Hyperbolic Monopoles

Ruth Hawksley

Doctor of Philosophy
University of Edinburgh
1998

Abstract

A Euclidean $SU(2)$ monopole consists of a connection and Higgs field on an $SU(2)$ bundle over $\mathbb{R}^3$, satisfying certain partial differential equations. Monopoles may equivalently be described in terms of holomorphic vector bundles on twistor space, algebraic curves in twistor space, rational maps, or solutions to Nahm's equations (a set of ODEs for matrix-valued functions), all satisfying some further conditions. Research by Atiyah, Donaldson, Hitchin, Nahm and others has provided a beautiful and relatively complete picture of these different viewpoints and the links between them.

Monopoles have also been studied on hyperbolic space $\mathbb{H}^3$, although the corresponding picture in this case is less well understood. One difficulty is that the conditions which must be imposed in order for all the various correspondences to be valid have not yet been completely determined. A partial answer is given in Chapter 2, where it is proved that any hyperbolic monopole arising from a spectral curve satisfies a certain natural boundary condition. The proof uses the algebraic geometry of the spectral curve and is similar to Hurtubise's proof of the analogous result in the Euclidean case.

A large part of this thesis concentrates on the "Braam-Austin" description of hyperbolic monopoles. This is the hyperbolic version of Nahm's description of Euclidean monopoles; a monopole corresponds to a pair of discrete matrix-valued functions satisfying some difference equations. Euclidean monopoles appear as limits of hyperbolic monopoles as the curvature of $\mathbb{H}^3$ tends to zero. This "Euclidean limit" is described geometrically and is studied in terms of Braam-Austin data. Explicit conditions are given for such a sequence to have a subsequence converging to a Euclidean monopole. The result depends on a conjecture (§4.5) about properties of Braam-Austin monopole solutions.

Explicit solutions are given in the case of charge 2, based on Ward's solution of the discrete Toda equations. The geometry of hyperbolic 2-monopoles is discussed in some detail, including their spectral curves and the case of "widely separated" monopoles. The Euclidean limit for charge 2 is illustrated explicitly.

Acknowledgements

I would like to thank my supervisor, Michael Singer, whose help and patience have made this thesis possible. I would also like to thank the numerous other people, in Edinburgh and beyond, who have helped me mathematically or who have been good friends over the past few years - and especially those who have done both! My thanks go to the EPSRC for financial support.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Ruth Hawksley)

Table of Contents

Chapter 1

Introduction

A monopole on a 3-manifold M is a soliton solution (meaning it behaves in some sense like a particle) to the Bogomolny equations (§1.2). These are partial differential equations for a connection and Higgs field defined on a bundle over M and are a dimensional reduction of the Yang-Mills equations for a 4-manifold. For nontrivial solutions to exist M must be noncompact, and for the purposes of this thesis M will always be either Euclidean space $\mathbb{R}^3$ or hyperbolic space $\mathbb{H}^3$.

A monopole has two topological invariants, the **charge** $k \in \mathbb{Z}$ and the **mass** $m \in \mathbb{R}_{>0}$. The mass may be rescaled, at the cost of rescaling the curvature of M, so the mass of a Euclidean monopole is generally assumed to be 1. On $\mathbb{H}^3$ however, or any space with nonzero curvature, the mass is a significant parameter (the curvature of hyperbolic space is usually fixed to be -1). On the other hand, scaling the curvature of M is equivalent to scaling the mass. It was conjectured by Atiyah [4] and proved by Jarvis and Norbury [20] that in a precise sense the moduli space of hyperbolic monopoles becomes the Euclidean monopole moduli space as the curvature of $\mathbb{H}^3$ tends to zero. This corresponds to the mass tending to infinity. Their result used a description of monopoles in terms of rational maps $f : \mathbb{C}P^1 \to \mathbb{C}P^1$. A large part of this thesis is devoted to studying the same Euclidean limit in terms of more explicit descriptions; it is not easy to recover any other description of a monopole from its rational map. The aim here is not to study the moduli spaces but instead to understand the convergence in terms of sequences of hyperbolic monopoles whose limit is a Euclidean monopole.

There are several equivalent ways to study monopoles, and the relations between them are interesting in their own right. The correspondence between "extended" monopole moduli spaces and spaces of rational maps was proved by Donaldson [13] in the Euclidean case, by Atiyah [4] for hyperbolic monopoles with integral mass and by Munari [22] for general hyperbolic monopoles. If the mass of a hyperbolic monopole is an integer, the monopole defines a circle-invariant instanton on $\mathbb{R}^4$; if not the same process gives an instanton on $\mathbb{R}^4 \setminus \mathbb{R}^2$ with nontrivial holonomy around the removed $\mathbb{R}^2$. Thus results about integral hyperbolic monopoles may be obtained using known facts about instantons, in particular the "ADHM" description; but these are not easily generalised to the nonintegral case. A Euclidean monopole may be thought of as a translation-invariant "instanton", though it will have infinite action (the action is the integral of the norm-squared of the curvature over $\mathbb{R}^4$).

Using twistor theory, both Euclidean and hyperbolic monopoles have descriptions in terms of holomorphic bundles over a "minitwistor" space Z [16], which is the total space of $T\mathbb{P}^1$ in the Euclidean case and $\mathbb{P}^1 \times \mathbb{P}^1$ in the hyperbolic case (where $\mathbb{P}^1$ denotes the complex projective line). A consequence of this is that a monopole determines and is determined by an algebraic curve $\mathcal{S}$ in Z, called the **spectral curve**. The procedure for obtaining $\mathcal{S}$ is explained by Hitchin in [16], and by Murray and Singer in [23] for the hyperbolic case. It will be discussed in more detail later.

The other very useful description of monopoles is related to the ADHM construction for instantons and was first used by Nahm [25]. The ADHM construction represents instantons in terms of algebraic objects: vector spaces and a linear map between them which depends linearly on $x \in \mathbb{R}^4$. In Nahm's version, for certain instantons with infinite action, the vector spaces are infinite dimensional and the linear map is replaced by a differential operator. This operator is determined by three functions $T_1(s)$, $T_2(s)$ and $T_3(s)$ on the interval $(0, 2)$, which have values in the 2×2 matrices. They must satisfy certain conditions, including Nahm's equations:

$$\frac{dT_i}{ds} = [T_j, T_k]$$

for each cyclic permutation (i, j, k) of $(1, 2, 3)$. In the hyperbolic case it is more obvious how to adapt the ADHM description for integral monopoles, by restricting to the circle-invariant case. This was done by Braam and Austin in [12]. The result is again a set of matrix-valued functions, this time on a discrete subset of the interval and satisfying a set of difference equations, which will be referred to as the "Braam-Austin equations". There is as yet no similar description for non-integral monopoles.

Although similar tools may be used to study Euclidean and hyperbolic monopoles, the picture in the hyperbolic case is less well understood. In particular it is not known what boundary conditions must be imposed on the connection and Higgs field of the monopole to ensure that all the correspondences between the various descriptions hold. It is shown in Chapter 2 that every monopole (with connection ∇, Higgs field Φ and mass $m \geq \frac{1}{2}$) which arises from a spectral curve satisfies the boundary condition

$$|\Phi| = m + O(e^{-2r})$$

where r is the geodesic distance from the origin in $\mathbb{H}^3$. This is a step towards finding a complete set of boundary conditions for (∇, Φ). The "Braam-Austin" description of integral monopoles, in terms of matrices and difference equations, is given in Chapter 3, where it is compared to the standard form of ADHM data. Solutions for 1-monopoles and an axially symmetric 2-monopole are given, together with some results on centring. Chapter 4 is concerned with the Euclidean limit in this picture and describes the way in which Braam-Austin data becomes Nahm data as the curvature of $\mathbb{H}^3$ tends to zero. In Chapter 5 a full solution for charge 2 monopoles with fixed centre and orientation is given, following Richard Ward's solution of the "discrete Toda" equations [29]; these are essentially the same as the Braam-Austin equations. The centre, spectral curve and Euclidean limit are discussed in detail for this solution. Some of the results on which the Braam-Austin description rests have proofs in terms of equivariant cohomology, and these are mentioned briefly in Appendix A. The rest of this chapter contains definitions of monopoles and instantons, together with some details about the link between them and the correspondence with holomorphic bundles via twistor theory.

1.1 Bundles and connections

Let E be a vector bundle over a manifold M. Then the p-forms $\Omega^p(M)$ on M and the E-valued p-forms $\Omega^p(E)$ are given by

$$\begin{aligned}\Omega^p(M) &= \Gamma(\Lambda^p T^*M) \\ \Omega^p(E) &= \Gamma(\Lambda^p T^*M \otimes E)\end{aligned}$$

where T^*M is the dual of the tangent bundle of M, $\Gamma(F)$ denotes the space of sections of the bundle F, and Λ^p refers to the pth exterior power. A **connection** A on E may be thought of as a covariant derivative D_A, which is a map from sections of E to bundle-valued one-forms:

$$D_A : \Omega^0(E) \to \Omega^1(E)$$

A will also sometimes be used to denote a one-form with values in End$E$, such that in a local frame for E, the map D_A is given by $d+A$, i.e. $D_A s = ds + As$ where d is the (de Rham) exterior derivative. From this point of view the **curvature** F_A of A is given by

$$F_A = dA + [A, A]$$

which is an endomorphism-valued 2-form. In fact, if E has structure group G, with Lie algebra $\mathfrak{g}$, then F_A has values in the **adjoint bundle** of E. For $G = SU(n)$, this is the subbundle of End$E$ consisting of trace-free endomorphisms. For general G, its fibres may be identified with $\mathfrak{g}$. The 1-form A may also be thought of as $\mathfrak{g}$-valued, or as a 1-form on the adjoint bundle of E. Note that the commutator $[A, A]$ in the above expression is a matrix commutator, but with ordinary multiplication replaced by exterior multiplication (wedge product).

The covariant derivative D_A extends to a **covariant exterior derivative**

$$d_A : \Omega^p(E) \to \Omega^{p+1}(E)$$

by imposing the Leibnitz rule. To be more precise, using the isomorphism

$$\Omega^p(E) \simeq \Omega^p(M) \otimes \Gamma(E)$$

the map d_A is given by

$$d_A(\zeta \otimes s) = d\zeta \otimes s + (-1)^{\deg(\zeta)} \zeta \otimes D_A s$$

where $\zeta \in \Omega^p(M)$, $s \in \Gamma(E)$ and d is the exterior derivative. There is an equivalent definition of curvature in terms of d_A, which is

$$F_A = d_A \circ d_A$$

and any connection A satisfies the **Bianchi identity**:

$$d_A F_A = 0$$

Given bundles E and E' on M, with connections A and A' respectively, there is a **direct sum connection** defined on $F = E \oplus E'$ by

$$D_{A \oplus A'}(s \oplus s') = D_A s \oplus D_{A'} s'$$

Any connection on F which may be written in this form as a direct sum is said to be **reducible**. On the other hand, a connection which does not fix any nontrivial subbundle is called **irreducible**.

If M is an n-dimensional, oriented Riemannian manifold (so that it has a metric $\langle , \rangle$ and a fixed n-form [vol]), then the Hodge star operator is defined. This is a map

$$* : \Omega^p(M) \to \Omega^{n-p}(M)$$

defined by the identity

$$a \wedge *b = \langle a, b \rangle . [\mathrm{vol}]$$

where $a, b \in \Omega^p(M)$. In future, $*_4$ will always denote the Hodge star on $\mathbb{R}^4$ with the Euclidean metric, and $*_3$ the star on $\mathbb{H}^3$ with the hyperbolic upper half space metric. Usually no subscript is needed as it should be clear which metric is being used. For clarity, connections on bundles over $\mathbb{R}^4$ will often be denoted by a bold $\mathbb{A}$, and connections on $\mathbb{R}^3$ or $\mathbb{H}^3$ by A.

1.2 Definitions of monopoles and instantons

Definition 1. *A* **monopole** *is a solution (A, Φ) to the Bogomolny equations:*

$$D_A\Phi = *F_A \tag{1.1}$$

over an oriented Riemannian 3-manifold M, where A is a connection on an $SU(2)$-bundle $E \to M$, F_A is its curvature, and Φ, the Higgs field, is a section of the adjoint bundle.

There are monopoles for other Lie groups, and most of the theory generalises to these cases, but the generalisations will not be considered here. The group will always be $SU(2)$, and so Φ may be thought of as an $\mathfrak{su}(2)$-valued section.

A **hyperbolic monopole** is a solution to these equations over hyperbolic 3-space $\mathbb{H}^3$, which will usually be thought of in terms of the upper half space model, with coordinates $(u, t) \in \mathbb{C} \oplus \mathbb{R}_{>0}$. Monopoles on $\mathbb{R}^3$ with its standard metric will always be referred to as Euclidean monopoles.

Monopoles are studied modulo the action of the **gauge group**, $\mathcal{G}$, which may be thought of as the group of bundle automorphisms of E fixing $\mathbb{H}^3$, or the group of transformations between local trivialisations of E. (Physicists refer to "fixing a gauge" when they mean choosing a trivialisation.) Two monopoles are equivalent if they are related by a bundle automorphism

$$g : \mathbb{H}^3 \to SU(2)$$

(a smooth map giving a "change of coordinates" in each fibre). If the action of $\mathcal{G}$ on sections of E is $s \mapsto gs$, then the action on the monopole is given by $(D_A, \Phi) \mapsto (gD_Ag^{-1}, g\Phi g^{-1})$. The action on F_A is $F_A \mapsto gF_Ag^{-1}$. Given a framing (or local trivialisation) of E this is, for Φ and F_A, conjugation of matrices in the usual sense (although the entries of F_A are 2-forms). D_A is a differential operator, thus if it has connection 1-form α, so that $D_A(f) = df + \alpha f$, then the action of $g \in \mathcal{G}$ on α is

$$\alpha \mapsto g\alpha g^{-1} + dgg^{-1}$$

There are several boundary conditions which are usually imposed on monopoles to ensure that the space of solutions is finite dimensional; one of these is that it must have **finite energy**, i.e. that

$$\int_{\mathbb{H}^3} (|F_A|^2 + |\Phi|^2)[\mathrm{vol}]$$

is finite. The norms are $\mathfrak{su}(2)$-norms, so $|\Phi|^2 = -\frac{1}{2}\mathrm{tr}\Phi^2$. The space of monopoles with fixed mass and charge is defined to be the space of solutions which have finite energy, modulo gauge.

F_A will sometimes be shortened to F. Another important boundary condition is that $|\Phi|$ has a constant (nonzero) limit at infinity, i.e. that if $x \in \partial\mathbb{H}^3$ is a boundary point, choose local coordinates (u, t) centred at x (so x is the point $(0, 0)$), then

$$\lim_{t\to 0} |\Phi(0, t)| = m$$

This limit $m \in \mathbb{R}_{>0}$ is called the **mass** of the monopole. For Euclidean monopoles m can always be taken to be 1, but in the hyperbolic case scaling the mass also scales the curvature. The assumption that $\mathbb{H}^3$ has constant curvature -1 means that the mass is a topological invariant. Usually, a slightly stronger boundary condition is imposed: that the errors in this boundary approximation decay like t^2. So there is a local frame near the boundary where

$$\Phi = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} + O(t^2) \tag{1.2}$$

One justification for this is that it holds for any monopole arising from a spectral curve. Further explanation of this and a proof will be given in Chapter 2.

The **charge** of a monopole is a topological invariant defined by

$$k = \lim_{r\to\infty} \frac{1}{4\pi m} \int_{S^2_r} \mathrm{tr}(\Phi F) \tag{1.3}$$

where S^2_r is the sphere radius r in $\mathbb{H}^3$, centred at the origin. The charge is always a positive integer which may be identified with the degree of the map

$$\frac{\Phi}{|\Phi|} : S^2_r \to S^2$$

where r is large and the image S^2 is the unit sphere in $\mathfrak{su}(2) \cong \mathbb{R}^3$ (for a proof, see [15], page 372).

Definition 2. *An* **instanton** *on an oriented Riemannian 4-manifold M_4 is a solution $\mathbb{A}$ to one of the equations*

$$*F_{\mathbb{A}} = \pm F_{\mathbb{A}} \tag{1.4}$$

where $\mathbb{A}$ is a connection on an SU(2)-bundle over M_4 and $F_{\mathbb{A}}$ is its curvature, which will often be written $\mathbb{F}$. An instanton is called **self-dual** *(SD) or* **anti-self-dual** *(ASD) depending whether the sign in (1.4) is plus or minus.*

Instantons are also considered up to gauge, where the gauge group $\mathcal{G}$ is the group of smooth maps $g : M_4 \mapsto SU(2)$. For the space of instantons on a manifold, modulo gauge, to be finite dimensional, instantons are required to have **finite action**. In other words, the integral

$$\int_{\mathbb{R}^4} |\mathbb{F}|^2 d^4x$$

must be finite. For $\mathbb{R}^4$ instantons this means that $\mathbb{A}$ and $\mathbb{F}$ extend from $\mathbb{R}^4$ to S^4, by a result of Uhlenbeck on removable singularities [28].

SU(2) instantons on S^4 have a single topological invariant, the **instanton number** $\kappa \in \mathbb{Z}_{>0}$, given by evaluating the second Chern class of the bundle on the fundamental class $[S^4]$.

1.3 Monopoles as circle-invariant instantons

When its mass is an integer, a hyperbolic monopole can be described as a circle-invariant instanton. These are relatively well understood, so this is a useful point of view for studying monopoles. It was first pointed out by Atiyah in [4] and is the hyperbolic analogue of the description of Euclidean monopoles as translation invariant solutions of (1.4) (see [7], p120). The hyperbolic case seems simpler in this respect than the Euclidean case, since it involves finite action instantons; Euclidean monopoles correspond to "instantons" with infinite action.

To see how the correspondence works, first take coordinates (x, y, t, θ) on $\mathbb{R}^4$ (where (t, θ) are polar coordinates on $\mathbb{R}^2$). Then the metric is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dt^2 + t^2 d\theta^2 \\ &= t^2 \left\{ \frac{dx^2 + dy^2 + dt^2}{t^2} + d\theta^2 \right\} \end{aligned}$$

Hence there is a conformal equivalence

$$\mathbb{R}^4 \setminus \mathbb{R}^2 \sim \mathbb{H}^3 \times S^1 \tag{1.5}$$

because $(dx^2 + dy^2 + dt^2)/t^2$ is the hyperbolic metric for the upper half space model of $\mathbb{H}^3$ $(t > 0)$. This means that $\mathbb{H}^3$ may be thought of as the quotient of $\mathbb{R}^4 \setminus \mathbb{R}^2$ by the natural circle action (given by θ). The fixed point set is the $\mathbb{R}^2$ spanned by x and y, which is the set $\{t = 0\}$, and this corresponds to the boundary of $\mathbb{H}^3$.

An instanton $\mathbb{A}$ is given by

$$\mathbb{A} = A_x dx + A_y dy + A_t dt + A_\theta d\theta$$

and $\mathbb{A}$ circle invariant just means that (in some gauge) the functions A_i must be independent of θ. Writing

$$\mathbb{A} = A + \Phi d\theta$$

gives a way to change between instantons and monopoles (suppressing the projection map $\tau : S^4 \to \mathbb{H}^3$), provided the following is true:

Lemma 1.1. *A pair (A, Φ) satisfies (1.1) iff $\mathbb{A} = A + \Phi d\theta$ satisfies (1.4) with a fixed sign.*

The sign depends on the orientations, so monopoles may correspond to SD or ASD instantons depending on the conventions chosen. The proof below uses the conventions of Atiyah in [4], starting from a given orientation of $\mathbb{R}^4$ and giving a correspondence with ASD instantons.

Proof Use the circle action to orient the t, θ plane. The orientation of $\mathbb{R}^4$ then induces an orientation on the fixed $\mathbb{R}^2$ (the x, y plane) and this, in turn, orients $\mathbb{H}^3$, which has the fixed $\mathbb{R}^2$ as boundary. The orientation of the fixed $\mathbb{R}^2$ corresponds to an "inward-pointing" normal, so that $\mathbb{H}^3$ inherits the opposite orientation by this method to the one it has from the equivalence (1.5). If $\mathbb{A} = A + \Phi d\theta$ then

$$\begin{aligned} \mathbb{F} &= F + (D_A + \Phi d\theta)\Phi d\theta - \Phi d\theta(D_A + \Phi d\theta) \\ &= F + D_A(\Phi) \wedge d\theta \end{aligned}$$

So, taking account of the orientations,

$$*_4\mathbb{F} = - *_3 F \wedge d\theta - *_3 D_A(\Phi)$$

Thus $\mathbb{F} = - *_4 \mathbb{F}$ if and only if $*_3 F = D_A(\Phi)$, i.e. $\mathbb{A}$ is an instanton if and only if (A, Φ) is a hyperbolic monopole. □

This shows that every hyperbolic monopole is equivalent to an instanton on $\mathbb{R}^4 \setminus \mathbb{R}^2$, though there is no reason to expect it to extend to the whole of $\mathbb{R}^4$. Generally it does not extend:

Lemma 1.2. *If $m \notin \mathbb{Z}$ then the instanton $\mathbb{A}$ given by the monopole (A, Φ) does not extend to the whole of $\mathbb{R}^4$.*

Proof The problem with extending $\mathbb{A}$ to $\mathbb{R}^4$ is that $d\theta$ is singular. If (x, y, u, v) are Euclidean coordinates on $\mathbb{R}^4$, where $u = t\cos\theta$ and $v = t\sin\theta$, then

$$\begin{aligned} dt &= du.\cos\theta + dv.\sin\theta \\ d\theta &= \frac{dv.\cos\theta - du.\sin\theta}{t} \end{aligned}$$

which shows that $d\theta$ is singular at $t = 0$, but $td\theta$ is not. The connection $\mathbb{A}$ cannot be extended across $t = 0$ unless there is a gauge where the $d\theta$ term is zero. By the boundary condition (1.2), there is a local frame where

$$\Phi = i \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} + O(t^2)$$

(recall that "∞" is at $t = 0$). If m is an integer, $g \in \mathcal{G}$ given by

$$g = \begin{pmatrix} e^{-im\theta} & 0 \\ 0 & e^{im\theta} \end{pmatrix} \tag{1.6}$$

is well defined on $\mathbb{R}^4 \setminus \mathbb{R}^2$. Changing gauge with g, the $d\theta$ part of $\mathbb{A}$ becomes

$$g^{-1}\Phi d\theta g + O(t^2)d\theta + g^{-1}\partial_\theta g = O(t^2)d\theta$$

which is nonsingular, so in this case A may extend.

On the other hand, any $g \in \mathcal{G}$ which removes the $d\theta$ term must involve $e^{\pm im\theta}$, since g must satisfy

$$\begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} g = -\frac{\partial g}{\partial \theta}$$

near $t = 0$. Solving these differential equations for the entries of g gives terms $Ce^{\pm im\theta}$. These functions are not well defined for $m \notin \mathbb{Z}$, and so instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$ with nonintegral mass do not extend to $\mathbb{R}^4$. □

It is a theorem of Sibner and Sibner [26] that the converse is true: when the mass is an integer, the instanton does extend to $\mathbb{R}^4$ (and so to S^4), provided the connection and curvature lie in certain Sobolev spaces.

This shows that the singularity at $t = 0$ is removable precisely for integral monopoles, which seems to imply that the only monopoles for which the S^4 picture makes sense are those with integral mass. However, looking for a gauge where everything is independent of θ is a fairly crude (and not gauge invariant) method of looking for circle invariance. Braam's more sophisticated approach ([11], pp 430-431) shows that the S^4 picture is in fact useful for $m \in \frac{1}{2}\mathbb{Z}$.

1.3.1 Half integral mass and double covers

To understand Braam's argument it is necessary to look more closely at actions of S^1 on SU(2)-bundles over S^4. The structure of SU(2) is important here - there are no such complications if, for example, the group is SO(3).

Let X be a manifold with a circle action $\mu : S^1 \to \text{Diff}(X)$, E an SU(2)-bundle on X (the construction is no more complicated in this general case than for S^4), and let $\mathcal{A}^*$ denote the set of irreducible connections on E. If $\xi \in S^1$, the diffeomorphism $\mu(\xi)$ of X is covered by some bundle automorphism of E (because

S^1 is connected). This automorphism is determined up to gauge because the group of automorphisms covering the identity is exactly $\mathcal{G}$. In other words, let $T \subset \mathrm{Diff}(X)$ be the group

$$T = \{\mu(\xi) : \xi \in S^1\}$$

and let $\mathcal{H}$ be the group of all bundle automorphisms covering some element of T. Then

$$e \to \mathcal{G} \to \mathcal{H} \to T \to e \tag{1.7}$$

is exact. Now the natural way to define an action of S^1 on E is as a homomorphism $\phi : S^1 \to \mathcal{H}$ which covers the action on X:

$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{\phi}{\nearrow} & \downarrow p \\ S^1 & \xrightarrow[\mu]{} & T \end{array}$$

where p is the natural projection.

Given such an action, S^1 acts on $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ and if $[A]$ is a fixed point of this action, A is fixed by the action on E up to gauge.

Lemma 1.3. *Let $\mathcal{G}_A \subset \mathcal{G}$ and $\mathcal{H}_A \subset \mathcal{H}$ be the stabilisers of a connection A. Then $[A] \in \mathcal{B}^*$ is a fixed point iff*

$$e \to \mathcal{G}_A \to \mathcal{H}_A \to T \to e \tag{1.8}$$

is exact.

Proof If $[A]$ is a fixed point then (1.8) is exact: By exactness of (1.7), the projection $\mathcal{H} \to T$ is onto; every $\mu(\xi) \in T$ is covered by something in $\mathcal{H}$. So if $\mathcal{H}_A \to T$ is not onto then there is some element of $\mathcal{H}$ which moves $[A]$, and $[A]$ is not a fixed point. The group $\mathcal{G}_A$ is the subgroup of $\mathcal{H}_A$ covering the identity, so (1.8) is exact for the same reasons that (1.7) is exact.

If (1.8) is exact then $[A]$ is a fixed point: In this case the map $\mathcal{H}_A \to T$ is onto, so any diffeomorphism of X induced by some $\xi \in S^1$ is covered by something in $\mathcal{H}_A$, which fixes A. Since the lift to $\mathcal{H}$ is unique up to gauge, any lift fixes $[A]$. □

For an irreducible connection A, the stabiliser $\mathcal{G}_A$ is $\{\pm 1\}$, so $\mathcal{H}_A$ is an extension

$$e \to \{\pm 1\} \to \mathcal{H}_A \to T \to e$$

For hyperbolic monopoles, as described above, $T \simeq S^1$. Thus $\mathcal{H}_A = \mathbb{Z}_2 \times S^1$ or $\mathcal{H}_A = \tilde{S}^1$, where $\tilde{S}^1$ is the double cover of S^1. The interpretation of this is that the natural way to lift an action of S^1 on X to an SU(2)-bundle E over X is as an $\tilde{S}^1$ action which covers the S^1 action on X, as this allows either possibility for $\mathcal{H}_A$. A circle invariant connection is then one which is invariant under such a lift. Note that if there is an S^1 action on the bundle as well, the weights with respect to this action will be half those of the $\tilde{S}^1$ weights. This is the fact which allows Braam and Austin in [12] to consider monopoles with half-integral weights, and corresponds to the fact that if m is half-integral (but not integral) then the matrix g in (1.6) is defined only up to sign, but gauge transformation with g is well-defined.

1.4 Mass, charge and instanton number

The aim of this section is to find interpretations of the mass and charge of a monopole, in terms of bundles and circle actions, and to prove

Proposition 1.4. *Let (A, Φ) be a monopole with charge k and integral mass $m \in \mathbb{Z}$. Let κ be the instanton number of the corresponding instanton $\mathbb{A}$ on S^4. Then*

$$\kappa = 2mk$$

First consider the definition of Φ. Let E be the bundle over S^4 (so $\kappa = c_2(E)$) and $\hat{E}$ the bundle on $\mathbb{H}^3$. Φ is the $d\theta$ component of a circle-invariant connection on E, so it encodes information about the circle action. More precisely, differentiating the circle action on E gives rise to a vector field X (a "lift of $\partial/\partial\theta$"). Contracting with the connection, $\mathbb{A}(X)$ is an $\mathfrak{su}(2)$-valued section over S^4, and $\Phi = f^*(\mathbb{A}(X))$ where $f : \mathbb{H}^3 \to S^4$ is the composition $\mathbb{H}^3 \to \mathbb{H}^3 \times S^1 \to S^4$. The second map is the conformal equivalence (together with the inclusion into the whole S^4) and $\mathbb{H}^3$ is identified with $\mathbb{H}^3 \times \{1\} \subset \mathbb{H}^3 \times S^1$.

Over a fixed point of S^4, the circle action is vertical, so Φ_∞ is given by the infinitesimal action on the fibre. Since S^2_∞ is pointwise fixed by the circle action, $E|_{S^2_\infty}$ splits by Grothendieck's theorem as a direct sum $L \oplus L^*$, with respect to which the circle acts via

$$\theta \mapsto \begin{pmatrix} e^{ip\theta} & 0 \\ 0 & e^{-ip\theta} \end{pmatrix}$$

some $p \in \mathbb{Z}$. The infinitesimal action on a fibre is given by $\begin{pmatrix} ip & 0 \\ 0 & -ip \end{pmatrix}$, so comparing with (1.2), $p = m$. In this way m appears as the weight of the circle action at infinity. The invariant k has an interpretation as the first Chern number of the positive weight bundle L, which will be a consequence of the proof of Proposition 1.4.

The next step is to find the action of the instanton and the energy of the monopole in terms of κ, m and k using the

Lemma 1.5 (Chern & Weil). *If F is a curvature 2-form on a bundle $B \to M$, then*

$$\det(1 + \frac{i}{2\pi}F) = 1 + \alpha_1(B) + \alpha_2(B) + \dots$$

where $\alpha_i(B) \in \Omega^{2i}(M)$ is a closed $2i$-form on M such that the cohomology classes $[\alpha_i(B)] \in H^{2i}(M)$ are the Chern classes $c_i(B)$ of B.

For a proof see, for example, [9]. □

Suppose $\mathbb{F} = \begin{pmatrix} g_1 & g_2 \\ g_3 & -g_1 \end{pmatrix}$, then by Lemma 1.5

$$\kappa = \frac{-1}{4\pi^2}\int_{S^4} \det \mathbb{F} = \frac{1}{4\pi^2}\int_{S^4} (g_1 \wedge g_1 + g_2 \wedge g_3)$$

So

$$\text{Action} = \int_{S^4} \text{tr}(\mathbb{F} \wedge *_4\mathbb{F}) = -\int_{S^4} \text{tr}(\mathbb{F} \wedge \mathbb{F}) \qquad \text{(since } \mathbb{F} \text{ is ASD)}$$

$$= -\int_{S^4} 2(g_1 \wedge g_1 + g_2 \wedge g_3) = -2(4\pi^2\kappa)$$

Thus Action $= -8\pi^2\kappa$.

Similarly for the Energy. Note first that $|F|^2 = |D_A\Phi|^2$ because of the monopole equations. Then

$$\begin{aligned} \text{Energy} &= \int_{\mathbb{H}^3} 2\text{tr}(F \wedge *_3F) = \int_{\mathbb{H}^3} 2\text{tr}(F \wedge D_A\Phi) \\ &= \int_{\mathbb{H}^3} 2d\text{tr}(\Phi F) \qquad (d_AF_A = 0 \text{ by the Bianchi identity}) \\ &= -\int_{S^2_\infty} 2\text{tr}(\Phi F) \end{aligned}$$

by Stokes' theorem (the sign is due to the orientation of S^2_∞). So Energy $= -8\pi mk$, by (1.3).

Proof of Proposition 1.4 All that remains for the proof is to relate the Action and the Energy. The map $\tau : S^4 \to \mathbb{H}^3$ will be written explicitly this time for clarity.

$$\text{Action} = \int_{S^4} \text{tr}(\mathbb{F}\wedge *_4\mathbb{F}) = \int_{S^4} \text{tr}\{\tau^*(F\wedge *_3F)\wedge d\theta + \tau^*(D_A\Phi \wedge *_3 D_A\Phi)\wedge d\theta\} \quad (1.9)$$

(cross terms cancel using the Bogomolny equations (1.1)). Everything in the trace is independent of θ, so doing the $d\theta$ integral (1.9) becomes

$$2\pi \int_{\mathbb{H}^3} \mathrm{tr}(F \wedge *_3 F + D_A\Phi \wedge *_3 D_A\Phi)$$

i.e. Action $= 2\pi$.Energy.

Substituting the expressions for the Action and Energy already found gives the result $\kappa = 2mk$, proving Proposition 1.4. □

A second proof of (1.4) using equivariant cohomology is given in Appendix A. As promised, there is also

Corollary 1.6. *The monopole charge k is the first Chern number of the positive weight line bundle L over S^2_∞.*

Proof From the proof of Proposition 1.4,

$$\int_{S^2_\infty} \mathrm{tr}(\Phi F) = 4\pi mk$$

On S_2^∞, $\Phi = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix}$ and $F = \begin{pmatrix} F_1 & 0 \\ 0 & -F_1 \end{pmatrix}$, diagonal because F is circle-invariant. So F_1 is a curvature form on L, $\mathrm{tr}(\Phi F) = 2imF_1$ and

$$2im \int_{S^2_\infty} F_1 = 4\pi mk \tag{1.10}$$

Let c be the first Chern number of L, then by (1.5)

$$c = \frac{i}{2\pi} \int_{S^2_\infty} F_1$$

Comparing this with (1.10) gives the result. □

1.5 Twistor descriptions

Roughly speaking, Penrose's "twistor programme" provides a method for transforming vector bundles with connections into holomorphic bundles over a different space. Twistor theory is a vast subject with many applications, and has proved an essential tool for the study of instantons and monopoles. One reason for describing hyperbolic monopoles as circle-invariant instantons is to allow the use of the twistor correspondence for S^4.

1.5.1 Twistors for S^4

In [3], Atiyah describes the twistor correspondence for S^4 in terms of the quaternions $\mathbf{H}$. Recall that

$$\mathbf{H} = \{x_1 + ix_2 + jx_3 + kx_4 : x_t \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. $\mathbf{H}$ can also be identified with $\mathbb{C}^2$:

$$\mathbf{H} = \{w_1 + jw_2 : w_1, w_2 \in \mathbb{C}\}$$

The 4-sphere S^4 is isomorphic to $\mathbf{H}P^1$, the quotient of $\mathbf{H}^2$ by scalars acting on the right. The **twistor fibration** : $\mathbb{C}P^3 \to \mathbf{H}P^1$ is the map

$$\pi : [z_1, z_2, z_3, z_4]_{\mathbb{C}} \mapsto [z_1 + jz_2, z_3 + jz_4]_{\mathbf{H}}$$

and $\mathbb{C}P^3$ is called the **twistor space** of S^4. The fibre over a point of $\mathbf{H}P^1$ is a complex line $\mathbb{C}P^1$ in $\mathbb{C}P^3$; for example the fibre over $[0,1]_{\mathbf{H}}$ is $\{[0,0,z_3,z_4]_{\mathbb{C}} : z_3, z_4 \in \mathbb{C}\}$. The fibre over a general point $[a + jb, 1]_{\mathbf{H}} \in S^4$ is given by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \tag{1.11}$$

Right multiplication by j on $\mathbf{H}P^1$ induces an antiholomorphic involution σ on $\mathbb{C}P^3$, namely

$$\sigma : [z_1, z_2, z_3, z_4] \mapsto [-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3]$$

This has no fixed points, but there are fixed lines; these are the fibres of π and are called **real lines**. The map σ gives a "real structure" on $\mathbb{C}P^3$, which can be thought of as an alternative version of the conjugation map.

A vector bundle E on S^4 lifts to a bundle $\tilde{E}$ on $\mathbb{C}P^3$, and there are lifts of the connection and curvature forms. The reason twistors are important is that a 2-form ω on S^4 is ASD iff its lift $\tilde{\omega}$ to $\mathbb{C}P^3$ is of type $(1,1)$ (see [3], Chapter IV, Proposition 2.7). Since the curvature of a holomorphic bundle always has type $(1,1)$ and any bundle with curvature of type (1,1) has a unique holomorphic structure compatible with the connection, E is ASD iff $\tilde{E}$ is holomorphic.

To be more precise about the relation between E and $\tilde{E}$, suppose P_x is the real line in $\mathbb{C}P^3$ corresponding to $x \in S^4$. Then because $\tilde{E}$ is a pullback, $\tilde{E}|_{P_x}$ is trivial. So a basis of E_x defines a holomorphic basis for $\tilde{E}|_{P_x}$. Conversely the fibre E_x is the space of holomorphic sections of $\tilde{E}|_{P_x}$. The bundle E has a unitary structure, or positive Hermitian form. This may be thought of as an antilinear isomorphism $\tau : E \to E^*$ such that the Hermitian form on E is given by $\langle u, v\rangle = (u, \tau v)$. Then τ defines a lifting of σ to $\tilde{E}$ via

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\tau}} & \tilde{E}^* \\ \downarrow & & \downarrow \\ \mathbb{C}P^3 & \xrightarrow{\sigma} & \mathbb{C}P^3 \end{array}$$

(because σ fixes real lines P_x, and $\tilde{E}|_{P_x} \simeq E_x \times P_x$). τ is recovered from $\tilde{\tau}$ as the map induced by $\tilde{\tau}$ on holomorphic sections of $\tilde{E}|_{P_x}$.

Proposition 1.7. *There is a natural 1-1 correspondence between anti-self-dual $SU(2)$-instantons on S^4 (up to gauge) and holomorphic $SL(2,\mathbb{C})$ vector bundles on $\mathbb{C}P^3$ with an antiholomorphic lift $\tilde{\tau}$ of σ (up to isomorphism).*

This is Theorem 2.9 of [3], Chapter IV. □

Note that the bundles $\tilde{E}$ each have a natural symplectic structure.

1.5.2 Circle action

In this picture, the circle action on $S^4 = \mathbf{H}P^1$ can be written

$$\theta : [q_1, q_2]_{\mathbf{H}} \mapsto [e^{-\frac{i\theta}{2}} q_1, e^{-\frac{i\theta}{2}} q_2]_{\mathbf{H}} \tag{1.12}$$

$\mathbb{R}^4$ is included in S^4 via

$$q \mapsto [q, 1]_{\mathbf{H}}$$

so the action (1.12) corresponds to the action $q \mapsto e^{-\frac{i\theta}{2}} q e^{\frac{i\theta}{2}}$ on $\mathbb{R}^4$, i.e.

$$z_1 + jz_2 \mapsto z_1 + je^{i\theta} z_2$$

which is rotation in the second factor, as before. The action lifts to $\mathbb{C}P^3$ and complexifies to a $\mathbb{C}^*$ action:

$$\lambda : [z_1, z_2, z_3, z_4] \mapsto [\lambda^{-\frac{1}{2}} z_1, \lambda^{\frac{1}{2}} z_2, \lambda^{-\frac{1}{2}} z_3, \lambda^{\frac{1}{2}} z_4]$$

There are two lines which are fixed pointwise:

$$\mathbb{P}^1_- = \{z_1 = 0 = z_3\} \text{ and } \mathbb{P}^1_+ = \{z_2 = 0 = z_4\}$$

and each of these projects back to the fixed S^2 in S^4. They are interchanged by σ. The $\mathbb{C}^*$ action on $\mathbb{C}P^3$ defines a flow, where $\mathbb{P}^1_+$ is the stable manifold and $\mathbb{P}^1_-$ the unstable manifold. Given $\mathbf{z} \in \mathbb{C}P^3$, and multiplying by $\lambda^{\pm\frac{1}{2}}$,

$$\lambda.\mathbf{z} = [z_1, \lambda z_2, z_3, \lambda z_4] = [\lambda^{-1} z_1, z_2, \lambda^{-1} z_3, z_4]$$

So

$$\lambda.\mathbf{z} \to [0, z_2, 0, z_4] \in \mathbb{P}^1_+ \text{ as } |\lambda| \to \infty$$
$$\lambda.\mathbf{z} \to [z_1, 0, z_3, 0] \in \mathbb{P}^1_- \text{ as } |\lambda| \to 0$$

Because of this,

$$\frac{\mathbb{C}P^3 - (\mathbb{P}^1_+ \cup \mathbb{P}^1_-)}{\mathbb{C}^*} \cong \mathbb{P}^1_+ \times \mathbb{P}^1_-$$

which is the twistor space analogue of the isomorphism $(\mathbb{R}^4 \setminus \mathbb{R}^2)/S^1 \cong \mathbb{H}^3$. The space $\mathbb{P}^1_+ \times \mathbb{P}^1_-$ is called the **minitwistor space** of $\overline{\mathbb{H}^3}$ (the closure of $\mathbb{H}^3$). The minitwistor space for $\mathbb{H}^3$ is the same with the points corresponding to $S^2_\infty = \partial\mathbb{H}^3$ removed. The points of S^2_∞ in S^4 are

$$\{[z_1, 1]_\mathbf{H} : z_1 \in \mathbb{C}\} \cup \{[1, 0]\}$$

and the fibres over these points are the complex projective lines

$$L_{z_1} = \{[z_1 z_3, \bar{z}_1 z_4, z_3, z_4] : z_3, z_4 \in \mathbb{C} \setminus 0\}$$

together with $L_\infty = \{[z_1, z_2, 0, 0]\}$. Looking at the limit points as $|\lambda|$ tends to 0 and ∞, the lines L_{z_1} correspond to $[0, \bar{z}_1, 0, 1] \in \mathbb{P}^1_+$ and $[z_1, 0, 1, 0] \in \mathbb{P}^1_-$; and these points are related by σ. L_∞ corresponds to $[0, 1, 0, 0]$ and $[1, 0, 0, 0]$ in $\mathbb{P}^1_+$ and $\mathbb{P}^1_-$ respectively and these are also related by σ. This set of points represents all pairs of points related by σ. So let $\bar{\Delta}$ be the antidiagonal, the set of points (w_+, w_-) in $\mathbb{P}^1_+ \times \mathbb{P}^1_-$ such that $w_+ = \sigma(w_-)$, then the minitwistor space of $\mathbb{H}^3$ is $Z = (\mathbb{P}^1_+ \times \mathbb{P}^1_-) \setminus \bar{\Delta}$. The space Z is the hyperbolic analogue of Hitchin's space TP^1 in the Euclidean case ([16], §3).

1.5.3 Spectral curve

A consequence of Hitchin's minitwistor correspondence is an expression for the holomorphic bundle $\tilde{E} \to T\mathbb{P}^1$ as an extension in two ways. These extensions may be used to define the spectral curve. A similar result ([4], §4) holds in the hyperbolic case, where the monopole defines a holomorphic bundle $\tilde{E} \to (\mathbb{P}^1_+ \times \mathbb{P}^1_-) \setminus \bar{\Delta} = Z$, and this will be of use. Let $\pi_\pm : Z \to \mathbb{P}^1_\pm$ be the natural projections and let $\mathcal{O}(r)$ be the complex line bundle on $\mathbb{P}^1$ of degree r. Then the line bundle $\mathcal{O}(p, q) \to Z$ is defined to be the bundle

$$\mathcal{O}(p, q) = \pi_+^*(\mathcal{O}(p)) \otimes \pi_-^*(\mathcal{O}(q))$$

Using this notation, the two extensions of [4] for $\tilde{E}$ are

$$0 \to \mathcal{O}(-m - k, m) \to \tilde{E} \to \mathcal{O}(m + k, -m) \to 0$$
$$0 \to \mathcal{O}(m, -m - k) \to \tilde{E} \to \mathcal{O}(-m, m + k) \to 0$$

The involution σ induces an involution on Z which lifts to a quaternionic structure on $\tilde{E}$. Pullback under this lift followed by complex conjugation exchanges the two extensions. The extensions may be combined to produce a map $\varphi : \mathcal{O}(-m-k, m) \to \mathcal{O}(-m, m+k)$. This map is multiplication by a section $\mathcal{F}$ of $\mathcal{O}(k,k)$, and the spectral curve of the monopole is defined to be its zero set $\{\mathcal{F} = 0\}$. From the definition, the spectral curve is the set of points in Z over which the two bundles $\mathcal{O}(-m-k, m)$ and $\mathcal{O}(m, -m-k)$ coincide and, in particular, $\mathcal{O}(-2m-k, 2m+k)$ is trivial on the spectral curve.

The points of the spectral curve correspond to geodesics in $\mathbb{H}^3$. Atiyah ([4], pp29,30) showed that these geodesics l are precisely those on which the solution s to

$$\nabla_l s + i\Phi s = 0$$

(where ∇_l is the component of ∇ along l) decays exponentially as a function of geodesic distance from the origin, in both directions along l.

It is necessary for the next chapter to make it clear which curves will be considered as spectral curves of monopoles. It was shown in [23] that the spectral curve of a hyperbolic monopole with mass m and charge k is a compact algebraic curve $\mathcal{S} \subset Z$ in the linear system $|\mathcal{O}(k,k)|$ (so it has genus $(k-1)^2$) such that

- $\mathcal{S}$ is real with respect to σ
- the restriction of $\mathcal{O}(-2m-k, 2m+k)$ to $\mathcal{S}$ is trivial

There are two more conditions suggested by those satisfied by the spectral curves of Euclidean monopoles. One is that $\mathcal{S}$ has no multiple components, and the other is an analogue of Hitchin's nonsingularity condition [17], that $H^0(\mathcal{S}, L^z(k-2)) = 0$ for $z \in (0,2)$ where L is a canonically defined line bundle. Murray and Singer are currently working on these as part of their programme [24]. What follows does not use them, so the monopoles obtained from spectral curves in the next chapter may have singularities.

Chapter 2

Boundary conditions

The aim of this chapter is to prove that the boundary condition (1.2), describing the rate of decay of the Higgs field, is satisfied for hyperbolic monopoles arising from spectral curves. A similar result was proved by Hurtubise [19] in the Euclidean case, by analysing the algebraic geometry of the spectral curve and its relation to the bundle E. His strategy will be used to prove

Theorem 2.1. *If (∇, Φ) is a hyperbolic monopole coming from a spectral curve, with mass $m \geq \frac{1}{2}$, then Φ is asymptotically constant and the error terms decay like e^{-2r} (where r denotes geodesic distance from the origin). In other words*

$$|\Phi_{as}| = m + O(e^{-2r})$$

The proof uses the structure of the minitwistor correspondence, and so the first task is to understand this in more detail. The reason for the condition $m \geq \frac{1}{2}$ will become clear towards the end of the chapter. All the results of this chapter hold for monopoles with nonintegral as well as integral mass.

2.1 The minitwistor correspondence

Hitchin ([16], §3) constructs the Euclidean minitwistor space $T\mathbb{P}^1$ as the space of oriented lines in $\mathbb{R}^3$. In the same way, $Z = \mathbb{P}^1_+ \times \mathbb{P}^1_- \setminus \bar{\Delta}$ is the space of oriented geodesics in $\mathbb{H}^3$. This is intuitively clear, thinking of $\mathbb{H}^3$ as the interior of the unit ball, since a geodesic is determined by its past and future endpoints, i.e. by a point of $S^2 \times S^2$ (minus the diagonal, since the past and future endpoints must be different). The complex structures arising from Hitchin's construction give the second sphere the opposite orientation to the first, so that in fact it is the antidiagonal $\bar{\Delta} = \{(\eta, \sigma\eta)\}$ which is removed. Coordinates on Z will sometimes be written (η, ζ) and sometimes $([w_0, w_1], [z_0, z_1])$, where $\eta = w_1/w_0$ and $\zeta = z_1/z_0$.

These are related to the $\mathbb{C}P^3$ coordinates of the previous chapter by

$$\eta = \frac{z_3}{z_1} \qquad \zeta = -\frac{z_2}{z_4}$$

The twistor correspondence gives a formula relating points of Z and geodesics in $\mathbb{H}^3$ via the formula (1.11):

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} u & -te^{-i\theta} \\ te^{i\theta} & \bar{u} \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$$

where $(u, t) \in \mathbb{C} \oplus \mathbb{R}_{>0} \cong \mathbb{H}^3$ are upper half space coordinates on $\mathbb{H}^3$. Eliminating θ, this pair of equations reduces to

$$t^2 z_3 z_4 = (uz_3 - z_1)(z_2 - \bar{u}z_4)$$

which, in terms of the coordinates (η, ζ) on Z, is

$$t^2\eta = (1 - u\eta)(\zeta + \bar{u}) \tag{2.1}$$

The relationship between $\mathbb{H}^3$ and Z may be thought of in terms of a *correspondence space* $C \subset Z \times \mathbb{H}^3$ consisting of those points of $Z \times \mathbb{H}^3$ satisfying (2.1). Since Z is the space of oriented geodesics in $\mathbb{H}^3$, an equivalent way to define C is as the space whose points consist of a geodesic in $\mathbb{H}^3$ and a point of $\mathbb{H}^3$ which lies on it. There are natural maps from C to Z and to $\mathbb{H}^3$.

It is a straightforward check that the real structure σ on $\mathbb{C}P^3$ descends to a real structure γ on Z given by:

$$\gamma(\eta, \zeta) = \left(-\frac{1}{\bar{\zeta}}, -\frac{1}{\bar{\eta}}\right)$$

which corresponds to changing the orientation of the geodesics.

Fixing a point $x = (u, t) \in \mathbb{H}^3$ fixes a complex projective line L_x in Z via (2.1), corresponding to the set of oriented geodesics through x (the antidiagonal does not meet any of the lines L_x). What is more, L_x is real, in the sense of being fixed by γ, and all real lines in Z arise in this way. Two of these real lines L_x and L_y, corresponding to $x, y \in \mathbb{H}^3$, meet in exactly two points in Z, which represent the two oriented geodesics joining x and y.

Thus a geodesic in $\mathbb{H}^3$ defines a pencil of real lines in Z (with one real parameter), all of which pass through two points (corresponding to the two orientations of the geodesic) and no two of which meet at any other point. For example, suppose the geodesic is given by $\{u = 0\}$, the t-axis in the upper half space model. Then for fixed t the real line $L_{(0,t)}$ is given by (2.1) to be $\eta = b\zeta$, where $b = 1/t^2 \in \mathbb{R}_{>0}$ (or equivalently $\zeta = b'\eta$ with $b' = t^2$). So the pencil of real lines defined by the geodesic $\{u = 0\}$ is

$$\mathcal{P} = \{\eta = b\zeta : b \in (0, \infty)\}$$

The lines in $\mathcal{P}$ all meet in the points $(\eta, \zeta) = (0, 0)$ and $(\eta, \zeta) = \gamma(0, 0)$.

2.1.1 A different viewpoint

A neat way of writing the twistor correspondence for $\mathbb{H}^3$ is described by Murray and Singer in ([23], §2); their argument will be summarised in this section. Twistor theory was originally developed for Minkowski space M, and hyperbolic 3-space is isomorphic to the quotient $U/\mathbb{R}_{>0}$, where U is the open cone of future-pointing timelike vectors in M and $\mathbb{R}_{>0}$ acts by scalar multiplication. Geodesics in $\mathbb{H}^3$ are represented by null geodesics in U, i.e. geodesics which have null direction. The $\mathbb{R}_{>0}$-orbit of such a geodesic is a time-like 2-plane in U, and such a 2-plane Σ contains two families of null geodesics. These two families provide a coordinate system on Σ (see Figures 1 and 2).

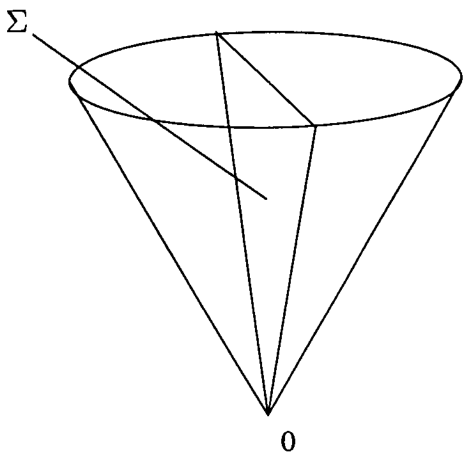


Fig 1

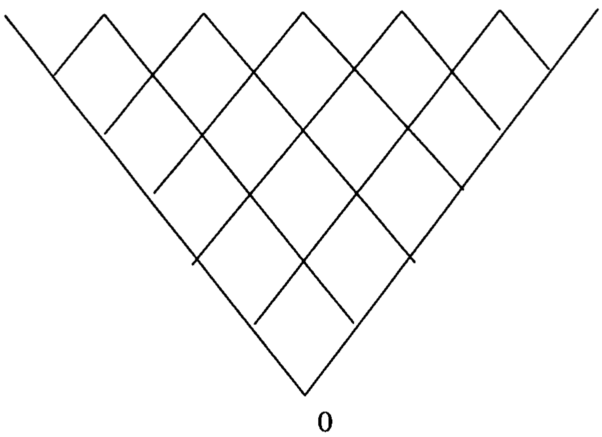


Fig 2

Figure 1 shows the time-like 2-plane Σ in U (with one dimension suppressed) and Figure 2 illustrates the two families of null geodesics contained in Σ. Note that each family may be thought of as the $\mathbb{R}_{>0}$ orbit of any one geodesic in the family. The limiting geodesics form the boundary of Σ.

Matrix version

Minkowski space M may be identified with the space of 2×2 complex Hermitian matrices, under

$$(X_0, X_1, X_2, X_3) \mapsto X = \begin{pmatrix} X_0 + X_3 & X_1 + iX_2 \\ X_1 - iX_2 & X_0 - X_3 \end{pmatrix}$$

Then $|X|^2 = \det X$ and U becomes the cone of positive definite matrices in M. The lift of (2.1) to this picture is

$$w = Xz \tag{2.2}$$

where $w = (w_0, w_1)^t$, $z = (z_0, z_1)^t$ (and still $\eta = w_1/w_0$, $\zeta = z_1/z_0$). There is a correspondence space diagram

$$\begin{array}{ccccc} & & C' & & \\ & \swarrow & & \searrow & \\ P & & & & U \\ \downarrow & & & & \downarrow \\ Z & & & & \mathbb{H}^3 \end{array}$$

where the bottom two arrows both represent taking the quotient by a $\mathbb{R}_{>0}$ action. P is the space

$$P = \{[w_0, w_1, z_0, z_1] \in \mathbb{C}P^3 : \bar{w}_0 z_0 + \bar{w}_1 z_1 \in \mathbb{R}_{>0}\}$$

and C' is the the subset of $U \times P$ given by

$$C' = \{(X : w, z) \in U \times P : w = Xz\}$$

A point $[w, z]$ of P corresponds via (2.2) to the null geodesic

$$\gamma_{[w,z]}(s) = \frac{ww^*}{\langle w, z\rangle} + s\frac{Jz(Jz)^*}{|z|^2}$$

in U, where $s \in \mathbb{R}_{>0}$, * denotes conjugate transpose, $J(z_0, z_1)^t = (-\bar{z}_1, \bar{z}_0)^t$ and $\langle w, z\rangle = w^* z$. ($\gamma_{[w,z]}(s)$ is a matrix since w and z are column vectors.) Thus P may be identified with the set of null geodesics in U.

The $\mathbb{R}_{>0}$-action on P is

$$\lambda : [w_0, w_1, z_0, z_1] \mapsto [\lambda^{\frac{1}{2}} w_0, \lambda^{\frac{1}{2}} w_1, \lambda^{-\frac{1}{2}} z_0, \lambda^{-\frac{1}{2}} z_1]$$

and taking the quotient recovers the twistor space Z of $\mathbb{H}^3$:

$$\begin{aligned} P/\mathbb{R}_{>0} &= \{([w], [z]) \in \mathbb{P}^1 \times \mathbb{P}^1 : \langle w, z\rangle \neq 0\} \\ &= Z \end{aligned}$$

A point $([w], [z])$ of Z corresponds to the $\mathbb{R}_{>0}$-orbit of the null geodesic $\gamma_{[w,z]}$, which is a null 2-plane in U. So it defines a geodesic in $\mathbb{H}^3$. Thus the quotient of the correspondence recovers the original twistor correspondence. The advantage of the Minkowski space picture is that it makes it easier to find a formula for the Higgs field Φ along a fixed geodesic $l = \{u = 0\}$. This geodesic l is now represented by the time-like 2-plane

$$\Pi = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \text{ are real and positive} \right\}$$

which is the intersection of the 2-plane of real, diagonal matrices with U.

2.1.2 Correspondence for Π

The restriction of the twistor correspondence to the timelike 2-plane Π will now be explained in more detail. For a point $X \in \Pi$, the line $\tilde{L}_X \in P$ is given by $w = Xz$, i.e.

$$w_0 = uz_0 \qquad w_1 = vz_1 \tag{2.3}$$

(so the quotient is the line $\eta = b\zeta$ in Z where $b = v/u$). The points of Π correspond to a pencil $\hat{\mathcal{P}}$ of real lines in P. If $[w, z] \in P$ lies on one of the lines in this pencil, then (2.3) may be solved for u and v:

$$u = \frac{w_0}{z_0} \qquad v = \frac{w_1}{z_1}$$

except at points of P of the form $[0, w_1, 0, z_1]$ or $[w_0, 0, z_0, 0]$. In the first case u is left undefined, and in the second case the same happens for v. The reason is that these special points of P correspond to null geodesics in U which lie completely inside Π. All the other geodesics corresponding to points of P lying on lines in $\hat{\mathcal{P}}$ intersect Π exactly once, and u and v give the coordinates of the intersection. So the two families of null geodesics on Π are the families $\{u = u_0\}$ and $\{v = v_0\}$.

A point $X = (u, v) \in \Pi$ defines a real projective line $\tilde{L}_X$ in P, given by

$$\tilde{L}_X = \{[uz_0, vz_1, z_0, z_1] : z_0, z_1 \in \mathbb{C}\} \tag{2.4}$$

Any two such lines $\tilde{L}_X$ and $\tilde{L}_Y$ descend to real lines L_X and L_Y in Z, which intersect in the two points

$$p_1 = ([1, 0], [1, 0]) \qquad p_2 = ([0, 1], [0, 1])$$

(coming from $z_1 = 0$ and $z_0 = 0$ respectively in (2.4)). This fact will become important for the calculation of the parallel transport coming from the Ward transform.

2.2 Ward transform

The Ward transform relates bundles with connection on $\mathbb{H}^3$ to holomorphic bundles on Z. This section contains a brief explanation, which is then used to find a formula for Φ (near $\partial\mathbb{H}^3$, along the fixed geodesic l) in terms of the geometry of Π and U. Given a bundle $E \to \mathbb{H}^3$, its *Ward transform* $\tilde{E} \to Z$ is the holomorphic bundle with fibres

$$\tilde{E}_{(\eta,\zeta)} = \{e \in \Gamma(l, E) : (\nabla_l - i\Phi)e = 0\}$$

where l is the geodesic in $\mathbb{H}^3$ corresponding to $(\eta, \zeta) \in Z$ and ∇_l is the covariant derivative in the direction of l. E is recovered from $\tilde{E}$ by

$$E_x = H^0(L_x, \mathcal{O}(\tilde{E}))$$

(certainly, E thus defined is a vector bundle, since the restriction of $\tilde{E}$ to any real line L_x is trivial). The connection and Higgs field on E are also defined by $\tilde{E}$, in a more involved way which will be explained in §2.2.1, §2.2.2 and §2.2.3.

There is a similar relation between the pullbacks $E_{(U)} \to U$ and $\tilde{E}_{(P)} \to P$ of E and $\tilde{E}$ to U and P respectively. P is a 5-dimensional real manifold, so $\tilde{E}_{(P)}$ cannot be holomorphic, but it is a CR-bundle. A definition of CR-bundles will not be given here; it is enough to know that the restriction of $\tilde{E}_{(P)}$ to any real line $\tilde{L}_X \cong \mathbb{P}^1$, corresponding to a point X of U, has a complex structure with respect to which the restricted bundle is holomorphically trivial. In addition, if $E_{(U)}$ is the pullback of a bundle on $\mathbb{H}^3$, with a pullback connection, then $\tilde{E}_{(P)}$ is the pullback of a holomorphic bundle on Z. For details of this correspondence see ([23], §3).

2.2.1 Parallel transport

Let $\tilde{E} \to Z$, a holomorphic bundle, be the Ward transform of the bundle $E \to \mathbb{H}^3$ with connection A, and suppose $x, y \in \mathbb{H}^3$ lie on the geodesic l. Then the corresponding projective real lines L_x and L_y in Z meet in two points. Let p be the one which represents l, when it is oriented so that the positive direction is from x to y. Let $e \in E_x$ be a point in the fibre of E over x. To define the parallel transport of e along l from x to y, first recall that

$$E_x = H^0(L_x, \mathcal{O}(\tilde{E})) \qquad\qquad E_y = H^0(L_y, \mathcal{O}(\tilde{E}))$$

So e is represented by a section s of $\tilde{E}$ over L_x. The restriction of $\tilde{E}$ to any real line is trivial, so there is a unique holomorphic section s' of $\tilde{E}$ over L_y such that $s(p) = s'(p)$. This section s' represents a point $e' \in E_y$, which is defined to be the parallel transport of e along l to E_y. The fact that this parallel transport does determine a connection A is a consequence of the existence of a Ward transform.

2.2.2 $\mathbb{R}_{>0}$-invariant bundles over Π

Suppose $E_{(U)} \to U$, with connection $A_{(U)}$, is the pullback of $E \to \mathbb{H}^3$, connection A. Then $(E_{(U)}, A_{(U)})$ is $\mathbb{R}_{>0}$-invariant, as is its transform $\tilde{E}_{(P)} \to P$, and $\tilde{E}_{(P)}$ is the pullback of some holomorphic $\tilde{E} \to Z$. The plane Π in U corresponds to the geodesic l in $\mathbb{H}^3$. By §2.1.2, evaluation at p_1 and at p_2 each define a connection on

$E|_l$, because if $x, y \in l$ then the real lines L_x, L_y in Z intersect at p_1 and p_2. But these connections also lift to connections ∇^1, ∇^2 on $E_{(U)}|_\Pi$. To see what these are, note from (2.4) that if $(u, v_1), (u, v_2) \in \Pi$ then the corresponding lines in P intersect at $[u, 0, 1, 0] \in P$, which is in the preimage of $([1, 0], [1, 0]) = p_1 \in Z$. So parallel transport with ∇^1, the lift of the connection on E defined by evaluation at p_1, is the same as parallel transport along one of the null geodesics $u = u_0$ in Π with $A_{(U)}$ (i.e. in the v direction). In the same way, parallel transport with ∇^2 is parallel transport along $v = v_0$ with $A_{(U)}$. They are defined for parallel transport between any pair of points in Π, because the orbit of any point of Π has a representative on any given one of the null geodesics in either of the two families. It makes no difference which geodesic in the family is chosen.

2.2.3 A formula for Φ

The coordinates (u, v) on Π give homogeneous coordinates on the quotient $\Pi/\mathbb{R}_{>0}$, since the action of $\mathbb{R}_{>0}$ is given by $(u, v) \mapsto (\lambda u, \lambda v)$. This symmetry is generated by the Euler field

$$\chi = u\partial_u + v\partial_v$$

A 1-form on Π is **horizontal** if its contraction with χ is zero (i.e. if it is pulled back from $\mathbb{H}^3$) so any horizontal 1-form Ω may be written

$$\Omega = \alpha\left(\frac{du}{u} - \frac{dv}{v}\right) = \alpha d\log(u/v)$$

for some function α which is homogeneous of degree 0 in (u, v).

Suppose $A_{(U)}$ has covariant derivative

$$D_A = d + Q\left(\frac{du}{u}\right) + R\left(\frac{dv}{v}\right)$$

Any connection may be written in this way, for some matrix-valued functions Q and R. Because $A_{(U)}$ is $\mathbb{R}_{>0}$-invariant, Q and R are homogeneous of degree 0 in (u, v) in an appropriate gauge. The Higgs field is $i.\psi$, where ψ is the component of $A_{(U)}$ in the direction of the symmetry. So ψ is the contraction of $A_{(U)}$ with χ:

$$\psi = \langle D_A, \chi\rangle = Q + R$$

Because ∇^1 and ∇^2 are pulled back from $\mathbb{H}^3$, they may be written

$$\nabla^1 = d + \alpha\left(\frac{du}{u} - \frac{dv}{v}\right) \qquad \nabla^2 = d + \beta\left(\frac{du}{u} - \frac{dv}{v}\right)$$

Parallel transport with ∇^2 is just parallel transport with $A_{(U)}$ in the u direction, so the contractions of ∇^2 and D_A with ∂_u (i.e. their du components) must be the same. Thus $Q = \beta$ and by a similar argument, $R = -\alpha$. Substituting these,

$$\nabla^1 = d - Rd\log(u/v) \qquad \nabla^2 = d + Qd\log(u/v)$$

so

$$\nabla^2 - \nabla^1 = (Q + R)d\log(u/v) = \psi d\log(u/v)$$

It remains to show that $\log(u/v) = 2r$, where r is geodesic distance in $\mathbb{H}^3$ along the t-axis l from $t = 1$, to obtain the result:

$$2\Phi dr = i(\nabla^2 - \nabla^1) \tag{2.5}$$

But a point $(u, v) \in \Pi$ corresponds to the line $\eta = b\zeta$ in Z, where $b = v/u$ (see §2.1.2), and $b = 1/t^2$. Geodesic distance along the t-axis is given by the coordinate $r = \log t$, measured from the point $t = 1$. So

$$\log(u/v) = \log(t^2) = 2r$$

completing the proof that (2.5) gives a formula for Φ. □

2.3 Blowing up Z

The idea for the proof of Theorem 2.1 is to use this formula for Φ and look at what happens as $t \to 0$, that is, as $b \to \infty$. This section contains the necessary groundwork, including finding a good coordinate system on (a blowup of) Z. Recall that l is the geodesic $\{u = 0\}$ in $\mathbb{H}^3$, which defines a pencil $\mathcal{P}$ of real lines $\{\eta = b\zeta\}$ in Z (taking the quotient of the pencil defined in P).

The twistor space Z can be covered by the two charts: $V_1 = \{\eta, \zeta \neq \infty\} \cap Z$ and $V_2 = \{\eta, \zeta \neq 0\} \cap Z$, since the two points of $\mathbb{P}^1_+ \times \mathbb{P}^1_-$ not contained in $V_1 \cup V_2$ lie on the antidiagonal $\bar{\Delta}$. The transition map from V_1 to V_2 is given by $(\eta, \zeta) \mapsto (1/\eta, 1/\zeta)$.

All the lines in the pencil $\mathcal{P}$ meet in the two points $p_1 = (0, 0)$ and $p_2 = \gamma(0, 0) = (\infty, \infty)$, and these are the only two points where a pair of lines in $\mathcal{P}$ meet. So blowing up Z at these two points turns the pencil into a ruling of lines $\mathbb{R}_{>0} \times \mathbb{P}^1$. Using coordinates (b, ζ) on the blowup $\tilde{Z}$, the map $\eta = b\zeta$ describes the projection $\pi_Z : \tilde{Z} \to Z$. Pulling everything back to $\tilde{Z}$ is equivalent to pulling back to the correspondence space C, because it separates all pairs $\{(L, x) : x \in L\}$, where $x \in \mathbb{H}^3$ is such that $L_x \in \mathcal{P}$ and L is a geodesic in $\mathbb{H}^3$. This is useful, since the natural coordinates (b, ζ) on $\mathcal{P}$ are much more convenient than (η, ζ) on Z. They are nonsingular (on the pullback $\tilde{\mathcal{P}}$ of $\mathcal{P}$ to $\tilde{Z}$), and some coordinate system including b is needed to make the behaviour as $b \to \infty$ clear. Also, the lines

$\{b = \text{constant}\}$ are real, whereas the lines $\{\zeta = \text{constant}\}$ and $\{\eta = \text{constant}\}$ are not. Calculating the transition matrix for the lift of $\tilde{E}|_{\mathcal{P}}$ to $\tilde{Z}$ is the same as finding the transition matrix on Z from $V_1' = V_1 - \{\zeta = 0, \eta \neq 0\}$ to $\gamma(V_1') = V_2'$. The idea is to find an explicit formula for this matrix. The notation $L_b = \{\eta = b\zeta\}$, $E_b = E|_{(0,b)}$ and $\tilde{E}_b = \tilde{E}|_{L_b}$ will be used frequently.

2.3.1 Intersection of $\mathcal{P}$ with the spectral curve

The aim of this section is to describe the intersection points $L_b \cap \mathcal{S}$, and their behaviour as $b \to \infty$. These results underpin all that follows.

The spectral curve $\mathcal{S}$ of the monopole is given by $\mathcal{F} \in \Gamma(Z, \mathcal{O}(k,k))$, so it is an algebraic curve on Z of bidegree (k,k). This means that for each $b \in (0,\infty)$ there are $2k$ intersection points of L_b with $\mathcal{S}$. As $b \to \infty$ half of these intersection points approach the line $\{\zeta = 0\}$ and the other half tend to limits on $\{\eta = \infty\}$ (see picture). Reality of $\mathcal{S}$ and of L_b means that the whole picture must be symmetric under γ, which, in Figure 3, may be thought of as reflection in the antidiagonal $\bar{\Delta}$. So the intersection points must come in pairs related by γ, and for large b there will be a natural way to split the points into two sets, with one from each pair in each set. Similarly as $b \to 0$ the intersection points approach $\{\eta = 0\}$ and $\{\zeta = \infty\}$. For large b, label the intersection points $L_b \cap \mathcal{S}$ by $\{(\eta_i(b), \zeta_i(b))\}$, where $\eta_i(b) = b\zeta_i(b)$ and, choose the ordering so that

$$(\eta_i, \zeta_i) \to (c_i, 0) \text{ for } 1 \leq i \leq k \text{ as } b \to \infty \tag{2.6}$$

and

$$(\eta_{k+i}, \zeta_{k+i}) = \gamma(\eta_i, \zeta_i) \tag{2.7}$$

Assume that $c_i \neq 0 \; \forall i$, i.e. that $(0,0) \notin \mathcal{S}$. This can always be arranged by choice of coordinates so there is no loss of generality. Then by (2.7)

$$(\eta_{k+i}, \zeta_{k+i}) = (-1/\bar{\zeta}_i, -1/\bar{\eta}_i)$$

so as $b \to \infty$,

$$\eta_{k+i} \to \infty \text{ and } \zeta_{k+i} \to -1/\bar{c}_i$$

The relation $\eta_r = b\zeta_r$ implies that

$$\eta_{k+i} = O(b) \text{ and } \zeta_i = O(b^{-1}) \tag{2.8}$$

when $1 \leq i \leq k$.

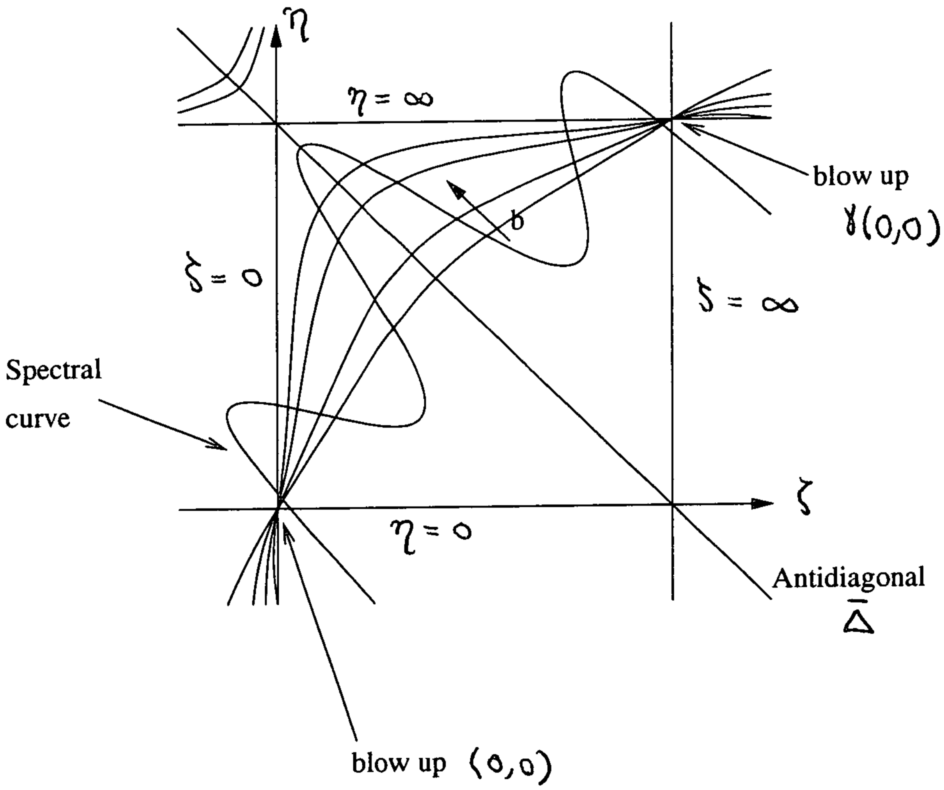


Fig 3: An intuitive picture to illustrate the pencil $\mathcal{P}$. The diagram shows the four (horizontal and vertical) lines $\{\zeta = 0\}$, $\{\zeta = \infty\}$, $\{\eta = 0\}$ and $\{\eta = \infty\}$ and also the points $(0,0)$ and $\gamma(0,0)$ where all the lines in the pencil intersect, which are to be blown up. Despite appearances, $\bar{\Delta}$ does not intersect $\mathcal{S}$ or any of the lines in $\mathcal{P}$.

2.4 An expression for the transition map of $\tilde{E}$

The next task is to make explicit the dependence of $\tilde{E}$ on the spectral curve, by finding the transition map from V_1' to V_2'.

First recall (§1.5.3) that $\tilde{E}$ can be described as an extension in two ways:

$$0 \to \mathcal{O}(-m-k, m) \to \tilde{E} \to \mathcal{O}(m+k, -m) \to 0 \tag{2.9}$$

$$0 \to \mathcal{O}(m, -m-k) \to \tilde{E} \to \mathcal{O}(-m, m+k) \to 0 \tag{2.10}$$

and $\mathcal{O}(-m-k, m)$ and $\mathcal{O}(m, -m-k)$ coincide on the spectral curve $\mathcal{S}$, given by $\{\mathcal{F} = 0\}$. Here $\mathcal{F}$ is the composition of the maps from $\mathcal{O}(-m-k, m)$ to $\tilde{E}$ to $\mathcal{O}(-m, m+k)$. Because $\mathcal{O}(-m-k, m)$ and $\mathcal{O}(m, -m-k)$ coincide on $\mathcal{S}$, the bundle $\mathcal{O}(-2m-k, 2m+k)|_{\mathcal{S}}$ is trivial.

2.4.1 Note on extensions

In general, extensions E given by an exact sequence

$$0 \to L \to E \to M \to 0$$

of bundles over a manifold X, are classified by $H^1(X, L \otimes M^*)$. To see this, first let $\{U_i\}$ be an open cover of X, so that $E|_{U_i} = L|_{U_i} \oplus M|_{U_i}$ and choose trivialisations

$$\begin{aligned} s_i &: X \times \mathbb{C}^n \xrightarrow{\simeq} L|_{U_i} \\ t_i &: X \times \mathbb{C}^m \xrightarrow{\simeq} M|_{U_i} \end{aligned}$$

for each i. These give a trivialisation of $E|_{U_i}$ for each i; and the transition functions of L and M are

$$\begin{aligned} \phi_{ij} &= s_i s_j^{-1} \\ \psi_{ij} &= t_i t_j^{-1} \end{aligned}$$

respectively. Because L is a subbundle of E and M is a quotient (of E by L), the transition function g_{ij} on $U_{ij} = U_i \cap U_j$ must be a matrix of the form

$$g_{ij} = \begin{pmatrix} \phi_{ij} & \xi_{ij} \\ 0 & \psi_{ij} \end{pmatrix}$$

mapping $L|_{U_{ij}} \oplus M|_{U_{ij}}$ (in the j trivialisation) to itself (in the i trivialisation). To make the trivialisations clearer, $\xi_{ij} : M|_{U_{ij}} \to L|_{U_{ij}}$ may be written

$$\xi_{ij} = s_i \eta_{ij} t_j^{-1}$$

where $\eta_{ij} : X \times \mathbb{C}^m \to X \times \mathbb{C}^n$ is the identity on the first factor (note no trivialisations are seen by η). Then the cocycle condition $g_{ij} g_{jk} g_{ki} = 1$ translates to the condition

$$s_i \eta_{ij} t_i^{-1} + s_i \eta_{jk} t_i^{-1} + s_i \eta_{ki} t_i^{-1} = 0$$

so $\{s_i \eta_{jk} t_i^{-1}\}$ is a Čech cocycle representing an element of $H^1(X, L \otimes M^*)$ (in the i trivialisation). The other elements of the same equivalence class are obtained by changing the trivialisations of L and M. This shows that extensions are classified by $H^1(X, L \otimes M^*)$. Conversely, if ν is a representative of an extension class and ν_{ij} is the (ij) part (in the j trivialisation) then, comparing definitions, the entry ξ_{ij} of the transition matrix g_{ij} is given by $s_i s_j^{-1} \nu_{ij} = \phi_{ij} \nu_{ij}$. This correspondence between bundles and extension classes via the transition map will be used now in the case of extension (2.9).

2.4.2 Calculating the transition map

Extensions of type (2.9) are determined by elements of $H^1(Z, \mathcal{O}(-2m-2k, 2m))$. Atiyah ([4], pp 14-15) explains how to obtain the extension class for $\tilde{E}$ from $\mathcal{S}$ as follows. Given $\mathcal{F}$, there is a short exact sequence:

$$0 \to \mathcal{O}(-2m-2k, 2m) \stackrel{\times\mathcal{F}}{\to} \mathcal{O}(-2m-k, 2m+k) \to \mathcal{O}(-2m-k, 2m+k)|_{\mathcal{S}} \to 0$$

which has coboundary map

$$\delta : H^0(\mathcal{S}, \mathcal{O}(-2m-k, 2m+k)) \to H^1(Z, \mathcal{O}(-2m-2k, 2m)).$$

The space $H^1(Z, \mathcal{O}(-2m-2k, 2m))$ determines the extensions (2.9).

Lemma 2.2. *All nonzero sections $f \in H^0(\mathcal{S}, \mathcal{O}(-2m-k, 2m+k))$ define the same extension (as in (2.9)) via the extension class $\delta(f) \in H^1(Z, \mathcal{O}(-2m-2k, 2m))$.*

Proof: The space, $H^0(\mathcal{S}, \mathcal{O}(-2m-k, 2m+k))$, is one dimensional, since the bundle is trivial on $\mathcal{S}$, so the result is true provided δ is injective. The kernel of δ is

$$\ker\delta = H^0(Z, \mathcal{O}(-2m-k, 2m+k))$$

(with equality since $H^0(Z, \mathcal{O}(-2m-2k, 2m)) = 0$).

The restriction of the bundle $\mathcal{O}(-2m-k, 2m+k)$ to a generator $\{\zeta = 0\}$ is $\mathcal{O}(-2m-k)$, which has no sections. But any section of $\mathcal{O}(-2m-k, 2m+k)$ over Z restricts to a section over the generator. Thus $H^0(Z, \mathcal{O}(-2m-k, 2m+k)) = 0$, so $\ker\delta = 0$ and the lemma is proved. □

The extension $\tilde{E}$ defined in this way is the holomorphic bundle corresponding to the spectral curve $\mathcal{S}$ (see [4], p15).

The transition function of $\tilde{E}$ is obtained by combining this extension class with the transition functions of the other bundles in (2.9). The transition function of $\mathcal{O}(-2m-k, 2m+k)$ from V_1 to V_2 is $(\eta/\zeta)^{(2m+k)}$ (since the transition function of $\mathcal{O}(-1,1)$ is $\eta/\zeta = b$, which is real on $\mathcal{P}$). Write $f = f_J(b, \zeta)$ for the restriction of f to $V_J' \cap \mathcal{S}$, so that $f_2(b, \zeta) = b^{2m+k} f_1(b, \zeta)$. Then the transition matrix is

$$T_{21} = \begin{pmatrix} b^{m+k}\zeta^k & b^{m+k}\zeta^k\delta(f)_{12} \\ 0 & b^{-m-k}\zeta^{-k} \end{pmatrix} \tag{2.11}$$

Here $\delta(f)_{12}$ is the (12) part of a Čech representative of $\delta(f)$ in $H^1(Z, \mathcal{O}(-2m-2k, 2m))$. Write $f_{iJ}(b) = f_J(b, \zeta_i(b))$, where $i \in \{1, \ldots, 2k\}$, $J \in \{1, 2\}$. Then

Lemma 2.3. *If the $\zeta_i(b)$ are all distinct, the* (12) *entry of the transition matrix* T_{21} *is given by*

$$b^{m+k}\zeta^k\delta(f)_{12} = b^{m+2k}\sum_{i=1}^{2k} a_i(b)(\zeta^k + \zeta_i(b)\zeta^{k-1} + \ldots + \zeta_i(b)^{2k}\zeta^{-k}),$$

where

$$a_i(b) = \frac{f_{i1}(b)}{b^k\zeta_i(b)\prod_{j\neq i}(\zeta_i(b) - \zeta_j(b))}.$$

The main part of the proof is calculating $\delta(f)$, which extends f from $\mathcal{S}$ to the whole of Z. The f_J have to be extended to functions F_J on V_J'. One way to do this is to use Lagrange interpolation, which gives a formula for a polynomial g of degree $r-1$ taking the values $a_1, \ldots, a_r$ at specified distinct points $\zeta_1, \ldots, \zeta_r$ respectively. The formula is

$$g(\zeta) = \sum_{i=1}^{r} a_i g_i(\zeta), \qquad \text{where } g_i(\zeta) = \frac{\prod_{j=1,j\neq i}^{r}(\zeta - \zeta_j)}{\prod_{j=1,j\neq i}^{r}(\zeta_i - \zeta_j)}$$

(see, for example, [21]).

Proof of lemma First interpolate for fixed b, and then regard the coefficients of the polynomial as functions of b. The resulting polynomials are supposed to be functions pulled back to the blowup $\tilde{Z}$ from Z, which means that they must be constant on the exceptional divisor. So impose the additional condition

$$F_1(b, 0) = 0 = F_2(b, \infty).$$

Then the extended functions F_i are given by

$$F_1(b,\zeta) = \zeta\sum_{i=1}^{2k}\frac{f_{i1}}{\zeta_i}\phi_i(\zeta), \text{ where } \phi_i(\zeta) = \frac{\prod_{j\neq i}(\zeta - \zeta_j)}{\prod_{j\neq i}(\zeta_i - \zeta_j)}$$

$$F_2(b,\zeta) = \zeta'\sum_{i=1}^{2k}\frac{f_{i2}}{\zeta_i'}\phi_i'(\zeta), \text{ where } \phi_i'(\zeta) = \frac{\prod_{j\neq i}(\zeta' - \zeta_j')}{\prod_{j\neq i}(\zeta_i' - \zeta_j')}$$

Here $\zeta' = 1/\zeta$ is a coordinate on V_2', and $\zeta_i' = 1/\zeta_i$, so substituting $1/\zeta$ for ζ',

$$\phi_i'(\zeta) = (\frac{\zeta_i}{\zeta})^{2k-1}\phi_i(\zeta).$$

The coboundary map is constructed from F_1 and F_2 by

$$\delta(f)_{12} = \frac{F_1 - b^{-2m-k}F_2}{\prod(\zeta - \zeta_i)}$$

since $\prod(\zeta - \zeta_i(b)) = \mathcal{F}$. Putting all this together, remembering that $f_{i2} = b^{2m+k}f_{i1}$, and simplifying,

$$\delta(f)_{12} = \sum_{1}^{2k}\frac{f_{i1}(\zeta^{2k} + \zeta^{2k-1}\zeta_i + \ldots + \zeta\zeta_i^{2k-1} + \zeta_i^{2k})}{\zeta_i\zeta^{2k}\prod_{j\neq i}(\zeta_i - \zeta_j)}.$$

Multiplying by $b^{m+k}\zeta^k$, the result follows from (2.11). □

2.4.3 Limit of $a_i(b)$

The reason for the factor b^{-k} in the definition of $a_i(b)$ is so that the following result holds:

Lemma 2.4. *Suppose the limits $\{c_i\}$ are all distinct. Then the function $a_i(b)$ has a finite nonzero limit as $b \to \infty$ when $1 \le i \le k$ and is $O(b^{-2m-2k})$ when $k+1 \le i \le 2k$.*

Proof Recall that the points (η_i, ζ_i) tend to $(c_i, 0)$ for $1 \le i \le k$ as b tends to infinity. Because the section f is continuous, f_2 is bounded near $\zeta = \infty$, so for large b and $i > k$,

$$f_{i1}(b) = O(b^{-2m-k}). \tag{2.12}$$

Since

$$\frac{1}{\zeta_i \prod_{j \ne i}(\zeta_i - \zeta_j)} \to \frac{(-\bar{c}_{i+k})^{k+1}}{\prod_{j>k, j \ne i-k}(1/\bar{c}_{j-k} - 1/\bar{c}_{i-k})},$$

which is finite and nonzero, and using (2.12),

$$a_i(b) = \frac{f_{i1}}{b^k \zeta_i \prod_{j \ne i}(\zeta_i - \zeta_j)} = b^{-k} O(b^{-2m-k}) = O(b^{-2m-2k}). \tag{2.13}$$

Now suppose that $i \le k$, then $f_{i1} = f_1(\eta_i, \zeta_i) \to f_1(c_i, 0)$, as $b \to \infty$, which is finite and nonzero. Also:

$$\frac{1}{b^k \zeta_i \prod_{j \ne i}(\zeta_i - \zeta_j)} = \frac{1}{b\zeta_i \prod_{j \ne i, j \le k}(b\zeta_i - b\zeta_j) \prod_{j>k}(\zeta_i - \zeta_j)},$$

which tends to

$$\frac{1}{c_i \prod_{j \ne i, j \le k}(c_i - c_j) \prod_{j=1}^{k}(1/\bar{c}_j)} = \frac{\prod_{j=1}^{k}(\bar{c}_j)}{c_i \prod_{j \ne i}(c_i - c_j)}$$

as $b \to \infty$. So $a_i(b)$ has a finite limit in this case, given by

$$a_i(\infty) = \frac{f_1(c_i, 0) \prod_{j=1}^{k} \bar{c}_j}{c_i \prod_{j \ne i}(c_i - c_j)}$$

and the proof is complete. □

2.5 A nonsingularity condition

Starting from the spectral curve, there is no guarantee that the monopole obtained will be nonsingular (i.e. that $\tilde{E}$ will be holomorphically trivial on all real lines). One use of the above results is to prove

Proposition 2.5. *Let (A, Φ) be the monopole corresponding to the spectral curve $\mathcal{S}$, or equivalently to the holomorphic bundle $\tilde{E}$, as above. Then (A, Φ) is non-singular outside a compact set in $\mathbb{H}^3$.*

The proof goes by first finding a condition for the monopole to be nonsingular at a given point on the t-axis (i.e. restricting to the pencil $\mathcal{P}$ in Z), then showing that this condition holds as $b \to \infty$, and finally extending to other directions in $\mathbb{H}^3$.

Each point $(0, t) \in \mathbb{H}^3$ corresponds to a line $L_b \in \mathcal{P}$, where $b = 1/t^2$; and the solution (A, Φ) is nonsingular at $(0, t)$ iff $\tilde{E}_b$ is holomorphically trivial. Since $c_1(\tilde{E}) = 0$, $\tilde{E}$ is holomorphically trivial on all lines except the jumping lines, the exceptional lines on which the splitting type of the bundle changes. So the first part of the problem is reduced to finding necessary and sufficient conditions on b for $L_b \in \mathcal{P}$ to be a jumping line.

Lemma 2.6. *Using the notation of the preceding sections, the line L_b is a jumping line for $\tilde{E}$ iff the matrix*

$$M(b) = \begin{pmatrix} \sum a_i\zeta_i^k & \sum a_i\zeta_i^{k+1} & \cdots & \sum a_i\zeta_i^{2k-1} \\ \sum a_i\zeta_i^{k-1} & \sum a_i\zeta_i^k & \cdots & \sum a_i\zeta_i^{2k-2} \\ \vdots & \vdots & & \vdots \\ \sum a_i\zeta_i & \sum a_i\zeta_i^2 & \cdots & \sum a_i\zeta_i^k \end{pmatrix}$$

is singular, where all the sums are over i between 1 and $2k$.

Proof of Lemma The restriction of $\tilde{E}$ to L_b is $\tilde{E}_b = \mathcal{O}(m) \oplus \mathcal{O}(-m)$ for some $m \in \mathbb{Z}$, the jumping lines being those where $m \neq 0$. Thus L_b is a jumping line iff $H^0(\tilde{E}_b(-1)) \neq 0$. Restrict (2.9) to L_b and twist with $\mathcal{O}(-1)$, to see that the extension

$$0 \to \mathcal{O}(-k-1) \to \tilde{E}_b(-1) \to \mathcal{O}(k-1) \to 0 \tag{2.14}$$

is given by the transition matrix ζT_{21} (restricted to L_b), since $\mathcal{O}(-1)$ has transition function ζ. The condition for triviality of $\tilde{E}_b$ is that $\tilde{E}_b(-1)$ has no sections. If there is a section, given by $f = (f_1, f_2)^t$ on V_1' and $g = (g_1, g_2)^t$ on V_2', where $g = \zeta T_{21} f$ on $V_1' \cap V_2'$, then (putting $b^{m+k}\zeta^k\delta(f)_{12} = K$)

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} b^{m+k}\zeta^{k+1} f_1 + \zeta K f_2 \\ b^{-m-k}\zeta^{-k+1} f_2 \end{pmatrix}$$

is holomorphic in ζ^{-1}. Because f_2 is holomorphic in ζ and g_2 is holomorphic in ζ^{-1}, f_2 must be a polynomial of degree $\leq (k-1)$ in ζ, say $f_2 = \sum_{j=0}^{k-1} \lambda_j \zeta^j$. Then looking at g_1, since f_1 is holomorphic, $\zeta K f_2$ must have no terms of order

$1, 2, \ldots, k$ in ζ; i.e. Kf_2 can have no terms of order $0, \ldots, k-1$.

$$Kf_2 = b^{m+2k} \sum_{i=1}^{2k} a_i(b)(\zeta^k + \zeta_i(b)\zeta^{k-1} + \ldots + \zeta_i(b)^{2k}\zeta^{-k}) \sum_{j=0}^{k-1} \lambda_j \zeta^j,$$

and the coefficients of $1, \zeta, \zeta^2, \ldots, \zeta^{k-1}$ in this (ignoring the factor b^{m+2k} and collecting powers of ζ) are given by the entries of the column vector

$$M\lambda = \begin{pmatrix} \sum a_i\zeta_i^k & \sum a_i\zeta_i^{k+1} & \ldots & \sum a_i\zeta_i^{2k-1} \\ \sum a_i\zeta_i^{k-1} & \sum a_i\zeta_i^k & \ldots & \sum a_i\zeta_i^{2k-2} \\ \vdots & \vdots & & \vdots \\ \sum a_i\zeta_i & \sum a_i\zeta_i^2 & \ldots & \sum a_i\zeta_i^k \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{k-1} \end{pmatrix}$$

where this is the matrix M of the lemma. There can only be a section of $\tilde{E}_b(-1)$ if $M\lambda = 0$, and this can only happen if all the λ_i are zero (i.e. $f_2 = 0$) or the matrix M is singular. But $f_2 = 0 \Rightarrow f_1 = 0$ (from the expression for g_1). So $E_b(-1)$ has no sections (and hence L_b is not a jumping line for $\tilde{E}$) unless the matrix M is singular. □

Note that this condition extends to the case where the $\zeta_i(b)$ are not all distinct.

An invariant version: Part of the long exact sequence associated to (2.14) is

$$0 \to H^0(L_b, \tilde{E}(-1)) \to H^0(L_b, \mathcal{O}(k-1)) \xrightarrow{\delta} H^1(L_b, \mathcal{O}(-k-1))$$

where δ is a map between k-dimensional vector spaces. This map has matrix M (with appropriate choices of coordinates), so clearly $H^0(L_b, \tilde{E}(-1)) = 0$ iff M is nonsingular.

Proof of Proposition It is not possible to take the $b \to \infty$ limit of this matrix as it stands, since all the entries tend to zero. Row and column operations do not affect the singularity of M, so multiply row i by b^{k-i} and column j by b^j. This just makes explicit the dependence of the λ_j on b. It is natural to put in the extra factor b^k, so that $g_1 = b^{m+k}\hat{g}_1$. So if M_{ij} is the (i, j) entry of the transformed M,

$$M_{ij}(b) = \sum_{n=1}^{2k} a_n(b\zeta_n(b))^{k-i+j} = \hat{M}_{ij} + \epsilon_{ij},$$

where $\hat{M}_{ij} = \sum_{n=1}^{k} a_n(b\zeta_n)^{k-i+j}$ and $\epsilon_{ij} = \sum_{n=k+1}^{2k} a_n(b\zeta_n)^{k-i+j}$. The matrix $\hat{M}$ has a finite limit, since a_n and $b\zeta_n$ have finite limits:

$$\hat{M}_{ij}(\infty) = \sum_{n=1}^{k} a_n(\infty)c_n^{k-i+j}.$$

Furthermore, this limit is nonsingular. The determinant of $\hat{M}$ is calculated in [19] by expressing it in terms of Vandermonde matrices. Let V, W and D be the matrices $V_{ij} = c_j^{k-i}$, $W_{ij} = c_i^j$ and $D = \mathrm{diag}\{a_i(\infty)\}$. Then $\hat{M}(\infty) = VDW$, and

$$\begin{aligned} \det \hat{M}(\infty) &= \det V \det D \det W \\ &= (-1)^k \prod_{k\geq i>j\geq 1} (c_i - c_j) \prod a_i(\infty) \prod c_i \prod_{k\geq i>j\geq 1} (c_i - c_j), \end{aligned}$$

which is finite and nonzero (Lemma 2.4). M can be approximated by $\hat{M}$ for large b, since by (2.13), and the calculations in (2.3.1), $\epsilon_{ij} = O(b^{-2m-2k}).O(b^{k-i+j}) = O(b^{-2m-k-i+j})$. The entry which decays slowest is in the top right hand corner, where $i = 1$ and $j = k$. This term is $O(b^{-2m-1})$. So ϵ decays as $b \to \infty$. This shows that for large enough b, say $b > b_0$, the matrix $M(b)$ is nonsingular and so L_b is not a jumping line of $\tilde{E}$. The proof extends to show that the monopole defined by $\tilde{E}$ is nonsingular outside a compact set: varying the choice of coordinates gives the result for an open patch of directions away from the origin in $\mathbb{H}^3$. Then the (compact) sphere of directions has a covering of these patches, with a bound for b on each. (It was assumed that the c_i were distinct, but even if this is not the case the result still holds because of the factor $\prod(c_i - c_j)$ in the denominator of $a_i(\infty)$.) □

2.6 The asymptotic expression for Φ

This is the section which leads to the proof of Theorem 2.1. Suppose that $b > b_0$, then L_b is not a jumping line and the restriction $\tilde{E}_b$ is holomorphically trivial. Then any section is completely determined by its value at a point. Remember that $E_b = H^0(L_b, \mathcal{O}(\tilde{E}))$, so that sections of $\tilde{E}$ over a real line correspond to points in the fibre E_b. By Proposition (1.7) and the argument before it, there is a symplectic form on these sections of $\tilde{E}$, given by

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = u_1 v_2 - u_2 v_1$$

evaluated at a point of the line. It does not depend on the point chosen. Let

$$s(b,\zeta) = (s_1(b,\zeta), s_2(b,\zeta))^t, \quad t(b,\zeta) = (t_1(b,\zeta), t_2(b,\zeta))^t$$

be the sections in the V_1' trivialisation with $s(b,0) = (1,0)^t$ and $t(b,0) = (0,1)^t$. Then $\{s,t\}$ is a ∇^1-flat basis for E_b. Recall that ∇^1 is the connection corresponding to evaluation at $(\eta,\zeta) = (0,0) \in Z$ and ∇^2 corresponds to evaluation at $\gamma(0,0)$ (i.e. the two connections come from evaluation at $\zeta = 0$ and $\zeta = \infty$ on the blowup). Let $u(b,\zeta)$ and $v(b,\zeta)$ be the functions corresponding to s and t

in the V_2' trivialisation, so that $\{u, v\}$ is a ∇^2-flat basis and the change of basis matrix is

$$g = \begin{pmatrix} u_1(b,\infty) & v_1(b,\infty) \\ u_2(b,\infty) & v_2(b,\infty) \end{pmatrix}. \tag{2.15}$$

In a ∇^1-flat basis, $\nabla^1 = d$ and $\nabla^2 = d + g^{-1}\partial_r g$, where $r = \frac{1}{2}\log b$ is the geodesic distance along the line in $\mathbb{H}^3$. Substituting these into the formula for Φ,

$$\Phi = \frac{i}{2} g^{-1}\partial_r g \tag{2.16}$$

Using the symplectic structure,

$$\langle u(b,\zeta), v(b,\zeta)\rangle = \langle u(b,0), v(b,0)\rangle = 1$$

for all ζ. But

$$\langle u(b,\infty), v(b,\infty)\rangle = \det g$$

So $\det g = 1$. The idea now is to use similar arguments to those of the previous section to find formulae for u and v at $\zeta = \infty$ in terms of the points ζ_i, and then use these in the formula for Φ.

Using the transition matrix T_{21} to relate s and u,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} b^{m+k}\zeta^k & K(b,\zeta) \\ 0 & b^{-m-k}\zeta^{-k} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} b^{m+k}\zeta^k s_1 + K(b,\zeta)s_2 \\ b^{-m-k}\zeta^{-k}s_2 \end{pmatrix} \tag{2.17}$$

Put

$$u_1(b,\zeta) = b^{m+k}\hat{u}_1(b,\zeta), \quad u_2(b,\zeta) = b^{-m-k}\hat{u}_2(b,\zeta), \tag{2.18}$$

and let $s_l(b,\zeta) = \sum s_{ln}(b)\zeta^n$. Then

$$\begin{aligned} \hat{u}_1 &= \sum s_{1n}\zeta^{k+n} + \sum_{i=1}^{2k} b^k a_i(\zeta^k + \cdots + \zeta_i^{2k}\zeta^{-k}) \sum s_{2n}\zeta^n \\ \hat{u}_2 &= \sum s_{2n}\zeta^{n-k} \end{aligned}$$

The section $\hat{u}_2$ is continuous at $\zeta = \infty$ so only negative powers of ζ can have nonzero coefficients. Hence, from (2.17), s_2 only has powers of ζ which are less than or equal to k, and $s_2(b,\zeta) = \sum_0^k s_{2n}\zeta^n$ because s is continuous at $\zeta = 0$. Similarly, looking at $\hat{u}_1$, there can be no positive powers of ζ. The positive powers of ζ coming from s_1 must cancel with those from the other term in the sum and hence also $s_1(b,\zeta) = \sum_0^k s_{1n}\zeta^n$.

2.6.1 A matrix formula

The constant term of $\hat{u}_2$ is $s_{2k} = \hat{u}_2(b,\infty)$. The coefficients of $\zeta, \ldots, \zeta^k$ in $\hat{u}_1$ must all vanish, and the coefficient of 1 is $\hat{u}_1(b,\infty) = \hat{u}_{10}$. Writing these as a matrix equation,

$$\begin{pmatrix} \sum b^k a_i\zeta_i^k & \sum b^k a_i\zeta_i^{k+1} & \cdots & \sum b^k a_i\zeta_i^{2k} \\ \sum b^k a_i\zeta_i^{k-1} & \sum b^k a_i\zeta_i^k & \cdots & \sum b^k a_i\zeta_i^{2k-1} \\ \vdots & \vdots & & \vdots \\ \sum b^k a_i & \sum b^k a_i\zeta_i & \cdots & \sum b^k a_i\zeta_i^k \end{pmatrix} \begin{pmatrix} s_{20} \\ s_{21} \\ \vdots \\ s_{2k} \end{pmatrix} = \begin{pmatrix} \hat{u}_{10} \\ 0 \\ \vdots \\ -s_{10} \end{pmatrix} \tag{2.19}$$

where the sums are all from $i = 1$ to $2k$, and $s_{10} = 1$ from the definition of s. If there is a form of (2.19) which has a good limit as $b \to \infty$, it will provide a formula for $\hat{u}_{10}$ in terms of the ζ_i. This problem is similar to that of §2.5. To check that the matrix has a good limit, write $\tilde{s}_{2n} = b^{k-n}s_{2n}$ and do the row operations $r_i \mapsto b^{k-i+1}r_i$, so that (2.19) becomes:

$$\begin{pmatrix} \sum a_i(b\zeta_i)^k & \sum a_i(b\zeta_i)^{k+1} & \cdots & \sum a_i(b\zeta_i)^{2k} \\ \sum a_i(b\zeta_i)^{k-1} & \sum a_i(b\zeta_i)^k & \cdots & \sum a_i(b\zeta_i)^{2k-1} \\ \vdots & \vdots & & \vdots \\ \sum a_i & \sum a_i(b\zeta_i) & \cdots & \sum a_i(b\zeta_i)^k \end{pmatrix} \begin{pmatrix} \tilde{s}_{20} \\ \tilde{s}_{21} \\ \vdots \\ \tilde{s}_{2k} \end{pmatrix} = \begin{pmatrix} b^k\hat{u}_{10} \\ 0 \\ \vdots \\ -1 \end{pmatrix} \tag{2.20}$$

It is also necessary to know the behaviour of the $\tilde{s}_{2n}$ as $b \to \infty$. Note that $\tilde{s}_{20} = 0$ (since $s_{20} = 0$), so leaving out the first row and first column of the matrix,

$$\begin{pmatrix} \sum a_i(b\zeta_i)^k & \cdots & \sum a_i(b\zeta_i)^{2k-1} \\ \vdots & & \vdots \\ \sum a_i(b\zeta_i) & \cdots & \sum a_i(b\zeta_i)^k \end{pmatrix} \begin{pmatrix} \tilde{s}_{21} \\ \vdots \\ \tilde{s}_{2k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ -1 \end{pmatrix}$$

This matrix is the matrix M of Lemma 2.6 (after row and column operations), so, from the proof of Proposition 2.5, $M = \hat{M} + \epsilon$ where $\hat{M}$ has a finite nonsingular limit as $b \to \infty$ and $\epsilon = O(b^{-2m-1})$. Hence

$$\begin{pmatrix} \tilde{s}_{21} \\ \vdots \\ \tilde{s}_{2k} \end{pmatrix} = \hat{M}^{-1} \begin{pmatrix} 0 \\ \vdots \\ -1 \end{pmatrix} + O(b^{-2m-1})$$

Because $\hat{M}$ has a finite nonsingular limit, so does $\hat{M}^{-1}$ and then each $\tilde{s}_{2n}$ must also have a finite limit as $b \to \infty$. In particular, $\tilde{s}_{2k} = s_{2k} = \hat{u}_2(b, \infty)$ has a finite limit. Now let H denote the $(k+1) \times (k+1)$ matrix of (2.20), then

$$\begin{aligned} H_{ij} &= \tilde{H}_{ij} + \mu_{ij} \\ &= \sum_{n=1}^{k} a_n(b\zeta_n)^{k-i+j} + \sum_{n=k+1}^{2k} a_n(b\zeta_n)^{k-i+j} \end{aligned}$$

where $\mu_{ij} = O(b^{-2m-2k}).O(b^{k-i+j})$. Again, the slowest decaying entry is in the top right hand corner: $\mu_{1,k+1} = O(b^{-2m})$, which tends to zero as $b \to \infty$. So H can be approximated by $\tilde{H}$ for large b, in other words by the same matrix but with sums from 1 to k instead of 1 to $2k$. Using this approximation and row reducing

(2.20) with the row operations $\{r_j \mapsto r_j - (b\zeta_1)r_{j+1}\}_{j=1}^{k}$, $\{r_j \mapsto r_j - (b\zeta_2)r_{j+1}\}_{j=1}^{k-1}$, $\ldots$, $r_1 \mapsto r_1 - (b\zeta_k)r_2$, the top row is

$$b^k \hat{u}_{10} = (-1)^k b^k \prod_{i=1}^{k} \zeta_i + O(b^{-2m})$$

so

$$\hat{u}_{10} = (-1)^k \prod_{i=1}^{k} \zeta_i + O(b^{-2m-k}).$$

(this is OK because the $\tilde{s}_{2n}$ have finite limits). For the other entries, put

$$v_1(b,\zeta) = b^{m+k}\hat{v}_1(b,\zeta), \quad v_2(b,\zeta) = b^{-m-k}\hat{v}_2(b,\zeta) \tag{2.21}$$

then a calculation like the one above shows that $\hat{v}_1(b,\infty) = \hat{v}_{10} = O(b^{-2m-k})$ (because $t_{10} = 0$). Using these results and the symplectic structure, $\hat{v}_{20} = \hat{u}_{10}^{-1}$.
Note: The approximations are all valid up to terms of order b^{-2m-k}. Calculating Φ involves differentiating, but with respect to the geodesic distance $r = -1/2 \log b$, rather than with respect to b. Since $\frac{d}{dr}(b^{-a}) = O(b^{-a})$, the derivatives of the approximation will also be valid up to terms of order b^{-2m-k}, and differentiating the approximations will not cause any problems.

2.6.2 Calculating Φ

From (2.16):

$$\Phi = \frac{i}{2}\begin{pmatrix} v_2(b,\infty) & -v_1(b,\infty) \\ -u_2(b,\infty) & u_1(b,\infty) \end{pmatrix} \partial_r \begin{pmatrix} u_1(b,\infty) & v_1(b,\infty) \\ u_2(b,\infty) & v_2(b,\infty) \end{pmatrix}$$

(recall $\det g = 1$) where r is geodesic distance and $b = e^{2r}$. Substituting, Φ is given by the matrix

$$\frac{i}{2}\begin{pmatrix} 2(m+k) + \partial_r \log \prod \zeta_i + O(e^{-4mr}) & O(e^{-4mr}) \\ O(e^{-2kr}) & -2(m+k) - \partial_r \log \prod \zeta_i + O(e^{-4mr}) \end{pmatrix}$$

which means that

$$|\Phi|_{as} = \frac{1}{2}\{2(m+k) + \partial_r \log \prod_{i=1}^{k} \zeta_i\}$$

The formula for Φ is correct up to terms of order b^{-2m}; this is where the condition $m \geq \frac{1}{2}$ becomes important. Otherwise the error terms will be larger than the approximations. But provided the condition is satisfied there is no problem in what follows. Note that the formula makes sense whether the $\zeta_i(b)$ are distinct or not. To see that the limit as $b \to \infty$ is the right one, use the fact that $\prod \zeta_i$, as a function of b, has a zero of order k at $b = \infty$ (from (2.8)). Write

$$\prod \zeta_n(b) = b^{-k} \sum_{n=0}^{\infty} a_n b^{-n} = e^{-2kr} \sum_{n=0}^{\infty} a_n e^{-2nr}$$

where $a_0 \neq 0$. Then, since $\partial \log f = \partial f / f$,

$$\frac{\partial_r \prod \zeta_n}{\prod \zeta_n} = \frac{-2ke^{-2kr} \sum a_n e^{-2nr} + e^{-2kr} \sum (-2n) a_n e^{-2nr}}{e^{-2kr} \sum a_n e^{-2nr}} = -2k + O(e^{-2r})$$

So

$$\begin{aligned} |\Phi|_{as} &= \frac{1}{2}\{2(m+k) - 2k + O(e^{-2r})\} \\ &= m + O(e^{-2r}) \end{aligned}$$

The error in this approximation is $O(e^{-4mr})$, which is smaller than the $O(e^{-2r})$ error because $4m \geq 2$. Thus $|\Phi|_{as} \to m$ as $r \to \infty$. This calculation proves Theorem 2.1. $\square$

2.6.3 Comparison with the Euclidean case

Although the method used here is very similar to Hurtubise's method in the Euclidean case, there are some significant differences, besides the fact that the boundary conditions for monopoles are different in the two cases. The most obvious difference is that the rates of decay of the errors must be kept track of much more carefully here; in the Euclidean case all errors decay exponentially in b instead of only polynomially. The Euclidean case may be thought of as the limit of the hyperbolic case as $m \to \infty$ (see Chapter 4) and clearly as m increases so does the rate of decay of the errors. In the Euclidean case the limits of the points ζ_i are either zero or infinity, there are no finite limits. Another difference is that the Euclidean minitwistor space $T\mathbb{P}^1$ is already a fibration, whereas Z is not. This is no problem when restricting to $\mathcal{P}$ and finding a formula for Φ along a single geodesic, but it does become an obstacle to extending results "horizontally". The same method as used above gives a formula for the component ∇_3 of the connection in the direction of l, namely $\nabla_3 = d + \Phi dr$, but not for the components ∇_1, ∇_2 perpendicular to it.

Chapter 3

Braam and Austin's description

Braam and Austin's description of hyperbolic monopoles [12] is a circle-invariant version of the ADHM construction (see below). It will be used to study monopoles with large mass in the hope of understanding the Euclidean limit $m \to \infty$. It only holds for instantons on S^4, i.e. monopoles with $m \in \frac{1}{2}\mathbb{Z}$, but is still useful as it consists of algebraic data, matrices and difference equations, which are relatively tractable. What follows is a review of Braam and Austin's construction, from [12], together with some details not in that paper. From now on, hyperbolic monopoles will be assumed to have mass $m \in \frac{1}{2}\mathbb{Z}$.

3.1 The ADHM construction

There is a by now well-known description of instantons on S^4 in terms of linear algebra [6] using the twistor correspondence and results of Horrocks and Barth. "ADHM data" for an instanton with invariant κ consists of complex vector spaces W and V, dimension κ and $2\kappa + 2$ respectively, and a map $A(\mathbf{z}) : W \to V$ which depends linearly on $\mathbf{z} \in \mathbb{C}^4$ (i.e. $A(\mathbf{z}) = \sum A_i z_i$ for constant matrices A_i).

There is also an antilinear map σ, such that $\sigma^2 = 1$ on W; $\sigma^2 = -1$ on V, and σ on $\mathbb{C}^4$ is given by $\sigma(\mathbf{z}) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)$. The conditions on these objects are that

- V has a fixed skew, nondegenerate bilinear form compatible with σ (in other words, $(\sigma v_1, \sigma v_2) = \overline{(v_1, v_2)})$
- The Hermitian form $\langle , \rangle$ defined by $\langle v_1, \sigma v_2 \rangle = (v_1, v_2)$ must be positive definite
- W has a real structure (given by σ)
- $A(\mathbf{z})$ is injective and $U_{\mathbf{z}} = A(\mathbf{z})W$ is isotropic with respect to the skew form on V, $\forall \mathbf{z} \neq 0$

- A(z) is compatible with σ, i.e. $\sigma[A(\mathbf{z})w] = A(\sigma\mathbf{z})\sigma w$

The first three conditions ensure that V may be identified with a quaternionic vector space, and W with $W_{\mathbb{R}} \otimes \mathbb{C}$, in a natural way. The map σ represents multiplication by the quaternion j on V and complex conjugation on W. This data describes a holomorphic bundle $\tilde{E}$ on $\mathbb{C}P^3$ which corresponds to an instanton on S^4, by $\tilde{E}_{\mathbf{z}} = U_{\mathbf{z}}^0/U_{\mathbf{z}}$, where $U_{\mathbf{z}}^0$ is the annihilator of $U_{\mathbf{z}}$. The fibre $\tilde{E}_{\mathbf{z}}$ has complex dimension 2 and is the cohomology of

$$W \xrightarrow{A(\mathbf{z})} V \xrightarrow{A(\mathbf{z})^*} W^*$$

with A^* defined using the quaternionic structure on V. The connection on $\tilde{E}$ is the projection onto $\tilde{E}$ of the trivial connection on $V \times \mathbb{C}P^3$. The fourth condition above is that $\tilde{E}$ really is a bundle; A must be injective so that the fibre dimension does not jump, and the isotropy condition implies that $U_{\mathbf{z}} \subset U_{\mathbf{z}}^0$. The last condition is that the whole construction respects σ, so $\tilde{E}$ inherits a quaternionic structure.

The group $G = \mathrm{GL}(W_{\mathbb{R}}) \times \mathrm{Sp}(V)$ acts on (W, V, A) by

$$(g, u) : (W, V, A) \mapsto (gW, uV, uAg^{-1})$$

where $\mathrm{GL}(W_{\mathbb{R}})$ is the subgroup of $\mathrm{GL}(W)$ preserving the real structure and $\mathrm{Sp}(V)$ preserves the quaternionic structure on V. This action defines an equivalence relation on the set of ADHM data. Because of the equivalence between instantons and holomorphic bundles, there is the

Theorem 3.1 (ADHM). *There is a 1-1 correspondence between*

(i) equivalence classes of ADHM data, and

(ii) ASD instantons on S^4 up to gauge. □

3.2 Adaptation for monopoles

When the instanton is circle-invariant, the $\tilde{S}^1$-action lifts in a natural way to an action on all the ADHM data. Note that the circle acting now is the double cover $\tilde{S}^1$, so it acts with weight 2 on $\mathbb{R}^4$. With this natural action, the map A is $\tilde{S}^1$-equivariant and the vector spaces V and W are representation spaces for $\tilde{S}^1$, which split into direct sums of weight spaces. The integral and half-integral cases are slightly different; in what follows, M will always denote an odd number ($m = M/2$ in the case where $m \notin \mathbb{Z}$). Let $\mathbb{Z}_o$ denote the odd integers. The first thing to do is to find the characters of the representations, by proving

Lemma 3.2. *The vector spaces V and W, written as complex representation spaces, are given by:*

$$V = \begin{cases} \mathbb{C}^{k+1}_{-2m} \oplus \mathbb{C}^{2k}_{-2m+2} \oplus \mathbb{C}^{2k}_{-2m+4} \cdots \oplus \mathbb{C}^{2k}_{0} \oplus \mathbb{C}^{2k}_{2} \oplus \ldots \oplus \mathbb{C}^{k+1}_{2m} & : m \in \mathbb{Z} \\ \mathbb{C}^{k+1}_{-M} \oplus \mathbb{C}^{2k}_{-M+2} \oplus \mathbb{C}^{2k}_{-M+4} \cdots \oplus \mathbb{C}^{2k}_{-1} \oplus \mathbb{C}^{2k}_{1} \oplus \ldots \oplus \mathbb{C}^{k+1}_{M} & : 2m = M \in \mathbb{Z}_o \end{cases}$$

and

$$W = \begin{cases} \mathbb{C}^{k}_{-2m+1} \oplus \mathbb{C}^{k}_{-2m+3} \oplus \ldots \oplus \mathbb{C}^{k}_{-1} \oplus \mathbb{C}^{k}_{1} \oplus \ldots \oplus \mathbb{C}^{k}_{2m-1} & : m \in \mathbb{Z} \\ \mathbb{C}^{k}_{-M+1} \oplus \mathbb{C}^{k}_{-M+3} \oplus \ldots \oplus \mathbb{C}^{k}_{0} \oplus \mathbb{C}^{k}_{2} \oplus \ldots \oplus \mathbb{C}^{k}_{M-1} & : 2m = M \in \mathbb{Z}_o \end{cases}$$

where $\mathbb{C}^s_r$ is the s-dimensional complex vector space on which $\tilde{S}^1$ acts with weight r.

The proof of this lemma involves the use of the equivariant Atiyah-Singer index formula, and some details are given in Appendix A. The most important point is that V and W are kernels of Dirac operators (explained in [14]). The quaternionic structure on V and the real structure on W arise from the quaternionic structures on E and on the spin bundles of S^4. Then the equivariant index theorem may be used to calculate the equivariant Chern characters of W and V, which give the weight space decompositions.

Write $V = \bigoplus V_j$ and $W = \bigoplus W_j$, where j is the weight of the $\tilde{S}^1$ representation, and note that $\dim_{\mathbb{C}}(W) = 2mk$ and $\dim_{\mathbb{C}}(V) = 4mk + 2$. Fix isomorphisms $W_j \to \mathbb{C}^k$ such that the real structure on W is given by $\sigma(w) = R\bar{w}$, where R is the $(2mk) \times (2mk)$ matrix with $k \times k$ blocks:

$$R = \begin{pmatrix} & & I_k \\ & \cdot^{\cdot^{\cdot}} & \\ I_k & & \end{pmatrix}$$

For V fix unitary bases such that $\sigma(v) = Jv$ where J is the matrix

$$J = \begin{pmatrix} & & & & & & I_{k+1} \\ & & & & & I_{2k} & \\ & & & & I_{2k} & & \\ & & & \cdot^{\cdot^{\cdot}} & & & \\ & & -I_{2k} & & & & \\ & -I_{2k} & & & & & \\ -I_{k+1} & & & & & & \end{pmatrix}$$

If $2m = M$, where $M \in \mathbb{Z}_o$, this definition is fine. If $m \in \mathbb{Z}$ there is a block J_0 in the middle of the matrix giving the symplectic structure on V_0. Let $J_0 = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$. This extra slight complication is why Braam and Austin only treat the case $2m$ odd in their paper.

The map $A(z) = \sum A_i z_i$ must respect the decompositions of V and W, in other words it may only change the weight of a representation by ± 1. So $A_i z_i W_j \subset V_{j\pm 1}$, with the sign depending on the circle action on z_i. The only other conditions on A are the last two listed in §3.1, which are that $A(z)$ is injective for all z, compatible with σ (quaternion linear), and that $A(z)W$ is isotropic with respect to the skew form on V.

The $\tilde{S}^1$ action on $\mathbb{C}^4$ is

$$\lambda : \mathbf{z} \mapsto (\lambda^{-1} z_1, \lambda z_2, \lambda^{-1} z_3, \lambda z_4)$$

i.e. the double cover of the S^1 action defined earlier. Circle equivariance

$$\lambda.(A(\mathbf{z})w) = A(\lambda.\mathbf{z})(\lambda.w)$$

means that

$$\begin{aligned} A_1, A_3 &: W_j \to V_{j-1} \\ A_2, A_4 &: W_j \to V_{j+1} \end{aligned}$$

and the bases for W and V may be chosen (using the injectivity of A_1) so that

$$A_1 = \begin{pmatrix} I_k & & & & & & \\ 0 & & & & & & \\ & I_k & & & & & \\ & 0_k & & & & & \\ & & \ddots & & & & \\ & & & I_k & & & \\ & & & 0_k & & & \\ & & & & 0_k & & \\ & & & & I_k & & \\ & & & & & \ddots & \\ & & & & & & 0_k \\ & & & & & & I_k \\ & & & & & & 0_k \\ & & & & & & 0 \end{pmatrix}$$

where the blocks are the weight spaces, with the largest negative weight at the top. The two middle blocks are

$$\begin{pmatrix} I_k \\ 0_k \end{pmatrix} : \begin{cases} W_{-1} \to V_{-2} \text{ when } m \in \mathbb{Z} \\ W_0 \to V_{-1} \text{ when } 2m = M \in \mathbb{Z}_o \end{cases}$$

$$\begin{pmatrix} 0_k \\ I_k \end{pmatrix} : \begin{cases} W_1 \to V_0 \text{ when } m \in \mathbb{Z} \\ W_2 \to V_1 \text{ when } 2m = M \in \mathbb{Z}_o \end{cases}$$

The quaternionic linearity condition $A(\sigma\mathbf{z})(\sigma w) = \sigma(A(\mathbf{z})w)$, expressed in terms of R and J, is

$$(-A_1\bar{z}_2 + A_2\bar{z}_1 - A_3\bar{z}_4 + A_4\bar{z}_3)R\bar{w} = J(\overline{A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4})\bar{w}$$

Equating coefficients of $\bar{z}_i$, this is equivalent to

$$A_2 = J\bar{A}_1 R \qquad A_4 = J\bar{A}_3 R$$

Using this,

$$A_2 = \begin{pmatrix}
0_k & & & & & & \\
0 & & & & & & \\
0_k & & & & & & \\
I_k & & & & & & \\
 & 0_k & & & & & \\
 & I_k & & & & & \\
 & & \ddots & & & & \\
 & & & 0_k & & & \\
 & & & I_k & & & \\
 & & & & -I_k & & \\
 & & & & 0_k & & \\
 & & & & & \ddots & \\
 & & & & & & -I_k \\
 & & & & & & 0
\end{pmatrix}$$

where this time the two middle blocks are

$$\begin{pmatrix} 0_k \\ I_k \end{pmatrix} : \begin{cases} W_{-3} \to V_{-2} \text{ when } m \in \mathbb{Z} \\ W_{-2} \to V_{-1} \text{ when } 2m = M \in \mathbb{Z}_o \end{cases}$$

$$\begin{pmatrix} -I_k \\ 0_k \end{pmatrix} : \begin{cases} W_{-1} \to V_0 \text{ when } m \in \mathbb{Z} \\ W_0 \to V_1 \text{ when } 2m = M \in \mathbb{Z}_o \end{cases}$$

With A_1 and A_2 fixed, A_3 must have the same diagonal form as A_1, and A_4 is related to it by quaternion linearity (and has the same shape as A_2), so they may be taken to be

$$A_3 = \begin{pmatrix}
\beta_{-2m+1} & & & & & & \\
v & & & & & & \\
 & \beta_{-2m+3} & & & & & \\
 & \gamma_{-2m+2} & & & & & \\
 & & \ddots & & & & \\
 & & & \beta_{-1} & & & \\
 & & & \gamma_{-2} & & & \\
 & & & & -\gamma_0 & & \\
 & & & & \beta_1 & & \\
 & & & & & \ddots & \\
 & & & & & & -\gamma_{2m-2} \\
 & & & & & & \beta_{2m-1} \\
 & & & & & & 0_k \\
 & & & & & & 0
\end{pmatrix}$$

where $\beta_j : W_j \to V_{j-1}$ and $\gamma_j : W_{j+1} \to V_j$, and

$$A_4 = \begin{pmatrix} \begin{matrix} 0_k \\ 0 \\ -\bar{\gamma}_{2m-2} \\ \bar{\beta}_{2m-1} \end{matrix} & & & & & & \\ & \begin{matrix} -\bar{\gamma}_{2m-4} \\ \bar{\beta}_{2m-3} \end{matrix} & & & & & \\ & & \ddots & & & & \\ & & & \begin{matrix} -\bar{\gamma}_{2} \\ \bar{\beta}_{3} \end{matrix} & & & \\ & & & & \begin{matrix} -\bar{\beta}_{1} \\ -\bar{\gamma}_{0} \end{matrix} & & \\ & & & & & \ddots & \\ & & & & & & \begin{matrix} -\bar{\beta}_{-2m+1} \\ -\bar{v} \end{matrix} \end{pmatrix}$$

These are matrices for A_3 and A_4 when $m \in \mathbb{Z}$. The matrices in the half-integral case are similar, but the numbering of the β_j and γ_j changes slightly. The gauge fixing means that the bundle is trivialised over P_∞, the fibre in $\mathbb{C}P^3$ over $\infty = [1, 0] \in S^4$ (this is the line $z_3 = z_4 = 0$).

3.3 Gauge group

The group acting on this data is $G^{\tilde{S}^1} = \mathrm{GL}(W_\mathbb{R})^{\tilde{S}^1} \times \mathrm{Sp}(V)^{\tilde{S}^1}$, the $\tilde{S}^1$-equivariant subgroup of G. The aim of this section is to find the subgroup of $G^{\tilde{S}^1}$ which preserves the chosen form of A.

If $(g, u) \in G^{\tilde{S}^1}$, then g and u must have the form $g = \mathrm{diag}\{g_i\}$ and $u = \mathrm{diag}\{u_i\}$, where $g_j : W_j \to W_j$ and $u_j : V_j \to V_j$. In addition, the blocks u_j must be unitary and $u_j = \bar{u}_{-j}$, from the condition $Ju = \bar{u}J$.

The condition that the action of (g, u) preserves A_1 and A_2 is

$$u_j = \begin{pmatrix} g_{j+1} & 0_k \\ 0_k & g_{j-1} \end{pmatrix}$$

Hence the group acting in this case is

$$\mathcal{G}_m = \{\text{maps}\{-2m+1, -2m+3, \ldots, 2m-3, 2m-1\} \to U(k) : g_j = \bar{g}_{-j}\}$$

acting on Braam and Austin's matrices by

$$\beta_j \mapsto g_j \beta_j g_j^{-1} \tag{3.1}$$

$$\gamma_j \mapsto g_{j-1} \gamma_j g_{j+1}^{-1} \tag{3.2}$$

$$v \mapsto v g_{-2m+1}^{-1} \tag{3.3}$$

There is also a circle action on v on the left, which corresponds to changing the trivialisation of $\tilde{E}$ at infinity. It is sometimes useful to include this (but see Theorem 3.3 part (ii)).

3.4 Isotropy conditions

The isotropy condition on $A(\mathbf{z})W = U_{\mathbf{z}}$ is that $U_{\mathbf{z}} \subset U_{\mathbf{z}}^0$, i.e. that $A(\mathbf{z})^* A(\mathbf{z}) = 0$ for all $\mathbf{z}$. The adjoint A^* of A is with respect to the symplectic structure on V, and in terms of matrices $A^* = A^t J$, using the diagram

$$\begin{array}{ccc} W & \xrightarrow{A} & V \\ & \swarrow_{A^*} & \downarrow J \\ W^* & \xleftarrow[A^t]{} & V^* \end{array}$$

The equation $\sum z_i A_i^t J A_j z_j = 0$, $\forall \mathbf{z}$, is the same as the conditions

$$\begin{aligned} A_i^t J A_i &= 0 \\ \text{and} \quad A_i^t J A_j + A_j^t J A_i &= 0 \text{ for } i \neq j \end{aligned}$$

Once these are multiplied out (using the explicit matrices defined above) their entries give the difference equations:

$$\gamma_j - \gamma_{-j}^t = 0 \tag{3.4}$$

$$\beta_j - \beta_{-j}^t = 0 \tag{3.5}$$

$$\beta_{j-1}\gamma_j - \gamma_j \beta_{j+1} = 0 \tag{3.6}$$

$$[\beta_j^*, \beta_j] + \gamma_{j-1}^* \gamma_{j-1} - \gamma_{j+1}\gamma_{j+1}^* = 0 \tag{3.7}$$

$$[\beta_{2m-1}, \beta_{2m-1}^*] + v^t \bar{v} - \gamma_{2m-2}^* \gamma_{2m-2} = 0 \tag{3.8}$$

These will be referred to as the **Braam-Austin equations**. If $m \in \mathbb{Z}$, β_j is defined for odd j and γ_j is defined for even j, with $-2m+1 \leq j \leq 2m-1$. If $2m = M$ is odd, β_j is defined for even j and γ_j for odd j, with $-M+1 \leq j \leq M-1$.

So Braam and Austin's data consists of a set of matrices $\{\beta_j, \gamma_j, v\}$ for $0 \leq j \leq 2m-1$ (using (3.4) and (3.5)) which satisfy (3.6), (3.7) and (3.8), up to the action of $\mathcal{G}_m$. There is a circle-invariant version of Theorem 3.1:

Theorem 3.3 (Braam and Austin). *There is a 1-1 correspondence between*

(i) Equivalence classes of Braam-Austin data

(ii) Gauge equivalence classes of monopoles with a fixed $U(1)$ trivialisation at infinity □

3.5 Standard form

Atiyah ([3], Chapter IV, §§2,3) has a formulation of the ADHM data as a quaternionic linear map, for a bundle on $S^4 = \mathbf{H}P^1$ in terms of homogeneous coordinates (q_1, q_2):

$$v(q_1, q_2) = Cq_1 + Dq_2$$

The conditions on v are that $v(q_1, q_2)$ has maximal rank for all $(q_1, q_2) \neq (0,0)$ and that

$$v^*v \text{ is real} \tag{3.9}$$

The gauge group acting is $\mathrm{Sp}(2mk+1) \times \mathrm{GL}(2mk, \mathbb{R})$, where

$$(x, y) : v \mapsto xvy^{-1}$$

Let $q = q_1/q_2$. Then the map v may be put into a "standard form" (see [3])

$$v(q) = \begin{pmatrix} \Lambda \\ B - qI_k \end{pmatrix}$$

where Λ is a k-row vector and B is a $k \times k$ matrix. The condition (3.9) translates to

$$\begin{aligned} &i) \quad \Lambda^*\Lambda + B^*B \text{ is real, and} \\ &ii) \quad B \text{ is symmetric} \end{aligned}$$

Writing the ADHM data this way brings the bundle back from twistor space to $\mathbb{R}^4$, and so describes the monopole more "directly". The Braam-Austin matrix $A(\mathbf{z})$ may also be written in this form after some changes of basis. This will be done for the case where $m = M/2$ with M odd; the case $m \in \mathbb{Z}$ is similar. The idea is to change bases so that Braam and Austin's real and quaternionic structures on W and V become the standard ones.

3.5.1 Quaternion linear map

First, the vector space V must be identified with a quaternionic vector space, and W with a real vector space

$$W = (\mathbb{R}_0^k \oplus \mathbb{R}_2^{2k} \oplus \ldots \oplus \mathbb{R}_{M-1}^{2k}) \times_{\mathbb{R}} \mathbb{C}$$

using the quaternionic and real structures on W and V. In the current basis, the circle acts on W_r as $e^{2ir\theta}$, whereas in terms of a real basis W_r is split into $\mathbb{R}^2$ factors on which the circle acts by

$$\begin{pmatrix} \cos 2r\theta & \sin 2r\theta \\ -\sin 2r\theta & \cos 2r\theta \end{pmatrix}$$

If $\{e_1, e_2\}$ is such a real basis, put

$$\begin{aligned} \eta_1 &= e_1 + ie_2 \\ \eta_2 &= e_1 - ie_2 \end{aligned}$$

then the circle acts in terms of the new basis by

$$\begin{pmatrix} e^{2ir\theta} & 0 \\ 0 & e^{-2ir\theta} \end{pmatrix}$$

making explicit the isomorphism $\mathbb{R}^2_r \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}_r \oplus \mathbb{C}_{-r}$. So a change of basis matrix from the complex to the real coordinates is given by

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and the Braam-Austin matrices must be postmultiplied by the matrix

$$\frac{1}{2}\begin{pmatrix} I_k & -iI_k & & & & & & \\ & & I_k & -iI_k & & & & \\ & & & & \ddots & & & \\ & & & & & I_k & -iI_k & \\ & & & & & & & 2I_k \\ & & & & & I_k & iI_k & \\ & & & & \therefore & & & \\ & & I_k & iI_k & & & & \\ I_k & iI_k & & & & & & \end{pmatrix}$$

The other identification is $V \simeq \mathbf{H}^{2mk+1}$ using $\mathbf{H}_r = \mathbb{C}_r \oplus \mathbb{C}_{-r}$ where $\mathbb{C}_r$ may be thought of as the "complex part" and $\mathbb{C}_{-r}$ as the "j part". This corresponds to premultiplying the matrices by blocks $(\, I_{2k} \quad -jI_{2k} \,)$, i.e. multiplying the full matrices by

$$\begin{pmatrix} I_{k+1} & & & & & & -jI_{k+1} \\ & I_{2k} & & & & -jI_{2k} & \\ & & \ddots & & \therefore & & \\ & & & I_{2k} \quad -jI_{2k} & & & \end{pmatrix}$$

(this matrix is not square because pairs of basis vectors are being added together to form quaternionic basis vectors). The transformed matrices $\tilde{A}_i$ satisfy

$$\begin{aligned} \tilde{A}_2 &= \tilde{A}_1 j \\ \tilde{A}_4 &= \tilde{A}_3 j \end{aligned}$$

so that writing $q_1 = z_1 + jz_2$ and $q_2 = z_3 + jz_4$, Braam and Austin's map is now

$$\tilde{A}(q_1, q_2) = \tilde{A}_1 q_1 + \tilde{A}_3 q_2$$

The matrix coefficient $\tilde{A}_1$ is

$$\tilde{A}_1 = \begin{pmatrix}
I_k & -iI_k & & & & & & \\
0 & 0 & & & & & & \\
0_k & 0_k & I_k & -iI_k & & & & \\
-jI_k & kI_k & 0_k & 0_k & & & & \\
 & & 0_k & 0_k & & & & \\
 & & -jI_k & kI_k & & & & \\
 & & & & \ddots & & & \\
 & & & & & 0_k & 0_k & 2I_k \\
 & & & & & -jI_k & kI_k & 0_k
\end{pmatrix}$$

and $\tilde{A}_3$ is the matrix

$$\begin{pmatrix}
\beta_{-2m+1} & -i\beta_{-2m+1} & & & & & & \\
v & -iv & & & & & & \\
j\gamma_{2m-2} & -k\gamma_{2m-2} & \beta_{-2m+3} & -i\beta_{-2m+3} & & & & \\
-j\beta_{2m-1} & k\beta_{2m-1} & \gamma_{-2m+2} & -i\gamma_{-2m+2} & & & & \\
 & & j\gamma_{2m-4} & -k\gamma_{2m-4} & & & & \\
 & & -j\beta_{2m-3} & k\beta_{2m-3} & & & & \\
 & & & & \ddots & & & \\
 & & & & & \beta_{-2} & -i\beta_{-2} & \\
 & & & & & \gamma_{-3} & -i\gamma_{-3} & \\
 & & & & & j\gamma_1 & -k\gamma_1 & 2\beta_0 \\
 & & & & & -j\beta_2 & k\beta_2 & 2\gamma_{-1}
\end{pmatrix}$$

The gauge group may be used to transform $\tilde{A}_1$ to

$$\begin{pmatrix} 0 \\ -I_{2mk} \end{pmatrix},$$

the necessary transformations being a permutation (so that the nonzero entries are in 2×2 blocks on the diagonal), a matrix which multiplies each of the blocks by its inverse, and a rescaling. $\tilde{A}_3$ is then (using (3.4), (3.5))

$$\begin{pmatrix}
\frac{v}{\sqrt{2}} & -\frac{iv}{\sqrt{2}} & & & & & \\
\mathfrak{s}_{2m-1} & \mathfrak{d}_{2m-1} & \frac{j}{2}\gamma^t_{2m-2} & \frac{k}{2}\gamma^t_{2m-2} & & & \\
\mathfrak{d}^t_{2m-1} & \mathfrak{s}_{2m-1} & -\frac{k}{2}\gamma^t_{2m-2} & \frac{j}{2}\gamma^t_{2m-2} & & & \\
\frac{j}{2}\gamma_{2m-2} & -\frac{k}{2}\gamma_{2m-2} & \ddots & & & & \\
\frac{k}{2}\gamma_{2m-2} & \frac{j}{2}\gamma_{2m-2} & & & & & \\
 & & & & \mathfrak{s}_2 & \mathfrak{d}_2 & \frac{j}{\sqrt{2}}\gamma^t_1 \\
 & & & & \mathfrak{d}^t_2 & \mathfrak{s}_2 & -\frac{k}{\sqrt{2}}\gamma^t_1 \\
 & & & & \frac{j}{\sqrt{2}}\gamma_1 & -\frac{k}{\sqrt{2}}\gamma_1 & \beta_0
\end{pmatrix}$$

where

$$\begin{aligned}
\mathfrak{s}_j &= \frac{1}{2}(\beta_j + \beta^t_j) \\
\mathfrak{d}_j &= \frac{i}{2}(\beta_j - \beta^t_j))
\end{aligned}$$

This is in Atiyah's standard form $\begin{pmatrix} \Lambda \\ B \end{pmatrix}$ with B symmetric. Because the γ_j are injective, there is no gauge in which this is a "t'Hooft type" solution with Λ real and $B = \text{diag}\{b_i\}$ diagonal. (The b_i should also be distinct and the entries of Λ positive. These solutions were discussed in ([3], p26) and represent a superposition of instantons with centres b_i.) Atiyah's ADHM condition (3.9) on these matrices corresponds precisely to the Braam-Austin equations (3.4)-(3.8). The calculation is not hard, but it is tedious, and has therefore been omitted.

Thus Braam-Austin data may be viewed as matrix blocks in ADHM data for an instanton with large κ. Alternatively it may be thought of as a set of discrete matrix valued functions (which have interpretations as "discrete endomorphisms" and "discrete connections" on a "discrete vector bundle" of rank k). This second approach will be the more useful in finding explicit solutions and for considering the Euclidean limit.

3.6 Solutions of the difference equations

Finding explicit solutions to the Braam-Austin difference equations is not at all easy, though there are some existence and uniqueness results. Braam and Austin [12] proved that hyperbolic monopoles are "determined by their boundary data", and it may be seen that fixed boundary data determine a unique solution of the difference equations, modulo $\mathcal{G}_m$. To see what this boundary data is, restrict to the negative weight space $\mathbb{C}^2_- \subset \mathbb{C}^4$, where $z_2 = z_4 = 0$. The projectivisation $\mathbb{P}^1_-$ covers $S^2_\infty \subset S^4$,which is the boundary of $\mathbb{H}^3$, with $E|_{S^2_\infty} = L \oplus L^*$. The ADHM data has no cohomology here except at the "ends", the spaces with largest weight, so ([12])

$$\begin{aligned} L^*_{\mathbf{z}} &= V_{-2m}/(A(\mathbf{z})W_{-2m+1}) \\ L_{\mathbf{z}} &= \ker(A(\mathbf{z})^* : V_{2m} \to W_{2m-1}) \end{aligned}$$

and the "boundary value" is the map

$$\begin{pmatrix} \beta_{-2m+1} - z \\ v \end{pmatrix} : W_{-2m+1} \to V_{-2m}$$

where here $z = -z_1/z_3$ is an inhomogeneous coordinate on S^2_∞. The condition $A(\mathbf{z})$ injective for all $\mathbf{z}$ implies that the matrices γ_j must all be injective (see [12], Lemma 4.2). Using this, and fixing β_{-2m+1} and v, the Braam-Austin equations may be solved successively to find the other β_j and γ_j. Equation (3.5) gives β_{2m-1} and then (3.8) gives an expression for $\gamma^*_{2m-2}\gamma_{2m-2}$. To fix gauge, assume $\gamma^*_j = \gamma_j$. There is always such a choice, provided $\gamma^*_{2m-2}\gamma_{2m-2}$ is positive definite (a necessary condition for a solution). Then $\gamma_{2m-2}^*\gamma_{2m-2} = h^2$ for some Hermitian h, and $h^{-1}\gamma_{2m-2}^*\gamma_{2m-2}h^{-1} = I$, which implies that $\gamma_j h^{-1} = u$, unitary. So $\gamma_j = hu$ and taking $\gamma_j = h$ provides a canonical way to choose the γ_j. Once γ_{2m-2} is fixed (since it is known to be invertible), β_{2m-3} is determined uniquely by (3.6). Continuing this procedure the full solution may be found. Note that if the condition $\gamma_j^* = \gamma_j$ is imposed, the only remaining gauge freedom is conjugation by a constant matrix $g_j = g \in O(k,\mathbb{R})$.

So $\{\beta_{-2m+1}, v\}$ determines a unique solution, i.e. a unique monopole, provided it determines a solution at all. There is no guarantee, starting from arbitrary $\{\beta_{-2m+1}, v\}$, that all the γ_j will be injective, so there may be no solution. It is known that v must be a cyclic vector for β, in other words that the set $\{v, \beta v, \ldots, \beta^{k-1}v\}$ must span $\mathbb{C}^k$, but it is still an open problem to determine which $\{\beta_{-2m+1}, v\}$ are boundary data for a hyperbolic monopole. Donaldson [13] showed that Euclidean monopoles correspond to pairs $\{\beta, v\}$ with v cyclic for β, up to the action of the complexified gauge group $GL(k,\mathbb{C})$. The rational map associated to the monopole is then

$$f(z) = v^t(z-\beta)^{-1}v$$

The rational map is defined in the same way in the hyperbolic case [12]. So although hyperbolic monopoles correspond to pairs $\{\beta, v\}$ with β symmetric and v cyclic for β up to the action of the complexified gauge group, it is not easy to see how to get back from this to solutions of the Braam-Austin equations with group $U(k)$.

3.6.1 Example: the 1-monopole

The case where all solutions can be found easily is that of the 1-monopole, where $k = 1$. It is known via other methods that there is a unique such monopole (up to its position), and that it is spherically symmetric. The Braam-Austin data gives another proof of this, since in this case the matrices are all scalars, so the commutators in the difference equations vanish. From (3.6), the β_j are all equal to some fixed complex number β. Using the $\gamma_j^* = \gamma_j$ gauge, the γ_j may all be set equal to a fixed positive real number γ, and by changing the trivialisation at infinity, v may be chosen so that $v = \gamma$ (otherwise $|v| = \gamma$). Thus a 1-monopole is determined by $(\beta,\gamma) \in \mathbb{C} \times \mathbb{R}_+$ and it will be shown shortly that these are precisely the centre coordinates of the monopole in the upper half space model of $\mathbb{H}^3$.

3.6.2 Example: 2-monopoles

In this case the form of the boundary data is less obvious. What follows demonstrates a method of calculating it in the case where the monopole is axially symmetric.

A second circle action

Braam-Austin data is already invariant under the $\tilde{S^1}$ action

$$\lambda.\mathbf{z} = (\lambda^{-1}z_1, \lambda z_2, \lambda^{-1}z_3, \lambda z_4)$$

To find an axially symmetric monopole, impose invariance under another action:

$$\rho(\mu)\mathbf{z} = (\mu z_1, \mu^{-1}z_2, \mu^{-1}z_3, \mu z_4)$$

On $\mathbb{R}^4$ the original action corresponds to a rotation in the second factor (the (t,θ)-plane). This new action is rotation in the first factor (the (x,y)-plane). It commutes with both the original action and with the real structure σ, and so it descends to a circle action on $\mathbb{H}^3$. Thus a solution invariant under both these circle actions will describe a circle-invariant instanton on $\mathbb{H}^3$. To say that a solution $\{\beta_j, \gamma_j, v\}$ to the Braam-Austin equations is invariant under both actions means that it may be recovered from its pullback $\rho(\mu)^*(\{\beta_j, \gamma_j, v\})$ by a gauge transformation. The transformation will not be in $\mathcal{G}_m$, since $\rho(\mu)$ does not preserve A_1 and A_2; instead there will be a representation of $\tilde{S^1}$ defining a circle subgroup of the full gauge group $\mathrm{GL}(W_\mathbb{R})^{\tilde{S^1}} \times \mathrm{Sp}(V)^{\tilde{S^1}}$, and this will contain the required map. The pullback is

$$\rho(\mu)^* : \begin{cases} A_1 \mapsto \mu A_1 \\ A_2 \mapsto \mu^{-1}A_2 \\ A_3 \mapsto \mu^{-1}A_3 \\ A_4 \mapsto \mu A_4 \end{cases}$$

Because the gauge transformation (u, g) must restore A_1 and A_2, the block diagonal matrices $u = \mathrm{diag}\{u_i\}$ and $g = \mathrm{diag}\{g_i\}$ must satisfy

$$u_{-2m} = \begin{pmatrix} \mu^{-1}g_{-2m+1} & 0 \\ 0 & S \end{pmatrix} \text{ and } u_{-j} = \begin{pmatrix} \mu^{-1}g_{-j+1} & 0 \\ 0 & \mu g_{-j-1} \end{pmatrix}$$

for positive j, where all diagonal blocks are 2×2 except S, which is a complex number with modulus 1. There are similar formulae for u_j when $j > 0$. Looking at u_{-2m} and the first block of A_3, the pullback condition is

$$u_{-2m} \begin{pmatrix} \mu^{-1}\beta_{-2m+1} \\ \mu^{-1}v \end{pmatrix} g_{-2m+1}^{-1} = \begin{pmatrix} \beta_{-2m+1} \\ v \end{pmatrix}$$

so that

$$\mu^{-2} g_{-2m+1} \beta_{-2m+1} g_{-2m+1}^{-1} = \beta_{-2m+1} \quad (3.10)$$

$$S \mu^{-1} v g_{-2m+1}^{-1} = v \quad (3.11)$$

The matrices g_j are representations of the circle, so for some choice of coordinates

$$g_j = \begin{pmatrix} \mu^x & 0 \\ 0 & \mu^y \end{pmatrix}$$

Putting this into (3.10), either

$$\beta_{-2m+1} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \text{ and } x - y = 2$$

or

$$\beta_{-2m+1} = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \text{ and } y - x = 2$$

and these two possibilities for β_{-2m+1} are gauge equivalent (by gauge fixing d may also be taken to be real and positive). Suppose $\beta_{-2m+1} = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}$. Then by (3.11), v is a left eigenvector of g_{-2m+1} (with eigenvalue $S^{-1}\mu$), so $v = (\, c \quad 0\,)$ or $v = (\, 0 \quad c\,)$. The condition $\begin{pmatrix} \beta_{-2m+1} \\ v \end{pmatrix}$ injective implies $v = (\, 0 \quad c\,)$.

Summary

This shows that boundary data for a symmetric $k = 2$ monopole must (in some gauge) have the form

$$\beta_{-2m+1} = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \qquad v = (\, 0 \quad c\,)$$

In fact, the full solution may be found in this case, which fixes the dependence of c on d.

3.7 Solution for the symmetric charge 2 monopole

The general solution of Braam and Austin's equations for $k = 2$, based on Ward's solution [29] to the discrete Toda equations, will be given in Chapter 5. The method will be illustrated here for the simpler case of the axially symmetric monopole, in the hope of making the general case more transparent later. Instead of solving iteratively from the boundary values, the whole system will be considered at once. Because of this, it is convenient to use a more index-free notation. Knowing the form of the boundary data is useful for guessing a good gauge in which to solve the equations. Once the boundary condition is imposed, the solution prescribes the relationship between the constants c and d above.

3.7.1 Ward's notation

The matrices β and γ are now to be thought of as functions of j, rather than as blocks inside a larger matrix. The subscripts "+" and "−" are used to denote a shift of 2 in j. Thus

$$\begin{aligned} \gamma_j &= \gamma = \gamma(j) \\ \gamma_+ &= \gamma(j+2) \\ \gamma_- &= \gamma(j-2) \end{aligned}$$

where j is an odd number. Because β_j is defined for even j there is a slight change in the labelling:

$$\beta_j = \beta(j+1)$$

with $\beta_+ = \beta(j+3)$ etc. defined in the obvious way. The Braam-Austin equations (3.6) and (3.7) are now

$$\beta\gamma - \gamma\beta_+ = 0 \tag{3.12}$$

$$\gamma_-^*\gamma_- - \gamma\gamma^* + [\beta^*, \beta] = 0 \tag{3.13}$$

3.7.2 Gauge choice

Since $\mathrm{tr}(\gamma^*\gamma)$ is independent of j, looking at the boundary conditions it seems reasonable to guess that there is a gauge in which β and γ have the form

$$\beta = bw\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \gamma = b\begin{pmatrix} \sqrt{1-g} & 0 \\ 0 & \sqrt{1+g} \end{pmatrix} \qquad v = \begin{pmatrix} 0 & a \end{pmatrix}$$

where w and g are functions of j and the solution is trivialised so that a and b are real, positive constants. This is a special case of Ward's choice of matrices for the "discrete Toda" equations. Substituting these into (3.12) and (3.13), the difference equations in terms of w and g are:

$$w\sqrt{1+g} = w_+\sqrt{1-g} \tag{3.14}$$

$$g = w^2 + g_- \tag{3.15}$$

3.7.3 A conserved quantity

Conserved quantities such as $\mathrm{tr}\beta$ and $\mathrm{tr}(\gamma^*\gamma)$ are relatively easy to spot from the Braam-Austin equations, but there is another, less obvious one:

Lemma 3.4. *Let*

$$\Omega = w^2(1+g) - g^2$$

Then Ω is independent of j.

Proof:

$$\begin{aligned}
\Omega_- &= w_-^2(1+g_-) - g_-^2 \\
&= \frac{1-g_-}{1+g_-}w^2(1+g_-) - (g-w^2)^2 \\
&= w^2(1-g+w^2) - g^2 + 2gw^2 - w^4 \\
&= w^2(1+g) - g^2 \\
&= \Omega
\end{aligned}$$

Thus Ω does not depend on j. □

3.7.4 Solution for g

Writing w in terms of Ω,

$$w^2 = \frac{\Omega + g^2}{1+g} \tag{3.16}$$

the equation (3.15) becomes a difference equation for g only:

$$g_+ = \frac{\Omega + g}{1-g} \tag{3.17}$$

The way to solve this is to guess a solution and then show that it satisfies the equation. However, by analogy with the Euclidean case [27], there should be a solution in terms of trigonometric functions and the difference equation should be essentially an addition formula. So guess that

$$g = A\tan(Bj) \tag{3.18}$$

then

$$g_+ = A\tan(B(j+2)) = \frac{A(\tan(Bj)+\tan(2B))}{1-\tan(Bj)\tan(2B)}$$

Clearly if $A = \tan(2B)$ and $\Omega = A^2$, this g satisfies (3.17).

3.7.5 Solution for w

Putting g back into (3.16) and taking the square root shows that

$$w = \frac{A\sec(Bj)}{\sqrt{1+A\tan(Bj)}} \tag{3.19}$$

3.7.6 Change of gauge

The above g and w certainly lead to solutions of (3.6) and (3.7) but these do not obviously satisfy the other Braam-Austin equations. Assume (3.18) and (3.19) give the solution for $j > 1$, then changing to a gauge in which β is symmetric makes it clearer how the solution extends to negative j. The required element of the gauge group is

$$g_j = g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and (using (3.1), (3.2) and (3.3)) the new matrices are

$$\beta = \frac{bw}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \qquad v = \frac{a}{\sqrt{2}} \begin{pmatrix} i & 1 \end{pmatrix},$$

$$\gamma = \frac{b}{2} \begin{pmatrix} (\sqrt{1+g} + \sqrt{1-g}) & i(\sqrt{1+g} - \sqrt{1-g}) \\ i(\sqrt{1-g} - \sqrt{1+g}) & (\sqrt{1+g} + \sqrt{1-g}) \end{pmatrix}$$

Because g is odd, the condition (3.4) is satisfied when the solution is extended to negative j. Condition (3.5), that $\beta_{-j} = \beta_j^t$ is also satisfied: First note that

$$\cos(2B)\sqrt{1+A^2} = \sqrt{\cos^2(2B) + \sin^2(2B)} = 1$$

Now

$$\beta_j = \frac{bw(j+1)}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$
$$\beta_{-j} = \frac{bw(-j+1)}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$

but

$$\begin{aligned} w(-j+1) &= \frac{A\sec(-B(j+1)+2B)}{\sqrt{1+A\tan(-B(j+1)+2B)}} \\ &= \frac{A(\cos(B(j+1))\cos(2B) + \sin(B(j+1))\sin(2B))^{-1}}{\left(1 + A\frac{\tan(2B)-\tan(B(j+1))}{1+A\tan(B(j+1))}\right)^{\frac{1}{2}}} \\ &= \frac{A\sqrt{1+A\tan(B(j+1))}}{\cos(B(j+1))\cos(2B)(1+A\tan(B(j+1)))\sqrt{1+A^2}} \\ &= \frac{A\sec(B(j+1))}{\sqrt{1+A\tan(B(j+1))}} \\ &= w(j+1) \end{aligned}$$

so β satisfies (3.5).

3.7.7 Boundary conditions

Substituting the solution into the boundary condition (3.8) yields

$$b^2w^2(2m)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a^2}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} - b^2\begin{pmatrix} 1 & ig(2m-2) \\ -ig(2m-2) & 1 \end{pmatrix} = 0$$

thus $a^2 = 2b^2$ and $w^2(2m) - 1 + g(2m-2) = 0$. Substituting for w and g and simplifying, the second equation is

$$A\tan(2Bm) = 1$$

i.e.

$$\begin{aligned}\tan(2B)\tan(2Bm) &= 1 \\ &\Rightarrow \cos((2m+2)B) = 0 \\ &\Rightarrow B = \frac{\pi}{2(2m+2)} = \frac{\pi}{4(m+1)}\end{aligned}$$

All this leads to the result:

Proposition 3.5. *The solution*

$$\beta = \frac{bw}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$

$$\gamma = \frac{b}{2}\begin{pmatrix} \sqrt{1+g}+\sqrt{1-g} & i(\sqrt{1+g}-\sqrt{1-g}) \\ i(\sqrt{1-g}-\sqrt{1+g}) & \sqrt{1+g}+\sqrt{1-g} \end{pmatrix}$$

$$v = b\,(\,i \quad 1\,)$$

where

$$\begin{aligned} w(j) &= \frac{A\sec(Bj)}{\sqrt{1+A\tan(Bj)}} \\ g(j) &= A\tan(Bj) \\ A &= \tan(2B) \\ B &= \frac{\pi}{4(m+1)} \end{aligned}$$

(for fixed b) is the Braam-Austin data for an axially symmetric charge 2 hyperbolic monopole (with fixed centre and orientation).

The proposition follows because it is known that there is a unique such monopole, up to the position of the centre and the axis of symmetry, and that sets of Braam-Austin data correspond precisely to monopoles and are completely determined by their boundary data $\{\beta_{-2m+1}, v\}$. The scaling b corresponds to a dilation of $\mathbb{H}^3$. □

Solving the difference equations in this way is much better than solving iteratively from the boundary data because it gives formulae for β and γ in terms of discretised analytic functions. The fact that these functions exist in general will become important in Chapter 4. This method does not rely on knowing the boundary conditions, as will become clear in Chapter 5.

3.8 Centre of a monopole

It is known ([12], p821) that hyperbolic monopoles have a well-defined centre (at least for $m \in \frac{1}{2}\mathbb{Z}$), but it is not easy to find the centre of a given monopole explicitly from its Braam-Austin description. The answer is known in the case $k = 1$:

Proposition 3.6. *The centre of the 1-monopole (β, γ), where the solution is in the gauge with $v = \gamma \in \mathbb{R}_{>0}$, is the point*

$$(\beta, \gamma) \in \mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}_{>0}$$

This is proved in §3.8.2. Recall that for a 1-monopole, β and γ are independent of j. These centre coordinates may be written gauge invariantly as $(\beta, \sqrt{\gamma^*\gamma})$.

3.8.1 Definition of centre

There is more than one way to calculate the centre of a monopole. One method is to use the boundary data. The restriction of the bundle E to the sphere at infinity splits as $E|_{\partial\mathbb{H}^3} = L \oplus L^*$. The connection on E induces $U(1)$ connections on L and L^* whose curvatures are nowhere vanishing (see [12]). Let A_∞ be the connection on L^* and let F_∞ be its curvature. Then the 2-form F_∞ defines a measure on $\partial\mathbb{H}^3 = S^2_\infty$. Any measure on a sphere has an associated "conformal centre of mass", which is a point in $\mathbb{H}^3$ (using the ball model) defined by the measure and behaving naturally under isometries of $\mathbb{H}^3$ (any isometry extends conformally to $\partial\mathbb{H}^3$). The centre of the monopole is defined to be this conformal centre. The definition of the conformal centre of mass may be found in [8], §5, where its equivariance under isometries is discussed. The proofs of existence and uniqueness are given in Appendix A of that paper. If the monopole has any symmetry (this includes all monopoles of charge 1 or 2), an alternative definition of the centre is to take the centre of symmetry, which may be found by considering symmetries of the spectral curve (see Chapter 5). The centre obtained this way agrees with the conformal centre because of the naturality of the conformal centre under symmetries of $\mathbb{H}^3$. Another consequence of naturality is the fact that, identifying $\mathbb{H}^3$ with the unit ball in $\mathbb{R}^3$, the conformal centre of mass is at the origin if and only if the Euclidean centre of mass is at the origin. The Euclidean centre of mass is given by the integrals of the coordinate functions restricted to S^2 against the measure.

3.8.2 Proof of Proposition 3.6

Braam and Austin have a way of calculating F_∞ as the pullback of the Kähler form on $\mathbb{C}P^k$ under a map $f : S^2_\infty \to \mathbb{C}P^k$. The map f is induced by the "boundary monad map"

$$\begin{pmatrix} \beta_{-2m+1} - u \\ v \end{pmatrix} : \mathbb{C}^k_{-2m+1} \to \mathbb{C}^{k+1}_{-2m}$$

with u a coordinate on S^2_∞. Then $f(u) = [\mathbf{n}(u)]$, where $\mathbf{n}(u)$ is the normal in $\mathbb{C}^{k+1}_{-2m}$ to the image $\begin{pmatrix} \beta_{-2m+1} - u \\ v \end{pmatrix} \mathbb{C}^k_{-2m+1}$. This image is always k-dimensional by injectivity.

When $k = 1$ the boundary map is

$$\begin{pmatrix} \beta - u \\ \gamma \end{pmatrix} : S^2_\infty \to \mathbb{C}P^1$$

so $f(u)$ is the point $\begin{bmatrix} \beta - u \\ \gamma \end{bmatrix}^\perp \in \mathbb{C}P^1$ and a representative is

$$f(u) = \begin{bmatrix} \gamma \\ -(\beta - u) \end{bmatrix}$$

In inhomogeneous coordinates $f(u) = -\frac{(\beta - u)}{\gamma}$ and the pullback of the Kähler form is

$$F_\infty = \frac{df(u) \wedge d\overline{f(u)}}{(1 + |f(u)|^2)^2} = \frac{|\gamma|^2 du \wedge d\bar{u}}{(|\gamma|^2 + |\beta - u|^2)^2}$$

The idea is to pull back the form F_∞ to the ball model $\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$ of $\mathbb{H}^3$ by a conformal map which sends $0 \in \mathcal{B}$ to $(\beta, \gamma) \in \mathcal{U}$. Then (β, γ) is the centre of mass of F_∞ iff the integrals of the coordinate functions of $\mathcal{B}$ against the pulled-back F_∞ vanish. It is enough to do this for $(\beta, \gamma) = (0, 1)$, since any point may be moved to $(0, 1)$ by an isometry of $\mathcal{U}$, and it is easier in practice to pull the coordinate functions back from $\mathcal{B}$ to $\mathcal{U}$ and to do the integration there.

The correspondence between $\mathcal{U}$ with coordinates (u, t) and $\mathcal{B}$ with coordinates (x, y, z), which restricts to stereographic projection on the boundary ($t = 0$ or $x^2 + y^2 + z^2 = 1$) and sends $0 \in \mathcal{B}$ to $(0, 1) \in \mathcal{U}$, is given by

$$x + iy = \frac{2u}{|u|^2 + (1 + t)^2} \qquad u = \frac{2(x + iy)}{x^2 + y^2 + (z - 1)^2}$$

$$z = \frac{|u|^2 + t^2 - 1}{|u|^2 + (1 + t)^2} \qquad t = \frac{1 - x^2 - y^2 - z^2}{x^2 + y^2 + (z - 1)^2}$$

So the integrals of the pulled-back coordinate functions are

$$\int_{u \in \mathbb{C}} \frac{2u}{(1 + |u|^2)^3} du d\bar{u} = 0 \text{ by symmetry}$$

and

$$\begin{aligned}
\int_{u \in \mathbb{C}} \frac{(|u|^2 - 1)}{(1 + |u|^2)^3} du d\bar{u} &= \int_0^{2\pi} \int_0^\infty \frac{2ir(r^2 - 1)}{(1 + r^2)^3} dr d\theta \quad \text{where } u = re^{i\theta} \\
&= 2\pi i \left\{ \left[\frac{1 - r^2}{2(1 + r^2)^2} \right]_0^\infty + \int_0^\infty \frac{2r}{(1 + r^2)^2} dr \right\} \\
&= 2\pi i \left\{ -\frac{1}{2} + \left[\frac{-1}{2(1 + r^2)} \right]_0^\infty \right\} = 2\pi i \left\{ -\frac{1}{2} + \frac{1}{2} \right\} = 0
\end{aligned}$$

completing the proof of the proposition, provided the coordinates behave in the expected way under isometries of $\mathbb{H}^3$. This is shown in the next section. □

3.8.3 Isometries of $\mathbb{H}^3$

It remains to check the effect of isometries on Braam-Austin matrices. The Braam-Austin formulation fixes a point at infinity, so the only relevant isometries are those that fix this point. They are generated by

$$\begin{array}{lll} (u,t) \overset{\phi_1}{\longmapsto} & (u+\alpha, t) & \alpha \in \mathbb{C} \\ (u,t) \overset{\phi_2}{\longmapsto} & (\lambda u, \lambda t) & \lambda \in \mathbb{R}_{>0} \end{array}$$

Including $\mathbb{H}^3$ in S^4 and using the fact (equation 1.11) that the fibre in $\mathbb{C}P^3$ over $a + jb \in S^4 = \mathbf{H}P^1$ is the real line

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$$

the isometries ϕ_1 and ϕ_2 lift to $\mathbb{C}P^3$. For ϕ_1,

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a+\alpha & -\bar{b} \\ b & \bar{a}+\bar{\alpha} \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \alpha z_3 \\ \bar{\alpha} z_4 \end{pmatrix}$$

So

$$\tilde{\phi_1}[\mathbf{z}] = [z_1 + \alpha z_3, z_2 + \bar{\alpha} z_4, z_3, z_4]$$

and similarly

$$\begin{aligned} \tilde{\phi_2}[\mathbf{z}] &= [\lambda^{-\frac{1}{2}} z_1, \lambda^{-\frac{1}{2}} z_2, \lambda^{\frac{1}{2}} z_3, \lambda^{\frac{1}{2}} z_4] \\ &= [z_1, z_2, \lambda z_3, \lambda z_4] \end{aligned}$$

Pulling back the matrices A_i by these maps,

$$\tilde{\phi_1}^* \left(\sum A_i z_i \right) = A_1 z_1 + A_2 z_2 + (A_3 + \alpha A_1) z_3 + (A_4 + \bar{\alpha} A_2) z_4$$

which means that

$$\begin{aligned} \beta_j &\mapsto \beta_j + \alpha I \\ \gamma_j &\mapsto \gamma_j \end{aligned}$$

and for ϕ_2,

$$\tilde{\phi_2}^* \left(\sum A_i z_i \right) = A_1 z_1 + A_2 z_2 + \lambda A_3 z_3 + \lambda A_4 z_4$$

so in this case $\beta_j \mapsto \lambda \beta_j$ and $\gamma_j \mapsto \lambda \gamma_j$. Thus when $k = 1$ the coordinates (β, γ) do indeed behave correctly under isometries of $\mathbb{H}^3$.

3.8.4 Higher charge

In general, the centre coordinates must be given by some expressions in β and γ which do not depend on j (of course these expressions will involve traces so that the result is a scalar). They must also be independent of gauge and transform in the right way under isometries of $\mathbb{H}^3$. Conversely, any invariant formula for the centre which is correct for centred, oriented monopoles and transforms as it should will be valid for all monopoles of that charge. This fact was used in the proof of Proposition 3.6.

Using coordinates $(u,t) \in \mathbb{H}^3 \cong \mathbb{C} \oplus \mathbb{R}_{>0}$, the obvious choice for the complex coordinate is to take $u = \frac{1}{k}\mathrm{tr}\beta$. This will be correct provided a general monopole (with a fixed orientation), centred at $(0,t)$ for some t, is defined by Braam-Austin data with $\mathrm{tr}\beta = 0$. The other coordinate is more complicated, as the following result illustrates.

Proposition 3.7. *The centre coordinates of the 2-monopole specified by Braam-Austin data $\{\beta,\gamma,v\}$, given in the upper half space model, are*

$$(u,t) = (\frac{1}{2}tr\beta, \frac{1}{\sqrt{2}}\chi^{\frac{1}{4}})$$

where

$$\begin{aligned}\chi &= (tr\beta^2)^2 - 2tr(\gamma^*\gamma(\gamma^*\gamma - 2\beta^*\beta)) + 2(tr\gamma^*\gamma)^2 + \frac{1}{4}(tr\beta)^4 - (tr\beta)^2 tr\beta^2 \\ &\quad -2tr\beta tr\beta^*\gamma^*\gamma - 2tr\beta^* tr\beta\gamma^*\gamma + tr\beta tr\beta^* tr\gamma^*\gamma\end{aligned}$$

A proof will be given in Chapter 5 (see §5.6.2). It may be checked that in the case of the centred symmetric 2-monopole solution given in Proposition 3.5, the formula for the centre simplifies to

$$(u,t) = (0, \frac{b}{\sqrt{2}}\{\sec(2B)\}^{\frac{1}{2}})$$

where as before, $B = \frac{\pi}{4(m+1)}$. The check is relatively easy because $\mathrm{tr}\beta$ and $\mathrm{tr}\beta^2$ both vanish so there are only two terms of χ to calculate.

This formula for the centre is only valid for $k = 2$. It was obtained by using explicit solutions and knowing the invariants of the Braam-Austin equations for $k = 2$. Finding a formula for t when $k > 2$ seems very hard at present.

Chapter 4

Euclidean limit

The aim of this chapter is to understand how the Braam-Austin description of monopoles becomes the Euclidean (Nahm) description for large mass. Euclidean space $\mathbb{R}^3$ may be thought of as a limit of hyperbolic spaces with curvatures tending to zero. It is conjectured that Braam-Austin data should be discretisations of analytic functions and, on the strength of this, conditions for a sequence of hyperbolic monopoles of fixed charge and increasing mass to have a subsequence converging to a Euclidean monopole are given.

4.1 Changing the curvature of $\mathbb{H}^3$

Recall that the correspondence between hyperbolic monopoles and instantons was based on the conformal equivalence $\mathbb{R}^4 \setminus \mathbb{R}^2 \sim \mathbb{H}^3 \times S^1$, which was illustrated by writing the $\mathbb{R}^4$ metric as

$$ds^2 = t^2 \left\{ \frac{dx^2 + dy^2 + dt^2}{t^2} + d\theta^2 \right\}$$

There is a similar formula for any $R \in \mathbb{R}_{>0}$, given by

$$ds^2 = \frac{t^2}{R^2} \left\{ R^2 \frac{dx^2 + dy^2 + dt^2}{t^2} + R^2 d\theta^2 \right\}$$

and so a conformal equivalence $\mathbb{R}^4 \setminus \mathbb{R}^2 \sim \mathbb{H}^3(R) \times S^1(R)$ where $S^1(R)$ is the circle of radius R and $\mathbb{H}^3(R)$ is hyperbolic space with constant curvature $-1/R^2$. The charge of a monopole is not changed by this rescaling of $\mathbb{H}^3$, but because

$$\mathbb{A} = A + \Phi d\theta = A + \frac{\Phi}{R} R d\theta \tag{4.1}$$

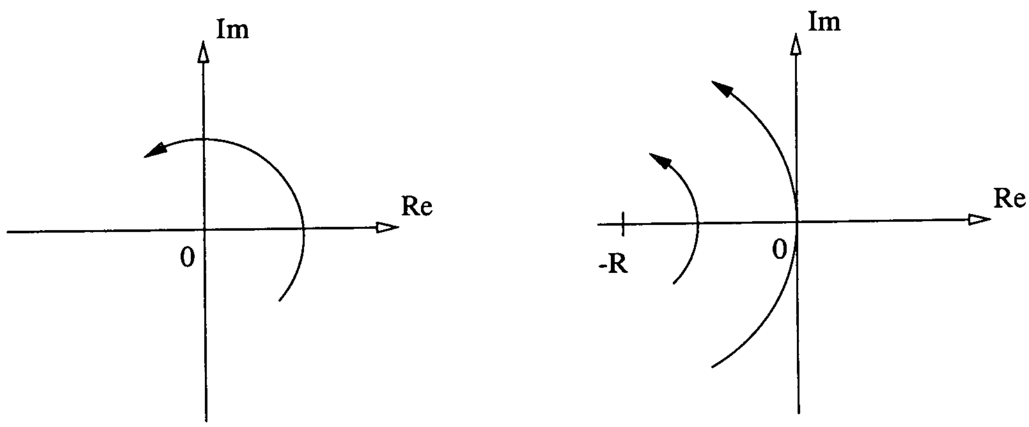
it is clear that a monopole of mass m on $\mathbb{H}^3$ corresponds to one with mass m/R on $\mathbb{H}^3(R)$. Euclidean monopoles may always be scaled to have mass 1. As R increases, the curvature of $\mathbb{H}^3(R)$ tends to zero, so it is reasonable to hope that a sequence of monopoles with increasing mass has some sensible limit as a Euclidean monopole of finite mass, using a rescaling of this sort.

4.2 Moving the boundary of $\mathbb{H}^3$

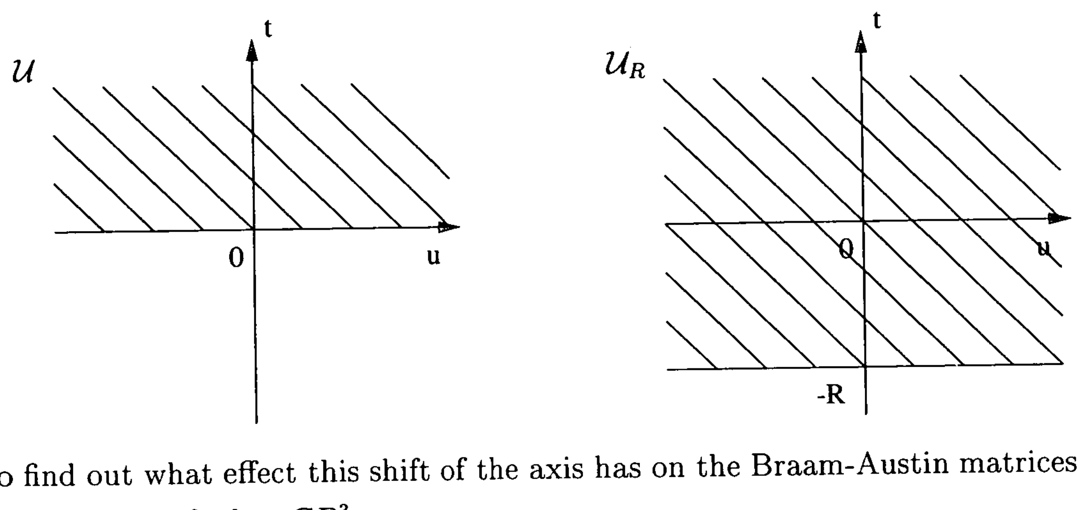
Atiyah's picture ([4], §5) of the limiting process is to consider shifting the axis of rotation along the real axis in the (t, θ)-plane. Recall that if $\mathbb{R}^4$ is identified with the quaternions $\mathbf{H}$, with coordinate $q = a + jb = x + iy + jte^{i\theta}$, then the circle action is

$$\lambda : a + jb \mapsto a + j\lambda b$$

As the axis of rotation moves away from the origin in the (t, θ)-plane the circle action near the origin becomes closer to the translation action (along the imaginary axis). This is illustrated in the following diagrams of the (t, θ)-plane:



In the upper half space picture, this corresponds to changing from $\mathcal{U} = \{t > 0\}$ to $\mathcal{U}_R = \{t > -R\}$ and rescaling the metric, so the boundary of hyperbolic space drops away from the origin; and on any compact set containing the origin the metric tends to the Euclidean metric as $R \to \infty$.



To find out what effect this shift of the axis has on the Braam-Austin matrices it must first be lifted to $\mathbb{C}P^3$.

4.2.1 Lift to $\mathbb{C}P^3$

Translation along the real axis (in the (t,θ)-plane) by R corresponds in $\mathbb{R}^4$ to $a+jb \mapsto a+j(b+R)$. Since

$$\begin{pmatrix} a & -\bar{b}-R \\ b+R & \bar{a} \end{pmatrix}\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} -Rz_4 \\ Rz_3 \end{pmatrix}$$

(and using the equation (1.11) for the fibre over a point of S^4) a lift of this translation to twistor space is given by

$$\mathbf{z} \mapsto [z_1 - Rz_4, z_2 + Rz_3, z_3, z_4]$$

Conjugating the circle action by this lift, the new rotation action with axis at $-R$ is given by

$$\rho_R(\lambda) : \mathbf{z} \mapsto [\lambda^{-\frac{1}{2}}z_1 + R(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})z_4, \lambda^{\frac{1}{2}}z_2 + R(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})z_3, \lambda^{-\frac{1}{2}}z_3, \lambda^{\frac{1}{2}}z_4]$$

Given matrices solving the Braam-Austin equations, their pullbacks under translation by $-R$ will be invariant under the new action. In other words if $\{A_i\}$ solves the Braam-Austin equations and is invariant under the original circle action, then

$$A_1z_1 + A_2z_2 + (A_3 - RA_2)z_3 + (A_4 + RA_1)z_4$$

is invariant under ρ_R. This formula may be used to convert between ρ_R-invariant data and data invariant under the original circle action.

4.2.2 Limit of this action

To confirm that the action has the expected limit for large R, put $\lambda = e^{iu/R}$, where u will remain finite. Then $\lambda^{\frac{1}{2}} \to 1$ as $R \to \infty$ and

$$\begin{aligned} R(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}) &= 2iR\sin\left(\frac{u}{2R}\right) \\ &\to iu \text{ as } R \to \infty \end{aligned}$$

So in the limit, the action $\rho_R(\lambda)$ becomes

$$\mathbf{z} \mapsto [z_1 + iuz_4, z_2 + iuz_3, z_3, z_4]$$

When this is restricted to $\mathbb{R}^4 = \{[q,1]\}$ it is just translation by iu in the (t,θ)-plane, which is the limit of the actions previously described in pictures.

4.3 Nahm's equations

From the method of construction, it would seem reasonable to expect Braam and Austin's difference equations to be related to the Nahm equations for Euclidean monopoles. This section contains a summary of Nahm's results, and Donaldson's reformulation of them.

Euclidean monopoles correspond to ASD connections on $\mathbb{R}^4$, invariant under a translation action (translation in the x_4 direction, say). This is less pleasant than a circle invariant connection since the "instanton" no longer has finite action. In the "ADHMN" description of monopoles (see [17], §2, for details), Nahm replaced the vector spaces V and W by infinite dimensional spaces and the map $A(\mathbf{z})$ by a differential operator $\Delta(\mathbf{z})$. To be more precise, let $\mathbb{C}^2$ represent the quaternions and $H^0 := \mathcal{L}^2[-1,1]$ with real structure $\sigma(f)(s) = \bar{f}(-s)$; then $V := H^0 \otimes \mathbb{C}^k \otimes \mathbb{C}^2$ and

$$W := \{f \in H^1 \otimes \mathbb{C}^k : f(-1) = f(1) = 0\}$$

where H^1 is the Sobolev space of differentiable functions on $[-1,1]$ whose derivatives are in $\mathcal{L}^2$. Note that W has a real structure and V a quaternionic structure arising from similar structures on the individual fibres. Using homogeneous coordinates and letting e_1, e_2 and e_3 be right multiplication by the quaternions i, j and k respectively, the map $\Delta(\mathbf{z})$ may be taken to have the form

$$\Delta(\mathbf{z})f = (z_0 + \sum_{i=1}^{3} z_i e_i)f + i\frac{df}{ds} + i\sum_{i=1}^{3} T_i(s)e_i f$$

where $T_j(s)$ is a $k \times k$ matrix depending analytically on $s \in (-1,1)$, with simple poles at the endpoints (the $\mathbb{C}^4$ coordinate is now (z_0, z_1, z_2, z_3) as this is more convenient here).

In this setup, the conditions for Δ to define a monopole become

$$\begin{aligned} T_j + T_j^* &= 0 \\ T_i(s) &= -\bar{T}_i(-s) \end{aligned}$$

and **Nahm's equations:**

$$\begin{aligned} \frac{dT_1}{ds} &= [T_2, T_3] \\ \frac{dT_2}{ds} &= [T_3, T_1] \\ \frac{dT_3}{ds} &= [T_1, T_2] \end{aligned}$$

Then $E_x := \ker\Delta^*(x)$ is a 1-dimensional quaternionic vector bundle on $\mathbb{R}^4$, with a connection defined by orthogonal projection of the trivial connection on V. It was proved by Nahm [25] and Hitchin [17] that such bundles correspond to Euclidean monopoles.

In fact Hitchin (as well as Nahm) uses the interval $(0,2)$ instead of $(-1,1)$, but this latter interval will prove more convenient later.

4.3.1 Donaldson's description

Donaldson, in [13], has a way of rewriting Nahm's equations, by noticing that they are equivalent to the ASD equations for a connection $A = \sum_{i=1}^{3} T_i(s)dp_i$ on the $\mathbb{R}^4$ with coordinates (s, p_1, p_2, p_3). From this point of view, it is natural to introduce a fourth matrix $T_0(s)$ and extend Nahm's equations to the full ASD equations (in the case where the coefficients depend on only one variable):

$$\begin{aligned} \frac{dT_1}{ds} + [T_0, T_1] - [T_2, T_3] &= 0 \\ \frac{dT_2}{ds} + [T_0, T_2] - [T_3, T_1] &= 0 \\ \frac{dT_3}{ds} + [T_0, T_3] - [T_1, T_2] &= 0 \end{aligned}$$

considered up to the action of the gauge group

$$\mathcal{G} = \{u : (-1,1) \to U(k) : u(-s) = u^t(s)^{-1}\}$$

$\mathcal{G}$ acts via

$$\begin{cases} u(T_i) = uT_iu^{-1} \text{ for } i = 1,2,3 \\ u(T_0) = uT_0u^{-1} - \left(\frac{du}{ds}\right)u^{-1} \end{cases}$$

so there is always a gauge where $T_0 = 0$. Donaldson's point of view is that of complex coordinates, since there is already a splitting $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$. So he defines the coordinates $s + ip_1$ and $p_2 + ip_3$, and the matrices

$$\alpha = \frac{1}{2}(T_0 + iT_1) \qquad \beta = \frac{1}{2}(T_2 + iT_3)$$

The original matrices T_i may be recovered from α and β by taking the adjoint and skew adjoint parts (because $T_i \in \mathfrak{su}(2)$, skew adjoint). Nahm's equations are now

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0 \tag{4.2}$$

$$\frac{d}{ds}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \tag{4.3}$$

4.3.2 Poles and residues

Nahm's matrices $T_i(s)$ have simple poles at the endpoints $s = \pm 1$. Hitchin, in ([17], p150), argues that near $s = 1$, the matrix $T_i(s)$ may be written

$$T_i(s) = \frac{a_i}{1-s} + b_i(s)$$

where a_i is constant and $b_i(s)$ is analytic in a neighbourhood of $s = 1$. Then

$$\frac{dT_i}{ds} = -\frac{a_i}{(1-s)^2} + \frac{db_i}{ds}$$

Putting this into Nahm's equations and taking the coefficient of $1/(1-s)^2$,

$$\begin{aligned} -a_1 &= [a_2, a_3] \\ &\quad \text{plus cyclic permutations} \end{aligned}$$

so that the residues a_i form a k-dimensional representation of $\mathfrak{su}(2)$. The fact that this representation must be the unique irreducible one of dimension k was proved in Proposition 5.24 of [17]. By the property $T_i(-s) = -\bar{T}_i(s)$, the representation of the residues at the other pole will then also be irreducible.

4.4 Shift of axis

It is pointed out in ([12], p817) that if $\gamma_j = \frac{m}{2}I_k + \alpha_j$ then the continuum limit of the Braam-Austin equations is exactly the Nahm equations in Donaldson's form. For example, using this substitution in (3.6) and rearranging produces the equation

$$\frac{1}{2}\frac{\beta\left(\frac{j-1}{2m}+\frac{1}{m}\right)-\beta\left(\frac{j-1}{2m}\right)}{1/m} = \beta\left(\frac{j-1}{2m}\right)\alpha\left(\frac{j}{2m}\right) - \alpha\left(\frac{j}{2m}\right)\beta\left(\frac{j+1}{2m}\right) \qquad (4.4)$$

Because Nahm's equations depend on a variable $s \in (-1,1)$, it is sometimes convenient to rescale j by setting $j/2m = s$. Using this, it is possible to take continuum limits, by letting j and m tend to infinity in such a way that s remains fixed. Doing this to (4.4), the left hand side becomes a derivative and (4.4) becomes equation (4.2) above. Similarly, substituting into (3.7), rearranging and taking the limit gives (4.3). These are Donaldson's form of Nahm's equations. In future, the relation $s = j/2m$ will often be used to switch between the consideration of β and γ as functions on a lattice and as discretised functions on $(-1,1)$.

To obtain a Euclidean monopole from a sequence $\{A^{(m)}\}$, where $A^{(m)} = \{A_1^{(m)}, \dots A_4^{(m)}\}$ is Braam-Austin data representing a hyperbolic monopole of mass m, the curvature of hyperbolic space is changed by moving the axis of rotation to $-R_m$ in the (t,θ)-plane, i.e. to the point $(t,\theta) = (R_m, -\frac{\pi}{2})$ (§4.2). It turns out from the above that the correct axis shift is $R_m = \frac{m}{2}$.

Justification: The following is a formal argument, in that it is assumed there are no difficulties with existence of derivatives or the continuum limit, but it is nevertheless useful and provides the motivation for the rest of the chapter. If $A^{(m)}$ is pulled back to ρ_{R_m}-invariant data $\tilde{A}^{(m)}$, then

$$\tilde{A}^{(m)}\mathbf{z} = A_1^{(m)}z_1 + A_2^{(m)}z_2 + (A_3^{(m)} - R_m A_2^{(m)})z_3 + (A_4^{(m)} + R_m A_1^{(m)})z_4$$

by §4.2.1, where the matrices $A_i^{(m)}$ of $A^{(m)}$ are in Braam and Austin's standard form, $A_3^{(m)}$ having blocks $\beta_j^{(m)}$, $\gamma_j^{(m)}$ and $v^{(m)}$. Recall $A^{(m)}\mathbf{z} : W^{(m)} \to V^{(m)}$, where $W^{(m)}$ and $V^{(m)}$ have the weight space decompositions given in Lemma 3.2. A vector $f^{(m)} \in W^{(m)}$ may be thought of as a discrete function $\{f_j^{(m)}\}$ with $f_j^{(m)}$ a k-vector in the weight space W_j of W. Consider the coefficient of z_3 in $A^{(m)}\mathbf{z}f^{(m)}$:

$$A_3^{(m)} - R_m A_2^{(m)} : f^{(m)} \mapsto \begin{pmatrix} \beta_{-2m+1}^{(m)} f_{-2m+1}^{(m)} \\ v^{(m)} f_{-2m+1}^{(m)} \\ \beta_{-2m+3}^{(m)} f_{-2m+3}^{(m)} \\ -R_m f_{-2m+1}^{(m)} + \gamma_{-2m+2}^{(m)} f_{-2m+3}^{(m)} \\ \vdots \\ \beta_0^{(m)} f_0^{(m)} \\ R_m f_0^{(m)} - \gamma_1^{(m)} f_2^{(m)} \\ \vdots \\ \beta_{2m-1}^{(m)} f_{2m-1}^{(m)} \\ R_m f_{2m-1}^{(m)} \\ 0 \end{pmatrix} \in V \qquad (4.5)$$

If there is to be a nonsingular limit as $m \to \infty$, the entries of this column vector, and in particular the functions

$$-R_m f_{j-1}^{(m)} + \gamma_j^{(m)} f_{j+1}^{(m)} \qquad (4.6)$$

must have finite nonzero limits (for a general $f^{(m)}$) as $m \to \infty$. To keep them from becoming arbitrarily large, $\gamma_j^{(m)}$ must have the form

$$\gamma_j^{(m)} = R_m I_k + \alpha_j^{(m)}$$

where $\alpha_j^{(m)}$ has a limit as a finite matrix-valued function α. Thus it is natural to use the pair $(\alpha_j^{(m)}, \beta_j^{(m)})$ to represent Braam-Austin data, especially when considering the Euclidean limit. Comparing with the substitution $\gamma_j = \frac{m}{2} I_k + \alpha_j$ gives $R_m = \frac{m}{2}$. The expression (4.6) becomes

$$\alpha(s) f(s) + \frac{1}{2}\frac{df}{ds}(s)$$

in the continuum limit.

Note that a shift of the axis by $m/2$ for a mass m monopole corresponds to scaling so that the monopoles all have mass 2, by equation (4.1).

4.5 Analytic functions

Recall that solutions (γ_j, β_j) to the Braam-Austin equations may be thought of as discrete functions $(\gamma(s), \beta(s))$ on the interval $(-1, 1)$ by setting $s = j/2m$. This is the viewpoint taken in the following conjecture:

Conjecture 1. *For each solution $(\gamma_{BA}, \beta_{BA}, v)$ to the Braam-Austin equations for hyperbolic monopoles with mass m and charge $k > 1$, there exist matrix-valued functions $\alpha(s)$ and $\beta(s)$ such that*

i) α and β are analytic on the intervals $(-1 - \frac{1}{m}, 1 + \frac{1}{m})$ and $(-1, 1 + \frac{1}{m})$ respectively with simple poles at the endpoints

ii) The functions $\gamma = \frac{m}{2} I_k + \alpha$ and β satisfy the Braam-Austin equations for mass m for all $s \in (-1, 1)$

iii) γ_{BA} and β_{BA} are discretisations of γ and β respectively, so that $(\gamma_{BA})_j = \gamma(j/(2m))$ and $(\beta_{BA})_j = \beta((j+1)/(2m))$

iv) The matrix $\gamma(s)$ is invertible for all s away from the poles, except at $s = \pm 1$ where it is singular

When considering $(\gamma_{BA}, \beta_{BA})$ as discrete functions it is convenient to use the notation of §3.7.1. There, the domain of definition of β was shifted so that β and γ were defined at the same points. This convention has been adopted here. The "unshifted" domain for β is thus $(-1 - \frac{1}{2m}, 1 + \frac{1}{2m})$.

The conjecture is satisfied for $k = 2$, which may be verified from the solutions given in Chapter 5. In the case $k = 1$, the matrices are constant and there are no poles.

While Conjecture 1 remains a conjecture for $k \geq 3$, there is very good evidence that it, or something very similar, is true. This evidence is mostly based on work of Murray and Singer [24], who describe how to construct "discrete Nahm" data (i.e. a solution to the Braam-Austin equations) from the spectral curve of a monopole, following Hitchin's construction [17] for the Euclidean case.

4.5.1 Murray and Singer's construction

Starting from the spectral curve $\mathcal{S}$, construct vector spaces V_r as spaces of sections of line bundles over $\mathcal{S}$:

$$V_r = H^0(\mathcal{S}, L^r(k-1, 0)) \tag{4.7}$$

where L is the restriction of the line bundle $\mathcal{O}(1, -1)$ over Z to $\mathcal{S}$ and $L^r(k-1, 0)$ is a shorthand for $L^r \otimes \mathcal{O}(k-1, 0)|_{\mathcal{S}} = \mathcal{O}(r+k-1, -r)|_{\mathcal{S}}$. The bundle $L^r(k-1, 0)$

represents a point on the Jacobian, $\mathrm{Jac}_{k(k-1)}(\mathcal{S})$, of line bundles on $\mathcal{S}$ with degree $k(k-1)$. The theta divisor Θ of $\mathcal{S}$ lies in $\mathrm{Jac}_{g-1}(\mathcal{S}) = \mathrm{Jac}_{k(k-2)}(\mathcal{S})$ (recall $\mathcal{S}$ has genus $(k-1)^2$). Murray and Singer ([24], Definition 2.1) make the following definition

Definition 3. *A bundle $\mathcal{L} \in Jac_{k(k-1)}(\mathcal{S})$ is* **regular** *if*

$$\mathcal{L}(-1,0) \in Jac_{k(k-2)}(\mathcal{S}) \setminus \Theta \qquad \text{and} \qquad \mathcal{L}(0,-1) \in Jac_{k(k-2)}(\mathcal{S}) \setminus \Theta$$

The degrees work out because $\mathcal{S}$ has bidegree (k,k), so tensoring with $\mathcal{O}(-1,0)$ or $\mathcal{O}(0,-1)$ reduces the degree by k. Murray and Singer then prove

Proposition 4.1. *If $\mathcal{L} \in Jac_{k(k-1)}(\mathcal{S})$ is regular, then*

$$h^0(\mathcal{S},\mathcal{L}) = k = h^0(\mathcal{S},\mathcal{L}\otimes L^{\pm 1})$$

(this is true for generic $\mathcal{L}$ by Riemann-Roch), and there is an exact sequence

$$0 \to H^0(\mathcal{S},\mathcal{L}) \to H^0(\mathcal{S},\mathcal{L}) \otimes H^0(\mathcal{S},\mathcal{O}(1,1)) \overset{m}{\to} H^0(\mathcal{S},\mathcal{L}(1,1)) \to 0 \tag{4.8}$$

where m is the multiplication map. □

Thus V_r has dimension k and depends holomorphically on r, for r in some region of $\mathbb{C}$ such that $L^r(k-1,0)$ is regular. For integral r, $L^r(k-1,0)$ is certainly not regular for $2m+1 \leq r \leq 2m+k$. To see this note that

$$\begin{aligned} L^r(k-1,0)\otimes\mathcal{O}(-1,0) &\cong L^r(k-2,0) \\ &\cong \mathcal{O}(r+k-2,-r)|_{\mathcal{S}} \\ &\cong \mathcal{O}(r-2m-2,-r+2m+k)|_{\mathcal{S}} \end{aligned}$$

because $\mathcal{O}(-2m-k,2m+k)|_{\mathcal{S}} \cong \mathcal{O}|_{\mathcal{S}}$. When $2m+2 \leq r \leq 2m+k$, both of $r-2m-2$ and $-r+2m+k$ are positive and so the bundle $L^r(k-2,0)$ has sections (for example, the restrictions of global sections, since $r \in \mathbb{Z}$). A generic bundle of degree $k(k-2)$ has no sections, and so $L^r(k-2,0)$ must lie on the theta divisor. Thus $L^r(k-1,0)$ is not regular for $2m+2 \leq r \leq 2m+k$. The same argument using $\mathcal{O}(0,-1)$ instead of $\mathcal{O}(-1,0)$ shows that $L^r(k-1,0)$ is not regular for $2m+1 \leq r \leq 2m+k-1$. This region where $L^r(k-1,0)$ is not regular occurs periodically, since $L^0(k-1,0) \cong L^{2m+k}(k-1,0)$. The claim is that $L^r(k-1,0)$ is regular for real r outside $[2m+1,2m+k]$ and its translates, and so for r in some region $\Omega \in \mathbb{C}$ containing the interval $[1,2m]$.

The kernel K_r of the multiplication map

$$V_r \otimes H^0(\mathcal{S},\mathcal{O}(1,1)) \to H^0(\mathcal{S},L^r(k,1))$$

is then k dimensional and may be identified with V_r, by Proposition 4.1. An element of K_r is a set of four elements $\{s_{00}, s_{01}, s_{10}, s_{11}\}$ of V_r such that

$$w_1 z_1 s_{00} + w_1 z_0 s_{01} + w_0 z_1 s_{10} + w_0 z_0 s_{11} = 0 \tag{4.9}$$

on $\mathcal{S}$, where $\eta = w_1/w_0$ and $\zeta = z_1/z_0$ are coordinates on Z. So (4.9) implies that

$$\begin{aligned}(w_1 s_{00} + w_0 s_{10}, w_1 s_{01} + w_0 s_{11}) &= (z_0, -z_1)t_+ \\ (z_1 s_{00} + z_0 s_{01}, z_1 s_{10} + z_0 s_{11}) &= (w_0, -w_1)t_-\end{aligned}$$

on $\mathcal{S}$, for some sections $t_\pm$. In fact, from these formulae $t_\pm \in V_{r\pm 1}$ so the procedure defines maps $P_r^\pm : K_r \to V_{r\pm 1}$. These maps depend holomorphically on $r \in \Omega$. Murray and Singer prove that for $r \in \Omega$, the $P_r^\pm$ are isomorphisms. The identification of K_r with V_r is by projecting to s_{10}, and so there are endomorphisms A_r, B_r, D_r of V_r such that (reverting to inhomogeneous coordinates)

$$(A_r \eta\zeta + B_r \eta + \zeta + D_r)s_r = 0 \qquad \forall s_r \in V_r$$

These endomorphisms also vary holomorphically with r. Fixing a basis for the V_r (and making the identification with K_r), the maps $P_r^\pm$, A_r and D_r may be written as matrices. It turns out [24] that these matrices satisfy the Braam-Austin equations, with the $P_r^\pm$ taking the roles of the γ_j and γ_j^*, and A_r and D_r the roles of β_j and β_j^*.

This construction makes it clear that difference equations, rather than differential equations, arise naturally from the geometry of $\mathcal{S}$. Although the difference equations are satisfied at all points of Ω, there is a natural discrete set of points to choose, namely those corresponding to real integral r. In this case, the points of $\mathrm{Jac}_{k(k-1)}(\mathcal{S})$ represented by the $L^r(k-1, 0)$ meet the boundary of the non-regular region (after a finite number of steps and without crossing non-regular points). This should give rise to the poles at the endpoints and the singular behaviour of γ there.

In the case $k = 2$, the spectral curve is an elliptic curve and $\mathrm{Jac}(\mathcal{S})$ may be identified with $\mathcal{S}$. Writing $\mathcal{S}$ as $\mathbb{C}/\Lambda$, the lattice Λ must be rectangular (i.e. must be generated by $\{1, it\}$ with t real) because of the reality conditions on $\mathcal{S}$. Thus $\mathcal{S}$ is determined as an abstract curve by one positive real number. The theta divisor is the single point $0 \in \mathrm{Jac}_0(\mathcal{S})$. Choose coordinates so that the $L^r(1,0) \in \mathrm{Jac}_2(\mathcal{S})$ (with r real) lie along the real axis in $\mathbb{C}$. Then looking at integral values of r divides the real interval $[0,1] \subset \mathbb{C}$ into equal steps, with the $L^r(1,0)$ only non-regular at the endpoints.

Murray and Singer [24] prove that the above matrix data may be constructed from the spectral curve of any monopole. They have a procedure (see §5.6) for finding the spectral curve from the matrices and prove that this recovers the original spectral curve. They also prove that for any such solution, V_r has the form

$$V_r = H^0(\mathcal{S}, \mathcal{L}_j)$$

where $\mathcal{L}_j \in \mathrm{Jac}_{k(k-1)}(\mathcal{S})$ is isomorphic to $\mathcal{L}_0 \otimes L^j$. The original definition of V_r fits this pattern, but calculating $\mathcal{L}_0$ remains a problem. So it remains a conjecture at present that the V_r are always given by (4.7). In any case, the maps $P_r^{\pm}$, A_r and D_r depend analytically on $r \in \Omega$.

4.5.2 Note

The particular endpoint conditions of Conjecture 1 are satisfied by the symmetric 2-monopole and by the 2-monopole solutions given in the next chapter. For larger k the lattice Λ is still rectangular, but the theta divisor is more complicated and the situation less easy to understand. The precise endpoint condition does not affect the results deduced from the conjecture, however, provided (α, β) are analytic on an interval containing $(-1, 1)$ with poles at the endpoints and that the positions of the poles tend to ± 1 as m tends to infinity. This much should follow from Murray and Singer's work. Part iv) will not be used in what follows. The rest of this chapter will be based on the assumption that Conjecture 1 is true.

4.6 Convergence definition

It is now possible to make sense of the statement that "hyperbolic monopoles of increasing mass tend to Euclidean monopoles". The idea is to let $A^{(m)}$ denote a set of Braam-Austin data for a monopole of mass m, and to look at sequences $\{A^{(m)}\}$ of such monopoles of fixed charge k, one for each integral mass. The rescaling is equivalent to writing $\gamma_j^{(m)} = \frac{m}{2} I_k + \alpha_j^{(m)}$ and considering the functions $(\alpha_j^{(m)}, \beta_j^{(m)})$ instead of $(\gamma_j^{(m)}, \beta_j^{(m)})$ (using the arguments of §4.4). By Conjecture 1, $(\alpha_j^{(m)}, \beta_j^{(m)}, v^{(m)})$ corresponds to a pair of analytic functions $(\alpha^{(m)}, \beta^{(m)})$ on an interval containing $(-1, 1)$, together with the vector $v^{(m)}$. If these functions converge to analytic functions on $(-1, 1)$ with poles at the endpoints, the limit functions (α, β) will satisfy Nahm's equations (again by §4.4) and so describe a Euclidean monopole.

Most sequences $\{A^{(m)}\}$ of monopoles do have a Euclidean limit in this sense. To make this precise,

Definition 4. *A* **sequence** $\{A^{(m)}\}$ **of monopoles of increasing mass** *is a sequence of Braam-Austin data, where* $A^{(m)} = (\gamma_{BA}^{(m)}, \beta_{BA}^{(m)}, v^{(m)})$ *represents a monopole of mass* m *and the sequence contains one such* $A^{(m)}$ *for each integral mass* m*. The sequence is indexed by* m*.*

The following theorem gives conditions for such a sequence to converge.

Theorem 4.2. *Assume Conjecture 1 is true. Let* $\{A^{(m)}\}$ *be a sequence of monopoles of increasing mass with fixed charge* k*, such that the corresponding sequences* $\{\alpha^{(m)}\}$*,* $\{\beta^{(m)}\}$ *of analytic functions and* $\{v^{(m)}\}$ *of vectors satisfy*

i) The functions $\alpha^{(m)}$*,* $\beta^{(m)}$ *all extend as meromorphic functions on some fixed region* $\Omega \subset \mathbb{C}$ *containing the interval* $[-1, 1]$ *with at most two simple poles (depending whether* $\pm(1+\frac{1}{m}) \in \Omega$*) and are uniformly bounded on any compact subset of* Ω *excluding the poles*

ii) There is a subsequence of the boundary data $\{(\beta^{(m)}(1)/m, v^{(m)}/m)\}$ *converging to* (b, v) *where* v *is a cyclic vector for* b *(i.e.* $\{v, bv, \ldots, b^{k-1}v\}$ *span* $\mathbb{C}^k$*)*

then there is a subsequence $\{A^{(m_n)}\}$ *converging to a Euclidean monopole of charge* k*.*

Notes

- Condition i) of Theorem 4.2 is the condition that no charge escapes to infinity, or in other words that "the monopoles remain inside a compact set" in $\mathbb{H}^3$. It will become clear in Chapter 5 that in the case of 2-monopoles the condition that the functions extend to meromorphic functions on a fixed set Ω corresponds to the "monopole separation" remaining finite, and that for a monopole with "finite separation", the uniformly bounded condition corresponds to the centre of the monopole remaining inside a compact set. An example will be given in Chapter 5 of a sequence of 2-monopoles where some, but not all, of the charge "escapes to infinity". The result is a Euclidean monopole of charge 1.

- Any analytic function on a real interval extends as a holomorphic function on some region Ω of $\mathbb{C}$. The important part of condition i) is that this Ω is the same for all $\alpha^{(m)}$ and $\beta^{(m)}$.

- Condition ii) that v is cyclic for b in the limit ensures that the representation formed by the residues of the limit is irreducible, and so the limit really is an $SU(2)$ monopole and does not degenerate in any way.

- In fact the theorem almost gives both necessary and sufficient conditions for a sequence of monopoles of increasing mass and fixed charge to have a subsequence with limit a Euclidean monopole of the same charge: such a limit exists if and only if i) and ii) of Theorem 4.2 hold for some infinite subset of the $\{A^{(m)}\}$.

4.6.1 Proof of Theorem 4.2

Assume Conjecture 1 is true and suppose the sequence $\{A^{(m)}\}$ satisfies i) and ii) of Theorem 4.2. Then by the conjecture, the poles of the corresponding analytic functions $\alpha^{(m)} =: \gamma^{(m)} - \frac{m}{2} I_k$ and $\beta^{(m)}$ are at $\pm(1+\frac{1}{m})$ and -1, $1+\frac{1}{m}$ respectively. Assume m is large enough that $\pm(1+\frac{1}{m}) \in \Omega$, and for convenience that

$$\left\{\left(\frac{\beta^{(m)}(1)}{m}, \frac{v^{(m)}}{m}\right)\right\} \to (b, v)$$

as $m \to \infty$, with v cyclic for b. If this were not true, it would be true for some subsequence by ii) of the theorem. This kind of assumption will be made again during the proof because it is notationally cumbersome to take subsequences and renumber when the original sequence is indexed by the mass. The proof is in two main parts; the first is that limit functions (α, β) exist and the second that they represent a Euclidean monopole. The first part subdivides into the proof that the functions minus their poles converge, and the proofs that the residues and hence the original functions converge.

Existence of limit

Both $\alpha^{(m)}$ and $\beta^{(m)}$ extend to meromorphic functions on Ω (given by part i) of the theorem) which have two simple poles and are holomorphic otherwise. So let $R^m_\pm$ be the residues of $\alpha^{(m)}$ at $\pm(1+\frac{1}{m})$ respectively and $S^m_\mp$ the residues of $\beta^{(m)}$ at -1 and $1+\frac{1}{m}$. Define holomorphic functions $\hat{\alpha}^{(m)}$ and $\hat{\beta}^{(m)}$ by

$$\hat{\alpha}^{(m)}(s) = \alpha^{(m)}(s) - \frac{R^m_+}{1+\frac{1}{m}-s} - \frac{R^m_-}{1+\frac{1}{m}+s} \tag{4.10}$$

$$\hat{\beta}^{(m)}(s) = \beta^{(m)}(s) - \frac{S^m_+}{1+\frac{1}{m}-s} - \frac{S^m_-}{1+s} \tag{4.11}$$

Now by the conditions of the theorem, the families $\{\hat{\alpha}^{(m)}\}$ and $\{\hat{\beta}^{(m)}\}$ are uniformly bounded on all compact subsets of Ω. Any such sequence has a subsequence converging to an analytic function on Ω, the convergence is uniform on compact sets and the derivatives also converge uniformly to the derivative of the limit (see, for example, [2]). So there is a subsequence $\{m_n\}$ such that the pair

$(\hat{\alpha}^{(m_n)}, \hat{\beta}^{(m_n)})$ converges to a pair of holomorphic functions $(\hat{\alpha}, \hat{\beta})$ defined on Ω, with the derivatives also converging. Again, for notational reasons, assume this is in fact the whole sequence. Thus $(\alpha^{(m)}, \beta^{(m)})$ converges to a pair of meromorphic functions on Ω with simple poles at ± 1 provided the residues converge.

Convergence of residues Note first the obvious fact that if $\{a_n\}$ is a sequence of points in $D := \{|z| < R\} \subset \mathbb{C}$ with $a_n \to 0$ as $n \to \infty$, then for any compact set $\Omega \subset D^* = D \setminus \{0\}$ there exists $N_\Omega \in \mathbb{N}$ such that $a_n \notin \Omega$ for $n > N_\Omega$. The idea is to consider a sequence $\{f_n\}$ of functions meromorphic on D with simple poles at a_n, residues c_n, and no other poles. Using this framework we can state:

Lemma 4.3. *Suppose $\{f_n\}$ is a sequence of meromorphic functions on D as described above, and that there is a holomorphic function f on D^* such that for any compact set $\Omega \subset D^*$, the sequence $\{f_n\}_{n>N_\Omega}$ converges uniformly to f. Then f extends to a meromorphic function on D with at worst a simple pole at $0 \in \mathbb{C}$, and the residues c_n converge to the residue c of f at the pole.*

Proof of Lemma 4.3

A holomorphic function g on $\{0 < |z| < R\}$ has Laurent expansion

$$g(z) = \sum_{n=-\infty}^{\infty} t_n z^n$$

where

$$t_n = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w^{n+1}} dw$$

and γ is a small loop around $w = 0$ in $\mathbb{C}$. Choose $\epsilon < R$, then there exists $N_\epsilon \in \mathbb{N}$ such that $a_n \in \{|z| < \epsilon\}$ for all $n > N_\epsilon$. Then for $n > N_\epsilon$ the residue c_n of f_n at a_n is

$$c_n = \frac{1}{2\pi i} \int_{|w|=\epsilon} f_n(w) dw$$

But f_n converges to f uniformly on the compact set $\{|z| = \epsilon\}$, so

$$c_n \to \frac{1}{2\pi i} \int_{|w|=\epsilon} f(w) dw = c$$

It remains to check that f has at worst a simple pole (it may have none at all if $c = 0$). Clearly the only possible bad point is $z = 0$. The coefficient of z^{-k} (where $k \geq 2$) in the Laurent expansion of f is given by

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} w^{k-1} f(w) dw = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{|w|=\epsilon} (w - a_n)^{k-1} f_n(w) dw = 0$$

since the functions f_n all have only simple poles. □

This lemma may be applied to neighbourhoods of ± 1 to show that the sequences of residues converge, say $R_{\pm}^m \to R_{\pm}$ and $S_{\pm}^m \to S_{\pm}$. Thus the functions $(\alpha^{(m)}, \beta^{(m)})$ converge to analytic functions (α, β) with simple poles at ± 1 (the fact that the limit residues are nonzero, and so the functions do have simple poles, will be shown in the last part of the proof of the theorem).

Reality and boundary conditions

Because each $A^{(m)}$ satisfies the Braam-Austin equations for mass m, it follows from §4.4 that the pair (α, β) satisfy Donaldson's form of Nahm's equations. It remains to check the other conditions which, in this formulation, are

- $\alpha(-s) = \alpha(s)^t$ and $\beta(-s) = \beta(s)^t$
- In a gauge where α is Hermitian, the residue of α at 1 and the Hermitian and skew-Hermitian parts of the residue of β at 1 define an irreducible representation of $\mathfrak{su}(2)$

The first condition is satisfied because $\alpha^{(m)}$ and $\beta^{(m)}$ satisfy the Braam-Austin equations (3.4) and (3.5), namely $\alpha^{(m)}(-s) = \alpha^{(m)}(s)^t$ and $\beta^{(m)}(-s + 1/2m) = \beta^{(m)}(s + 1/2m)^t$ for all $s \in (-1, 1)$.

If the second condition is to be satisfied, the residues must in particular be nonzero. To see this note that

Lemma 4.4. *The residue S_+ of β at 1 satisfies*

$$S_+ = b$$

where b is the limit as $m \to \infty$ of the boundary data $\beta^{(m)}(1)/m$.

Proof of Lemma 4.4

The function $\hat{\beta}^{(m)}$ is bounded on Ω, so evaluating (4.11) at $s = 1$ and dividing by m,

$$\frac{\hat{\beta}^{(m)}(1)}{m} = \frac{\beta^{(m)}(1)}{m} - S_+^m - \frac{S_-^m}{2m}$$

Taking the limit as $m \to \infty$ gives the result, since $b = \lim_{m\to\infty} \beta^{(m)}(1)/m$. □

Then the condition follows from the argument of §4.3.2, which proves that any solution to Nahm's equations which satisfies all the other conditions must define a k-dimensional representation of $\mathfrak{su}(2)$ via the residues at its poles. By condition ii) of the theorem, $b = S_+$ has a cyclic vector and so the representation is irreducible. This completes the proof of Theorem 4.2. Note that this proof is also valid (with only small modifications) assuming the slightly weaker version of Conjecture 1 given in §4.5.2. □

Chapter 5

Monopoles of charge 2

The general solution of Nahm's equations for a charge 2 monopole may be expressed in terms of Jacobi elliptic functions, and the same is true in the hyperbolic case. Ward's solution of the discrete Toda equations provides explicit solutions to the Braam-Austin equations. The Euclidean limit may be calculated explicitly as well as the spectral curve, and the behaviour of "widely separated" monopoles studied.

5.1 Jacobi elliptic functions

This section lists some basic facts about elliptic functions, for reference, as they will be needed in what follows. These and further details may be found in [1].

An elliptic function is a doubly periodic meromorphic function on $\mathbb{C}$. The Jacobi elliptic functions are defined by the rectangular lattice:

$$\begin{array}{cccc} .n & .d & .n & .d \\ .s & .c & .s & .c \\ .n & .d & .n & .d \\ .s & .c & .s & .c \end{array}$$

in $\mathbb{C}$, where zero is one of the points s, the real axis is horizontal and the imaginary axis is vertical. The horizontal distance between s and c is

$$K = K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}$$

and the vertical distance between s and n is $K' = K(\kappa')$ with $\kappa'^2 = 1 - \kappa^2$. If p and q are any two of the letters s, c, d, n and $p \neq q$, then the function $pq(u)$ is defined by :

1. pq has a simple zero at p and a simple pole at q and no other zeros or poles
2. The step from p to q is a half-period, the steps from p to the other two corners of the square are quarter-periods

3. The leading coefficient in the expansion about $u = 0$ is 1 (i.e. the first term is u, 1 or $1/u$ according as 0 is a zero, regular point or pole of pq)

As might be expected, $\mathrm{sn}(x)/\mathrm{cn}(x) = \mathrm{sc}(x)$ and $\mathrm{cn}(x)\mathrm{nc}(x) = 1$ etc., so formulae will be given only for sn, cn and dn. The equivalents for the other functions may be deduced from these. The functions where one of p and q is s are odd, and the others are even. All the functions $pq(x)$ are real for real x. In this case $\mathrm{sn}(x)$ is positive for $x \in (0, 2K)$ and $\mathrm{cn}(x)$ is positive for $x \in (-K, K)$, both vanishing at the endpoints of these intervals. $\mathrm{dn}(x)$ is always positive for real x, and all three functions have modulus less than 1. Note that, in particular, this means $|\mathrm{nc}(x)|$ and $\mathrm{dc}(x)$ are greater than or equal to 1 for real x.

The number κ is called the elliptic modulus, and to denote this dependence sn will sometimes be written sn_κ. The elliptic functions have properties similar to the trigonometric functions, indeed when κ is 0 or 1 they become trigonometric or hyperbolic functions:

$$\mathrm{sn}_0 = \sin \qquad \mathrm{sn}_1 = \tanh$$
$$\mathrm{cn}_0 = \cos \qquad \mathrm{cn}_1 = \mathrm{sech}$$
$$\mathrm{dn}_0 = 1 \qquad \mathrm{dn}_1 = \mathrm{sech}$$

Relations between squares

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \qquad \kappa^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1$$

Addition formulae

$$\mathrm{sn}(a+b) = \frac{\mathrm{sn}(a)\mathrm{cn}(b)\mathrm{dn}(b) + \mathrm{sn}(b)\mathrm{cn}(a)\mathrm{dn}(a)}{1 - \kappa^2\mathrm{sn}^2(a)\mathrm{sn}^2(b)}$$
$$\mathrm{cn}(a+b) = \frac{\mathrm{cn}(a)\mathrm{cn}(b) - \mathrm{sn}(a)\mathrm{sn}(b)\mathrm{dn}(a)\mathrm{dn}(b)}{1 - \kappa^2\mathrm{sn}^2(a)\mathrm{sn}^2(b)}$$
$$\mathrm{dn}(a+b) = \frac{\mathrm{dn}(a)\mathrm{dn}(b) - \kappa^2\mathrm{sn}(a)\mathrm{sn}(b)\mathrm{cn}(a)\mathrm{cn}(b)}{1 - \kappa^2\mathrm{sn}^2(a)\mathrm{sn}^2(b)}$$

Special arguments

$$\mathrm{sn}(0) = 0 \qquad \mathrm{sn}(K) = 1$$
$$\mathrm{cn}(0) = 1 \qquad \mathrm{cn}(K) = 0$$
$$\mathrm{dn}(0) = 1 \qquad \mathrm{dn}(K) = \kappa'$$

Imaginary transform

$$\mathrm{sn}_{\kappa'}(iu) = i\mathrm{sc}_\kappa(u) \qquad \mathrm{cn}_{\kappa'}(iu) = \mathrm{nc}_\kappa(u) \qquad \mathrm{dn}_{\kappa'}(iu) = \mathrm{dc}_\kappa(u)$$

Taylor series

$$\begin{aligned}
\mathrm{sn}_\kappa u &= u - (1+\kappa^2)\frac{u^3}{3!} + O(u^5) \\
\mathrm{cn}_\kappa u &= 1 - \frac{u^2}{2} + O(u^4) \\
\mathrm{dn}_\kappa u &= 1 - \kappa^2\frac{u^2}{2} + O(u^4)
\end{aligned}$$

All these properties are needed to verify the calculations in this chapter.

5.2 The general solution

The general solution to the Braam-Austin equations for charge 2 is essentially Ward's $k = 2$ discrete Toda solution [29] and is found by the same method as the symmetric solution (§3.7). Using the notation of §3.7.1, write $\gamma_j = \gamma(j)$, $\beta_j = \beta(j+1)$ and look for a solution of the form

$$\gamma = b\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \qquad \beta = b\begin{pmatrix} 0 & u+x \\ u-x & 0 \end{pmatrix}$$

where $f_1 = \sqrt{1-g}$, $f_2 = \sqrt{1+g}$. Then the Braam-Austin equations

$$\begin{aligned}
\beta_+ &= \gamma^{-1}\beta\gamma \\
\gamma_+\gamma_+^* &= \gamma^*\gamma + [\beta_+^*, \beta_+]
\end{aligned}$$

are equivalent to

$$u_+ = \frac{u+xg}{\sqrt{1-g^2}} \tag{5.1}$$

$$x_+ = \frac{x+ug}{\sqrt{1-g^2}} \tag{5.2}$$

$$g_+ = g + 4u_+x_+ \tag{5.3}$$

The discrete Toda equations are the same except that the first two have numerators $u - xg$ and $x - ug$ on the right hand sides. The solution given here is essentially the same as Ward's discrete Toda solution, but because those solutions were bounded with fixed centring it is necessary to change to an imaginary variable and convenient to have the scaling b clearly visible. Thus it is easier to solve the equations again from the beginning, incorporating the differences from the start.

5.2.1 Conserved quantities

As in the symmetric case, the equations have conserved quantities (where Θ no longer denotes the theta divisor):

Lemma 5.1. *Let*

$$\begin{aligned}\Theta &= x^2 - u^2 \\ \Omega &= g^2 - 2u^2 - 2x^2 - 4gux\end{aligned}$$

then Θ and Ω are independent of j.

Proof

$$\begin{aligned}\Omega_+ &= (g + 4u_+x_+)^2 - 2u_+^2 - 2x_+^2 - 4(g + 4u_+x_+)u_+x_+ \\ &= g^2 + 4gu_+x_+ - 2u_+^2 - 2x_+^2 \\ &= g^2 + \frac{4g(u + gx)(x + gu) - 2(u + gx)^2 - 2(x + gu)^2}{1 - g^2} \\ &= g^2 - 2u^2 - 2x^2 - 4gux \\ &= \Omega\end{aligned}$$

This shows that Ω is conserved. The proof for Θ is simpler, following the same lines, and so is omitted. □

5.2.2 An equation for g

In order to solve the equations, first use the change of variable $p = x-u$, $q = x+u$, so that

$$\begin{aligned}\Theta &= pq \\ \Omega &= g^2 - p^2 - q^2 + g(p^2 - q^2)\end{aligned}$$

Solving these (by substituting $q = \Theta/p$ and solving a quadratic equation for p^2, and likewise for q^2),

$$\begin{aligned}p^2 &= \frac{g^2 - \Omega + R}{2(1 - g)} \\ q^2 &= \frac{g^2 - \Omega - R}{2(1 + g)}\end{aligned}$$

with

$$R^2 = (g^2 - \Omega)^2 - 4(1 - g^2)\Theta^2 \tag{5.4}$$

These can be solved for u and x to get

$$u = \frac{1}{2}(gg_- - \Omega - 2\Theta)^{1/2} \tag{5.5}$$

$$x = \frac{1}{2}(gg_- - \Omega + 2\Theta)^{1/2} \tag{5.6}$$

Note that these are interchangeable since the difference equations do not distinguish between u and x. This is reflected by the fact that the sign of Θ is not determined by the equations. It will be shown that p^2 and q^2 are real and positive, over a suitable range for j, so that u and x are well-defined.

Substituting into equation (5.3) leads to a difference equation in terms of g alone:

$$\begin{aligned} g_+ &= g + 4u_+x_+ \\ &= g - p_+^2 + q_+^2 \\ &= g + \frac{g_+\Omega - g_+^3 - R_+}{1 - g_+^2} \end{aligned}$$

which (translating j) simplifies to

$$g - (1 - g^2)g_- - g\Omega + R = 0 \tag{5.7}$$

5.2.3 Solution for g

The discrete Toda equations, which are almost identical to Braam and Austin's equations, also lead to (5.7). Ward [29] has a solution in terms of the Jacobi elliptic function sn, although because his variable is real, the solution is bounded. For a monopole, the function g must have poles at the endpoints of the interval and behave something like tan. The most obvious way to accomplish this is to use the fact that $\mathrm{sn}_{\kappa'}(iz) = i\mathrm{sc}_\kappa(z)$, in other words to make the argument purely imaginary. It is easier to work with sn than with sc for the moment (because of the form of the addition formula), so start by assuming that

$$g = A\mathrm{sn}_{\kappa'}(iBj)$$

where B is real. Then

$$g_- = A\frac{\mathrm{sn}(iBj)\mathrm{cn}(2iB)\mathrm{dn}(2iB) - \mathrm{sn}(2iB)\mathrm{cn}(iBj)\mathrm{dn}(iBj)}{1 - \kappa'^2\mathrm{sn}^2(2iB)\mathrm{sn}^2(iBj)}$$

and

$$1 - g^2 = 1 - A^2\mathrm{sn}^2(iBj)$$

Set $A = -\kappa' \mathrm{sn}_{\kappa'}(2iB)$ (the sign is chosen so that $g = \kappa' \mathrm{sc}_\kappa(2B)\mathrm{sc}_\kappa(Bj)$ using the imaginary transform), which simplifies the g_- term:

$$\begin{aligned}(1-g^2)g_- &= -\kappa' \mathrm{sn}(2iB)\{\mathrm{sn}(iBj)\mathrm{cn}(2iB)\mathrm{dn}(2iB) - \mathrm{sn}(2iB)\mathrm{cn}(iBj)\mathrm{dn}(iBj)\} \\ &= \kappa' \mathrm{sc}(2B)\{\mathrm{sc}(Bj)\mathrm{nc}(2B)\mathrm{dc}(2B) - \mathrm{sc}(2B)\mathrm{nc}(Bj)\mathrm{dc}(Bj)\}\end{aligned}$$

Using this expression for g in (5.7) and cancelling two of the terms by setting $\Omega = 1 - \mathrm{nc}(2B)\mathrm{dc}(2B)$, the equation simplifies to

$$R = -\kappa' \mathrm{sc}^2(2B)\mathrm{nc}(Bj)\mathrm{dc}(Bj) \tag{5.8}$$

Equation (5.8) fixes which root of R is needed, provided the squares of the two sides are equal (which must be checked since (5.8) must agree with (5.4)). This is equivalent to the condition $4\Theta^2 = (\mathrm{nc}(2B) - \mathrm{dc}(2B))^2$. The check is messy but not hard, using properties of the elliptic functions. The calculation is given in §5.2.4. Thus $g = \kappa' \mathrm{sc}(2B)\mathrm{sc}(Bj)$ solves the difference equations, since (5.5) and (5.6) are formulae for u and x in terms of g.

5.2.4 Proof of (5.8)

It remains to check that

$$R^2 - \kappa'^2 \mathrm{sc}^4(2B)\mathrm{nc}^2(Bj)\mathrm{dc}^2(Bj) = 0 \tag{5.9}$$

if and only if $4\Theta^2 = (\mathrm{nc}(2B) - \mathrm{dc}(2B))^2$. Substituting from (5.4), the left hand side of (5.9) is

$$(g^2 - \Omega)^2 - 4(1-g^2)\Theta^2 - \kappa'^2 \mathrm{sc}^4(2B)\mathrm{nc}^2(Bj)\mathrm{dc}^2(Bj) \tag{5.10}$$

Multiplying out the brackets, using the formulae for g and Ω and the identities

$$\mathrm{nc}^2(u) = 1 + \mathrm{sc}^2(u) \qquad \mathrm{dc}^2(u) = 1 + \kappa'^2 \mathrm{sc}^2(u)$$

the expression (5.10) simplifies to

$$\begin{aligned}&\{1 - \kappa'^2 \mathrm{sc}^2(2B)\mathrm{sc}^2(Bj)\}\{2 + \mathrm{sc}^2(2B) + \kappa'^2 \mathrm{sc}^2(2B) - 2\mathrm{nc}(2B)\mathrm{dc}(2B) - 4\Theta^2\} \\ &= \{1 - \kappa'^2 \mathrm{sc}^2(2B)\mathrm{sc}^2(Bj)\}\{\mathrm{nc}^2(2B) + \mathrm{dc}^2(2B) - 2\mathrm{nc}(2B)\mathrm{dc}(2B) - 4\Theta^2\} \\ &= \{1 - \kappa'^2 \mathrm{sc}^2(2B)\mathrm{sc}^2(Bj)\}\{(\mathrm{nc}(2B) - \mathrm{dc}(2B))^2 - 4\Theta^2\}\end{aligned}$$

which is identically zero if and only if

$$4\Theta^2 = (\mathrm{nc}(2B) - \mathrm{dc}(2B))^2$$

The above calculation verifies that the solution for g is consistent with all the equations for this (and only this) choice of Θ^2.

5.3 Other conditions

The solution is not yet in a form satisfying all the Braam-Austin equations. It remains to impose the boundary condition and to find a gauge where the reality conditions (3.4): $\gamma_{-j} = \gamma_j^t$ and (3.5): $\beta_{-j} = \beta_j^t$ hold.

5.3.1 Change of gauge

As in the symmetric case, change to a more convenient gauge using the constant unitary matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

so that

$$\beta = b\begin{pmatrix} -ix & u \\ u & ix \end{pmatrix} \qquad \gamma = \frac{b}{2}\begin{pmatrix} f_1+f_2 & i(f_2-f_1) \\ i(f_1-f_2) & f_1+f_2 \end{pmatrix}$$

(where $f_1^2 = 1-g$ and $f_2^2 = 1+g$ as before). It is now clear that (3.4) holds, since replacing j by $-j$ exchanges f_1 and f_2, as sc is odd, and that (3.5) reduces to the conditions $u(j+1) = u(-j+1)$ and $x(j+1) = x(-j+1)$, which by (5.5) and (5.6) is equivalent to $gg_-(j+1) = gg_-(-j+1)$. This is straightforward to check from the formula for g, verifying that the solution satisfies (3.4) and (3.5).

Note also that in this gauge

$$\gamma\gamma^* = b^2\begin{pmatrix} 1 & ig \\ -ig & 1 \end{pmatrix} \qquad [\beta,\beta^*] = 4b^2iux\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The vector v appears in the boundary condition and nowhere else. Let $v = b(x \quad y)$.

5.3.2 Boundary condition

The boundary condition (3.8) is

$$[\beta(2m),\beta^*(2m)] + v^t\bar{v} - \gamma^*(2m-2)\gamma(2m-2) = 0$$

the left-hand side of which, in terms of matrices, is

$$4ib^2u(2m)x(2m)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + b^2\begin{pmatrix} |x|^2 & x\bar{y} \\ y\bar{x} & |y|^2 \end{pmatrix} - b^2\begin{pmatrix} 1 & ig(2m-2) \\ -ig(2m-2) & 1 \end{pmatrix}$$

Thus $|x|^2 = |y|^2 = 1$, $y\bar{x} = -x\bar{y} = \pm i$, and

$$4u(2m)x(2m) \pm 1 + g(2m-2) = 0 \tag{5.11}$$

Using the difference equation for g, (5.11) becomes

$$g(2m) = \mp 1$$

i.e. $\kappa'\text{sc}(2B)\text{sc}(2mB) = \mp 1$.

Proposition 5.2. *The boundary condition $\kappa' sc(2B)sc(2mB) = \mp 1$ is satisfied when*

$$B = \frac{K}{2(m \mp 1)}$$

Proof: First note that

$$\kappa'\mathrm{sc}(2B)\mathrm{sc}(2mB) = 1 \Leftrightarrow \frac{1}{\kappa'}\mathrm{cs}(2B)\mathrm{cs}(2mB) = 1$$

and that

$$\mathrm{cs}(a+b) = \frac{\mathrm{cs}(a)\mathrm{cs}(b) - \mathrm{dn}(a)\mathrm{dn}(b)}{\mathrm{cs}(a)\mathrm{dn}(a) + \mathrm{cs}(b)\mathrm{dn}(b)}$$

(from the addition formulae for sn and cn). So when $B = \frac{K}{2(m+1)}$,

$$\begin{aligned} \mathrm{cs}(2mB) &= cs\left(\frac{Km}{m+1}\right) = \mathrm{cs}\left(K - \frac{K}{m+1}\right) \\ &= \frac{\mathrm{cs}(K)\mathrm{cs}\left(\frac{-K}{m+1}\right) - \mathrm{dn}(K)\mathrm{dn}\left(\frac{-K}{m+1}\right)}{\mathrm{cs}(K)\mathrm{dn}(K) + \mathrm{cs}\left(\frac{-K}{m+1}\right)\mathrm{dn}\left(\frac{-K}{m+1}\right)} \\ &= \frac{\kappa'}{\mathrm{cs}\left(\frac{K}{m+1}\right)} = \frac{\kappa'}{\mathrm{cs}(2B)} \end{aligned}$$

thus $\frac{1}{\kappa'}\mathrm{cs}(2B)\mathrm{cs}(2mB) = 1$. The check for the other case is almost identical. □

The constant B can take no other values since sc has a pole at K, so g would then have poles in the interval where β and γ are defined (which can't happen by Conjecture 1).

Because of the shift $\beta_j = \beta(j+1)$, g must be defined for all j between $-2m+2$ and $2m$, so $B = \frac{K}{2(m-1)}$ is no good either and the only possibility is

$$B = \frac{K}{2(m+1)}$$

corresponding to $g(2m) = 1$. The vector v may be taken to be $v = b\,(\,i \quad 1\,)$ (it is easy to check that the matrix $\begin{pmatrix} \beta_{-2m+1} \\ v \end{pmatrix}$ is injective for this choice).

5.3.3 Reality of u and x

It was assumed from the beginning that u and x were real, so it remains to check that this is true; or equivalently that p^2 and q^2 are always positive, as g, Ω, Θ and R are all real. Recall that

$$p^2 = \frac{g^2 - \Omega + R}{2(1-g)} \qquad q^2 = \frac{g^2 - \Omega - R}{2(1+g)}$$

and that $R^2 = (g^2 - \Omega)^2 - 4(1-g^2)\Theta^2$, so $p^2q^2 = \Theta^2 > 0$ and hence p^2 and q^2 have the same sign. For real z, $\mathrm{nc}(z)\mathrm{dc}(z) \geq 1$, so

$$\Omega = 1 - \mathrm{nc}(2B)\mathrm{dc}(2B) \leq 0$$

Let I_m be the interval $(-2m+2, 2m)$. By (5.8), $R = -\kappa' \mathrm{sc}^2(2B)\mathrm{nc}(Bj)\mathrm{dc}(Bj)$, which is always negative for $j \in I_m$. On that interval $-1 < g \leq 1$. These facts together show that q^2 is always positive on I_m, and hence so is p^2. Since g reaches the value 1 (at $j = 2m$), it looks as though p^2 is undefined there. In fact, taking the limit,

$$p^2 \to \frac{2\Theta^2}{1-\Omega}$$

as $j \to 2m$ (remembering to use the negative root of R^2). So u and x are defined and real on I_m, and this solution solves the Braam-Austin equations for a hyperbolic monopole of charge 2.

5.3.4 Summary

The general solution to the Braam-Austin equations for a charge 2 monopole (with a fixed centre and orientation; it will be explained later exactly what these are) is

$$\beta = b\begin{pmatrix} -ix & u \\ u & ix \end{pmatrix} \qquad v = b\,(\,i \quad 1\,)$$

$$\gamma = \frac{b}{2}\begin{pmatrix} \sqrt{1-g}+\sqrt{1+g} & i(\sqrt{1+g}-\sqrt{1-g}) \\ i(\sqrt{1-g}-\sqrt{1+g}) & \sqrt{1-g}+\sqrt{1+g} \end{pmatrix}$$

where

$$\begin{aligned} g &= \kappa' \mathrm{sc}(2B)\mathrm{sc}(Bj) \\ B &= \frac{K}{2(m+1)} \\ u &= \frac{1}{2}(gg_- - \Omega - 2\Theta)^{1/2} \\ x &= \frac{1}{2}(gg_- - \Omega + 2\Theta)^{1/2} \\ \Omega &= 1 - \mathrm{nc}(2B)\mathrm{dc}(2B) \\ \Theta &= \frac{1}{2}(\mathrm{nc}(2B) - \mathrm{dc}(2B)) \end{aligned}$$

and the number $\kappa \in [0,1)$ is the modulus of the elliptic functions, $\kappa'^2 = 1 - \kappa^2$ and K is the complete elliptic integral of the first kind with parameter κ.

All 2-monopoles, up to centre and orientation, are obtained this way, since an elliptic curve is determined as an abstract curve by the modulus $\kappa \in [0,1)$ and there is a solution for each κ. Changing the embedding of the curve in Z corresponds to changing the centre and orientation of the monopole. The general solution may be used to demonstrate various properties of monopoles and to begin to clarify the role of κ. For example, as $\kappa \to 0$, we recover the symmetric solution. On the other hand, as $\kappa \to 1$, the solution becomes an approximate superposition of two widely separated 1-monopoles. The Euclidean limit of the solution is the general solution for (centred, oriented) Euclidean 2-monopoles, and by modifying the data to incorporate a translation it is possible to illustrate the phenomenon noted by Jarvis and Norbury [20] of charge "escaping to infinity". More precisely, this will be done by giving an explicit sequence of 2-monopoles on hyperbolic spaces with decreasing curvature, whose limit is a Euclidean monopole of charge 1. Note that the functions α and β of Conjecture 1 are obvious in this case. It may be verified that this solution satisfies all the conditions of the conjecture. In addition, the vector v appears as

$$v = \mathrm{im}\gamma(2m) = \ker\gamma(-2m)$$

so that the degeneracy of γ at $\pm 2m$ defines v. Murray and Singer conjecture that something similar to this should be true in general.

5.4 Euclidean limit

The method for taking the Euclidean limit was described in Chapter 4. The general Euclidean 2-monopole solution will be briefly reviewed first, and then shown to be the same as the limit of this hyperbolic solution.

5.4.1 Euclidean monopoles

It is known (see Sutcliffe [27]) that the Nahm data for a charge 2 monopole (with fixed centre and orientation) may be expressed in the form

$$T_1 = \frac{f_1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_2 = \frac{f_2}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad T_3 = \frac{f_3}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and then the Nahm equations are

$$\dot{f}_1 = f_2 f_3 \qquad \dot{f}_2 = f_3 f_1 \qquad \dot{f}_3 = f_1 f_2 \tag{5.12}$$

To solve these for a symmetric monopole two of these functions are equal, say $f_2 = f_3$. Then equations (5.12) become

$$\dot{f}_1 = f_2^2 \qquad \dot{f}_2 = f_1 f_2$$

Using these,

$$\begin{aligned} \ddot{f}_1 &= 2f_2\dot{f}_2 = 2f_1 f_2^2 = 2f_1\dot{f}_1 \\ &\Rightarrow\ \dot{f}_1 = f_1^2 + C^2 \quad \text{(the constant must be positive for a solution in terms of trigonometric functions)} \\ &\Rightarrow\ \int \frac{1}{f_1^2 + C^2}\, df_1 = \int ds \\ &\Rightarrow\ f_1 = C\tan C(s-K) \end{aligned}$$

for constants C and K. Because f_1 must be defined for $s \in (-1, 1)$, with poles at -1 and 1, the constants are $K = 0$ and $C = \frac{\pi}{2}$. Putting this into the first of the Nahm equations,

$$f_2^2 = \dot{f}_1 = (\frac{\pi}{2})^2 \sec^2 \frac{\pi s}{2}$$

So the solution, in this gauge, of Nahm's equations for a symmetric charge 2 monopole is:

$$\begin{aligned} f_1 &= \frac{\pi}{2} \tan \frac{\pi s}{2} \\ f_2 &= \frac{\pi}{2} \sec \frac{\pi s}{2} \end{aligned}$$

In general, the equations (5.12) are the Euler top equations, which have a solution in terms of elliptic functions (see [27]):

$$\begin{aligned} f_1 &= -K\mathrm{cs}(Ks') \\ f_2 &= -K\mathrm{ns}(Ks') \\ f_3 &= K\mathrm{ds}(Ks') \end{aligned}$$

As before, ns, cs and ds are Jacobi elliptic functions with modulus κ and K is the complete elliptic integral of the first kind with parameter κ. The variable s' lies in the interval $(0, 2)$, which is the standard interval for solving Nahm's equations. Braam and Austin's construction leads us to use $s \in (-1, 1)$ instead, and the two versions may be compared by setting $s' = s + 1$. The symmetric solution is the special case $\kappa = 0$.

5.4.2 Limit of the discrete solution

If κ is fixed, the corresponding family $\{A^{(m)}\}$ of 2-monopoles satisfies Conjecture 1 and i) and ii) of Theorem 4.2, so a limit exists. The first step in finding a limit is to replace the parameter $j \in (-2m + 2, 2m)$ by $s \in (-1, 1)$, where $s = j/2m$. Then $j = 2ms$ and

$$\begin{aligned} Bj &= \frac{2mKs}{2(m+1)} \sim Ks \quad \text{for large } m \\ \mathrm{sc}(2B) &= \mathrm{sc}\left(\frac{K}{m+1}\right) \sim \frac{K}{m+1} \quad \text{for large } m \end{aligned}$$

For a limit to exist, γ must have the form $\gamma = \frac{m}{2}I + \alpha$. Since g is small for large m, at least near the centre of the interval I_m, choose the constant $b_{(m)} = \frac{m}{2}$. This corresponds to shifting the axis of rotation in $\mathbb{R}^4$ by the right amount (as in §4.4). For fixed s and large m,

$$g = \kappa' \mathrm{sc}(2B)\mathrm{sc}(2mBs) = O(1/m)$$

and so

$$\sqrt{1+g} = 1+\frac{1}{2}g+O(1/m^2)$$
$$\sqrt{1-g} = 1-\frac{1}{2}g+O(1/m^2)$$

Thus, putting this into the formula for γ,

$$\gamma = \frac{m}{2}I + \frac{img}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O\left(\frac{1}{m}\right)$$

with

$$\frac{img}{4} = \frac{im}{4}\kappa' \mathrm{sc}(2B)\mathrm{sc}(2mBs) \sim \frac{i\kappa' K}{4}\mathrm{sc}(Ks)$$

for large m.

For the limits of u and x, it is easiest to start by using the Taylor expansions of sn, cn and dn. Observe that

$$\begin{aligned}
\Omega &= 1-\mathrm{cn}_{\kappa'}(2iB)\mathrm{dn}_{\kappa'}(2iB) \\
&= 1-\left(1-\frac{(2iB)^2}{2}+O(B^4)\right)\left(1-\frac{\kappa'^2(2iB)^2}{2}+O(B^4)\right) \\
&= -2B^2(1+\kappa'^2)+O(B^4)
\end{aligned}$$

and

$$\begin{aligned}
2\Theta &= \mathrm{cn}_{\kappa'}(2iB)-\mathrm{dn}_{\kappa'}(2iB) \\
&= 2(1-\kappa'^2)B^2+O(B^4)
\end{aligned}$$

Using the fact that $gg_- = \frac{\kappa'^2K^2}{(m+1)^2}\mathrm{sc}^2(Ks) + O(B^4)$ and putting these into the expressions for u and x,

$$\begin{aligned}
2u &= (gg_- - \Omega - 2\Theta)^{1/2} \\
&= \left(\frac{\kappa'^2K^2}{(m+1)^2}\mathrm{sc}^2(Ks)+2B^2(1+\kappa'^2)-2B^2(1-\kappa'^2)+O(1/m^2)\right)^{1/2} \\
&= \left(\frac{\kappa'^2K^2}{(m+1)^2}(\mathrm{sc}^2(Ks)+1)+O(1/m^2)\right)^{1/2} \\
&= \left(\frac{\kappa'^2K^2}{m+1}\mathrm{nc}^2(Ks)+O(1/m^2)\right)^{1/2}
\end{aligned}$$

There is a similar calculation for x so that (in terms of elliptic functions with modulus κ),

$$u = \frac{\kappa' K}{2(m+1)}\mathrm{nc}(Ks)+O(1/m) \tag{5.13}$$
$$x = \frac{K}{2(m+1)}\mathrm{dc}(Ks)+O(1/m) \tag{5.14}$$

So the limits as $m \to \infty$ of $\alpha^{(m)}$ and $\beta^{(m)}$ are matrix-valued functions $\alpha^{(\infty)}$ and $\beta^{(\infty)}$, given by

$$\alpha^{(\infty)} = \frac{i\kappa' K}{4}\mathrm{sc}(Ks)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\beta^{(\infty)} = \frac{K}{4}\begin{pmatrix} -i\mathrm{dc}(Ks) & \kappa'\mathrm{nc}(Ks) \\ \kappa'\mathrm{nc}(Ks) & i\mathrm{dc}(Ks) \end{pmatrix}$$

5.4.3 Recovering the Nahm matrices

Recall that (in a gauge where α is Hermitian) the Donaldson matrices α and β are related to Nahm data by

$$\alpha = \frac{i}{2}T_1 \qquad \beta = \frac{1}{2}(T_2 + iT_3)$$

Thus T_2 is given by the skew Hermitian part of β and T_3 by the Hermitian part, and the Nahm data corresponding to $\{\alpha^{(\infty)}, \beta^{(\infty)}\}$ is:

$$T_1 = \frac{\kappa' K}{2}\mathrm{sc}(Ks)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$T_2 = \frac{K}{2}\mathrm{dc}(Ks)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$T_3 = -\frac{\kappa' K}{2}\mathrm{nc}(Ks)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

in other words

$$f_1 = \kappa' K\mathrm{sc}(Ks) \qquad f_2 = K\mathrm{dc}(Ks) \qquad f_3 = -\kappa' K\mathrm{nc}(Ks)$$

Using the addition formulae to evaluate $f_j(s+1)$,

$$f_1(s+1) = -K\mathrm{cs}(Ks) \qquad f_2(s+1) = -K\mathrm{ns}(Ks) \qquad f_3(s+1) = K\mathrm{ds}(Ks)$$

which is precisely the solution in Sutcliffe's paper.

5.5 Small and large separation of centres

The aim of the rest of this chapter is to illustrate properties of the solution given in §5.3.4 and its Euclidean limit. In this section the limiting situations $\kappa = 0$ and $\kappa \to 1$ are studied. The case $\kappa = 0$ corresponds to the symmetric solution and as κ becomes close to 1 the solution approximates a superposition of two widely spaced 1-monopoles. The approximate separation in this case is found and the Euclidean limit compared to results of Atiyah and Hitchin.

The corresponding spectral curves are given in the final section, which lead to results on 2-monopole centring. The behaviour in the "κ close to 1" case is studied and shown to agree with results found directly from the Braam-Austin data.

5.5.1 $\kappa = 0$

The check that the general solution reduces to the symmetric case already found in Proposition 3.5 is fairly straightforward. Recall

$$\begin{aligned} \mathrm{sc}_0 &= \tan \\ \mathrm{nc}_0 &= \sec \\ \mathrm{dc}_0 &= \sec \end{aligned}$$

Also $K \to \pi/2$ and $\kappa' \to 1$ as $\kappa \to 0$. So

$$g = \kappa' \mathrm{sc}\left(\frac{K}{m+1}\right) \mathrm{sc}\left(\frac{Kj}{2(m+1)}\right) \to \tan\left(\frac{\pi}{2(m+1)}\right) \tan\left(\frac{\pi j}{4(m+1)}\right)$$

and γ becomes the symmetric γ in the limit. The limits of the constants Θ and Ω are

$$\begin{aligned} \Theta^0 &= \frac{1}{2}(\mathrm{nc}_0(2B) - \mathrm{dc}_0(2B))^{1/2} = 0 \\ \Omega^0 &= 1 - \mathrm{nc}_0(2B)\mathrm{dc}_0(2B) = 1 - \sec^2(2B_0) = \tan^2(2B_0) \end{aligned}$$

where $B_0 = \frac{\pi}{4(m+1)}$. So (by (5.5) and (5.6)) u and x are both equal when $\kappa = 0$:

$$u^0 = x^0 = \frac{1}{2}(\tan^2(2B_0)\tan(B_0 j)\tan(B_0 j - 2B_0) - \tan^2(2B_0))^{1/2}$$

Expanding using the addition formula for tan and simplifying, this is equal to

$$\frac{1}{2}\frac{\tan(2B_0)\sec(B_0 j)}{\sqrt{1+\tan(2B_0)\tan(B_0 j)}}$$

leading to exactly the symmetric solution for β. As expected, the Euclidean limit of this symmetric solution is the Euclidean symmetric monopole.

5.5.2 "Widely separated" monopoles

The limit as $\kappa \to 1$ corresponds to the monopoles becoming infinitely separated. The limit itself is meaningless, because the monopoles have disappeared to infinity in opposite directions, but it is possible to see how the solution is an approximate superposition of widely spaced 1-monopoles for κ close to 1. Note that $\kappa' \to 0$ and $K \to \infty$ as $\kappa \to 1$. There are also approximations for sc, nc and dc in terms of hyperbolic functions when κ is close to 1 (see [1]):

$$\begin{aligned} \mathrm{sc}(z) &= \sinh(z) + O(\kappa'^2) \\ \mathrm{nc}(z) &= \cosh(z) + O(\kappa'^2) \\ \mathrm{dc}(z) &= 1 + O(\kappa'^2) \end{aligned}$$

Recall $s = j/2m$, so that for any m the variable s lies in the interval $[-1, 1]$. Although the function $g = \kappa' \mathrm{sc}(2B)\mathrm{sc}(Bj) = \kappa' \mathrm{sc}(2B)\mathrm{sc}(2mBs)$ takes the value ± 1 at $s = \pm 1$, for other values of s it is "small" for κ close to 1. More precisely:

Lemma 5.3. *For fixed mass m and fixed $s_0 \in (-1, 1)$,*

$$g_\kappa(s_0) \to 0 \qquad \textit{as} \quad \kappa \to 1$$

The rate of convergence increases as $|s_0|$ decreases and as m increases.

Proof

Using the approximations above, and the fact [7] that $K \sim -\log \kappa'$ when κ is close to 1,

$$\begin{aligned} g(s_0) &\sim \kappa' \sinh(2B) \sinh(2mBs_0) \\ &\sim \frac{\kappa'}{4}\left(e^{\frac{-\log\kappa'}{m+1}} - e^{\frac{\log\kappa'}{m+1}}\right)\left(e^{\frac{-ms_0\log\kappa'}{m+1}} - e^{\frac{ms_0\log\kappa'}{m+1}}\right) \\ &\sim \frac{1}{4}\left(\kappa'^{\frac{m(1-s_0)}{m+1}} + \kappa'^{\frac{2+m(1+s_0)}{m+1}} - \kappa'^{\frac{2+m(1-s_0)}{m+1}} - \kappa'^{\frac{m(1+s_0)}{m+1}}\right) \end{aligned}$$

All the powers of κ' in this expression are positive, so $g(s_0) \to 0$ as $\kappa' \to 0$, i.e. as $\kappa \to 1$. Note that this expression tends to zero as $|s_0| \to 0$ or as $m \to \infty$ regardless of κ', hence the increased rates of convergence when $|s_0|$ is small or m is large. □

This lemma shows the sense in which for κ close to 1, the matrix γ is approximately bI_2 (I_2 denoting the 2×2 identity matrix).

An approximation for β

The matrix β also becomes approximately diagonal when κ is close to 1. First approximate Θ and Ω in terms of hyperbolic functions:

$$\begin{aligned} \Omega &= 1 - \mathrm{nc}(2B)\mathrm{dc}(2B) \\ &= 1 - \cosh(2B) + O(\kappa'^2) \\ &= -2\sinh^2(B) + O(\kappa'^2) \end{aligned}$$

and similarly $2\Theta = 2\sinh^2(B) + O(\kappa'^2)$. So, by (5.5) and (5.6),

$$\begin{aligned} u &\sim 0 \\ x &\sim \sinh(B) \end{aligned}$$

when κ is close to 1.

Thus when κ is close to 1 both γ and β are approximately diagonal, and so correspond to an approximate superposition of widely spaced 1-monopoles with centres $(\pm ibx, b) \sim (\pm ib\sinh(B), b)$ in upper half space coordinates $(\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}_{>0})$. The separation of the two centres is then approximately $2\sinh(B)$, determined by K and m (it is independent of b because of the factor $1/t^2$ in the hyperbolic metric).

5.5.3 Euclidean limit in the widely-separated case

If κ is fixed close to 1 we may take $m \to \infty$ in this picture. As before, b must be set equal to $m/2$, so in this case $\alpha \sim 0$. For β,

$$\beta^{(m)} \sim \frac{ixm}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} \frac{ixm}{2} &\sim \frac{im}{2}\sinh\left(\frac{K}{2(m+1)}\right) \\ &\sim \frac{imK}{4(m+1)} + O(1/m) \end{aligned}$$

Thus the limit is an approximate superposition of Euclidean monopoles with centres $(\pm iK/4, 0)$ in $\mathbb{R}^3 \simeq \mathbb{C} \oplus \mathbb{R}$. The separation of the centres is $K/2$ in the limiting Euclidean metric. Recall from §4.4 that the Euclidean limit gives monopoles of mass 2. If the metric is rescaled so that the limit has mass 1, then the separation of centres is precisely K, as calculated by Atiyah and Hitchin ([7], p62).

Example of charge "escaping to infinity"

It has been shown that the $\kappa = 1$ solution is not a monopole because the charge (concentrated at the monopole "centres") has escaped to the boundary $\partial\mathbb{H}^3$. This highlights one of the problems in trying to find a Euclidean limit for a general sequence of hyperbolic monopoles, one for each integer mass. It may be that some or all of the charge "escapes" in the limit. This was addressed in condition i) of Theorem 4.2. The following is an example where that condition is violated. The example shows that it is possible for only a part of the charge to escape, in particular that it is possible to have a sequence of hyperbolic 2-monopoles converging to a Euclidean 1-monopole.

The idea is simple; to take a sequence of 2-monopoles which approximate a pair of widely separated 1-monopoles, with the spacing increasing with m, and to translate so that one of the "centres" is fixed at the origin and the other disappears to infinity. Recall that the effect on the matrices of translations in $\mathbb{H}^3$ was given in §3.8.3.

Let $\tilde{A}^{(m)} = (\tilde{\gamma}^{(m)}, \tilde{\beta}^{(m)}, \tilde{v}^{(m)})$ be the monopole solution of §5.3.4 with modulus κ_m such that $\kappa'_m = \frac{1}{m}$. Let $A^{(m)}$ be the translated data

$$\begin{aligned}
\beta^{(m)} &= \tilde{\beta}^{(m)} + \frac{im}{2}\sinh(B_m)I_2 \\
\gamma^{(m)} &= \tilde{\gamma}^{(m)} \\
v^{(m)} &= \tilde{v}^{(m)}
\end{aligned}$$

As m tends to infinity, κ'_m tends to zero, κ_m tends to 1 and K_m tends to infinity. So $\gamma^{(m)}$ behaves as before in the limit and $\alpha^{(m)} \to 0_2$. For large m, the modulus κ_m is close to 1, so (putting $b = m/2$)

$$\beta^{(m)} \sim \frac{m}{2}\begin{pmatrix} O(1/m^2) & 0 \\ 0 & 2i\sinh(B_m) \end{pmatrix}$$

As $m \to \infty$, the top left-hand entry of $\beta^{(m)}$ tends to 0 while the bottom right-hand entry tends to infinity. Thus the Euclidean limit is a 1-monopole situated at the origin. The "other charge" has escaped.

This is an example of a sequence violating the conditions of Theorem 4.2 but which has a limit of lower charge. The sequence clearly does not satisfy condition i) of the theorem, as the functions are not uniformly bounded on any fixed Ω. (As a consequence of this, condition ii) is not satisfied either.) It should be the case that for larger k any number of the charges can escape in this way, and the rest have a limit as a Euclidean monopole.

Thus, geometrically, the reason for condition i) of Theorem 4.2 becomes clear, at least in the case of charge 2. The spectral curve $\mathcal{S} \simeq \mathbb{C}/\Lambda$ is defined by the lattice Λ, generated by $\{1, it\}$. The real number t is given by $t = K'/K$, so the monopoles becoming infinitely separated corresponds to t tending to zero. In this case the lattice "collapses". Recall that the theta divisor is the single point $0 \in \mathbb{C}$ modulo Λ, or in other words the lattice points in $\mathbb{C}$. As the lattice collapses, the shortest distance between lattice points becomes arbitrarily small. On the other hand, the functions α and β can only be extended to meromorphic functions on $\Omega \subset \mathbb{C}$ with at most two poles if Ω contains at most two lattice points. So the first part of condition i) is precisely that the "monopole spacing" remains finite.

5.6 Spectral curve

Murray and Singer ([24], p5, equation (1.19)) have a formula for the spectral curve $\mathcal{S}$ of a monopole in terms of "discrete Nahm" data, which is essentially the condition for a real line in $\mathbb{P}^3$ (the fibre over a point of S^4) to be a jumping line (in the sense described in [4], Proposition 3.5). The formula in terms of Braam-Austin matrices is:

$$\det(\beta\eta\zeta + (\gamma^*\gamma + \beta_+^*\beta_+)\eta + \zeta + \beta^*) = 0 \tag{5.15}$$

where η and ζ are inhomogeneous coordinates on $Z = \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \bar{\Delta}$. Recall (from (2.1)) that the correspondence with $\mathbb{H}^3$ is given by

$$t^2\eta = (1 - u\eta)(\zeta + \bar{u}) \tag{5.16}$$

Murray and Singer prove that (5.15) is independent of j. Substituting the $k = 2$ solution into the formula and simplifying, the spectral curve of this solution is given by $\mathcal{F}(\eta, \zeta) = 0$ where

$$\mathcal{F}(\eta, \zeta) = b^2\Theta\eta^2\zeta^2 + \eta^2 b^4(\Theta^2 + 1 - \Omega) + \zeta^2 + 2b^2\eta\zeta + b^2\Theta$$

In the last part of this chapter, the behaviour of the spectral curve under symmetries of $\mathbb{H}^3$ will be examined in some detail. A formula for the centre will be found and a proof of Proposition 3.7 given. The case $\kappa \to 1$ is studied and the results compared to those of §5.5.2. There is also a comparison with results of Atiyah and Hitchin [7].

5.6.1 Symmetries of S and centring

Note that $\overline{\sigma^*\mathcal{F}} = 0 \Leftrightarrow \mathcal{F} = 0$, so the spectral curve is invariant under the real structure σ. It is also invariant under the involution

$$I : (\eta, \zeta) \mapsto (\lambda^{-1}\zeta, \lambda\eta)$$

where $\lambda = b^2(\Theta^2 + 1 - \Omega)^{1/2}$. This gives a way to calculate the centre. The involution I has a fixed line $\zeta = \lambda\eta$ which via (5.16) corresponds to the point $(u, t) = (0, \lambda^{1/2})$; hence the monopole is centred at this point. If the solution is scaled by setting $b = (\Theta^2 + 1 - \Omega)^{-1/4}$, the monopoles are all centred at the point $(0, 1)$. The factor $(\Theta^2 + 1 - \Omega)^{1/4}$ is $\{\frac{1}{2}\sec(2B_0)\}^{\frac{1}{2}}$ when $\kappa = 0$, which is the factor in the formula for the centre of a symmetric monopole mentioned in §3.8.4. It is now possible to prove Proposition 3.7 of Chapter 3.

5.6.2 Proof of Proposition 3.7

Recall that this was the proposition that the centre coordinates of a 2-monopole are

$$(u, t) = (\frac{1}{2}\mathrm{tr}\beta, \frac{1}{\sqrt{2}}\chi^{\frac{1}{4}})$$

where

$$\begin{aligned}\chi &= (\mathrm{tr}\beta^2)^2 - 2\mathrm{tr}(\gamma^*\gamma(\gamma^*\gamma - 2\beta^*\beta)) + 2(tr\gamma^*\gamma)^2 + \frac{1}{4}(\mathrm{tr}\beta)^4 - (\mathrm{tr}\beta)^2\mathrm{tr}\beta^2 \\ &\quad -2\mathrm{tr}\beta\mathrm{tr}\beta^*\gamma^*\gamma - 2\mathrm{tr}\beta^*\mathrm{tr}\beta\gamma^*\gamma + \mathrm{tr}\beta\mathrm{tr}\beta^*\mathrm{tr}\gamma^*\gamma\end{aligned}$$

It was remarked in Chapter 3 that it is enough to show the formula holds for a general centred, oriented monopole and transforms the right way under isometries of $\mathbb{H}^3$. It has now been shown that in the special case $u = 0$ the centre is given by

$$\chi^0 = 2b^4(\Theta^2 + 1 - \Omega) \tag{5.17}$$

Ward [29] has pointed out that the conserved quantities of the Braam-Austin equations for charge 2 are

$$\mathrm{tr}\beta, \quad \mathrm{tr}\beta^*, \quad \mathrm{tr}\beta^2, \quad \mathrm{tr}\beta^{*2}, \quad \mathrm{tr}\gamma^*\gamma, \quad \mathrm{tr}\beta\gamma^*\gamma, \quad \mathrm{tr}\beta^*\gamma^*\gamma, \quad \mathrm{tr}(\gamma^*\gamma(\gamma^*\gamma - 2\beta^*\beta))$$

These are independent of j and gauge invariant. They may be used to express b, Θ and Ω in terms of β and γ:

$$\begin{aligned}2b^2 &= \mathrm{tr}\gamma^*\gamma \\ 2b^4(1+\Omega) &= \mathrm{tr}(\gamma^*\gamma(\gamma^*\gamma - 2\beta^*\beta)) \\ -2b^2\Theta &= \mathrm{tr}\beta^2\end{aligned}$$

Substituting, (5.17) becomes

$$\chi^0 = (\mathrm{tr}\beta^2)^2 - 2\mathrm{tr}(\gamma^*\gamma(\gamma^*\gamma - 2\beta^*\beta)) + 2(tr\gamma^*\gamma)^2$$

This is a candidate for χ in general, but unfortunately it is not invariant under $\beta \mapsto \beta + \alpha I$. By adding combinations of order 4 (so that the scaling still works) of the conserved quantities which vanish when $\mathrm{tr}\beta = 0$, it is possible to arrive at a formula which is invariant under translating β. This is the formula of the proposition, which was obtained by the ugly method of adding combinations of the invariant quantities and solving for their coefficients using the invariance condition. □

5.6.3 Inversions

The symmetries of the spectral curve will now be studied in more detail. Hitchin, Manton and Murray [18] look at the effect on $\mathcal{S}$ of inversions of $\mathbb{H}^3$ to find Euclidean monopoles with symmetry. This approach is also useful to clarify the geometry of hyperbolic 2-monopoles. An inversion I of $\mathbb{H}^3$ changes the orientation, so it induces an antiholomorphic map (which will also be denoted I) on

Z. Since $\mathcal{S}$ is invariant under the real structure, the method of [18] is to study the holomorphic map $\sigma^* I$, the pullback of I under the real structure. Then the condition for $\mathcal{S}$ to be invariant under I is

$$\mathcal{F} \circ I \circ \sigma(\eta, \zeta) = 0 \Leftrightarrow \mathcal{F}(\eta, \zeta) = 0$$

so that all points of $\mathcal{S}$ also lie on the pullback of $\mathcal{S}$ under $I \circ \sigma$. All 2-monopoles are invariant under some inversion; for example

Proposition 5.4. *Let h be the geodesic hemisphere in $\mathbb{H}^3$ which is symmetric about the t-axis and contains the point $(0,1)$. If the 2-monopole solution of §5.3.4 is scaled by $b = (\Theta^2 + 1 - \Omega)^{-1/4}$, then it is invariant under inversion in h.*

Proof: Inversion in h is the map

$$(u, t) \mapsto \left(\frac{u}{|u|^2 + t^2}, \frac{t}{|u|^2 + t^2} \right)$$

in hyperbolic 3-space. This is transferred to Z via (5.16), by finding (η', ζ') such that

$$\left(\frac{t}{|u|^2 + t^2} \right)^2 \eta' = \left(1 - \frac{u\eta'}{|u|^2 + t^2} \right) \left(\zeta' + \frac{\bar{u}}{|u|^2 + t^2} \right)$$

which simplifies to $t^2 \zeta' = (\eta' - \bar{u})(1 + u\zeta)$. Comparing with the complex conjugate of (5.16), the antiholomorphic map induced on Z is $(\eta, \zeta) \mapsto (1/\bar{\eta}, 1/\bar{\zeta})$. The pullback with σ is the holomorphic map

$$I_h : (\eta, \zeta) \mapsto (-\zeta, -\eta)$$

When $b = (\Theta^2 + 1 - \Omega)^{-1/4}$, the spectral curve is

$$\frac{\Theta \eta^2 \zeta^2}{(\Theta^2 + 1 - \Omega)^{1/2}} + \eta^2 + \zeta^2 + \frac{2\eta\zeta}{(\Theta^2 + 1 - \Omega)^{1/2}} + \frac{\Theta}{(\Theta^2 + 1 - \Omega)^{1/2}} = 0$$

which is clearly invariant under I_h. □

The solutions of §5.3.4 are also invariant in this sense under inversions in certain planes perpendicular to the boundary $\{t = 0\}$ of $\mathbb{H}^3$. Inversion in the plane making an angle ψ with the real axis $\{\mathrm{Im}(u) = 0\}$ is given by $(u, t) \mapsto (e^{2i\psi}\bar{u}, t)$, which corresponds to $(\eta, \zeta) \mapsto (e^{-2i\psi}\bar{\eta}, e^{-2i\psi}\bar{\zeta})$. The holomorphic version is

$$I_\psi : (\eta, \zeta) \mapsto \left(-\frac{e^{-2i\psi}}{\zeta}, -\frac{e^{-2i\psi}}{\eta} \right)$$

The monopole is invariant under the inversion if $I_\psi^* \mathcal{F} = 0$ for all points on the original spectral curve $\mathcal{S}$, i.e. if

$$\frac{e^{4i\psi} \Theta \eta^2 \zeta^2}{(\Theta^2 + 1 - \Omega)^{1/2}} + \eta^2 + \zeta^2 + \frac{2\eta\zeta}{(\Theta^2 + 1 - \Omega)^{1/2}} + \frac{e^{-4i\psi}\Theta}{(\Theta^2 + 1 - \Omega)^{1/2}} = 0$$

for all points (η, ζ) of $\mathcal{S}$. Thus of all the I_ψ, $\mathcal{S}$ is only invariant under inversion in $\{\mathrm{Im}(u) = 0\}$ (when $\psi = 0$) or in $\{\mathrm{Re}(u) = 0\}$ (when $\psi = \pi/2$) unless $\Theta = 0$. This is precisely the symmetric monopole, and is invariant under all the I_ψ.

These calculations show that all the monopoles of §5.3.4 with the given scaling are invariant under inversion in h, $\{\mathrm{Im}(u) = 0\}$ and $\{\mathrm{Re}(u) = 0\}$. The only point fixed by of all of these is the centre, $(0, 1)$.

Symmetry group

It was shown in ([7], p58) that the symmetry group of a Euclidean 2-monopole is $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times S^1$ in the axially symmetric case. This is also true for hyperbolic monopoles. The inversions discussed above can be composed in pairs to make orientation-preserving maps. Still keeping the fixed scaling for b, the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the two maps

$$\begin{aligned} (\eta, \zeta) &\mapsto (-\eta, -\zeta) \\ (\eta, \zeta) &\mapsto \left(\frac{1}{\eta}, \frac{1}{\zeta}\right) \end{aligned}$$

The first corresponds to a rotation by π about the t-axis and the second to a rotation by π about the geodesic $c_1 := h \cap \{\mathrm{Re}(u) = 0\}$. Their composition corresponds to rotation by π about $c_2 := h \cap \{\mathrm{Im}(u) = 0\}$. In the axially symmetric case there are also all other rotations about the t-axis as described earlier. The geodesics c_1 and c_2 are fixed (setwise) by the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the only fixed point is the centre $c_1 \cap c_2 = (0, 1)$.

5.6.4 Wide separation

Again, it is possible to check that $\mathcal{S}$ has the correct behaviour as $\kappa \to 1$. Using the approximations

$$\begin{aligned} \Omega &\sim -2\sinh^2 B \\ \Theta &\sim \sinh^2 B \end{aligned}$$

of §5.5.2,

$$b = \frac{1}{(\Theta^2 + 1 - \Omega)^{1/4}} \sim \frac{1}{\cosh(B)}$$

Putting this into the equation for $\mathcal{S}$ (and using the fact that $\tanh(B) \sim 1$ and $1/\cosh(B) \sim 0$ for large K) the spectral curve is approximated by

$$\eta^2\zeta^2 + \eta^2 + \zeta^2 + 1 = 0$$

The left hand side factorises as $(1 - i\eta)(1 + i\eta)(\zeta + i)(\zeta - i)$, so $\mathcal{S}$ is approximately the product of two real lines $(1 - i\eta)(\zeta - i) = 0$ and $(1 + i\eta)(\zeta + i) = 0$. These are the points $(i, 0)$ and $(-i, 0)$ on $\partial\mathbb{H}^3$. The monopoles of this particular solution are in some sense constrained to the two fixed geodesics c_1 and c_2, so it is to be expected that the limit as $\kappa \to 1$ will be either the pair $(\pm i, 0)$ or $(\pm 1, 0)$ on $\partial\mathbb{H}^3$. Some choices were made during the solution regarding the sign of Θ; if these had been made differently the limit would have been the second pair of points. This is the phenomenon noted by Atiyah and Hitchin ([7], p63) where two of the axes may swap roles. Using Atiyah and Hitchin's terminology, the t-axis is the "main axis" of these monopoles, c_1 is the "Higgs axis" and c_2 is the "third axis". Changing the sign of Θ exchanges c_1 and c_2, which happens as the solution passes through the symmetric monopole $\Theta = 0$.

Unscaled version

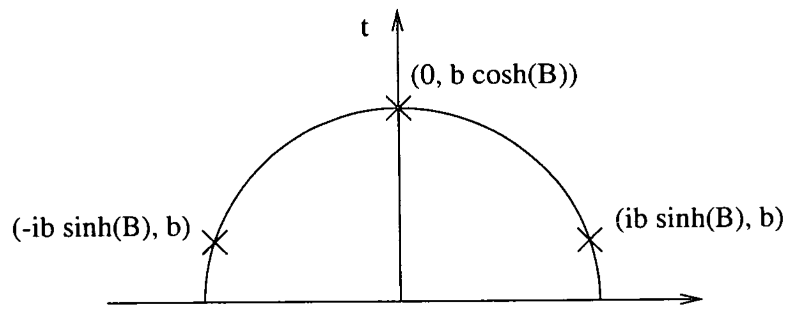
If b is not scaled, the result for widely separated monopoles agrees with that obtained in §5.5.2. The spectral curve calculation shows that the centre of the monopole is at the point

$$(u, t) = (0, b(\Theta^2 + 1 - \Omega)^{1/4}) \sim (0, b\cosh(B))$$

for κ close to 1, whereas the calculation via the matrices demonstrates that for κ close to 1 the monopole is an approximate superposition of 1-monopoles at the points

$$(u, t) = (\pm ib\sinh(B), b)$$

It is a short calculation to check that these points and the centre all lie on the intersection of the hemisphere $|u|^2 + t^2 = b^2\cosh^2(B)$ with the plane $\{\mathrm{Re}(u) = 0\}$, which is a geodesic symmetric about the t-axis (see picture).



So the centre of mass of the two widely separated 1-monopoles is the same as the centre calculated from the spectral curve.

Appendix A

Equivariant cohomology

This appendix contains the equivariant cohomology proof of Proposition 1.4, that $\kappa = 2mk$, and the proof of Lemma 3.2. The aim is not to develop the theory of equivariant cohomology, so only some basic definitions will be given. The details may be found at the beginning of [5].

A.1 Background

Let G be a topological group. Then EG denotes a contractible space (not necessarily unique) on which G acts freely, with quotient $BG = EG/G$, so that

$$G \to EG \to BG$$

is a fibration. The space BG is also not necessarily unique, but for fixed G all possible BG are homotopy equivalent.

Example: EG and BG when $G = S^1$. The case $G = S^1$ is the only one which will be needed. To find ES^1 note that S^1 acts freely on odd-dimensional spheres. To obtain the contractible space ES^1, take the direct limit of all these spheres, which can be included inside one another. Taking the quotient, BS^1 is $\mathbb{C}P^\infty$:

$$\begin{array}{ccccccccc} S^3 & \hookrightarrow & S^5 & \hookrightarrow \cdots \hookrightarrow & S^{2n+1} & \hookrightarrow \cdots & & & ES^1 \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathbb{C}P^1 & \hookrightarrow & \mathbb{C}P^2 & \hookrightarrow \cdots \hookrightarrow & \mathbb{C}P^n & \hookrightarrow \cdots & & & BS^1 \end{array}$$

The first vertical arrow is the Hopf fibration, and the others are higher dimensional analogues of it.

Note that if G acts on X, then G acts freely on $X \times_G EG$, where if X and Y are spaces with G-actions, the space $X \times_G Y$ is the quotient of $X \times Y$ by the relations

$$(x.g, y) = (x, g.y) \qquad g \in G$$

These spaces fit into a "mixing diagram":

$$\begin{array}{ccccc} X & \longleftarrow & X \times EG & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ X/G & \xleftarrow{\sigma} & X \times_G EG & \xrightarrow{\pi} & BG \end{array}$$

The map π is a fibration; $X \times_G EG$ is a fibre bundle over BG with fibre X. On the other hand, σ is not usually a fibration. If G_x is the stabiliser of x in G then the fibre of σ is

$$\sigma^{-1}(xG) = EG/G_x = BG_x$$

If G acts freely on an orbit xG then $\sigma^{-1}(xG) = EG$, whereas if x is a fixed point of the action, $\sigma^{-1}(xG) = BG$. If G acts freely, σ is a fibration.

Definition 5. *The* **equivariant cohomology** $H^*_G(X)$ *of a space* X *with a* G*-action is defined by*

$$H^*_G(X) = H^*(X \times_G EG)$$

so that $H^*_G(X)$ *is a* $H^*(BG)$*-module, where* H^* *denotes ordinary cohomology.*

This is a good way to define cohomology for quotients X/G when the action of G is not free. If the action is free, then $H^*_G(X) = H^*(X/G \times EG) = H^*(X/G)$ since EG is contractible. Using these definitions, most of the constructions of cohomology extend to the equivariant case, including the theory of characteristic classes.

A.2 Proof of Proposition 1.4

Recall that $\mathbb{A} = A + \Phi d\theta$, where $\mathbb{A}$ is an instanton on $S^4 \setminus S^2$ with invariant κ, and (A, Φ) is a monopole on $\mathbb{H}^3$ with charge k and mass m. The proposition $\kappa = 2mk$ will be proved here in the case $m \in \mathbb{Z}$, so that the instanton $\mathbb{A}$ extends to the whole of S^4. The connections $\mathbb{A}$ and A are defined on vector bundles E and $\hat{E}$ over S^4 and $\mathbb{H}^3$ respectively, where E is the pullback of $\hat{E}$ to S^4. The second Chern number of E is κ and the restriction $\hat{E}|_{S^2_\infty} \cong E|_{S^2_\infty}$ of $\hat{E}$ to the sphere at infinity splits as $L \oplus L^*$, with $\int_{S^2_\infty} c_1(L) = k$, and the circle action defined by Φ has weight m when restricted to L.

Let x be a generator of $H^2(S^2_\infty)$ and u a generator of $H^2(BS^1) = H^2(\mathbb{C}P^1) = H^2_{S^1}$, where $H^*_{S^1}$ is the equivariant cohomology of a point. Then the equivariant first Chern class of L is given by

$$(c_1)_{S^1}(L) = kx - mu$$

To see this, first note that S^2_∞ is the fixed point set of the action, so

$$H^*_{S^1}(S^2_\infty) = H^*(S^2_\infty \times BS^1) = \bigoplus H^*(S^2_\infty) \otimes H^*(BS^1)$$

by Kunneth's formula. Therefore $H^2_{S^1}(S^2_\infty) = H^2(S^2_\infty) \oplus H^2(BS^1)$, and the "ordinary" and equivariant parts can be treated separately. The ordinary part of $(c_1)_{S^1}(L)$ is the first Chern class $c_1(L) = kx$. The equivariant part is given by the circle action. Because the orientation of S^2_∞ is opposite to the one it inherits from S^4, this part will be $-mu$. The claim that $(c_1)_{S^1}(L) = kx - mu$ follows.

The Chern class of the dual L^* is the negative of this, and so

$$(c_2)_{S^1}(\hat{E}|_{S^2_\infty}) = -(kx - mu)^2 = 2mkxu - m^2u^2$$

because $\hat{E}$ is a direct sum on S^2_∞ and $x^2 = 0$ (x is the volume form on S^2_∞).

Integration formula

As in the mixing diagram, let π^X be the fibration $\pi^X : X \times_G EG \to BG$, so that

$$\pi^X_* : H^*_G(X) \to H^*_G$$

is given by integration over X; and let i be the inclusion

$$i : S^2_\infty \hookrightarrow S^4$$

Then

$$(c_2)_{S^1}(\hat{E}|_{S^2_\infty}) = i^*(c_2)_{S^1}(E)$$

Atiyah and Bott ([5] p9) deduce the integration formula

$$\pi^{S^4}_* \varphi = \pi^{S^2_\infty}_* \left(\frac{i^*\varphi}{e}\right)$$

where e is the equivariant Euler class of $\nu_{S^2_\infty}$, the normal bundle of S^2_∞ in S^4. Since S^1 acts on the (trivial) normal bundle with weight 1, in this case e is just the generator u. Applying the integration formula to the restricted second Chern class provides a way to calculate the equivariant second Chern number of E:

$$\begin{aligned}
\int_{S^4} (c_2)_{S^1}(E) &= \pi^{S^4}_* (c_2)_{S^1}(E) = \pi^{S^2_\infty}_* \left(\frac{i^*(c_2)_{S^1}(E)}{e}\right) \\
&= \int_{S^2_\infty} \frac{2mkxu - m^2u^2}{u} \\
&= \int_{S^2_\infty} (2mkx - m^2u) = 2mk
\end{aligned}$$

Because $H^*_{S^1}(S^4) \cong H^*(S^4) \otimes H^*(\mathbb{C}P^\infty)$ (see §A.2.1), $H^4_{S^1}(S^4)$ is generated by y and u^2, where y is a volume form on S^4. So $(c_2)_{S^1}(E) = \kappa y + \lambda u^2$. Integrating over S^4, the u^2 term vanishes and

$$\int_{S^4} (c_2)_{S^1}(E) = \kappa = \int_{S^4} c_2(E)$$

So the equivariant second Chern number of E is equal to the ordinary one, κ. Thus $\kappa = 2mk$, which proves the proposition. □

A.2.1 $H^*_{S^1}(S^4)$

This section contains a brief explanation of why

$$H^*_{S^1}(S^4) \cong H^*(S^4) \otimes H^*(\mathbb{C}P^\infty)$$

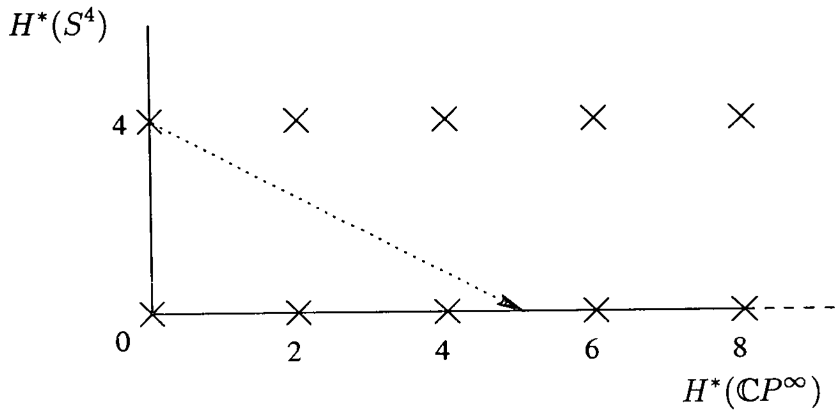
which was needed above. The map π of the mixing diagram is a fibration, which in this case is

$$S^4 \to S^4 \times_{S^1} ES^1 \xrightarrow{\pi} \mathbb{C}P^\infty$$

There is a corresponding Serre spectral sequence, with $E_r \Rightarrow H^*(S^4 \times_{S^1} ES^1)$. The E_2 term is

$$E_2^{p,q} = H^p(S^4) \otimes H^q(\mathbb{C}P^\infty)$$

since $\mathbb{C}P^\infty$ is simply connected and $H^*(S^4)$ is finite dimensional (see [10], page 170). A diagram for the E_2 term is



since $H^*(S^4)$ has one generator in dimension 4 and one in dimension 0, and $H^*(\mathbb{C}P^\infty)$ has a generator in each even (real) dimension. But all the differentials are zero, since

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

The only one which maps the 4th row to the baseline is d_5, one of which is marked in the diagram, and there is nothing in the image of those differentials d_5 which start somewhere nonzero. Thus $E_2^{p,q} = E_\infty^{p,q}$ and

$$H^*_{S^1}(S^4) = H^*(S^4 \times_{S^1} ES^1) \cong H^*(S^4) \otimes H^*(\mathbb{C}P^\infty).$$

A.3 Proof of Lemma 3.2

The methods of equivariant cohomology are needed for the proof of

Lemma 3.2. *Let $\mathbb{Z}_o$ denote the odd integers. The vector spaces V and W, written as complex representation spaces, are given by:*

$$V = \begin{cases} \mathbb{C}^{k+1}_{-2m} \oplus \mathbb{C}^{2k}_{-2m+2} \oplus \mathbb{C}^{2k}_{-2m+4} \cdots \oplus \mathbb{C}^{2k}_{0} \oplus \mathbb{C}^{2k}_{2} \oplus \ldots \oplus \mathbb{C}^{k+1}_{2m} & : m \in \mathbb{Z} \\ \mathbb{C}^{k+1}_{-M} \oplus \mathbb{C}^{2k}_{-M+2} \oplus \mathbb{C}^{2k}_{-M+4} \cdots \oplus \mathbb{C}^{2k}_{-1} \oplus \mathbb{C}^{2k}_{1} \oplus \ldots \oplus \mathbb{C}^{k+1}_{M} & : 2m = M \in \mathbb{Z}_o \end{cases}$$

and

$$W = \begin{cases} \mathbb{C}^{k}_{-2m+1} \oplus \mathbb{C}^{k}_{-2m+3} \oplus \ldots \oplus \mathbb{C}^{k}_{-1} \oplus \mathbb{C}^{k}_{1} \oplus \ldots \oplus \mathbb{C}^{k}_{2m-1} & : m \in \mathbb{Z} \\ \mathbb{C}^{k}_{-M+1} \oplus \mathbb{C}^{k}_{-M+3} \oplus \ldots \oplus \mathbb{C}^{k}_{0} \oplus \mathbb{C}^{k}_{2} \oplus \ldots \oplus \mathbb{C}^{k}_{M-1} & : 2m = M \in \mathbb{Z}_o \end{cases}$$

where $\mathbb{C}^s_r$ is the s-dimensional complex vector space on which $\tilde{S}^1$ acts with weight r.

To prove the lemma, some further theory is needed, and then the proof is the character calculation below. Remember that the circle acting now is the double cover $\tilde{S}^1$ of S^1.

Braam and Austin have the formulae

$$\begin{aligned} V &= \ker\{\mathcal{D}_{\mathbb{A}} : \Gamma(S^4, E \otimes S_- \otimes S_-) \to \Gamma(S^4, E \otimes S_+ \otimes S_-)\} \\ W &= (\ker\{\mathcal{D}_{\mathbb{A}} : \Gamma(S^4, E \otimes S_-) \to \Gamma(S^4, E \otimes S_+)\})^* \end{aligned}$$

for V and W in terms of the spin bundles $S_\pm$ on S^4, the bundle E and $\mathcal{D}_{\mathbb{A}}$, the adjoint of the Dirac operator with coefficients in $S_- \otimes E$ and E (see [12], p813).

Note: Because $\mathbb{A}$ is an SU(2)-connection and $SU(2) = Sp(1)$, the bundles E and S_- each carry a quaternionic structure σ such that $\sigma^2 = -1$. These induce the real structure on W and the quaternionic structure on V.

A.3.1 Character calculation

Following [11], using the fact that V is the kernel of a Dirac operator and applying the equivariant version of the Atiyah-Singer index theorem,

$$\mathrm{ch}_{\tilde{S}^1}(V) = \int_{S^2_\infty} e_{\tilde{S}^1}(N)^{-1} \mathrm{ch}_{\tilde{S}^1}(E \otimes S_-)|_{S^2_\infty} . \hat{A}_{\tilde{S}^1}(S^4)|_{S^2_\infty} \tag{A.1}$$

where S^2_∞ is the fixed point set of the action, N is the normal bundle of S^2_∞ in S^4 and $\mathrm{ch}_{\tilde{S}^1}$ is the equivariant Chern character. The $\hat{A}$ genus of a bundle E is given by

$$\hat{A}(E) = \prod_{i=1}^{n} \frac{y_i/2}{\sinh y_i/2}$$

where E has total Pontrjagin class

$$p(E) = 1 + p_1(E) + \ldots = \prod_{i=1}^{n}(1 + y_i^2)$$

For a manifold X, $\hat{A}(X)$ means $\hat{A}(TX)$, and the equivariant genus $\hat{A}_{\tilde{S}^1}$ is calculated in the same way but using the equivariant Pontrjagin classes $p_{\tilde{S}^1}(E)$. It remains to calculate the various parts of (A.1) and substitute back (noting that the Pontrjagin classes of S^4 vanish except for p_1).

Using (as before) the notation x for a generator of $H^2(S^2_\infty, \mathbb{Z})$ (so that $x^2 = 0$) and $\tilde{u}$ for a generator of $H^2(B\tilde{S^1})$, the equivariant Euler class $e_{\tilde{S}^1}(N)$ is $2\tilde{u}$, because N is trivial and the circle acts on it with weight 1 (so the double cover $\tilde{S}^1$ acts with weight 2).

Calculation of $\hat{A}_{\tilde{S}^1}(S^4)|_{S^2_\infty}$:

The class $\hat{A}_{\tilde{S}^1}(S^4)|_{S^2_\infty}$ involves knowing $(p_1)_{\tilde{S}^1}(TS^4)|_{S^2_\infty} = (p_1)_{\tilde{S}^1}(TS^4|_{S^2_\infty})$, which may be calculated using the following facts:

$$\begin{aligned} (p_1)_{\tilde{S}^1} &= (c_1)^2_{\tilde{S}^1} - 2(c_2)_{\tilde{S}^1} && \text{(A.2)} \\ TS^4|_{S^2_\infty} &\cong TS^2_\infty \oplus N && \text{(A.3)} \end{aligned}$$

Because $c_1(T^*S^2_\infty) = 2x$ (and there is no equivariant part), $(c_1)_{\tilde{S}^1}(TS^2_\infty) = -2x$. So, using (A.3),

$$\begin{aligned} (c)_{\tilde{S}^1}(TS^4|_{S^2_\infty}) &= (c)_{\tilde{S}^1}(TS^2_\infty).(c)_{\tilde{S}^1}(N) = (1-2x)(1+2\tilde{u}) \\ &= 1 - 2x + 2\tilde{u} - 4\tilde{u}x \end{aligned}$$

(take the parts in dimensions 2 and 4)

$$\Rightarrow \begin{cases} (c_1)_{\tilde{S}^1}(TS^4|_{S^2_\infty}) = 2\tilde{u} - 2x \\ (c_2)_{\tilde{S}^1}(TS^4|_{S^2_\infty}) = -4\tilde{u}x \end{cases}$$

Then by (A.2), $(p_1)_{\tilde{S}^1}(TS^4|_{S^2_\infty}) = 4\tilde{u}^2$.

Hence

$$\hat{A}_{\tilde{S}^1}(S^4)|_{S^2_\infty} = \frac{\tilde{u}}{\sinh \tilde{u}}$$

Calculation of $\mathbf{ch}_{\tilde{S}^1}(E \otimes S_-)|_{S^2_\infty}$:

This will be done by first working out the Chern characters of the two parts, since $\text{ch}_{\tilde{S}^1}(E \otimes S_-)|_{S^2_\infty} = \text{ch}_{\tilde{S}^1}(E)|_{S^2_\infty}.\text{ch}_{\tilde{S}^1}(S_-)|_{S^2_\infty}$. The Chern character of $S_+|_{S^2_\infty}$ may be calculated using the result from ([11], p437):

$$S_+ \otimes S_+|_{S^2_\infty} = \mathbb{C} \oplus (\Lambda^{1,0}(S^2_\infty) \otimes_{\mathbb{C}} N^*) \oplus (\Lambda^{0,1}(S^2_\infty) \otimes_{\mathbb{C}} N) \oplus \mathbb{C}$$

so that, substituting results already calculated,

$$\begin{aligned} \mathrm{ch}_{\tilde{S}^1}(S_+ \otimes S_+)|_{S^2_\infty} &= 2 + e^{2x-2\tilde{u}} + e^{-2x+2\tilde{u}} = (e^{x-\tilde{u}} + e^{-x+\tilde{u}})^2 \\ &\Rightarrow \mathrm{ch}_{\tilde{S}^1}(S_+|_{S^2_\infty}) = e^{x-\tilde{u}} + e^{x+\tilde{u}} = 2\cosh\tilde{u} - 2x\sinh\tilde{u} \end{aligned}$$

because $x^2 = 0$. This also calculates $\mathrm{ch}_{\tilde{S}^1}(S_-)|_{S^2_\infty} = -\mathrm{ch}_{\tilde{S}^1}(S_+)|_{S^2_\infty}$.

The Chern character of $E|_{S^2_\infty}$ may be calculated easily, using the splitting $E|_{S^2_\infty} = L \oplus L^*$ and the additive property of the Chern character:

$$\begin{aligned} \mathrm{ch}_{\tilde{S}^1}(E)|_{S^2_\infty} &= \mathrm{ch}_{\tilde{S}^1}(L) + \mathrm{ch}_{\tilde{S}^1}(L^*) = e^{kx-2m\tilde{u}} + e^{-kx+2m\tilde{u}} \\ &= 2\cosh 2m\tilde{u} - 2kx\sinh 2m\tilde{u} \end{aligned}$$

Putting these results together,

$$\mathrm{ch}_{\tilde{S}^1}(E \otimes S_-)|_{S^2_\infty} = -4\cosh 2m\tilde{u}\cosh\tilde{u} + 4x\cosh 2m\tilde{u}\sinh\tilde{u} + 4kx\cosh\tilde{u}\sinh 2m\tilde{u}$$

Back to the index formula

Putting all this back into the RHS of (A.1) and integrating gives

$$\mathrm{ch}_{\tilde{S}^1}(V) = 2\cosh 2m\tilde{u} + \frac{2k\cosh\tilde{u}\sinh 2m\tilde{u}}{\sinh\tilde{u}} \tag{A.4}$$

because integrating over S^2_∞ the terms not containing x vanish and the integral of x over S^2_∞ is 1. It is not obvious how to write out a formula for (A.4) as a polynomial in $\tilde{u}$, but it isn't too hard to multiply it out in specific cases and to see from that what the formula will be. Repeated use of the hyperbolic double angle formulae produces the useful identities (for $t \in \mathbb{Z}_{>0}$):

$$\begin{aligned} \sinh 2tx &= \sum_{r=1}^{t} \binom{2t}{2r-1} \sinh^{2t-2r+1} x \cosh^{2r-1} x \\ \sinh(2t+1)x &= \sum_{r=1}^{t+1} \binom{2t+1}{2r-1} \sinh^{2r-1} x \cosh^{2t-2r+2} x \\ \cosh 2tx &= \sum_{r=0}^{t} \binom{2t}{2r} \sinh^{2t-2r} x \cosh^{2r} x \\ \cosh(2t+1)x &= \sum_{r=0}^{t} \binom{2t+1}{2r+1} \sinh^{2t-2r} x \cosh^{2r+1} x \end{aligned}$$

which allow the first part of the calculation to be done in general. When $m \in \mathbb{Z}$,

$$\begin{aligned} \sinh 2m\tilde{u} &= 2m\sinh^{2m-1}\tilde{u}\cosh\tilde{u} + \binom{2m}{3}\sinh^{2m-3}\tilde{u}\cosh^3\tilde{u} + \dots \\ &+ \binom{2m}{2m-3}\sinh^3\tilde{u}\cosh^{2m-3}\tilde{u} + 2m\sinh\tilde{u}\cosh^{2m-1}\tilde{u} \end{aligned}$$

This has a factor of $\sinh \tilde{u}$, so substituting into the expression for $\mathrm{ch}_{\tilde{S}^1}(V)$ and writing sinh and cosh in terms of exponentials,

$$\begin{aligned}\mathrm{ch}_{\tilde{S}^1}(V) &= e^{2m\tilde{u}} + e^{-2m\tilde{u}} + \frac{k}{2^{2m}}(e^{\tilde{u}} + e^{-\tilde{u}})^2 \{2m(e^{\tilde{u}} - e^{-\tilde{u}})^{2m-2} \\ &+ \binom{2m}{3}(e^{\tilde{u}} - e^{-\tilde{u}})^{2m-4}(e^{\tilde{u}} + e^{-\tilde{u}})^2 + \ldots \\ &+ \binom{2m}{2m-3}(e^{\tilde{u}} - e^{-\tilde{u}})^2(e^{\tilde{u}} + e^{-\tilde{u}})^{2m-4} + 2m(e^{\tilde{u}} + e^{-\tilde{u}})^{2m-2}\}\end{aligned}$$

When $m = 1$ this is just

$$\begin{aligned}\mathrm{ch}_{\tilde{S}^1}(V) &= e^{2\tilde{u}} + e^{-2\tilde{u}} + \frac{k}{4}(e^{2\tilde{u}} + e^{-2\tilde{u}} + 2)(2+2) \\ &= (k+1)e^{-2\tilde{u}} + 2k + (k+1)e^{2\tilde{u}}\end{aligned}$$

which corresponds to the representation $\mathbb{C}_{-2}^{k+1} \oplus \mathbb{C}_0^{2k} \oplus \mathbb{C}_2^{k+1}$.

When $m = 2$, (A.4) becomes

$$\begin{aligned}\mathrm{ch}_{\tilde{S}^1}(V) &= e^{4\tilde{u}} + e^{-4\tilde{u}} + \frac{k}{2^4}(e^{\tilde{u}} + e^{-\tilde{u}})^2(8(e^{\tilde{u}} - e^{-\tilde{u}})^2 + 8(e^{\tilde{u}} + e^{-\tilde{u}})^2) \\ &= (k+1)e^{-4\tilde{u}} + 2ke^{-2\tilde{u}} + 2k + 2ke^{2\tilde{u}} + (k+1)e^{4\tilde{u}}\end{aligned}$$

i.e., the representation is

$$\mathbb{C}_{-4}^{k+1} \oplus \mathbb{C}_{-2}^{2k} \oplus \mathbb{C}_0^{2k} \oplus \mathbb{C}_2^{2k} \oplus \mathbb{C}_4^{k+1}$$

In general, for $m \in \mathbb{Z}$, this pattern continues, and V is as given in Lemma 3.2. Similarly for $m = M/2$ where $M \in \mathbb{Z}$ is odd, the calculation yields the result of the lemma, expanding $\sinh M\tilde{u}$ in terms of $\sinh \tilde{u}$ and $\cosh \tilde{u}$. The calculation for W is almost the same, starting from

$$\mathrm{ch}_{\tilde{S}^1}(W^*) = \int_{S^2_\infty} e_{\tilde{S}^1}(N)^{-1}\mathrm{ch}_{\tilde{S}^1}(E)|_{S^2_\infty}.\hat{A}_{\tilde{S}^1}(S^4)|_{S^2_\infty}$$

This completes the proof of Lemma 3.2. □

Bibliography

[1] Abramowitz M and Stegun I A (eds): Handbook of Mathematical Functions; Dover 1972

[2] Ahlfors L V : Complex analysis; 3rd ed, McGraw-Hill, 1979; [Chapter 5, theorems 1 and 12]

[3] Atiyah M F : The Geometry of Yang Mills Fields; Fermi Lectures, Scuola Normale Pisa (1979)

[4] Atiyah M F : Magnetic monopoles in hyperbolic spaces; in Proceedings of Bombay Colloquium 1984 on vector bundles in algebraic varieties (OUP) 1–34, (1987).

[5] Atiyah M F and Bott R : The moment map and equivariant cohomology; Topology **23** (1984) 1–28

[6] Atiyah M F, Drinfeld V G, Hitchin N J, Manin Yu I : Construction of Instantons; Phys. Lett. **65A** (1978) 185–187

[7] Atiyah M F and Hitchin N J : The Geometry and Dynamics of Magnetic Monopoles; Princeton University Press 1988

[8] Besson G, Courtois G, Gallot S : Entropies et rigidités des espaces localement symétriques de courbure strictement négative; Geometric and Functional Analysis **5** (1995) 731–799

[9] Bott R and Chern SS : Hermitian Vector bundles and the equidistribution of the zeroes of their holomorphic sections, §2; Acta Math. **114** (1965); p71

[10] Bott R and Tu L W : Differential Forms in Algebraic Topology, Springer-Verlag 1982

[11] Braam P J : Magnetic monopoles on 3-manifolds; J. Diff. Geom. **30** (1989) 425–464

[12] Braam P J and Austin D M : Boundary values of hyperbolic monopoles; Nonlinearity **3** (1990) 809–823

[13] Donaldson S K : Nahm's Equations and the Classification of Monopoles; Comm. Math. Phys. **96** (1984) 387–407

[14] Donaldson S K and Kronheimer P B : The Geometry of Four-Manifolds; OUP 1990; §3.3

[15] Groisser D : Integrality of the monopole number in SU(2) Yang-Mills-Higgs theory on $\mathbb{R}^3$* ; Comm. Math. Phys. **93** (1984) 367–378

[16] Hitchin N J : Monopoles and Geodesics; Comm. Math. Phys. **83** (1982) 579–602

[17] Hitchin N J : On the Construction of Monopoles; Comm. Math. Phys. **89** (1983) 145–190

[18] Hitchin N J, Manton N S and Murray M K: Symmetric monopoles; Nonlinearity **8** (1995) 661–692; §4

[19] Hurtubise J : The Asymptotic Higgs Field of a Monopole; Comm. Math. Phys. **97** (1985) 381–389

[20] Jarvis S and Norbury P : Zero and infinite curvature limits of hyperbolic monopoles; Bull. LMS **29** (1997) 737–744

[21] Kreyszig E : Advanced Engineering Mathematics; 7th ed, Wiley 1993; §18.3

[22] Munari A : DPhil thesis, University of Oxford 1993

[23] Murray M K and Singer M A : Spectral curves of non-integral hyperbolic monopoles; Nonlinearity **9** (1996) 973–997

[24] Murray M K and Singer M A : preprint, Edinburgh 1998

[25] Nahm W : The construction of all self-dual multimonopoles by the ADHM method; in Monopoles in Quantum Field Theory, Proceedings of the Monopole meeting, Trieste December 1981 (eds: N S Craigie, P Goddard, W Nahm) World Scientific, Singapore 1982

[26] Sibner L M and Sibner R J : Classification of Singular Sobolev Connections by their Holonomy; Comm. Math. Phys. **144** (1992) 337–350; [Theorem 5.2]

[27] Sutcliffe P M : BPS Monopoles; review article to appear in the International Journal of Modern Physics A; [p10]

[28] Uhlenbeck K : Removable singularities in Yang-Mills fields; Comm. Math. Phys. **83** 11–29; [Corollary 4.3 p24]

[29] Ward R : preprint, Durham 1998