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**Well-posedness of nonlinear  
Schrödinger equations from  
deterministic and probabilistic  
viewpoints**

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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

*(Ruoyuan Liu)*



# Abstract

In this thesis, we study the Cauchy problem for nonlinear Schrödinger equations (NLS) in various settings.

Firstly, we consider NLS with a quadratic nonlinearity  $|u|^2$  on the two-dimensional torus. By separately estimating the contributions from the nearly resonant and highly non-resonant interactions, we prove its sharp deterministic local well-posedness, thus resolving an open problem of thirty years since Bourgain (1993).

Secondly, we investigate the well-posedness issues of NLS with a quadratic nonlinearity  $\bar{u}^2$  in negative Sobolev spaces on the one-dimensional and the two-dimensional tori. By introducing modified versions of the Fourier restriction norm spaces and overcome the failure of the crucial bilinear estimates, we establish deterministic local well-posedness in negative Sobolev spaces.

Thirdly, we come back to study NLS with the quadratic nonlinearity  $|u|^2$  on the two-dimensional torus with random initial data distributed according to a fractional derivative of the Gaussian free field. We prove almost sure local well-posedness below the deterministic threshold and a probabilistic ill-posedness result when the random initial data becomes too irregular.

Finally, we consider the dispersive Anderson model, namely NLS with a multiplicative spatial white noise, on the two-dimensional Euclidean space. We prove its global well-posedness by using a gauge-transform and constructing the solution as a limit of solutions to a family of approximating equations.



# Lay summary

In this thesis, we study nonlinear Schrödinger equations in various settings. Nonlinear Schrödinger equations are typical examples of nonlinear dispersive partial differential equations (PDEs), which model wave-like phenomena that appear ubiquitously in physics and engineering such as quantum mechanics, plasma physics, nonlinear optics, water waves, etc.

We explore the fundamental aspects on nonlinear Schrödinger equations: existence (the PDE has a solution given a certain initial condition), uniqueness (the solution of the PDE is the only solution), and stability (the solution has a small change if we perturb the initial condition). Our goal is to find the lowest condition on the initial data so that the above properties hold for a certain equation.

Moreover, we incorporate randomness into nonlinear Schrödinger equations, either via random initial condition or via a random forcing in the equation. From the physical point of view, the randomness models the disturbance in a physical system in real world applications. Under certain level of randomness, we are still able to establish existence, uniqueness, and stability of nonlinear Schrödinger equations.



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# Chapter 1

## Introduction

In this thesis, we study well-posedness issues of nonlinear dispersive partial differential equations (PDEs). In particular, we focus on the Cauchy problem for the nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u = \Delta u + F(u, \xi) \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathcal{M}, \quad (1.1)$$

where  $\xi$  denotes a stochastic forcing,  $F$  is a general forcing term depending on  $u$  and  $\xi$ , and  $\mathcal{M} = \mathbb{R}^d$  or  $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$  for some dimension  $d \geq 1$ .

The Schrödinger equations are typical examples of dispersive PDEs, which have wave-like solutions with speed depending on their frequencies; see [118, 111, 38] for more discussion on dispersive PDEs. Other examples of dispersive PDEs include wave equations, Korteweg-de Vries equations, beam equations, etc. These dispersive PDEs appear ubiquitously in physics and engineering, such as quantum mechanics, plasma physics, nonlinear optics, water waves, etc.

One of the central questions in the field of dispersive PDEs is *well-posedness* of a PDE: existence of a solution, uniqueness of the solution, and continuous dependence of solutions on initial data. A first step for studying well-posedness of a dispersive PDE, such as NLS in (1.1), is to construct a local-in-time solution, which we refer to as *local well-posedness*. An important tool for studying local well-posedness of NLS (1.1) is the  $L^2$ -based Sobolev space  $H^s(\mathcal{M})$  defined via the norm

$$\|f\|_{H^s(\mathcal{M})} = \|\langle \nabla \rangle^s f\|_{L^2(\mathcal{M})},$$

where  $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$ . Such a choice is natural for NLS because the linear Schrödinger flow  $S(t) = e^{-it\Delta}$ , denoting the Fourier multiplier by  $e^{it|\xi|^2}$  (when  $\mathcal{M} = \mathbb{R}^d$ ) or  $e^{it|n|^2}$  (when  $\mathcal{M} = \mathbb{T}^d$ ), is an isometry under the  $H^s(\mathcal{M})$ -norm for any  $s \in \mathbb{R}$ .

One important approach for studying local well-posedness of dispersive PDEs is through perturbative argument. In other words, by writing out the Duhamel formulation of (1.1):

$$u(t) = S(t)u_0 - i \int_0^t S(t-t')F(u, \xi)dt', \quad (1.2)$$

we view the nonlinear solution  $u$  as a perturbation of the linear solution  $S(t)u_0$ . When the initial data  $u_0$  is regular enough and the nonlinearity  $F(u, \xi)$  is nice enough, one can use the Banach fixed-point theorem to construct a unique solution to (1.2) on the function space  $C(\mathbb{R}; H^s(\mathcal{M}))$ . However, when the initial data  $u_0$  becomes irregular, one encounters difficulty in closing the contraction argument. This is mainly due to the fact that at a fixed time, there is no smoothing property of the linear Schrödinger propagator. Nevertheless, by using the Strichartz spaces  $L^p(\mathbb{R}; L^q(\mathcal{M}))$  for  $1 \leq p, q < \infty$ , one can exploit smoothing of the linear Schrödinger propagator by considering the time averaging effect via the Strichartz estimates. It turns out that the Strichartz spaces work well when  $\mathcal{M} = \mathbb{R}^d$ , but it is more difficult for  $\mathcal{M} = \mathbb{T}^d$  if

one wants to cover well-posedness for the full subcritical regime. This issue was addressed by Bourgain in [9], where he introduced the Fourier restriction norm method via the following  $X^{s,b}$ -space:

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}^d)} = \|\langle \nabla \rangle^s \langle \partial_t \rangle^b S(-t)u(t)\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{T}^d)} = \|\langle n \rangle^s \langle \tau - |n|^2 \rangle^b \widehat{u}(\tau, \xi)\|_{L_\tau^2 \ell_n^2(\mathbb{R} \times \mathbb{Z}^d)},$$

where  $\widehat{u}$  denotes the space-time Fourier transform of  $u$ . This space captures the dispersive nature of the Schrödinger equations by penalizing functions that are away from being the linear Schrödinger solution (supported on the curve  $\tau = |\xi|^2$ ). See Section A.4 for more details on this space.

Recently, there is also a growing interest in the study of *random dispersive PDEs*. Namely, one can consider dispersive PDEs with random initial data, which was first initiated by Bourgain [10]. Such study is natural from the physical point of view since physical systems are often perturbed by the random noise in many applications. From an analytic point of view, this study allows us to go beyond the deterministic analysis of dispersive equations. Indeed, in the case of rough initial data, by combining deterministic analysis and probabilistic tools, one can prove almost sure well-posedness of a dispersive PDE by removing an exceptional set of initial data with probability measure zero.

In addition to the difficulty in dealing with rough initial data, the nonlinearity  $F(u, \xi)$  may also not be nice in some settings, especially when the stochastic forcing  $\xi$  is very rough. In this case, a contraction argument based on the Banach fixed-point theorem may not be possible. An alternative approach to deal with this issue is to use the classical energy method to prove local well-posedness via a compactness argument. In other words, one starts with a smoothed equation, establishes a priori bounds of approximating solutions in suitable function spaces, and shows the convergence of the approximating solutions with the limit solving the original equation.

A next natural question after establishing local-in-time well-posedness of a dispersive PDE is how to prove global-in-time well-posedness. In order to extend the local-in-time solution globally-in-time, we usually need to provide an a priori control of the  $H^s(\mathcal{M})$ -norm of the (smoothed) solution. For the compactness argument mentioned above, this a priori control allows us to extract a convergent sequence of approximating solutions. As for the contraction argument, the local time of existence, at least in the subcritical regime, depends on the size of the initial data, and so the  $H^s(\mathcal{M})$  a priori bound of the solution allows us to iterate the local well-posedness argument arbitrarily many times. In the field of dispersive PDEs, such  $H^s(\mathcal{M})$  a priori bound is usually shown by using the conservation laws. For example, for the classical cubic NLS

$$i\partial_t u = \Delta u - |u|^2 u, \tag{1.3}$$

assuming that  $u$  has sufficient regularity, we have conservation of mass

$$M(u(t)) = \int_{\mathcal{M}} |u(t)|^2 dx = M(u(0))$$

and conservation of energy (Hamiltonian)

$$H(u(t)) = \frac{1}{2} \int_{\mathcal{M}} |\nabla u(t)|^2 dx + \frac{1}{4} \int_{\mathcal{M}} |u(t)|^4 dx = H(u(0)).$$

In this thesis, we study well-posedness of NLS (1.1) in the following settings.

- In Chapter 2, we consider the two-dimensional periodic NLS with a quadratic nonlinearity  $|u|^2$ . Specifically, we prove sharp local well-posedness of (1.1) with  $\mathcal{M} = \mathbb{T}^2$ ,  $F(u, \xi) = |u|^2$ , and  $u_0 \in L^2(\mathbb{T}^2)$ . This is based on the following joint work with my supervisor Tadahiro Oh (University of Edinburgh):  
[85] R. Liu, T. Oh, *Sharp local well-posedness of the two-dimensional periodic nonlinear Schrödinger equation with a quadratic nonlinearity  $|u|^2$* , to appear in *Math. Res. Lett.*
- In Chapter 3, we consider the one-dimensional and two-dimensional periodic NLS with

a quadratic nonlinearity  $\bar{u}^2$  in negative Sobolev spaces. More precisely, we prove local well-posedness of (1.1) with  $\mathcal{M} = \mathbb{T}$  or  $\mathbb{T}^2$ ,  $F(u, \xi) = \bar{u}^2$ , and  $u_0 \in H^s(\mathcal{M})$  for  $s > -\frac{2}{3}$ . This is based on the following work:

[84] R. Liu, *Local well-posedness of the periodic nonlinear Schrödinger equation with a quadratic nonlinearity  $\bar{u}^2$  in negative Sobolev spaces*, to appear in J. Dyn. Diff. Equat.

- In Chapter 4, we again consider the two-dimensional periodic NLS with a quadratic nonlinearity  $|u|^2$ , but this time with random initial data distributed according to a fractional derivative (of order  $\alpha \geq 0$ ) of the Gaussian free field. More precisely, for (1.1) with  $\mathcal{M} = \mathbb{T}^2$  and  $F(u, \xi) = |u|^2$ , we prove almost sure local well-posedness for  $\alpha < \frac{1}{2}$  and we prove a probabilistic ill-posedness result for  $\alpha \geq \frac{3}{4}$  in a suitable sense. This is based on the following work:

[83] R. Liu, *On the probabilistic well-posedness of the two-dimensional periodic nonlinear Schrödinger equation with the quadratic nonlinearity  $|u|^2$* , J. Math. Pures Appl. 171 (2023), 75-101.

- In Chapter 5, we consider NLS with a multiplicative spatial white noise and a polynomial nonlinearity on the two-dimensional full space domain. Specifically, we prove global well-posedness of (1.1) with  $\mathcal{M} = \mathbb{R}^2$ ,  $F(u, \xi) = \xi u - \lambda |u|^{p-1} u$  for some  $\lambda \in \mathbb{R}$ , and  $u_0$  satisfying some structural assumptions. This is based on the following joint work with Arnaud Debussche (ENS Rennes), Nikolay Tzvetkov (ENS Lyon), and Nicola Visciglia (University of Pisa):

[28] A. Debussche, R. Liu, N. Tzvetkov, N. Visciglia, *Global well-posedness of the 2D nonlinear Schrödinger equation with multiplicative spatial white noise on the full space*, to appear in Probab. Theory Related Fields.

In the rest of the introduction, we discuss the literature and backgrounds on each setting listed above and state our main results in a more detailed manner. In Chapters 2-5, we present the proofs of these results. For notations, functions spaces, and some preliminary lemmas that are useful for proving our main results, we introduce them in the Appendix.

## 1.1 Deterministic local well-posedness of quadratic NLS with nonlinearity $|u|^2$

Let us first provide some background on the following quadratic NLS

$$i\partial_t u = \Delta u \pm \mathcal{N}(u, u), \quad (1.4)$$

where  $\mathcal{N}(u, u)$  can be  $u^2$ ,  $\bar{u}^2$ , or  $|u|^2$ . Note that on  $\mathbb{R}^d$ , if  $u$  is a solution to (1.4), then for any  $\lambda > 0$ ,  $u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x)$  is also a solution to (1.4). This scaling symmetry induces the following scaling critical Sobolev regularity:

$$s_{\text{crit}} = \frac{d}{2} - 2.$$

When  $d = 1, 2, 3$ , the scaling critical regularity becomes negative, which often fails to predict well-posedness and ill-posedness issues. In this thesis, we mainly focus on the cases when  $d = 1$  and  $d = 2$ .

We now review some previous results on the quadratic NLS (1.4). On the real line  $\mathbb{R}$ , Kenig-Ponce-Vega [70] used the Fourier restriction norm method via the  $X^{s,b}$ -spaces (see Section A.4) to prove local well-posedness of (1.4) for all types of nonlinearities  $u^2$ ,  $\bar{u}^2$ , and  $|u|^2$ . In particular, they established the crucial bilinear estimates

$$\begin{aligned} \|uv\|_{X^{s, -\frac{1}{2}+}} &\lesssim \|u\|_{X^{s, \frac{1}{2}+}} \|v\|_{X^{s, \frac{1}{2}+}}, \\ \|\bar{u}v\|_{X^{s, -\frac{1}{2}+}} &\lesssim \|u\|_{X^{s, \frac{1}{2}+}} \|v\|_{X^{s, \frac{1}{2}+}}, \\ \|u\bar{v}\|_{X^{s, -\frac{1}{2}+}} &\lesssim \|u\|_{X^{s, \frac{1}{2}+}} \|v\|_{X^{s, \frac{1}{2}+}}, \end{aligned} \quad (1.5)$$

Setting	$\mathbb{R}$			$\mathbb{R}^2$		
	$u^2$	$\bar{u}^2$	$ u ^2$	$u^2$	$\bar{u}^2$	$ u ^2$
Scaling critical regularity	$s_{\text{crit}} = -\frac{3}{2}$			$s_{\text{crit}} = -1$		
$X^{s,b}$ -bilinear estimate	$s > -\frac{3}{4}$	$s > -\frac{3}{4}$	$s > -\frac{1}{4}$	$s > -\frac{3}{4}$	$s > -\frac{3}{4}$	$s > -\frac{1}{4}$
Failure of $X^{s,b}$ -bilinear estimate	$s \leq -\frac{3}{4}$	$s \leq -\frac{3}{4}$	$s \leq -\frac{1}{4}$	$s \leq -\frac{3}{4}$	$s \leq -\frac{3}{4}$	$s \leq -\frac{1}{4}$
Local well-posedness	$s \geq -1$	$s \geq -1$	$s \geq -\frac{1}{4}$	$s > -1$	$s > -1$	$s \geq -\frac{1}{4}$
Ill-posedness	$s < -1$	$s < -1$	$s < -\frac{1}{4}$	$s \leq -1$	$s \leq -1$	$s < -\frac{1}{4}$

Table 1.1: Known results for quadratic NLS on  $\mathbb{R}$  and  $\mathbb{R}^2$ . Note that in all cases, local well-posedness results and ill-posedness results are sharp.

where the range of admissible  $s$  varies in each case and can be found in Table 1.1. These bilinear estimates fail when  $s$  gets below certain threshold; see [70, 91]. Despite the failure of the bilinear estimate, Bejenaru-Tao [5] showed sharp local well-posedness of (1.4) on  $\mathbb{R}$  with  $\mathcal{N}(u, u) = u^2$  by introducing weighted  $X^{s,b}$ -spaces. Later, sharp local well-posedness results of (1.4) on  $\mathbb{R}$  with  $\mathcal{N}(u, u) = \bar{u}^2$  and  $\mathcal{N}(u, u) = |u|^2$  were shown by Kishimoto [72, 73]. As for ill-posedness results for (1.4) on  $\mathbb{R}$ , see [5, 72, 73, 79, 64, 65, 78]. See Table 1.1 for the specific ranges of  $s$  for these results, where we note that for all nonlinearities, ill-posedness occurs before  $s$  reaches the scaling critical regularity of (1.4) on  $\mathbb{R}$ :  $s_{\text{crit}} = -\frac{3}{2}$ . Also, all local well-posedness and ill-posedness results are sharp on  $\mathbb{R}$ .

Let us also mention well-posedness results of (1.4) on  $\mathbb{R}^2$ , again summarized in Table 1.1. The  $X^{s,b}$ -bilinear estimates (1.5) were established in [107, 22]. The failure of these  $X^{s,b}$ -bilinear estimates for lower  $s$  was shown in [22, 91]. Local well-posedness results of (1.4) on  $\mathbb{R}^2$  were shown in [4, 73, 74]. Ill-posedness results of (1.4) on  $\mathbb{R}^2$  were shown in [64, 65, 78]. As in the case of  $\mathbb{R}$ , all local well-posedness and ill-posedness results of (1.4) on  $\mathbb{R}^2$  are sharp.

We now turn to well-posedness issues of (1.4) on periodic domains  $\mathbb{T}$  and  $\mathbb{T}^2$ . The results are summarized in Table 1.2, where the boldface texts refer to the results to be presented in this thesis. On the circle  $\mathbb{T}$ , the  $X^{s,b}$ -bilinear estimates (1.5) for  $s \geq 0$  follow immediately from the  $L^3$ -Strichartz estimate:

$$\|e^{-it\Delta} f\|_{L_t^3 L_x^3([-1,1] \times \mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})},$$

which is obtained by interpolating the  $L^4$ -Strichartz estimate on  $\mathbb{T}$  (see [119, 9]) and the trivial  $L^2$ -bound. For nonlinearities  $u^2$  and  $\bar{u}^2$ , Kenig-Ponce-Vega [70] showed  $X^{s,b}$ -bilinear estimates for negative values of  $s$ . In the same paper, they also showed the failure of these bilinear estimates when  $s$  gets below a threshold. For ill-posedness of (1.4) on  $\mathbb{T}$ , see [78]. From Table 1.2, we see that for nonlinearities  $u^2$  and  $\bar{u}^2$ , there are gaps between local well-posedness and ill-posedness results. Also, the nonlinearity  $|u|^2$  behaves worse on  $\mathbb{T}$  than on  $\mathbb{R}$ , since ill-posedness on  $\mathbb{T}$  holds for a wider range of  $s$  than on  $\mathbb{R}$ .

For (1.4) on  $\mathbb{T}^2$ , fewer results are known. In [47], Grünrock showed the  $X^{s,b}$ -bilinear estimate for the nonlinearity  $\bar{u}^2$  and proved the corresponding local well-posedness result. For the failure of the  $X^{s,b}$ -bilinear estimates for nonlinearities  $u^2$  and  $\bar{u}^2$ , see also [47]. For ill-posedness of (1.4) on  $\mathbb{T}^2$ , see [78], where ill-posedness for the nonlinearity  $|u|^2$  implies the failure of its  $X^{s,b}$ -bilinear estimate. As in the case of  $\mathbb{T}$ , there are some gaps between local well-posedness and ill-posedness results for (1.4) on  $\mathbb{T}^2$ .

There has also been studies on long-time behaviors of the quadratic NLS (1.4). For global existence and scattering, see [40, 42, 59, 67, 87, 103]. For finite-time blowup results, see [63, 94].

Let us now focus our attention on the following quadratic NLS with nonlinearity  $|u|^2$  posed

Setting	$\mathbb{T}$			$\mathbb{T}^2$		
	$u^2$	$\bar{u}^2$	$ u ^2$	$u^2$	$\bar{u}^2$	$ u ^2$
Scaling critical regularity	$s_{\text{crit}} = -\frac{3}{2}$			$s_{\text{crit}} = -1$		
$X^{s,b}$ -bilinear estimate	$s > -\frac{1}{2}$	$s > -\frac{1}{2}$	$s \geq 0$	<b><math>s \geq 0</math></b>	$s > -\frac{1}{2}$	<b><math>s \geq 0</math></b>
Failure of $X^{s,b}$ -bilinear estimate	$s < -\frac{1}{2}$	$s < -\frac{1}{2}$	$s < 0$	$s < 0$	$s < -\frac{1}{2}$	$s < 0$
Local well-posedness	$s > -\frac{1}{2}$	<b><math>s &gt; -\frac{2}{3}</math></b>	$s \geq 0$	<b><math>s \geq 0</math></b>	<b><math>s &gt; -\frac{2}{3}</math></b>	<b><math>s \geq 0</math></b>
Ill-posedness	$s < -1$	$s < -1$	$s < 0$	$s \leq -1$	$s \leq -1$	$s < 0$

Table 1.2: Current results for the quadratic NLS on  $\mathbb{T}$  and  $\mathbb{T}^2$ . The boldface texts in the table refer to the results of this thesis. Note that for the nonlinearity  $|u|^2$ , local well-posedness and ill-posedness results are sharp on both  $\mathbb{T}$  and  $\mathbb{T}^2$ . For nonlinearities  $u^2$  and  $\bar{u}^2$  on either  $\mathbb{T}$  or  $\mathbb{T}^2$ , there are gaps between local well-posedness and ill-posedness results.

on the two-dimensional torus  $\mathbb{T}^2$ :

$$\begin{cases} i\partial_t u = \Delta u \pm |u|^2 \\ u|_{t=0} = u_0. \end{cases} \quad (1.6)$$

Our goal is to prove local well-posedness of (1.6) in the low regularity setting. As mentioned above, in the case of a periodic domain  $\mathbb{T}^d$ , Bourgain [9] introduced the Fourier restriction norm method via the  $X^{s,b}$ -spaces that allowed him to prove local well-posedness of NLS in the low regularity setting. In particular, on the two-dimensional torus  $\mathbb{T}^2$ , he proved the following  $L^4$ -Strichartz estimate with a derivative loss:

$$\|e^{-it\Delta} f\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \lesssim \|f\|_{H^s(\mathbb{T}^2)} \quad (1.7)$$

for any  $s > 0$ , which allowed him to prove local well-posedness of the cubic NLS (1.3) in  $H^s(\mathbb{T}^2)$  for any  $s > 0$ . We point out that the  $L^4$ -Strichartz estimate (1.7) fails when  $s = 0$  (see [9, 109, 77]) and that the well-posedness issue of the cubic NLS in  $L^2(\mathbb{T}^2)$  remains a challenging open problem.

As for the quadratic NLS (1.6), by interpolating the  $L^4$ -Strichartz estimate (1.7) with the trivial  $L^2$ -bound, we obtain the  $L^3$ -Strichartz estimate with a derivative loss. Then, by proceeding as in [9], we immediately obtain local well-posedness of (1.6) in  $H^s(\mathbb{T}^2)$  for any  $s > 0$ . However, due to the derivative loss in the  $L^3$ -Strichartz estimate, local well-posedness of (1.6) in  $L^2(\mathbb{T}^2)$  has been open for the last thirty years. In this thesis, we prove that (1.6) is indeed locally well-posed in  $L^2(\mathbb{T}^2)$ .

**Theorem 1.1.1.** *The quadratic NLS (1.6) is locally well-posed in  $L^2(\mathbb{T}^2)$ . More precisely, given any initial data  $u_0 \in L^2(\mathbb{T}^2)$ , there exists  $T = T(\|u_0\|_{L^2}) > 0$  and a unique solution  $u \in C([-T, T]; L^2(\mathbb{T}^2))$  to (1.6) with  $u|_{t=0} = u_0$ , and the solution  $u$  depends continuously on the initial data  $u_0$ .*

Note that our proof is based on the Fourier restriction norm method, so that the uniqueness in the statement of Theorem 1.1.1 holds only in (the local-in-time version of) the relevant  $X^{s,b}$ -space; see Section A.4 below.

Our local well-posedness result for the quadratic NLS (1.6) is a sharp result. On the one hand, in [78], Kishimoto proved ill-posedness of (1.6) in  $H^s(\mathbb{T}^2)$  for  $s < 0$ , so that our local well-posedness result completes the picture of well-posedness of (1.6). On the other hand, Theorem 1.1.1 is also sharp in the sense that the local-in-time solution constructed in Theorem 1.1.1 cannot be in general extended globally-in-time. Indeed, in [94], Oh proved a finite-time blowup result for the quadratic NLS (1.6). The argument in [94] can be easily extended to  $L^2(\mathbb{T}^2)$ , which results in the following proposition.

**Proposition 1.1.2.** *Let  $s \geq 0$  and  $u_0 \in H^s(\mathbb{T}^2)$ . If the initial data  $u_0$  satisfies*

$$\operatorname{Im} \int_{\mathbb{T}^2} u_0 dx < 0 \quad \text{or} \quad \operatorname{Re} \int_{\mathbb{T}^2} u_0 dx \neq 0,$$

*then the forward maximal existence time  $T^*$  of the solution  $u$  to (1.6) with  $u|_{t=0} = u_0$  is finite and  $\liminf_{t \nearrow T^*} \|u(t)\|_{H^s} = \infty$ .*

See [41] for the lifespan of solutions to (1.6). We point out that Fujiwara-Georgiev [40] proved the criterion for global existence of (1.6) on the circle  $\mathbb{T}$ . Namely, they proved that there exists a global  $L^2$ -solution to (1.6) on  $\mathbb{T}$  if and only if  $\operatorname{Re} u_0 = 0$  and  $\operatorname{Im} u_0 = \mu$  for some  $\mu \geq 0$  and that any global  $L^2$ -solution is necessarily constant in space. It would be of interest to investigate this property in the two-dimensional setting.

As mentioned before, our proof of local well-posedness of (1.6) in Theorem 1.1.1 is based on the Fourier restriction norm method. In particular, Theorem 1.1.1 follows from a standard contraction argument (see Section 2.2) given that the following bilinear estimate can be established.

**Proposition 1.1.3.** *Let  $0 < T \leq 1$ ,  $s \geq 0$ , and  $0 < \delta_2 < \delta_1$  be sufficiently small. Then, we have*

$$\|u\bar{v}\|_{X_T^{s, -\frac{1}{2} + \delta_1}} \lesssim \|u\|_{X_T^{s, \frac{1}{2} + \delta_2}} \|v\|_{X_T^{s, \frac{1}{2} + \delta_2}}. \quad (1.8)$$

Here, the  $X_T^{s,b}$ -norm denotes a local-in-time version of the  $X^{s,b}$ -norm; see Section A.4. As mentioned above, the  $L^3$ -Strichartz estimate is only known with a derivative loss and so cannot be used directly to prove the bilinear estimate (1.8). Instead, we separate the proof into two main cases: non-resonant interaction and nearly resonant interaction. For the non-resonant interaction, we have gain of derivative from multilinear dispersion, so that we can make up for the loss of derivative in the  $L^3$ -Strichartz estimate. For the nearly resonant interaction, we notice that the angle between the second incoming wave and the outgoing wave is almost perpendicular, which allows us to prove the estimate without any derivative loss; see also [110, 24]. See Chapter 2 for details.

We conclude this section with several remarks.

**Remark 1.1.4.** Unlike [9], our proof does not rely on number theoretic properties and so Theorem 1.1.1 also holds on a general torus  $\mathbb{T}_{\alpha}^2 = (\mathbb{R}/\alpha_1\mathbb{Z}) \times (\mathbb{R}/\alpha_2\mathbb{Z})$  for any ratio  $\alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1, \alpha_2 > 0$ . This is essentially due to the fact that the bilinear estimate in Lemma A.4.6, the counting estimate in Lemma A.5.3, and the  $L^4$ -Strichartz estimate in Lemma A.4.5 hold on a general torus; see [76, Lemma 2.5], [76, Lemma 2.9], and [14, Theorem 2.4]. For further discussions on the Strichartz estimates and well-posedness of NLS on general tori, see [13, 19, 52, 14, 71, 31].

**Remark 1.1.5.** The bilinear estimate (1.8) also holds with  $u\bar{v}$  on the left-hand-side replaced by  $uv$  or  $\overline{uv}$ . Indeed, for the bilinear term  $\overline{uv}$ , Grünrock [47] proved the corresponding bilinear estimate for  $s > -\frac{1}{2}$ . As for the bilinear term  $uv$ , one can a slight modification of the proof of Proposition 1.1.3 to obtain the corresponding bilinear estimate for  $s \geq 0$ ; see Remark 2.1.3 below. These  $X^{s,b}$ -bilinear estimates yield local well-posedness for corresponding  $s$  via a standard contraction argument.

**Remark 1.1.6.** It is conjectured in [9] that on  $\mathbb{T}^2$ , the  $L^p$ -Strichartz estimate on  $\mathbb{T}^2$  for  $2 < p < 4$  holds without any derivative loss. At this point, this conjecture remains open since the  $L^p$ -Strichartz estimate on  $\mathbb{T}^2$  for  $2 < p < 4$  is only known to hold with a slight loss of derivative. Nevertheless, by considering a multilinear version of the Strichartz estimate, we obtain the following trilinear  $L^3$ -Strichartz estimate without any derivative loss:

$$\left| \int_{-1}^1 \int_{\mathbb{T}^2} \prod_{j=1}^3 (e^{-it\Delta} f_j)^* dx dt \right| \lesssim \prod_{j=1}^3 \|f_j\|_{L^2}, \quad (1.9)$$

where  $u^*$  denotes  $u$  or  $\bar{u}$ . The estimate (1.9) follows directly from the bilinear estimate (1.8) (also with  $wv$  and  $\bar{w}\bar{v}$  on the left-hand-side) and duality. We point out that the product structure on the left-hand-side of (1.9) is crucial.

**Remark 1.1.7.** Lastly, we consider the following quadratic NLS with a gauge-invariant nonlinearity on  $\mathbb{T}^2$ :

$$i\partial_t u = \Delta u \pm |u|u. \quad (1.10)$$

The equation (1.10) is interesting in view of the mass and energy conservations. In particular, if one can prove local well-posedness of (1.10) in  $L^2(\mathbb{T}^2)$ , then one can immediately obtain global well-posedness of (1.10) in  $L^2(\mathbb{T}^2)$  by using the mass conservation.

When  $s > 0$ , we easily obtain local well-posedness of (1.10) in  $H^s(\mathbb{T}^2)$  by using the  $L^4$ -Strichartz estimate (1.7). When  $s = 0$ , however, local well-posedness problem of (1.10) becomes challenging due to the non-algebraic nature of the nonlinearity. For instance, the trilinear estimate (1.9) is not applicable to study (1.10). In fact, multilinear analysis via the Fourier restriction norm method is not useful for studying (1.10) due to the presence of  $|u|$ . While there are well-posedness results for NLS on periodic domains with non-algebraic gauge-invariant nonlinearities [96, 82], one needs much more intricate analysis to prove well-posedness of (1.10) in  $L^2(\mathbb{T}^2)$ .

## 1.2 Deterministic local well-posedness of quadratic NLS with nonlinearity $\bar{u}^2$

In this section, we consider the following quadratic NLS with nonlinearity  $\bar{u}^2$  posed on a periodic domain  $\mathbb{T}^d$  with  $d = 1$  or  $d = 2$ :

$$\begin{cases} i\partial_t u = \Delta u \pm \bar{u}^2 \\ u|_{t=0} = u_0. \end{cases} \quad (1.11)$$

Our main goal is to establish local well-posedness of (1.11) in negative Sobolev spaces.

In the previous section, we discussed well-posedness results for quadratic NLS with nonlinearities  $u^2$ ,  $\bar{u}^2$ , and  $|u|^2$ . As can be seen from Table 1.2 above, these three nonlinearities have different behaviors in terms of  $X^{s,b}$ -bilinear estimates and well-posedness issues on periodic domains. This difference is closely related to their distinct phase functions. By letting  $n_1$  and  $n_2$  be the frequencies of the two incoming waves and  $n$  be the frequency of the outgoing wave, we can write out the frequency interactions and phase functions for these three nonlinearities in Table 1.3. When the phase function is large, we expect some gain of regularities in bilinear estimates. For instance, for nonlinearity  $\bar{u}^2$  on  $\mathbb{T}^2$ , the phase function  $|n|^2 + |n_1|^2 + |n_2|^2$  is always large compared to the three frequencies  $n$ ,  $n_1$ , and  $n_2$ , and so provides gain of derivatives. As a result, one can establish local well-posedness for nonlinearity  $\bar{u}^2$  with very rough initial data. On the other hand, for nonlinearity  $u^2$  on  $\mathbb{T}^2$ , the phase function  $|n|^2 - |n_1|^2 - |n_2|^2 = 2n_1 \cdot n_2$  can be very small if the frequencies  $n_1$  and  $n_2$  are almost perpendicular to each other. Consequently, it is much harder to prove local well-posedness for nonlinearity  $u^2$  with rough initial data.

We now focus on low regularity local well-posedness of quadratic NLS (1.11) and state our main result as follows.

**Theorem 1.2.1.** *Let  $\mathcal{M} = \mathbb{T}$  or  $\mathbb{T}^2$ . Then, the quadratic NLS (1.11) is locally well-posed in  $H^s(\mathcal{M})$  for  $s > -\frac{2}{3}$ . More precisely, given any initial data  $u_0 \in H^s(\mathcal{M})$ , there exists  $T = T(\|u_0\|_{H^s}) > 0$  and a unique solution  $u \in C([-T, T]; H^s(\mathcal{M}))$  to (1.11) with  $u|_{t=0} = u_0$ , and the solution  $u$  depends continuously on the initial data  $u_0$ .*

We present the proof of Theorem 1.2.1 in Chapter 3. Our local well-posedness results in Theorem 1.2.1 improve previous local well-posedness results in [70] (where Kenig-Ponce-Vega proved local well-posedness for (1.11) on  $\mathbb{T}$  for  $s > -\frac{1}{2}$ ) and [47] (where Grünrock proved local well-posedness for (1.11) on  $\mathbb{T}^2$ ). Also, as mentioned above, the usual  $X^{s,b}$ -bilinear estimates

Nonlinearity $\mathcal{N}(u, u)$	$u^2$	$\bar{u}^2$	$ u ^2$
Frequency interaction	$n - n_1 - n_2 = 0$	$n + n_1 + n_2 = 0$	$n - n_1 + n_2 = 0$
Phase function	$ n ^2 -  n_1 ^2 -  n_2 ^2$	$ n ^2 +  n_1 ^2 +  n_2 ^2$	$ n ^2 -  n_1 ^2 +  n_2 ^2$

Table 1.3: Frequency interactions and phase functions for the quadratic NLS with nonlinearities  $u^2$ ,  $\bar{u}^2$ , and  $|u|^2$ .

for (1.11) on  $\mathbb{T}$  and  $\mathbb{T}^2$  fail when  $s < -\frac{1}{2}$  (see also Table 1.2), and our results seem to be the first local well-posedness result for the quadratic NLS on periodic domains below the regularity thresholds where the usual  $X^{s,b}$ -bilinear estimates fail.

Since local well-posedness of (1.11) for  $s > -\frac{1}{2}$  was already shown in [70] on  $\mathbb{T}$  and in [47] on  $\mathbb{T}^2$ , we mainly focus on the case  $-\frac{2}{3} < s \leq -\frac{1}{2}$  in our proof. In proving Theorem 1.2.1, we construct the solution in modified  $X^{s,b}$ -spaces, and so the uniqueness statement in Theorem 1.2.1 holds only in the relevant function space; see the  $Z_T^{s,b}$ -norm in Section 3.1. Also, for the proof of Theorem 1.2.1, we will mainly focus on the case  $\mathcal{M} = \mathbb{T}^2$ , since the proof for  $\mathcal{M} = \mathbb{T}$  follows from the proof for  $\mathcal{M} = \mathbb{T}^2$  with minor modifications.

We now explain our strategy for proving Theorem 1.2.1. In [5], Bejenaru-Tao reduced the well-posedness problem of the quadratic NLS (1.4) in  $H^s(\mathcal{M})$  (with  $\mathcal{M} = \mathbb{R}^d$  or  $\mathbb{T}^d$ ) to finding a space-time norm  $\|\cdot\|_{W^s}$  that satisfy the following properties:

(i) (Monotonicity) If  $|\widehat{f}| \leq |\widehat{g}|$  pointwise with  $\widehat{f}$  denoting the space-time Fourier transform of  $f$ , then

$$\|f\|_{W^s} \leq \|g\|_{W^s}. \quad (1.12)$$

(ii) ( $H^s$ -energy estimate) The following inequality holds true:

$$\|\langle \xi \rangle^s \widehat{f}(\tau, \xi)\|_{L_\xi^2 L_\tau^1} \lesssim \|f\|_{W^s}. \quad (1.13)$$

(iii) (Homogeneous linear estimate) There exists  $b \in \mathbb{R}$  such that

$$\|f\|_{W^s} \lesssim \|f\|_{X^{s,b}}. \quad (1.14)$$

(iv) (Bilinear estimate) The following inequality holds true:

$$\|\langle \tau - |\xi|^2 \rangle^{-1} \mathcal{B}(\widehat{f}, \widehat{g})\|_{\widehat{W}^s} \lesssim \|f\|_{W^s} \|g\|_{W^s}, \quad (1.15)$$

where  $\widehat{W}^s$  is the same norm  $W^s$  on the Fourier side and  $\mathcal{B}(\widehat{f}, \widehat{g})$  is  $\widehat{f} * \widehat{g}$  if  $\mathcal{N}(u, u) = u^2$ ,  $\widetilde{\widehat{f}} * \widetilde{\widehat{g}}$  if  $\mathcal{N}(u, u) = \bar{u}^2$ , and  $\widehat{f} * \widetilde{\widehat{g}}$  if  $\mathcal{N}(u, u) = |u|^2$ . Here,  $\widetilde{f}(\tau, \xi) = f(-\tau, -\xi)$ .

The main task is to find a suitable function space satisfying the above properties. From now on, we restrict our focus on the nonlinearity  $\mathcal{N}(u, u) = \bar{u}^2$  and  $\mathcal{M} = \mathbb{T}^2$  in this section. As mentioned above, the usual  $X^{s,b}$ -bilinear estimate fails when the regularity  $s$  is very low. This failure is due to certain ‘‘dangerous’’ frequency interactions. Thus, we need to introduce modifications on the  $X^{s,b}$ -norm in order to reduce the effect from those ‘‘dangerous’’ frequency interactions. Below, we discuss some examples of such frequency interactions and our strategy to deal with them.

**Example 1.** For a large number  $N \in \mathbb{N}$ , we let

$$\begin{aligned} \widehat{u}_N(\tau, n) &= \mathbf{1}_{\{n=Ne_1\}} \mathbf{1}_{[-1,1]}(\tau - N^2), \\ \widehat{v}_N(\tau, n) &= \mathbf{1}_{\{n=-Ne_1\}} \mathbf{1}_{[-1,1]}(\tau - N^2), \end{aligned}$$

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On  $\mathbb{T}^d$ , this framework only works for small initial data. See Remark 1.2.2 below for a discussion on local well-posedness on  $\mathbb{T}^d$  for large initial data.

where  $e_1 = (1, 0)$ . In this case, we have  $\|u_N\|_{X^{s,b}} \sim N^s$  and  $\|v_N\|_{X^{s,b}} \sim N^s$ . A direct computation yields

$$\widehat{u_N v_N}(\tau, n) \gtrsim \mathbf{1}_{\{n=0\}} \mathbf{1}_{[-1,1]}(\tau + 2N^2),$$

and so  $\|\widehat{u_N v_N}\|_{X^{s,b-1}} \gtrsim N^{2b-2}$ . Thus, the bilinear estimate (1.5) holds only if  $2b-2 \leq 2s$ , which is equivalent to  $s \geq b-1$ . Since we need  $b > \frac{1}{2}$  to ensure that the solution lies in  $C(\mathbb{R}; H^s(\mathbb{T}^2))$ , we require that  $s > -\frac{1}{2}$ .

In the above example, the frequency interaction is “high-high to low” and the modulation interaction (i.e.  $\tau - |n|^2$ ) is “low-low to high”. However, the modulation for  $\widehat{u_N v_N}$  is not high enough for the usual  $X^{s,b}$ -bilinear estimate to hold when  $s \leq -\frac{1}{2}$ . To deal with the above interaction for  $s \leq -\frac{1}{2}$ , we consider the following  $Y^{s,b}$ -norm introduced by Kishimoto [74]:

$$\|u\|_{Y^{s,b}} \stackrel{\text{def}}{=} \|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1(\mathbb{R} \times \mathbb{Z})} + \|\langle \tau - |n|^2 \rangle^{\frac{s}{2}+b} \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2(\mathbb{R} \times \mathbb{Z})},$$

and we define the space  $Z^{s,b} = X^{s,b} + Y^{s,b}$  via the norm

$$\|u\|_{Z^{s,b}} \stackrel{\text{def}}{=} \inf\{\|u_1\|_{X^{s,b}} + \|u_2\|_{Y^{s,b}} : u_1 + u_2 = u\}.$$

The  $\ell_n^2 L_\tau^1$ -norm in the  $Y^{s,b}$ -norm is needed in order to ensure that the  $Z^{s,b}$ -norm satisfies the  $H^s$ -energy estimate (1.13). It is not hard to see that the  $Z^{s,b}$ -norm also satisfies the monotonicity property (1.12) and the homogeneous linear estimate (1.14). Note that for  $s \leq 0$  and  $b > \frac{1}{2}$ , if  $\text{supp } \widehat{u} \subset \{|\tau - |n|^2| \lesssim |n|^2\}$ , then we have

$$\|u\|_{Z^{s,b}} \sim \|u\|_{X^{s,b}} \lesssim \|u\|_{Y^{s,b}};$$

if  $\text{supp } \widehat{u} \subset \{|\tau - |n|^2| \gtrsim |n|^2\}$ , then we have

$$\|u\|_{Z^{s,b}} \sim \|u\|_{Y^{s,b}} \lesssim \|u\|_{X^{s,b}}.$$

In Section 3.1 below, we will revisit this  $Z^{s,b}$ -norm, which will be defined in a more precise manner for practical purposes.

In Example 1, due to the high modulation of  $\widehat{u_N v_N}$ , the desired bilinear estimate (1.15) holds for the  $Z^{s,b}$ -norm since the  $\widehat{Z}^{s,b}$ -norm (i.e. the  $Z^{s,b}$ -norm on the Fourier side) of the term  $\langle \tau - |n|^2 \rangle \widehat{u_N v_N}$  is small enough. One can easily check that the bilinear estimate for Example 1 holds for the  $Z^{s,b}$ -norm for  $s \geq 2b - 2$ . This is better than the usual  $X^{s,b}$ -bilinear estimate as long as  $\frac{1}{2} < b \leq \frac{3}{4}$ .

Let us now take a look at another example using the  $Z^{s,b}$ -norm with  $s \leq 0$ .

**Example 2.** For a large number  $N \in \mathbb{N}$ , we let

$$\begin{aligned} \widehat{u_N}(\tau, n) &= \mathbf{1}_{\{n=Ne_1\}} \mathbf{1}_{[-1,1]}(\tau - N^2), \\ \widehat{v_N}(\tau, n) &= \mathbf{1}_{\{n=-Ne_1\}} \mathbf{1}_{[-1,1]}(\tau + N^2). \end{aligned}$$

A direct computation yields

$$\widehat{u_N v_N}(\tau, n) \gtrsim \mathbf{1}_{\{n=0\}} \mathbf{1}_{[-1,1]}(\tau).$$

In this case, the frequency interaction is “high-high to low” and the modulation interaction is “low-high to low”. Their corresponding  $Z^{s,b}$ -norms can be computed as follows:

$$\begin{aligned} \|u_N\|_{Z^{s,b}} &\sim \|u_N\|_{X^{s,b}} \sim N^s, \\ \|v_N\|_{Z^{s,b}} &\sim \|v_N\|_{Y^{s,b}} \sim N^{s+2b}, \\ \|\langle \tau - |n|^2 \rangle^{-1} \widehat{u_N v_N}\|_{\widehat{Z}^{s,b}} &\gtrsim 1. \end{aligned}$$

Thus, the bilinear estimate (1.15) with  $W = Z^{s,b}$  holds only if  $0 \leq 2s + 2b$ , which is equivalent to  $s \geq -b$ .

Combining Example 1 and Example 2, we note that the regularity  $s$  needs to satisfy  $s \geq 2b-2$  and  $s \geq -b$ . These two lower bounds of  $s$  become optimal when  $b = \frac{2}{3}$ , so that  $s = -\frac{2}{3}$  seems to be the threshold regularity of the bilinear estimate (1.15) with  $W^s = Z^{s,b}$ . In Section 3.2, we show that the bilinear estimate (1.15) with  $W^s = Z^{s,b}$  holds when  $s > -\frac{2}{3}$ . See Remark 1.2.3 below for a discussion on the end point  $s = -\frac{2}{3}$ .

We conclude this section with several remarks.

**Remark 1.2.2.** On  $\mathbb{T}^d$ , suppose that one can obtain local well-posedness with small initial data, one can use a scaling argument to prove local well-posedness for large initial data; see [23]. Nevertheless, we do not pursue the scaling argument in this thesis and rely instead on the time localization lemma for the  $X^{s,b}$ -space (see Lemma A.4.4 below) to prove local well-posedness for large initial data.

**Remark 1.2.3.** In [5, 72], a Besov refinement was considered in constructing function spaces in order to cover the endpoint regularity (i.e.  $s = -1$  for the quadratic NLS (1.4) on  $\mathbb{R}$  with  $\mathcal{N}(u, u) = u^2$  or  $\bar{u}^2$ ). Similar Besov refinements were used by [51, 75] in the context of the Korteweg-de Vries equation.

For the quadratic NLS on  $\mathbb{T}$ , it seems possible to adapt the Besov modification to our estimate so that the endpoint case  $s = -\frac{2}{3}$  can be covered. For the quadratic NLS (1.11) on  $\mathbb{T}^2$ , however, due to the derivative loss of the  $L^4$ -Strichartz estimate on  $\mathbb{T}^2$  (see (1.7) or Lemma A.4.5 below), such Besov modification does not seem to be enough to cover the case when  $s = -\frac{2}{3}$ .

**Remark 1.2.4.** For the quadratic NLS (1.11) on  $\mathbb{T}$  and  $\mathbb{T}^2$ , there are still gaps between local well-posedness and ill-posedness results; see Table 1.2. Specifically, well-posedness of (1.11) on  $\mathbb{T}$  is open for  $-1 \leq s \leq -\frac{2}{3}$  and well-posedness of (1.11) on  $\mathbb{T}^2$  is open for  $-1 < s \leq -\frac{2}{3}$ . One possible strategy for covering these ranges of regularity is to introduce weighted function spaces as in [4, 5, 72, 74] in the context of Euclidean spaces.

**Remark 1.2.5.** Let us consider the quadratic NLS (1.4) with nonlinearity  $\mathcal{N}(u, u) = u^2$ . On the one-dimensional torus  $\mathbb{T}$ , local well-posedness is known in the range  $s > -\frac{1}{2}$  and ill-posedness holds for  $s < -1$ ; see Table 1.2. It is possible to use modified function spaces to cover well-posedness in the range  $-1 \leq s \leq -\frac{1}{2}$ , but one may need to use the weighted spaces as in [4, 5] to obtain the desired bilinear estimate.

For nonlinearity  $\mathcal{N}(u, u) = u^2$  on  $\mathbb{T}^2$ , local well-posedness of (1.4) is known to hold for  $s \geq 0$  and ill-posedness holds for  $s \leq -1$ ; see Table 1.2. However, it seems unlikely that the method of finding modified function spaces  $W^s$  as illustrated above works in the range  $s < 0$ . This is mainly due to the following example from [47]. For a large number  $N \in \mathbb{N}$ , we let

$$\begin{aligned}\widehat{u}_N(\tau, n) &= \mathbf{1}_{\{n=Ne_1\}} \mathbf{1}_{[-1,1]}(\tau - N^2), \\ \widehat{v}_N(\tau, n) &= \mathbf{1}_{\{n=Ne_2\}} \mathbf{1}_{[-1,1]}(\tau - N^2),\end{aligned}$$

where  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ . A direct computation yields

$$\begin{aligned}\widehat{u}_N \widehat{v}_N &= \mathbf{1}_{\{n=N(e_1+e_2)\}} \max\{0, \min\{2 + \tau - 2N^2, 2 - \tau + 2N^2\}\} \\ &\geq \mathbf{1}_{\{n=N(e_1+e_2)\}} \mathbf{1}_{[-1,1]}(\tau - 2N^2).\end{aligned}$$

In this example, the frequency interaction is “high-high to high” and the modulation interaction is “low-low to low”. This means that there seems to be no way to use the modulation to improve the bilinear estimate. Note that this “low-low to low” interaction does not occur for the nonlinearity  $\mathcal{N}(u, u) = \bar{u}$  studied in this thesis, which can be seen from the computations at the beginning of Subcase 2.3 of Lemma 3.2.2 below and Case 3 of Lemma 3.2.3 below.

For any  $s \in \mathbb{R}$  and  $b \in \mathbb{R}$ , we have

$$\|u_N\|_{X^{s,b}} \sim \|v_N\|_{X^{s,b}} \sim \|\langle \tau - |n|^2 \rangle^{-1} \widehat{u}_N \widehat{v}_N\|_{\widehat{X}^{s,b}} \sim N^s,$$

where the  $\widehat{X}^{s,b}$ -norm is the  $X^{s,b}$ -norm on the Fourier side. Thus, in view of the homogeneous linear estimate (1.14) and the similar structures of  $\widehat{u}_N$ ,  $\widehat{v}_N$ , and  $\widehat{u}_N \widehat{v}_N$ , we observe that any

valid norm  $\|\cdot\|_{W^s}$  should decrease the corresponding norms of  $\widehat{u}_N$ ,  $\widehat{v}_N$ , and  $\langle \tau - |n|^2 \rangle^{-1} \widehat{u}_N \widehat{v}_N$  with the same rate with respect to  $N$ . Suppose that there exists  $a \geq 0$  such that

$$\|u_N\|_{W^s} \sim \|v_N\|_{W^s} \sim \|\langle \tau - |n|^2 \rangle^{-1} \widehat{u}_N \widehat{v}_N\|_{\widehat{W}^s} \sim N^{s-a}.$$

Then, in order for the bilinear estimate (1.15) to hold, we need

$$N^{s-a} \lesssim N^{2s-2a},$$

so that we must have  $s \geq a \geq 0$ . Therefore, we do not expect that the method of finding the suitable  $W^s$ -norm works for the quadratic NLS with  $\mathcal{N}(u, u) = u^2$  on  $\mathbb{T}^2$  for  $s < 0$ . It is possible that some ill-posedness results may hold in this range.

### 1.3 Probabilistic well-posedness of quadratic NLS with random initial data

In this section, we go back to the quadratic NLS with nonlinearity  $|u|^2$ , but with random initial data. More precisely, we consider the Cauchy problem for the following quadratic NLS on the two-dimensional torus  $\mathbb{T}^2$ :

$$\begin{cases} i\partial_t u = \Delta u \pm (|u|^2 - f|u|^2) \\ u|_{t=0} = u_0^\omega. \end{cases} \quad (1.16)$$

Here,  $f = (2\pi)^{-1} \int f$  and  $u_0^\omega$  is the following Gaussian random initial data:

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle^{1-\alpha}} e^{in \cdot x}, \quad (1.17)$$

where  $\alpha \in \mathbb{R}$  and  $\{g_n\}_{n \in \mathbb{Z}^2}$  is a set of independent standard complex-valued Gaussian random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\mathbb{E}(g_n) = 0$  and  $\mathbb{E}(|g_n|^2) = 1$ . Note that when  $\alpha = 0$ , the initial data  $u_0^\omega$  is distributed according to the massive Gaussian free field on  $H^s(\mathbb{T}^2)$  with  $s < 0$ .

The idea of constructing solutions of NLS using random initial data was first introduced by Bourgain in [10], where he proved almost sure local well-posedness of the (renormalized) cubic NLS on  $\mathbb{T}^2$  with random initial data (1.17) with  $\alpha = 0$ . See also [11, 25, 33, 39, 34] for more results on almost sure local well-posedness of NLS with other power-type nonlinearities on periodic domains with random initial data of the form (1.17). In this thesis, we choose to work with quadratic NLS (1.16) with random initial data (1.17); see Remark 1.3.2 below for the necessity of removing the mean of the nonlinearity.

Note that the initial data  $u_0^\omega$  in (1.17) almost surely belongs to  $H^{-\alpha-\varepsilon}(\mathbb{T}^2) \setminus H^{-\alpha}(\mathbb{T}^2)$  for any  $\varepsilon > 0$ ; see [18, Lemma B.1]. When  $\alpha < 0$ ,  $u_0^\omega$  almost surely belongs to  $H^s(\mathbb{T}^2)$  for some  $s > 0$ , so that we can easily prove almost sure local well-posedness of (1.16) using purely deterministic methods as described in Section 1.1. Our goal in this thesis is to obtain probabilistic local well-posedness of (1.16) with  $\alpha \geq 0$  and to identify bad behaviors of (1.16) when  $\alpha$  becomes too large. Specifically, we first show that (1.16) is almost surely locally well-posed when  $0 \leq \alpha < \frac{1}{2}$ ; see Subsection 1.3.1 below. Then, we show that (1.16) is probabilistically ill-posed in a suitable sense when  $\alpha \geq \frac{3}{4}$ ; see Subsection 1.3.2 below.

#### 1.3.1 Almost sure local well-posedness of the renormalized quadratic NLS

Let us first consider probabilistic well-posedness of the quadratic NLS (1.16). We define

$$z(t) = z^\omega(t) \stackrel{\text{def}}{=} e^{-it\Delta} u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle^{1-\alpha}} e^{it|n|^2 + in \cdot x}, \quad (1.18)$$

which is the solution to the linear Schrödinger equation with random initial data  $u_0^\omega$ :

$$\begin{cases} i\partial_t z = \Delta z \\ z|_{t=0} = u_0^\omega. \end{cases}$$

We are now ready to state our almost sure local well-posedness result.

**Theorem 1.3.1.** *Let  $0 \leq \alpha < \frac{1}{2}$  and  $s > 0$ . Then, the quadratic NLS (1.16) is almost surely locally well-posed in the class  $z + C([-T, T]; H^s(\mathbb{T}^2))$ . More precisely, there exist  $T_0 > 0$  and constants  $C, c, \theta > 0$  such that for all  $0 < T \leq T_0$ , there exists a set  $\Omega_T \subset \Omega$  with the following properties:*

- (i)  $\mathbb{P}(\Omega \setminus \Omega_T) \leq C \exp(-\frac{c}{T^\theta})$ .
- (ii) For each  $\omega \in \Omega_T$ , there exists a unique solution  $u = u^\omega$  to (1.16) on  $[-T, T]$  with  $u|_{t=0} = u_0^\omega$  in the class  $z + C([-T, T]; H^s(\mathbb{T}^2))$ .

To prove Theorem 1.3.1, we use the following first order expansion [86, 10, 26]:

$$u = z + v, \tag{1.19}$$

where  $z$  is as defined in (1.18) and  $v$  is a residual term satisfying the following equation:

$$\begin{cases} i\partial_t v = \Delta v \pm (|z + v|^2 - f|z + v|^2) \\ v|_{t=0} = 0. \end{cases} \tag{1.20}$$

Note that the uniqueness statement in Theorem 1.3.1 refers to the uniqueness of  $v$  as a solution to the equation (1.20) in the corresponding function space (i.e. the local-in-time  $X^{s,b}$ -space in Section A.4).

To prove local well-posedness of the perturbed quadratic NLS (1.20), we need to establish bilinear estimates to the quadratic terms  $|v|^2$ ,  $v\bar{z}$ ,  $z\bar{v}$ , and  $|z|^2$ . For this purpose, we use the operator norm approach based on the random tensor theory developed by Deng-Nahmod-Yue in [34]; see Subsection 4.1. See also [34, 15, 100, 102, 16] for more applications of the random tensor theory in the study of local well-posedness of random dispersive PDEs. We show the details of bilinear estimates in Section 4.2 and we prove Theorem 1.3.1 in Section 4.3.

**Remark 1.3.2.** Let us consider the following quadratic NLS:

$$\begin{cases} i\partial_t u = \Delta u \pm |u|^2 \\ u|_{t=0} = u_0^\omega, \end{cases} \tag{1.21}$$

where  $u_0^\omega$  is the Gaussian random initial data given by (1.17). For  $N \in \mathbb{N}$ , we denote  $u_{0,N}^\omega$  as the sharp frequency truncation of  $u_0^\omega$  onto  $\{|n| \leq N\}$  and we define  $z_N(t) = e^{-it\Delta} u_{0,N}^\omega$ . Then, one can easily check that the zeroth frequency of the following Picard second iterate

$$\int_0^t e^{-i(t-t')\Delta} (|z_N(t')|^2) dt'$$

diverges almost surely when  $\alpha \geq 0$ ; see, for example, [95, Subsection 4.4]. Thus, in order to establish well-posedness, we need to remove this singular behavior of (1.21) at the zeroth frequency.

A more natural way of dealing with the above issue is to consider the following renormalized quadratic NLS:

$$\begin{cases} i\partial_t u_N = \Delta u_N \pm (|u_N|^2 - \sigma_N) \\ u_N|_{t=0} = u_{0,N}^\omega, \end{cases} \tag{1.22}$$

where  $\sigma_N = \mathbb{E}[|u_{0,N}^\omega|^2]$ . The renormalization in (1.22) was used by Bourgain [10] when studying

the cubic NLS

$$i\partial_t u_N = \Delta u_N \pm (|u_N|^2 u_N - 2\sigma_N u_N), \quad (1.23)$$

and he used a Gauge transform  $u_N = \exp(2i(f|u_N|^2 - \sigma_N)t) \cdot v_N$  to show that (1.23) is equivalent to the following cubic NLS:

$$i\partial_t v_N = \Delta v_N \pm \left( |v_N|^2 - 2 \int |v_N|^2 \right) v_N.$$

Here, in the case of the cubic NLS, we note that the quantity  $f|u_N|^2 - \sigma_N$  is time invariant and one can easily recover  $u_N$  from  $v_N$  since  $f|u_N|^2 = f|v_N|^2$ . However, in the case of quadratic NLS, since the nonlinearity  $|u|^2$  is not gauge invariant, there seems to be no way to establish the equivalence between the quadratic NLS (1.16) and the equation (1.22).

One can also directly proceed with the quadratic NLS (1.22), but there are some issues. Firstly, after using the first order expansion  $u_N = z_N + v_N$ , there are some problems in bounding the zeroth frequency of the bilinear terms  $v_N \bar{z}_N$ ,  $z_N \bar{v}_N$ , and  $|z_N|^2$  when  $\alpha \geq 0$ . Secondly, the remainder term  $v_N$  is not necessarily of mean zero, which makes it difficult to estimate the bilinear term  $z_N \bar{v}_N$  when  $\alpha \geq 0$ . See Remark 4.2.3 for more details. Therefore, in this thesis, we choose to focus on the quadratic NLS (1.16), i.e. with nonlinearity  $|u|^2 - f|u|^2$ .

**Remark 1.3.3.** Let  $\rho \in C(\mathbb{R}^2)$  be smooth mollification kernel such that  $\int \rho = 1$ ,  $\rho \geq 0$ , and we define  $\rho_\varepsilon(x) = \varepsilon^{-2} \rho(\varepsilon^{-1}x)$ . Using a slight modification of the proof of Theorem 1.3.1, we can show that when  $0 \leq \alpha < \frac{1}{2}$ , if  $u_\varepsilon$  is the solution to the quadratic NLS with a mollified initial data:

$$\begin{cases} i\partial_t u_\varepsilon = \Delta u_\varepsilon \pm (|u_\varepsilon|^2 - f|u_\varepsilon|^2) \\ u_\varepsilon|_{t=0} = \rho_\varepsilon * u_0^\omega, \end{cases}$$

then  $u_\varepsilon$  converges in probability to some unique limiting distribution  $u \in C([-T_\omega, T_\omega]; H^{-\alpha-\varepsilon}(\mathbb{T}^2))$  with  $\varepsilon > 0$  arbitrarily small and  $T_\omega > 0$  for almost sure  $\omega \in \Omega$ . Here, the limiting distribution  $u$  is independent of the choice of the mollification kernel  $\rho$ .

**Remark 1.3.4.** Let us consider NLS with other quadratic nonlinearities with random initial data:

$$\begin{cases} i\partial_t u = \Delta u \pm \mathcal{N}(u, u) \\ u|_{t=0} = u_0^\omega, \end{cases} \quad (1.24)$$

where  $\mathcal{N}(u, u) = u^2$  or  $\bar{u}^2$  and  $u_0^\omega$  is as defined in (1.17). We point out that these nonlinearities have different phase functions. By letting  $n_1$  and  $n_2$  be the frequencies of the two incoming waves and letting  $n$  be the frequency of the outgoing wave, we have the following phase functions for different quadratic nonlinearities:  $2n \cdot n_2$  for  $|u|^2$ ,  $2n_1 \cdot n_2$  for  $u^2$ , and  $-|n|^2 - |n_1|^2 - |n_2|^2$  for  $\bar{u}^2$ .

For nonlinearity  $\mathcal{N}(u, u) = u^2$ , by using similar steps from the proof of Theorem 1.3.1, we can obtain almost sure local well-posedness of (1.24) with  $\alpha < \frac{1}{2}$ . Note that in this case, we do not need to remove any singularities from the zeroth frequency as compared to the case of  $\mathcal{N}(u, u) = |u|^2$ .

For nonlinearity  $\mathcal{N}(u, u) = \bar{u}^2$ , since the phase function is always large compared to any single frequency, we expect that one can establish almost sure local well-posedness of (1.24) beyond the range  $\alpha < \frac{1}{2}$  for nonlinearities  $|u|^2$  and  $u^2$ . However, the method for proving Theorem 1.3.1 based on the first order expansion (1.19) does not seem to be enough to go beyond  $\alpha < \frac{1}{2}$ , since the crucial bilinear estimate for the produce of two random linear solutions (Proposition 4.2.2 (iii)) is only valid for  $\alpha < \frac{1}{2}$ . In this case, it is possible to establish almost sure local well-posedness for some range of  $\alpha \geq \frac{1}{2}$  using higher order expansions as in [6, 97, 99].

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This also includes the paracontrolled approach in [49, 15, 16]

### 1.3.2 Probabilistic ill-posedness of the renormalized quadratic NLS

Let us now consider probabilistic ill-posedness issues of the quadratic NLS (1.16) with random initial data (1.17) for large values of  $\alpha$ . Given  $N \in \mathbb{N}$ , we consider the following Picard second iterate:

$$z_N^{(2)}(t) = \int_0^t e^{i(t-t')\Delta} \left( |z_N(t')|^2 - \int |z_N(t')|^2 \right) dt', \quad (1.25)$$

where  $z_N$  is the truncation of the random linear solution  $z$  in (1.18) onto frequencies  $\{|n| \leq N\}$ :

$$z_N(t) = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{g_n(\omega)}{\langle n \rangle^{1-\alpha}} e^{it|n|^2 + in \cdot x}.$$

Our probabilistic ill-posedness result refers to the non-convergence of every non-zero Fourier coefficient of the Picard second iterate  $z_N^{(2)}$ , which we state in the following theorem.

**Theorem 1.3.5.** *Let  $\alpha \geq \frac{3}{4}$ ,  $n \neq 0$ , and  $t \neq 0$ . Then, the sequence  $\{\mathbb{E}[|\mathcal{F}_x z_N^{(2)}(t, n)|^2]\}_{N \in \mathbb{N}}$  increases to infinity as  $N \rightarrow \infty$ . Consequently, any subsequence of the sequence of random variables  $\{\mathcal{F}_x z_N^{(2)}(t, n)\}_{N \in \mathbb{N}}$  is not tight.*

The non-convergence of the Picard second iterate  $z_N^{(2)}$  is mainly caused by the ‘‘high-high to low’’ frequency interaction. See Section 4.4 for the proof of Theorem 1.3.5.

Theorem 1.3.5 implies that when  $\alpha \geq \frac{3}{4}$ , any subsequence of the Fourier coefficients  $\{\mathcal{F}_x z_N^{(2)}(t, n)\}_{N \in \mathbb{N}}$  (with  $n \neq 0$ ) does not converge in law. This means that standard methods for establishing probabilistic local well-posedness, such as the first order expansion [86, 10, 26] or its higher order variants [6, 97, 99] do not work for  $\alpha \geq \frac{3}{4}$ .

We now discuss an important consequence of our probabilistic ill-posedness result. In [33], Deng-Nahmod-Yue introduced the probabilistic scaling and the associated critical regularity of NLS with random initial data. Their notion is based on the observation that, in order to obtain local well-posedness, the Picard second iterate should not be rougher than the random linear solution. This observation allowed them to compute the probabilistic scaling critical regularity of NLS without too much difficulty, and they conjectured in [33] that for NLS with nonlinearities  $|u|^{p-1}u$  ( $p \in 2\mathbb{N}+1$ ), almost sure local well-posedness should hold for all subcritical regularities. Indeed, later in [34], Deng-Nahmod-Yue proved almost sure local well-posedness for NLS with nonlinearities  $|u|^{p-1}u$  ( $p \in 2\mathbb{N}+1$ ) on  $\mathbb{T}^d$  in the full subcritical range with respect to the probabilistic scaling.

As for NLS with a quadratic nonlinearity  $|u|^2$ , however, the probabilistic scaling does not seem to provide a useful prediction for probabilistic well-posedness issues. Let us compute the probabilistic scaling critical regularity for NLS with nonlinearity  $|u|^2$  in the following, which is similar to the procedure in [33]. Given a dyadic number  $N \in 2^{\mathbb{N} \cup \{0\}}$ , we consider the initial data  $u_0^\omega$  in (1.17) truncated on frequencies  $\{|n| \sim N\}$ :

$$P_N u_0^\omega = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \sim N}} \frac{g_n(\omega)}{\langle n \rangle^{1-\alpha}} e^{in \cdot x}.$$

Note that  $\|P_N u_0^\omega\|_{H^{-\alpha}(\mathbb{T}^2)} \sim 1$ . We now consider the following Picard second iterate:

$$u_N^{(2)}(t) = \int_0^t e^{-i(t-t')\Delta} (|e^{-it'\Delta} u_0^\omega|^2) dt',$$

where we do not need to subtract the zeroth frequency as in (1.25) since later we only focus on

the case when  $|n| \sim N$ . We can compute the  $n$ th Fourier coefficient as follows:

$$\mathcal{F}_x u_N^{(2)}(t, n) = \int_0^t e^{it|n|^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \\ |n_1| \sim N, |n_2| \sim N}} e^{-it'(|n|^2 - |n_1|^2 + |n_2|^2)} \frac{g_{n_1}(\omega) \overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^{1-\alpha} \langle n_2 \rangle^{1-\alpha}} dt'.$$

Thus, restricting our attention to  $\{|n| \sim N\}$ , by using the Wiener chaos estimate (Lemma A.6.1 below along with Chebyshev's inequality) and a counting estimate (Lemma A.5.2 (i) below), we have the following estimate for  $u_N^{(2)}$ :

$$\begin{aligned} \|u_N^{(2)}\|_{H^{-\alpha}(\mathbb{T}^2)}^2 &\sim \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \sim N}} \langle n \rangle^{-2\alpha} \left( \sum_{\substack{n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \\ |n_1| \sim N, |n_2| \sim N}} \frac{g_{n_1}(\omega) \overline{g_{n_2}(\omega)}}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle \langle n_1 \rangle^{1-\alpha} \langle n_2 \rangle^{1-\alpha}} \right)^2 \\ &\leq C_\omega \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \\ |n|, |n_1|, |n_2| \sim N}} \frac{\langle n \rangle^{-2\alpha}}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle^2 \langle n_1 \rangle^{2-2\alpha} \langle n_2 \rangle^{2-2\alpha}} \\ &\sim C_\omega N^{2\alpha-4} \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \\ |n|, |n_1|, |n_2| \sim N}} \frac{1}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle^2} \\ &\leq C_\omega N^{2\alpha-2+\varepsilon}, \end{aligned}$$

where  $C_\omega > 0$  is a constant and  $\varepsilon > 0$  is arbitrarily small. In order for  $u_N^{(2)}$  to be smoother than the random linear solution, we need  $2\alpha - 2 + \varepsilon \leq 0$ , which is equivalent to  $\alpha < 1$ .

The above computation suggests that the probabilistic scaling critical regularity is  $\alpha_* = 1$ . However, Theorem 1.3.5 shows that every nonzero Fourier coefficient of the Picard second iterate diverges when  $\alpha \geq \frac{3}{4}$ , which happens before  $\alpha$  reaches the critical value  $\alpha_* = 1$ . This means that the probabilistic scaling introduced in [33] fails in the case of the quadratic nonlinearity  $|u|^2$ . We point out that this discrepancy is mainly due to the fact that the probabilistic scaling does not take into account the ‘‘high-high to low’’ frequency interaction, which is the main issue that results in probabilistic ill-posedness. Indeed, in a recent work [35], Deng-Nahmod-Yue refined their probabilistic scaling paradigm by taking into account the ‘‘high-high to low’’ interactions for quadratic nonlinearities, and their new scaling critical value matches the index for our probabilistic ill-posedness result. Also, this discrepancy is closely related to the fact that we are considering *very* rough random initial data, which is in particular relevant in studying NLS with a power-type nonlinearity of low degree and in low dimensions; see Remark 1.3.7 below.

We end this section by stating several remarks.

**Remark 1.3.6.** We point out that there is a gap  $\frac{1}{2} \leq \alpha < \frac{3}{4}$  between our almost sure local well-posedness and probabilistic ill-posedness of the quadratic NLS (1.16). If some well-posedness results of the quadratic NLS (1.16) can be established in the range  $\frac{1}{2} \leq \alpha < \frac{3}{4}$ , this will mean that NLS behaves better than the nonlinear wave equation (NLW) for a quadratic nonlinearity; see Remark 1.3.8 below. This will be an interesting result, because usually NLW behaves better than NLS in terms of well-posedness.

**Remark 1.3.7.** The proof of Theorem 1.3.5, probabilistic ill-posedness of the quadratic NLS (1.16), also works in general dimensions. Specifically, on  $\mathbb{T}^d$ , given  $\alpha \geq \frac{5}{4} - \frac{d}{4}$ ,  $n \neq 0$ , and  $t \neq 0$ , any subsequence of  $\{\mathcal{F}_x z_N^{(2)}(t, n)\}_{N \in \mathbb{N}}$  is not tight.

The probabilistic scaling for the quadratic NLS (1.16) presented above can also be extended to general  $\mathbb{T}^d$ , in which the critical regularity is  $\alpha_* = 2 - \frac{d}{2}$ . Note that when  $d = 1$  or  $d = 2$ , probabilistic ill-posedness occurs before  $\alpha$  reaches the critical value  $\alpha_*$ .

**Remark 1.3.8.** In [95], Oh-Okamoto studied well-posedness and ill-posedness of the stochastic nonlinear wave equation (NLW) with a quadratic nonlinearity on  $\mathbb{T}^2$ . Let us consider the

following quadratic NLW on  $\mathbb{T}^2$  with random initial data:

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u = u^2 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega), \end{cases} \quad (1.26)$$

where

$$(u_0^\omega, u_1^\omega) = \left( \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle^{1-\alpha}} e^{in \cdot x}, \sum_{n \in \mathbb{Z}^2} \langle n \rangle^\alpha h_n(\omega) e^{in \cdot x} \right).$$

Here,  $\alpha \in \mathbb{R}$  and  $\{g_n, h_n\}_{n \in \mathbb{Z}^2}$  is a sequence of independent standard complex Gaussian random variables conditioned such that  $g_{-n} = \overline{g_n}$  and  $h_{-n} = \overline{h_n}$  for all  $n \in \mathbb{Z}^2$ . We point out that, after a suitable renormalization, well-posedness and ill-posedness results in [95] also apply to (1.26): almost sure local well-posedness holds when  $\alpha < \frac{1}{2}$  and probabilistic ill-posedness (in the sense that every Fourier coefficient of the Picard second iterate diverges almost surely) holds when  $\alpha \geq \frac{1}{2}$ .

We note that the quadratic NLS (1.16) and the quadratic NLW (1.26) are both almost surely locally well-posed when  $\alpha < \frac{1}{2}$ . The probabilistic ill-posedness, however, holds for the quadratic NLS (1.16) when  $\alpha \geq \frac{3}{4}$  and holds for the quadratic NLW (1.26) when  $\alpha \geq \frac{1}{2}$ . The difference of the ill-posedness behaviors of the two quadratic equations is mainly due to the different structures of the corresponding Duhamel operators.

**Remark 1.3.9.** Let us also mention some failures of scaling analysis that happen in parabolic PDEs with stochastic forcing. In the past decade, there has been a tremendous progress in the study of stochastic parabolic PDEs using the theory of regularity structures introduced by Hairer [53, 54, 55, 56]. In particular, the theory of regularity structures can be used to solve a wide range of parabolic PDEs with an additive space-time white noise forcing that are subcritical according to the notion of local subcriticality as introduced by Hairer [54]. However, when the stochastic forcing is *very* rough, the scaling analysis may fail to provide a valid prediction for well-posedness issues. Indeed, in [61], Hoshino showed that for the KPZ equation forced by a fractional derivative of the space-time white noise, the standard solution theory breaks down before reaching the predicted critical regularity. Also, in [95], Oh-Okamoto showed a similar phenomenon occurring in the context of the stochastic nonlinear heat equation driven by a fractional derivative of the space-time white noise.

## 1.4 Global well-posedness of the dispersive Anderson model

In this section, we study the following Cauchy problem for NLS with a multiplicative spatial white noise on  $\mathbb{R}^2$ :

$$\begin{cases} i\partial_t u = \Delta u + \xi u - \lambda|u|^{p-1}u \\ u|_{t=0} = u_0, \end{cases} \quad (1.27)$$

where  $p > 1$  and  $\lambda \geq 0$ . Here,  $\xi$  stands for a real-valued spatial white noise. More precisely, given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\xi : \Omega \rightarrow \mathcal{S}'(\mathbb{R}^2)$  is a random variable such that for each  $f \in \mathcal{S}(\mathbb{R}^2)$ ,  $(\xi, f)$  is a real-valued centered Gaussian random variable such that  $\mathbb{E}[(\xi, f)^2] = \|f\|_{L^2}^2$ .

One can view the equation (1.27) as a stochastic version of the deterministic NLS with a power-type nonlinearity. On the other hand, the equation (1.27) can be viewed as the dispersive counterpart of the well-studied parabolic Anderson model (i.e. with  $i\partial_t u$  replaced by  $\partial_t u$ ), and so we also refer to (1.27) as the dispersive Anderson model.

The dispersive Anderson model (1.27) was first considered by Debussche-Weber in [30] on  $\mathbb{T}^2$ . To deal with the ill-defined nature of the term  $\xi u$ , they used a gauge transform  $v = e^Y u$  with  $Y = \Delta^{-1} \xi$ . This gauge transform is called the Doss-Sussmann transform [36, 108] and was later used by Hairer-Labbé [57] in the context of the parabolic Anderson model on  $\mathbb{R}^2$  (the definition

of  $Y$  is slightly different on  $\mathbb{R}^2$ ). We can then write out the equation for  $v$ :

$$\begin{cases} i\partial_t v = \Delta v - 2\nabla Y \cdot \nabla v + |\nabla Y|^2 v - \lambda e^{-(p-1)Y} |v|^{p-1} v \\ v|_{t=0} = e^Y u_0, \end{cases} \quad (1.28)$$

which is easier to deal with since the most singular term is canceled. However, the term  $\nabla Y$  is merely a distribution, so that  $|\nabla Y|^2$  needs to be replaced by a Wick ordered product  $:|\nabla Y|^2:$  as in [57, 30]; see below for more detailed explanations.

In [30], Debussche-Weber showed global well-posedness of (1.28) on  $\mathbb{T}^2$  in the cubic case  $p = 3$ . They considered a mollified noise  $\xi_\varepsilon$  and a smoothed process  $Y_\varepsilon = \Delta^{-1}\xi_\varepsilon$ , and then constructed the solution  $v$  as a limit of  $v_\varepsilon$  in probability, where  $v_\varepsilon$  satisfies (1.28) with  $Y$  replaced by  $Y_\varepsilon$ . The key ingredient for the convergence of  $v_\varepsilon$  is an  $H^2$  a priori bound for  $v_\varepsilon$  with a logarithmic loss in  $\varepsilon$ , which was obtained by exploiting the mass conservation and the energy conservation of the gauge-transformed NLS (1.28). Later on, Tzvetkov-Visciglia [113] improved the result in [30] by proving global well-posedness of (1.28) on  $\mathbb{T}^2$  for  $3 < p \leq 4$ . In particular, they introduced modified energies that allow them to obtain the  $H^2$  a priori bound for a larger range of  $p$ . In a subsequent work [114], Tzvetkov-Visciglia further improved their global well-posedness of (1.28) on  $\mathbb{T}^2$  by covering all  $p > 1$ . Specifically, they exploited the time averaging effect via the Strichartz estimates to obtain the  $H^2$  a priori bound for the whole range of  $p > 1$ . Moreover, the authors in [113, 114] proved almost sure convergence of  $v_\varepsilon$  to  $v$ , which improved the convergence in probability in [30].

We now turn our attention to the dispersive Anderson model (1.27) on  $\mathbb{R}^2$ , which is the main concern in this thesis. The additional difficulty on  $\mathbb{R}^2$  comes from the logarithmic growth of the (mollified) spatial white noise. In [29], Debussche-Martin showed global well-posedness of a gauge-transformed NLS similar to (1.28) (see (1.30) below) for  $1 < p < 2$  via convergence of  $v_\varepsilon$  to  $v$  in probability. In particular, they used weighted Sobolev and Besov spaces to overcome the growth of the (mollified) noise and obtained a weighted  $H^2$  a priori bound for  $v_\varepsilon$ . In this situation, we require more assumptions on the regularity of the initial data than those on the  $\mathbb{T}^2$  setting. This approach of using the weighted Sobolev and Besov spaces had also been used in [57, 88, 58, 48, 98] in the study of stochastic PDEs. In this thesis, we extend the result in [29] by proving global well-posedness for all  $p > 1$ , using an intricate combination of the methods mentioned above. Moreover, we prove almost sure convergence of  $v_\varepsilon$  to  $v$ , which is a stronger convergence result than that in [29].

We now describe our setup in details and state our main results. We proceed as in [57] and use a truncated Green's function  $G \in C^\infty(\mathbb{R}^2 \setminus \{0\})$  such that  $G$  is supported on  $\{|x| < 1\}$  and  $G(x) = -\frac{1}{2\pi} \log|x|$  for  $|x|$  small enough. We define  $Y \stackrel{\text{def}}{=} G * \xi$ , so that  $Y$  satisfies

$$\Delta Y = \xi + \varphi * \xi$$

for some  $\varphi \in C_c^\infty(\mathbb{R}^2)$ . By writing  $v = e^Y u$ , we convert (1.27) into the following gauge-transformed NLS for  $v$ :

$$\begin{cases} i\partial_t v = \Delta v - 2\nabla Y \cdot \nabla v + (|\nabla Y|^2 - \varphi * \xi)v - \lambda e^{-(p-1)Y} |v|^{p-1} v \\ v|_{t=0} = e^Y u_0. \end{cases}$$

As mentioned above, the term  $|\nabla Y|^2$  needs to be replaced by a meaningful object  $:|\nabla Y|^2:$ , which is defined almost surely as a distribution:

$$:|\nabla Y|^2: (\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \nabla G(x - y_1) \nabla G(x - y_2) \boldsymbol{\xi}(dy_1) \boldsymbol{\xi}(dy_2) \quad (1.29)$$

with  $\phi \in \mathcal{S}(\mathbb{R}^2)$  and  $\boldsymbol{\xi}$  denoting the Gaussian measure on  $\mathbb{R}^2$  induced by the white noise  $\xi$ ; see [66, page 95-99]. We recall that for  $f_1, f_2 \in L^2(\mathbb{R}^2)$  and Gaussian random variables  $X_1 = \int_{\mathbb{R}^2} f_1(y) \boldsymbol{\xi}(dy)$  and  $X_2 = \int_{\mathbb{R}^2} f_2(y) \boldsymbol{\xi}(dy)$ , we have the following identity from [66, Theorem 7.26]:

$$:X_1 X_2: = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(y_1) f_2(y_2) \boldsymbol{\xi}(dy_1) \boldsymbol{\xi}(dy_2),$$

where  $:X_1X_2:$  denotes the Wick product between  $X_1$  and  $X_2$ . From this perspective, the object  $:|\nabla Y|^2:$  can be viewed as a Wick product of the distribution  $\nabla Y$  with itself. For more details on Wick calculus, see [60, 66, 93].

From now on, we focus on the following equation

$$\begin{cases} i\partial_t v = \Delta v - 2\nabla Y \cdot \nabla v + :|\widetilde{\nabla Y}|^2: v - \lambda e^{-(p-1)Y} |v|^{p-1} v \\ v|_{t=0} = v_0, \end{cases} \quad (1.30)$$

where  $v_0 \stackrel{\text{def}}{=} e^Y u_0$  and

$$:\widetilde{|\nabla Y|^2}: \stackrel{\text{def}}{=} :|\nabla Y|^2: - \varphi * \xi. \quad (1.31)$$

In order to construct a solution to (1.30), we consider the following approximating equation:

$$\begin{cases} i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla Y_\varepsilon \cdot \nabla v_\varepsilon + :|\widetilde{\nabla Y_\varepsilon}|^2: v_\varepsilon - \lambda e^{-(p-1)Y_\varepsilon} |v_\varepsilon|^{p-1} v_\varepsilon \\ v_\varepsilon|_{t=0} = v_0. \end{cases} \quad (1.32)$$

Here,  $Y_\varepsilon$  is defined as

$$Y_\varepsilon = \rho_\varepsilon * Y = \rho_\varepsilon * G * \xi = G * \xi_\varepsilon,$$

where  $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\varepsilon^{-1}x)$  with  $\rho \in C_c^\infty(\mathbb{R}^2)$  supported on  $\{|x| < 1\}$ ,  $\rho \geq 0$ ,  $\int_{\mathbb{R}^2} \rho = 1$ , and  $\xi_\varepsilon = \rho_\varepsilon * \xi$  is a mollification of the spatial white noise. Also,  $:|\widetilde{\nabla Y_\varepsilon}|^2:$  is defined as

$$:\widetilde{|\nabla Y_\varepsilon|^2}: = :|\nabla Y_\varepsilon|^2: - \varphi * \xi_\varepsilon \quad (1.33)$$

where the Wick product  $:|\nabla Y_\varepsilon|^2:$  is defined as

$$:|\nabla Y_\varepsilon|^2: = |\nabla Y_\varepsilon|^2 - c_\varepsilon \quad \text{with} \quad c_\varepsilon = \mathbb{E}[|\nabla Y_\varepsilon|^2] = \|\rho_\varepsilon * \nabla G\|_{L^2}^2. \quad (1.34)$$

We will show in Section 5.1 that  $Y_\varepsilon$  converges to  $Y$ ,  $\nabla Y_\varepsilon$  converges to  $\nabla Y$ , and  $:|\nabla Y_\varepsilon|^2:$  converges to  $:|\nabla Y|^2:$  almost surely in corresponding function spaces.

To state our main result, we need weighted Sobolev spaces, whose definitions and properties are presented in Section A.2. For now, we consider the following equivalent norm  $H_\mu^s(\mathbb{R}^2)$  for  $s, \mu \in \mathbb{R}$ :

$$\|f\|_{H_\mu^s(\mathbb{R}^2)} \sim \| \langle x \rangle^\mu f \|_{H^s(\mathbb{R}^2)}.$$

We now state the following result regarding global well-posedness of the approximating equation (1.32) for  $v_\varepsilon$ .

**Theorem 1.4.1.** *Let  $p > 1$ ,  $T > 0$ ,  $\delta_0 > 0$ ,  $\delta > 0$ ,  $0 < \varepsilon < \frac{1}{2}$ , and  $1 < s < 2$ . Then, there exists  $\Omega_0 \subset \Omega$  with  $\mathbb{P}(\Omega_0) = 1$  such that for each  $\omega \in \Omega_0$ , there exists  $\delta_1 > 0$  and a unique solution  $v_\varepsilon$  to the equation (1.32) with initial data  $v_0 \in H_{\delta_0}^2(\mathbb{R}^2)$  in the class*

$$L^\infty(\mathbb{R}; H_{-\delta}^2(\mathbb{R}^2)) \cap C(\mathbb{R}; H_{\delta_1}^s(\mathbb{R}^2)).$$

Moreover, there exist constants  $C, C(\omega) > 0$  independent of  $\varepsilon$  such that

$$\|v_\varepsilon\|_{L^\infty([-T, T]; H_{-\delta}^2(\mathbb{R}^2))} \leq C(\omega) |\log \varepsilon|^C.$$

Well-posedness of the approximating equation (1.32) is not obvious since the stochastic objects  $\nabla Y_\varepsilon$ ,  $:|\nabla Y_\varepsilon|^2:$ ,  $\xi_\varepsilon$ , and  $e^{-(p-1)Y_\varepsilon}$  are smooth but unbounded. Because of this, the classical Strichartz estimates are not available and so we cannot apply a standard contraction argument. In fact, we will establish some weighted Strichartz estimates that will enable us to perform a compactness argument as in [29] and construct a unique global solution  $v_\varepsilon$  as a limit

of solutions  $v_{\varepsilon,n}$  satisfying

$$\begin{cases} i\partial_t v_{\varepsilon,n} = \Delta v_{\varepsilon,n} - 2\nabla(\theta_n Y_\varepsilon) \cdot \nabla v_{\varepsilon,n} + \theta_n: \widetilde{|\nabla Y_\varepsilon|^2}: v_{\varepsilon,n} - \lambda e^{-(p-1)\theta_n Y_\varepsilon} |v_{\varepsilon,n}|^{p-1} v_{\varepsilon,n} \\ v_{\varepsilon,n}|_{t=0} = v_0, \end{cases} \quad (1.35)$$

where  $\theta_n(x) = \theta(x/n)$  with  $\theta \in C_c^\infty(\mathbb{R}^2)$ ,  $\theta \geq 0$ ,  $\eta \equiv 1$  on  $\{|x| \leq 1\}$ . To see that (1.35) is globally well-posed, we define  $u_{\varepsilon,n} = e^{-\theta_n Y_\varepsilon} v_{\varepsilon,n}$ , which satisfies the following equation:

$$\begin{cases} i\partial_t u_{\varepsilon,n} = \Delta u_{\varepsilon,n} + (\theta_n: \widetilde{|\nabla Y_\varepsilon|^2}: - \nabla(\theta_n Y_\varepsilon)^2 + \Delta(\theta_n Y_\varepsilon)) u_{\varepsilon,n} - \lambda |u_{\varepsilon,n}|^{p-1} u_{\varepsilon,n} \\ u_{\varepsilon,n}|_{t=0} = e^{-\theta_n Y_\varepsilon} v_0. \end{cases} \quad (1.36)$$

Note that  $\theta_n Y_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ , so that  $e^{-\theta_n Y_\varepsilon} v_0 \in H^2(\mathbb{R}^2)$  given that  $v_0 \in H^2(\mathbb{R}^2) \subset H_{\delta_0}^2(\mathbb{R}^2)$ . Since the equation (1.36) is a nonlinear Schrödinger equation containing only bounded and smooth terms, by classical results in [44, 68, 20], there exists a unique global solution  $u_{\varepsilon,n}$  to (1.36) in  $C(\mathbb{R}; H^2(\mathbb{R}^2))$ . This shows that there exists a unique global solution  $v_{\varepsilon,n}$  to (1.35) in  $C(\mathbb{R}; H^2(\mathbb{R}^2))$ . Then, once we establish a weighted  $H^2$  a priori bound for  $v_{\varepsilon,n}$  independent of  $n$ , we can use a similar argument in [29] to prove Theorem 1.4.1. We present the proof in Chapter 5 (in particular Subsection 5.6.1).

Next, once we have the global unique solution  $v_\varepsilon$  to (1.35), we would like to take the limit  $\varepsilon \rightarrow 0$ . We remark that this can also be interpreted as the convergence (up to a phase shift) of  $u_\varepsilon$  satisfying the following smoothed version of (1.27):

$$\begin{cases} i\partial_t u_\varepsilon = \Delta u_\varepsilon + \xi_\varepsilon u_\varepsilon - \lambda |u_\varepsilon|^{p-1} u_\varepsilon \\ u_\varepsilon|_{t=0} = e^{-Y_\varepsilon} v_0. \end{cases} \quad (1.37)$$

Note that it is not obvious to prove well-posedness of the equation (1.37). Nevertheless, by a direct computation, one can easily show that

$$u_\varepsilon = e^{-ic_\varepsilon t} e^{-Y_\varepsilon} v_\varepsilon$$

solves (1.37) given that  $v_\varepsilon$  solves (1.32).

We are now ready to state our global well-posedness result of the limiting equation (1.30).

**Theorem 1.4.2.** *Let  $p > 1$ ,  $T > 0$ ,  $\delta_0 > 0$ ,  $\delta > 0$ , and  $1 < s < 2$ . Then, there exists  $\Omega_0 \subset \Omega$  with  $\mathbb{P}(\Omega_0) = 1$  such that for each  $\omega \in \Omega_0$ , there exists  $\delta_1 > 0$  and  $v \in C(\mathbb{R}; H_{\delta_1}^s(\mathbb{R}^2))$  such that the following convergence holds:*

$$\|v_\varepsilon - v\|_{C([-T,T]; H_{\delta_1}^s(\mathbb{R}^2))} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , where  $v_\varepsilon$  is given by Theorem 1.4.1. In particular,  $u_\varepsilon = e^{-ic_\varepsilon t} e^{-Y_\varepsilon} v_\varepsilon$  solves (1.37) and

$$\|e^{ic_\varepsilon t} e^{Y_\varepsilon} u_\varepsilon - v\|_{C([-T,T]; H_{\delta_1}^s(\mathbb{R}^2))} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , where  $c_\varepsilon \sim |\log \varepsilon|$  is the constant defined in (1.34). Moreover,  $v$  is the unique global solution to (1.30) in the class  $C([-T, T]; H_{\delta_1}^s(\mathbb{R}^2))$ .

Theorem 1.4.2 extends the result proved in [29] in the case  $1 < p < 2$ . We will focus on the situation  $p \geq 2$  in the proof of Theorem 1.4.2. Also, note that we prove almost sure convergence of  $v_\varepsilon$  in Theorem 1.4.2, which is stronger than the convergence in probability of  $v_\varepsilon$  in [29]. We present the proof of Theorem 1.4.2 in Chapter 5 (in particular Subsection 5.6.2).

We conclude this section by stating several remarks.

**Remark 1.4.3.** In Theorem 1.4.2, we are mainly concerned with the defocusing case  $\lambda \geq 0$ . For the focusing case  $\lambda < 0$ , the only place that requires a different proof is Proposition 5.3.1. In particular, Theorem 1.4.2 also holds with  $\lambda < 0$  and  $1 < p < 3$ . If  $\lambda < 0$  and  $p \geq 3$ , one

need to impose a smallness assumption on  $\|v_0\|_{H_{\delta_0}^1}$  to obtain Theorem 1.4.2. For details, see Remark 5.3.2.

**Remark 1.4.4.** It is not clear whether our approach based on the gauge-transform works for the dispersive Anderson model (1.27) in dimension 3. The main challenge in dimension 3 is that the spatial white noise is too rough (with regularity  $< -\frac{3}{2}$ ). One can compare the situation with the 3-dimensional parabolic Anderson model in [58], where Hairer-Labbé used the theory of regularity structures introduced by Hairer [54].

**Remark 1.4.5.** The authors in [50, 115, 90] introduced another approach to study the dispersive Anderson model (1.27). Their methods are based on the realization of the Anderson Hamiltonian, formally written as  $H = \Delta + \xi$ , as a self-adjoint operator on  $L^2$ . In their settings, the initial data  $u_0$  is required to belong to the domain of  $H$ . One can compare the initial condition in [50, 115] and the initial condition in Theorem 1.4.2 and also in [30, 29, 113, 114], where the initial data  $u_0$  is required to have a specific structure  $e^{-Y}v_0$  with  $v_0$  belonging to a weighted  $H^2$  space. See also [1, 81, 21, 3, 89] for more discussions on the Anderson Hamiltonian.

## Chapter 2

# Deterministic local well-posedness of quadratic NLS with nonlinearity $|u|^2$

In this chapter, we study local well-posedness of the quadratic NLS with nonlinearity  $|u|^2$  on the two-dimensional torus  $\mathbb{T}^2$ . In particular, we prove Theorem 1.1.1, local well-posedness of the quadratic NLS (1.6) in  $L^2(\mathbb{T}^2)$ .

### 2.1 Proof of the bilinear estimate

In this section, we present the proof of Proposition 1.1.3. By the triangle inequality  $\langle n \rangle^s \lesssim \langle n_1 \rangle^s \langle n_2 \rangle^s$  under  $n = n_1 + n_2$  for  $s \geq 0$ , our goal is to prove the following bilinear estimate (with  $s = 0$ ):

$$\|u\bar{v}\|_{X_T^{0, -\frac{1}{2} + \delta_1}} \lesssim \|u\|_{X_T^{0, \frac{1}{2} + \delta_2}} \|v\|_{X_T^{0, \frac{1}{2} + \delta_2}}, \quad (2.1)$$

where  $0 < T \leq 1$  and  $\delta_1 > \delta_2 > 0$  are sufficiently small. In view of the definitions (A.26) and (A.27) of the  $X^{s,b}$ -spaces, the bilinear estimate (2.1) follows once we prove

$$\left\| \sum_{\substack{n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \int_{\tau = \tau_1 - \tau_2} \frac{\widehat{u}(\tau_1, n_1) \overline{\widehat{v}(\tau_2, n_2)}}{\langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}} d\tau_1 \right\|_{L_\tau^2 \ell_n^2} \lesssim \|u\|_{L_t^2 L_x^2} \|v\|_{L_t^2 L_x^2}. \quad (2.2)$$

By duality, the estimate (2.2) follows once we prove the following estimate:

$$\left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} \frac{\widehat{u}(\tau_1, n_1) \widehat{v}(\tau_2, n_2) \overline{\widehat{w}(\tau, n)}}{\langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}} d\tau_1 d\tau \right| \lesssim \|u\|_{L_t^2 L_x^2} \|v\|_{L_t^2 L_x^2} \|w\|_{L_t^2 L_x^2}. \quad (2.3)$$

We first note that if  $n = 0$ ,  $n_1 = 0$ , or  $n_2 = 0$ , then the estimate (2.3) follows easily from the Cauchy-Schwarz inequality, provided that  $\delta_1 > 0$  is sufficiently small. Thus, for the remaining part of this section, we assume that  $n \neq 0$ ,  $n_1 \neq 0$ , and  $n_2 \neq 0$ .

Before proceeding further, we recall the following key algebraic relation:

$$(\tau - |n|^2) - (\tau_1 - |n_1|^2) + (\tau_2 - |n_2|^2) = 2n \cdot n_2, \quad (2.4)$$

where  $n = n_1 - n_2$  and  $\tau = \tau_1 - \tau_2$ . The main difficulty in proving (2.3) comes from the failure of the  $L^4$ -Strichartz estimate without a derivative loss. In order to overcome this difficulty, we

separately estimate the contributions coming from (i) non-resonant case:  $|n \cdot n_2| \gtrsim |n|^\varepsilon |n_2|^\varepsilon$  and (ii) nearly resonant case:  $|n \cdot n_2| \ll |n|^\varepsilon |n_2|^\varepsilon$  for some small  $\varepsilon > 0$ . In the non-resonant case, we can use the multilinear dispersion to make up for the derivative loss in the  $L^4$ -Strichartz estimate (Lemma A.4.5). In the nearly resonant case, by noting that the frequencies  $n$  and  $n_2$  are almost perpendicular, we make use of this angular restriction to prove a multilinear estimate without a derivative loss in the spirit of [110, 24].

### 2.1.1 Non-resonant interaction

In this subsection, we consider the non-resonant case

$$|n \cdot n_2| \gtrsim |n|^\varepsilon |n_2|^\varepsilon \quad (2.5)$$

for some small  $\varepsilon > 0$  sufficiently small. From (2.4) and (2.5), we have

$$\begin{aligned} \text{MAX} &\stackrel{\text{def}}{=} \max(\langle \tau - |n|^2 \rangle, \langle \tau_1 - |n_1|^2 \rangle, \langle \tau_2 - |n_2|^2 \rangle) \\ &\gtrsim \langle n \cdot n_2 \rangle \gtrsim \langle n \rangle^\varepsilon \langle n_2 \rangle^\varepsilon. \end{aligned} \quad (2.6)$$

We then consider the following three cases.

**Case 1:**  $\text{MAX} = \langle \tau - |n|^2 \rangle$ .

In this case, we directly prove (2.2). Then, with  $\varepsilon_1 = \frac{\varepsilon}{2}(\frac{1}{2} - \delta_1) > 0$ , it follows from (2.6) that

$$\langle n_1 \rangle^{\varepsilon_1} \langle n_2 \rangle^{\varepsilon_1} = \langle n + n_2 \rangle^{\varepsilon_1} \langle n_2 \rangle^{\varepsilon_1} \lesssim \langle n \rangle^{2\varepsilon_1} \langle n_2 \rangle^{2\varepsilon_1} \lesssim \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}. \quad (2.7)$$

We let

$$\begin{aligned} \widehat{U}(\tau_1, n_1) &\stackrel{\text{def}}{=} \langle n_1 \rangle^{-\varepsilon_1} \langle \tau_1 - |n_1|^2 \rangle^{-\frac{1}{2} - \delta_2} \widehat{u}(\tau_1, n_1) \\ \widehat{V}(\tau_2, n_2) &\stackrel{\text{def}}{=} \langle n_2 \rangle^{-\varepsilon_1} \langle \tau_2 - |n_2|^2 \rangle^{-\frac{1}{2} - \delta_2} \widehat{v}(\tau_2, n_2). \end{aligned}$$

Then, by (2.7), Plancherel's theorem, Hölder's inequality, and the  $L^4$ -Strichartz estimate (Lemma A.4.5), we obtain that the left-hand-side of (2.2) is bounded by

$$\|\widehat{U\overline{V}}\|_{L_t^2 L_x^2} \leq \|U\|_{L_t^4 L_x^4} \|V\|_{L_t^4 L_x^4} \lesssim \|U\|_{X^{\varepsilon_1, \frac{1}{2} + \delta_2}} \|V\|_{X^{\varepsilon_1, \frac{1}{2} + \delta_2}} = \|u\|_{L_t^2 L_x^2} \|v\|_{L_t^2 L_x^2},$$

as desired.

**Case 2:**  $\text{MAX} = \langle \tau_1 - |n_1|^2 \rangle$ .

In this case, we prove (2.3). With  $\varepsilon_2 = \varepsilon(\frac{1}{2} - \delta_1) > 0$ , it follows from (2.6) that

$$\langle n \rangle^{\varepsilon_2} \langle n_2 \rangle^{\varepsilon_2} \lesssim \langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} - \delta_1}. \quad (2.8)$$

Then, letting

$$\begin{aligned} \widehat{V}(\tau_2, n_2) &\stackrel{\text{def}}{=} \langle n_2 \rangle^{-\varepsilon_2} \langle \tau_2 - |n_2|^2 \rangle^{-\frac{1}{2} - \delta_2} \widehat{v}(\tau_2, n_2), \\ \widehat{W}(\tau, n) &\stackrel{\text{def}}{=} \langle n \rangle^{-\varepsilon_2} \langle \tau - |n|^2 \rangle^{-\frac{1}{2} - \delta_2} \widehat{w}(\tau, n), \end{aligned}$$

it follows from the Cauchy-Schwarz inequality in  $\tau_1$  and  $n_1$ , (2.8), Plancherel's theorem, Hölder's inequality, and the  $L^4$ -Strichartz estimate (Lemma A.4.5) that the left-hand-side of (2.3) is bounded by

$$\begin{aligned} &\|u\|_{L_t^2 L_x^2} \|V\|_{L_t^4 L_x^4} \|W\|_{L_t^4 L_x^4} \\ &\lesssim \|u\|_{L_t^2 L_x^2} \|V\|_{X^{\varepsilon_2, \frac{1}{2} + \delta_2}} \|W\|_{X^{\varepsilon_2, \frac{1}{2} + \delta_2}} \\ &= \|u\|_{L_t^2 L_x^2} \|v\|_{L_t^2 L_x^2} \|w\|_{L_t^2 L_x^2}, \end{aligned}$$

as desired.

**Case 3:**  $\text{MAX} = \langle \tau_2 - |n_2|^2 \rangle$ .

This case follows from proceeding as in Case 2 and thus we omit details.

## 2.1.2 Resonant interaction

We now consider the resonant case  $|n \cdot n_2| \ll |n|^\varepsilon |n_2|^\varepsilon$ . With  $\theta = \theta(n, n_2) = \frac{1}{|n|^{1-\varepsilon} |n_2|^{1-\varepsilon}}$ , we rewrite this condition as

$$|n \cdot n_2| \ll \theta |n| |n_2|.$$

In the following, we divide the argument into three main cases, depending on the sizes of  $n$  and  $n_2$ .

**Case 1:**  $|n|^{\frac{1}{2}} \lesssim |n_2| \lesssim |n|^2$ .

In this case, we follow the idea from the proof of [110, Proposition 10.1] (see also [24]).

**Subcase 1.1:**  $|n| \sim |n_2|$ .

By applying the dyadic decompositions to the frequencies  $n$  and  $n_2$ , we obtain that the left-hand-side of (2.3) is bounded by

$$\begin{aligned} & \sum_{\substack{N \sim N_2 \geq 1 \\ \text{dyadic}}} \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} \mathbf{1}_{\{|\cos \angle(n, n_2)| \ll \theta\}} \right. \\ & \quad \times \left. \frac{\widehat{u}(\tau_1, n_1) \widehat{P_{N_2} v}(\tau_2, n_2) \widehat{P_N w}(\tau, n)}{\langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}} d\tau_1 d\tau_2 \right|. \end{aligned} \quad (2.9)$$

Note that when  $|n| \sim |n_2| \sim N$ , we have  $\theta = \frac{1}{|n|^{1-\varepsilon} |n_2|^{1-\varepsilon}} \sim N^{-2+2\varepsilon}$ . We now restrict  $n$  and  $n_2$  to the following angular sectors:

$$\begin{aligned} A_\ell & \stackrel{\text{def}}{=} \{n \in \mathbb{Z}^2 : |n| \sim N, \arg(n) = \ell N^{-2+2\varepsilon} + O(N^{-2+2\varepsilon})\}, \\ A_{\ell_2} & \stackrel{\text{def}}{=} \{n_2 \in \mathbb{Z}^2 : |n_2| \sim N_2, \arg(n_2) = \ell_2 N^{-2+2\varepsilon} + O(N^{-2+2\varepsilon})\}, \end{aligned} \quad (2.10)$$

where  $\ell, \ell_2 \in I_N \stackrel{\text{def}}{=} [1, 2\pi N^{2-\varepsilon}] \cap \mathbb{Z}$ . Since  $|\cos \angle(n, n_2)| \ll \theta \sim N^{-2+2\varepsilon}$ , we have  $|\arg(n) - \arg(n_2)| = \frac{\pi}{2} + O(N^{-2+2\varepsilon})$  or  $|\arg(n) - \arg(n_2)| = \frac{3\pi}{2} + O(N^{-2+2\varepsilon})$ . This means that for each fixed  $\ell$ , there exists a set  $L_2(\ell)$  of size  $O(1)$  such that  $\mathbf{1}_{A_\ell}(n) \cdot \mathbf{1}_{A_{\ell_2}}(n_2) = 0$  unless  $\ell_2 \in L_2(\ell)$ .

Continuing with the right-hand-side of (2.9), we insert the angular restrictions (2.10) and set

$$\widehat{v_{N_2, \ell_2}}(\tau_2, n_2) = \mathbf{1}_{A_{\ell_2}}(n_2) \cdot \widehat{P_{N_2} v}(\tau_2, n_2) \quad \text{and} \quad \widehat{w_{N, \ell}}(\tau, n) = \mathbf{1}_{A_\ell}(n) \cdot \widehat{P_N w}(\tau, n).$$

Then, by applying the Cauchy-Schwarz inequalities in  $\tau_1$  and  $n_1$ , the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , and Hölder's inequality in  $\tau_1$  and  $n_1$ , we obtain that the left-hand-side of (2.3)

is bounded by

$$\begin{aligned}
& \sum_{\substack{N \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I_N} \sum_{\ell_2 \in L_2(\ell)} \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} \mathbf{1}_{\{|\cos \angle(n, n_2)| \ll N^{-2+2\varepsilon}\}} \right. \\
& \quad \times \frac{\widehat{u}(\tau_1, n_1) \widehat{v_{N_2, \ell_2}}(\tau_2, n_2) \widehat{w_{N, \ell}}(\tau, n)}{\langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}} d\tau_1 d\tau \left. \right| \\
& \leq \sum_{\substack{N \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I_N} \sum_{\ell_2 \in L_2(\ell)} \left\| \widehat{u} \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \left\| \sum_{n \in \mathbb{Z}^2} \int \mathbf{1}_{\{|\cos \angle(n, n_1 - n)| \ll N^{-2+2\varepsilon}\}} \right. \\
& \quad \times \frac{\widehat{v_{N_2, \ell_2}}(\tau_1 - \tau, n_1 - n) \widehat{w_{N, \ell}}(\tau, n)}{\langle \tau_1 - \tau - |n_1 - n|^2 \rangle^{\frac{1}{2} + \delta_2}} d\tau \left. \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \tag{2.11} \\
& \leq \|u\|_{L_t^2 L_x^2} \sum_{\substack{N \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I_N} \sum_{\ell_2 \in L_2(\ell)} \|\widehat{v_{N_2, \ell_2}}\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|\widehat{w_{N, \ell}}\|_{L_{\tau}^2 \ell_n^2} \\
& \quad \times \sup_{\tau_1, n_1} \left( \sum_n \int \frac{\mathbf{1}_{\{|\cos \angle(n, n_1 - n)| \ll N^{-2+2\varepsilon}\}} \mathbf{1}_{A_{\ell_2}}(n_1 - n) \cdot \mathbf{1}_{A_\ell}(n)}{\langle \tau_1 - \tau - |n_1 - n|^2 \rangle^{1+2\delta_2}} d\tau \right)^{\frac{1}{2}} \\
& \lesssim \|u\|_{L_t^2 L_x^2} \sum_{\substack{N \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I_N} \sum_{\ell_2 \in L_2(\ell)} \|\widehat{v_{N_2, \ell_2}}\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|\widehat{w_{N, \ell}}\|_{L_{\tau}^2 \ell_n^2} \sup_{n_1, \ell, \ell_2} |\mathcal{A}_{N, \ell, \ell_2}(n_1)|^{\frac{1}{2}},
\end{aligned}$$

where  $\mathcal{A}_{N, \ell, \ell_2}(n_1)$  is defined by

$$\mathcal{A}_{N, \ell, \ell_2}(n_1) \stackrel{\text{def}}{=} \{n \in \mathbb{Z}^2 : |\cos \angle(n, n_1 - n)| \ll N^{-2+2\varepsilon}, n \in A_\ell, n_1 - n \in A_{\ell_2}\}.$$

In view of (2.10), for fixed  $n_1 \in \mathbb{Z}^2$  and fixed  $\ell, \ell_2 \in I_N$ , we note that any  $n \in \mathcal{A}_{N, \ell, \ell_2}$  is contained in a rectangle with sides of length  $\sim$

$$|n|N^{-2+2\varepsilon} \sim N^{-1+2\varepsilon} \quad \text{and} \quad |n_1 - n|N^{-2+2\varepsilon} \sim N^{-1+2\varepsilon}.$$

Hence, for  $0 < \varepsilon < \frac{1}{2}$ , we have

$$\sup_{n_1, \ell, \ell_2} |\mathcal{A}_{N, \ell, \ell_2}(n_1)| \lesssim 1. \tag{2.12}$$

Therefore, by estimating (2.11) with (2.12), the Cauchy-Schwarz inequality in  $\ell$ , and the Cauchy-Schwarz inequality in  $N \sim N_2$ , we obtain the desired inequality (2.3).

**Subcase 1.2:**  $|n_2| \ll |n| \lesssim |n_2|^2$ .

Since  $|n| \gg |n_2|$  and  $n_1 = n + n_2$ , we have  $|n_1| \sim |n|$ . Note that when  $|n| \sim N \lesssim |n_2|^2$ , we have  $\theta = \frac{\pi}{|n|^{1-\varepsilon}|n_2|^{1-\varepsilon}} \lesssim N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}$ . We proceed as in Subcase 1.1 by restricting  $n$  and  $n_2$  to the following angular sectors:

$$\begin{aligned}
B_\ell & \stackrel{\text{def}}{=} \{n \in \mathbb{Z}^2 : |n| \sim N, \arg(n) = \ell N^{-\frac{3}{2} + \frac{3}{2}\varepsilon} + O(N^{-\frac{3}{2} + \frac{3}{2}\varepsilon})\}, \\
B_{\ell_2} & \stackrel{\text{def}}{=} \{n_2 \in \mathbb{Z}^2 : N^{\frac{1}{2}} \lesssim |n_2| \ll N, \arg(n_2) = \ell_2 N^{-\frac{3}{2} + \frac{3}{2}\varepsilon} + O(N^{-\frac{3}{2} + \frac{3}{2}\varepsilon})\},
\end{aligned} \tag{2.13}$$

where  $\ell, \ell_2 \in I'_N \stackrel{\text{def}}{=} [1, 2\pi N^{\frac{3}{2} - \frac{3}{2}\varepsilon}] \cap \mathbb{Z}$ . Under  $|\cos \angle(n, n_2)| \ll N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}$ , we have  $|\arg(n) - \arg(n_2)| = \frac{\pi}{2} + O(N^{-\frac{3}{2} + \frac{3}{2}\varepsilon})$  or  $|\arg(n) - \arg(n_2)| = \frac{3\pi}{2} + O(N^{-\frac{3}{2} + \frac{3}{2}\varepsilon})$ . Hence, for each fixed  $\ell$ , there exists a set  $L'_2(\ell)$  of size  $O(1)$  such that  $\mathbf{1}_{B_\ell}(n) \cdot \mathbf{1}_{B_{\ell_2}}(n_2) = 0$  unless  $\ell_2 \in L'_2(\ell)$ . With a slight abuse of notation, we set

$$\widehat{v_{N, \ell_2}}(\tau_2, n_2) = \mathbf{1}_{B_{\ell_2}}(n_2) \cdot \widehat{v}(\tau_2, n_2) \quad \text{and} \quad \widehat{w_{N, \ell}}(\tau, n) = \mathbf{1}_{B_\ell}(n) \cdot \widehat{P_N w}(\tau, n).$$

Then, by proceeding as in (2.11), i.e. by applying the Cauchy-Schwarz inequalities in  $\tau_1$  and  $n_1$ , the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , and Hölder's inequality in  $\tau_1$  and  $n_1$ , we obtain

that the left-hand-side of (2.3) is bounded by

$$\begin{aligned}
& \sum_{\substack{N \sim N_1 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I'_N} \sum_{\ell_2 \in L'_2(\ell)} \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} \mathbf{1}_{\{|\cos \angle(n, n_2)| \ll N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}\}} \right. \\
& \quad \times \frac{\widehat{P_{N_1} u}(\tau_1, n_1) \widehat{v_{N, \ell_2}}(\tau_2, n_2) \widehat{w_{N, \ell}}(\tau, n)}{\langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2} + \delta_2} \langle \tau - |n|^2 \rangle^{\frac{1}{2} - \delta_1}} d\tau_1 d\tau \left. \right| \\
& \leq \sum_{\substack{N \sim N_1 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I'_N} \sum_{\ell_2 \in L'_2(\ell)} \left\| \widehat{P_{N_1} u} \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \left\| \sum_{n \in \mathbb{Z}^2} \int \mathbf{1}_{\{|\cos \angle(n, n_1 - n)| \ll N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}\}} \right. \\
& \quad \times \frac{\widehat{v_{N, \ell_2}}(\tau_1 - \tau, n_1 - n) \widehat{w_{N, \ell}}(\tau, n)}{\langle \tau_1 - \tau - |n_1 - n|^2 \rangle^{\frac{1}{2} + \delta_2}} d\tau \left. \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \\
& \leq \sum_{\substack{N \sim N_1 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I'_N} \sum_{\ell_2 \in L'_2(\ell)} \left\| \widehat{P_{N_1} u} \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \left\| \widehat{v_{N, \ell_2}} \right\|_{L_{\tau_2}^2 \ell_{n_2}^2} \left\| \widehat{w_{N, \ell}} \right\|_{L_{\tau}^2 \ell_n^2} \\
& \quad \times \sup_{\tau_1, n_1} \left( \sum_n \int \frac{\mathbf{1}_{\{|\cos \angle(n, n_1 - n)| \ll N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}\}} \mathbf{1}_{B_{\ell_2}}(n_1 - n) \cdot \mathbf{1}_{B_{\ell}}(n)}{\langle \tau_1 - \tau - |n_1 - n|^2 \rangle^{1 + 2\delta_2}} d\tau \right)^{1/2} \\
& \lesssim \sum_{\substack{N \sim N_1 \geq 1 \\ \text{dyadic}}} \sum_{\ell \in I'_N} \sum_{\ell_2 \in L'_2(\ell)} \left\| \widehat{P_{N_1} u} \right\|_{L_{\tau_1}^2 \ell_{n_1}^2} \left\| \widehat{v_{N, \ell_2}} \right\|_{L_{\tau_2}^2 \ell_{n_2}^2} \left\| \widehat{w_{N, \ell}} \right\|_{L_{\tau}^2 \ell_n^2} \sup_{n_1} |\mathcal{B}_{N, \ell, \ell_2}(n_1)|^{\frac{1}{2}},
\end{aligned}$$

where  $\mathcal{B}_{N, \ell, \ell_2}(n_1)$  is defined by

$$\mathcal{B}_{N, \ell, \ell_2}(n_1) \stackrel{\text{def}}{=} \{n \in \mathbb{Z}^2 : |\cos \angle(n, n_1 - n)| \ll N^{-\frac{3}{2} + \frac{3}{2}\varepsilon}, n \in B_{\ell}, n_1 - n \in B_{\ell_2}\}.$$

Then, the desired bound (2.3) follows from the Cauchy-Schwarz inequality in  $\ell$  and then in the Cauchy-Schwarz inequality in  $N \sim N_1$ , once we prove

$$\sup_{n_1, \ell, \ell_2} |\mathcal{B}_{N, \ell, \ell_2}(n_1)| \lesssim 1. \quad (2.14)$$

In view of (2.13), for fixed  $n_1 \in \mathbb{Z}^2$  and fixed  $\ell, \ell_2 \in I'_N$ , we note that any  $n \in \mathcal{B}_{N, \ell, \ell_2}$  is contained in a rectangle with sides of length  $\sim$

$$|n|N^{-\frac{3}{2} + \frac{3}{2}\varepsilon} \sim N^{-\frac{1}{2} + \frac{3}{2}\varepsilon} \quad \text{and} \quad |n_1 - n|N^{-\frac{3}{2} + \frac{3}{2}\varepsilon} \ll N^{-\frac{1}{2} + \frac{3}{2}\varepsilon}.$$

Therefore, as long as  $0 < \varepsilon < \frac{1}{3}$ , there are at most  $O(1)$  many  $n$ 's in  $\mathcal{B}_{N, \ell, \ell_2}$ , which implies (2.14).

**Subcase 1.3:**  $|n| \ll |n_2| \lesssim |n|^2$ .

This subcase follows from Subcase 1.2 by switching the roles of  $(\tau, n)$  and  $(\tau_2, n_2)$  and thus we omit details.

**Case 2:**  $|n|^2 \ll |n_2|$ .

We divide the argument into two subcases, depending on the sizes of  $N_2$  and the largest modulation  $L_{\max} = \max(L_0, L_1, L_2)$ .

**Subcase 2.1:**  $L_{\max} \gtrsim N_2$  (high-modulation case).

In this subcase, recalling the notations in Section A.1, we dyadically decompose the spatial frequencies and modulations of  $u, v$ , and  $w$  so that  $\text{supp } \widehat{P_{N_1, L_1} u} \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}$ ,  $\text{supp } \widehat{P_{N_2, L_2} v} \subset \mathfrak{P}_{N_2} \cap \mathfrak{S}_{L_2}$ , and  $\text{supp } \widehat{P_{N_0, L_0} w} \subset \mathfrak{P}_{N_0} \cap \mathfrak{S}_{L_0}$  for dyadic  $N_j, L_j \geq 1, j = 0, 1, 2$ . Note that we have  $N_0^2 \ll N_2 \sim N_1$ . Our main goal is to show the following lemma.

**Lemma 2.1.1.** *Let  $N_j, L_j \geq 1, j = 0, 1, 2$ , be dyadic numbers with  $N_0^2 \ll N_2 \sim N_1$  and  $L_{\max} \gtrsim N_2$ . Suppose that  $f, g, h \in L^2(\mathbb{R} \times \mathbb{Z}^2)$  are real-valued nonnegative functions such that*

$$\text{supp } f \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}, \quad \text{supp } g \subset \mathfrak{P}_{N_2} \cap \mathfrak{S}_{L_2}, \quad \text{and} \quad \text{supp } h \subset \mathfrak{P}_{N_0} \cap \mathfrak{S}_{L_0}. \quad (2.15)$$

Then, we have

$$\begin{aligned} & \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} f(\tau_1, n_1) g(\tau_2, n_2) h(\tau, n) d\tau d\tau_1 \right| \\ & \lesssim L_1^{\frac{1}{2}+} L_2^{\frac{1}{2}+} L_0^{\frac{1}{4}+} N_2^{0-} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2}. \end{aligned} \quad (2.16)$$

We first present the proof of (2.3) in this case by assuming Lemma 2.1.1. Under  $N_0^2 \ll N_2 \sim N_1$  and  $L_{\max} \gtrsim N_2$ , by applying Lemma 2.1.1, we obtain that the left-hand-side of (2.3) is bounded by

$$\begin{aligned} & \sum_{\substack{N_1, N_2, N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{L_1, L_2, L_0 \geq 1 \\ \text{dyadic}}} L_1^{-\frac{1}{2}-\delta_2} L_2^{-\frac{1}{2}-\delta_2} L_0^{-\frac{1}{2}+\delta_1} \\ & \quad \times L_1^{\frac{1}{2}+} L_2^{\frac{1}{2}+} L_0^{\frac{1}{4}+} N_2^{0-} \|P_{N_1, L_1} u\|_{L_{\tau}^2 L_x^2} \|P_{N_2, L_2} v\|_{L_{\tau}^2 L_x^2} \|P_{N_0, L_0} w\|_{L_{\tau}^2 L_x^2} \\ & \leq \sum_{\substack{N_1, N_2, N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{L_1, L_2, L_0 \geq 1 \\ \text{dyadic}}} L_1^{0-} L_2^{0-} L_0^{0-} N_2^{0-} \|u\|_{L_{\tau}^2 L_x^2} \|v\|_{L_{\tau}^2 L_x^2} \|w\|_{L_{\tau}^2 L_x^2} \\ & \lesssim \|u\|_{L_{\tau}^2 L_x^2} \|v\|_{L_{\tau}^2 L_x^2} \|w\|_{L_{\tau}^2 L_x^2}, \end{aligned}$$

as desired.

We now present the proof of Lemma 2.1.1.

*Proof of Lemma 2.1.1.* By the Cauchy-Schwarz inequality in  $\tau$  and  $n$ , Lemma A.4.6, and the condition  $L_{\max} \gtrsim N_2 \gg N_0^2$ , we obtain that the left-hand-side of (2.16) is bounded by

$$\begin{aligned} & \|h\|_{L_{\tau}^2 \ell_n^2} \min\{L_1, L_2\}^{\frac{1}{2}} \left( \frac{\max\{L_1, L_2\}}{N_0} + 1 \right)^{\frac{1}{2}} N_0^{\frac{1}{2}} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \\ & \lesssim (L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} + \min\{L_1, L_2\}^{\frac{1}{2}} N_2^{\frac{1}{4}}) \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2} \\ & \lesssim L_1^{\frac{1}{2}+} L_2^{\frac{1}{2}+} L_0^{\frac{1}{4}+} N_2^{0-} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2}, \end{aligned}$$

which gives (2.16).  $\square$

**Subcase 2.2:**  $L_{\max} \ll N_2$  (low-modulation case).

Our goal in this subcase is to show the following lemma.

**Lemma 2.1.2.** *Let  $N_1, N_2, L_0, L_1, L_2 \geq 1$  be dyadic numbers with  $N_1 \sim N_2$  and  $L_{\max} \ll N_2$ . Suppose that  $f, g, h \in L^2(\mathbb{R} \times \mathbb{Z}^2)$  are real-valued nonnegative functions such that*

$$\begin{aligned} & \text{supp } f \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}, \quad \text{supp } g \subset \mathfrak{P}_{N_2} \cap \mathfrak{S}_{L_2}, \quad \text{and} \\ & \text{supp } h \subset \{n \in \mathbb{Z}^2 : 1 \leq |n|^2 \ll N_2\} \cap \mathfrak{S}_{L_0}. \end{aligned}$$

Then, we have

$$\begin{aligned} & \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} f(\tau_1, n_1) g(\tau_2, n_2) h(\tau, n) \cdot \mathbf{1}_{\{|\cos \angle(n, n_2)| \ll \theta\}} d\tau d\tau_1 \right| \\ & \lesssim L_{\text{med}}^{\frac{3}{8}} L_{\text{max}}^{\frac{3}{8}} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2}, \end{aligned} \quad (2.17)$$

where  $\theta = \frac{1}{|n|^{1-\varepsilon} |n_2|^{1-\varepsilon}} \ll 1$  and  $L_{\text{med}}$  is the second largest among  $L_0, L_1$ , and  $L_2$ .

We first assume Lemma 2.1.2 and prove (2.3). Recall that  $|n|^2 \ll |n_2| \sim |n_1|$  in this case. Then, by Lemma 2.1.2 (with  $\theta$  as above) and the Cauchy-Schwarz inequality in  $N_1 \sim N_2$ , we

obtain that the left-hand-side of (2.3) is bounded by

$$\begin{aligned}
& \sum_{\substack{N_1 \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\substack{L_1, L_2, L_0 \geq 1 \\ \text{dyadic}}} L_1^{-\frac{1}{2}-\delta_2} L_2^{-\frac{1}{2}-\delta_2} L_0^{-\frac{1}{2}+\delta_1} \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2}} \iint_{\tau = \tau_1 - \tau_2} \mathbf{1}_{\{|\cos \angle(n, n_2)| \ll \theta\}} \right. \\
& \quad \left. \times \mathbf{1}_{|n|^2 \ll N_2} \widehat{P_{N_1, L_1} u}(\tau_1, n_1) \widehat{P_{N_2, L_2} v}(\tau_2, n_2) \widehat{Q_{L_0} w}(\tau, n) d\tau_1 d\tau \right| \\
& \lesssim \sum_{\substack{N_1 \sim N_2 \geq 1 \\ \text{dyadic}}} \sum_{\substack{L_1, L_2, L_0 \geq 1 \\ \text{dyadic}}} L_1^{0-} L_2^{0-} L_0^{0-} \| \widehat{P_{N_1, L_1} u} \|_{L_{\tau_1}^2 \ell_{n_1}^2} \| \widehat{P_{N_2, L_2} v} \|_{L_{\tau_2}^2 \ell_{n_2}^2} \| \widehat{Q_{L_0} w} \|_{L_{\tau}^2 \ell_n^2} \\
& \lesssim \| \widehat{u} \|_{L_{\tau_1}^2 \ell_{n_1}^2} \| \widehat{v} \|_{L_{\tau_2}^2 \ell_{n_2}^2} \| \widehat{w} \|_{L_{\tau}^2 \ell_n^2},
\end{aligned}$$

as desired.

We now present the proof of Lemma 2.1.2, where we use the idea from the proofs of [76, Proposition 3.6, Proposition 3.9].

*Proof of Lemma 2.1.2.* We first consider the case when  $|n| \lesssim L_{\max}$ . In this case, we decompose the spatial frequencies of  $h$  on the left-hand side of (2.17) into dyadic blocks  $\{|n| \sim N_0\}$ , where the dyadic number  $N_0 \geq 1$  satisfies  $N_0^2 \ll N_2$ . For fixed  $N_0$ , we decompose the spatial frequencies of  $f$  and  $g$  into balls of radius  $\sim N_0$ , indexed by  $J_1 \in \mathcal{J}_1$  and  $J_2 \in \mathcal{J}_2$ , respectively. With a slight abuse of notation, we also use  $J_1$  and  $J_2$  to denote the balls themselves. Note that for each fixed  $J_1 \in \mathcal{J}_1$ , the product  $\mathbf{1}_{J_1}(\tau_1, n_1) \cdot \mathbf{1}_{J_2}(\tau_1 - \tau, n_1 - n)$  is nonzero for at most  $O(1)$  many  $J_2 \in \mathcal{J}_2$ , and we denote the set of these indices  $J_2$ 's as  $\mathcal{J}_2(J_1)$ .

Since

$$||n_1|^2 - |n_2|^2| = ||n|^2 + (\tau - |n|^2) - (\tau_1 - |n_1|^2) + (\tau_2 - |n_2|^2)| \lesssim N_0^2 + L_{\max},$$

we have

$$||n_1| - |n_2|| \lesssim N_2^{-1}(N_0^2 + L_{\max}) \ll 1. \quad (2.18)$$

Given  $j \in \mathbb{N}$ , we let  $A_j = \{(\tau_1, n_1) : j < |n_1| \leq j + 1\}$  and  $B_j = \{(\tau_2, n_2) : j < |n_2| \leq j + 1\}$ . In view of (2.18), we see that the product  $\mathbf{1}_{A_{j_1}}(\tau_1, n_1) \cdot \mathbf{1}_{B_{j_2}}(\tau_2, n_2)$  is nonzero if and only if  $|j_1 - j_2| \leq 1$ . Hence, by letting  $f_{j_1, J_1}(\tau_1, n_1) = \mathbf{1}_{A_{j_1} \cap J_1}(\tau_1, n_1) \cdot f(\tau_1, n_1)$  and  $g_{j_2, J_2}(\tau_2, n_2) = \mathbf{1}_{B_{j_2} \cap J_2}(\tau_2, n_2) \cdot g(\tau_2, n_2)$ , we can estimate the left-hand-side of (2.17) by

$$\begin{aligned}
& \sum_{\substack{N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{J_1 \in \mathcal{J}_1 \\ J_2 \in \mathcal{J}_2(J_1)}} \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ |j_1 - j_2| \leq 1}} \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 - n_2 \\ |n| \sim N_0}} \iint_{\tau = \tau_1 - \tau_2} f_{j_1, J_1}(\tau_1, n_1) g_{j_2, J_2}(\tau_2, n_2) \right. \\
& \quad \left. \times h(\tau, n) \cdot \mathbf{1}_{S_{\tau, n, j_1, j_2, J_1, J_2}}(\tau_1, n_1) d\tau d\tau_1 \right|,
\end{aligned}$$

where the set  $S_{\tau, n, j_1, j_2, J_1, J_2}$  is defined by

$$\begin{aligned}
S_{\tau, n, j_1, j_2, J_1, J_2} \stackrel{\text{def}}{=} \{ & (\tau_1, n_1) \in \mathbb{R} \times \mathbb{Z}^2 : (\tau_1, n_1) \in \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1} \cap A_{j_1}, n_1 \in J_1 \\
& (\tau_1 - \tau, n_1 - n) \in \mathfrak{P}_{N_2} \cap \mathfrak{S}_{L_2} \cap B_{j_2}, n_1 - n \in J_2, \\
& |\cos \angle(n, n_1 - n)| \ll \theta \}.
\end{aligned} \quad (2.19)$$

By the Cauchy-Schwarz inequality in  $\tau_1$  and  $n_1$ , Hölder's inequality in  $\tau$  and  $n$ , and Hölder's inequality in  $j$ , we then bound the left-hand-side of (2.17) by

$$\begin{aligned}
& \sum_{\substack{N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{J_1 \in \mathcal{J}_1 \\ J_2 \in \mathcal{J}_2(J_1)}} \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ |j_1 - j_2| \leq 1}} \sum_{n: |n| \sim N_0} \int \| \mathbf{1}_{S_{\tau, n, j_1, j_2, J_1, J_2}} \|_{L_{\tau_1}^2 \ell_{n_1}^2} \\
& \quad \times \| f_{j_1, J_1}(\tau_1, n_1) g_{j_2, J_2}(\tau_1 - \tau, n_1 - n) \|_{L_{\tau_1}^2 \ell_{n_1}^2} |h(\tau, n)| d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\substack{N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{J_1 \in \mathcal{J}_1 \\ J_2 \in \mathcal{J}_2(J_1)}} \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ |j_1 - j_2| \leq 1}} \min\{L_1, L_2\}^{\frac{1}{2}} \|f_{j_1, J_1}\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g_{j_2, J_2}\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2} \\
&\quad \times \sup_{\substack{\tau, \tau_1, n \\ |n| \sim N_0}} |\tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}|^{\frac{1}{2}} \\
&\lesssim \sum_{\substack{N_0 \geq 1 \\ \text{dyadic}}} \sum_{\substack{J_1 \in \mathcal{J}_1 \\ J_2 \in \mathcal{J}_2(J_1)}} \min\{L_1, L_2\}^{\frac{1}{2}} \|\mathbf{1}_{J_1} \cdot f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|\mathbf{1}_{J_2} \cdot g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_{\tau}^2 \ell_n^2} \\
&\quad \times \sup_{\substack{\tau, \tau_1, n, j_1, j_2 \\ |n| \sim N_0}} |\tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}|^{\frac{1}{2}},
\end{aligned}$$

where the set  $\tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}$  is defined by

$$\tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2} \stackrel{\text{def}}{=} \{n_1 \in \mathbb{Z}^2 : (\tau_1, n_1) \in \mathcal{S}_{\tau, n, j_1, j_2, J_1, J_2}\}. \quad (2.20)$$

We claim that the following bound holds:

$$\sup_{\substack{J_1 \in \mathcal{J}_1 \\ J_2 \in \mathcal{J}_2(J_1)}} \sup_{\substack{\tau, \tau_1, n, j \\ |n| \sim N_0}} |\tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}| \lesssim \min\left\{N_0, \frac{L_{\max}}{N_0}\right\}. \quad (2.21)$$

For now, let us assume (2.21). Note that the right-hand-side of (2.21) is bounded by  $L_{\max}^{\frac{1}{2}}$ . Then, by applying the Cauchy-Schwarz inequality in  $J_1 \in \mathcal{J}_1$  and summing over dyadic  $N_0 \geq 1$  together with (2.21), we obtain the desired bound (2.17).

It remains to prove (2.21). For  $n_1 \in \tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}$ , it follows from (2.20) with (2.19) that

$$\tau_1 \in (|n_1|^2 + [-CL_1, CL_1]) \cap (\tau + |n_1 - n|^2 + [-CL_2, CL_2]),$$

for some constant  $C > 0$ . Since this intersection of the two sets is nonempty, we must have  $||n_1|^2 - (\tau + |n_1 - n|^2)| = O(\max\{L_1, L_2\})$ , which in turn implies

$$n_1 \cdot \frac{n}{|n|} = \frac{\tau}{2|n|} + \frac{|n|}{2} + O\left(\frac{\max\{L_1, L_2\}}{N_0}\right). \quad (2.22)$$

On the other hand, for  $n_1 \in \tilde{\mathcal{S}}_{\tau, \tau_1, n, j_1, j_2, J_1, J_2}$ , we also have  $n_1 \in J_1$  where  $J_1$  is a ball of radius  $\sim N_0$ . Hence, together with (2.22), we see that the component of  $n_1$  which is parallel to  $n$  is restricted to an interval of length  $\sim \min\{N_0, L_{\max}/N_0\}$ . Furthermore, we have

$$\begin{aligned}
|\cos \angle(n_1, n)| &= \frac{|n_1 \cdot n|}{|n_1||n|} \leq \frac{|n|}{|n_1|} + \frac{|(n_1 - n) \cdot n|}{|n_1||n|} \\
&\leq \frac{|n|}{|n_1|} + |\cos \angle(n_1 - n, n)| \frac{|n_1 - n|}{|n_1|} \\
&\ll \frac{|n|}{|n_1|} + \frac{|n_1 - n|^\varepsilon}{|n|^{1-\varepsilon}|n_1|} \ll 1,
\end{aligned}$$

since  $|n_1| \sim |n_1 - n| \gg |n|^2$ . Hence, by Lemma A.5.3 with  $\mu = 1$ ,  $\nu \sim \min\{N_0, L_{\max}/N_0\}$ , and  $\beta = \frac{\pi}{4}$ , we obtain (2.21).

The case  $L_{\max} \ll |n|$  is similar and much simpler. We apply the same steps as in the previous case ( $L_{\max} \gtrsim |n|$ ), except that we do not need to decompose the spatial frequencies of  $n$  into dyadic piece or to localize  $n_1$  and  $n_2$  on balls of smaller radii. In this case, we apply Lemma A.5.3 with  $\nu \sim 1$  to obtain the desired bound.  $\square$

**Case 3:**  $|n_2|^2 \ll |n|$ .

This case is similar to Case 2, so we will be brief here. For the low-modulation case (i.e.  $L_{\max} \ll N_0$ ), the same argument works by switching the roles of  $(\tau, n)$  and  $(\tau_2, n_2)$ .

For the high-modulation case (i.e.  $L_{\max} \gtrsim N_0$ ), simply switching the roles of  $(\tau, n)$  and  $(\tau_2, n_2)$  does not work, since we need to ensure that on the right-hand-side of (2.16) in Lemma 2.1.1, the power of  $L_0$  is less than  $\frac{1}{2}$ . With  $f, g$ , and  $h$  as in the statement of Lemma 2.1.1, when  $N_2^2 \ll N_0 \sim N_1$  and  $L_{\max} \gtrsim N_0$ , we use the Cauchy-Schwarz inequality in  $\tau$  and  $n$  and apply Lemma A.4.6 to bound the left-hand-side of (2.16) by

$$\begin{aligned} & \|h\|_{L_\tau^2 \ell_n^2} \min\{L_1, L_2\}^{\frac{1}{2}} \left( \frac{\max\{L_1, L_2\}}{N_0} + 1 \right)^{\frac{1}{2}} N_2^{\frac{1}{2}} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \\ & \lesssim (L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} + \min\{L_1, L_2\}^{\frac{1}{2}} N_0^{\frac{1}{4}}) \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_\tau^2 \ell_n^2} \\ & \lesssim (L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_{\max}^{0+} + \min\{L_1, L_2\}^{\frac{1}{2}} L_{\max}^{\frac{1}{4}+}) N_0^{0-} \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_\tau^2 \ell_n^2}, \end{aligned}$$

which suffices for our purpose.

**Remark 2.1.3.** The bilinear estimate (1.8) in Proposition 1.1.3 also holds if we replace  $u\bar{v}$  by  $uv$ . In fact, a slight modification of the argument allows us to show

$$\|uv\|_{X_T^{0, -\frac{1}{2} + \delta_1}} \lesssim \|u\|_{X_T^{0, \frac{1}{2} + \delta_2}} \|v\|_{X_T^{0, \frac{1}{2} + \delta_2}}, \quad (2.23)$$

where  $0 < T \leq 1$  and  $\delta_1 > \delta_2 > 0$  are sufficiently small. Note that, in this case, by denoting  $n_1, n_2$ , and  $n$  as the frequency of  $u, v$ , and the duality term  $w$ , respectively, we have  $n = n_1 + n_2$  and  $\tau = \tau_1 + \tau_2$  so that

$$(\tau - |n|^2) - (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2) = -2n_1 \cdot n_2.$$

Hence, we need to perform case-by-case analysis, depending on the interaction between  $n_1$  and  $n_2$ .

The proof of (2.23) follows essentially as in the proof of (1.8) except for Lemma 2.1.1. With  $f, g$ , and  $h$  as in the statement of Lemma 2.1.1, when  $N_1^2 \ll N_2 \sim N_0$  and  $L_{\max} \gtrsim N_2$ , the Cauchy-Schwarz inequality in  $\tau_2$  and  $n_2$  and Lemma A.4.6 yield

$$\begin{aligned} & \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 + n_2}} \iint_{\tau = \tau_1 + \tau_2} f(\tau_1, n_1) g(\tau_2, n_2) h(\tau, n) d\tau d\tau_1 \right| \\ & \lesssim (L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} N_2^{-\frac{1}{4}} + \min\{L_0, L_1\}^{\frac{1}{2}} N_2^{\frac{1}{4}}) \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_\tau^2 \ell_n^2}. \end{aligned} \quad (2.24)$$

In order to repeat the argument in Subcase 2.2 in the proof of Proposition 1.1.3, we need the power of  $L_0$  to be less than  $\frac{1}{2}$ , especially when  $L_{\max} = L_0$ . Namely, the bound (2.24) can not be used directly as it is.

By Hölder's inequality in  $\tau$  and  $n$ , Young's inequality, and Hölder's inequality with (2.15), we obtain

$$\begin{aligned} & \left| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n = n_1 + n_2}} \iint_{\tau = \tau_1 + \tau_2} f(\tau_1, n_1) g(\tau_2, n_2) h(\tau, n) d\tau d\tau_1 \right| \\ & \lesssim \|f\|_{\ell_{n_1}^2 L_{\tau_1}^{\frac{8}{7}}} \|g\|_{\ell_{n_2}^1 L_{\tau_2}^2} \|h\|_{\ell_n^2 L_\tau^{\frac{8}{5}}} \\ & \lesssim L_0^{\frac{1}{8}} L_1^{\frac{3}{8}} N_2 \|f\|_{L_{\tau_1}^2 \ell_{n_1}^2} \|g\|_{L_{\tau_2}^2 \ell_{n_2}^2} \|h\|_{L_\tau^2 \ell_n^2}. \end{aligned} \quad (2.25)$$

Then, by interpolating (2.24) and (2.25), we obtain the desired inequality. A similar inequality also holds when  $N_2^2 \ll N_1 \sim N_0$  and  $L_{\max} \gtrsim N_1$  by switching the roles of  $(\tau_1, n_1)$  and  $(\tau_2, n_2)$ .

## 2.2 Local well-posedness of the quadratic NLS with non-linearity $|u|^2$

In this section, we present the proof of Theorem 1.1.1, local well-posedness of the quadratic NLS (1.6) in  $L^2(\mathbb{T}^2)$ . By writing (1.6) in the following Duhamel formulation, we have

$$u(t) = \Gamma(u) \stackrel{\text{def}}{=} e^{-it\Delta}u_0 - i \int_0^t e^{-i(t-t')\Delta}|u|^2(t')dt'. \quad (2.26)$$

Let  $\varepsilon > 0$  be sufficiently small. Then, by (2.26), Lemma A.4.1 with  $k = 0$ , Lemma A.4.2, Lemma A.4.4, and Proposition 1.1.3, we have

$$\begin{aligned} \|\Gamma(u)\|_{X_T^{0, \frac{1}{2} + \varepsilon}} &\leq \|e^{-it\Delta}u_0\|_{X_T^{0, \frac{1}{2} + \varepsilon}} + \left\| \int_0^t e^{-i(t-t')\Delta}u(t')\bar{u}(t')dt' \right\|_{X_T^{0, \frac{1}{2} + \varepsilon}} \\ &\lesssim \|u_0\|_{L^2} + T^\varepsilon \|u\bar{u}\|_{X_T^{0, -\frac{1}{2} + 2\varepsilon}} \\ &\lesssim \|u_0\|_{L^2} + T^\varepsilon \|u\|_{X_T^{0, \frac{1}{2} + \varepsilon}}^2. \end{aligned}$$

Similarly, we obtain the following difference estimate:

$$\|\Gamma(u) - \Gamma(v)\|_{X_T^{0, \frac{1}{2} + \varepsilon}} \lesssim T^\varepsilon \left( \|u\|_{X_T^{0, \frac{1}{2} + \varepsilon}} + \|v\|_{X_T^{0, \frac{1}{2} + \varepsilon}} \right) \|u - v\|_{X_T^{0, \frac{1}{2} + \varepsilon}}.$$

Therefore, by choosing  $T = T(\|u_0\|_{L^2}) > 0$  sufficiently small, we conclude that  $\Gamma$  is a contraction on the ball  $B_R \subset X_T^{0, \frac{1}{2} + \varepsilon}$  of radius  $R \sim \|u_0\|_{L^2}$ . This proves Theorem 1.1.1.

## Chapter 3

# Deterministic local well-posedness of quadratic NLS with nonlinearity $\overline{u}^2$

In this chapter, we study local well-posedness of the quadratic NLS with nonlinearity  $\overline{u}^2$  on the one-dimensional torus  $\mathbb{T}$  and the two-dimensional torus  $\mathbb{T}^2$ . Specifically, we prove Theorem 1.2.1, local well-posedness of the quadratic NLS (1.11) in  $H^s(\mathcal{M})$  for  $\mathcal{M} = \mathbb{T}$  or  $\mathbb{T}^2$  and  $-\frac{2}{3} < s \leq -\frac{1}{2}$ . As mentioned after the statement of Theorem 1.2.1, we mainly focus on proving local well-posedness for the case  $\mathcal{M} = \mathbb{T}^2$ .

### 3.1 Modified function spaces

In this section, we define our solution space for the quadratic NLS (1.11) in the low regularity setting and establish corresponding linear estimates.

Given  $s, b \in \mathbb{R}$ , we define the space  $Y^{s,b} = Y^{s,b}(\mathbb{R} \times \mathbb{T}^2)$  to be the completion of functions that are smooth in space and Schwartz in time with respect to the norm

$$\|u\|_{Y^{s,b}} \stackrel{\text{def}}{=} \|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} + \|\langle \tau - |n|^2 \rangle^{\frac{s}{2}+b} \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2}. \quad (3.1)$$

The idea of this modification comes from Kishimoto [74].

We now define the space  $Z^{s,b} = Z^{s,b}(\mathbb{R} \times \mathbb{T}^2)$  via the norm

$$\|u\|_{Z^{s,b}} \stackrel{\text{def}}{=} \|P_{\text{lo}} u\|_{X^{s,b}} + \|P_{\text{hi}} u\|_{Y^{s,b}}, \quad (3.2)$$

where  $P_{\text{lo}}$  is the space-time frequency projector onto the frequencies  $\{|\tau - |n|^2| < 2^{-10}|n|^2\}$  and  $P_{\text{hi}}$  is the space-time frequency projector onto the frequencies  $\{|\tau - |n|^2| \geq 2^{-10}|n|^2\}$ . From the definition, we observe that the  $Z^{s,b}$ -norm has the monotonicity property: if  $|\widehat{u}_1| \leq |\widehat{u}_2|$ , then

$$\|u_1\|_{Z^{s,b}} \leq \|u_2\|_{Z^{s,b}}. \quad (3.3)$$

For  $T > 0$ , we define the space  $Z_T^{s,b}$  as the restriction of the  $Z^{s,b}$ -space onto the time interval  $[-T, T]$  via the norm

$$\|u\|_{Z_T^{s,b}} \stackrel{\text{def}}{=} \inf \{ \|v\|_{Z^{s,b}} : v|_{[-T,T]} = u \}. \quad (3.4)$$

Note that the  $Z_T^{s,b}$ -space is complete.

For convenience and conciseness, later on we may use the notations  $\widehat{X}^{s,b}$ ,  $\widehat{Y}^{s,b}$ , and  $\widehat{Z}^{s,b}$  to denote the corresponding norms on the Fourier side. In other words, for a complex-valued

function  $f$  defined on  $\mathbb{R} \times \mathbb{Z}^2$ , we write

$$\begin{aligned}\|f\|_{\widehat{X}^{s,b}} &= \|\mathcal{F}_{t,x}^{-1}(f)\|_{X^{s,b}}, \\ \|f\|_{\widehat{Y}^{s,b}} &= \|\mathcal{F}_{t,x}^{-1}(f)\|_{Y^{s,b}}, \\ \|f\|_{\widehat{Z}^{s,b}} &= \|\mathcal{F}_{t,x}^{-1}(f)\|_{Z^{s,b}}.\end{aligned}$$

We now establish some linear estimates of the  $Z^{s,b}$ -norm, starting with the following  $H^s$ -energy estimate.

**Lemma 3.1.1.** *Let  $s \in \mathbb{R}$  and  $b > \frac{1}{2}$ . Then, we have*

$$\|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|u\|_{Z^{s,b}}.$$

*Proof.* By the definition of the  $Z^{s,b}$ -norm in (3.2), we know that it suffices to show the following two estimates:

$$\|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|u\|_{X^{s,b}}, \quad (3.5)$$

$$\|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|u\|_{Y^{s,b}}. \quad (3.6)$$

Since  $b > \frac{1}{2}$ , we use the Cauchy-Schwarz inequality in  $\tau$  to obtain

$$\|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|\langle n \rangle^s \langle \tau - |n|^2 \rangle^b \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2} \leq \|u\|_{X^{s,b}},$$

so that we obtain (3.5). Also, note that (3.6) is easily obtained from the definition of the  $Y^{s,b}$ -norm in (3.1).  $\square$

The above lemma implies the following embedding result.

**Lemma 3.1.2.** *Let  $s \in \mathbb{R}$ ,  $b > \frac{1}{2}$ , and  $T > 0$ . Then, we have*

$$\|u\|_{C_T H^s} \lesssim \|u\|_{Z_T^{s,b}}.$$

*Consequently, the embedding*

$$Z_T^{s,b} \hookrightarrow C([-T, T]; H^s(\mathbb{T}^2))$$

*holds.*

*Proof.* Let  $\varepsilon > 0$  and let  $v$  be an extension of  $u$  outside of  $[-T, T]$  such that

$$\|v\|_{Z^{s,b}} \leq \|u\|_{Z_T^{s,b}} + \varepsilon. \quad (3.7)$$

Note that we have the following embedding

$$\|v\|_{C_T H^s} \lesssim \|\langle n \rangle^s \widehat{v}(\tau, n)\|_{\ell_n^2 L_\tau^1}. \quad (3.8)$$

Thus, by (3.8), Lemma 3.1.1, and (3.7), we obtain

$$\|u\|_{C_T H^s} = \|v\|_{C_T H^s} \lesssim \|\langle n \rangle^s \widehat{v}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|v\|_{Z^{s,b}} \leq \|u\|_{Z_T^{s,b}} + \varepsilon,$$

and so the desired estimate follows since  $\varepsilon > 0$  can be arbitrarily small.  $\square$

Lastly, we show the following lemma, which shows that the  $X^{s,b}$ -space is embedded in the  $Z^{s,b}$ -space.

**Lemma 3.1.3.** *Let  $s \leq 0$  and  $b > \frac{1}{2}$ . Then, we have*

$$\|u\|_{Z^{s,b}} \lesssim \|u\|_{X^{s,b}}.$$

*Proof.* We recall from (3.2) that

$$\|u\|_{Z^{s,b}} = \|P_{\text{lo}}u\|_{X^{s,b}} + \|P_{\text{hi}}u\|_{Y^{s,b}},$$

where  $P_{\text{lo}}$  projects the space-time frequencies onto  $\{|\tau - |n|^2| < 2^{-10}|n|^2\}$  and  $P_{\text{hi}}$  projects the space-time frequencies onto  $\{|\tau - |n|^2| \geq 2^{-10}|n|^2\}$ . Note that we have

$$\|P_{\text{lo}}u\|_{X^{s,b}} \leq \|u\|_{X^{s,b}}.$$

For the  $\|P_{\text{hi}}u\|_{Y^{s,b}}$  term, note that by the Cauchy-Schwarz inequality, we have

$$\|\langle n \rangle^s \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1} \lesssim \|\langle n \rangle^s \langle \tau - |n|^2 \rangle^b \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2} = \|u\|_{X^{s,b}},$$

since  $b > \frac{1}{2}$ . Also, we have

$$\|\langle \tau - |n|^2 \rangle^{\frac{s}{2}+b} \widehat{u}(\tau, n) \mathbf{1}_{\{|\tau - |n|^2| \geq 2^{-10}|n|^2\}}\|_{\ell_n^2 L_\tau^2} \lesssim \|\langle n \rangle^s \langle \tau - |n|^2 \rangle^b \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2} = \|u\|_{X^{s,b}}.$$

Thus, we obtain that  $\|P_{\text{hi}}u\|_{Y^{s,b}} \lesssim \|u\|_{X^{s,b}}$ , so that we achieve the desired inequality.  $\square$

## 3.2 Bilinear estimate

In this section, we establish the crucial bilinear estimate with respect to the  $Z^{s,b}$ -norm introduced in the previous section. Specifically, we show the following proposition.

**Proposition 3.2.1.** *Let  $-\frac{2}{3} < s \leq -\frac{1}{2}$  and  $0 < T \leq \frac{1}{2}$ . Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\varphi \equiv 1$  on  $[-1, 1]$  and  $\varphi \equiv 0$  outside of  $[-2, 2]$ , and let  $\varphi_T(t) = \varphi(t/T)$ . Then, we have*

$$\|\langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \bar{u} \cdot \varphi_T \bar{v})(\tau, n)\|_{\widehat{Z}^{s, \frac{2}{3}}} \lesssim_\varphi T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}$$

for some  $\theta > 0$ .

Let us first consider two particular cases of Proposition 3.2.1. We start with the following ‘‘high-low interaction’’ estimate.

**Lemma 3.2.2.** *Let  $-\frac{2}{3} < s \leq -\frac{1}{2}$ ,  $0 < T \leq \frac{1}{2}$ , and  $N, N_1, N_2 \geq 1$  be dyadic numbers. Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\varphi \equiv 1$  on  $[-1, 1]$  and  $\varphi \equiv 0$  outside of  $[-2, 2]$ , and let  $\varphi_T(t) = \varphi(t/T)$ .*

(i) *If  $2^{-5}N \leq N_1 \leq 2^5N$  and  $N_2 \leq 2^6N$ , we have*

$$\|\langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \bar{u}_{N_1} \cdot \varphi_T \bar{v}_{N_2})(\tau, n)\|_{\widehat{Z}^{s, \frac{2}{3}}(\mathfrak{P}_N)} \lesssim_\varphi N_2^{-\delta} T^\theta \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}$$

for some  $\delta > 0$  and  $\theta > 0$ .

(ii) *If  $2^{-5}N \leq N_2 \leq 2^5N$  and  $N_1 \leq 2^6N$ , we have*

$$\|\langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \bar{u}_{N_1} \cdot \varphi_T \bar{v}_{N_2})(\tau, n)\|_{\widehat{Z}^{s, \frac{2}{3}}(\mathfrak{P}_N)} \lesssim_\varphi N_1^{-\delta} T^\theta \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}$$

for some  $\delta > 0$  and  $\theta > 0$ .

*Proof.* By the symmetry of  $u$  and  $v$ , it suffices to prove (i). Below we use  $(\tau_1, n_1)$  as the variables of  $\widehat{\varphi_T u_{N_1}}$  or  $\widehat{u_{N_1}}$  and  $(\tau_2, n_2)$  as the variables of  $\widehat{\varphi_T v_{N_2}}$  or  $\widehat{v_{N_2}}$ . Note that we have the relations  $\tau + \tau_1 + \tau_2 = 0$  and  $n + n_1 + n_2 = 0$ . We also recall the notation  $\widetilde{f}(x) = f(-x)$ .

We divide the argument into two main cases depending on the relationship between the modulation function  $\tau - |n|^2$  and  $|n|^2$ .

**Case 1:**  $|\tau - |n|^2| \geq 2^{-10}|n|^2$ .

In this case, we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \bar{u}_{N_1} \cdot \varphi_T \bar{v}_{N_2})$  term using the  $\widehat{Y}^{s, \frac{2}{3}}$ -norm, and we need to evaluate both the  $\ell_n^2 L_\tau^1$  term and the  $\ell_n^2 L_\tau^2$  term. We consider the following three subcases.

**Subcase 1.1:**  $|\tau_1 - |n_1|^2| \geq 2^{-10}|n_1|^2$ .

In this subcase, we need to estimate  $u_{N_1}$  using the  $Y^{s, \frac{2}{3}}$ -norm. By Young's convolution inequality, Lemma A.4.4, the Cauchy-Schwarz inequality, and Lemma 3.1.1, we obtain

$$\begin{aligned}
& \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2} - \frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
&= \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2} - \frac{1}{3}} \widehat{\varphi_T^2 u_{N_1}} * \widehat{v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
&\lesssim N^{s - \frac{2}{3}} \left\| \widehat{\varphi_T^2 u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widehat{v_{N_2}} \right\|_{\ell_{n_2}^1 L_{\tau_2}^1} \\
&\lesssim_{\varphi} N^{s - \frac{2}{3}} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} N_2^{-s+1} \left\| \langle n_2 \rangle^s \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^1} \\
&\lesssim N^{s - \frac{2}{3}} T^\varepsilon N_1^{-s - \frac{4}{3} + 2\varepsilon} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} N_2^{-s+1} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}} \\
&\lesssim N^{-2+2\varepsilon} N_2^{-s+1} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}},
\end{aligned} \tag{3.9}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $-s+1 > 0$  given  $s \leq -\frac{1}{2}$ , the above estimate is acceptable if  $-s-1+2\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

Also, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&\lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)},
\end{aligned}$$

which can be estimated similarly as in (3.9). Combining the above two estimates, we obtain the desired inequality.

**Subcase 1.2:**  $|\tau_2 - |n_2|^2| \geq 2^{-10}|n_2|^2$ .

In this subcase, we need to estimate  $v_{N_2}$  using the  $Y^{s, \frac{2}{3}}$ -norm. By Young's convolution inequality, the Cauchy-Schwarz inequality, Lemma A.4.4, and Lemma 3.1.1, we obtain

$$\begin{aligned}
& \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2} - \frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
&= \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2} - \frac{1}{3}} \widehat{u_{N_1}} * \widehat{\varphi_T^2 v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
&\lesssim N^{s - \frac{2}{3}} \left\| \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^1} \left\| \widehat{\varphi_T^2 v_{N_2}} \right\|_{\ell_{n_2}^1 L_{\tau_2}^2} \\
&\lesssim_{\varphi} N^{s - \frac{2}{3}} N_1^{-s} \left\| \langle n_1 \rangle^s \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^1} T^\varepsilon N_2 \left\| \langle \tau_2 - |n_2|^2 \rangle^\varepsilon \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
&\lesssim N^{s - \frac{2}{3}} N_1^{-s} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} T^\varepsilon N_2^{-s - \frac{1}{3} + 2\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\
&\lesssim N^{-\frac{2}{3}} N_2^{-s - \frac{1}{3} + 2\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}},
\end{aligned} \tag{3.10}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $-s - \frac{1}{3} + 2\varepsilon > 0$  given  $s \leq -\frac{1}{2}$ , the above estimate is acceptable if  $-s-1+2\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

Also, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&\lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)},
\end{aligned}$$

which can be estimated similarly as in (3.10). Combining the above two estimates, we obtain the desired inequality.

**Subcase 1.3:**  $|\tau_1 - |n_1|^2| < 2^{-10}|n_1|^2$  and  $|\tau_2 - |n_2|^2| < 2^{-10}|n_2|^2$ .

In this subcase, we need to estimate both  $u_{N_1}$  and  $v_{N_2}$  using the  $X^{s, \frac{2}{3}}$ -norm. Using the fact that  $\varphi_T$  is supported on  $[-1, 1]$  given  $0 < T \leq \frac{1}{2}$ , by the Plancherel theorem, Hölder's

inequality, Lemma A.4.5, and Lemma A.4.4, we obtain

$$\begin{aligned}
& \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2} - \frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim N^{s - \frac{2}{3}} \left\| \varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}} \right\|_{L_t^2 L_x^2([-1,1] \times \mathbb{T}^2)} \\
& \lesssim N^{s - \frac{2}{3}} \left\| \varphi_T u_{N_1} \right\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \left\| \varphi_T v_{N_2} \right\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \\
& \lesssim N^{s - \frac{2}{3}} N_1^{4\varepsilon} \left\| \varphi_T u_{N_1} \right\|_{X^{0, \frac{1}{2} - \varepsilon}} N_2^{4\varepsilon} \left\| \varphi_T v_{N_2} \right\|_{X^{0, \frac{1}{2} - \varepsilon}} \\
& \lesssim_\varphi N^{s - \frac{2}{3}} N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s, \frac{1}{2} - \frac{\varepsilon}{2}}} N_2^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|v_{N_2}\|_{X^{s, \frac{1}{2} - \frac{\varepsilon}{2}}} \\
& \lesssim N^{-\frac{2}{3} + 4\varepsilon} N_2^{-s+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}},
\end{aligned} \tag{3.11}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $s \leq -\frac{1}{2} < 0$ , the above estimate is acceptable if  $-s - \frac{2}{3} + 8\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough.

Also, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
& \lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)},
\end{aligned}$$

which can be estimated similarly as in (3.11). Combining the above two estimates, we obtain the desired inequality.

**Case 2:**  $|\tau - |n|^2| < 2^{-10}|n|^2$ .

In this case, we need to evaluate the  $\mathcal{F}_{x,t}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $\widehat{X}^{s, \frac{2}{3}}$ -norm.

We assume that  $n \neq 0$ . Note that if  $n = 0$ , we have  $N = 1$  which then implies that  $N_1 \leq 2^5$  and  $N_2 \leq 2^6$ , and so the estimate will follow in a similar (and much easier) manner.

We consider the following three subcases.

**Subcase 2.1:**  $|\tau_1 - |n_1|^2| \geq 2^{-10}|n_1|^2$  and  $|\tau_2 - |n_2|^2| \geq 2^{-10}|n_2|^2$ .

In this subcase, we need to estimate both  $u_{N_1}$  and  $v_{N_2}$  using the  $Y^{s, \frac{2}{3}}$ -norm. By Hölder's inequality, Young's convolution inequality, and Lemma A.4.4, we have

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& = \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \widehat{u_{N_1}} * \widehat{\varphi_T^2 v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \left\| \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widehat{\varphi_T^2 v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
& \lesssim_\varphi N^{s+1} N^{\frac{1}{3}} N_1^{-s - \frac{4}{3}} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} T^\varepsilon \left\| \langle \tau_2 - |n_2|^2 \rangle^\varepsilon \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
& \lesssim T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} N_2^{-s - \frac{4}{3} + 2\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\
& \lesssim N_2^{-s - \frac{4}{3} + 2\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}},
\end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. The above estimate is acceptable if  $-s - \frac{4}{3} + 2\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

**Subcase 2.2:**  $|\tau_1 - |n_1|^2| \geq 2^{-10}|n_1|^2$  and  $|\tau_2 - |n_2|^2| < 2^{-10}|n_2|^2$ .

In this subcase, we need to estimate  $\widehat{u_{N_1}}$  using the  $Y^{s, \frac{2}{3}}$ -norm and estimate  $v_{N_2}$  using the

$X^{s, \frac{2}{3}}$ -norm. By duality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x} (\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim N^s \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \iint_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \widehat{\varphi_T v_{N_2}}(\tau_2, n_2) \right. \\
& \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau d\tau_1 \right| \\
& \leq N^s \left\| \widehat{\varphi_T u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left\| \sum_{\substack{n, n_2 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T v_{N_2}}(\tau_2, n_2) \right. \\
& \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau \right\|_{\ell_{n_1}^2 L_{\tau_1}^2}. \tag{3.12}
\end{aligned}$$

Let  $w_N$  be a space-time distribution that satisfy  $\widehat{w_N}(\tau, n) = h(\tau, n) / \langle \tau - |n|^2 \rangle^{\frac{1}{3}}$ . Then, using the fact that  $\varphi_T$  is supported on  $[-1, 1]$  given  $0 < T \leq \frac{1}{2}$ , by the Plancherel theorem, Hölder's inequality, Lemma A.4.5, and Lemma A.4.4, we have

$$\begin{aligned}
& \left\| \sum_{\substack{n, n_2 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T v_{N_2}}(\tau_2, n_2) \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \\
& = \left\| \varphi_T v_{N_2} \widehat{w_N} \right\|_{L_t^2 L_x^2([-1, 1] \times \mathbb{T}^2)} \\
& \lesssim \left\| \varphi_T v_{N_2} \right\|_{L_t^4 L_x^4([-1, 1] \times \mathbb{T}^2)} \left\| w_N \right\|_{L_t^4 L_x^4([-1, 1] \times \mathbb{T}^2)} \\
& \lesssim N_2^{4\varepsilon} \left\| \varphi_T v_{N_2} \right\|_{X^{0, \frac{1}{2}-\varepsilon}} N^{\frac{1}{3}+\varepsilon} \left\| w_N \right\|_{X^{0, \frac{1}{3}}} \\
& \lesssim_\varphi N_2^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \left\| v_{N_2} \right\|_{X^{s, \frac{2}{3}}} N^{\frac{1}{3}+\varepsilon} \left\| h \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)},
\end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Thus, continuing with (3.12), we use Lemma A.4.4 to obtain

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x} (\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} \left\| \widehat{\varphi_T u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} N_2^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \left\| v_{N_2} \right\|_{X^{s, \frac{2}{3}}} \\
& \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_2^{-s+4\varepsilon} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^{\frac{\varepsilon}{2}} \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| v_{N_2} \right\|_{Z^{s, \frac{2}{3}}} \\
& \lesssim N^{s+\frac{1}{3}+\varepsilon} N_2^{-s+4\varepsilon} T^\varepsilon N_1^{-s-\frac{4}{3}+\varepsilon} \left\| u_{N_1} \right\|_{Y^{s, \frac{2}{3}}} \left\| v_{N_2} \right\|_{Z^{s, \frac{2}{3}}} \\
& \lesssim N^{-1+2\varepsilon} N_2^{-s+4\varepsilon} T^\varepsilon \left\| u_{N_1} \right\|_{Z^{s, \frac{2}{3}}} \left\| v_{N_2} \right\|_{Z^{s, \frac{2}{3}}}.
\end{aligned}$$

Since  $s < 0$ , the above estimate is acceptable if  $-s - 1 + 6\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough.

**Subcase 2.3:**  $|\tau_1 - |n_1|^2| < 2^{-10}|n_1|^2$ .

In this subcase, we first note that

$$\tau > |n|^2 - 2^{-10}|n|^2 \quad \text{and} \quad \tau_1 > |n_1|^2 - 2^{-10}|n_1|^2.$$

Note that since we assumed  $n \neq 0$ , we have

$$\tau_2 = -\tau - \tau_1 < 2^{-10}|n|^2 - |n|^2 + 2^{-10}|n_1|^2 - |n_1|^2 < -\frac{1}{2}|n|^2.$$

Thus, we have

$$|\tau_2 - |n_2|^2| \gtrsim N^2 \tag{3.13}$$

and  $|\tau_2 - |n_2|^2| \geq |n_2|^2 > 2^{-10}|n_2|^2$ .

We need to estimate  $u_{N_1}$  using the  $X^{s, \frac{2}{3}}$ -norm and estimate  $v_{N_2}$  using the  $Y^{s, \frac{2}{3}}$ -norm. By using similar steps as in Subcase 2.2 by switching the roles of  $u_{N_1}$  and  $v_{N_2}$  along with the additional condition (3.13), we obtain

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s, \frac{2}{3}}} \left\| \widehat{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
& \lesssim_\varphi N^{\frac{1}{3}+5\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \left\| \langle \tau_2 - |n_2|^2 \rangle^{\frac{\varepsilon}{2}} \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
& \lesssim N^{\frac{1}{3}+5\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} N^{-s-\frac{4}{3}+\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\
& \lesssim N^{-s-1+6\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}.
\end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. The above estimate is acceptable if  $-s - 1 + 6\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough.

Thus, we have finished our proof.  $\square$

We now show the following ‘‘high-high interaction’’ estimate.

**Lemma 3.2.3.** *Let  $-\frac{2}{3} < s \leq -\frac{1}{2}$  and  $0 < T \leq \frac{1}{2}$ . Let  $N, N_1, N_2 \geq 1$  be dyadic numbers such that  $\frac{1}{2}N_1 \leq N_2 \leq 2N_1$  and  $N < 2^{-5}N_1$ . Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\varphi \equiv 1$  on  $[-1, 1]$  and  $\varphi \equiv 0$  outside of  $[-2, 2]$ , and let  $\varphi_T(t) = \varphi(t/T)$ . Then, we have*

$$\left\| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})(\tau, n) \right\|_{\widehat{Z}^{s, \frac{2}{3}}(\mathfrak{P}_N)} \lesssim_\varphi N^{-\delta} T^\theta \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}$$

for some  $\delta > 0$  and  $\theta > 0$ .

*Proof.* As in the proof of the previous lemma, we use  $(n_1, \tau_1)$  as the variables of  $\overline{\varphi_T u_{N_1}}$  or  $\overline{u_{N_1}}$ , and  $(n_2, \tau_2)$  as the variables of  $\overline{\varphi_T v_{N_2}}$  or  $\overline{v_{N_2}}$ . Note that we have the relations  $\tau + \tau_1 + \tau_2 = 0$  and  $n + n_1 + n_2 = 0$ . Also, the assumptions on the sizes of  $N, N_1$ , and  $N_2$  ensure that  $n_1 \neq 0$  and  $n_2 \neq 0$ . We also recall the notation  $\tilde{f}(x) = f(-x)$ .

We consider the following four main cases.

**Case 1:**  $|\tau - |n|^2| \geq 2^{-10}|n_1|^2$ .

In this case, we have  $|\tau - |n|^2| \geq 2^{-10}|n_1|^2 \geq 2^{-10}|n|^2$  given  $N < 2^{-5}N_1$ , so that we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $\widehat{Y}^{s, \frac{2}{3}}$ -norm, and we need to evaluate both the  $\ell_n^2 L_\tau^1$  term and the  $\ell_n^2 L_\tau^2$  term. We consider the following three subcases.

**Subcase 1.1:**  $|\tau_1 - |n_1|^2| \geq 2^{-10}|n_1|^2$ .

In this subcase, we need to estimate  $u_{N_1}$  using the  $Y^{s, \frac{2}{3}}$ -norm. By Young’s convolution inequality, Lemma A.4.4, the Cauchy-Schwarz inequality, and Lemma 3.1.1, we obtain

$$\begin{aligned}
& \left\| \langle \tau - |n|^2 \rangle^{\frac{\varepsilon}{2}-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& = \left\| \langle \tau - |n|^2 \rangle^{\frac{\varepsilon}{2}-\frac{1}{3}} \widehat{\varphi_T^2 u_{N_1}} * \widehat{v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
& \lesssim N_1^{s-\frac{2}{3}} \left\| \widehat{\varphi_T^2 u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widehat{v_{N_2}} \right\|_{\ell_{n_2}^1 L_{\tau_2}^1} \\
& \lesssim_\varphi N_1^{s-\frac{2}{3}} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} N_2^{-s+1} \left\| \langle n_2 \rangle^s \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^1} \\
& \lesssim N_1^{s-\frac{2}{3}} T^\varepsilon N_1^{-s-\frac{4}{3}+2\varepsilon} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} N_2^{-s+1} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}} \\
& \lesssim N_1^{-s-1+2\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}},
\end{aligned}$$

which is acceptable given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

Also, by Hölder’s inequality, Young’s convolution inequality, Lemma A.4.4, and Lemma

3.1.1, we have

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x} (\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&= \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \widehat{\varphi_T^2 u_{N_1}} * \widehat{v_{N_2}} \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&\lesssim N_1^{-1+2\varepsilon} \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{2}-\varepsilon} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \left\| \widehat{\varphi_T^2 u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^1} \\
&\lesssim_\varphi N_1^{-1+2\varepsilon} N^{s+1} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} N_2^{-s} \left\| \langle n_2 \rangle^s \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^1} \\
&\lesssim N^{s+1} N_1^{-s-1+2\varepsilon} T^\varepsilon N_1^{-s-\frac{4}{3}+2\varepsilon} \|u_{N_1}\|_{Y^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}} \\
&\lesssim N^{s+1} N_1^{-2s-\frac{7}{3}+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}},
\end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $s+1 > 0$  given  $s > -\frac{2}{3}$ , the above estimate is acceptable if  $-s - \frac{4}{3} + 4\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough. Combining the above two estimates, we obtain the desired inequality.

**Subcase 1.2:**  $|\tau_2 - |n_2|^2| \geq 2^{-10}|n_2|^2$ .

This subcase is similar to Subcase 1.1 by switching the roles of  $u_{N_1}$  and  $v_{N_2}$ , and so we omit details.

**Subcase 1.3:**  $|\tau_1 - |n_1|^2| < 2^{-10}|n_1|^2$  and  $|\tau_2 - |n_2|^2| < 2^{-10}|n_2|^2$ .

In this subcase, we need to estimate both  $u_{N_1}$  and  $v_{N_2}$  using the  $X^{s,\frac{2}{3}}$ -norm. Using the fact that  $\varphi_T$  is supported on  $[-1, 1]$  given  $0 < T \leq \frac{1}{2}$ , by the Plancherel theorem, Hölder's inequality, Lemma A.4.5, and Lemma A.4.4, we obtain

$$\begin{aligned}
& \left\| \langle \tau - |n|^2 \rangle^{\frac{2}{3}-\frac{1}{3}} \mathcal{F}_{t,x} (\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\
&\lesssim N_1^{s-\frac{2}{3}} \|\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}\|_{L_t^2 L_x^2([-1,1] \times \mathbb{T}^2)} \\
&\lesssim N_1^{s-\frac{2}{3}} \|\varphi_T u_{N_1}\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \|\varphi_T v_{N_2}\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \\
&\lesssim N_1^{s-\frac{2}{3}} N_1^{4\varepsilon} \|\varphi_T u_{N_1}\|_{X^{0,\frac{1}{2}-\varepsilon}} N_2^{4\varepsilon} \|\varphi_T v_{N_2}\|_{X^{0,\frac{1}{2}-\varepsilon}} \\
&\lesssim_\varphi N_1^{s-\frac{2}{3}} N_1^{-s+4\varepsilon} T^{\frac{5}{2}} \|u_{N_1}\|_{X^{s,\frac{2}{3}}} N_2^{-s+4\varepsilon} T^{\frac{5}{2}} \|v_{N_2}\|_{X^{s,\frac{2}{3}}} \\
&\lesssim N_1^{-s-\frac{2}{3}+8\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}},
\end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. The above estimate is acceptable if  $-s - \frac{2}{3} + 8\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough.

Regarding the  $\ell_n^2 L_\tau^1$  norm of the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term, we first let  $\varepsilon_1, \varepsilon_2 > 0$  satisfying

$$1 + \frac{1}{1 + \varepsilon_1} = \frac{2}{1 + \varepsilon_2}.$$

Note that both  $\varepsilon_1$  and  $\varepsilon_2$  can be arbitrarily small. By Hölder's inequality, Young's convolution inequality, Hölder's inequalities twice, and Lemma A.4.4, we have

$$\begin{aligned}
& \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x} (\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&= \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \widehat{\varphi_T^2 u_{N_1}} * \widehat{\varphi_T v_{N_2}} \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\
&\lesssim N_1^{-2+2\varepsilon_1} \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\varepsilon_1} \right\|_{\ell_n^2 L_\tau^{(1+\varepsilon_1)/\varepsilon_1}(\mathfrak{P}_N)} \left\| \widehat{\varphi_T^2 u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^{1+\varepsilon_2}} \left\| \widehat{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^{1+\varepsilon_2}} \\
&\lesssim N_1^{-2+2\varepsilon_1} N^{s+1} \left\| \langle \tau_1 - |n_1|^2 \rangle^{\frac{1}{2+2\varepsilon_1} + \varepsilon_1} \widehat{\varphi_T^2 u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \langle \tau_2 - |n_2|^2 \rangle^{\frac{1}{2+2\varepsilon_1} + \varepsilon_1} \widehat{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\
&\lesssim_\varphi N^{s+1} N_1^{-2+2\varepsilon_1} T^\theta N_1^{-s} \|u_{N_1}\|_{X^{s,\frac{2}{3}}} T^\theta N_2^{-s} \|v_{N_2}\|_{X^{s,\frac{2}{3}}} \\
&\lesssim N^{s+1} N_1^{-2s-2+2\varepsilon_1} T^{2\theta} \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}}
\end{aligned}$$

for some  $\theta > 0$ . Since  $s+1 > 0$  given  $s > -\frac{2}{3}$ , the above estimate is acceptable if  $-s-1+2\varepsilon_1 < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon_1 > 0$  sufficiently small. Combining the above two estimates, we obtain the desired inequality.

**Case 2:**  $|\tau - |n|^2| < 2^{-10}|n_1|^2$ ,  $|\tau_1 - |n_1|^2| \geq 2^{-10}|n_1|^2$ , and  $|\tau_2 - |n_2|^2| \geq 2^{-10}|n_2|^2$ .

In this case, we need to estimate both  $u_{N_1}$  and  $v_{N_2}$  using the  $Y^{s, \frac{2}{3}}$ -norm. We consider the following two subcases.

**Subcase 2.1:**  $|\tau - |n|^2| < 2^{-10}|n|^2$ .

In this subcase, we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $\widehat{X}^{s, \frac{2}{3}}$ -norm. By Hölder's inequality, Young's convolution inequality, and Lemma A.4.4, we have

$$\begin{aligned} & \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &= \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \widetilde{\varphi_T u_{N_1}} * \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &\lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \left\| \widetilde{\varphi_T u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim_\varphi N^{s+1} N^{\frac{1}{3}} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} T^\varepsilon \left\| \langle \tau_2 - |n_2|^2 \rangle^\varepsilon \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim N^{s+\frac{4}{3}} T^\varepsilon N_1^{-s-\frac{4}{3}+2\varepsilon} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} T^\varepsilon N_2^{-s-\frac{4}{3}+2\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\ &\lesssim N^{s+\frac{4}{3}} N_1^{-2s-\frac{8}{3}+4\varepsilon} T^{2\varepsilon} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $s > -\frac{2}{3}$ , we have  $s + \frac{4}{3} > 0$ . Thus, the above estimate is acceptable if  $-s - \frac{4}{3} + 4\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$ .

**Subcase 2.2:**  $2^{-10}|n|^2 \leq |\tau - |n|^2| < 2^{-10}|n_1|^2$ .

In this subcase, we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $\widehat{Y}^{s, \frac{2}{3}}$ -norm. By Hölder's inequality, Young's convolution inequality, and Lemma A.4.4, we have

$$\begin{aligned} & \left\| \langle \tau - |n|^2 \rangle^{\frac{2}{3}-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &= \left\| \langle \tau - |n|^2 \rangle^{\frac{2}{3}-\frac{1}{3}} \widetilde{\varphi_T u_{N_1}} * \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &\lesssim \left\| \langle \tau - |n|^2 \rangle^{\frac{2}{3}-\frac{1}{3}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \left\| \widetilde{\varphi_T u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim_\varphi N^{s+\frac{1}{3}+2\varepsilon} \left\| \langle \tau - |n|^2 \rangle^{-\frac{1}{2}-\varepsilon} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &\quad \times T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} T^\varepsilon \left\| \langle \tau_2 - |n_2|^2 \rangle^\varepsilon \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim N^{s+\frac{4}{3}+2\varepsilon} T^{2\varepsilon} N_1^{-s-\frac{4}{3}+2\varepsilon} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} N_2^{-s-\frac{4}{3}+2\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\ &\lesssim N^{s+\frac{4}{3}+2\varepsilon} N_1^{-2s-\frac{8}{3}+4\varepsilon} T^{2\varepsilon} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Note that the second inequality is valid since  $s + \frac{1}{3} + 2\varepsilon < 0$  given  $s \leq -\frac{1}{2}$  and  $\varepsilon > 0$  small enough. Since  $s + \frac{4}{3} + 2\varepsilon > 0$  given  $s > -\frac{2}{3}$ , the above estimate is acceptable if  $-s - \frac{4}{3} + 6\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$ .

Also, by the Cauchy-Schwarz inequality, Hölder's inequality, Young's convolution inequality, and Lemma A.4.4, we get

$$\begin{aligned} & \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\ &\lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{2}+\varepsilon} \widetilde{\varphi_T u_{N_1}} * \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ &\lesssim N^s N_1^{4\varepsilon} \left\| \langle \tau - |n|^2 \rangle^{-\frac{1}{2}-\varepsilon} \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \left\| \widetilde{\varphi_T u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \left\| \widetilde{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim_\varphi N^{s+1} N_1^{4\varepsilon} T^\varepsilon \left\| \langle \tau_1 - |n_1|^2 \rangle^\varepsilon \widehat{u_{N_1}} \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} T^\varepsilon \left\| \langle \tau_2 - |n_2|^2 \rangle^\varepsilon \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ &\lesssim N^{s+1} N_1^{4\varepsilon} T^{2\varepsilon} N_1^{-s-\frac{4}{3}+2\varepsilon} \|u_{N_1}\|_{Y^{s, \frac{2}{3}}} N_2^{-s-\frac{4}{3}+2\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \end{aligned}$$

$$\lesssim N^{s+1} N_1^{-2s-\frac{8}{3}+8\varepsilon} T^{2\varepsilon} \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}},$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $s+1 > 0$  given  $s > -\frac{2}{3}$ , the above estimate is acceptable if  $-s - \frac{5}{3} + 8\varepsilon < 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  small enough. Combining the above two estimates, we obtain the desired inequality.

**Case 3:**  $|\tau - |n|^2| < 2^{-10}|n_1|^2$  and  $|\tau_1 - |n_1|^2| < 2^{-10}|n_1|^2$ .

In this case, we need to estimate  $u_{N_1}$  using the  $X^{s,\frac{2}{3}}$ -norm. Note that we have

$$\begin{aligned} \tau_2 &= -\tau - \tau_1 \\ &= (-\tau + |n|^2) + (-\tau_1 + |n_1|^2) - |n|^2 - |n_1|^2 \\ &< 2^{-10}|n_1|^2 + 2^{-10}|n_1|^2 - |n|^2 \\ &< 0, \end{aligned}$$

and so  $|\tau_2 - |n_2|^2| > |n_2|^2 > 2^{-10}|n_2|^2$ . Thus, we need to estimate  $v_{N_2}$  using the  $Y^{s,\frac{2}{3}}$ -norm. We consider the following two subcases.

**Subcase 3.1:**  $|\tau - |n|^2| < 2^{-10}|n|^2$ .

In this subcase, we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $\widehat{X}^{s,\frac{2}{3}}$ -norm. By duality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ & \lesssim N^s \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left| \sum_{\substack{n,n_1,n_2 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \iint_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \widehat{\varphi_T v_{N_2}}(\tau_2, n_2) \right. \\ & \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau d\tau_2 \right| \\ & \leq N^s \left\| \widehat{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left\| \sum_{\substack{n,n_1 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \right. \\ & \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau \right\|_{\ell_{n_2}^2 L_{\tau_2}^2}. \end{aligned} \tag{3.14}$$

Let  $w_N$  be a space-time distribution that satisfy  $\widehat{w}_N(\tau, n) = h(\tau, n)/\langle \tau - |n|^2 \rangle^{\frac{1}{3}}$ . Then, using the fact that  $\varphi_T$  is supported on  $[-1, 1]$  given  $0 < T \leq \frac{1}{2}$ , by the Plancherel theorem, Hölder's inequality, Lemma A.4.5, and Lemma A.4.4, we have

$$\begin{aligned} & \left\| \sum_{\substack{n,n_1 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{3}}} d\tau \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & = \|\varphi_T u_{N_1} \widetilde{w}_N\|_{L_t^2 L_x^2([-1,1] \times \mathbb{T}^2)} \\ & \lesssim \|\varphi_T u_{N_1}\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \|w_N\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \\ & \lesssim N_1^{4\varepsilon} \|\varphi_T u_{N_1}\|_{X^{0,\frac{1}{2}-\varepsilon}} N^{\frac{1}{3}+\varepsilon} \|w_N\|_{X^{0,\frac{1}{3}}} \\ & \lesssim_\varphi N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s,\frac{2}{3}}} N^{\frac{1}{3}+\varepsilon} \|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)}, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Thus, continuing with (3.14), we use Lemma A.4.4 to obtain

$$\begin{aligned} & \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ & \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s,\frac{2}{3}}} \left\| \widehat{\varphi_T v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \left\| \langle \tau_2 - |n_2|^2 \rangle^{\frac{\varepsilon}{2}} \widehat{v_{N_2}} \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & \lesssim N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} N_2^{-s-\frac{4}{3}+\varepsilon} \|v_{N_2}\|_{Y^{s,\frac{2}{3}}} \end{aligned}$$

$$\lesssim N^{s+\frac{1}{3}+\varepsilon} N_1^{-2s-\frac{4}{3}+5\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s,\frac{2}{3}}} \|v_{N_2}\|_{Z^{s,\frac{2}{3}}}.$$

Since  $s \leq -\frac{1}{2}$ , we have  $s + \frac{1}{3} + \varepsilon < 0$  for  $\varepsilon > 0$  small enough. Thus, the above estimate is acceptable if  $-2s - \frac{4}{3} + 5\varepsilon \leq 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

**Subcase 3.2:**  $2^{-10}|n|^2 \leq |\tau - |n|^2| < 2^{-10}|n_1|^2$ .

In this subcase, we need to evaluate the  $\mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}})$  term using the  $Y^{s,\frac{2}{3}}$ -norm. By duality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2}-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ & \lesssim N^{s+\frac{1}{3}} \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \iint_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \widehat{\varphi_T v_{N_2}}(\tau_2, n_2) \right. \\ & \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{2}}} d\tau d\tau_2 \right| \\ & \leq N^{s+\frac{1}{3}} \|\widehat{\varphi_T v_{N_2}}\|_{\ell_{n_2}^2 L_{\tau_2}^2} \sup_{\|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \leq 1} \left\| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \right. \\ & \quad \left. \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{2}}} d\tau \right\|_{\ell_{n_2}^2 L_{\tau_2}^2}. \end{aligned} \tag{3.15}$$

Note that the first inequality is valid since  $s + \frac{1}{3} < 0$  given  $s \leq -\frac{1}{2}$ . Let  $w_N$  be a space-time distribution that satisfy  $\widehat{w}_N(\tau, n) = h(\tau, n)/\langle \tau - |n|^2 \rangle^{\frac{1}{2}}$ . Then, using the fact that  $\varphi_T$  is supported on  $[-1, 1]$  given  $0 < T \leq \frac{1}{2}$ , by the Plancherel theorem, Hölder's inequality, Lemma A.4.5, and Lemma A.4.4, we have

$$\begin{aligned} & \left\| \sum_{\substack{n, n_1 \in \mathbb{Z}^2 \\ n+n_1+n_2=0}} \int_{\tau+\tau_1+\tau_2=0} \widehat{\varphi_T u_{N_1}}(\tau_1, n_1) \times \frac{h(\tau, n)}{\langle \tau - |n|^2 \rangle^{\frac{1}{2}}} d\tau \right\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & = \|\varphi_T u_{N_1} \widetilde{w}_N\|_{L_t^2 L_x^2([-1, 1] \times \mathbb{T}^2)} \\ & \lesssim \|\varphi_T u_{N_1}\|_{L_t^4 L_x^4([-1, 1] \times \mathbb{T}^2)} \|w_N\|_{L_t^4 L_x^4([-1, 1] \times \mathbb{T}^2)} \\ & \lesssim N_1^{4\varepsilon} \|\varphi_T u_{N_1}\|_{X^{0, \frac{1}{2}-\varepsilon}} N^\varepsilon \|w_N\|_{X^{0, \frac{1}{2}}} \\ & \lesssim_\varphi N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s, \frac{2}{3}}} N^\varepsilon \|h\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)}, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Thus, continuing with (3.15), we use Lemma A.4.4 to obtain

$$\begin{aligned} & \left\| \langle \tau - |n|^2 \rangle^{\frac{s}{2}-\frac{1}{3}} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)} \\ & \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^{\frac{\varepsilon}{2}} \|u_{N_1}\|_{X^{s, \frac{2}{3}}} \|\widehat{\varphi_T v_{N_2}}\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & \lesssim_\varphi N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|\langle \tau_2 - |n_2|^2 \rangle^{\frac{\varepsilon}{2}} \widehat{v_{N_2}}\|_{\ell_{n_2}^2 L_{\tau_2}^2} \\ & \lesssim N^{s+\frac{1}{3}+\varepsilon} N_1^{-s+4\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} N_2^{-s-\frac{4}{3}+\varepsilon} \|v_{N_2}\|_{Y^{s, \frac{2}{3}}} \\ & \lesssim N^{s+\frac{1}{3}+\varepsilon} N_1^{-2s-\frac{4}{3}+5\varepsilon} T^\varepsilon \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}. \end{aligned}$$

Since  $s \leq -\frac{1}{2}$ , we have  $s + \frac{1}{3} + \varepsilon < 0$  for  $\varepsilon > 0$  small enough. Thus, the above estimate is acceptable if  $-2s - \frac{4}{3} + 5\varepsilon \leq 0$ , which is valid given  $s > -\frac{2}{3}$  and  $\varepsilon > 0$  sufficiently small.

Also, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^1(\mathfrak{P}_N)} \\ & \lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-\frac{1}{2}+\varepsilon} \mathcal{F}_{t,x}(\varphi_T \overline{u_{N_1}} \cdot \varphi_T \overline{v_{N_2}}) \right\|_{\ell_n^2 L_\tau^2(\mathfrak{P}_N)}, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. The above term can be estimated similarly as above (along

with  $\langle \tau - |n|^2 \rangle^\varepsilon \lesssim N_1^{2\varepsilon}$ ) for the  $\ell_n^2 L_\tau^2$  term. Combining the above two estimates, we obtain the desired inequality.

**Case 4:**  $|\tau - |n|^2| < 2^{-10}|n_1|^2$  and  $|\tau_2 - |n_2|^2| < 2^{-10}|n_2|^2$ .

This case follows similarly from Case 3 by switching the roles of  $u_{N_1}$  and  $v_{N_2}$ . We thus omit details.

Thus, we have finished our proof.  $\square$

Before moving on to the proof of our main bilinear estimate in Proposition 3.2.1, we first observe that by definition of the  $X^{s,b}$ -norm in (A.26) and the  $Y^{s,b}$ -norm in (3.1), we have the following decompositions:

$$\begin{aligned} \|u\|_{X^{s,b}}^2 &= \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|u_N\|_{X^{s,b}}^2, \\ \|u\|_{Y^{s,b}}^2 &\sim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|u_N\|_{Y^{s,b}}^2. \end{aligned}$$

Thus, it follows that we have the following decomposition regarding the  $Z^{s,b}$  norm:

$$\begin{aligned} \|u\|_{Z^{s,b}}^2 &\sim \|P_{\text{lo}}u\|_{X^{s,b}}^2 + \|P_{\text{hi}}u\|_{Y^{s,b}}^2 \\ &\sim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} (\|P_{\text{lo}}u_N\|_{X^{s,b}}^2 + \|P_{\text{hi}}u_N\|_{Y^{s,b}}^2) \\ &\sim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|u_N\|_{Z^{s,b}}^2. \end{aligned} \tag{3.16}$$

*Proof of Proposition 3.2.1.* By (3.16), we have

$$\begin{aligned} &\| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \bar{u} \cdot \varphi_T \bar{v})(\tau, n) \|_{\dot{Z}^{s, \frac{2}{3}}}^2 \\ &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{\substack{N_1, N_2 \geq 1 \\ \text{dyadic}}} \| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\varphi_T \bar{u}_{N_1} \cdot \varphi_T \bar{v}_{N_2})(\tau, n) \|_{\dot{Z}^{s, \frac{2}{3}}(\mathfrak{B}_N)} \right)^2. \end{aligned} \tag{3.17}$$

For each nonzero summand on the right-hand-side of (3.17), we know that  $N$ ,  $N_1$ , and  $N_2$  must satisfy one of the following:

1.  $2^{-5}N \leq N_1 \leq 2^5N$  and  $N_2 \leq 2^6N$ ,
2.  $2^{-5}N \leq N_2 \leq 2^5N$  and  $N_1 \leq 2^6N$ ,
3.  $\frac{1}{2}N_1 \leq N_2 \leq 2N_1$  and  $N < 2^{-5}N_1$ .

We now treat the above three cases separately.

**Case 1:**  $2^{-5}N \leq N_1 \leq 2^5N$  and  $N_2 \leq 2^6N$ .

In this case, by Lemma 3.2.2, the Cauchy-Schwarz inequality, and (3.16) twice, we have

$$\begin{aligned}
(3.17) &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{\substack{N_1 \geq 1 \text{ dyadic} \\ 2^{-5} \leq N_1/N \leq 2^5}} \sum_{\substack{N_2 \geq 1 \text{ dyadic} \\ N_2 \leq 2^6 N}} N_2^{-\delta} T^\theta \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}} \right)^2 \\
&\lesssim T^{2\theta} \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{\substack{N_1 \geq 1 \text{ dyadic} \\ 2^{-5} \leq N_1/N \leq 2^5}} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \left( \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}}^2 \right)^{1/2} \right)^2 \\
&\lesssim T^{2\theta} \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \sum_{\substack{N_1 \geq 1 \text{ dyadic} \\ 2^{-5} \leq N_1/N \leq 2^5}} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}}^2 \|v\|_{Z^{s, \frac{2}{3}}}^2 \\
&\lesssim T^{2\theta} \|u\|_{Z^{s, \frac{2}{3}}}^2 \|v\|_{Z^{s, \frac{2}{3}}}^2,
\end{aligned}$$

where in the right-hand side of the first inequality we have  $\delta > 0$ .

**Case 2:**  $2^{-5}N \leq N_2 \leq 2^5N$  and  $N_1 \leq 2^6N$ .

This case can be treated in the same way as Case 1, and so we omit details.

**Case 3:**  $\frac{1}{2}N_1 \leq N_2 \leq 2N_1$  and  $N < 2^{-5}N_1$ .

In this case, by Lemma 3.2.3, the Cauchy-Schwarz inequality, and (3.16) twice, we have

$$\begin{aligned}
(3.17) &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{\substack{N_1 \geq 1 \text{ dyadic} \\ N_1 > 2^5 N}} \sum_{\substack{N_2 \geq 1 \text{ dyadic} \\ 1/2 \leq N_2/N_1 \leq 2}} N^{-\delta} T^\theta \|u_{N_1}\|_{Z^{s, \frac{2}{3}}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}} \right)^2 \\
&\lesssim T^{2\theta} \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{-2\delta} \left( \sum_{\substack{N_1 \geq 1 \\ \text{dyadic}}} \|u_{N_1}\|_{Z^{s, \frac{2}{3}}}^2 \right) \sum_{\substack{N_1 \geq 1 \\ \text{dyadic}}} \left( \sum_{\substack{N_2 \geq 1 \text{ dyadic} \\ 1/2 \leq N_2/N_1 \leq 2}} \|v_{N_2}\|_{Z^{s, \frac{2}{3}}} \right)^2 \\
&\lesssim T^{2\theta} \|u\|_{Z^{s, \frac{2}{3}}}^2 \sum_{\substack{N_1 \geq 1 \\ \text{dyadic}}} \|v_{N_1}\|_{Z^{s, \frac{2}{3}}}^2 \\
&\lesssim T^{2\theta} \|u\|_{Z^{s, \frac{2}{3}}}^2 \|v\|_{Z^{s, \frac{2}{3}}}^2,
\end{aligned}$$

where in the right-hand side of the first inequality we have  $\delta > 0$ .

Combining the above three cases, we have thus finished our proof.  $\square$

### 3.3 Local well-posedness of the quadratic NLS with non-linearity $\bar{u}^2$

In this section, we present the proof of Theorem 1.2.1, local well-posedness of the quadratic NLS (1.11) in the low regularity setting. As mentioned in Section 1.2, we mainly focus our attention on local well-posedness of (1.11) on  $H^s(\mathbb{T}^2)$  for  $-\frac{2}{3} < s \leq -\frac{1}{2}$ , using the estimates of the  $Z^{s,b}$ -norm in Section 3.1 and Section 3.2.

By writing (1.11) in the Duhamel formulation, we have

$$u(t) = \Gamma[u](t) \stackrel{\text{def}}{=} e^{-it\Delta} u_0 - i \int_0^t e^{-i(t-t')\Delta} \bar{u}^2(t') dt'. \quad (3.18)$$

Since we are only interested in local well-posedness, we can insert time cut-off functions. For  $0 < T \leq \frac{1}{2}$ , we let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\eta \equiv 1$  on  $[-1, 1]$  and  $\eta \equiv 0$  outside of  $[-2, 2]$  and let  $\eta_{2T}(t) = \eta(t/2T)$ . We first replace the two  $\bar{u}$ 's on the right-hand side of (3.18) by  $\eta_{2T}\bar{u}$ . Also, note that for any function  $F$  that is smooth in space and Schwartz in

time, we have

$$\begin{aligned}
\int_0^t e^{-i(t-t')\Delta} F(t', x) dt' &= \int_0^t \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} e^{i(t-t')|n|^2} \int_{\mathbb{R}} e^{it'\tau} \widehat{F}(\tau, n) d\tau dt' \\
&= \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it|n|^2} \widehat{F}(\tau, n) \frac{e^{it(\tau-|n|^2)} - 1}{i(\tau - |n|^2)} d\tau \\
&= \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it|n|^2} \widehat{F}(\tau, n) \psi(\tau - |n|^2) \frac{e^{it(\tau-|n|^2)} - 1}{i(\tau - |n|^2)} d\tau \\
&\quad - \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it|n|^2} \widehat{F}(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau \\
&\quad + \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it\tau} \widehat{F}(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau,
\end{aligned}$$

where  $\psi : \mathbb{R} \rightarrow [0, 1]$  is a smooth cut-off function such that  $\psi \equiv 1$  on  $[-1, 1]$  and  $\psi \equiv 0$  outside of  $[-2, 2]$ . Let us define the following nonlinear terms.

$$\begin{aligned}
\mathcal{N}_1(u, v) &\stackrel{\text{def}}{=} -i\eta(t) \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it|n|^2} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \\
&\quad \times \psi(\tau - |n|^2) \frac{e^{it(\tau-|n|^2)} - 1}{i(\tau - |n|^2)} d\tau, \\
\mathcal{N}_2(u, v) &\stackrel{\text{def}}{=} i\eta(t) \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it|n|^2} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau, \\
\mathcal{N}_3(u, v) &\stackrel{\text{def}}{=} -i \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} e^{it\tau} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau.
\end{aligned} \tag{3.19}$$

We consider the following formulation of the quadratic NLS (1.11):

$$u(t) = \Gamma_1[u](t) \stackrel{\text{def}}{=} \eta(t) e^{-it\Delta} u_0 + \mathcal{N}_1(u, u) + \mathcal{N}_2(u, u) + \mathcal{N}_3(u, u). \tag{3.20}$$

### 3.3.1 Relevant estimates

In this subsection, we present some relevant estimates for proving our local well-posedness result. We first show the following homogeneous linear estimate.

**Lemma 3.3.1.** *Let  $s \leq 0$ ,  $b > \frac{1}{2}$ , and  $0 < T \leq 1$ . Then, we have*

$$\|\eta(t) e^{-it\Delta} \phi\|_{Z_T^{s,b}} \lesssim_{\eta} \|\phi\|_{H^s(\mathbb{T}^2)},$$

*Proof.* By the definition of the  $Z_T^{s,b}$ -norm in (3.4), Lemma 3.1.3, and Lemma A.4.1 with  $k = 0$ , we have

$$\|\eta(t) e^{-it\Delta} \phi\|_{Z_T^{s,b}} \leq \|\eta(t) e^{-it\Delta} \phi\|_{Z^{s,b}} \lesssim \|\eta(t) e^{-it\Delta} \phi\|_{X^{s,b}} \lesssim_{\eta} \|\phi\|_{H^s(\mathbb{T}^2)},$$

as desired.  $\square$

We now take  $b = \frac{2}{3}$  and show the following bilinear estimate.

**Lemma 3.3.2.** *Let  $-\frac{2}{3} < s \leq -\frac{1}{2}$  and  $0 < T \leq \frac{1}{4}$ . Then, we have*

$$\begin{aligned}\|\mathcal{N}_1(u, v)\|_{Z_T^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z_T^{s, \frac{2}{3}}} \|v\|_{Z_T^{s, \frac{2}{3}}}, \\ \|\mathcal{N}_2(u, v)\|_{Z_T^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z_T^{s, \frac{2}{3}}} \|v\|_{Z_T^{s, \frac{2}{3}}}, \\ \|\mathcal{N}_3(u, v)\|_{Z_T^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z_T^{s, \frac{2}{3}}} \|v\|_{Z_T^{s, \frac{2}{3}}}\end{aligned}$$

for some  $\theta > 0$ , where  $\mathcal{N}_1$ ,  $\mathcal{N}_2$ , and  $\mathcal{N}_3$  are as defined in (3.19).

*Proof.* The idea of the proof comes from [9]. As in the proof of Lemma 3.1.2, by working with the extensions of  $u$  and  $v$  outside  $[-T, T]$ , it suffices to show the following three estimates:

$$\begin{aligned}\|\mathcal{N}_1(u, v)\|_{Z^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}, \\ \|\mathcal{N}_2(u, v)\|_{Z^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}, \\ \|\mathcal{N}_3(u, v)\|_{Z^{s, \frac{2}{3}}} &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}},\end{aligned}$$

for some  $\theta > 0$ .

To deal with the  $\mathcal{N}_1$  term, by Lemma 3.1.3, the Taylor expansion, Lemma A.4.1, Lemma 3.1.1, and Proposition 3.2.1, we obtain

$$\begin{aligned}\|\mathcal{N}_1(u, v)\|_{Z^{s, \frac{2}{3}}} &\lesssim \|\mathcal{N}_1(u, v)\|_{X^{s, \frac{2}{3}}} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{k!} \left\| t^k \eta(t) e^{-it\Delta} \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} \mathcal{F}_{x,t}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right. \\ &\quad \left. \times \psi(\tau - |n|^2) \langle \tau - |n|^2 \rangle^{k-1} d\tau \right\|_{X^{s, \frac{2}{3}}} \\ &\lesssim_\eta \sum_{k=1}^{\infty} \frac{3^k}{k!} \left\| \langle n \rangle^s \int_{\mathbb{R}} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \psi(\tau - |n|^2) \langle \tau - |n|^2 \rangle^{k-1} d\tau \right\|_{\ell_n^2} \\ &\lesssim \sum_{k=1}^{\infty} \frac{6^k}{k!} \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right\|_{\ell_n^2 L^1} \\ &\lesssim \left\| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right\|_{\widehat{Z}^{s, \frac{2}{3}}} \\ &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}\end{aligned}$$

for some  $\theta > 0$ .

For the  $\mathcal{N}_2$  term, using Lemma 3.1.3, Lemma A.4.1 with  $k = 0$ , the fact that  $1 - \psi$  is bounded by 1 and supported outside of  $[-1, 1]$ , Lemma 3.1.1, and Proposition 3.2.1, we have

$$\begin{aligned}\|\mathcal{N}_2(u, v)\|_{Z^{s, \frac{2}{3}}} &\lesssim \|\mathcal{N}_2(u, v)\|_{X^{s, \frac{2}{3}}} \\ &\lesssim \left\| \eta(t) e^{-it\Delta} \sum_{n \in \mathbb{Z}^2} e^{in \cdot x} \int_{\mathbb{R}} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right. \\ &\quad \left. \times \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau \right\|_{X^{s, \frac{2}{3}}} \\ &\lesssim_\eta \left\| \langle n \rangle^s \int_{\mathbb{R}} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{i(\tau - |n|^2)} d\tau \right\|_{\ell_n^2} \\ &\lesssim \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right\|_{\ell_n^2 L^1} \\ &\lesssim \left\| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \right\|_{\widehat{Z}^{s, \frac{2}{3}}} \\ &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}\end{aligned}$$

for some  $\theta > 0$ .

For the  $\mathcal{N}_3$  term, since  $1 - \psi$  is bounded by 1 and supported outside of  $[-1, 1]$ , by the

monotonicity property (3.3) and Proposition 3.2.1, we have

$$\begin{aligned}\|\mathcal{N}_3(u, v)\|_{Z^{s, \frac{2}{3}}} &= \left\| \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v})(\tau, n) \frac{1 - \psi(\tau - |n|^2)}{\tau - |n|^2} \right\|_{\widehat{Z}^{s, \frac{2}{3}}} \\ &\lesssim \left\| \langle \tau - |n|^2 \rangle^{-1} \mathcal{F}_{t,x}(\eta_{2T}\bar{u} \cdot \eta_{2T}\bar{v}) \right\|_{\widehat{Z}^{s, \frac{2}{3}}} \\ &\lesssim_\eta T^\theta \|u\|_{Z^{s, \frac{2}{3}}} \|v\|_{Z^{s, \frac{2}{3}}}\end{aligned}$$

for some  $\theta > 0$ . Thus, we finish our proof.  $\square$

### 3.3.2 Local well-posedness

We now use the formulation (3.20) and the estimates in Subsection 3.3.1 to prove our local well-posedness result. As mentioned in Subsection 1.2, we only focus on the case  $\mathcal{M} = \mathbb{T}^2$ .

We let  $0 < T \leq \frac{1}{4}$  and fix  $-\frac{2}{3} < s \leq -\frac{1}{2}$ . By (3.20), Lemma 3.3.1, and Lemma 3.3.2, we have

$$\begin{aligned}\|\Gamma_1[u]\|_{Z_T^{s, \frac{2}{3}}} &\lesssim \|\eta(t)e^{-it\Delta}u_0\|_{Z_T^{s, \frac{2}{3}}} + \sum_{j=1}^3 \|\mathcal{N}_j(u, u)\|_{Z_T^{s, \frac{2}{3}}} \\ &\lesssim_\eta \|u_0\|_{H^s(\mathbb{T}^2)} + T^\theta \|u\|_{Z_T^{s, \frac{2}{3}}}^2,\end{aligned}\tag{3.21}$$

for some  $\theta > 0$ . Similarly, we obtain the following difference estimate:

$$\begin{aligned}\|\Gamma_1[u] - \Gamma_1[v]\|_{Z_T^{s, \frac{2}{3}}} &\lesssim \sum_{j=1}^3 \left( \|\mathcal{N}_j(u, u-v)\|_{Z_T^{s, \frac{2}{3}}} + \|\mathcal{N}_j(u-v, v)\|_{Z_T^{s, \frac{2}{3}}} \right) \\ &\lesssim_\eta T^\theta \left( \|u\|_{Z_T^{s, \frac{2}{3}}} + \|v\|_{Z_T^{s, \frac{2}{3}}} \right) \|u-v\|_{Z_T^{s, \frac{2}{3}}}.\end{aligned}\tag{3.22}$$

Thus, by choosing  $T = T(\|u_0\|_{H^s(\mathbb{T}^2)}) > 0$  sufficiently small, we have that  $\Gamma_1$  is a contraction on the ball  $B_R \subset Z^{s, \frac{2}{3}}$  of radius  $R \sim \|u_0\|_{H^s(\mathbb{T}^2)}$ . This gives the existence part of Theorem 1.2.1 when  $\mathcal{M} = \mathbb{T}^2$  and the uniqueness in the ball  $B_R$ . Also, the continuous dependence of solutions on the initial data follows easily from the formulation (3.20), Lemma 3.3.1, (3.21), and (3.22).

It remains to extend the uniqueness of solutions to (1.6) to the entire  $Z_T^{s, \frac{2}{3}}$ -space. We let  $u$  and  $v$  be two solutions of (1.6) in  $Z_T^{s, \frac{2}{3}}$ . Note that  $u$  and  $v$  satisfy the formulation (3.20) for  $t \in [-T, T]$ . For  $0 < T_0 \leq T$ , we use (3.22) to obtain

$$\begin{aligned}\|u-v\|_{Z_{T_0}^{s, \frac{2}{3}}} &\lesssim_\eta T_0^\theta \left( \|u\|_{Z_{T_0}^{s, \frac{2}{3}}} + \|v\|_{Z_{T_0}^{s, \frac{2}{3}}} \right) \|u-v\|_{Z_{T_0}^{s, \frac{2}{3}}} \\ &\leq T_0^\theta \left( \|u\|_{Z_T^{s, \frac{2}{3}}} + \|v\|_{Z_T^{s, \frac{2}{3}}} \right) \|u-v\|_{Z_{T_0}^{s, \frac{2}{3}}}.\end{aligned}$$

Thus, by choosing

$$T_0 = T_0 \left( \|u\|_{Z_T^{s, \frac{2}{3}}}, \|v\|_{Z_T^{s, \frac{2}{3}}} \right) > 0$$

sufficiently small, we can use Lemma 3.1.2 to obtain

$$\|u-v\|_{C([-T_0, T_0]); H^s(\mathbb{T}^2)} \lesssim \|u-v\|_{Z_{T_0}^{s, \frac{2}{3}}} = 0,$$

so that  $u \equiv v$  on  $[-T_0, T_0]$ . Since  $T_0$  depends only on  $\|u\|_{Z_T^{s, \frac{2}{3}}}$  and  $\|v\|_{Z_T^{s, \frac{2}{3}}}$ , we can iterate the above argument on  $[-T, -T_0]$  and  $[T_0, T]$ . This shows that  $u \equiv v$  on  $[-T, T]$  after a finite number of iterations, and so the uniqueness of (1.6) on the entire  $Z_T^{s, \frac{2}{3}}$ -space follows.

## Chapter 4

# Probabilistic well-posedness of quadratic NLS with random initial data

In this chapter, we study probabilistic well-posedness of the quadratic NLS with nonlinearity  $|u|^2$  with Gaussian random initial data on the two-dimensional torus  $\mathbb{T}^2$ . Specifically, we prove almost sure local well-posedness (Theorem 1.3.1) and probabilistic ill-posedness (Theorem 1.3.5) of the quadratic NLS (1.16).

### 4.1 Random tensor and deterministic tensor estimates

In this section, we recall some useful results of random tensor estimates developed in [34] and also show some deterministic tensor estimates.

Let us first recall the definition of (random) tensors. Let  $A$  be a finite index set. We denote  $n_A$  as the tuple  $(n_j : j \in A)$ . A *tensor*  $h = h_{n_A}$  is a function from  $(\mathbb{Z}^2)^A$  to  $\mathbb{C}$  with  $n_A$  being the input variables. The *support* of  $h$  is the set of  $n_A$  such that  $h_{n_A} \neq 0$ . Note that  $h$  may also depend on  $\omega \in \Omega$ , in which case  $h$  is called a *random tensor*.

Given a finite index set  $A$ , we define the norm  $\|\cdot\|_{n_A}$  by

$$\|h\|_{n_A} \stackrel{\text{def}}{=} \|h\|_{\ell_{n_A}^2} = \left( \sum_{n_A \in (\mathbb{Z}^2)^A} |h_{n_A}|^2 \right)^{\frac{1}{2}}.$$

For any partition  $(B, C)$  of  $A$ , i.e.  $B \cup C = A$  and  $B \cap C = \emptyset$ , we define the norm  $\|\cdot\|_{n_B \rightarrow n_C}$  by

$$\|h\|_{n_B \rightarrow n_C}^2 \stackrel{\text{def}}{=} \sup \left\{ \sum_{n_C \in (\mathbb{Z}^2)^C} \left| \sum_{n_B \in (\mathbb{Z}^2)^B} h_{n_A} \cdot f_{n_B} \right|^2 : \sum_{n_B \in (\mathbb{Z}^2)^B} |f_{n_B}|^2 = 1 \right\}.$$

For any tensor  $h$ , by duality, we have  $\|h\|_{n_B \rightarrow n_C} = \|h\|_{n_C \rightarrow n_B} = \|\bar{h}\|_{n_B \rightarrow n_C}$ . If either  $B = \emptyset$  or  $C = \emptyset$ , we have  $\|h\|_{n_B \rightarrow n_C} = \|h\|_{n_A}$ .

We also need the following definitions to state the random tensor estimates. For a complex number  $a$ , we define  $a^+ = a$  and  $a^- = \bar{a}$ . Let  $A$  be a finite index set. For each  $j \in A$ , we associate  $j$  with a sign  $\zeta_j \in \{\pm\}$ . For  $j_1, j_2 \in A$ , we say that  $(n_{j_1}, n_{j_2})$  is a *pairing* if  $n_{j_1} = n_{j_2}$  and  $\zeta_{j_1} = -\zeta_{j_2}$ . Also, recall that  $\{g_n\}_{n \in \mathbb{Z}^2}$  is a set of independent standard complex-valued Gaussian random variables. For each  $n \in \mathbb{Z}^2$ , we can write

$$g_n(\omega) = \rho_n(\omega) \eta_n(\omega),$$

where  $\rho_n = |g_n|$  and  $\eta_n = \rho_n^{-1} g_n$  are independent. Note that each  $\eta_n$  is uniformly distributed on the unit circle of  $\mathbb{C}$ .

We now record the following random tensor estimate. For a proof, see [34, Proposition 4.14].

**Lemma 4.1.1.** *Let  $0 < T \leq 1$ . Let  $h_{a_1 a_2 n_A} = h_{a_1 a_2 n_A}(\omega)$  be a random tensor, where each  $n_j \in \mathbb{Z}^2$  and  $(a_1, a_2) \in (\mathbb{Z}^2)^q$  for some integer  $q \geq 2$ . Given a dyadic number  $M \geq 1$ , we assume that  $\langle a_1 \rangle, \langle a_2 \rangle \lesssim M$  and  $\langle n_j \rangle \lesssim M$  for all  $j \in A$ . We also assume that in the support of  $h_{a_1 a_2 n_A}$ , there is no pairing in  $n_A$ . Moreover, we assume that  $\{h_{a_1 a_2 n_A}\}$  is independent with  $\{\eta_n\}_{n \in \mathbb{Z}^2}$ . Define the tensor*

$$H_{a_1 a_2} = \sum_{n_A} h_{a_1 a_2 n_A} \prod_{j \in A} \eta_{n_j}^{\zeta_j}.$$

*Then, there exists constants  $C, c > 0$  such that outside an exceptional set of probability  $\leq C \exp(-\frac{cM}{T^\theta})$  with  $\theta > 0$ , we have*

$$\|H_{a_1 a_2}\|_{a_1 \rightarrow a_2} \lesssim T^{-\theta} M^\theta \cdot \max_{(A_1, A_2)} \|h\|_{a_1 n_{A_1} \rightarrow a_2 n_{A_2}},$$

where  $(A_1, A_2)$  runs over all partitions of  $A$ .

We also record the following variant of Lemma 4.1.1. For a proof, see [34, Proposition 4.15].

**Lemma 4.1.2.** *Consider the same setting as in Lemma 4.1.1 with the following differences:*

- (1) *We only restrict  $\langle n_j \rangle \lesssim M$  for all  $j \in A$  but do not impose any condition on  $\langle a_1 \rangle$  or  $\langle a_2 \rangle$ .*
- (2) *We assume that  $a_1, a_2 \in \mathbb{Z}^2$  and that in the support of the random tensor  $h_{a_1 a_2 n_A}$  we have  $|a_1 - \zeta a_2| \lesssim M$  where  $\zeta \in \{\pm\}$ .*
- (3) *The random tensor  $h_{a_1 a_2 n_A}$  only depends on  $a_1 - \zeta a_2$ ,  $|a_1|^2 - \zeta |a_2|^2$ , and  $n_A$ , and is supported in the set where  $||a_1|^2 - \zeta |a_2|^2| \lesssim M^{10}$ .*

*Then, there exists constants  $C, c > 0$  such that outside an exceptional set of probability  $\leq C \exp(-\frac{cM}{T^\theta})$  with  $\theta > 0$ , we have*

$$\|H_{a_1 a_2}\|_{a_1 \rightarrow a_2} \lesssim T^{-\theta} M^\theta \cdot \max_{(A_1, A_2)} \|h\|_{a_1 n_{A_1} \rightarrow a_2 n_{A_2}},$$

where  $(A_1, A_2)$  runs over all partitions of  $A$ .

We now turn our attention to some deterministic tensor estimates. Given  $m \in \mathbb{Z}$ , we define the base tensor  $h_{nn_1 n_2}^m$  as

$$h_{nn_1 n_2}^m = \mathbf{1}_{n-n_1+n_2=0} \mathbf{1}_{|n|^2-|n_1|^2+|n_2|^2=m} \mathbf{1}_{n \neq 0} \mathbf{1}_{n_2 \neq 0}. \quad (4.1)$$

We now show the following estimates regarding the base tensor  $h_{nn_1 n_2}^m$ .

**Lemma 4.1.3.** *Let  $N, N_1, N_2 \geq 1$  be dyadic numbers and  $\varepsilon > 0$ . Let  $J$  be a ball of radius  $\sim N$ ,  $J_1$  be a ball of radius  $\sim N_1$ , and  $J_2$  be a ball of radius  $\sim N_2$ . We define*

$$S \stackrel{\text{def}}{=} \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \in J, n_1 \in J_1, n_2 \in J_2\}.$$

*Thus, we have the following estimates:*

$$\|h_{nn_1 n_2}^m \cdot \mathbf{1}_S\|_{nn_1 n_2} \lesssim N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \max\{N_1^\varepsilon, N_2^\varepsilon\}, \quad (4.2)$$

$$\|h_{nn_1 n_2}^m \cdot \mathbf{1}_S\|_{n_1 \rightarrow nn_2} \lesssim \max\{N_1^\varepsilon, N_2^\varepsilon\}, \quad (4.3)$$

$$\|h_{nn_1 n_2}^m \cdot \mathbf{1}_S \cdot \mathbf{1}_{n_2 \neq 0}\|_{n_2 \rightarrow nn_1} \lesssim \min\{N^{\frac{1}{2}}, N_1^{\frac{1}{2}}\}, \quad (4.4)$$

$$\|h_{nn_1 n_2}^m \cdot \mathbf{1}_S\|_{n \rightarrow n_1 n_2} \lesssim \min\{N_1^{\frac{1}{2}}, N_2^{\frac{1}{2}}\}. \quad (4.5)$$

*Proof.* For (4.2), we use Lemma A.5.2 (i) to obtain

$$\|h_{nn_1 n_2}^m \cdot \mathbf{1}_S\|_{nn_1 n_2} \lesssim N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \max\{N_1^\varepsilon, N_2^\varepsilon\}.$$

For (4.3), we use Schur's test and Lemma A.5.2 (ii) to obtain

$$\begin{aligned} \|h_{nn_1n_2}^m \cdot \mathbf{1}_S\|_{n_1 \rightarrow nn_2} &\leq \left( \sup_{n_1} \sum_{n, n_2} h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \left( \sup_{n, n_2} \sum_{n_1} h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \\ &\lesssim \max\{N^\varepsilon, N_2^\varepsilon\}. \end{aligned}$$

For (4.4), we use Schur's test and Lemma A.5.2 (iii) to obtain

$$\begin{aligned} \|h_{nn_1n_2}^m \cdot \mathbf{1}_S\|_{n_2 \rightarrow nn_1} &\leq \left( \sup_{n_2} \sum_{n, n_1} h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \left( \sup_{n, n_1} \sum_{n_2} h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \\ &\lesssim \min\{N^{\frac{1}{2}}, N_1^{\frac{1}{2}}\}. \end{aligned}$$

For (4.5), we use Schur's test and Lemma A.5.2 (iv) to obtain

$$\begin{aligned} \|h_{nn_1n_2}^m \cdot \mathbf{1}_S\|_{n \rightarrow n_1n_2} &\leq \left( \sup_n \sum_{n_1, n_2} h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \left( \sup_{n_1, n_2} \sum_n h_{nn_1n_2}^m \cdot \mathbf{1}_S \right)^{\frac{1}{2}} \\ &\lesssim \min\{N_1^{\frac{1}{2}}, N_2^{\frac{1}{2}}\}. \end{aligned}$$

We thus finish our proof.  $\square$

**Remark 4.1.4.** The conditions  $n \neq 0$  and  $n_2 \neq 0$  in the base tensor (4.1) are necessary for the tensor estimates (4.4) and (4.5) to hold in view of the restriction  $n_2 \neq 0$  in Lemma A.5.1 (iii) and the restriction  $n \neq 0$  in Lemma A.5.1 (iv).

## 4.2 Bilinear estimates

In this section, we establish several bilinear estimates that are crucial for proving Theorem 1.3.1, the almost sure local well-posedness result of the quadratic NLS (1.16). Specifically, we need to estimate the following term

$$\|\mathcal{I}(v^{(1)}\overline{v^{(2)}})\|_{X_T^{s, \frac{1}{2}+\delta}},$$

where  $s, \delta > 0$  are sufficiently small and  $0 < T \leq 1$ , and  $\mathcal{I} = \mathcal{I}_\chi$  is the truncated Duhamel operator as defined in (A.28) with  $\chi$  being a smooth cut-off function such that  $\chi \equiv 1$  on  $[-1, 1]$  and  $\chi \equiv 0$  outside of  $[-2, 2]$ . Here, each of  $v^{(1)}$  and  $v^{(2)}$  is either an arbitrary space-time function supported on  $[-1, 1] \times \mathbb{T}^2$  or the random linear solution with a time cut-off  $\chi \cdot z$ , where  $z$  is as defined in (1.18).

We first consider the case when neither  $v^{(1)}$  nor  $v^{(2)}$  is  $\chi \cdot z$ . Specifically, we show the following bilinear estimate.

**Proposition 4.2.1.** *Let  $s > 0$ ,  $\delta > 0$  be sufficiently small, and  $0 < T \leq 1$ . Then, we have*

$$\|\mathcal{I}(v^{(1)}\overline{v^{(2)}})\|_{X_T^{s, \frac{1}{2}+\delta}} \lesssim T^\delta \|v^{(1)}\|_{X_T^{s, \frac{1}{2}+\delta}} \|v^{(2)}\|_{X_T^{s, \frac{1}{2}+\delta}}.$$

*Proof.* By Lemma A.4.4 and Lemma A.4.2, we have

$$\|\mathcal{I}(v^{(1)}\overline{v^{(2)}})\|_{X_T^{s, \frac{1}{2}+\delta}} \lesssim T^\delta \|\mathcal{I}(v^{(1)}\overline{v^{(2)}})\|_{X_T^{s, \frac{1}{2}+2\delta}} \lesssim T^\delta \|v^{(1)}\overline{v^{(2)}}\|_{X_T^{s, -\frac{1}{2}+2\delta}}. \quad (4.6)$$

In the following, we work on extensions of  $v^{(1)}$  and  $v^{(2)}$  outside of  $[-T, T]$ , we can ignore the

subscript  $T$  for the  $X^{s,b}$ -norm. By duality and dyadic decomposition, we have

$$\begin{aligned} \|v^{(1)}\overline{v^{(2)}}\|_{X^{s,-\frac{1}{2}+2\delta}} &= \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}} \leq 1} \left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^s (v^{(1)}\overline{v^{(2)}}) \overline{w} \, dx dt \right| \\ &\lesssim \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} \left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} \overline{P_{N_2} v^{(2)}}) \overline{P_N w} \, dx dt \right|. \end{aligned} \quad (4.7)$$

Let  $n_1, n_2, n$  be the frequencies corresponding to the three terms  $P_{N_1} v^{(1)}, P_{N_2} v^{(2)}, P_N w$ , respectively. In order for the above integral on  $\mathbb{T}^2$  to be nonzero, we must have  $n_1 - n_2 - n = 0$ . This leads us to the following three cases.

**Case 1:**  $N_1 \sim N_2$ .

In this case, we have  $N \lesssim N_1 \sim N_2$ . By Hölder's inequality and Lemma A.4.5, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} \overline{P_{N_2} v^{(2)}}) \overline{P_N w} \, dx dt \right| \\ &\lesssim N^s \|P_{N_1} v^{(1)}\|_{L_t^4 L_x^4} \|P_{N_2} v^{(2)}\|_{L_t^4 L_x^4} \|P_N w\|_{L_t^2 L_x^2} \\ &\lesssim N_1^{0-} \|N_1^{\frac{s}{2}+} P_{N_1} v^{(1)}\|_{L_t^4 L_x^4} \|N_2^{\frac{s}{2}} P_{N_2} v^{(2)}\|_{L_t^4 L_x^4} \|P_N w\|_{L_t^2 L_x^2} \\ &\lesssim N_1^{0-} \|P_{N_1} v^{(1)}\|_{X^{s,\frac{1}{2}+\delta}} \|P_{N_2} v^{(2)}\|_{X^{s,\frac{1}{2}+\delta}} \|P_N w\|_{X^{0,0}} \\ &\leq N_1^{0-} \|v^{(1)}\|_{X^{s,\frac{1}{2}+\delta}} \|v^{(2)}\|_{X^{s,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}-2\delta}}. \end{aligned} \quad (4.8)$$

Combining (4.6), (4.7), (4.8) and summing over  $N_1 \sim N_2 \gtrsim N$ , we obtain the desired estimate.

**Case 2:**  $N_1 \gg N_2$ .

In this case, we have  $N \sim N_1 \gg N_2$ . We partition the annulus  $\{|n_1| \sim N_1\}$  into balls of radius  $\sim N_2$  and denote the set of these balls as  $\mathcal{J}_1$ , and we partition the annulus  $\{|n| \sim N\}$  into balls of radius  $\sim N_2$  and denote the set of these balls as  $\mathcal{J}$ . Note that for each fixed  $J_1 \in \mathcal{J}_1$ , the product  $\mathbf{1}_{J_1}(n_1) \cdot \mathbf{1}_J(n)$  is nonzero for at most  $O(1)$  many  $J \in \mathcal{J}$ , and we denote the set of these  $J$ 's as  $\mathcal{J}(J_1)$ . Thus, by Hölder's inequality, Lemma A.4.5, and the Cauchy-Schwarz inequality in  $J_1$ , we have

$$\begin{aligned} &\left| \int_{\mathbb{R}} \int_{\mathbb{T}^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} \overline{P_{N_2} v^{(2)}}) \overline{P_N w} \, dx dt \right| \\ &\lesssim \sum_{J_1 \in \mathcal{J}_1} \sum_{J \in \mathcal{J}(J_1)} N_1^s \|P_{J_1} P_{N_1} v^{(1)}\|_{L_{t,x}^4} \|P_{N_2} v^{(2)}\|_{L_t^4 L_x^4} \|P_J P_N w\|_{L_t^2 L_x^2} \\ &\lesssim \sum_{J_1 \in \mathcal{J}_1} \sum_{J \in \mathcal{J}(J_1)} N_1^s N_2^{0+} \|P_{J_1} P_{N_1} v^{(1)}\|_{X^{0,\frac{1}{2}+\delta}} \|P_{N_2} v^{(2)}\|_{X^{0,\frac{1}{2}+\delta}} \|P_J P_N w\|_{X^{0,\frac{1}{2}-2\delta}} \\ &\lesssim N_1^s N_2^{0-} \|P_{N_1} v^{(1)}\|_{X^{0,\frac{1}{2}+\delta}} \|P_{N_2} v^{(2)}\|_{X^{s,\frac{1}{2}+\delta}} \|P_N w\|_{X^{0,\frac{1}{2}-2\delta}} \\ &\sim N_2^{0-} \|P_{N_1} v^{(1)}\|_{X^{s,\frac{1}{2}+\delta}} \|P_{N_2} v^{(2)}\|_{X^{s,\frac{1}{2}+\delta}} \|P_N w\|_{X^{0,\frac{1}{2}-2\delta}}. \end{aligned} \quad (4.9)$$

Combining (4.6), (4.7), (4.9), applying the Cauchy-Schwarz inequality in  $N_1 \sim N$ , and summing over  $N_1 \sim N \gg N_2$ , we obtain the desired estimate.

**Case 3:**  $N_1 \ll N_2$ .

The steps in this case are the same as those in Case 2 by switching the roles of  $v^{(1)}$  and  $v^{(2)}$ , so that we omit details.  $\square$

We now consider the case when at least one of  $v^{(1)}$  and  $v^{(2)}$  is the random linear solution with a time cut-off  $\chi \cdot z$ . Our goal is to prove the following estimates. The idea of the computations in the proof comes from [117].

**Proposition 4.2.2.** *Let  $0 \leq \alpha < \frac{1}{2}$ ,  $s, \delta > 0$  be sufficiently small, and  $0 < T \leq 1$ .*

(i) We have

$$\|P_{\neq 0}(\mathcal{I}(v \cdot \overline{\chi \cdot z}))\|_{X_T^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \|v\|_{X_T^{s, \frac{1}{2} + \delta}} \quad (4.10)$$

outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$  with  $C, c > 0$  being constants and  $0 < \theta \ll \delta$ .

(ii) If  $v$  has mean zero (i.e. has no zeroth frequency term), we have

$$\|P_{\neq 0}(\mathcal{I}(\chi \cdot z \cdot \overline{v}))\|_{X_T^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \|v\|_{X_T^{s, \frac{1}{2} + \delta}} \quad (4.11)$$

outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$  with  $C, c > 0$  being constants and  $0 < \theta \ll \delta$ .

(iii) We have

$$\|P_{\neq 0}(\mathcal{I}(\chi \cdot z \cdot \overline{\chi \cdot z}))\|_{X_T^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \quad (4.12)$$

outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$  with  $C, c > 0$  being constants and  $0 < \theta \ll \delta$ .

*Proof.* As in the proof of Proposition 4.2.1, we can drop the subscript  $T$  for the  $X^{s,b}$ -norm. We first do the following general setup. Let  $v^{(1)}$  and  $v^{(2)}$  be two space-time functions. By Lemma A.4.4, Lemma A.4.3, duality, and dyadic decomposition, we have

$$\begin{aligned} & \|P_{\neq 0}(\mathcal{I}(v^{(1)} \overline{v^{(2)}}))\|_{X_T^{s, \frac{1}{2} + \delta}} \\ & \lesssim T^\delta \|P_{\neq 0}(\mathcal{I}_\chi(v^{(1)} \overline{v^{(2)}}))\|_{X^{s, \frac{1}{2} + 2\delta}} \\ & = T^\delta \left\| \mathbf{1}_{n \neq 0} \langle n \rangle^s \langle \tau \rangle^{\frac{1}{2} + 2\delta} \int_{\mathbb{R}} K(\tau, \tau') \widehat{v^{(1)} \overline{v^{(2)}}}(\tau' + |n|^2, n) d\tau' \right\|_{\ell_n^2 L_\tau^2} \\ & = T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1} \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \neq 0}} \langle n \rangle^s \iiint K(\tau, -|n|^2 + (\tau_1 + |n_1|^2) - (\tau_2 + |n_2|^2)) \right. \\ & \quad \left. \times \langle \tau \rangle^{\frac{1}{2} + 2\delta} \widehat{v^{(1)}}(\tau_1 + |n_1|^2, n_1) \overline{\widehat{v^{(2)}}}(\tau_2 + |n_2|^2, n_2) \overline{\widehat{w}}(\tau, n) d\tau d\tau_1 d\tau_2 \right| \\ & \lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \neq 0}} \langle n \rangle^s \right. \\ & \quad \times \iiint K(\tau, -|n|^2 + (\tau_1 + |n_1|^2) - (\tau_2 + |n_2|^2)) \langle \tau \rangle^{\frac{1}{2} + 2\delta} \\ & \quad \left. \times \widehat{P_{N_1} v^{(1)}}(\tau_1 + |n_1|^2, n_1) \overline{\widehat{P_{N_2} v^{(2)}}}(\tau_2 + |n_2|^2, n_2) \overline{\widehat{P_N w}}(\tau, n) d\tau d\tau_1 d\tau_2 \right|, \end{aligned} \quad (4.13)$$

where the kernel  $K$  given by Lemma A.4.3 satisfies

$$|K(\tau, \tau')| \lesssim \frac{1}{\langle \tau \rangle \langle \tau - \tau' \rangle}. \quad (4.14)$$

We now separately discuss (i), (ii), and (iii).

(i) We consider the following two cases.

**Case 1:**  $\langle \tau \rangle \gg N_2^{10}$ .

In this case, by (4.14), the Cauchy-Schwarz inequalities in  $\tau_1, \tau$ , and  $n$ , and Lemma A.5.4,

we have

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_2^{-5+20\delta} N_2^2 N_2^{-1+\alpha} \\
&\times \sup_{\langle n_2 \rangle \sim N_2} \sum_{\langle n \rangle \sim N} \iiint \langle (\tau + |n|^2) - (\tau_1 + |n_1|^2) + (\tau_2 + |n_2|^2) \rangle^{-1} \\
&\times \left| \widehat{P_{N_1} v}(\tau_1 + |n + n_2|^2, n + n_2) \right| |g_{n_2}(\omega) \widehat{\chi}(\tau_2)| \left| \widehat{P_N w}(\tau, n) \right| d\tau_1 d\tau_2 d\tau \quad (4.15) \\
&\lesssim \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_2^{-4+20\delta+\alpha} \sup_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} |g_{n_2}(\omega)| \left\| \langle \tau_1 \rangle^{\frac{1}{2}+\delta} \widehat{P_{N_1} v}(\tau_1 + |n_1|^2, n_1) \right\|_{\ell_{n_1}^2 L_{\tau_1}^2} \\
&\lesssim \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-s} N_2^{-4+20\delta+\alpha} \sup_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} |g_{n_2}(\omega)| \left\| P_{N_1} v \right\|_{X^{s, \frac{1}{2}+\delta}}.
\end{aligned}$$

Note that we have the following Gaussian tail bound:

$$\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} P(|g_{n_2}| > T^{-\theta} N_2^\delta) < C \exp\left(-c \frac{N_2^\delta}{T^\theta}\right) \quad (4.16)$$

for some constants  $C, c > 0$  and  $0 < \theta \ll \delta$ , so that (4.15) gives

$$(4.13) \lesssim T^{-\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-s} N_2^{-4+21\delta+\alpha} \left\| P_{N_1} v \right\|_{X^{s, \frac{1}{2}+\delta}} \quad (4.17)$$

outside an exceptional set of probability  $\leq C \exp(-c N_2^\delta / T^\theta)$ . Recall that  $\delta$  and  $s$  can be made sufficiently small and  $\alpha < 1$ . If  $N \gg N_1$ , we have  $N \sim N_2$ , so that we can use  $N^s \sim N^{0-N_2^{s+}}$  and sum up dyadic  $N, N_1, N_2$  in (4.17) to obtain (4.10). If  $N \ll N_1$ , we have  $N_1 \sim N_2$ , so that we can use  $N^s \ll N^{0-N_2^{s+}}$  and sum up dyadic  $N, N_1, N_2$  in (4.17) to obtain (4.10). If  $N \sim N_1$ , we can use the Cauchy-Schwarz inequality in  $N \sim N_1$  and sum up dyadic  $N, N_1, N_2 \geq 1$  in (4.17) to obtain (4.10).

**Case 2:**  $\langle \tau \rangle \lesssim N_2^{10}$ .

We further split this case into two subcases.

**Subcase 2.1:**  $N \lesssim N_2$ .

In this case, we have  $N_1 \lesssim N_2$ . By the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , (4.14), and Minkowski's inequality, we have

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_2^{30\delta} \\
&\times \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \neq 0}} \iiint K(\tau, -|n|^2 + (\tau_1 + |n_1|^2) - (\tau_2 + |n_2|^2)) \langle \tau \rangle^{\frac{1}{2}-\delta} \right. \\
&\times \left. \widehat{P_{N_1} v}(\tau_1 + |n_1|^2, n_1) \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \widehat{\chi}(\tau_2) \widehat{P_N w}(\tau, n) d\tau d\tau_1 d\tau_2 \right| \quad (4.18) \\
&\lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_2^{30\delta} \left[ \int \langle \tau \rangle^{-1-2\delta} \times \left( \sum_{m \in \mathbb{Z}} \iint \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \right. \right. \\
&\times \left. \left. \langle \tau_1 \rangle^{-\frac{1}{2}-\delta} \widehat{\chi}(\tau_2) \right\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1 n_2}^m \mathbf{1}_{S_1} \cdot \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2}+\delta} \right. \\
&\times \left. \left. \widehat{P_{N_1} v}(\tau_1 + |n_1|^2, n_1) \right\| d\tau_1 d\tau_2 \right]^{1/2},
\end{aligned}$$

where  $h_{nn_1n_2}^m$  is the base tensor as defined in (4.1) and  $S_1$  is a set defined by

$$\begin{aligned} S_1 &\stackrel{\text{def}}{=} S_1(N, N_1, N_2) \\ &= \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : |n| \sim N, |n_1| \sim N_1, |n_2| \sim N_2\}. \end{aligned} \quad (4.19)$$

Note that for  $(n, n_1, n_2)$  restricted in  $S_1$ , we have  $\lesssim N_2^2$  choices for the value

$$m = |n|^2 - |n_1|^2 + |n_2|^2,$$

which implies that

$$\sum_{m \in \mathbb{Z}} \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \lesssim \log(1 + N_2^2) \lesssim N_2^\delta. \quad (4.20)$$

Thus, continuing with (4.18), by (4.20) and the Cauchy-Schwarz inequality in  $\tau_1$ , we obtain

$$(4.13) \lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_2^{31\delta} \left\| \sum_{n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_1} \cdot \frac{\overline{g_{n_2}}(\omega)}{\langle n_2 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_1} \|P_{N_1} v\|_{X^{s, \frac{1}{2}+\delta}}. \quad (4.21)$$

By Lemma 4.1.1, the Gaussian tail bound (4.16), and Lemma 4.1.3, we have

$$\begin{aligned} &\left\| \sum_{n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_1} \cdot \frac{\overline{g_{n_2}}(\omega)}{\langle n_2 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_1} \\ &\lesssim T^{-2\theta} N_2^{-1+2\delta+\alpha} \max \{ \|h_{nn_1n_2}^m \mathbf{1}_{S_1}\|_{nn_2 \rightarrow n_1}, \|h_{nn_1n_2}^m \mathbf{1}_{S_1}\|_{n \rightarrow n_1 n_2} \} \\ &\lesssim T^{-2\theta} N_2^{-\frac{1}{2}+2\delta+\alpha} \end{aligned} \quad (4.22)$$

outside an exceptional set of probability  $\leq C \exp(-cN_2^\delta/T^\theta)$  for some universal constants  $C, c > 0$ . Thus, combining (4.21) and (4.22), using the fact that  $\alpha < \frac{1}{2}$ ,  $N \lesssim N_2$ ,  $N_1 \lesssim N_2$ ,  $\delta, s > 0$  are sufficiently small, and summing over dyadic  $N, N_1, N_2 \geq 1$ , we obtain the desired inequality (4.10).

**Subcase 2.2:**  $N \gg N_2$ .

In this subcase, note that due to (4.14), we can assume that  $\langle (\tau + |n|^2) - (\tau_1 + |n_1|^2) + (\tau_2 + |n_2|^2) \rangle \lesssim N_2^{10}$ , since otherwise we can conclude by using similar steps as in Case 1. Similarly, we can assume that  $\langle \tau_1 \rangle \lesssim N_2^{10}$  and also  $\langle \tau_2 \rangle \lesssim N_2^{10}$ . Thus, we have  $||n|^2 - |n_1|^2 + |n_2|^2| \lesssim N_2^{10}$ , so that  $||n|^2 - |n_1|^2| \lesssim N_2^{10}$ .

We now perform an orthogonality argument. Note that we have  $N_1 \sim N \gg N_2$  in this subcase. We decompose the set  $\{|n| \sim N\}$  into balls of radius  $\sim N_2$  and denote the set of these balls as  $\mathcal{J}$ , and we decompose the set  $\{|n_1| \sim N_1\}$  into balls of radius  $\sim N_2$  and denote the set of these balls as  $\mathcal{J}_1$ . Note that for each fixed  $J \in \mathcal{J}$ , the product  $\mathbf{1}_J(n) \cdot \mathbf{1}_{J_1}(n_1)$  is non-zero for at most  $O(1)$  many  $J_1 \in \mathcal{J}_1$ , and we denote the set of these  $J_1$ 's as  $\mathcal{J}_1(J)$ . Thus, by the

Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , (4.14), and Minkowski's inequality, we have

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sup_{\substack{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1 \\ \text{dyadic}}} \sum_{N, N_1, N_2 \geq 1} \sum_{J \in \mathcal{J}} \sum_{J_1 \in \mathcal{J}_1(J)} N^s N_2^{30\delta} \\
&\times \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \neq 0}} \iint K(\tau, -|n|^2 + (\tau_1 + |n_1|^2) - (\tau_2 + |n_2|^2)) \langle \tau \rangle^{\frac{1}{2} - \delta} \right. \\
&\times \widehat{P_{J_1} v}(\tau_1 + |n_1|^2, n_1) \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \widehat{\chi}(\tau_2) \overline{\widehat{P_J w}}(\tau, n) d\tau d\tau_1 d\tau_2 \left. \right| \\
&\lesssim T^\delta \sup_{\substack{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1 \\ \text{dyadic}}} \sum_{N, N_1, N_2 \geq 1} \sum_{J \in \mathcal{J}} \sum_{J_1 \in \mathcal{J}_1(J)} N^s N_2^{30\delta} \|\widehat{P_J w}\|_{\ell_n^2 L_\tau^2} \\
&\times \left[ \int \langle \tau \rangle^{-1-2\delta} \left( \sum_{m \in \mathbb{Z}} \iint \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \langle \tau_1 \rangle^{-\frac{1}{2} - \delta} \widehat{\chi}(\tau_2) \right. \right. \\
&\times \left. \left. \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_2} \cdot \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2} + \delta} \widehat{P_{J_1} v}(\tau_1 + |n_1|^2, n_1) \right\|_n d\tau_1 d\tau_2 \right)^2 d\tau \right]^{1/2},
\end{aligned} \tag{4.23}$$

where  $h_{nn_1n_2}^m$  is the base tensor as defined in (4.1) and  $S_2$  is a set defined by

$$S_2 \stackrel{\text{def}}{=} S_2(N_2) = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : |n|^2 - |n_1|^2 \lesssim N_2^{10}, |n_2| \sim N_2\}.$$

Note that for  $(n, n_1, n_2)$  restricted in  $S_2$ , we have  $\lesssim N_2^{10}$  choices for the value

$$m = |n|^2 - |n_1|^2 + |n_2|^2,$$

which implies that

$$\sum_{m \in \mathbb{Z}} \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \lesssim \log(1 + N_2^{10}) \lesssim N_2^\delta. \tag{4.24}$$

Thus, continuing with (4.23), by (4.24) and the Cauchy-Schwarz inequalities in  $\tau_1$ ,  $J$ , and  $N_1 \sim N$ , we obtain

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sup_{\substack{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1 \\ \text{dyadic}}} \sum_{N, N_1, N_2 \geq 1} \sum_{J \in \mathcal{J}} \sum_{J_1 \in \mathcal{J}_1(J)} N_2^{31\delta} \|\widehat{P_J w}\|_{\ell_n^2 L_\tau^2} \\
&\times \left\| \sum_{n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_2} \cdot \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_1} \|P_{J_1} v\|_{X^{s, \frac{1}{2} + \delta}} \\
&\lesssim T^\delta \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \left\| \sum_{n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_2} \cdot \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_1} \|v\|_{X^{s, \frac{1}{2} + \delta}}.
\end{aligned} \tag{4.25}$$

By Lemma 4.1.2, the Gaussian tail bound (4.16), and Lemma 4.1.3, we have

$$\begin{aligned}
&\left\| \sum_{n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_2} \cdot \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_1} \\
&\lesssim T^{-2\theta} N_2^{-1+2\delta+\alpha} \max \{ \|h_{nn_1n_2}^m \mathbf{1}_{S_2}\|_{nn_2 \rightarrow n_1}, \|h_{nn_1n_2}^m \mathbf{1}_{S_2}\|_{n \rightarrow n_1 n_2} \} \\
&\lesssim T^{-2\theta} N_2^{-\frac{1}{2}+2\delta+\alpha}
\end{aligned} \tag{4.26}$$

outside an exceptional set of probability  $\leq C \exp(-cN_2^\delta/T^\theta)$  for some universal constants  $C, c > 0$ . Thus, combining (4.25) and (4.26), using the fact that  $\alpha < \frac{1}{2}$  and  $\delta, s > 0$  are sufficiently small, and summing over dyadic  $N_2 \geq 1$ , we obtain the desired inequality (4.10).

(ii) This part follows similarly from part (i), so that we will be brief here. Using similar steps

as in Case 1 of part (i), we can assume that  $\langle \tau \rangle \lesssim N_1^{10}$ .

When  $N \lesssim N_1$ , we use the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , (4.14), and Minkowski's inequality to obtain

$$(4.13) \lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{31\delta} \left[ \int \langle \tau \rangle^{-1-2\delta} \left( \sum_{m \in \mathbb{Z}} \iint \langle \tau_2 \rangle^{-\frac{1}{2}-\delta} \widehat{\chi}(\tau_1) \right. \right. \\ \left. \left. \times \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_3} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2}+\delta} \widehat{P_{N_2} v}(\tau_2 + |n_2|^2, n_2) \right\|_n d\tau_1 d\tau_2 \right)^2 d\tau \right]^{1/2}, \quad (4.27)$$

where  $h_{nn_1n_2}^m$  is the base tensor as defined in (4.1) and  $S_3$  is a set defined by

$$S_3 \stackrel{\text{def}}{=} S_3(N, N_1, N_2) \\ = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n_2 \neq 0, |n| \sim N, |n_1| \sim N_1, |n_2| \sim N_2\}.$$

Then, by (4.27), the Cauchy-Schwarz inequality in  $\tau_2$ , Lemma 4.1.1, the Gaussian tail bound, and Lemma 4.1.3, we obtain

$$(4.13) \lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{31\delta} \left\| \sum_{n_1 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_3} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_2} \|P_{N_2} v\|_{X^{s, \frac{1}{2}+\delta}} \\ \lesssim T^{\delta-2\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-1+33\delta+\alpha} \max \{ \|h_{nn_1n_2}^m \mathbf{1}_{S_3}\|_{nn_1 \rightarrow n_2}, \|h_{nn_1n_2}^m \mathbf{1}_{S_3}\|_{n \rightarrow n_1 n_2} \} \\ \lesssim T^{\delta-2\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-\frac{1}{2}+33\delta+\alpha}$$

outside an exceptional set of probability  $\leq C \exp(-cN_1^\delta/T^\theta)$  for some universal constants  $C, c > 0$ . Thus, since  $\alpha < \frac{1}{2}$ ,  $N \lesssim N_1$ ,  $N_2 \lesssim N_1$ , and  $\delta, s > 0$  are sufficiently small, we can sum over dyadic  $N, N_1, N_2 \geq 1$  to obtain the desired inequality (4.11).

When  $N \gg N_1$ , as in Subcase 2.2 in part (i), we can assume that  $\langle (\tau + |n|^2) - (\tau_1 + |n_1|^2) + (\tau_2 + |n_2|^2) \rangle \lesssim N_1^{10}$ ,  $\langle \tau_1 \rangle \lesssim N_1^{10}$ , and  $\langle \tau_2 \rangle \lesssim N_1^{10}$ , so that  $|n|^2 + |n_2|^2 \lesssim N_1^{10}$ . We perform an orthogonality argument as in Subcase 2.2 in part (i) to decompose  $\{|n| \sim N\}$  into a set of balls (denoted as  $\mathcal{J}$ ) of radius  $\sim N_1$  and decompose  $\{|n_2| \sim N_2\}$  into a set of balls (denoted as  $\mathcal{J}_2$ ) of radius  $\sim N_1$ . For each  $J \in \mathcal{J}$ ,  $\mathbf{1}_J(n) \cdot \mathbf{1}_{J_2}(n_2)$  is non-zero for at most  $O(1)$  many  $J_2 \in \mathcal{J}_2$ , and we denote the set of these  $J_2$ 's as  $\mathcal{J}_2(J)$ . By the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , (4.14), Minkowski's inequality, and the Cauchy-Schwarz inequalities in  $\tau_2$ ,  $J$ , and  $N_2 \sim N$ , we have

$$(4.13) \lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_\tau^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} \sum_{J \in \mathcal{J}} \sum_{J_2 \in \mathcal{J}_2(J)} N^s N_1^{30\delta} \|\widehat{P_J w}\|_{\ell_n^2 L_\tau^2} \\ \times \left[ \int \langle \tau \rangle^{-1-2\delta} \left( \sum_{m \in \mathbb{Z}} \iint \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \langle \tau_1 \rangle^{-\frac{1}{2}-\delta} \widehat{\chi}(\tau_1) \right. \right. \\ \left. \left. \times \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_4} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2}+\delta} \widehat{P_{J_2} v}(\tau_2 + |n_2|^2, n_2) \right\|_n d\tau_1 d\tau_2 \right)^2 d\tau \right]^{1/2} \quad (4.28) \\ \lesssim T^\delta \sum_{\substack{N_1 \geq 1 \\ \text{dyadic}}} \left\| \sum_{n_1 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_4} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_2} \|v\|_{X^{s, \frac{1}{2}+\delta}},$$

where  $h_{nn_1n_2}^m$  is the base tensor as defined in (4.1) and  $S_4$  is a set defined by

$$S_4 \stackrel{\text{def}}{=} S_4(N_1) = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n_2 \neq 0, |n|^2 + |n_2|^2 \lesssim N_1^{10}, |n_1| \sim N_1\}.$$

By Lemma 4.1.2, the Gaussian tail bound, and Lemma 4.1.3, we have

$$\begin{aligned}
& \left\| \sum_{n_1 \in \mathbb{Z}^2} h_{nm_1 n_2}^m \mathbf{1}_{S_4} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \right\|_{n \rightarrow n_2} \\
& \lesssim T^{-2\theta} N_1^{-1+2\delta+\alpha} \max \left\{ \|h_{nm_1 n_2}^m \mathbf{1}_{S_4}\|_{n n_1 \rightarrow n_2}, \|h_{nm_1 n_2}^m \mathbf{1}_{S_4}\|_{n \rightarrow n_1 n_2} \right\} \\
& \lesssim T^{-2\theta} N_1^{-\frac{1}{2}+2\delta+\alpha}
\end{aligned} \tag{4.29}$$

outside an exceptional set of probability  $\leq C \exp(-cN_1^\delta/T^\theta)$  for some universal constants  $C, c > 0$ . Thus, since  $\alpha < \frac{1}{2}$  and  $\delta, s > 0$  are sufficiently small, we can combine (4.28) and (4.29) and sum over dyadic  $n_2 \geq 1$  to obtain the desired inequality (4.11).

(iii) We consider the following two cases.

**Case 1:**  $\langle \tau \rangle \gg N_1^{10} N_2^{10}$ .

In this case, by (4.14) and Lemma A.5.4, we have

$$\begin{aligned}
(4.13) & \lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_n^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-5+20\delta} N_2^{-5+20\delta} N_1^2 N_2^2 N_1^{-1+\alpha} N_2^{-1+\alpha} \\
& \quad \times \sup_{\substack{n_1 \in \mathbb{Z}^2 \\ \langle n_1 \rangle \sim N_1}} \sup_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} |g_{n_1}(\omega)| |g_{n_2}(\omega)| \int \frac{|\widehat{P_N w}(\tau, n_1 - n_2)|}{\langle \tau + |n_1 - n_2|^2 - |n_1|^2 + |n_2|^2 \rangle} d\tau \\
& \lesssim \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-4+20\delta+\alpha} N_2^{-4+20\delta+\alpha} \sup_{\substack{n_1 \in \mathbb{Z}^2 \\ \langle n_1 \rangle \sim N_1}} \sup_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} |g_{n_1}(\omega)| |g_{n_2}(\omega)|.
\end{aligned}$$

By using the following Gaussian tail bounds:

$$\sum_{\substack{n_1 \in \mathbb{Z}^2 \\ \langle n_1 \rangle \sim N_1}} P(|g_{n_1}| > T^{-\theta} N_1^\delta) < C \exp\left(-c \frac{N_1^\delta}{T^\theta}\right), \tag{4.30}$$

$$\sum_{\substack{n_2 \in \mathbb{Z}^2 \\ \langle n_2 \rangle \sim N_2}} P(|g_{n_2}| > T^{-\theta} N_2^\delta) < C \exp\left(-c \frac{N_2^\delta}{T^\theta}\right), \tag{4.31}$$

we obtain

$$(4.13) \lesssim T^{\delta-\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-4+21\delta+\alpha} N_2^{-4+21\delta+\alpha}.$$

outside an exceptional set of probability  $\leq C \exp(-cN_1^\delta/T^\theta) + C \exp(-cN_2^\delta/T^\theta)$ . Note that we have either  $N \lesssim N_1$  or  $N \lesssim N_2$ . Thus, since  $\alpha < \frac{1}{2}$  and  $\delta, s > 0$  are sufficiently small, we can sum over dyadic  $N, N_1, N_2 \geq 1$  to obtain (4.12).

**Case 2:**  $\langle \tau \rangle \lesssim N_1^{10} N_2^{10}$ .

In this case, by the Cauchy-Schwarz inequalities in  $\tau$  and  $n$ , (4.14), and Minkowski's in-

equality, we have

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sup_{\|\widehat{w}\|_{\ell_n^2 L_n^2} \leq 1} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{30\delta} N_2^{30\delta} \\
&\times \left| \sum_{\substack{n, n_1, n_2 \in \mathbb{Z}^2 \\ n_1 - n_2 = n \neq 0}} \iiint K(\tau, -|n|^2 + (\tau_1 + |n_1|^2) - (\tau_2 + |n_2|^2)) \langle \tau \rangle^{\frac{1}{2} - \delta} \right. \\
&\times \left. \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \widehat{\chi}(\tau_1) \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \widehat{\chi}(\tau_2) \overline{P_N w}(\tau, n) d\tau d\tau_1 d\tau_2 \right| \\
&\lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{30\delta} N_2^{30\delta} \left[ \int \langle \tau \rangle^{-1-2\delta} \right. \\
&\times \left( \sum_{m \in \mathbb{Z}} \iint \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \widehat{\chi}(\tau_1) \widehat{\chi}(\tau_2) \right. \\
&\times \left. \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_1} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \Big|_n d\tau_1 d\tau_2 \right\|^2 d\tau \right]^{1/2}, \tag{4.32}
\end{aligned}$$

where  $h_{nn_1n_2}^m$  is the base tensor as defined in (4.1) and  $S_1$  is as defined in (4.19). Note that for  $(n, n_1, n_2)$  restricted in  $S_1$ , we have  $\lesssim \max\{N_1^2, N_2^2\}$  choices for the value

$$m = |n|^2 - |n_1|^2 + |n_2|^2,$$

which implies that

$$\sum_{m \in \mathbb{Z}} \langle \tau - \tau_1 + \tau_2 + m \rangle^{-1} \lesssim \log(1 + N_1^2 N_2^2) \lesssim N_1^\delta N_2^\delta. \tag{4.33}$$

Thus, continuing with (4.32), by (4.33), Lemma 4.1.1, the Gaussian tail bounds (4.30) and (4.31), and Lemma 4.1.3, we have

$$\begin{aligned}
(4.13) &\lesssim T^\delta \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{31\delta} N_2^{31\delta} \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{nn_1n_2}^m \mathbf{1}_{S_1} \cdot \frac{g_{n_1}(\omega)}{\langle n_1 \rangle^{1-\alpha}} \frac{\overline{g_{n_2}(\omega)}}{\langle n_2 \rangle^{1-\alpha}} \Big|_n \right\| \\
&\lesssim T^{\delta-2\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-1+33\delta+\alpha} N_2^{-1+33\delta+\alpha} \|h_{nn_1n_2}^m \mathbf{1}_{S_1}\|_{nn_1n_2} \\
&\lesssim T^{\delta-2\theta} \sum_{\substack{N, N_1, N_2 \geq 1 \\ \text{dyadic}}} N^s N_1^{-\frac{1}{2}+34\delta+\alpha} N_2^{-\frac{1}{2}+34\delta+\alpha}
\end{aligned}$$

outside an exceptional set of probability  $\leq C \exp(-cN_1^\delta N_2^\delta / T^\theta)$  for some universal constants  $C, c > 0$ . Note that we have either  $N \lesssim N_1$  or  $N \lesssim N_2$ . Thus, since  $\alpha < \frac{1}{2}$  and  $\delta, s > 0$  are sufficiently small, we can sum over dyadic  $N, N_1, N_2 \geq 1$  to obtain the desired inequality (4.12).  $\square$

**Remark 4.2.3.** The frequency projectors  $P_{\neq 0}$  in all bilinear estimates in Proposition 4.2.2 are necessary in our approach. For (4.10) and (4.11), we need to avoid the zeroth frequencies in view of the condition  $n \neq 0$  in the base tensor (4.1); see also Remark 4.1.4.

Moreover, for (4.11), the assumption that that  $v$  has mean zero is important for the desired bilinear estimate to hold. If we do not have this assumption, we can let  $v$  be a non-zero constant, so that the left-hand-side of (4.11) essentially becomes  $\|\mathcal{I}(\chi \cdot z)\|_{X_T^{s, \frac{1}{2}+\delta}}$ , which is equal to infinity almost surely for  $\alpha \geq 0$  and  $s > 0$ .

### 4.3 Almost sure local well-posedness

In this section, we prove Theorem 1.3.1, the almost sure local well-posedness result of the quadratic NLS (1.16). We fix  $0 \leq \alpha < \frac{1}{2}$  throughout this section.

We recall from (1.19) the following first order expansion:

$$u = z + v.$$

Here,  $z$  is the random linear solution as in (1.18) and  $v$  is the remainder term that satisfies (1.20), which we can write in the following Duhamel formulation:

$$v(t) = \Gamma[v](t) \stackrel{\text{def}}{=} -i\mathcal{I}\left(|z + v|^2 - \int |z + v|^2\right)(t), \quad (4.34)$$

where  $0 < t \leq 1$  and  $\mathcal{I}$  is the Duhamel operator as defined in (A.28). We note from (4.34) that  $v$  has mean zero (i.e. has no zeroth frequency term). Our goal is to show that  $\Gamma$  is a contraction map on a ball of the space  $X_T^{s,b} \subset C([-T, T]; H^s(\mathbb{T}^2))$  for some  $s > 0$  and  $b > \frac{1}{2}$  outside an exceptional set of exponentially small probability.

Let  $s, \delta > 0$  be sufficiently small. By the definition of  $X_T^{s,b}$ -norm, (4.34), Proposition 4.2.1, and Proposition 4.2.2, we have that for every  $0 < T \leq 1$ ,

$$\begin{aligned} \|\Gamma[v]\|_{X_T^{s, \frac{1}{2} + \delta}} &\leq \|P_{\neq 0}(\mathcal{I}(|\chi \cdot z + v|^2))\|_{X_T^{s, \frac{1}{2} + \delta}} \\ &\leq \|\mathcal{I}(|v|^2)\|_{X_T^{s, \frac{1}{2} + \delta}} + \|P_{\neq 0}(\mathcal{I}(v \cdot \overline{\chi \cdot z}))\|_{X_T^{s, \frac{1}{2} + \delta}} \\ &\quad + \|P_{\neq 0}(\mathcal{I}(\chi \cdot z \cdot \overline{v}))\|_{X_T^{s, \frac{1}{2} + \delta}} + \|P_{\neq 0}(\mathcal{I}(|\chi \cdot z|^2))\|_{X_T^{s, \frac{1}{2} + \delta}} \\ &\lesssim T^{\delta - \theta} \left( \|v\|_{X_T^{s, \frac{1}{2} + \delta}}^2 + \|v\|_{X_T^{s, \frac{1}{2} + \delta}} + 1 \right), \end{aligned}$$

outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$  with  $C, c > 0$  being constants and  $0 < \theta \ll \delta$ . Similarly, we obtain the following difference estimate outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$ :

$$\|\Gamma[v_1] - \Gamma[v_2]\|_{X_T^{s, \frac{1}{2} + \delta}} \lesssim T^{\frac{\delta}{2}} \|v_1 - v_2\|_{X_T^{s, \frac{1}{2} + \delta}} \left( \|v_1\|_{X_T^{s, \frac{1}{2} + \delta}} + \|v_2\|_{X_T^{s, \frac{1}{2} + \delta}} + 1 \right).$$

Therefore, for a fixed  $R > 0$ , by choosing  $T = T(R) > 0$  sufficiently small, we obtain that  $\Gamma$  is a contraction on the ball  $B_R \subset X_T^{s, \frac{1}{2} + \delta}$  of radius  $R$  outside an exceptional set of probability  $\leq C \exp(-\frac{c}{T^\theta})$ . This finishes the proof of Theorem 1.3.1.

### 4.4 Probabilistic ill-posedness

In this section, we prove Theorem 1.3.5, the non-convergence of the Picard second iterate  $z_N^{(2)}$  as defined in (1.25).

We fix  $n \neq 0$ ,  $t \neq 0$ , and  $\alpha \geq \frac{3}{4}$ . Let us first show that  $\lim_{N \rightarrow \infty} \mathbb{E}[|\mathcal{F}_x z_N^{(2)}(t, n)|^2] = \infty$ . A direct computation yields

$$\begin{aligned} \mathcal{F}_x z_N^{(2)}(t, n) &= \int_0^t e^{i(t-t')|n|^2} \sum_{\substack{k \in \mathbb{Z}^2 \\ 0 < |k| \leq N \\ 0 < |n+k| \leq N}} e^{it'|n+k|^2 - it'|k|^2} \frac{g_{n+k}(\omega) \overline{g_k(\omega)}}{\langle n+k \rangle^{1-\alpha} \langle k \rangle^{1-\alpha}} dt' \\ &= \sum_{\substack{k \in \mathbb{Z}^2 \\ 0 < |k| \leq N \\ 0 < |n+k| \leq N}} \frac{g_{n+k}(\omega) \overline{g_k(\omega)}}{\langle n+k \rangle^{1-\alpha} \langle k \rangle^{1-\alpha}} e^{it|n|^2} \frac{e^{2itn \cdot k} - 1}{2in \cdot k}. \end{aligned} \quad (4.35)$$

By independence, we can compute that

$$\mathbb{E}[|\mathcal{F}_x z_N^{(2)}(t, n)|^2] = \sum_{\substack{k \in \mathbb{Z}^2 \\ 0 < |k| \leq N \\ 0 < |n+k| \leq N}} \frac{1}{\langle n+k \rangle^{2-2\alpha} \langle k \rangle^{2-2\alpha}} \frac{2 \sin(tn \cdot k)^2}{|n \cdot k|^2}. \quad (4.36)$$

We focus on the case when  $n \cdot k = 0$ , so that (4.36) is bounded from below (up to some constant depending only on  $n$  and  $t$ ) by

$$\sum_{\substack{k \in \mathbb{Z}^2 \\ n \cdot k = 0 \\ 0 < |k| \leq N}} \frac{1}{\langle k \rangle^{4-4\alpha}}. \quad (4.37)$$

We write  $n = (n^1, n^2)$ . Note that if either  $n^1 = 0$  or  $n^2 = 0$ , then we can easily see that (4.37) diverges as  $N \rightarrow \infty$  when  $\alpha \geq \frac{3}{4}$ . If  $n^1 \neq 0$  and  $n^2 \neq 0$ , we note that all  $k$ 's that satisfy  $n \cdot k = 0$  are of the form  $k = ak'$ , where  $a \in \mathbb{Z}$  and

$$k' = \left( -\frac{n^2}{\gcd(n^1, n^2)}, \frac{n^1}{\gcd(n^1, n^2)} \right).$$

Thus, (4.37) is bounded from below by

$$\sum_{\substack{a \in \mathbb{Z} \\ 0 < |a| \leq N/|k'|}} \frac{1}{|a|^{4-4\alpha} \langle k' \rangle^{4-4\alpha}},$$

which increases to infinity as  $N \rightarrow \infty$  when  $\alpha \geq \frac{3}{4}$ . This shows that

$$\mathbb{E}[|\mathcal{F}_x z_N^{(2)}(t, n)|^2] \rightarrow \infty \quad (4.38)$$

as  $N \rightarrow \infty$ .

We now show that for any sequence  $\{N_\ell\}_{\ell \in \mathbb{N}}$ , the sequence of random variables  $\{\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)\}_{\ell \in \mathbb{N}}$  is not tight. Assume for the sake of contradiction that  $\{\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)\}_{\ell \in \mathbb{N}}$  is tight. Using the explicit formula of  $\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)$  in (4.35), we can write  $\mathcal{F}_x z_{N_\ell}^{(2)}(t, n) = X_\ell + iY_\ell$ , where  $X_\ell, Y_\ell \in \mathcal{H}_{\leq 2}$  are real-valued. Here, we recall that the space  $\mathcal{H}_{\leq 2}$  is as defined in (A.29). By Lemma A.6.1, we have

$$\begin{aligned} \mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^4]^{\frac{1}{4}} &\leq \mathbb{E}[|X_\ell|^4]^{\frac{1}{4}} + \mathbb{E}[|Y_\ell|^4]^{\frac{1}{4}} \\ &\leq 3\mathbb{E}[|X_\ell|^2]^{\frac{1}{2}} + 3\mathbb{E}[|Y_\ell|^2]^{\frac{1}{2}} \\ &\leq 3\sqrt{2}\mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^2]^{\frac{1}{2}}. \end{aligned} \quad (4.39)$$

By the Paley-Zygmund inequality and (4.39), we have

$$P\left(|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^2 > \frac{\mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^2]}{2}\right) \geq \frac{1}{4} \frac{(\mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^2])^2}{\mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^4]} \geq \frac{1}{1296}. \quad (4.40)$$

By tightness, we know that there exists a constant  $A > 0$  such that for all  $\ell \in \mathbb{N}$ ,

$$P(|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)| > A) < \frac{1}{1296}. \quad (4.41)$$

Due to (4.40) and (4.41), we must have  $\mathbb{E}[|\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)|^2] \leq 2A^2$  for all  $\ell \in \mathbb{N}$ , which is a contradiction to (4.38). Therefore, the sequence  $\{\mathcal{F}_x z_{N_\ell}^{(2)}(t, n)\}_{\ell \in \mathbb{N}}$  is not tight. This finishes the proof of Theorem 1.3.5.

**Remark 4.4.1.** In the proof above, although we only considered the case when  $n \cdot k = 0$ , we point out that the range  $\alpha \geq \frac{3}{4}$  for the divergence of  $\mathbb{E}[|\mathcal{F}_x z_N^{(2)}(t, n)|^2]$  is sharp. More precisely, suppose that we have  $\alpha < \frac{3}{4}$ . Note that the right-hand-side of (4.36) converges as  $N \rightarrow \infty$  if and only if the following integral converges:

$$\int_{\{x \in \mathbb{R}^2; |x| \leq N\}} \frac{1}{\langle x \rangle^{4-4\alpha}} \frac{\sin(tn \cdot x)^2}{|n \cdot x|^2} dx. \quad (4.42)$$

By using a change of variable, we note that the convergence of (4.42) is equivalent to the convergence of the following term:

$$\int_0^N \int_0^N \frac{1}{(1 + |y_1|^2 + |y_2|^2)^{2-2\alpha}} \frac{\sin(ty_1)^2}{|y_1|^2} dy_1 dy_2,$$

which can easily be seen to converge when  $\alpha < \frac{3}{4}$ .

# Chapter 5

## Global well-posedness of the dispersive Anderson model

In this chapter, we study global well-posedness of the dispersive Anderson model (1.27). Specifically, we prove Theorem 1.4.1 and Theorem 1.4.2, global well-posedness of the equation (1.32) for  $v_\varepsilon$  and global well-posedness of the equation (1.30) for  $v$ , respectively.

### 5.1 Bounds for stochastic terms

In this section, we prove some useful bounds for stochastic terms mentioned in Section 1.4. These estimates hold almost surely, and so improve some results from [57] and [29], where similar estimates were given in terms of moments.

#### 5.1.1 Estimates in classical spaces

Let us first show some estimates in classical spaces.

**Proposition 5.1.1.** *For  $0 < \delta < 1$ ,  $2 < r < \infty$  with  $\delta r > 2$ , and  $0 < \varepsilon < \frac{1}{2}$ , we have*

$$\|\nabla Y_\varepsilon\|_{L^r_{-\delta}}^2 + \| |\nabla Y_\varepsilon|^2 \|_{L^r_{-\delta}} \leq C(\omega) |\log \varepsilon| \quad (5.1)$$

for almost sure  $\omega \in \Omega$ . Moreover, for  $0 < \alpha < 1$ ,  $\delta > 0$ ,  $\beta \in \mathbb{R}$ ,  $0 < \varepsilon < 1$ , and  $\varphi \in C_c^\infty(\mathbb{R}^2)$ , there exists  $0 < \kappa < 1$  such that

$$\|Y_\varepsilon - Y\|_{C^\alpha_{-\delta}} + \|\varphi * \xi_\varepsilon - \varphi * \xi\|_{C^\beta_{-\delta}} \leq C(\omega) \varepsilon^\kappa \quad (5.2)$$

for almost sure  $\omega \in \Omega$ .

To prove Proposition 5.1.1, we shall use the following result; see [62, Proposition 3.1] and [116, Proposition 2.3].

**Lemma 5.1.2.** *Let  $(X, \|\cdot\|_X)$  be a separable Banach space and  $\{\eta_n\}_{n \in \mathbb{N}}$  be a sequence of  $X$ -valued random variables. Assume that there exists a sequence  $\{\sigma_n\}_{n \in \mathbb{N}}$  of real numbers such that for all  $f \in X^*$ ,*

$$\mathbb{E}(\langle \eta_n, f \rangle^2) \leq \sigma_n^2 \|f\|_{X^*}^2,$$

Then, we have

$$\mathbb{E}\left(\sup_{n \in \mathbb{N}} \|\eta_n\|_X\right) \leq \sup_{n \in \mathbb{N}} \mathbb{E}(\|\eta_n\|_X) + 3\rho(\sigma_n),$$

where

$$\rho(\sigma_n) = \inf \left\{ \delta > 0 : \sum_{n \in \mathbb{N}} \sigma_n^2 \delta^{-2} \exp(-2^{-1}(\delta \sigma_n^{-1})^2) \leq 1 \right\}.$$

**Remark 5.1.3.** As recalled in [116], if  $\sigma_n \leq \alpha^n$  then  $\rho(\sigma_n) \sim \sqrt{\log(1-\alpha)^{-1}}$ .

We also need the following result, which is an immediate consequence of [29, Lemma 2.5].

**Lemma 5.1.4.** *Let  $0 < \alpha < 1$  and  $\delta > 0$ . Then, we have*

$$\|Y\|_{C_{-\delta}^\alpha} + \|\xi\|_{C_{-\delta}^{\alpha-2}} \leq C(\omega) \quad (5.3)$$

for almost sure  $\omega \in \Omega$ .

*Proof of Proposition 5.1.1.* Let us first show that

$$\mathbb{E}(\|\nabla Y\|_{B_{r,\infty,-\delta}^0}) < \infty, \quad (5.4)$$

whose proof is a generalization of [116, Theorem 3.4]. Let  $N^2 K(N \cdot)$  be the kernel corresponding to the Littlewood-Paley projection  $\Delta_N$ . We know that  $\Delta_N \xi(x)$  is a Gaussian random variable for any fixed  $x \in \mathbb{R}^2$ , so that

$$\begin{aligned} \mathbb{E}(\|\Delta_N \xi\|_{L_{-\delta}^r}^r) &= \int_{\mathbb{R}^2} \mathbb{E}(|\Delta_N \xi(x)|^r) \langle x \rangle^{-\delta r} dx \\ &\leq C \int_{\mathbb{R}^2} \left( \mathbb{E}(|\Delta_N \xi(x)|^2) \right)^{\frac{r}{2}} \langle x \rangle^{-\delta r} dx \end{aligned}$$

for some constant  $C > 0$ . Moreover, we have

$$\mathbb{E}(|\Delta_N \xi(x)|^2) = N^4 \mathbb{E}[\langle \xi, K(N(x - \cdot)) \rangle^2] = N^4 \|K(N(x - \cdot))\|_{L^2}^2 = N^2 \|K\|_{L^2}^2$$

and hence

$$\mathbb{E}(\|\Delta_N \xi\|_{L_{-\delta}^r}) \leq CN \quad (5.5)$$

provided that  $\delta r > 2$ . Also, for any  $f \in L_{\delta}^{r'}$  with  $\frac{1}{r} + \frac{1}{r'} = 1$ , we have

$$\begin{aligned} \mathbb{E}(\langle \Delta_N \xi, f \rangle^2) &= \mathbb{E}(\langle \xi, \Delta_N f \rangle^2) \\ &= N^4 \|K(N \cdot) * f\|_{L^2}^2 \\ &\leq N^4 \|K(N \cdot)\|_{L^{\frac{2r}{r+2}}}^2 \|f\|_{L^{r'}}^2 \\ &\leq CN^{\frac{2(r-2)}{r}} \|f\|_{L_{\delta}^{r'}}^2. \end{aligned} \quad (5.6)$$

By using the fact that  $L_{\delta}^{r'} = (L_{-\delta}^r)^*$ , (5.5), and (5.6), we can apply Lemma 5.1.2, where we take  $X = L_{-\delta}^r$ ,  $\eta_N = N^{-1} \Delta_N \xi$ ,  $\sigma_N = N^{-\frac{1}{r}}$ . Hence, we obtain

$$\mathbb{E}(\|\xi\|_{B_{r,\infty,-\delta}^{-1}}) = \mathbb{E} \left( \sup_{\substack{N \geq 1 \\ \text{dyadic}}} N^{-1} \|\Delta_N \xi\|_{L_{-\delta}^r} \right) \leq C + C\rho(N^{-\frac{2}{r}}) \lesssim C. \quad (5.7)$$

where we used Remark 5.1.3 in the last inequality. The estimate (5.4) then follows from (5.7).

Next, we claim that

$$\|\rho_\varepsilon * \nabla Y\|_{B_{r,2,-\delta}^0} \lesssim \|\rho_\varepsilon\|_{B_{1,2,\delta}^0} \|\nabla Y\|_{B_{r,\infty,-\delta}^0}. \quad (5.8)$$

Once this estimate is established, by combining (5.4) with Lemma A.3.6, we can obtain

$$\|\nabla Y_\varepsilon\|_{L_{-\delta}^r}^2 \leq C(\omega) |\log \varepsilon|$$

and also

$$\| :|\nabla Y_\varepsilon|^2 : \|_{L^r_{-\delta}}^2 \leq C(\omega) |\log \varepsilon|$$

since  $:|\nabla Y_\varepsilon|^2 := \nabla Y_\varepsilon^2 - c_\varepsilon$  with  $c_\varepsilon \sim |\log \varepsilon|$ . Hence, (5.1) follows from (5.8), which we now prove as follows. By using  $(\Delta_{\frac{N}{2}} + \Delta_N + \Delta_{2N})\Delta_N = \Delta_N$  for any dyadic  $N \geq 1$ , we have

$$\begin{aligned} \|\rho_\varepsilon * \nabla Y\|_{B^0_{r,2,-\delta}}^2 &= \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|\Delta_N(\rho_\varepsilon * \nabla Y)\|_{L^r_{-\delta}}^2 \\ &= \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \rho_\varepsilon * \Delta_N(\nabla Y)\|_{L^r_{-\delta}}^2 \\ &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \rho_\varepsilon\|_{L^1_\delta}^2 \|\Delta_N \nabla Y\|_{L^r_{-\delta}}^2 \\ &\lesssim \|\rho_\varepsilon\|_{B^0_{1,2,\delta}}^2 \|\nabla Y\|_{B^0_{r,\infty,-\delta}}^2, \end{aligned}$$

where we have used the inequality  $\langle x \rangle^{-1} \leq 2\langle y \rangle \langle x - y \rangle^{-1}$ . Thus, (5.8) follows.

We now consider (5.2). The bound

$$\|\varphi * \xi_\varepsilon - \varphi * \xi\|_{C^\beta_{-\delta}} \leq C(\omega) \varepsilon^\kappa$$

follows by combining Lemma A.3.8 with Lemma A.3.5 and (5.3). Also, by Lemma A.3.8 and (5.3), we obtain

$$\|Y_\varepsilon - Y\|_{C^\alpha_{-\delta}} \leq C(\omega) \varepsilon^\kappa,$$

which then gives (5.2).  $\square$

We now establish the following uniform bound and convergence result for  $e^{aY_\varepsilon}$  for any  $a \in \mathbb{R}$ .

**Proposition 5.1.5.** *For  $0 < \alpha < 1$ ,  $\delta > 0$ , and  $a \in \mathbb{R}$ , we have*

$$\sup_{\varepsilon \in (0,1)} \|e^{aY_\varepsilon}\|_{C^\alpha_{-\delta}} \leq C(\omega) \tag{5.9}$$

for almost sure  $\omega \in \Omega$ . Moreover, there exists  $0 < \kappa < 1$  such that

$$\|e^{aY_\varepsilon} - e^{aY}\|_{L^\infty_{-\delta}} \leq C(\omega) \varepsilon^\kappa \tag{5.10}$$

for almost sure  $\omega \in \Omega$ .

To prove Proposition 5.1.5, we shall use the following lemma (see [29, Corollary 2.6]).

**Lemma 5.1.6.** *For any  $0 < \alpha < 1$ ,  $a \in \mathbb{R}$ , and  $\delta > 0$ , we have*

$$\|e^{aY}\|_{C^\alpha_{-\delta}} \leq C(\omega)$$

for almost sure  $\omega \in \Omega$ .

We also need the following result (see [29, Lemma 2.3] and [1, Lemma 5.3]). Below we shall use the functions  $\chi_k \in C_c^\infty(\mathbb{R}^2)$  with  $k \in \mathbb{N}$ ,  $\text{supp } \chi_k \subseteq [-k-1, k+1]^2$ , and  $\chi_k = 1$  on  $[-k, k]^2$ .

**Lemma 5.1.7.** *For  $\alpha < 1$ , there exist  $\lambda, \lambda' > 0$  such that*

$$\sup_{k \in \mathbb{N}} \frac{\mathbb{E}(\exp(\lambda \|\chi_k \xi\|_{C^{\alpha-2}}^2))}{k^{\lambda'}} < \infty.$$

We are now ready to prove Proposition 5.1.5.

*Proof of Prop. 5.1.5.* We first establish the uniform bound (5.9), whose proof is similar to that of [29, Corollary 2.6]. For every  $k \in \mathbb{N}$ , we have

$$\begin{aligned} \|Y_\varepsilon\|_{C^\alpha([-k,k]^2)} &= \|\rho_\varepsilon * Y\|_{C^\alpha([-k,k]^2)} \\ &\leq \|Y\|_{C^\alpha([-k-1,k+1]^2)} \\ &\lesssim \|G * \chi_{k+2}\xi\|_{C^\alpha} \\ &\lesssim \|\chi_{k+2}\xi\|_{C^{\alpha-2}} \end{aligned} \quad (5.11)$$

uniformly in  $0 < \varepsilon < 1$ , where in the second inequality we used  $Y = G * \xi$  and the fact that  $G$  is supported on  $\{|x| < 1\}$ , and in the last inequality we used a Schauder estimate (see [106]). We also note that

$$\|e^{aY_\varepsilon}\|_{C_{-\delta}^\alpha} \lesssim \sup_{k \in \mathbb{N}} \frac{\|e^{aY_\varepsilon}\|_{C^\alpha([-k,k]^2)}}{k^\delta} \lesssim \sup_{k \in \mathbb{N}} \frac{\exp(C|a|\|Y_\varepsilon\|_{C^\alpha([-k,k]^2)})}{k^\delta}. \quad (5.12)$$

Thus, for  $p > 1$  large enough, by (5.12), (5.11), and Lemma 5.1.7, we get

$$\begin{aligned} \mathbb{E}\left(\sup_{\varepsilon \in (0,1)} \|e^{aY_\varepsilon}\|_{C_{-\delta}^\alpha}^p\right) &\lesssim \mathbb{E}\left(\left|\sup_{\varepsilon \in (0,1)} \sup_{k \in \mathbb{N}} \frac{\exp(C|a|\|Y_\varepsilon\|_{C^\alpha([-k,k]^2)})}{k^\delta}\right|^p\right) \\ &\lesssim \mathbb{E}\left(\left|\sup_{k \in \mathbb{N}} \frac{\exp(C|a|\|\chi_{k+2}\xi\|_{C^{\alpha-2}})}{k^\delta}\right|^p\right) \\ &\lesssim \sum_{k=1}^{\infty} \frac{\mathbb{E}\left(\exp(pC|a|\|\chi_k\xi\|_{C^{\alpha-2}})\right)}{k^{\delta p}} \\ &\lesssim \sum_{k=1}^{\infty} \frac{\mathbb{E}\left(\exp(\lambda\|\chi_k\xi\|_{C^{\alpha-2}}^2)\right)}{k^{2+\lambda'}} \\ &< \infty, \end{aligned}$$

where in the last step we used  $\exp(pC|a|x) \leq C \exp(\lambda x^2)$  and we picked  $p$  large enough such that  $\delta p \geq 2 + \lambda'$ . The bound (5.9) then follows.

To prove (5.10), by (5.2), (5.9), and Lemma 5.1.6, we get

$$\|e^{aY_\varepsilon} - e^{aY}\|_{L_{-\delta}^\infty} \lesssim \|Y_\varepsilon - Y\|_{L_{-\frac{\delta}{2}}^\infty} \left(\|e^{aY_\varepsilon}\|_{L_{-\frac{\delta}{2}}^\infty} + \|e^{aY}\|_{L_{-\frac{\delta}{2}}^\infty}\right) \leq C(\omega)\varepsilon^\kappa,$$

which is the desired estimate.  $\square$

## 5.1.2 Estimates in spaces at negative regularity

In this subsection, we aim to prove the following convergence result in negative regularity.

**Proposition 5.1.8.** *For  $0 < \alpha < 1$ ,  $\delta > 0$ , and  $0 < \varepsilon < \frac{1}{2}$ , there exists  $0 < \kappa < 1$  such that*

$$\|\nabla Y_\varepsilon - \nabla Y\|_{C_{-\delta}^{\alpha-1}} + \|\cdot|\nabla Y_\varepsilon|^2 - \cdot|\nabla Y|^2\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)\varepsilon^\kappa \quad (5.13)$$

for almost sure  $\omega \in \Omega$ . Moreover, we have

$$\left\|\widetilde{|\nabla Y_\varepsilon|^2} - \widetilde{|\nabla Y|^2}\right\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)\varepsilon^\kappa \quad (5.14)$$

for almost sure  $\omega \in \Omega$ , where  $\widetilde{|\nabla Y_\varepsilon|^2}$  and  $\widetilde{|\nabla Y|^2}$  are defined in (1.33) and (1.31), respectively.

To prove Proposition 5.1.8, we need the following result which follows from [57] and [29, Lemma 2.7].

**Lemma 5.1.9.** For any  $0 < \alpha < 1$  and  $\delta > 0$ , we have the bound

$$\|Y\|_{C_{-\delta}^\alpha} + \|\nabla Y\|_{C_{-\delta}^{\alpha-1}} + \|:|\nabla Y|^2:\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)$$

for almost sure  $\omega \in \Omega$ .

We are now ready to prove Proposition 5.1.8.

*Proof of Proposition 5.1.8.* The estimate  $\|\nabla Y_\varepsilon - \nabla Y\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)\varepsilon^\kappa$  follows by combining Lemma A.3.8 with  $\|\nabla Y\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)$  (by Lemma 5.1.9). Also, the estimate (5.14) follows immediately from (5.13) and (5.2)

Hence, we focus on the proof of  $\|:|\nabla Y_\varepsilon|^2:-:|\nabla Y|^2:\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)\varepsilon^\kappa$ . The argument is a little more complicated since Wick products cannot be estimated pathwise. It is shown in [57] that there exists  $\kappa_0 > 0$  such that for all  $k \geq 1$  we have:

$$\mathbb{E}\left(\|:|\nabla Y_\varepsilon|^2:-:|\nabla Y|^2:\|_{C_{-\delta}^{\alpha-1}}^k\right) \lesssim \varepsilon^{k\kappa_0}, \quad (5.15)$$

By (1.34) and Young's convolution inequality, we have that for any  $0 < \varepsilon < \eta < \frac{1}{2}$  and  $1 < q < \infty$ ,

$$\begin{aligned} |c_\varepsilon - c_\eta| &\leq \|(\rho_\varepsilon - \rho_\eta) * \nabla G\|_{L^2} \left( \|\rho_\varepsilon * \nabla G\|_{L^2} + \|\rho_\eta * \nabla G\|_{L^2} \right) \\ &\lesssim |\log \varepsilon|^{\frac{1}{2}} \|\rho_\varepsilon - \rho_\eta\|_{L^q}, \end{aligned} \quad (5.16)$$

where we used  $\nabla G \in L^p$  for any  $p \in (1, 2)$  and the fact that  $\|\rho_\varepsilon * \nabla G\|_{L^2}^2 = c_\varepsilon \sim |\log \varepsilon|$  as  $\varepsilon \rightarrow 0$ . On the other hand, we have for any  $0 < \varepsilon < \eta < \frac{1}{2}$  and  $q \geq 1$ ,

$$\|\rho_\varepsilon - \rho_\eta\|_{L^q} \lesssim \varepsilon^{-3+2/q} |\varepsilon - \eta|. \quad (5.17)$$

Combining (5.16) and (5.17), we obtain

$$|c_\varepsilon - c_\eta| \lesssim \varepsilon^{-\gamma} |\varepsilon - \eta| \quad (5.18)$$

for some  $\gamma > 1$ . Moreover, by Lemma A.2.4 and the estimate  $\|\nabla G * f\|_{C_\mu^s} \leq C\|f\|_{C_\mu^{s-1}}$  for any  $s, \mu \in \mathbb{R}$  (see [57]), we obtain that for  $1 - \alpha < \beta < 1$  and  $p > \frac{2}{2-\alpha}$ ,

$$\begin{aligned} &\| |\nabla Y_\varepsilon|^2 - |\nabla Y_\eta|^2 \|_{C_{-\delta}^{\alpha-1}} \\ &\lesssim \|\nabla Y_\varepsilon - \nabla Y_\eta\|_{C_{-\frac{\delta}{2}}^{\alpha-1}} \left( \|\nabla Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^\beta} + \|\nabla Y_\eta\|_{C_{-\frac{\delta}{2}}^\beta} \right) \\ &\lesssim \|\xi_\varepsilon - \xi_\eta\|_{C_{-\frac{\delta}{2}}^{\alpha-2}} \left( \|Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^{\beta+1}} + \|Y_\eta\|_{C_{-\frac{\delta}{2}}^{\beta+1}} \right) \\ &\lesssim \|\xi_\varepsilon - \xi_\eta\|_{L_{-\frac{\delta}{2}}^p} \left( \|Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^{\beta+1}} + \|Y_\eta\|_{C_{-\frac{\delta}{2}}^{\beta+1}} \right), \end{aligned} \quad (5.19)$$

where we used  $L_{-\frac{\delta}{2}}^p \subset C_{-\frac{\delta}{2}}^{\alpha-2}$  for  $p > \frac{2}{2-\alpha}$  in the last inequality. In fact, this embedding comes from the following estimate by (A.1) and the Sobolev embedding:

$$\sup_{\substack{N \geq 1 \\ \text{dyadic}}} N^{\alpha-2} \|\Delta_N(\langle x \rangle^{-\frac{\delta}{2}} f)\|_{L^\infty} \lesssim \sup_{\substack{N \geq 1 \\ \text{dyadic}}} N^{\alpha-2} \|\Delta_N(\langle x \rangle^{-\frac{\delta}{2}} f)\|_{W^{2-\alpha,p}} \lesssim \|f\|_{L_{-\frac{\delta}{2}}^p}.$$

Then, by the Gaussian hypercontractivity, Minkowski's integral inequality, and (5.17), we have

that for  $k \geq p$ ,

$$\begin{aligned}
\mathbb{E} \left( \|\xi_\varepsilon - \xi_\eta\|_{L^p_{-\frac{\delta}{2}}}^k \right) &\leq \left( \int_{\mathbb{R}^2} \langle x \rangle^{-\frac{p\delta}{2}} \mathbb{E} (|\xi_\varepsilon(x) - \xi_\eta(x)|^k)^{\frac{p}{k}} dx \right)^{\frac{k}{p}} \\
&\lesssim \left( \int_{\mathbb{R}^2} \langle x \rangle^{-\frac{p\delta}{2}} \mathbb{E} (|\xi_\varepsilon(x) - \xi_\eta(x)|^2)^{\frac{p}{2}} dx \right)^{\frac{k}{p}} \\
&= \|\rho_\varepsilon - \rho_\eta\|_{L^2}^k \\
&\lesssim \varepsilon^{-2k} |\varepsilon - \eta|^k.
\end{aligned} \tag{5.20}$$

Then, by (5.19), (5.20), and Lemma A.3.7, the Cauchy-Schwarz inequality, and Lemma 5.1.9, we have that for any  $k$  large enough,

$$\mathbb{E} \left( \left\| |\nabla Y_\varepsilon|^2 - |\nabla Y_\eta|^2 \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k. \tag{5.21}$$

By (5.21), (5.18) and (1.34), we obtain

$$\mathbb{E} \left( \left\| |\nabla Y_\varepsilon|^2 : - : |\nabla Y_\eta|^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k. \tag{5.22}$$

On the other hand, by (5.15), we have

$$\mathbb{E} \left( \left\| \nabla Y_\varepsilon^2 : - : \nabla Y_\eta^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim \eta^{k\kappa_0}. \tag{5.23}$$

We now consider the following several cases.

**Case 1:**  $2\varepsilon < \eta$ . Then, we have  $\eta < 2|\varepsilon - \eta|$ , so that by (5.23), we obtain

$$\mathbb{E} \left( \left\| \nabla Y_\varepsilon^2 : - : \nabla Y_\eta^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim |\varepsilon - \eta|^{k\kappa_0} < |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

**Case 2:**  $\varepsilon < \eta \leq 2\varepsilon$  and  $\varepsilon < |\varepsilon - \eta|^{\frac{1}{\kappa_0+2+\beta+\zeta}}$ . Then, again by (5.23), we obtain

$$\mathbb{E} \left( \left\| \nabla Y_\varepsilon^2 : - : \nabla Y_\eta^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim \varepsilon^{k\kappa_0} \lesssim |\varepsilon - \eta|^{\frac{k\kappa_0}{\kappa_0+2+\beta+\zeta}}.$$

**Case 3:**  $\varepsilon < \eta \leq 2\varepsilon$  and  $\varepsilon \geq |\varepsilon - \eta|^{\frac{1}{\kappa_0+2+\beta+\zeta}}$ . In this case, we use (5.22) to obtain

$$\mathbb{E} \left( \left\| \nabla Y_\varepsilon^2 : - : \nabla Y_\eta^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \lesssim \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k \lesssim |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

Summarizing the above cases, we obtain that for any  $0 < \varepsilon < \eta < \frac{1}{2}$ ,

$$\mathbb{E} \left( \left\| \nabla Y_\varepsilon^2 : - : \nabla Y_\eta^2 : \right\|_{\mathcal{C}_{-\delta}^{\alpha-1}}^k \right) \leq C |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

It remains to choose  $k$  large enough so that  $\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta} > 1$  and we can invoke the Kolmogorov continuity criterion (see [27, Theorem 3.3]) to deduce that  $\varepsilon \mapsto |\nabla Y_\varepsilon|^2 : - :$  from  $[0, 1]$  to  $\mathcal{C}_{-\delta}^{\alpha-1}$  is almost surely Hölder continuous of exponent less than  $\frac{\kappa_0}{2+\kappa_0} - \frac{1}{k}$  on  $[0, 1]$ . This finishes the proof of Proposition 5.1.8.  $\square$

## 5.2 Linear estimates

In this section, we study an abstract linear equation and prove some useful linear estimates.

We introduce the propagator  $S_{A,V}(t)$  of the equation

$$i\partial_t w = \Delta w - 2\nabla A \cdot \nabla w + Vw. \tag{5.24}$$

We also denote for simplicity

$$H_{A,V} = \Delta - 2\nabla A \cdot \nabla + V. \quad (5.25)$$

In the following, we assume that for all  $\delta > 0$ , there exists  $C > 0$  such that for any  $\varphi \in H_{\delta}^{\frac{1}{2}}$ ,

$$\|\langle x \rangle^{-\delta} e^{\pm A}\|_{L^{\infty}} \leq C \quad (5.26)$$

and

$$\left| \int_{\mathbb{R}^2} V |\varphi|^2 e^{-2A} dx \right| \leq C \|\varphi\|_{H_{\delta}^{\frac{1}{2}}}^2. \quad (5.27)$$

By using the assumption (5.26) and (A.1), we have for any  $\delta > 0$ ,

$$\|u\|_{H_{-\delta}^2} \lesssim \|\Delta u e^{-A}\|_{L^2} + \|u e^{-A}\|_{L^2} \lesssim \|u\|_{H_{\delta}^2}. \quad (5.28)$$

One can easily check that any solution to (5.24) satisfies the following conservation laws:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |w(t)|^2 e^{-2A} dx = 0 \quad (5.29)$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left( |\nabla w(t)|^2 e^{-2A} - V |w(t)|^2 e^{-2A} \right) dx = 0. \quad (5.30)$$

We also define the following quantity for any  $\delta > 0$  and  $2 < r < \infty$ :

$$\begin{aligned} |(A, V)|_{\delta, r} &= \|\langle x \rangle^{-\delta} \nabla A e^{-A}\|_{L^r} + \|\langle x \rangle^{-2\delta} \nabla A e^{-2A} V\|_{L^{\frac{r}{2}}} + \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r} \\ &\quad + \|\langle x \rangle^{-\delta} e^{-(p+2)A}\|_{L^{\infty}} + \|\langle x \rangle^{-\delta} V e^{-(p+2)A}\|_{L^r} \\ &\quad + \|\langle x \rangle^{-\delta} \nabla A e^{-(p+2)A}\|_{L^r} + \|\langle x \rangle^{-\delta} e^{-(p+1)A}\|_{L^{\infty}} \\ &\quad + \|\langle x \rangle^{-\delta} \nabla A e^{-(p+1)A}\|_{L^r} + \|\langle x \rangle^{-\delta} e^{-pA}\|_{L^{\infty}} \\ &\quad + \|\langle x \rangle^{-\delta} \nabla A e^{-pA}\|_{L^r}. \end{aligned} \quad (5.31)$$

### 5.2.1 Linear energy estimates

In this subsection, we first prove some weighted  $L^2$  estimates for the linear propagator  $S_{A,V}(t)$  associated with (5.24).

**Proposition 5.2.1.** *Let  $A$  and  $V$  satisfy (5.26) and (5.27).*

(i) *For any  $\delta > 0$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L_t^{\infty} L_{-\delta}^2} \lesssim \|\varphi\|_{L_{\delta}^2}. \quad (5.32)$$

(ii) *For any  $T > 0$  and  $0 < \delta < \delta^+$  satisfying  $\frac{\delta}{2} + 2\delta^+ < 1$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L_T^{\infty} L_{\delta}^2} \lesssim \|\varphi\|_{H_{\delta^+}^1}. \quad (5.33)$$

(iii) *For any  $T > 0$ ,  $0 < s < 1$ , and  $0 < \delta < \delta^+$  satisfying  $\delta + 9\delta^+ < 4s$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L_T^{\infty} L_{\delta}^2} \leq C \|\varphi\|_{H_{\frac{\delta^+}{s}}^{s+}}. \quad (5.34)$$

(iv) *For any  $T > 0$ ,  $2 < r < \infty$ , and  $0 < \delta < \delta^+$  satisfying  $\frac{\delta}{2} + 2\delta^+ < \frac{r-2}{3r+2}$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L_T^{\infty} H_{-\delta}^2} \leq \mathcal{P}(|(A, V)|_{\delta, r}) \|\varphi\|_{H_{(\frac{3r+2}{r-2})\delta^+}^2}, \quad (5.35)$$

where we recall from Section A.1 that  $\mathcal{P}(a_1, \dots, a_n)$  is a polynomial function depending on  $a_1, \dots, a_n$ .

*Proof.* For simplicity, we denote  $w(t) = S_{A,V}(t)\varphi$ .

(i) The conservation of mass (5.29) along with (5.26) imply (5.32).

(ii) To prove (5.33), we need to use both conservation laws (5.29) and (5.30). By integration in time, (5.27), interpolation (Lemma A.2.1), and (5.26), we have

$$\begin{aligned} & \|e^{-A}w(t)\|_{L^2}^2 + \|e^{-A}\nabla w(t)\|_{L^2}^2 \\ & \lesssim \|e^{-A}w(0)\|_{L^2}^2 + \|e^{-A}\nabla w(0)\|_{L^2}^2 + \|w(0)\|_{H^{\frac{1}{2}, \frac{\delta}{4}}}^2 + \|w(t)\|_{H^{\frac{1}{2}, \frac{\delta}{4}}}^2 \\ & \lesssim \|w(0)\|_{H^{\frac{1}{2}}}^2 + \|w(t)\|_{L^2_\delta} \|w(t)\|_{H^{-\frac{\delta}{2}}}. \end{aligned}$$

By using (5.26) again, we obtain the bound

$$\|w(t)\|_{H^{-\frac{\delta}{2}}} \lesssim \|w(0)\|_{H^{\frac{1}{2}_+}} + \|w(t)\|_{L^2_\delta}. \quad (5.36)$$

Next, by letting  $\bar{\delta} \in (\delta, \delta^+)$  and following the proof of [29, Lemma 3.1], we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \langle x \rangle^{2\bar{\delta}} |w(t)|^2 e^{-2A} \\ & = 2 \operatorname{Re} \int_{\mathbb{R}^2} \langle x \rangle^{2\bar{\delta}} \partial_t w(t) \bar{w}(t) e^{-2A} dx \\ & = 2 \operatorname{Im} \int_{\mathbb{R}^2} \langle x \rangle^{2\bar{\delta}} (\Delta w(t) - 2\nabla w(t) \cdot \nabla A) \bar{w}(t) e^{-2A} dx \\ & = -2 \operatorname{Im} \int_{\mathbb{R}^2} \nabla \langle x \rangle^{2\bar{\delta}} \cdot \nabla w(t) \bar{w}(t) e^{-2A} dx \\ & \lesssim \int_{\mathbb{R}^2} \langle x \rangle^{2\bar{\delta}-1} |\nabla w(t)| |w(t)| e^{-2A} dx. \end{aligned} \quad (5.37)$$

By integration in time, (5.26), and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \|w(t)e^{-A}\|_{L^2_\delta}^2 & \leq \|w(0)e^{-A}\|_{L^2_\delta}^2 + C \int_0^t \|\nabla w(t')\|_{L^2_{2\delta-1}} \|w(t')\|_{L^2_\delta} dt' \\ & \leq \|w(0)e^{-A}\|_{L^2_\delta}^2 + C \int_0^t \|\nabla w(t')\|_{L^2_{-\frac{\delta}{2}}}^2 dt' + C \int_0^t \|w(t')\|_{L^2_\delta}^2 dt' \end{aligned} \quad (5.38)$$

for some constant  $C > 0$ , where we used the condition  $\frac{\delta}{2} + 2\delta^+ < 1$  in order to guarantee  $\langle x \rangle^{2\bar{\delta}-1} \leq \langle x \rangle^{-\frac{\delta}{2}}$ . By inserting in (5.38) the estimate (5.36) and using (5.26) again, we deduce

$$\|w(t)\|_{L^2_\delta}^2 \leq C(1+T)\|w(0)\|_{H^{\frac{1}{2}_+}}^2 + C \int_0^t \|w(t')\|_{L^2_\delta}^2 dt',$$

and so we can conclude (5.33) using Gronwall's inequality.

(iii) To prove (5.34), we let  $\eta, \mu > 0$  be such that

$$s\eta - (1-s)\mu = \delta, \quad \eta = \frac{\delta}{s} + \frac{(\delta^+ - \delta)}{2s}. \quad (5.39)$$

Then, for any  $0 < t \leq T$ , we use interpolation (Lemma A.2.1) and (5.32) to obtain

$$\|S_{A,V}(t)\varphi\|_{L^2_\delta} \leq \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N \varphi\|_{L^2_\delta}$$

$$\begin{aligned}
&\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N \varphi\|_{L^2_{\eta}}^s \|S_{A,V}(t)\Delta_N \varphi\|_{L^2_{-\mu}}^{1-s} \\
&\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N \varphi\|_{L^2_{\eta}}^s \|\Delta_N \varphi\|_{L^2_{\frac{\delta}{s}}}^{1-s}.
\end{aligned}$$

In view of the conditions  $\delta + 9\delta^+ < 4s$  and  $\delta^+ > \delta$ , by (5.33), we continue the estimate above as

$$\begin{aligned}
\|S_{A,V}(t)\varphi\|_{L^2_{\delta}} &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|\Delta_N \varphi\|_{H^1_{\frac{\delta^+}{s}}}^s \|\Delta_N \varphi\|_{L^2_{\frac{\delta}{s}}}^{1-s} \\
&\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^s \|\Delta_N \varphi\|_{L^2_{\frac{\delta^+}{s}}}^s \|\Delta_N \varphi\|_{L^2_{\frac{\delta}{s}}}^{1-s} \lesssim \|\varphi\|_{H^s_{\frac{\delta^+}{s}}},
\end{aligned}$$

where we used Lemma A.3.1 and Lemma A.3.3.

(iv) We postpone the proof of (5.35) to Section 5.4 since the proof uses a special case of modified energies. However, we emphasize that the tools from modified energies that are needed to prove (5.35) follow from direct computations and elementary elements (such as Hölder's inequality). In particular, these tools do not rely on any estimates in Subsection 5.2.2 and Section 5.3.  $\square$

## 5.2.2 Linear Strichartz estimates

In this subsection, we show some linear Strichartz estimates for the linear propagator  $S_{A,V}(t)$  associated with (5.24). For this purpose, we will need the following norm for any  $\delta > 0$  and  $1 \leq k < \infty$ :

$$\|(A, V)\|_{\delta, k} = \|V\|_{L^k_{-\delta}} + \|\nabla A\|_{L^k_{-\delta}}. \quad (5.40)$$

We first prove some useful lemmas.

**Lemma 5.2.2.** *Let  $0 < s < 1$  and  $\delta > 0$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, we have*

$$\|(H_{A,V} - \Delta)u\|_{L^2} \lesssim \|(A, V)\|_{\delta, \frac{2}{s}} \|u\|_{H^s_{\delta^{1+s}}}.$$

*Proof.* For any  $0 < s < 1$  and  $\delta > 0$ , by using the embedding  $H^s_{\delta} \subset L^{\frac{2}{1-s}}_{\delta}$  (Lemma A.2.2), the commutator bound (A.2), and (A.1), we have

$$\|\nabla u \cdot \nabla A\|_{L^2} \leq \|\nabla A\|_{L^{\frac{2}{s}}_{-\delta}} \|\nabla u\|_{L^{\frac{2}{1-s}}_{\delta}} \lesssim \|\nabla A\|_{L^{\frac{2}{s}}_{-\delta}} \|u\|_{H^s_{\delta^{1+s}}},$$

Similarly, we have

$$\|Vu\|_{L^2} \lesssim \|V\|_{L^{\frac{2}{s}}_{-\delta}} \|u\|_{H^s_{\delta}},$$

which finishes the proof.  $\square$

**Lemma 5.2.3.** *Let  $T > 0$  and  $4 \leq r < \infty$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L^{\infty}_T H^{\frac{1+\delta}{2}(1-3\delta)}_{\frac{\delta}{2}}} \leq \mathcal{P}(|(A, V)|_{\delta, r}) \|\varphi\|_{H^{\frac{\sqrt{\delta}(1-\delta)}{4\sqrt{\delta}}}_{\frac{\sqrt{\delta}}{2}} + 1 + \delta^+}.$$

*Proof.* By Proposition 5.2.1 (iii) with  $s = \sqrt{\delta}$ , we know that there exists  $\bar{\delta} > 0$  such that for

$0 < \delta < \bar{\delta}$ , we have

$$\|S_{A,V}(t)\varphi\|_{L_T^\infty L_{2\delta}^2} \lesssim \|\varphi\|_{H_{4\sqrt{\delta}}^{\sqrt{2\delta}}}. \quad (5.41)$$

From Proposition 5.2.1 (iv) and the assumption  $r \geq 4$ , we have

$$\|S_{A,V}(t)\varphi\|_{L_T^\infty H_{-\delta}^2} \leq \mathcal{P}(|(A,V)|_{\delta,r}) \|\varphi\|_{H_{8\delta}^2}. \quad (5.42)$$

Next, we note that by interpolation (Lemma A.2.1), (5.41), and (5.42), we obtain

$$\begin{aligned} \|S_{A,V}(t)\varphi\|_{L_T^\infty H_{\frac{\delta}{2}(1-3\delta)}^{1+\delta}} &\leq \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty H_{\frac{\delta}{2}(1-3\delta)}^{1+\delta}} \\ &\leq \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty L_{2\delta}^2}^{\frac{1-\delta}{2}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty H_{-\delta}^2}^{\frac{1+\delta}{2}} \\ &\leq \mathcal{P}(|(A,V)|_{\delta,r}) \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|\Delta_N\varphi\|_{H_{4\sqrt{\delta}}^{\frac{1-\delta}{2}}} \|\Delta_N\varphi\|_{H_{8\delta}^2}^{\frac{1+\delta}{2}} \\ &\leq \mathcal{P}(|(A,V)|_{\delta,r}) \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{\frac{\sqrt{\delta}(1-\delta)}{\sqrt{2}}+1+\delta} \|\Delta_N\varphi\|_{L_{4\sqrt{\delta}}^2}. \end{aligned}$$

The desired estimate then follows from Lemma A.3.1.  $\square$

**Lemma 5.2.4.** *Let  $T > 0$  and  $4 \leq r < \infty$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , we have*

$$\|S_{A,V}(t)\varphi\|_{L_T^\infty H^{\frac{3}{2}}} \leq \mathcal{P}(|(A,V)|_{\delta,r}) \|\varphi\|_{H_{\sqrt{\delta+}}^{\frac{3}{2}+\frac{\sqrt{\delta}}{4}}}$$

*Proof.* By interpolation (Lemma A.2.1), Proposition 5.2.1 (iv), Proposition 5.2.1 (iii) with  $\delta > 0$  sufficiently small, we have

$$\begin{aligned} \|S_{A,V}(t)\varphi\|_{H^{\frac{3}{2}}} &\leq \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty H^{\frac{3}{2}}} \\ &\lesssim \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty H_{-\frac{\delta}{3}}^2}^{\frac{3}{4}} \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\infty L_{\delta}^2}^{\frac{1}{4}} \\ &\leq \mathcal{P}(|(A,V)|_{\delta,r}) \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|\Delta_N\varphi\|_{H_{8\delta}^2}^{\frac{3}{4}} \|\Delta_N\varphi\|_{H_{\sqrt{\delta+}}^{\frac{1}{4}}} \\ &\leq \mathcal{P}(|(A,V)|_{\delta,r}) \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{\frac{3}{2}+\frac{\sqrt{\delta+}}{4}} \|\Delta_N\varphi\|_{L_{\sqrt{\delta+}}^2}. \end{aligned}$$

The desired estimate then follows from Lemma A.3.1.  $\square$

We are now ready to prove the Strichartz estimates associated with  $S_{A,V}(t)$ .

**Proposition 5.2.5.** *Let  $T > 0$ ,  $\delta, s > 0$ , and  $4 \leq r < \infty$ . Let  $2 < \ell, q < \infty$  be such that  $\frac{1}{\ell} + \frac{1}{q} = \frac{1}{2}$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exist  $\tilde{\delta}, \delta_1, s_1 > 0$  with  $\frac{\delta_1}{s_1} > 1$  such that*

$$\|S_{A,V}(t)\varphi\|_{L_T^\ell L^q} \leq \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) \mathcal{P}(\|(A,V)\|_{\delta_1, \frac{2}{s_1}}) \|\varphi\|_{H_{\delta}^{\frac{1}{\ell}+s}}.$$

*Proof.* Note that by Lemma 5.2.3, for any  $\delta_2, s_2 > 0$  and  $4 \leq r < \infty$ , there exist  $\tilde{\delta}, \delta_1, s_1 > 0$

such that

$$\|S_{A,V}(t)\varphi\|_{L_T^\infty H_{\delta_1}^{1+s_1}} \leq \mathcal{P}(|(A,V)|_{\delta,r}) \|\varphi\|_{H_{\delta_2}^{1+\frac{s_2}{2}}}. \quad (5.43)$$

Moreover, by (A.3) we can assume that  $\frac{\delta_1}{s_1}$  is larger than 1. By Proposition 5.2.1 (iii), we also have the following bound for any given  $s_2, \delta_2 > 0$ :

$$\|S_{A,V}(t)\varphi\|_{L_T^\infty L^2} \lesssim \|\varphi\|_{H_{\delta_2}^{\frac{s_2}{2}}}. \quad (5.44)$$

Next, following [17, 80], we split the interval  $[-T, T]$  into an essentially disjoint union of intervals of size  $N^{-1}$ , and we denote

$$[-T, T] = \bigcup_j I_j. \quad (5.45)$$

We aim to estimate  $\|S_{A,V}(t)\Delta_N\varphi\|_{L_{I_j}^\ell L^q}$ . Suppose that  $I_j = [a, b]$ . Then, for  $t \in [a, b]$  we can write

$$\begin{aligned} S_{A,V}(t)\Delta_N\varphi &= e^{i(t-a)\Delta} S_{A,V}(a)\Delta_N\varphi \\ &\quad + i \int_a^t e^{i(t-t')\Delta} (H_{A,V} - \Delta) S_{A,V}(t')\Delta_N\varphi dt'. \end{aligned} \quad (5.46)$$

We now estimate each term on the right-hand-side of (5.46). By using the Strichartz estimates on  $\mathbb{R}^2$  (see [20, 45, 69]), (5.44), and Lemma A.3.4, we obtain

$$\begin{aligned} &\|e^{i(t-a)\Delta} S_{A,V}(a)\Delta_N\varphi\|_{L_{I_j}^\ell L^q} \\ &\lesssim \|S_{A,V}(a)\Delta_N\varphi\|_{L^2} \\ &\lesssim \|\Delta_N\varphi\|_{H_{\delta_2}^{\frac{s_2}{2}}} \\ &\lesssim N^{-\frac{1}{\ell} - \frac{s_2}{2}} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_N\varphi\|_{H_{\delta_2}^{\frac{1}{\ell} + s_2}} \\ &\lesssim N^{-\frac{1}{\ell} - \frac{s_2}{2}} \|\varphi\|_{H_{\delta_2}^{\frac{1}{\ell} + s_2}}. \end{aligned}$$

By using Minkowski's inequality, the Strichartz estimates on  $\mathbb{R}^2$ , Lemma 5.2.2, and (5.43) (with  $s_2$  small), and Lemma A.3.4, we obtain

$$\begin{aligned} &\left\| \int_a^t e^{i(t-t')\Delta} (H_{A,V} - \Delta) S_{A,V}(t')\Delta_N\varphi dt' \right\|_{L_{I_j}^\ell L^q} \\ &\lesssim \int_{I_j} \|(H_{A,V} - \Delta) S_{A,V}(t')\Delta_N\varphi\|_{L^2} dt' \\ &\lesssim \|(A, V)\|_{\delta_1, \frac{2}{s_1}} \mathcal{P}(|(A, V)|_{\delta,r}) N^{-1} \|\Delta_N\varphi\|_{H_{\delta_2}^{1+\frac{s_2}{2}}} \\ &\lesssim \|(A, V)\|_{\delta_1, \frac{2}{s_1}} \mathcal{P}(|(A, V)|_{\delta,r}) N^{-1} N^{1-\frac{1}{\ell} - \frac{s_2}{2}} \|\varphi\|_{H_{\delta_2}^{\frac{1}{\ell} + s_2}}. \end{aligned}$$

Summarizing the above two estimates, we obtain

$$\|S_{A,V}(t)\Delta_N\varphi\|_{L_{I_j}^\ell L^q} \lesssim \mathcal{P}(|(A, V)|_{\delta,r}) (1 + \|(A, V)\|_{\delta_1, \frac{2}{s_1}}) N^{-\frac{1}{\ell} - \frac{s_2}{2}} \|\varphi\|_{H_{\delta_2}^{\frac{1}{\ell} + s_2}}.$$

Note that the number of intervals  $I_j$  in (5.45) is  $O(TN)$ . Thus, by taking the  $\ell$ 'th power of the

previous bound and summing over  $j$ , we obtain

$$\|S_{A,V}(t)\Delta_N\varphi\|_{L_T^\ell L^q} \lesssim T^{\frac{1}{\ell}} \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) (1 + \|(A,V)\|_{\delta_1, \frac{2}{s_1}}) N^{-\frac{s_2}{2}} \|\varphi\|_{H_{\delta_2}^{\frac{1}{2}+s_2}}$$

and hence we obtain the desired estimate by summation over dyadic  $N \geq 1$ .  $\square$

In later sections, we will need the following Strichartz estimates.

**Proposition 5.2.6.** *Let  $T > 0$ ,  $\delta, s > 0$  be sufficiently small, and  $4 \leq r < \infty$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exist  $\tilde{\delta}, \delta_1, s_1 > 0$  with  $\frac{\delta_1}{s_1} > 1$  such that*

$$\|S_{A,V}(t)\varphi\|_{L_T^4 W^{\frac{3}{4}-s,4}} \leq \mathcal{P}(|(A,V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) \mathcal{P}(\|(A,V)\|_{\delta_1, \frac{2}{s_1}}) \|\varphi\|_{H_\delta^1} \quad (5.47)$$

and

$$\begin{aligned} & \left\| \int_0^t S_{A,V}(t-t')f(t')dt' \right\|_{L_T^4 W^{\frac{3}{4}-s,4}} \\ & \leq \mathcal{P}(|(A,V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) \mathcal{P}(\|(A,V)\|_{\delta_1, \frac{2}{s_1}}) \|f\|_{L_T^1 H_\delta^1}. \end{aligned} \quad (5.48)$$

*Proof.* Note that (5.48) follows from (5.47) and Minkowski's integral inequality. Thus, we focus on the proof of (5.47).

For any  $0 < s < \frac{1}{2}$ , there exists  $2 < q < \infty$  such that the following Gagliardo-Nirenberg inequality holds:

$$\|u\|_{W^{\frac{3}{4}-s,4}} \lesssim \|u\|_{L^q}^{\frac{1}{2}} \|u\|_{H^{\frac{3}{2}}}^{\frac{1}{2}}.$$

Hence, by integration in time, Hölder's inequality in time, and Lemma 5.2.4, we have

$$\begin{aligned} \|S_{A,V}(t)\varphi\|_{L_T^4 W^{\frac{3}{4}-s,4}}^4 & \lesssim \|S_{A,V}(t)\varphi\|_{L_T^2 L^q}^2 \|S_{A,V}(t)\varphi\|_{L_T^\infty H^{\frac{3}{2}}}^2 \\ & \lesssim \mathcal{P}(|(A,V)|_{\frac{\delta^2}{4},r}) \|S_{A,V}(t)\varphi\|_{L_T^\ell L^q}^2 \|\varphi\|_{H^{\frac{3}{2}+\frac{\delta}{8}+}}^2, \end{aligned}$$

where  $\ell > 2$  satisfies  $\frac{1}{\ell} + \frac{1}{q} = \frac{1}{2}$ . By using Proposition (5.2.5), we continue the estimate above and obtain

$$\|S_{A,V}(t)\varphi\|_{L_T^4 W^{\frac{3}{4}-s,4}}^4 \leq \mathcal{P}(|(A,V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) \mathcal{P}(\|(A,V)\|_{\delta_1, \frac{2}{s_1}}) \|\varphi\|_{H_\delta^{\frac{1}{2}+s}}^2 \|\varphi\|_{H^{\frac{3}{2}+\frac{\delta}{8}+}}^2,$$

where  $\tilde{\delta}, \delta_1, s_1 > 0$  depend on  $s, \delta$ . Replacing  $\varphi$  by  $\Delta_N\varphi$  for a dyadic number  $N \geq 1$  in the previous estimate, we obtain

$$\begin{aligned} & \|S_{A,V}(t)\Delta_N\varphi\|_{L_T^4 W^{\frac{3}{4}-s,4}} \\ & \leq \mathcal{P}(|(A,V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(|(A,V)|_{\tilde{\delta},r}) \mathcal{P}(\|(A,V)\|_{\delta_1, \frac{2}{s_1}}) \|\Delta_N\varphi\|_{H_\delta^{\frac{1}{2\ell}+\frac{s}{2}+\frac{3}{4}+\frac{\delta}{16}+}}. \end{aligned}$$

We conclude (5.47) by summing over dyadic  $N \geq 1$  and using Lemma A.3.1, as long as  $\frac{1}{2\ell} + \frac{s}{4} < 1$  for  $\ell > 2$  and  $s, \delta > 0$  sufficiently small.  $\square$

### 5.3 Nonlinear estimates

In this section, we focus on the following nonlinear equation

$$i\partial_t v = H_{A,V}v - \lambda e^{-(p-1)A}|v|^{p-1}v, \quad (5.49)$$

where  $\lambda > 0$  and  $H_{A,V}$  is defined in (5.25).

### 5.3.1 Nonlinear energy estimates

In this subsection, we prove some nonlinear energy estimates for any solution  $v$  to the equation (5.49) with  $v|_{t=0} = v_0$ . One can easily check that any solution to (5.49) satisfies the following conservation laws:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |v(t)|^2 e^{-2A} dx = 0 \quad (5.50)$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla v|^2 e^{-2A} - \frac{1}{2} |v(t)|^2 V e^{-2A} + \frac{\lambda}{p+1} |v(t)|^{p+1} e^{-(p+1)A} \right) dx = 0. \quad (5.51)$$

**Proposition 5.3.1.** *Let  $T > 0$  and let  $A$  and  $V$  satisfy (5.26) and (5.27). Let  $v$  be a solution to (5.49). Then, for any  $0 < \delta < \frac{1}{9}$ , we have*

$$\|v\|_{L_T^\infty L_\delta^2} \lesssim \left( 1 + \|v_0\|_{H_{4\delta}^{\frac{p+1}{2}}} \right) \quad (5.52)$$

and

$$\|v\|_{L_T^\infty H_{-\delta}^1} \lesssim \left( 1 + \|v_0\|_{H_{8\delta}^{\frac{p+1}{2}}} \right). \quad (5.53)$$

Moreover, for any  $2 \leq q < \infty$  and  $0 < \delta < \frac{1}{36(2q-1)}$ , we have

$$\|v\|_{L_T^\infty W_\delta^{1,q}} \lesssim \|v\|_{L_T^\infty H_{-\delta}^2}^{1-\frac{1}{q}} \mathcal{P}(\|v_0\|_{H_{4(2q-1)\delta}^1}). \quad (5.54)$$

*Proof.* By using conservations (5.50) and (5.51) and the fact that  $\lambda > 0$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx - \int_{\mathbb{R}^2} \frac{1}{2} |v(t)|^2 V e^{-2A} dx \\ & \leq \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v_0|^2 + |v_0|^2) e^{-2A} dx - \int_{\mathbb{R}^2} \frac{1}{2} |v_0|^2 V e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+1} |v_0|^{p+1} e^{-(p+1)A} dx. \end{aligned}$$

Then, by (5.27), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx \\ & \lesssim \|v(t)\|_{H_{\frac{\delta}{2}}^{\frac{1}{2}}}^2 + \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v_0|^2 + |v_0|^2) e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+1} |v_0|^{p+1} e^{-(p+1)A} dx. \end{aligned} \quad (5.55)$$

By (5.26), the Sobolev embedding  $H_\delta^1 \subset L_\delta^{p+1}$  (Lemma A.2.2), and interpolation (Lemma A.2.1), we obtain

$$\|v(t)\|_{H_{-\delta}^1}^2 \lesssim (1 + \|v_0\|_{H_\delta^1}^{p+1}) + \|v(t)\|_{H_{-\delta}^1} \|v(t)\|_{L_{2\delta}^2}. \quad (5.56)$$

Hence, by Cauchy's inequality, we obtain

$$\|v(t)\|_{H_{-\delta}^1}^2 \lesssim (1 + \|v_0\|_{H_\delta^1}^{p+2}) + \|v(t)\|_{L_{2\delta}^2}^2. \quad (5.57)$$

Next, by following the computation in [29, Lemma 3.1] (see also (5.37)), we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\langle x \rangle^\delta v(t)|^2 e^{-2A} dx \lesssim \int_{\mathbb{R}^2} \langle x \rangle^{2\delta-1} |\nabla v(t)| |v(t)| e^{-2A} dx.$$

Thus, by integration in time, (5.26), and the Cauchy-Schwartz inequality, we obtain

$$\|v(t)e^{-A}\|_{L_{\frac{\delta}{2}}^2}^2 \lesssim \|e^{-A}v_0\|_{L_{\frac{\delta}{2}}^2}^2 + \int_0^t \|\nabla v(t')\|_{L_{2\delta-1}^2} \|v(t')\|_{L_{\frac{\delta}{2}}^2} dt',$$

so that by using (5.26) again, we get

$$\|v(t)\|_{L_{\frac{\delta}{2}}^2}^2 \lesssim \|v_0\|_{L_{2\delta}^2}^2 + \int_0^t \|\nabla v(t')\|_{L_{-\frac{\delta}{4}}^2}^2 dt' + \int_0^t \|v(t')\|_{L_{\frac{\delta}{2}}^2}^2 dt',$$

where we used  $\langle x \rangle^{2\delta-1} \leq \langle x \rangle^{-\frac{\delta}{4}}$  given that  $0 < \delta < \frac{1}{5}$ . By combining this estimate with (5.57), we obtain

$$\|v(t)\|_{L_{\frac{\delta}{2}}^2}^2 \lesssim \|v_0\|_{L_{2\delta}^2}^2 + t \left(1 + \|v_0\|_{H_{\frac{\delta}{4}}^1}^{p+1}\right) + \int_0^t \|v(t')\|_{L_{\frac{\delta}{2}}^2}^2 dt'.$$

We deduce (5.52) by Gronwall's inequality. The estimate (5.53) follows from (5.52) and (5.57). The estimate (5.54) follows from Lemma A.2.3 and (5.52).  $\square$

**Remark 5.3.2.** We consider (5.49) with  $\lambda < 0$ . If  $1 < p < 3$ , (5.52) follows from [29, Corollary 3.3] and (5.53) follows from [29, Proposition 3.2]. If  $p \geq 3$ , we further assume that  $\|v_0\|_{H_{\delta_0}^1} \ll 1$  for some  $\delta_0 > 0$ . In this case, (5.55) needs to be replaced by

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx \\ & \lesssim \|v(t)\|_{H_{\frac{\delta}{2}}^{\frac{1}{2}}}^2 + \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v_0|^2 + |v_0|^2) e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+1} |v_0|^{p+1} e^{-(p+1)A} dx - \int_{\mathbb{R}^2} \frac{\lambda}{p+1} |v(t)|^{p+1} e^{-(p+1)A} dx. \end{aligned}$$

Thus, instead of (5.56), we obtain from (5.26), the Sobolev embedding  $H_{\mu}^{\frac{p-1}{p+1}} \subset L_{\mu}^{p+1}$  for any  $\mu \in \mathbb{R}$  (Lemma A.2.2), and interpolation (Lemma A.2.1) that

$$\begin{aligned} \|v(t)\|_{H_{-\delta}^1}^2 & \leq C \|v_0\|_{H_{\delta}^1}^2 + C \|v(t)\|_{H_{-\delta}^1} \|v(t)\|_{L_{2p\delta}^2} + C \|v(t)\|_{L_{\frac{3p\delta+\delta}{p+1}}^{p+1}}^{p+1} \\ & \leq C \|v_0\|_{H_{\delta}^1}^2 + \frac{1}{2} \|v(t)\|_{H_{-\delta}^1}^2 + C \|v(t)\|_{L_{2p\delta}^2}^2 + C \|v(t)\|_{H_{\frac{3p\delta+\delta}{p+1}}^{\frac{p-1}{p+1}}}^{p+1} \\ & \leq C \|v_0\|_{H_{\delta}^1}^2 + \frac{1}{2} \|v(t)\|_{H_{-\delta}^1}^2 + C \|v(t)\|_{L_{2p\delta}^2}^2 + C \|v(t)\|_{H_{-\delta}^1}^{p-1} \|v(t)\|_{L_{2p\delta}^2}^2, \end{aligned}$$

where  $C > 0$  is a constant and we need to take  $\delta > 0$  sufficiently small. Assume that  $\|v(t)\|_{H_{-\delta}^1} \leq 1$ , so that we have the bound

$$\|v(t)\|_{H_{-\delta}^1}^2 \leq C \|v_0\|_{H_{\delta}^1}^2 + C \|v(t)\|_{L_{2p\delta}^2}^2. \quad (5.58)$$

Then, by arguing as in the proof of Proposition 5.3.1 and using (5.58) instead of (5.57), we get

$$\|v(t)\|_{L_{2p\delta}^2}^2 \leq C(1+t) \|v_0\|_{H_{4p\delta}^1}^2 + C \int_0^t \|v(t')\|_{L_{2p\delta}^2}^2 dt'.$$

By applying the Gronwall inequality and taking  $\|v_0\|_{H_{4p\delta}^1}^2$  to be sufficiently small, we obtain

$\|v(t)\|_{L_{2p\delta}^2}^2 \leq \frac{1}{100C}$ , so that (5.58) gives  $\|v(t)\|_{H_{-\delta}^1} \leq \frac{1}{2}$ . Thus, by a standard continuity argument, we get the following two uniform bounds for  $\delta > 0$  sufficiently small:

$$\begin{aligned}\|v\|_{L_T^\infty L_\delta^2} &\leq 1, \\ \|v\|_{L_T^\infty H_{-\delta}^1} &\leq 1.\end{aligned}$$

The next proposition will also be useful.

**Proposition 5.3.3.** *Let  $t > 0$  and let  $A$  and  $V$  satisfy (5.26) and (5.27). Let  $v$  be a solution to (5.49). Then, for any  $0 < \bar{\eta} < 1$  and  $0 < \delta < \min\{\frac{1}{9}, \frac{\bar{\eta}}{9(6-4\bar{\eta})}\}$ , we have the bound*

$$\|v(t)\|_{H_\delta^{\frac{2}{2-\bar{\eta}}}} \lesssim \mathcal{P}\left(\|v_0\|_{H_{4\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^1}\right) \|v(t)\|_{H_{-\delta(\frac{1-2\bar{\eta}}{\bar{\eta}})}^2}^{\bar{\eta}}. \quad (5.59)$$

In particular, for any  $0 < \eta_0 < \frac{1}{3}$  and  $\delta_0 > 0$ , there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ ,

$$\|v(t)\|_{H_\delta^{\frac{2}{2-\eta_0}}} \lesssim \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0}. \quad (5.60)$$

*Proof.* By interpolation (Lemma A.2.1), for any  $\delta > 0$  and  $0 < s < 1$ , we obtain

$$\|f\|_{H_\delta^{1+s}} \lesssim \|f\|_{H_{2\delta}^{\frac{1-s}{1+s}}}^{\frac{1-s}{1+s}} \|f\|_{H_{-\delta(\frac{1-3s}{2s})}^2}^{\frac{2s}{1+s}}$$

and

$$\|f\|_{H_{2\delta}^{1-s}} \lesssim \|f\|_{H_{-\delta}^{1-s}} \|f\|_{L_{\delta(\frac{3-s}{s})}^2}^s.$$

Summarizing the above two estimates, we obtain

$$\|f\|_{H_\delta^{1+s}} \lesssim \|f\|_{H_{-\delta}^{\frac{(1-s)^2}{1+s}}}^{\frac{(1-s)^2}{1+s}} \|f\|_{L_{\delta(\frac{3-s}{s})}^2}^{\frac{s(1-s)}{1+s}} \|f\|_{H_{-\delta(\frac{1-3s}{2s})}^2}^{\frac{2s}{1+s}}.$$

Next, we select  $0 < s < 1$  such that  $\bar{\eta} = \frac{2s}{s+1}$ , namely  $s = \frac{\bar{\eta}}{2-\bar{\eta}}$ . By Proposition 5.3.1, we obtain

$$\|v(t)\|_{H_\delta^{\frac{2}{2-\bar{\eta}}}} \leq C\mathcal{P}(\|v_0\|_{H_{8\delta}^1}) \mathcal{P}\left(\|v_0\|_{H_{4\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^1}\right) \|v(t)\|_{H_{-\delta(\frac{1-2\bar{\eta}}{\bar{\eta}})}^2}^{\bar{\eta}}.$$

The estimate (5.59) follows since  $8\delta \leq 4\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})$  for any  $0 < \bar{\eta} < 1$ . The estimate (5.60) is an easy consequence of (5.59).  $\square$

### 5.3.2 Nonlinear Strichartz estimates

In this subsection, we establish some local-in-time Strichartz estimates for any solution  $v$  to the equation (5.49) with initial data  $v_0 \in H_{\delta_0}^2$  for a fixed  $\delta_0 > 0$ . We will also use the quantities  $|(A, V)|_{\delta, r}$  and  $\|(A, V)\|_{\delta, k}$  introduced in (5.31) and (5.40), respectively.

**Proposition 5.3.4.** *Let  $T > 0$ ,  $4 \leq r < \infty$ , and  $s > 0$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, for any  $0 < \bar{\eta} < 1$  and  $0 < \delta < \min\{\frac{1}{18}, \frac{\bar{\eta}}{18(6-4\bar{\eta})}\}$ , there exist  $s_1, \delta_1, \bar{\delta} > 0$  with  $\frac{\delta_1}{s_1} > 1$  such that*

$$\begin{aligned}\|v\|_{L_T^4 W^{\frac{3}{4}-s, 4}} &\leq \mathcal{P}(|(A, V)|_{\delta, r}) \mathcal{P}(|(A, V)|_{\bar{\delta}, r}) \mathcal{P}(|(A, V)|_{\frac{\delta^2}{4}, r}) \mathcal{P}(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}) \\ &\quad \times \left( \|v_0\|_{H_\delta^1} + \mathcal{P}\left(\|v_0\|_{H_{8\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^1}\right) \|v\|_{L_T^\infty H_{-2\delta(\frac{1-2\bar{\eta}}{\bar{\eta}})}^2}^{p\bar{\eta}} \right).\end{aligned} \quad (5.61)$$

In particular, for any  $0 < \eta_0 < \frac{2}{5}$ , there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , there are

$s_1, \delta_1, \tilde{\delta} > 0$  with  $\frac{\delta_1}{s_1} > 1$  such that

$$\begin{aligned} \|v\|_{L_T^4 W^{\frac{3}{4}-s,4}} &\leq \mathcal{P}(|(A, V)|_{\delta,r}) \mathcal{P}(|(A, V)|_{\tilde{\delta},r}) \mathcal{P}(|(A, V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}) \\ &\quad \times \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \|v\|_{L_T^\infty H_{-\delta}^2}^{p\eta_0}. \end{aligned} \quad (5.62)$$

*Proof.* By Proposition 5.2.6, we have

$$\begin{aligned} \|v\|_{L_T^4 W^{\frac{3}{4}-s,4}} &\leq \mathcal{P}(|(A, V)|_{\delta,r}) \mathcal{P}(|(A, V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}) \\ &\quad \times \left( \|v_0\|_{H_\delta^1} + |\lambda| \int_0^T \|e^{-(p-1)A} |v(t')|^{p-1} v(t')\|_{H_\delta^1} dt' \right). \end{aligned}$$

We also note that by the Sobolev embedding  $H_{2\delta}^{\frac{2}{2-\eta}} \subset L_{2\delta}^q$  for any  $2 \leq q \leq \infty$  (Lemma A.2.2), we have

$$\begin{aligned} &\|e^{-(p-1)A} |v|^{p-1} v\|_{H_\delta^1} \\ &\lesssim \|e^{-(p-1)A} |v|^{p-1} v\|_{L_\delta^2} + \|\nabla A e^{-(p-1)A} |v|^{p-1} v\|_{L_\delta^2} + \|e^{-(p-1)A} \nabla v |v|^{p-1}\|_{L_\delta^2} \\ &\lesssim \|\langle x \rangle^{-\delta} e^{-(p-1)A}\|_{L^\infty} \|v\|_{H_{2\delta}^1} \|v\|_{L^\infty}^{p-1} + \|\langle x \rangle^{-\delta} \nabla A e^{-(p-1)A}\|_{L^r} \|v\|_{L_{2\delta}^{\frac{2r}{r-2}}} \|v\|_{L^\infty}^{p-1} \\ &\lesssim |(A, V)|_{\delta,r} \|v\|_{H_{2\delta}^{\frac{2}{2-\eta}}}^p. \end{aligned}$$

Thus, By using Proposition 5.3.3, we obtain

$$\|e^{-(p-1)A} |v|^{p-1} v\|_{L_T^\infty H_\delta^1} \lesssim \mathcal{P}\left(\|v_0\|_{H_{8\delta(\frac{6-4\bar{\eta}}{\eta})}^1}\right) \|v\|_{L_T^\infty H_{-2\delta(\frac{1-2\bar{\eta}}{\eta})}^2}^{p\bar{\eta}},$$

which gives (5.61). The estimate (5.62) follows directly from (5.61).  $\square$

As a consequence, we can show the following estimates.

**Proposition 5.3.5.** *Let  $T > 0$ ,  $4 \leq r < \infty$ ,  $0 < s < \frac{1}{8}$ , and  $0 < \eta_0 < \frac{2}{5}$ . Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , there exist  $s_1, \delta_1, \tilde{\delta} > 0$  with  $\frac{\delta_1}{s_1} > 1$  such that*

$$\begin{aligned} \|v\|_{L_T^2 W^{1,4}} &\leq \mathcal{P}(|(A, V)|_{\delta,r}) \mathcal{P}(|(A, V)|_{\tilde{\delta},r}) \mathcal{P}(|(A, V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}) \\ &\quad \times \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \|v\|_{L_T^\infty H_{-\delta}^2}^{\frac{2(1-2s)p\eta_0}{3}} \|v\|_{L_T^\infty H_{-\frac{s\delta_0}{4(4-s)}}^2}^{\frac{(1+4s)(2-s)}{6}}. \end{aligned} \quad (5.63)$$

*In particular, by choosing  $s = \frac{1}{16}$  and  $0 < \delta < \min\left\{\bar{\delta}, \frac{\delta_0}{64(4-\frac{1}{16})}\right\}$ , we get*

$$\begin{aligned} \|v\|_{L_T^2 W^{1,4}} &\leq \mathcal{P}(|(A, V)|_{\delta,r}) \mathcal{P}(|(A, V)|_{\tilde{\delta},r}) \mathcal{P}(|(A, V)|_{\frac{\delta^2}{4},r}) \mathcal{P}(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}) \\ &\quad \times \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \|v\|_{L_T^\infty H_{-\delta}^2}^{\frac{7}{12}p\eta_0 + \frac{155}{384}} \end{aligned} \quad (5.64)$$

*Proof.* By interpolation, we have

$$\|v(t)\|_{W^{1,4}} \lesssim \|v(t)\|_{W^{\frac{3}{4}-s,4}}^{\frac{2(1-2s)}{3}} \|v(t)\|_{H^{2-s}}^{\frac{1+4s}{3}}$$

Thus, by integration in time and Hölder's inequality in time, we obtain

$$\|v(t)\|_{L_T^2 W^{1,4}} \lesssim \|v(t)\|_{L_T^4 W^{\frac{3}{4}-s,4}}^{\frac{2(1-2s)}{3}} \|v(t)\|_{L_T^\infty H^{2-s}}^{\frac{1+4s}{3}}. \quad (5.65)$$

We also note that by interpolation (Lemma A.2.1), we have

$$\|v(t)\|_{H^{2-s}} \lesssim \|v(t)\|_{L^2}^{\frac{s}{2}} \|v(t)\|_{H^2}^{\frac{2-s}{2}}. \quad (5.66)$$

Thus, the estimate (5.63) follows from (5.62) in Proposition 5.3.4, (5.65), (5.66), and (5.52) in Proposition 5.3.1. The estimate (5.64) follows from (5.63) since the condition  $0 < \delta < \frac{\delta_0}{64(4-\frac{1}{16})}$  implies the embedding  $H_{-\delta}^2 \subset H_{-\frac{s\delta_0}{4(4-s)}}^2$  for  $s = \frac{1}{16}$ , so that one can bound the term  $\|v\|_{L_T^\infty H_{-\frac{s\delta_0}{4(4-s)}}^2}$  by  $\|v\|_{L^\infty((0,T);H_{-\delta}^2)}$  on the right-hand-side of (5.63).  $\square$

## 5.4 Modified energies

In this section, we introduce modified energies (originally introduced in [113]) and establish some useful estimates for these energies. We denote by  $v$  a solution to the following equation with time-independent  $A$  and  $V$ :

$$i\partial_t v = \Delta v - 2\nabla A \cdot \nabla v + Vv - \lambda e^{-(p-1)A} |v|^{p-1} v, \quad (5.67)$$

where  $\lambda \geq 0$  and the initial data  $v_0 \in H_{\delta_0}^2$  for some fixed  $\delta_0 > 0$ .

We first show the following conservation of modified energies.

**Proposition 5.4.1.** *Let  $v$  be a solution to (5.67), then we have the following identity:*

$$\frac{\partial}{\partial t} \mathcal{E}_{A,V}(v(t)) = -\lambda \mathcal{H}_{A,V}(v(t)), \quad (5.68)$$

where

$$\mathcal{E}_{A,V}(v(t)) = \int_{\mathbb{R}^2} |\Delta v(t)|^2 e^{-2A} dx + \mathcal{F}_{A,V}(v(t)) - \lambda \mathcal{G}_{A,V}(v(t))$$

and the energies  $\mathcal{F}_{A,V}, \mathcal{G}_{A,V}, \mathcal{H}_{A,V}$  are defined as follows:

$$\begin{aligned} \mathcal{F}_{A,V}(v(t)) &\stackrel{\text{def}}{=} -4\text{Re} \int_{\mathbb{R}^2} \Delta v(t) \nabla A \cdot \nabla \bar{v}(t) e^{-2A} dx \\ &\quad - 4 \int_{\mathbb{R}^2} (\nabla A \cdot \nabla v)^2 e^{-2A} dx + 2\text{Re} \int_{\mathbb{R}^2} v(t) V \nabla \bar{v}(t) \cdot \nabla (e^{-2A}) dx \\ &\quad + 2\text{Re} \int_{\mathbb{R}^2} \Delta v(t) \bar{v}(t) V e^{-2A} dx + \int_{\mathbb{R}^2} |v(t)|^2 V^2 e^{-2A} dx, \\ \mathcal{G}_{A,V}(v(t)) &\stackrel{\text{def}}{=} - \int_{\mathbb{R}^2} |\nabla v(t)|^2 |v(t)|^{p-1} e^{-(p+1)A} dx \\ &\quad - 2\text{Re} \int_{\mathbb{R}^2} v(t) \nabla (|v(t)|^{p-1}) \cdot \nabla \bar{v}(t) e^{-(p+1)A} dx \\ &\quad + \frac{p-1}{4} \int_{\mathbb{R}^2} |\nabla (|v(t)|^2)|^2 |v(t)|^{p-3} e^{-(p+1)A} dx \\ &\quad + \frac{2}{p+1} \int_{\mathbb{R}^2} |v(t)|^{p+1} V e^{-(p+1)A} dx \\ &\quad + 2(p-1)\text{Re} \int_{\mathbb{R}^2} v(t) |v(t)|^{p-1} \nabla A \cdot \nabla \bar{v}(t) e^{-(p+1)A} dx, \\ \mathcal{H}_{A,V}(v(t)) &\stackrel{\text{def}}{=} - \int_{\mathbb{R}^2} |\nabla v(t)|^2 \partial_t (|v(t)|^{p-1}) e^{-(p+1)A} dx \\ &\quad - 2\text{Re} \int_{\mathbb{R}^2} \partial_t v(t) \nabla (|v(t)|^{p-1}) \cdot \nabla \bar{v}(t) e^{-(p+1)A} dx \\ &\quad - \frac{p-1}{4} \int_{\mathbb{R}^2} |\nabla (|v(t)|^2)|^2 \partial_t (|v(t)|^{p-3}) e^{-(p+1)A} dx \\ &\quad + 2(p-1)\text{Re} \int_{\mathbb{R}^2} \partial_t (|v(t)|^{p-1} v(t)) \nabla A \cdot \nabla \bar{v}(t) e^{-(p+1)A} dx. \end{aligned}$$

*Proof.* We denote by  $(\cdot, \cdot)$  the usual scalar product in  $L^2$ . By using the equation (5.67), we have

$$\begin{aligned}
\frac{d}{dt}(\Delta v, \Delta v e^{-2A}) &= 2\operatorname{Re}(\partial_t \Delta v, \Delta v e^{-2A}) \\
&= 2\operatorname{Re}(\partial_t \Delta v, i\partial_t v e^{-2A}) + 4\operatorname{Re}(\partial_t \Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
&\quad - 2\operatorname{Re}(\partial_t \Delta v, v V e^{-2A}) + 2\lambda \operatorname{Re}(\partial_t \Delta v, |v|^{p-1} v e^{-(p+1)A}) \\
&= I + II + III + IV.
\end{aligned} \tag{5.69}$$

Note that

$$\begin{aligned}
I &= -2\operatorname{Re}(\partial_t \nabla v, i\partial_t \nabla v e^{-2A}) - 2\operatorname{Re}(\partial_t \nabla v, i\partial_t v \nabla(e^{-2A})) \\
&= -2\operatorname{Im}(\partial_t \nabla v, \partial_t v \nabla(e^{-2A})).
\end{aligned} \tag{5.70}$$

Moreover, we have

$$\begin{aligned}
II &= 4\operatorname{Re}(\partial_t \Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
&= 2\operatorname{Re}(\Delta v, \partial_t \nabla v \cdot \nabla(e^{-2A})) + 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}),
\end{aligned}$$

and by using again the equation (5.67), we obtain

$$\begin{aligned}
II &= 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) - 2\operatorname{Im}(\partial_t v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&\quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2\operatorname{Re}(Vv, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&\quad + 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&= 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) + 2\operatorname{Im}(\partial_t \nabla v, \partial_t v \nabla(e^{-2A})) \\
&\quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2\operatorname{Re}(Vv, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&\quad + 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

Hence, by (5.70), we obtain

$$\begin{aligned}
II &= 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) - I + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&\quad - 2\operatorname{Re}(Vv, \partial_t \nabla v \cdot \nabla(e^{-2A})) + 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&= 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) - I + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&\quad - 2\frac{d}{dt}\operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) + 2\operatorname{Re}(V\partial_t v, \nabla v \cdot \nabla(e^{-2A})) \\
&\quad + 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

Namely, we have

$$\begin{aligned}
I + II &= 2\operatorname{Re}(V\partial_t v, \nabla v \cdot \nabla(e^{-2A})) + 4\frac{d}{dt}\operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
&\quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2\frac{d}{dt}\operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) \\
&\quad + 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

On the other hand, we can compute the last term in the above identity as

$$\begin{aligned}
&2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&= -2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}|v|^{p-1}v), \partial_t \nabla v e^{-2A}) - 2\lambda \operatorname{Re}(e^{-(p-1)A}|v|^{p-1}v, \partial_t \Delta v e^{-2A}) \\
&= -2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}|v|^{p-1}v), \partial_t \nabla v e^{-2A}) - IV
\end{aligned}$$

$$\begin{aligned}
&= -2\lambda \operatorname{Re}(e^{-(p-1)A} \nabla v |v|^{p-1}, \partial_t \nabla v e^{-2A}) - 2\lambda \operatorname{Re}(e^{-(p-1)A} v \nabla(|v|^p), \partial_t \nabla v e^{-2A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda(\partial_t(|\nabla v|^2) |v|^{p-1}, e^{-(p+1)A}) - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(v \nabla \partial_t(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda \frac{d}{dt}(|\nabla v|^2 |v|^{p-1}, e^{-(p+1)A}) + \lambda(|\nabla v|^2 \partial_t(|v|^{p-1}), e^{-(p+1)A}) \\
&\quad - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad + \frac{\lambda(p-1)}{2} (\partial_t(\nabla(|v|^2) |v|^{p-3}), \nabla(|v|^2) e^{-(p+1)A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda \frac{d}{dt}(|\nabla v|^2 |v|^{p-1}, e^{-(p+1)A}) + \lambda(|\nabla v|^2 \partial_t(|v|^{p-1}), e^{-(p+1)A}) \\
&\quad - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad + \frac{\lambda(p-1)}{4} (\partial_t(|\nabla(|v|^2)|^2) |v|^{p-3}, e^{-(p+1)A}) + \frac{\lambda(p-1)}{2} (\nabla(|v|^2) \partial_t(|v|^{p-3}), \nabla(|v|^2) e^{-(p+1)A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda \frac{d}{dt}(|\nabla v|^2 |v|^{p-1}, e^{-(p+1)A}) + \lambda(|\nabla v|^2 \partial_t(|v|^{p-1}), e^{-(p+1)A}) \\
&\quad - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad + \frac{\lambda(p-1)}{4} \frac{d}{dt}(|\nabla(|v|^2)|^2 |v|^{p-3}, e^{-(p+1)A}) + \frac{\lambda(p-1)}{4} (|\nabla(|v|^2)|^2 \partial_t(|v|^{p-3}), e^{-(p+1)A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}) - IV.
\end{aligned}$$

By combining the identities above, we obtain

$$\begin{aligned}
&I + II + IV \\
&= 2\operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) + 2 \frac{d}{dt} \operatorname{Re}(\Delta v, 2\nabla A \cdot \nabla v e^{-2A}) \\
&\quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2 \frac{d}{dt} \operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) \\
&\quad - \lambda \frac{d}{dt} \operatorname{Re}(|\nabla v|^2 |v|^{p-1}, e^{-(p+1)A}) + \lambda \operatorname{Re}(|\nabla v|^2 \partial_t(|v|^{p-1}), e^{-(p+1)A}) \\
&\quad - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
&\quad + \frac{\lambda(p-1)}{4} \frac{d}{dt}(|\nabla(|v|^2)|^2 |v|^{p-3}, e^{-(p+1)A}) + \frac{\lambda(p-1)}{4} (|\nabla(|v|^2)|^2 \partial_t(|v|^{p-3}), e^{-(p+1)A}) \\
&\quad - 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A}) |v|^{p-1} v, \partial_t \nabla v e^{-2A}).
\end{aligned}$$

Since  $III = -2 \frac{d}{dt} \operatorname{Re}(\Delta v, Vv e^{-2A}) + 2\operatorname{Re}(\Delta v, \partial_t v V e^{-2A})$ , we obtain the following identity:

$$\begin{aligned}
&I + II + III + IV \\
&= 2\operatorname{Re}(V \partial_t v, \nabla v \cdot \nabla(e^{-2A})) + 4 \frac{d}{dt} \operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
&\quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2 \frac{d}{dt} \operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) \\
&\quad - \lambda \frac{d}{dt} \operatorname{Re}(|\nabla v|^2 |v|^{p-1}, e^{-(p+1)A}) + \lambda \operatorname{Re}(|\nabla v|^2 \partial_t(|v|^{p-1}), e^{-(p+1)A})
\end{aligned}$$

$$\begin{aligned}
& -2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+1)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^{p-1}), \nabla v e^{-(p+1)A}) \\
& + \frac{\lambda(p-1)}{4} \frac{d}{dt} (|\nabla(|v|^2)|^2 |v|^{p-3}, e^{-(p+1)A}) \\
& + \frac{\lambda(p-1)}{4} (|\nabla(|v|^2)|^2 \partial_t(|v|^{p-3}), e^{-(p+1)A}) \\
& - 2\lambda \frac{d}{dt} \operatorname{Re}(\nabla(e^{-(p-1)A})|v|^{p-1}v, \nabla v e^{-2A}) \\
& + 2\lambda \operatorname{Re}(\nabla(e^{-(p-1)A})\partial_t(|v|^{p-1}v), \nabla v e^{-2A}) \\
& - 2 \frac{d}{dt} \operatorname{Re}(\Delta v, v V e^{-2A}) + 2 \operatorname{Re}(\Delta v, \partial_t v V e^{-2A}).
\end{aligned}$$

Next, by using the equation (5.67) again, we compute the first, the third, and the last term on the right-hand-side above as

$$\begin{aligned}
& 2\operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
& + 2\operatorname{Re}(\Delta v, \partial_t v V e^{-2A}) \\
& = 2\operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) - 8\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla A e^{-2A}) \\
& + 4\operatorname{Re}(\nabla v \cdot \nabla A, \partial_t v V e^{-2A}) - 2\operatorname{Re}(V v, \partial_t v V e^{-2A}) \\
& + 2\lambda \operatorname{Re}(e^{-(p-1)A} |v|^{p-1} v, \partial_t v V e^{-2A}) \\
& = -4\operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla A e^{-2A}) - 4 \frac{d}{dt} ((\nabla A \cdot \nabla v)^2, e^{-2A}) \\
& + 4\operatorname{Re}(\nabla v \cdot \nabla A, \partial_t v V e^{-2A}) - 2\operatorname{Re}(V v, \partial_t v V e^{-2A}) \\
& + 2\lambda \operatorname{Re}(|v|^{p-1} v, \partial_t v V e^{-(p+1)A}) \\
& = -4 \frac{d}{dt} ((\nabla A \cdot \nabla v)^2, e^{-2A}) - \frac{d}{dt} (V |v|^2, V e^{-2A}) \\
& + \frac{2\lambda}{p+1} \frac{d}{dt} \operatorname{Re}(|v|^{p+1}, V e^{-(p+1)A}).
\end{aligned}$$

The proof of (5.68) then follows by combining the above two identities with (5.69).  $\square$

We now provide some useful estimates for the energies  $\mathcal{F}_{A,V}, \mathcal{G}_{A,V}, \mathcal{H}_{A,V}$ , starting with  $\mathcal{F}_{A,V}$ . We recall the quantity  $|(A, V)|_{\delta, r}$  from (5.31).

**Proposition 5.4.2.** *Let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, for any  $\delta > 0$  and  $2 < r < \infty$ , we have*

$$|\mathcal{F}_{A,V}(w)| \leq \mathcal{P}(|(A, V)|_{\delta, r}) \left( \|e^{-A} \Delta w\|_{L^2} \|w\|_{W_\delta^{1, \frac{2r}{r-2}}} + \|w\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right). \quad (5.71)$$

Moreover, let  $v$  be a solution to (5.67). Then, for any  $T > 0$  and  $4 \leq r < \infty$ , there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , we have

$$\begin{aligned}
\sup_{t \in [-T, T]} |\mathcal{F}_{A,V}(v(t))| & \leq \mathcal{P}(|(A, V)|_{\delta, r}) \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \\
& \times \left( \|e^{-A} \Delta v\|_{L_T^\infty L^2} \|v\|_{L_T^\infty H_{-\delta}^{\frac{r+2}{2r}}} + \|v\|_{L_T^\infty H_{-\delta}^{\frac{r+2}{r}}} \right).
\end{aligned} \quad (5.72)$$

*Proof.* By Hölder's inequalities, we obtain

$$\begin{aligned}
|\mathcal{F}_{A,V}(w)| & \lesssim \|e^{-A} \Delta w\|_{L^2} \|\langle x \rangle^\delta \nabla w(t)\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^{-\delta} \nabla A e^{-A}\|_{L^r} \\
& + \|\langle x \rangle^{-\delta} e^{-A} \nabla A\|_{L^r}^2 \|\langle x \rangle^\delta \nabla w\|_{L^{\frac{2r}{r-2}}}^2 \\
& + \|\langle x \rangle^{-2\delta} \nabla A e^{-2A} V\|_{L^{\frac{r}{2}}} \|\langle x \rangle^\delta \nabla w\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}} \\
& + \|e^{-A} \Delta w\|_{L^2} \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r}
\end{aligned}$$

$$+ \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r}^2 \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}}^2,$$

which in turn implies (5.71). On the other hand, by (5.54) in Proposition 5.3.1, for  $0 < \delta < \frac{r-2}{36(3r+2)}$ , we have

$$\begin{aligned} \|v\|_{L_T^\infty W_\delta^{1, \frac{2r}{r-2}}} &\lesssim \|v\|_{L_T^\infty H_{-\delta}^{\frac{r+2}{2r}}} \mathcal{P}\left(\|v(0)\|_{H_{4\delta(\frac{3r+2}{r-2})}^1}\right) \\ &\lesssim \|v\|_{L_T^\infty H_{-\delta}^{\frac{r+2}{2r}}} \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right), \end{aligned} \quad (5.73)$$

where at the last step we used that for  $r \geq 4$  we have  $4\delta(\frac{3r+2}{r-2}) \leq 28\delta < \delta_0$  provided that  $\delta > 0$  is small enough. This proves (5.72).  $\square$

Recall we have not provided a proof for part (iv) in Proposition 5.2.1. Now we have the necessary tools for proving it.

*Proof of Proposition 5.2.1 (iv).* We denote for simplicity  $w(t) = S_{A,V}(t)\varphi$ . Concerning (5.35), we need to use the conservation (5.68) with  $\lambda = 0$ :

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left( |\Delta w(t)|^2 e^{-2A} + \mathcal{F}_{A,V}(w(t)) \right) dx = 0,$$

which in turn after integration in time implies

$$\int_{\mathbb{R}^2} |\Delta w(t)|^2 e^{-2A} dx = \int_{\mathbb{R}^2} |\Delta w(0)|^2 e^{-2A} dx - \mathcal{F}_{A,V}(w(t)) + \mathcal{F}_{A,V}(w(0)).$$

By (5.71) in Proposition 5.4.2 (whose proof only involves Hölder's inequality) along with the conservation (5.29), we obtain that for any  $\mu > 0$ ,

$$\begin{aligned} &\int_{\mathbb{R}^2} (|w(t)|^2 + |\Delta w(t)|^2) e^{-2A} dx \\ &\leq \int_{\mathbb{R}^2} (|w(0)|^2 + |\Delta w(0)|^2) e^{-2A} dx \\ &\quad + \mathcal{P}(|(A,V)|_{\delta,r}) \left( \mu \|e^{-A} \Delta w(t)\|_{L^2}^2 + \left(1 + \frac{1}{2\mu}\right) \|w(t)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right) \\ &\quad + \mathcal{P}(|(A,V)|_{\delta,r}) \left( \|e^{-A} \Delta w(0)\|_{L^2} \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}} + \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right). \end{aligned}$$

In particular, we can choose  $\mu = \frac{1}{2\mathcal{P}(|(A,V)|_{\delta,r})}$  and obtain

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^2} (|w(t)|^2 + |\Delta w(t)|^2) e^{-2A} dx \\ &\leq \int_{\mathbb{R}^2} (|w(0)|^2 + |\Delta w(0)|^2) e^{-2A} dx + \mathcal{P}(|(A,V)|_{\delta,r}) \|w(t)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \\ &\quad + \mathcal{P}(|(A,V)|_{\delta,r}) \left( \|e^{-A} \Delta w(0)\|_{L^2} \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}} + \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right) \end{aligned} \quad (5.74)$$

Then, by Lemma A.2.3 with  $q = \frac{2r}{r-2}$  and Proposition 5.2.1 (ii), we get

$$\|w(t)\|_{W_\delta^{1, \frac{2r}{r-2}}} \leq C \|w(t)\|_{H_{-\delta}^{\frac{r+2}{2r}}}^{\frac{r+2}{2r}} \|w(t)\|_{L_{\delta(\frac{3r+2}{r-2})}^2}^{\frac{r-2}{2r}} \leq C \|w(t)\|_{H_{-\delta}^{\frac{r+2}{2r}}}^{\frac{r+2}{2r}} \|w(0)\|_{H_{\delta^+(\frac{3r+2}{r-2})}^1}^{\frac{r-2}{2r}}, \quad (5.75)$$

where we used the assumptions  $\frac{\delta}{2} + 2\delta^+ < \frac{r-2}{3r+2}$  and  $\delta^+ > \delta$ . We conclude the desired estimate (5.35) by combining (5.74), (5.75), and (5.28).  $\square$

We now move on and prove the following estimate for the energy  $\mathcal{G}_{A,V}$ .

**Proposition 5.4.3.** *Let  $T > 0$ ,  $4 \leq r < \infty$ ,  $0 < \eta_0 < \frac{1}{3}$ , and let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , we have*

$$\sup_{t \in [-T, T]} |\mathcal{G}_{A, V}(v(t))| \leq |(A, V)|_{\delta, r} \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \left(1 + \|v\|_{L_T^\infty H_{-\delta}^2}^{\frac{1}{r} + \frac{1}{2} + \eta_0(p+1)}\right). \quad (5.76)$$

*Proof.* By Hölder's inequalities, we have

$$\begin{aligned} |\mathcal{G}_{A, V}(v(t))| &\lesssim \|\nabla v(t)\|_{L^{\frac{2r}{r-2}}}^2 \|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^{\frac{r}{2}}} \|\langle x \rangle^{-\delta} e^{-(p+1)A}\|_{L^\infty} \\ &\quad + \|\langle x \rangle^\delta |v(t)|^{p+1}\|_{L^{\frac{r}{r-1}}} \|\langle x \rangle^{-\delta} V e^{-(p+1)A}\|_{L^r} \\ &\quad + \|\nabla v(t)\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^\delta |v(t)|^p\|_{L^2} \|\langle x \rangle^{-\delta} \nabla A e^{-(p+1)A}\|_{L^r}. \end{aligned} \quad (5.77)$$

By the Sobolev embedding  $L_\mu^q \subset H_\mu^{\frac{2}{2-\eta_0}}$  for  $2 \leq q \leq \infty$  and  $\mu \in \mathbb{R}$  (Lemma A.2.2) and (5.60) in Proposition 5.3.3, we obtain

$$\begin{aligned} \|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^{\frac{r}{2}}} &= \|v(t)\|_{L^{\frac{r(p-1)}{2}}}^p \leq \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p-1)}, \\ \|\langle x \rangle^\delta |v(t)|^{p+1}\|_{L^{\frac{r}{r-1}}} &= \|v(t)\|_{L^{\frac{r(p+1)}{r-1}}}^{p+1} \leq \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p+1)}, \\ \|\langle x \rangle^\delta |v(t)|^p\|_{L^2} &= \|v(t)\|_{L^{\frac{2}{p}}}^p \leq \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0 p}. \end{aligned}$$

By combining the estimates above with (5.77) and (5.54) in Proposition 5.3.1 with  $q = \frac{2r}{r-2}$ , we conclude our desired estimate.  $\square$

Lastly, we show the following estimate for the energy  $\mathcal{H}_{A, V}$ .

**Proposition 5.4.4.** *Let  $T > 0$ ,  $4 \leq r < \infty$ ,  $0 < \eta_0 < \frac{1}{3}$ , and let  $A$  and  $V$  satisfy (5.26) and (5.27). Then, there exists  $\bar{\delta} > 0$  such that for any  $0 < \delta < \bar{\delta}$ , we have*

$$\begin{aligned} \int_{-T}^T |\mathcal{H}_{A, V}(v(t'))| dt' &\leq |(A, V)|_{r, \delta} \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|\partial_t v e^{-A}\|_{L_T^\infty L^2} \\ &\quad \times (1 + \|\nabla v\|_{L_T^2 L^4}^2) (1 + \|v\|_{L_T^\infty H_{-\delta}^2})^{\eta_0(p-1)}. \end{aligned} \quad (5.78)$$

*Proof.* We only focus on the case when  $p > 2$ . The case  $p = 2$  will follow in a similar (and easier) manner. By Hölder's inequalities, we have

$$\begin{aligned} |\mathcal{H}_{A, V}(v(t))| &\lesssim \|\nabla v(t)\|_{L^4}^2 \|\partial_t v e^{-A}\|_{L^2} \|\langle x \rangle^\delta |v(t)|^{p-2}\|_{L^\infty} \|\langle x \rangle^{-\delta} e^{-pA}\|_{L^\infty} \\ &\quad + \|\nabla v(t)\|_{L^4} \|\partial_t v e^{-A}\|_{L^2} \|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^{\frac{4r}{r-4}}} \|\langle x \rangle^{-\delta} |\nabla A| e^{-pA}\|_{L^r}. \end{aligned}$$

By (5.60) in Proposition 5.3.3 and the Sobolev embedding  $H_\rho^{\frac{2}{2-\eta_0}} \subset L_\rho^q$  for  $2 \leq q \leq \infty$  and  $\mu \in \mathbb{R}$  (Lemma A.2.2), we obtain

$$\|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^{\frac{4r}{r-4}}} = \|v(t)\|_{L^{\frac{4r(p-1)}{r-4}}}^{p-1} \leq \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p-1)}$$

and

$$\|\langle x \rangle^\delta |v(t)|^{p-2}\|_{L^\infty} = \|v(t)\|_{L^{\frac{4r}{p-2}}}^{p-2} \leq \mathcal{P}(\|v(0)\|_{H_{\delta_0}^1}) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p-2)}.$$

After integrating in time and applying Hölder's inequality in time, we obtain (5.78).  $\square$

## 5.5 $H^2$ a priori bounds

In this section, we use the results from previous sections to prove key  $H^2$  a priori bounds.

Let us first introduce the following family of regularized and localized potentials, for  $\varepsilon > 0$  and  $n \in \mathbb{N}$ :

$$A_{\varepsilon,n} = \theta_n Y_\varepsilon, \quad V_{\varepsilon,n} = \theta_n : |\nabla Y_\varepsilon|^2 :$$

where  $\theta_n(x) = \theta(\frac{x}{n})$ ,  $\theta \in C_c^\infty(\mathbb{R}^2)$ ,  $\theta \geq 0$ , and  $\theta(0) = 1$ . Note that due to (5.9) in Proposition 5.1.5, if we choose  $A = A_{\varepsilon,n}$  and  $V = V_{\varepsilon,n}$ , then the condition (5.26) holds uniformly in  $n \in \mathbb{N}$  and  $\varepsilon > 0$  with the constant  $C$  replaced by a random constant  $C(\omega)$ . The same holds true for (5.27) uniformly in  $n \in \mathbb{N}$  and  $\varepsilon > 0$  with a random constant  $C(\omega)$ , which can be proved by (5.13) in Proposition 5.1.8, Lemma 5.1.9, (5.2) in Proposition 5.1.1, Lemma 5.1.4, and [29] (more precisely, see at page 1160 three lines below (33)).

Following the notations in (5.25), we introduce the operators

$$H_{\varepsilon,n} = H_{A_{\varepsilon,n}, V_{\varepsilon,n}}$$

and the associated nonlinear Cauchy problem

$$\begin{cases} i\partial_t v_{\varepsilon,n} = H_{\varepsilon,n} v_{\varepsilon,n} + \lambda e^{-(p-1)A_{\varepsilon,n}} |v_{\varepsilon,n}|^{p-1} v_{\varepsilon,n} \\ v_{\varepsilon,n}|_{t=0} = v_0, \end{cases} \quad (5.79)$$

where  $\lambda \geq 0$  and  $v_0 \in H_{\delta_0}^2$  for some fixed  $\delta_0 > 0$ . The main goal in this section is to establish the following a priori bound for any  $T > 0$  and any sufficiently small  $\delta > 0$ :

$$\sup_{n \in \mathbb{N}} \|v_{\varepsilon,n}\|_{L_T^\infty H_{-\delta}^2} \leq C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}) \quad (5.80)$$

for almost sure  $\omega \in \Omega$ .

We now focus on proving the bound (5.80). Recall the quantity  $|(A, V)|_{\delta,r}$  defined in (5.31). We fix  $\delta > 0$  in such a way that (5.72) in Proposition 5.4.2, (5.76) in Proposition 5.4.3, and (5.78) in Proposition 5.4.4 are satisfied with  $A = A_{\varepsilon,n}$  and  $V = V_{\varepsilon,n}$ . Note that by (5.31), (5.1) in Proposition 5.1.1, and (5.9) in Proposition 5.1.5, we can choose  $r \geq 4$  large enough in such a way that

$$\sup_{n \in \mathbb{N}} |(A_{\varepsilon,n}, V_{\varepsilon,n})|_{\delta,r} \leq C(\omega) |\log \varepsilon|^C \quad (5.81)$$

for some constant  $C > 0$  and almost sure  $\omega \in \Omega$ , where the cutoff  $\theta_n$  can be easily handled by an elementary argument and the bounds are uniform in  $n \in \mathbb{N}$ . Moreover, by Proposition 5.4.1, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} |\Delta v_{\varepsilon,n}(t)|^2 e^{-2A_{\varepsilon,n}} dx \\ &= \int_{\mathbb{R}^2} |\Delta v_0|^2 e^{-2A_{\varepsilon,n}} dx - \mathcal{F}_{\varepsilon,n}(v_{\varepsilon,n}(t)) + \mathcal{F}_{\varepsilon,n}(v_0) \\ & \quad - \lambda \mathcal{G}_{\varepsilon,n}(v_{\varepsilon,n}(t)) + \lambda \mathcal{G}_{\varepsilon,n}(v_0) + \int_0^t \mathcal{H}_{\varepsilon,n}(v_{\varepsilon,n}(t')) dt' \end{aligned} \quad (5.82)$$

where

$$\mathcal{F}_{\varepsilon,n} = \mathcal{F}_{A_{\varepsilon,n}, V_{\varepsilon,n}}, \quad \mathcal{G}_{\varepsilon,n} = \mathcal{G}_{A_{\varepsilon,n}, V_{\varepsilon,n}}, \quad \mathcal{H}_{\varepsilon,n} = \mathcal{H}_{A_{\varepsilon,n}, V_{\varepsilon,n}}$$

are the energies defined in Proposition 5.4.1 with  $A = A_{\varepsilon,n}$  and  $V = V_{\varepsilon,n}$ .

Next, by (5.72) in Proposition 5.4.2 along with (5.81) and  $r \geq 4$  sufficiently large, we have

$$\begin{aligned} \sup_{t \in [0, T]} |\mathcal{F}_{\varepsilon,n}(v_{\varepsilon,n}(t))| &\leq C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}) \\ &\times \left( \|e^{-A_{\varepsilon,n}} \Delta v_{\varepsilon,n}\|_{L_T^\infty L^2} \|v_{\varepsilon,n}\|_{L_T^\infty H_{-\delta}^2}^{1-} + \|v_{\varepsilon,n}\|_{L_T^\infty H_{-\delta}^2}^{2-} \right). \end{aligned} \quad (5.83)$$

We also choose  $\eta_0 > 0$  sufficiently small that

$$\frac{1}{r} + \frac{1}{2} + \eta_0(p+2) < 1, \quad (5.84)$$

$$\frac{7}{6}p\eta_0 + \frac{155}{192} + \eta_0(p-1) < 1, \quad (5.85)$$

$$\eta_0 p < 1. \quad (5.86)$$

By Proposition 5.4.3, (5.81), and (5.84), we obtain

$$\begin{aligned} & \sup_{t \in [-T, T]} |\mathcal{G}_{\varepsilon, n}(v_{\varepsilon, n}(t))| \\ & \leq C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) (1 + \|v_{\varepsilon, n}(t)\|_{L_T^\infty H_{-\delta}^2}^{1-}). \end{aligned} \quad (5.87)$$

Note that with sufficiently large  $r \geq 4$ , we have

$$\begin{aligned} & \sup_{n \in \mathbb{N}} |(A_{\varepsilon, n}, V_{\varepsilon, n})|_{\frac{\delta^2}{4}, r} \leq C(\omega) |\log \varepsilon|^C, \\ & \sup_{n \in \mathbb{N}} |(A_{\varepsilon, n}, V_{\varepsilon, n})|_{\delta, r} \leq C(\omega) |\log \varepsilon|^C \end{aligned} \quad (5.88)$$

for almost sure  $\omega \in \Omega$ , where  $\tilde{\delta} > 0$  is the one that appears in (5.64) in Proposition 5.3.5. Also, by (5.1) in Proposition 5.1.1 along with the fact that in (5.64) we can assume  $\frac{\delta_1}{s_1} > 1$ , we also have (see (5.40))

$$\sup_{n \in \mathbb{N}} \|(A_{\varepsilon, n}, V_{\varepsilon, n})\|_{\delta_1, \frac{2}{s_1}} \leq C(\omega) O(|\log \varepsilon|^C) \quad (5.89)$$

for almost sure  $\omega \in \Omega$ . Thus, by Proposition 5.4.4, (5.64) in Proposition 5.3.5, (5.85), (5.88), and (5.89) we obtain

$$\begin{aligned} & \int_{-T}^T |\mathcal{H}_{\varepsilon, n}(v_{\varepsilon, n}(t'))| dt' \leq C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) \\ & \quad \times \|\partial_t v_{\varepsilon, n} e^{-A_{\varepsilon, n}}\|_{L_T^\infty L^2} (1 + \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{1-}), \end{aligned} \quad (5.90)$$

for almost sure  $\omega \in \Omega$ . Then, we note that by using the equation (5.79), Hölder's inequality, the Sobolev embedding  $H_\delta^{\frac{2}{2-\eta_0}} \subset L_\delta^q$  with  $2 \leq q \leq \infty$  (Lemma A.2.2), (5.54) in Proposition 5.3.1, (5.60) in Proposition 5.3.3, (5.89), and (5.86),

$$\begin{aligned} & \|\partial_t v_{\varepsilon, n}(t) e^{-A_{\varepsilon, n}}\|_{L^2} \\ & \leq \|\Delta v_{\varepsilon, n}(t) e^{-A_{\varepsilon, n}}\|_{L^2} + C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{1-} \\ & \quad + C(\omega) |\log \varepsilon|^C \| |v_{\varepsilon, n}(t)|^{p-1} v_{\varepsilon, n}(t) \|_{L_\delta^3} \\ & \leq \|\Delta v_{\varepsilon, n}(t) e^{-A_{\varepsilon, n}}\|_{L^2} + C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{1-} \\ & \quad + C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{\eta_0 p} \\ & \leq \|\Delta v_{\varepsilon, n}(t) e^{-A_{\varepsilon, n}}\|_{L^2} + C(\omega) |\log \varepsilon|^C \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}) \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{1-} \end{aligned} \quad (5.91)$$

for almost sure  $\omega \in \Omega$ . Hence, we can gather together (5.82), (5.83), (5.87), (5.90), and (5.91) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} |\Delta v_{\varepsilon, n}(t)|^2 e^{-2A_{\varepsilon, n}} dx \leq C \int_{\mathbb{R}^2} |\Delta v_{\varepsilon, n}(0)|^2 e^{-2A_{\varepsilon, n}} dx \\ & \quad + C \|v_{\varepsilon, n}\|_{L_T^\infty H_{-\delta}^2}^{2-} + C(\omega) |\log \varepsilon|^C + \mathcal{P}(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1}), \end{aligned}$$

which in turn by (5.28) implies that

$$\|v_{\varepsilon,n}\|_{L_T^\infty H_{-\delta}^2}^2 \leq \mathcal{P}(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^2}) + C\|v_{\varepsilon,n}\|_{L_T^\infty H_{-\delta}^2}^{2-} + C(\omega)|\log \varepsilon|^C.$$

By Young's inequality, we obtain (5.80).

## 5.6 Global well-posedness

In this section, we first prove Theorem 1.4.1, global well-posedness of the mollified equation (1.32). After that, we prove Theorem 1.4.2, the convergence of the solutions of the mollified problem to a unique solution of (1.30).

### 5.6.1 Global well-posedness of the mollified equation

We first follow the steps from [29, Proposition 2.11] to show the existence of a solution  $v_\varepsilon$  to (1.32), where  $0 < \varepsilon < \frac{1}{2}$  is fixed. Fix  $\delta_0 > 0$ ,  $T > 0$ , and let  $v_0 \in H_{\delta_0}^2$ . By the  $H^2$  a priori bound (5.80), (5.60) in Proposition 5.3.3, and interpolation (Lemma A.2.1), we have the following bound for any  $1 < \gamma < 2$  and for some  $\delta > 0$ :

$$\sup_{n \in \mathbb{N}} \|v_{\varepsilon,n}\|_{C_T H_\delta^\gamma} \leq C(\omega)|\log \varepsilon|^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}) \quad (5.92)$$

for almost sure  $\omega \in \Omega$ . Also, by (5.2) in Proposition 5.1.1, (5.13) in Proposition 5.1.8, and (5.9) in Proposition 5.1.5, for any  $0 < \alpha < 1$ ,  $0 < \delta^- < \delta$ , and  $\varepsilon \in (0, \frac{1}{2})$  we have the following bounds:

$$\sup_{n \in \mathbb{N}} \left( \|\theta_n Y_\varepsilon\|_{C_{-\delta^-}^\alpha} + \left\| \theta_n : \widetilde{|\nabla Y_\varepsilon|^2} : \right\|_{C_{-\delta^-}^{\alpha-1}} + \|e^{-p\theta_n Y_\varepsilon}\|_{L_{-\delta^-}^\infty} \right) \leq C(\omega) \quad (5.93)$$

for almost sure  $\omega \in \Omega$ . Using the equation (1.35), (5.92), (5.93), and the Sobolev embedding (Lemma A.2.2), we can easily deduce that  $\{\partial_t v_{\varepsilon,n}\}_{n \in \mathbb{N}}$  is bounded (uniformly in  $n \in \mathbb{N}$ ) in  $C([-T, T]; H_{\delta'}^{\gamma-2})$  for any  $0 < \delta' < \delta$ . By the Arzelà-Ascoli theorem along with the compact embedding (A.3), we obtain a convergent subsequence  $\{v_{\varepsilon,n_k}\}_{k \in \mathbb{N}}$  in  $C([-T, T]; H_{\delta''}^{\gamma_1-2})$  for any  $\gamma_1 < \gamma$  and  $\delta'' < \delta$ , and we denote the limit as  $v_\varepsilon$ . By the  $H^2$  a priori bound (5.80) and interpolation (Lemma A.2.1), the convergence also holds in  $C([-T, T]; H_{\delta_1}^s)$  for any  $1 < s < 2$  and some  $\delta_1 > 0$ . Also, by the  $H^2$  a priori bound (5.80), the Banach-Alaoglu theorem, and taking a further subsequence if necessary, we obtain the following bound:

$$\|v_\varepsilon\|_{L_T^\infty H_{-\delta}^2} \leq C(\omega)|\log \varepsilon|^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}) \quad (5.94)$$

for some  $\tilde{\delta} > 0$  and almost sure  $\omega \in \Omega$ . Furthermore,  $v_\varepsilon$  satisfies the equation (1.32).

Next, we show the uniqueness of  $v_\varepsilon$  in  $C([-T, T]; H_{\delta_1}^s)$ . Assume  $v_\varepsilon^1$  and  $v_\varepsilon^2$  are two solutions to (1.32). For  $-T \leq t \leq T$ , we define

$$r_\varepsilon(t) = v_\varepsilon^1(t) - v_\varepsilon^2(t).$$

Then,  $r_\varepsilon$  satisfies the equation:

$$\begin{cases} i\partial_t r_\varepsilon = \Delta r_\varepsilon - 2\nabla Y_\varepsilon \cdot \nabla r_\varepsilon + : \widetilde{|\nabla Y_\varepsilon|^2} : r_\varepsilon - \lambda e^{-(p-1)Y_\varepsilon} (|v_\varepsilon^1|^{p-1} v_\varepsilon^1 - |v_\varepsilon^2|^{p-1} v_\varepsilon^2) \\ r_\varepsilon|_{t=0} = 0. \end{cases}$$

Using the equation for  $r$ , we can deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_\varepsilon(t)|^2 e^{-2Y_\varepsilon} dx = \lambda \operatorname{Im} \int_{\mathbb{R}^2} \overline{r_\varepsilon(t)} (|v_\varepsilon^1(t)|^{p-1} v_\varepsilon^1(t) - |v_\varepsilon^2(t)|^{p-1} v_\varepsilon^2(t)) e^{-(p+1)Y_\varepsilon} dx.$$

Thus, using the Sobolev embedding  $H_{\delta_1}^s \subset L_{\delta_1}^\infty$  (Lemma A.2.2) and (5.9) in Proposition 5.1.5,

there exists  $\delta > 0$  small enough such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_\varepsilon(t)|^2 e^{-2Y_\varepsilon} dx &\leq \|r_\varepsilon e^{-Y_\varepsilon}\|_{L^2}^2 (\|v_\varepsilon(t)\|_{L^\infty_{(p-1)\delta}}^{p-1} + \|w_\varepsilon(t)\|_{L^\infty_{(p-1)\delta}}^{p-1}) \|e^{-Y_\varepsilon}\|_{L^\infty_{-\delta}}^{p-1} \\ &\leq C(\omega) \|r_\varepsilon e^{-Y_\varepsilon}\|_{L^2}^2 (\|v_\varepsilon(t)\|_{H_{\delta_1}^{s_1}}^{p-1} + \|w_\varepsilon(t)\|_{H_{\delta_1}^{s_1}}^{p-1}). \end{aligned}$$

By Gronwall's inequality, we obtain  $r_\varepsilon(t) = 0$  for any  $-T \leq t \leq T$ , so the uniqueness result follows. This implies that the whole sequence  $\{v_{\varepsilon,n}\}_{n \in \mathbb{N}}$  converges to  $v_\varepsilon$  in  $C([-T, T]; H_{\delta_1}^s)$ .

## 5.6.2 Global well-posedness of the main equation

We now consider the equation (1.30) and prove Theorem 1.4.2. The scheme of the proof is similar to the previous works on NLS equation with white noise potential.

We first note that by taking

$$A = Y_\varepsilon \quad \text{and} \quad V = :|\widetilde{\nabla Y_\varepsilon}|^2:,$$

the conditions (5.26) and (5.27) hold almost surely uniformly in  $\varepsilon \in (0, \frac{1}{2})$  by using the same reasoning as in the beginning of Section 5.5. In particular, we can use Proposition 5.3.1 and Proposition 5.3.3 with  $v$  replaced by  $v_\varepsilon$ .

Fix  $\delta_0 > 0$  and assume that  $v_0 \in H_{\delta_0}^2$ . Let  $0 < \varepsilon_2 < \varepsilon_1 < \frac{1}{2}$ . For  $-T \leq t \leq T$ , we define

$$r(t) = v_{\varepsilon_1}(t) - v_{\varepsilon_2}(t).$$

Then,  $r$  satisfies the equation:

$$\begin{cases} i\partial_t r = \Delta r + :|\widetilde{\nabla Y_{\varepsilon_1}}|^2: r + \left( :|\widetilde{\nabla Y_{\varepsilon_1}}|^2: - :|\widetilde{\nabla Y_{\varepsilon_2}}|^2: \right) v_{\varepsilon_2} - 2\nabla Y_{\varepsilon_1} \cdot \nabla r \\ \quad - 2(\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}) \cdot \nabla v_{\varepsilon_2} - \lambda |v_{\varepsilon_1} e^{-Y_{\varepsilon_1}}|^p v_{\varepsilon_1} + \lambda |v_{\varepsilon_2} e^{-Y_{\varepsilon_2}}|^p v_{\varepsilon_2} \\ r|_{t=0} = 0. \end{cases}$$

Using the equation for  $r$ , we can deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r(t)|^2 e^{-2Y_{\varepsilon_1}} dx \\ &= \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) \left( :|\widetilde{\nabla Y_{\varepsilon_1}}|^2: - :|\widetilde{\nabla Y_{\varepsilon_2}}|^2: \right) v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}} dx \\ &\quad - 2 \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) (\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}) \cdot \nabla v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}} dx \\ &\quad - \lambda \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) (|v_{\varepsilon_1}(t) e^{-Y_{\varepsilon_1}}|^{p-1} v_{\varepsilon_1}(t) - |v_{\varepsilon_2}(t) e^{-Y_{\varepsilon_2}}|^{p-1} v_{\varepsilon_2}(t)) e^{-2Y_{\varepsilon_1}} dx \\ &\stackrel{\text{def}}{=} I + II + III. \end{aligned} \tag{5.95}$$

We now estimate  $I$ ,  $II$ , and  $III$  in (5.95). By duality (Lemma A.2.5), the product estimate (Lemma A.2.4), (5.14) in Proposition 5.1.8, (5.9) in Proposition 5.1.5, interpolation (Lemma A.2.1), and Proposition 5.3.1, we get that for  $0 < \alpha < \frac{1}{100}$ , there exists  $\delta > 0$  small enough and  $\kappa > 0$  such that

$$\begin{aligned} |I| &\leq C \|\bar{r}(t) v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}}\|_{B_{1,1,\delta}^\alpha} \left\| :|\widetilde{\nabla Y_{\varepsilon_1}}|^2: - :|\widetilde{\nabla Y_{\varepsilon_2}}|^2: \right\|_{C_{-\delta}^{-\alpha}} \\ &\leq C(\omega) \varepsilon_1^\kappa \|r(t)\|_{H_\delta^{2\alpha}} \|v_{\varepsilon_2}(t)\|_{H_\delta^{3\alpha}} \|e^{-2Y_{\varepsilon_1}}\|_{C_{-\delta}^{3\alpha}} \\ &\leq C(\omega) \varepsilon_1^\kappa \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}). \end{aligned} \tag{5.96}$$

Similarly, by duality (Lemma A.2.5), the product estimate (Lemma A.2.4), (5.13) in Proposition 5.1.8, (5.9) in Proposition 5.1.5, (5.60) in Proposition 5.3.3, and the  $H^2$  a priori bound

(5.94), there exists  $\delta > 0$  small enough and  $\kappa > 0$  such that

$$\begin{aligned}
|II| &\leq C \|\bar{r}(t) \nabla v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}}\|_{B_{1,1,\delta}^\alpha} \|\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}\|_{C_{-\delta}^{-\alpha}} \\
&\leq C(\omega) \varepsilon_1^\kappa \|r(t)\|_{H_\delta^{2\alpha}} \|\nabla v_{\varepsilon_2}(t)\|_{H_\delta^{3\alpha}} \|e^{-2Y_{\varepsilon_1}}\|_{C_{-\delta}^{3\alpha}} \\
&\leq C(\omega) \varepsilon_1^\kappa \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \|v_{\varepsilon_2}(t)\|_{H_{-\delta}^2} \\
&\leq C(\omega) \varepsilon_1^\kappa |\log \varepsilon_2|^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}).
\end{aligned} \tag{5.97}$$

Concerning *III*, using the Sobolev embedding  $H_\delta^{\frac{2}{2-\eta}} \subset L_\delta^\infty$  with  $\eta, \tilde{\delta} > 0$  small enough (Lemma A.2.2), (5.9) in Proposition 5.1.5, (5.52) in Proposition 5.3.1, (5.10) in Proposition 5.1.5, (5.60) in Proposition 5.3.3, and the  $H^2$  a priori bound (5.94), there exists  $\delta > 0$  small enough and  $\kappa > 0$  such that

$$\begin{aligned}
|III| &\leq C \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 (\|v_{\varepsilon_1}(t)\|_{L_{(p-1)\delta}^{p-1}}^{p-1} + \|v_{\varepsilon_2}(t)\|_{L_{(p-1)\delta}^{p-1}}^{p-1}) \|e^{-Y_{\varepsilon_1}}\|_{L_{-\delta}^\infty}^{p-1} \\
&\quad + \|r\|_{L_\delta^2} \|v_{\varepsilon_2}(t)\|_{L_\delta^{2p}}^p \|e^{-(p-1)Y_{\varepsilon_1}} - e^{-(p-1)Y_{\varepsilon_2}}\|_{L_{-(p-1)\delta}^\infty} \|e^{-Y_{\varepsilon_1}}\|_{L_{-\delta}^\infty} \\
&\leq C(\omega) \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 \left( \|v_{\varepsilon_1}(t)\|_{H_{(p-1)\delta}^{\frac{2}{2-\eta}}}^{p-1} + \|v_{\varepsilon_2}(t)\|_{H_{(p-1)\delta}^{\frac{2}{2-\eta}}}^{p-1} \right) \\
&\quad + C(\omega) \varepsilon_1^\kappa \|v_{\varepsilon_2}(t)\|_{H_\delta^{\frac{2}{2-\eta}}} \mathcal{P}(\|v_0\|_{H_{\delta_0}^1}) \\
&\leq C(\omega) |\log \varepsilon_2|^\gamma \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}) \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 + C(\omega) \varepsilon_1^\kappa |\log \varepsilon_2|^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}),
\end{aligned} \tag{5.98}$$

for some  $0 < \gamma < 1$ .

Now, by letting  $\varepsilon_1 = 2^{-k}$  and  $\varepsilon_2 = 2^{-(k+1)}$  for  $k \in \mathbb{N}$ , we can combine (5.95), (5.96), (5.97), and (5.98) and apply Gronwall's inequality to obtain

$$\begin{aligned}
&\sup_{t \in [-T, T]} \int_{\mathbb{R}^2} |r(t)|^2 e^{-2Y_{2^{-k}}} dx \\
&\leq C(\omega) 2^{-k\kappa} (k+1)^C \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}) e^{C(\omega)T |\log 2^{-(k+1)}|^\gamma \mathcal{P}(\|v_0\|_{H_{\delta_0}^2})} \\
&\leq C(\omega) 2^{-\frac{k\kappa}{2}} 2^{\tilde{\gamma}(k+1)C(\omega)T \mathcal{P}(\|v_0\|_{H_{\delta_0}^2})} \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}),
\end{aligned}$$

where we used  $e^{|\log(1+x)|^\gamma} \leq Cx^{\tilde{\gamma}}$  for  $0 < \gamma < 1$  with  $\tilde{\gamma} > 0$  arbitrarily small and  $x > 0$  large. Thus, by (5.9) in Proposition 5.1.5, we obtain that for any  $\delta > 0$ ,

$$\|v_{2^{-k}} - v_{2^{-(k+1)}}\|_{C_T L_{-\delta}^2}^2 \leq C(\omega) 2^{-\frac{k\kappa}{4}}.$$

Using interpolation (Lemma A.2.1) along with (5.60) in Proposition 5.3.3 and the  $H^2$  a priori bound (5.94), we can deduce that for any  $1 < s < 2$ , there exists  $\delta_1 > 0$  such that  $\{v_{2^{-k}}\}_{k \in \mathbb{N}}$  is a Cauchy sequence in  $C([-T, T]; H_{\delta_1}^s)$  and converges to some function  $v \in C([-T, T]; H_{\delta_1}^s)$ . By using similar steps as above, we can also deduce that

$$\sup_{\varepsilon \in (2^{-(k+1)}, 2^{-k}]} \|v_\varepsilon - v_{2^{-k}}\|_{C_T H_{\delta_1}^s} \leq C(\omega) 2^{-k\tilde{\kappa}} \mathcal{P}(\|v_0\|_{H_{\delta_0}^2}),$$

for some  $\tilde{\kappa} > 0$ , so that the whole sequence  $\{v_\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})}$  converges to  $v$  in  $C([-T, T]; H_{\delta_1}^s)$  as  $\varepsilon \rightarrow 0$ . This finishes the proof of the convergence part of Theorem 1.4.2.

Lastly, we prove the uniqueness of the solution  $v$  in  $C([-T, T]; H_{\delta_1}^s)$  to the equation (1.30). Assume that  $v^1$  and  $v^2$  are two solutions to (1.30). For  $-T \leq t \leq T$ , we define

$$r_0(t) = v^1(t) - v^2(t).$$

Then,  $r_0$  satisfies the equation:

$$\begin{cases} i\partial_t r_0 = \Delta r_0 + :|\widetilde{\nabla Y}|^2: r_0 - 2\nabla Y \cdot \nabla r_0 - \lambda e^{-(p-1)Y} (|v^1|^{p-1}v^1 - |v^2|^{p-1}v^2) \\ r_0|_{t=0} = 0. \end{cases}$$

Using the equation for  $r_0$ , we can deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_0(t)|^2 e^{-2Y} dx = \lambda \text{Im} \int_{\mathbb{R}^2} \overline{r_0}(t) (|v^1(t)|^{p-1}v^1(t) - |v^2(t)|^{p-1}v^2(t)) e^{-(p+1)Y} dx.$$

Thus, using the Sobolev embedding  $H_{\delta_1}^s \subset L_{\delta_1}^\infty$  (Lemma A.2.2) and Lemma 5.1.6, there exists  $\delta > 0$  small enough such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_0(t)|^2 e^{-2Y} dx &\leq \|r_0 e^{-Y}\|_{L^2}^2 (\|v^1(t)\|_{L_{(p-1)\delta}^\infty}^{p-1} + \|v^2(t)\|_{L_{(p-1)\delta}^\infty}^{p-1}) \|e^{-Y}\|_{L_{-\delta}^\infty}^{p-1} \\ &\leq C(\omega) \|r_0 e^{-Y}\|_{L^2}^2 (\|v^1(t)\|_{H_{\delta_1}^s}^{p-1} + \|v^2(t)\|_{H_{\delta_1}^s}^{p-1}). \end{aligned}$$

By Gronwall's inequality, we obtain  $r_0(t) = 0$  for any  $-T \leq t \leq T$ , so the uniqueness result follows.

# Appendix A

## Notations, function spaces, and preliminary lemmas

### A.1 Notations

Throughout the thesis, we drop the harmless factor  $2\pi$ . We use  $A \lesssim B$  to denote  $A \leq CB$  for some constant  $C > 0$ . We write  $A \sim B$  if we have  $A \lesssim B$  and  $B \lesssim A$ . We may use subscripts to denote dependence on external parameters. Also, we use  $a_+$  and  $a_-$  to denote  $a + \varepsilon$  and  $a - \varepsilon$ , respectively, for sufficiently small  $\varepsilon > 0$ .

In this chapter, we let  $\mathcal{M} = \mathbb{R}^2$  or  $\mathbb{T}^2$ , and we let  $\widehat{\mathcal{M}} = \mathbb{R}^2$  or  $\mathbb{Z}^2$ , respectively. For a space-time distribution  $u$ , we write  $\widehat{u}$  or  $\mathcal{F}_{t,x}u$  to denote the space-time Fourier transform of  $u$ . We also write  $\mathcal{F}_x u$  to denote the spatial Fourier transform of  $u$ . If a function  $\phi$  only has a space (or time) variable, then we use  $\widehat{\phi}$  to denote the Fourier transform of  $\phi$  with respect to the space (or time, respectively) variable. We also denote  $\mathcal{F}^{-1}$  as the inverse Fourier transform. For any function  $f$ , we denote  $\widetilde{f}$  as the reflection of  $f$ , i.e.  $\widetilde{f}(x) = f(-x)$ . We set  $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$ . For  $x \in \mathcal{M}$  and  $r > 0$ , we also denote  $B(x, r)$  as the ball in  $\mathcal{M}$  centered at  $x$  of radius  $r$ .

Given a dyadic number  $N \in 2^{\mathbb{N} \cup \{0\}}$ , we let  $P_N$  be the spatial frequency projector onto the frequencies

$$\mathfrak{P}_N \stackrel{\text{def}}{=} \{\xi \in \widehat{\mathcal{M}} : \frac{N}{2} < \langle \xi \rangle \leq N\}.$$

Also, given a dyadic number  $L \in 2^{\mathbb{N} \cup \{0\}}$ , we define  $Q_L$  to be the modulation projector onto the space-time frequencies

$$\mathfrak{G}_L \stackrel{\text{def}}{=} \{(\tau, \xi) \in \mathbb{R} \times \widehat{\mathcal{M}} : \frac{L}{2} < \langle \tau - |n|^2 \rangle \leq L\}.$$

For simplicity, we set  $P_{N,L} = P_N Q_L$ . For a space-time distribution  $u$ , we write  $u_N = P_N u$  for simplicity. We also use  $P_{\neq 0}$  to denote the restriction to nonzero frequencies.

In dealing with space-time functions, we usually use the short-hand notations such as  $L_T^q L_x^r = L^q([0, T]; L^r(\mathcal{M}))$ . We also denote by  $\mathcal{P}(a_1, \dots, a_n)$  a polynomial function depending on  $a_1, \dots, a_n$ .

### A.2 Sobolev and Besov spaces

Let  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . We define the  $L^2$ -based Sobolev space  $H^s(\mathcal{M})$  by the norm

$$\|f\|_{H^s} \stackrel{\text{def}}{=} \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^2_\xi(\widehat{\mathcal{M}})}$$

and we define the  $L^p$ -based Sobolev space  $W^{s,p}(\mathcal{M})$  by the norm

$$\|f\|_{W^{s,p}} \stackrel{\text{def}}{=} \|\langle \nabla \rangle^s f\|_{L^p} = \|\mathcal{F}^{-1}[\langle \xi \rangle^s \widehat{f}(\xi)]\|_{L^p}.$$

When  $p = 2$ , we have  $H^s(\mathcal{M}) = W^{s,2}(\mathcal{M})$ .

We will need weighted function spaces on  $\mathbb{R}^2$ . Given  $1 \leq p \leq \infty$  and  $\mu \in \mathbb{R}$ , we define the weighted  $L^p$  space as

$$\|f\|_{L_\mu^p} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^2} \langle x \rangle^{\mu p} |f|^p dx \right)^{\frac{1}{p}},$$

with the usual interpretation if  $p = \infty$ . We also define

$$\|f\|_{W_\mu^{1,p}} \stackrel{\text{def}}{=} \|f\|_{L_\mu^p} + \|\nabla f\|_{L_\mu^p}.$$

If  $\mu = 0$ , we write  $L^p = L_0^p$  and  $W^{1,p} = W_0^{1,p}$ .

We will also need the Besov spaces. Let  $\phi : \mathbb{R} \rightarrow [0, 1]$  be a smooth bump function supported on  $[-\frac{8}{5}, \frac{8}{5}]$  and  $\phi \equiv 1$  on  $[-\frac{5}{4}, \frac{5}{4}]$ . For  $\xi \in \mathbb{R}^2$ , we set  $\varphi_0(\xi) = \phi(|\xi|)$  and

$$\varphi_j(\xi) = \phi\left(\frac{|\xi|}{2^j}\right) - \phi\left(\frac{|\xi|}{2^{j-1}}\right)$$

for  $j \in \mathbb{N}$ . Then, for  $j \in \mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$ , we define the Littlewood-Paley projector  $\Delta_{2^j}$  as the Fourier multiplier operator with a symbol  $\varphi_j$ . Note that we have

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$$

for each  $\xi \in \mathbb{R}^2$ . Thus, we have

$$f = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \Delta_N f.$$

For  $s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$ , and  $\mu \in \mathbb{R}$ , we define the Besov space  $B_{p,q,\mu}^s = B_{p,q,\mu}^s(\mathbb{R}^2)$  by the norm

$$\|f\|_{B_{p,q,\mu}^s} = \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{sq} \|\Delta_N f\|_{L_\mu^p}^q \right)^{\frac{1}{q}}.$$

If  $\mu = 0$ , we write  $B_{p,q}^s = B_{p,q,0}^s$ . A convenient property of the weighted Besov spaces  $B_{p,q,\mu}^s$  is that the weight can be ‘‘pulled in’’, i.e.

$$\|f\|_{B_{p,q,\mu}^s} \sim \|\langle x \rangle^\mu f\|_{B_{p,q}^s}; \tag{A.1}$$

see [37, Theorem 4.2.2]. The relation (A.1) can be used to translate results from the unweighted spaces to their weighted analogues. We will also need the following commutator bound

$$\|[\nabla, \langle x \rangle^\delta] f\|_{H^\sigma} \lesssim \|f\|_{H^\sigma} \tag{A.2}$$

with  $0 \leq \sigma \leq 1$ , which follows from interpolating the  $L^2 \rightarrow L^2$  bound and the  $H^1 \rightarrow H^1$  bound of the commutator  $[\nabla, \langle x \rangle^\delta]$ .

When  $(p, q) = (2, 2)$ , the weighted Besov spaces are generalizations of weighted Sobolev spaces  $B_{2,2,\mu}^s = H_\mu^s$ , where

$$\|f\|_{H_\mu^s} \stackrel{\text{def}}{=} \|\langle \nabla \rangle^s f\|_{L_\mu^2}.$$

Note that we have the following obvious continuous embedding for  $s_1 \geq s_2$  and  $\mu_1 \geq \mu_2$ :

$$H_{\mu_1}^{s_1} \subset H_{\mu_2}^{s_2}. \tag{A.3}$$

The embedding (A.3) is compact if  $s_1 > s_2$  and  $\mu_1 > \mu_2$ ; see [37, Section 4.2.3]. When  $(p, q) = (\infty, \infty)$ , we denote  $\mathcal{C}_\mu^s = B_{\infty, \infty, \mu}^s$ ; in the case  $s \geq 0$  and  $s \notin \mathbb{Z}$ , the space  $\mathcal{C}^s$  is equivalent to the classical weighted Hölder-Zygmund space:

$$\|f\|_{\mathcal{C}_\mu^s} \sim \sum_{|k| \leq \lfloor s \rfloor} \sup_{s \in \mathbb{R}^2} \langle x \rangle^s |\partial^k f(x)| + \sum_{|k| = \lfloor s \rfloor} \sup_{\substack{x, y \in \mathbb{R}^2 \\ 0 < |x-y| \leq 1}} \langle x \rangle^\mu \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{s-\lfloor s \rfloor}},$$

which can be seen from (A.1) and [46].

We now present and recall some useful properties of the weighted Besov spaces, starting with the following interpolation inequality. For a proof, see [104, Theorem 3.8].

**Lemma A.2.1.** *Let  $s, s_1, s_2, \mu, \mu_1, \mu_2 \in \mathbb{R}$ ,  $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$ , and  $\theta \in [0, 1]$  be such that*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2, \quad \mu = (1-\theta)\mu_1 + \theta\mu_2.$$

Then, we have

$$\|f\|_{B_{p, q, \mu}^s} \lesssim \|f\|_{B_{p_1, q_1, \mu_1}^{s_1}}^{1-\theta} \|f\|_{B_{p_2, q_2, \mu_2}^{s_2}}^\theta,$$

where the underlying constant may depend on all the parameters but independent of  $f$ .

Next, we state a Sobolev embedding for weighted Sobolev spaces, whose proof follows from (A.1) and the usual Sobolev embedding for unweighted Sobolev spaces.

**Lemma A.2.2.** *Let  $2 \leq p < \infty$ ,  $s \geq 1 - \frac{2}{p}$ , and  $\mu, \nu \in \mathbb{R}$  with  $\nu \leq \mu$ . Then, we have*

$$\|f\|_{L_\nu^p} \lesssim \|f\|_{H_\mu^s}.$$

Moreover, for any  $s > 1$ , we have

$$\|f\|_{L_\nu^\infty} \lesssim \|f\|_{H_\mu^s}.$$

As a consequence, we have the following estimate.

**Lemma A.2.3.** *Let  $2 \leq q < \infty$  and  $0 < \delta \leq 1$ . Then, we have*

$$\|f\|_{W_\delta^{1, q}} \lesssim \|f\|_{H_{-\delta}^{2-\frac{1}{q}}}^{1-\frac{1}{q}} \|f\|_{L_{(2q-1)\delta}^2}^{\frac{1}{q}}.$$

*Proof.* By the commutator bound (A.2), the Sobolev embedding (Lemma A.2.2), and interpolations (Lemma A.2.1), we obtain

$$\begin{aligned} \|f\|_{W_\delta^{1, q}} &\lesssim \|f\|_{H_\delta^{2-\frac{2}{q}}} \\ &\lesssim \|f\|_{H_{-\delta}^{2-\frac{2}{q}}}^{1-\frac{2}{q}} \|f\|_{H_{(q-1)\delta}^1}^{\frac{2}{q}} \\ &\lesssim \|f\|_{H_{-\delta}^{2-\frac{2}{q}}}^{1-\frac{2}{q}} \|f\|_{H_{-\delta}^2}^{\frac{1}{q}} \|f\|_{L_{(2q-1)\delta}^2}^{\frac{1}{q}}, \end{aligned}$$

which is the desired estimate.  $\square$

We will also need the following product estimate. For a proof, see [2] or [101].

**Lemma A.2.4.** *Let  $s, s_1, s_2, \mu, \mu_1, \mu_2 \in \mathbb{R}$  and  $1 \leq p, p_1, p_2 \leq \infty$  be such that*

$$s_1 + s_2 > 0, \quad s = \min(s_1, s_2, s_1 + s_2), \quad \mu = \mu_1 + \mu_2, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then, for any  $\kappa > 0$ , we have

$$\|f_1 \cdot f_2\|_{B_{p,p,\mu}^{s-\kappa}} \lesssim \|f_1\|_{B_{p_1,p_1,\mu_1}^{s_1}} \|f_2\|_{B_{p_2,p_2,\mu_2}^{s_2}}.$$

Also, we have

$$\|f_1 \cdot f_2\|_{C_\mu^s} \lesssim \|f_1\|_{C_{\mu_1}^{s_1}} \|f_2\|_{C_{\mu_2}^{s_2}}.$$

Finally, we record the following duality property. For a proof, see [112, Theorem 2.11.2].

**Lemma A.2.5.** *Let  $s, \mu \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ . Let  $p'$  and  $q'$  be the Hölder conjugates of  $p$  and  $q$ , respectively. Then, we have*

$$\left| \int_{\mathbb{R}^2} f_1 \cdot f_2 dx \right| \lesssim \|f_1\|_{B_{p,q,\mu}^s} \|f_2\|_{B_{p',q',-\mu}^{-s}}.$$

### A.3 Preliminary estimates on weighted spaces

In this section, we gather some properties of the weighted Besov spaces and will be used later.

#### A.3.1 Some properties of the Littlewood-Paley projection

In this subsection, we present some properties of the Littlewood-Paley projection on weighted Besov spaces. We start with the following elementary but useful property.

**Lemma A.3.1.** *Let  $s, \delta \geq 0$  and  $s_0 > 0$ . Then, we have*

$$\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^s \|\Delta_N f\|_{L_\delta^2} \lesssim \|f\|_{H_\delta^{s+s_0}}.$$

*Proof.* By the Cauchy-Schwarz inequality, we have

$$\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^s \|\Delta_N f\|_{L_\delta^2} \lesssim \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2(s+s_0)} \|\Delta_N f\|_{L_\delta^2}^2 \right)^{\frac{1}{2}} = \|f\|_{H_\delta^{s+s_0}},$$

which is the desired estimate.  $\square$

We now show the following commutator estimates.

**Lemma A.3.2.** *Let  $0 < \delta < 1$ ,  $1 \leq p \leq \infty$ , and  $N \geq 1$  be a dyadic number. Then, we have*

$$\|[\Delta_N, \langle x \rangle^\delta] f\|_{L^p} \lesssim N^{-1} \|f\|_{L^p}, \quad (\text{A.4})$$

$$\|[\nabla, [\Delta_N, \langle x \rangle^\delta]] f\|_{L^p} \lesssim N^{-1} \|f\|_{L^p}. \quad (\text{A.5})$$

*Proof.* Let  $N^2 K(N \cdot)$  be the kernel that corresponds to the Littlewood-Paley projection  $\Delta_N$ . Note that we have

$$[\Delta_N, \langle x \rangle^\delta] = N^2 \int_{\mathbb{R}^2} (\langle y \rangle^\delta - \langle x \rangle^\delta) K(N(x-y)) f(y) dy. \quad (\text{A.6})$$

Since  $|\langle y \rangle^\delta - \langle x \rangle^\delta| \lesssim |x-y|$  for  $\delta < 1$ , we have

$$\begin{aligned} \sup_{x \in \mathbb{R}^2} \left( N^2 \int_{\mathbb{R}^2} |\langle y \rangle^\delta - \langle x \rangle^\delta| |K(N(x-y))| dy \right) &\lesssim N^{-1} \| |x| K(x) \|_{L^1}, \\ \sup_{y \in \mathbb{R}^2} \left( N^2 \int_{\mathbb{R}^2} |\langle y \rangle^\delta - \langle x \rangle^\delta| |K(N(x-y))| dy \right) &\lesssim N^{-1} \| |x| K(x) \|_{L^1}, \end{aligned}$$

so that by (A.6) and Schur's test, we obtain the desired estimate (A.4).

For (A.5), we denote by  $\tilde{K}_N(x, y) = N^2(\langle y \rangle^\delta - \langle x \rangle^\delta)$  the kernel of the commutator  $[\Delta_N, \langle x \rangle^\delta]$ . Thus, we have

$$[\nabla, [\Delta_N, \langle x \rangle^\delta]]f = \int_{\mathbb{R}^2} (\nabla_x \tilde{K}_N(x, y) + \nabla_y \tilde{K}_N(x, y)) f(y) dy.$$

Since

$$\nabla_x \tilde{K}_N(x, y) + \nabla_y \tilde{K}_N(x, y) = N^2(-\nabla_x \langle x \rangle^\delta + \nabla_y \langle y \rangle^\delta) K(N(x - y))$$

and  $|\nabla_x \langle x \rangle^\delta + \nabla_y \langle y \rangle^\delta| \lesssim |x - y|$ , we can then conclude (A.5) by using Schur's test as above.  $\square$

Next, we show a useful property of the Littlewood-Paley projector  $\Delta_N$  in weighted Sobolev spaces.

**Lemma A.3.3.** *Let  $0 < \delta < 1$ ,  $0 \leq s \leq 2$ , and  $N \geq 1$  be a dyadic number. Then, we have*

$$\|\Delta_N f\|_{H_\delta^s} \lesssim N^s (\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2}) \quad (\text{A.7})$$

with the understanding that  $\Delta_M = 0$  if  $M < 1$ .

*Proof.* The case  $s = 0$  is obvious. We consider the following three cases.

**Case 1:**  $0 < s < 1$ .

By using (A.1) and (A.4) in Lemma A.3.2, we obtain

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^s}^2 &\lesssim \sum_{\substack{M \geq 1 \\ \text{dyadic}}} M^{2s} \|\Delta_M(\langle x \rangle^\delta \Delta_N f)\|_{L^2}^2 \\ &\lesssim \sum_{\substack{M \geq 1 \\ \text{dyadic}}} M^{2s} \|\langle x \rangle^\delta \Delta_M(\Delta_N f)\|_{L^2}^2 + \sum_{\substack{M \geq 1 \\ \text{dyadic}}} M^{2s-2} \|\Delta_N f\|_{L^2}^2 \\ &\lesssim \sum_{M=\frac{N}{2}, N, 2N} M^{2s} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + \|\Delta_N f\|_{L^2}^2, \end{aligned} \quad (\text{A.8})$$

where we used  $2s - 2 < 0$  for  $0 < s < 1$ . Thus, we obtain

$$\|\Delta_N f\|_{H_\delta^s}^2 \lesssim N^{2s} (\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2}) + N^{-2s} \|\Delta_N f\|_{H_\delta^2}^2. \quad (\text{A.9})$$

From (A.9), we obtain the desired estimate (A.7) provided that  $N > N_0$ , with  $N_0$  chosen in such a way that the right-hand-side of (A.9) can be absorbed on the left-hand-side. Moreover, we have finitely many dyadic numbers  $N$  with  $1 \leq N \leq N_0$ , so that (A.7) holds for these values of  $N$  provided that the underlying constant on the right-hand-side of (A.7) is large enough.

**Case 2:**  $1 \leq s < 2$ .

We first note that by (A.1) and (A.2), we have

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^s} &\lesssim \|\langle x \rangle^\delta \Delta_N f\|_{H^s} \\ &\lesssim \|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\nabla(\langle x \rangle^\delta \Delta_N f)\|_{H^{s-1}} \\ &\lesssim \|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\langle x \rangle^\delta (\Delta_N \nabla f)\|_{H^{s-1}} + \|[\nabla, \langle x \rangle^\delta] \Delta_N f\|_{H^{s-1}} \\ &\lesssim \|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\langle x \rangle^\delta (\Delta_N \nabla f)\|_{H^{s-1}} + \|\Delta_N f\|_{H^{s-1}}. \end{aligned} \quad (\text{A.10})$$

Thus, by (A.10) and (A.7) from Case 1, we have

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^s} &\lesssim N^{s-1} (\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2}) \\ &\quad + N^{s-1} (\|\Delta_{\frac{N}{2}} \nabla f\|_{L_\delta^2} + \|\Delta_N \nabla f\|_{L_\delta^2} + \|\Delta_{2N} \nabla f\|_{L_\delta^2}). \end{aligned}$$

Then, for any dyadic number  $M \geq 1$ , by the fact that  $(\Delta_{\frac{M}{2}} + \Delta_M + \Delta_{2M})\Delta_M = \Delta_M$ , (A.2),

and Lemma A.3.2, we have

$$\begin{aligned}
\|\Delta_M \nabla f\|_{L_\delta^2} &\lesssim \|\nabla(\langle x \rangle^\delta \Delta_M f)\|_{L^2} + \|[\langle x \rangle^\delta, \nabla] \Delta_M f\|_{L^2} \\
&\lesssim \sum_{M'=\frac{M}{2}, M, 2M} \left( \|\nabla \Delta_{M'}(\langle x \rangle^\delta \Delta_M f)\|_{L^2} + \|\nabla([\Delta_{M'}, \langle x \rangle^\delta] \Delta_M f)\|_{L^2} \right) \\
&\quad + \|\Delta_M f\|_{L^2} \\
&\lesssim \sum_{M'=\frac{M}{2}, M, 2M} \left( \|[\Delta_{M'}, \langle x \rangle^\delta] \Delta_M \nabla f\|_{L^2} + \|[\nabla, [\Delta_{M'}, \langle x \rangle^\delta]] \Delta_M f\|_{L^2} \right) \\
&\quad + M \|\Delta_M f\|_{L_\delta^2} \\
&\lesssim M \|\Delta_M f\|_{L_\delta^2}.
\end{aligned} \tag{A.11}$$

The desired estimate (A.7) then follows from (A.10) and (A.11).

**Case 3:**  $s = 2$ .

Note that (A.10) also holds for  $s = 2$ . Thus, by (A.10) and (A.7) from Case 2, we obtain

$$\begin{aligned}
\|\Delta_N f\|_{H_\delta^2} &\lesssim \|\Delta_N f\|_{H_\delta^1} + \|\Delta_N \nabla f\|_{H_\delta^1} \\
&\lesssim N(\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2}) \\
&\quad + N(\|\Delta_{\frac{N}{2}} \nabla f\|_{L_\delta^2} + \|\Delta_N \nabla f\|_{L_\delta^2} + \|\Delta_{2N} \nabla f\|_{L_\delta^2}).
\end{aligned}$$

We can then conclude (A.7) by using (A.11).  $\square$

We will also need the following estimate of the Littlewood-Paley projector  $\Delta_N$  in weighted Sobolev spaces.

**Lemma A.3.4.** *Let  $\delta > 0$ ,  $0 < s < 2$ ,  $s_0 \in \mathbb{R}$ , and  $N \geq 1$  be a dyadic number. Then, we have*

$$\|\Delta_N f\|_{H_\delta^s} \lesssim N^{s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M f\|_{H_\delta^{s-s_0}} \tag{A.12}$$

with the understanding that  $\Delta_M = 0$  if  $M < 1$ .

*Proof.* We consider the following two cases.

**Case 1:**  $0 < s < 1$ .

By (A.8) and (A.4) in Lemma A.3.2, we obtain

$$\begin{aligned}
\|\Delta_N f\|_{H_\delta^s}^2 &\lesssim \sum_{M=\frac{N}{2}, N, 2N} M^{2s} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + \|\Delta_N f\|_{L^2}^2 \\
&\lesssim N^{2s_0} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + \|\Delta_N f\|_{L^2}^2 \\
&\lesssim N^{2s_0} \sum_{M'=\frac{M}{2}, M, 2M} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0} \|\Delta_M(\langle x \rangle^\delta \Delta_{M'} f)\|_{L^2}^2 \\
&\quad + N^{2s_0} \sum_{M'=\frac{M}{2}, M, 2M} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0-2} \|\Delta_{M'} f\|_{L^2}^2 + \|\Delta_N f\|_{L^2}^2.
\end{aligned}$$

We can then conclude as in the proof of Lemma A.3.3 by absorbing  $\|\Delta_N f\|_{L^2}^2$  on the left-hand-side.

**Case 2:**  $1 \leq s < 2$ .

From (A.10) and (A.12) from Case 1, we have

$$\begin{aligned}
\|\Delta_N f\|_{H_\delta^s}^2 &\lesssim \|\Delta_N f\|_{H_\delta^{s-1}}^2 + \|\Delta_N \nabla f\|_{H_\delta^{s-1}}^2 \\
&\lesssim N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M f\|_{H_\delta^{s-1-s_0}}^2 + \|\Delta_N f\|_{L^2}^2 \\
&\quad + N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M \nabla f\|_{H_\delta^{s-1-s_0}}^2 + \|\Delta_N \nabla f\|_{L^2}^2.
\end{aligned}$$

As in the proof of Lemma A.3.3, we can absorb  $\|\Delta_N f\|_{L^2}^2$  and  $\|\Delta_N \nabla f\|_{L^2}^2$  on the left-hand-side, so that by (A.11) we obtain

$$\begin{aligned}
\|\Delta_N f\|_{H_\delta^s}^2 &\lesssim N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M f\|_{H_\delta^{s-1-s_0}}^2 + N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M \nabla f\|_{H_\delta^{s-1-s_0}}^2 \\
&\lesssim N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} \|\Delta_M f\|_{H_\delta^{s-1-s_0}}^2 + N^{2s_0} \sum_{\substack{\frac{N}{4} \leq M \leq 4N \\ \text{dyadic}}} M^{2s-2s_0} \|\Delta_M f\|_{L_\delta^2}^2.
\end{aligned}$$

We can then conclude the desired estimate.  $\square$

We close this subsection with the following result.

**Lemma A.3.5.** *Let  $s_1, s_2 \in \mathbb{R}$ ,  $\delta \geq 0$ , and  $\varphi \in C_c^\infty(\mathbb{R}^2)$ . Then, we have*

$$\|\varphi * f\|_{C_{-\delta}^{s_1}} \lesssim \|f\|_{C_{-\delta}^{s_2}}.$$

*Proof.* For any dyadic  $N \geq 1$ , since  $(\Delta_{\frac{N}{2}} + \Delta_N + \Delta_{2N})\Delta_N = \Delta_N$  and  $\langle x \rangle^{-1} \lesssim \langle y \rangle \langle x-y \rangle^{-1}$ , by Hölder's inequality we obtain

$$\begin{aligned}
N^{s_1} \|\Delta_N(\varphi * f)\|_{L_{-\delta}^\infty} &= N^{s_1} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \varphi * \Delta_N f\|_{L_{-\delta}^\infty} \\
&\leq N^{s_1} \|\Delta_N f\|_{L_{-\delta}^\infty} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \varphi\|_{L_\delta^1} \\
&\leq N^{s_2} \|\Delta_N f\|_{L_{-\delta}^\infty} \sum_{M=\frac{N}{2}, N, 2N} M^{s_1-s_2} \|\Delta_M \varphi\|_{L_{2+\delta+\varepsilon}^\infty},
\end{aligned}$$

where  $\varepsilon > 0$ . We can then conclude since  $\varphi \in C_{2+\delta+\varepsilon}^{s_1-s_2}$ .  $\square$

### A.3.2 Some estimates on the approximation identity

In this subsection, we present some useful estimates for  $\rho_\varepsilon(x) = \varepsilon^{-2} \rho(\varepsilon^{-1}x)$ , where  $\rho \in C_c^\infty(B(0,1))$ ,  $\rho \geq 0$ , and  $\int_{\mathbb{R}^2} \rho = 1$ . We start with the following estimate, which was proved in [7] and in [8, Lemma 8] (in a different case with a similar proof).

**Lemma A.3.6.** *Let  $\delta > 0$  and  $0 < \varepsilon < \frac{1}{2}$ . Then, we have*

$$\|\rho_\varepsilon\|_{B_{1,2,\delta}^0} \lesssim \sqrt{|\log \varepsilon|}.$$

*Proof.* We first note that

$$\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)(\varepsilon x) = N^2 \int_{\mathbb{R}^2} K(\varepsilon N(x-y)) \langle \varepsilon y \rangle^\delta \rho(y) dy.$$

Thus, we deduce that

$$\begin{aligned}
\varepsilon^{-2} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} &\lesssim N^2 \|K(\varepsilon N x)\|_{L^1} \|\langle \varepsilon x \rangle^\delta \rho\|_{L^1} \\
&= \varepsilon^{-2} \|K\|_{L^1} \|\langle \varepsilon x \rangle^\delta \rho\|_{L^1} \\
&\leq \varepsilon^{-2} \|K\|_{L^1} \|\rho\|_{L^1_\delta}.
\end{aligned} \tag{A.13}$$

Let  $N_0$  be the dyadic number such that  $N_0 \leq \frac{1}{\varepsilon} < 2N_0$ . Thus, from (A.13), we get

$$\sum_{1 \leq N \leq 8N_0} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 \lesssim |\log \varepsilon|. \tag{A.14}$$

We now focus on the case  $N \geq 8N_0$ . Note that

$$\varepsilon^2 \langle x \rangle^\delta \rho_\varepsilon = \sum_{\substack{M \geq 1 \\ \text{dyadic}}} \Delta_M(\langle \varepsilon x \rangle^\delta \rho)\left(\frac{x}{\varepsilon}\right), \tag{A.15}$$

where the Fourier transform of each summand on the right-hand-side of (A.15) is supported in

$$\left\{ \frac{M}{2\varepsilon} \leq |\xi| \leq \frac{2M}{\varepsilon} \right\} \subset \left\{ \frac{MN_0}{2} \leq |\xi| \leq 4MN_0 \right\}.$$

Thus, we obtain

$$\varepsilon^2 \Delta_N(\langle x \rangle^\delta \rho_\varepsilon) = \sum_{\substack{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0} \\ \text{dyadic}}} \Delta_N\left(\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\left(\frac{x}{\varepsilon}\right)\right), \tag{A.16}$$

which implies that

$$\begin{aligned}
\|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 &\leq \left( \sum_{\substack{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0} \\ \text{dyadic}}} \varepsilon^{-2} \left\| \Delta_M(\langle \varepsilon x \rangle^\delta \rho)\left(\frac{x}{\varepsilon}\right) \right\|_{L^1} \right)^2 \\
&\leq 6 \sum_{\substack{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0} \\ \text{dyadic}}} \|\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1}^2.
\end{aligned}$$

This shows that

$$\sum_{\substack{N > 8N_0 \\ \text{dyadic}}} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 \leq 36 \sum_{\substack{M \geq 1 \\ \text{dyadic}}} \|\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1}^2 \lesssim \|\langle \varepsilon x \rangle^\delta \rho\|_{B_{1,2}^0}^2. \tag{A.17}$$

Combining (A.14) and (A.17) along with the obvious bound  $\sup_{\varepsilon \in (0, \frac{1}{2})} \|\langle \varepsilon x \rangle^\delta \rho\|_{B_{1,2}^0} < \infty$ , we can conclude our estimate.  $\square$

We will also need the following estimate.

**Lemma A.3.7.** *Let  $\delta \geq 0$ ,  $0 < \varepsilon < 1$ ,  $0 < s_1 < 1$ , and  $s_2 > 0$  with  $s_1 + s_2 < 1$ . Then, we have*

$$\|\rho_\varepsilon * f\|_{\mathcal{C}_{-\delta}^{s_1+1}} \lesssim \varepsilon^{-s_1-s_2} \|f\|_{\mathcal{C}_{-\delta}^{1-s_2}}.$$

*Proof.* As in the proof of Lemma A.3.5, we get

$$N^{1+s_1} \|\Delta_N(\rho_\varepsilon * f)\|_{L_{-\delta}^\infty} \leq N^{1-s_2} \|\Delta_N f\|_{L_{-\delta}^\infty} \sum_{M=\frac{N}{2}, N, 2N} M^{s_1+s_2} \|\Delta_M \rho_\varepsilon\|_{L^1_\delta}.$$

Thus, we only need to show that

$$(\varepsilon N)^{s_1+s_2} \|\Delta_N \rho_\varepsilon\|_{L^1_\delta} \leq C < \infty \quad (\text{A.18})$$

uniformly in  $\varepsilon$  and  $N$ .

To prove (A.18), since  $0 < s_1 + s_2 < 1$ , by (A.4) in Lemma A.3.2, we only need to prove

$$(\varepsilon N)^{s_1+s_2} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \leq C < \infty \quad (\text{A.19})$$

uniformly in  $\varepsilon$  and  $N$ . Let  $N_0$  be the dyadic number such that  $N_0 \leq \frac{1}{\varepsilon} < 2N_0$ . When  $N \leq 8N_0$ , we can easily obtain (A.19). When  $N > 8N_0$ , by (A.16), we obtain

$$\|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \leq \sum_{\substack{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0} \\ \text{dyadic}}} \|\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1},$$

so that

$$\left(\frac{N}{N_0}\right)^{s_1+s_2} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \lesssim \sup_{\varepsilon \in (0,1)} \|\langle \varepsilon x \rangle^\delta \rho\|_{B_{1,\infty}^{s_1+s_2}}.$$

This implies (A.19) since  $\sup_{\varepsilon \in (0,1)} \|\langle \varepsilon x \rangle^\delta \rho\|_{B_{1,\infty}^{s_1+s_2}} < \infty$ .  $\square$

We close this subsection with the following convergence result.

**Lemma A.3.8.** *Let  $s \in \mathbb{R}$ ,  $0 < \varepsilon, \eta < 1$ , and  $\delta \geq 0$ . Then, we have*

$$\|\rho_\varepsilon * f - f\|_{C_{-\delta}^s} \lesssim \varepsilon^\eta \|f\|_{C_{-\delta}^{s+\eta}}.$$

*Proof.* Let  $N \geq 1$  be a dyadic number. If  $\varepsilon N \geq 1$ , by the fact that  $\langle x \rangle^{-1} \lesssim \langle y \rangle \langle x - y \rangle^{-1}$  and Hölder's inequality, we have

$$\begin{aligned} N^s \|\Delta_N(\rho_\varepsilon * f) - \Delta_N f\|_{L_{-\delta}^\infty} &\leq N^s \|\rho_\varepsilon * \Delta_N f\|_{L_{-\delta}^\infty} + N^s \|\Delta_N f\|_{L_{-\delta}^\infty} \\ &\leq N^s \|\rho_\varepsilon\|_{L^1_\delta} \|\Delta_N f\|_{L_{-\delta}^\infty} + N^s \|\Delta_N f\|_{L_{-\delta}^\infty} \\ &\lesssim \varepsilon^\eta N^{s+\eta} \|\Delta_N f\|_{L_{-\delta}^\infty}. \end{aligned} \quad (\text{A.20})$$

If  $\varepsilon N < 1$ , we write

$$\langle x \rangle^{-\delta} (\Delta_N(\rho_\varepsilon * f) - \Delta_N f) = \int_{\mathbb{R}^2} \tilde{K}_{\varepsilon,N}(x,y) \langle y \rangle^{-\delta} f(y) dy,$$

where

$$\begin{aligned} \tilde{K}_{\varepsilon,N}(x,y) &= N^2 \frac{\langle y \rangle^\delta}{\langle x \rangle^\delta} \left( \int_{\mathbb{R}^2} K(Nx - Nz) \rho_\varepsilon(z - y) dz - K(Nx - Ny) \right) \\ &= N^2 \frac{\langle y \rangle^\delta}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} (K(Nx - Nz) - K(Nx - Ny)) \rho_\varepsilon(z - y) dz. \end{aligned}$$

For fixed  $x$ , we can compute that

$$\begin{aligned} &\int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon,N}(x,y)| dy \\ &\leq \frac{N^2}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - Nz) - K(Nx - Ny)| \rho_\varepsilon(z - y) \langle y \rangle^\delta dy dz \\ &= \frac{(\varepsilon N)^2}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - \varepsilon Nz) - K(Nx - \varepsilon Ny)| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz \\ &\leq \frac{(\varepsilon N)^3}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \sup_{w \in [Nx - \varepsilon Nz, Nx - \varepsilon Ny]} |\nabla K(w)| \right) |z - y| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz, \end{aligned} \quad (\text{A.21})$$

where we denoted by  $[a, b]$  the segment between  $a$  and  $b$ . We define

$$G(x) = \sup_{w \in B(x,1)} |\nabla K(w)|.$$

Note that for any  $\bar{x} \in \mathbb{R}^2$  and  $0 < \lambda < 1$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \sup_{w \in [\bar{x} - \lambda z, \bar{x} - \lambda y]} |\nabla K(w)| \right) |z - y| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz \\ & \lesssim \int_{\mathbb{R}^2} \int_{|z-y| < 1} \left( \sup_{w \in B(\bar{x} - \lambda z, \lambda)} |\nabla K(w)| \right) \langle \varepsilon z \rangle^\delta dz dy \\ & \lesssim \int_{\mathbb{R}^2} G(\bar{x} - \lambda z) \langle \varepsilon z \rangle^\delta dz \\ & = \lambda^{-2} \int_{\mathbb{R}^2} G(u) \langle \varepsilon \lambda^{-1} (\bar{x} - u) \rangle^\delta du. \end{aligned} \tag{A.22}$$

Thus, from (A.21) and (A.22) with  $\bar{x} = Nx$  and  $\lambda = \varepsilon N$ , we obtain

$$\int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon, N}(x, y)| dy \lesssim \varepsilon N \langle x \rangle^{-\delta} \int_{\mathbb{R}^2} G(u) \langle x - N^{-1}u \rangle^\delta du \lesssim \varepsilon N. \tag{A.23}$$

uniformly in  $x \in \mathbb{R}^2$ . Also, for fixed  $y$ , we use similar steps to obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon, N}(x, y)| dx \\ & = N^2 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - \varepsilon Nz - Ny) - K(Nx - Ny)| \rho(z) \langle x \rangle^{-\delta} dx dz \\ & \lesssim \varepsilon N^3 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \sup_{w \in [Nx - \varepsilon Nz - Ny, Nx - Ny]} |\nabla K(w)| \right) |z| \rho(z) \langle x \rangle^{-\delta} dx dz \\ & \lesssim \varepsilon N^3 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{|z| < 1} G(N(x - y)) \langle x \rangle^{-\delta} dz dx \\ & \lesssim \varepsilon N \langle y \rangle^\delta \int_{\mathbb{R}^2} G(x) \langle N^{-1}x + y \rangle^{-\delta} dx \lesssim \varepsilon N. \end{aligned} \tag{A.24}$$

Combining (A.23) and (A.24) and using Schur's test, we obtain

$$\|\Delta_N(\rho_\varepsilon * f) - \Delta_N f\|_{L^\infty_\delta} \lesssim \varepsilon N \|\Delta_N f\|_{L^\infty_\delta} \lesssim \varepsilon^\eta N^\eta \|\Delta_N f\|_{L^\infty_\delta}. \tag{A.25}$$

We can then conclude by combining (A.20) and (A.25).  $\square$

## A.4 Fourier restriction norm method

In this section, we recall the definition and properties of  $X^{s,b}$ -spaces for the Schrödinger equations, known as the Fourier restriction norm method introduced by Bourgain [9]. For  $s, b \in \mathbb{R}$ , we define the space  $X^{s,b} = X^{s,b}(\mathbb{R} \times \mathbb{T}^2)$  to be the completion of functions that are smooth in space and Schwartz in time with respect to the norm

$$\|u\|_{X^{s,b}} \stackrel{\text{def}}{=} \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^b \hat{u}(\tau, n) \right\|_{L^2_\tau \ell^2_n(\mathbb{R} \times \mathbb{Z}^2)}. \tag{A.26}$$

For  $T > 0$ , we define the space  $X_T^{s,b}$  to be the restriction of the  $X^{s,b}$ -space onto the time interval  $[-T, T]$  via the norm

$$\|u\|_{X_T^{s,b}} \stackrel{\text{def}}{=} \inf \{ \|v\|_{X^{s,b}} : v|_{[-T, T]} = u \}. \tag{A.27}$$

Note that  $X_T^{s,b}$  is complete. Given any  $s \in \mathbb{R}$  and  $b > \frac{1}{2}$ , we have  $X_T^{s,b} \hookrightarrow C([-T, T]; H^s(\mathbb{T}^2))$ .

We now present and recall some useful properties related to  $X^{s,b}$ -spaces, starting with the following homogeneous linear estimate of the  $X^{s,b}$ -norm as in [9].

**Lemma A.4.1.** *Let  $s \in \mathbb{R}$ ,  $b \leq 1$ , and  $k \in \mathbb{Z}_{\geq 0}$ . Let  $\eta$  be a smooth function supported on  $[-2, 2]$ . Then, we have*

$$\|t^k \eta(t) e^{-it\Delta} \phi\|_{X^{s,b}} \lesssim_{\eta} 3^k \|\phi\|_{H^s(\mathbb{T}^2)}.$$

*Proof.* Note that since  $b \leq 1$ , we have

$$\begin{aligned} \|t^k \eta(t) e^{it\Delta} \phi\|_{X^{s,b}} &= \left\| \widehat{(\cdot)^k \eta}(\tau - |n|^2) \langle \tau - |n|^2 \rangle^b \langle n \rangle^s \widehat{\phi}(n) \right\|_{L^2_{\tau} L^2_n} \\ &= \|t^k \eta(t)\|_{H^b(\mathbb{R})} \|\phi\|_{H^s(\mathbb{T}^2)} \\ &\lesssim (\|t^k \eta(t)\|_{L^2(\mathbb{R})} + \|\partial_t(t^k \eta(t))\|_{L^2(\mathbb{R})}) \|\phi\|_{H^s(\mathbb{T}^2)} \\ &\lesssim (2^k + k2^{k-1})(\|\eta\|_{L^2(\mathbb{R})} + \|\partial_t \eta\|_{L^2(\mathbb{R})}) \|\phi\|_{H^s(\mathbb{T}^2)} \\ &\lesssim_{\eta} 3^k \|\phi\|_{H^s(\mathbb{T}^2)}, \end{aligned}$$

which is the desired estimate.  $\square$

Let us define the following Duhamel operator

$$\mathcal{I}F(t) = \mathcal{I}_{\chi} F(t) \stackrel{\text{def}}{=} \chi(t) \int_0^t \chi(t') e^{-i(t-t')\Delta} F(t') dt', \quad (\text{A.28})$$

where  $\chi : \mathbb{R} \rightarrow [0, 1]$  is a smooth function such that  $\chi \equiv 1$  on  $[-1, 1]$  and  $\chi \equiv 0$  outside of  $[-2, 2]$ . We now record the inhomogeneous linear estimate of the  $X^{s,b}$ -norm for the Duhamel operator  $\mathcal{I}$ . For a proof, see [9, 43, 111].

**Lemma A.4.2.** *Let  $s \in \mathbb{R}$  and  $b > \frac{1}{2}$ . Then, we have*

$$\|\mathcal{I}F\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}.$$

We also record the following formula. For a proof, see [32, Lemma 3.1].

**Lemma A.4.3.** *For any  $\tau \in \mathbb{R}$  and  $n \in \mathbb{Z}^2$ , we have the formula*

$$\widehat{\mathcal{I}F}(\tau + |n|^2, n) = \int_{\mathbb{R}} K(\tau, \tau') \widehat{F}(\tau' + |n|^2, n) d\tau',$$

where the kernel  $K$  satisfies

$$|K(\tau, \tau')| \lesssim \left( \frac{1}{\langle \tau \rangle^3} + \frac{1}{\langle \tau - \tau' \rangle^3} \right) \frac{1}{\langle \tau' \rangle} \lesssim \frac{1}{\langle \tau \rangle \langle \tau - \tau' \rangle}.$$

Next, we recall the following time localization estimate. For a proof, see [9, 111].

**Lemma A.4.4.** *Let  $s \in \mathbb{R}$ ,  $-\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}$ , and  $0 < T \leq 1$ . Let  $\eta$  be a Schwartz function in time and let  $\eta_T = \eta(t/T)$ . Then, we have*

$$\|\eta_T u\|_{X^{s,b_1}} \lesssim_{\eta} T^{b_2-b_1} \|u\|_{X^{s,b_2}}.$$

Lastly, we record the following  $L^4$ -Strichartz estimate on  $\mathbb{T}^2$ . For a proof, see [9, 12].

**Lemma A.4.5.** *Let  $0 < s < \frac{1}{2}$  and  $b > \frac{1-s}{2}$ . Let  $Q$  be a ball of radius  $N \geq 1$  (not necessarily centered at the origin) and let  $P_Q$  be the spatial frequency projector onto  $\{n \in \mathbb{Z}^2 : n \in Q\}$ . Then, we have*

$$\|P_Q u\|_{L_t^4 L_x^4([-1,1] \times \mathbb{T}^2)} \lesssim N^s \|u\|_{X^{s,b}}.$$

We conclude this section by recording the following lemma. For a proof, see [76, Lemma 2.5].

**Lemma A.4.6.** *Let  $N_j, L_j \geq 1$ ,  $j = 0, 1, 2$ , be dyadic numbers. Suppose that  $u_1, u_2 \in L_t^2 L_x^2(\mathbb{R} \times \mathbb{T}^2)$  satisfy*

$$\text{supp } \widehat{u_1} \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1} \quad \text{and} \quad \text{supp } \widehat{u_2} \subset \mathfrak{P}_{N_2} \cap \mathfrak{S}_{L_2}.$$

Then, we have

$$\begin{aligned} \|\mathcal{F}_{t,x}(u_1 \overline{u_2})\|_{L_t^2 \ell_n^2(\mathfrak{P}_{N_0})} &\lesssim \min(L_1, L_2)^{\frac{1}{2}} \min(N_0, N_1, N_2)^{\frac{1}{2}} \\ &\times \left( \frac{\max(L_1, L_2)}{N_0} + 1 \right)^{\frac{1}{2}} \|u_1\|_{L_t^2 L_x^2} \|u_2\|_{L_t^2 L_x^2}. \end{aligned}$$

## A.5 Counting estimates and a convolution lemma

In this section, we recall some counting estimates and a convolution lemma. We start with the following fact from number theory. For a proof, see [33, Lemma 4.3].

**Lemma A.5.1.** *Let  $a_0, b_0 \in \mathbb{C}$ ,  $m \in \mathbb{Z}[i] \setminus \{0\}$ , and  $M_1, M_2 > 0$ . Then, the number of tuples  $(a, b) \in (\mathbb{Z}[i])^2$  that satisfies*

$$ab = m, \quad |a - a_0| \leq M_1, \quad |b - b_0| \leq M_2$$

is  $O(M_1^\varepsilon M_2^\varepsilon)$  for any small  $\varepsilon > 0$ , where the underlying constant depends only on  $\varepsilon$ .

We now show the following counting estimates.

**Lemma A.5.2.** *Let  $N_0, N_1, N_2 \geq 1$  be dyadic numbers and let  $n_0, n_1, n_2 \in \mathbb{Z}^2$  be such that  $n_j$  lies in a ball of radius  $N_j$  for  $j = 0, 1, 2$ ,  $n_0 - n_1 + n_2 = 0$ , and  $|n_0|^2 - |n_1|^2 + |n_2|^2 = m$  for some fixed  $m \in \mathbb{Z}$ .*

- (i) *The number of tuples  $(n_0, n_1, n_2) \in (\mathbb{Z}^2)^3$  that satisfy the above conditions is  $O(N_1 N_2 \max\{N_1^\varepsilon, N_2^\varepsilon\})$  for any small  $\varepsilon > 0$ , where the underlying constant depends only on  $\varepsilon$ .*
- (ii) *If  $n_1$  is fixed, then the number of tuples  $(n_0, n_2) \in (\mathbb{Z}^2)^2$  that satisfy the above conditions is  $O(\max\{N_0^\varepsilon, N_2^\varepsilon\})$  for any small  $\varepsilon > 0$ , where the underlying constant depends only on  $\varepsilon$ .*
- (iii) *If  $n_2$  is fixed and  $n_2 \neq 0$ , then the number of tuples  $(n_0, n_1) \in (\mathbb{Z}^2)^2$  that satisfy the above conditions is  $O(\min\{N_0, N_1\})$ .*
- (iv) *If  $n$  is fixed and  $n \neq 0$ , then the number of tuples  $(n_1, n_2) \in (\mathbb{Z}^2)^2$  that satisfy the above conditions is  $O(\min\{N_1, N_2\})$ .*

*Proof.* (i) See [33, Lemma 4.3] for the proof of this part.

(ii) Since  $n_1$  is fixed, we know that  $n_0 + n_2 = n_1$  is fixed. Let  $k = (k_1, k_2) = n_0 - n_2$ , so that we have

$$(k_1 + ik_2)(k_1 - ik_2) = |k|^2 = 2|n_0|^2 + 2|n_2|^2 - |n_0 + n_2|^2 = 2m + |n_1|^2$$

is fixed. Since  $k$  lies in a ball of radius  $\leq N_0 + N_2$ , by Lemma A.5.1, we know that the number of choices for  $k$  is  $O(\max\{N_0^\varepsilon, N_2^\varepsilon\})$  for any small  $\varepsilon > 0$ .

(iii) Note that since  $n_0 = n_1 - n_2$ , we have

$$m = |n_1 - n_2|^2 - |n_1|^2 + |n_2|^2 = -2n_1 \cdot n_2 + 2|n_2|^2.$$

This shows that  $n_1 \cdot n_2$  is fixed, which means that  $n_1$  is restricted to a line. Also, we have

$$m = |n|^2 - |n_0 + n_2|^2 + |n_2|^2 = -2n_0 \cdot n_2.$$

This shows that  $n_0 \cdot n_2$  is fixed, which means that  $n_0$  is restricted to a line. Thus, the number of choices for  $(n_0, n_1)$  is  $O(\min\{N_0, N_1\})$ .

(iv) The proof of this part is the same as that in part (iii). Thus, we omit details.  $\square$

We also record the following counting lemma. For a proof, see [76, Lemma 2.9(ii)], which was stated in a general dimension  $d \geq 2$  but we only need the  $d = 2$  case.

**Lemma A.5.3.** *Let  $N \gg 1$ ,  $N^{-1} \leq \mu, \nu \ll N$ ,  $M \geq 0$ , and*

$$D \stackrel{\text{def}}{=} \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : N \leq |\xi| \leq N + \mu, M \leq \xi_1 \leq M + \nu\}.$$

*Let  $\mathcal{R}$  be an arbitrary rotation operator on  $\mathbb{R}^2$ . Moreover, with  $e_1 = (1, 0) \in \mathbb{R}^2$ , we set*

$$K \stackrel{\text{def}}{=} \left\{ \xi \in \mathbb{R}^2 : \frac{\beta}{2} \leq \angle(\xi, e_1) \leq 2\beta \right\},$$

*where  $\angle(\xi, e_1)$  denotes the angle between  $\xi$  and  $e_1$  and  $\beta > 0$  satisfies*

$$\left( \frac{\mu + \min\{\nu, 1\}}{N} \right)^{\frac{1}{2}} \ll \beta \leq \frac{\pi}{4}.$$

*Then, we have*

$$|\mathbb{Z}^2 \cap \mathcal{R}(D \cap K)| \lesssim \max\{\nu, 1\}(\beta^{-1}(\mu + \min\{\nu, 1\}) + 1).$$

We end this section by recording the following convolution inequality. For a proof, see [43, Lemma 4.2].

**Lemma A.5.4.** *Let  $0 \leq \beta \leq \gamma$  with  $\gamma > 1$ . Then, for any  $a \in \mathbb{R}$ , we have*

$$\int_{\mathbb{R}} \frac{1}{\langle x \rangle^\beta \langle x - a \rangle^\gamma} \lesssim \frac{1}{\langle a \rangle^\beta}.$$

## A.6 Wiener chaos estimate

In this section, we recall the Wiener chaos estimate. Let  $(H, B, \mu)$  be an abstract Wiener space, where  $\mu$  is a Gaussian measure on a separable Banach space  $B$  and  $H \subset B$  is its Cameron-Martin space. Let  $\{e_j\}_{j \in \mathbb{N}} \subset B$  be an orthonormal system of  $H^* = H$ . We define a polynomial chaos of order  $k$  as an element of the form  $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle)$ , where  $x \in B$ ,  $k_j \neq 0$  for finitely many  $j$ 's,  $k = \sum_{j=1}^{\infty} k_j$ ,  $H_{k_j}$  is the Hermite polynomial of degree  $k_j$ , and  $\langle \cdot, \cdot \rangle = {}_B \langle \cdot, \cdot \rangle_{B^*}$  denotes the  $B$ - $B^*$  duality pairing. We denote by  $\mathcal{H}_k$  the closure of all polynomial chaoses of order  $k$  in the space  $L^2(B, \mu)$ . We also denote

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j \tag{A.29}$$

for  $k \in \mathbb{N}$ .

Let  $L$  be the Ornstein-Uhlenbeck operator. It is known that any element in  $\mathcal{H}_k$  is an eigenfunction of  $L$  with eigenvalue  $-k$ . Then, we have the following Wiener chaos estimate as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup  $U(t) = e^{tL}$  due to Nelson [92]. For a proof, see [105, Theorem I.22].

**Lemma A.6.1.** *Let  $k \in \mathbb{N}$  and  $p \geq 2$ . Then, for any  $X \in \mathcal{H}_{\leq k}$ , we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}.$$



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