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THE SECOND CURVATURE OF A SUB-SPACE

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1. Introduction. This paper is mainly a discussion of the first and second curvatures of a Riemannian sub-space. The first curvature is well known as the mean curvature, and has been examined by Eisenhart and other writers; the second curvature is introduced directly as a generalization of the squared torsion of a twisted curve. The notation employed is, in general, that used by Eisenhart,* and we shall quote many of his results without giving explicit references.

2. Notation. Consider a flat space V_p, coordinates z^t $(t = 1, 2, ..., p)$. Let V_m, coordinates y^α $(\alpha = 1, 2, ..., m)$, be a sub-space of V_p, and let V_n, coordinates x^i $(i = 1, 2, ..., n)$, be a sub-space of V_m. Then V_m, V_n are given by equations of the form

$$z^t = z^t(y); \qquad y^\alpha = y^\alpha(x), \qquad \text{or } z^t = z^t(x), \qquad (t = 1, 2, ..., p). \tag{1.1}$$

As V_p is a flat space, we can use the generalized vector analysis when referring to this space.† Let $\mathbf{r}$ be the position vector of a point of V_p. Then $\mathbf{r}$ is a function of the y's for points of V_m, and of the x's for points of V_n. We shall use the Greek letters α, β, γ, δ, and the roman letters h, i, j, k for suffixes when referring to V_m and V_n respectively. Thus we shall write $f_\alpha = \partial f/\partial y^\alpha$, and $f_i = \partial f/\partial x^i$. Also, $f_{,\alpha\beta}$ and $f_{,ij}$ will denote the second covariant derivatives of f with respect to V_m and V_n respectively.

The fundamental tensors of V_m, V_n are given by

$$a_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta, \qquad g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j, \tag{1.2}$$

and from the relation $\mathbf{r}_i = \mathbf{r}_\alpha y^\alpha_{,i}$, we have at once $g_{ij} = a_{\alpha\beta} y^\alpha_{,i} y^\beta_{,j}$.

There are $m-n$ independent normals to V_n in V_m, and $p-m$ normals to V_m in V_p. We shall choose these to be mutually orthogonal, writing them $\mathbf{N}_{\sigma|}$ $(\sigma = 1, 2, ..., m-n)$, and $\mathbf{N}_{r|}$ $(r = m-n+1, ..., p-n)$. Thus we have

$$\mathbf{N}_{\sigma|} \cdot \mathbf{N}_{\mu|} = 0, \quad \mathbf{N}^2_{\sigma|} = e_\sigma = \pm 1; \quad \mathbf{N}_{r|} \cdot \mathbf{N}_{s|} = 0, \quad \mathbf{N}^2_{r|} = e_r = \pm 1,$$

and
$$\mathbf{N}_{\sigma|} \cdot \mathbf{r}_i = 0, \qquad \mathbf{N}_{r|} \cdot \mathbf{r}_\alpha = 0, \tag{1.3}$$

$$(\sigma, \mu = 1, 2, ..., m-n, \quad r, s = m-n+1, ..., p-n, \quad i = 1, 2, ..., n, \quad \alpha = 1, 2, ..., m, \quad \sigma \neq \mu; r \neq s).$$

* *Riemannian Geometry* (1926).

† For an introduction to this use of the vector analysis, see the writer's paper 'On deformation of sub-spaces': *Proc. Edinburgh Math. Soc.* (1932).

As the normals $\mathbf{N}_{\sigma|}$ are tangent to V_m, we can write

$$\mathbf{N}_{\sigma|} = \xi_{\sigma|}{}^{\alpha}\mathbf{r}_{\alpha} \qquad (\sigma = 1, 2,..., m-n) \tag{1.31}$$

where $\xi_{\sigma|}{}^{\alpha}$ are the contravariant components of these normals in V_m.

The second fundamental tensors of V_n in V_m are given by

$$\left.\begin{aligned} \Omega_{\sigma|ij} &= \mathbf{N}_{\sigma|}\cdot\mathbf{r}_{,ij} = -\mathbf{N}_{\sigma|,i}\cdot\mathbf{r}_j = -\mathbf{N}_{\sigma|,j}\cdot\mathbf{r}_i \\ \mu_{\sigma\mu|i} &= \mathbf{N}_{\sigma|}\cdot\mathbf{N}_{\mu|,i} = -\mathbf{N}_{\sigma|,i}\cdot\mathbf{N}_{\mu|} \end{aligned}\right\}. \tag{1.4}$$

For V_n in V_p, we have the above tensors, together with the tensors $\Omega_{r|ij}$, $\mu_{rs|i}$ similarly defined, and also the tensors

$$\mu_{\sigma r|i} = \mathbf{N}_{\sigma|}\cdot\mathbf{N}_{r|,i}. \tag{1.41}$$

From (1.4) and (1.41) we see that the fundamental equations for V_n in V_p may now be written

$$\left.\begin{aligned} \mathbf{r}_{,ij} &= \sum_{\sigma} e_{\sigma}\Omega_{\sigma|ij}\mathbf{N}_{\sigma|} + \sum_{r} e_r\Omega_{r|ij}\mathbf{N}_{r|} \\ \mathbf{N}_{\sigma|,i} &= -\Omega_{\sigma|ij}g^{jk}\mathbf{r}_k - \sum_{\mu} e_{\mu}\mu_{\sigma\mu|i}\mathbf{N}_{\mu|} - \sum_{r} e_r\mu_{\sigma r|i}\mathbf{N}_{r|} \end{aligned}\right\}, \tag{1.5}$$

and the equations for V_n in V_m are found from these by omitting vector components normal to V_m.

For V_n in V_p, we have introduced the vector operator $\nabla = g^{ij}\mathbf{r}_i\dfrac{\partial}{\partial x^j}$. We can now extend this for V_n in V_m, and define the *relative operator*

$$\nabla = g^{ij}\mathbf{r}_i\frac{\partial_m}{\partial x^j},$$

where $\partial_m/\partial x^j$ is $\partial/\partial x^j$ followed by an operator for annihilating vector components normal to V_m.* In this way, we shall extend the operator $\nabla^2 = \nabla\cdot\nabla$, etc.

LEMMA. *To show that*

$$g^{ij}y^{\alpha}{}_{,i}y^{\beta}{}_{,j} = a^{\alpha\beta} - \sum_{\sigma} e_{\sigma}\xi_{\sigma|}{}^{\alpha}\xi_{\sigma|}{}^{\beta}.$$

Let $\lambda_{h|}{}^{i}$ $(i, h = 1, 2,..., n)$ be the components of an orthogonal ennuple in V_n. Then the components in V_m are $\lambda_{h|}{}^{\alpha} = \lambda_{h|}{}^{i}y^{\alpha}{}_{,i}$, and these n vectors, together with the $(m-n)$ vectors $\xi_{\sigma|}{}^{\alpha}$, form an orthogonal m-uple in V_m. Hence, we have

$$a^{\alpha\beta} = \sum_{h} e_h\lambda_{h|}{}^{\alpha}\lambda_{h|}{}^{\beta} + \sum_{\sigma} e_{\sigma}\xi_{\sigma|}{}^{\alpha}\xi_{\sigma|}{}^{\beta}. \tag{2.1}$$

But

$$\sum_{h} e_h\lambda_{h|}{}^{\alpha}\lambda_{h|}{}^{\beta} = y^{\alpha}{}_{,i}y^{\beta}{}_{,j}\sum_{h} e_h\lambda_{h|}{}^{i}\lambda_{h|}{}^{j} = g^{ij}y^{\alpha}{}_{,i}y^{\beta}{}_{,j}. \tag{2.11}$$

* For a discussion of the vector operator, see the writer's paper above mentioned. We shall use the same notation ∇ for the relative operator: for this is the only one with which we shall be concerned.

Therefore $$g^{ij}y^{\alpha}{}_{,i}y^{\beta}{}_{,j} = a^{\alpha\beta} - \sum_{\sigma} e_{\sigma}\xi_{\sigma|}{}^{\alpha}\xi_{\sigma|}{}^{\beta}. \tag{2.2}$$

If $m = n+1$, this becomes

$$\xi^{\alpha}\xi^{\beta} = e(a^{\alpha\beta} - g^{ij}y^{\alpha}{}_{,i}y^{\beta}{}_{,j}) = c^{\alpha\beta}, \tag{2.3}$$

say. Hence the components of the normal to V_n in V_{n+1} are given by

$$\xi^{\alpha} = e_{\alpha}(c^{\alpha\alpha})^{1/2}, \tag{2.4}$$

where $e_1 = 1$, and e_{α} is the sign of $c^{1\alpha}$.

For a surface in V_3, (2.4) becomes, with the usual notation,

$$X = \{1-(Gx_1^2 - 2Fx_1x_2 + Ex_2^2)\}^{1/2}, \text{ etc.}, \tag{2.5}$$

where $x_1 = \partial x/\partial u$, $x_2 = \partial x/\partial v$.

3. For a curve in V_3, the vector operator reduces to $\nabla = \mathbf{t}d/ds$, where $\mathbf{t}$ is the unit-tangent vector. We find that the curvature vector of the curve can be written

$$\kappa\mathbf{n} = \nabla^2\mathbf{r},$$

where $\mathbf{n}$ is the principal normal, and κ the curvature. Generalizing this expression to V_n in V_m, we write

$$M\mathbf{N} = \nabla^2\mathbf{r} = g^{ij}\sum_{\sigma} e_{\sigma}\Omega_{\sigma|ij}\mathbf{N}_{\sigma|}. \tag{3.1}$$

We see that $\mathbf{N}$ so defined is the mean-curvature normal, and M is the mean curvature of V_n in V_m, these, then, being the generalizations of the principal normal and the curvature of a curve. We shall call $\mathbf{N}$ the *principal normal*, and M the *first curvature* of V_n in V_m. The second fundamental tensor for the normal $\mathbf{N}$ we shall write as Ω_{ij}. Hence,

$$M\Omega_{ij} = \sum_{\sigma} e_{\sigma}\Omega_{\sigma|}\Omega_{\sigma|ij}; \qquad \Omega_{\sigma|} = g^{ij}\Omega_{\sigma|ij}, \tag{3.11}$$

and

$$M = e\Omega, \qquad M^2 = e\sum_{\sigma} e_{\sigma}\Omega_{\sigma|}{}^2; \qquad \Omega = g^{ij}\Omega_{ij}, \qquad \mathbf{N}^2 = e = \pm 1.$$

Also, for a curve, we find that the torsion is given by

$$\tau^2 = -\mathbf{n}\cdot\nabla^2\mathbf{n} - (\nabla\cdot\mathbf{n})^2,$$

where $\mathbf{n}$ is the principal normal. Hence, generalizing, we shall define the *second curvature* T of V_n in V_m by

$$T = -\mathbf{N}\cdot\nabla^2\mathbf{N} - (\nabla\cdot\mathbf{N})^2, \tag{3.2}$$

where $\mathbf{N}$ is the principal normal of V_n in V_m. We have*

$$\mathbf{N}\cdot\nabla^2\mathbf{N} = g^{ij}\mathbf{N}\cdot[\mathbf{N}_{,ij}]_m = -g^{ij}[\mathbf{N}_{,i}\cdot\mathbf{N}_{,j}]_m \tag{3.21}$$

* The suffix m denotes that, after each operation on the vectors in the brackets, the components normal to V_m must be omitted. This follows from the definition of the relative operators.

and
$$(\nabla\cdot\mathbf{N})^2 = (g^{ij}\mathbf{r}_i\cdot\mathbf{N}_{,j})^2 = M^2.$$

Hence, choosing the remaining $m-n-1$ normals $\mathbf{N}_{\sigma|}$ to be orthogonal to $\mathbf{N}$, and writing $\mu_{\sigma|i} = \mathbf{N}\cdot\mathbf{N}_{\sigma|,i}$, we have, from (3.5),

$$T+M^2 = g^{ih}g^{jk}\Omega_{ij}\Omega_{hk}+g^{ij}\sum_{\sigma}\mu_{\sigma|i}\mu_{\sigma|j},$$

and hence,

$$T = g^{ih}g^{jk}(\Omega_{ij}\Omega_{hk}-\Omega_{ih}\Omega_{jk})+g^{ij}\sum_{\sigma}\mu_{\sigma|i}\mu_{\sigma|j}. \tag{3.22}$$

From (3.21), we see that an alternative definition would be

$$T+M^2 = S(\nabla\mathbf{N}\cdot\mathbf{N}\nabla), \tag{3.23}$$

where $\mathbf{N}\nabla$ is the dyadic conjugate to $\nabla\mathbf{N}$, and S denotes the scalar.

If $m = n+1$, we have

$$T = g^{ih}g^{jk}(\Omega_{ij}\Omega_{hk}-\Omega_{ih}\Omega_{jk}). \tag{3.3}$$

The Gauss equation for V_n in V_{n+1} is

$$e(\Omega_{ij}\Omega_{hk}-\Omega_{ih}\Omega_{jk}) = R_{ikjh}-R'_{\alpha\beta\gamma\delta}\,y^{\alpha}{}_{,i}\,y^{\beta}{}_{,k}\,y^{\gamma}{}_{,j}\,y^{\delta}{}_{,h}, \tag{3.31}$$

where R, R' are the curvature tensors of V_n, V_{n+1} respectively. Hence, (3.3) becomes

$$T = eR-eg^{ih}g^{jk}y^{\alpha}{}_{,i}\,y^{\beta}{}_{,k}\,y^{\gamma}{}_{,j}\,y^{\delta}{}_{,h}\,R'_{\alpha\beta\gamma\delta},$$

i.e. by (2.3), $\quad T = eR-e(a^{\alpha\delta}-e\xi^{\alpha}\xi^{\delta})(a^{\beta\gamma}-e\xi^{\beta}\xi^{\gamma})R'_{\alpha\beta\gamma\delta},$

i.e.
$$T = eR-eR'+2R'_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}, \tag{3.32}$$

where $R'_{\alpha\beta}$ is the Ricci tensor for V_m, and R, R' the scalar curvatures of V_n, V_m.

If V_{n+1} is an Einstein space, we have $R'_{\alpha\beta} = \lambda a_{\alpha\beta}$, and (3.32) reduces to

$$T = eR-e(n-1)\lambda. \tag{3.33}$$

If the Ricci tensor vanishes in V_{n+1}, we have

$$T = eR. \tag{3.34}$$

In particular, this is satisfied if V_{n+1} is a flat space. For the case of a surface in V_3, we have

$$T = R = -2K, \tag{3.35}$$

where K is the total curvature. Weatherburn* showed that $-2K$ was a generalization of the squared torsion, and the mean curvature a generalization of the curvature of a curve.

We observe that, for a curve in hyper-space, T is the squared second curvature, and, from (3.33), this vanishes for a curve in V_2, the curvature tensor of a curve being zero.

* *Messenger of Math.* 56 (1927), 173.

4. If the second fundamental tensors of V_n, V_m in V_p for the normals $\mathbf{N}_{r|}$ are $\Omega_{r|ij}$, $\Omega_{r|\alpha\beta}$, we have

$$\Omega_{r|\alpha\beta} = -\mathbf{N}_{r|,\alpha}\cdot\mathbf{r}_\beta$$

and $$\Omega_{r|ij} = -\mathbf{N}_{r|,i}\cdot\mathbf{r}_j = -\mathbf{N}_{r|,\alpha}\cdot\mathbf{r}_\beta\, y^\alpha{}_{,i}\, y^\beta{}_{,j}.$$

Hence $$\Omega_{r|ij} = \Omega_{r|\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j}. \tag{4.1}$$

Also, from (1.41), we have

$$\mu_{\sigma r|i} = \mathbf{N}_{\sigma|}\cdot\mathbf{N}_{r|,i} = \xi_{\sigma|}{}^\alpha\mathbf{r}_\alpha\cdot\mathbf{N}_{r|,\beta}\, y^\beta{}_{,i},$$

i.e. $$\mu_{\sigma r|i} = -\Omega_{r|\alpha\beta}\,\xi_{\sigma|}{}^\alpha y^\beta{}_{,i}. \tag{4.11}$$

If $\mathbf{N}$, $\mathbf{N}'$ are the principal normals and M, M' the first curvatures of V_n in V_m and V_n in V_p respectively, we have

$$M\mathbf{N} = \sum_\sigma e_\sigma \Omega_{\sigma|}\mathbf{N}_{\sigma|},$$

and $$M'\mathbf{N}' = \sum_\sigma e_\sigma \Omega_{\sigma|}\mathbf{N}_{\sigma|} + \sum_r e_r \Omega_{r|}\mathbf{N}_{r|}. \tag{4.2}$$

Hence, the principal normal of V_n in V_p is tangent to V_m if

$$\Omega_{r|} = 0 \qquad (r = m-n+1,\dots,p-n). \tag{4.21}$$

In this case, we see that

$$\mathbf{N}' = \mathbf{N}, \qquad M' = M, \qquad \Omega'_{ij} = \Omega_{ij}, \tag{4.22}$$

and we may say that V_n is an *asymptotic sub-space* of V_m. These results are evidently true if V_p is not necessarily a flat space.

Writing $\mu_{r|i} = \mathbf{N}\cdot\mathbf{N}_{r|,i}$, from (3.22), (4.22) we see that, if T, T' are the second curvatures of V_n in V_m, V_n in V_p, we have

$$T'-T = g^{ij}\sum_r e_r\,\mu_{r|i}\,\mu_{r|j}, \tag{4.23}$$

i.e. from (4.11),

$$T'-T = g^{ij}y^\alpha{}_{,i}\,y^\beta{}_{,j}\,\xi^\gamma\xi^\delta\sum_r e_r\Omega_{r|\alpha\gamma}\Omega_{r|\beta\delta}, \tag{4.24}$$

where $\mathbf{N} = \xi^\alpha\mathbf{r}_\alpha$. From (4.1), the equations (4.21) can be written

$$g^{ij}y^\alpha{}_{,i}\,y^\beta{}_{,j}\,\Omega_{r|\alpha\beta} = 0.$$

Hence, (4.24) can be written

$$T'-T = g^{ij}y^\alpha{}_{,i}\,y^\beta{}_{,j}\,\xi^\gamma\xi^\delta\sum_r e_r(\Omega_{r|\alpha\gamma}\Omega_{r|\beta\delta}-\Omega_{r|\alpha\beta}\Omega_{r|\gamma\delta}). \tag{4.25}$$

This equation is true if V_p is not a flat space. If V_p is a flat space, the Gauss equations for V_m in V_p are

$$\sum_r e_r(\Omega_{r|\alpha\gamma}\Omega_{r|\beta\delta}-\Omega_{r|\alpha\beta}\Omega_{r|\gamma\delta}) = R'_{\alpha\delta\gamma\beta}. \tag{4.26}$$

Hence, from (2.2), (4.25) becomes

$$T'-T = R'_{\alpha\beta}\,\xi^\alpha\xi^\beta - R'_{\alpha\delta\gamma\beta}\,\xi^\gamma\xi^\delta\sum_\sigma e_\sigma\,\xi_{\sigma|}{}^\alpha\xi_{\sigma|}{}^\beta. \tag{4.27}$$

If $m = n+1$, we have $\xi_{1|}{}^{\alpha} = \xi^{\alpha}$, and (4.27) reduces to

$$T'-T = R'_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}. \tag{4.28}$$

Hence, from (3.32), we have

$$T' = eR-eR'+3R'_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}. \tag{4.29}$$

We observe that the direction of the vector ξ^{α} is the principal normal direction of V_n in V_p, and T' is the second curvature of V_n in V_p. Also, R depends only on V_n. Hence

Given a space V_n in a flat space V_p, then all the spaces V_{n+1} which contain V_n and the principal normal direction of V_n in V_p at points of V_n have the same value for the expression $3R'_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}-eR'$ at points of V_n, where $R'_{\alpha\beta}$, R' refer to V_{n+1}, and ξ^{α} are the components in V_{n+1} of the principal normal of V_n in V_p.

This could be written

All spaces V_{n+1} of which V_n is an asymptotic sub-space have the same value for the above expression at points of V_n.

For a curve on a surface in V_3, we have $n = 1$, $m = 2$, $p = 3$, and $R'_{\alpha\beta} = -Ka_{\alpha\beta}$ where K is the total curvature. Hence we have

$$T' = -K,$$

R being zero for a curve. This is a well-known relation, for T' is the squared torsion of the curve, and the curve is an asymptotic line on the surface.

If V_p is any space, not necessarily flat, and if V_m is a sub-space of V_p, V_m is said to be *totally geodesic* if all geodesics in V_m are geodesics in V_p. The necessary and sufficient conditions for this are easily shown to be

$$\Omega_{r|\alpha\beta} = 0 \qquad \begin{pmatrix} r = 1,2,\ldots,p-m \text{ or } = m-n+1,\ldots,p-n, \\ \alpha,\beta = 1,2,\ldots,m. \end{pmatrix}. \tag{4.3}$$

From (4.26), we see that if V_p is a flat space, then all totally geodesic sub-spaces are flat spaces.

If V_m is a totally geodesic sub-space of V_p, then (4.24) becomes

$$T'-T = 0, \tag{4.31}$$

i.e. *if V_m is a geodesic sub-space of a space V_p, and if V_n is immersed in V_m, then the second curvature of V_n in V_p is the second curvature of V_n in V_m.*

For the case $n = 1$, $m = 2$, $p = 3$, we have $T = 0$ and hence

$T' = 0$. If V_3 is Euclidean, V_2 is a plane, and the above theorem reduces to the property that the torsion of a plane curve is zero.

5. Consider a system of ∞^{m-n} spaces V_n in V_m, such that one passes through each point of V_m. Then the normals $\mathbf{N}_{\sigma|}$ may be considered as functions of position in V_m. We have $\mathbf{r}_i = \mathbf{r}_\alpha y^\alpha{}_{,i}$, $\mathbf{N}_{\sigma|,j} = \mathbf{N}_{\sigma|,\beta} y^\beta{}_{,j}$, and hence, for the normal $\mathbf{N}_{\sigma|} = \xi_{\sigma|}{}^\alpha \mathbf{r}_\alpha$, we have

$$\Omega_{\sigma|} = -g^{ij}\mathbf{r}_i \cdot \mathbf{N}_{\sigma|,j} = -g^{ij} y^\alpha{}_{,i} y^\beta{}_{,j} \mathbf{r}_\alpha \cdot \mathbf{N}_{\sigma|,\beta}, \tag{5.1}$$

i.e. substituting for $\mathbf{N}_{\sigma|}$,

$$\Omega_{\sigma|} = -g^{ij} y^\alpha{}_{,i} y^\beta{}_{,j} \xi_{\sigma|\alpha,\beta}. \tag{5.11}$$

We have $$\xi_{\mu|}{}^\alpha \xi_{\mu|}{}^\beta \xi_{\sigma|\alpha,\beta} = -\xi_{\mu|}{}^\alpha{}_{,\beta} \xi_{\mu|}{}^\beta \xi_{\sigma|\alpha} = -\eta_{\mu|}{}^\alpha \xi_{\sigma|\alpha},$$

where $\eta_{\mu|}{}^\alpha = \xi_{\mu|}{}^\alpha{}_{,\beta} \xi_{\mu|}{}^\beta$ is the principal or geodesic curvature vector of the system of curves $\xi_{\mu|}{}^\alpha$ in V_m. Hence, from (2.2), we have

$$\Omega_{\sigma|} = -\xi_{\sigma|}{}^\alpha{}_{,\alpha} - \sum_\mu e_\mu a_{\alpha\beta} \xi_{\sigma|}{}^\alpha \eta_{\mu|}{}^\beta. \tag{5.12}$$

We have thus found the mean curvatures for the normals as functions of position of V_m. Obtaining the $m-n$ mean curvatures in this way, we can now find the principal normal and the first curvature from the relations

$$M\xi^\alpha = \sum e_\sigma \Omega_{\sigma|} \xi_{\sigma|}{}^\alpha, \qquad M^2 = e \sum e_\sigma \Omega_{\sigma|}{}^2. \tag{5.13}$$

Having found the principal normal and first curvature, we can find the second curvature from the equation

$$T + M^2 = g^{ij}[\mathbf{N}_{,i} \cdot \mathbf{N}_{,j}]_m = g^{ij} y^\alpha{}_{,i} y^\beta{}_{,j} [\mathbf{N}_{,\alpha} \cdot \mathbf{N}_{,\beta}]_m, \tag{5.2}$$

i.e., substituting from $\mathbf{N} = \xi^\alpha \mathbf{r}_\alpha$, we have

$$T = -M^2 + \Big(a^{\alpha\beta} - \sum_\mu e_\mu \xi_{\mu|}{}^\alpha \xi_{\mu|}{}^\beta\Big) a_{\gamma\delta} \xi^\gamma{}_{,\alpha} \xi^\delta{}_{,\beta}. \tag{5.21}$$

If $m = n+1$, there is one normal ξ^α, this being the principal normal, and we have

$$\Omega = -\xi^\alpha{}_{,\alpha}, \qquad M = e\Omega, \tag{5.3}$$

and $$T = -M^2 + (a^{\alpha\beta} - e\xi^\alpha \xi^\beta) a_{\gamma\delta} \xi^\gamma{}_{,\alpha} \xi^\delta{}_{,\beta}, \tag{5.31}$$

i.e. $$T = -M^2 + a^{\alpha\beta} a_{\gamma\delta} \xi^\gamma{}_{,\alpha} \xi^\delta{}_{,\beta} - ee'\kappa^2, \tag{5.32}$$

where κ is the principal curvature of the curves of congruence ξ^α, and $e'\kappa^2 = a_{\gamma\delta} \eta^\gamma \eta^\delta$, $\eta^\alpha = \xi^\alpha{}_{,\beta} \xi^\beta$.

From the above equations, we see that, given $m-n$ directions, which we may take to be mutually orthogonal, at each point of V_m, then, if these satisfy the conditions of integrability of the system of equations $a_{\alpha\beta} \xi_{\sigma|}{}^\alpha dy^\beta = 0$ $(\sigma = 1, 2, \ldots, m-n)$, we can find the principal normal and the curvatures of the spaces V_n defined as being normal to these $m-n$ directions.

Consider the case $m = 3$, $n = 2$, V_3 being referred to Cartesian axes. Let the given direction be (X, Y, Z), the components being functions of x, y, z.

The condition of integrability of $\sum X\,dx = 0$ is

$$X(Y_3-Z_2)+Y(Z_1-X_3)+Z(X_2-Y_1) = 0 \tag{5.4}$$

where $X_1 = \partial X/\partial x$, etc. If this is satisfied, the above directions are normal to a family of surfaces, and, from (5.3), (5.31), we see that the mean curvature K_m, and the total curvature, K, of these surfaces are given by

$$K_m = -(X_1+Y_2+Z_3), \tag{5.41}$$

$$2K = (X_1+Y_2+Z_3)^2+\sum[(XX_1+YX_2+ZX_3)^2-(X_1^2+X_2^2+X_3^2)]. \tag{5.42}$$

As an example, consider

$$X = \frac{x}{r}, \qquad Y = \frac{y}{r}, \qquad Z = \frac{z}{r}, \qquad r^2 = x^2+y^2+z^2.$$

In this case, (5.41) reduces to $K_m = -2/r$, and (5.42) reduces to $K = 1/r^2$. These are the results we should expect, for the given directions are evidently normal to a family of concentric spheres.

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SPATIAL DISTANCE IN GENERAL RELATIVITY

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A PROBLEM introduced recently is that of finding a suitable definition of spatial distance, i.e. the distance of a star from an observer, in a general Riemannian space-time. Professor E. T. Whittaker* gave a definition based on the method of comparing absolute with apparent brightness of the star, and this has been modified† to allow for the motion of the star. In this paper is introduced a similar definition, made possible by a new theorem on null geodesics. It can easily be verified that this definition is equivalent to that given by Etherington.

Relative coordinates are frequently used in this paper, these having already been introduced by the author.‡

1. Relative coordinates

If C is a curve in a Riemannian space V_n, and if an orthogonal ennuple is given at points of C, then at each point P of C is defined a system of normal coordinates with P as origin, the parametric directions at P in these coordinates being the directions of the vectors of the given ennuple at this point. Thus there is a system of reference at each point of the curve. These coordinates are called *relative coordinates*, and are written $(z^0, z^1, ..., z^{n-1})$. If the points of C are defined in terms of a parameter s, taken to be the arc if C is not null, any other curve C' may be given by the z's as functions of s, a $(1, 1)$ correspondence being set up between the points of C and C'. In general, we only consider curves in the neighbourhood of C, so that the third, and sometimes the second, powers of z may be neglected. It is usually convenient to take a particular correspondence, the most useful being such that $z^0 = 0$, i.e. if $\lambda_{\sigma|}{}^i$ $(\sigma, i = 0, 1, ..., n-1)$ are the components of the vectors of the given ennuple, the point Q of C' corresponding to P of C is the point where C' meets the geodesic surface orthogonal to the direction $\lambda_{0|}{}^i$ at P.

If x'^i are the given coordinates of V_n, the curve C is defined by equations of the form $x'^i = x^i(s)$, and if g_{ij} is the fundamental tensor,

* *Proc. Roy. Soc.*, A, 133 (1931), 93.

† Etherington (not yet published).

‡ *Proc. Roy. Soc. Edin.*, 52 (1932), 345.

the tangent vector of C is λ^i, where

$$\lambda^i = dx^i/ds, \qquad g_{ij}\lambda^i\lambda^j = e, \tag{1.1}$$

and $e = 0, \pm 1$ according as C is, or is not, null. Invariants u^σ, $\gamma_{\sigma\nu}$ are defined by the equations

$$\lambda^i = u^\sigma \lambda_{\sigma|}{}^i, \qquad \gamma_{\sigma\nu} = -\gamma_{\nu\sigma} = \lambda_{\sigma|i,j}\lambda_{\nu|}{}^i\lambda^j, \tag{1.2}$$

an index appearing as a subscript and also as a superscript indicating summation, unless attached to $e = \pm 1$, the summation being from 0 to $n-1$ unless otherwise stated. With this notation, the arc $\bar{s}$ of a curve C' near C is given by

$$\pm\left(\frac{d\bar{s}}{ds}\right)^2 = e + 2\gamma_{\sigma h}u^h z^\sigma + 2\sum_\sigma e_\sigma u^\sigma \dot{z}^\sigma + \sum_\sigma e_\sigma (\dot{z}^\sigma)^2 + 2\gamma_{\sigma\nu} z^\sigma \dot{z}^\nu + \\ + \sum_\sigma e_\sigma \gamma_{\sigma\mu}\gamma_{\sigma\nu} z^\mu z^\nu + \Gamma_{\sigma\nu} z^\sigma z^\nu, \tag{1.3}$$

where $\qquad \dot{z}^\sigma = dz^\sigma/ds, \qquad e_\sigma = g_{ij}\lambda_{\sigma|}{}^i\lambda_{\sigma|}{}^j = \pm 1,$

and

$$\Gamma_{\sigma\nu} = \Gamma_{\nu\sigma} = R_{hijk}\lambda^h\lambda^k\lambda_{\sigma|}{}^i\lambda_{\nu|}{}^j. \tag{1.4}$$

If C is a geodesic, then, from (1.2),

$$\gamma_{\sigma h} u^h = \lambda_{\sigma|i,j}\lambda^i\lambda^j = e_\sigma \dot{u}^\sigma,$$

and hence

$$\gamma_{\sigma h} u^h z^\sigma + \sum_\sigma e_\sigma u^\sigma \dot{z}^\sigma = \dot{J},$$

where

$$J = \sum_\sigma e_\sigma u^\sigma z^\sigma. \tag{1.5}$$

If C is a null geodesic,* and C' a null curve, $e = 0$, and $d\bar{s}/ds = 0$, and from (1.3) we have

$$J = \text{const.} \tag{1.6}$$

to the first order of approximation.

2. A theorem on null geodesics

If P_0, P are points on a null geodesic C, let p denote the thin† pencil of ∞^{n-2} null geodesics through P_0 passing near P. We shall prove that *the volume of the $(n-2)$-dimensional cross-section of p by a surface orthogonal to a vector V^i at P is independent of the direction of V^i.*

* It is assumed that when C is a null geodesic, the parameter s is so chosen that the geodesic equations of C take the usual simplified form. The parameter is then defined except for arbitrary additive and multiplicative constants.

† There should be no ambiguity in the use of the word 'thin'. Although the inner product of the tangent vectors at P_0 of two members of the pencil is zero to the first order, the angle between their projections on a space not orthogonal to C at P_0 is small, i.e. the solid angle of the pencil as measured by an observer at P_0 is small.

The pencil p being thin, the cross-section is evidently independent of the particular surface orthogonal to V^i. It is assumed that the direction of V^i is such that the square of distances in the cross-section may be neglected. Referring to relative coordinates along C, we shall take $z^0 = 0$, as explained in § 1, and we shall show that the volume when $V^i = \lambda_{0|}{}^i$ is equal to the volume for any other cross-section.

The null geodesics belonging to the pencil p are given by equations of the form

$$z^r = z^r(s, \alpha^1, \alpha^2, ..., \alpha^{n-2}) \qquad (r = 1, 2, ..., n-1),$$

where $z^r = 0$ at P_0, and the α's are constants, varying for the different geodesics. A surface orthogonal to $\lambda_{0|}{}^i$ is $z^0 = 0$ in relative coordinates, and, writing $Dz^r = (\partial z^r/\partial\alpha^p)d\alpha^p$, the linear element of the cross-section at P by this surface is given by

$$\pm d\sigma^2 = \sum_{r=1}^{n-1} e_r(Dz^r)^2.$$

If the surface $z^0 = 0$ at P meets a null geodesic C' of p at the point Q, it has been shown that the tangent vector of C' at Q is μ^σ, where

$$\mu^\sigma = \frac{\delta z^\sigma}{ds} = u^\sigma + \frac{dz^\sigma}{ds} - e_\sigma \gamma_{\sigma\nu} z^\nu.$$

Hence, a point Q' of C', near Q, has coordinates $\bar{z}^\sigma = z^\sigma + \phi\mu^\sigma$, where ϕ is an arbitrary function of s, of the order of smallness of the z's. Choosing ϕ so that Q' is the point of intersection of C' with a surface orthogonal to a vector V^i at P, and neglecting second-order terms, the points of the cross-section of p by this surface are $\bar{z}^\sigma = z^\sigma + \phi u^\sigma$, where ϕ is now a function of s and the α's. The element of length of this cross-section is given by

$$\pm d\bar{\sigma}^2 = \sum_{\sigma=0}^{n-1} e_\sigma(D\bar{z}^\sigma)^2 = \sum_\sigma e_\sigma(Dz^\sigma + u^\sigma D\phi)^2.$$

We have $\sum e_\sigma(u^\sigma)^2 = 0$, and $\sum e_\sigma u^\sigma Dz^\sigma = 0$, the latter following from (1.6), in which the constant is zero as the z's all vanish at P_0. Hence

$$d\bar{\sigma}^2 = \pm \sum e_\sigma(Dz^\sigma)^2 = d\sigma^2.$$

Thus the linear element of the cross-section is independent of the particular cross-section through P, from which the theorem follows immediately.

It has been pointed out to me that this theorem is a direct generalization of a simple property of surfaces on a null cone in a flat space. In space of the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

the null cone through the origin is given by

$$x = rl, \qquad y = rm, \qquad z = rn, \qquad t = r; \qquad l^2 + m^2 + n^2 = 1,$$

and a surface on this cone is given by an expression for r as a function of l, m, n. Its line element is

$$d\sigma^2 = -dr^2 + \sum (r\,dl + l\,dr)^2 = r^2 \sum dl^2,$$

since $\sum l^2 = 1$, $\sum l\,dl = 0$. Thus the elements of distance and area at any point are the same for all surfaces through this point.

3. The equations of geodesics

If C is a geodesic, not null, and the vectors $\lambda_{\sigma|}{}^i$ are given by parallel displacement along C, then $\gamma_{\sigma\nu} = 0$, and, from (1.3),

$$T = \frac{d\bar{s}}{ds} = 1 + e\dot{J} - \tfrac{1}{2}\dot{J}^2 + \tfrac{1}{2}e \sum_{\sigma} e_\sigma (\dot{z}^\sigma)^2 + \tfrac{1}{2}e\Gamma_{\sigma\nu} z^\sigma z^\nu. \tag{3.1}$$

Varying the z's as functions of s, the geodesics near C are given by $\delta \int T\,ds = 0$, i.e. by

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{z}^\sigma}\right) - \frac{\partial T}{\partial z^\sigma} = 0 \qquad (\sigma = 0, 1, \ldots, n-1). \tag{3.2}$$

Substituting from (3.1), these equations become

$$\ddot{z}^\sigma - e_\sigma \Gamma_{\sigma\nu} z^\nu = eu^\sigma \ddot{J}.$$

If, as before, we take $z^0 = 0$, then

$$eu^0 \ddot{J} = -e_0 \Gamma_{0\nu} z^\nu,$$

and hence, if $\lambda_{0|}{}^i$ is not orthogonal to C, the geodesics are given by

$$\ddot{z}^r - \left(e_r \Gamma_{rs} - \frac{e_0}{u^0} u^r \Gamma_{0s}\right) z^s = 0 \qquad (r, s = 1, 2, \ldots, n-1) \tag{3.3}$$

to the first order in the z's.

We observe that these equations are unaltered when $e = 0$, although the method of obtaining them is not valid in this case. We should therefore expect (3.3) to be the equations of geodesics when C is null. This is true, and can be verified by an actual transformation of coordinates in V_n. In what follows, we take C to be a null geodesic, the vectors $\lambda_{\sigma|}{}^i$ being given by parallel transport along C. From a property of null vectors we may take the vectors $\lambda_{2|}{}^i, \ldots, \lambda_{n-1|}{}^i$ to be independent of λ^i, and orthogonal to C. In this case $u^2, \ldots, u^{n-1}$ vanish, and (1.6) becomes

$$z^1 = \text{const.} \tag{3.4}$$

From the identities $\Gamma_{\sigma\nu}u^\sigma = 0$, $\sum e_\sigma(u^\sigma)^2 = 0$, equation (3.3) for $r = 1$ is now satisfied identically, and the remaining equations give

$$\ddot{z}^p - e_p\,\Gamma_{pr}z^r = 0 \qquad (p = 2, 3,..., n-1), \tag{3.5}$$

where r takes the values 1 to $n-1$. Hence, the null geodesics near C are given by equations (3.4), (3.5), together with $z^0 = 0$.

4. Spatial distance in general relativity

Let A be a star, B an observer, and, in order that light may pass from A to B, let A and B lie on a null geodesic. We shall adopt the following definition:

The spatial distance of A from B is proportional to the square root of the two-dimensional cross-section of a thin pencil of null geodesics issuing from A and passing near B, made by the three-dimensional instantaneous space of the observer. The definition is made complete by the requirement that when the observer is near and has the motion of the star, the spatial distance must reduce to the element of length in the observer's instantaneous space.

The instantaneous space of an observer is orthogonal to his direction of motion, and hence, from the theorem of § 2, the above cross-section is independent of the direction of motion of the observer. Thus we see that *spatial distance is independent of the observer's motion*. It follows, from the definition, that the formula for spatial distance will consist of two factors, the first being an invariant Θ depending only on the positions of the star and observer, the second being independent of the position of the observer. The main problem is to find the above invariant.

If λ^i is the tangent vector of the null geodesic C, let $\lambda_{2|}{}^i$, $\lambda_{3|}{}^i$ be two orthogonal vectors independent of λ^i, both being orthogonal to C and given by parallel transport along C. Then, from (3.4) and (3.5), the null geodesics through A passing near B are given by $z^1 = 0$, and

$$\ddot{z}^p + \Gamma_{pq}z^q = 0 \qquad (p, q = 2, 3), \tag{4.1}$$

where Γ_{pq} are defined by (1.4), and are functions of s.* We have assumed that the vectors are chosen so that $e_2 = e_3 = -1$, these vectors being supposed to lie in the instantaneous space of some observer at B. The problem therefore reduces to the solution of the simultaneous linear differential equations (4.1).

* It is assumed that the parameter s is chosen so that it increases when passing from A to B along the null geodesic.

If the position of A is given by $s = s_0$, it is required that the solutions of (4.1) should vanish when $s = s_0$. They can therefore be written in the form

$$z^2 = \alpha\phi_1+\beta\phi_2, \qquad z^3 = \alpha\psi_1+\beta\psi_2, \tag{4.2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are functions of s and s_0 vanishing at s_0, and α, β are small arbitrary constants. The cross-section of the pencil of null geodesics by the plane $z^0 = 0$ at B is given by equations (4.2) and $z^1 = 0$, the variables being α, β, and s having its value at B. The area of cross-section is at once found to be proportional to $(\phi_1\psi_2-\phi_2\psi_1)$, and the required invariant Θ must be proportional to the square root of this function. We therefore define

$$\pm\Theta^2 = K(\phi_1\psi_2-\phi_2\psi_1), \tag{4.3}$$

where the arbitrary constant K is chosen so that $\Theta \sim s-s_0$ as $s \to s_0$. Thus

$$\frac{1}{K} = \lim_{s\to s_0}\frac{\phi_1\psi_2-\phi_2\psi_1}{(s-s_0)^2}.$$

From the definition, the spatial distance between the points s_0 and s is given by an equation of the form

$$\Delta = \nu\Theta, \tag{4.4}$$

where ν is independent of s. For an observer B' at the point $s_0+\epsilon$, where ϵ is small, we have $\Theta = \epsilon$, from (4.3), and $\Delta' = \nu\epsilon$. If the star and the observer B' both have the motion v^i, the distance δ between the star and B' in the instantaneous space of this observer is

$$\delta = \epsilon|g_{ij}v^i\lambda^j|_{s=s_0}.$$

Hence, to satisfy the second part of the definition, we must have

$$\nu = |g_{ij}v^i\lambda^j|_{s=s_0}. \tag{4.5}$$

Thus the spatial distance between the star and the observer is $\Delta = \nu\Theta$, where ν is the inner product of the direction of motion of the star and the tangent vector of the null geodesic, and Θ is the invariant defined as above. We observe that Θ and ν are not completely determined by the positions of the star and observer and the motion of the star, owing to the arbitrary multiplicative constant attached to the parameter s. This constant, however, is cancelled in the product $\Delta = \nu\Theta$.

5. Spatial distance in space of constant curvature

In a space of constant curvature K_0, the components of the curvature tensor satisfy

$$R_{hijk} = K_0(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

We have $g_{ij}\lambda^i\lambda^j = 0$, and $g_{ij}\lambda^i\lambda_{p|}{}^j = 0$ for $p = 2, 3$, and hence, from (1.4),

$$\Gamma_{pq} = 0 \qquad (p, q = 2, 3).$$

From (4.1), the null geodesics through s_0 are given by

$$z^2 = \alpha(s-s_0), \qquad z^3 = \beta(s-s_0),$$

and from (4.3) we have

$$\Theta = s-s_0.$$

Hence, *in a space of constant curvature, if the parameter s of the null geodesic is chosen to vanish at the position of the star, the spatial distance is proportional to s.**

6. A more general method of obtaining spatial distance

The method of § 4 depends on finding two solutions of the equations of parallel transport along the given null geodesic. In some cases this may be impracticable; it can be avoided as follows:

Let $\lambda_{2|}{}^i$, $\lambda_{3|}{}^i$ be two orthogonal vectors at points of the geodesic, both being orthogonal to the curve and satisfying $e_2 = e_3 = -1$. As before, we have $u^2 = u^3 = 0$, and $z^1 = 0$ for the null geodesics through A. Completing an orthogonal 4-uple at points of C, we find, by a method similar to that of § 3, that the null geodesics through A are given by

$$\ddot{z}^p + 2\gamma_{pq}\dot{z}^q + \rho_{pq}z^q = 0, \tag{6.1}$$

where

$$\begin{aligned}\rho_{pq} &= \Gamma_{pq} + \dot{\gamma}_{pq} + \sum_{\sigma=0}^{3} e_\sigma \gamma_{\sigma p}\gamma_{\sigma q} \\ &= \Gamma_{pq} - \lambda_{q|i,jk}\lambda_{p|}{}^i\lambda^j\lambda^k.\end{aligned} \tag{6.2}$$

The invariant Θ can now be found as before in terms of the solutions of the linear equations (6.1).

If one of the vectors, say $\lambda_{2|}{}^i$, is a solution of the parallel transport equations, we have $\gamma_{2\sigma} = 0$ from (1.2). Also, it can easily be verified that for any vector $\lambda_{3|}{}^i$ orthogonal to such a vector and to the curve,

$$\sum_\sigma e_\sigma(\gamma_{3\sigma})^2 = g_{ij}\lambda_{3|}{}^i{}_{,h}\lambda_{3|}{}^j{}_{,k}\lambda^h\lambda^k = 0,$$

* This has been proposed by Professor E. T. Whittaker as an alternative definition of spatial distance in a general space. Another definition by H. S. Ruse, *Proc. Roy. Soc. Edin.*, 52 (1932), 183, leads to this result in a general space.

and hence, from (6.2),

$$\rho_{pq} = \Gamma_{pq} \qquad (p, q = 2, 3). \tag{6.3}$$

Thus equations (4.1) are still true when only one of the vectors is given by parallel transport along the geodesic.

7. The application to an important class of spaces

We shall now consider the application of the foregoing methods to space of the common form*

$$\gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{7.1}$$

where γ is a function of r alone. It is easily shown that, by a suitable choice of s, the equations of a typical null geodesic can be written in the form

$$\frac{dt}{ds} = \frac{1}{\gamma}, \qquad \frac{dr}{ds} = e\psi, \qquad \theta = \tfrac{1}{2}\pi, \qquad \frac{d\phi}{ds} = \frac{h}{r^2}. \tag{7.2}$$

where

$$\psi^2 = 1 - \frac{h^2\gamma}{r^2}, \qquad e = \pm 1, \tag{7.3}$$

and h is a constant. If the geodesic is through the origin, h is zero. A solution of the transport equations along this curve is

$$\lambda_{2|}{}^{i} \equiv (0, 0, 1/r, 0), \tag{7.4}$$

and a vector orthogonal to this vector and to the curve is

$$\lambda_{3|}{}^{i} \equiv (0, -eh\gamma/r, 0, \psi/r). \tag{7.5}$$

The non-vanishing components of the curvature tensor are

$$\begin{aligned} &R_{0110} = \tfrac{1}{2}\gamma'', \qquad && R_{0220} = \tfrac{1}{2}r\gamma\gamma', \qquad && R_{0330} = \tfrac{1}{2}r\gamma\gamma' \sin^2\theta, \\ &R_{1221} = -\tfrac{1}{2}r\frac{\gamma'}{\gamma}, \qquad && R_{1331} = -\tfrac{1}{2}r\frac{\gamma'}{\gamma}\sin^2\theta, \qquad && R_{2332} = -r^2(\gamma - 1)\sin^2\theta, \end{aligned} \tag{7.6}$$

and hence, from (1.4), (7.2), (7.4), and (7.5), we find

$$\begin{aligned} \Gamma_{22} &= \frac{h^2}{r^4}(\tfrac{1}{2}r\gamma' - \gamma + 1), \\ \Gamma_{23} &= 0, \\ \Gamma_{33} &= \frac{h^2}{2r^3}(r\gamma'' - \gamma'). \end{aligned} \tag{7.7}$$

* This form includes some well-known spaces in general relativity, e.g. the gravitational fields of a material particle and an electron.

If the geodesic passes through the origin, $h = 0$, and from (7.7),

$$\Gamma_{pq} = 0 \qquad (p, q = 2, 3).$$

The null geodesics through the point $s = s_0$ are therefore given by

$$z^2 = \alpha(s-s_0), \qquad z^3 = \beta(s-s_0),$$

and we have

$$\Theta = s-s_0. \tag{7.8}$$

In this case $r \pm s = \text{const.}$, and hence $\Theta = r+r_0$, $|r-r_0|$, according as the star and observer are, or are not, separated by the origin.

If $h \neq 0$, write $z^2 = rx$, $z^3 = r\psi y$. Substituting (7.7) in (4.1), and changing the independent variable from s to ϕ by (7.2), these equations become

$$\frac{d^2x}{d\phi^2}+x = 0, \tag{7.9}$$

$$\frac{d^2y}{d\phi^2}+\frac{2}{\psi}\frac{d\psi}{d\phi}\frac{dy}{d\phi} = 0. \tag{7.10}$$

Hence we have

$$z^2 = \alpha r\sin(\phi-\phi_0), \tag{7.11}$$

$$z^3 = \beta r\psi(V-V_0), \tag{7.12}$$

where

$$V = \int^{\phi} \frac{d\phi}{\psi^2},$$

and the subscript $_0$ indicates the value at the position of the star. The required invariant is now given by

$$\pm\Theta^2 = Kr^2\psi(V-V_0)\sin(\phi-\phi_0),$$

where

$$\frac{1}{K} = \lim_{s\to s_0}\frac{r^2\psi(V-V_0)\sin(\phi-\phi_0)}{(s-s_0)^2}.$$

We have

$$\frac{ds}{d\phi} = \frac{r^2}{h}, \qquad \lim_{\phi\to\phi_0}\frac{\sin(\phi-\phi_0)}{\phi-\phi_0} = 1, \qquad \lim_{\phi\to\phi_0}\frac{V-V_0}{\phi-\phi_0} = \frac{1}{\psi_0^2}.$$

Hence $K = r_0^2\psi_0/h^2$, and we have

$$\pm\Theta^2 = \frac{1}{h^2}r^2r_0^2\psi\psi_0(V-V_0)\sin(\phi-\phi_0). \tag{7.13}$$

It is interesting to observe that, in certain cases, this method of defining spatial distance will give unexpected results. These would be due to the term $\sin(\phi-\phi_0)$ in the above formula, for it may be possible for the null geodesics to be sufficiently curved so that ϕ_0 and $\pi+\phi_0$ give distinct points on the curve for some values of ϕ_0. In such a case, if the observer is placed near the point $\pi+\phi_0$, the star would appear to be very bright, and the above definition would indicate only a small spatial distance. This could quite possibly occur when light from a star passes near another heavy body, and it should be possible to find two stars, satisfying the required conditions, such that the earth is near the critical point of observation. It is seen that the above peculiarities arise from equation (7.11), which shows that it is possible for a pencil of null geodesics to converge again in one direction.

In conclusion we may remark that the above method of calculating spatial distance has been successfully applied to space of the more general form

$$ds^2 = \nu dt^2 - R^2(\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2),$$

where R is any function of t, and ν, γ are any functions of r.

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Relative Co-ordinates.

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Introduction.

If C is a given curve in a Riemannian space V_n, a system of co-ordinates $(z^1, z^2, \ldots, z^n)$ can be set up at each point P of C, thus generalising moving axes along a twisted curve. If in these co-ordinates a point Q is defined relative to the point P, then Q traces some curve as P moves along C; any curve C′ can be defined in this way by setting up a (1, 1) correspondence between points of C and C′. We shall take the relative co-ordinates at P to be normal co-ordinates with origin at P, the parametric directions at P being the directions of an orthogonal ennuple defined at points of C. In general, this work is too heavy except for the consideration of points within a certain distance from the curve C, and we shall therefore consider only points the cube of whose distance from C may be neglected. This is sufficient for the application to such problems as the motion of a small rigid body in space-time.

§ 1. Relative Co-ordinates.

Let V_n be a Riemannian space, co-ordinates $(x'^1, x'^2, \ldots, x'^n)$, and let C be a curve in V_n, defined by the equations

$$x'^i = x^i(s), \qquad (i = 1, 2, \ldots, n) \quad . \quad . \quad . \quad . \quad (1.1)$$

where s is the arc, or a parameter if C is null. Let $\lambda_{\sigma|}{}^i$ be the components of the vectors of an orthogonal ennuple* defined at points of C. The ennuple chosen will depend in general on the particular problem under consideration. The vectors of the ennuple may be given explicitly, or they may be solutions of a set of differential equations. An example of the former is obtained when the ennuple is formed by the tangent and principal normals of C, and an example of the latter when the ennuple is defined by Levi-Civita parallel transport from a given ennuple at some point of C. In both cases the vectors of the ennuple are solutions of displacement equations of the form

$$dV^i + C^i_{jk} V^j dx^k = 0, \qquad C^i_{jk} = \Gamma^i_{jk} + A^i_{jk} \quad . \quad . \quad . \quad (1.2)$$

* The notation is that of Eisenhart, *Riemannian Geometry* (1926). In $\lambda_{\sigma|}{}^i$, σ indicates the vector of the ennuple and i the component.

where Γ^i_{jk} are the Christoffel symbols of the second kind, and A^i_{jk} are the components of a tensor. For if the vectors are given explicitly, and invariants $\gamma_{\theta\sigma}$ are defined by

$$\gamma_{\theta\sigma} = -\lambda_{\theta|}{}^i\left(\frac{d\lambda_{\sigma|}{}^i}{ds} + \Gamma^i_{jk}\lambda_{\sigma|}{}^j\frac{dx^k}{ds}\right) \quad . \quad . \quad . \quad . \quad (1.3)$$

then, multiplying by $e_\theta\lambda_{\theta|}{}^l$, and summing for θ, we have

$$\frac{d\lambda_{\sigma|}{}^l}{ds} + \Gamma^l_{jk}\lambda_{\sigma|}{}^j\frac{dx^k}{ds} = -\sum_\theta e_\theta\gamma_{\theta\sigma}\lambda_{\theta|}{}^l$$
$$= -\sum_{\theta,\,\phi} e_\theta e_\phi\gamma_{\theta\phi}\lambda_{\theta|}{}^l\lambda_{\phi|j}\lambda_{\sigma|}{}^j.$$

Hence, the vectors $\lambda_{\sigma|}{}^i$ are solutions of (1.2), where

$$A^i_{jk}\dot{x}^k = \sum e_\theta e_\phi\gamma_{\theta\phi}\lambda_{\theta|}{}^i\lambda_{\phi|j}, \qquad \dot{x}^k = dx^k/ds.$$

From this equation we see that if the vectors are given as solutions of the equations (1.2), the invariants $\gamma_{\theta\phi}$ are given by

$$\gamma_{\theta\phi} = A^i_{jk}\dot{x}^k\lambda_{\theta|i}\lambda_{\phi|}{}^j. \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

By differentiating the equations of orthogonality, $g_{ij}\lambda_{\theta|}{}^i\lambda_{\phi|}{}^j = e_\phi\delta^\theta_\phi$, it can be seen at once from (1.3) that the invariants satisfy

$$\gamma_{\theta\phi} + \gamma_{\phi\theta} = 0, \qquad (\theta,\, \phi = 1,\, 2,\, .\, .\, .,\, n). \quad . \quad . \quad . \quad (1.5)$$

It may here be of interest to discuss the forms (1.2) must take in order that the vectors should satisfy certain conditions. It is always necessary that the magnitude of a vector and the angle between two vectors should be unaltered by the displacement, *i.e.*, if u^i, v^i are any two solutions, the expression $g_{ij}u^iv^j$ must be constant. Differentiating this expression along the curve, and substituting from (1.2), the required conditions become

$$(g_{ij}A^i_{kl} + g_{ik}A^i_{jl})\dot{x}^l = 0.$$

Thus, if this is all that is required, the most simple solution is $A^i_{jk} = 0$, giving the ordinary parallel transport.

If it is also required that the vectors should make constant angles with the curve C, then $\dot{x}^i$ must be a solution of the equations, *i.e.* substituting in (1.2),

$$A^i_{jk}\dot{x}^j\dot{x}^k = -\eta^i = -(\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k),$$

η^i being the curvature vector of the curve. The most simple solution of these two sets of equations is evidently

$$A^i_{jk}\,\dot{x}^k = eg_{jk}(\dot{x}^i\eta^k - \dot{x}^k\eta^i), \qquad e = g_{ij}\dot{x}^i\dot{x}^j. \quad . \quad . \quad . \quad (1.6)$$

Let a point P of C be given by $s = s_0$ in (1.1), and let the subscript 0 indicate the value at this point. Then with P as origin, a system of Riemannian co-ordinates $(y^1, y^2, \ldots, y^n)$ is defined by the equations

$$x'^i = x_0{}^i + y^i - \frac{1}{2!}(\Gamma^{\ i}_{\alpha\beta})_0 y^\alpha y^\beta - \frac{1}{3!}(\Gamma^{\ i}_{\alpha\beta\gamma})_0 y^\alpha y^\beta y^\gamma - \ldots, \qquad (1.7)$$

where the coefficients are certain well-known functions of the Christoffel symbols and their derivatives. Any other system of Riemannian co-ordinates is now given by a linear transformation of the y's with constant coefficients, and a system of normal co-ordinates is given by such a transformation so that the parametric directions at P are mutually orthogonal. We now wish to find the transformation of the y's giving normal co-ordinates with directions $(\lambda_{\sigma|}{}^i)_0$ at P.

If $(\lambda'_{\sigma|}{}^\alpha)_0$ are the components in the y's of the vectors $(\lambda_{\sigma|}{}^i)_0$, the required normal co-ordinates $(z^1, z^2, \ldots, z^n)$ are evidently given by

$$y^\alpha = (\lambda'_{\sigma|}{}^\alpha)_0 z^\sigma.$$

We have $(\lambda'_{\sigma|}{}^\alpha)_0 = (\lambda_{\sigma|}{}^i \partial y^\alpha / \partial x'^i)_0$, and from (1.7), $(\partial y^\alpha / \partial x'^i)_0 = \delta^\alpha_i$. Hence $(\lambda'_{\sigma|}{}^\alpha)_0 = (\lambda_{\sigma|}{}^\alpha)_0$, and the normal co-ordinates are given by

$$y^\alpha = (\lambda_{\sigma|}{}^\alpha)_0 z^\sigma. \qquad (1.8)$$

Substituting in (1.7), and dropping the subscript, we have at the point P(s) of C a transformation of the form

$$x'^i = x^i + \mathrm{F}^i(s, z), \qquad (1.9)$$

this being given by (1.7), which can be written in the form

$$x'^i = x^i + f^i(x, y); \qquad (1.10)$$

and by

$$y^\alpha = \lambda_{\sigma|}{}^\alpha z^\sigma, \qquad (1.11)$$

the x's and λ's being known functions of s. The relative co-ordinates are now completely defined at each point of C.

§ 2. Fundamental Formulæ.

Let Q be the point (z) in the co-ordinates at P(s), and let Q$'$ be the point $(z+dz)$ in the co-ordinates at P$(s+ds)$, and $(z+\delta z)$ in the co-ordinates at P. We proceed to find the relation between δz and dz where dz is small. If Q$'$ is the point $(\xi^1, \xi^2, \ldots, \xi^n)$ in the co-ordinates of V_n, we have from (1.9),

$$\xi^i = x^i(s) + \mathrm{F}^i(s, z+\delta z)$$

and

$$\xi^i = x^i(s+ds) + \mathrm{F}^i(s+ds, z+dz).$$

Expanding to the first order and equating, we get

$$\frac{\partial \mathrm{F}^i}{\partial z^\sigma}\frac{\mathrm{D}z^\sigma}{ds} = \dot{x}^i + \frac{\partial \mathrm{F}^i}{\partial s}, \qquad (2.1)$$

where

$$Dz^\sigma = \delta z^\sigma - dz^\sigma.$$

From (1.10) and (1.11) we have, writing $f^i_\alpha = \partial f^i/\partial y$

$$\frac{\partial F^i}{\partial z^\sigma} = f^i_\alpha \lambda_{\sigma|}{}^\alpha, \qquad (2.2)$$

and

$$\frac{\partial F^i}{\partial s} = \frac{\partial f^i}{\partial x^k}\dot{x}^k + f^i_\alpha \frac{d}{ds}(\lambda_{\sigma|}{}^\alpha) z^\sigma;$$

i.e. from (1.2),

$$\frac{\partial F^i}{\partial s} = \frac{\partial f^i}{\partial x^k}\dot{x}^k - f^i_\alpha C^{\,\alpha}_{jk}\dot{x}^k y^j. \qquad (2.3)$$

If $\bar{f}^\alpha_i$ is the minor of f^i_α in $|f^j_\rho|$, divided by the determinant, then substituting (2.2) and (2.3) in (2.1) and multiplying by $\bar{f}^l_i$, we get

$$\lambda_{\sigma|}{}^l \frac{Dz^\sigma}{ds} = \bar{f}^l_i\left(\dot{x}^i + \frac{\partial f^i}{\partial x^k}\dot{x}^k\right) - C^{\,l}_{jk} y^j \dot{x}^k. \qquad (2.4)$$

In expanding this expression, we shall neglect terms of order greater than the second in the z's (or y's). The definition of relative co-ordinates being independent of the particular system of original co-ordinates of V_n, the coefficients in the expansion are invariants with respect to these co-ordinates. We may therefore assume that the x co-ordinates are geodesic at P, and if we express the result in invariant form, the expression so obtained will be true whatever the original co-ordinates may have been.

We have

$$f^i(x, y) = y^i - \frac{1}{2\,!}\Gamma^{\ \,i}_{\alpha\beta} y^\alpha y^\beta - \frac{1}{3\,!}\Gamma^{\ \ \,i}_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma, \qquad (2.5)$$

and hence at P,

$$f^i_\alpha = \delta^i_\alpha - \tfrac{1}{2}\Gamma^{\ \ \,i}_{\alpha\beta\gamma} y^\beta y^\gamma, \qquad (2.6)$$

for the Christoffel symbols now vanish at P. From $f^i_\alpha \bar{f}^\alpha_j = \delta^i_j$ we have

$$\bar{f}^\alpha_i = \delta^\alpha_i + \tfrac{1}{2}\Gamma^{\ \ \,\alpha}_{i\beta\gamma} y^\beta y^\gamma. \qquad (2.7)$$

Also, from (2.5),

$$\frac{\partial f^i}{\partial x^k} = -\tfrac{1}{2}\Gamma^{\ \ i}_{\alpha\beta\,.\,k} y^\alpha y^\beta, \qquad \Gamma^{\ \ i}_{\alpha\beta\,.\,k} = \frac{\partial}{\partial x^k}\Gamma^{\ \,i}_{\alpha\beta},$$

and hence

$$\bar{f}^l_i\left(\dot{x}^i + \frac{\partial f^i}{\partial x^k}\dot{x}^k\right) = \dot{x}^l + \tfrac{1}{2}(\Gamma^{\ \ \,l}_{k\beta\gamma} - \Gamma^{\ \ l}_{\beta\gamma\,.\,k}) y^\beta y^\gamma \dot{x}^k.$$

From the expression* for $\Gamma^{\ \ \,i}_{k\beta\gamma}$, we have, at P,

$$\begin{aligned}\Gamma^{\ \ \,l}_{k\beta\gamma} - \Gamma^{\ \ l}_{\beta\gamma\,.\,k} &= \tfrac{1}{3}(\Gamma^{\ \ l}_{k\beta\,.\,\gamma} + \Gamma^{\ \ l}_{\beta\gamma\,.\,k} + \Gamma^{\ \ l}_{\gamma k\,.\,\beta}) - \Gamma^{\ \ l}_{\beta\gamma\,.\,k} \\ &= \tfrac{1}{3}[(\Gamma^{\ \ l}_{\beta k\,.\,\gamma} - \Gamma^{\ \ l}_{\beta\gamma\,.\,k}) + (\Gamma^{\ \ l}_{\gamma k\,.\,\beta} - \Gamma^{\ \ l}_{\gamma\beta\,.\,k})] \\ &= \tfrac{1}{3}(R^{\ \ \ \,l}_{\beta\gamma k} + R^{\ \ \ \,l}_{\gamma\beta k}). \qquad (2.8)\end{aligned}$$

* Eisenhart, *op. cit.*, p. 52.

Substituting from (1.3) and (2.8) in (2.4),

$$\lambda_{\sigma|}{}^{l}\frac{Dz^{\sigma}}{ds}=\dot{x}^{l}-A^{l}_{jk}\dot{x}^{k}y^{j}+\tfrac{1}{3}R_{\beta\gamma k}{}^{l}\dot{x}^{k}y^{\beta}y^{\gamma}, \qquad (2.9)$$

this being true whatever the original co-ordinates may be. From (2.9) we get at once

$$e_{\sigma}\frac{Dz^{\sigma}}{ds}=\dot{x}^{i}\lambda_{\sigma|i}-A^{i}_{jk}\dot{x}^{k}\lambda_{\sigma|i}y^{j}+\tfrac{1}{3}R_{i\beta\gamma k}\dot{x}^{k}\lambda_{\sigma|}{}^{i}y^{\beta}y^{\gamma},$$

a repeated index not indicating summation if it is attached to $e=\pm 1$. From (1.4) and (1.11) we now have, writing $u^{\sigma}=e_{\sigma}\lambda_{\sigma|i}\dot{x}^{i}$,

$$\frac{\delta z^{\sigma}}{ds}=\frac{dz^{\sigma}}{ds}+u^{\sigma}-e_{\sigma}\gamma_{\sigma\nu}z^{\nu}+\tfrac{1}{3}e_{\sigma}\gamma_{\sigma\mu\nu h}u^{h}z^{\mu}z^{\nu}. \qquad (\sigma=1,2,\ldots,n), \qquad (2.10)$$

where $\gamma_{\sigma\mu\nu h}$ is the well-known invariant $R_{lijk}\lambda_{\sigma|}{}^{l}\lambda_{\mu|}{}^{i}\lambda_{\nu|}{}^{j}\lambda_{h|}{}^{k}$.

If C is a geodesic, we can use Levi-Civita parallel transport along C, taking $\lambda_{1|}{}^{i}=\dot{x}^{i}$, in which case we have $\gamma_{\sigma\nu}=0$, $\mu^{\sigma}=\delta^{1}_{\sigma}$, and hence

$$\frac{\delta z^{\sigma}}{ds}=\frac{dz^{\sigma}}{ds}+\delta^{1}_{\sigma}+\tfrac{1}{3}e_{\sigma}\gamma_{\sigma\mu\nu 1}z^{\mu}z^{\nu}. \qquad (2.11)$$

If C is not a geodesic, an interesting set of relative co-ordinates is found by taking the vectors of reference to be in the principal directions of the curve. In this case the Serret-Frenet formulæ give

$$\gamma_{\sigma\nu}=-\kappa_{\tau-1}\delta^{\nu}_{\sigma-1}+\kappa_{\sigma}\delta^{\nu}_{\sigma+1},$$

and hence

$$\frac{\delta z^{\sigma}}{ds}=\frac{dz^{\sigma}}{ds}+\delta^{1}_{\sigma}+e_{\sigma}\kappa_{\sigma-1}z^{\sigma-1}-e_{\sigma}\kappa_{\sigma}z^{\sigma+1}+\tfrac{1}{3}e_{\sigma}\gamma_{\sigma\mu\nu 1}z^{\mu}z^{\nu}, \qquad (2.12)$$

where the κ's are the principal curvatures of C. If V_n is a flat space, we have without any approximation

$$\frac{\delta z^{\sigma}}{ds}=\frac{dz^{\sigma}}{ds}+\delta^{1}_{\sigma}+e_{\sigma}\kappa_{\sigma-1}z^{\sigma-1}-e_{\sigma}\kappa_{\sigma}z^{\sigma+1}. \qquad (2.13)$$

This is an evident generalisation of the moving axes formulæ for a twisted curve, where $n=3$, and κ_1 is the curvature, κ_2 the torsion.

§ 3. The Application to Neighbouring Curves.

For the further consideration of relative co-ordinates we require the metric of V_n referred to normal co-ordinates with a given point P as origin. Let $a_{\alpha\beta}$ be the fundamental tensor for the y co-ordinates defined by (1.7), and $b_{\sigma\nu}$ the tensor for the z co-ordinates defined by (1.7) and (1.8). Then from the law of transformation, we have

$$a_{\alpha\beta}=g_{ij}\frac{\partial x'^{i}}{\partial y^{\alpha}}\frac{\partial x'^{j}}{\partial y^{\beta}}, \qquad b_{\sigma\nu}=a_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial z^{\sigma}}\frac{\partial y^{\beta}}{\partial z^{\nu}}. \qquad (3.1)$$

By a method similar to that used in expanding (2.4) it can easily be shown that we get

$$a_{\alpha\beta}=(g_{\alpha\beta})_0+\tfrac{1}{3}(\mathrm{R}_{\alpha\gamma\delta\beta})_0 y^\gamma y^\delta \qquad (3.2)$$

to the required order of approximation, and hence from (3.1) and (1.8),

$$b_{\sigma\nu}=e_\sigma\delta^\nu_\sigma+\tfrac{1}{3}(\gamma_{\sigma rs\nu})_0 z^r z^s. \qquad (3.3)$$

If the co-ordinates of a point Q relative to the point P(s) of C are given as functions of s, then as P moves along C, Q traces a curve C′. The tangent vector of this curve at Q in the co-ordinates at P is $\delta z^\sigma/d\bar{s}$, where $\bar{s}$ is the arc, and is given by

$$\pm\left(\frac{d\bar{s}}{ds}\right)^2=b_{\sigma\nu}\frac{\delta z^\sigma}{ds}\frac{\delta z^\nu}{ds}. \qquad (3.4)$$

Substituting from (2.10) and (3.3) we get

$$\pm\left(\frac{d\bar{s}}{ds}\right)^2=e+2\gamma_{\sigma h}u^h z^\sigma+2\sum_\sigma e_\sigma u^\sigma\dot{z}^\sigma+\sum_\sigma e_\sigma(\dot{z}^\sigma)^2+2\gamma_{\sigma\nu}z^\sigma\dot{z}^\nu$$
$$+\sum_\sigma e_\sigma\gamma_{\sigma\mu}\gamma_{\sigma\nu}z^\mu z^\nu+\gamma_{h\mu\nu k}u^h u^k z^\mu z^\nu, \qquad (3.5)$$

where $e=\sum\limits_\sigma e_\sigma(u^\sigma)^2=\pm1$, 0 according as C is not, or is null. If C is not null, the sign on the left is evidently that of e.

If C is not null, and C′ is a curve near C, we can take C′ to be the locus of a point Q where the geodesic QP is orthogonal to C at P. We now take $\lambda_{1|}{}^i=\dot{x}^i$, and C′ is given by $z^1=0$, and $z^2, z^3, \ldots, z^n$ as functions of s. If C is not a geodesic, the simplest displacement of vectors along C which transports the tangent at one point to the tangent at any other point was found in § 1. This is defined by

$$\mathrm{A}^i_{jk}\dot{x}^k=eg_{jk}(\dot{x}^i\eta^k-\dot{x}^k\eta^i);\qquad \eta^i=\dot{x}^i{}_{,k}\dot{x}^k, \qquad (3.6)$$

from which we get

$$\gamma_{\sigma\nu}=\delta^1_\sigma v_\nu-\delta^1_\nu v_\sigma;\qquad v_\sigma=\eta^i\lambda_{\sigma|i}. \qquad (3.7)$$

Substituting in (2.10) and (3.5), we have

$$\frac{\delta z^1}{ds}=1-ev_r z^r+\tfrac{1}{3}e\gamma_{1rs1}z^r z^s, \qquad (3.8)$$

$$\frac{\delta z^r}{ds}=\dot{z}^r+\tfrac{1}{3}e_r\gamma_{rst1}z^s z^t,$$

$$\left(\frac{d\bar{s}}{ds}\right)^2=1-2ev_r z^r+e\sum_r e_r(\dot{z}^r)^2+(v_r z^r)^2+e\gamma_{1rs1}z^r z^s$$

i.e.

$$\frac{d\bar{s}}{ds}=1-ev_r z^r+\tfrac{1}{2}e\sum_r e_r(\dot{z}^r)^2+\tfrac{1}{2}e\Gamma_{rs}z^r z^s \qquad (3.9)$$

where $\Gamma_{rs}=\Gamma_{sr}=\gamma_{1rs1}=R_{hijk}\dot{x}^h\dot{x}^k\lambda_{r|}{}^i\lambda_{s|}{}^j$, and r, s, t take the values $2, 3, \ldots, n$. These equations hold if C is a geodesic, for in this case we have $v_r=0$, $(r=1, 2, \ldots, n)$, and the above transport reduces to Levi-Civita parallel transport.

§ 4. Geodesics in Relative Co-ordinates.

To find the equations of the geodesics near C (not null) we use the variational definition

$$\delta\int T ds=0, \qquad (4.1)$$

where $T=d\bar{s}/ds$, and $z^2, z^3, \ldots, z^n$ are varied as functions of s. This equation gives the differential equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{z}^r}\right)-\frac{\partial T}{\partial z^r}=0, \qquad (4.2)$$

and substituting from (3.9), we get

$$e_r\ddot{z}^r-\Gamma_{rs}z^s=-v_r, \qquad (r=2, 3, \ldots, n). \qquad (4.3)$$

These equations are only true when $(z)^2$ may be neglected. They show, as expected, that v_r must be of the order of smallness of z, *i.e.* C cannot lie near a geodesic unless the first curvature of C is small.

If C is a geodesic, we have $v_r=0$, and the geodesics near C are given by

$$\ddot{z}^r-e_r\Gamma_{rs}z^s=0 \qquad (r=2, 3, \ldots, n). \qquad (4.4)$$

We can at once find the general solution of equations (4.4) when V_n is a space of constant curvature. In this case we have

$$R_{hijk}=K(g_{hj}g_{ik}-g_{hk}g_{ij}),$$

where K is constant, and hence

$$\Gamma_{rs}=K\lambda_{r|i}\lambda_{s|j}(\dot{x}^i\dot{x}^j-eg^{ij})=-ee_rK\delta^s_r. \qquad (4.5)$$

Equations (4.4) now become

$$\ddot{z}^r+eKz^r=0$$

and the solution is

$$z^r=A^r\cos(\sqrt{eK}s+\alpha^r), \qquad A^r\cosh(\sqrt{-eK}s+\alpha^r), \qquad (4.6)$$

according as eK is $+ve$ or $-ve$, where A^r, α^r are arbitrary constants.

§ 5. Rigid Motion in a Riemannian Space.

Let $L_p, L_q, \ldots$ be the paths, or world-lines, of a system of neighbouring particles. Then we may say that the system of particles moves as a rigid body if the orthogonal distance between any two world-lines is constant along these lines. We shall suppose that the lines are sufficiently near together so that the cube of the greatest distance between them may be neglected, and we shall refer to one line C and use relative co-ordinates with the transport (1.6) in referring to the other lines.

In the co-ordinates at a point P of C it is evident from (3.3) that the geodesics are straight lines to the required order of smallness. Consider any two lines L, L′ of the system; let L meet the V_{n-1} orthogonal to C at P at the point $Q(z^2, z^3, \ldots, z^n)$, and let L′ meet this space at the point $Q'(z'^2, z'^3, \ldots, z'^n)$. Let $Q\bar{Q}$ be the geodesic orthogonal to L at Q, meeting L′ at the point $\bar{Q}$. Then it can easily be verified that $Q\bar{Q} = QQ'$ to the required order of approximation, *i.e.* we have

$$\pm (Q\bar{Q})^2 = \sum_{r=2}^{n} e_r (z'^r - z^r)^2. \qquad (5.1)$$

Hence, for rigid motion, this expression must be independent of s, and so for each pair of world-lines. Thus if L_p is the locus of $Q_p(z_p{}^2, z_p{}^3, \ldots, z_p{}^n)$, the conditions are

$$\frac{d}{ds}\sum_{r=2}^{n} e_r (z_p^r - z_q^r)^2 = 0 \qquad (5.2)$$

for all values of p, q. We have taken C to be the world-line of one of the particles, and hence, from (5.2), the conditions are

$$\frac{d}{ds}\sum_r e_r (z_p^r)^2 = 0, \qquad \frac{d}{ds}\sum_r e_r z_p^r z_q^r = 0\,; \qquad (5.3)$$

i.e. the geodesic distance PQ_p, and the angle at P between the geodesics PQ_p, PQ_q are constant along C. Hence, we can write

$$z_p^r = \xi_s^r v_p^s \qquad (5.4)$$

where the v's are constants, and ξ_s^r, $(r, s = 2, 3, \ldots, n)$ are the coefficients of an orthogonal transformation in V_{n-1} and are dependent only on s. *Equations* (5.4) *are the necessary and sufficient conditions that the lines* (z_p) *should be the world-lines of the particles of a small rigid body, referred to relative co-ordinates along one of the world-lines.*

Thomsen * has discussed the equations of motion of a rigid body in space-time by considering the variation of the energy integral. For this we require the energy of the particles, which is at once given by (3.9) and

* Thomsen, *Math. Zeitschrift*, 29, 1929, 96.

(5.4). For the particle $Q(z)$ the energy is $m d\bar{s}/dt$, and we can write $d\bar{s}/dt$ in the form

$$\frac{d\bar{s}}{dt}=\frac{ds}{dt}(1-e\mathrm{T}_r\xi'^r_s v^s)+\tfrac{1}{2}e\frac{dt}{ds}(\Sigma e_r\xi'^r_s\xi'^r_t+\mathrm{R}_{hijk}x'^h x'^k\lambda_{u|}{}^i\lambda_{v|}{}^j\xi^u_s\xi^v_t)v^s v^t, \quad . \quad (5.5)$$

where a dash denotes differentiation with respect to t, and T_r is written for $\eta^i\lambda_{r|i}$, $\eta^i=\dot{x}^i{}_{,k}\dot{x}^k$. In space-time we have $n=4$, $e=1$, $e_r=-1$ $(r=2, 3, 4)$, and we have Thomsen's form, with a slight change of notation.

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On Small Deformation of Sub-Spaces of a Flat Space

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The object of this paper is to introduce the differential operator, ∇, generalised for a Riemannian space V_n immersed in a flat space V_p, and then to discuss the general small deformation of V_n.

§ 1. *Notation.*

We shall use the notation of vector analysis in the flat space, and tensor calculus in the Riemannian space. Consider a Riemannian space V_n immersed in a flat space V_p, $p > n$. Let $\mathbf{r} = (z^1, z^2, \ldots . z^p)$ be the position vector of a point of V_p, the fundamental form of V_p being

$$\phi = \sum_{a=1}^{p} e_a (dz^a)^2, \quad e_a = \pm 1. \tag{1.1}$$

The scalar product of two vectors $\mathbf{a}$, $\mathbf{b}$ in V_p is defined to be

$$\mathbf{a} \cdot \mathbf{b} = \sum_{a=1}^{p} e_a a^a b^a. \tag{1.2}$$

The space V_n is given by equations of the form $z^a = z^a(x)$, where x^i $(i = 1, 2, \ldots, n)$ are the coordinates of V_n, and, substituting for the z's, we have $\mathbf{r}$ as a function of x for points of V_n. From the form (1.1), which can now be written $\phi = (d\mathbf{r})^2$, we find that the fundamental tensor of V_n is given by

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j, \qquad \mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial x^i}. \tag{1.3}$$

We may consider $\mathbf{r}$ as an invariant in V_n, and we can differentiate the vector covariantly with respect to g_{ij}, obtaining vectors in V_p which have tensor forms in V_n.

By considering a small displacement in V_n of a point of V_n, we find that the n vectors $\mathbf{r}_i$ are tangent to V_n; they must also be independent in order that the coordinates x^i should be independent.

Hence a vector tangent to V_n may be written in the form

$$\mathbf{t} = \lambda^i \mathbf{r}_i \tag{1.4}$$

where λ^i are the components of a contravariant vector in V_n. Thus a vector tangent to V_n can be defined either by a vector in V_p, or by a contravariant vector in V_n. It can easily be verified that these definitions define the same magnitude of such a vector and also the same angle between two such vectors. These results are important as showing some of the relations between the two methods of discussing a Riemannian space.

Differentiating (1.3) covariantly, we get $\mathbf{r}_{,ik} \cdot \mathbf{r}_j + \mathbf{r}_i \cdot \mathbf{r}_{,jk} = 0$, where $\mathbf{r}_{,ij}$ is the second covariant derivative[1] of $\mathbf{r}$. Permuting i, j, k, we at once find that

$$\mathbf{r}_i \cdot \mathbf{r}_{,jk} = 0, \quad (i, j, k = 1, 2, \ldots, n). \tag{1.5}$$

Hence $\mathbf{r}_{,jk}$ is orthogonal to every direction tangent to V_n, that is, is normal to V_n.

The normals to V_n are given by $\mathbf{N} \cdot \mathbf{r}_i = 0$, $(i = 1, 2, \ldots, n)$. There are $p - n$ independent normals, and these can be chosen to be mutually orthogonal, such a set of unit normals being written $\mathbf{N}_{\sigma|}$, $(\sigma = 1, 2, \ldots, p - n)$.

We can define tensors $\Omega_{\sigma|ij}$, $\mu_{\sigma\nu|i}$ by the equations

$$\begin{aligned} \Omega_{\sigma|ij} &= \mathbf{N}_{\sigma|} \cdot \mathbf{r}_{,ij} = -\mathbf{N}_{\sigma|,i} \cdot \mathbf{r}_j = -\mathbf{N}_{\sigma|,j} \cdot \mathbf{r}_i, \\ \mu_{\sigma\nu|i} &= \mathbf{N}_{\sigma|} \cdot \mathbf{N}_{\nu|,i} = -\mathbf{N}_{\nu|} \cdot \mathbf{N}_{\sigma|,i}. \end{aligned} \tag{1.6}$$

These tensors can easily be identified with the second fundamental tensors[2].

From (1.5) and (1.6), it follows that $\mathbf{r}_{,ij}$, $\mathbf{N}_{\sigma|,i}$ can be written in the forms

$$\begin{aligned} \mathbf{r}_{,ij} &= \sum_{\sigma=1}^{p-n} e_\sigma \Omega_{\sigma|ij} \mathbf{N}_{\sigma|}, \\ \mathbf{N}_{\sigma|,i} &= -\Omega_{\sigma|ij} g^{jk} \mathbf{r}_k - \sum_{\nu=1}^{p-n} e_\nu \mu_{\sigma\nu|i} \mathbf{N}_{\nu|}, \end{aligned} \tag{1.6}$$

where $e_\sigma = \mathbf{N}_{\sigma|}{}^2 = \pm 1$.

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[1] This is the usual notation for covariant derivatives. With this notation, we could write $\mathbf{r}_{,i}$ for $\mathbf{r}_i$.

[2] Eisenhart, *Riemannian Geometry*, § 47. The notation used by Eisenhart will be used throughout the paper.

§ 2. *Differential Operators.*

Generalising the operator, ∇, we define

$$\nabla = \sum_{h=1}^{n} e_h \, \mathbf{t}_{h|} \frac{\partial}{\partial s_h} \tag{2.1}$$

where $\mathbf{t}_{h|}$, $(h = 1, 2, \ldots, n)$ are the vectors of an orthogonal ennuple in V_n, $e_h = \mathbf{t}_{h|}{}^2 = \pm 1$, and $\partial f / \partial s_h$ is the intrinsic derivative of f in the direction $\mathbf{t}_{h|}$. From (1.4), using the usual notation for orthogonal ennuples, we have $\mathbf{t}_{h|} = \lambda_{h|}{}^i \, \mathbf{r}_i$, where $\lambda_{h|}{}^i$ are the contravariant components of the vectors in V_n. With this notation, we have $\partial / \partial s_h = \lambda_{h|}{}^i \, \partial / \partial x^i$; hence, using the equation

$$\sum_{h=1}^{n} e_h \, \lambda_{h|}{}^i \, \lambda_{h|}{}^j = g^{ij},$$

(2.1) becomes

$$\nabla = g^{ij} \, \mathbf{r}_i \, \frac{\partial}{\partial x^j}. \tag{2.2}$$

It is evident that this operator is independent of the ennuple chosen in the definition.

Operating on a scalar function, f, we get a vector $\nabla f = g^{ij} f_{,j} \, \mathbf{r}_i$ called the *gradient* of f. This vector is tangent to V_n, and is in the direction of critical variation of f, the magnitude being the variation.

Operating with closed product on a vector $\mathbf{R}$, we get a scalar, $\nabla \cdot \mathbf{R} = g^{ij} \mathbf{r}_i \cdot \mathbf{R}_{,j}$, called the *divergence* of $\mathbf{R}$. For $\mathbf{t} = \lambda^i \mathbf{r}_i$, we have, from (1.5),

$$\operatorname{div} \mathbf{t} = \lambda^i_{\ ,i}\,.$$

Operating with open product on a vector $\mathbf{R}$, we get a dyadic, $\nabla \mathbf{R} = g^{ij} \mathbf{r}_i \, \mathbf{R}_{,j}$.

It is easily shown that, if $\mathbf{s}$, $\mathbf{t}$ are unit vectors tangent to V_n at points of V_n, *the necessary and sufficient condition that the vectors* $\mathbf{s}$ *should be parallel in* V_n *along the curves of congruence defined by* $\mathbf{t}$, *is that* $\mathbf{t} \cdot \nabla \mathbf{s}$ *should be normal to* V_n. An equivalent condition is that $(\nabla \mathbf{s}) \cdot \mathbf{R}$ should be orthogonal to $\mathbf{t}$ for all vectors $\mathbf{R}$. In particular, $\mathbf{t}$ *defines a geodesic congruence if* $\mathbf{t} \cdot \nabla \mathbf{t}$ *is normal to* V_n.

If $\mathbf{t}_{h|}$ $(h = 1, 2, \ldots, n)$ are the vectors of an orthogonal ennuple in V_n, we find that the coefficients of rotation are given by

$$\gamma_{hkl} = \mathbf{t}_{l|} \cdot \nabla \, \mathbf{t}_{h|} \cdot \mathbf{t}_{k|}\,. \tag{2.3}$$

Hence, if we define *normal coefficients of rotation* by

$$I_{hk\sigma} = \mathbf{t}_{k\,|} \cdot \nabla\, \mathbf{t}_{h\,|} \cdot \mathbf{N}_{\sigma\,|} = \Omega_{\sigma\,|\,ij}\, \lambda_{h\,|}{}^{i} \lambda_{k\,|}{}^{j},$$

we have

$$\nabla\, \mathbf{t}_{h\,|} = \sum_{\theta,\,\phi} e_\theta\, e_\phi\, \gamma_{h\phi\theta}\, \mathbf{t}_{\theta\,|}\; \mathbf{t}_{\phi\,|} + \Sigma\, e_\theta\, e_\sigma\, I_{h\theta\sigma}\, \mathbf{t}_{\theta\,|}\; \mathbf{N}_{\sigma\,|}\,, \tag{2.4}$$

where $e_\theta = \mathbf{t}_{\theta\,|}{}^2 = \pm 1$; $e_\sigma = \mathbf{N}_{\sigma\,|}{}^2 = \pm 1$.

Prof. C. E. Weatherburn[1] has introduced an operator $\overline{\nabla}$, similar to ∇, in the study of a surface V_2. This can be generalised by considering some normal $\mathbf{N}$ of V_n, and defining

$$\overline{\nabla} = \sum_h e_h\, \kappa_h\, \mathbf{t}_{h\,|}\, \frac{\partial}{\partial s_h}, \tag{2.5}$$

where the ennuple $\mathbf{t}_{h\,|}$ is the principal ennuple for the normal $\mathbf{N}$, and κ_h are the corresponding principal curvatures. From the theory of principal directions, we have

$$\Sigma\, e_h\, \kappa_h\, \lambda_{h\,|}{}^{i}\, \lambda_{h\,|}{}^{j} = g^{il} g^{jm}\, \Omega_{lm} = \Omega^{ij},$$

where Ω_{ij} is the tensor associate to the normal $\mathbf{N}$.
Hence we have

$$\overline{\nabla} = \Omega^{ij}\, \mathbf{r}_i \frac{\partial}{\partial x^j}. \tag{2.51}$$

It can easily be verified that

$$\overline{\nabla} = -\,(\nabla\, \mathbf{N}) \cdot \nabla. \tag{2.52}$$

A second order operator may be defined by $\nabla^2 = \nabla \cdot \nabla$. For an invariant V, we have

$$\nabla^2\, V = g^{ij}\, V_{,\,ij}. \tag{2.6}$$

Thus ∇^2 is the Beltrami operator Δ_2.

We see that

$$\nabla^2\, \mathbf{r} = g^{ij}\, \mathbf{r}_{,\,ij} = M\mathbf{N}, \tag{2.7}$$

where $\mathbf{N}$ is the mean curvature normal[2], and M is the mean curvature of V_n. This shows that *the mean curvature normal, and the mean curvature are generalisations of the principal normal and curvature of a curve*, for we have, for a curve, $\nabla = \mathbf{t}\, d\,/\,ds$ where $\mathbf{t}$ is the unit tangent,

[1] *Quart. Journ. of Maths.*, **50** (1927), 277.

[2] *Cf.* Eisenhart, *loc. cit.*, p. 169.

and hence

$$\nabla^2 \mathbf{r} = \kappa \mathbf{n} \tag{2.71}$$

where $\mathbf{n}$ is the principle normal, and κ is the curvature.

Another second order operator is $\overline{\nabla} \cdot \nabla$. For an invariant V, we have

$$\overline{\nabla} \cdot \nabla V = \Omega^{ij} V_{,ij}. \tag{2.8}$$

§ 3. *The general small deformation.*

We shall now examine the space V'_n obtained by deforming V_n in V_h.

Let ϵ be a constant of the order of magnitude of the greatest displacement of points of V_n, and let the deformation be such that ϵ^2 may be neglected. Then the position vector of a point of V'_n is given by

$$\mathbf{r}' = \mathbf{r} + \epsilon \mathbf{s} \tag{3.1}$$

where $\epsilon\mathbf{s}$ is the displacement vector of the point $\mathbf{r}$, $\mathbf{s}$ being a finite function of position on V_n. Let dashes refer to V'_n.

We have at once

$$\mathbf{r}'_i = \mathbf{r}_i + \epsilon \mathbf{s}_i \tag{3.11}$$

and hence,

$$g'_{ij} = \mathbf{r}'_i \cdot \mathbf{r}'_j = g_{ij} + \epsilon c_{ij} \tag{3.12}$$

where

$$c_{ij} = \mathbf{r}_i \cdot \mathbf{s}_j + \mathbf{r}_j \cdot \mathbf{s}_i. \tag{3.13}$$

From (3.12) and the identities $g'_{ij}\, g'^{jk} = \delta_i^k$, we get

$$g'^{ij} = g^{ij} - \epsilon c^{ij} \tag{3.14}$$

where

$$c^{ij} = g^{il}\, g^{jm}\, c_{lm}.$$

From (3.12) we have

$$g' = | g'_{ij} | = g\,(1 + \epsilon g^{ij}\, c_{ij})$$

$$\textit{i.e.} \quad \sqrt{g'} = \sqrt{g}\,(1 + \epsilon\, \nabla \cdot \mathbf{s}). \tag{3.15}$$

If dV, dV' are corresponding elements of volume of V_n, V'_n respectively, the *dilation* is defined to be the ratio $(dV' - dV)/dV$. Hence, from (3.15), the dilation is given by

$$\frac{dV' - dV}{dV} = \epsilon\, \nabla \cdot \mathbf{s}, \tag{3.16}$$

i.e. the dilation is the divergence of the displacement vector.

Writing

$$2C_{ijk} = (c_{ij,k} + c_{ik,j} - c_{jk,i}); \quad C^h_{jk} = g^{ih} C_{ijk}, \tag{3.2}$$

we have

$$\Gamma'^h_{jk} = \Gamma^h_{jk} + \epsilon C^h_{jk} \tag{3.21}$$

where Γ^h_{jk}, Γ'^h_{jk} are the Christoffel symbols of the second kind. Hence the curvature tensor is given by

$$R'^h_{ijk} = R^h_{ijk} + \epsilon(C^h_{ik,j} - C^h_{ij,k}) \tag{3.22}$$

and from (3.12), we have

$$R'_{hijk} = R_{hijk} + \epsilon(c_{hl} R^l_{ijk} + C_{hik,j} - C_{hij,k}). \tag{3.23}$$

From this equation and (3.14), we get

$$R' = R - \epsilon(c^{ij} R_{ij} + c^{ij}{}_{,ij} - g^{ij} c_{,ij}) \tag{3.24}$$

where R_{ij} is the Ricci tensor, and $c = g^{ij} c_{ij} = 2 \nabla \cdot \mathbf{s}$.

Let $\mathbf{N}$ be a unit normal of V_n, and let $\mathbf{N}'$ be a corresponding unit normal of V'_n. We have $\mathbf{N}' \cdot \mathbf{r}'_i = 0$ $(i = 1, 2, \ldots, n)$, and writing $\mathbf{N}' = \mathbf{N} + \epsilon\overline{\mathbf{N}}$, we find

$$\overline{\mathbf{N}} = -(\nabla \mathbf{s}) \cdot \mathbf{N} \tag{3.3}$$

where $\overline{\mathbf{N}}$ is taken to be tangent[1] to V_n. Hence

$$\mathbf{N}' = \mathbf{N} - \epsilon(\nabla \mathbf{s}) \cdot \mathbf{N}. \tag{3.31}$$

If Ω_{ij} is the second fundamental tensor in V_n associate to the normal $\mathbf{N}$, and Ω'_{ij} the corresponding tensor for $\mathbf{N}'$, we have

$$\Omega'_{ij} = -\mathbf{N}'_{,i} \cdot \mathbf{r}'_j = \Omega_{ij} + \epsilon \mathbf{N} \cdot \mathbf{s}_{,ij}, \tag{3.32}$$

and hence, the mean curvature for the normal $\mathbf{N}'$ is given by

$$\Omega' = g'^{ij} \Omega'_{ij} = \Omega + \epsilon(\mathbf{N} \cdot \nabla^2 \mathbf{s} - 2\overline{\nabla} \cdot \mathbf{s}) \tag{3.33}$$

where $\overline{\nabla}$ is the operator given by the normal $\mathbf{N}$.

The linear element of V'_n is given by

$$eds'^2 = eds^2 + \epsilon c_{ij}\, dx^i\, dx^j; \quad eds^2 = g_{ij}\, dx^i\, dx^j. \tag{3.4}$$

Hence, the *extension* for the direction $\mathbf{t} = \lambda^i \mathbf{r}_i$ is given by

$$\epsilon E = \frac{ds' - ds}{ds} = \tfrac{1}{2} e\epsilon \lambda^i \lambda^j c_{ij} = e\epsilon\, \mathbf{t} \cdot \nabla\, \mathbf{s} \cdot \mathbf{t} \tag{3.41}$$

where $e = \mathbf{t}^2 = \pm 1$.

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[1] We need not take $\overline{\mathbf{N}}$ tangent to V_n, but we do so to define the particular normal $\mathbf{N}'$. All we actually know is that $\overline{\mathbf{N}}$ is orthogonal to $\mathbf{N}$, and satisfies $\overline{\mathbf{N}} \cdot \mathbf{r}_i + \mathbf{N} \cdot \mathbf{s}_i = 0$.

If ϵE_h $(h = 1, 2, \ldots, n)$ are the extensions for the directions of an orthogonal ennuple, we have $E_h = e_h \lambda_{h|}{}^i \lambda_{h|}{}^j \mathbf{r}_i \cdot \mathbf{s}_j$, and hence

$$\sum_h E_h = \nabla \cdot \mathbf{s}. \tag{3.42}$$

Thus *the sum of the extensions for n mutually orthogonal directions is independent of these directions and is equal to the dilation.*

From (3.41), we see that *the extension has critical values for the principal directions*[1] *determined by the tensor c_{ij}, and if ρ_h are the corresponding invariants, then $2E_h = \rho_h$.*

Writing

$$\begin{aligned} E_{hk} &= \lambda_{h|}{}^i \lambda_{k|}{}^j \mathbf{r}^i \cdot \mathbf{s}_j = \mathbf{t}_{k|} \cdot \nabla \mathbf{s} \cdot \mathbf{t}_{h|}, \\ \overline{E}_{h\sigma} &= \lambda_{h|}{}^i \mathbf{s}_i \cdot \mathbf{N}_{\sigma|} \quad = \mathbf{t}_{h|} \cdot \nabla \mathbf{s} \cdot \mathbf{N}_{\sigma|}, \end{aligned} \tag{3.43}$$

where $\mathbf{t}_{h|}$ are the vectors of any orthogonal ennuple, we have $E_{hh} = e_h E_h$, and

$$\nabla \mathbf{s} = \sum_{\theta, \phi} e_\theta e_\phi E_{\phi\theta} \mathbf{t}_{\theta|} \mathbf{t}_{\phi|} + \sum_{\theta, \sigma} e_\theta e_\sigma \overline{E}_{\theta\sigma} \mathbf{t}_{\theta|} \mathbf{N}_{\sigma|}. \tag{3.44}$$

From (3.11), it is easily shown that a direction $\mathbf{t}$ tangent to V_n becomes the direction $\mathbf{t}'$ tangent to V_n' where

$$\mathbf{t}' = \mathbf{t} + \epsilon(\mathbf{t} \cdot \nabla \mathbf{s} - E\mathbf{t}), \tag{3.5}$$

ϵE being the extension in the direction $\mathbf{t}$.

Hence, for two directions $\mathbf{t}_{1|}$, $\mathbf{t}_{2|}$ making an angle ω, the angle between the new directions is $\omega + \epsilon\,\theta$ where

$$\theta \sin \omega = \lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} - (E_1 + E_2) \cos \omega. \tag{3.51}$$

In particular, if $\omega = \pi/2$, we have

$$\theta = \lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} = \mathbf{t}_{1|} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \mathbf{t}_{2|}, \tag{3.52}$$

where $\mathbf{s}\nabla$ is the dyadic conjugate to $\nabla \mathbf{s}$, and hence, two orthogonal directions remain orthogonal if they satisfy

$$\lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} = 0. \tag{3.53}$$

From this condition, we see that *if two directions are orthogonal, and if one of them is a principal direction of c_{ij}, the directions remain orthogonal.*

Also *the only orthogonal ennuple remaining orthogonal is the principal ennuple of c_{ij}.*

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[1] An account of the principal directions of a tensor is given by Eisenhart, *loc. cit.* § 33.

If the principal ennuple of c_{ij} is also the principal ennuple given by a normal $\mathbf{N}$, it becomes the principal ennuple in V_n' of the normal $\mathbf{N}'$ if

$$(\Omega'_{ij} - \kappa'_h g'_{ij}) \lambda_{h|}{}^i = 0. \tag{3.54}$$

Writing $\kappa'_h = \kappa_h + \epsilon\bar{\kappa}_h$, and substituting from (3.12), (3.32), these conditions become

$$\{(2\kappa_h E_h + \bar{\kappa}_h) g_{ij} - k_{ij}\} \lambda_{h|}{}^i = 0 \tag{3.55}$$

where $k_{ij} = \mathbf{N} \cdot \mathbf{s}_{,ij}$. Hence the ennuple must also be the principal ennuple of the tensor k_{ij}, and if ρ_h are the principal invariants for this tensor, the principal curvatures for the normal $\mathbf{N}'$ are $\kappa_h + \epsilon\kappa_h$ where

$$\bar{\kappa}_h = \rho_h - 2\kappa_h E_h. \tag{3.56}$$

Let us now find the conditions that a geodesic congruence λ^i in V_n becomes geodesic in V_n'. We have

$$\lambda^i \mathbf{r}_i \rightarrow \lambda'^i \mathbf{r}'_i, \quad \lambda'^i = (1 - \epsilon E)\lambda^i, \quad E = e\lambda^i \lambda^j c_{ij}. \tag{3.6}$$

Differentiating covariantly with respect to g'_{ij}, and substituting $\lambda^i{}_{,j} \lambda^j = 0$ in V_n, we find

$$\lambda'^i{}_{,j} \lambda'^j = \epsilon\lambda^j \lambda^k (C^i_{jk} - 2e\lambda^i \lambda^l C_{ljk}). \tag{3.61}$$

Hence the congruence remains geodesic if

$$\lambda^j \lambda^k (C^i_{jk} - 2e\lambda^i \lambda^l C_{ljk}) = 0. \tag{3.62}$$

Multiplying by λ_i and summing, we get

$$\lambda^l \lambda^j \lambda^k C_{ljk} = 0. \tag{3.63}$$

Substituting in (3.62), we have *the necessary and sufficient conditions that the geodesic congruence* λ^i *should remain geodesic are*

$$\lambda^j \lambda^k C^i_{jk} = 0, \quad (i = 1, 2, \ldots, n). \tag{3.64}$$

We at once see that *the necessary and sufficient conditions that all geodesics of* V_n *should become geodesics of* V_n' *are*

$$c_{ij,k} = 0 \quad (i, j, k = 1, 2, \ldots, n). \tag{3.65}$$

A more general theorem is as follows.

If the vectors μ^i *are parallel along the curves of the congruence* λ^i *in* V_n, *the corresponding vectors are parallel along the corresponding curves in* V_n' *if*

$$\lambda^j \mu^k C^i_{jk} = 0. \tag{3.66}$$

The differential equations (3.65) have been studied by Eisenhart[1] and Levy[2]. A particular result is that when V_n has constant Riemannian curvature, the tensor c_{ij} must be a constant multiple of the fundamental tensor g_{ij}. In this case V_n, V_n' are conformal, and the extension is constant in all directions and at all points of V_n, being $\epsilon\rho$ where $c_{ij} = \rho\, g_{ij}$.

§ 4. *Some particular types of deformation.*

An *inextensible deformation* is such that all lengths remain unaltered. For this, we must have $g'_{ij} = g_{ij}$. Hence, the necessary and sufficient conditions for an inextensible deformation are

$$c_{ij} = 0 \quad (i, j = 1, 2, \ldots, n). \tag{4.1}$$

In this case, we have $\nabla \cdot \mathbf{s} = 0$, $\overline{\nabla} \cdot \mathbf{s} = 0$, and (3.33) reduces to

$$\Omega' = \Omega + \epsilon\, \mathbf{N} \cdot \nabla^2 \mathbf{s}. \tag{4.11}$$

From the definition, the curvature tensors remain unaltered.

A *normal deformation*, is such that all points of V_n are displaced in directions normal to V_n. If $\mathbf{N}$ is the normal direction of displacement of the point $\mathbf{r}$, the deformation is given by

$$\mathbf{s} = s\, \mathbf{N} \tag{4.2}$$

where s is a function of position of V_n.

If Ω_{ij} is the tensor associate to $\mathbf{N}$, and Ω the corresponding mean curvature, we have

$$c_{ij} = -\, 2s\, \Omega_{ij}, \tag{4.21}$$

and

$$\nabla \cdot \mathbf{s} = -\, s\, \Omega. \tag{4.22}$$

The normal to V_n' corresponding to $\mathbf{N}$ is now

$$\mathbf{N}' = \mathbf{N} - \epsilon \nabla s. \tag{4.23}$$

If $p = n + 1$, (3.24), (3.32), and (3.33) reduce to

$$R' = R + 2\epsilon\, (s\, \Omega^{ij} R_{ij} + \overline{\nabla} \cdot \nabla s - \Omega\, \nabla^2 s), \tag{4.24}$$

$$\Omega'_{ij} = \Omega_{ij} + \epsilon\, \{s_{,\,ij} - s\, (\Omega\, \Omega_{ij} + eR_{ij})\}, \tag{4.25}$$

$$\Omega' = \Omega + \epsilon\, \{\nabla^2 s + s\, (\Omega^2 + eR)\}, \tag{4.26}$$

where $e = \mathbf{N}^2 = \pm 1$.

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[1] *Trans. of the Amer. Math. Soc.*, 25 (1923), 297.

[2] *Annals of Math.*, 27 (1926), 91.

The only orthogonal ennuple remaining orthogonal for a normal deformation is now the principal ennuple of the normal $\mathbf{N}$, and substituting for k_{ij} in (3.55), we find that this becomes the principal ennuple of $\mathbf{N}'$ if it is also the ennuple given by the tensor $s_{,ij} - \sum_{\sigma} e_{\sigma}\, \mu_{\sigma\,|\,i}\, \mu_{\sigma\,|\,j}$ where $\mu_{\sigma\,|\,i} = \mathbf{N} \cdot \mathbf{N}_{\sigma\,|\,i}$, $\mathbf{N}_{\sigma\,|}$ ($\sigma = 1, 2, \ldots, p - n - 1$) being orthogonal to $\mathbf{N}$, and $e_{\sigma} = \mathbf{N}_{\sigma\,|}{}^{2} = \pm 1$. Also, if ρ_h are the invariants of this tensor, the principal curvatures κ_h of $\mathbf{N}$ become $\kappa_h + \epsilon\bar{\kappa}_h$ where

$$\bar{\kappa}_h = \rho_h + s\kappa_h^2. \tag{4.27}$$

A *tangent deformation* is such that all points of V_n are displaced in directions tangent to V_n. Writing

$$\mathbf{s} = \lambda^i\, \mathbf{r}_i \tag{4.3}$$

we have

$$\mathbf{r}' = \mathbf{r} + \epsilon\lambda^i\, \mathbf{r}_i = \mathbf{r}\,(x^i + \epsilon\lambda^i) \tag{4.31}$$

to the first order of approximation.
Hence this *tangent deformation is equivalent to a point transformation of V_n, given by*

$$x'^i - x^i = \epsilon\lambda^i. \tag{4.32}$$

Tangent deformations have been discussed intrinsically from this point of view by McConnell.[1]

In concluding, we may remark that, writing δt for ϵ, and considering $\mathbf{s}$ also as a function of the parameter t, the spaces V_n, V_n' may be considered as members of a family of such spaces in V_p, *i.e.* hypersurfaces of a V_{n+1}. Many of the above results may then be interpreted as giving the variation with respect to t of the tensors, etc., connected with V_n.

[1] *Annali di Mat.*, 6 (1928-1929), 207.

4 enclosures

THE THEORY OF RELATIVE CO-ORDINATES

in

RIEMANNIAN GEOMETRY

by

A. G. WALKER

Thesis for the Degree of Ph. D. 1933

C O N T E N T S.

Part III.

Spatial Distance in General Relativity.

Introduction.

The theory of relative co-ordinates was first introduced by the author (1), but only a few of the results of Part I of this work were given. The theory was afterwards applied to a study of distance in relativity (2), the method outlined being fundamentally that used in Part III. In the following work, the notation of the above two papers has been considerably altered so that the same notation can be used consistently throughout this work. With very few exceptions, symbols will retain their meanings in order to avoid continual references to previous definitions.

In the study of twisted curves in ordinary Euclidean space, a very useful theory is that of moving axes.[x] At a point of a given curve, a set of axes is formed by the tangent, principal normal and binormal of the curve, so, as the point moves along the curve, we have a set of axes moving in space. Any curves or surfaces associated with the given curve can now be referred to these moving axes: e.g., a curve is described by a point with given co-ordinates relative to these axes: thus, the locus of centres of curvature of the given curve is described by the point $(0, \rho, 0)$ where ρ is the radius of curvature.

The object of the following work is to develope systems of reference similar to the above at points of a curve in a general Riemannian space V_n. The first difficulty is

x Eisenhart, (3), Chap. I, §16.

to find the most convenient system of reference at a given point, and to overcome this, we shall use the theory of normal co-ordinates (§3). The second difficulty is to decide how the system of reference at one point of the given curve is to be related to the system of reference at any other point. This leads us to examine the theory of transport along the curve and this will be discussed at length in §2.

Part I will be devoted entirely to the geometrical developement and application of the theory of relative co-ordinates. We believe that these co-ordinates have a purely geometrical interest, and several methods and results will be discussed at length on this assumption. With their aid, we shall give short proofs of several known results and we shall also state and prove some new theorems.

Part II is concerned with the study of motion in relativity. From the theory of observation in relativity, it will be seen that relative co-ordinates play a very important part, and observable quantities are naturally expressed in terms of these co-ordinates.

In Part III, relative co-ordinates play a secondary part. The problem of defining distance in relativity has been greatly discussed recently, and we shall not attempt to criticise the work already published on the subject. We shall be mainly concerned with two of the definitions of distance and with the aid of some of the results of ~~chap~~ PART I, we shall formulate these definitions more naturally and give a practical method of calculating the results in a given space-time. To demonstrate the advantage of this method we shall find the formulae for distance in a very general form of space-time, but the subject of this work is not concerned with a detailed discussion of these results.

PART I.

GEOMETRICAL THEORY.

1. Orthogonal n-uples.

Let V_n be a Riemannian space of n dimensions, co-ordinates $(x^0, x^1, \cdots, x^{n-1})$. Then at each point of V_n there exist $\infty^{n(n-1)/2}$ sets of n mutually orthogonal non-null vectors each of unit length. Such a set of vectors is called an <u>orthogonal n-uple</u>, or, for short, an <u>n-uple</u>, and the contravariant components of these vectors are usually written $\lambda_{\sigma|}{}^i$, where the subscript before the vertical bar indicates the particular vector of the n-uple, and the superscript indicates the particular ~~vector~~ component of this vector.[x] From the definition of an n-uple, these components must satisfy the relations

$$g_{ij}\lambda_{\sigma|}{}^i\lambda_{\nu|}{}^j = \delta_{\sigma\nu}, \qquad (\sigma, \nu = 0, 1, \cdots, n-1), \tag{1.1}$$

where g_{ij} is the fundamental tensor of V_n, and the δ's are defined by

$$\delta_{\sigma\nu} = 0, \qquad \delta_{\sigma\sigma} = e_\sigma = \pm 1. \quad (\sigma \neq \nu;\ \sigma, \nu = 0, 1, \cdots, n-1). \tag{1.2}$$

If the above components are functions of the x's and satisfy these conditions for all values of the x's we have an n-uple at each point of V_n.

Given an n-uple $\lambda_{\sigma|}{}^i$, a new n-uple $\lambda'_{\sigma|}{}^i$ can be defined by equations of the form

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[x] The notation, unless otherwise stated, is that used by <u>Eisenhart</u>, (4), and several of his results will be quoted without giving explicit references. An account of orthogonal n-uples is given in chap. III.

$$\lambda'_{\sigma|}{}^{i} = \sum_{\nu=0}^{n-1} \xi_{\sigma}^{\nu} \lambda_{\nu|}{}^{i} \quad , \qquad (1.3)$$

where the ξ's satisfy the relations

$$\sum_{\lambda,\mu} \delta_{\lambda\mu} \xi_{\sigma}^{\lambda} \xi_{\nu}^{\mu} = \delta'_{\sigma\nu} \quad , \qquad (\sigma, \nu = 0, 1, \cdots, n-1) \qquad (1.4)$$

and $\delta'_{\sigma\nu}$ is defined as in (1.2), with e'_{σ} for e_{σ}. The ξ's may be called the <u>coefficients of an orthogonal transformation in V_n</u>.

If an n-uple $\lambda_{\sigma|}{}^{i}$ is given at points of V_n, then to a tensor $T_{i_1 i_2 \cdots}^{j_1 j_2 \cdots}$ there corresponds a set of invariants defined by

$$t_{\sigma_1 \sigma_2 \cdots \nu_1 \nu_2 \cdots} = T_{i_1 i_2 \cdots}^{j_1 j_2 \cdots} \lambda_{\sigma_1|}{}^{i_1} \lambda_{\sigma_2|}{}^{i_2} \cdots \lambda_{\nu_1|j_1} \lambda_{\nu_2|j_2} \cdots \quad . \qquad (1.5)$$

It can be shown that the conditions (1.1) are equivalent to the conditions

$$\sum_{\sigma=0}^{n-1} e_{\sigma} \lambda_{\sigma|}{}^{i} \lambda_{\sigma|}{}^{j} = g^{ij} \quad , \qquad (i, j = 0, 1, \cdots, n-1) \qquad (1.6)$$

and, using these relations, we find, from (1.5)

$$T_{i_1 i_2 \cdots}^{j_1 j_2 \cdots} = \sum_{\sigma \cdots \nu \cdots} e_{\sigma_1} e_{\sigma_2} \cdots e_{\nu_1} e_{\nu_2} \cdots t_{\sigma_1 \sigma_2 \cdots \nu_1 \nu_2 \cdots} \lambda_{\sigma_1|i_1} \lambda_{\sigma_2|i_2} \cdots \lambda_{\nu_1|}{}^{j_1} \lambda_{\nu_2|}{}^{j_2} \cdots \quad . \qquad (1.7)$$

We shall often be dealing with invariants of the form given in (1.5), i.e., invariants associated with a given n-uple, and for this reason, we shall introduce what we consider to be a natural and useful notation, a notation that will enable us to use the standard summation convention when dealing with such invariants.

It will be observed that when a tensor is obtained from a set of n-uplet invariants, the terms of the sum always include the unit e's, given by the n-uple. Let us therefore make the following conventions:

(a) When an invariant is formed as in (1.5), write the n-uplet suffixes as subscripts.

(b) Let an n-uplet subscript of an invariant be raised to become a superscript when the invariant is multiplied by the corresponding unit e, and let a superscript be lowered in the same way. Thus,

$$e_\sigma t_{..\sigma..} = t_{..}{}^{\sigma}{}_{..} \quad ; \quad e_\sigma t_{..}{}^{\sigma}{}_{..} = t_{..\sigma..} \quad . \tag{1.8}$$

We observe that these two operations are consistent, for $(e_\sigma)^2 = 1$.

(c) If an n-uplet index occurs in any term as a subscript and superscript, let this indicate summation, unless the index is attached to a unit e. Thus, (1.7) can be written

$$T_{i_1 i_2 ..}^{j_1 j_2 ..} = t^{\sigma_1 \sigma_2 .. \nu_1 \nu_2 ..} \lambda_{\sigma_1 | i_1} \lambda_{\sigma_2 | i_2} .. \lambda_{\nu_1 |}{}^{j_1} \lambda_{\nu_2 |}{}^{j_2} .. \quad . \tag{1.9}$$

The exception should be noted, e.g., there is no summation in the second equation of (1.8).

From (1.1), we see that $\delta_{\sigma\nu}$ are the n-uplet invariants associated with the tensor g_{ij}. Writing $\delta^\sigma_\nu = e_\sigma \delta_{\sigma\nu}$, then, from (1.2), δ^σ_ν is the Kronecker delta, i.e., $\delta^\sigma_\nu = 1, 0$ according as σ is or is not equal to ν. Also, $\delta^{\sigma\nu} = e_\sigma e_\nu \delta_{\sigma\nu}$, and hence, $\delta^{\sigma\nu}$ has the same value as $\delta_{\sigma\nu}$. (1.6) can therefore be written

$$\delta^{\sigma\nu} \lambda_{\sigma |}{}^{i} \lambda_{\nu |}{}^{j} = g^{ij} \quad . \tag{1.10}$$

2. <u>N-uples at points of a curve.</u>

Let C be a curve in V_n, given by the equations

$$x^i = x^i(s) \quad , \quad (i = 0, 1, \cdots, n-1) , \tag{2.1}$$

where s is the arc, or a parameter if C is null, and let $\lambda_{\sigma|}{}^i$ be an n-uple at points of C. Then the components are functions of s, satisfying (1.1) for all values of s. The vectors of the n-uple may be given explicitly at each point of C, or they may be obtained by some given law of transport along C from an n-uple given at a particular point of C. An example of the former is obtained when the n-uple is formed by the tangent and principal normals of C, and an example of the latter when the n-uple is displaced by parallel transport[x] along C. In the second general case, the vectors are solutions of transport equations of the form

$$dV^i + C^i_{jk} V^j dx^k = 0 \qquad , \qquad C^i_{jk} = \Gamma^i_{jk} + A^i_{jk} \quad , \tag{2.2}$$

where Γ^i_{jk} is a Christoffel symbol of the second kind, and A^i_{jk} is the component of a tensor. It can be shown that this case includes the first general case, for if the vectors are given explicitly as functions of s, let invariants $\gamma_{\sigma\nu}$ be defined by

$$\gamma_{\sigma\nu} = -\lambda_{\sigma|l}\left(\frac{d\lambda_{\nu|}{}^l}{ds} + \Gamma^l_{jk}\lambda_{\nu|}{}^j\frac{dx^k}{ds}\right) \quad , \quad (\sigma, \nu = 0, 1, \cdots, n-1) . \tag{2.3}$$

Then, multiplying by $e_\sigma \lambda_{\sigma|}{}^i$ and summing for σ, we have, from (1.6),

$$\frac{d\lambda_{\nu|}{}^i}{ds} + \Gamma^i_{jk}\lambda_{\nu|}{}^j\frac{dx^k}{ds} = -\gamma^\sigma{}_\nu \lambda_{\sigma|}{}^i = -(\gamma^{\sigma\mu}\lambda_{\sigma|}{}^i\lambda_{\mu|j})\lambda_{\nu|}{}^j ,$$

where $\gamma^\sigma{}_\nu = e_\sigma \gamma_{\sigma\nu}$, $\gamma^{\sigma\mu} = e_\sigma e_\mu \gamma_{\sigma\mu}$. Hence, the vectors $\lambda_{\nu|}{}^i$ are

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[x] By ~~parallel transport~~, we shall always mean Levi-Civita parallel transport.

solutions of (2.2) where

$$A_{jk}^{i}\dot{x}^{k}=\gamma^{\sigma\mu}\lambda_{\sigma|}{}^{i}\lambda_{\mu|j}\quad,\quad \dot{x}^{k}=\frac{dx^{k}}{ds}. \tag{2.4}$$

Thus, when dealing with an n-uple at points of C, we shall consider the vectors as solutions of equations of the form (2.2). The law of transport is given by the tensor A_{jk}^{i}, but in our work, we shall not so much consider this tensor as the invariants $\gamma_{\sigma\nu}$ defined above. From (2.4), we see that, given A_{jk}^{i}, the invariants are given by

$$\gamma_{\sigma\nu}=A_{jk}^{i}\lambda_{\sigma|i}\lambda_{\nu|}{}^{j}\dot{x}^{k}. \tag{2.5}$$

By differentiating (1.1), it can be seen at once from (2.3) that the invariants satisfy

$$\gamma_{\sigma\nu}+\gamma_{\nu\sigma}=0\quad,\qquad (\sigma,\nu=0,1,\cdots,n-1). \tag{2.6}$$

Although it has only been assumed that the vectors are defined at points of C, we can, from (2.3), write, for convenience

$$\gamma_{\sigma\nu}=-\gamma_{\nu\sigma}=\lambda_{\sigma|}{}^{i}{}_{,k}\lambda_{\nu|i}\dot{x}^{k} \tag{2.7}$$

where the comma denotes covariant differentiation. The vector $\lambda_{\sigma|}{}^{i}{}_{,k}\dot{x}^{k}$ can be considered as <u>the derivative of $\lambda_{\sigma|}{}^{i}$ along C</u>, and may be written $\frac{\Delta\lambda_{\sigma|}{}^{i}}{ds}$, or simply $\dot{\lambda}_{\sigma|}{}^{i}$, when there is no ambiguity. Thus, $\gamma_{\sigma\nu}=\dot{\lambda}_{\sigma|}{}^{i}\lambda_{\nu|i}$.

If the equations (2.2) give an n-uple at points of C, as required, the tensor A_{jk}^{i} must satisfy certain conditions, arising from the fact that the magnitude of a vector and the angle between two vectors must be conserved during the transport. These conditions are equivalent to saying that if U^i, V^i are any two solutions, the expression $g_{ij}U^iV^j$ must be constant along the curve. Differentiating along C, we must have

$$g_{ij}(\dot{U}^iV^j + U^i\dot{V}^j) = 0$$

i.e. substituting for $\dot{U}^i, \dot{V}^j$ from (2.2),

$$(g_{ij}A^i_{\ell k} + g_{i\ell}A^i_{jk})U^jV^\ell\dot{x}^k = 0 .$$

This equation must be satisfied for all vectors U^i, V^i, and hence A^i_{jk} must satisfy

$$(g_{ij}A^i_{\ell k} + g_{i\ell}A^i_{jk})\dot{x}^k = 0 \qquad (j,\ell = 0,1,\cdots,n-1) \qquad (2.8)$$

From (2.5), we see that these conditions are equivalent to (2.6). If the tensor satisfies these conditions, then, given an n-uple at a particular point of C, the vectors of this n-uple, when displaced by the transport (2.2), give an n-uple at each point of C.

If the n-uple need satisfy no further conditions, the most simple solution of (2.8) is $A^i_{jk} = 0$, giving parallel transport along C. It is often required, however, that the n-uple should satisfy certain conditions. The most common condition is that at each point of C a vector of the n-uple should be tangent to C. It is evident that the further condition for this is that the tangent vector at one point of C must be transported to become the tangent vector at any other point, i.e. the vector $\dot{x}^i$ must be a solution of (2.2).
Substituting $\dot{x}^i$ in (2.2), we get

$$-A^i_{jk}\dot{x}^j\dot{x}^k = \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \eta^i \qquad , \qquad (2.9)$$

η^i being the curvature vector of the curve. The most simple solution of the above two sets of equations is easily found to be

$$A^i_{jk}\dot{x}^k = e\,g_{jk}(\dot{x}^i\eta^k - \dot{x}^k\eta^i) \quad , \quad e = g_{ij}\dot{x}^i\dot{x}^j = \pm 1 \tag{2.10}$$

where it is assumed that C is not null. With this transport, a vector V^i orthogonal to C remains orthogonal to C, and satisfies the equations

$$\frac{dV^i}{ds} + \Gamma^i_{jk}V^j\dot{x}^k + e(g_{jk}V^j\eta^k)\dot{x}^i = 0 \tag{2.11}$$

i.e. Fermi's equations[x1] for transporting a vector orthogonal to C. Thus, the transport given by (2.10) reduces to Fermi transport in the case of a vector orthogonal to C, and, as Fermi does not consider the behaviour of other vectors, we shall call the transport given by (2.10) <u>Fermi transport</u>. We now have $\lambda_{01}{}^i = \dot{x}^i$, and hence, by (2.5), the invariants $\gamma_{\sigma\nu}$ are given by

$$\gamma_{\sigma\nu} = \delta^0_\sigma v_\nu - \delta^0_\nu v_\sigma \quad . \quad v_\sigma = \eta^i\lambda_{\sigma|i}\,. \tag{2.12}$$

We observe that when C is a geodesic, Fermi transport becomes parallel transport.

Many other interesting laws of transport can be found by the above method, but these will not be used in this work. There is, however, a set of n-uples of some geometrical interest, these n-uples being formed by the tangent and principal normals[x2] of C at each point of C. For such an n-uple,

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[x1] <u>Fermi,</u>(5).

[x2] <u>Eisenhart,</u>(4), §32.

we have $\lambda_{0|}^{i} = \dot{x}^{i}$, and the Serret-Frenet formulae give

$$\dot{\lambda}_{\sigma|}^{i} = -e_{\sigma-1} K_{\sigma-1} \lambda_{\sigma-1|}^{i} + e_{\sigma+1} K_{\sigma} \lambda_{\sigma+1|}^{i}, \quad (\sigma = 0, 1, \cdots, n-1)$$

where $K_{-1} = K_{n-1} = 0$, and $K_0, K_1, \cdots, K_{n-2}$ are the principal curvatures of C.

Hence, from (2.7), the non-vanishing invariants are

$$\gamma_{\sigma.\sigma+1} = -\gamma_{\sigma+1.\sigma} = K_\sigma, \quad (\sigma = 0, 1, \cdots, n-2) \qquad (2.19)$$

3. Relative co-ordinates.

Given a point P of V_n, co-ordinates $(y^0, y^1, \cdots, y^{n-1})$ can be found such that $y^i = 0$ at P, and the geodisics through P are given by $y^i = \xi^i s$, where s is the arc, and the ξ's are constants, being the components of a unit vector at P. Such co-ordinates are called Riemannian co-ordinates, and, with the same point P as origin, any other system of Riemannian co-ordinates is given by a linear transformation with constant coefficients.[x] In particular, for a given n-uple at P, we can find a system of Riemannian co-ordinates such that the parametric directions at P are the directions of the vectors of the n-uple. These co-ordinates are called normal co-ordinates, and are completely determined by the given n-uple. From the definition, it is evident that normal co-ordinates in a Riemannian space correspond to cartesian co-ordinates in Euclidean space.

Let P be the point $s = s_0$ of the curve C defined by (2.1),

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[x] Eisenhart, (4), §18.

and let $\lambda_{\sigma|}^{\;i}$ be an n-uple given at points of C, the n-uple at P being $(\lambda_{\sigma|}^{\;i})_0$. Then, with P as origin we can find normal co-ordinates $(z^0, z^1, \cdots, z^{n-1})$ having the parametric directions $(\lambda_{\sigma|}^{\;i})_0$ at P. If the subscript 0 indicates values at P, then with P as origin, a system of Riemannian co-ordinates is given by the transformation

$$X^i = (x^i)_0 + y^i - \frac{1}{2!}(\Gamma^i_{\alpha\beta})_0 y^\alpha y^\beta - \frac{1}{3!}(\Gamma^i_{\alpha\beta\gamma})_0 y^\alpha y^\beta y^\gamma - \cdots \qquad (3.1)$$

where the coefficients are certain well-known functions of the Christoffel symbols and their derivatives. The required normal co-ordinates are now evidently given by $y^\alpha = (\lambda'^{\;\alpha}_{\sigma|})_0 z^\sigma$ where $(\lambda'^{\;\alpha}_{\sigma|})_0$ are the components in the y's of the vectors $(\lambda_{\sigma|}^{\;i})_0$. We have $(\lambda'^{\;\alpha}_{\sigma|})_0 = \left(\lambda_{\sigma|}^{\;i} \frac{\partial y^\alpha}{\partial X^i}\right)_0$, and from (3.1), $\left(\frac{\partial y^\alpha}{\partial X^i}\right)_0 = \delta^\alpha_i$, giving $(\lambda'^{\;\alpha}_{\sigma|})_0 = (\lambda_{\sigma|}^{\;\alpha})_0$. Hence, the normal co-ordinates are given by (3.1) and

$$y^\alpha = (\lambda_{\sigma|}^{\;\alpha})_0 z^\sigma. \qquad (3.2)$$

Dropping the subscript 0, the above equations give a system of normal co-ordinates at each point P(s) of C. These co-ordinates, which we shall call <u>relative co-ordinates</u>, give a direct generalisation of the theory of moving axes along a curve in Euclidean space. We observe that the coefficients in (3.1) are now functions of $x^i(s)$ and we can write

$$X^i = \xi^i(x, y) \qquad (3.3)$$

where

$$\xi^i(x,y) = x^i + y^i - \frac{1}{2!}\Gamma^i_{\alpha\beta} y^\alpha y^\beta - \frac{1}{3!}\Gamma^i_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma - \cdots \qquad (3.4)$$

The relative co-ordinates are thus given by (3.3) and

$$\gamma^\alpha = \lambda_{\sigma|}{}^\alpha z^\sigma \quad . \tag{3.5}$$

The components $\lambda_{\sigma|}{}^\alpha$ are functions only of s, so we can write

$$x^i = F^i(s,z) \tag{3.6}$$

where $F^i(s,z)$ is given by (3.4) and (3.5). We shall say that Q is the point (z) referred to P(s) when Q has co-ordinates $(z^0, z^1, \cdots, z^{n-1})$ in the system of relative co-ordinates with P as origin.

4. Fundamental Formulae.

Let Q be the point (z) referred to P(s), and let Q' be the point $(z+dz)$ referred to $P'(s+ds)$, and $(z+\delta z)$ referred to P. We proceed to find the relation between δz and dz, where ds and dz are small. If Q' is the point $(x'^0, x'^1, \cdots, x'^{n-1})$ in the co-ordinates of V_n, we have from (3.6),

$$x'^i = F^i(s+ds, z+dz) = F^i(s, z+\delta z).$$

Expanding to the first order, and writing F^i for $F^i(s,z)$, we get

$$F^i + \frac{\partial F^i}{\partial s} ds + \frac{\partial F^i}{\partial z^\sigma} dz^\sigma = F^i + \frac{\partial F^i}{\partial z^\sigma} \delta z^\sigma ,$$

i.e., writing $Dz^\sigma = \delta z^\sigma - dz^\sigma$,

$$\frac{\partial F^i}{\partial z^\sigma} \cdot \frac{Dz^\sigma}{ds} = \frac{\partial F^i}{\partial s} .$$

From (3.4) and (3.5)

$$\frac{\partial F^i}{\partial z^\sigma} = \xi^i{}_\alpha \lambda_{\sigma|}{}^\alpha \tag{4.1}$$

where $\xi^i{}_\alpha = \frac{\partial \xi^i}{\partial \gamma^\alpha}$, and also

$$\frac{\partial F^i}{\partial s} = \frac{\partial \xi^i}{\partial x^k} \dot{x}^k + \xi^i{}_\alpha \frac{d}{ds}(\lambda_{\sigma|}{}^\alpha) z^\sigma \quad . \tag{4.2}$$

i.e. from (2.2)

$$\frac{\partial F^i}{\partial s} = \left(\frac{\partial \xi^i_\alpha}{\partial x^k} - \xi^i_\alpha C^\alpha_{jk} \gamma^j\right) \dot{x}^k \qquad (4.3)$$

If $\bar{\xi}^\alpha_i$ is the normalized co-factor of ξ^i_α i.e. is given by $\bar{\xi}^\alpha_i \xi^i_\beta = \delta^\alpha_\beta$, then, substituting (4.2) and (4.3) in (4.1) and multiplying by $\bar{\xi}^\alpha_i$,

$$\lambda_{\sigma|}{}^\alpha \frac{Dz^\sigma}{ds} = \left(\bar{\xi}^\alpha_i \frac{\partial \xi^i}{\partial x^k} - C^\alpha_{jk} \gamma^j\right) \dot{x}^k$$

Changing the dummy suffixes, and multiplying both sides by $\lambda_{\sigma|\alpha}$, we get

$$\frac{Dz^\sigma}{ds} = e_\sigma \left(\bar{\xi}^i_j \frac{\partial \xi^j}{\partial x^k} - C^i_{jk} \gamma^j\right) \lambda_{\sigma|i} \dot{x}^k . \qquad (4.4)$$

By (3.4), the expression in brackets can be expanded as a series in the y's, so, substituting from (3.5), $\frac{Dz^\sigma}{ds}$ can be expressed as a series in the z's.

From (2.2), the expression on the right of (4.4) can be written

$$-e_\sigma A^i_{jk} \lambda_{\sigma|i} \gamma^\alpha \dot{x}^k + e_\sigma \left(\bar{\xi}^i_j \frac{\partial \xi^i}{\partial x^k} - \Gamma^i_{jk} \gamma^j\right) \lambda_{\sigma|i} \dot{x}^k \qquad (4.5)$$

and, from the nature of relative co-ordinates, this expression does not depend on the particular co-ordinate system (X) of V_n. Hence, when expanded as a series in the z's, the coefficients must be invariants in V_n. This is already satisfied by the first term of the expression, A^i_{jk} being a tensor, and so the same must be true of the remaining part. Hence, from the form of this part of the expression, we have at once that <u>when</u>

$$\bar{\xi}^i_j \frac{\partial \xi^j}{\partial x^k} - \Gamma^i_{jk} \gamma^j \qquad (4.6)$$

<u>is expanded by (3.4) as a series in the y's, the coefficients, when X^i is written for x^i, are components of tensors in V_n.</u>

This is an unusual type of expression for generating a sequence of tensors, and an independent proof of the above property should be of some interest in the theory of tensors.

The first four tensors of the above sequence can easily be found, but after this stage, the calculation becomes very complicated. Writing the expansion in the form

$$\bar{f}^i_j \frac{\partial f^j}{\partial x^k} - \Gamma^i_{jk}\zeta^j = T^i_k + T^i_{\alpha k}\zeta^\alpha + \frac{1}{2!}T^i_{\alpha\beta k}\zeta^\alpha\zeta^\beta + \ldots + \frac{1}{m!}T^i_{\alpha\beta\ldots\mu k}\zeta^\alpha\zeta^\beta\ldots\zeta^\mu + \ldots, \qquad (4.7)$$

where $T^i_{\alpha\beta\ldots\mu k}$ are the components of a tensor symmetric in $\alpha, \beta, \ldots, \mu$, then to evaluate these tensors we can choose any convenient system of co-ordinates in V_n providing the expressions so obtained are written in tensor form. Let us therefore evaluate the tensors at some given point P, the co-ordinates of V_n being Riemannian with P as origin. In this system of co-ordinates, we have

$$\Gamma^i_{\alpha\beta} = 0 \quad , \ \Gamma^i_{\alpha\beta\gamma} = 0 \ , \ \ldots , \quad \Gamma^i_{\alpha\beta\ldots\mu} = 0 \ , \ \ldots , \qquad (4.8)$$

at P, and hence, from (3.4),

$$f^i_\alpha = \delta^i_\alpha \quad , \ \bar{f}^\alpha_i = \delta^\alpha_i \ ,$$

and

$$\frac{\partial f^i}{\partial x^k} = \delta^i_k - \frac{1}{2!}\Gamma^i_{\alpha\beta . k}\zeta^\alpha\zeta^\beta - \frac{1}{3!}\Gamma^i_{\alpha\beta\gamma . k}\zeta^\alpha\zeta^\beta\zeta^\gamma - \ldots \quad , \quad \Gamma^i_{\alpha\beta\ldots\mu . k} = \frac{\partial}{\partial x^k}\Gamma^i_{\alpha\beta\ldots\mu} \ ,$$

where the coefficients are evaluated at P. Substituting in (4.7), we get

$$T^i_k = \delta^i_k \quad , \ T^i_{\alpha k} = 0 \quad , \ T^i_{\alpha\beta\ldots\mu k} = -\Gamma^i_{\alpha\beta\ldots\mu . k} \ . \qquad (4.9)$$

The required tensors can now be found by writing the expressions on the right in tensor form, using the relations (4.8). The first two are already in the required form, and we shall proceed to evaluate the next two tensors.

The Γ's are given by[x]

$$\Gamma^{i}_{\alpha\beta\cdots\lambda\mu} = \frac{1}{m} \underset{(\alpha\beta\cdots\mu)}{P} \left(\Gamma^{i}_{\alpha\beta\cdots\lambda\cdot\mu} + \Gamma\Gamma \right)$$

where $\Gamma\Gamma$ denotes the sum of products of two Γ's, m is the number of subscripts, and P denotes the sum of terms obtained by permuting the subscripts. Hence, at P,

$$\Gamma^{i}_{\alpha\beta\cdot k} = \Gamma^{i}_{\alpha\beta\cdot k} - \Gamma^{i}_{\alpha\beta k} = \tfrac{1}{3}\left[\left(\Gamma^{i}_{\alpha\beta\cdot k} - \Gamma^{i}_{\alpha k\cdot\beta} \right) + \left(\Gamma^{i}_{\beta\alpha\cdot k} - \Gamma^{i}_{\beta k\cdot\alpha} \right) \right].$$

At P, the curvature tensor is given by

$$-R^{i}_{\alpha\beta k} = \Gamma^{i}_{\alpha\beta\cdot k} - \Gamma^{i}_{\alpha k\cdot\beta}$$

and so we have

$$T^{i}_{\alpha\beta k} = \tfrac{1}{3}\left(R^{i}_{\alpha\beta k} + R^{i}_{\beta\alpha k} \right). \qquad (4.10)$$

Similarly,

$$\Gamma^{i}_{\alpha\beta\gamma\cdot k} = \Gamma^{i}_{\alpha\beta\gamma\cdot k} - \Gamma^{i}_{\alpha\beta\gamma k} = \tfrac{1}{4} \underset{(\alpha\beta\gamma)}{P} \left[\Gamma^{i}_{\beta\gamma\alpha\cdot k} - \Gamma^{i}_{\beta\gamma k\cdot\alpha} \right].$$

Also

$$\Gamma^{i}_{\beta\gamma\alpha} - \Gamma^{i}_{\beta\gamma\cdot\alpha} = T^{i}_{\beta\gamma\alpha} + \Gamma\Gamma .$$

Hence, at P

$$\Gamma^{i}_{\beta\gamma\alpha\cdot k} - \Gamma^{i}_{\beta\gamma\cdot\alpha k} = T^{i}_{\beta\gamma\alpha\cdot k} = T^{i}_{\beta\gamma\alpha,k} ,$$

the covariant derivative at P of a tensor being the ordinary derivative, and therefore

$$\Gamma^{i}_{\beta\gamma\alpha\cdot k} - \Gamma^{i}_{\beta\gamma k\cdot\alpha} = T^{i}_{\beta\gamma\alpha,k} - T^{i}_{\beta\gamma k,\alpha}$$

giving

$$T^{i}_{\alpha\beta\gamma k} = \tfrac{1}{4} \underset{(\alpha\beta\gamma)}{P} \left[T^{i}_{\beta\gamma k,\alpha} - T^{i}_{\beta\gamma\alpha,k} \right].$$

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[x] Eisenhart, (4), §17.

From the properties of the curvature tensor,

$$\underset{(\alpha\beta\gamma)}{P}\left(T^{\;i}_{\beta\gamma\alpha}\right) = 0 .$$

Also,

$$R^{\;i}_{\beta\gamma k,\alpha} = R^{\;i}_{\beta\gamma\alpha,k} + R^{\;i}_{\beta\alpha k,\gamma}$$

i.e.

$$\underset{(\alpha\beta\gamma)}{P}\left(R^{\;i}_{\beta\gamma k,\alpha}\right) = P\left(R^{\;i}_{\beta\alpha k,\gamma}\right) = P\left(R^{\;i}_{\gamma\beta k,\alpha}\right)$$

Hence,

$$T^{\;i}_{\alpha\beta\gamma k} = \tfrac{1}{6}\left(R^{\;i}_{\beta\gamma k,\alpha} + R^{\;i}_{\gamma\alpha k,\beta} + R^{\;i}_{\alpha\beta k,\gamma}\right). \qquad (4.11)$$

The first terms of the expansion of (4.6) can thus be written

$$\delta^i_k + \tfrac{1}{3} R^{\;i}_{\alpha\beta k}\, \eta^\alpha \eta^\beta + \tfrac{1}{12} R^{\;i}_{\alpha\beta k,\gamma}\, \eta^\alpha \eta^\beta \eta^\gamma \ldots \qquad (4.12)$$

In all the following applications of relative co-ordinates, we shall consider only the space in the neighbourhood of a given curve C such that the cube, and sometimes the square, of distances from C can be neglected. It is therefore only necessary to expand (4.4) to the second order in the z's. From (3.5) and(2.5),

$$e_\sigma A^{\;i}_{j k}\, \lambda_{\sigma|i}\, \eta^j \dot{x}^k = \gamma^\sigma_{\;\nu} z^\nu .$$

Hence, from (4.4), (4.5), (4.12), and (3.5), writing

$$u_\sigma = \dot{x}^i \lambda_{\sigma|i} \quad , \gamma_{\sigma\mu\nu\lambda} = R_{lijk}\, \lambda_{\sigma|}{}^{l} \lambda_{\mu|}{}^{i} \lambda_{\nu|}{}^{j} \lambda_{\lambda|}{}^{k} \quad , \quad \dot{z}^\sigma = \frac{dz^\sigma}{ds}, \qquad (4.13)$$

we have, to the required order

$$\frac{\delta z^\sigma}{ds} = u^\sigma + \dot{z}^\sigma - \gamma^\sigma_{\;\nu} z^\nu + \tfrac{1}{3}\gamma^\sigma_{\;\mu\nu\lambda}\, u^\lambda z^\mu z^\nu . \qquad (4.14)$$

5. Application to neighbouring curves.

It is first required to find the fundamental tensor of V_n referred to the normal co-ordinates with the point P(s) of C as origin. Let g^{x}_{ij} be the fundamental tensor for the y co-ordinates defined by (3.1) with s for s_0, and $a_{\sigma\nu}$ for the z co-ordinates defined by (3.1) and (3.5). Then from the law of transformation,

$$a_{\sigma\nu} = g^{x}_{ij} \frac{\partial y^i}{\partial z^\sigma} \frac{\partial y^j}{\partial z^\nu} = g^{x}_{ij} \lambda_{\sigma|}{}^{i} \lambda_{\nu|}{}^{j}.$$

It is known that, to the second order,[x]

$$g^{x}_{ij} = g_{ij} + \tfrac{1}{3} R_{i\alpha\beta j} y^\alpha y^\beta \qquad (5.1)$$

where g_{ij}, $R_{i\alpha\beta j}$ are evaluated at P. Hence, to the required order,

$$a_{\sigma\nu} = \delta_{\sigma\nu} + \tfrac{1}{3} \gamma_{\sigma\mu\lambda\nu} z^\mu z^\lambda. \qquad (5.2)$$

If the co-ordinates of a point Q referred to P(s) of C are given as functions of s, then as P moves along C, Q traces a curve C'. Referred to P, a small displacement along C' at Q is δz^σ and hence the tangent vector of C' at Q, referredto P is $\delta z^\sigma / ds'$ where s' is the arc of C', and is therefore given by

$$\pm\left(\frac{ds'}{ds}\right)^2 = a_{\sigma\nu} \frac{\delta z^\sigma}{ds} \frac{\delta z^\nu}{ds}. \qquad (5.3)$$

Substituting from (4.14) and (5.2), we find

$$\begin{aligned} \pm\left(\frac{ds'}{ds}\right)^2 = e + 2u_\sigma \dot z^\sigma + 2\gamma_{\sigma\kappa} u^\kappa z^\sigma + \delta_{\sigma\nu} \dot z^\sigma \dot z^\nu + 2\gamma_{\sigma\nu} z^\sigma \dot z^\nu \\ + \gamma^{\mu}{}_{\sigma} \gamma_{\mu\nu} z^\sigma z^\nu + \Gamma_{\sigma\nu} z^\sigma z^\nu \end{aligned} \qquad (5.4)$$

where $e = u_\kappa u^\kappa = g_{ij} \dot x^i \dot x^j = \pm 1, 0$ according as C is not, or is

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[x] Veblen, (6), p.97.

null, and

$$\Gamma_{\sigma\nu} = \Gamma_{\nu\sigma} = \gamma_{h\sigma\nu k} u^h u^k = R_{hijk}\, \dot{x}^h \dot{x}^k \lambda_{\sigma|}{}^i \lambda_{\nu|}{}^j \,. \qquad (5.5)$$

If C is not null, the sign on the left is evidently that of e.

Any curve C' near C can be defined as above for any system of relative co-ordinates along C, by fixing a (1,1) correspondence between the points of C' and C. The most useful correspondence, for a given system of reference, is found by taking $z^0 = 0$ for all values of s, i.e. the point Q of C' corresponding to P of C is the point where C' meets the geodesic surface orthogonal to $\lambda_{0|}{}^i$ at P. In particular, if C is not null and the n-uple of reference is chosen so that $\lambda_{0|}{}^i = \dot{x}^i$, the point Q of C' is such that the geodesic PQ is orthogonal to C at P. The most convenient system of reference satisfying this condition is given by Fermi transport along C. With this system, $\gamma_{\sigma\nu}$ is given by (2.12), and also, $u^\sigma = \delta^\sigma_0$, $e_0 = e = \pm 1$, and hence, taking $z^0 = 0$, we have

$$\frac{\delta z^0}{ds} = 1 - e v_s z^s + \tfrac{1}{3} e\, \Gamma_{st} z^s z^t$$
$$\frac{\delta z^r}{ds} = \dot{z}^r + \tfrac{1}{3} \gamma^r{}_{st0}\, z^s z^t \qquad (5.6)$$

where r, s, t take the values 1,2, ---, $n-1$. Also, from (5.4),

$$\frac{ds'}{ds} = 1 - e v_r z^r + \tfrac{1}{2} e\, \delta_{rs} \dot{z}^r \dot{z}^s + \tfrac{1}{2} e\, \Gamma_{rs} z^r z^s \,. \qquad (5.7)$$

From (5.6), (5.7), we see that, to the first order, the unit vector tangent to C' is

$$\frac{\delta z^0}{ds'} = 1 \quad , \quad \frac{\delta z^r}{ds'} = \dot{z}^r \,. \qquad (5.8)$$

If V_n is a flat space, it can easily be seen that, without any approximation,

$$\frac{\delta z^\sigma}{ds} = u^\sigma + \dot{z}^\sigma - \gamma^\sigma{}_\nu z^\nu . \qquad (5.9)$$

If the n-uple of reference is formed by the principal directions of C, $\gamma_{\sigma\nu}$ is given by (2.13), and we have

$$\frac{\delta z^\sigma}{ds} = \delta^\sigma_0 + \dot{z}^\sigma + e_\sigma \kappa_{\sigma-1} z^{\sigma-1} - e_\sigma \kappa_\sigma z^{\sigma+1} . \qquad (5.10)$$

In the case of a Euclidean space V_3, $e_\sigma = +1$, κ_0 is the curvature, κ_1 is the torsion, and (5.10) are the well-known formulae for moving axes along a twisted curve.[x]

6. Formulae for vector displacements.

If $Q(z^0, z^1, \cdots, z^{n-1})$ is a point of a curve C' near C, let μ^σ be a given vector at Q, referred to the point P(s) of C, the μ's being functions of s. Thus there is a vector at each point of C', and if $\mu^\sigma + d\mu^\sigma$ is the vector at Q' referred to P'(s+ds), it is required to refer this vector, referred to P. Let this vector at Q' be $\mu^\sigma + \delta\mu^\sigma$ referred to P.

Consider the point $R(\bar{z})$ referred to P, where $\bar{z}^\sigma = z^\sigma + \varepsilon\mu^\sigma$, ε being a small constant, and let $R'(\bar{z} + d\bar{z})$ be the corresponding point referred to P'(s+ds), so that the vector Q'R' is $\varepsilon(\mu^\sigma + d\mu^\sigma)$ referred to P', and $\varepsilon(\mu^\sigma + \delta\mu^\sigma)$ referred to P. Referring to P, the point Q' is $(z + \delta z)$ and R' is $(\bar{z} + \delta\bar{z})$, and hence

$$\varepsilon(\mu^\sigma + \delta\mu^\sigma) = \bar{z}^\sigma + \delta\bar{z}^\sigma - (z^\sigma + \delta z^\sigma). \qquad (6.1)$$

Writing $\phi^\sigma = \frac{Dz^\sigma}{ds}$, we have $\delta z^\sigma = dz^\sigma + \phi^\sigma ds$, and

$$\delta\bar{z}^\sigma = d\bar{z}^\sigma + \bar{\phi}^\sigma ds = dz^\sigma + \varepsilon\, d\mu^\sigma + (\phi^\sigma + \varepsilon \frac{\partial\phi^\sigma}{\partial z^\nu}\mu^\nu) ds$$

$$= \delta z^\sigma + \varepsilon\left(\frac{d\mu^\sigma}{ds} + \frac{\partial\phi^\sigma}{\partial z^\nu}\mu^\nu\right) ds$$

to the first order in ε. Substituting in (6.1),

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[x] Eisenhart (3), §16

$$\Sigma(\mu^\sigma+\delta\mu^\sigma) = \Sigma\left\{\mu^\sigma + \left(\frac{d\mu^\sigma}{ds} + \frac{\partial\phi^\sigma}{\partial z^\nu}\mu^\nu\right)ds\right\}$$

and hence,

$$\frac{\delta\mu^\sigma}{ds} = \frac{d\mu^\sigma}{ds} + \frac{\partial\phi^\sigma}{\partial z^\nu}\mu^\nu \tag{6.2}$$

Substituting for ϕ^σ from (4.14),

$$\frac{\delta\mu^\sigma}{ds} = \frac{d\mu^\sigma}{ds} - \gamma^\sigma{}_\nu\mu^\nu + \tfrac{1}{3}\left(\gamma^\sigma{}_{\lambda\nu\kappa} + \gamma^\sigma{}_{\nu\lambda\kappa}\right)u^\kappa z^\lambda\mu^\nu \tag{6.3}$$

where second order terms in z are now neglected. A higher order of approximation in these formulae will not be required for this work.

From the definition of the derivative of a vector along a curve, (§2), the derivative of μ^σ along C' is given by

$$\frac{\Delta\mu^\sigma}{ds'} = \frac{\delta\mu^\sigma}{ds'} + \left(\Gamma^\sigma_{\nu\kappa}\right)_a\mu^\nu\frac{\delta z^\kappa}{ds'} \tag{6.4}$$

where $(\Gamma^\sigma_{\nu\kappa})_a$ are the Christoffel symbols formed from the fundamental tensor $a_{\sigma\nu}$ at P. From (5.2),

$$\Gamma^\sigma_{\nu\kappa} = \tfrac{1}{2}a^{\sigma\mu}\left(\frac{\partial a_{\mu\nu}}{\partial z^\kappa} + \frac{\partial a_{\mu\kappa}}{\partial z^\nu} - \frac{\partial a_{\nu\kappa}}{\partial z^\mu}\right) = -\tfrac{1}{3}\left(\gamma^\sigma{}_{\nu\kappa\lambda} + \gamma^\sigma{}_{\kappa\nu\lambda}\right)z^\lambda \tag{6.5}$$

using the well-known relations between the invariants $\gamma_{\sigma\mu\nu\kappa}$. Substituting (4.14), (6.3), and (6.5) in (6.4), we find, to the first order, using the above relations again,

$$\frac{\Delta\mu^\sigma}{ds} = \frac{\Delta\mu^\sigma}{ds'}\frac{ds'}{ds} = \frac{d\mu^\sigma}{ds} - \gamma^\sigma{}_\nu\mu^\nu + \gamma^\sigma{}_{\nu\lambda\kappa}u^\kappa z^\lambda\mu^\nu. \tag{6.6}$$

From (6.6), we see at once that <u>μ^σ is displaced by parallel transport along C' if</u>

$$\frac{d\mu^\sigma}{ds} = \gamma^\sigma{}_\nu\mu^\nu - \gamma^\sigma{}_{\nu\lambda\kappa}u^\kappa z^\lambda\mu^\nu. \tag{6.7}$$

It must be remembered that C' is now assumed to be sufficiently near C so that the squares of distances between C and C' can be neglected.

If C is not null, consider as an application of (6.6) the derivative along C' of the vector $u'^{\sigma} = \delta z^{\sigma}/ds'$. This vector is the unit vector tangent to C', so the derivative, v'^{σ}, is the curvature vector of C' at the point Q. For shortness, write $\frac{ds'}{ds} = 1+k$ to the first order. From (2.7), $y_{\sigma\lambda}u^{\lambda} = \lambda_{\sigma|i,k}\dot{x}^i\dot{x}^k$, and if v_{σ} is the curvature vector of C at P, $\lambda_{\sigma|i}\dot{x}^i_{,k}\dot{x}^k = v_{\sigma}$. Hence

$$y_{\sigma\lambda}u^{\lambda} = \dot{u}_{\sigma} - v_{\sigma} \quad , \tag{6.8}$$

and, from (5.4),

$$k = e\dot{J} - e v_{\sigma} z^{\sigma} \quad , \quad J = u_{\sigma} z^{\sigma}. \tag{6.9}$$

From (4.14), the vector u'^{σ} is, to the first order,

$$\begin{aligned} u'^{\sigma} &= (u^{\sigma} + \dot{z}^{\sigma} - y^{\sigma}{}_{\nu} z^{\nu})/(1+k) \\ &= (1-k)u^{\sigma} + \dot{z}^{\sigma} - y^{\sigma}{}_{\nu} z^{\nu}. \end{aligned} \tag{6.10}$$

Substituting in (6.6), we get, to the first order,

$$\begin{aligned} (1+k)v'^{\sigma} &= \frac{d}{ds}\{(1-k)u^{\sigma} + \dot{z}^{\sigma} - y^{\sigma}{}_{\nu} z^{\nu}\} - y^{\sigma}{}_{\mu}\{(1-k)u^{\mu} + \dot{z}^{\mu} - y^{\mu}{}_{\nu} z^{\nu}\} + y^{\sigma}{}_{\nu\lambda k} u^{k} z^{\lambda} u^{\nu} \\ &= (1-k)(\dot{u}^{\sigma} - y^{\sigma}{}_{\mu} u^{\mu}) - \dot{k}u^{\sigma} + \ddot{z}^{\sigma} - 2y^{\sigma}{}_{\nu}\dot{z}^{\nu} - \rho^{\sigma}{}_{\nu} z^{\nu} \end{aligned}$$

where

$$\rho_{\sigma\nu} = \Gamma_{\sigma\nu} + \dot{y}_{\sigma\nu} - y_{\sigma\mu} y^{\mu}{}_{\nu} \quad , \quad \rho^{\sigma}{}_{\nu} = e_{\sigma}\rho_{\sigma\nu} \, , \tag{6.11}$$

and $\Gamma_{\sigma\nu}$ is given by (5.5). Hence, from (6.8),

$$v'^{\sigma} = (1-2k)v^{\sigma} - \dot{k}u^{\sigma} + \ddot{z}^{\sigma} - 2y^{\sigma}{}_{\nu}\dot{z}^{\nu} - \rho^{\sigma}{}_{\nu} z^{\nu}. \tag{6.12}$$

If the system of reference is given by Fermi transport along C, $u^{\sigma} = \delta^{\sigma}_{0}$, $v_0 = 0$, $y_{\sigma\nu} = \delta^{0}_{\sigma} v_{\nu} - \delta^{0}_{\nu} v_{\sigma}$. and $\Gamma_{\sigma 0} = 0$. Substituting in (6.11),

$$\rho_{0r} = \dot{v}_r \quad , \quad \rho_{rs} = \Gamma_{rs} + e v_r v_s \quad . (r,s \neq 0)$$

Taking $z^0 = 0$ as explained before, $J = 0$, $k = -e v_r z^r$, and the curvature vector is now given by

$$v'^0 = -e v_s \dot{z}^s \quad , \quad v'^r = (1 + e v_s z^s) v^r + \ddot{z}^r - \Gamma^r_s z^s . \qquad (6.13)$$

If κ is the principal curvature of C at P, $e'\kappa^2 = v_\sigma z^\sigma$, $e' = \pm 1$, and from (6.13), the principal curvature of C' at Q is κ' where

$$e'\kappa'^2 = v'_\sigma v'^\sigma = e'\kappa^2(1 + 2e v_r z^r) + 2 v_r \ddot{z}^r - 2\Gamma_{rs} v^r z^s ,$$

$$\text{i.e.} \qquad \kappa' = \kappa(1 + e v_r z^r) + \frac{e'}{\kappa}(v_r \ddot{z}^r - \Gamma_{rs} z^r z^s) \qquad (6.14)$$

Similarly, for the general case we have, from (6.12)

$$\kappa' = \kappa(1 - 2h) + \frac{e'}{\kappa}(v_\sigma \ddot{z}^\sigma - 2\gamma_{\sigma\nu} v^\sigma \dot{z}^\nu - \rho_{\sigma\nu} v^\sigma z^\nu) \qquad (6.15)$$

For future reference, let us now find the conditions that a vector μ^σ orthogonal to C', should be displaced by Fermi transport along C'. Using Fermi transport along C with $z^0 = 0$, the condition that μ^σ should be orthogonal to C' is, from (5.8),

$$\mu^0 = -e \mu_r \dot{z}^r . \qquad (6.16)$$

From (2.11), the components μ^r must therefore satisfy

$$\frac{\Delta \mu^r}{ds} + e(v'_\sigma \mu^\sigma)\frac{\delta z^r}{ds} = 0 .$$

Substituting from (5.6), (6.6), and (6.13), we get

$$\frac{d\mu^r}{ds} + \mu^0 v^r + \gamma^r_{\nu t 0} z^t \mu^\nu + e(v_s \mu^s)\dot{z}^r = 0$$

i.e.

$$\frac{d\mu^r}{ds} + e\mu_s(v^s \dot{z}^r - v^r \dot{z}^s) + \gamma^r_{st0} z^t \mu^s = 0 \qquad (r = 1, 2, \cdots, n-1). \qquad (6.17)$$

where s,t take the values $1, 2, \cdots, n-1$.

7. Geodesics in relative co-ordinates.

If the curve C' near C is a geodesic, the components of the curvature vector v'^{σ} (§6) at each point of C' all vanish. If C is not null, then from (6.12), the co-ordinates (z) must satisfy

$$(1-2k)v^{\sigma} - \dot{k}u^{\sigma} + \ddot{z}^{\sigma} - 2\gamma^{\sigma}_{\nu}\dot{z}^{\nu} - \rho^{\sigma}_{\nu}z^{\nu} = 0 .$$

It has been assumed that $\dot{z}$, $\ddot{z}$ are of the order of smallness of z, so we see that, as we should expect, the above equations give geodesics only when the v's are of the order of smallness of z, i.e. when the principal curvature of C is small. Assuming that the v's are small as required, k reduces to $e\dot{J}$ from (6.9), and kv^{σ} can be neglected. The equations for geodesics near C can therefore be written

$$\ddot{z}^{\sigma} - 2\gamma^{\sigma}_{\nu}\dot{z}^{\nu} - \rho^{\sigma}_{\nu}z^{\nu} - e\ddot{J}u^{\sigma} + v^{\sigma} = 0 \qquad (\sigma = 0,1,\cdots,n-1) . \qquad (7.1)$$

When Fermi transport is used along C, and $z^0 = 0$, the geodesic equations reduce to

$$\ddot{z}^{r} - \Gamma^{r}_{s}z^{s} + v^{r} = 0 , \qquad (r = 1,2,\cdots,n-1) . \qquad (7.2)$$

The above geodesic equations can be obtained at once by using the variational definition of geodesics:

$$\delta\int\left(\frac{ds'}{ds}\right)ds = 0 \qquad (7.3)$$

where ds'/ds is given by (5.4) or (5.7), the z's being varied as functions of s.

Writing $U = ds'/ds$, (7.3) gives the equations

$$\frac{d}{ds}\left(\frac{\partial U}{\partial \dot{z}^{\sigma}}\right) - \frac{\partial U}{\partial z^{\sigma}} = 0 \qquad (7.4)$$

and substituting for U, we get the above geodesic equations.

When C is a geodesic, $v^r = 0$ and Fermi transport becomes parallel transport. In this case, equations (7.2) reduce to

$$\ddot{z}^r - \Gamma^r_s z^s = 0 \quad , \qquad (r = 1, 2, \cdots, n-1) . \qquad (7.5)$$

As a particular case, consider a surface S in Euclidean V_3 and let C be a curve on S, the geodesic curvature κ_g of C being small. We have $n = 2$ and there is only one co-ordinate z, which may be measured in the direction of the geodesic normal of C. If K is the total curvature of S then $R_{1212} = K(EG - F^2)$ with the usual notation, and hence at points of C the remaining invariant Γ'_1 is $-K$. Thus from (7.2), the geodesics near C are given by

$$\ddot{z} + Kz + \kappa_g = 0 . \qquad (7.6)$$

If C is a geodesic, this equation becomes

$$\ddot{z} + Kz = 0 \qquad (7.7)$$

which is the classical formula of Jacobi. The above geodesic equations in V_n when C is a geodesic are equivalent to equations given by Levi-Civita.[x]

If V_n is a space of constant curvature K, the curvature tensor is given by

$$R_{hijk} = K(g_{hj} g_{ik} - g_{hk} g_{ij}) . \qquad (7.8)$$

Using Fermi transport along C, we have, from (5.5),

$$\Gamma_{rs} = K(\dot{x}^i \dot{x}^j - e g^{ij}) \lambda_{r|i} \lambda_{s|j} = -e K \delta_{rs} \quad , \qquad (7.9)$$

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[x] (7). pp.208-220.

and the geodesic equations (7.5) become

$$\ddot{z}^{r} + eKz^{r} = 0. \tag{7.10}$$

These equations can be solved at once, K being constant, and we find

$$z^{r} = A^{r}\sin(\sqrt{eK}\,s) + B^{r}\cos(\sqrt{eK}\,s)$$

if eK is positive, and

$$z^{r} = A^{r}\exp.(\sqrt{-eK}\,s) + B^{r}\exp(-\sqrt{-eK}\,s) \tag{7.11}$$

if eK is negative, where A^{r}, B^{r} are small arbitrary constants. Thus, if K has the sign of e, the neighbouring geodesics keep together, and if K has the sign of $-e$, the geodesics diverge.

We observe that in a space of constant curvature, the geodesic equations (7.5) separate into $(n-1)$ independent equations, and it is an interesting problem in tensor geometry to find the conditions that (7.5) should separate in this way. We have $\dot{x}^{i} = \lambda_{0|}{}^{i}$, $\Gamma_{0\sigma} = 0$, and it is therefore necessary that

$$r_{ij}\lambda_{\sigma|}{}^{i}\lambda_{\nu|}{}^{j} = 0, \quad (\sigma \neq \nu,\ \sigma, \nu = 0, 1, \cdots, n-1), \tag{7.12}$$

where $r_{ij} = R_{kijh}\dot{x}^{k}\dot{x}^{h}$. These equations show that $\lambda_{\sigma|}{}^{i}$ must be the principal vectors[x] of the tensor r_{ij}, and hence, the required necessary and sufficient conditions are that the principal vectors of r_{ij} should be displaced by parallel transport along the geodesic C. Equations (7.12) are

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[x] For an account of principal directions of a tensor, see Eisenhart, (4). §33.

equivalent to the standard equations

$$(r_{ij} - k_\sigma g_{ij})\lambda_{\sigma|}{}^{j} = 0 \qquad (i, \sigma = 0, 1, \dots, n-1) \qquad (7.13)$$

where k_σ are some invariants. Differentiating these equations along C, and writing $\dot{r}_{ij} = R_{hijk,l}\,\dot{x}^h \dot{x}^k \dot{x}^l$, then, assuming that $\lambda_{\sigma|}{}^{i}$ are displaced by parallel transport, we get

$$(\dot{r}_{ij} - \dot{k}_\sigma g_{ij})\lambda_{\sigma|}{}^{j} = 0 \qquad (7.14)$$

Thus, the principal n-uple of r_{ij} must be the principal n-uple of $\dot{r}_{ij}$. These conditions are necessary and also sufficient, for assuming that $\lambda_{\sigma|}{}^{i}$ satisfy

$$(\dot{r}_{ij} - k'_\sigma g_{ij})\lambda_{\sigma|}{}^{j} = 0$$

we get, by differentiating (7.13)

$$(r_{ij} - k_\sigma g_{ij})\dot{\lambda}_{\sigma|}{}^{j} = (\dot{k}_\sigma - k'_\sigma)\lambda_{\sigma|i} .$$

Multiplying by $\lambda_{\sigma|}{}^{i}$ and using (7.13), we see that $k'_\sigma = \dot{k}_\sigma$, and hence from (7.13), $\dot{\lambda}_{\sigma|}{}^{j}$ is in the direction of $\lambda_{\sigma|}{}^{j}$. But $\lambda_{\sigma|}{}^{j}$ is a unit vector, giving $\dot{\lambda}_{\sigma|}{}^{j}\lambda_{\sigma|j} = 0$, and hence $\dot{\lambda}_{\sigma|}{}^{j} = 0$, i.e. the vectors $\lambda_{\sigma|}{}^{j}$ are displaced by parallel transport.

<u>Lemma.</u>

The necessary and sufficient conditions that two symmetric tensors a_{ij} , b_{ij} should have the same principal n-uple are

$$g^{hk} a_{hi} b_{kj} = g^{hk} a_{hj} b_{ki} \quad , \qquad (i, j = 0, 1, \dots, n-1). \qquad (7.15)$$

These conditions are necessary, for if $\lambda_{\sigma|}{}^{i}$ is this common n-uple, then

$$a_{hi}\lambda_{\sigma|}{}^{h} = \alpha_\sigma \lambda_{\sigma|i}$$
$$b_{hj}\lambda_{\sigma|}{}^{h} = \beta_\sigma \lambda_{\sigma|j}$$

and hence,

$$g^{hk} a_{hi} b_{kj} = \sum_\sigma e_\sigma \alpha_\sigma \beta_\sigma \lambda_{\sigma|i}\lambda_{\sigma|j}.$$

the expression on the right is symmetric in i, j, giving the above equations.

The conditions are also in general sufficient, for assuming they are satisfied, then if $\lambda_{\sigma|}{}^{i}$ is the principal n-uple of a_{ij},

$$g^{hk}(a_{hi}\lambda_{\sigma|}{}^{i})\, b_{kj}\lambda_{\nu|}{}^{j} = g^{hk}(a_{kj}\lambda_{\nu|}{}^{j})\, b_{hi}\lambda_{\sigma|}{}^{i}$$

i.e. $$\alpha_\sigma b_{kj}\lambda_{\sigma|}{}^{k}\lambda_{\nu|}{}^{j} = \alpha_\nu b_{ki}\lambda_{\nu|}{}^{k}\lambda_{\sigma|}{}^{i}.$$

Thus, if $\sigma \neq \nu$, and $\alpha_\sigma \neq \alpha_\nu$, we have

$$b_{ij}\lambda_{\sigma|}{}^{i}\lambda_{\nu|}{}^{j} = 0,$$

showing that $\lambda_{\sigma|}{}^{i}$ is the principal n-uple of b_{ij}.

We can now say at once that <u>the necessary and sufficient conditions that the equations (7.5) should separate are</u>

$$g^{hk}(r_{hi}\dot{r}_{kj} - r_{kj}\dot{r}_{hi}) = 0, \qquad (i,j = 0,1,\dots,n-1), \tag{7.16}$$

<u>where</u>

$$r_{ij} = R_{hijk}\dot{x}^h\dot{x}^k, \qquad \dot{r}_{ij} = R_{hijk,l}\dot{x}^h\dot{x}^k\dot{x}^l. \tag{7.17}$$

8. Null geodesics.

We shall proceed to consider null curves near C where C is now a null geodesic. It will be assumed that the parameter s, no longer the arc of C, is chosen so that the geodesic equations of C in V_n take the usual simplified form, s then being determined except for arbitrary additive and multiplicative constants.[x] The vector dx^i/ds which we shall always write as λ^i is determined when s is chosen, and will be calledthe tangent vector of C. It is at once evident from the geodesic equations that the vector λ^i at a point of C is displaced by parallel transport along C to become the vector λ^i at any other point, with the usual definition of parallel transport along a null geodesic. Relative co-ordinates can be defined as before with respect to C, and many of the above general results are still true. One important difference, however, is that we cannot now choose a convenient n-uple of reference with one vector of the n-uple tangent to C. We shall use the same notation as employed above, e.g., $u_\sigma = \lambda^i \lambda_{\sigma|i}$, etc.

We now have $e = u_\sigma u^\sigma = 0$, and from (5.4), the arc s' of a curve C' near C is given, to the first order, by

$$\pm\left(\frac{ds'}{ds}\right)^2 = \gamma_{\sigma h} u^h z^\sigma + u_\sigma \dot{z}^\sigma = \dot{J} \qquad (8.1)$$

by (6.8), where, as before,

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[x] We shall call such a parameter of a null geodesic a <u>null parameter</u>.

$$J = u_\sigma z^\sigma = g_{ij}\xi^i\lambda^j \quad , \quad \xi^i = z^\sigma\lambda_{\sigma|}{}^i . \tag{8.2}$$

If C' is a null curve, $ds'/ds = 0$ and hence,

$$J = \text{const.} \tag{8.3}$$

Thus, <u>if C' is a null curve near a null geodesic C, and if ξ^i is the vector joining a point P of C to a corresponding point Q of C', then $g_{ij}\xi^i\lambda^j$ is constant as P moves along C.</u> This property of null geodesics is due to J. L. Synge and A. J. M'Connell, (8).

The above invariant property has been generalised by Prof. E. T. Whittaker[x] who proved that J, as defined above, is independent of the particular correspondence between points of C and C', so that J depends only on the two curves as a whole, and not on any particular points on them. The following is a short proof of this generalisation.

If the correspondence between points of C and C' is given a small arbitrary variation, the point of C' now corresponding to P of C is $\bar{Q}$, where $\bar{Q}$ is found from Q by a small arbitrary displacement along C'. Thus, the co-ordinates of $\bar{Q}$ referred to P are $\bar{z}^\sigma = z^\sigma + \varepsilon\frac{\delta z^\sigma}{ds}$, where ε is a small arbitrary function of s.

Hence

$$J + \delta J = \bar{J} = u_\sigma \bar{z}^\sigma = u_\sigma z^\sigma + \varepsilon u_\sigma \frac{\delta z^\sigma}{ds}$$

i.e.

$$\delta J = \varepsilon u_\sigma \frac{\delta z^\sigma}{ds} . \tag{8.4}$$

From (4.14), to the first order,

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[x] (9), theorem I.

$$u_\sigma \frac{\delta z^\sigma}{ds} = u_\sigma u^\sigma + u_\sigma \dot{z}^\sigma + y_{\sigma h} u^h z^\sigma = 0$$

from (8.1) and $u_\sigma u^\sigma = 0$. Hence, $\delta J = 0$ for all values of ε, i.e. J is unaltered for small arbitrary variations in the correspondence, which proves the theorem.

We shall now state and prove a new theorem of great importance in the study of spatial distance in Relativity. If P_0, P are points on the null geodesic C, let p denote a thin pencil of ∞^{n-2} null geodesics through P_0 passing near P. Then the volume of the (n-2)-dimensional cross-section of p by the locally flat sub-space S_{n-1} orthogonal to a vector V^i at P is independent of V^i, i.e. the volume of cross-section at P is independent of the particular section chosen at P.

Referring to relative co-ordinates along C, we shall define the correspondence between the points of C and any curve C' of p by taking $z^0 = 0$, and we shall show that the volume when $V^i = \lambda_{01}{}^i$ is equal to the volume for any other cross-section.

The null geodesics belongingto the pencil p are given by equations of the form

$$z^r = z^r(s;\alpha^2, \alpha^3, \dots, \alpha^{n-1}) \qquad (r = 1, 2, \dots, n-1)$$

where $z^r = 0$ at P_0, and the α's are constants, varying for the different geodesics. The space S_{n-1} at P orthogonal to $\lambda_{01}{}^i$ is $z^0 = 0$ in relative co-ordinates, and, writing $Dz^r = (\partial z^r/\partial \alpha^p)\, d\alpha^p$, the linear element of the cross-section at P by this space is given by

$$\pm d\sigma^2 = \sum_{r=1}^{m-1} e_r (Dz^r)^2 . \tag{8.5}$$

If the space $z^0 = 0$ at P meets a null geodesic C'(z) of p at the point Q, the vector $\delta z^\sigma/ds$ is tangent to C' at Q, and hence, a point $\bar{Q}$ of C', near Q, has co-ordinates $\bar{z}^\sigma = z^\sigma + \varepsilon \frac{\delta z^\sigma}{ds}$ where ε is some small function of s. Choosing ε so that $\bar{Q}$ is the point where C' meets the space orthogonal to a vector V^i at P, and neglecting second-order terms, the points of the cross-section of p by this space are $\bar{z}^\sigma = z^\sigma + \varepsilon u^\sigma$, where ε is now a function of s and the α's depending on V^i.

The element of length of this cross-section is given by

$$\pm (d\bar{\sigma})^2 = \sum_{\sigma=0}^{m-1} e_\sigma (D\bar{z}^\sigma)^2 = \sum_{\sigma=0}^{m-1} e_\sigma (Dz^\sigma + u^\sigma D\varepsilon)^2$$
$$= \sum_{r=1}^{m-1} e_r (Dz^r)^2 + 2D\varepsilon\, u_\sigma Dz^\sigma + (D\varepsilon)^2 u_\sigma u^\sigma .$$

C is null, giving $u_\sigma u^\sigma = 0$. Also, $J = 0$ for each member of p, the z's all vanishing at P_0. Hence $DJ = u_\sigma Dz^\sigma = 0$, and we have, from (8.5),

$$d\bar{\sigma}^2 = d\sigma^2 .$$

Thus the linear element of the cross-section at P is independent of the particular cross-section, from which the theorem follows immediately.

The above theorem is a direct generalisation of a simple property of surfaces on a null cone in a flat space. For example, in space of the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

the null cone through the origin is given by

$$t = r, \quad x = rl, \quad y = rm, \quad z = rn \quad ; \quad l^2 + m^2 + n^2 = 1 ,$$

and a surface on this cone is given by an expression for r as a function of l,m,n.

Its line element is

$$d\sigma^2 = -dr^2 + \Sigma(r\,dl + l\,dr)^2 = r^2\Sigma dl^2$$

since $\Sigma l^2 = 1$, $\Sigma l\,dl = 0$. Thus the elements of distance and area at any point are the same for all surfaces through this point.

9. Equations of null geodesics.

When C is not null, a convenient system of reference is obtained by taking (n-1) vectors of the n-uple orthogonal to C at each point. This cannot be done, however, when C is null, and we shall now find the most convenient system of reference in this case. We shall first prove some elementary properties of null vectors and curves.

Let $\lambda_{\sigma|}{}^i$ be an orthogonal n-uple at a point P, and let λ^i be a null vector at this point. If $\lambda_{r|}{}^i$, $(r = 1,2,\cdots,n-1)$ are (n-1) vectors orthogonal to λ^i, we have $\lambda^i\lambda_i = 0$; $\lambda^i\lambda_{r|i} = 0$, and hence $\lambda^i, \lambda_{r|}{}^i$ cannot all be independent. We have assumed that $\lambda_{r|}{}^i$ are independent, so, if (n-1) independent vectors are orthogonal to a null vector, this null vector can be expressed in terms of the (n-1) vectors. We can, however, choose the (n-2) vectors $\lambda_{p|}{}^i$ $(p = 2,3,\cdots,n-1)$ to be orthogonal to and independent of λ^i and if this is done, we see at once that any vector orthogonal to λ^i can be expressed in terms of

λ^i and λ_{p1}^i $(p=2,\cdots,n-1)$. Hence, any other set of (n-2) mutually orthogonal vectors orthogonal to and independent of λ^i is given by equations of the form

$$\mu_{p1}^i = \xi_p^q \lambda_{q1}^i + \alpha_p \lambda^i \tag{9.1}$$

where q takes the values $2,3,\cdots,n-1$, and from the conditions of orthogonality, the ξ's must be coefficients of an orthogonal transformation in V_{n-2}.

If, at P, λ_{01}^i is an n-uple such that λ_{01}^i is not orthogonal to λ^i and the (n-2) vectors λ_{p1}^i are orthogonal to and independent of λ^i, then λ_{11}^i is determined to within sign by λ_{01}^i and λ_{p1}^i. Also, λ^i can be expressed in terms of λ_{01}^i and λ_{11}^i, i.e. λ_{11}^i can be expressed in terms of λ_{01}^i and λ^i, and as λ_{11}^i is orthogonal to λ_{01}^i, we see that λ_{11}^i is determined completely to within sign by λ^i and λ_{01}^i. Hence, fixing the sign, the vector λ_{11}^i, which may be called the <u>conjugate</u> of λ_{01}^i, is given by

$$\lambda_{11}^i = \lambda_{01}^i - \frac{e_0}{(\lambda^i \lambda_{01 i})} \lambda^i , \tag{9.2}$$

and we have

$$e_1 = -e_0 , \qquad \lambda^i \lambda_{11 i} = \lambda^i \lambda_{01 i} .$$

When C is a null geodesic, the tangent vector λ^i at each point is null, and we can assume that the n-uple of reference satisfies the above conditions at each point of C. Hence, with the usual notation, we have

$$e_1 = -e_0 \;; \quad u_1 = u_0 \;, \quad u_p = 0 \qquad (p=2,3,\cdots,n-1) \tag{9.3}$$

Also, if λ_{01}^i is displaced by parallel transport along C, then, from (9.2), $\lambda_{1}{}^{i}$ is also parallel along C, and $u_1 = u_0 = const.$.

If C' is a null curve near C, we find, with the above system of reference, taking $z^0 = 0$,

$$J = u_1 z^1 = u_0 z^1. \tag{9.4}$$

Hence, all the null curves giving the same value to J meet the locally flat space S_{n-1} orthogonal to $\lambda_{01}{}^{i}$ at P at points of a flat space S_{n-2} given by

$$z^1 = J/u_0 \tag{9.5}$$

in the above co-ordinates at P.[x] The normal to this sub-space of S_{n-1} is the direction conjugate to $\lambda_{01}{}^{i}$ and the distance from the origin is $J/\lambda^i \lambda_{01i}$.

If t is any parameter, the null geodesics in V_n are given by

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = \psi \frac{dx^i}{dt} \tag{9.6}$$

where ψ is a function of t, and a null parameter is given by $\tau = \varphi(t)$ where $\psi = \frac{d}{dt}\left(\log \frac{d\varphi}{dt}\right)$. In finding the equations for a null geodesic C' near C, it must be remembered that s, though a null parameter of C, will not in general be a null parameter of C'.

If, with the usual notation, $u'^{\sigma} = \delta z^{\sigma}/ds$, then, from (9.6), the null geodesics near C are given by

$$\frac{\Delta u'^{\sigma}}{ds} = \psi u'^{\sigma} \tag{9.7}$$

where ψ is some function of s. Using a general system of reference and substituting from (4.14) and (6.6), these equations become

$$\ddot{z}^{\sigma} - 2\gamma^{\sigma}{}_{\nu}\dot{z}^{\nu} - \rho^{\sigma}{}_{\nu} z^{\nu} = \psi u^{\sigma} \qquad (\sigma = 0, 1, \cdots, n-1) \tag{9.8}$$

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[x] C.F. (9), theorem II.

where $\rho_{\sigma\nu}$ is defined as in (6.11). We have used the fact that $\dot{u}_\sigma = \gamma_{\sigma h} u^h$, C being a geodesic, and also assumed, as is evident, that ψ is small. If $\lambda_{0|}{}^i$ is not orthogonal to C, we can choose ψ so that $z^0 = 0$ for all values of s, and we get

$$\psi u^0 = -2\gamma^0{}_s \dot{z}^s - \rho^0{}_s z^s$$

where (s) takes the values $1, 2, \cdots, n-1$). Hence the geodesics are given by $z^0 = 0$ and

$$\ddot{z}^r - 2\left(\gamma^r{}_s - \frac{u^r}{u^0}\gamma^0{}_s\right)\dot{z}^s - \left(\rho^r{}_s - \frac{u^r}{u^0}\rho^0{}_s\right)z^s = 0 \qquad (r = 1, 2, \cdots, n-1) \qquad (9.9)$$

It must also be remembered that $J = u_r z^r = \text{const.}$, for this is not given as an integral of the equations (9.9),

If the system of reference is chosen so that $u_p = 0$ $(p = 2, 3, \cdots, n-1)$, the null geodesics are given by $z^0 = 0$, $u_1 z^1 = \text{const.}$, and

$$\ddot{z}^p - 2\gamma^p{}_s \dot{z}^s - \rho^p{}_s z^s = 0 \qquad (p = 2, 3, \cdots, n-1). \qquad (9.10)$$

An important set of null geodesics near C are those for which $J = 0$. With the last system of reference, $u_1 \neq 0$ and hence $z^1 = 0$. These null geodesics are therefore given by

$z^0 = z^1 = 0$, and

$$\ddot{z}^p - 2\gamma^p{}_q \dot{z}^q - \rho^p{}_q z^q = 0 \qquad (p = 2, 3, \cdots, n-1) \qquad (9.11)$$

where q takes the values $2, 3, \cdots, n-1$. From (2.7) and (6.11),

$$\rho_{pq} = \Gamma_{pq} - \frac{d}{ds}(\dot{\lambda}_{q|}{}^i \lambda_{p|i}) + g_{ij}\dot{\lambda}_{p|}{}^i \dot{\lambda}_{q|}{}^j$$
$$= \Gamma_{pq} - \ddot{\lambda}_{q|}{}^i \lambda_{p|i}. \qquad (9.12)$$

Hence, if $\lambda_{p|}{}^i$ $(p = 2, 3, \cdots, n-1)$ are given by parallel transport along C,

$$\gamma_{pq} = 0, \quad \rho_{pq} = \Gamma_{pq} \qquad (p, q = 2, 3, \cdots, n-1)$$

and equations (9.11) reduce to

$$\ddot{z}^p - \Gamma^p_q z^q = 0 \qquad (p = 2, 3, \cdots, n-1). \qquad (9.13)$$

It may happen that the geodesic equations reduce to this simplified form for some more general system of reference satisfying $u_p = 0$, $p = 2, 3, \cdots, n-1$. Let $\mu_{p|}{}^i$ be (n-2) mutually orthogonal vectors orthogonal to and independent of λ^i and given by parallel transport along C. Then from (9.1), we must have

$$\lambda_{p|}{}^i = \xi^{p'}_p \mu_{p'|}{}^i + \alpha_p \lambda^i$$

where the ξ's are the coefficients of an orthogonal transformation, and the α's are some functions of s. If $\bar{\delta}_{p'q'}$ refers to the μ's, we have

$$\gamma_{pq} = \dot{\lambda}_{p|}{}^i \lambda_{q|i} = \bar{\delta}_{p'q'} \dot{\xi}^{p'}_p \xi^{q'}_q .$$

It is required that $\gamma_{pq} = 0$, and hence, $\bar{\delta}_{p'q'} \dot{\xi}^{p'}_p \xi^{q'}_q = 0$, from which we get at once

$$\dot{\xi}^{p'}_p = 0 \qquad (p, p' = 2, 3, \cdots, n-1).$$

Thus the vectors $\xi^{p'}_p \mu_{p'|}{}^i$ are displaced by parallel transport along C, and we can write

$$\lambda_{p|}{}^i = \mu_{p|}{}^i + \alpha_p \lambda^i \qquad (9.14)$$

where $\mu_{p|}{}^i$ are given by parallel transport, and the α's are any functions of s. From (9.14), $\dot{\lambda}_{p|}{}^i = \dot{\alpha}_p \lambda^i$, and hence, $\rho_{pq} = \Gamma_{pq}$. Thus, <u>the necessary and sufficient conditions that the geodesic equations should take the simplified form (9.13) are that the (n-2) vectors $\lambda_{p|}{}^i$ should be given by (9.14).</u> This result is very useful in applications, e.g. when $n = 4$ and $\lambda_{2|}{}^i$ is given by parallel transport along C,

$\lambda_{31|}{}^{i}$ can be any vector orthogonal to $\lambda_{2|}{}^{i}$ and λ^{i}.

If, however, $\lambda_{0|}{}^{i}$ is given by parallel transport along C, we find, by differentiating $\lambda_{0|}{}^{i}\lambda_{p|i} = 0$ that $\alpha_p = \text{const.}$, $p = 2, 3, \cdots, n-1$, and hence, $\lambda_{p|}{}^{i}$ must be parallel along C.

In the following work, it will be assumed that $\lambda_{0|}{}^{i}$ is not orthogonal to C, and the vectors $\lambda_{p|}{}^{i}$ are given by (9.14).

In space of constant curvature, the curvature tensor is given by (7.8), and we find

$$\Gamma_{pq} = K\lambda^{i}\lambda^{i}\lambda_{p|i}\lambda_{q|j} = 0 \qquad (p, q = 2, 3, \cdots, n-1). \qquad (9.15)$$

The null geodesics for which $J = 0$ are therefore given by

$z^{0} = 0$ and

$$z' = 0\ , \qquad z^{p} = \alpha^{p}s + \beta^{p} \qquad (p = 2, 3, \cdots, n-1) \qquad (9.16)$$

where the α's, β's are small arbitrary constants.

10. Pencils of null geodesics.

We shall now return to the consideration of a pencil p of null geodesics through the point $P_0(s_0)$ of the null geodesic C, and passing near the point P(s). All the members of p satisfy $J = 0$, so, with the convenient system of reference given in §9, the null geodesics of p are given by $z^{0} = z' = 0$ and

$$\ddot{z}^{p} - \Gamma^{p}{}_{q} z^{q} = 0\ , \qquad (10.1)$$

where $z^{p} = 0$ at $s = s_0$. If $z^{p} = \phi^{p}_{p'}$ $(p' = 2, 3, \cdots, n-1)$ are (n-2) independent sets of solutions of (10.1), each vanishing at $s = s_0$, the general solution of (10.1) satisfying the required conditions, can evidently be written

$$z^p = \phi^p_{p'} \alpha^{p'} \qquad (p=2,3,\cdots,n-1) \qquad (10.2)$$

where the α's are (n-2) arbitrary constants, varying for the different members of the pencil p. Hence, from (8.5), the linear element of the cross-section of p by the space $z^0 = 0$ at P is given by

$$\pm d\sigma^2 = C_{p'q'} \, d\alpha^{p'} d\alpha^{q'} \qquad (10.3)$$

where

$$C_{p'q'} = \sum_{p=2}^{n-1} e_p \phi^p_{p'} \phi^p_{q'} \, , \qquad (10.4)$$

and if $|C|$ denotes the determinant of the C's, the volume of the cross-section of p at P is

$$V_{(p)} = \int \sqrt{\pm|C|} \, d\alpha^2 d\alpha^3 \cdots d\alpha^{n-1} = \sqrt{\pm|C|} \int d\alpha^2 d\alpha^3 \cdots d\alpha^{n-1} \, . \qquad (10.5)$$

Thus the limits of the pencil p only occur in $V_{(p)}$ as a constant factor, and, for each pencil, $V_{(p)}$ is proportional to $\sqrt{\pm|C|}$ as s varies. Each ϕ vanishes at s_0, and $V_{(p)}$ is therefore of the order $(s-s_0)^{n-2}$ as $s \to s_0$. Hence, if we define

$$V = e K V_{(p)}$$

where

$$\frac{1}{K} = \lim_{s \to s_0} \frac{V_{(p)}}{(s-s_0)^{n-2}} \qquad (10.6)$$

and e is chosen so that V is positive, then $V \sim |s-s_0|^{n-2}$ as $s \to s_0$, and also, V is independent of the particular pencil p chosen above. We have shown (§8) that the volume of cross section at P is independent of the particular section. Thus, V, as defined above, is completely determined by the null geodesic C, and the points P_0, P, except for variations in the null parameter of C.

We shall now prove that <u>if s_0, s are the values of the</u>

parameter s at P_0, P respectively, then V is unaltered by interchanging s_0, s, i.e. the same value of V is obtained by considering the volume of cross-section at P_0 of a pencil of null geodesics through P.

To prove this, we shall use the matrix notation. Thus, A is written for the square matrix with terms A^i_j where i denotes the row and j the column. The product of two matrics A,B is written $D=AB$ where $D^i_j = A^i_k B^k_j$, the product being row into column. It must be remembered that, in general, $AB \neq BA$. The reciprocal of a matrix A is written A^{-1} and is given by $AA^{-1} = I$ where I is the unit matrix. It can be shown that $(A^{-1})^{-1} = A$, and hence, $A^{-1}A = I = AA^{-1}$. The value of the determinant formed from a square matrix A is written $|A|$. If $D = AB$, then $|D| = |AB| = |A||B|$, and hence, $|A^{-1}| = 1/|A|$. The divergence of a matrix A is the scalar A^i_i and is written $div.A$. Also, if α is the vector α^i written as a column, the vector $\beta^i = A^i_j \alpha^j$ can be written $\beta = A\alpha$.

Let φ denote the matrix φ^p_q and Γ the matrix Γ^p_q, etc. Then, from (10.1), we have the matrix equation

$$\ddot{\varphi} = \Gamma\varphi. \qquad (10.7)$$

From (10.4), we see that

$$|C| = e_2 e_3 \cdots e_{n-1} |\varphi|^2 = \pm |\varphi|^2.$$

Hence,

$$V = e K |\varphi|$$

where

$$\frac{1}{K} = \lim_{s \to s_0} \frac{|\varphi|}{(s-s_0)^{n-2}}, \quad e = \pm 1. \qquad (10.8)$$

The complete solution of (10.1) in terms of 2(n-2) arbitrary constants α^p, β^p can be written, in matrix notation

$$z = \theta\alpha + \psi\beta \tag{10.9}$$

where θ^p_q, ψ^p_q are some functions of s satisfying

$$\ddot{\theta} = \Gamma\theta \quad , \quad \ddot{\psi} = \Gamma\psi. \tag{10.10}$$

It is required that $z = 0$ at $s = s_0$, and hence, if a bar denotes the value at $s = s_0$,

$$0 = \bar{\theta}\alpha + \bar{\psi}\beta.$$

Pre-multiplying by $\bar{\psi}^{-1}$, we get at once

$$\beta = -\bar{\psi}^{-1}\bar{\theta}\alpha$$

giving

$$z = (\theta - \psi\bar{\psi}^{-1}\bar{\theta})\alpha$$

Hence,

$$\varphi = \theta - \psi\bar{\psi}^{-1}\bar{\theta}. \tag{10.11}$$

Pre-multiplying by ψ^{-1},

$$\psi^{-1}\varphi = A \quad , \quad A = \psi^{-1}\theta - \bar{\psi}^{-1}\bar{\theta} \quad , \tag{10.12}$$

and therefore

$$|\varphi| = |\psi| . |A|.$$

From the form of A, we see that $|A|$ is at most altered in sign when s_0 and s are interchanged. Substituting in (10.8),

$$\frac{1}{K} = \lim_{s\to s_0} \frac{|\psi|.|A|}{(s-s_0)^{n-2}} = |\bar{\psi}| \lim_{s\to s_0} \frac{|A|}{(s-s_0)^{n-2}}.$$

Now each term of A vanishes at $s = s_0$, and hence,

$$\lim_{s\to s_0} \frac{|A|}{(s-s_0)^{n-2}} = |\bar{B}| \quad , \quad B = \frac{d}{ds}A = \frac{d}{ds}(\psi^{-1}\theta). \tag{10.13}$$

Thus,

$$V = e\,\frac{|\psi|.|A|}{|\bar{\psi}|.|\bar{B}|} = e\,\frac{|\bar{\psi}|.|\psi|.|A|}{|\bar{\psi}|^2.|\bar{B}|}. \tag{10.14}$$

We see that $|\bar{\psi}|.|\psi|.|A|$ is at most changed in sign when

s_0 and s are interchanged and $|\bar{\Psi}|^2.|\bar{B}|$ is a function of s_0 only. To complete the proof of the theorem, we must therefore show that $|\psi|^2.|B|$ is independent of s.

Writing $F = \psi^{-1}\theta$, we have

$$B = \dot{F}, \qquad \dot{F} = \frac{d}{ds}F.$$

Also, from the formula for the derivative of a determinant,

$$\frac{d}{ds}|B| = |B|.\,div(\dot{B}B^{-1}) = |B|.\,div(\ddot{F}B^{-1}).$$

We have $\psi F = \theta$ and therefore,

$$\psi\ddot{F} + 2\dot{\psi}\dot{F} + \ddot{\psi}F = \ddot{\theta}.$$

From (10.10), $\ddot{\psi}F = \Gamma\psi F = \Gamma\theta = \ddot{\theta}$, and hence,

$$\psi\ddot{F} + 2\dot{\psi}\dot{F} = 0$$

i.e.

$$\ddot{F} = -2\psi^{-1}\dot{\psi}\dot{F} = -2GB$$

where

$$G = \psi^{-1}\dot{\psi}. \tag{10.15}$$

Hence

$$\frac{d}{ds}|B| = |B|.\,div(-2GBB^{-1}) = -2|B|.\,div\,G. \tag{10.16}$$

Also,

$$\frac{d}{ds}|\psi| = |\psi|\,div(\dot{\psi}\psi^{-1}) = |\psi|\,div(\psi^{-1}\dot{\psi})$$

$$= |\psi|\,div\,G, \tag{10.17}$$

so $$\frac{d}{ds}(|\psi|^2.|B|) = 2|\psi|.|B|\frac{d}{ds}|\psi| + |\psi|^2\frac{d}{ds}|B| = 0, \tag{10.18}$$

i.e. $|\psi|^2.|B|$ is independent of s. Hence, from (10.14), V is unaltered when s_0 and s are interchanged.

As a matter of interest, we shall now give another proof of the theorem of §8, which states that the volume $V_{(p)}$ is independent of the particular section of the pencil p at the point P(s). Changing the section at P is equivalent to taking a new set of (n-2) vectors $\bar\lambda_{p|}{}^{i}$ and by (9.1), these vectors can be written in terms of the original set of vectors and λ^{i} thus

$$\bar\lambda_{p|}{}^{i} = \xi_{p}^{p'} \lambda_{p'|}{}^{i} + \varsigma_{p} \lambda^{i} \qquad (10.19)$$

where the ξ's are coefficients of an orthogonal transformation and the ς's are any functions of s. We observe that the ξ's must be constants in order that the new set of vectors should satisfy the required conditions of transport along C. The new Γ's are given by

$$\bar\Gamma_{pq} = R_{hijk} \lambda^{h} \lambda^{k} \bar\lambda_{p|}{}^{i} \bar\lambda_{q|}{}^{j} = R_{hijk} \lambda^{h} \lambda^{k} \xi_{p}^{p'} \xi_{q}^{q'} \lambda_{p'|}{}^{i} \lambda_{q'|}{}^{j}$$

i.e. $$\bar\Gamma_{pq} = \Gamma_{p'q'} \xi_{p}^{p'} \xi_{q}^{q'} . \qquad (10.20)$$

The new co-ordinates $\bar z$ are therefore solutions of the equations

$$\ddot{\bar z}^{p} = \bar\Gamma^{p}_{q} \bar z^{q} = \Gamma^{p'}_{q'} \xi^{\cdot p}_{p'} \xi^{q'}_{q} \bar z^{q}$$

which can be written

$$\frac{d^2}{ds^2}\left(\xi^{p'}_{p} \bar z^{p}\right) = \Gamma^{p'}_{q'} \left(\xi^{q'}_{p} \bar z^{p}\right) .$$

Comparing with (10.1), we can write

$$\xi^{p'}_{p} \bar z^{p} = z^{p'} \quad ; \quad \bar z^{p} = \xi^{\cdot p}_{p'} z^{p'} , \qquad (10.21)$$

and hence, the new functions $\bar\varphi^{p}_{q}$ are given by $\bar\varphi^{p}_{q} = \xi^{\cdot p}_{p'} \varphi^{p'}_{q}$, which can be written

$$\bar\varphi = \xi \varphi \qquad (10.22)$$

when ξ is the matrix $\xi^{\cdot p}_{p'}$. Hence

$$|\bar{\phi}| = |\xi| . |\phi| = \pm |\phi| \tag{0.23}$$

from the fact that $|\xi| = \pm 1$ when ξ is an orthogonal transformation; and the theorem follows immediately.

PART II.

KINEMATICS IN RELATIVITY.

11. Fundamental ideas of Kinematics.

A particle or observer can be considered as a curve, or world-line[x1], in 4-dimensional space-time; in fact, the world-line is the observer. When talking about the observerat a particular 'instant', we are considering a particular point P of his world-line C.

At the instant P, the instantaneous space of the observer is the local 3-dimensional Euclidean sub-space S orthogonal to C at P, and as P moves along C, the observer appears to make observations in the series of sub-spaces so defined. At each instant P, the most convenient frame of reference of the observer is formed by a set of three mutually orthogonal axes in S(P),[x2] and the observer is then referring to a system of Cartesian co-ordinates. In space-time, the observer's frame of reference is thus given by three vectors $\lambda_{r|}^{i}$ $(r=1,2,3)$, mutually orthogonal and orthogonal to C, and these, together with the tangent vector $\dot{x}^i = \lambda^i = \lambda_{0|}^{i}$ of C, define a system of relative co-ordinates (z^0, z^1, z^2, z^3) along C. The instantaneous spaces S are the sub-spaces $z^0 = 0$ and the remaining co-ordinates (z^1, z^2, z^3) are the Cartesian co-ordinates of S.

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x1. Eddington, (10), p. 87.

x2. When there is no ambiguity, S will always denote an instantaneous space, and S(P) the space at the instant or point P.

The proper time of the above observer C is the arc-length of his world-line. This time has the dimension of length, and the unit of time is the astronomical unit, which is such that the velocity of light is unity.

The observer C can now make observations. Suppose he is observing a particle C', the world-line C' of this particle being near C. The curve C' will meet S(P) at a point Q, and at the instant P, C will say that the particle is at the position Q. As P moves along C, Q will move in S, and this motion will be the motion of the particle as observed by C. Thus C can measure the velocity, acceleration, etc. of the particle with respect to some chosen frame of reference, and his own proper time. In the above system of relative co-ordinates, the correspondence between points of the curves C and C' can be defined by taking $z^0 = 0$, and C' is then given by z^1, z^2, z^3 as functions of s, (§5). Hence, the motion of the particle as observed by C is at once seen to be given by the Cartesian co-ordinates (z^1, z^2, z^3) as functions of s, where s is the time.

12. Proper, relative, and observed motion.

We see that, in the above system, C can make no observations on himself, in fact, he would say that he is stationary at the origin of his frame of reference. Yet, from the fact that his world-line has certain invariant properties such as curvature, it would seem that he ought to have some motion, which we may term proper motion, defined completely by the space time and his world-line. This motion, if it exists, must therefore be measured by some observer or observers whose world-lines are determined by these data, and the most natural observers to do this are those moving freely in space-time, i.e. observers whose world-lines are geodesics.[x]

At each point P of C is a geodesic $\bar{C}(P)$ tangent to C, and the free observer corresponding to this world-line can be said to have the same instantaneous velocity as the observer C at the instant P, for any other observer will observe C and $\bar{C}$ to be moving together for a short interval at P. Let us therefore define the proper motion of C at an instant P as being the motion of C as measured by the free observer at P having the same instantaneous velocity as C.

We at once see that a free observer has no proper motion, and the proper velocity of any particle or observer is zero.

It is now necessary to decide what frame of reference a free observer will use. As his world-line is a geodesic, the most natural frame is formed by any set of three vectors, mutually orthogonal and orthogonal to the geodesic, which are displaced by parallel transport along the geodesic. It is not necessary to specify any particular set of vectors, for the resulting measurements when expressed in terms of the

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x Eddington, (11), §15.

space-time are either scalar or vector quantities, and are therefore independent of the particular frame of reference.

Having decided upon this system of observation, we can use it for more general purposes. We have already explained the motion of C' as observed by C. Now, at the instant P, instead of C measuring the motion of C' let $\bar{C}(P)$ measure this motion. We shall call this <u>the motion of C' relative to C</u>; it can be considered as the motion of C' as observed by C with some allowance being made for C's proper motion. If T is the frame of reference used by C, and $\bar{T}(P)$ the frame used by $\bar{C}(P)$, we shall generally assume, for convenience, that $\bar{T}(P)$ coincides with T at P. Thus, when T is given at all points of C, the frames of reference of all the associated free observers are completely defined.

If a unit vector in $S(P)$ is given at each point P of C, we can define the proper angular velocity, acceleration, etc. of this vector in a manner similar to the above. If the vector is ξ^{σ} in the relative co-ordinates, defined above, $\xi^{0} = 0$ and ξ^{τ} $(\tau = 1, 2, 3)$ are the direction cosines of the vector referred to C's frame of reference, the ξ's being functions of s. Let us find the proper angular velocity of this vector, when the frame of reference is displaced by Fermi transport along C. Although, from the theory of relativity, we know that $e_0 = e = +1$, $e_1 = e_2 = e_3 = -1$, we shall retain the e's for completeness.

If ξ^σ is the vector at P of C, let Q be a point a small arbitrary distance from P along ξ^σ and let C' be the locus of such points as P moves along C. Then referring to P, Q has co-ordinates

$$z^0 = 0\ , \quad z^r = r\xi^r \ . \quad r^2 = -\sum_{r=1}^{3} e_r (z^r)^2 \ . \qquad (12.1)$$

When dealing with problems such as the above, we shall consider a point $P'(s)$ of C, and the geodesic $\bar{C}$ will be the world-line of a free observer, near C at the point P'. We shall find the results as observed by $\bar{C}$, and then proceed to the limit, P' becoming the point P of C, and $\bar{C}$ the geodesic tangent to C at P. We shall consider the above problem in detail in order to demonstrate this method of attacking such problems.

Let $C', \bar{C}$ meet $S(P')$ at the points $Q', \bar{P}'$ respectively, and let Q'', $\bar{P}''$ be points of C', $\bar{C}$ such that the vectors $P'Q''$, $\bar{P}''P'$ are both orthogonal to $\bar{C}$ at P'. Then when $\bar{C}$ observes the event P' (at his instant $\bar{P}''$), he also observes a particle C' to be at the position Q''. Thus, although the given vector at P' is in the direction $P'Q'$, the vector observed by $\bar{C}$ is in the direction $P'Q''$.

If $\bar{z}^\sigma$ are the co-ordinates of $\bar{P}'$ referred to P', then in the limit when P' is at P, $\bar{z}^\sigma = \dot{\bar{z}}^\sigma = 0$, $\bar{C}$ being tangent to C at P, and the same is evidently true of the co-ordinates of $\bar{P}''$ referred to P'. Thus, from (5.4), $\frac{d\bar{s}'}{ds} = 1$, $\frac{d^2\bar{s}'}{ds^2} = 0$ when P' is at P, where $d\bar{s}'$ is the element of arc of $\bar{C}$ at $\bar{P}''$. Similar results exist in all such problems, and we deduce that <u>in all problems dealing only with velocity and acceleration, we can take s, the arc of C, to be the proper time of the observer $\bar{C}$.</u>

The point Q'' of C' is near Q'. Hence, if Q' has co-ordinates z^σ, $(z^0=0)$, referred to P', the tangent vector of C' at Q' is, from (5.8), $(1, \dot{z}^r)$, and the co-ordinates of Q'' are

$$z'^0 = \phi \quad , \quad z'^r = z^r \tag{12.2}$$

where ϕ is some small function of s, vanishing when P' is at P. The unit vector in the direction $P'Q''$ is therefore $\xi'^\sigma = \frac{z'^\sigma}{r'}$, $r'^2 = -\Sigma e_\sigma (z'^\sigma)^2$, i.e. from (12.1)

$$\xi'^0 = \frac{\phi}{r'} \quad , \xi'^r = \frac{z^r}{r'}, \quad r'^2 = r^2 + \phi^2, \tag{12.3}$$

Let μ^σ referred to points of C, be a vector orthogonal to $\bar{C}$ at points of $\bar{C}$, and given by parallel transport along $\bar{C}$. Then from (6.7), if μ^σ is the vector at $\bar{P}''$,

$$\mu^0 = 0 \quad , \dot{\mu}^r = 0 \quad (r=1,2,3) \tag{12.4}$$

when P' is at P. The cosine of the angle between this vector and ξ'^σ is $l = \mu_\sigma \xi'^\sigma$, and hence, when P' is at P,

$$\frac{dl}{ds} = \mu_r \dot{\xi}'^r = \mu_r \frac{d}{ds}\left(\frac{z^r}{r'}\right) = \mu_r \dot{\xi}^r . \left[\text{For in the limit, } \phi = 0;\ r' = r;\ \dot{r}' = \dot{r} \text{ from 12.3, above, from (12.1), etc}\right] \tag{12.5}$$

Choosing vectors μ^σ to coincide with the frame of reference of C at P, we see at once that the proper rate of change of the direction cosine ξ^r is simply $\dot{\xi}^r$.

If $\xi^r = \text{const.}$, $(r=1,2,3)$, the vector is displaced by Fermi transport along C. Hence, <u>if a vector at points of C has no proper angular velocity, it is displaced by Fermi transport along C.</u> This indicates the most suitable frame of reference to be used by any observer, for if the three vectors are given by Fermi transport, the frame of reference so formed has no proper angular velocity. We shall therefore define <u>the motion of C' as observed by C</u> to be the motion as measured with respect to such a frame of reference. The above frame of reference reduces to the natural frame when C is a free observer.

13. <u>Acceleration relative to an observer.</u>

We shall now find the acceleration of a particle C' relative to the observer C. Referring to relative co-ordinates with Fermi transport along C, we shall use a method similar to that used above for angular velocity.

As before, let P' be a point of C ~~[illegible]~~, and let $S(P')$ meet C', $\bar{C}$ at the points Q', $\bar{P}'$ respectively. Referring to P', let (z) be the co-ordinates of Q' and $(\bar{z})$ the co-ordinates of $\bar{P}'$. Then if $\bar{P}$ is the point of $\bar{C}$ such that $\bar{P}Q'$ is orthogonal to $\bar{C}$ at $\bar{P}$, when the particle C' is observed by C at the instant P', it is also observed by $\bar{C}$ at the instant $\bar{P}$, and the motion of the particle relative to C at the instant P is the limit when P' is at P of the motion of Q' referred to $\bar{P}$.

From (5.8), the tangent to $\bar{C}$ at $\bar{P}'$, referred to P', has the direction~~s~~ $(1, \dot{\bar{z}}^{\tau})$ and as $\bar{P}$ is near $\bar{P}'$, the co-ordinates of $\bar{P}$ can be written $\bar{z}'^{\sigma}$ where

$$\bar{z}'^{0} = \phi, \qquad \bar{z}'^{\tau} = \bar{z}^{\tau} + \phi\dot{\bar{z}}^{\tau} \tag{13.1}$$

ϕ being some small function of s. The point Q' is (z), and hence $\bar{P}Q'$ is $(z^{\sigma} - \bar{z}'^{\sigma})$. The condition that $\bar{P}Q'$

should be orthogonal to $\bar{C}$ is therefore

$$e(z^0-\bar{z}'^0) + \sum_{r=1}^{3} e_r \dot{\bar{z}}^r (z^r-\bar{z}'^r) = 0$$

i.e. $$-e\phi + \sum e_r \dot{\bar{z}}^r (z^r-\bar{z}^r-\phi\dot{\bar{z}}^r) = 0$$

When P' is at P, we have

$$\bar{z}^r = \dot{\bar{z}}^r = 0 \quad , \quad \ddot{\bar{z}}^r = -v^r \qquad (13.2)$$

from (7.2), and hence, at P,

$$\phi = 0 \quad , \quad \dot{\phi} = -e v_r z^r . \qquad (13.3)$$

If μ^σ referred to points of C, is a vector orthogonal to and parallel along $\bar{C}$ at points of $\bar{C}$, we have, from (6.7), when evaluated at P,

$$\dot{\mu}_\sigma = \gamma_{\sigma\nu}\mu^\nu = \delta_\sigma^0 (v_s\mu^s) - \mu^0 v_\sigma .$$

Hence, at P,

$$\mu_0 = 0 \quad , \quad \dot{\mu}_0 = v_s\mu^s \quad . \quad \dot{\mu}_r = 0 \quad . \quad \ddot{\mu}_r = -e v_r (v_s\mu^s). \qquad (13.4)$$

The vector $\bar{P}Q'$ is $(z^\sigma-\bar{z}'^\sigma)$ and the projection on to the vector μ^σ is $\mu_\sigma(z^\sigma-\bar{z}'^\sigma)$.

Therefore if $f_{(R)}^r$ referred to P is the acceleration of C' as observed by $\bar{C}$, the component in the direction μ^σ is

$$\begin{aligned} \mu_r f_{(R)}^r &= \frac{d^2}{ds^2}\left\{\mu_\sigma(z^\sigma-\bar{z}'^\sigma)\right\} \\ &= \mu_\sigma(\ddot{z}^\sigma-\ddot{\bar{z}}'^\sigma) + 2\dot{\mu}_\sigma(\dot{z}^\sigma-\dot{\bar{z}}'^\sigma) + \ddot{\mu}_\sigma(z^\sigma-\bar{z}'^\sigma) \end{aligned}$$

where the right hand side must be evaluated at P.

Substituting from (13.1), (13.2), (13.3) and (13.4), we get

$$\mu_r f_{(R)}^r = \mu_r\left\{\ddot{z}^r + v^r - e v^r(v_s z^s)\right\}.$$

Hence, the acceleration of the particle C' relative to C at the instant P is

$$f_{(R)}^r = \ddot{z}^r + v^r(1-e v_s z^s). \qquad (13.5)$$

If C' coincides with C, the acceleration of C' relative to C reduces to the proper acceleration of C. Writing $z^r=0$

in (13.5), we find

$$f^r = v^r, \qquad (13.6)$$

i.e. the proper acceleration of a particle at an instant is the curvature vector of its world-line at this point. From this, we see that the scalar acceleration is the principal curvature of the world-line.

The proper acceleration of C' at the point Q is therefore v'^{σ} given by (6.13).

The component $v'^0 = -e v_r \dot{z}^r$ is orthogonal to S(P) and is small. The remaining components v'^r give the projection onto S(P), and writing $f'^r = v'^r$, we have

$$f'^r = (1 + e v_s z^s) v^r + \ddot{z}^r - \Gamma^r_s z^s. \qquad (13.7)$$

Hence, from (13.5), the acceleration of C' relative to C can be written

$$f^r_{(R)} = f'^r (1 - e v_s z^s)^2 + \Gamma^r_s z^s$$

i.e. from (5.7),

$$f^r_{(R)} = f'^r \left(\frac{ds'}{ds}\right)^2 + \Gamma^r_s z^s. \qquad (13.8)$$

This is the type of relation we should expect, for ds'/ds is the factor relating the proper times of C' and C, and the term $\Gamma^r_s z^s$ is due to the curvature of the space.

The acceleration of C' as observed by C is $f^r_{(0)} = \ddot{z}^r$, and (13.5) can be written

$$f^r_{(R)} = f^r + f^r_{(0)} - e f^r (f_s z^s) \qquad (13.9)$$

The term $-e f^r (f_s z^s)$ gives the deviation from the addition law of accelerations. Actually, the world-line of a particle or observer differs only very slightly from a geodesic, so the v's are usually small, and the term $-e v^r (v_s z^s)$ is very small compared to the other terms. Hence, in general, the addition law of acceleration is a very good approximation.

The acceleration of C′ relative to C can be expressed in terms of V_n by a vector F^i given by $F^i = f_{(R)}^r \lambda_{r|}{}^i$. If ξ^i is the vector PQ, we have $z^r = e_r \xi^i \lambda_{r|i}$ and $\xi^i \lambda_i = 0$. Also, from (2.11), $\dot{\lambda}_{r|}{}^i = -e v_r \lambda^i$.

Hence,

$$\dot{z}^r = e_r \dot{\xi}^i \lambda_{r|i} + e_r \xi^i \dot{\lambda}_{r|i} = e_r \dot{\xi}^i \lambda_{r|i},$$

and

$$\ddot{z}^r = e_r \ddot{\xi}^i \lambda_{r|i} + e_r \dot{\xi}^i \dot{\lambda}_{r|i} = e_r \ddot{\xi}^i \lambda_{r|i} - e v^r \dot{\xi}^i \lambda_i .$$

From $\xi^i \lambda_i = 0$, we have $\dot{\xi}^i \lambda_i = -\xi^i \eta_i$, where $\eta^i = \dot{\lambda}^i$ is the curvature vector, and we get

$$\ddot{z}^r = e_r \ddot{\xi}^i \lambda_{r|i} + e v^r (\xi^i \eta_i).$$

Also, $v_r z^r = \xi^i \eta_i$ and $\sum_{r=1}^{3} e_r \lambda_{r|j} \lambda_{r|}{}^i = \delta_j^i - e \lambda^i \lambda_j$. Hence, from (13.5),

$$F^i = \ddot{\xi}^i - e(\ddot{\xi}^j \lambda_j) \lambda^i + \eta^i . \tag{13.10}$$

The term η^i is the component of the curvature vector of C, and the remaining terms give the component of the part of the vector $\ddot{\xi}^i = \Delta^2 \xi^i / \Delta s^2$ normal to C.

14. Uniform proper acceleration.

If the proper acceleration is constant at points of C, the observer C can be said to be moving with uniform proper acceleration. The conditions for this are $\dot{v}^r = 0$, $(r=1,2,3)$, i.e.

$$\frac{d}{ds}(\eta^i \lambda_{r|i}) = 0$$

We have $\dot{\lambda}_{r|}{}^i = -e v_r \lambda^i$ and $\eta^i \lambda_i = 0$. Hence, we must have $\dot{\eta}^i \lambda_{r|i} = 0$, i.e. $\dot{\eta}^i$ must be in the direction λ^i. The Serret-Frenet formulae give

$$\eta^i = \dot{\lambda}^i = e_1' \kappa \lambda'_{1|}{}^i$$

and

$$\dot{\eta}^i = e_1' \dot{\kappa} \lambda'_{1|}{}^i + e_1' \kappa \dot{\lambda}'_{1|}{}^i$$
$$= e_1' \dot{\kappa} \lambda'_{1|}{}^i - e e_1' \kappa^2 \lambda^i + e_1' e_2' \kappa \kappa_1 \lambda'_{2|}{}^i$$

where $\lambda'_{1|}{}^i$, $\lambda'_{2|}{}^i$ are the first two principal normals, and κ is the principal curvature, κ_1 the second curvature. Hence if κ is not zero, the necessary and sufficient conditions are

$$\dot{\kappa} = 0 \quad , \quad \kappa_1 = 0 \qquad (14.1)$$

i.e. If an observer is moving with uniform proper acceleration the principal curvature of his world-line is constant, and the second curvature is zero.

In the flat space of special relativity, we must have

$$\dot{\eta}^i = \frac{d^2}{ds^2}(\lambda^i) = \kappa^2 \lambda^i$$

i.e. $$\lambda^i = \alpha^i \cosh \kappa s + \beta^i \sinh \kappa s$$

where $\alpha^i \alpha_i = -\beta^i \beta_i = 1$, $\alpha^i \beta_i = 0$. The four-dimensional position vector is therefore given by

$$\underline{r} = \underline{a} + \frac{1}{\kappa}(\underline{\alpha} \sinh \kappa s + \underline{\beta} \cosh \kappa s) \qquad (14.2)$$

where $\underline{a}$ is an arbitrary constant vector, and $\underline{\alpha}, \underline{\beta}$ are two unit orthogonal vectors, $\underline{\alpha}$ being time-like and $\underline{\beta}$ space-like. The curve is therefore a hyperbola in a two dimensional plane.

For rectilinear motion, take $\underline{a}=0$ and $y=z=0$. We can write

$$\underline{\alpha} = (\cosh\alpha, \sinh\alpha, 0, 0), \quad \underline{\beta} = (\sinh\alpha, \cosh\alpha, 0, 0)$$

α being constant, and we get

$$\kappa t = \cosh\alpha \sinh\kappa s + \sinh\alpha \cosh\kappa s$$
$$\kappa x = \sinh\alpha \sinh\kappa s + \cosh\alpha \cosh\kappa s$$

i.e. $$x^2 - t^2 = \frac{1}{\kappa^2}. \qquad (14.3)$$

To find the limiting case for classical mechanics, we write $t = cT$ and, as κ is the scalar acceleration, $\kappa = \frac{F}{c^2}$ where c is the velocity of light, T is the classical time, and F the acceleration. The curve is therefore,

$$x^2 = c^2T^2 + \frac{c^4}{F^2} = \frac{c^4}{F^2}\left(1 + \frac{F^2T^2}{c^2}\right)$$

and hence

$$x = \frac{c^2}{F}\left\{1 + \frac{1}{2}\frac{F^2T^2}{c^2} + O\left(\frac{1}{c^4}\right)\right\} = \text{const} + \frac{1}{2}FT^2 + O\left(\frac{1}{c^2}\right).$$

Making $c \to \infty$, we get

$$x = \text{const.} + \frac{1}{2}FT^2, \qquad (14.4)$$

i.e. the parabola of classical mechanics.

15. Inertial frames of reference.

When the special theory of relativity was extended to the first general theory, it was assumed that at each point of a geodesic in a Riemannian space an inertial frame of reference can be found with respect to which all neighbouring geodesics have no acceleration. We shall now prove that inertial frames of reference exist along all geodesics only when the space is flat.[x]

Referring to relative co-ordinates using parallel transport along a geodesic C, the neighbouring geodesics are given by

$z^0 = 0$ and

$$\ddot{z}^r - \Gamma^r_s z^s = 0 . \qquad (15.1)$$

If an inertial frame of reference is given by $\mu_{p|}{}^i = \xi^r_p \lambda_{r|}{}^i \quad (p = 1,2,3)$, where ξ^r_p are the coefficients of an orthogonal transformation in V_3 the new system of relative co-ordinates (w) is given by $z^r = \xi^r_p w^p$, and the geodesics are solutions of

$$\xi^r_p \ddot{w}^p + 2\dot{\xi}^r_p \dot{w}^p + (\ddot{\xi}^r_p - \Gamma^r_s \xi^s_p) w^p = 0 \qquad (15.2)$$

It is therefore required that all curves given by equations of the form $w^p = \alpha^p s + \beta^p$ should be solutions of (15.2). Substituting in (15.2) and equating to zero the coefficients of α^p, β^p, we find that we must have

$$\ddot{\xi}^r_p - \Gamma^r_s \xi^s_p = 0 , \quad \dot{\xi}^r_p = 0 . \qquad (r, p = 1,2,3) \qquad (15.3)$$

The second equation shows that <u>the vectors of an inertial frame of reference must be displaced by parallel transport along the geodesic</u>. Substituting in the first equation, we find that we must have $\Gamma_{rs} = 0$, which, from (5.5), leads at once to

$$R_{\lambda i j k} \lambda^\lambda \lambda^k = 0 \qquad (i, j = 0,1,2,3) \qquad (15.4)$$

These, then, are the conditions that such a frame exists along the geodesic with tangent vector λ^i. If these conditions are required to be satisfied by all geodesics, the equations (15.4) must hold at every point and for all unit vectors λ^i. The

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[x] This result was first obtained by Prof. E. T. Whittaker.

necessary and sufficient conditions for this are

$$R_{kijh} + R_{hijk} = 0 \qquad (h,i,j,k = 0,1,2,3) \qquad (15.5)$$

We have

$$R_{hijk} = -R_{hjki} - R_{hkij} = R_{kijh} + R_{khij}$$

and by (15.5),

$$R_{khij} = -R_{jhik} = R_{kijh}.$$

hence, the conditions are

$$R_{kijh} = 0 \quad , \qquad (h,i,j,k = 0,1,2,3) \qquad (15.6)$$

i.e. the space must be flat.

16. Rigid motion.

Let $C_p, C_q, \cdots,$ be the world-lines of a system of particles sufficiently near together so that the cubes of distances between them can be neglected. Then we may say that the system of particles moves as a rigid body if an observer on each particle can find a frame of reference with respect to which all the other particles are stationary. This is evidently equivalent to saying that the geodesic distance between any two world-lines must be constant along these lines.

If C is one of the world-lines, then, referring to such a frame of reference as the above along C, the co-ordinates of points of C_p must satisfy

$$z^0_{(p)} = 0 \quad , \quad z^r_{(p)} = \text{const.}, \qquad (r = 1,2,3) \qquad (16.1)$$

and we must now see if the condition that the distance between C_p, C_q should be constant follows from (16.1).

Let the geodesic space orthogonal to C at P meet C_p at $Q_p(z_{(p)})$ and C_q at $Q_q(z_{(q)})$, and let the space orthogonal to C_p at Q_p meet C_q at Q'_q. The distances Q_pQ_q, $Q_pQ'_q$, and the angle $(Q_pQ_q, Q_pQ'_q)$ are first order quantities, and therefore, to the required order, the distance $Q_pQ'_q$ is equal to the distance Q_pQ_q. From (6.5), we see at once that, to the required order, the geodesics in normal co-ordinates are straight lines, and we therefore have

$$(Q_pQ'_q)^2 = -\sum_{r=1}^{3} e_r(z^r_{(p)} - z^r_{(q)})^2. \qquad (16.2)$$

Hence, from (16.1), the distance $Q_pQ'_q$ is constant along C_p, C_q, and this is true for all values of p,q. Thus, the necessary and sufficient conditions that the system should move as a rigid body are that, along any world-line C of the system, a frame of reference can be found with respect to which the remaining world-lines are given by equations of the form (16.1).

The above frame of reference can be considered to be fixed in the body. If it is given by Fermi transport along C, it has no proper angular velocity, and we can say that the body has no rotation relative to the particle C.

If a frame of reference is given along C, it is not necessarily fixed in the body, and the conditions that the body should be rigid become, from (16.1),

$$z^r_{(p)} = \xi^r_s w^s_{(p)} \qquad (16.3)$$

where $z^r_{(p)}$ refers to the given frame of reference, the w's are constants, and the ξ's are coefficients of an orthogonal transformation in V_3. If this frame of reference is given by Fermi transport along C, the ξ's give the rotation of the body and the components of angular velocity are, from a well-known result in classical kinematics,

$$\omega_1 = \sum_{r=1}^{3} \xi^2_r \dot{\xi}^3_r \quad , \quad \omega_2 = \sum \xi^3_r \dot{\xi}^1_r \quad , \quad \omega_3 = \sum \xi^1_r \dot{\xi}^2_r . \qquad (16.4)$$

17. Non-rotating bodies.

If C is one of the world-lines and Fermi transport is used along C, we have said that the body has no rotation relative to the particle C if the other world-lines are given by (16.1). If these conditions are satisfied, let us find the conditions that the body should have no rotation relative to another particle C′ of the body. We shall now assume that the body is small so that the squares of distances between the world-lines can be neglected.

If P(s) is a point of C, let S(P) meet C′ at the point Q(z) and the world-line $\bar{C}$ of any other particle of the body at the point $\bar{Q}(\bar{z})$. Then, from (16.1), $z^r = \text{const.}$, $\bar{z}^r = \text{const.}$, and the tangent vector of C′ at Q is therefore $(1, 0, 0, 0)$ i.e. $Q\bar{Q}$ is orthogonal to C′ at Q. Hence, if μ^σ is any vector at points of C′ orthogonal to, and given by Fermi transport along C′, we require that the projection of $Q\bar{Q}$ onto this vector at Q should be constant as P moves along C, i.e.

$$\frac{d}{ds}\{\mu_r(\bar{z}^r - z^r)\} = 0 \qquad (17.1)$$

From (6.17), we have

$$\frac{d\mu^r}{ds} = -e\mu_s(v^s\dot{z}^r - v^r\dot{z}^s) - \gamma^r_{st0}\, z^t\mu^s, \qquad (r=1,2,3)$$

and hence, as $z^r = \text{const.}$, $\bar{z}^r = \text{const.}$, we require

$$\gamma_{rst0}\, z^t\mu^s(\bar{z}^r - z^r) = 0 .$$

This must be satisfied for all such vectors μ^σ and for all curves $\bar{z}^r$ and we must therefore have

$$\gamma_{rst0}\, z^t = 0 \qquad (r,s=1,2,3). \qquad (17.2)$$

These, then, are the necessary and sufficient conditions that the body should have no rotation relative to the particle C′.

From (17.2), we see that the body has no rotation relative to any of the particles if

$$\gamma_{rst0} = 0 \qquad (r,s,t=1,2,3) \qquad (17.3)$$

referred to one of the world-lines C. We have

$\sum_{r=1}^{3} e_r \lambda_{r|i}\lambda_{r|}{}^{j} = \delta_i^j - e\lambda_i\lambda^j$, and hence, multiplying (17.3) by $e_r e_s e_t \lambda_{r|l}\lambda_{s|i}\lambda_{t|j}$ and summing for $r,s,t=1,2,3$, we finally get the conditions in the form

$$e R_{lijk}\lambda^k - \lambda_l\rho_{ij} + \lambda_i\rho_{lj} = 0 \qquad (17.4)$$

where $\rho_{ij} = R_{lijk}\lambda^l\lambda^k$, and λ^i is the tangent vector of C.

In space of constant curvature K,

$$R_{lijk} = K(g_{lj}g_{ik} - g_{lk}g_{ij}), \qquad \rho_{ij} = K(\lambda_i\lambda_j - e g_{ij})$$

and substituting in (17.4), we see that these conditions are satisfied identically. Hence, <u>if, in space of constant curvature, a body has no rotation relative to one of its particles, it has no rotation relative to any other of its particles.</u>

We shall now show that <u>if a space is such that all rigid</u>

bodies having no rotation relative to one particle have no rotation relative to the other particles, the space must be one of constant curvature.

The required conditions are that (17.4) should be satisfied for all unit vectors λ^i, i.e. substituting $e = g_{ij}\lambda^i\lambda^j$, we must have

$$(g_{\beta\gamma} R_{kij\alpha} - g_{\alpha k} R_{\beta ij\gamma} + g_{\alpha i} R_{\beta kj\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = 0$$

for all λ's. Hence,

$$\sum_{\alpha,\beta,\gamma}\left[g_{\beta\gamma} R_{kij\alpha} - g_{\alpha k} R_{\beta ij\gamma} + g_{\alpha i} R_{\beta kj\gamma}\right] = 0 \qquad (17.5)$$

where the sum is taken over all possible arrangements of α, β, γ. Multiplying by $g^{\beta\gamma}$ and summing for β, γ, we at once get

$$R_{kij\alpha} = -\tfrac{1}{3}(R_{kj}\, g_{i\alpha} - R_{ij}\, g_{k\alpha}) \qquad (17.6)$$

where $R_{ij} = g^{\beta\gamma}R_{\beta ij\gamma}$ is the Ricci tensor. Multiplying by g^{ij} and summing again, we find

$$R_{k\alpha} = \tfrac{1}{4} R\, g_{k\alpha} \qquad (17.7)$$

where $R = g^{ij}R_{ij}$ is the scalar curvature. Hence, substituting in (17.6),

$$R_{kij\alpha} = -\tfrac{1}{12} R\,(g_{kj}\, g_{i\alpha} - g_{k\alpha}\, g_{ij}) \qquad (17.8)$$

i.e. the space must be one of constant curvature.

18. A particular rigid motion.

A problem that arises naturally is to find the conditions that a system of world-lines giving rigid motion should be geodesics. Assuming the world-lines are geodesics, and using parallel transport along one of these, C, the other world-lines are solutions of

$$\ddot{z}^r - \Gamma^r{}_s z^s = 0 \qquad (18.1)$$

Thus, from (16.3),

$$z^r = \xi^r{}_p w^p \qquad (r=1,2,3)$$

must be solutions for all values of the constants w^p, i.e. the coefficients $\xi^r{}_p$ must satisfy

$$\ddot{\xi}^r{}_p - \Gamma^r{}_s \xi^s{}_p = 0 \qquad (r,p=1,2,3) \qquad (18.2)$$

We have, substituting $e_0 = e = 1$, $e_1 = e_2 = e_3 = -1$,

$$\sum_{p=1}^{3} (\xi^r{}_p)^2 = 1 \quad , \quad \sum_p \xi^r{}_p \xi^s{}_p = 0 \qquad (r \neq s;\ r,s=1,2,3) \qquad (18.3)$$

and hence, from $\Gamma_{rs} = -\Gamma^r_s$,

$$\Gamma_{rs} = -\sum_p \ddot{\xi}^r{}_p \xi^s{}_p = -\sum_p \xi^r{}_p \ddot{\xi}^s{}_p = \sum_p \dot{\xi}^r{}_p \dot{\xi}^s{}_p \,. \qquad (18.4)$$

The second form arises from $\Gamma_{rs} = \Gamma_{sr}$, and the third is found by differentiating (18.3) twice. Hence, we see that

$$\sum_p \dot{\xi}^r{}_p \dot{\xi}^s{}_p = \text{const.} \qquad (18.5)$$

i.e. from (16.4), the components $\omega_1, \omega_2, \omega_3$ of the angular velocity are constant. Thus, <u>the motion must be one with constant angular velocity about an axes fixed relative to C.</u>

Substituting from (16.4) in (18.4), we find

$$\Gamma_{rr} = \omega^2 - (\omega_r)^2 \quad , \quad \Gamma_{rs} = -\omega_r \omega_s \quad , \quad \omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \ ; \ (r \neq s,\ r,s=1,2,3). \qquad (18.6)$$

Hence, if $\Gamma_{ij} = R_{kijh} \lambda^k \lambda^h$, we have

$$\rho_{ij} = -\omega^2 \sum_{r=1}^{3} e_r \lambda_{r|i} \lambda_{r|j} - \sum_{r,s=1}^{3} e_r e_s \omega_r \omega_s \lambda_{r|i} \lambda_{s|j} .$$

The vector $\mu^i = \sum e_r \omega_r \lambda_{r|}{}^i$ is the angular velocity vector, and $\sum e_r \lambda_{r|i} \lambda_{r|j} = g_{ij} - \lambda_i \lambda_j$. We therefore have

$$\rho_{ij} = -\omega^2 g_{ij} + \omega^2 \lambda_i \lambda_j - \mu_i \mu_j . \qquad (18.7)$$

Now $g^{ij} \mu_i \mu_j = -\omega^2$, and multiplying by g^{ij} we get

$$R_{ij} \lambda^i \lambda^j = -4\omega^2 + \omega^2 + \omega^2 = -2\omega^2 .$$

The magnitude of the angular velocity is therefore given by

$$\omega^2 = -\tfrac{1}{2} R_{ij} \lambda^i \lambda^j , \qquad (18.8)$$

and the unit vector $\bar{\mu}^i$ in the direction of the axis of rotation is given by

$$\bar{\mu}_i \bar{\mu}_j = \lambda_i \lambda_j - g_{ij} - \frac{1}{\omega^2} R_{\alpha ijk} \lambda^\alpha \lambda^k . \qquad (18.9)$$

The space V_4 and the null geodesic C must satisfy certain conditions in order that the above rigid body should exist. The first set of conditions is found by eliminating the four components $\bar{\mu}_i$ from the equations (18.9), and the second set of conditions is that the ω's should be constants. The former conditions are complicated, but it may be of interest to examine the latter.

If the ω's are constant, then from (18.6)

$$\Gamma_{rs} = \text{const.} \qquad (r,s = 1,2,3) \qquad (18.10)$$

i.e. remembering that the vectors $\lambda^i, \lambda_{r|}{}^i$ are displaced by parallel transport,

$$R_{\alpha ijk,l} \lambda^\alpha \lambda^k \lambda^l \lambda_{r|}{}^i \lambda_{s|}{}^j = 0$$

from which we get

$$R_{\alpha ijk,l} \lambda^\alpha \lambda^k \lambda^l = 0 \qquad (i,j = 1,2,3) . \qquad (18.11)$$

If these conditions are satisfied by all geodesics, (18.11) must hold at each point and for all vectors λ^i. Hence, from $R_{hijk} = R_{ihkj}$, we must have

$$\sum_{h,k,l} R_{ihkj,l} = 0 \qquad (18.12)$$

where the sum is over all possible arrangements of h, k, l.

We have

$$R_{ihkj,l} = R_{ihlj,k} + R_{ihkl,j} \quad ; \quad \underset{hkl}{P}(R_{ihkl}) = 0 ,$$

so (18.12) reduce to

$$R_{ihkj,l} + R_{iklj,h} + R_{ilhj,k} = 0 \qquad (18.13)$$

Interchanging j, l and adding,

$$R_{ihkj,l} + R_{ihkl,j} + R_{ilhj,k} + R_{ijhl,k} = 0 .$$

Substituting $R_{ihkl,j} = -R_{ihlj,k} + R_{ihkj,l}$; $R_{ijhl,k} = -R_{ihlj,k} + R_{ilhj,k}$;

$$R_{ihkj,l} - R_{ihlj,k} + R_{ilhj,k} = 0 .$$

Substituting for first and third terms by (18.13),

$$R_{iklj,h} + R_{ihlj,k} = 0 \qquad (18.14)$$

Replacing the first term of (18.13) by using (18.14),

$$-R_{ilkj,h} + R_{iklj,h} + R_{ilhj,k} = 0$$

i.e.

$$-R_{ijlk,h} + R_{ilhj,k} = 0 .$$

Interchanging i, j, and adding,

$$R_{ilhj,k} + R_{jlhi,k} = 0 . \qquad (18.15)$$

But

$$R_{jlhi} = -R_{jhil} - R_{jilh} = R_{ilhj} + R_{ijlh} ,$$

and from (18.15),

$$R_{ijlh,k} = - R_{hjli,k} = R_{ilhj,k} .$$

Hence, the conditions reduce to

$$R_{ilhj,k} = 0 \qquad (i,j,h,k,l = 0,1,2,3) \qquad (18.16)$$

In space of constant curvature K, (18.16) are satisfied, but as $\Gamma_{rs} = -K\delta_{rs}$, we find from (18.6) that if any such body as the above exists, $\omega_1 = \omega_2 = \omega_3 = 0$ and we must have $K = 0$. Hence, <u>the only space of constant curvature in which such a body can exist is a flat space, and the body has no rotation relative to any of its particles</u>.

PART III.

SPATIAL DISTANCE IN GENERAL RELATIVITY.

19. Definitions of distance.

In the study of kinematics, we have already taken for granted the conception of distance in an observer's instantaneous space; such distances are small and can be considered to be immediately measurable by the observer. We now come to the problem of deciding upon the meaning of 'distance' when the object is not accessible to the observer.

The observer's knowledge of the object, or star, must be obtained from observations made by him upon rays of light issuing from the star and intersecting his instantaneous space. The tracks of rays of light in space-time are null geodesics, so if A is a position of the star and B the position of the observer when receiving light emitted from the star at the position A, the points A B must lie on a null geodesic in the space-time. We are therefore faced with the problem of defining distance along a null geodesic. We see at once that such 'distance' differs from 'interval' for the interval is zero along a null geodesic.

This problem can be treated in two different ways. The first is by reducing to mathematical terms the practical methods used by astronomers when calculating such distances. This method was first suggested and examined by Prof. E. T. Whittaker (12) and later, the finer points of his definition were discussed and improved upon by I. M. H. Etherington (13), the original idea, however, being retained. The second procedure is more fundamentally mathematical, and consists of finding expressions in terms of the positions of two points on a null geodesic, these expressions satisfying certain limiting conditions and reducing to ordinary conceptions of distance in the more elementary forms of space-time. Some work on the subject by Dr. H. S. Ruse (14 and 15) belongs to this latter class, though he deals mainly with observable quantities. Another definition of such a distance was given by Whittaker (~~1~~9).

The following work is based upon Whittaker's original definition with the modifications due to Etherington. The definition adopted is, in fact, equivalent to that of Etherington, though, by a theorem already proved on null geodesics, we are able to state the definition in a more precise way comformable to the usual conceptions of distance.

20. Apparent luminosity definition of distance.

Following Whittaker, let us examine the apparent luminosity of a star A as observed by the observer at the position B. Let C be the null geodesic AB, and let p be the thin pencil of null geodesics issuing from A and passing near B. The instantaneous space of the observer will intersect p in a two-dimensional cross section, and the apparent luminosity is evidently inversely proportional to the area of this cross section. By the last theorem of §8, this area is independent of the pafticular section at B, and hence, the apparent luminosity is independent of the observer's motion.

We can now adopt the inverse square law of light as used by astronomers, and define the distance of A from B as being proportional to the square root of the area of cross section of p at B for varying positions of B along the null geodesic C. We observe that this would be meaningless but for the above theorem. The factor of proportionality is thus independent of the particular position of B on C. To determine this factor, we shall say that when the observer is near and has the motion of the star, the above distance must reduce to the ordinary distance as measured by the observer.

With Etherington's definition of absolute luminosity, it can now easily be verified that distance as defined above is exactly that obtained by comparing apparent with absolute luminosity, giving a true translation of the practical methods used by astronomers.

With the notation of §§ 8,9,10, let s be a null parameter of C, the values at A, B being s_0, s_1 respectively. Then, defining the function V as in § 10, with s_1 for s, we see at once from the above definition that the distance so defined can be writtem

$$\Delta_{(e)} = \mu_{(e)} \sqrt{V} \tag{20.1}$$

where $\mu_{(e)}$ is independent of s_1. Let B′ be an observer at the point $s_0 + \varepsilon$ of C, where ε is small. Then from the definition of V, the distance of A from B reduces to

$$\Delta'_{(e)} = \mu_{(e)}\,\mathcal{E}. \tag{20.2}$$

If the star and the observer B′ both have the motion A^i where A^i is the unit vector at A tangent to the star's world-line, the distance δ between the star and B in the instantaneous space of this observer is

$$\delta = \mathcal{E}\,|g_{ij}A^i\lambda^j|_{s=s_0} \tag{20.3}$$

where λ^i is the tangent vector of C. But from the definition of distance $\Delta'_{(e)} = \delta$. Hence,

$$\mu_{(e)} = |g_{ij}A^i\lambda^j|_{s=s_0} \tag{20.4}$$

and from (20.1), we now have the full expression for spatial distance as defined above.

We observe that V and $\mu_{(e)}$ are not completely determined owing to the arbitrary multiplicative constant that can be attached to the parameter s. This constant, however, is cancelled in the product $\mu_{(e)}\sqrt{V}$ and leaves us with an invariant expression for $\Delta_{(e)}$.

From the fact that the apparent luminosity of the star is independent of the observer's motion, we see that distance as defined above is also independent of this motion, but is dependent on that of the star.

21. Apparent magnitude definition of distance.

Another practical method of calculating distances is by using the inverse square law for magnitudes instead of luminosities. When reducing this method to mathematical terms, we need not consider the star to have a definite magnitude, or rather, we need only consider any arbitrary small part of the star. We therefore consider the thin pencil of null geodesics passing near A and converging to pass through the point B, and define distance to be proportional to the square root of the cross section of this pencil at A. The factor of proportionality will now be determined by saying that for a star near and with the motion of the observer, the above distance must reduce to that measured by the observer in his instantaneous space. Thus, distance is now independent of the motion of the star, but dependent on the motion of the observer.

To calculate this distance, we need only repeat the previous calculation, interchanging the star and the observer. From the theorem of § 10, V is unaltered when s_0 and s_1 are interchanged. We can therefore write at once the expression for this distance in the form

$$\Delta_{(m)} = \mu_{(m)} \sqrt{V} \qquad (21.1)$$

where

$$\mu_{(m)} = | g_{ij} B^i \lambda^j |_{s=s_1} \qquad (21.2)$$

and B^i is the unit vector giving the motion of the observer at B. This, then, is the distance found in practice by comparing the apparent with the actual magnitude of the star.

From (20.1), (20.2), we see at once that the two expressions for distance are connected by the relation

$$\frac{\Delta_{(l)}}{\Delta_{(m)}} = \frac{\mu_{(l)}}{\mu_{(m)}} \qquad (21.3)$$

It has been proved[x] that the *Doppler* effect observed by the observer is, with the above notation,

$$\frac{|g_{ij} A^i \lambda^j|_{s=s_0}}{|g_{ij} B^i \lambda^j|_{s=s_1}} \qquad (21.4)$$

Hence, the ratio of the distances as given by the luminosity and magnitude definitions is the *Doppler* effect. This result was also obtained by Etherington (13).

22. Calculation of distance.

From the above results, the main problem is to calculate the function V. From §9, the simplest method is to find a vector $\lambda_{2|}{}^i$ orthogonal to and displaced by parallel transport along C, and then choose $\lambda_{3|}{}^i$ to be orthogonal to $\lambda_{2|}{}^i$ and λ^i, the tangent vector of C. We then calculate the three invariants Γ_{22}, Γ_{23}, Γ_{33} from

$$\Gamma_{rs} = R_{kijh} \lambda^l \lambda^h \lambda_{r|}{}^i \lambda_{s|}{}^j \qquad (22.1)$$

and find solutions of the equations

$$\begin{aligned} \ddot{z}^2 + \Gamma_{22} z^2 + \Gamma_{23} z^3 &= 0 \\ \ddot{z}^3 + \Gamma_{23} z^2 + \Gamma_{33} z^3 &= 0 \end{aligned} \qquad (22.2)$$

in the form

$$z^2 = \alpha\varphi_1 + \beta\psi_1 , \qquad z^3 = \alpha\varphi_2 + \beta\psi_2 \qquad (22.3)$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ vanish when $s = s_0$, and α, β are arbitrary constants. Then, from §10, we have

$$V = K(\varphi_1\psi_2 - \varphi_2\psi_1) \qquad (22.4)$$

where $$\frac{1}{K} = \lim_{s \to s_0} \frac{\varphi_1\psi_2 - \varphi_2\psi_1}{(s-s_0)^2}.$$

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x (9), p.39.

Here we have assumed that s increases from A to B, i.e. in the direction of the light.

As an example, consider a geodesic C in space of constant curvature K. We have

$$\Gamma_{rs} = K(g_{\lambda j}g_{ik} - g_{\lambda k}g_{ij})\lambda^{j}\lambda^{k}\lambda_{r|}{}^{\lambda}\lambda_{s|}{}^{i} = 0 \qquad (22.5)$$

and hence, at once,

$$z^2 = \alpha(s - s_0) \qquad\qquad z^3 = \beta(s - s_0)$$

and

$$v = (s_1 - s_0)^2$$

Hence

$$\Delta = \mu(s_1 - s_0). \qquad (22.6)$$

Thus <u>in space of constant curvature, the element of spatial distance is proportional to the element of the null parameter.</u>

Distance as defined by Ruse leads to this result in general space.[x1] It was also proposed as an alternative definition of distance by E. T. Whittaker.[x2] It may be of interest to prove Ruse's result by using relative co-ordinates along C.

A definition of distance as proposed by Ruse is as follows: Let $B_0, B_1, B_2 \cdots, B$ be a series of neighbouring observers at positions along the null geodesic C, extending from A to B. Also, let the directions of their world-lines at points of C be obtained by parallel transport along C from the given

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[x1] (15), p.86.

[x2] (9), p.35.

direction at B. Then if each observer measures the distance from himself to the next observer in his instantaneous space, define the distance from A to B as being the sum of all such small distances.

Let $\lambda_{01}^{\,i}$ at points C be the directions of the above world-lines, and choose relative co-ordinates as in §9, so that $u_2 = u_3 = 0$, $u_0 = u_1 = \text{const.}$, the latter following from the fact that $\lambda_{01}^{\,i}$ is given by parallel transport along C. Then the distance between observers at points s, $s + ds$ as measured by the observer at s is simply $u_1 ds$ and integrating along C from s_0 to s_1, we have, as $u_1 = u_0 = (g_{ij} B^i \lambda^j)_{s=s_1} = \mu_{(m)}$ for $(B^i = \lambda_{01}^{\,i})$

$$\Delta = \mu_{(m)} (s_1 - s_0). \tag{22.7}$$

23. Spatial distance in an expanding universe.

Consider space of the form

$$\nu\, dt^2 - a^2 \left(\frac{1}{y} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) \tag{23.1}$$

where ν, y are functions of the co-ordinate r, and (a) is a function of t. Such space has spherical symmetry in the 'spatial' co-ordinates r, θ, ϕ, and this form includes all forms considered up to date in work on relativity, the older statical universes being given by $a = \text{const.} = 1$. For $a = 1$; $\nu = 1$, $y = 1 - r^2/R^2$, we have the Einstein universe, and for $\nu = y = 1 - r^2/R^2$ we have the De Sitter universe. For other values of ν, y we have various forms, including those with a singularity at the origin.[x]

We shall now calculate the formula for spatial distance in the above general space-time.

First we must find the null geodesics, these being given in terms of a null parameter s by $T=0$ and

$$\frac{d}{ds}\left(\frac{\partial T}{\partial t'}\right) - \frac{\partial T}{\partial t} = 0$$

and the three other similar equations, where $t' = dt/ds$, and

$$T = \tfrac{1}{2}\nu\left(\frac{dt}{ds}\right)^2 - a^2\left\{\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right\} \qquad (23.2)$$

The third equation is

$$\frac{d}{ds}\left(a^2r^2\frac{d\theta}{ds}\right) - a^2r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 .$$

This is satisfied identically by $\theta = \pi/2$ and hence, from the spherical symmetry of the space, all null geodesics lie in such sub-spaces. Thus, when examining a null geodesic, we can take it to lie in this space $\theta = \pi/2$ this being a typical geodesic from the above symmetry. Substituting $\theta = \pi/2$, the fourth equation becomes

$$\frac{d}{ds}\left(a^2r^2\frac{d\phi}{ds}\right) = 0$$

so we have

$$\frac{d\phi}{ds} = \frac{h}{a^2r^2} \qquad (23.3)$$

where h is a constant, taken to be positive so that ϕ increases with s. The first equation is now

$$\frac{d}{ds}\left(\nu\frac{dt}{ds}\right) + a\dot{a}\left\{\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2\right\} = 0 \qquad , \dot{a} = da/dt ,$$

but, from $T=0$,

$$\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 = \frac{\nu}{a^2}\left(\frac{dt}{ds}\right)^2 \qquad (23.4)$$

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[x] See (11), (16), (17), (18) and (19).

Hence,

$$\frac{d}{ds}\left(\nu\frac{dt}{ds}\right) + \frac{\nu}{a}\cdot\frac{da}{ds}\cdot\frac{dt}{ds} = 0$$

i.e.

$$a\nu\frac{dt}{ds} = \text{const.}.$$

We can choose s so that this constant is unity, s then increasing with t as required, and we have

$$\frac{dt}{ds} = \frac{1}{a\nu}. \tag{23.5}$$

Substituting (23.3), (23.5) in (23.4), we find

$$\frac{dr}{ds} = e\frac{\psi}{a^2}\sqrt{\frac{\gamma}{\nu}}, \qquad \psi^2 = 1 - h^2\frac{\nu}{r^2}, \quad e = \pm 1. \tag{23.6}$$

Thus, the tangent vector λ^i is given by

$$\lambda^0 = \frac{dt}{ds} = \frac{1}{a\nu}\ ; \quad \lambda^1 = \frac{dr}{ds} = \frac{e\psi}{a^2}\sqrt{\frac{\gamma}{\nu}}\ ; \quad \lambda^2 = \frac{d\theta}{ds} = 0\ ; \quad \lambda^3 = \frac{d\phi}{ds} = \frac{h}{a^2r^2}. \tag{23.7}$$

The constant h varies for the different geodesics in the space $\theta = \pi/2$.

Following the method outlined in §22, we now require a vector $\lambda_{2|}{}^i$ orthogonal to λ^i and satisfying the parallel transport equations along this geodesic. From the geometry it is evident that the normal to the space $\theta = \pi/2$ will satisfy these conditions,* so we have, making this vector of unit length by (23.1),

$$\lambda_{2|}{}^0 = 0\ ; \quad \lambda_{2|}{}^1 = 0\ ; \quad \lambda_{2|}{}^2 = \frac{1}{ar}\ ; \quad \lambda_{2|}{}^3 = 0. \tag{23.8}$$

All that now remains is to find a vector $\lambda_{3|}{}^i$ orthogonal to λ^i and $\lambda_{2|}{}^i$. Such a vector is at once found to be given by

$$\lambda_{3|}{}^0 = \frac{h}{r\psi}, \quad \lambda_{3|}{}^1 = 0, \quad \lambda_{3|}{}^2 = 0, \quad \lambda_{3|}{}^3 = \frac{1}{ar\psi} \tag{23.9}$$

where ψ is given by (23.6).

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* This can easily be verified by calculating the Christoffel symbols from (23.1), and substituting in the equations

$$\frac{d\nu^i}{ds} + \Gamma^i_{jk}\nu^j\lambda^k = 0.$$

From (23.1), the non-vanishing components of the curvature tensor are[×]

$$\left.\begin{array}{ll} R_{0221} = \frac{1}{2} r^2 \frac{\nu'}{\nu} a \dot{a} , & R_{0331} = \frac{1}{2} r^2 \frac{\nu'}{\nu} a \dot{a} \sin^2\theta , \\ R_{1001} = \frac{1}{4}\left(2\nu'' - \frac{\nu'^2}{\nu} + \frac{\nu' y'}{y}\right) - \frac{1}{y} a \ddot{a} , & R_{3223} = r^2 a^2 \left(1 - y + \frac{r^2}{\nu} \dot{a}^2\right) \sin^2\theta , \\ R_{2002} = r\left(\frac{1}{2} y \nu' - r a \ddot{a}\right) , & R_{3003} = r\left(\frac{1}{2} y \nu' - r a \ddot{a}\right) \sin^2\theta , \\ R_{2112} = \frac{r}{y} a^2 \left(-\frac{1}{2} y' + \frac{r}{\nu} \dot{a}^2\right) , & R_{3113} = \frac{r}{y} a^2 \left(-\frac{1}{2} y' + \frac{r}{\nu} \dot{a}^2\right) \sin^2\theta . \end{array}\right\} \quad (23.10)$$

Substituting (23.10), (23.7), (23.8) and (23.9) in (22.1), we can find the invariants $\Gamma_{22}, \Gamma_{23}, \Gamma_{33}$. We shall omit this purely algebraic work. The terms so obtained can very conveniently be collected and re-expressed by using the differential relations (23.7). We find

$$\left.\begin{array}{l} \Gamma_{23} = 0 , \\ \Gamma_{22} = -\frac{1}{ar} \frac{d^2}{ds^2}(ar) + \frac{h^2}{a^4 r^4} , \\ \Gamma_{33} = -\frac{1}{ar} \frac{d^2}{ds^2}(ar) - \frac{h^2}{a^4 r^4} \cdot \frac{1}{\psi} \frac{d^2 \psi}{d\varphi^2} , \end{array}\right\} \quad (23.11)$$

and if $h=0$, the Γ's are given correctly by writing $h=0$ in the above expressions.

I. $\underline{h \neq 0}$.

The geodesic does not pass through origin $(r=0)$ so we can take φ as the independent variable. Writing $z = arx$ we find, from (23.7),

$$\frac{d^2 z}{ds^2} = \frac{h^2}{a^3 r^3} \frac{d^2 x}{ds^2} + x \frac{d^2}{ds^2}(ar) .$$

Hence, from (22.2) and (23.11), $z^2 = arx$ where

$$\frac{d^2 x}{d\varphi^2} + x = 0$$

× These can easily be calculated by using (4), p.44.

Thus, in the required form,

$$Z^2 = \alpha . \, a r \sin(\phi-\phi_0) . \tag{23.12}$$

Also, $Z^3 = a r y$ where

$$\frac{d^2 y}{d\phi^2} - \frac{y}{\psi}\frac{d^2\psi}{d\phi^2} = 0$$

Thus,

$$Z^3 = \beta . \, a r \psi \int_{\phi_0}^{\phi} \frac{d\phi}{\psi^2} . \tag{23.13}$$

Writing

$$U = \int^{\phi} \frac{d\phi}{\psi^2} , \tag{23.14}$$

we have

$$V = K a_1^2 r_1^2 \psi_1 (U_1 - U_0) \sin(\phi_1 - \phi_0)$$

where

$$\frac{1}{K} = \lim_{s \to s_0} \frac{a^2 r^2 \psi (U - U_0) \sin(\phi - \phi_0)}{(s-s_0)^2}$$

$$= a_0^2 r_0^2 \psi_0 \left(\frac{dU}{ds}\right)_0 \left(\frac{d\phi}{ds}\right)_0 .$$

From (23.7), (23.14), $\frac{dU}{ds} = \frac{1}{\psi^2}\frac{d\phi}{ds}$, $\frac{d\phi}{ds} = \frac{h}{a^2 r^2}$, and hence,

$$K = \frac{1}{h^2} a_0^2 r_0^2 \psi_0 .$$

Thus,

$$V = \frac{1}{h^2} a_1^2 a_0^2 r_1^2 r_0^2 \psi_1 \psi_0 (U_1 - U_0) \sin(\phi_1 - \phi_0) \tag{23.15}$$

and

$$\Delta = \mu . \frac{1}{h} a_1 a_0 r_1 r_0 \sqrt{\psi_1 \psi_0 (U_1 - U_0) \sin(\phi_1 - \phi_0)} \tag{23.16}$$

where μ is determined by (23.1), (23.7) and the given motion of the star at $s = s_0$.

II. $h = 0$

In this case, $d\theta/ds = 0$ and the geodesic passes through the origin. We have $\Gamma_{23} = 0$ and

$$\Gamma_{22} = \Gamma_{33} = -\frac{1}{ar}\frac{d^2}{ds^2}(ar).$$

We must therefore solve

$$\frac{d^2 z}{ds^2} - \frac{z}{ar}\frac{d^2}{ds^2}(ar) = 0$$

and we get

$$z = \alpha . \, ar \int_{s_0}^{s} \frac{ds}{a^2 r^2} . \qquad (23.17)$$

Hence

$$V = K \, a_1^2 r_1^2 (w_1 - w_0)^2$$

where

$$w = \int^{s} \frac{ds}{a^2 r^2} \qquad (23.18)$$

and

$$\frac{1}{K} = \lim_{s \to s_0} \frac{a^2 r^2 (w - w_0)^2}{(s - s_0)^2}$$

$$= a_0^2 r_0^2 \left(\frac{dw}{ds}\right)_0^2 = \frac{1}{a_0^2 r_0^2} .$$

Thus,

$$V = a_1^2 a_0^2 r_1^2 r_0^2 (w_1 - w_0)^2 \qquad (23.19)$$

and

$$\Delta = \mu . \, a_1 a_0 r_1 r_0 (w_1 - w_0) \qquad (23.20)$$

We now have $h = 0$ so $\psi = 1$ and $\frac{dr}{ds} = e \frac{1}{a^2}\sqrt{\frac{\gamma}{\nu}}$. Hence, from (23.18),

$$w = e \int^{r} \sqrt{\frac{\nu}{\gamma}} \frac{dr}{r^2} . \qquad (23.21)$$

where it is assumed that the star and observer are not separated by the origin, so e does not change sign.

It may happen that the observer is at the origin, so that $r_1 = 0$. In this case, $e = -1$ and we must find

$$w' = \lim_{\varepsilon \to 0} \varepsilon \int_\varepsilon^{r_0} \sqrt{\frac{v}{y}} \frac{dr}{r^2}.$$

We have

$$\int_\varepsilon^{r_0} \sqrt{\frac{v}{y}} \frac{dr}{r^2} = \left[-\sqrt{\frac{v}{y}} \cdot \frac{1}{r} \right]_\varepsilon^{r_0} + \int_\varepsilon^{r_0} \frac{d}{dr}\sqrt{\frac{v}{y}} \cdot \frac{dr}{r}.$$

If $\left| \frac{d}{dr}\sqrt{\frac{v}{y}} \right| \leq M$ for $0 \leq r \leq r_0$,

$$\left| \int_\varepsilon^{r_0} \frac{d}{dr}\sqrt{\frac{v}{y}} \frac{dr}{r} \right| \leq M \left| \int_\varepsilon^{r_0} \frac{dr}{r} \right| = M \log \frac{r_0}{\varepsilon}.$$

Hence, as $\lim_{\varepsilon \to 0} \varepsilon \log \varepsilon = 0$, we have

$$w' = \lim_{\varepsilon \to 0} \varepsilon \left[-\sqrt{\frac{v}{y}} \frac{1}{r} \right]_\varepsilon^{r_0} = \left(\sqrt{\frac{v}{y}} \right)_{r=0}.$$

Thus, writing r for r_0,

$$\Delta = \mu \left(a\sqrt{\frac{v}{y}} \right)_{r=0} a r. \qquad (23.22)$$

A similar result holds if the star is at the origin, and the observer at the point r.

It is of interest to examine the above general results before proceeding to discuss particular examples. From (22.7), $\frac{dr}{d\varphi} = \frac{c}{h} r^2 \psi \sqrt{\frac{v}{y}}$. Hence, if the path of the ray of light in space (r, θ, φ) is known, the time function (a) occurs in the formula (23.20) for distance only in the part $\mu a_1 a_0$, and a similar result holds if $h = 0$.

If the star is 'stationary in space', its world-line satisfies $r, \theta, \varphi = \text{const}$ and hence, from (23.1)

$$A^0 = \frac{1}{\sqrt{v_0}}, \quad A^1 = A^2 = A^3 = 0.$$

Thus, from (20.4)

$$\mu_{(e)} = \frac{1}{a_0 \sqrt{v_0}},$$

and, if $h \neq 0$,

$$\Delta_{(e)} = \frac{1}{h} a_1 r_1 \frac{r_0}{\sqrt{\nu_0}} \sqrt{\psi_1 \psi_0 (v_1 - v_0) \sin(\theta_1 - \theta_0)} \qquad (23.23)$$

If $h = 0$,

$$\Delta_{(e)} = a_1 r_1 \frac{r_0}{\sqrt{\nu_0}} (w_1 - w_0) \qquad (27.24)$$

and if the observer is at the origin and the star at the point r,

$$\Delta_{(e)} = \left(a\sqrt{\frac{\nu}{y}}\right)_{r=0} \frac{r}{\sqrt{\nu}}. \qquad (27.25)$$

Similarly, if the star is at the origin and the observer at the point r,

$$\Delta_{(e)} = \left(\frac{1}{\sqrt{y}}\right)_{r=0} a r. \qquad (27.26)$$

If the observer is stationary in space, we have

$$B^0 = \frac{1}{\sqrt{\nu_1}}, \quad B^1 = B^2 = B^3 = 0$$

and

$$\mu_{(m)} = \frac{1}{a_1\sqrt{\nu_1}}.$$

Thus, if $h \neq 0$,

$$\Delta_{(m)} = \frac{1}{h} a_0 r_0 \frac{r_1}{\sqrt{\nu_1}} \sqrt{\psi_1 \psi_0 (v_1 - v_0) \sin(\theta_1 - \theta_0)} \qquad (27.27)$$

If $h = 0$,

$$\Delta_{(m)} = a_0 r_0 \frac{r_1}{\sqrt{\nu_1}} (w_1 - w_0) \qquad (27.28)$$

and if the star is at the origin and the observer at the point r,

$$\Delta_{(m)} = \left(a\sqrt{\frac{\nu}{y}}\right)_{r=0} \frac{r}{\sqrt{\nu}}. \qquad (27.29)$$

Similarly, if the observer is at the origin, the star at the point r,

$$\Delta_{(m)} = \left(\frac{1}{\sqrt{y}}\right)_{r=0} a r. \qquad (27.30)$$

If the universe is static, $a = 1$ and a typical null geodesic is given by $\theta = \pi/2$ and

$$\frac{dt}{ds} = \frac{1}{\nu}, \quad \frac{dr}{ds} = e\psi\sqrt{\frac{y}{\nu}}. \quad \frac{d\phi}{ds} = \frac{h}{r^2}.$$

We now get, if $h \neq 0$,

$$\Delta = \mu \cdot \frac{1}{h} r_1 r_0 \sqrt{\psi_1 \psi_0 (V_1 - V_0) \sin(\phi_1 - \phi_0)} \qquad (23.31)$$

If $h = 0$,

$$\Delta = \mu \cdot r_1 r_0 (w_1 - w_0) \qquad (23.32)$$

and if the observer is at the origin, the star at the point r,

$$\Delta = \mu \left(\sqrt{\frac{\nu}{y}}\right)_{r=0} r \qquad (23.33)$$

with a similar result if the star is at the origin and the observer at the point r.

If the observer is at the origin, $\mu_{(m)}$ is evaluated at the origin, and hence, from (23.33), the distance $\Delta_{(m)}$ is proportion to the value of r at the position of the star. Similarly, if the star is at the origin, $\Delta_{(e)}$ is proportional to the value of r at the position of the observer.

24. Particular cases of the general form.

I. $$dt^2 - a^2\left(\frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \qquad (24.1)$$

This is the form usually considered in discussions on the expanding universe,[x1] and when $a = \text{const.} = 1$, it reduces to the Einstein cylindrical universe.[x2]

When $h \neq 0$, we have $\nu = 1$, $y = 1 - \frac{r^2}{R^2}$, $\psi^2 = 1 - \frac{h^2}{r^2}$, and $\frac{dr}{d\phi} = \frac{e}{h} \psi r^2 \sqrt{y}$. Hence

$$V = \int^{\phi} \frac{d\phi}{\psi^2} = \frac{eh}{R} \int^{r} \sqrt{\frac{R^2 - r^2}{r^2 - h^2}} \cdot r^3 \, dr. \qquad (24.2)$$

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x1. See Lemaître, (20) & Eddington, (21).

x2. R is the 'radius' of the Einstein world, and in the expanding form, [a(t) R] is the radius at time t.

This integral can easily be evaluated, and hence Δ can be found explicitly when the motion of the star (or observer) is known.

When $k=0$, then from (23.21) it is required to evaluate

$$w = e\int^r \frac{dr}{r^2\sqrt{1-\frac{r^2}{R^2}}}.$$

Writing $r = R\sin\chi$,

$$w = \frac{e}{R}\int\frac{d\chi}{\sin^2\chi} = -\frac{e}{R}\cot\chi$$

and hence,

$$r_1 r_0\,|w_1 - w_0| = R\,|\sin(\chi_1-\chi_0)|.$$

Thus

$$\Delta = \mu R\,a_1 a_0\,|\sin(\chi_1-\chi_0)| \qquad (24.3)$$

We observe that χ is the co-ordinate often used instead of r, the metric (24.1) then being

$$dt^2 - R^2a^2(d\chi^2 + \sin^2\chi\,d\theta^2 + \sin^2\chi\sin^2\theta\,d\phi^2) \qquad (24.4)$$

It has been pointed out[x] that r is a misleading co-ordinate to use in describing the whole universe, for it is required that χ should take the values 0 to π. The other co-ordinate proposed is $\rho = 2R\tan\frac{\chi}{2}$, when (24.1) takes the form

$$dt^2 - \frac{a^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2}\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta\,d\phi^2\right). \qquad (24.5)$$

When an observer or star is at the origin, the other being at the point χ,

$$\Delta = \mu R\,a_1 a \sin\chi \qquad ,\ a_1 = (a)_{r=0} \qquad (24.6)$$

or, using the co-ordinate ρ,

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[x] McCrea and McVittie, (17), p. 9. These authors use q and r where we use r and ρ respectively.

$$\Delta = \mu\, a_1 a \frac{\rho}{1+\frac{\rho^2}{4R^2}} \,. \qquad (24.7)$$

If the observer is stationary at the origin, and the star stationary at the point χ,

$$\Delta_{(e)} = R a_1 \sin\chi = a_1 \rho/(1+\tfrac{\rho^2}{4R^2}) \,, \qquad (24.8)$$

$$\Delta_{(m)} = R a \sin\chi = a \rho/(1+\tfrac{\rho^2}{4R^2}) \,. \qquad (24.9)$$

From $\rho = 2R\tan\frac{\chi}{2}$, we see that ρ can take the values $0-\infty$. But keeping the time of observation fixed, a_1 is constant and from (24.8), the maximum value of $\Delta_{(e)}$ is given by a star in the position $\rho = 2R$, i.e. $\chi = \pi/2$. For this star

$$\Delta_{(e)} = a_1 R \,, \qquad \Delta_{(m)} = a R \qquad (24.10)$$

i.e. <u>the maximum observable distance $\Delta_{(e)}$ is equal to the radius of the universe at the time of observation, and the distance $\Delta_{(m)}$ is then the radius of the universe at the time of emission of the observed light.</u>

Again, from (24.8), there are two positions in which a star gives an asigned distance $\Delta_{(e)} < a_1 R$; for one position, $\rho < 2R$ and for the other, $\rho > 2R$. The two stars can be distinguished, however, by observing the Doppler effect now equal to a_1/a. If the universe is continually expanding and the time of observation remains fixed, (a) decreases as ρ increases, so the Doppler effect for the star $\rho > 2R$ is greater than that for the stars $\rho < 2R$.

II. Statical universes.

All the usual forms of statical universes are obtained from (23.1) by writing $a=1$. The two most common are the Einstein and the De Sitter forms. The Einstein universe is given by

$$dt^2 - \left(\frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \qquad (24.11)$$

and distances in this space can be obtained from those immediately above by writing $a=1$. Thus, from (24.8) and (24.9), if the star and observer are both stationary and the observer is at the origin, $\Delta_{(e)} = \Delta_{(m)} = \rho/\left(1+\frac{\rho^2}{4R^2}\right)$, and the maximum value of either is R, given by $\rho = 2R$, or $\chi = \pi/2$.

The De Sitter universe is one of constant curvature, and has the form

$$\left(1-\frac{r^2}{R^2}\right)dt^2 - \left(\frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \qquad (24.12)$$

Thus $\gamma = y = 1 - r^2/R^2$. Along the null geodesic through the origin, we find at once, from (23.21) and (23.20)

$$\Delta = \mu_\cdot |r_0 - r_1|$$

it being assumed that the star and observer are not separated by the origin. This result is also given by (22.6), for we now have $dr/ds = \pm 1$. If the star and observer are both fixed in space,

$$\mu_{(e)} = \frac{1}{\sqrt{1-\frac{r_0^2}{R^2}}} \quad , \quad \mu_{(m)} = \frac{1}{\sqrt{1-\frac{r_1^2}{R^2}}}$$

and, if $r_0 > r_1$,

$$\Delta_{(e)} = \frac{r_0 - r_1}{\sqrt{1-\frac{r_0^2}{R^2}}} \quad , \quad \Delta_{(m)} = \frac{r_0 - r_1}{\sqrt{1-\frac{r_1^2}{R^2}}} . \qquad (24.13)$$

Hence, if the observer is at the origin and the star at the point r,

$$\Delta_{(l)} = \frac{r}{\sqrt{1-\frac{r^2}{R^2}}}, \quad \Delta_{(m)} = r. \qquad (24.14)$$

Thus when r has its maximum value R, $\Delta_{(m)} = R$ but $\Delta_{(l)}$ is infinite.

The above are only a few of the general results obtained when the astronomical method of measuring distance is analysed mathematically in the light of modern relativity. Several interesting results are obtained when distance is discussed in more detail in particular forms of space-time; for example, peculiar results arise when the space is the gravitational field of the sun and we consider the distance of the stars as observed on the earth when the light grazes the sun. In this case, the gravitational field of the sun acts as a convex cylindrical lens and the pencil of rays of light issuing from a star begins to converge after passing the sun. It may be noticed that results such as this should be quite common, for in (23.12), the expression for z^2 contains the term $\sin(\phi-\phi_0)$. Thus we may expect z^2 to decrease after some point, and the cross-section of the pencil will become elongated, the area decreasing.

REFERENCES

1. A. G. Walker: Relative co-ordinates. Proc.Roy.Soc. Edin., 52(1932),345-353.
2. A. G. Walker: Spatial distance in general relativity. Quart.Journ.of Maths.,4(1933),71-80.
3. L. P. Eisenhart: Differential geometry of curves and surfaces, (1909).
4. L. P. Eisenhart: Riemannian geometry, (1926).
5. E. Fermi: Sopra i fenomeni che avvengono in vicinanza di una linea oraria. Rendiconti dei Lincei, 31 (1922).
6. O. Veblen: Invariants of quadratic differential forms. Cambridge Tract, No.24.
7. T. Levi-Civita: Absolute differential calculus, (1927).
8. J. L. Synge and A. J. M'Connell: Phil.Mag., (7),5(1928),241-263
9. W. O. Kermack, W. H. M'Crea, and E. T. Whittaker: On properties of null geodesics, and their application to the theory of radiation. Proc.Roy.Soc.Edin., 53(1933),31-47.
10. Sir A. S. Eddington: Time, space, and gravitation, (1929).
11. " " Mathematical theory of relativity,(1930
12. E. T. Whittaker: On the definition of distance in curved space, and the displacement of the spectral lines of distant sources. Proc.Roy.Soc.,A, 133(1931),93-105.
13. I. M. H. Etherington: On the definition of distance in general relativity. Phil.Mag., (1933).

14. H. S. Ruse: On the definition of spatial distance in general relativity. Proc.Roy.Soc.Edin.,52(1932),184-196.

15. H. S. Ruse: On the measurement of spatial distance in a curved space-time. Proc.Roy.Soc.Edin., 53(1933),79-88.

16. W. H. McCrea, and G. C. McVittie: On the contraction of the universe. M.N.R.A.S., 91(1930),128-133.

17. W. H. McCrea, and G. C. McVittie: The expanding universe. M.N.R.A.S., 92(1931),7-12.

18. G. C. McVittie: The problem of n bodies and the expansion of the universe. M.N.R.A.S., 91(1931),274-283.

19. G. C. McVittie: Condensations in an expanding universe. M.N.R.A.S., 92(1932),500-518.

20. G. Lemaître: Ann.Soc.Scientifique de Bruxelles,47A,(1927),49.

21. Sir A. S. Eddington: M.N.R.A.S., 90(1930),668.