quad (4.21)$$

then we have

$$\phi^\eta = \frac{g_X^0 v^8 + g_X^8 v^0}{g_X^0 g_\eta^8 + g_X^8 g_\eta^0} \quad (4.22)$$

$$\phi^X = \frac{-g_\eta^0 v^8 + g_\eta^8 v^0}{g_X^0 g_\eta^8 + g_X^8 g_\eta^0} \quad (4.23)$$

It is clear that, since all the g 's are unknown, and since we cannot now resort to the trick outlined at the end of the previous section, it may not be possible to determine all the linear expansions uniquely. It turns out that only those amplitudes in which one SU_2 singlet field appears are determined uniquely, namely $K\pi \rightarrow K\eta$ (X).

We sketch through the way in which these amplitudes are obtained. There are five unknown constants in the linear expansion in each amplitude, which, from crossing and C-invariance take the form

$$M(K\pi \rightarrow K\eta) = a + b(s+u) + ct + dq_{\pi}^2 + eq_{\eta}^2 \quad (4.24)$$

$$M(K\pi \rightarrow KX) = A + B(s+u) + Ct + Dq_{\pi}^2 + Eq_X^2 \quad (4.25)$$

In eqs. (4.22) to (4.25) there are fourteen unknowns to be determined. Surprisingly enough, there are fourteen independent off-mass-shell constraints on the amplitudes⁺, namely four Adler conditions, and ten equations from the following σ -commutator terms; in eq. (4.24), $[A^K, \partial A^K]$, $[A^K, \partial A^{\pi}]$, $[A^K, \phi^{\eta}]$, $[A^{\pi}, \partial A^K]$, $[A^{\pi}, \phi^{\eta}]$, and similarly in eq. (4.25). The equations result in the following solution.

$$g_{\eta}^8 = \sqrt{\frac{3}{2}} f m^2 \cos \theta \quad (4.26)$$

$$g_X^8 = \sqrt{\frac{3}{2}} f m^2 \sin \theta \quad (4.27)$$

$$g_X^0 = Ig_{\eta}^8, \quad g_{\eta}^0 = Ig_X^8 \quad (4.28)$$

⁺ Many constraints may also be obtained from the two- and three-point functions alone, but none of these provides anything new.

Here θ is the conventional mixing angle between octet and singlet states,

$$\tan^2 \theta = \frac{(m_8^2 - m_\eta^2)}{(m_X^2 - m_8^2)} \quad (4.29)$$

and
$$\overline{m^2} = \frac{1}{3}(2m_K^2 + m_\pi^2) \quad (4.30)$$

$$m_8^2 = \frac{1}{3}(4m_K^2 - m_\eta^2) \quad (4.31)$$

$$I \tan \theta = \frac{3}{2\sqrt{2}} \frac{(m_8^2 - m_\eta^2)}{(m_K^2 - m_\pi^2)} \quad (4.32)$$

Numerically⁽³⁵⁾, $\theta = 10.4 \pm 0.2^\circ$ and $I = .52$, so that the η and X fields are not orthogonal combinations of v_0 and v_8 , which corresponds (eqs. (4.26) to (4.28)) to $I = 1$.

A further interesting point arises if we write the divergence of A_μ^8 as a combination of ϕ^η and ϕ^X by inverting eqs. (4.22) and (4.23)

$$\begin{aligned} \partial A^8 &= \left(\sqrt{\frac{2}{3}} - \frac{c}{\sqrt{3}}\right)v^8 + \sqrt{\frac{2}{3}} cv^0 \\ &= f(m_\eta^2 \phi^\eta \cos \theta - m_X^2 \phi^X \sin \theta) \quad (4.33) \end{aligned}$$

We see that we may use ∂A^8 as interpolating field for η only if (a) $\theta = 0$, which is the case of no mixing, previously considered, or (b) $m_X^2 = 0$. However, in the latter case it is still not possible to obtain other scattering amplitudes uniquely, since the terms $q_\nu \langle \alpha | A_\nu^8 | \beta \rangle$ do not vanish in the limit $q_\nu \rightarrow 0$, the matrix element going to the (zero mass) X pole in this limit. Thus it is not possible to make a clever choice of interpolating fields so that we can use PCAC for η and so obtain uniquely the

amplitudes involving more than one SU_2 singlet particle.

Eq. (4.33) can be used to obtain a relationship amongst the 2γ decays of π , η and X , by using the anomalous PCAC relation introduced by Adler⁽⁷⁹⁾. By evaluating in a gauge-invariant fashion the triangle graph in spinor electrodynamics, Fig. (4.3), he shows that the usual Ward identity fails, and that an additional "anomalous" term must appear in the divergence of the axial current.⁺

$$\partial A^3 = f_{\pi} m_{\pi}^2 \phi^{\pi} + \frac{g_a^{(3)}}{4\pi} F^{\mu\nu} F^{\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \quad (4.34)$$

where $F_{\mu\nu}$ is the usual antisymmetric tensor of the photon field and $a^{(3)}$ is a constant determined by the average (charge)² of the fermions participating in the loop in the triangle graph.

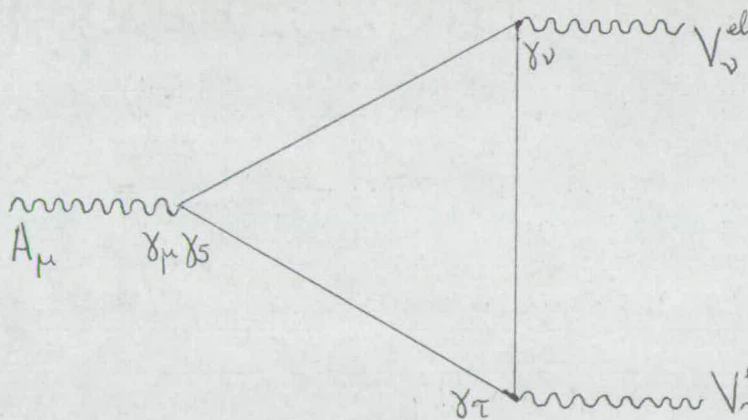


Fig. 4.3.

Eq. (4.34) gives excellent agreement with the $\pi^0 \rightarrow 2\gamma$ width, 9.7 eV, compared with the experimental value $(7.4 \pm 1.5)eV$. When

⁺ The extra term in eq. (4.34) has no effect in $\eta \rightarrow 3\pi$ decay in second order perturbation theory of the electromagnetic Hamiltonian, as its contribution vanishes if the internal photons form a loop.

it is extended to η decay, assuming SU_3 ($f_\pi = f_\eta$) and neglecting $\eta - X$ mixing, i.e.

$$\partial A^8 = f_\pi m_\eta^2 \phi^\eta + \frac{g_a(8)}{4\pi} F^{\mu\nu} F^{\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \quad (4.35)$$

then the predicted width of $\eta \rightarrow 2\gamma$ is about a factor of 8 smaller than the experimental value⁽⁷⁹⁾.

If we use eq. (4.35) to modify eq. (4.33), we obtain a relationship among π , η and X decays. Denoting by Γ_π the partial width $\pi \rightarrow 2\gamma$, etc., it is

$$\sqrt{\frac{\Gamma_X}{m_X^3}} \cdot \sin \theta = \sqrt{\frac{\Gamma_\eta}{m_\eta^3}} \cos \theta - \sqrt{\frac{\Gamma_\pi}{m_\pi^3}} \cdot \frac{1}{\sqrt{3}} \quad (4.36)$$

This is the same equation as obtained by allowing for standard $\eta - X$ mixing theory in the SU_3 framework⁽⁸⁰⁾. The factor m^{-3} arises purely from kinematics and phase-space. With Γ_π and Γ_η as input, eq. (4.36) predicts an $X \rightarrow 2\gamma$ width of

$$\Gamma(X \rightarrow 2\gamma) \approx 60 \text{ keV} \quad (4.37)$$

which, with the experimental ratio⁽³⁵⁾

$$\Gamma(X \rightarrow 2\gamma) / \Gamma_X(\text{total}) = .05 \pm .03 \quad (4.38)$$

$$\text{implies } \Gamma(X \rightarrow \eta\pi\pi) = 0(1 \text{ MeV}) \quad (4.39)$$

Similar calculations have been performed by Glashow et al.⁽⁸¹⁾. These authors attempt to go beyond the first order symmetry-breaking contained in the linear expansion approach, by allowing $f_\pi \neq f_K$, etc. In order to obtain an equation equivalent to (4.33), they must introduce an SU_3 singlet axial current, A_μ^0 . Although

they obtain a suitable value for f_K/f_π , this approach has two major drawbacks; firstly the relationship $m_\eta^2 = m_\pi^2$ appears as a result of introducing A^0 , with $(3, \bar{3})$ breaking and, secondly, the predictions depend in an essential manner on the average charge of the fermions in the loop of the triangle graph, Fig. 4.3. Neither of these criticisms arises when using eq. (4.33) (with (4.35)) since the charge dependence vanishes in the ratio $a^{(3)}/a^{(8)} = \sqrt{3}$. However, both the predictions of ref. (81) (which give Γ_X similarly large, essentially because Γ_η is much larger than the octet SU_3 value) and eqs. (4.37) and (4.39) could be significantly altered by PCAC corrections, particularly to ∂A^0 .

We end this section by quoting the $K\pi \rightarrow K\eta(X)$ amplitudes, eqs. (4.24) and (4.25)

$$M(K\pi \rightarrow K\eta) = \frac{\cos \theta}{2\sqrt{3}f^2} \left\{ (\sqrt{2} I \tan \theta - 1/2)(s+u - 2m_K^2) + 2(\sqrt{2} I \tan \theta + 1)(t - m_\eta^2) \right. \\ \left. + \sqrt{2} \cdot \frac{1-I^2}{I} \cdot \tan \theta (q_\eta^2 - m_\eta^2) \right\} \quad (4.40)$$

$$M(K\pi \rightarrow KX) = \frac{\sin \theta}{2\sqrt{3}f^2} \left\{ (\sqrt{2} I \cot \theta + 1/2)(s+u - 2m_K^2) + 2(\sqrt{2} I \cot \theta - 1)(t - m_X^2) \right. \\ \left. + \sqrt{2} \cdot \frac{1-I^2}{I} \cdot \cot \theta (q_X^2 - m_X^2) \right\} \quad (4.41)$$

Here I and θ are as defined in eqs. (4.29) to (4.32). It is clear firstly that, apart from the overall sign, there is no ambiguity in either of these amplitudes from the sign of θ . Secondly, unless $I^2 = 1$, the amplitudes are not "smooth", in the sense of the previous section and contain essentially dependence on q_η^2 and q_X^2 . We elaborate more fully on this point in the

next section.

4.4 Introduction of an SU_3 Singlet Axial Current

The idea for the work in this section begins with the deduction by Glashow⁽⁸²⁾ of the result $m_\eta^2 = m_\pi^2$, in a study of the effects of $(3, \bar{3})$ breaking on a $U_3 \times U_3$ current algebra, i.e. the case where there exists an SU_3 singlet axial current A_μ^0 , such that, at equal times

$$\begin{aligned} \left[\int A_0^0(x) d^3x, u^i \right] &= -i d^{0ik} v^k \\ \left[\int A_0^0(x) d^3x, v^i \right] &= +i d^{0ik} u^k \end{aligned} \quad (4.42)$$

The question which we wish to ask, then, is 'Is there a $U(3) \times U(3)$ algebra which gives scattering amplitudes linear in s , t and u , which are equivalent to Veneziano model amplitudes, in the sense that they are the same as low s , t , u expansions of the latter?' It is answered in the affirmative in the course of this section, and an interpretation of results like eq. (4.6) is given as a possible alternative to those discussed in Section 4.2.

The result $m_\eta^2 = m_\pi^2$ is obtained within the framework of dominance of pseudoscalar and scalar densities, u^i and v^i , by single particle poles. The same approach is discussed more fully by Arvil and Deshpande⁽⁸³⁾ (see also ref. (116)). Two principal assumptions are made; (a) the two point functions of the u 's and v 's are dominated by suitable intermediate states, and (b) the three point functions of two v 's and one

u are as smooth as possible, i.e. constant, when the poles of relevant single particle intermediate states are removed from the amplitudes.

Let us begin with the results of having only an $SU_3 \times SU_3$ algebra, with no A_{μ}^0 . Two points of relevance emerge from this study (82, 83)

- (i) The η and X are orthogonal combinations of v^0 and v^8 .
- (ii) A mass relationship appears amongst the mesons π , K , η , X and χ .

Both of these results require essential use of χ -pole dominance of ∂v^k , so it is hardly surprising that they are more restrictive than those obtained in the previous section from current algebra and PCAC alone.

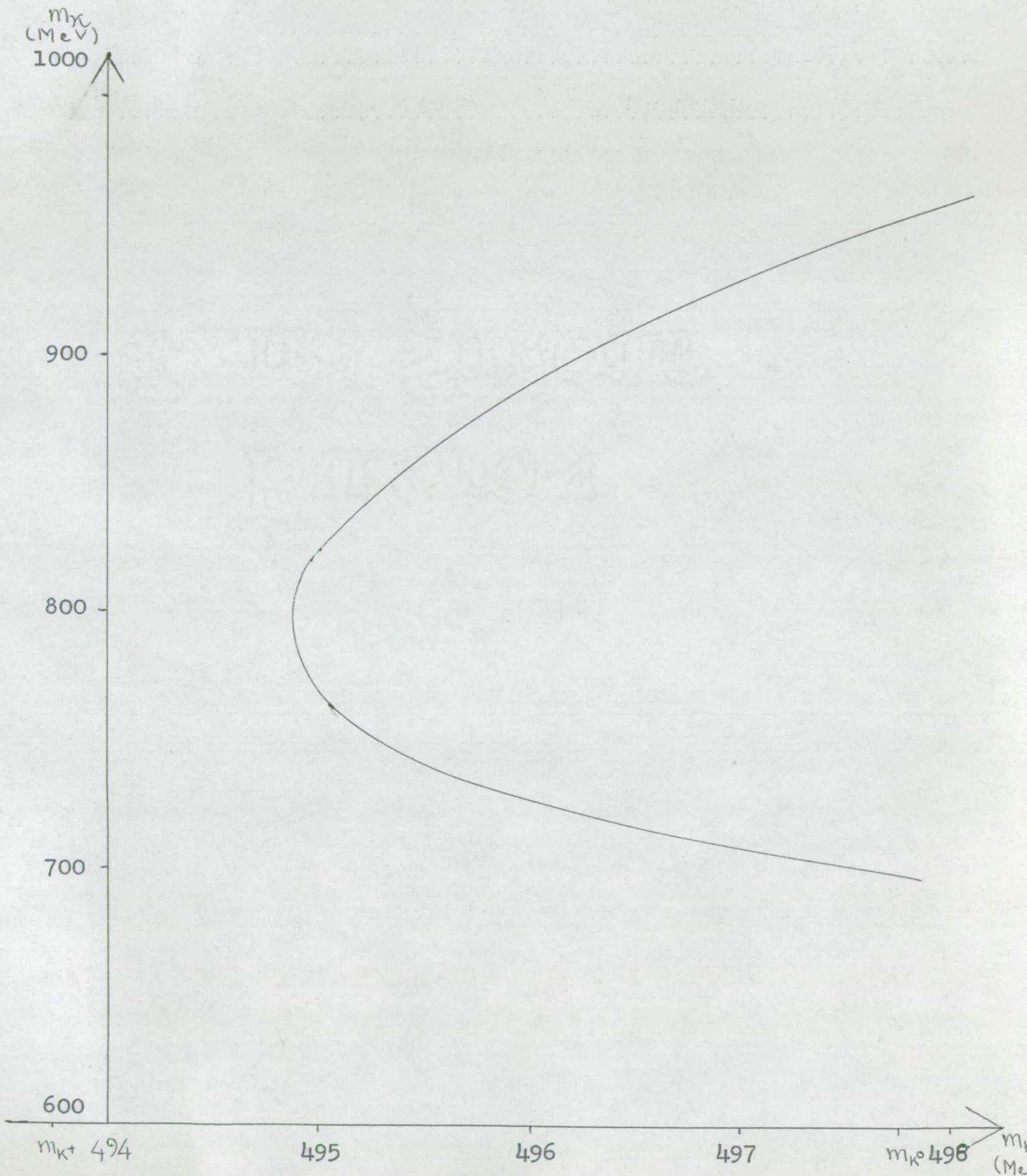
The mass formula predicted is in reasonable agreement with experiment, although the output of m_{χ} is critically dependent on the input of m_K as the rather interesting graph 4.4 shows! In the limit $m_{\chi} \rightarrow \infty$, which corresponds to the constancy of the $\langle M^i | u^j | M^k \rangle$ vertices, inherent in the linear expansion approach, the mass formula reduces to the Schwinger mass formula⁽⁸⁴⁾, predicting $m_X \simeq 1600$ MeV,

$$9(m_X^2 - m_8^2)(m_8^2 - m_{\eta}^2) = 8(m_K^2 - m_{\pi}^2)^2 \quad (4.43)$$

In this limit, it can be seen that conditions (i) and (ii) above are both realised simply by taking $I^2 = 1$ in eqs. (4.26) to (4.32). As commented at the end of the previous section, the $K\pi \rightarrow K\eta(X)$ amplitudes are functions of s , t and u only, when

FIGURE 4.4

Graph of m_{χ} v. m_K from the mass formula of Glashow⁽⁸²⁾ and Auvil and Deshpande⁽⁸³⁾.



$I^2 = 1$, and hence become suitable candidates for comparison with dual amplitudes such as the Veneziano model. With the clear understanding that we are making assumptions beyond current algebra, PCAC and linear expansions we consider the special case $I = 1$ for the remainder of this chapter; the more general case will be discussed in the framework of chiral Lagrangians in the next chapter.

The results of introducing an SU_3 singlet axial current, with

$$\partial A^0 = \sqrt{\frac{2}{3}} (v_0 + c v_8) \quad (4.44)$$

are easily obtained from the equations in refs. (82) and (83), (taking the limit of infinite scalar meson mass, if necessary).

They are

$$\begin{aligned} m_\eta^2 &= m_\pi^2 \\ 2m_K^2 &= m_X^2 + m_\pi^2 \\ \tan^2 \theta &= 2 \end{aligned} \quad (4.45)$$

and the interpolating fields for η and X are

$$\phi^\eta = (\sqrt{\frac{1}{3}} v^8 + \sqrt{\frac{2}{3}} v^0) / \sqrt{\frac{3}{2}} f_m^2 = (\sqrt{\frac{1}{3}} \partial A^8 + \sqrt{\frac{2}{3}} \partial A^0) / f_m^2 \quad (4.46)$$

$$\phi^X = (-\sqrt{\frac{2}{3}} v^8 + \sqrt{\frac{1}{3}} v^0) / \sqrt{\frac{3}{2}} f_m^2 = (-\sqrt{\frac{2}{3}} \partial A^8 + \sqrt{\frac{1}{3}} \partial A^0) / f_m^2 \quad (4.47)$$

Thus the pseudoscalar mesons are required in this treatment to exhibit ideal nonet mixing.

All pseudoscalar meson scattering amplitudes can now be calculated in the linear expansion approach, using eqs. (4.46) and (4.47). They are exhibited in Table III for reference.

TABLE III $U_3 \times U_3$ Chiral Amplitudes Exhibiting Ideal Nonet
Mixing

$$\begin{aligned}
 \left. \begin{array}{l} \eta\pi \rightarrow \eta\pi \\ \eta\eta \rightarrow \eta\eta \end{array} \right\} & : \quad \frac{1}{f^2} (s + t + u - 3m_\pi^2) \\
 \eta K \rightarrow \eta K & : \quad \frac{1}{4f^2} (s + u + 2t - 2m_\pi^2 - 2m_K^2) \\
 XK \rightarrow \eta K & : \quad \frac{1}{2\sqrt{2}f^2} (s + u - m_\eta^2 - m_X^2) \\
 XK \rightarrow XK & : \quad \frac{1}{2f^2} (s + u + 2t - 2m_K^2 - 2m_X^2) \\
 K\eta \rightarrow K\pi & : \quad \frac{1}{4f^2} (s + u + 2t - 2m_K^2 - 2m_\pi^2) \\
 KX \rightarrow K\pi & : \quad \frac{1}{2\sqrt{2}f^2} (s + u - 2m_K^2) \\
 XX \rightarrow XX & : \quad \frac{2}{f^2} (s + t + u - 3m_X^2) \\
 X\pi \rightarrow \eta\pi, \quad X\pi \rightarrow X\pi, \quad X\eta \rightarrow \eta\eta, \quad X\eta \rightarrow X\eta, \quad XX \rightarrow X\eta & : \quad 0
 \end{aligned}$$

Of course, the reason for doing this is not that we believe that they approximate the physical amplitudes in any way, but to compare them with the subset of the leading Veneziano terms which obey the Adler condition for pseudoscalar mesons exhibiting ideal nonet mixing⁽⁸⁵⁾. These Veneziano terms are listed in Table IV. Each amplitude is normalised relative to the others by SU_3 , although this normalisation could be relaxed in principle. A

TABLE IV Leading Term Veneziano Amplitudes Satisfying the Adler Condition, and exhibiting Ideal Nonet Mixing

$\eta\pi \rightarrow \eta\pi$	}	:	$\frac{1}{2}(V(\alpha(s), \alpha(t)) + V(\alpha(s), \alpha(u)) + V(\alpha(u), \alpha(t)))$
$\eta\eta \rightarrow \eta\eta$			
$\eta K \rightarrow \eta K$:	:	$\frac{1}{4}(V(\alpha_{K^*}(s), \alpha(t)) + s \leftrightarrow u)$
$XK \rightarrow \eta K$:	:	$\frac{1}{2\sqrt{2}} \cdot V(\alpha_{K^*}(s), \alpha_{K^*}(u))$
$XK \rightarrow XK$:	:	$\frac{1}{2}(V(\alpha_{K^*}(s), \alpha_1(t)) + s \leftrightarrow u)$
$K\eta \rightarrow K\pi$:	:	$\frac{1}{4}(V(\alpha_{K^*}(s), \alpha(t)) + s \leftrightarrow u)$
$KX \rightarrow K\pi$:	:	$\frac{1}{2\sqrt{2}} \cdot V(\alpha_{K^*}(s), \alpha_{K^*}(u))$
$XX \rightarrow XX$:	:	$V(\alpha_1(s), \alpha_1(t)) + V(\alpha_1(s), \alpha_1(u)) + V(\alpha_1(t), \alpha_1(u))$

The remainder vanish as in Table III.

$$\text{Here } V(\alpha_i(s), \alpha_j(t)) = \lambda \frac{\Gamma(1 - \alpha_i(s)) \Gamma(1 - \alpha_j(t))}{\Gamma(1 - \alpha_i(s) - \alpha_j(t))}$$

$\alpha(t)$ is the ρ, f, ω, A_2 trajectory, α_{K^*} the K^*, K^{**} trajectory and α_1 the ϕ, f' trajectory.

comparison between Tables III and IV shows that corresponding amplitudes have the same s, t, u structure (and are of course smooth, i.e. independent of q_1^2) and if the Veneziano terms are expanded about an Adler point they reproduce to lowest order in s, t and u , the linear expansions. In this sense, we have demonstrated the "equivalence" between Veneziano terms and $U_3 \times U_3$

algebra. It appears not insignificant that the chiral amplitudes are those determined as the limit of a framework involving scalar mesons.

It only remains in this chapter to comment on whether this statement has any fundamental significance, or whether it is largely due to the application of the Adler zero to the Veneziano model, as mentioned in Section 2. The attitude we would like to take is that if one accepts relation (4.3) and its extensions as a success of the Veneziano model, then one should also regard its "equivalence" with $U_3 \times U_3$ algebra and the associated nonet symmetry for the pseudoscalars, as a failure. Our reasons for adopting this viewpoint are principally (i) the constraint of nonet symmetry for internal trajectories with exchange-degeneracy mechanisms⁽⁶⁹⁾ and its realisation in the Veneziano model⁽⁶⁷⁾ and (ii), in order to counterbalance the impressive arguments of Ellis and Renner⁽⁷⁴⁾, that the properties of amplitudes based on the β function should not be classed with a field theory, linear in Kinematic Lorentz algebra. Indeed, it is perfectly clear that this is not the case for the Veneziano model⁽⁷³⁾.

CHAPTER V

DECAYS OF η AND X IN THE CHIRAL LAGRANGIAN FRAMEWORK

1. The Chiral Lagrangian Approach to Pseudoscalar meson Scattering

It will be appreciated that the algebraic manipulations required to produce all the results of the previous chapter are long and tedious, particularly since individual manipulations must be performed for each amplitude. The same results can be obtained much more readily within the framework of non-linear chiral Lagrangians (9).

A chiral Lagrangian is a model field theory based on a set of fields representing (more or less) physical particles, and in which there exist vector and axial vector currents which are functions of these fields and obey current algebra, as a consequence of the canonical field commutation relations. In general the fields themselves obey specified commutation relations with the currents and are regarded as belonging to the corresponding realisation of the (broken) chiral symmetry group generated by the currents.

The non-linear chiral Lagrangians are based on the fields of the eight pseudoscalar mesons, π , K and η , which are given suitable transformation properties under the chiral group. The pseudoscalar mesons transform linearly under the SU_3 group but the axial transformations are clearly non-linear since the commutator of a pseudoscalar meson field with the axial charge

must give an even function of the pseudoscalar fields by parity. Several approaches to the mathematical properties of these non-linear realisations exist⁽⁹⁾; we restrict ourselves to a brief review of the results of Callan, Coleman, Wess and Zumino⁽¹²⁾.

The first feature discussed is the use of the tree-graph approximation, in which one sums over all possible Feynman diagrams which contain no loops. At present this is the best that one can do with these non-polynomial Lagrangians; if the Wick expansion of the time-ordered product in the perturbation series converges, then non-renormalisable singularities appear in the integrals over all x-space⁽⁸⁹⁾ (although chiral Lagrangians may have some attractive properties in this respect⁽⁹⁰⁾) and if the Wick expansion must be formally summed by Borel's method, then arbitrary entire functions appear in the real part of the scattering amplitude⁽⁹¹⁾. The result of ref. (12) is that the tree graph approximation is invariant on the mass shell under point transformations of the field, i.e.

$$\Phi(x) = \phi(x) F(\phi(x)) \dots, \quad F(0) = 1 \quad (5.1)$$

and that it maintains the invariance (and P.C.A.C.) properties of the full Lagrangian, and the full s-matrix (if this could be calculated!). Both of these properties hold because the tree graph approximation is the lowest term in a systematic expansion in the parameter α , defined by (the $1/\alpha^2$ is introduced to provide the correct normalisation for the kinetic terms)

$$(\phi, \alpha) = (\alpha\phi)/\alpha^2. \quad (5.2)$$

Such a parameter appears naturally as the inverse of the pion

decay constant, f_π , in chiral theories. Any diagram with loops is associated with a higher power of α than a tree graph for the same number of external lines.

For the remaining properties, a specific non-linear realisation is chosen, in which the pseudoscalar mesons transform in the so-called standard form. This is justified because it is shown that any non-linear realisation is equivalent to this particular one, since it is related by a point transformation of the form of eq. (5.1), and hence gives the same results on mass shell.

The basic part of the Lagrangian is to be a chiral invariant function of the fields, according to the attitude outlined in the first chapter. It is not possible to form an invariant function from the pseudoscalar meson fields, ξ^i , alone. However, the derivative $\partial_\mu \xi^i$ transforms linearly under the vector SU_3 group and it is possible to define the covariant derivative $D_\mu \xi^i$, transforming in the standard form, so that $D_\mu \xi^i D^\mu \xi^i$ is a chiral invariant. Thus the basic chiral invariant Lagrangian is

$$\mathcal{L}_0 = \frac{f^2}{2} D_\mu (\xi^i/f) D^\mu (\xi^i/f) \quad (5.3)$$

$$\text{where } D_\mu (\xi^i) = [\sinh (\xi, t)/\xi \cdot t]^{ij} \partial_\mu \xi^j \quad (5.4)$$

the matrix $(t^{ijk}) = -f^{ijk}$, and f is the pion (and kaon) decay constant.

Eq. (5.3) describes a system of massless pseudoscalar mesons; in order to produce a physical system, with massive mesons, we must add to it functions of the ξ^i , which will break the chiral symmetry. In the spirit outlined in Chapter I, we must construct

suitable linearly transforming functions of the non-linearly transforming ξ^1 . It turns out that only those representations may be constructed which contain an SU_3 singlet, i.e. only the (n, \bar{n}) type of representations. We single out only the $(3, \bar{3})$ representations, where, in the "standard form"⁺,

$$\sqrt{\frac{2}{3}}(u^0(\xi) + i\gamma_5 v^0(\xi)) + \lambda^i(u^i(\xi) + i\gamma_5 v^i(\xi)) = \exp i\gamma_5 \xi^i \lambda^i \quad (i = 1, \dots, 8) \quad (5.5)$$

The expansions of the u's and v's in terms of the ξ 's are given by

$$u^0(\xi) = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{6}} \xi^2 + \frac{1}{4!} \frac{2}{3} \xi^4 + \dots \quad (5.6)$$

$$v^0(\xi) = -\frac{1}{3!} \sqrt{\frac{2}{3}} d^{ijk} \xi^i \xi^j \xi^k + \dots \quad (5.7)$$

$$u^i(\xi) = -\frac{1}{2} d^{ijk} \xi^j \xi^k + \frac{1}{4!} \left[d^{ijk} \xi^j \xi^k \xi^2 + \frac{2}{3} \xi^i (d^{jkl} \xi^j \xi^k \xi^l) \right] \quad (5.8)$$

$$v^i(\xi) = \xi^i - \frac{1}{3!} \xi^i \xi^2 + \dots \quad (5.9)$$

where we have simplified the terms by using expressions for products of the d coefficients given in e.g. ref. (92).

With the symmetry-breaking term added, eq. (5.3) becomes

$$\mathcal{L} = f^2 \left[\frac{1}{2} D_\mu(\xi^1/f) D^\mu(\xi^1/f) + \epsilon(u^0(\xi/f) + c u^8(\xi/f)) \right]. \quad (5.10)$$

In the tree-graph approximation to the four meson amplitudes, one simply picks out the coefficient of order ξ^4 from this expression.

⁺ It is seen from this equation that the standard form is just the exponential realisation discussed by Cronin⁽⁹³⁾.

The resultant amplitudes are at most quadratic in momentum, and, since they have the same chiral properties as the linear expansions discussed in the previous chapter, will be the same as them, on the mass shell⁺. In this way one obtains all the amplitudes in Table I from the one equation - clearly a more compact and economic statement of content than the linear expansion approach.

Finally, we wish to introduce into eq. (5.10) an SU_3 singlet field ϕ , which will mix with ξ^8 to produce the physical η and X fields. In general, if a field ψ_i transforms linearly under the SU_3 group according to

$$\psi_i' = D_{ij}(\theta) \psi_j \quad (5.11)$$

where the D 's form some linear representation of SU_3 , then we can define the ψ 's to transform, in the standard form, according to

$$\psi_i' = D_{ij}(\chi(\xi)) \psi_j \quad (5.12)$$

The argument of the transformation matrix D_{ij} is now a (non-linear) function of the parameters χ of the transformation and

⁺ Off-shell, the corresponding amplitudes are different because the exponential realisation does not prescribe the same off-shell continuation as P.C.A.C. Since $\partial A \sim [\int A_0 d^3x, \mathcal{L}] \sim v$, the P.C.A.C. continuation is obtained by eliminating the ξ^i fields in favour of the v^i by a point transformation, as in eq. (5.1).

the meson fields ξ . As a specific case of eqs. (5.11) and (5.12), if ϕ is an SU_3 singlet field, then in the standard realisation it is defined to be a chiral $SU_3 \times SU_3$ singlet.

We may thus introduce into eq. (5.10) the field ϕ in the most general way possible, safe in the knowledge that the chiral transformation properties of the Lagrangian will remain unaltered. We note that the functions $\phi v^0(\xi)$ and $\phi v^8(\xi)$ are further suitable candidates for symmetry-breaking terms. The most general Lagrangian with a symmetry-breaking term belonging to a single $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU_3 \times SU_3$ may be written,

$$\begin{aligned} \mathcal{L}(\xi, \phi)/f^2 &= \frac{1}{2} D_\mu(\xi^1/f) D^\mu(\xi^1/f)(1 + a_4 \phi^2/f^2 + \dots) \\ &+ \frac{1}{2} \partial_\mu \phi/f \partial^\mu \phi/f (1 + a_1 \phi^2/f^2 + \dots) \\ &+ a_2 \phi^2/f^2 + a_3 \phi^4/f^4 + \dots \\ &+ \epsilon(U^0(\xi/f, \phi/f) + c U^8(\xi/f, \phi/f)) \end{aligned} \quad (5.13)$$

$$\text{where } U^0(\xi, \phi) = u_0(\xi)(1 + a_5 \phi^2/f^2) + v_0(\xi)\phi(a_6 + a_7 \phi^2/f^2) \quad (5.14)$$

and similarly for U^8 . In eq. (5.13), we have exhibited explicitly only those terms of up to fourth order in the fields and have introduced the scaling parameter $1/f$ for the ϕ field as well, so that all the a 's are dimensionless. All the results of the linear expansion approach to $\eta - X$ mixing may be obtained from

these two equations[†] in the tree graph approximation.

The terms of second order in the fields give the following expressions for the masses;

$$m_{\pi}^2 = \epsilon \left(\sqrt{\frac{2}{3}} + \frac{c}{\sqrt{3}} \right) \quad (5.15)$$

$$m_K^2 = \epsilon \left(\sqrt{\frac{2}{3}} - \frac{c}{2\sqrt{3}} \right) \quad (5.16)$$

$$m_8^2 = \epsilon \left(\sqrt{\frac{2}{3}} - \frac{c}{\sqrt{3}} \right) \quad (5.17)$$

These equations give the Gell-Mann - Okubo mass formula. The remaining quadratic terms are

$$\Delta = -\epsilon c a_6 \quad (5.18)$$

$$m_0^2 = -2a_2 - 2\epsilon a_5 \sqrt{\frac{3}{2}} \quad (5.19)$$

The last three terms occur in the Lagrangian in the form $= -\frac{1}{2} m_8^2 (\xi^8)^2 - \Delta \xi^8 \phi - \frac{1}{2} m_0^2 \phi^2 + \dots$. The physical η and X fields are eigenstates of the free Lagrangian, with eigenvalues $-\frac{1}{2} m_{\eta}^2, -\frac{1}{2} m_X^2$. If we write

$$\phi_{\eta} = \cos \theta \xi^8 + \sin \theta \phi \quad (5.20)$$

$$\phi_X = -\sin \theta \xi^8 + \cos \theta \phi \quad (5.21)$$

[†] The agreement on-shell found between the amplitudes from eq. (5.13) and from ref. (28), which uses the P.C.A.C. off-shell continuation may provide the reader with some confidence in Coleman, Wess and Zumino's theorem, and, more appropriately, in the absence of mistakes in our calculations!

we find

$$\tan^2 \theta = (m_8^2 - m_\eta^2)/(m_X^2 - m_8^2) \quad (5.21)$$

$$a_6 = -\frac{\sqrt{3}}{2} \sin \theta \cos \theta (m_X^2 - m_\eta^2)/(m_K^2 - m_\pi^2) \quad (5.22)$$

$$a_2 + \epsilon a_5 \frac{\sqrt{3}}{2} = -\frac{1}{2} (m_X^2 + m_\eta^2 - m_8^2) \quad (5.23)$$

Eq. (5.21) gives the usual mixing angle (c.f. eq. (4.29)). We note that there are a sufficient number of arbitrary constants to avoid predicting a mass formula.

It is further to be noted that the physical fields are not related to the V^0 and V^8 , which are defined from the U's in eq. (5.14) and which are the meson fields in the P.C.A.C. off-shell continuation, by an orthogonal transformation, since

$$\begin{aligned} \langle 0 | V_8 | \eta \rangle &= \cos \theta & \langle 0 | V_0 | \eta \rangle &= I \sin \theta \\ \langle 0 | V_8 | X \rangle &= -\sin \theta & \langle 0 | V_0 | X \rangle &= I \cos \theta \end{aligned} \quad (5.24)$$

This is just the corresponding result to eqs. (4.26) to (4.28).

As we go to processes involving more external lines, more arbitrary constants appear; this corresponds to our inability to calculate some scattering amplitudes uniquely in the linear expansion approach. Now, however, the arbitrariness is contained concisely in the arbitrary constants, rather than in complicated relationships between η and X amplitudes implied by eq. (4.33).

The amplitudes containing one SU_2 singlet $K\pi \rightarrow K\eta(X)$ have

only the additional parameter a_6 and are uniquely determined, from eq. (5.22). Explicitly

$$M(K^+\pi^0 \rightarrow K^+\eta) = \frac{1}{f^2} \left[\epsilon \frac{c \cos \theta}{12} - \frac{aa_6}{2} \sqrt{\frac{2}{3}} \sin \theta - \frac{\sqrt{3}}{12} (s + u - 2t) \cos \theta \right] \quad (5.25)$$

This gives the same on shell as eq. (4.40).

For amplitudes containing more than one SU_2 singlet particle, undetermined constants enter. Thus amplitudes containing two of X and/or η are described in terms of the two unknowns, a_4 and a_5 , and amplitudes containing η and X only appear to involve three new constants a_1 , a_3 and a_7 . In fact this statement is not strictly true because one is at liberty to eliminate a_3 , for example, by redefining the ϕ field by a point transformation, i.e. amplitudes involving only SU_2 singlets are determined in terms of two more arbitrary constants. The relevance of the discussion after eq. (4.33) to the equivalence of the linear expansion and chiral Lagrangian approaches should now be clear.

The specific forms of the $X\pi \rightarrow \eta\pi$ and related amplitudes are given in the following equations. In order to avoid confusion over the signs of amplitudes, we revert to the phase of the S-matrix, so that, e.g. $M(X\pi \rightarrow \eta\pi) = \langle \eta\pi | i\mathcal{L} | X\pi \rangle$.

$$X\pi^i \rightarrow \eta\pi^j : i \frac{\delta^{ij}}{f^2} \sin \theta \cos \theta m_\pi^2 \left(-\frac{1}{3} - \frac{a_6 (\cos^2 \theta - \sin^2 \theta)}{\sqrt{3} \sin \theta \cos \theta} - 2a_5 + \frac{2a_4}{m_\pi^2} p^{(i)} \cdot p^{(j)} \right) \quad (5.26)$$

$$\eta\pi^i \rightarrow \eta\pi^j : i \frac{\delta^{ij}}{f^2} m_\pi^2 \left(\frac{1}{3} \cos^2\theta - \frac{2}{\sqrt{3}} a_6 \sin\theta \cos\theta - 2a_5 \sin^2\theta + \frac{2a_4}{m_\pi^2} \sin^2\theta p^{(i)} \cdot p^{(j)} \right) \quad (5.27)$$

$$X\pi^i \rightarrow X\pi^j : i \frac{\delta^{ij}}{f^2} m_\pi^2 \left(\frac{1}{3} \sin^2\theta + \frac{2a_6 \sin\theta \cos\theta}{\sqrt{3}} - 2a_5 \cos^2\theta + \frac{2a_4}{m_\pi^2} \cos^2\theta p^{(i)} \cdot p^{(j)} \right) \quad (5.28)$$

From eq. (5.22) we see that there is no ambiguity within these amplitudes from the sign of θ .

Eqn. (5.26) describes the physically observable decay $X \rightarrow \eta\pi\pi$. Experimentally this decay is described by two quantities, its width⁽³⁵⁾, and its slope in the Dalitz plot, as a function of the η energy⁽⁹⁴⁾. Neither of these quantities is well known,

$$\Gamma(X \rightarrow \eta\pi\pi) \lesssim 3 \text{ MeV} \quad (5.29)$$

$$a = -0.28 \pm 0.06 \quad (5.30)$$

where⁽⁹⁴⁾ $X \rightarrow \eta\pi\pi = M(1 + ay)$ (5.31)

with $y = \frac{m_\eta + 2m_\pi}{m_\pi Q} \cdot T_\eta - 1$ (5.32)

It is clear that it is not possible to make predictions of the decay $X \rightarrow \eta\pi\pi$ from $SU_3 \times SU_3$ current algebra, P.C.A.C. and $(3, \bar{3})$ symmetry-breaking. In order to obtain some testable consequences from the Lagrangian, one must make further assumptions. We take two approaches; first, we assume that the effective electromagnetic Hamiltonian discussed in Chapter II, and particularly Section 2.2, may be used to describe the decays $\eta \rightarrow 3\pi$ and $X \rightarrow 3\pi$. To this end, we give a brief review of relevant

theoretical discussions of $\eta \rightarrow 3\pi$ decays in the next section, and in Section 3 present our calculations and results. Secondly, we give a treatment in Section 4 of the introduction of an SU_3 singlet axial current in the chiral Lagrangian framework; this provides further constraints on amplitudes, but, as must be expected from our discussions in the previous chapter, these are too tight to be of physical interest.

2. The $\eta \rightarrow 3\pi$ Decay

The contents of this section are largely in the nature of a review, to discuss previous papers and to act as an introduction to the next section. Our starting point is the paper by Sutherland,⁽⁹⁵⁾ who shows that the decay $\eta \rightarrow 3\pi$ by the conventional electromagnetic Hamiltonian, eq. (2.3), will vanish in the limit of one pion going to zero four-momentum. This result is modified if one takes more than one particle off-shell and retains the σ -commutator terms. The slope in the Dalitz plot of $\eta \rightarrow \pi^+\pi^-\pi^0$ is then determined, from use of a linear expansion and isoscalar σ -terms, to be in good agreement with experiment; on mass shell,

$$M(\eta \rightarrow \pi^+\pi^-\pi^0) = A(1 - 2E_3/m_\eta) \quad (5.33)$$

where E_3 is the centre of mass energy of the odd pion π^0 . The $\eta \rightarrow 3\pi^0$ decay can have no slope in the linear expansion approach.

The problem with eq. (5.33) arises because A is of order m_π^2 , since it contains the $\pi\pi$ σ -commutator. Bell and Sutherland⁽²⁶⁾

show that a consistent treatment⁺ necessarily implies a very small A (assuming η is pure octet) and consequently a width smaller than the experimental value by a factor of order 1/1500.

The effective u_3 Hamiltonian, discussed in Section 2.2 has been used by several authors. Its different algebraic properties ensure that the amplitude does not vanish in the limit of zero pion mass. Typically⁽¹⁹⁾, in the calculation of $\eta \rightarrow 3\pi$, if two pions are taken to zero four momentum, neglecting the π - π σ -commutator, and the $\langle \eta | u_3 | \pi \rangle$ matrix element is related to the kaon "tadpole" mass splitting, then one obtains an amplitude

$$M(\eta \rightarrow 3\pi^0) = \frac{1}{\sqrt{3}f^2} \Delta_t(m_K^2) \quad (5.34)$$

The resultant width is⁺⁺

$$\Gamma(\eta \rightarrow 3\pi^0) \simeq 100 \text{ eV}. \quad (5.35)$$

This is to be compared with the experimental value⁽³⁵⁾ of 800 ± 200 eV. As regards the slope of $\eta \rightarrow \pi^+\pi^-\pi^0$, a linear

⁺ In order to obtain their large result, Bardeen et al.⁽⁹⁶⁾ manage to spirit an extra factor of m_η^2/m_π^2 into the amplitude by (apparently) applying SU_3 to the quantities $(q^2 - m_1^2) \langle 0 | T \{ \partial A^i(x) H^{el}(0) \} | \xi^j \rangle / m_{U_j}^2$. One must regard this application of SU_3 to matrix elements of ∂A^i with considerable reserve in view of the known magnitude of SU_3 breaking in $\langle 0 | \partial A^\pi | \pi \rangle$ and $\langle 0 | \partial A^K | K \rangle$. Their method has also been criticised by Weinberg⁽⁹⁸⁾.

⁺⁺ This is larger than the result quoted in ref. (19), where $\Delta(m_K^2)$ is used, rather than $\Delta_t(m_K^2)$.

expansion approach may be employed to obtain the same energy dependence⁺⁺ as in eq. (5.33). Thus the u_3 Hamiltonian appears to provide a reasonable description of these decays, although the widths obtained are not quite large enough.

Specifically in the calculation we take the linear expansion as $a + bs + c(t + u) + d q_0^2$ where $s = (q_+ + q_-)^2$ etc., the suffices +, - and 0 denoting the charge of the associated pion. We assume that the spurion is never allowed to carry off four-momentum in the reduction and that the η always remains on shell. The amplitude is reduced to one unknown, with energy dependence as in eq. (5.33), by noting that (i) with π^+ , π^0 off-shell and $q^+ \rightarrow 0$, the amplitude vanishes and (ii) with $\pi^+ \pi^-$ off-shell, $q^+ \rightarrow 0$, the amplitude is proportional to $(m_\pi^2 - q_-^2)$. The overall scale is determined by taking $q^0 \rightarrow 0$. It agrees with the SU_2 relationship to $\eta \rightarrow 3\pi^0$ up to terms $O(m_\pi^2/m_\eta^2)$, which are related, for example, to the neglect of σ -terms in the Bose-Zimmerman reduction. We give the details because of the existence of previous unsuccessful calculations⁽⁹⁸⁾, which do not allow for sufficient structure to incorporate, e.g. the vanishing of the amplitude when $q^+ \rightarrow 0$. The correct energy dependence has been obtained by Chiu et al.⁽⁹⁹⁾, although in somewhat more restricted fashion than the above.

We turn now to calculations which take into account one or other of the pole diagrams, Figs. 5.1(a) and (b). Historically, the pion pole diagram^(17, 100) and the related idea of final state π - π interactions⁽¹⁰¹⁾ were discussed before the current algebra calculations. These calculations generally assume a

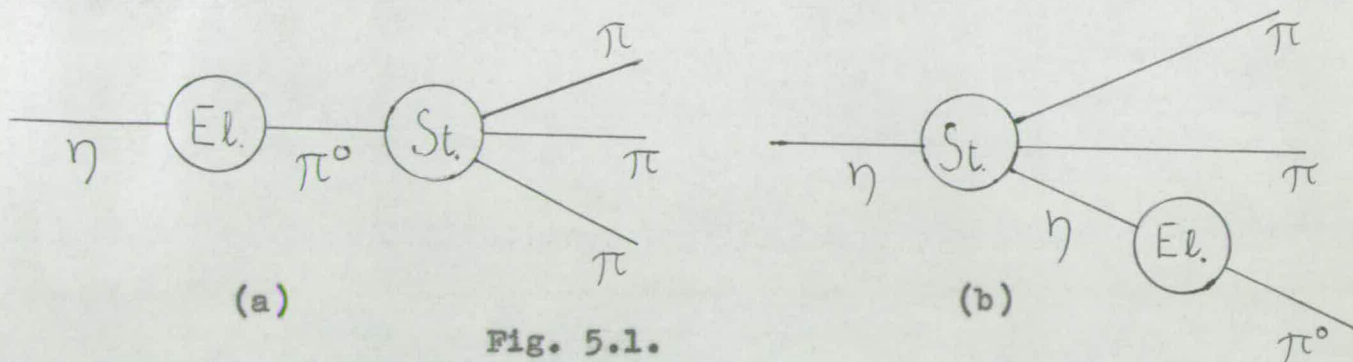


Fig. 5.1.

λ_p^4 strong interaction vertex and a constant electromagnetic matrix element, with no momentum dependence. In these circumstances, the sum of Figs. 5.1(a) and (b) are shown to cancel in the SU_3 symmetry limit⁽¹⁰²⁾, due to the change in sign of the propagator; a cancellation of this nature is essential, since the propagator is infinite if $m_\eta^2 = m_\pi^2$.

We are more interested in calculations which combine the pole models with current algebra. The first point to note is that if we relate the $\eta - \pi^0$ mixing to pseudoscalar meson mass splittings and use the Weinberg $\pi\pi$ scattering amplitude (see Table I), we obtain the same results from Fig. 5.1(a) as the linear expansion approach outlined above; the slope in $\eta \rightarrow \pi^+\pi^-\pi^0$ is the same as eq. (5.33) and the magnitude of the amplitudes are the same, to order m_π^2/m_η^2 . There is thus an apparent paradox, where Fig. 5.1(a) alone appears to describe the decay amplitudes reasonably well.

An incomplete advance towards a consistent solution incorporating current algebra and pole models was made by Sarker⁽¹⁰³⁾, who showed that on shell, the graph of Fig. 5.1(b) gives a contribution of order $(m_\pi^2)/(m_\eta^2)$ compared with Fig. 5.1(a)

and hence the good results of the pion pole graph are effectively unaltered. The qualitative reason for this suppression of Fig. 5.1(b) is easily understood; the η intermediate state is $O(m_\pi^2)$ from the Adler zero point.

There are two criticisms one may make of this paper. Firstly the sum of the pole graphs does not obey the condition that $\eta \rightarrow \pi^+ \pi^- \pi^0$ vanishes when $q_+ \rightarrow 0$ (with both the standard and effective electromagnetic Hamiltonians) and that no allowance is made for a "contact" term. With a u_3 Hamiltonian we can combine these two features to produce what we regard as a "complete" treatment. It does not seem possible to repeat the calculation for the standard electromagnetic Hamiltonian unless we assume

$$\langle \eta | H^{el} | \pi^0 \rangle = 0 \quad (5.36)$$

The way we do this is analogous to the $K \rightarrow 3\pi$ calculation of Macnamee⁽¹⁰⁴⁾ (compare also refs. 37, 22 and 60), i.e. we assume that the $\eta \rightarrow 3\pi$ amplitude is given by a constant "contact" term, plus the sum of the pion and η pole diagrams, with scattering amplitudes given in Table I. The contact term is determined by $\lim \{ \langle \pi^i \pi^j \pi^k | u_3 | \eta \rangle - \sum \text{pole diagrams} \}$ as any one of the pions tends to zero. It is determined in terms of $\langle \pi^i \pi^j | v^l | \eta \rangle$ from which a further application of the same technique, exactly as for $K \rightarrow 2\pi$ in ref. (22), yields the contact term, totally symmetric in the pion indices, namely

$$C = \frac{M_{\eta\pi}}{3f^2} \cdot \frac{m_\pi^2}{m_\eta^2} (\delta^{ij} \delta^{k3} + \delta^{ik} \delta^{j3} + \delta^{jk} \delta^{i3}) \quad (5.37)$$

When this expression is added to the sum of the on-shell pole

diagrams, we obtain the on-shell decay amplitude. The attractive feature of this calculation is that it includes fully the current algebra constraints and provides an answer which maintains Bose symmetry explicitly. The factor of m_π^2/m_η^2 in eq. (5.37) ensures that the amplitudes are the same as the linear expansion amplitudes, to this order. Explicitly, they are

$$M(\eta \rightarrow 3\pi^0) = -\frac{M\eta\pi}{f^2} \quad (5.38)$$

$$M(\eta \rightarrow \pi^+\pi^-\pi^0) = -\frac{M\eta\pi}{f^2} \cdot \frac{m_\eta^2}{m_\eta^2 - m_\pi^2} \cdot \left(1 - \frac{1}{3} \frac{m_\pi^2}{m_\eta^2} - \frac{2E_3}{m_\eta}\right) \quad (5.39)$$

These amplitudes exhibit exactly the SU_2 relationship at the centre of the Dalitz plot ($E_3 = m_\eta/3$).

A further discussion in the same vein but in the chiral Lagrangian framework is given in ref. (28). There the independence of the on-shell result on the η field, and the coincidental smallness of the contact term eq. (5.37) for the P.C.A.C. off-shell continuation are explicitly exhibited.

Here we digress to comment on papers which use Veneziano model expressions for the scattering amplitudes^(66,105). Our first point is that, in view of the close similarity of the Veneziano and Weinberg $\pi\pi$ amplitudes, as long as Fig. 5.1(a) dominates Fig. 5.1(b), a reasonable approximation to the matrix elements will result. Secondly, one should not expect to obtain reasonable predictions for $K \rightarrow 3\pi$ from the pole contributions alone, since the contact terms are not negligible in this case^(22,104); indeed, a

leading term Veneziano, plus pole, model for $K \rightarrow 2\pi$ gives zero amplitude because the kaon in Fig. (2.5) is at zero four-momentum.

Returning to our main theme, we may summarise the conclusions of this section, relevant to our calculations in the next, as

- (i) the standard electromagnetic Hamiltonian may be ignored in $\eta \rightarrow 3\pi$ decays, according to ref. (26) and to eq. (5.36), and
- (ii) even with a u_3 effective Hamiltonian, the theoretical width is still too small with pure octet, so that $\eta - X$ mixing should be studied as a possible mechanism for enhancing the width.

As regards the statement (i), we finally note a recent calculation⁽¹⁰⁶⁾ purporting to obtain a large width for $\eta \rightarrow 3\pi$, using the standard H^{el} . This paper is based on a pion pole model for the decay, and on identifying the σ -terms in the low energy limits of the matrix elements $\langle \eta | H^{el} | \pi^0 \rangle$ and $\langle K | H^{el} | K \rangle$ with the tadpole contribution to electromagnetic effects. The dubiety of this procedure has already been pointed out in a footnote in Section 2.2.

3. Chiral Lagrangian Constraints on η and X Decays

We return now to the calculation of η and X decays from the chiral Lagrangian defined in eqs. (5.13) and (5.14). According to our previous discussion, we further assume that $\Delta I = 1$ electromagnetic decays may be described by an additional term $\epsilon \times U_3$, with U_3 defined as in eq. (5.14). We are then in a position to describe the widths and slopes in the Dalitz plot of the decays $X \rightarrow \eta \pi\pi$, $\eta \rightarrow 3\pi$ and $X \rightarrow 3\pi$ in terms of the two parameters a_4 and a_5 .

Conventionally, it is not unreasonable to expect that linear expansions have sufficient structure to describe the decays $X \rightarrow \eta \pi\pi$ and $\eta \rightarrow 3\pi$, where $m_{\pi\pi} \leq 3m_\pi$ and $m_{\eta\pi}$ is reasonably far from the narrow δ resonance. In contrast it is not entirely plausible to describe $X \rightarrow 3\pi$ by a linear expansion, since the $\pi\pi$ channel goes up to the $\epsilon(700)$ resonance region. However, we feel that the proof of our phenomenological approach will lie in the fitting and remain presently optimistic, in view of the remarkably linear fit⁽¹⁰⁷⁾ of $\eta \rightarrow \pi^+\pi^-\pi^0$.

The $X \rightarrow \eta \pi\pi$ amplitude is already given in eq. (5.26). The electromagnetic η and X decays are given as sums of the contact term plus the π and η pole graphs, as in Fig. 5.1(a) and (b) and the X pole graph similar to Fig. 5.1(b). The expressions required are eqs. (5.26) to (5.28), the $\eta\pi$ and $X\pi$ transition amplitudes (normalised to the kaon tadpole mass-splitting in eq. (2.4)) and the $\pi\pi$ amplitude from eq. (5.13). These are

$$\langle \pi^0 | \epsilon x U_3 | \eta \rangle = - \frac{\Delta_t(m_K^2)}{\sqrt{3}} (\cos \theta + \sqrt{2} I \sin \theta) \quad (5.40)$$

$$\langle \pi^0 | \epsilon x U_3 | X \rangle = - \frac{\Delta_t(m_K^2)}{\sqrt{3}} (-\sin \theta + \sqrt{2} I \cos \theta) \quad (5.41)$$

$$\pi^i \pi^j \rightarrow \pi^k \pi^l : \frac{1}{3f^2} (2s - t - u + m_\pi^2) \delta^{ij} \delta^{kl} + \text{terms } \delta^{ik} \delta^{jl}, \delta^{il}, \delta^{jk} \quad (5.42)$$

With numerical values inserted for the masses, the amplitudes may be written as follows. We use $f = 84.5$ MeV, from the Goldberger Treiman relation.

$$\underline{\eta \rightarrow \pi^+ \pi^- \pi^0} : \quad \frac{A}{3} (1 + \alpha y) ; \quad \eta \rightarrow \pi^0 \pi^0 \pi^0 : \quad A$$

where y is defined by $y = \frac{3T^0}{Q} - 1$ ($Q = m_\eta - \sum m_\pi$,
 $T^0 = \text{K.E. of } \pi^0$)

$$A = \frac{\Delta t (m_K^2)}{\sqrt{3} f^2} \{ 1.094 + .027a_5 + .059a_4 \} \quad (5.43)$$

$$\alpha = -.538 \cdot \frac{1 + .060a_4}{1 + .054a_4 + .025a_5} \quad (5.44)$$

$$\underline{X \rightarrow \pi^+ \pi^- \pi^0} : \quad \frac{B}{3} (1 + \beta y) \quad X \rightarrow \pi^0 \pi^0 \pi^0 : \quad B$$

$$y = \frac{3T^0}{Q} - 1 \quad (Q = m_X - \sum m_\pi)$$

$$B = \frac{\Delta t (m_K^2)}{\sqrt{3} f^2} \{ 0.515 + .14a_5 + 1.11a_4 \} \quad (5.45)$$

$$\beta = - \frac{1.15 + 2.6a_4}{1 + .28a_5 + 2.15a_4} \quad (5.46)$$

$$\underline{X \rightarrow \eta \pi \pi} : \quad M(1 + \gamma y)$$

$$y = \frac{2m_\pi + m_\eta}{m_\pi \cdot Q} \cdot T^\eta - 1 \quad (Q = m_X - m_\eta - 2m_\pi)$$

$$M = .44 - .95a_5 - 2.18a_4 \quad (5.47)$$

$$\gamma = \frac{a_4}{.41 - .88a_5 - 2.02a_4} \quad (5.48)$$

The phase spaces are evaluated relativistically⁽¹⁰⁸⁾ with

physical masses and corrections for non-zero Dalitz plot slopes⁺.

In units of keV, we have

$$\begin{aligned}
 \Gamma(\eta \rightarrow \pi^+\pi^-\pi^0) &= .489(1 + 0.02\alpha + .22\alpha^2) |A|^2 \\
 \Gamma(\eta \rightarrow 3\pi^0) &= .827 |A|^2 \\
 \Gamma(X \rightarrow \pi^+\pi^-\pi^0) &= 3.6(1 - .03\beta + .24\beta^2) |B|^2 \\
 \Gamma(X \rightarrow 3\pi^0) &= 5.5 |B|^2 \\
 \Gamma(X \rightarrow \eta\pi^+\pi^-) &= 1.91(1 + .22\gamma + .27\gamma^2) |M|^2 \\
 \Gamma(X \rightarrow \eta\pi^0\pi^0) &= 1.08(1 + .28\gamma + .28\gamma^2) |M|^2
 \end{aligned} \tag{5.49}$$

Finally, we list the relevant experimental data^(94,102,35).

Widths are in keV.

$$\begin{aligned}
 \Gamma(\eta \rightarrow \pi^+\pi^-\pi^0) &= 0.60 \pm .15 & \Gamma(\eta \rightarrow 3\pi^0) &= .83 \pm .20 \\
 \Gamma(X \rightarrow \eta\pi\pi) &\leq 2.7 \times 10^3 \\
 \Gamma(X \rightarrow 3\pi)/\Gamma(X \rightarrow \eta\pi\pi) &\leq 0.1 \\
 \alpha &= -0.52 \pm 0.03 \\
 \gamma &= -0.28 \pm 0.06
 \end{aligned} \tag{5.50}$$

⁺ For $\pi^+\pi^-\pi^0$ modes, the term linear in α or β is very sensitive to the exact definition of the centre of the Dalitz plot. The value quoted is for the centre defined by $s_{\pi^+\pi^-} = (m_\eta - m_{\pi^0})^2 - 2/3 m_\eta Q$; for $s_{\pi^+\pi^-} = 1/3(m_\eta^2 + m_{\pi^0}^2 + 2m_{\pi^+}^2)$ it would be $-.09\alpha$. The value of Q for physical masses is used in all cases; hence the difference in the $\eta\pi\pi$ modes. There is clearly a small problem here as to where precisely in the chiral Lagrangian formalism one should distinguish between m_{π^+} and m_{π^0} . We have adopted the convention that the Lagrangian gives the correct dependence on the π^0 kinetic energy with other pion masses taken as the average of $m_{\pi^+}^2$ and $m_{\pi^0}^2$.

a has been taken as the average of spark chamber and bubble chamber experiments⁽¹⁰²⁾.

The restrictions imposed on a_4 and a_5 by eqs. (5.43) to (5.50) are exhibited in Fig. (5.2). The known values of a and γ are used to evaluate the phase-space corrections in eq. (5.49). The relevant allowed regions are appropriately shaded. It is clear from the large errors and the dearth of data on X decays that a_4 and a_5 are not very tightly constrained. Indeed it is possible for a_4 and a_5 to assume values as large as 8, producing considerable enhancement of the $\eta \rightarrow 3\pi$ width. Such large values are not necessarily surprising since these parameters are normalised with respect to f^2 , which is the scale appropriate to the pseudoscalar octet. In principle, one does not wish too large a_4 and a_5 since this might jeopardise somewhat the original assumptions of small scattering lengths which helps to justify the use of a linear amplitude, but eqs. (5.26) to (5.28) are already $O(m_\pi^2)$ so the problem is not too serious.

With the specific values $a_4 = 9$, $a_5 = 8$, we have

$$\begin{aligned}
 \Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0) &= .28 \text{ keV}, & \Gamma(\eta \rightarrow 3\pi^0) &= .45 \text{ keV} \\
 \Gamma(X \rightarrow \eta \pi\pi) &= 2.3 \text{ keV} \\
 \Gamma(X \rightarrow \pi^+ \pi^- \pi^0) \gamma \Gamma(X \rightarrow \eta \pi\pi) &= .05 & & (5.51) \\
 a &= -.49 \\
 \beta &= -1.1 \\
 \gamma &= -.36
 \end{aligned}$$

These results are in reasonable agreement with (5.50) and provide a considerable enhancement of $\eta \rightarrow 3\pi$, so that non-linear

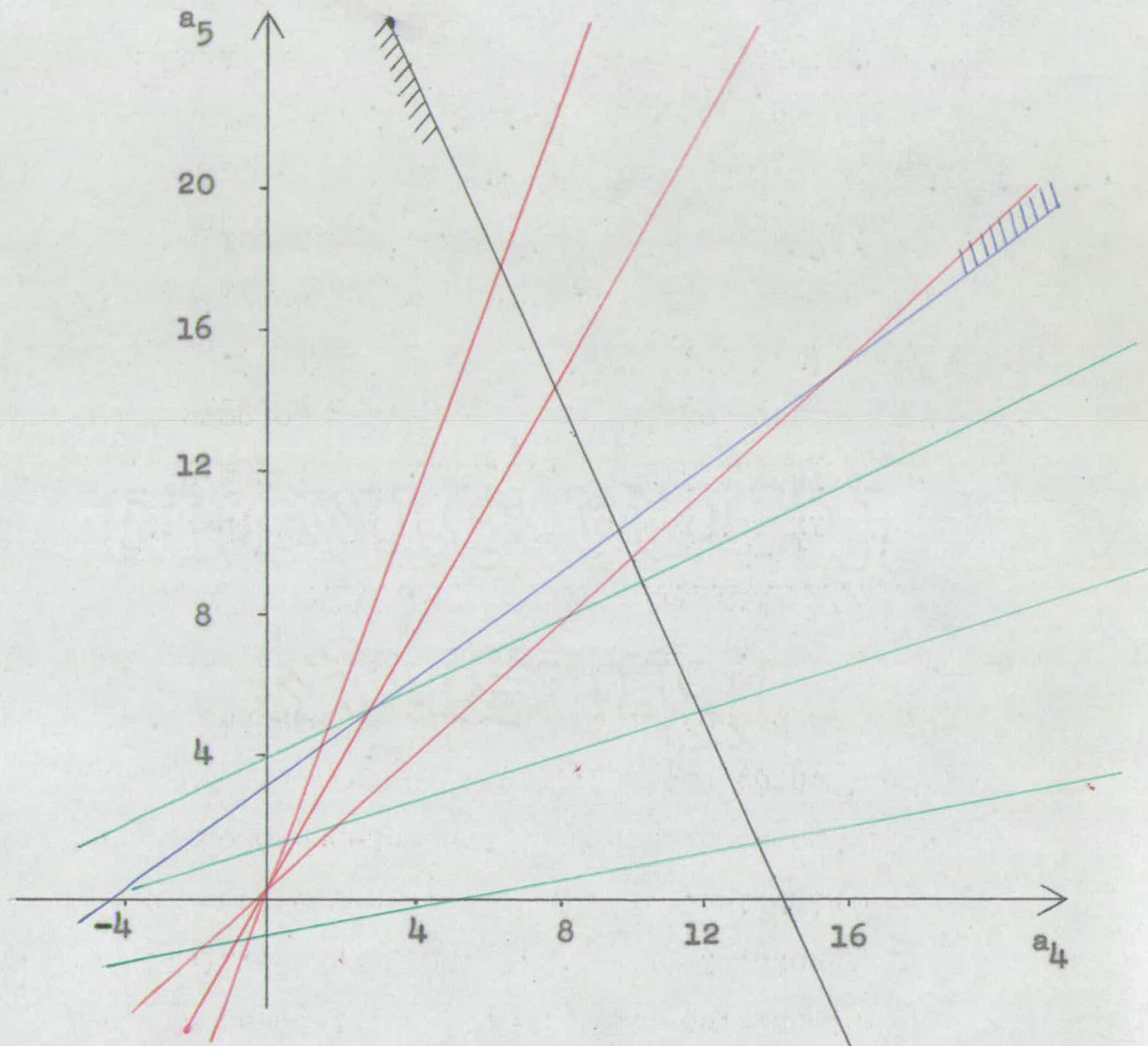


Fig. 5.2 : Restrictions on a_4 and a_5 from data on η and X decays. The area between the red (green) lines represents the allowed region for one standard deviation error from the slope of $X \rightarrow \eta\pi^+\pi^-$ ($\eta \rightarrow \pi^+\pi^-\pi^0$) decay. The shaded side of the black (blue) line represents the allowed region from the width of $X \rightarrow \eta\pi\pi$ (the branching ratio $X \rightarrow 3\pi/X \rightarrow \eta\pi\pi$). To enhance $\eta \rightarrow 3\pi$, a_4 and a_5 should be positive and as large as possible.

effects⁺ could bring the width close to its experimental value.

An alternative method of analysing the result is to eliminate one of the parameters using the experimental slope of $X \rightarrow \eta \pi\pi$

$$a_5 = (1.8 \pm 1) a_4 + .47 \quad (5.52)$$

A particular advantage of this approach is that it shows that β becomes essentially independent of the precise values of a_4 and a_5 ;

$$\beta = - \frac{1.15 + 2.6a_4}{1.13 + 2.65a_4 \pm (.28a_4)} \approx -1 \quad (5.53)$$

unless the denominator happens to be very close to zero⁺⁺.

The use of chiral Lagrangians to describe the $X - \eta$ system has been undertaken before, with differing results due to lack of generality. Cronin⁽⁹³⁾ uses essentially a $U_3 \times U_3$ chiral Lagrangian, with non-simple symmetry-breaking properties. As a result, he obtains zero slope for $X \rightarrow \eta \pi\pi$ (as will be emphasized in the next section) and a large width ≈ 7 MeV from his amplitude, which is proportional to m_X^2 . Schwinger⁽¹⁰⁹⁾ predicts the slope of $X \rightarrow \eta \pi\pi$, $\gamma = - .40$, although he himself recognises that his treatment is not the most general. Majumdar⁽¹¹⁰⁾ is able to obtain a relationship between the width and slope of $X \rightarrow \eta \pi\pi$ by omitting a possible term $\partial_\mu M \partial^\mu M^\dagger (\det M + \text{h.c.})$, in his notation. Ambiguities occur in both refs. (93) and (110) from the sign of θ .

A current algebra calculation has recently been given by

⁺ Note that the non-linear effects will in general not be as large as given in ref. (28), since an imaginary part in the pole term will reduce the effect.

⁺⁺ It is interesting to note that a slope of the same magnitude as

Mathur et al.⁽¹¹¹⁾, using a $U_3 \times U_3$ algebra. The η and two pions are taken simultaneously off-mass-shell to zero four-momentum, without explicitly extrapolating the X particle. This calculation has the three drawbacks that it will not give any slope for the decay, that it is dependent on the particular interpolating field chosen and that no allowance is made for off-shell extrapolation corrections.

for $X \rightarrow \eta\pi\pi$ based on the
 An amplitude based on the Veneziano model has been discussed by Baacke et al.⁽¹¹²⁾. The leading terms are modified by the introduction of two satellite terms whose coefficients are fixed up by eliminating the unphysical daughter poles in the $\eta\pi$ channels at the ρ mass, and by demanding that the pion Adler condition is satisfied. The overall scale is determined from the $A_2 \rightarrow \eta\pi$ width and predicts an X width of the order of 1 MeV. (Note that eq. (15) in ref. (112) is wrong, but eq. (16) is still correct as to order of magnitude). However the slope predicted from the amplitude is around -1, so clearly this particular addition of satellite terms has failed to produce a realistic amplitude. Moen and Moffat⁽¹¹³⁾ fit the decay satisfactorily with a sum of interference model terms of the form

$$\frac{\Gamma(1 - \alpha(s))}{\Gamma(n - \alpha(s))} (\alpha(t))^{\alpha(s)} \quad (5.54)$$

but somewhat adhoc assumptions are required to do so.

We may summarise the results of this section by saying that

++ (Contd.) (5.53) is reproduced from the pion pole model alone with the P.C.A.C. off-shell continuation $\beta \approx -8/7$.

we have six experimental quantities in terms of two parameters. With the poor quality of the present data, it is not possible to make a strict test of these equations. The slope in $X \rightarrow \pi^+ \pi^- \pi^0$ is fairly unambiguously predicted to be around -1, and it is possible to enhance the $\eta \rightarrow 3\pi$ width to a value approaching experiment. This situation could be radically altered by an improvement in the experimental data, particularly the X width and $X \rightarrow 3\pi$ branching ratio: indeed, if the X width turned out to be very small, we should regard this as a success of eq. (5.26), where the amplitude is proportional to $m_\pi^2 \sin \theta$; we would then not be in a position to enhance $\eta \rightarrow 3\pi$, of course.

4. Introduction of an SU_3 Singlet Axial Current

The contents of the last section may all be found in ref. (28), but they were given fully because of the different presentation adopted. The results of this section are all contained in the same reference and we avoid verbatim reproduction by omitting all the details of the calculations.

The most general form of an SU_3 singlet axial current is

$$A_\mu^0 = -g(\phi) \partial_\mu \phi \quad (5.55)$$

The covariant derivative term in the Lagrangian, of the form $D_\mu \xi^i D^\mu \xi^i \cdot f_3(\phi)$ gives rise to the SU_3 currents, which contain a factor $f_3(\phi)$. These currents must commute with A_μ^0 to produce $U_3 \times U_3$ algebra, so that $f_3(\phi) = 1$ i.e. $a_4 = 0$ and hence $X \rightarrow \eta \pi\pi$ has zero slope. Thus we have another parallel between

the leading term Veneziano amplitude, which is completely symmetric in s , t and u , and a $U_3 \times U_3$ algebra, irrespective of the nature of the symmetry breaking term.

One may go further and assume that the symmetry-breaking should have $(3, \bar{3})$ properties in $U_3 \times U_3$, as in Section IV.4. The commutators are turned into differential equations for the unknown functions of the field ϕ , by using the canonical commutation relations. The differential equations may be solved, and a redefinition of the ϕ field by point transformation enables the strong interaction Lagrangian (5.13) to be written in the simple form,

$$\begin{aligned} \mathcal{L}(\xi, \phi) = & \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{f^2}{2} D_\mu \xi^{i/f} D^\mu \xi^{i/f} \\ & + \varepsilon(U_0(\xi/f, \phi/f) + c U_8(\xi/f, \phi/f)) f^2 \end{aligned} \quad (5.56)$$

$$\text{where } U^0(\xi, \phi) = u_0(\xi) \cos(\sqrt{\frac{2}{3}} \phi f/g(0)) + v_0(\xi) \sin(\sqrt{\frac{2}{3}} \phi f/g(0)) \quad (5.57)$$

(and similarly for U^1 , etc.) and

$$A_\mu^0 = -g(0) \partial_\mu \phi \quad (5.58)$$

The four masses depend on three parameters ε , c and $f/g(0)$. Thus there is a mass formula among the pseudoscalars;

$$9(m_8^2 - m_\eta^2)(m_X^2 - m_8^2) \bar{m}^2 = 8(m_X^2 + m_\eta^2 - m_8^2)(m_K^2 - m_\pi^2)^2 \quad (5.59)$$

It is very badly broken indeed.

It is perhaps possible to overcome both of these difficulties. For the mass problem, one may introduce a function $e \cos \delta \phi$

which provides an additional term with simple symmetry breaking properties. Also higher derivative terms in the Lagrangian can describe the $X \rightarrow \eta \pi \pi$ amplitude, e.g.

$$D_\mu \xi^i \cdot D_\mu \xi^i (a \partial_\mu \phi \partial^\mu \phi + b D_\mu \xi^j D^\mu \xi^j) \quad (5.60)$$

gives a term of the form

$$a + \beta p_x \cdot p_y \quad q_\pi^1 \cdot q_\pi^2 \quad (5.61)$$

The term a is expected to be "small" since it contains a factor m_π^2 (eq. (5.26)). If it is neglected, we have

$$M(X \rightarrow \eta \pi \pi) \propto 1 - .46y - .02y^2 \quad (5.62)$$

The quadratic term in y (defined before in eq. (5.47)) is small and the linear term is not too far from the experimental result. In view of their rather ad hoc nature, neither of these possibilities is considered further.

Finally, we note that eq. (5.57) may be written in the very compact form

$$U^a = \frac{1}{4} \text{Tr}(\lambda^a e^{i \xi^\beta \lambda^\beta}) + \text{h.c.} \quad (5.63)$$

where $\alpha, \beta = 0, 1, \dots, 8$ and $\xi^\alpha = (\phi, \xi^i)$, with the understanding that the scaling parameter for ϕ ($g(0)^{-1}$) is different from that for ξ^i (f^{-1}). Thus in this case of non-simple, non-linear realisations, we see that in eqs. (5.56) and (5.63) we have simply the Coleman, Wess and Zumino prescription⁽¹²⁾ for the Lagrangian, with different scaling parameters for the particles associated with commuting axial transformations. The special case $f = g(0)$ just

corresponds to I (eq. 4.32) equal to one. In this case the amplitudes will give exactly the same on mass shell as those in Table III (what else?).

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APPENDIX A: LORENTZ AND AMPLITUDE CONVENTIONS

Throughout the thesis we use the conventions for Lorentz invariants as in Bjorken and Drell⁽⁴¹⁾. A useful "translation dictionary" is given in ref. (1)). Thus the metric $g_{\mu\nu} = \text{diag.}(1, -1, -1, -1)$, so that $K \cdot x = Et - \underline{k} \cdot \underline{x}$.

The amplitudes listed in Tables I to IV have been given the phase of the T-matrix, in $S = 1 + i(2\pi)^4 \delta^4(P_i - P_f) T$, (S is the S-matrix and P_i (P_f) is the total initial (final) four-momentum). However, in calculating the more complicated pole-diagrams in Chapters II and V, where it is crucial to get the correct signs, it was found easier to work from first principles, giving amplitudes the phase of the S-matrix. In perturbation theory, then

$$M(\alpha \rightarrow \beta) = \langle \beta | T \left\{ \exp + i \int d^4x \mathcal{I}_I(x) \right\} | \alpha \rangle \quad (A1)$$

The propagator functions which appear in a Wick expansion of the time-ordered product in this equation are given, for a scalar field, for example, by

$$\begin{aligned} \Delta_F(x) &= \langle T \{ \phi(x) \phi(0) \} \rangle_0 \\ &= (2\pi)^{-4} \int d^4k e^{-ik \cdot x} \frac{i}{k^2 - m^2 + i\epsilon} \end{aligned} \quad (A2)$$

i.e. in momentum space the propagator is given by $i/(k^2 - m^2 + i\epsilon)$. Similarly, the spin-one propagator function is given by $i(-g_{\mu\nu} + k_\mu k_\nu / m^2) / (k^2 - m^2 + i\epsilon)$.

The differential decay rate for a particle of mass M is then

$$d\Gamma = d\left(\frac{1}{\tau}\right) = \frac{s}{2M} |m|^2 \prod_{i=1}^n \left(\frac{d^3 k_i}{2E_i (2\pi)^3} \right) (2\pi)^4 \delta^4(P_i - P_f) \quad (A3)$$

where s is the statistical factor for identical particles and the k_i are the three-momenta of the n final particles.

As suggested in eq. (A3), the states are given Lorentz invariant normalisation, e.g. for single particle states,

$$\langle p' | p \rangle = (2\pi)^3 2p_0 \delta^3(\underline{p} - \underline{p}') \quad (A4)$$

and the sum over intermediate states of momentum k is realised through the Lorentz invariant factor $\int \frac{d^3 k}{(2\pi)^3 2E_k}$.

Finally, consistency with the propagator function in (A2) means that

$$\langle 0 | \phi(x) | p \rangle = e^{-ip \cdot x} \quad (A5)$$

and the connected part of the L.S.Z. reduction formula takes the form

$$\langle \alpha, p | H(0) | \beta \rangle = i(m^2 - p^2) \int d^4 x e^{ip \cdot x} \langle \alpha | T \{ \phi(x) H(0) \} | \beta \rangle \quad (A6)$$

Here H is an arbitrary operator, the state β does not contain a particle of momentum p , and ϕ is the renormalised pion field.

APPENDIX B : SU₂ AND SU₃ CONVENTIONS

The charges generating SU₃ x SU₃ transformations are normalised according to eqs. (1.3), (1.6) and (1.7). The totally antisymmetric f^{abc} in these equations are tabulated in the review article by Gasiorowicz and Geffen⁽⁹⁾, as are also the totally symmetric d^{abc} in eq. (1.13).

The pion decay constant, f_π , is defined according to

$$\langle 0 | A_\mu^a | \pi^b(q) \rangle = i q_\mu \delta^{ab} f_\pi \quad (B1)$$

so that, if A is assumed to be the weak interaction axial current, then f_π obtained from the $\pi_{\mu 2}$ decay width $\simeq 93$ MeV. (This is to be compared with the value of f_π from the Goldberger-Treiman relation; $f_\pi \simeq 85$ MeV). The kaon (and eta) decay constants are defined with the same normalisation.

Throughout the thesis, crossing-symmetric phase conventions are used. Thus in performing L.S.Z. reductions on π^+ and π^- , no relative minus sign appears compared with reductions for π^0 . When soft pion techniques are combined with SU₂, or SU₃, the Clebsch-Gordan coefficients with the crossing-symmetric phase convention must be used. These are, for example, the tensors δ^{ab} , ϵ^{abc} , f^{abc} , d^{abc} , and not the coefficients of e.g. the Rosenfeld Tables⁽³⁵⁾ or de Swart⁽¹¹⁷⁾. (The latter contains a useful review, however.)

Matrix elements are normalised so that statistical factors appear, in the expressions for the widths, only for identical particles. A simple example is given by $K_1^0 \rightarrow 2\pi$ decays. For a $\Delta I = \frac{1}{2}$ weak Hamiltonian, we have in the crossing-symmetric convention $\langle \pi^a \pi^b | H_W | K_1^0 \rangle = R \delta^{ab}$ so that

$$\langle \pi^+ \pi^- | \mathcal{H}_W | K_1^0 \rangle = \langle \pi^0 \pi^0 | \mathcal{H}_W | K_1^0 \rangle$$

Only the $K_1^0 \rightarrow \pi^0 \pi^0$ width has the statistical factor $(2!)^{-1}$.

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ABSTRACT OF THESIS

Name of Candidate DAVID JAMES WALLACE
Address Department of Mathematical Physics, 1 Roxburgh Street.
Degree DOCTOR OF PHILOSOPHY Date JUNE 1970.
Title of Thesis APPLICATIONS OF CURRENT ALGEBRAS AND CHIRAL SYMMETRY BREAKING

The work in this thesis revolves around applications to pseudoscalar meson interactions of $SU_3 \times SU_3$ current algebra and of the way in which this symmetry group of the strong interactions should be broken in theory, as it is to a small extent in the physical world. The first chapter introduces these topics mainly from the point of view of operator commutators in quantum mechanics, of which current algebra is believed to be a particular example. In the same spirit, the $(3, \bar{3})$ symmetry breaking model is introduced as giving a physical system with the simplest algebraic properties.

In Chapter II, we study $\Delta I = 3/2$ electromagnetic corrections to a purely $\Delta I = 1/2$ weak Hamiltonian in $K \rightarrow 2\pi$ decays. Specifically, we estimate, through soft pion techniques the effect of a dynamical mechanism (i.e. Feynman diagram). When suitable account is taken of the nature of the electromagnetic interactions of the hadrons through the use of an effective electromagnetic Hamiltonian to describe the $\Delta I = 1$ "tadpole" effects, a result is obtained of the correct sign and giving about 50% of the $\Delta I = 3/2$ amplitude in $K \rightarrow 2\pi$ decays.

In the third chapter, we study soft meson reductions in $K \rightarrow 2\pi$ decays in the intermediate vector boson model. When all mesons are taken to zero four-momentum, the decays are related to integrals of the spectral functions of the vector and axial current propagators, which are convergent if certain spectral function sum rules hold. Our contribution is a study of the results of retaining all possible σ -commutators in the reductions; this approach has several attractive features, and the numerical results are significantly different from those of previous authors.