Stochastic analysis and partial differential equations: theory and numerics

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or professional qualification.

William John Trenberth

Abstract

This thesis is concerned with problems at the interface of stochastic analysis and partial differential equations (PDEs). In particular, we focus on two different classes of problem: the transport of measures under the flow of nonlinear PDEs and the global well-posedness of singular stochastic PDEs. First we present a work, joint with J. Forlano, studying the transport of Gaussian measures on periodic functions under the flow of the fractional nonlinear Schrödinger equation on the 1-dimensional torus. In particular, we prove that Gaussian measures are quasi-invariant under this flow. We present another work, joint with P. Sosoe and T. Xiao, demonstrating that, in the 3-dimensional setting, Gaussian measures on periodic functions are quasi-invariant under the flow of nonlinear wave equations with polynomial nonlinearity. With regard to singular stochastic PDEs, under certain conditions on the coefficients, we prove the global well-posedness of the renormalised stochastic complex Ginzburg-Landau equation on the 2-dimensional torus. To conclude this thesis, from a numerical perspective we study the transport of Gaussian measures, under the flow of nonlinear PDEs, by way of Monte-Carlo simulation. More specifically, for several PDEs, we generate a large number of solutions, with initial data sampled from a Gaussian measure, and then examine statistical properties of the ensemble of solutions. These simulations illustrate the equations and problems studied in this thesis and give insight into conjectures beyond current theoretical techniques.

Lay summary

Partial differential equations (PDEs) are some of the most studied mathematical objects. PDEs are important because they are the language many important physical laws are written in; Maxwell's equations for electromagnetism and the Navier-Stokes equations for fluids being two such examples. Studying PDEs from a mathematical perspective leads to precise understanding of the different ways in which our world operates through these equations.

Stochastic analysis is another active area of mathematics. It is concerned with the study of functions that can only be made sense of probabilistically, that are often very rough. Stochastic analysis is key to understanding the evolution of rough, random things we encounter in the world. For example, the evolution of the price of a financial security or the motion of a particle suspended in liquid.

This thesis is concerned with topics at the intersection of these two areas of mathematics.

For example, one of the most commonly studied problems in PDEs is the initial value problem. That is, we start with some initial function and we want to find out how the function changes over time, under the influence of a PDE. What about if we give the PDE a random initial function? What, statistically, can we say about solutions to the PDE, starting from random initial data? Chapters 2 and 3 investigate topics related to this.

Another way to introduce randomness to PDEs is to start from a fixed, non random, initial condition and then kick the PDE randomly at each time. This is essentially what stochastic PDEs (SPDEs) are. SPDEs often have much rougher, more jagged, solutions than regular PDEs. Sometimes solutions of SPDEs can be so rough, that the equation doesn't make sense. One then has to modify the equation in a way so it makes sense. We call SPDEs of this type singular SPDEs. These are studied in Chapter 4 of this thesis.

To conclude, we discuss some Monte-Carlo simulations we performed. Essentially, we generated many random solutions to certain PDEs and then calculated statistical properties, such as means and variances, of the ensemble of solutions. These numerical simulations stand in contrast to the theoretical methods used in previous chapters and give another way of looking at some of the topics studied in this thesis.

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Notation

In this thesis, unless otherwise stated, we use the following notation.

$\mathbb{Z}$: the set of integers
$\mathbb{R}$: the set of real numbers
$\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, the 1-dimensional torus
$\mathcal{F}f(n) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-in\cdot x} f(x)\,dx$, for $n \in \mathbb{Z}^d$, the nth Fourier coefficient of f
$\widehat{f}(n) := \mathcal{F}f(n)$, the nth Fourier coefficient of f
$f_n := \mathcal{F}f(n)$, the nth Fourier coefficient of f
$\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$, the Japanese bracket
$A \lesssim B$: $A \leq CB$ where C is an unspecified constant whose exact value is unimportant
$\mathbf{P}_{\leq N}$: the sharp Fourier truncation operator onto frequencies less than N
δ_j : the smooth Fourier restriction operator onto frequencies approximately 2^j

Chapter 1

Introduction

In this thesis, we study problems at the intersection of the fields of stochastic analysis and partial differential equations. We study the fractional nonlinear Schrödinger equation (FNLS),

$$\begin{cases} i\partial_t u + (-\partial_x^2)^\alpha u = \pm\lambda |u|^2 u, \\ u|_{t=0} = u_0 \end{cases} \quad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.0.1}$$

the nonlinear wave equation (NLW),

$$\begin{cases} \partial_t^2 u - \Delta u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, v_0) \end{cases} \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}, \tag{1.0.2}$$

the stochastic complex Ginzburg-Landau equation (SCGL),

$$\begin{cases} \partial_t u = (a_1 + ia_2)\Delta u + (b_1 + ib_2)u - (c_1 + ic_2)|u|^{2m-2}u + \sqrt{2\gamma}\xi \\ u|_{t=0} = u_0 \end{cases} \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}, \tag{1.0.3}$$

and the dispersion generalised BBM equation (gBBM),

$$\begin{cases} \partial_t u + \partial_t (-\partial_x^2)^{\frac{\gamma}{2}} u + \partial_x u + \lambda \partial_x (u^2) = 0 \\ u|_{t=0} = u_0 \end{cases} \quad (x,t) \in \mathbb{T} \times \mathbb{R}. \tag{1.0.4}$$

We theoretically study the transport of Gaussian measures under the flow of dispersive PDEs for FNLS (1.0.1) in Chapter 2 and NLW (1.0.2) in Chapter 3. In Chapter 4 we study singular SPDEs, focusing on SCGL (1.0.3). In Chapter 5, we perform numerical simulations studying the transport of Gaussian measures under the flow of FNLS (1.0.1) and gBBM (1.0.4).

For the rest of this introductory chapter, we give introductions to the problems studied in this thesis, give a summary of other works in the literature, and discuss how the results of this thesis fit in with these works.

We note that Chapter 2, Chapter 3 and Chapter 4 of this thesis are based on works the author has contributed to. In particular, Chapter 2 is based on the paper [30], which is a collaboration with J. Forlano. Chapter 3 is based on [89], a collaboration with P. Sosoe and T. Xiao. Chapter 4 is based on the sole authored work [92].

1.1 Quasi-invariant measures for dispersive PDEs

In Chapters 2 and 3 of this thesis, we study quasi-invariant measures for dispersive PDEs. In particular, we focus on two models; FNLS (1.0.1) and NLW (1.0.2).

We study *Gaussian measures* supported on distributions. For the purposes of this thesis, a Gaussian measure is the law of a distribution valued random variable of the form

$$\omega \mapsto u_\omega(x) = \sum_{n\in\mathbb{Z}^d} c_n g_n(\omega) e^{in\cdot x} \tag{1.1.1}$$

where $\{c_n\}_{n\in\mathbb{Z}^d}$ is some sequence of constants and $\{g_n\}_{n\in\mathbb{Z}^d}$ is some sequence of standard identically distributed complex-valued Gaussian random variables with $\text{Var}(g_n) = 2$. For a more detailed discussion on Gaussian measures, we refer the reader to [21].

In studying quasi-invariant Gaussian measures for PDEs, we view a PDE, and an associated space of functions, say X, as a dynamical system. That is, we associate to the PDE the flow $\Phi : \mathbb{R} \times X \to X$, defined by $\Phi(t)u_0 = u(t)$, where $u(t)$ is the solution to the PDE, with initial data $u_0 \in X$.

The concept of quasi-invariance can be defined as follows. Given a measure space (X, μ), and a flow $\Phi : \mathbb{R} \times X \to X$ on X, we say that μ is quasi-invaraint under the flow Φ, if for each $t \in \mathbb{R}$, the measures μ and $\Phi(t)_\#\mu$ are mutually absolutely continuous. Here $\Phi(t)_\#\mu$ is the push forward measure defined by

$$\Phi(t)_\#\mu(A) = \mu(\{\Phi(-t, x) : x \in A\})$$

for measurable $A \subset X$.

The existence of a quasi-invariant measure is a delicate property of the flow of a PDE. Indeed, quasi-invariance does not hold for many well known PDEs and Gaussian measures. For example, consider a flow, Φ, induced by a heat type equation on some space of functions containing C^∞. Heat equations usually have some instantaneous smoothing property and so any measure μ, with the property that $\mu(C^\infty) = 0$, cannot be quasi-invariant under the flow of Φ. This is because if one starts with a set of positive measure, the flow instantly shrinks this set to a set contained in C^∞, and so to a set of measure 0.

In contrast to heat type equations, dispersive equations tend not to have smoothing properties and so the question of quasi-invariance is not easily dismissed. For linear dispersive equations, one often expects Gaussian measures to be quasi-invariant. Indeed, on the Fourier side, one can usually write a linear dispersive PDE in the form

$$\partial_t u_n = i\psi(n) u_n. \tag{1.1.2}$$

For example, for the linear Schrödinger equation, $i\partial_t u = \Delta u$ we have $\psi(n) = |n|^2$. The push forward, under the flow of (1.1.2), measure of a Gaussian measure given by (1.1.1) is the measure which is the law of

$$\omega \mapsto \sum_{n\in\mathbb{Z}^d} e^{i\psi(n)} c_n g_n(\omega) e^{in\cdot x}. \tag{1.1.3}$$

If g_n is rotationally symmetric, that is $e^{i\theta} g_n \overset{\text{law}}{=} g_n$ for all $\theta \in \mathbb{R}$, for all $n \in \mathbb{Z}^d$, then the law of (1.1.1) is identical to the law of (1.1.3). Hence in fact one often expects Gaussian measures to be *invariant* under the flow of linear dispersive PDEs.

What about nonlinear dispersive equations? Starting with the work of Bourgain, [9], methods have been developed in the literature to prove the invariance of Gibbs measures under the flow of dispersive PDEs. This automatically implies quasi-invariance for Gaussian measures that are mutually absolutely continuous to invariant Gibbs measures.

This raises the question: can quasi-invariance be proven for Gaussian measures under the flow of nonlinear dispersive PDEs when there is no mutually absolutely continuous Gibbs measure? The answer to this question is yes. We will give an overview of methods used to prove results in this direction in the next subsection.

We believe quasi-invariant measures for dispersive PDEs is an interesting topic of study in its own right, but it also has interesting applications to the growth of Sobolev norms, see [96, Remark 7.4], and to persistence of regularity, see [78, pg. 6].

Since Chapters 2 and 3 are both concerned with quasi-invariant Gaussian measures for dispersive PDEs, we merge the introductions to these chapters into this, single section, Section 1.1. In the remainder of this section we will first give a brief introduction to the methods used to prove quasi-invariance in the literature. We then discuss the quasi-invariance results we prove for FNLS and NLW.

1.1.1 Methods in the Literature

The earliest work we know of studying quasi-invariance of Gaussian measures, is the work of Cameron and Martin [14], dating to 1940, studying the quasi-invariance of Gaussian measures under linear transformations. Later in 1974, Ramer [84] studied the quasi-invariance of Gaussian measures under general nonlinear transformations in the context of abstract Weiner spaces. In the setting of d-dimensional Hamiltonian PDEs, the hypotheses in these works can be interpreted as requiring a $(d+\varepsilon)$-degree of smoothing on the nonlinear part of the flow generated by a given PDE, see [96].

Later, in 1983, Cruzeiro [20] proved another abstract result showing that Gaussian measures are quasi-invariant under the flow induced by a general equation of the form $\partial_t u = X(u)$, if an exponential integrability condition on X is satisfied.

Recently, in 2015, Tzvetkov [96] introduced a general methodology, for proving the quasi-invariance of Gaussian measures under the flow of nonlinear Hamiltonian PDEs, which does not appeal to Ramer [84] or Cruzeiro [20] (although Tzvetkov says his method is of a similar sprit to Cruzeiro [20], see [96, pg. 4]). Tzvetkov [96] spawned interest in, and was the genesis of many, methods related to the quasi-invariance of Gaussian measures under the flow of dispersive PDEs. The results proven in Chapter 2 and Chapter 3 of this thesis are additions to this recent program attempting to understand the role dispersion has on the transport properties of Gaussian measures under the flow of nonlinear dispersive PDEs. See also [40, 75, 76, 78, 79, 82] for more works in this program.

We now go over the methods recently developed to study the quasi-invariance of Gaussian measures for nonlinear dispersive PDEs. For simplicity, we will describe the methods as applied to the Gaussian measure μ_s given by

$$\omega \in \Omega \longmapsto u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}, \tag{1.1.4}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables, i.e $\mathrm{Var}(g_n) = 2$. Formally we have,

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|^2_{H^s(\mathbb{T}^d)}}\, du.$$

Here for $s \geq 0$, $H^s(\mathbb{T}^d)$ is the L^2 based Sobolev space defined as the completion of $C^\infty(\mathbb{T}^d)$ under the norm

$$\|u\|^2_{H^s(\mathbb{T}^d)} = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |u_n|^2,$$

see also Appendix A. We note that the measure (1.1.4) is supported on $H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)\backslash H^{s-\frac{d}{2}}(\mathbb{T}^d)$ almost surely for any $\varepsilon > 0$.

In order to rigorously justify the methods below, it is often necessary to consider a suitably truncated version of the equation being studied (see for example (2.2.2) and (2.2.5)).

• **Method 1: ('Ramer's argument')** The first method is to directly verify the hypothesis of Ramer's result [84] on the quasi-invariance of Gaussian measures under general nonlinear transformations. In the d-dimensional context, this essentially reduces to demonstrating a $(d + \varepsilon)$-degree of smoothing for the nonlinear part of the flow. This approach was first applied in [75, 76, 96]. Typically, we find that this method is too restrictive.

• **Method 2:** Introduced by Tzvetkov [96], the second method involves both nonlinear PDE techniques and stochastic analysis. We give an overview of the method here; see also [77]. Let Ψ be the flow of a given PDE. Given a measurable set $A \subset H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$ satisfying $\mu_s(A) = 0$, one aims to show

$$\mu_s(\Psi(t)(A)) = Z_s^{-1} \int_{\Phi(t)(A)} e^{-\frac{1}{2}\|u\|^2_{H^s(\mathbb{T}^d)}} du = 0 \text{ for all } t \in \mathbb{R}, \tag{1.1.5}$$

by obtaining a differential inequality of the form

$$\frac{d}{dt}\mu_s(\Psi(t)(A)) \leq Cp^\beta \{\mu_s(\Psi(t)(A))\}^{1-\frac{1}{p}}, \tag{1.1.6}$$

where $0 \leq \beta \leq 1$ and $p < \infty$. Then, applying Yudovich's argument (or a variant of, see [78, 96]) to (1.1.6) implies (1.1.5) for small times. The argument can then be iterated to give (1.1.5) for all times. Thus, matters reduce to obtaining (1.1.6). By Liouville's theorem and the bijectivity of the flow Ψ, one has the following 'change of variables' formula (see for example Lemma 2.4.6):

$$\mu_s(\Psi(t)(A)) = Z_s^{-1} \int_A e^{-\frac{1}{2}\|\Psi(t)u\|^2_{H^s}} du \text{ for all } t \in \mathbb{R}.$$

Taking a time derivative, evaluating at a fixed $t_0 \in \mathbb{R}$ and using the group property of the flow, $\Psi(t + t_0) = \Psi(t)\Psi(t_0)$, one obtains

$$\begin{aligned} \frac{d}{dt}\mu_s(\Psi(t)(A))\bigg|_{t=t_0} &= -\frac{1}{2}Z_s^{-1} \int_{\Psi(t_0)(A)} \frac{d}{dt}\Big(\|\Psi(t)(u)\|^2_{H^s}\Big) e^{-\frac{1}{2}\|\Psi(t)u\|^2_{H^s}}\bigg|_{t=0} du \\ &\leq C \left\| \frac{d}{dt}\Big(\|\Phi(t)(u)\|^2_{H^s}\Big)\bigg|_{t=0} \right\|_{L^p(\mu_s)} \{\mu_s(\Psi(t_0)(A))\}^{1-\frac{1}{p}}. \end{aligned} \tag{1.1.7}$$

Thus, one is lead to the following energy estimate (with smoothing):

$$\frac{d}{dt}\Big(\|\Psi(t)(u)\|^2_{H^s}\Big)\bigg|_{t=0} \leq C(\|u\|_B)\|u\|^\theta_X, \tag{1.1.8}$$

where $\theta \leq 2$. Here, there is freedom in the choice of the X-norm above provided it captures the regularity of the random distribution (1.1.4) almost surely; for example, we may take $X = H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$, the Bessel potential space $W^{s-\frac{d}{2}-\varepsilon,\infty}(\mathbb{T}^d)$ or the Fourier-Lebesgue space $\mathcal{F}L^{s-\varepsilon,\infty}(\mathbb{T}^d)$ (which we choose in (1.1.20)). On the other hand, we must choose the weaker B-norm so that it can be controlled in terms of conserved quantities of the given PDE. The inequality (1.1.6) then follows from (1.1.7), (1.1.8) and estimates on higher moments of Gaussian random random variables (see for example (2.4.16)). Indeed, the reduction to time $t = 0$ in the above analysis allows us to use stochastic tools on the explicit random distribution (1.1.4).

For the gBBM (1.0.4), Tzvetkov [96] was able to obtain a suitable energy estimate of the form (1.1.8). Unfortunately, for general dispersive PDE such an estimate does not always hold. The key modification is to instead consider a 'modified energy' E of the form

$$E(u) = \|u\|^2_{H^s} + \text{correction terms}$$

and obtain the following estimate (with smoothing):

$$\left|\frac{d}{dt}E(\Psi(t)u)\Big|_{t=0}\right| \leq C(\|u\|_B)\|u\|_X^\theta. \tag{1.1.9}$$

Now, provided we show the measure ρ_s with density

$$d\rho_s = Z_s^{-1}e^{-E(u)}du \tag{1.1.10}$$

can be normalised into a probability measure, we can repeat the above argument for ρ_s and conclude the quasi-invariance of ρ_s under the flow Ψ. Finally, we appeal to the mutual absolute continuity of ρ_s and μ_s to conclude the quasi-invariance for μ_s under the flow Ψ. To summarise, Method 2 requires two crucial ingredients: (i) a modified energy estimate of the form (1.1.9) and (ii) the construction of the weighted Gaussian measure ρ_s in (1.1.10).

• **Method 3:** Introduced by Planchon, Tzvetkov and Visciglia [82], who studied the quasi-invariance of Gaussian measures under the flow of the (super-)quintic NLS on $\mathbb{T}$, the third approach is similar in spirit to Method 2. The fundamental feature of this method is the use of deterministic growth bounds on the $H^{s-\frac{d}{2}-\varepsilon}$-norm of solutions (see for example Proposition 2.3.6), so that the analysis can be restricted to a closed ball $B_R \subset H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T})$. The benefit of this idea over Method 2 is that a softer energy estimate is required,

$$\left|\frac{d}{dt}E(\Psi(t)u)\right| \leq C\big(1 + \|\Psi(t)u\|^k_{H^{s-\frac{d}{2}-\varepsilon}}\big) \tag{1.1.11}$$

for some $k \geq 0$. Through the use of (1.1.11) and the growth bound on solutions coming from standard local well-posedness theory, for any $A \subset B_R$ with $\mu_s(A) = 0$, we have

$$\frac{d}{dt}\tilde{\rho}_s(\Psi(t)(A)) \leq C(R,T)^k\tilde{\rho}_s(\Psi(t)(A)) \quad \text{for all } t \in [0,T],$$

where $\tilde{\rho}_s$ is the measure with density

$$d\tilde{\rho}_s = e^{-E(u)}du.$$

Gronwall's inequality and soft arguments then imply $\tilde{\rho}_s(\Psi(t)(A)) = 0$ and hence $\mu_s(\Psi(t)(A)) = 0$ for every $A \subset B_R$. We then take $R \to \infty$ to obtain the quasi-invariance of μ_s under the flow Ψ. Further differences to Method 2 are: (i) there is no need to reduce to time $t = 0$ to access stochastic tools and (ii) we do not need to normalise the measure $\tilde{\rho}_s$. Notice that the above argument works even when the flow is only locally-in-time well-defined, which leads to a local-in-time quasi-invariance result as in Theorem 1.1.3 and Theorem 1.1.5. See [10, p. 28] where it is asked if it is possible to prove local-in-time quasi-invariance results for Hamiltonian PDEs.

• **Method 4:** This approach combines aspects of Methods 2 and 3 and was introduced by Gunaratnam, Oh, Tzvetkov and Weber [40] for handling the cubic nonlinear wave equation (NLW) on $\mathbb{T}^3$. Namely, by arguing locally within $H^{s-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$ and returning the analysis to time $t = 0$, one needs an even softer energy estimate taking the following form:

$$\left| \frac{d}{dt} E(\Psi(t)v) \Big|_{t=0} \right| \leq C(\|v\|_{H^{s-\frac{d}{2}-\varepsilon}}) \|v\|_X^\theta, \tag{1.1.12}$$

for $\theta \leq 2$ and where the X-norm may be chosen as in Method 2. Analogously to Method 2, we must also construct a suitable auxiliary probability measure adapted to the modified energy.

• **Method 5:** This new method, released on arXiv early 2020, by Debussche and Tsutsumi [24] is completely different to the previous methods. It relies on an explicit formula for the Radon-Nikodym derivative of the transported Gaussian measure with respect to the original measure.

In this thesis we do not apply Method 1 as we find it tends to give significantly worse results than other methods, see the discussion in Section 1.1.2. Further, in some cases it is not possible to apply Method 1 due to insufficient smoothing on the nonlinear part of a PDE. This is the case for the 3-dimensional NLW studied in Chapter 3. We call Method 2, Method 3 and Method 4 'Energy Methods' because, at their core, they involve establishing a PDE energy estimate. We do not use Method 2 in this thesis as we have found it to be completely superseded, in terms of regularity restrictions, by Method 3 and Method 4. This is partly due to the fact that Method 2 requires an energy estimate involving a conversed quantity. Requiring a conserved quantity to be finite in the support of the Gaussian measures being studied can immediately put an unsatisfactory regularity condition on the Gaussian measures being studied. See the discussion in Section 1.1.2 for more information in the context of FNLS (1.0.1). Finally, we do not apply Method 5 as it was submitted to the arXiv after the work in this thesis was completed. However, the simplicity of this method seems very promising. It is possible that this method could lead to simplified proofs, of increased generality, of the results in Chapter 2 and Chapter 3 of this thesis.

In the following subsections, we will explain how we use the methods described above to prove quasi-invariance results for FNLS and NLW.

1.1.2 Main results for FNLS

- The results we present here, and later prove in Chapter 2, are based on the joint work with J. Forlano:
  J. Forlano, W. J. Trenberth, *On the transport of Gaussian measures under*

the one-dimensional fractional nonlinear Schrödinger equations, Ann. Inst. H. Poincare Anal. Non Lineaire 36 (2019), no. 7, 1987–2025.

We study the cubic fractional nonlinear Schrödinger equation (FNLS) on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$,

$$\begin{cases} i\partial_t u + (-\partial_x^2)^\alpha u = \pm|u|^2 u, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1.13}$$

Here $u : \mathbb{T} \times \mathbb{R} \longmapsto \mathbb{C}$ is the unknown function and for $\alpha > 0$, $(-\partial_x^2)^\alpha$ is the Fourier multiplier operator defined by $((-\partial_x^2)^\alpha f)\widehat{\ }(n) := |n|^{2\alpha}\widehat{f}(n)$, $n \in \mathbb{Z}$. FNLS (1.1.13) is FNLS (1.0.1) with $\lambda = 1$. We say (1.1.13) is defocusing when the sign on the nonlinearity is negative and focusing when the sign on the nonlinearity is positive.

Before we can state the quasi-invariance results we prove for FNLS (1.1.13), we need to review the well-posedness literature for this equation. This informs us of the spaces where FNLS (1.1.13) is known to have a well-defined flow. The well-posedness theory of FNLS (1.1.13) in the L^2-based Sobolev spaces $H^\sigma(\mathbb{T})$ (see Appendix A for the definition of these spaces) crucially depends upon the strength of the dispersion; namely, if $\alpha \geq 1$ or $\frac{1}{2} < \alpha < 1$. In [8], Bourgain proved local well-posedness of NLS in $L^2(\mathbb{T})$, which immediately extends to global well-posedness in $L^2(\mathbb{T})$ as a consequence of mass conservation:

$$M(u)(t) = \int_{\mathbb{T}} |u(x,t)|^2 \, dx = M(u)(0) \quad \text{for all } t \in \mathbb{R}. \tag{1.1.14}$$

A persistence-of-regularity argument then implies global well-posedness of NLS in $H^\sigma(\mathbb{T})$ for any $\sigma \geq 0$. This result is sharp in the sense that NLS is ill-posed if $\sigma < 0$. More precisely, the solution map[1] $\Phi : u_0 \in H^\sigma(\mathbb{T}) \mapsto u \in C([-T,T]; H^\sigma(\mathbb{T}))$, if it even exists in view of the non-existence of solutions in [41], is discontinuous [62] (see also [13, 17, 51, 71, 80]). In [76, Appendix A], the 4NLS was shown to be globally well-posed in $H^\sigma(\mathbb{T})$ for any $\sigma \geq 0$. In [30, Appendix B], the work Chapter 2 is based on, this global well-posedness result is extended for all $\alpha > 1$.

The well-posedness situation for FNLS (1.1.13) is somewhat less complete in the setting $\frac{1}{2} < \alpha < 1$. In [16], Cho, Hwang, Kwon and Lee proved local well-posedness of FNLS (1.1.13) in $H^\sigma(\mathbb{T})$ for $\sigma \geq \frac{1-\alpha}{2}$ by a contraction mapping argument; see also [25]. Following the argument in [76, Appendix A], we can show the solution map for FNLS (1.1.13) fails to be locally uniformly continuous in $H^\sigma(\mathbb{T})$ for any $\sigma < 0$. Thus we can not construct solutions below $L^2(\mathbb{T})$ using a contraction mapping argument. See also [17].

As for global well-posedness, the flow of FNLS (1.1.13) conserves the energy $H(u)$; that is,

$$H(u)(t) = \frac{1}{2}\int_{\mathbb{T}} \left|(-\partial_x^2)^{\frac{\alpha}{2}} u(x,t)\right|^2 \, dx \pm \frac{1}{4}\int_{\mathbb{T}} |u(x,t)|^4 \, dx = H(u)(0). \tag{1.1.15}$$

[1] For clarity of presentation, we neglect to explicitly show the dependence of the solution map Φ on α. Unless otherwise stated, the precise value of α will be clear from the context.

In the defocusing case, the energy controls the H^α-norm and hence energy conservation can be used to globalise (in time) all solutions with regularity at or above the energy space, that is, for when $\sigma \geq \alpha$. This result also holds in the focusing case, as the H^α-norm can be controlled in terms of both the energy and the mass by using the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \lesssim \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{L^2}^{\frac{1}{\alpha}} \|u\|_{L^2}^{4-\frac{1}{\alpha}}$$

for $\alpha \geq \frac{1}{4}$. Below the energy space $H^\alpha(\mathbb{T})$, global well-posedness of FNLS (1.1.13) (for both defocusing and focusing nonlinearities) in $H^\sigma(\mathbb{T})$ was obtained for $\sigma > \frac{10\alpha+1}{12}$ [25] by using the high-low frequency decomposition of Bourgain [12].

In summary, the flow of FNLS (1.1.13) is well-defined under the following conditions:

Proposition 1.1.1 (Well-posedness of the cubic FNLS (1.1.13) in $H^\sigma(\mathbb{T})$ [8, 16, 25, 76])**.**

(i) *Let* $\alpha \geq 1$. *Then, the cubic FNLS* (1.1.13) *is globally well-posed in* $H^\sigma(\mathbb{T})$ *for* $\sigma \geq 0$.

(ii) *Let* $\frac{1}{2} < \alpha < 1$. *Then, the cubic FNLS* (1.1.13) *is locally well-posed in* $H^\sigma(\mathbb{T})$ *for* $\sigma \geq \frac{1-\alpha}{2}$. *Moreover, the cubic FNLS* (1.1.13) *is globally well-posed for* $\sigma > \frac{10\alpha+1}{12}$.

In the following, we make no distinction between the defocusing or focusing nature of (1.1.13) and henceforth we assume that (1.1.13) is defocusing. For future use, we define $\Phi(t) : u_0 \in H^\sigma(\mathbb{T}) \mapsto u(t) \in H^\sigma(\mathbb{T})$ to be the solution map of FNLS (1.1.13) (when it exists) at time t.

Our goal in Chapter 2 is to study the transport property of Gaussian measures on periodic functions under the flow of FNLS (1.1.13). Given $s \in \mathbb{R}$, we define the Gaussian measure μ_s to be the induced probability measure under the map[2]:

$$\omega \in \Omega \longmapsto u^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}, \tag{1.1.16}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables, i.e $\mathrm{Var}(g_n) = 2$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Formally, μ_s has density

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n\in\mathbb{Z}} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2} d\widehat{u}_n.$$

A computation shows that the random distribution (1.1.16) belongs to $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ and not to $H^{s-\frac{1}{2}}(\mathbb{T})$ almost surely. It follows that the Gaussian measure μ_s is supported on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{s-\frac{1}{2}}(\mathbb{T})$ for any $\varepsilon > 0$. Thus, in order to discuss the transport

[2]From now on, we drop the factor 2π as it plays no role in our analysis.

property of these measures under the flow of (1.1.13), Proposition 1.1.1 restricts us to the range:

$$s > \max\left(\frac{1}{2}, 1 - \frac{\alpha}{2}\right),$$

which ensures there exists well-defined dynamics within the support of μ_s.

The quasi-invariance of Gaussian measures supported on periodic functions under the flow of 4NLS was recently studied by Oh and Tzvetkov [76] and Oh, Sosoe and Tzvetkov [79]. See also the recent work [75] on Schrödinger-type equations. Our main goal is to extend these quasi-invariance results to more general values of dispersion α. Thus, in this direction we establish the following:

Theorem 1.1.2. *Let $s \in \mathbb{R}$ and $\alpha > \frac{1}{2}$ be such that*

(i) $s > \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right)$ *if $\alpha \geq 1$, or*

(ii) $s > \frac{10\alpha+7}{12}$ *if $\frac{1}{2} < \alpha < 1$.*

Then, the Gaussian measure μ_s is quasi-invariant under the flow of the cubic FNLS (1.1.13)*. More precisely, given any measurable set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ satisfying $\mu_s(A) = 0$, we have $\mu_s(\Phi(-t)(A)) = 0$ for every $t \in \mathbb{R}$.*

Our proof of quasi-invariance of μ_s in the case $\frac{1}{2} < \alpha < 1$ also holds for any

$$\min\left(1, \frac{11}{6} - \alpha\right) < s \leq \frac{10\alpha + 7}{12}. \tag{1.1.17}$$

The restriction in Theorem 1.1.2 (ii) is due to a lack of globally well-defined dynamics within the support of μ_s for s satisfying (1.1.17) (see Proposition 1.1.1 (ii)). However, our arguments in Chapter 2 allow us to recover the following *local-in-time quasi-invariance* result:

Theorem 1.1.3 (Local-in-time quasi-invariance)**.** *Let $\frac{1}{2} < \alpha < 1$ and s satisfy* (1.1.17)*. Then, for every $R > 0$, there exists $T > 0$ such that for every measurable*

$$A \subset \{u \in H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) : \|u\|_{H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})} < R\}$$

satisfying $\mu_s(A) = 0$, we have $\mu_s(\Phi(-t)(A)) = 0$ for every $t \in [-T, T]$.

We note that any lowering of the global well-posedness regularity threshold, to say σ_0, for the FNLS (1.1.13) as stated in Proposition 1.1.1 (ii), will immediately imply a corresponding improvement to Theorem 1.1.2 (ii) and Theorem 1.1.3. That is, we can 'upgrade' from local-in-time quasi-invariance to quasi-invariance as in Theorem 1.1.2, provided

$$\sigma_0 - \frac{1}{2} > \min\left(1, \frac{11}{6} - \alpha\right).$$

This should be contrasted with the local-in-time quasi-invariance result in [82, Theorem 1.5] for the focusing quintic NLS on $\mathbb{T}$. In that setting, a global flow does not exist in view of the presence of finite-time blow-up solutions (see for example [70]). Thus, it is impossible to remove the 'local-in-time' restriction.

Our proof of Theorem 1.1.2 and Theorem 1.1.3 is split into two parts. In the first, we employ the recent argument in [82] (see Method 3 above) to obtain quasi-invariance for all $\alpha > \frac{1}{2}$, for some range of regularities s. We then apply the argument in [40] (see Method 4 above) to improve upon the previous regularity restriction for $\alpha > \frac{5}{6}$.

In the following, we survey how the methods in Section 1.1.1 may be implemented within the context of FNLS (1.1.13).

In our situation of FNLS (1.1.13) with $\alpha \geq 1$, we could demonstrate $(2\alpha - 1)$-degrees of nonlinear smoothing, provided $s > 1$. Hence, Method 1 yields quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1.13), provided that $\alpha > 1$ and $s > 1$. For FNLS (1.1.13) with $\frac{1}{2} < \alpha \leq 1$, a $(1+\varepsilon)$-degree of nonlinear smoothing is not expected [27] and thus we do not know at this point if Ramer's argument can be applied in this case. In addition, Methods 2, 3 and 4 usually give lower, and so better, regularity restrictions compared to using Ramer's argument (e.g. [79]). We found this to indeed be the case for FNLS (1.1.13). For this reason, we do not present Method 1 here.

For application to FNLS (1.1.13), it turns out that Method 3 gives an improved result in terms of regularity over Method 2. Indeed, in terms of the energy estimate itself, the rigidity in the choice of the B-norm in (1.1.9) leads to far less flexibility compared to the energy estimate (1.1.11) in Method 3. In the regime $\frac{1}{2} < \alpha < 1$, we established an energy estimate of the form (1.1.9) with $B = H^{\alpha}(\mathbb{T})$ and $X = H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$. Thus, energy conservation (1.1.15) immediately places the regularity restriction $s > \alpha + \frac{1}{2}$, in this use of Method 2. This restriction is unnatural since it goes against our intuition that greater dispersion gives a lower regularity threshold. In Chapter 2, we use Method 3 which allows us to remove the restrictions coming from using conservation laws and thus lower the regularity threshold.

We now describe our application of Method 3. The main goals are to establish (i) a suitable modified energy (see (2.3.5)) and (ii) a corresponding energy estimate of the form (1.1.11) (see Proposition 2.3.1). For this purpose, we apply gauge transformations to convert (1.1.13) into a form more amenable to apply the normal form reductions used to define the modified energy (see Sections 2.2 and 2.3). In this approach, the phase function

$$\phi(\overline{n}) = |n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n|^{2\alpha} \tag{1.1.18}$$

naturally arises as the source of dispersion. In order to exploit this for a smoothing benefit, which is required to achieve (ii), we crucially rely on the following lower bound: for $\alpha > \frac{1}{2}$, we have

$$|\phi(\overline{n})| \gtrsim |n - n_1||n - n_3| n_{\max}^{2\alpha-2} \qquad \text{when} \qquad n = n_1 - n_2 + n_3.$$

Here, $n_{\max} := \max(|n_1|, |n_2|, |n_3|, |n|)$. This lower bound first appeared in the setting $\frac{1}{2} < \alpha \leq 1$ in [25]. With minor modifications, its proof extends easily to the case $\alpha > 1$; see Lemma 2.1.3 and Appendix B. It can be viewed as a replacement of the explicit factorisations available for the phase function (1.1.18) of NLS

$$\phi(\overline{n}) = n_1^2 - n_2^2 + n_3^2 - n^2 = -2(n - n_1)(n - n_3) \qquad \text{when } n = n_1 - n_2 + n_3,$$

and 4NLS (see [76, Lemma 3.1]). Following the argument in [82], we obtain quasi-invariance of Gaussian measures μ_s under the flow of (1.1.13) for

$$\text{(i) } s > 1, \text{ when } \frac{1}{2} < \alpha \leq 1, \text{ and (ii) } s > s_\alpha, \text{ for some } s_\alpha \leq 1, \text{ when } \alpha > 1. \tag{1.1.19}$$

See (2.3.7) for a precise statement of s_α.

Our next goal is to attempt to lower the regularity restriction from Method 3 by using Method 4. This requires us to construct a suitable weighted Gaussian measure (Section 2.4.2) and establish an effective energy estimate of the form (1.1.12). In establishing the energy estimate, we have some freedom in the choice of the X-norm. One choice is the Hölder-Besov norm as used in [40]. Since we work intimately on the Fourier side, we use the *Fourier-Lebesgue* $X = \mathcal{F}L^{\sigma,\infty}(\mathbb{T})$-norm for $\sigma < s$. Here, given $q \geq 1$ and $s \in \mathbb{R}$, the Fourier-Lebesgue $\mathcal{F}L^{s,q}(\mathbb{T})$-norm is defined by:

$$\|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} := \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^q(\mathbb{Z})}. \tag{1.1.20}$$

It is easy to check that the random distribution in (1.1.16) belongs almost surely to $\mathcal{F}L^{\sigma,\infty}(\mathbb{T})$ for any $\sigma < s$ (Lemma 2.4.5). Moreover, Hölder's inequality implies the embedding

$$H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) \supset \mathcal{F}L^{\sigma,\infty}(\mathbb{T}) \tag{1.1.21}$$

for σ sufficiently close to s. This fact allows us to further relax the energy estimate we obtained in Method 3 (see Proposition 2.4.1). We then follow the argument in [40] to conclude quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1.13) for the following regularities:

$$\begin{aligned} &\text{(i) } \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right) < s \leq 1, \text{ when } \alpha \geq 1 \text{ and} \\ &\text{(ii) } \frac{11}{6} - \alpha < s \leq 1, \text{ when } \frac{5}{6} < \alpha < 1. \end{aligned} \tag{1.1.22}$$

Notice that in (1.1.22), we improve the regularity restriction (1.1.19) we obtained using Method 3 only when $\alpha > \frac{5}{6}$. The reason for this is that our use of the stronger $\mathcal{F}L^{\sigma,\infty}$-norm in the energy estimate for Method 4 (Proposition 2.4.1) yields a regularity gain over the energy estimate in Method 3 (Proposition 2.3.1) provided

$\alpha > \frac{5}{6}$. Furthermore, the upper bound $s \leq 1$ in (1.1.22) is necessary for our construction of the weighted Gaussian measure.

When $\alpha \geq \frac{7}{6}$, the regularity restriction in Theorem 1.1.2 (i) achieves the largest range of $s > \frac{2}{3}$. In particular, when $\alpha = 2$, this improves upon the result in [76] of $s > \frac{3}{4}$. However, as remarked in [75], this same result of $s > \frac{2}{3}$ for 4NLS could be obtained by using Method 1 and an additional novel gauge transformation introduced in that same paper. For FNLS (1.1.13) with $\alpha > 1$ (and large enough), we expect the optimal result $s > \frac{1}{2}$ could be obtained by using a finer modified energy arising from an infinite sequence of normal form reductions. See [79] where this approach led to the optimal result for 4NLS.

We end this subsection with a few possible future research directions.

It would be of interest to study how our approach for the cubic FNLS (1.1.13) may extend to higher order nonlinearities, say, for the quintic nonlinearity $|u|^4u$. For instance, the relevant phase function is now

$$|n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n_4|^{2\alpha} + |n_5|^{2\alpha} - |n|^{2\alpha}, \tag{1.1.23}$$

which is restricted to the hyperplane $n_1 - n_2 + n_3 - n_4 + n_5 = n$. When $\alpha = 1$, there is no factorisation for (1.1.23). Therefore, an appropriate analogue of Lemma 2.1.3 is not obvious (and likewise for Lemma 2.1.2). However, even if such results were proved, the formulation of the modified energies and the appropriate nonlinear estimates would still have to be verified. We note that the method in [82] introduces a modified energy functional which is not derivable from differentiation by parts, and hence their analysis is not based on factorisations of the phase function (1.1.23).

It would also be of interest to determine whether Theorem 1.1.2 holds in the limiting case of $\alpha = \frac{1}{2}$. In this case, FNLS corresponds to the half-wave equation. The half wave equation does not exhibit dispersion and hence we do not expect quasi-invariance.

1.1.3 Main results for NLW

- The results we present here, and later prove in Chapter 3, are based on the joint work with P. Sosoe and T. Xiao (which at the time of writing has been accepted for publication in Differential and Integral Equations subject to minor changes):
  P. Sosoe, W. J. Trenberth, T. Xiao *Quasi-invariance of fractional Gaussian fields nonlinear wave equation with polynomial nonlinearity* arXiv:1906.02257 [math.AP].

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We study the nonlinear wave equation (NLW), on $\mathbb{T}^3$

$$\begin{cases} \partial_t^2 u - \Delta u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, v_0) \end{cases} \tag{1.1.24}$$

where k is a positive, odd integer. This is NLW (1.0.2) with $d = 3$. We can rewrite (1.1.24) as a first order system:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u^k = 0 \\ (u, v)|_{t=0} = (u_0, v_0) \in \vec{H}^\sigma(\mathbb{T}^3) \end{cases} \tag{1.1.25}$$

where

$$\vec{H}^\sigma(\mathbb{T}^3) := H^\sigma(\mathbb{T}^3) \times H^{\sigma-1}(\mathbb{T}^3).$$

The system (1.1.25) is a Hamiltonian system with Hamiltonian

$$E(u, v) := \frac{1}{2} \int_{\mathbb{T}^d} \left(|\nabla u|^2 + v^2 \right) \mathrm{d}x + \frac{1}{k+1} \int_{\mathbb{T}^d} u^{k+1} \, \mathrm{d}x. \tag{1.1.26}$$

Before we discuss the results we prove in Chapter 3, we first give a brief overview of some well-posedness results for NLW. We do not go into detail of the exact range that local and global well-posedness is known, like we did in Section 1.1.2. This is because the regularity we need to prove the results for NLW in Chapter 3 is far greater than that where local and global well-posedness is known. In the 3-dimensional setting, global wellposedness is known for NLW for $k = 3$ and $k = 5$ for certain regularities, at least for $s > \frac{3}{2}$. A pedagogical proof of global well-posedness in the 3-dimensional case appears in [97]. The global well-posedness situation in the 3-dimensional setting stands in contrast to the 2-dimensional setting where NLW is globally well-posed for all odd k, above a certain regularity threshold. The difference in difficulty between the 2 and 3-dimensional settings is due to the scaling critical index

$$s_c = \frac{d}{2} - \frac{2}{k-1}$$

associated to NLW. When $d = 2$, $s_c < 1$ for all k, and so the conservation of the Hamiltonian (1.1.26) can be used to extend local solutions to global solutions. When $d = 3$, $s_c > 1$ for $k > 5$ and so the conservation of the Hamiltonian cannot be directly used.

The goal of Chapter 3 is to study the transport properties of Gaussian measures on periodic functions under the flow of NLW (1.1.25). In particular, we study the transport properties of the Gaussian measure on initial data $\vec{u} = (u, v)$ formally given by

$$\vec{\mu}_s(\mathrm{d}\vec{u}) = Z_s^{-1} e^{-\frac{1}{2}\|(u,v)\|^2_{\vec{H}^{s+1}}} \mathrm{d}\vec{u}. \tag{1.1.27}$$

The expression (1.1.27) can be given meaning as a product measure on the Fourier coefficients of the pair (u, v):

$$\vec{\mu}_s(\mathrm{d}u\mathrm{d}v) = Z_s^{-1} \prod_{n \in \mathbb{Z}^3} e^{-\frac{1}{2}\langle n \rangle^{2(s+1)}|\widehat{u}_n|^2} e^{-\frac{1}{2}\langle n \rangle^{2s}|\widehat{v}_n|^2} \, \mathrm{d}\widehat{u}_n \mathrm{d}\widehat{v}_n.$$

Equivalently, μ_s is the law of the pair of function-valued random variables given by $\omega \mapsto (u^\omega, v^\omega)$, where u^ω, v^ω are the functions on $\mathbb{T}^3$ defined by

$$\begin{aligned} u^\omega(x) &:= \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{s+1}} e^{in\cdot x}, \\ v^\omega(x) &:= \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{s}} e^{in\cdot x}. \end{aligned} \tag{1.1.28}$$

Here, g_n, h_n, $n \in \mathbb{Z}^3$ are standard complex Gaussian random variables, independent except for the conditions:

$$g_n = \bar{g}_{-n}, \quad h_n = \bar{h}_{-n}, n \neq 0, \tag{1.1.29}$$

and g_0, h_0 real-valued. By inspection of the series (1.1.28), it is clear that μ_s is supported on $\vec{H}^\sigma$ for $\sigma < s - \frac{1}{2}$.

The quasi-invariance of Gaussian measures supported on periodic functions under the flow of NLW was recently studied in the 2-dimensional setting, for s an even integer, by Oh and Tzvertkov [78] and in the 3-dimensional setting, but only for $k = 3$, by Gunaratnam, Oh, Tzvetkov and Weber [40]. The main aim of the work in Chapter 3 is to address a number of questions mentioned in the introduction of [40]. In particular, we generalise the result of [40] to non-integer s and to odd integer $k > 3$.

The main quasi-invariance result we prove in Chapter 3 is the following.

Theorem 1.1.4. *Let $s > \frac{5}{2}$. Let $k = 3$ or $k = 5$. Then for any time $t > 0$, the distribution of the solution $\Phi(t)(u, v) = (u(t), v(t))$ of the system* (1.1.25) *is absolutely continuous with respect to the distribution μ_s of the initial data, given by* (1.1.28).

For odd integer $k > 5$, as mentioned above, it is not known if NLW (1.1.25) is globally well-posed. Hence in this case, it is not known if there is even a global flow associated to NLW (1.1.25) for which the measures given by (1.1.28) can be quasi-invaraint under. However, our estimates also yield a local-in-time quasi-invariance result in the spirit of [82, Theorem 1.5] concerning the transport of bounded subsets of $\vec{H}^\sigma$ in settings where global well-posedness is not known.

Theorem 1.1.5. *Let $s > 3$, and $\frac{3}{2} < \sigma < s - \frac{1}{2}$ sufficiently close to $s - \frac{1}{2}$. For each $R > 0$, let*

$$B_R(0; \vec{H}^\sigma) := \{(u, v) : \|(u, v)\|_{\vec{H}^\sigma} < R\}$$

denote the ball of radius R centered at the origin in $\vec{H}^\sigma$. There exists $T = T(R)$ and $C(R) > 0$ such that for $(u_0, v_0) \in B_R(0; \vec{H}^\sigma)$, there is a solution $(u(t), v(t))$ of (1.1.25) *such that*

$$\sup_{[-t,t]\leq T} \|(u, v)(t)\|_{\vec{H}^\sigma} < C(R). \tag{1.1.30}$$

Let A be a Borel subset of $\vec{H}^\sigma$ such that

$$A \subset \{u \in \vec{H}^\sigma : \|u\|_{\vec{H}^\sigma} < R\}$$

and $\vec{\mu}_s(A) = 0$. Then

$$\vec{\mu}_s(\{u(t) : u(t) \textit{ solution of } (1.1.24) \textit{ with initial data } u_0 \in A\}) = 0$$

for all $t \in [-T, T]$.

The proofs of Theorem 1.1.4 and Theorem 1.1.5 in Chapter 3 are more straightforward than the proofs in Chapter 2. We only use Method 4, described in Section 1.1.1. This method was developed for proving quasi-invariance results for NLW in the 3-dimensional setting in [40]. As we are trying to extend the results of [40], Method 4 is the most natural method to use. We recall that there are two main parts to Method 4: the construction of a measure mutually absolutely continuous with the measure given by (1.1.28) and the the establishment of an energy estimate.

As in [40, 78], we study an energy with a formally "infinite" term. More precisely, the energy we consider contains the term

$$\int_{\mathbb{T}^3} (D^s u_N)^2 u_N^{k-1} \tag{1.1.31}$$

where $u_N = \mathbf{P}_{\leq N} u$ and u is given by (1.1.28). This term does not converge as the truncation is removed. Indeed,

$$\mathbb{E}_{\vec{\mu}_s}[|D^s u_N|^2] \sim N.$$

Hence this term requires renormalisation. As in [40,78] we use a renormalisation based on an argument akin to Nelson's argument for the construction of $P(\phi)_2$ quantum fields [67]. We essentially replace the term (1.1.31) by

$$\int_{\mathbb{T}^3} ((D^s u_N)^2 - \mathbb{E}_{\vec{\mu}_s}[|D^s u_N|^2]) u_N^{k-1}. \tag{1.1.32}$$

This type of renormalisation is commonly known as Wick ordering.

One of the major advancements of Gunaratnam-Oh-Tzvetkov-Weber [40] was the application of ideas in [4], which describes a way to construct measures using a variational approach, in the construction of the measure required by Method 4. The somewhat surprising aspect of [40] is that although the weighted measure involves the quartic quantity ($k = 3$ in (1.1.31))

$$\int_{\mathbb{T}^3} (D^s u_N)^2 u_N^2,$$

no renormalisation other than the Wick type subtraction (1.1.32) is necessary, in contrast to the Φ_3^4 model. In Chapter 3, we also find this to be the case but for super quartic quantities. We find this equally surprising.

Using the renormalisation discussed above and following calculations similar to that in [40], we are lead to consider the energy

$$\mathscr{E}_{s,N}(u,v) := \frac{1}{2}\bigg(\int_{\mathbb{T}^3}(D^s v_N)^2 + \int_{\mathbb{T}^3}(D^{s+1}u_N)^2 + \bigg(\int_{\mathbb{T}^3} u_N\bigg)^2\bigg) - \frac{k}{2}\int_{\mathbb{T}^3} Q_{s,N}(u_N)u_N^{k-1},$$

where $Q_{s,N}$ is given by (3.1.5). One of the main novelties of the work in Chapter 3 is the addition of a power of the Hamiltonian (1.1.26), $E(u,v)^q$, where $q = q(s,k)$ is sufficiently large, to the energy $\mathscr{E}_{s,N}(u,v)$. To construct the weighted measure, formally given by,

$$\vec{\rho}_s(du) = Z^{-1}e^{-\mathscr{E}_s(u,v)-E(u,v)^q}du$$

one needs uniform control of the partition function of the truncated measures $\vec{\rho}_{s,N}$. As in [4,40], we achieve this by pathwise bounds on the terms in a stochastic optimisation problem involving a measure perturbed by a "control drift". In our case, the relevant expression involves higher powers of the control terms. These terms appear because of the higher order nonlinearity: u^k for $k > 3$ instead of $k = 3$ as in [40]. Because of this, we found we could not get uniform control on the partition function of the truncated measures $\vec{\rho}_{s,N}$ without the presence of $E(u,v)^q$ in the energy to assist in controlling higher powers of the drift. As $E(u,v)$ is conserved under the flow of NLW (1.1.25), adding $E(u,v)^q$ to the energy does not impact the energy estimate required by Method 4.

Another major difference of the results of we prove for NLW (1.1.25) with previous works is that is that the key energy estimates in [40, 78] depend on an initial integration by parts (see [40, Equations (3.5)-(3.6)]). This integration by parts removes the most singular term in the derivative of the energy. It is also in this step of the argument that the correction needed to define the weighted measure is identified. As pointed out in [40], when s is not an integer, we cannot integrate by parts and obtain exact cancellation. The main tools here are paraproducts and an expansion of the relevant multiplier. Since we do not require s to be an integer, we automatically lower the restriction on the regularity of the measure $\vec{\mu}_s$.

We conclude this subsection with a future research direction.

The assumptions on s and σ in the statement of Theorem 1.1.4 and Theorem 1.1.5 are not optimal. For example, it is well known that a basic short-time well-posedness result for NLW (1.1.25) can be proved in the range $s > \frac{3}{2} - \frac{1}{k-1}$ using Strichartz estimates. We do not attempt to optimise s in Theorem 1.1.4 and Theorem 1.1.5. However, this would be interesting to do because, as remarked in [78], [40], it is of interest to consider quasi-invariance in low regularity settings.

1.2 Singular SPDEs

Singular SPDEs are SPDEs with noise so rough that the equation itself does not make sense. To give meaning to an equation of this type, one needs to perform a surgery on the equation. In effect, one needs to abandon the study of the original equation and study a closely related 'renormalised' equation. In this introductory section, we will explain some of the different renormalisation procedures that have been used in the literature. We focus in particular on Wick ordering for the stochastic quantisation equation (SQE), (1.2.1), as we use a similar renormalisation for the stochastic complex Ginzburg-Landau equation (SCGL), studied in Chapter 4. We then give a brief introduction to SCGL, explain the results we prove in Chapter 4 and then, finally, we explain how the results we prove fit into the wider singular SPDE literature.

1.2.1 Methods in the Literature

The interest in singular SPDEs goes back to an article by Parisi and Wu, [81], (see also [68]) where it was proposed that the Euclidean Φ^k_d quantum field theory on finite volume could be constructed as the invariant measure of the *stochastic quantisation equation* (SQE)

$$\begin{cases} \partial_t u = \Delta u - u + u^k + \xi \\ u|_{t=0} = u_0 \end{cases} \qquad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+, \tag{1.2.1}$$

where ξ denotes a real-valued space-time white noise. More precisely, the Euclidean quantum field theory on finite volume is the invariant measure of an appropriately renormalised version of (1.2.1). SQE (1.2.1) is also referred to as the Φ^k_d-model in the literature. Due to its importance in physics, this equation has been the primary motivator for most major advancements in the field of singular SPDEs.

As it is stated, SQE (1.2.1) does not make sense in dimension 2 and above. For example, in the 2-dimensional setting, the real-valued space-time noise ξ has spatial regularity $-2-\varepsilon$, for $\varepsilon > 0$. Heuristically, solutions of heat type equations gain 2 derivatives. Hence, if a solution u exists to SQE (1.2.1), one expects u to have negative spatial regularity, $-\varepsilon$, for $\varepsilon > 0$. This is an issue because the nonlinearity, u^3, then does not make sense. The product of distributions of negative regularity is not always well-defined.

One of the first works studying singular SPDEs was the work [22], by Da Prato and Debussche. In the 2-dimensional setting, Da Prato and Debussche proved the local well-posedness of a suitably renormalised SQE. We will briefly explain the method of [22] as we use the same general method in Chapter 4. We note that to make the steps below fully rigorous, one would need to consider a version of SQE (1.2.1) with mollified ξ.

Instead of studying SQE (1.2.1), which does not make sense in 2-dimensions as explained above, Da Prato and Debussche [22] studied the Wick ordered SQE (WSQE),

$$\begin{cases} \partial_t u = \Delta u - u + :u^k: + \xi \\ u|_{t=0} = u_0 \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+ \tag{1.2.2}$$

where

$$:u^k:= H_k(u,\infty)$$

is the Wick ordered non-linearity.

Here $H_k(\cdot,\ell)$ are the generalised Hermite polynomials and can be defined through the generating function:

$$G_H(t,x;\ell) = e^{tx-\frac{1}{2}\ell t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\ell).$$

When $\ell = 1$ we simply call $H_k(x;1)$ the Hermite polynomials and write $H_k(x;1) = H_k(x)$. The first few generalised Hermite polynomials are:

$$H_0(x;\ell) = 1, \qquad H_1(x;\ell) = x, \qquad H_2(x;\ell) = x^2 - \ell,$$
$$H_3(x;\ell) = x^3 - 3\ell x, \qquad H_4(x;\ell) = x^4 - 6\ell x^2 + 3\ell^2.$$

In the cubic setting, $k = 3$, the WSQE (1.2.2) essentially amounts to replacing u^3 by $u^3 - 3\infty u$. Of course, this replacement does not make sense rigorously but, by considering a mollified noise, it is possible to justify this process.

Da Prato and Debussche [22] were able to prove the local well-posedness of WSQE (1.2.2) by using a technique now known in the singular SPDE literature as the Da Prato-Debussche trick. However, this idea was first used by McKean [61] and Bourgain [11] in the context of PDEs with random initial data.

The Da Prato-Debussche trick involves making the ansatz

$$u = v + \Psi,$$

where Ψ is the stochastic convolution defined by

$$\Psi(t) = \int_{-\infty}^{t} e^{-(t-t')\Delta}\xi(t')\,dt',$$

and then studying the resulting equation for v,

$$\begin{cases} \partial_t v = \Delta v - \sum_{j=0}^{k} v^j :\Psi^{k-j}: \\ v|_{t=0} = u_0 - \Psi(0) \end{cases} \qquad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+ \tag{1.2.3}$$

where

$$:\Psi^{k-j}:= H_{k-j}(\Psi,\infty)$$

is the Wick ordered monomial of order $k-j$. To go from WSQE (1.2.2) to (1.2.3) one uses the well known Hermite polynomial sum formula:

$$H_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} x^j H_{k-j}(y).$$

Assuming the following, one can make sense of the equation for v, (1.2.3),

• **Assumption 1:** If a function f has regularity $\alpha > 0$ and a function g has regularity $\beta < 0$ and $\alpha + \beta > 0$, then the product fg is well-defined.

• **Assumption 2:** The Wick ordered monomials, $: \Psi^{k-j} :$ have regularity $-\varepsilon$, for $\varepsilon > 0$.

Hence if one postulates that v has regularity 2ε, all of the products on the right hand side of (1.2.3) make sense, unlike SQE (1.2.1). Da Prato and Debussche [22] then used standard PDE techniques to prove the local well-posedness of WSQE (1.2.2).

The groundbreaking work of Da Prato and Debussche [22] lead to many other works in the singular SPDE literature. In [63], an energy estimate was used to prove deterministic global well-posedness for WSQE (1.2.2), improving the almost sure global well-posedness result in [22]. Here by deterministic, we mean for all initial data, rather than for almost all initial data. The energy estimate in [63] essentially amounts to multiplying the equation (1.2.3) by v^{p-1} and integrating in space to get a bound on the growth of the L^p-norm of v. See [85, 93] for more papers on the 2-dimensional SQE involving similar energy estimates.

The situation for the 3-dimensional SQE proved to be much more difficult. The Da Prato-Debussche trick and Wick ordering are not sufficient to make sense of SQE in the 3-dimensional setting. In the groundbreaking work of [43], Hairer invented the theory of regularity structures, a general framework for making sense of singular SPDEs. Using this machinery, Hairer was able to make sense of, and prove the local well-posedness of, SQE in 3-dimensions. In [64] Mouratt and Weber used PDE techniques to show that the 3-dimensional SQE 'comes down from infinity'. This is a strong result which implies deterministic global well-posedness. Since the work of Hairer, there has been a menagerie of results giving alternative frameworks in which to study singular SPDEs. In [15], Catellier and Chouk used the theory of paracontrolled distributions developed by Gubinelli, Imkeller and Perkowski in [36], and gave an alternative way to make sense of, and prove local well-posedness for, the 3-dimensional SQE. In [53], Kupiainen developed a renormalisation group approach, using techniques from the physics literature, to solve SQE.

Many other singular SPDEs fit nicely into the frameworks listed above. For example the KPZ equation

$$\partial_t h = \partial_{xx} h + (\partial_x h)^2 + \xi$$

can be solved in the context of regularity structures, see [43], and paracontrolled distributions, see [37]. The 2 and 3-dimensional parabolic Anderson model

$$\partial_t u = \Delta u + u\xi$$

can be solved in the context of regularity structures, see [43], or paracontrolled distributions, see [36]. In [47], the 3-dimensional SCGL was solved in the context of regularity structures and paracontrolled distributions.

1.2.2 Main results for SCGL

- The results we present here, and later prove in Chapter 4, are based on: W. J. Trenberth, *Global well-posedness for the two-dimensional stochastic complex Ginzburg-Landau equation*, arXiv:1911.09246 [math.AP].

We study the following stochastic complex Ginzburg-Landau equation on $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with an additive space-time white noise forcing:

$$\begin{cases} \partial_t u = (a_1 + ia_2)\Delta u + (b_1 + ib_2)u - (c_1 + ic_2)|u|^{2m-2}u + \sqrt{2\gamma}\xi \\ u|_{t=0} = u_0 \end{cases} \tag{1.2.4}$$

where $(x,t) \in \mathbb{T}^2 \times \mathbb{R}_+$, $a_1, \gamma, > 0$, $a_2, b_1, b_2, c_2, c_3 \in \mathbb{R}$, $m \geq 2$ is an integer and $\xi(x,t)$ denotes a complex-valued, Gaussian, space-time white noise on $\mathbb{T}^2 \times \mathbb{R}_+$. This is (1.0.3) with $d = 2$.

Formally, $\xi(x,t)$ is defined by

$$\frac{dW(t)}{dt} = \xi(x,t)$$

where $W(t)$ is a cylindrical Weiner process on $L^2(\mathbb{T}^2)$,

$$W(t) = \sum_{n \in \mathbb{Z}^2} \beta_n(t) e^{in\cdot x},$$

where $\{\beta_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions on some fixed probability space, with $\mathrm{Var}(\beta_n(t)) = t$.

For $s > -\frac{2}{2m-1}$, we consider SCGL with initial data in the space $C^s(\mathbb{T}^2)$. Here $C^s(\mathbb{T}^2)$ is the Hölder space of regularity s. See Appendix A for the definition and some basic properties of these spaces.

The complex Ginzburg-Landau equation (CGL), namely equation (1.2.4) without the white noise forcing, is one of the most studied equations in physics. CGL is used to describe phenomenon such as nonlinear waves, second order phase transitions, superconductivity, Bose-Einstein condensation and liquid crystals. For more information on complex Ginzburg-Landau equations in physics, see the survey paper [1].

Due to its physical importance, CGL has been heavily studied from a mathematical perspective. See for example [26] where an energy estimate was used to show that, with twice differentiable initial data, if the ratio $|\frac{a_2}{a_1}|$ is small enough, CGL on $\mathbb{T}^d$ is globally well-posed. See [34] for a similar result on $\mathbb{R}^d$.

There has also been a substantial amount of research on complex Ginzburg-Landau equations with random forcing. See for example [5, 42, 46, 47, 49, 54–57, 86].

We note in particular the work [42] where Hairer studied a complex Ginzburg-Landau equation driven by a real-valued space-time white noise in 1-dimension and the work [47] where Hoshino, Inahama and Naganuma studied a complex Ginzburg-Landau equation driven by a complex-valued space-time white noise in 3-dimensions. To the authors knowledge, no work has studied complex Ginzburg-Landau equations driven by a space-time white noise in 2-dimensions [3]. The current work aims to fill this gap in the literature.

In Chapter 4 we will only consider SCGL (1.2.4), with $b_1 = -a_1$ and $b_2 = -a_2$. That is, we consider SCGL in the form:

$$\partial_t u = (a_1 + ia_2)[\Delta - 1]u - (c_1 + ic_2)|u|^{2m-2}u + \sqrt{2\gamma}\xi. \tag{1.2.5}$$

We do this to avoid issues occurring at the zero frequency that arise as $-\Delta$ is not a strictly positive operator.

As it is stated, the equation SCGL (1.2.4) does not make sense. Similar to SQE (1.2.1), discussed in Section 1.2.1, if solutions to this equation did exist they would have regularity $-\varepsilon$, for $\varepsilon > 0$. If this were the case, it would not be possible to make sense of the nonlinearity $|u|^{2m-2}u$ as the product of distributions of negative regularity is not always well-defined.

To get around this issue, we perform a surgery on SCGL (1.2.4), similar to the one discussed in Section 1.2.1. It turns out that by subtracting certain counter terms, it is possible to give meaning to SCGL. Instead of studying SCGL (1.2.4), we study the Wick ordered stochastic complex Ginzburg-Landau equation (WSCGL),

$$\begin{cases} \partial_t u = (a_1 + ia_2)[\Delta - 1]u - (c_1 + ic_2) :|u|^{2m-2}u: + \sqrt{2\gamma}\xi \\ u|_{t=0} = u_0 \end{cases} \tag{1.2.6}$$

where

$$:|u|^{2m-2}u: = (-1)^{m-1}(m-1)! L^{(1)}_{m-1}(|u|^2; \infty)u$$

is the Wick ordered nonlinearity.

Here $L^{(\ell)}_k(x;\sigma) = \sigma^k L^{(\ell)}_k(\frac{x}{\sigma})$ where $L^{(\ell)}_k(x)$ are the generalised Laguerre polynomials. These polynomials can be defined using the recursion relation

$$L^{(\ell)}_{k+1}(x) = \frac{(2k+1+\ell-x)L^{(\ell)}_k(x) - (k+\ell)L^{(\ell)}_{k-1}(x)}{k+1}$$

after initialising

$$L^{(\ell)}_0(x) = 1, \quad L^{(\ell)}_1(x) = 1 + \ell - x.$$

Alternatively, the generalised Laguerre polynomials can be defined using a generating function:

$$G_\ell(t,x) = \frac{1}{(1-t)^{\ell+1}} e^{-\frac{tx}{1-t}} = \sum_{n=0}^{\infty} t^n L^{(\ell)}_n(x). \tag{1.2.7}$$

[3] Very recently, after the paper this chapter is based on was completed, the work [60] studied the cubic case.

The first few generalised Laguerre polynomials are:

$$
\begin{aligned}
L_0^{(\ell)}(x) &= 1 \\
L_1^{(\ell)}(x) &= 1 + \ell - x \\
L_1^{(\ell)}(x) &= \frac{1}{2}x^2 - (\ell+2)x + \frac{(\ell+1)(\ell+2)}{2} \\
L_3^{(\ell)}(x) &= \frac{-1}{6}x^3 + \frac{(\ell+3)}{2}x^2 - \frac{(\ell+2)(\ell+3)}{2}x + \frac{(\ell+1)(\ell+2)(\ell+3)}{6}.
\end{aligned}
$$

As an example, in the cubic setting, we study the nonlinearity $|u|^2 - 2\infty u$ instead of $|u|^2u$. We have subtracted the counter term $2\infty u$. Of course, WSCGL (1.2.6) also does not make any sense. Subtracting $2\infty u$ does not make sense rigorously. However, morally WSCGL (1.2.6) is the renormalised equation we study in Chapter 4. In Section 4.1 of this thesis, we will rigorously describe how we make sense of solutions to WSCGL (1.2.6).

The main aim of Chapter 4, is to study the existence of solutions for WSCGL (1.2.6). First, we prove the following local well-posedness result.

Theorem 1.2.1. *Let $a_1 > 0$, $m \geq 2$ be an integer. Let $s_0 > -\frac{2}{2m-1}$ and $\varepsilon > 0$ be sufficiently small. Then WSCGL* (1.2.6) *is pathwise locally well-posed for initial data in $C^{s_0}(\mathbb{T}^2)$.*

Using an energy estimate, we are then able to upgrade this local well-posedness result to deterministic global well-posedness, provided that the dispersion, a_2, is small compared to the dissipation, a_1, and that the heat part of the nonlinearity is defocusing, $c_1 > 0$.

Theorem 1.2.2. *Let $a_1, c_1 > 0$ and $s_0 > -\frac{2}{2m-1}$. Set $r = \left|\frac{a_1}{a_2}\right|$ and let $m \geq 2$ be an integer such that*

$$2m - 1 < 2 + 2\left(r^2 + 2r\sqrt{1+r^2}\right).$$

Then WSCGL, (4.1.7) *is pathwise globally well-posed for initial data in $C^{s_0}(\mathbb{T}^2)$.*

We now describe how our results for WSCGL (1.2.6) in this thesis, and the methods used to prove them, fit into the wider singular SPDE literature.

SCGL can be viewed as an equation interpolating the parabolic SPDE setting corresponding to $a_2 = c_2 = 0$, SQE, and the dispersive SPDE setting corresponding to $a_1 = c_1 = 0$, a stochastic nonlinear Schrödinger equation (SNLS) with white noise forcing. As explained at the beginning of this subsection, SQE is well understood in the 1, 2 and 3-dimensional settings. Contrastingly, almost nothing is known about SNLS with white noise forcing. Local well-posedness is an open problem even in the 1-dimensional setting. Local well posedness of SNLS with a smoothed noise has been studied in many papers, see for example [31] where local well-posedness of SNLS with an almost space-time white noise forcing is proven. More is understood for other dispersive SPDEs with white noise forcing. See for example [38, 39, 71]. The results in Chapter 4 of this thesis can be seen as interpolating the easy dissipative setting and the difficult, unsolved, dispersive setting.

The main aim of this work is to fill in a gap in the literature by solving the 2-dimensional SCGL. As noted earlier, SCGL has been studied in dimensions 1 and 3, but not dimension 2. This problem of course is much easier than the 3-dimensional equation solved in [47]. We do not need to use the frameworks of regularity structures or paracontrolled distributions to give meaning to (1.2.5). We use a simple application of the method outlined in Section 1.2.2, developed for SQE.

Even though technically the 2-dimensional setting is simpler than the 3-dimensional setting, we believe this gap in the literature is an interesting problem as the renormalisation procedure needed is slightly different to that in other papers. Consider for example, SQE (1.2.1). Due to physical considerations, one often wants to only study real-valued solutions of SQE. To do this one puts the 'reality' condition

$$\widehat{\xi}(n) = \overline{\widehat{\xi}(-n)}, \qquad \text{for all } n \in \mathbb{Z}^d$$

on the white noise ξ. In contrast to SQE, SCGL has to be studied in the complex-valued setting. This is due to the complexifying nature of the Schrödinger, $ia_2\Delta u - ic_2|u|^{2m-2}u$, part of the equation. Because of this, the renormalisation procedure for the real-valued 2-dimensional SQE, discussed in Section 1.2.2, then differs to the renormalisation procedure for the complex-valued 2-dimensional SCGL, which we will describe in Section 4.1. For SCGL, in the complex-valued setting, we use the Laguerre polynomials. For SQE, in the real-valued setting, one uses the Hermite polynomials. For the real-valued cubic SQE one essentially replaces the nonlinearity u^3 by

$$H_3(u; \infty) = u^3 - 3\infty u.$$

Compare this to the complex-valued cubic SCGL were we replace $|u|^2u$ by

$$-L_1^{(1)}(|u|^2; \infty)u = |u|^2u - 2\infty u.$$

Note the difference in the amount of 'counter terms' subtracted in the above two equations. Formally, this is because in the real-valued setting there are three distinct ways to pair u and u in the nonlinearity $u^3 = uuu$, while in the complex-valued setting, there are only 2 distinct ways to pair u and $\overline{u}$ in the nonlinearity $|u|^2u = u\overline{u}u$.

Wick ordering has previously been studied in the complex-valued setting, see for example [74] where the complex-valued Wick ordered nonlinear Schrödinger equation was studied in the context of random initial data. However, to the authors knowledge, this is the first time Wick ordering and Laguerre polynomials have been applied together in the complex-valued setting in the context of singular SPDEs.

To prove Theorem 1.2.1, we use the method of Da Prato-Debussche [22] outlined in Section 1.2.1. One of the steps in the local well-posedness argument of Da Prato-Debussche [22] is proving the regularity of the Wick ordered monomials, namely

Assumption 2 in Section 1.2.2. In Chapter 4, we use a Fourier analytic proof, inspired by the analysis in [38, Proposition 2.1] to do this, see Proposition 4.1.1. One key formula used to prove the regularity of the Wick ordered monomials in [22,38,63,93] and other papers studying real-valued equations is the following Hermite polynomial expectation formula:

$$\mathbb{E}[H_k(f;\sigma_f)H_\ell(g;\sigma_g)] = k!\delta_{k\ell} \tag{1.2.8}$$

where f and g are mean-zero complex-valued Gaussian random variables with variances σ_f and σ_g respectively. In Chapter 4, as we work in the complex-valued setting and use Laguerre polynomials, we need the following Laguerre polynomial analogue of (1.2.8):

$$\mathbb{E}\left[L_k^{(\ell)}(|f|^2;\sigma_f)f^\ell\overline{L_m^{(\ell)}(|g|^2;\sigma_g)g^\ell}\right] = \delta_{km}\frac{(k+\ell)!}{k!}\left|\mathbb{E}[f\overline{g}]\right|^{2k}\mathbb{E}[f\overline{g}]^\ell. \tag{1.2.9}$$

See [28] for more information on (1.2.9).

The rest of the argument used to prove local well-posedness (Theorem 1.2.1) is virtually the same as that in [22]: Once we know the Wick ordered monomials have regularity $C^{-\varepsilon}(\mathbb{T}^2)$, we can close a contraction mapping argument to solve the equation for $v = u - \Psi$ by postulating that $v \in C^{2\varepsilon}(\mathbb{T}^2)$ and using the following product estimate for Besov-Hölder spaces:

$$\|fg\|_{C^{-\varepsilon}(\mathbb{T}^2)} \le \|f\|_{C^{-\varepsilon}(\mathbb{T}^2)}\|g\|_{C^{2\varepsilon}(\mathbb{T}^2)}.$$

See Appendix A for a more general statement of this formula.

The argument used to prove global well-posedness in Chapter 4 is similar to the papers [63, 93]. From local well-posedness theory it can be shown that the time of local existence, T, satisfies $T \gtrsim_\omega \|v_0\|_{L^p(\mathbb{T}^2)}^{-\theta}$ for some $\theta > 0$ if

$$2m - 1 < p. \tag{1.2.10}$$

It then suffices to get an a priori bound on the growth of $\|v(t)\|_{L^p}$. This argument relies on the nonlinearity of (1.2.6) having a good sign, that is $c_1 > 0$. However, compared to the real-valued setting in [63, 93] a complication arises due to the dispersion, the $ia_2\Delta$ term, in (1.2.6). Instead of getting a nice a priori bound of the form

$$\partial_t\|v\|_{L^p(\mathbb{T}^2)}^p \le C \tag{1.2.11}$$

as is obtained in [63,93], we get a bound of the form

$$\partial_t\|v\|_{L^p(\mathbb{T}^2)}^p + 4A(-2\operatorname{Im}(\overline{v}\nabla v), \nabla|v|^2) \le C$$

where A is a quadratic form with coefficients depending on p, a_1 and a_2. Using ideas originating in [26], see also [34,46], we can show that A is a positive definite quadratic form if

$$2 < p < 2 + 2\left(r^2 + 2r\sqrt{1+r^2}\right) \tag{1.2.12}$$

where $r = |\frac{a_1}{a_2}|$. The positivity of A gives us an a priori bound of the form (1.2.11). The conditions (1.2.10) and (1.2.12) give the restriction on the dissipation-dispersion ratio in Theorem 1.2.2.

We end this section with a few possible future research directions.

The Theorem 1.2.2 leaves global well-posedness open for small dissipation-dispersion ratios, $r = |\frac{a_1}{a_2}|$. It is possible extend it to small values of r using an invariant measure argument, albeit only almost everywhere, see [92] for details. Further, it should be possible to adapt the arguments in [93] or [44] to show that the transition semi-group associated to WSCGL (1.2.6) satisfies the strong Feller property. One should then be able to extend this almost sure GWP result of [92] to deterministic GWP. See for example [23] where this was done for the 1-dimensional Gross-Pitaevskii equation.

Using the estimates used to prove Theorem 1.2.2, it should be possible to prove a coming down from infinity result like in [64, 93]. However, as we had no use for this result, in this Chapter 4 we did not pursue this.

1.3 Numerical Simulations

In Chapter 5, the final chapter of this thesis, we describe some numerical simulations that we performed. These simulations illustrate the transport of Gaussian measure results in Chapter 2 and give insight into potential future research directions. The work in Chapter 5 is exploratory in nature. We aim to get a heuristical idea of the behaviour of the objects studied and to illustrate concepts previously studied in thesis without being overly concerned with error analysis. For this reason, any conclusion reached in Chapter 5 should be viewed skeptically.

1.3.1 Motivation for simulations

Over the last 30 years, there has been a wealth of research, studying the transport of measures under the flow of partial differential equations, in particular dispersive PDEs. This was initiated by results concerning invariant measures for the nonlinear Schrödinger equation. See for example the work [59], dating to 1988, where Lebowitz, Rose and Speer gave a construction of a Gibbs measure for the nonlinear Schrödinger equation and the work [9], dating to 1994, where Bourgain proved that this measure is invariant under the flow of the nonlinear Schrödinger equation.

More recently, in the last 5 years, there has been a surge of activity studying the transport of Gaussian measures under the flow of nonlinear PDEs. The genesis of this was the paper [96] where Tzvetkov developed a general method, based on PDE type energy estimates, to prove that certain Gaussian measures are quasi-invariant under the flow of Hamiltonian PDEs. Chapters 2 and 3 are examples of results in this program.

To our knowledge, there has been no work in the literature studying the transport of measures under the flow of dispersive PDEs from a numerical perspective. This is likely due to to the technical difficulties this would entail. To numerically study the transport of measures, one needs to generate a large amount of samples, each sample being a solution of a given PDE with initial data sampled from some measure. Numerically solving PDEs can be very computationally expensive and doing so, say 10^6 times, is even more still.

The aim of Chapter 5 is to make an attempt at studying the transport of Gaussian measures under the flow of dispersive PDEs numerically, as opposed to theoretically. The simplest method we are familiar with in the literature, that can numerically study measures, is Monte-Carlo simulation. That is, we numerically generate a large amount of functions, sampled from the initial measure being studied. From the ensemble of initial functions, we can approximate any statistical property of the initial measure we wish. Then, for each function, we numerically solve the given PDE up to some time t. From the ensemble of solutions, at time t, we can approximate any statistical property of the push forward, to time t, measure. We describe in technical detail how we perform these Monte-Carlo simulations in Section 5.1 of this thesis.

We have three main motivations for performing these simulations. Firstly, as mentioned above, we do not know of any similar simulations performed in the literature. Our second motivation is to visualise how quasi-invariance, a topic studied in Chapter 2 and Chapter 3, manifests itself in terms of statistical properties of Fourier coefficients. We are in particular interested in the evolution of the variance of the nth Fourier coefficient,

$$\mathrm{Var}(u_n(t)) = \mathbb{E}[|u_n(t)|^2] - |\mathbb{E}[u_n(t)]|^2$$

and the evolution of the covariance between the nth and mth Fourier coefficients,

$$\mathrm{Covar}(u_n(t), u_m(t)) = \mathbb{E}[u_n(t)\overline{u_m(t)}] - \mathbb{E}[u_n(t)]\overline{\mathbb{E}[u_m(t)]}.$$

Our final motivation for performing these simulations, is that we believe some mathematical properties can be opaque when looked through a shroud of equations and symbols but transparent with the right visualisation. The hope is that this work could lead to some interesting theoretical questions or give insight into problems that are beyond current theoretical techniques. Below we make Conjecture 1.3.1 which we believe to be one such interesting research direction.

1.3.2 Simulations performed for FNLS and gBBM

In the remainder of this introductory section we describe the simulations we perform.

First, we study the defocusing FNLS ((1.0.1) with a '$-$' sign) on the 1-dimensional torus[4],

$$\begin{cases} i\partial_t u + (-\partial_x^2)^\alpha u = -\lambda |u|^2 u, \\ u|_{t=0} = u_0, \end{cases} \tag{1.3.1}$$

with initial data, u_0, sampled form the Gaussian measure formally given by

$$d\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du.$$

This Gaussian measure is induced by the map

$$\omega \mapsto u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle 2\pi n\rangle^s} e^{2\pi i n x} \tag{1.3.2}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ are mutually independent standard Gaussian complex random variables.

The equation (1.3.1) is similar to the equation studied in Chapter 2. The only difference is the presence of λ on the nonlinearity. This does not affect the proofs in Chapter 2. We expect all of the results in Chapter 2 to also hold for FNLS (1.3.1).

We study FNLS (1.3.1) with $\alpha = 0.55$ and the measure (1.3.2) with $s = 4.2$, as we found these values were approachable computationally, and with $\lambda = 10$ to increase the strength of the nonlinearity. As the Gaussian measure (1.3.2) is invariant under the flow of the linear Schrödinger equation, if λ were close to 0, we would not expect to see interesting nonlinear behaviour. Further, we only study FNLS (1.3.1) in the 1-dimensional setting. This is mainly because of the 'curse of dimensionality'. In the context of numerically solving PDEs, it is often found that as dimensionality goes up, it becomes increasingly computationally expensive to numerically solve PDEs.

We note that the following quantity is perserved by the flow of FNLS (1.3.1)

$$M(u(t)) = \int_{\mathbb{T}} |u(t)|^2 = \sum_{n\in\mathbb{Z}} |u_n(t)|^2.$$

Later, in Section 5.3.2, we will use this conservation law to check the accuracy of the Monte-Carlo simulation performed for FNLS. We note that FNLS also has a Hamiltonian that is conserved, see (1.1.15). However, this quantity is not easily computed numerically, and so we do not use it in Chapter 5.

In Section 5.3, we perform a Monte-Carlo simulation of the Gaussian measure (1.3.2) under the flow of FNLS (1.3.1). We describe in detail how we perform this simulation in Sections 5.1 and 5.2. Based on the results of the simulation performed for FNLS (1.3.1), we hypothesize that the Gaussian measure (1.3.2) under the flow of FNLS (1.3.1) converges to some limiting measure.

Encouraged by this result, we perform another simulation where we believe there is a natural candidate for the limit measure.

[4]Here we depart from a convention established in the Notation section of this thesis. In this section we define $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$, instead of $\mathbb{T} = \mathbb{R}\backslash 2\pi\mathbb{Z}$. We do this as it leads to a simpler application of the fast Fourier transform algorithm.

We study, the real-valued, dispersion generalised Benjamin-Bona-Mahony equation (gBBM) on the 1-dimensional torus,

$$\begin{cases} \partial_t u + \partial_t(-\partial_x^2)^{\frac{\gamma}{2}} u + \partial_x u + \lambda \partial_x(u^2) = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.3.3}$$

with initial data, u_0, sampled form the Gaussian measure induced by the map

$$\omega \mapsto \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+(2\pi n)^\gamma}} e^{2\pi i n x}, \tag{1.3.4}$$

Here,

$$\begin{aligned} g_1(\omega) &= \sqrt{0.225}(h_1(\omega) + h_2(\omega) + h_3(\omega) + h_5(\omega)) \\ g_2(\omega) &= \sqrt{0.275}(h_2(\omega) + h_3(\omega) + h_4(\omega) + h_6(\omega)) \\ g_3(\omega) &= \sqrt{0.225}(h_3(\omega) + h_4(\omega) + h_1(\omega) + h_7(\omega)) \\ g_4(\omega) &= \sqrt{0.275}(h_4(\omega) + h_1(\omega) + h_2(\omega) + h_8(\omega)) \end{aligned} \tag{1.3.5}$$

where $h_1, h_2, \dots, h_8, g_5, g_6, \cdots$ are mutually independent standard Gaussian random variables. We define $g_1, \dots g_4$ the way we did in (1.3.5) to introduce correlations between the Fourier coefficients of the random series (1.3.4), to see how correlation between Fourier coefficients is transported. Further, $g_0 = \frac{1}{2}$ $\overline{g_n} = g_{-n}$ so that the random series (1.3.4) is real-valued.

The equation gBBM (1.3.3) was first introduced by Tzvetkov [96] as a generalisation of the well known Benjamin-Bona-Mahony (BBM) equation, as a model equation in the study of quasi-invariant measures under the flow of Hamiltonian PDEs. The equation (1.3.3) can be written in the form

$$\partial_t u = -\lambda \frac{\partial_x}{1+(-\partial_x^2)^{\frac{\gamma}{2}}}(u+u^2).$$

In this form it is clear that, unlike other dispersive equations, the dispersion relation of (1.3.3) is bounded. This makes (1.3.3) easy to study from a theoretical perspective and in fact in Chapter 5 we find that it is easy to study this equation from a numerical perspective for the same reason.

We do not study (1.3.3) directly. We instead study the dispersion generalised Benjamin-Bona-Mahony type equation (which we also label gBBM)

$$\begin{cases} \partial_t u = -\lambda \frac{\partial_x}{1+(-\partial_x^2)^{\frac{\gamma}{2}}}(u^2), \\ u|_{t=0} = u_0, \end{cases} \tag{1.3.6}$$

with initial data u_0 coming from inital measure given by (1.3.4) but with $g_0 = 0$. Going from (1.3.3) to gBBM (1.3.6) amounts to substituting $u' = u - \frac{1}{2}$ into (1.3.3).

We choose to to study gBBM (1.3.6) instead of (1.3.3) as we found the former is faster to solve numerically.

We note that gBBM (1.3.6) conserves the mean,

$$\int_{\mathbb{T}} u(t)\, dx$$

and the $\dot{H}^{\frac{\gamma}{2}}$-norm of solutions, the quantity

$$\|u\|^2_{\dot{H}^{\frac{\gamma}{2}}} = \sum_{n\in\mathbb{Z}} |n|^\gamma |u_n|^2. \tag{1.3.7}$$

From the conservation law (1.3.7) the Gibbs type measure, formally given by the following equation, is invariant under the flow of gBBM (1.3.6)

$$d\mu_{\frac{\gamma}{2}} = Z^{-1} e^{-\beta^2 \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}}\, du. \tag{1.3.8}$$

This is the measure induced by the following random Fourier series,

$$\omega \mapsto \beta \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{|2\pi n|^{\frac{\gamma}{2}}} e^{2\pi i n x}, \tag{1.3.9}$$

were $g_0 = 0$ and g_n are mutually independent standard Gaussian random variables except for $\overline{g_n} = g_{-n}$.

We chose to study the equation gBBM (1.3.6) because it possesses the Gibbs measure (1.3.8) which has readily available statistical information. For example, the variance of the nth Fourier coefficient of samples from (1.3.8) is simply $\beta^2 \frac{1}{|2\pi n|^\gamma}$. This is not the case for Gibbs measures for other equations. We believe invariant Gibbs measures are natural candidates for convergence. Further, the initial Gaussian measure (1.3.4) and the Gibbs measure (1.3.8) are supported on functions of the same regularity.

We choose $\gamma = 5$ in gBBM (1.3.6) as we found this value to be computationally approachable. See Section 5.4 for more information on the simulation we perform for gBBM (1.3.6).

Based on the results of this Monte-Carlo simulation, we do not find conclusive evidence confirming or rejecting our hypothesis of convergence to the Gibbs measure (1.3.8). However, we still find evidence of convergence to some other, unknown, limiting measure.

Based on the results of the simulations performed in Chapter 5 we make the following conjecture.

Conjecture 1.3.1. *The push forward of the Gaussian measure* (1.3.2) *under the flow of FNLS* (1.3.1) *converges in some suitable sense as $t \to \infty$. The push forward of the Gaussian measure*(1.3.4) *under the flow of gBBM* (1.3.6) *converges in some suitable sense as $t \to \infty$.*

We conclude this section with a few possible future research directions.

It would be interesting to test Conjecture 1.3.1 for different PDEs and initial measures to see how widely behaviour of this type is expected.

In Conjecture 1.3.1, we do not make a guess at what the limiting measures may be. If one can show theoretically that the flow of measures does converge, it would be interesting to investigate properties of the limiting measure.

It could also be of interest to plot the evolution of more complicated properties of the flow of measures. For example the L^p-norm of the Radon-Nikodym derivative with respect to the initial measure. From the recent work of [24], there is a way to find an exact formula for the Radon-Nikodym derivative, so this should be possible.

We observe in our simulation for FNLS (1.3.1) that $\mathrm{Var}(u_n)$ for large n seems to increase more as a percentage than for small n. It would be interesting to numerically investigate this in more detail. This has a connection to the growth of higher Sobolev norms.

Chapter 2

On the transport of Gaussian measures under the 1-dimensional fractional nonlinear Schrödinger equations

This chapter is dedicated to the proof of Theorem 1.1.2 and Theorem 1.1.3, as stated in Section 1.1.2.

2.1 Preliminary estimates

In this section we record some elementary estimates that will be useful in the coming analysis. The first result we need is the double mean value theorem (DMVT) from [18, Lemma 2.3].

Lemma 2.1.1 (DMVT). *Let $\xi, \eta, \lambda \in \mathbb{R}$ and $f \in C^2(\mathbb{R})$. Then, we have*

$$f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi) = \lambda\eta \int_0^1 \int_0^1 f''(\xi + s\lambda + t\eta)\, dsdt.$$

We have the following consequence of DMVT:

Lemma 2.1.2. *Fix $s > 1$ and let $n_1, n_2, n_3, n \in \mathbb{Z}$ be such that $n = n_1 - n_2 + n_3$. Then, we have*

$$\big|\langle n_1\rangle^{2s} - \langle n_2\rangle^{2s} + \langle n_3\rangle^{2s} - \langle n\rangle^{2s}\big| \lesssim |n - n_1||n - n_3|\langle n_{\max}\rangle^{2s-2},$$

where $n_{\max} = \max(|n_1|, |n_2|, |n_3|, |n|)$ and the implicit constant depends only on s.

Proof. This is a simple application of DMVT upon setting $n_1 = \xi+\eta+\lambda$, $n_2 = \xi+\eta$, $n = \xi + \lambda$ and $n_3 = \xi$. □

The next lemma states a crucial lower bound on the phase function $\phi(\overline{n})$ of (1.1.18) which we use repeatedly throughout. It was proved for the case $\frac{1}{2} < \alpha \le 1$ in [25]. Their proof easily extends to the case $\alpha > 1$; see Appendix B.

Lemma 2.1.3. *Fix $\alpha > \frac{1}{2}$ and let $n_1, n_2, n_3, n \in \mathbb{Z}$ be such that $n = n_1 - n_2 + n_3$. Then, we have*

$$\begin{aligned} |\phi(\overline{n})| &\gtrsim |n - n_1||n - n_3| \left(|n - n_1| + |n - n_3| + |n|\right)^{2\alpha-2} \\ &\gtrsim |n - n_1||n - n_3| n_{\max}^{2\alpha-2} \end{aligned}$$

where $n_{\max} = \max(|n_1|, |n_2|, |n_3|, |n|)$ and the implicit constant depends only on α.

We next state a useful summing estimate, a proof of which can be found in, for example, [33, Lemma 4.2].

Lemma 2.1.4. *If $\beta \ge \gamma \ge 0$ and $\beta + \gamma > 1$, then we have*

$$\begin{aligned} \sum_n \frac{1}{\langle n - k_1 \rangle^\beta \langle n - k_2 \rangle^\gamma} &\lesssim \frac{\varphi_\beta(k_1 - k_2)}{\langle k_1 - k_2 \rangle^\gamma}, \\ \int_{\mathbb{R}} \frac{1}{\langle x - k_1 \rangle^\beta \langle x - k_2 \rangle^\gamma} dx &\lesssim \frac{\varphi_\beta(k_1 - k_2)}{\langle k_1 - k_2 \rangle^\gamma}, \end{aligned}$$

where

$$\varphi_\beta(k) := \sum_{1 \le |n| \le |k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1, & \text{if } \beta > 1, \\ \log(1 + \langle k \rangle), & \text{if } \beta = 1, \\ \langle k \rangle^{1-\beta}, & \text{if } \beta < 1. \end{cases}$$

Finally, we will require the following fact from elementary number theory [45]: Given $n \in \mathbb{N}$ and any $\delta > 0$, there exists a constant $C_\delta > 0$ such that the number of divisors $d(n)$ of n satisfies

$$d(n) \le C_\delta n^\delta. \tag{2.1.1}$$

2.2 Reformulation of FNLS

In this section, we reformulate FNLS (1.1.13) into a more amenable form for the normal form reductions in the next section. Given $t \in \mathbb{R}$, we consider the gauge transform $\mathcal{G}_t$ on $L^2(\mathbb{T})$ defined by

$$\mathcal{G}_t[f] = e^{2it \fint |f|^2 dx} f,$$

where $\fint_{\mathbb{T}} f(x)\, dx := \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\, dx$. Furthermore, given $u \in C(\mathbb{R}; L^2(\mathbb{T}))$, we define $\mathcal{G}$ by

$$\mathcal{G}[u](t) := \mathcal{G}_t[u(t)].$$

It is easy to check that $\mathcal{G}$ is invertible with inverse

$$\mathcal{G}^{-1}[u](t) = \mathcal{G}_{-t}[u(t)].$$

Now, let $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ be a solution to (1.1.13) and define v by

$$v(t) = \mathcal{G}[u](t).$$

Then it follows from mass conservation (1.1.14) that v satisfies

$$i\partial_t v + (-\partial_x^2)^\alpha v = \left(|v|^2 - 2 \fint_{\mathbb{T}} |v|^2 \, dx \right) v. \tag{2.2.1}$$

Namely, v satisfies (1.1.13) but with a more favourable nonlinearity. We define $\Phi(t) : u_0 \in H^\sigma(\mathbb{T}) \mapsto v(t) \in H^\sigma(\mathbb{T})$ to be the solution map of (2.2.1) (when it exists) at time t.

In order to make the following calculations secure, we consider the following truncated equation:

$$i\partial_t v + (-\partial_x^2)^\alpha v = \mathbf{P}_{\leq N} \left[\left(|\mathbf{P}_{\leq N} v|^2 - 2 \fint_{\mathbb{T}} |\mathbf{P}_{\leq N} v|^2 \, dx \right) \mathbf{P}_{\leq N} v \right]. \tag{2.2.2}$$

Here, $\mathbf{P}_{\leq N}$ is the projection onto frequencies $\{n : |n| \leq N\}$ for $N \in \mathbb{N}$. We let $\Phi_N(t)$ denote the solution map of (2.2.2) at time t (when it exists).

To exploit the dispersive nature of (2.2.1), we will need another gauge transform. We define the *interaction representation* of v as

$$w(t) = S(-t)v(t), \tag{2.2.3}$$

where $S(t) = e^{it(-\partial_x^2)^\alpha}$. On the Fourier side, we have[1]

$$\widehat{w}_n(t) = e^{-it|n|^{2\alpha}} \widehat{v}_n(t).$$

Then, the equation (2.2.1) becomes the following equation for the Fourier coefficients $\{\widehat{w}_n\}_{n\in\mathbb{Z}}$:

$$\partial_t \widehat{w}_n = -i \sum_{\Gamma(n)} e^{it\phi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} + i|\widehat{w}_n|^2 \widehat{w}_n, \tag{2.2.4}$$

where the phase function $\phi(\overline{n})$ and the plane $\Gamma(n)$ are given by

$$\phi(\overline{n}) = \phi(n_1, n_2, n_3, n) = |n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n|^{2\alpha}$$

and

$$\Gamma(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\}.$$

[1] For clarity, we will sometimes write $\widehat{f}(n)$ as $\widehat{f}_n$.

Similarly, the truncated equation (2.2.2) becomes the following equation for the Fourier coefficients $\{\widehat{w}_n\}_{n\in\mathbb{Z}}$:

$$\partial_t \widehat{w}_n = \mathbf{1}_{|n|\le N} \left[-i \sum_{\Gamma_N(n)} e^{it\phi(\overline{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} + i|\widehat{w}_n|^2 \widehat{w}_n \right], \tag{2.2.5}$$

where the plane $\Gamma_N(n)$ is given by

$$\Gamma_N(n) = \Gamma(n) \cap \{(n_1, n_2, n_3) : |n_j| \le N,\ j = 1, 2, 3\}.$$

From now on, for ease of notation, we will typically ignore the 'hats' on the Fourier coefficients.

The following lemma shows that it suffices to prove the quasi-invariance of μ_s under the flow of (2.2.1).

Lemma 2.2.1. *The following is true:*

(i) *Let $s > \frac{1}{2}$. Then, for any $t \in \mathbb{R}$, the Gaussian measure μ_s is invariant under the map $\mathcal{G}_t$.*

(ii) *Let (X, μ) be a measure space and suppose that T_1 and T_2 are maps from X to itself such that μ is quasi-invariant under T_1 and is quasi-invariant under T_2. Then, μ is quasi-invariant under the composition $T_1 \circ T_2$.*

For a proof of these, see [76, Lemmas 4.4 and 4.5]. We note in the particular case $s = 1$, (i) follows from the results in [66].

2.3 Quasi-invariance for $\alpha > \frac{1}{2}$

In this section, we present part of the proof of Theorem 1.1.2 and Theorem 1.1.3 by applying the argument in [82] (Method 3). Namely, we establish quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1.13) for regularities s given in (1.1.19). We begin in Section 2.3.1 by deriving a suitable modified energy and obtaining the key energy estimate of the form (1.1.11). Then, in Section 2.3.2, we use this energy estimate to conclude the quasi-invariance of μ_s.

2.3.1 Energy estimate

Given a smooth solution v to (2.2.1), let w be as in (2.2.3). Then from (2.2.4), we have

$$\begin{aligned} \frac{d}{dt}\|v(t)\|_{H^s}^2 &= \frac{d}{dt}\|w(t)\|_{H^s}^2 \\ &= -2\operatorname{Re} i \sum_{n\in\mathbb{Z}} \sum_{\Gamma(n)} \langle n\rangle^{2s} e^{it\phi(\overline{n})} w_{n_1}\overline{w_{n_2}} w_{n_3}\overline{w_n} \\ &= \frac{1}{2}\operatorname{Re} i \sum_{\Gamma(\overline{n})} \psi_s(\overline{n}) e^{it\phi(\overline{n})} w_{n_1}\overline{w_{n_2}} w_{n_3}\overline{w_n}, \end{aligned} \tag{2.3.1}$$

where $\overline{n} = (n_1, n_2, n_3, n)$,

$$\Gamma(\overline{n}) := \{(n_1, n_2, n_3, n) \in \mathbb{Z}^4 : n_1 - n_2 + n_3 = n \text{ and } n_1, n_3 \neq n\}$$

and

$$\psi_s(\overline{n}) = \langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n \rangle^{2s}. \tag{2.3.2}$$

The second equality in (2.3.1) follows by a symmetrisation argument. Indeed, a relabelling of the sum implies

$$\operatorname{Re} i \sum_{\Gamma(\overline{n})} \langle n \rangle^{2s} e^{it\phi(\overline{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} = \operatorname{Re} i \sum_{\Gamma(\overline{n})} \langle n_2 \rangle^{2s} e^{i\phi(\overline{n})t} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}.$$

Using the fact that $\operatorname{Re} ia = -\operatorname{Re} i\overline{a}$ for all $a \in \mathbb{C}$, a relabelling also shows

$$\operatorname{Re} i \sum_{\Gamma(\overline{n})} \langle n \rangle^{2s} e^{it\phi(\overline{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} = -\operatorname{Re} i \sum_{\Gamma(\overline{n})} \langle n_1 \rangle^{2s} e^{it\phi(\overline{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}.$$

This symmetrisation puts us in a position to apply Lemma 2.1.2 later. Writing

$$\frac{d}{dt}\left(\frac{e^{it\phi(\overline{n})}}{i\phi(\overline{n})}\right) = e^{it\phi(\overline{n})},$$

and applying the product rule in reverse, (2.3.1) implies

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{H^s}^2 =& \frac{1}{2} \operatorname{Re} \frac{d}{dt} \left[\sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} \right] \\ & - \frac{1}{2} \operatorname{Re} \sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} \partial_t (w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}). \end{aligned} \tag{2.3.3}$$

We now define the modified energy:

$$\begin{aligned} E_{s,t}(z) &= \|z\|_{H^s}^2 - \frac{1}{2} \operatorname{Re} \sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} z_{n_1} \overline{z_{n_2}} z_{n_3} \overline{z_n} \\ &=: \|z\|_{H^s}^2 + R_{s,t}(z). \end{aligned}$$

Then, it follows from (2.3.3) that for any solution w to (2.2.4), we have

$$\frac{d}{dt} E_{s,t}(w) = -\frac{1}{2} \operatorname{Re} \sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} \partial_t (w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}). \tag{2.3.4}$$

At first glance it seems like the modified energy (2.3.5) is non-autonomous in time. However, this time dependence is only superficial. Writing the modified energy in terms of $y := S(t)z$, we have

$$E_s(y) := E_{s,t}(S(-t)y) = \|y\|_{H^s}^2 + R_s(y), \tag{2.3.5}$$

where

$$R_s(y) := -\frac{1}{2}\operatorname{Re}\sum_{\Gamma(\overline{n})}\frac{\psi_s(\overline{n})}{\phi(\overline{n})}y_{n_1}\overline{y_{n_2}}y_{n_3}\overline{y_n}. \tag{2.3.6}$$

Now, the nonlinear functionals E_s and R_s are clearly autonomous in time.

We now state the following key energy estimate which is of the form (1.1.11).

Proposition 2.3.1. *Let (s, α) belong to one of the following regions:*

$$\begin{aligned}
&\text{(i)} && s > 1, && \text{when} && \alpha > \frac{1}{2},\\
&\text{(ii)} && \max\left(\frac{2}{3}, \frac{25}{12} - \alpha\right) < s \leq 1, && \text{when} && \alpha \geq \frac{5}{4},\\
&\text{(iii)} && \frac{3-\alpha}{2} < s \leq 1, && \text{when} && 1 < \alpha < \frac{5}{4}.
\end{aligned} \tag{2.3.7}$$

Then, for sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left|\frac{d}{dt}E_s(\mathbf{P}_{\leq N}v(t))\right| \leq C\|v(t)\|^6_{H^{s-\frac{1}{2}-\varepsilon}}, \tag{2.3.8}$$

for all $N \in \mathbb{N}$ and any solution v to (2.2.2), uniformly in $t \in \mathbb{R}$.

Proof. Using (2.3.5) and the unitarity of $S(t)$ on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, it suffices to prove, that for small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left|\frac{d}{dt}E_{s,t}(\mathbf{P}_{\leq N}w(t))\right| \leq C\|w(t)\|^6_{H^{s-\frac{1}{2}-\varepsilon}} \tag{2.3.9}$$

for all $N \in \mathbb{N}$ and any solution w to (2.2.5), uniformly in $t \in \mathbb{R}$.

Using (2.3.5), (2.2.5) and the symmetry between n_1 and n_3 and between n_2 and n in (the appropriate version of) (2.3.4), we have

$$\frac{d}{dt}E_{s,t}(\mathbf{P}_{\leq N}w) = \mathcal{N}_1(w) + \mathcal{R}_1(w) + \mathcal{N}_2(w) + \mathcal{R}_2(w), \tag{2.3.10}$$

where

$$\mathcal{N}_1(w) := -\mathrm{Re}\, i \sum_{\Gamma_N(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} \left(\sum_{\Gamma_N(n_1)} e^{it\phi(\overline{m},n_1)} w_{m_1} \overline{w_{m_2}} w_{m_3} \right) \overline{w_{n_2}} w_{n_3} \overline{w_n}, \tag{2.3.11}$$

$$\mathcal{R}_1(w) := -\mathrm{Re}\, i \sum_{\Gamma_N(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} |w_{n_1}|^2 w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}, \tag{2.3.12}$$

$$\mathcal{N}_2(w) := -\mathrm{Re}\, i \sum_{\Gamma_N(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} w_{n_1} \left(\sum_{\Gamma_N(n_2)} e^{-it\phi(\overline{m},n_2)} \overline{w_{m_1}} w_{m_2} \overline{w_{m_3}} \right) w_{n_3} \overline{w_n}, \tag{2.3.13}$$

$$\mathcal{R}_2(w) := -\mathrm{Re}\, i \sum_{\Gamma_N(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} e^{it\phi(\overline{n})} w_{n_1} |w_{n_2}|^2 \overline{w_{n_2}} w_{n_3} \overline{w_n}. \tag{2.3.14}$$

Here,

$$\phi(\overline{m}, n_1) := |m_1|^{2\alpha} - |m_2|^{2\alpha} + |m_3|^{2\alpha} - |n_1|^{2\alpha}$$

and

$$\Gamma_N(\overline{n}) := \{(n_1, n_2, n_3, n) \in \Gamma(\overline{n}) : |n_j|, |n| \le N,\ j = 1, 2, 3\}.$$

From now on, we will simply write $\Gamma(\overline{n})$ instead of $\Gamma_N(\overline{n})$ as N plays no further role. In the following, we heavily make use of the Fourier lattice property of $H^s(\mathbb{T})$; namely, that the H^s-norm depends only on the absolute value of the Fourier coefficients. In particular, we assume all the Fourier coefficients w_n are real and non-negative. Moreover, as we never make use of the oscillatory factors such as $e^{it\phi(\overline{n})}$, we will neglect explicitly writing them.

Consider the first scenario (i). For $s > 1$, Lemma 2.1.3 and Lemma 2.1.2 imply we have

$$\frac{|\psi_s(\overline{n})|}{|\phi(\overline{n})|} \le \langle n_{\max} \rangle^{2s-2\alpha}. \tag{2.3.15}$$

We first estimate $\mathcal{N}_1(w)$ by decomposing the sum into two cases depending on which frequency attains $n_{\max}$.

• **Case 1:** $n_{\max} = |n_1|$

From the conditions $n = n_1 - n_2 + n_3$ and $n_1 = m_1 - m_2 + m_3$ we have $\max(|n_2|, |n_3|, |n|) \gtrsim |n_1|$ and $\max_{j=1,2,3} |m_j| \gtrsim |n_1|$, respectively. We assume $|n_2| \gtrsim |n_1|$ and $|m_1| \gtrsim |n_1|$ as the other cases are similar. Hence, we have

$$\langle n_{\max} \rangle^{2s-2\alpha} \lesssim \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_{\max} \rangle^{-2\alpha+1+2\varepsilon} \lesssim \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon}. \tag{2.3.16}$$

Using (2.3.15), (2.3.16) and Young's inequality for the convolution of sequences we have

$$\begin{aligned}|\mathcal{N}_1(w)| &\lesssim \sum_{\Gamma(\overline{n})} \left(\sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right) \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} w_{n_3} w_n \\ &\lesssim \left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell^2_{n_1}} \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|_{\ell^2_n} \|w_n\|^2_{\ell^1_n} .\end{aligned}$$

A further application of Young's inequality gives

$$\left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell^2_{n_1}} \lesssim \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|_{\ell^2_n} \|w_n\|^2_{\ell^1_n} .$$

By Hölder's inequality and choosing ε small enough so that $\frac{1}{2} + \varepsilon \leq s - \frac{1}{2} - \varepsilon$, we have

$$\|w_n\|_{\ell^1_n} \lesssim \|w\|_{H^{\frac{1}{2}+\varepsilon}} \lesssim \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}.$$

Putting this together we get

$$|\mathcal{N}_1(w)| \lesssim \|v\|^6_{H^{s-\frac{1}{2}-\varepsilon}},$$

which is the desired estimate for $\mathcal{N}_1(w)$.

- **Case 2:** $n_{\max} \in \{|n_2|, |n_3|, |n|\}$

It suffices to assume $n_{\max} = |n_2|$ as the remaining cases follow analogously as below. Similar to Case 1, we have $\max(|n_1|, |n_3|, |n|) \gtrsim |n_2|$. If $|n_1| \gtrsim |n_2|$, we proceed in exactly the same way as Case 1. If instead $|n_3| \gtrsim |n_2|$ or $|n| \gtrsim |n_2|$, say the former as both subcases are similar, we use Young's inequality

$$\begin{aligned}|\mathcal{N}_1(w)| &\lesssim \sum_{\Gamma(\overline{n})} \left(\sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right) \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} \langle n_3 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_3} w_n \\ &\lesssim \left\| \sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell^1_{n_1}} \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|^2_{\ell^2_n} \|w_n\|_{\ell^2_n} .\end{aligned}$$

A further application of Young's inequality and Hölder's inequality gives

$$\left\| \sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell^1_{n_1}} \lesssim \|w_n\|^3_{\ell^1_n} \lesssim \|w\|^3_{H^{\frac{1}{2}+\varepsilon}}.$$

This completes the case $n_{\max} = |n_2|$ and hence the estimate for $\mathcal{N}_1(w)$. The estimate for $\mathcal{N}_2(w)$ follows from similar arguments. Now we estimate $\mathcal{R}_1(w)$.

- **Case 1:** $n_{\max} = |n_1|$

  As before, $\max(|n_2|, |n_3|, |n|) \gtrsim |n_1|$. It suffices to assume $|n_2| \gtrsim |n_1|$, as the subcases $|n_3| \gtrsim |n_1|$ and $|n| \gtrsim |n_1|$ are similar. We have,

$$
\begin{aligned}
|\mathcal{R}_1(w)| &\lesssim \sum_{\Gamma(\overline{n})} \langle n_1\rangle^{s-\frac{1}{2}-\varepsilon} w_{n_1}^3 \langle n_2\rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} w_{n_3} w_n \\
&\lesssim \|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n^3\|_{\ell_n^2} \|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^1}^2.
\end{aligned}
$$

Using Hölders inequality and then the embedding $\ell_n^\infty \subset \ell_n^2$ we have

$$
\|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n^3\|_{\ell_n^2} \lesssim \|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^\infty}^2 \lesssim \|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^2}^2.
$$

Putting everything together, we have shown

$$
|\mathcal{R}_1(w)| \lesssim \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}^4 \|w\|_{L^2}^2.
$$

- **Case 2:** $n_{\max} \in \{|n_2|, |n_3|, |n|\}$

  It suffices to assume $n_{\max} = |n_2|$ as the remaining cases follow analogously as below. As before, $\max(|n_1|, |n_3|, |n|) \gtrsim |n_2|$. We apply the argument of Case 1 if $|n_1| \gtrsim |n_2|$. Instead, if $|n_3| \gtrsim |n_2|$, the remaining case being similar, Young's inequality, the embedding $\ell_n^2 \subset \ell_n^3$ and Hölder's inequality yield

$$
\begin{aligned}
|\mathcal{R}_1(w)| &\lesssim \sum_{\Gamma(\overline{n})} w_{n_1}^3 \langle n_2\rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} \langle n_3\rangle^{s-\frac{1}{2}-\varepsilon} w_{n_3} w_n \\
&\lesssim \|w_n\|_{\ell_n^3}^3 \|\langle n\rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2}^2 \|w_n\|_{\ell_n^1} \\
&\lesssim \|w\|_{L^2}^3 \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}^3.
\end{aligned}
$$

This completes the estimate for $\mathcal{R}_1(w)$ and the estimate for $\mathcal{R}_2(w)$ is similar. Thus, we have established (2.3.8) in the region (i).

We now move onto establishing (2.3.8) when $\frac{1}{2} < s \leq 1$. This regime is responsible for the regions (ii) and (iii) in (2.3.7). As before, we begin with $\mathcal{N}_1(w)$. Notice that since $s \leq 1$, we can no longer apply Lemma 2.1.2. We set $\sigma = s - \frac{1}{2} - \varepsilon$ and define $\tilde{w}_n = \langle n\rangle^\sigma w_n$. Without loss of generality, we suppose $|n_1| \lesssim |m_1|$. The regularity restriction of $s > \frac{2}{3}$ arises from the following estimate:

$$
\Bigg\| \sum_{\Gamma(n_1)} \langle m_1\rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \Bigg\|_{\ell_{n_1}^\infty} \lesssim \|w\|_{H^{\frac{1}{6}}}^2 \|w\|_{H^\sigma} \lesssim \|w\|_{H^\sigma}^3, \tag{2.3.17}
$$

where the second inequality holds provided $s > \frac{2}{3}$. We decompose the sum in $\mathcal{N}_1(w)$ into a few cases depending on which frequency attains $n_{\max}$.

• **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

In this case, it is clear from (1.1.18) and Lemma 2.1.3 that

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n - n_1| \tag{2.3.18}$$

and from (2.3.2) and the mean value theorem,

$$|\psi_s(\overline{n})| \lesssim n_{\max}^{2s-1}|n - n_1|. \tag{2.3.19}$$

Hence with (2.3.17), we have

$$\begin{aligned}\text{RHS of (2.3.11)} &\lesssim \sum_{\Gamma(\overline{n})} \frac{n_{\max}^{2s-1}}{n_{\max}^{2\alpha-1}} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{\langle n\rangle^{\sigma}\langle n_1\rangle^{\sigma-\frac{1}{6}}\langle n_2\rangle^{\sigma}\langle n_3\rangle^{\sigma}} \bigg\| \sum_{\Gamma(n_1)} \langle m_1\rangle^{\sigma-\frac{1}{6}} w_{m_1}\overline{w_{m_2}}w_{m_3}\bigg\|_{\ell^\infty_{n_1}} \\ &\lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\overline{n})} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{n_{\max}^{\nu}\langle n_2\rangle^{\sigma}\langle n_3\rangle^{\sigma}},\end{aligned}$$

where

$$\nu = 2\alpha - 2s + 2\sigma - \frac{1}{6} = 2\alpha - \frac{7}{6} - 2\varepsilon > 0.$$

By the Cauchy-Schwarz inequality, we bound this by

$$\|w\|_{H^\sigma}^6 \bigg(\sum_{\Gamma(\overline{n})} \frac{1}{n_{\max}^{2\nu}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{2\sigma}} \bigg)^{\frac{1}{2}}$$

and we are done, provided we show

$$\sum_{\Gamma(\overline{n})} \frac{1}{n_{\max}^{2\nu}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{2\sigma}} \lesssim 1. \tag{2.3.20}$$

For $\delta > 0$ sufficiently small, we have

$$\sum_{\Gamma(\overline{n})} \frac{1}{n_{\max}^{2\nu}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{2\sigma}} \sim \sum_{\substack{n_1, n_2, n_3 \\ |n_2|,|n_3| \ll |n_1|}} \frac{1}{\langle n_1\rangle^{1+\delta}\langle n_2\rangle^{1+\delta}\langle n_3\rangle^{1+\delta}} \frac{\langle n_2\rangle^{1+\delta-2\sigma}\langle n_3\rangle^{1+\delta-2\sigma}}{\langle n_1\rangle^{2\nu-1-\delta}}.$$

Thus, provided

$$2\nu > 1 \qquad \text{and} \qquad 4\sigma + 2\nu > 3, \tag{2.3.21}$$

we have

$$\frac{\langle n_2\rangle^{1+\delta-2\sigma}\langle n_3\rangle^{1+\delta-2\sigma}}{\langle n_1\rangle^{2\nu-1-\delta}} \lesssim \frac{1}{\langle n_1\rangle^{2\nu+4\sigma-3-3\delta}} \lesssim 1,$$

and hence (2.3.20) follows. The first condition in (2.3.21) requires $\alpha > \frac{5}{6}$, while the last condition requires $s > \frac{11}{6} - \alpha$.

- **Case 2:** $|n| \sim |n_2| \gg |n_1|, |n_3|$

  We have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha}, \tag{2.3.22}$$

and using (2.3.17) leads to

$$\text{RHS of (2.3.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\overline{n})} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{n_{\max}^{\nu}\langle n_1\rangle^{\sigma-\frac{1}{6}}\langle n_3\rangle^{\sigma}},$$

where

$$\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon > 0.$$

Using Cauchy-Schwarz as in the previous case, we sum over n_1, n_3 and n_2 provided

$$2\nu > 1 \qquad \text{and} \qquad 2\nu + 4\sigma - \frac{1}{3} > 3$$

and hence $s > \frac{11}{6} - \alpha$. Notice the first condition above requires $\alpha > \frac{3}{4}$.

- **Case 3:** $|n| \sim |n_3| \gg |n_1|, |n_2|$

  We have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n - n_3| \qquad \text{and} \qquad |\psi_s(\overline{n})| \lesssim n_{\max}^{2s-1}|n - n_3|.$$

Thus

$$\text{RHS of (2.3.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\overline{n})} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{n_{\max}^{\nu}\langle n_1\rangle^{\sigma-\frac{1}{6}}\langle n_2\rangle^{\sigma}},$$

where

$$\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon > 0.$$

Using Cauchy-Schwarz as in the previous cases, we sum over n_1, n_2 and n_3 provided

$$2\nu > 1 \qquad \text{and} \qquad 2\nu + 4\sigma - \frac{1}{3} > 3,$$

and hence $s > \frac{11}{6} - \alpha$. The first condition above requires $\alpha > \frac{3}{4}$.

- **Case 4:** $|n_1| \sim |n_2| \gg |n_3|, |n|$

  We have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n - n_3| \qquad \text{and} \qquad |\psi_s(\overline{n})| \lesssim n_{\max}^{2s-1}|n - n_3|$$

and we proceed as in Case 1 as long as $s > \frac{11}{6} - \alpha$ and $\alpha > \frac{5}{6}$.

- **Case 5:** $|n_1| \sim |n_3| \gg |n_2|, |n|$
  We have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha},$$

and hence

$$\text{RHS of (2.3.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\overline{n})} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{n_{\max}^{\nu}\langle n\rangle^{\sigma}\langle n_2\rangle^{\sigma}},$$

where

$$\nu = 2\alpha - 2s + 2\sigma - \frac{1}{6} = 2\alpha - \frac{7}{6} - 2\varepsilon > 0.$$

By Cauchy-Schwarz and summing in n, n_2 and n_1 as long as

$$2\nu > 1 \qquad \text{and} \qquad 2\nu + 4\sigma > 3,$$

and hence $s > \frac{11}{6} - \alpha$. The first condition above is satisfied provided $\alpha > \frac{5}{6}$.

- **Case 6:** $|n_2| \sim |n_3| \gg |n_1|, |n|$
  We have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n - n_1| \qquad \text{and} \qquad |\psi_s(\overline{n})| \lesssim n_{\max}^{2s-1}|n - n_1|,$$

and we proceed as in Case 4 as long as $s > \frac{11}{6} - \alpha$ and $\alpha > \frac{3}{4}$.

- **Case 7:** $|n_1| \sim |n_2| \sim |n_3| \gg |n|$
  From Lemma 2.1.3, we have $|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha}$ and hence

$$\text{RHS of (2.3.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\overline{n})} \frac{n_{\max}^{2s}\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{n_{\max}^{2\alpha} n_{\max}^{3\sigma - \frac{1}{6}}\langle n\rangle^{\sigma}}.$$

Applying Cauchy-Schwarz in n_2, n_3 and n, we sum provided

$$2\bigg(2\alpha + 3\sigma - \frac{1}{6} - 2s\bigg) > 2 \qquad \text{and} \qquad 2\alpha - 2s + 3\sigma - \frac{1}{6} + \sigma > \frac{3}{2},$$

which requires $\alpha + s > \frac{11}{6}$ and $2\alpha + s > \frac{8}{3}$. Notice that when $\alpha > \frac{5}{6}$, this latter condition is superseded by the former. The remaining cases of the form $|n_{j_1}| \sim |n_{j_2}| \sim |n_{j_3}| \gg |n_{j_4}|$ with distinct $j_k \in \{1, 2, 3, 4\}$ ($n_{j_4} := n$) are similar and are thus omitted.

- **Case 8:** $|n| \sim |n_1| \sim |n_2| \sim |n_3|$

We distinguish when α is 'close too' or 'far from' 1.

- **Subcase 8.1:** $1 < \alpha < \frac{5}{4}$

By Lemma 2.1.3, we have

$$\sum_{\Gamma(\overline{n})} \frac{n_{\max}^{2s}}{|n-n_1||n-n_3|n_{\max}^{2\alpha-2}\langle n\rangle^\sigma}\bigg(\prod_{j=1}^3 \frac{1}{\langle n_j\rangle^\sigma}\bigg)\bigg(\sum_{\Gamma(n_1)} \tilde{w}_{m_1}w_{m_2}w_{m_3}\bigg)\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n$$
$$\lesssim \sum_{\Gamma(\overline{n})} \frac{1}{|n-n_1||n-n_3|n_{\max}^{\nu}}\bigg(\sum_{\Gamma(n_1)} \tilde{w}_{m_1}w_{m_2}w_{m_3}\bigg)\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n,$$

where

$$\nu = 2\alpha - 2 - 2s + 4\sigma = 2(\alpha + s - 2 - 2\varepsilon) > 0, \tag{2.3.23}$$

provided

$$\alpha + s > 2. \tag{2.3.24}$$

By Cauchy-Schwarz in n, n_1 and n_3, followed by summing in n and n_3, we get

$$\lesssim \|w\|_{H^\sigma}^3 \bigg\| \frac{1}{\langle n-n_1\rangle\langle n-n_3\rangle\langle n_{\max}\rangle^\nu}\bigg(\sum_{\Gamma(n_1)} \tilde{w}_{m_1}w_{m_2}w_{m_3}\bigg)\bigg\|_{\ell^2_{n,n_1,n_3}}$$
$$\lesssim \|w\|_{H^\sigma}^3 \bigg\| \frac{1}{\langle n_1\rangle^\nu}\sum_{\Gamma(n_1)} \tilde{w}_{m_1}w_{m_2}w_{m_3}\bigg\|_{\ell^2_{n_1}}.$$

Imposing

$$\alpha + s < 2 + \frac{1}{4} \tag{2.3.25}$$

implies $\nu < \frac{1}{2}$ so that we can apply Hölder's inequality and then Young's inequality, with exponents[2]

$$\frac{1-2\nu+}{2} + 2 = \frac{1}{2} + \frac{2}{\frac{2}{2-\nu+}},$$

to obtain

$$\lesssim \|w\|_{H^\sigma}^3 \bigg\| \sum_{\Gamma(n_1)} \tilde{w}_{m_1}w_{m_2}w_{m_3}\bigg\|_{\ell_{n_1}^{\frac{2}{1-2\nu+}}}$$
$$\lesssim \|w\|_{H^\sigma}^4 \|w_n\|^2_{\ell_n^{\frac{2}{2-\nu+}}}.$$

Once more with Hölder's inequality, we have

$$\lesssim \|w\|_{H^\sigma}^6 \|\langle n\rangle^{-\sigma}\|^2_{\ell_n^{\frac{2}{1-\nu+}}} \lesssim \|w\|_{H^\sigma}^6,$$

[2]Here, we use the notation $a-$ (respectively, $a+$) to denote $a-\delta$ (respectively, $a+\delta$), where $0<\delta\ll 1$ is extremely small.

provided

$$\frac{2\sigma}{1-\nu+} > 1.$$

Using (2.3.23), this last condition requires

$$s > \frac{3-\alpha}{2}. \tag{2.3.26}$$

Putting the conditions (2.3.24), (2.3.25) and (2.3.26) together implies we must enforce in this subcase

$$\max\left(\frac{1}{2}, \frac{3-\alpha}{2}, 2-\alpha\right) < s \leq \min\left(1, \frac{9}{4}-\alpha\right),$$

where the upper bound is strict if the minimum is $\frac{9}{4}-\alpha$. Now as $\alpha \leq \frac{5}{4}$, $\min\big(1, \frac{9}{4}-\alpha\big) = 1$ and this implies the range

$$\frac{3-\alpha}{2} < s \leq 1.$$

- **Subcase 8.2:** $\alpha \geq \frac{5}{4}$

Given $n \in \mathbb{Z}$, let

$$\Gamma(n,\rho) = \Gamma(n) \cap \{(n_1,n_2,n_3) \in \mathbb{Z}^3 \,:\, \rho = (n-n_1)(n-n_3) \in \mathbb{Z}\}.$$

From (2.1.1) we have, for any $\delta > 0$, there exists a $C_\delta > 0$ such that

$$|\#\Gamma(n,\rho)| \lesssim C_\delta |\rho|^\delta. \tag{2.3.27}$$

By Lemma 2.1.3 and (2.3.17), we have

$$\begin{aligned}
|\mathcal{N}_1(w)| &\lesssim \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n,\rho)} \frac{n_{\max}^{2s}}{|\rho| n_{\max}^{2\alpha-2}} \frac{\tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n}{\langle n\rangle^\sigma \langle n_1\rangle^{\sigma-\frac{1}{6}} \langle n_2\rangle^\sigma \langle n_3\rangle^\sigma} \left\| \sum_{\Gamma(n)} \langle m_1\rangle^{\sigma-\frac{1}{6}} w_{m_1}\overline{w_{m_2}} w_{m_3} \right\|_{\ell^\infty_{n_1}} \\
&\lesssim \|w\|_{H^\sigma}^3 \sum_n \sum_{\rho\neq 0} \sum_{\Gamma(n,\rho)} \frac{1}{|\rho| n_{\max}^\nu} \tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n,
\end{aligned}$$

where

$$\nu = 2\alpha - 2 - 2s + 4\sigma - \frac{1}{6} = 2\alpha + 2s - \frac{25}{6} - 2\varepsilon,$$

which is positive provided

$$\alpha + s > \frac{25}{12}. \tag{2.3.28}$$

To continue, we follow the argument in [76, Proposition 6.1] and for completeness we detail it here. By Cauchy-Schwarz, (2.3.27) and Lemma 2.1.4, we have

$$\begin{aligned}
&\lesssim \|w\|_{H^\sigma}^4 \Bigg[\sum_n \bigg(\sum_{\rho\neq 0}\sum_{\Gamma(n,\rho)} \frac{1}{|\rho| n_{\max}^{\nu}} \tilde{w}_{n_2}\tilde{w}_{n_3}\bigg)^2\Bigg]^{\frac{1}{2}} \\
&\lesssim \|w\|_{H^\sigma}^4 \Bigg[\sum_n \bigg(\sum_{\rho\neq 0} \frac{1}{|\rho|^{1+2\delta}} \sum_{\Gamma(n,\rho)} 1\bigg) \sum_{\rho\neq 0}\sum_{\Gamma(n,\rho)} \frac{1}{|\rho|^{1-2\delta} n_{\max}^{2\nu}} \tilde{w}_{n_2}^2\tilde{w}_{n_3}^2\Bigg]^{\frac{1}{2}} \\
&\lesssim \|w\|_{H^\sigma}^4 \bigg(\sum_{n_2,n_3} \tilde{w}_{n_2}^2\tilde{w}_{n_3}^2 \sum_{n_1\neq n_2} \frac{1}{|n_1-n_2|^{1-2\delta}\langle n_1\rangle^{2\nu}}\bigg)^{\frac{1}{2}} \\
&\lesssim \|w\|_{H^\sigma}^6,
\end{aligned}$$

where from (2.3.28) we choose $\delta > 0$ small enough so that $\delta < \nu$.

The required range for s in this subcase is

$$\max\left(\frac{2}{3}, \frac{25}{12} - \alpha\right) < s \leq 1.$$

This completes the estimates for $\mathcal{N}_1(w)$. The estimate for $\mathcal{N}_2(w)$ follows analogously.

We now move onto bounding $\mathcal{R}_1(w)$. Writing

$$m(\overline{n}) := \frac{|\psi_s(\overline{n})|}{|\phi(\overline{n})|\langle n_1\rangle^{3\sigma}\langle n_2\rangle^{\sigma}\langle n_3\rangle^{\sigma}\langle n\rangle^{\sigma}},$$

it suffices to show

$$\sum_{\Gamma(\overline{n})} m(\overline{n})|\tilde{w}_{n_1}|^3 \tilde{w}_{n_2}\tilde{w}_{n_3}\tilde{w}_n \lesssim \|\tilde{w}\|_{L^2}^6. \tag{2.3.29}$$

As above, we divide into a few cases.

- **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

Using (2.3.18) and (2.3.19) we have

$$m(\overline{n}) \lesssim \frac{1}{n_{\max}^{\nu}\langle n_2\rangle^{\sigma}\langle n_3\rangle^{\sigma}},$$

where $\nu = 2\alpha - 2s + 4\sigma = 2\alpha + 2s - 2 - 4\varepsilon$. By Cauchy-Schwarz in n_2, n_3 and n and the embedding $\ell_n^2 \subset \ell_n^6$, we have

$$\text{LHS of (2.3.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \bigg(\sum_n \tilde{w}_n^2 \sum_{n_2,n_3} \frac{1}{n_{\max}^{2\nu}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{2\sigma}}\bigg)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6,$$

where we can sum provided $4\sigma + 2\nu > 2$ which requires $s > 1 - \frac{1}{2}\alpha$.

• **Case 2:** $|n| \sim |n_3| \gg |n_1|, |n_2|$
Using (2.3.22), we have

$$m(\overline{n}) \lesssim \frac{1}{n_{\max}^{\nu} \langle n_1 \rangle^{3\sigma} \langle n_2 \rangle^{\sigma}},$$

where $\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon$. By Cauchy-Schwarz in n_1, n_2 and n_3 and the embedding $\ell_n^2 \subset \ell_n^6$, we have

$$\text{LHS of (2.3.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \bigg(\sum_{n_3} \tilde{w}_{n_3}^2 \sum_{n_1, n_2} \frac{1}{n_{\max}^{2\nu} \langle n_1 \rangle^{6\sigma} \langle n_2 \rangle^{2\sigma}} \bigg)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6,$$

where we can sum provided $s > \max(\frac{2}{3}, 2 - 2\alpha)$.

• **Case 3:** $|n_1| \sim |n_2| \sim |n_3| \sim |n|$
From (1.1.18), we have

$$m(\overline{n}) \lesssim \frac{n_{\max}^{2s}}{|n - n_1||n - n_3| n_{\max}^{2\alpha - 2}} \frac{1}{\langle n_1 \rangle^{3\sigma} \langle n_2 \rangle^{\sigma} \langle n_3 \rangle^{\sigma} \langle n \rangle^{\sigma}} \sim \frac{1}{|n - n_1||n - n_3| n_{\max}^{\nu}},$$

where

$$\nu = 2\alpha - 2 - 2s + 6\sigma > 0,$$

provided $s > \frac{5}{4} - \frac{1}{2}\alpha$. An application of Cauchy-Schwarz then implies

$$\text{LHS of (2.3.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \bigg(\sum_{n} \tilde{w}_{n}^2 \sum_{n_1, n_3 \neq n} \frac{1}{|n - n_1|^2 |n - n_3|^2} \bigg)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6.$$

All remaining cases follow analogously from the methods in either Case 1 or Case 2 above and are thus omitted. This completes the bound for $\mathcal{R}_1$. Notice the condition from Case 3 supersedes the conditions from Cases 1 and 2. Furthermore, at least for every $\alpha \geq \frac{7}{6}$, we have

$$\frac{2}{3} \geq \frac{5}{4} - \frac{1}{2}\alpha$$

and hence we obtain the final conditions on s and α of (2.3.7).
Finally, this completes the proof of (2.3.9).

□

We also have the following difference estimate for $R_s(v)$. It will be convenient to view R_s as a multi-linear functional

$$R_s(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}) := \frac{1}{2} \text{Re} \sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} u_{n_1}^{(1)} \overline{u_{n_2}^{(2)}} u_{n_3}^{(3)} \overline{u_{n}^{(4)}},$$

where $R_s(v, v, v, v) = R_s(v)$.

Proposition 2.3.2. *Suppose*

$$\begin{aligned} &\text{(i)} \qquad & s > 1, \qquad & \text{when} \qquad \alpha > \frac{1}{2}, \text{ or} \\ &\text{(ii)} \qquad & s > \max\left(2-\alpha, \frac{1}{2}\right), \qquad & \text{when} \qquad \alpha \geq 1. \end{aligned} \tag{2.3.30}$$

Then, for sufficiently small $\varepsilon > 0$ there exists $C > 0$ such that

$$|R_s(u) - R_s(v)| \leq C\|u-v\|_{H^{s-\frac{1}{2}-\varepsilon}}(\|u\|^3_{H^{s-\frac{1}{2}-\varepsilon}} + \|v\|^3_{H^{s-\frac{1}{2}-\varepsilon}})$$

for all $u, v \in H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$.

Proof. By the multi-linearity of $R_s(u)$, it suffices to show

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}, \tag{2.3.31}$$

where $\sigma := s - \frac{1}{2} - \varepsilon$.

We first consider case (i) in (2.3.30). Using (2.3.6) and Lemmas 2.1.3 and 2.1.2 we have

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \sum_{\Gamma(\overline{n})} \langle n_{\max}\rangle^{2s-2\alpha}|u_{n_1}^{(1)}||u_{n_2}^{(2)}||u_{n_3}^{(3)}||u_n^{4)}|.$$

Similar to the proof of (2.3.1), it suffices to consider the following case when $n_{\max} = |n_1|$ and $|n_2| \gtrsim |n_1|$. Using Young's and Hölder's inequality, we get

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \|\langle n_1\rangle^\sigma u_{n_1}^{(1)}\|_{\ell^2_{n_1}}\|\langle n_2\rangle^\sigma u_{n_2}^{(2)}\|_{\ell^2_{n_2}}\|u_{n_3}^{(3)}\|_{\ell^1_{n_3}}\|u_n^{(4)}\|_{\ell^1_n} \lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}.$$

We now consider case (ii) in (2.3.30). As it is already contained within the case $s > 1$ and $\alpha > \frac{1}{2}$ proved above, we now bound $|R_s(\{u^{(j)}\}_{j=1}^4)|$ when $s \leq 1$ for $\alpha \geq 1$. Given such an s, let $\sigma = s - \frac{1}{2} - \varepsilon$. We have

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \sum_{\Gamma(\overline{n})} m(\overline{n})|\tilde{u}_{n_1}^{(1)}||\tilde{u}_{n_2}^{(2)}||\tilde{u}_{n_3}^{(3)}|\tilde{u}_n^{(4)}|, \tag{2.3.32}$$

where

$$m(\overline{n}) = \frac{|\psi_s(\overline{n})|}{|\phi(\overline{n})|\langle n\rangle^\sigma} \prod_{j=1}^3 \frac{1}{\langle n_j\rangle^\sigma}.$$

As in the proof of Proposition 2.3.1, we consider a few cases depending on $n_{\max}$.

- **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

In this case, we have

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n-n_1| \qquad \text{and} \qquad |\psi(\overline{n})| \lesssim n_{\max}^{2s-1}|n-n_1|,$$

and thus

$$m(\overline{n}) \lesssim \frac{1}{n_{\max}^{\nu}}\frac{1}{\langle n_2\rangle^{\sigma}\langle n_3\rangle^{\sigma}},$$

where $\nu = 2\alpha - 1 - 2\varepsilon > 0$. By Cauchy-Schwarz inequality,

$$\text{RHS of (2.3.32)} \lesssim \prod_{j=1}^{3}\|u^{(j)}\|_{H^\sigma}^3\bigg(\sum_{n,n_2,n_3\in\mathbb{Z}}\frac{1}{n_{\max}^{2\nu}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{2\sigma}}(\tilde{u}_n^{(4)})^2\bigg)^{\frac{1}{2}} \lesssim \prod_{j=1}^{4}\|u^{(j)}\|_{H^\sigma}^4,$$

where we can sum in n_2 and n_3 provided $\nu + 2\sigma > 1$, which requires

$$\alpha + s > \frac{3}{2}.$$

- **Case 2:** $|n| \sim |n_2| \gg |n_1|, |n_3|$

Here we use $|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha}$ which implies

$$m(\overline{n}) \lesssim \frac{1}{n_{\max}^{\nu}\langle n_1\rangle^{\sigma}\langle n_3\rangle^{\sigma}},$$

where $\nu = 2\alpha - 1 - 2\varepsilon > 0$. We proceed as in Case 1 by using Cauchy-Schwarz and summing in n_1 and n_3 with $\tilde{v}_{n_2}^2$ absorbing the remaining n_2 summation.

It is easy to check that all remaining Cases 3 through 7 as explicated in the proof of Proposition 2.3.1 follow analogously to the two cases above.

- **Case 3:** $|n| \sim |n_1| \sim |n_2| \sim |n_3|$

We can only use the lower bound of Lemma 2.1.3 and this implies

$$m(\overline{n}) \lesssim \frac{1}{|n-n_1||n-n_3|n_{\max}^{\nu}},$$

where $\nu = 2\alpha - 2 - 2s + 4\sigma$ which is non-negative provided

$$\alpha + s > 2.$$

Then Cauchy-Schwarz over $\Gamma(\overline{n})$ gives

$$\begin{aligned}\text{RHS of (2.3.32)} &\lesssim \prod_{j=1}^{3}\|u^{(j)}\|_{H^\sigma}^3\bigg(\sum_{n\in\mathbb{Z}}(\tilde{u}_n^{(4)})^2\sum_{n_1,n_3\in\mathbb{Z}}\frac{1}{|n-n_1|^2|n-n_3|^2}\bigg)^{\frac{1}{2}}\\ &\lesssim \prod_{j=1}^{4}\|u^{(j)}\|_{H^\sigma}^4.\end{aligned}$$

This completes the proof of (2.3.31).

□

Remark 2.3.3. Since $2-\alpha \leq \frac{25}{12} - \alpha$ and $2 - \alpha \leq \frac{3-\alpha}{2}$ for $\alpha \geq 1$, the restriction (2.3.7) for the energy estimate supersedes (2.3.30) (ii), which is the condition for the correction term $R_s(u)$.

2.3.2 Proof of Theorem 1.1.2 (ii)

In this subsection, we follow the argument introduced in [82] to conclude quasi-invariance of Gaussian measures μ_s for $\alpha > \frac{1}{2}$ and those s given in Proposition 2.3.1. In particular, we conclude Theorem 1.1.2 (ii) and the $s > 1$ portion of Theorem 1.1.3.

We define the following measures:

$$d\rho_s = F_s(u)d\mu_s \qquad \text{and} \qquad d\rho_{s,N} = F_{s,N}(u)d\mu_s,$$

where

$$F_s(u) = \exp\big(-\tfrac{1}{2}E_s(u) + \tfrac{1}{2}\|u\|_{H^s}^2\big) = \exp\big(-\tfrac{1}{2}R_s(u)\big) \quad \text{and} \quad F_{s,N}(u) = F_s(\mathbf{P}_{\le N}u).$$

The measure $\rho_{s,N}$ can also be expressed as

$$d\rho_{s,N} = Z_{s,N}^{-1}\exp\big(-\tfrac{1}{2}E_s(\mathbf{P}_{\le N}u)\big)\, du_{\le N} \times d\mu_{s,N}^{\perp}, \tag{2.3.33}$$

where $du_{\le N}$ denotes the Lebesgue measure on $\mathbb{C}^{2N+1}$. The constant $Z_{s,N}^{-1}$ is the normalisation constant associated to the measure $\mu_{s,N}$ which is given by

$$d\mu_{s,N} = Z_{s,N}^{-1} e^{-\frac{1}{2}\|\mathbf{P}_{\le N}u\|_{H^s}^2} du_N.$$

In particular, $\mu_{s,N}$ is the probability measure induced under the map

$$\omega \in \Omega \longmapsto u_{\le N}^{\omega}(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}.$$

Likewise, $\mu_{s,N}^{\perp}$ is the probability measure induced under the map

$$\omega \in \Omega \longmapsto u_{> N}^{\omega}(x) = \sum_{|n|> N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}.$$

Note that we do not require F_s and $F_{s,N}$ to be integrable with respect to μ_s. Hence ρ_s and $\rho_{s,N}$ are not necessarily probability measures. However, as the quasi-invariance argument is purely local (see the proof of Theorem 1.2 (i) below), it suffices to have $F_{s,N} \in L^1_{\text{loc}}(\mu_s)$ and with convergence to F_s. This is the content of the next proposition, whose proof can be found in [82, Proposition 2.1].

Proposition 2.3.4. *Let s be as in* (2.3.30). *Then, for every bounded set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, we have*

$$\lim_{N\to\infty} \int_A |F_{s,N}(u) - F_s(u)| d\mu_s(u) = 0$$

and in particular

$$\lim_{N\to\infty} |\rho_{s,N}(A) - \rho_s(A)| = 0$$

The next result states important properties of the truncated flow Φ_N.

Proposition 2.3.5. *Let s be as in Proposition 1.1.1 be such that the flow Φ of FNLS* (1.1.13) *is globally well-defined. Then, the following statements hold:*

(i) *For every $R > 0$ and $T > 0$, there exists $C(R,T) > 0$ such that*

$$\Phi_N(t)\,(B_R) \subset B_{C(R,T)}$$

for all $t \in [0,T]$ and for all $N \in \mathbb{N} \cup \{\infty\}$. Here, $\Phi_\infty := \Phi$ denotes the untruncated flow.

(ii) *Let $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ be a compact set and $t \in \mathbb{R}$. Then, for every $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$\|\Phi(t)(u) - \Phi_N(t)(u)\|_{H^s} < \delta,$$

for any $u \in A$ and any $N \geq N_0$. Furthermore, we have

$$\Phi(t)\,(A) \subset \Phi_N(t)\,(A + B_\delta)$$

for all $N \geq N_0$.

We also have the following local-in-time version of Proposition 2.3.5.

Proposition 2.3.6. *Let s be as in Proposition 1.1.1* (ii) *be such that the flow Φ of FNLS* (1.1.13) *is only locally well-defined. Then, the following statements hold:*

(i) *Then, for every $R > 0$, there exist $T(R) > 0$ and $C(R) > 0$ such that*

$$\Phi_N(t)\,(B_R) \subset B_{C(R)}$$

for all $t \in [0, T(R)]$ and for all $N \in \mathbb{N} \cup \{\infty\}$.

(ii) *Let $A \subset B_R \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ be a compact set and denote by $T(R) > 0$ the local existence time of the solution map Φ defined on B_R. Then, for every $\delta > 0$, there exists $N_0 \in \mathbb{N}$, such that*

$$\|\Phi(t)(u) - \Phi_N(t)(u)\|_{H^s} < \delta,$$

for any $u \in A$, $N \geq N_0$ and $t \in [0, T(R)]$. Furthermore, we have

$$\Phi(t)\,(A) \subset \Phi_N(t)\,(A + B_\delta)$$

for all $t \in [0, T(R)]$ and for all $N \geq N_0$.

The proof of Proposition 2.3.5 (i) follows from the global well-posedness of FNLS (1.1.13) when $\alpha > \frac{10\alpha+1}{12}$, while the proof of Proposition 2.3.6 (i) follows by the local existence theory (for short times) when $\frac{1}{2} < \alpha \leq \frac{10\alpha+1}{12}$. The proof of Proposition 2.3.5 (ii) follows from the arguments in [76, Appendix B] using the existence theory in [30], [8] and [16].

Proof of Theorem 1.1.2 (i). In the following we fix s and α satisfying the conditions of Propositions 2.3.1 and 2.3.2. As long as the conclusions of these propositions are satisfied, the following general argument due to [82] implies the quasi-invariance of μ_s (either globally or locally in time). For clarity, we will only detail the following arguments in the case when FNLS (1.1.13) admits a globally well-defined flow Φ (see Proposition 1.1.1). We obtain local-in-time quasi-invariance from the same arguments by suitably restricting to the local well-posedness lifetime where necessary.

Given $t > 0$, by the inner regularity of the measure μ_s, it is enough to show that

$$A \subset H^{s-\frac{1}{2}-\varepsilon} \text{ compact and } \mu_s(A) = 0 \implies \mu_s(\Phi(-t)A) = 0. \tag{2.3.34}$$

From Proposition 2.3.2 with $u = 0$, we have $0 < \exp(R_s(v)) < \infty$ for almost all $v \in A$. Hence the implication (2.3.34) is equivalent to the following implication:

$$A \subset H^{s-\frac{1}{2}-\varepsilon} \text{ compact and } \rho_s(A) = 0 \implies \rho_s(\Phi(-t)A) = 0.$$

As A is compact, there exists $R > 0$ such that $A \subset B_R$. Then, by Proposition 2.3.5, there exists a constant $C(R) > 0$ such that

$$\Phi(\tau)(B_{2R}) \cup \Phi_N(\tau)(B_{2R}) \subset B_{C(R)} \tag{2.3.35}$$

for all $\tau \in [0,t]$. For a measurable $D \subset B_{2R}$, it follows from (2.3.33), Liouville's theorem and the invariance of complex Gaussians under rotations, that

$$\begin{aligned}\left|\frac{d}{d\tau}\rho_{s,N}(\Phi_N(\tau)(D))\right| &= \left|\frac{d}{d\tau}Z_{s,N}^{-1}\int_{\Phi_N(\tau)(D)}\exp\left(-\tfrac{1}{2}E_s(\mathbf{P}_{\leq N}u)\right)du_{\leq N}\times d\mu_{s,N}^{\perp}\right| \\ &= \left|Z_{s,N}^{-1}\int_D \frac{d}{d\tau}\exp\left(-\tfrac{1}{2}E_s(\Phi(\tau)(\mathbf{P}_{\leq N}u))\right)du_{\leq N}\times d\mu_{s,N}^{\perp}\right|.\end{aligned}$$

Using the energy estimate of Proposition 2.3.1 along with (2.3.35) we have

$$\left|\frac{d}{d\tau}\exp\left(-\tfrac{1}{2}E_s(\Phi(\tau)(\mathbf{P}_{\leq N}u))\right)\right| \leq C(R)\exp\left(-\tfrac{1}{2}E_s(\Phi(\tau)(\mathbf{P}_{\leq N}u))\right)$$

for all $\tau \in [0,t]$ and for all $u \in D$. Combining the above we have

$$\begin{aligned}\left|\frac{d}{d\tau}\rho_{s,N}(\Phi_N(\tau)(D))\right| &\leq Z_{s,N}^{-1}C(R)\int_D \frac{d}{d\tau}\exp\left(-\tfrac{1}{2}E_s(\Phi(\tau)(\mathbf{P}_{\leq N}u))\right)du_{\leq N}\times d\mu_{s,N}^{\perp} \\ &= C(R)\rho_{s,N}(\Phi_N(\tau)(D)).\end{aligned}$$

From Gronwall's inequality, we get

$$\rho_{s,N}(\Phi_N(\tau)(D)) \leq e^{C(R)\tau}\rho_{s,N}(D) \tag{2.3.36}$$

for all $N \in N$ and for all $\tau \in [0,t]$. By Proposition 2.3.5 (ii), we have

$$\rho_s(\Phi_N(\tau)(A)) \leq \rho_s\left(\Phi_N(\tau)(A + B_\delta)\right)$$

for any fixed $\delta > 0$ and N large enough. Further, from Proposition 2.3.4 for N large enough, we have

$$\rho_s\left(\Phi_N(\tau)(A + B_\delta)\right) \leq \rho_{s,N}\left(\Phi_N(\tau)(A + B_\delta)\right) + \delta$$

and so

$$\rho_s(\Phi_N(\tau)(A)) \leq \rho_{s,N}(\Phi_N(\tau)(A + B_\delta)) + \delta.$$

Choosing $\delta < R$ so that $A + B_\delta \subset B_{2R}$ and (2.3.36), can be applied we get

$$\rho_s(\Phi_N(\tau)(A)) \leq e^{C(R)\tau}\rho_{s,N}(A + B_\delta) + \delta.$$

Using Proposition 2.3.4 to go from $\rho_{s,N}$ back to ρ_s, we have

$$\rho_s(\Phi_N(t)(A)) \leq e^{C(R)t}\rho_s(A + B_\delta) + 2\delta.$$

Letting δ approach 0 and using regularity properties of the measure μ_s, we finally obtain

$$\rho_s(\Phi_N(\tau)(A)) \leq e^{C(R)\tau}\lim_{\delta\to 0}\rho_s(A + B_\delta) = e^{C(R)\tau}\rho_s(A) = 0$$

for any $\tau \in [0, t]$. This completes the proof. □

2.4 Improvement for $\alpha > \frac{5}{6}$

In this section, we employ the hybrid argument (Method 4) from [40] in order to lower the regularity threshold we previously obtained using Method 3. Namely, we complete the proofs of Theorem 1.1.2 (i) and Theorem 1.1.3 by proving the quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1.13) for regularities satisfying (1.1.22).

2.4.1 Alternative energy estimate

Our first port of call is to obtain an energy estimate where we place two factors into the Fourier-Lebesgue space $\mathcal{F}L^{\sigma,\infty}(\mathbb{T})$, where $\sigma < s$. By placing these two factors into this stronger norm, we can lower the regularity restriction; compare (2.3.7) and (2.4.1).

Proposition 2.4.1. *Let $\alpha > \frac{5}{6}$ and*

$$\max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right) < s \leq 1. \tag{2.4.1}$$

Then, for sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left|\frac{d}{dt}E_s(\mathbf{P}_{\leq N}v(t))\Big|_{t=0}\right| \leq C\|\mathbf{P}_{\leq N}v(0)\|^2_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}}\|\mathbf{P}_{\leq N}v(0)\|^4_{H^{s-\frac{1}{2}-\varepsilon}}, \tag{2.4.2}$$

for any $N \in \mathbb{N}$, any solution v to (2.2.2) and for any $0 < \tilde{\varepsilon} < \varepsilon$, uniformly in $t \in \mathbb{R}$.

Proof. Using (2.3.5), the estimate (2.4.2) reduces to proving that for small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left| \frac{d}{dt} E_{s,t}(\mathbf{P}_{\leq N} w(t)) \Big|_{t=0} \right| \leq C \|\mathbf{P}_{\leq N} w(0)\|^2_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}} \|\mathbf{P}_{\leq N} w(0)\|^4_{H^{s-\frac{1}{2}-\varepsilon}}. \tag{2.4.3}$$

From (2.3.10), (2.3.11), (2.3.12), (2.3.13) and (2.3.14), (2.4.3) further reduces to showing

$$\left| \sum_{j=1}^{2} \mathcal{N}_j(\mathbf{P}_{\leq N} w(0)) + \sum_{j=1}^{2} \mathcal{R}_j(\mathbf{P}_{\leq N} w(0)) \right| \leq C \|\mathbf{P}_{\leq N} w(0)\|^2_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}} \|\mathbf{P}_{\leq N} w(0)\|^4_{H^{s-\frac{1}{2}-\varepsilon}} \tag{2.4.4}$$

for all $N \in \mathbb{N}$ and uniformly in $t \in \mathbb{R}$. Recalling the decomposition (2.3.10), we estimate $\mathcal{N}_1$ and $\mathcal{R}_1$, with estimates for $\mathcal{N}_2$ and $\mathcal{R}_2$ following analogously. In the following, we simply replace $w(0)$ by w. We consider $\mathcal{N}_1$ first. Recall that in Cases 1 through 7 of the proof of Proposition 2.3.1 (ii) and (iii), we obtained

$$|\mathcal{N}_1(w)| \lesssim \|w\|^6_{H^{s-\frac{1}{2}-\varepsilon}} \tag{2.4.5}$$

for any $\alpha > \frac{5}{6}$ and for any s satisfying

$$1 \geq s > \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right).$$

Then in these cases, we obtain (2.4.4) by using the embedding (1.1.21) to put two factors of (2.4.5) into the required Fourier-Lebesgue space. Note that we could certainly improve upon the regularity lower bound on s in these cases by proving (2.4.4) 'directly.' However, we find that consideration of the remaining case $|n| \sim |n_1| \sim |n_2| \sim |n_3|$ yields a restriction on s given by (2.4.1). We now describe this remaining case. To simplify notation, we drop the frequency projections $\mathbf{P}_{\leq N}$. Furthermore, we let $\sigma = s - \frac{1}{2} - \varepsilon$ and we set $\tilde{w}_n = \langle n \rangle^\sigma w_n$ and $\mathbf{w}_n = \langle n \rangle^{s-\tilde{\varepsilon}} w_n$. We employ the argument (and the notation) from subcase 8.2 in the proof of Proposition 2.3.1. We have

$$\begin{aligned}
|\mathcal{N}_1(w)| &\lesssim \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n,\rho)} \frac{n_{\max}^{2s}}{|\rho| n_{\max}^{2\alpha-2}} \frac{\mathbf{w}_{n_2} \mathbf{w}_{n_3} \tilde{w}_n}{\langle n \rangle^\sigma \langle n_1 \rangle^{\sigma - \frac{1}{6}} \langle n_2 \rangle^{s-\tilde{\varepsilon}} \langle n_3 \rangle^{s-\tilde{\varepsilon}}} \left\| \sum_{\Gamma(n)} \langle m_1 \rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \right\|_{\ell^\infty_{n_1}} \\
&\lesssim \|w\|^3_{H^\sigma} \|w\|^2_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}} \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n,\rho)} \frac{1}{|\rho| n_{\max}^{\nu}} \tilde{w}_n,
\end{aligned}$$

where

$$\nu = 2\alpha + 2s - \frac{19}{6} > 0,$$

provided $\alpha + s > \frac{19}{12}$. Then, by Cauchy-Schwarz and the divisor counting lemma (2.3.27), we bound the above by

$$\|w\|_{H^\sigma}^4 \|w\|_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}}^2 \left(\sum_n \sum_{\Gamma(n)} \frac{1}{|n-n_1|^{1-2\delta}|n-n_3|^{1-2\delta} n_{\max}^{2\nu}} \right)^{\frac{1}{2}}$$
$$\lesssim \|w\|_{H^\sigma}^4 \|w\|_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}}^2 \left(\sum_n \frac{1}{\langle n \rangle^{2\nu - 6\delta}} \right)^{\frac{1}{2}}.$$

Summing this requires $\nu > \frac{1}{2}$ which restricts us further to enforcing

$$\alpha + s > \frac{11}{6},$$

completing the proof for $\mathcal{N}_1$.

Now, recall from the proof of Proposition 2.3.1 that we obtained the estimate

$$|\mathcal{R}_1(w)| \lesssim \|\mathbf{P}_{\leq N} w\|_{H^{s-\frac{1}{2}-\varepsilon}}^6, \tag{2.4.6}$$

for $s > \max\left(\frac{5}{4} - \frac{1}{2}\alpha, \frac{2}{3}\right)$ and $\alpha > \frac{1}{2}$. Since

$$\max\left(\frac{2}{3}, \frac{11}{6} - \alpha, \frac{5}{4} - \frac{1}{2}\alpha\right) = \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right)$$

for any $\alpha \in \mathbb{R}$, then we may simply use (1.1.21) on two factors of (2.4.6) to obtain (2.4.4) for $\mathcal{R}_1$. This completes the proof of (2.4.4).

□

2.4.2 Construction of weighted Gaussian measures

In this section, we construct weighted Gaussian measures which are adapted to the modified energy $E_s(v)$. Our attention is only on the low regularity setting $\frac{1}{2} < s \leq 1$ and high enough dispersion ($\alpha \geq \frac{5}{6}$), since the results in Section 4 established quasi-invariance when $s > 1$ for any $\alpha > \frac{1}{2}$.

Given $r > 0$ and $N \geq 1$, we first wish to construct the measure

$$d\rho_{s,N,r}(v) = Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v)$$

and then, by taking $N \to \infty$, construct the measure

$$d\rho_{s,r}(v) = Z_{s,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_s(v)} d\mu_s(v),$$

where we recall

$$R_s(v) = -\frac{1}{2} \mathrm{Re} \sum_{\Gamma(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} v_{n_1} \overline{v}_{n_2} v_{n_3} \overline{v}_{n_4},$$

and we define

$$R_{s,N}(v) := -\frac{1}{2}\mathrm{Re}\sum_{\Gamma_N(\overline{n})} \frac{\psi_s(\overline{n})}{\phi(\overline{n})} v_{n_1}\overline{v}_{n_2}v_{n_3}\overline{v}_{n_4}.$$

We set

$$F_{N,r}(v) = \mathbf{1}_{\{\|v\|_{L^2}\le r\}} e^{-\frac{1}{2}R_{s,N}(\mathbf{P}_{\le N}v)} \quad \text{and} \quad F_r(v) = \mathbf{1}_{\{\|v\|_{L^2}\le r\}} e^{-\frac{1}{2}R_s(v)}.$$

The main result of this subsection is the following proposition which states, not only that the probability measure $\rho_{s,r}$ exists, but that we have 'good' uniform L^p bounds on the density for $\rho_{s,N,r}$ (see (2.4.7) below). Such higher L^p bounds are crucial for the hybrid argument in [40] (see Lemma 2.4.5 and Proposition 2.4.7).

Proposition 2.4.2. *Let $r > 0$, $\alpha \ge \frac{3}{4}$ and $\max(\frac{5-4\alpha}{2}, \frac{1}{2}) < s \le 1$. Then, given $p < \infty$, there exists $C > 0$ such that*

$$\|F_r(v)\|_{L^p(\mu_s)},\ \|F_{N,r}(v)\|_{L^p(\mu_s)} \le C_{p,r,s,\alpha}, \tag{2.4.7}$$

uniformly in $N \in \mathbb{N}$. Furthermore, there exists $R_s(v) \in L^p(\mu_s)$ such that

$$\lim_{N\to\infty} R_{s,N}(\mathbf{P}_{\le N}v) = R_s(v) \qquad \text{in } \ L^p(\mu_s) \tag{2.4.8}$$

and

$$\lim_{N\to\infty} F_{N,r}(v) = F_r(v) \qquad \text{in } \ L^p(\mu_s). \tag{2.4.9}$$

In order to prove Proposition 2.4.2 by employing the argument in [76, Proposition 6.2], we need the following bound. Note that we define $R_{s,\infty}(v) = R_s(v)$.

Lemma 2.4.3. *Let $\alpha > \frac{1}{2}$ and $\frac{1}{2} < s \le 1$. Then for any*

$$\gamma > \max\left(0, \frac{2s+1-2\alpha}{3}, \frac{1}{4} + s - \alpha\right),$$

we have

$$|R_{s,N}(\mathbf{P}_{\le N}v)| \lesssim \|\mathbf{P}_{\le N}v\|_{L^2} \|\mathbf{P}_{\le N}v\|_{H^\gamma}^3, \tag{2.4.10}$$

uniformly in $N \in \mathbb{N}\cup\{\infty\}$. In particular, if $\alpha > \frac{3}{2}$, we may take $\gamma \equiv 0$ in (2.4.10).

Proof. Notice that we have a symmetry with respect to the interchange of n_1 and n_3 and the interchange of n_2 and n. We split the proof of (2.4.10) into a few cases with the remaining cases following analogously by exploiting this symmetry. Below we prove (2.4.10) for $N = \infty$ as it is clear how to adjust the argument when $N \in \mathbb{N}$. We write $\tilde{v}_n := \langle n\rangle^\gamma v_n$.

• **Case 1:** $|n_1| \sim |n| \gg |n_2|, |n_3|$

In this case, it is clear from Lemma 2.1.3 that

$$|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha-1}|n-n_1|,$$

and from the mean value theorem,

$$|\psi(\overline{n})| \lesssim n_{\max}^{2s-1}|n-n_1|.$$

Hence by Cauchy-Schwarz,

$$|R_s(v)| \lesssim \Bigg(\sum_{\Gamma(\overline{n})} \frac{\tilde{v}_{n_1}^2}{n_{\max}^{2(2\alpha-2s+2\gamma)}\langle n_3\rangle^{2\gamma}}\Bigg)^{\frac{1}{2}} \|v\|_{H^\gamma}^2 \|v\|_{L^2} \lesssim \|v\|_{L^2}\|v\|_{H^\gamma}^3,$$

provided

$$2(2\alpha-2s+2\gamma) > 1 \qquad \text{and} \qquad 2\alpha-2s+3\gamma > 1. \tag{2.4.11}$$

• **Case 2:** $|n_2| \sim |n| \gg |n_1|, |n_3|$

Using $|\phi(\overline{n})| \gtrsim n_{\max}^{2\alpha}$ and applying Cauchy-Schwarz as in the previous case, we obtain (2.4.10) provided γ satisfies (2.4.11).

• **Case 3:** $|n_1| \sim |n_2| \sim |n_3| \sim |n|$

On $\Gamma(\overline{n})$, we have

$$|\psi_s(\overline{n})| \lesssim |n-n_3| n_{\max}^{2s-1} \qquad \text{and} \qquad |\phi(\overline{n})| \gtrsim |n-n_3||n-n_1| n_{\max}^{2\alpha-2}.$$

With $\gamma \geq 0$ to be determined, we have

$$\begin{aligned} |R_s(v)| &\lesssim \sum_{\Gamma(\overline{n})} \frac{n_{\max}^{2s-1}}{|n-n_1| n_{\max}^{2\alpha-2}} \frac{\tilde{v}_{n_1} v_{n_2} \tilde{v}_{n_3} \tilde{v}_n}{\langle n_1\rangle^\gamma \langle n_3\rangle^\gamma \langle n\rangle^\gamma} \\ &\lesssim \sum_{\Gamma(\overline{n})} \frac{1}{|n-n_1| n_{\max}^{\nu}} \tilde{v}_{n_1} v_{n_2} v_{n_3} \tilde{v}_n, \end{aligned}$$

where $\nu = 2\alpha - 1 - 2s + 3\gamma > 0$ provided

$$\gamma > \max\left(0, \frac{2s+1-2\alpha}{3}\right).$$

By the Cauchy-Schwarz inequality and Lemma 2.1.4, we have

$$\begin{aligned} &\lesssim \Bigg(\sum_{n,n_1,n_3} \frac{\tilde{v}_n^2 \tilde{v}_{n_3}^2}{\langle n-n_1\rangle^{1+\delta}}\Bigg)^{\frac{1}{2}} \Bigg(\sum_{n_2,n_1,n} \frac{v_{n_2}^2 \tilde{v}_{n_1}^2}{\langle n-n_3\rangle^{1-\delta}\langle n\rangle^{2\delta}}\Bigg)^{\frac{1}{2}} \\ &\lesssim \|v\|_{L^2}\|v\|_{H^\gamma}^3. \end{aligned}$$

Notice from the condition $2\alpha - 1 - 2s + 2\gamma > 0$, that if $\alpha > \frac{3}{2}$, we can take $\gamma = 0$. This completes the proof. □

We also require the following probabilistic estimate, see [76, Lemma 6.4].

Lemma 2.4.4. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be independent standard complex-valued Gaussian random variables. Then, there exist $c, C > 0$ such that, for any $M \geq 1$, we have*

$$\mathbb{P}\bigg[\bigg(\sum_{n=1}^{M} |g_n|^2\bigg)^{\frac{1}{2}} \geq K\bigg] \leq e^{-cK^2},$$

provided $K \geq CM^{\frac{1}{2}}$.

We now give the proof of Proposition 2.4.2.

Proof of Proposition 2.4.2. For $\alpha > \frac{3}{2}$, Lemma 2.4.3 implies we may take $\gamma \equiv 0$ in (2.4.10) and hence

$$\mathbf{1}_{\{\|v\|_{L^2}\leq r\}}|R_{s,N}(\mathbf{P}_{\leq N}v)| \lesssim \mathbf{1}_{\{\|v\|_{L^2}\leq r\}}\|\mathbf{P}_{\leq N}v\|_{L^2}^4 \lesssim r^4,$$

at which point, the bound (2.4.7) follows trivially. We make up the remaining case $\frac{3}{4} \leq \alpha \leq \frac{3}{2}$ in the following. Given $\frac{1}{2} < s \leq 1$, let γ be as in Lemma 2.4.3 whose precise value will be specified later. On $\{\|v\|_{L^2} \leq r\}$, (2.4.10) implies

$$|R_s(\mathbf{P}_{\leq M_0}v)| \leq C_0 r\|\mathbf{P}_{\leq M_0}v\|_{H^\gamma}^3 \leq C_0 M_0^{3\gamma} r^4.$$

We have

$$\|F_r(v)\|_{L^p(d\mu_s)}^p \leq C^p + p\int_{\max(e,e^{2^{\frac{3}{2}}C_0 r})}^{\infty} \lambda^{p-1}\mu_s(|R_s(v)| \geq \log\lambda,\ \ \|v\|_{L^2} < r)d\lambda. \tag{2.4.12}$$

We choose $M_0 > 0$ such that

$$\log\lambda = 2^{\frac{3}{2}}C_0 M_0^{3\gamma} r^4. \tag{2.4.13}$$

For $j \in \mathbb{N}$, let $M_j = 2^j M_0$ and $\sigma_j = C_\varepsilon 2^{-\varepsilon j} = CM_0^\varepsilon M_j^{-\varepsilon}$ for some small $\varepsilon > 0$ such that $\sum_{j=1}^\infty \sigma_j = \frac{1}{2}$. Then we have

$$\begin{aligned}
\mu_s(|R_s(v)| \geq \log\lambda,\ \ \|v\|_{L^2} < r) &\leq \mu_s(\|v\|_{H^\gamma}^2 \geq (C_0^{-1}r^{-1}\log\lambda)^{\frac{2}{3}}) \\
&\leq \sum_{j=1}^\infty \mu_s(\|\mathbf{P}_{M_j}v\|_{H^\gamma}^2 \geq \sigma_j(C_0^{-1}r^{-1}\log\lambda)^{\frac{2}{3}}) \\
&\lesssim \sum_{j=1}^\infty \mathbb{P}\bigg(\bigg(\sum_{|n|\sim M_j}|g_n|^2\bigg)^{\frac{1}{2}} \gtrsim L_j\bigg),
\end{aligned}$$

where $L_j := (C_0^{-1} r^{-1} \log\lambda)^{\frac{1}{3}} \sigma_j^{\frac{1}{2}} M_j^{s-\gamma} \gtrsim M_0^{\frac{1}{2}\varepsilon} M_j^{s-\gamma-\frac{1}{2}\varepsilon} \gg M_j^{\frac{1}{2}}$, provided $s-\gamma > \frac{1}{2}$. We used here that $\lambda > e^{2^{\frac{3}{2}} C_0 r}$ implies, from (2.4.13), $M_0^{\gamma} r \geq 1$ and hence $(C_0^{-1} r^{-1} \log\lambda)^{2/3} \sim M_0^{4\gamma} r^4 \gtrsim 1$. Therefore, by Lemma 2.4.4, we have

$$\mu_s(|R_s(v)| \geq \log\lambda,\ \|v\|_{L^2} < r) \lesssim \sum_{j=1}^{\infty} e^{-c_r 2^{j(2s-2\gamma-\frac{2}{3}\varepsilon)} (\log\lambda)^{\frac{2}{3}+\frac{2}{3}\frac{s-\gamma}{\gamma}}}$$
$$\lesssim e^{-c_r''(\log\lambda)^{\frac{2s}{3\gamma}}}.$$

Thus, from (2.4.12), we have

$$\|F_r(v)\|^p_{L^p(d\mu_s)} \lesssim C_p + p\int_C^{\infty} e^{p\lambda} e^{-c_r''\lambda^{\frac{2s}{3\gamma}}}\, d\lambda < C < \infty,$$

provided $\frac{2}{3}s > \gamma$. It is clear that the above arguments also apply to obtain the uniform bound (2.4.7) when $N \in \mathbb{N}$.

Thus we can construct the measure $\rho_{s,r}$ provided we can choose $\gamma \in \mathbb{R}$ satisfying

$$\max\left(0, \frac{2s+1-2\alpha}{3}, \frac{1}{4}+s-\alpha\right) < \gamma < \min\left(s-\frac{1}{2}, \frac{2}{3}s\right) = s - \frac{1}{2}.$$

As we wish to consider s close to 1, we must impose $\alpha > \frac{3}{4}$ to rule out the maximum on the left hand side being $\frac{1}{4}+s-\alpha$. Now, if $s+\frac{1}{2}-\alpha \leq 0$, it is clear we can pick a $\gamma > 0$. Otherwise, if $s+\frac{1}{2}-\alpha \geq 0$, we can choose $\gamma > 0$ as long as

$$\frac{2s+1-2\alpha}{3} < s - \frac{1}{2},$$

which upon rearranging yields the condition $\max(\frac{5-4\alpha}{2}, \frac{1}{2}) < s \leq 1$.

As for the $L^p(\mu_s)$ convergence of $R_{s,N}(\mathbf{P}_{\leq N} v)$ and $F_{N,r}(v)$, we note that when $\alpha > \frac{3}{2}$, we have

$$|R_{s,N}(\mathbf{P}_{\leq N}(v)) - R_s(v)| \lesssim \|\mathbf{P}_{>N} v\|_{L^2} \|v\|^3_{L^2}. \tag{2.4.14}$$

By a slight modification of (2.4.10), when $\frac{1}{2} < \alpha < \frac{3}{2}$, we also have

$$|R_{s,N}(\mathbf{P}_{\leq N}(v)) - R_s(v)| \lesssim \|\mathbf{P}_{>N} v\|_{L^2} \|v\|^3_{H^\gamma} + \|v\|_{L^2} \|v\|^2_{H^\gamma} \|\mathbf{P}_{>N} v\|_{H^\gamma}. \tag{2.4.15}$$

Taking $N \to \infty$ in (2.4.14) and (2.4.15) and noting that $s-\gamma > \frac{1}{2}$ shows $R_{s,N}(\mathbf{P}_{\leq N} v)$ converges almost surely with respect to μ_s to $R_s(v)$. Then because of the uniform in N bounds

$$\|R_{s,N}(\mathbf{P}_{\leq N} v)\|_{L^p(\mu_s)}, \|R_s(v)\|_{L^p(\mu_s)} \leq C_{p,s} < \infty,$$

which follow from Lemma 2.4.3, a standard argument using Egoroff's theorem implies (2.4.8) (see [76, Proposition 6.2]).

From (2.4.14) and (2.4.15), we have almost sure convergence of $F_{r,N}(v)$ to $F_r(v)$ with respect to μ_s. Using (2.4.7), the above standard argument implies convergence in $L^p(\mu_s)$; namely (2.4.9). This completes the proof of Proposition 2.4.2.

□

The next lemma shows that the two factors lying in $\mathcal{F}L^{\sigma,\infty}$ for $\sigma < s$ in the modified energy estimate (2.4.4) have moments indeed contributing a factor of $p^{\frac{1}{2}}$. Notice that as a consequence of Proposition 2.4.2 (namely, $Z_{s,N,r} \to Z_{s,r}$ as $N \to \infty$), $Z_{s,N,r}^{-1}$ is bounded uniformly in $N \in \mathbb{N}$.

Lemma 2.4.5. *Given $\varepsilon > 0$ and $r > 0$, there exists $C = C(\varepsilon, r) > 0$ such that*

$$\big\| \|f\|_{\mathcal{F}L^{s-\varepsilon,\infty}} \big\|_{L^p(\rho_{s,N,r})} \leq C p^{\frac{1}{2}}$$

for any $p \geq 1$ and $N \in \mathbb{N}$.

Proof. Applying the uniform bound (2.4.7), the uniform bound on $Z_{s,N,r}^{-1}$ and Minkowski's integral inequality, for any $q > \frac{1}{\varepsilon}$, we have

$$\begin{aligned}
\big\| \|f\|_{\mathcal{F}L^{s-\varepsilon,\infty}} \big\|_{L^p(\rho_{s,N,r})} &\leq \big\| \|f\|_{\mathcal{F}L^{s-\varepsilon,q}} \big\|_{L^p(\rho_{s,N,r})} \\
&\leq Z_{s,N,r}^{-\frac{1}{p}} \|F_{N,r}\|_{L^{q'}(\mu_s)}^{\frac{1}{p}} \big\| \|f\|_{\mathcal{F}L^{s-\varepsilon,q}} \big\|_{L^{pq}(\mu_s)}^{\frac{1}{p}} \\
&\lesssim_q \big\| \langle n \rangle^{-\varepsilon} \|g_n\|_{L^{pq}(\Omega)} \big\|_{\ell_n^q} \\
&\lesssim_q q^{\frac{1}{2}} p^{\frac{1}{2}},
\end{aligned}$$

where we have used the following well-known estimate on higher moments of Gaussian random variables in the last inequality:

$$\|g_n\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} \tag{2.4.16}$$

for any $p \geq 2$. □

2.4.3 Transport of the truncated weighted Gaussian measures

In this subsection, we study how the measures $\rho_{s,N,r}$ evolve under the flow of the truncated equation (2.2.2). We follow the method of [40] in which we use a 'change of variables formula' (see Lemma 2.4.6) to make the modified energy E_s along the truncated flow appear. Taking a time derivative and using the estimate (2.4.4) then gives a differential inequality for the evolution of $\rho_{s,N,r}$ under Φ_N (see Proposition 2.4.7).

Lemma 2.4.6 (Change of variables formula)**.** *Let $\alpha \geq \frac{5}{6}$, s be as in (2.4.1) and $r > 0$. Then for any $N \in \mathbb{N}$, $t \in \mathbb{R}$ and measurable set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, we have*

$$\rho_{s,N,r}(\Phi_N(t)(A)) = \widehat{Z}_{s,N}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \, e^{-E_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(v))} du_{\leq N} \times d\mu_{s,N}^{\perp}. \tag{2.4.17}$$

We omit the proof of Lemma 2.4.6 as it is identical to those in [40, 76, 78]. The core ingredients are the invariance of the Lebesgue measure L_N under the truncated flow Φ_N (because of Liouville's theorem), the invariance of Φ_N in the L^2-norm (mass conservation) and the bijectivity of the flow Φ_N. When $\alpha \geq 1$, (2.4.17) also holds for any measurable $A \subset L^2(\mathbb{T})$.

Proposition 2.4.7. *Let $\alpha \geq \frac{5}{6}$ and s be as in* (2.4.1)*. Then, given $r, R > 0$ and $T > 0$, there exists $C_{r,R,T} > 0$ such that*

$$\frac{d}{dt}\rho_{s,N,r}(\Phi_N(t)(A)) \leq C_{r,R,T} \cdot p \left\{\rho_{s,N,r}(\Phi_N(t)(A))\right\}^{1-\frac{1}{p}} \tag{2.4.18}$$

for any $p \geq 2$, any $N \in \mathbb{N}$, any $t \in [0,T]$ and any measurable set $A \subset B_R \subset H^\sigma(\mathbb{T})$.

Proof. Fix $r, R > 0$ and $T > 0$. As a preliminary step, we first note the following key estimate on the growth of the modified energy $E_{s,N}$: for α and s as in Proposition 2.4.7, we have

$$\left\| \mathbf{1}_{B_R} \partial_t E_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(v))\Big|_{t=0} \right\|_{L^p(\rho_{s,N,r})} \leq C_{r,R} \cdot p \tag{2.4.19}$$

for any $p \geq 2$ and for any $N \in \mathbb{N}$. This follows from (2.4.4), Lemma 2.4.5, the uniform bound (2.4.7) on $F_{N,r}$, the uniform bound on $Z_{s,N,r}^{-1}$ and Cauchy-Schwarz inequality, since

$$\begin{aligned}
\text{LHS of (2.4.19)} &\leq Z_{s,N,r}^{-\frac{1}{p}} \|F_{N,r}(v)\|_{L^2(\mu_s)}^{\frac{1}{p}} \|\mathbf{1}_{B_R}\partial_t E_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(v))|_{t=0}\|_{L^{2p}(\mu_s)} \\
&\leq C \Big\| \mathbf{1}_{B_R} \|\mathbf{P}_{\leq N} v\|^2_{\mathcal{F}L^{s-\frac{\varepsilon}{2},\infty}} \|\mathbf{P}_{\leq N} v\|^4_{H^\sigma} \Big\|_{L^{2p}(\mu_s)} \\
&\leq CR^4 \Big\| \|\mathbf{P}_{\leq N} v\|_{\mathcal{F}L^{s-\frac{\varepsilon}{2},\infty}} \Big\|^2_{L^{4p}(\mu_s)} \\
&\leq CR^4 p.
\end{aligned}$$

Now fix a measurable set $A \subset B_R \subset H^\sigma(\mathbb{T})$ and $t_0 \in [0,T]$. By the semigroup property of $\Phi_N(t)$ and the change of variables formula (Lemma 2.4.6), we have

$$\begin{aligned}
\frac{d}{dt}\rho_{s,N,r}(\Phi_N(t)(A))\Big|_{t=t_0} &= Z_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \Big|_{t=t_0} \\
&= Z_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t)(\Phi_N(t_0)(A))} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \Big|_{t=0} \\
&= \widehat{Z}_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t_0)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-E_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(v))} du_{\leq N} \times d\mu_{s,N}^{\perp} \Big|_{t=0} \\
&= -Z_{s,N,r}^{-1} \int_{\Phi_N(t_0)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \partial_t E_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(v))|_{t=0}\, e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v).
\end{aligned}$$

Now recall from Proposition 2.3.5 (i) that for any $t \in [0,T]$ and $N \in \mathbb{N}$, there exists $C(R,T) > 0$ such that $\Phi_N(t)(B_R) \subset B_{C(R,T)}$. Note that when the flow Φ is only well-defined locally-in-time, we use Proposition 2.3.5 (i). Hence, by Hölder's inequality we obtain

$$\begin{aligned}
&\frac{d}{dt}\rho_{s,N,r}(\Phi_N(t)(A))\bigg|_{t=t_0}\\
&\qquad \le Z_{s,N,r}^{-1}\int_{\Phi_N(t_0)(A)}\big|\partial_t E_{s,N}(\mathbf{P}_{\le N}\Phi_N(t)(u))|_{t=0}\big|\mathbf{1}_{\{\|v\|_{L^2}\le r\}}e^{-R_{s,N}(\mathbf{P}_{\le N}v)}d\mu_s(v)\\
&\qquad \le \bigg\|\mathbf{1}_{B_{C(R,T)}}\partial_t E_{s,N}(\mathbf{P}_{\le N}\Phi_N(t)(v))\Big|_{t=0}\bigg\|_{L^p(\rho_{s,N,r})}\{\rho_{s,N,r}(\Phi_N(t_0)(A))\}^{1-\frac{1}{p}}.
\end{aligned}$$

Applying (2.4.19) yields (2.4.18).

□

2.4.4 Proof of Theorem 1.1.2 (i)

In this section, we apply the argument in [78] to deduce from Proposition 2.4.7, the quasi-invariance of μ_s under the untruncated flow $\Phi(t)$. In what follows, we fix $\alpha \ge \frac{5}{6}$ and consider s satisfying (2.4.1). We show that for each fixed $R > 0$,

$$\text{if} \quad \mu_s(A) = 0, \qquad \text{then} \qquad \mu_s(\Phi(t)(A)) = 0 \tag{2.4.20}$$

for any $t \in [0, T(R)]$ and for any measurable set $A \subset B_R$. This implies local-in-time quasi-invariance of μ_s under (1.1.13) for any

$$s > \frac{11}{6} - \alpha.$$

When $\alpha \ge 1$, (2.4.20) is true for any $t \in \mathbb{R}$. As R is arbitrary, this implies quasi-invariance of μ_s under the dynamics of FNLS (1.1.13). For the rest of this section, we fix $R > 0$ and $\alpha \ge 1$ since the arguments below are easily modified to imply local-in-time quasi-invariance when $\frac{5}{6} < \alpha < 1$.

For the first step, we use Proposition 2.4.7 to show that $\rho_{s,N,r}$ is quasi-invariant under $\Phi_N(t)$; see Lemma 2.4.8. The proof of Lemma 2.4.8 follows exactly as in [78, Proposition 5.3].

Lemma 2.4.8. *Given $r > 0$, there exists $0 < t_{r,R} < T$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that if, for a measurable set $A \subset B_R \subset H^\sigma(\mathbb{T})$, there exists $N_0 \in \mathbb{N}$ such that*

$$\rho_{s,N,r}(A) < \delta$$

for any $N \ge N_0$, then we have

$$\rho_{s,N,r}(\Phi_N(t)(A)) < \varepsilon$$

for any $t \in [0, t_{r,R}]$ and any $N \ge N_0$.

Then a careful argument allows the previous statement to hold when $N = \infty$; that is, we have that $\rho_{s,r}$ is quasi-invariant under the untruncated flow $\Phi(t)$ (Lemma 2.4.9). The proof of Lemma 2.4.9 makes use of the approximation property of the dynamics of FNLS (1.1.13) as in Proposition 2.3.5 (ii) and follows the arguments in [78, Lemma 5.5].

Lemma 2.4.9. *Given $r > 0$, there exists $0 < t_{r,R} < T$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\rho_{s,r}(A) < \delta,$$

then we have

$$\rho_{s,r}(\Phi_N(t)(A)) < \varepsilon$$

for any $t \in [0, t_{r,R}]$.

Now, invoking the mutual absolute continuity of $\rho_{s,r}$ and $\mu_{s,r}$ implies $\mu_{s,r}$ is quasi-invariant under $\Phi(t)$. We then take $r \to \infty$ (as in [78, Theorem 1.2]) and iterate in time to obtain (2.4.20) for every $t \in \mathbb{R}$, for a fixed $R > 0$. This concludes the proof of Theorem 1.1.2 (i).

Chapter 3

Quasi-invariance of fractional Gaussian fields by the nonlinear wave equation with polynomial nonlinearity

This chapter is dedicated to the proof of Theorems 1.1.4 and 1.1.5, as stated in Section 1.1.3.

3.1 Outline of proof of Theorems 1.1.4 and 1.1.5

Our proof proceeds along the lines of the proofs in [40, 78], following a general methodology introduced by Tzvetkov in [96]. Tzvetkov's method is based on the construction of a measure $\vec{\rho}_s$ which is mutually absolutely continuous with respect to the Gaussian measure $\vec{\mu}_s$ of interest, but for which the time evolution of sets can be controlled effectively. Then, we need a suitable energy estimate on the support of the renormalised measure.

We start by replacing the measure μ_s by a mutually absolutely continuous measure more suitable to the analysis of the nonlinear wave flow.

Definition 3.1.1. *Let g_n, h_n, $n \in \mathbb{Z}^3$ be standard complex Gaussian random variables satisfying* (1.1.29) *such that g_0 and h_0 are real valued. We define $\mathrm{d}\vec{\nu}_s$ to be the distribution of the random series*

$$u^\omega(x) = g_0 + \sum_{n\in\mathbb{Z}^3\setminus\{0\}} \frac{g_n(\omega)}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} e^{in\cdot x}, \tag{3.1.1}$$

$$v^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{(1 + |n|^{2s})^{\frac{1}{2}}} e^{in\cdot x}. \tag{3.1.2}$$

For a proof that these measures are indeed absolutely continuous, we refer the reader to [78, Lemma 6.1] where this was done in the 2-dimensional setting using Kakutani's theorem [50].

For each $N \geq 1$, let $\Phi_N(t)$ denote the time t flow of the following approximation of the flow of the equation (1.1.25):

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - \mathbf{P}_{\leq N}((\mathbf{P}_{\leq N} u)^k) \\ (u, v)|_{t=0} = (u_0, v_0), \end{cases}$$

where by $\mathbf{P}_{\leq N}$ we denote the projection (A.1.1) onto frequencies less than N.

A change of variables formula (see [96, Proposition 4.1] or [78, Lemma 5.1]) then implies

$$\int_{\Phi_N(t)(A)} \vec{\nu}_s(\mathrm{d}\vec{u}) = Z_{s,N}^{-1} \int_A e^{-\frac{1}{2}\|\Phi_N(t)(\mathbf{P}_{\leq N} u)\|^2} \mathrm{d}L_N \otimes \vec{\nu}_{s,N}^{\perp}(\mathrm{d}\vec{u}). \tag{3.1.3}$$

In (3.1.3), we have used the notation

$$\|(u,v)\|^2 = \int_{\mathbb{T}^3} (D^s v)^2 \,\mathrm{d}x + \int_{\mathbb{T}^3} (D^{s+1} u)^2 \,\mathrm{d}x + \int_{\mathbb{T}^3} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{T}^3} v^2 \,\mathrm{d}x + \left(\int_{\mathbb{T}^3} u \,\mathrm{d}x \right)^2.$$

Here, $D^s u$ denotes the action on u of the Fourier multiplier with symbol $|n|^s$:

$$D^s u := |\nabla|^s u = \mathcal{F}^{-1}(|n|^s \widehat{u}).$$

The measure

$$L_N \otimes \vec{\nu}_{s,N}^{\perp}(\mathrm{d}\vec{u})$$

appearing in (3.1.3) is the product of Lebesgue measure L_N on

$$\mathcal{E}_N \times \mathcal{E}_N := (\mathbf{P}_{\leq N} L^2(\mathbb{T}^3))^2$$

and the projection $\vec{\nu}_{s,N}^{\perp} = (\mathrm{id} - \mathbf{P}_{\leq N})_* \vec{\nu}_{s,N}$ of $\vec{\nu}_{s,N}$ on

$$\mathcal{E}_N^{\perp} \times \mathcal{E}_N^{\perp} := ((\mathrm{id} - \mathbf{P}_{\leq N}) L^2(\mathbb{T}^3))^2.$$

$Z_{s,N}$ is a normalisation factor.

Differentiating (3.1.3) and using the invariance of the Hamiltonian, we obtain

$$-Z_{s,N}^{-1} \int_A \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Phi_N(t)(\mathbf{P}_{\leq N} u)\|^2 e^{-\frac{1}{2}\|\Phi_N(t)(\mathbf{P}_{\leq N} u)\|^2} \mathrm{d}L_N \otimes \vec{\nu}_{s,N}^{\perp}(\mathrm{d}u).$$

Denote by

$$u_N := \mathbf{P}_{\leq N} u, \quad v_N := \mathbf{P}_{\leq N} v$$

the projections of the solution on Fourier frequencies less than or equal to N. The derivative of the energy is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_N(t)(\mathbf{P}_{\leq N}u)\|^2 &= \partial_t\left(\frac{1}{2}\int_{\mathbb{T}^3}(D^s v_N)^2 + (D^{s+1}u_N)^2 + \frac{1}{2}\Big(\int_{\mathbb{T}^3} u_N\Big)^2\right) \\
&\quad + \partial_t\left(E(u_N, v_N) - \frac{1}{k+1}\int_{\mathbb{T}^3} u_N^{k+1}\right) \\
&= \int_{\mathbb{T}^3} D^{2s}v_N(-u_N^k) + \int_{\mathbb{T}^3} u_N \int_{\mathbb{T}^3} v_N - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{k+1}\int_{\mathbb{T}^3} u_N^{k+1}\Big).
\end{aligned}
\tag{3.1.4}
$$

We now rewrite the first quantity in (3.1.4) as

$$
\begin{aligned}
\int_{\mathbb{T}^3} u_N^k D^{2s} v_N &= k\int_{\mathbb{T}^3} D^s v_N D^s u_N u_N^{k-1} \\
&\quad + \int_{\mathbb{T}^3} D^s v_N (D^s u_N^k - k u_N^{k-1} D^s u_N) \\
&= \frac{k}{2}\partial_t \int_{\mathbb{T}^3} (D^s u_N)^2 u_N^{k-1} - \frac{k(k-1)}{2}\int_{\mathbb{T}^3}(D^s u_N)^2 v_N u_N^{k-2} \\
&\quad + \int_{\mathbb{T}^3} D^s v_N (D^s u_N^k - k u_N^{k-1} D^s u_N).
\end{aligned}
$$

The quantity $(D^s u_N)^2$ is divergent on the support of $\vec{\nu}_s$, and so requires a renormalisation. This was one of the innovations in [78]. Following the notation introduced there, we define

$$
Q_{s,N}(f) := (D^s f)^2 - \sigma_N, \tag{3.1.5}
$$

with

$$
\sigma_N := \mathbb{E}_{\vec{\nu}_s}[(D^s \mathbf{P}_{\leq N} u)^2] \sim N.
$$

Defining the *renormalised energy* by

$$
\mathscr{E}_{s,N}(u,v) := \frac{1}{2}\Big(\int_{\mathbb{T}^3}(D^s v_N)^2 + \int_{\mathbb{T}^3}(D^{s+1}u_N)^2 + \Big(\int_{\mathbb{T}^3} u_N\Big)^2\Big) - \frac{k}{2}\int_{\mathbb{T}^3} Q_{s,N}(u_N) u_N^{k-1}, \tag{3.1.6}
$$

the result of the above computations is the following expression for the time derivative of $\mathscr{E}_{s,N}(u,v)$:

$$
\begin{aligned}
\partial_t \mathscr{E}_{s,N}(\Phi_N(t)(u_N, v_N)) &= -\frac{k(k-1)}{2}\int_{\mathbb{T}^2} Q_{s,N}(u_N) v_N u_N^{k-2} \\
&\quad + \int_{\mathbb{T}^2} D^s v_N (D^s u_N^k - k u_N^{k-1} D^s u_N) + \int_{\mathbb{T}^3} u_N v_N.
\end{aligned}
\tag{3.1.7}
$$

The idea in [96] is to pass from $\vec{\nu}_s$ to the weighted measure

$$e^{-R_s(u,v)}\mathrm{d}\vec{\nu}_s,$$

where $R_s(u,v)$ is a limit of the terms

$$R_{k,s,N}(u,v) := \frac{k}{2}\int_{\mathbb{T}^3} Q_{s,N}(u_N)u_N^{k-1} + \frac{1}{k+1}\int_{\mathbb{T}^3} u_N^{k+1}$$

appearing in the renormalised energy (3.1.6). We must then estimate the time derivative (3.1.7).

One of the main novelties of this work is that we instead pass to the weighted measure

$$e^{-R_s(u,v)-E(u,v)^q}\mathrm{d}\vec{\nu}_s,$$

where $q = q(s,k)$ is sufficently large and $E(u,v)$ is the Hamiltonian (1.1.26). This measure is still absolutely continuous with respect to $\vec{\nu}_s$ (and hence $\vec{\mu}_s$). Further, no complications arrive in the time derivative formula (3.1.7) as the Hamiltonian is conserved.

The following two propositions contain the main technical results needed to close the argument to prove quasi-invariance.

Proposition 3.1.2. *Let $s > \frac{5}{2}$. Define*

$$E_N(u,v) := \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u_N|^2 + \frac{1}{2}\int_{\mathbb{T}^3} v_N^2 + \frac{1}{k+1}\int u_N^{k+1}.$$

Then for $p < \infty$ there exists $q > 0$ and $C_p > 0$ such that

$$\sup_{N\in\mathbb{N}} \|e^{-R_{k,s,N}(u)-E_N^q(u,v)}\|_{L^p(\vec{\nu}_s)} \leq C_p < \infty$$

and moreover,

$$\lim_{N\to\infty} e^{-R_{k,s,N}(u)-E_N^q(u,v)} = e^{F_{s,k}(u,v)} \quad \text{in } L^p(\vec{\nu}_s)$$

where $F_{s,k}(u,v)$ is as in Lemma 3.3.2.

In particular, for any $\sigma < s - \frac{1}{2}$, the restrictions to $\vec{H}^\sigma$ of the measures

$$\mathrm{d}\vec{\rho}_{s,N} = \mathcal{Z}_{s,N}^{-1} e^{-R_{k,s,N}(\vec{u})-E_N^q(u,v)}\, \vec{\nu}_s(\mathrm{d}\vec{u})$$

converge in total variation to a measure $\vec{\rho}_s$.

Proposition 3.1.3. *Let $s > \frac{5}{2}$. Given $R_0 > 0$ there exists $C = C(R_0)$ such that, for all $p \geq 1$ finite, we have*

$$\mathbb{E}_{\vec{\rho}_s}[\mathbf{1}_{B_{R_0}(\vec{u})}\, |\partial_t \mathscr{E}_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(\vec{u}))|_{t=0}|^p]^{\frac{1}{p}} \leq Cp \tag{3.1.8}$$

where C can be taken independent of N. Here B_{R_0} is the ball of radius R_0 in $\vec{H}^\sigma$.

As in [40], we obtain the estimates necessary for the construction of our measure from a variational bound introduced in [4].

Finishing the proof of Theorems 1.1.4 and 1.1.5. We now indicate how to finish the proofs given Propositions 3.1.2 and 3.1.3. Since this part of the argument requires no modification from [40], we refer the reader to that paper for details.

Let $\sigma < s - \frac{3}{2}$, and $k = 3$ or $k = 5$. Fix $R > 0$ and let $A \subset B_R \subset \vec{H}^\sigma$ be a Borel measurable set (for the topology induced by the norm) such that

$$\vec{\rho}_{s,N}(A) = 0.$$

When $k = 3$, a simple application of Gronwall's lemma and conservation of the energy[1] shows that for any $T > 0$, there is a radius $C(T, R)$ such that for $|t| \le T$,

$$\Phi_N(t)(B_R) \subset B_{C(T,R)}. \tag{3.1.9}$$

The estimate (3.1.9) in case $k = 5$ is proved in Appendix C.

By the change of variables formula, we have

$$\vec{\rho}_{s,N}(\Phi_N(t)(A)) = \mathcal{Z}_{s,N}^{-1} \int_A e^{-E_N(\mathbf{P}_{\le N}\Phi_N(t)(u,v)) - E_N^q(u,v)} \mathrm{d}L_N \otimes \vec{\nu}_{s,N}^{\perp}(\mathrm{d}u).$$

Differentiating and using (3.1.8), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\rho}_{s,N}(\Phi_N(t)(A)) \le C_R p \cdot \vec{\rho}_{s,N}(\Phi_N(t)(A))^{1-\frac{1}{p}}$$

This inequality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\rho}_s(\Phi_N(t)(A))^{\frac{1}{p}} \le C_R. \tag{3.1.10}$$

The linear dependence on p on the right side is essential in (3.1.8) plays an essential role here. Integrating (3.1.10), we find

$$\begin{aligned}\vec{\rho}_{s,N}(\Phi_N(t)(A)) &\le (\vec{\rho}_{s,N}(A)^{1/p} + C_R t)^p \\ &\le 2^p \vec{\rho}_{s,N}(A) + C_R^p 2^p t^p.\end{aligned}$$

Taking $t \le \frac{1}{4C_R}$ and p large enough, we find that

$$\vec{\rho}_{s,N}(\Phi_N(t)(A)) < \varepsilon, \tag{3.1.11}$$

uniformly in N, for any $A \subset B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$. Theorem 1.1.4 follows from (3.1.11) by a soft approximation argument, using Proposition 3.1.2, identical to that in [78, Section 5.2].

For Theorem 1.1.5, the estimate (3.1.9) is replaced by the growth bound (1.1.30) proved in Appendix C, where T now depends on R, but otherwise the proof proceeds as before.

[1] See [40, Lemma 2.5].

3.2 Energy estimate for fractional s

In this section, we make extensive use of Littlewood-Paley theory in its dyadic decomposition incarnation. We briefly outline some objects and concepts we use, see [3] for a thorough treatment. Following these authors, we $B(\xi, r)$ denote the ball in $\mathbb{R}^d$ of radius r around a point ξ in $\mathbb{R}^d$. Consider functions χ, $\tilde{\chi}$ such that

$$\begin{aligned}&\mathrm{supp}\tilde{\chi} \subset B\big(0, \tfrac{4}{3}\big),\\ &\mathrm{supp}\chi \subset B\big(0, \tfrac{4}{3}\big) \setminus B\big(0, \tfrac{3}{8}\big).\end{aligned}$$

We define $\chi_0 = \tilde{\chi}$ and

$$\psi_j(\cdot) = \chi(2^{-j}\cdot), \quad j \geq 1. \tag{3.2.1}$$

We also define δ_j, the *Littlewood-Paley projector*, associated to symbol ψ_j by

$$\delta_j u(x) = \big(\psi_j(\nabla)u\big)(x) = \sum_{n\in\mathbb{Z}^d} \phi_j(n)\widehat{u}(n)e^{in\cdot x}.$$

One can use δ_j to define the Besov spaces $B^s_{p,q}(\mathbb{T}^d)$ and Holder spaces $C^s(\mathbb{T}^d)$ used in this Chapter. We do this in Appendix A. We define the (low-high) paraproduct $T_a b$ by

$$T_a b := \sum_{j=-1}^{\infty} S_{j-1} a \delta_j b,$$

where

$$S_j := \sum_{k\leq j-1} \delta_k.$$

In this section, we derive Proposition 3.1.3, the energy estimate for fractional s. This is the analogue of [40, estimate 3] to general nonlinearities and fractional regularity $s > 5/2$.

The possibility of fractional s makes our derivation more involved, since we cannot integrate by parts as in [78, Equation (1.25)] and [40, Equation (3.5)] to remove the most singular spatial derivatives in the time derivative of the energy. Instead, we perform a higher order expansion to exploit the cancellation in the commutator term $ku_N^{k-1}D^s u_N - D^s u_N^k$ appearing in (3.2.2).

Recall that

$$\begin{aligned}\partial_t \mathscr{E}_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(\vec{u}))|_{t=0} = \frac{k(k-1)}{2}\int Q_{s,N}(u_N)v_N u_N^{k-2}\\ + \int D^s v_N(ku_N^{k-1}D^s u_N - D^s u_N^k) + \int u_N \int v_N.\end{aligned} \tag{3.2.2}$$

We aim to prove the following.

Proposition 3.2.1. *For $s > \frac{5}{2}$, there exists $\sigma < s - \frac{1}{2}$, such that, for ε sufficiently small,*

$$\begin{aligned}
&|\partial_t \mathscr{E}_{s,N}(\mathbf{P}_{\leq N}\Phi_N(t)(\vec{u}))|_{t=0}| \\
&\lesssim (1 + \|\vec{u_N}\|_{\vec{H}^\sigma}^{k-1})(1 + \|Q_{s,N}(u_N)\|_{C^{-1-\varepsilon}} + \|u_N\|_{C^{s-\frac{1}{2}-\varepsilon}}\|v_N\|_{C^{s-\frac{3}{2}-\varepsilon}} \\
&\quad + \sum_{j=1}^{3} \|D^s v_N \partial_j D^{s-2} u_N\|_{C^{-1-\varepsilon}} + \|D^s v_N \partial_j^2 D^{s-4} u_N\|_{C^{-\varepsilon}}).
\end{aligned} \tag{3.2.3}$$

The implicit constants are uniform in N.

Then Proposition 3.1.3 follows from (3.2.3) and Lemma 3.2.2 below.

Lemma 3.2.2. *We have*

$$\|Q_{s,N}(u)\|_{C^{-1-\varepsilon}(\mathbb{T}^3)} \lesssim p, \tag{3.2.4}$$

$$\|v_N\|_{L^p(dv_s)C^{s-\frac{3}{2}-\varepsilon}(\mathbb{T}^3)} \lesssim \sqrt{p}, \tag{3.2.5}$$

$$\|u_N\|_{L^p(\mathrm{d}\vec{\nu}_s)C^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)} \lesssim \sqrt{p}. \tag{3.2.6}$$

If $2s - \alpha - \beta > \frac{1}{2}$, then for ε sufficiently small,

$$\|D^\alpha v_N D^\beta u_N\|_{L^p(\mathrm{d}\vec{\nu}_s)C^{2s-\alpha-\beta-2-\varepsilon}(\mathbb{T}^3)} \lesssim p. \tag{3.2.7}$$

Moreover, the above terms converge to limits denoted by $Q_s(u)$, v, u, $D^\alpha v D^\beta u$ respectively in the same topologies.

Proof. The estimate for (3.2.4) was proven in [40, Proposition 4.3]. As for the other bounds, we only prove the fourth bound, (3.2.7). The first two can be proved in a similar manner. To simplify the notations, let $\gamma = 2s - \alpha - \beta - 2 - \varepsilon$, and let L^p_w, L^q_x denote $L^p(\mathrm{d}\vec{\nu}_s), L^q(\mathbb{T}^3)$ respectively. By Bernstein's inequality (A.1.11), Fubini's theorem and the Wiener chaos estimate (A.2.1),

$$\begin{aligned}
\|D^\alpha v_N D^\beta u_N\|_{L^p_\omega C^\gamma_x} &\leq \|D^\alpha v_N D^\beta u_N\|_{L^p_x B^{\gamma+\frac{3}{p}}_{p,1}} \\
&= \|\sum_j 2^{j(\gamma+\frac{3}{p})}\|\delta_j(D^\alpha v_N D^\beta u_N)\|_{L^p_x}\|_{L^p_\omega} \\
&\leq \sum_j 2^{j(\gamma+\frac{3}{p})}\|\delta_j(D^\alpha v_N D^\beta u_N)\|_{L^p_x L^p_\omega} \\
&\lesssim p \sum_j 2^{j(\gamma+\frac{3}{p})}\|\delta_j(D^\alpha v_N D^\beta u_N)\|_{L^p_x L^2_\omega}.
\end{aligned} \tag{3.2.8}$$

On the other hand, by series representation (1.1.28),

$$\begin{aligned}
\mathbb{E}_\omega[|\delta_j(D^\alpha v_N D^\beta u_N)|^2] &\lesssim \sum_{n\sim 2^j}\sum_{|n_1|\leq N} \frac{1}{\langle n_1\rangle^{2(s-\alpha)}\langle n-n_1\rangle^{2(s+1-\beta)}} \\
&\lesssim \sum_{n\sim 2^j} \frac{1}{\langle n\rangle^{2(2s+1-\alpha-\beta)-3}} \lesssim 2^{2(2s-\alpha-\beta-2)}.
\end{aligned}$$

The convolution estimate [65, Lemma 4.1] requires $2s - \alpha - \beta > \frac{1}{2}$. Hence (3.2.8) is bounded by p if p is large and $\gamma < 2s - \alpha - \beta - 2$. As for convergence, for $M \geq N \geq 1$ similar estimates show

$$\mathbb{E}_\omega[|\delta_j(D^\alpha v_N D^\beta u_N - D^\alpha v_M D^\beta u_M)|^2] \lesssim N^{-\theta}$$

for sufficiently small $\theta > 0$ if $2s - \alpha - \beta > \frac{1}{2}$ and so convergence follows. □

In the following, our goal is to prove uniform in N estimates for the derivative of the energy (3.2.2). For this reason and for simplicity of notation, in this section we sometimes omit the subscripts N on u and v in deriving the estimates.

To prove Proposition 3.2.1, we only need to bound the second term (the commuter term) on the RHS of (3.2.2). The other terms can be bounded exactly as in [40, Proposition 5.1].

To do this, we now collect some helpful lemmas involving paraproducts.

Lemma 3.2.3. *For any $\varepsilon > 0$, we have*

$$\|kT_{u^{k-1}}u - u^k\|_{B^{\alpha+\beta-\varepsilon}_{1,1}} \lesssim \|u\|_{B^\alpha_{p,\infty}} \|u\|_{B^\beta_{q,\infty}} \|u\|^{k-2}_{L^\infty_x},$$

provided $\alpha + \beta > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. A more general $\mathbb{R}^d$ version can be found in [3, Theorem 2.92]. For our case, note

$$\begin{aligned} u^k &= \sum_{j=-1}^{\infty} (S_{j+1}u)^k - (S_j u)^k \\ &= \sum_{j=-1}^{\infty} \delta_j u \sum_{\ell=0}^{k-1} (S_{j+1}u)^\ell (S_j u)^{k-\ell-1}. \end{aligned}$$

It suffices to bound $T_{u^{k-1}}u - \sum_j \delta_j u (S_{j+1}u)^{k-1}$, the other terms are similar. Since $\alpha + \beta > 0$, we may assume $\beta > 0$. Taking the L^1_x-norm, we have

$$\begin{aligned} &\|\delta_n \left(T_{u^{k-1}}u - \sum_j \delta_j u (S_{j+1}u)^{k-1} \right) \|_{L^1_x} \\ =&\|\delta_n \sum_{j \geq n-3} \delta_j u \left(S_{j-1}u^{k-1} - (S_{j+1}u)^{k-1} \right) \|_{L^1_x} \\ \lesssim& \sum_{j \geq n-3} \|\delta_j u\|_{L^p_x} \|S_{j-1}(u^{k-1} - (S_{j-k}u)^{k-1}) + \left((S_{j-k}u)^{k-1} - (S_{j+1}u)^{k-1} \right) \|_{L^q_x} \\ \lesssim& \sum_{j \geq n-3} \|\delta_j u\|_{L^p_x} \left(\|u^{k-1} - (S_{j-k}u)^{k-1}\|_{L^q_x} + \|(S_{j-k}u)^{k-1} - (S_{j+1}u)^{k-1}\|_{L^q_x} \right). \end{aligned}$$

Expressing $u^{k-1} - (S_{j-1}u)^{k-1}$ in terms of products of lower-order quantities, we find that the previous expression is bounded up to a constant factor by

$$\begin{aligned}&\sum_{j\geq n-3} \|\delta_j u\|_{L^p_x} \sum_{m\geq j-k} \|\delta_m u\|_{L^q_x} \sum_{\ell=0}^{k-2} \|S_{j-k}u\|^{\ell}_{L^\infty_x} \left(\|u\|^{k-\ell-2}_{L^\infty_x} + \|S_{j+1}u\|^{k-\ell-2}_{L^\infty_x}\right)\\
\lesssim &\sum_{j\geq n-3} 2^{-j\alpha}\|u\|_{B^\alpha_{p,\infty}} \sum_{m\geq j-k} 2^{-m\beta}\|u\|_{B^\beta_{q,\infty}}\|u\|^{k-2}_{L^\infty_x}\\
\lesssim &2^{-n(\alpha+\beta)}\|u\|_{B^\alpha_{p,\infty}}\|u\|_{B^\beta_{q,\infty}}\|u\|^{k-2}_{L^\infty_x}.\end{aligned}$$

□

The next lemma allows us to replace $u^{k-1}D^s u$ by $T_{u^{k-1}}D^s u$.

Lemma 3.2.4. $\|T_a b - ab\|_{B^{\alpha+\beta}_{1,1}} \lesssim \|a\|_{B^\alpha_{1,\infty}}\|b\|_{C^\beta}$, *provided* $\beta \neq 0$ *and* $\alpha+\beta > 0$.

Proof. The proof is a straightforward application of the definition of the paraproduct and Besov spaces. □

The difference $D^s T_{u^{k-1}}u - T_{u^{k-1}}D^s u$ cannot be bounded directly. The following decomposition is the main result describing the regularity of this commutator:

Lemma 3.2.5. $D^s(T_w u) - T_w(D^s u) = F_1 + F_2 + R$, *where*

$$\begin{aligned}F_1 &= s\sum_{j=1}^{3} T_{\partial_j w}(\partial_j D^{s-2}u),\\
F_2 &= \frac{s(s-2)}{2}\sum_{j=1}^{3} T_{\partial_j^2 w}(\partial_j^2 D^{s-4}u) + \frac{s}{2}T_{D^2 w}D^{s-2}u,\\
\|R\|_{B^{\frac{3}{2}+\varepsilon}_{1,1}} &\lesssim \|w\|_{B^{2+2\varepsilon}_{1,\infty}}\|u\|_{C^{s-\frac{1}{2}-\varepsilon}}, \quad \varepsilon > 0.\end{aligned}$$

Proof. Let $m \in C^\infty(\mathbb{R})$ be a bump function such that $m = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and is supported on $[-\frac{1}{2}-, \frac{1}{2}+]$. Then,

$$\begin{aligned}&(D^s(T_w u) - T_w(D^s u))(x)\\
=&\sum_{n_1,n_2\in\mathbb{Z}^3} (|n_1+n_2|^s - |n_2|^s)\widehat{w}(n_1)\widehat{u}(n_2)m(\tfrac{|n_1|}{|n_2|})e^{i(n_1+n_2)\cdot x}.\end{aligned}$$

By Multivariate Taylor's theorem,

$$\begin{aligned}&|n_1+n_2|^s - |n_2|^s\\
=&s|n_2|^{s-2}n_2\cdot n_1 + \frac{1}{2}\left(s(s-2)|n_2|^{s-4}(n_2\cdot n_1)^2 + s|n_2|^{s-2}|n_1|^2\right) + R_1\end{aligned}$$

where the first 2 terms correspond to F_1 and F_2, and

$$R_1(n_1,n_2) := \frac{s(s-2)}{2}\int_0^1 (1-t)^2\Big(3|n_2+tn_1|^{s-4}(n_2+tn_1)\cdot n_1|n_1|^2 \\ + (s-4)|n_2+tn_1|^{s-6}((n_2+tn_1)\cdot n_1)^3\Big)\mathrm{d}t.$$

Define R by

$$R = \sum_{n_1,n_2\in\mathbb{Z}^3} R_1(n_1,n_2)\widehat{u}(n_1)\widehat{w}(n_2)e^{i(n_1+n_2)\cdot x}.$$

We now estimate R. First, we can write $R_1 = \sum_{|\alpha|=3} C_\alpha(n_1,n_2)n_1^\alpha$, where α is a 3-dimensional multi-index and C_α can be extended by homogeneity to a function on $\mathbb{R}^6$ such that

$$C_\alpha(\lambda\xi_1,\lambda\xi_2) = \lambda^{s-3}C_\alpha(\xi_1,\xi_2)$$

for any $\lambda > 0$, and is smooth on the support of $m(\frac{|\xi_1|}{|\xi_2|})\psi_j(\xi_1+\xi_2)$ for any $j\in\mathbb{N}$. We do this because we aim to prove a bound in the Euclidean setting and pull it back to the periodic setting via the Poisson summation formula [35, Chapter 4].

Recall (3.2.1), the definition of ψ_j. To bound $\|\delta_j R\|_{L^1_x}$, set

$$K_{\alpha,j}(\xi_1,\xi_2) := \psi_j(\xi_1+\xi_2)m(\tfrac{|\xi_1|}{|\xi_2|})C_\alpha(\xi_1,\xi_2),$$

and define $h_j\in L^2(\mathbb{T}^3\times\mathbb{T}^3)$ and $H_j\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ by

$$h_j(y,z) := \sum_{|\alpha|=3}(\mathcal{F}^{-1}_{\mathbb{T}^6}K_{\alpha,j})(y,z),$$
$$H_j(y,z) := \sum_{|\alpha|=3}(\mathcal{F}^{-1}_{\mathbb{R}^6}K_{\alpha,j})(y,z).$$

By the Poisson summation formula,

$$h_j(y,z) = \sum_{(m_1,m_2)\in\mathbb{Z}^6} H_j(y+m_1,z+m_2).$$

Hence,

$$\begin{aligned}\|h_j\|_{L^1(\mathbb{T}^6)} &\le \|H_j\|_{L^1(\mathbb{R}^6)} \\ &= \|2^{j(s-3)}\cdot 2^{6j}H_0(2^j\cdot,2^j\cdot)\|_{L^1(\mathbb{R}^6)} \\ &= 2^{j(s-3)}\|H_0\|_{L^1(\mathbb{R}^6)}.\end{aligned}$$

Note $\|H_0\|_{L^1}$ is bounded since $K_{\alpha,0} \in C_0^\infty(\mathbb{R}^6)$. Then,

$$\begin{aligned}
\|\delta_j R\|_{L^1_x} &= \| \sum_{n_1,n_2} \psi_j(n_1+n_2) m(\tfrac{|n_1|}{|n_2|}) \sum_{|\alpha|=3} C_\alpha(n_1,n_2) n_1^\alpha \widehat{w}(n_1)\widehat{u}(n_2) e^{i(n_1+n_2)\cdot x}\|_{L^1_x} \\
&= \| \sum_{|\alpha|=3} \sum_{n_1,n_2} K_{\alpha,j}(n_1,n_2) \widehat{S_{j-1}\partial^\alpha} w(n_1) \widehat{\widetilde{\delta}_j u}(n_2) e^{i(n_1+n_2)\cdot x}\|_{L^1_x} \\
&= \| \iint h_j(y,z) S_{j-1}\partial^\alpha w(x-y) \widetilde{\delta}_j u(x-z) dydz \|_{L^1_x} \\
&\le \|h_j\|_{L^1_{y,z}} \|S_{j-1}\partial^\alpha w\|_{L^1_x} \|\widetilde{\delta}_j u\|_{L^\infty_x} \\
&\lesssim 2^{j(s-3)} 2^{j(1-2\varepsilon)} \|\partial^\alpha w\|_{B^{-1+2\varepsilon}_{1,\infty}} \cdot 2^{-j(s-\frac{1}{2}-\varepsilon)} \|u\|_{C^{s-\frac{1}{2}-\varepsilon}} \\
&\lesssim 2^{-j(\frac{3}{2}+\varepsilon)} \|w\|_{B^{2+2\varepsilon}_{1,\infty}} \|u\|_{C^{s-\frac{1}{2}-\varepsilon}}.
\end{aligned}$$

Here $\widetilde{\delta}_j$ is another Littlewood-Paley projector such that $\widetilde{\delta}_j \delta_j = \delta_j$. □

We can now give the proof of the energy estimate (3.2.3).

Proof of Proposition 3.2.1. Recall that we will only bound the commuter term. The other terms can be bounded exactly as in [40, Proposition 5.1]. We first write

$$\begin{aligned}
&\int_{\mathbb{T}^3} D^s v (k u^{k-1} D^s u - D^s u^k) \\
&= \int_{\mathbb{T}^3} D^s v \Big[(k u^{k-1} D^s u - k T_{u^{k-1}} D^s u) - k(D^s(T_{u^{k-1}} u) \\
&\qquad - T_{u^{k-1}}(D^s u)) + (k D^s T_{u^{k-1}} u - D^s u^k) \Big].
\end{aligned}$$

Using Duality, Lemma 3.2.4, 3.2.5, and 3.2.3, we estimate the right hand side by by

$$\begin{aligned}
&\|D^s v\|_{C^{-\frac{3}{2}-\varepsilon}} \Big(k\|u^{k-1} D^s u - T_{u^{k-1}} D^s u\|_{B^{\frac{3}{2}+\varepsilon}_{1,1}} + k\|R_1\|_{B^{\frac{3}{2}+\varepsilon}_{1,1}} + \|k D^s T_{u^{k-1}} u - D^s u^k\|_{B^{\frac{3}{2}+\varepsilon}_{1,1}} \Big) \\
&+ \left| \int D^s v (F_1 + F_2) \right| \\
&\lesssim \|v\|_{C^{s-\frac{3}{2}-\varepsilon}} \|u\|_{C^{s-\frac{1}{2}-\varepsilon}} \|u\|_{B^{2+2\varepsilon}_{1,\infty}} \|u\|^{k-2}_{L^\infty_x} + \left| \int D^s v (F_1 + F_2) \right| \\
&\lesssim \|v\|_{C^{s-\frac{3}{2}-\varepsilon}} \|u\|_{C^{s-\frac{1}{2}-\varepsilon}} \|u\|^{k-1}_{H^\sigma} + \left| \int D^s v (F_1 + F_2) \right|.
\end{aligned}$$

Here, R_1, F_1, F_2 are the terms in Lemma 3.2.5. We used (A.1.3) and (A.1.5) in the last step.

It remains to deal with $D^s v F_1$ and $D^s v F_2$. As in Lemma 3.2.4, we can replace F_1, F_2 by products. For example, we can add and subtract the following term to

$D^s v F_1$:

$$D^s v(s\sum_{j=1}^{3}\partial_j D^{s-2}u\partial_j u^{k-1}).$$

For $\varepsilon > 0$, we have.

$$\begin{aligned}\left|\int D^s v\partial_j D^{s-2}u\partial_j u^{k-1}\right| &\leq \|D^s v\partial_j D^{s-2}u\|_{C^{-1-\varepsilon}}\|\partial_j u^{k-1}\|_{B^{1+\varepsilon}_{1,1}}\\ &\lesssim \|D^s v\partial_j D^{s-2}u\|_{C^{-1-\varepsilon}}\|u\|_{H^{2+\varepsilon}}\|u\|^{k-2}_{L^\infty_x}\\ &\lesssim \|D^s v\partial_j D^{s-2}u\|_{C^{-1-\varepsilon}}\|u\|^{k-1}_{H^{2+\varepsilon}}.\end{aligned}$$

The difference can be bound using Lemma 3.2.4 and estimates similar to the above. □

3.3 Construction of the measure

In this section, we construct a measure $\vec{\rho}_s$ which is absolutely continuous with respect to μ_s and corresponds to the formal expression:

$$d\vec{\rho}_s = \mathcal{Z}_s^{-1}e^{-\mathscr{E}_s(u,v)-E^q(u,v)}dudv.$$

Here $\mathscr{E}_s(u,v)$ is the renormalised energy defined in (3.1.6), $E(u,v)$ is the Hamiltonian energy (1.1.26), and $q = q(s,k)$ is a large integer to be chosen later.

Define the truncated measures

$$\begin{aligned}d\vec{\rho}_{s,N} &= \mathcal{Z}_{s,N,r}^{-1}e^{-\mathscr{E}_{s,N}(u,v)-E^q_N(u,v)}dudv\\ &= \mathcal{Z}_{s,N}^{-1}e^{-R_{k,s,N}(u)-E^q_N(u,v)}d\nu_s(u,v),\end{aligned}\tag{3.3.1}$$

where the truncated energy $E_N(u,v)$ is defined by

$$E_N(u,v) := E(u_N,v_N) = \frac{1}{2}\int_{\mathbb{T}^3}(|\nabla\mathbf{P}_{\leq N}u|^2 + (\mathbf{P}_{\leq N}v)^2)\,\mathrm{d}x + \frac{1}{k+1}\int_{\mathbb{T}^3}(\mathbf{P}_{\leq N}u)^{k+1}\,\mathrm{d}x.$$

In this section, we prove Proposition 3.3.2, which asserts that the measures $\vec{\rho}_{s,N}$ converge to a limiting measure as $N\to\infty$.

The general method to establish convergence of the measures is standard (see for example [95, Remark 3.8]), and consists of two steps, corresponding to Lemma 3.3.2 and Proposition 3.1.2, respectively.

1. Convergence of $R_{k,s,N}(u)$ and $E^q_N(u,v)$ in L^p. This is a consequence of the regularity properties of the field $\vec{u}$ on the support of $\vec{\mu}_s$, since $R_{k,s,N}(u)$ and $E^q_N(u,v)$ are continuous functions of the Fourier truncated field $\mathbf{P}_{\leq N}\vec{u}$.

2. Uniform integrability of $e^{-R_{k,s,N}(u)-E^q_N(u,v)}$ with respect to $\vec{\nu}_s$. This will follow from a uniform bound in L^p, $p>1$. It is here that we make use of the variational representation of

$$\mathcal{Z}_{s,N} := \mathbb{E}_{\vec{\nu}_s}[e^{-R_{s,k,N}(u)-E^q_N(u,v)}].$$

Indeed, the uniform integrability resulting from the second point allows us to take the limit in the expectation

$$\mathbb{E}_{\vec{\nu}_s}[e^{-R_{k,s,N}(u)-E^q_N(u,v)}],$$

which is sufficient to define $\vec{\rho}_s$ as a measure.

Compared to the cubic case, $k=3$, in [40], the addition of $-E^q(u,v)$ makes the construction of the measure easier as it introduces more decay. Also, as the energy is conserved we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^q_N(u,v) = 0.$$

Consequently, no extra terms appear in the energy estimate.

Definition 3.3.1. *For u given by* (3.1.1)*, we define*

$$:(D^s u_N)^2 := (D^s u_N)^2 - \mathbb{E}_{\vec{\nu}_s}[(D^s u_N)^2].$$

This notation is inspired by an analogy with Wick ordering in Gaussian analysis and quantum field theory (see [48, Chapter 3]).

Lemma 3.3.2. *Let $s > \frac{3}{2}$. Set*

$$F_{s,k,N}(u,v) := -R_{k,s,N}(u) - E^q_N(u,v).$$

Then for $p<\infty$, $F_{s,k,N}$ converges to some $F_{s,k}$ in $L^p(\vec{\nu}_s)$ as $N\to\infty$.

Proof. This lemma can be proved by basic Besov space embeddings and the convergence results in Lemma 3.2.2. For example, from the estimate $\|u\|_{L^{k+1}} \lesssim \|u\|_{C^\varepsilon}$, the $\frac{1}{k+1}\|u_N\|^{k+1}_{L^{k+1}}$ component of E_N converges in $L^p(\vec{\nu}_s)$. See also [40, Lemma 4.1]. □

3.3.1 Variational formulation

In this section, we apply the Barashkov-Gubinelli variational approach to obtain uniform in N control over the quantity $e^{-R_{k,s,N}(u)-E^q_N(u,v)}$. This is equivalent to showing that the partition function is uniformly bounded, since higher L^p-norms of $e^{-R_{k,s,N}(u)-E^q_N(u,v)}$ introduce only constant factors in the representation (3.3.4). This approach was first applied in [40]. The idea is to write the partition function as an optimisation over time-dependent processes, so we begin by representing the measure $\vec{\nu}_s$ as the time 1 distribution of a pair of cylindrical processes. We refer to [4, 40] for more details.

The main aim of this subsection is to prove Proposition 3.3.4. First we state some notation.

Let $\Omega = C(\mathbb{R}_+, C^{-\frac{3}{2}-\varepsilon}(\mathbb{T}^3))$. We then define

$$\vec{X}(t) = (X_1(t), X_2(t)) = \left(\sum_{n\in\mathbb{Z}^3} B_1^n(t)e^{in\cdot x}, \sum_{n\in\mathbb{Z}^3} B_2^n(t)e^{in\cdot x} \right)$$

so that $\vec{X}(t)$ is a Brownian motion on $L^2(\mathbb{T}^3)\times L^2(\mathbb{T}^3)$. Here $\{B_1^n\}_{n\in\mathbb{Z}}$ and $\{B_2^n\}_{n\in\mathbb{Z}}$ are mutually independent complex valued Brownian motions, normalised to $\mathrm{Var}(B_j^n(t)) = t$ and such that $\overline{B_j^n(t)} = B_j^{-n}(t)$ for $j = 1, 2$. This implies B_j^0 is real valued.

Set $\vec{Y}(t) = (Y^1(t), Y^2(t))$ where

$$Y_1(t) = \mathcal{J}^{-s-1}X_1(t) := B_1^0(t) + \sum_{n\in\mathbb{Z}^3\setminus\{0\}} \frac{B_1^n(t)}{(|n|^2 + |n|^{2s} + 2)^{\frac{1}{2}}} e^{in\cdot x}$$

and

$$Y_2(t) = J^{-s}X_2(t) := \sum_{n\in\mathbb{Z}^3} \frac{B_2^n(t)}{(1 + |n|^{2s})^{\frac{1}{2}}} e^{in\cdot x}.$$

Note that

$$\mathrm{Law}(\vec{Y}(1)) := \vec{\nu}_s.$$

We let $\mathbb{H}_a$ be the set of progressively measurable processes belonging to

$$L^2([0,1], L^2(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$$

almost surely. For $\theta \in \mathbb{H}_a$, the classical Girsanov theorem [58, Section 5.5] describes the semimartingale decomposition of $\vec{X}(t)$ and $\vec{Y}(t)$ with respect to the measure $\mathbb{Q}^\theta$ defined by its relative density

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}} = e^{\int_0^1 \langle \theta(t), \mathrm{d}\vec{X}(t)\rangle - \frac{1}{2}\int_0^1 \|\theta(s)\|^2_{L^2_x\times L^2_x}\,\mathrm{d}s}.$$

We have the decompositions

$$\vec{X}(t) = \vec{X}^\theta(t) + \int_0^t \theta(s)\mathrm{d}s$$

and

$$\vec{Y}(t) = (Y_1^\theta(t), Y_2^\theta(t)) + \int_0^t (\mathcal{J}^{-s-1}\theta_1, J^{-s}\theta_2)(t')\,\mathrm{d}t',$$

where $\vec{X}^\theta$ is a $\mathbb{Q}^\theta$ L^2-cylindrical Brownian motion and

$$Y_1^\theta(t) := \mathcal{J}^{-s-1}X_1^\theta(t), \quad Y_2^\theta(t) := J^{-s}X_2^\theta(t)$$

where we have used the notation $(\theta_1, \theta_2) = \theta$. For convenience we have set

$$\vec{I}^{\theta}(t) := (I_1^{\theta}(t), I_2^{\theta}(t)) = \int_0^t (\mathcal{J}^{-s-1}\theta_1, J^{-s}\theta_2)(t')\,\mathrm{d}t',$$

$\vec{Y}^{\theta}(t) := (Y_1^{\theta}(t), Y_2^{\theta}(t))$ and $\vec{X}^{\theta}(t) := (X_1^{\theta}(t), X_2^{\theta}(t))$.

The following lemma is convenient in the proof of the variational formula Proposition 3.3.4.

Lemma 3.3.3. *Let $\theta \in \mathbb{H}_a$, $N \geq 1$. Suppose $\theta \in \mathbb{H}_a$ is such that*

$$\frac{e^{-R_{k,s,N}(Y_1) - E_N^q(\vec{Y}(1))}}{\mathcal{Z}_{s,N}} = e^{\int_0^1 \langle \theta(t), \mathrm{d}\vec{X}(t)\rangle - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2\times L^2}^2\,\mathrm{d}t} = \frac{\mathrm{d}\mathbb{Q}^{\theta}}{\mathrm{d}\mathbb{P}}.$$

Then the relative entropy *of $\mathbb{Q}^{\theta}$ with respect to $\mathbb{P}$ is finite:*

$$H(\mathbb{Q}^{\theta} \mid \mathbb{P}) = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{\theta}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\mathbb{Q}^{\theta}}{\mathrm{d}\mathbb{P}}\right] < \infty.$$

In particular,

$$\mathbb{E}_{\mathbb{Q}^{\theta}}\left[\int_0^1 \|\theta(s)\|_{L^2\times L^2}^2 \mathrm{d}s\right] < \infty. \tag{3.3.2}$$

Proof. Once we prove the finiteness of the relative entropy, the bound (3.3.2) follows from the inequality [29, Lemma 2.6],

$$\mathbb{E}_{\mathbb{Q}^{\theta}}\left[\int_0^1 \|\theta(s)\|_{L^2\times L^2}^2 \mathrm{d}s\right] \leq 2H(\mathbb{Q}^{\theta} \mid \mathbb{P}).$$

We turn to the relative entropy. In our case, it takes the following explicit form:

$$H(\mathbb{Q}^{\theta} \mid \mathbb{P}) = \mathbb{E}\left[\frac{e^{-R_{s,k,N} - E_N^q}}{\mathcal{Z}_{s,N}} \log \frac{e^{-R_{s,k,N} - E_N^q}}{\mathcal{Z}_{s,N}}\right].$$

For the partition function $\mathcal{Z}_{s,N}$, we have by Jensen's inequality:

$$\begin{aligned}
\mathcal{Z}_{s,N} &= \mathbb{E}[e^{-R_{s,k,N}} e^{-E_N^q}] \\
&\geq e^{-\mathbb{E}[R_{s,k,N}] - \mathbb{E}[E_N^q]} \\
&\geq c(N).
\end{aligned} \tag{3.3.3}$$

In the final step, we have used the integrability of $R_{s,k,N}$ and E_N^q, which follows directly from (3.2.6), (3.2.5) since $B_{\infty,\infty}^{\alpha} \subset L^{\infty}$ when $\alpha > 0$.

Using Hölder's inequality, Jensen's Inequality and Young's inequality, it is easy to see that for $q \geq 1$,

$$
\begin{aligned}
R_{s,k,N}(Y_1) + E_N^q(\vec{Y}) &\geq \frac{k}{2}\int_{\mathbb{T}}((D^s Y_1)^2 - \sigma_N)(\mathbf{P}_{\leq N}Y_1)^{k-1}\,\mathrm{d}x + E_N^q(\vec{Y}) \\
&\geq -\frac{k}{2}\sigma_N \int (\mathbf{P}_{\leq N}Y_1)^{k-1}\,\mathrm{d}x + \frac{1}{(k+1)^q}\left(\int (\mathbf{P}_{\leq N}Y_1)^{k+1}\,\mathrm{d}x\right)^q \\
&\geq -\frac{k}{2}\sigma_N|\mathbb{T}^3|^{\frac{2}{k+1}}\left(\int (\mathbf{P}_{\leq N}Y_1)^{k+1}\right)^{\frac{k-1}{k+1}} \\
&\quad + \frac{1}{(k+1)^q}\left(\int (\mathbf{P}_{\leq N}Y_1)^{k+1}\,\mathrm{d}x\right)^q \\
&\geq -\left(\frac{k}{2\varepsilon}\right)^r \frac{|\mathbb{T}^3|^{\frac{2r}{k+1}}}{r}\sigma_N^r + (\frac{1}{4} - \frac{\varepsilon^q}{r})\left(\int (\mathbf{P}_{\leq N}Y_1)^{k+1}\,\mathrm{d}x\right)^q \\
&> C(N) > -\infty,
\end{aligned}
$$

where $\frac{k-1}{q(k+1)} + \frac{1}{r} = 1$. From this lower bound and using (3.3.3) and Cauchy-Schwarz, we now have

$$
\begin{aligned}
\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}}\log\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}}\right] &\leq C(N)\mathbb{E}[e^{-R_{s,k,N}(Y_1) - E_N^q(\vec{Y})}(1 + R_{s,k,N}(Y_1) + E_N^q(\vec{Y})] \\
&\leq C(N)\mathbb{E}[e^{-2R_{s,k,N}(Y_1) - 2E_N^q(\vec{Y})} + |R_{s,k,N}(Y_1) + E_N^q(\vec{Y}|^2 + 1] < \infty.
\end{aligned}
$$

□

With this in place we have the following variational formula for $\mathcal{Z}_{s,N}$.

Proposition 3.3.4. *Recall the definition of the partition function $\mathcal{Z}_{s,N}$. For $N \in \mathbb{N}$ we have,*

$$
\begin{aligned}
-\log \mathcal{Z}_{s,N} = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}_{\mathbb{Q}^\theta}\Bigg[&R_{k,s,N}(Y_1^\theta(1) + I_1^\theta(1)) + E_N^q(\vec{Y}^\theta(1) + \vec{I}^\theta(1)) \\
&+ \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x \times L^2_x} dt\Bigg].
\end{aligned} \tag{3.3.4}
$$

Proof. Given $\theta \in \mathbb{H}_a$, Girsanov's theorem gives:

$$
\begin{aligned}
-\log \mathcal{Z}_{s,N} &= -\log \mathbb{E}[e^{-R_{k,s,N}(u) - E_N^q(u,v)}] \\
&= -\log \mathbb{E}_{\mathbb{Q}^\theta}[e^{-R_{k,s,N}(Y_1^\theta(1) + I_1^\theta(1)) - E_N^q(\vec{Y}^\theta(1) + \vec{I}^\theta(1))}e^{-\int_0^1 \langle \theta(t), \mathrm{d}\vec{X}(t)\rangle + \frac{1}{2}\int_0^1 \|\theta(s)\|^2_{L^2 \times L^2}\,\mathrm{d}s}].
\end{aligned}
$$

By Jensen's inequality, we have

$$
\begin{aligned}
-\log \mathcal{Z}_{s,N} \leq\ &\mathbb{E}_{\mathbb{Q}^\theta}[R_{k,s,N}(Y_1^\theta(1) + I_1^\theta(1)) - E_N^q(\vec{Y}^\theta(1) + \vec{I}^\theta(1))] \\
&+ \mathbb{E}_{\mathbb{Q}^\theta}[\int_0^1 \langle \theta(t), \mathrm{d}\vec{X}^\theta(t)\rangle - \frac{1}{2}\int_0^1 \|\theta(s)\|^2_{L^2 \times L^2}\,\mathrm{d}s]
\end{aligned}
$$

If

$$\mathbb{E}_{\mathbb{Q}^\theta}\left[\int_0^1 \|\theta(s)\|^2_{L^2\times L^2}\,\mathrm{d}s\right] < \infty,$$

the stochastic integral term is a martingale, so its expectation vanishes and we find

$$-\log \mathcal{Z}_{s,N} \leq \mathbb{E}_{\mathbb{Q}^\theta}\left[R_{k,s,N}(Y_1^\theta(1)+I_1^\theta(1)) - E_N^q(\vec{Y}^\theta(1)+\vec{I}^\theta(1)) + \frac{1}{2}\int_0^1 \|\theta(s)\|^2_{L^2\times L^2}\,\mathrm{d}s\right]. \tag{3.3.5}$$

If instead

$$\mathbb{E}_{\mathbb{Q}^\theta}\left[\int_0^1 \|\theta(s)\|^2_{L^2\times L^2}\,\mathrm{d}s\right] = \infty,$$

the inequality (3.3.5) holds trivially.

Now we show that there exists θ giving equality in (3.3.5). The measure

$$\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} = \frac{e^{-R_{k,s,N}(Y_1)-E_N^q(\vec{Y}(1))}}{\mathcal{Z}_{s,N}}$$

is absolutely continuous with respect to $\mathbb{P}$, so by the Brownian martingale representation theorem there is a $\tilde{\theta}^N \in \mathbb{H}_a$, such that

$$\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} = e^{\int_0^1 \langle \tilde{\theta}^N(t), \mathrm{d}\vec{X}(t)\rangle - \frac{1}{2}\int_0^1 \|\tilde{\theta}^N(t)\|^2_{L^2\times L^2}\,\mathrm{d}t}.$$

Combining the last two expressions gives

$$-\log \mathcal{Z}_{s,N} := R_{k,s,N}(Y_1) + E_N^q(\vec{Y}(1)) + \int_0^1 \langle \tilde{\theta}^N(t), \mathrm{d}\vec{X}(t)\rangle - \frac{1}{2}\int_0^1 \|\tilde{\theta}^N(t)\|^2_{L^2\times L^2}\,\mathrm{d}t. \tag{3.3.6}$$

By Lemma 3.3.3 we have

$$\mathbb{E}_{\mathbb{Q}^{\tilde{\theta}^N}}\left[\int_0^1 \|\tilde{\theta}^N(s)\|^2_{L^2\times L^2}\,\mathrm{d}s\right] < \infty.$$

We can take expectations in (3.3.6) and the martingale term vanishes, so

$$-\log \mathcal{Z}_{s,N} = \mathbb{E}_{\mathbb{Q}^{\tilde{\theta}^N}}\left[R_{k,s,N}(Y_1) + E_N^q(\vec{Y}(1)) + \frac{1}{2}\int_0^1 \|\tilde{\theta}^N(t)\|^2_{L^2\times L^2}\,\mathrm{d}t\right].$$

as required. □

3.3.2 Exponential integrability

We now prove Proposition 3.1.2 by estimating the quantity on the right side of (3.3.4) from below. Since the time $t = 1$ is fixed, for simplicity we set

$$\vec{Y}^\theta := (Y_1^\theta, Y_2^\theta) = (Y_1^\theta(1), Y_2^\theta(1)).$$

A simple application of Young's inequality gives

$$\frac{1}{2}E_N^q(\vec{I}^\theta) \leq CE_N^q(\vec{Y}^\theta) + E_N^q(\vec{Y}^\theta + \vec{I}^\theta).$$

for some large constant C. Hence, to prove Proposition 3.1.2 it suffices to bound from below the following quantity.

$$\mathbb{E}_{\mathbb{Q}^\theta}\left[R_{k,s,N}(Y_1^\theta + I_1^\theta) - CE_N^q(\vec{Y}^\theta) + \frac{1}{2}E_N^q(\vec{I}^\theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L_x^2\times L_x^2} dt\right]. \tag{3.3.7}$$

The following lemma gives the regularity of $D^sY_1^\theta$, $:(D^sY_1^\theta)^2:$ and Y_1^θ.

Lemma 3.3.5. *Let $2 < p < \infty$. Then for $\varepsilon > 0$,*

$$\sup_{\theta\in\mathbb{H}_a} \mathbb{E}_{\mathbb{Q}^\theta}\left[\|D^sY_1^\theta\|^p_{C^{-\frac{1}{2}-\varepsilon}} + \|:(D^sY_1^\theta)^2:\|^p_{C^{-1-\varepsilon}} + \|Y_1^\theta\|^p_{C^{s-\frac{1}{2}-\varepsilon}}\right] < \infty.$$

Proof. This follows directly from Proposition 4.3 in [40]. □

A direct application of Cauchy-Schwarz (see [40, Lemma 4.7]) gives

$$\|I_1^\theta\|^2_{H^{s+1}} \leq \int_0^1 \|\theta(t)\|^2_{L^2\times L^2} dt.$$

Lemma 3.3.6. *For $s > \frac{3}{2}$,*

$$\sup_{\theta\in\mathbb{H}_a} \mathbb{E}_{\mathbb{Q}^\theta}\left[E_N^q(\vec{Y}^\theta)\right] < \infty$$

independently of N.

Proof. Under $\mathbb{Q}^\theta$, $\vec{Y}^\theta(1) = (Y_1^\theta(1), Y_2^\theta(1))$ has the same distribution as the pair (u, v) under $\vec{\nu}_s$. The result then follows from (3.2.6), (3.2.5). □

We introduce some abbreviated notations for the most common terms appearing in the estimates below. We set:

$$\begin{aligned} Y &:= Y_1^\theta, \\ \Theta &:= I_1^\theta, \\ E &:= E_N(\vec{I}^\theta). \end{aligned}$$

From the definition of $R_{k,s,N}$ we have

$$\begin{aligned} R_{k,s,N}(Y+\Theta) =& \frac{k(k-1)}{2}\int_{\mathbb{T}^3} :(D^sY)^2: \sum_{m=0}^{k-1}\binom{k-1}{m} Y^{k-1-m}\Theta^m \\ &+ k(k-1)\int_{\mathbb{T}^3} D^sY D^s\Theta \sum_{m=0}^{k-1}\binom{k-1}{m} Y^{k-1-m}\Theta^m \\ &+ \frac{k(k-1)}{2}\int_{\mathbb{T}^3} (D^s\Theta)^2 \sum_{m=0}^{k-1}\binom{k-1}{m} Y^{k-1-m}\Theta^m \\ &+ \frac{1}{k+1}\int_{\mathbb{T}^3} (Y+\Theta)^{k+1}. \end{aligned}$$

We aim to bound (3.3.7) by using Young's inequality and the positive terms

$$\int_{\mathbb{T}^3} (D^s\Theta)^2\Theta^{k-1}, \quad \int_{\mathbb{T}^3} \Theta^{k+1}, \quad E^q, \quad \|\Theta\|_{H^{s+1}}^2$$

in (3.3.7). As in [40], this will yeild Proposition 3.1.2.

Lemma 3.3.7. *(Terms quadratic in D^sY). Let $s > \frac{3}{2}$ and $0 \le m \le k-1$. Then for sufficiently small $\delta > 0$ there exists small $\varepsilon > 0$ and large p and $c(\delta)$ such that*

$$\left|\int_{\mathbb{T}^3} :(D^sY)^2\!: Y^{k-1-m}\Theta^m\right| \le c(\delta)\left(\|:(D^sY)^2:\|_{C^{-1-\varepsilon}}^p + \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}^p + \|Y\|_{C^{s-\frac{1}{2}-\varepsilon}}^p\right) + \delta\left(\|\Theta\|_{H^{s+1}}^2 + E^q\right).$$

Proof. For $m = 0$, using duality (A.1.6), the embedding (A.1.3) and the algebra property (A.1.4) we have

$$\begin{aligned}\left|\int_{\mathbb{T}^3} :(D^sY)^2\!: Y^{k-1}\right| &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y^{k-1}\|_{B^{1+2\varepsilon}_{1,\infty}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y\|^{k-1}_{C^{1+2\varepsilon}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y\|^{k-1}_{C^{s-\frac{1}{2}-\varepsilon}}\end{aligned}$$

if $s > \frac{3}{2}$ and $\varepsilon > 0$ is small. The estimate then follows from Young's inequality. If $m = k-1$, using duality (A.1.6), the embedding (A.1.3), fractional Leibniz (A.1.8) and Sobolev (A.1.5) we have,

$$\begin{aligned}\left|\int_{\mathbb{T}^3} :(D^sY)^2\!:\Theta^{k-1}\right| &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|\Theta^{k-1}\|_{B^{1+2\varepsilon}_{1,\infty}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|\Theta^{k-2}\|_{L^1}\|\Theta\|_{B^{1+2\varepsilon}_{\infty,\infty}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|\Theta^{k+1}\|_{L^1}^{\frac{k-2}{k+1}}\|\Theta\|_{B^{\frac{5}{2}+3\varepsilon}_{2,2}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}E^{\frac{k-2}{k+1}}\|\Theta\|_{H^{s+1}}\end{aligned}$$

if $s > \frac{3}{2}$ and $\varepsilon > 0$ is small. If we choose q large enough so that

$$\frac{k-2}{q(k+1)} + \frac{1}{2} < 1$$

the stated inequality then follows from Young's inequality.

If $0 < m < k-1$ then similar to the above,

$$\begin{aligned}\left|\int_{\mathbb{T}^3} :(D^sY)^2\!:Y^{k-1-m}\Theta^m\right| &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y^{k-1-m}\Theta^m\|_{B^{1+2\varepsilon}_{1,\infty}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y^{k-1-m}\|_{C^{1+2\varepsilon}}\|\Theta^m\|_{B^{1+2\varepsilon}_{1,\infty}}\\ &\lesssim \|:(D^sY)^2:\|_{C^{-1-\varepsilon}}\|Y\|^{k-1-m}_{C^{s-\frac{1}{2}-\varepsilon}}E^{\frac{m-1}{k+1}}\|\Theta\|_{H^{s+1}}.\end{aligned}$$

Hence if we choose q large enough so that

$$\frac{m-1}{q(k+1)} + \frac{1}{2} < 1.$$

Young's inequality the gives the desired result. $\square$

Lemma 3.3.8. *(Terms linear in D^sY). Let $s > \frac{5}{2}$ and $0 \leq m \leq k-1$. Then for sufficiently small $\delta > 0$ there exists small $\varepsilon > 0$ and large p and $c(\delta)$ such that*

$$\left| \int_{\mathbb{T}^3} D^sY D^s\Theta Y^{k-1-m}\Theta^m \right| \leq c(\delta) \left(\|D^sY\|^p_{C^{-\frac{1}{2}-\varepsilon}} + \|Y\|^p_{C^{s-\frac{1}{2}-\varepsilon}} \right) + \delta \left(\|\Theta\|^2_{H^{s+1}} + E^q \right).$$

Proof. First we estimate the term corresponding to $m = k-1$. Using duality (A.1.6), the embedding (A.1.3) followed by fractional Leibniz (A.1.7),

$$\left| \int_{\mathbb{T}^3} D^sY D^s\Theta\Theta^{k-1} \right| \lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}} \|D^s\Theta\Theta^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}.$$

Hence it remains to estimate $\|D^s\Theta\Theta^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}$. Using fractional Leibniz (A.1.7), the embedding (A.1.3), Sobolev (A.1.5), (A.1.8), Sobolev (A.1.5) again, Jensen's inequality and interpolation (A.1.2), we have

$$\begin{aligned}
\|D^s\Theta\Theta^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}} &\lesssim \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} \|\Theta^{k-1}\|_{L^2} + \|D^s\Theta\|_{L^2} \|\Theta^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} \\
&\lesssim \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} \|\Theta^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} \\
&\lesssim \|\Theta\|_{H^{s+\frac{1}{2}+2\varepsilon}} \|\Theta^{k-1}\|_{B^{2+2\varepsilon}_{1,\infty}} \\
&\lesssim \|\Theta\|_{H^{s+\frac{1}{2}+2\varepsilon}} \|\Theta^{k-2}\|_{L^1} \|\Theta\|_{B^{2+2\varepsilon}_{\infty,\infty}} \\
&\lesssim \|\Theta\|_{H^{s+\frac{1}{2}+2\varepsilon}} \|\Theta^{k+1}\|^{\frac{k-2}{k+1}}_{L^1} \|\Theta\|_{B^{\frac{7}{2}+2\varepsilon}_{2,\infty}} \\
&\lesssim \|\Theta\|^{\gamma}_{H^{s+1}} \|\Theta\|^{1-\gamma}_{L^2} \|\Theta^{k+1}\|^{\frac{k-2}{k+1}}_{L^1} \|\Theta\|_{H^{s+1}} \\
&\lesssim \|\Theta\|^{1+\gamma}_{H^{s+1}} E^{1-\gamma+\frac{k-2}{k+1}}
\end{aligned} \tag{3.3.8}$$

for $s > \frac{5}{2}$ and $\varepsilon > 0$ small where $\gamma = \gamma(s, \varepsilon) = \frac{s+\frac{1}{2}+2\varepsilon}{s+1} < 1$. If we choose $\varepsilon = \varepsilon(s, k)$ small enough and $q = q(s, k)$ large enough so that

$$\frac{1+\gamma}{2} + \frac{1}{q}\left(1 - \gamma + \frac{k-2}{k+1}\right) < 1$$

the desired inequality follows from Young's inequality.

Now for the case $0 < m < k-1$, using duality (A.1.6), the embedding (A.1.3) and fractional Leibniz (A.1.7) we have,

$$\begin{aligned}\left|\int_{\mathbb{T}^3} D^sYD^s\Theta Y^{k-1-m}\Theta^m\right| &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|Y^{k-1-m}D^s\Theta\Theta^m\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}\\ &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|Y^{k-1-m}\|_{C^{\frac{1}{2}+2\varepsilon}}\|D^s\Theta\Theta^m\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}\\ &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|Y\|^{k-1-m}_{C^{s-\frac{1}{2}-\varepsilon}}\|D^s\Theta\Theta^m\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}\end{aligned}$$

for $s > 1$ and $\varepsilon > 0$ small. It remains to estimate the term $\|D^s\Theta\Theta^m\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}$. If $s > \frac{5}{2}$ and $\varepsilon > 0$ is small enough, this term can be estimated in a manner similar to (3.3.8).

Finally we estimate the term corresponding to $m = 0$. We have,

$$\begin{aligned}\left|\int_{\mathbb{T}^3} D^sYD^s\Theta Y^{k-1-m}\right| &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|D^s\Theta Y^{k-1}\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}}\\ &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|Y\|^{k-1}_{C^{\frac{1}{2}+2\varepsilon}}\|D^s\Theta\|_{B^{\frac{1}{2}+\varepsilon}_{1,\infty}}\\ &\lesssim \|D^sY\|_{C^{-\frac{1}{2}-\varepsilon}}\|Y\|^{k-1}_{C^{s-\frac{1}{2}+2\varepsilon}}\|D^s\Theta\|_{H^{s+1}}\end{aligned}$$

for $s > 1$ and $\varepsilon > 0$ small. The desired inequality then follows from Young's inequality. $\square$

Lemma 3.3.9. *(Terms quadratic in $D^s\Theta$). Let $s > \frac{1}{2}$ and $0 < m < k-1$. Then for sufficiently small $\delta > 0$ there exists small $\varepsilon > 0$ and large p and $c(\delta)$ such that*

$$\begin{aligned}\left|\int_{\mathbb{T}^3}(D^s\Theta)^2Y^{k-1-m}\Theta^m\right| \leq & c(\delta)\left(\|D^sY\|^p_{C^{-\frac{1}{2}-\varepsilon}} + \|Y\|^p_{C^{s-\frac{1}{2}+2\varepsilon}}\right)\\ &+ \delta\left(\|\Theta\|^2_{H^{s+1}} + \|D^s\Theta\Theta^{\frac{k-1}{2}}\|^2_{L^2} + E^q\right).\end{aligned}$$

Proof. Using Young's inequality,

$$\left|\int_{\mathbb{T}^3}(D^s\Theta)^2Y^{k-1-m}\Theta^m\right| \leq C(\delta)\int_{\mathbb{T}^3}(D^s\Theta)^2Y^{k-1} + \delta\int_{\mathbb{T}^3}(D^s\Theta)^2\Theta^{k-1}.$$

It remains to estimate the first term on the right hand side of the above inequality. Using Hölder's inequality, interpolation (A.1.2) and the fact that $s > \frac{1}{2}$ we have,

$$\begin{aligned}\left|\int_{\mathbb{T}^3}(D^s\Theta)^2Y^{k-1}\right| &\lesssim \|\Theta\|^2_{H^s}\|Y^{k-1}\|_{L^\infty}\\ &\lesssim \|\Theta\|^{2\frac{s}{s+1}}_{H^{s+1}}\|\Theta\|^{2\frac{1}{s+1}}_{L^2}\|Y\|^{k-1}_{L^\infty}\\ &\lesssim \|\Theta\|^{\frac{2s}{s+1}}_{H^{s+1}}\|\Theta\|^{\frac{2}{s+1}}_{L^{k+1}}\|Y\|^{k-1}_{C^{s-\frac{1}{2}-\varepsilon}}\\ &\lesssim \|\Theta\|^{\frac{2s}{s+1}}_{H^{s+1}}E^{\frac{2}{(s+1)(k+1)}}\|Y\|^{k-1}_{C^{s-\frac{1}{2}-\varepsilon}}.\end{aligned}$$

For q large enough,

$$\frac{s}{s+1} + \frac{2}{q(s+1)(k+1)} < 1$$

and so the desired inequality follows from Young's inequality. □

Lemma 3.3.10. *(Remaining Terms) Let $s > \frac{1}{2}$. Then for sufficiently small $\delta > 0$ there exists small $\varepsilon > 0$ and large $c(\delta)$ such that*

$$\int_{\mathbb{T}^3} (Y+\Theta)^{k+1} \leq C(\delta)\|Y\|_{C^{s-\frac{1}{2}-\varepsilon}} + \delta \int_{\mathbb{T}^3} \Theta^{k+1}.$$

Proof. Using Young's inequality and (A.1.3) we have,

$$\begin{aligned}\int_{\mathbb{T}^3} (Y+\Theta)^{k+1} &\leq C(\delta)\int_{\mathbb{T}^3} Y^{k+1} + \delta \int_{\mathbb{T}^3} \Theta^{k+1} \\ &\leq C(\delta)\|Y\|^{k+1}_{C^{s-\frac{1}{2}-\varepsilon}} + \delta \int_{\mathbb{T}^3} \Theta^{k+1},\end{aligned}$$

which completes the proof. □

Chapter 4

Global well-posedness for the 2-dimensional stochastic complex Ginzburg-Landau equation

This chapter is dedicated to the proof of Theorems 4.1.2 and 4.1.3, as stated in Section 1.2.2.

4.1 Renormalised SCGL

Before we can prove the main results of this chapter, first we need to rigorously describe the renormilisation procedure we use for SCGL (1.2.4). This will lead us to a way to rigorously define the concept of a solution to WSCGL (1.2.6).

To explain the renormalisation procedure used for this equation, we consider a truncated version of it. We do this so that the equation, and its solutions make sense. Recall that $\mathbf{P}_{\leq N}$ is the sharp Fourier truncation operator onto frequencies less than N

$$\mathbf{P}_{\leq N} f = \sum_{|n| \leq N} f_n e^{in\cdot x}.$$

Consider the following truncated version of SCGL (1.2.5):

$$\begin{cases} \partial_t u_N = (a_1 + ia_2)[\Delta - 1]u_N - (c_1 + ic_2)|u_N|^{2m-2}u_N + \sqrt{2\gamma}\mathbf{P}_{\leq N}\xi \\ u_N|_{t=0} = \mathbf{P}_{\leq N} u_0. \end{cases} \tag{4.1.1}$$

It can be shown that for each fixed N, this truncated equation is globally well-posed. We define the truncated stochastic convolution

$$\begin{aligned} \Psi_N(t) &\stackrel{\text{def}}{=} \sqrt{2\gamma}\int_{-\infty}^{t} S(t-t')d(\mathbf{P}_{\leq N}W(t')) \\ &= \sqrt{2\gamma}\sum_{|n|\leq N} e^{in\cdot x}\int_{-\infty}^{t} e^{-(t-t')(a_1+ia_2)(|n|^2+1)}d\beta_n(t'). \end{aligned}$$

Here $S(t)$ is the heat type semigroup

$$S(t) := e^{t(a_1+ia_2)[\Delta-1]} \tag{4.1.2}$$

interpreted as a Fourier multiplier and $\{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of mutually independent complex valued Brownian motions with $\mathrm{Var}(\beta_n(t)) = 2t$.

Then, using the fact that β_n and β_m are independent unless $n = m$, and using Itô's isometry we have,

$$\begin{aligned}\sigma_N &= \mathbb{E}\left[|\Psi_N(x,t)|^2\right]\\ &= 2\gamma \sum_{|n|\le N} \int_{-\infty}^t e^{-2(t-t')a_1(|n|^2+1)}dt'\\ &= \sum_{|n|\le N} \frac{\gamma}{a_1(|n|^2+1)}\\ &\sim_{a_1,\gamma} \log N.\end{aligned}$$

In particular σ_N is independent of $(x,t) \in \mathbb{T}^2 \times \mathbb{R}_+$. It follows that Ψ_N is a Gaussian random variable of mean zero and variance σ_N. We make the ansatz $u_N = v_N + \Psi_N$ and then study the resulting equation for v_N:

$$\begin{cases}\partial_t v_N = (a_1+ia_2)[\Delta-1]v_N - (c_1+ic_2)\sum\limits_{\substack{0\le i\le m\\ 0\le j\le m-1}} \binom{m}{i}\binom{m-1}{j} v_N^i \overline{v_N}^j \Psi_N^{m-i}\overline{\Psi_N}^{m-j-1}\\ u_N|_{t=0} = \mathbf{P}_{\le N}u_0 - \Psi_N(0).\end{cases} \tag{4.1.3}$$

The nonlinearity in the above equation comes from expanding

$$|x+y|^{2m-2}(x+y) = (x+y)^m\overline{(x+y)^{m-1}}$$

using the binomial theorem twice. This ansatz is one of the main ideas in [22] and has come to be known as the *Da Prato-Debussche trick* in the SPDE literature. However, this idea was first used by McKean [61] and Bourgain [11] in the context of PDEs with random initial data.

The equation (4.1.3) still has the problem that the monomials $\Psi_N^k\overline{\Psi_N}^\ell$ do not have good limiting behaviour as $N \to \infty$. Beleaguered by this lack of convergence, we consider instead the *Wick ordered* truncated monomials defined by[1]

$$:\Psi_N^k\overline{\Psi_N}^\ell: = \begin{cases}(-1)^k k! L_k^{(\ell-k)}(|\Psi_N(x,t)|^2;\sigma_N)\overline{\Psi_N}^{\ell-k}, & k<\ell,\\ (-1)^\ell \ell! L_\ell^{(k-\ell)}(|\Psi_N(x,t)|^2;\sigma_N)\Psi_N^{k-\ell}, & \ell\le k.\end{cases} \tag{4.1.4}$$

One can show that, for given $k,\ell \in \mathbb{N}$, the Wick ordered truncated monomial, $:\Psi_N^k\overline{\Psi_N}^\ell:$, converges to a well-defined distribution which we denote by $:\Psi^k\overline{\Psi}^\ell:$. In particular we have the following proposition.

[1] For the purposes of this chapter, it is enough to take this as a definition. See [68, 74] for information on how this relates to Fock spaces.

Proposition 4.1.1. *Let $k, \ell \in \mathbb{N}$, $T > 0$ and $p \geq 1$. Then, $\{: \Psi_N^\ell \overline{\Psi_N}^k :\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; C^{-\varepsilon}(\mathbb{T}^2)))$. Moreover, denoting the limit by $:\Psi^\ell\overline{\Psi}^k:$, we have $:\Psi^\ell\overline{\Psi}^k: \in C([0,T]; C^{-\varepsilon}(\mathbb{T}^2))$ almost surely.*

We will prove this proposition in Section 4.4. One can think of $:\Psi^k\overline{\Psi}^\ell:$ as being $\Psi^k\overline{\Psi}^\ell$ with infinite counter terms. For example, for $k = 2$ and $\ell = 1$ we can think of $: |\Psi|^2\Psi :$ as being $|\Psi|^2\Psi - 2\infty\Psi$. This heuristic is justified by looking at (4.1.4) and noting that $\sigma_N \to \infty$ as $N \to \infty$.

Consider now the following equation:

$$\begin{cases} \partial_t v = (a_1 + ia_2)[\Delta - 1]v - (c_1 + ic_2) \sum\limits_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1}} \binom{m}{i}\binom{m-1}{j} v^i\overline{v}^j :\Psi^{m-i}\overline{\Psi}^{m-j-1}: \\ v|_{t=0} = u_0 - \Psi(0). \end{cases} \tag{4.1.5}$$

We interpret this equation as the following integral equation

$$v(t) = S(t)(u_0 - \Psi(0)) - \int_0^t S(t-t')F(v, \vec{\Psi})(t')\, dt' \tag{4.1.6}$$

where

$$F(v, \vec{\Psi}) = (c_1 + ic_2) \sum_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1}} \binom{m}{i}\binom{m-1}{j} v^i\overline{v}^j :\Psi^{m-i}\overline{\Psi}^{m-j-1}: .$$

We call this integral equation the mild formulation of (4.1.5).

The equation for v (4.1.5) is an untruncated version of (4.1.3) with $\Psi^{m-i}\overline{\Psi}^{m-j-1}$ replaced by $:\Psi^{m-i}\overline{\Psi}^{m-j-1}:$. The point here is that although $\Psi^{m-i}\overline{\Psi}^{m-j-1}$ is not well defined, $:\Psi^{m-i}\overline{\Psi}^{m-j-1}:$ is by Proposition 4.1.1. Making the ansatz $v \in C^{2\varepsilon}(\mathbb{T}^2)$ one can make sense of the right hand side of (4.1.5), unlike SCGL (1.2.4).

If a solution v exists to (4.1.5) we *define*

$$u \overset{\text{def}}{=} v + \Psi$$

to be a solution of the WSCGL (1.2.6),

$$\begin{cases} \partial_t u = (a_1 + ia_2)[\Delta - 1]u - (c_1 + ic_2) :|u|^{2m-2}u: + \sqrt{2\gamma}\xi \\ u_N|_{t=0} = u_0 \end{cases} \tag{4.1.7}$$

This perhaps seems unusual because we are defining solutions to WSCGL (1.2.6) through another equation, (4.1.5). We do this because WSCGL only makes sense formally. WSCGL (1.2.6) is an abuse of notation because $L_{m-1}^{(1)}(x;\infty)$ makes no sense and is an abuse of definitions because Wick ordering is only defined for Gaussian random variables, see [68] for more information, and there is no reason for u to be a Gaussian random variable. As the nonlinearity in WSCGL (1.2.6) does not have any rigorous meaning, WSCGL (1.2.6) itself also does not have any rigorous meaning. However, morally, WSCGL (1.2.6) is the renormalised equation that we are trying to solve in this chapter.

This definition of u solving WSCGL (1.2.6) is of such importance to this Chapter that it is worth stating in a reverse manner for extra clarity: u solves WSCGL (1.2.6) if $v = u - \Psi$ solves (4.1.6).

The connection between the equation for v (4.1.5) and WSCGL (1.2.6) can be understood by looking at truncated versions of these equations. Looking at the following truncated version of (4.1.5):

$$\begin{cases} \partial_t v_N = (a_1 + ia_2)[\Delta - 1]v_N - (c_1 + ic_2) \sum_{\substack{0 \le i \le m \\ 0 \le j \le m-1}} \binom{m}{i}\binom{m-1}{j} v_N^i \overline{v_N^j} :\Psi_N^{m-i}\overline{\Psi_N}^{m-j-1}: \\ v|_{t=0} = \mathbf{P}_{\le N} u_0 - \Psi_N(0) \end{cases} \tag{4.1.8}$$

and using the following generalised Laguerre polynomial sum formula, see Section 4.3,

$$(-1)^m m! L_m^{(1)}(|x+y|^2;\sigma)(x+y) = \sum_{\substack{0 \le i \le m+1 \\ 0 \le j \le m}} \binom{m+1}{i}\binom{m}{j} P_{i,j}^{m,\sigma}(y,\overline{y}) x^i \overline{x}^j \tag{4.1.9}$$

where

$$P_{i,j}^{m,\sigma}(y,\overline{y}) = \begin{cases} (-1)^{m-j}(m-j)! L_{m-j}^{(j-i+1)}(|y|^2;\sigma) y^{j-i+1}, & j+1 \ge i \\ (-1)^{m-i+1}(m-i+1)! L_{m-i+1}^{(i-j-l)}(|y|^2;\sigma)\overline{y}^{i-j-l}, & j+1 \le i. \end{cases}$$

it follows that $u_N = v_N + \Psi_N$ satisfies the following equation:

$$\begin{cases} \partial_t u_N = (a_1 + ia_2)[\Delta - 1]u_N - (c_1 + ic_2)(-1)^{m-1}(m-1)! L_{m-1}^{(1)}(|u_N|^2;\sigma_N)u_N \\ \qquad\qquad + \sqrt{2\gamma}\mathbf{P}_{\le N}\xi \\ u_N|_{t=0} = \mathbf{P}_{\le N} u_0. \end{cases} \tag{4.1.10}$$

The Laguerre polynomial sum formula (4.1.9) intermediates (4.1.10) and (4.1.8). Formally taking a limit as $N \to \infty$, the relationship, $u_N = v_N + \Psi_N$, between the truncated equations (4.1.10) and (4.1.8) gives justification for defining solutions to the purely formal WSCGL (1.2.6) through the equation for v (4.1.5). To the authors knowledge the sum formula (4.1.9) has not appeared in the literature. An elementary proof is given in Section 4.3 of this chapter.

Our main goal in this Chapter is to study the well-posedness of WSCGL (1.2.6), in particular we prove Theorem 1.2.1 and Theorem 1.2.2. Based on what we have discussed in this section, we can give more precise statements of these results.

Theorem 4.1.2. *Let $a_1 > 0$, $m \geq 2$ be an integer, let $s_0 > -\frac{2}{2m-1}$ and $\varepsilon > 0$ be sufficiently small. Then WSCGL* (1.2.6) *is pathwise locally well-posed in $C^{s_0}(\mathbb{T}^2)$. More precisely there exists $\theta > 0$ such that given any $u_0 \in C^{s_0}(\mathbb{T}^2)$, there exists $T \sim_\omega \|u_0\|_{C^{s_0}(\mathbb{T}^2)}^{-\theta}$, which is positive almost surely, such that there is a unique solution to the mild formulation* (4.1.6) *of* (4.1.5) *on* $[0, T]$ *with*

$$v \in C((0,T]; C^{2\varepsilon}(\mathbb{T}^2)) \cap C([0,T]; C^{s_0}(\mathbb{T}^2)).$$

Theorem 4.1.3. *Let $a_1, c_1 > 0$ and $s_0 > -\frac{2}{2m-1}$. Set $r = \left|\frac{a_1}{a_2}\right|$ and let $m \geq 2$ be an integer such that*

$$2m - 1 < 2 + 2\left(r^2 + 2r\sqrt{1 + r^2}\right).$$

Then WSCGL, (1.2.6) *is pathwise globally well-posed in $C^{s_0}(\mathbb{T}^2)$. More precisely for any $T > 0$, and any $u_0 \in C^{s_0}(\mathbb{T}^2)$, there almost surely exists a unique solution to the mild formulation of* (4.1.5) *on* $[0, T]$ *with*

$$v \in C((0,T]; C^{2\varepsilon}(\mathbb{T}^2)) \cap C([0,T]; C^{s_0}(\mathbb{T}^2))$$

almost surely, for $\varepsilon > 0$ small enough.

4.2 Heat smoothing estimates

For the rest of this chapter, unless explicitly stated otherwise, all functions spaces are defined with $\mathbb{T}^2$ as the underlying space. For ease of notation we will omit writing $\mathbb{T}^2$ when referring to the function spaces. For example we write L^p instead of $L^p(\mathbb{T}^2)$.

The following three heat-type linear smoothing estimates on the semi-group $S(t)$ (4.1.2) are used to prove that WSCGL is locally well-posed. For proofs of these estimates we refer the reader to [3], where the results are proven for $a_2 = 0$. The proofs easily adapt to the case $a_2 \neq 0$.

Proposition 4.2.1. *Let $s_0 \leq s_1$. Then,*

$$\|S(t)f\|_{C^{s_1}} \lesssim t^{\frac{s_0 - s_1}{2}} \|f\|_{C^{s_0}}.$$

Proposition 4.2.2. *Let $s_0 \leq s_1$ be such that $s_1 - s_0 \leq 2$. Then,*

$$\|(1 - S(t))f\|_{C^{s_0}} \lesssim t^{\frac{s_1 - s_0}{2}} \|f\|_{C^{s_1}}.$$

The previous Proposition shows that, if $s_1 > s_0$ and $f \in C^{s_1}$, then the mapping $t \mapsto S(t)f$ is continuous as a mapping from $[0, \infty)$ to C^{s_0}. The proposition however, says nothing about continuity if $s_0 = s_1$. The following proposition states that this mapping is continuous, even though we do not have an explicit bound.

Proposition 4.2.3. *Suppose $s_0 \in \mathbb{R}$ and $f \in C^{s_0}$. Then the mapping $t \mapsto S(t)f$ is continuous as a mapping from $[0, \infty)$ to C^{s_0}.*

4.3 Laguerre polynomial formulae

This section is devoted to proving the Laguerre polynomial sum formula, (4.1.9) and the Laguerre polynomial expectation formula (1.2.9). We do not need the Laguerre polynomial sum formula (4.1.9) to prove Theorems 4.1.2 and 4.1.3 but it is important as it motivates the renormalisation we used. We critically use the Laguerre polynomial expectation formula Proposition 4.3.3 in the proof of Proposition 4.1.1.

4.3.1 Sum formula

The generalised Laguerre polynomials enjoy the following classical three point rules, see for example [83, Chapter 23].

$$(n+\ell)L_{n-1}^{(\ell)}(x) = nL_n^{(\ell)}(x) + xL_{n-1}^{(\ell+1)}(x), \quad L_n^{(\ell)}(x) - L_{n-1}^{(\ell)}(x) = L_n^{(\ell-1)}(x).$$

Together these relations imply

$$(n+\ell)L_n^{(\ell-1)}(x) = \ell L_n^{(\ell)}(x) - xL_{n-1}^{(\ell+1)}(x). \tag{4.3.1}$$

There is also a well known recurrence formula for derivatives of generalised Laguerre polynomials:

$$\frac{d}{dx}L_k^{\ell}(x) = -L_{k-1}^{\ell+1}(x), \quad \text{for } k \geq 1. \tag{4.3.2}$$

The aim of this subsection is to prove the following sum formula.

Lemma 4.3.1. *Let $m \geq 1$ and $\ell \geq 0$. Then the following is true:*

$$(-1)^m m! L_m^{(\ell)}(|x+y|^2;\sigma)(x+y)^\ell = \sum_{\substack{0\leq i\leq m+\ell \\ 0\leq j\leq m}} \binom{m+\ell}{i}\binom{m}{j} P_{i,j}^{m,\ell,\sigma}(y,\overline{y})x^i\overline{x}^j$$

where

$$P_{i,j}^{m,\ell,\sigma}(y,\overline{y}) = \begin{cases} (-1)^{m-j}(m-j)!L_{m-j}^{(\ell+j-i)}(|y|^2;\sigma)y^{\ell+j-i}, & \ell+j\geq i \\ \\ (-1)^{m+\ell-i}(m+\ell-i)!L_{m+\ell-i}^{(i-j-\ell)}(|y|^2;\sigma)\overline{y}^{i-j-\ell}, & \ell+j\leq i. \end{cases}$$

Proof. We view both sides of the equality as polynomials in x and $\overline{x}$ with coefficients depending on y and $\overline{y}$. By scaling it suffices to prove the lemma for $\sigma = 1$.

Note that the lemma is true for all $m, \ell \geq 0$ satisfying $2m+\ell \leq 1$. To prove the lemma for all m and ℓ we induct on $2m+\ell$.

Let $n \in \mathbb{N}$ and suppose the statement in the Lemma is true for all $m, \ell \geq 0$ satisfying $2m+\ell < n$. Then for $m, \ell \geq 0$ such that $2m+\ell = n$, using (4.3.2) and the inductive hypothesis, we have,

$$\frac{\partial}{\partial \overline{x}}\left[(-1)^m m! L_m^{(\ell)}(|x+y|^2)(x+y)^\ell\right] = (-1)^{m-1} m! L_{m-1}^{(\ell+1)}(|x+y|^2)(x+y)^{\ell+1}$$

$$= \sum_{\substack{0\le i\le m+\ell \\ 0\le j\le m-1}} m\binom{m+\ell}{i}\binom{m-1}{j} P_{i,j}^{m-1,\ell+1,\sigma}(y,\overline{y})x^i\overline{x}^j.$$

"Partially integrating" this expression with respect to $\overline{x}$ we get

$$\begin{aligned}
&(-1)^m m! L_m^{(\ell)}(|x+y|^2)(x+y)^\ell \\
&\quad = \sum_{\substack{0\le i\le m+l \\ 0\le j\le m-1}} \frac{m}{j+1}\binom{m+l}{i}\binom{m-1}{j} P_{i,j}^{m-1,\ell+1,\sigma}(y,\overline{y})x^i\overline{x}^{j+1} + C(y,\overline{y},x) \\
&\quad = \sum_{\substack{0\le i\le m+\ell \\ 1\le j\le m}} \binom{m+\ell}{i}\binom{m}{j} P_{i,j}^{m,\ell,\sigma}(y,\overline{y})x^i\overline{x}^j + C(y,\overline{y},x) \qquad (4.3.3)
\end{aligned}$$

where the last equality comes from relabeling j in the summation and noting $P_{i,j-1}^{m-1,\ell+1,\sigma}(y,\overline{y}) = P_{i,j}^{m,\ell,\sigma}(y,\overline{y})$. Differentiating the left hand side of (4.3.3) with respect to x and using the three point rule (4.3.1) we have,

$$\begin{aligned}
\frac{\partial}{\partial x}\left[(-1)^m m! L_m^{(\ell)}(|x+y|^2)(x+y)^\ell\right] &= -(-1)^m m! L_{m-1}^{(\ell+1)}(|x+y|^2)(x+y)^{\ell-1}|x+y|^2 \\
&\quad + \ell(-1)^m m! L_m^{(\ell)}(|x+y|^2)(x+y)^{\ell-1} \\
&= (m+\ell)(-1)^m m! L_m^{(\ell-1)}(|x+y|^2)(x+y)^{\ell-1}.
\end{aligned}$$

Equating this with the derivative of the right hand side of (4.3.3),

$$\begin{aligned}
&\sum_{\substack{0\le i\le m+\ell-1 \\ 0\le j\le m}} \binom{m+\ell-1}{i}\binom{m}{j} P_{i,j}^{m\ell-1}(y,\overline{y})x^i\overline{x}^j \\
&\quad = \sum_{\substack{1\le i\le m+\ell \\ 1\le j\le m}} \frac{i}{m+\ell}\binom{m+\ell}{i}\binom{m}{j} P_{i,j}^{m,\ell}(y,\overline{y})x^{i-1}\overline{x}^j + \frac{\partial}{\partial x}\, C(y,\overline{y},x) \\
&\quad = \sum_{\substack{0\le i\le m+\ell-1 \\ 1\le j\le m}} \binom{m+\ell-1}{i}\binom{m}{j} P_{i,j}^{m,\ell-1}(y,\overline{y})x^i\overline{x}^j + \frac{1}{m+\ell}\frac{\partial}{\partial x}\, C(y,\overline{y},x)
\end{aligned}$$

where in the last equality we used the fact that $P_{i,j}^{m,\ell-1}(y,\overline{y}) = P_{i+1,j}^{m,\ell}(y,\overline{y})$. Hence we get an expression for $\frac{\partial}{\partial x}\, C(y,\overline{y},x)$,

$$\frac{\partial}{\partial x}\, C(y,\overline{y},x) = (m+\ell) \sum_{0\le i\le m+\ell-1} \binom{m+\ell-1}{i} P_{i,0}^{m,\ell-1}(y,\overline{y})x^i.$$

"Partially integrating" this expression we get

$$C(y,\overline{y},x) = \sum_{0\le i\le m+\ell-1} \frac{m+1}{i+1}\binom{m+\ell-1}{i} P^{m,\ell-1}_{i,0}(y,\overline{y})x^{i+1} + C(y,\overline{y})$$
$$= \sum_{1\le i\le m+\ell} \binom{m+\ell}{i} P^{m,\ell}_{i,0}(y,\overline{y})x^{i} + C(y,\overline{y})$$

where we have relabeled the sum in the second inequality. This shows,

$$(-1)^m m! L^{(\ell)}_m(|x+y|^2;\sigma)(x+y)^\ell = \sum_{\substack{0\le i\le m+\ell\\ 0\le j\le m\\ (i,j)\neq(0,0)}} \binom{m+\ell}{i}\binom{m}{j} P^{m,\ell,\sigma}_{i,j}(y,\overline{y})x^i\overline{x}^j + C(y,\overline{y}). \tag{4.3.4}$$

For $C(y,\overline{y})$, note that when $x=0$ (4.3.4) reduces to,

$$(-1)^m m! L^{(\ell)}_m(|y|^2;\sigma)y^\ell = C(y,\overline{y}).$$

This completes the proof. □

4.3.2 Expectation formula

The aim of this subsection is to prove Proposition 4.3.3 stated below.

Before we proof this, we first state some elementary facts.

The generalised Hermite polynomials satisfy the following recurrence relation

$$H_{k+1}(x;\sigma) = xH_k(x;\sigma) - \sigma H_{k-1}(x;\sigma). \tag{4.3.5}$$

These polynomials also enjoy the following properties.

Proposition 4.3.2. *Let $k\ge 0$ and $\sigma,\beta\in\mathbb{R}$. Then the following are true.*

1. $$\int_{\mathbb{R}} H_k(x;\sigma)e^{ux-\frac{x^2}{2}}dx = \sqrt{2\pi}H_k(u;\sigma-1)e^{\frac{u^2}{2}}, \tag{4.3.6}$$

2. $$i^k H_k(x;-\sigma) = H_k(ix;\sigma), \tag{4.3.7}$$

3. $$H_n(x;\ell+\beta) = \sum_{k=0}^{n}\binom{n}{k}H_k(x;\ell)H_{n-k}(x;\beta), \tag{4.3.8}$$

4. *if $\sigma>0$,*
$$H_k(x;\sigma) = \sigma^{k/2}H_k(x/\sqrt{\sigma}).$$

Proof. These facts can be proven by using the recurrence relation (4.3.5) and a standard induction argument. We will just prove the first one to give the reader a taste of how to complete such an argument. Note that the generating function for the Hermite polynomials is

$$e^{xu-\frac{u^2}{2}} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} u^n.$$

Multiplying both sides by $H_n(x)e^{\frac{x^2}{2}}$ and using the fact that $\int_{\mathbb{R}} H_n(x)H_m(x)e^{\frac{x^2}{2}}\,dx = \sqrt{2\pi}n!\sigma_{nm}$,

$$e^{\frac{-u^2}{2}} \int_{\mathbb{R}} e^{xu-\frac{x^2}{2}} H_n(x)dx = \sqrt{2\pi}u^n$$

and so the result is true for $\sigma = 1$. As $H_0(x;\sigma) = 1$ and $H_1(x;\sigma) = x$ for all σ, the $k = 0$ and $k = 1$ cases are also true. From the recurrence relation (4.3.5) the result is then true for all σ. □

The key ingredient in the real valued analogue of Proposition 4.1.1 in the next section is the following well known identity:

$$\mathbb{E}[H_k(f;\sigma_f)H_\ell(g;\sigma_g)] = k!\delta_{k\ell} \tag{4.3.9}$$

where f and g are Gaussian random variables with variances σ_f and σ_g respectively. To prove Proposition 4.1.1 we need the following Laguerre polynomial analogue of (4.3.9).

Proposition 4.3.3. *Let f and g be mean-zero complex valued Gaussian random variables with variances σ_f and σ_g respectively. Then,*

$$\mathbb{E}\left[L_k^{(\ell)}(|f|^2;\sigma_f)f^\ell\overline{L_m^{(\ell)}(|g|^2;\sigma_g)g^\ell}\right] = \delta_{km}\frac{(k+\ell)!}{k!}\left|\mathbb{E}[f\overline{g}]\right|^{2k}\mathbb{E}[f\overline{g}]^\ell.$$

The above proposition was proven in [74] for $\ell = 0$ and $\ell = 1$. The proof for the general case proved in this section is the natural generalisation of the proof in [74]. We use the following elementary lemma.

Lemma 4.3.4. *Let g be a mean-zero complex valued random variable. Then*

$$\mathbb{E}\left[e^{\operatorname{Re} g}\right] = e^{\frac{1}{4}\mathbb{E}[|g|^2]}.$$

Proof of Proposition 4.3.3. Recall the generating function for the Laguerre polynomials (1.2.7). It suffices to prove the Lemma assuming $\sigma_f = \sigma_g = 1$. Let $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$. Using the binomial expansion formula for $(f_1+if_2)^\ell$ and then applying

(4.3.6) with $\sigma = 1$, $u = \sqrt{\frac{-2t}{1-t}} f_1$ and again with $\sigma = 1$, $u = \sqrt{\frac{-2t}{1-t}} f_2$,

$$\begin{aligned} G_\ell(t, |f|^2) f^\ell &= \frac{1}{(1-t)^{\ell+1}} (f_1 + i f_2)^\ell e^{\frac{-t}{1-t}(f_1^2+f_2^2)} \\ &= \sum_{k=0}^{l} \frac{1}{(1-t)^{\ell+1}} \binom{\ell}{k} i^{\ell-k} f_1^k f_2^{\ell-k} e^{\frac{-t}{1-t}(f_1^2+f_2^2)} \\ &= \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{i^{\ell-k}}{(\sqrt{-2t})^l (1-t)^{l/2+1}} \frac{1}{2\pi} \int_{\mathbb{R}^2} H_k(x_1) H_{\ell-k}(x_2) e^{-\frac{x_1^2+x_2^2}{2}} e^{\sqrt{\frac{-2t}{1-t}}(x_1 f_1 + x_x f_2)} dx_1 dx_2. \end{aligned} \tag{4.3.10}$$

Given $x_1, x_2, x_3, x_4 \in \mathbb{R}$, we set $x = x_1 + i x_2$ and $y = y_1 + i y_2$. For $s, t \in (-1, 0)$ applying (4.3.10) twice and taking an expectation gives,

$$\begin{aligned} &\int_\Omega G_\ell(t, |f|^2) f^\ell \overline{G_\ell(t, |g|^2) g^\ell} dP(\omega) \\ &\quad = \sum_{k,m=0}^{\ell} \binom{\ell}{k} \binom{\ell}{m} i^{2\ell-k-m} \frac{1}{(\sqrt{-2t})^l (1-t)^{l/2+1}} \frac{1}{(\sqrt{-2s})^\ell (1-s)^{l/2+1}} \frac{1}{4\pi^2} \\ &\quad \times \int_{\mathbb{R}^4} H_k(x_1) H_{\ell-k}(x_2) H_m(y_1) H_{\ell-m}(y_2) e^{-\frac{|x|^2+|y|^2}{2}} \\ &\quad \times \int_\Omega \exp\left(\mathrm{Re}\left(\sqrt{\frac{-2t}{1-t}} \overline{x} f + \sqrt{\frac{-2s}{1-s}} y \overline{g} \right) \right) dx_1 dx_2 dy_1 dy_2 \\ &\quad = \sum_{k,m=0}^{\ell} \binom{\ell}{k} \binom{\ell}{m} i^{2\ell-k-m} \frac{1}{(\sqrt{-2t})^\ell (1-t)^{l/2+1}} \frac{1}{(\sqrt{-2s})^\ell (1-s)^{\ell/2+1}} \frac{1}{4\pi^2} \\ &\quad \times \int_{\mathbb{R}^4} H_k(x_1) H_{\ell-k}(x_2) H_m(y_1) H_{\ell-m}(y_2) \\ &\quad \times e^{-\frac{|x|^2}{2(1-t)} - \frac{|y|^2}{2(1-s)}} e^{\frac{1}{2} \mathrm{Re}\left(\sqrt{\frac{-2t}{1-t}} \sqrt{\frac{-2t}{1-t}} \overline{x} y \mathbb{E}[f\overline{g}] \right)} dx_1 dx_2 dy_1 dy_2 \end{aligned}$$

where in the second inequality we used Lemma 4.3.4. Applying the change of variables $x, = \frac{1}{\sqrt{1-t}} x$ and $y, = \frac{1}{\sqrt{1-s}} y$ and then using Lemma 4.3.6 with $u = \sqrt{ts}\,\mathrm{Re}(y\mathbb{E}[f\overline{g}])$ and again with $u = \sqrt{ts}\,\mathrm{Im}(y\mathbb{E}[f\overline{g}])$ we have,

$$\begin{aligned} &\int_\Omega G_\ell(t, |f|^2) f^\ell \overline{G_\ell(t, |g|^2) g^\ell} dP(\omega) \\ &= \sum_{k,m=0}^{\ell} \binom{\ell}{k} \binom{\ell}{m} i^{2\ell-k-m} \frac{1}{(2ts)^{\ell/2}} \frac{1}{4\pi^2} \\ &\times \int_{\mathbb{R}^4} H_k(x_1; (1-t)^{-1}) H_{\ell-k}(x_2; (1-t)^{-1}) H_m(y_1; (1-s)^{-1}) H_{\ell-m}(y_2; (1-s)^{-1}) \\ &\times e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2}} e^{\sqrt{ts} x_1 \mathrm{Re}(y\mathbb{E}[f\overline{g}]) + \sqrt{ts} x_2 \mathrm{Im}(y\mathbb{E}[f\overline{g}])} dx_1 dx_2 dy_1 dy_2 \end{aligned}$$

$$= \sum_{k,m=0}^{\ell} \binom{\ell}{k}\binom{\ell}{m} \frac{i^{2\ell-k-m}}{2\pi(2ts)^{\ell/2}} \int_{\mathbb{R}^2} H_k(\sqrt{ts}\,\mathrm{Re}\,(y\mathbb{E}[f\overline{g}])\,; \tfrac{t}{1-t}) H_{\ell-k}(\sqrt{ts}\,\mathrm{Im}\,(y\mathbb{E}[f\overline{g}])\,; \tfrac{t}{1-t})$$
$$\times H_m(y_1; (1-s)^{-1}) H_{\ell-m}(y_2; (1-s)^{-1}) e^{-\frac{|y|^2}{2}} e^{\frac{1}{2}\sqrt{ts}|y|^2|\mathbb{E}[f\overline{g}]|^2} dy_1 dy_2$$
$$= \frac{1}{(2ts)^{\ell/2}} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\sum_{k=0}^{\ell} \binom{\ell}{k} H_k(\sqrt{ts}\,\mathrm{Re}\,(y\mathbb{E}[f\overline{g}])\,; \tfrac{t}{1-t}) i^{\ell-k} H_{\ell-k}(\sqrt{ts}\,\mathrm{Im}(y\mathbb{E}[f\overline{g}]); \tfrac{t}{1-t}) \right)$$
$$\times \left(\sum_{m=0}^{\ell} \binom{\ell}{m} H_m(y_1; (1-s)^{-1}) i^{\ell-m} H_{\ell-m}(y_2; (1-s)^{-1}) \right) e^{-\frac{1}{2}\left(1-\sqrt{ts}|\mathbb{E}[f\overline{g}]|^2\right)|y|^2} dy_1 dy_2.$$

Using (4.3.7) and then (4.3.8) we get,

$$\int_\Omega G_\ell(t, |f|^2) f^\ell \overline{G_\ell(t, |g|^2) g^\ell} dP(\omega) = \frac{\mathbb{E}[f\overline{g}]^\ell}{2^{\ell+1}\pi} \int_{\mathbb{R}^2} |y|^{2\ell} e^{-\frac{1}{2}\left(1-\sqrt{ts}|\mathbb{E}[f\overline{g}]|^2\right)|y|^2} dy.$$

Integrating the right hand side over $\mathbb{R}^2$ using the formula

$$\int_{\mathbb{R}^2} |y|^{2\ell} e^{-\beta|y|^2} dy = \frac{\ell!\pi}{\beta^{\ell+1}}$$

we get,

$$\begin{aligned}
\int_\Omega G_\ell(t, |f|^2) f^\ell \overline{G_\ell(t, |g|^2) g^\ell} dP(\omega) &= \frac{\mathbb{E}[f\overline{g}]^\ell}{2^{\ell+1}\pi} \frac{2^{\ell+1}\pi\ell!}{\left(1 - ts|\mathbb{E}[f\overline{g}]|^2\right)^{\ell+1}} \\
&= \ell! \frac{\mathbb{E}[f\overline{g}]^\ell}{\left(1 - ts|\mathbb{E}[f\overline{g}]|^2\right)^{\ell+1}} \\
&= \sum_{k=0}^{\infty} \ell! \binom{\ell+k}{\ell} t^k s^k |\mathbb{E}[f\overline{g}]|^{2k} \mathbb{E}[f\overline{g}]^\ell
\end{aligned}$$

where the last equality uses the Maclaurin series

$$\frac{1}{(1-x)^{\ell+1}} = \sum_{n=0}^{\infty} \binom{\ell+n}{\ell} x^n$$

which is valid for $|x| \leq 1$. On the other hand, from the the generating function G_ℓ we have,

$$\int_\Omega G_\ell(t, |f|^2) f^\ell \overline{G_\ell(t, |g|^2) g^\ell} dP(\omega) = \sum_{k,m=0}^{\infty} t^n s^m \mathbb{E}\left[L_k^{(\ell)}(|f|^2; \sigma_f) f^\ell \overline{L_m^{(\ell)}(|g|^2; \sigma_g) g^\ell} \right].$$

The proposition follows by comparing coefficients. □

4.4 On the stochastic convolution

In this section we will give a proof of Proposition 4.1.1, establishing regularity estimates for the Wick ordered powers $:\Psi_N^k \overline{\Psi_N}^\ell:$, using a Fourier analytic approach similar to that in [38, Proposition 2.1].

Proof of Proposition 4.1.1. We will assume $k \geq \ell$, the other case is similar. By Proposition A.1.3 it suffices to prove the proposition with $W^{-\varepsilon,\infty}$ in place of $C^{-\varepsilon}$. Further, as $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$ for $p_1 \leq p_2$, it suffices to prove the proposition for p sufficiently large.

First we drive a useful formula used throughout this proof. For $t_1 \leq t_2$, by the independence of β_n and β_m for $m \neq n$, the independent increment property of Brownian motion and the Itô isometry we have

$$\begin{aligned}
\mathbb{E}\big[\Psi_N(x,t_1)\overline{\Psi_N(y,t_2)}\big] &= \gamma \sum_{|n|,|m|\leq N} e_n(x)e_{-m}(y)\mathbb{E}\Big[\int_{-\infty}^{t_1} e^{-(t_1-t')(a_1+ia_2)(|n|^2+1)}d\beta_n(t') \\
&\qquad \times \overline{\int_{-\infty}^{t_2} e^{-(t_2-t')(a_1+ia_2)(|m|^2+1)}d\beta_m(t')}\Big] \\
&= \gamma \sum_{0\leq|n|\leq N} e_n(x-y)\mathbb{E}\Big[\int_{-\infty}^{t_1} e^{-(t_1-t')(a_1+ia_2)(|n|^2+1)}d\beta_n(t') \\
&\qquad \times \overline{\int_{-\infty}^{t_1} e^{-(t_2-t')(a_1+ia_2)(|n|^2+1)}d\beta_n(t')}\Big] \\
&= 2\gamma \sum_{0\leq|n|\leq N} e_n(x-y)e^{-(t_2-t_1)(a_1-ia_2)(|n|^2+1)} \int_{-\infty}^{t_1} e^{-2(t_1-t')(|n|^2+1)a_1}dt' \\
&= \sum_{0\leq|n|\leq N} e_n(x-y)e^{-(t_2-t_1)(a_1-ia_2)(|n|^2+1)}\frac{\gamma}{a_1(|n|^2+1)} \\
&= \sum_{0\leq|n|\leq N} e_n(x-y)\zeta(n,t_1,t_2) \qquad (4.4.1)
\end{aligned}$$

where

$$\zeta(n,t_1,t_2) = e^{-(t_2-t_1)(a_1-ia_2)(|n|^2+1)}\frac{\gamma}{a_1(|n|^2+1)}.$$

Note that

$$|\zeta(n,t_1,t_2)| \lesssim_{a_1,\gamma} \langle n\rangle^{-2}. \qquad (4.4.2)$$

When $t_1 = t_2$, $\zeta_n(n,t_1,t_2)$ is independent of t_1 and t_2 and so we write $\zeta(n)$ instead of $\zeta_n(n,t_1,t_2)$.

Now we will show that $\Psi_N(\cdot,t) \in W^{-\varepsilon,\infty}$ for a fixed t. Applying the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ to (4.4.1) with $t_1 = t_2 = t$ we have,

$$\mathbb{E}[|\langle\nabla_x\rangle^{-\varepsilon}\Psi_N(x,t)|^2] = \sum_{|n|\leq N}\frac{\zeta(n)}{\langle n\rangle^{2\varepsilon}} \lesssim \sum_{|n|\leq N}\frac{1}{\langle n\rangle^{2+2\varepsilon}} \lesssim 1 < \infty$$

uniformly in $N \in \mathbb{N}$, $x \in \mathbb{T}^2$ and $t \in \mathbb{R}$. Using Proposition A.1.2 (Sobolev), switching the order of integration and then using Proposition A.2.1 (Weiner chaos) we have

$$\begin{aligned}\mathbb{E}\left[\|\Psi_N(\cdot,t)\|^p_{W^{-\varepsilon,\infty}}\right] &\lesssim_{p,\varepsilon} \mathbb{E}\left[\|\Psi_N(\cdot,t)\|^p_{W^{-\varepsilon/2,p}}\right] \\ &= \int_{\mathbb{T}^2} \mathbb{E}[|\langle\nabla\rangle^{-\varepsilon/2}\Psi_N(x,t)|^p]dx \\ &\lesssim \int_{\mathbb{T}^2} \mathbb{E}[|\langle\nabla\rangle^{-\varepsilon/2}\Psi_N(x,t)|^2]dx \\ &\lesssim 1.\end{aligned} \tag{4.4.3}$$

Now we will show that $:\Psi_N(\cdot,t)^k\overline{\Psi_N(\cdot,t)}^\ell:$ is in $W^{-\varepsilon,\infty}$ for a fixed t. Using Lemma 4.3.3,

$$\begin{aligned}&\mathbb{E}\bigg[:\Psi_N(x,t)^k\overline{\Psi_N(x,t)}^\ell :\overline{:\Psi_N k(y,t)^k\overline{\Psi_N(y,t)}^\ell:}\bigg] \\ &\qquad= C_{k,l}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(x,t)}\big]^k\overline{\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(x,t)}\big]}^\ell \\ &\qquad= C_{k,l}\bigg(\sum_{|n|\le N} e_n(x-y)\zeta(n)\bigg)^k\bigg(\overline{\sum_{|n|\le N} e_n(x-y)\zeta(n)}\bigg)^\ell \\ &\qquad= C_{k,l}\sum_{|n_1|,\dots,|n_{k+\ell}|\le N} e_{n_1\cdots+n_{k+\ell}}(x-y)\prod_{j=1}^{k+\ell}\zeta(n_j)\end{aligned} \tag{4.4.4}$$

for some inessential constant $C_{k,\ell}$. Applying the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ and then setting $x = y$ we get,

$$\mathbb{E}\left[|\langle\nabla_x\rangle^{-\varepsilon}:\Psi_N k(x,t)^k\overline{\Psi_N(x,t)}^\ell:|^2\right] \lesssim \sum_{|n_1|,\dots,|n_{k+\ell}|\le N}\frac{1}{\langle n_1\cdots+n_{k+\ell}\rangle^{2\varepsilon}}\prod_{j=1}^{k+\ell}\frac{1}{\langle n_j\rangle^2}.$$

We want to use an argument similar to that in equation (4.4.3) but before we can do this we need to show the sum in the above equation is bounded independently of N. To do this we argue by induction. Note that it is obviously bounded when $k+\ell = 1$. When $k+\ell > 1$, we split the sum into two regions corresponding to

$$\langle n_1+\cdots+n_{k+\ell}\rangle \le \langle n_{k+\ell}\rangle \quad \text{and} \quad \langle n_1+\cdots+n_{k+\ell}\rangle > \langle n_{k+\ell}\rangle.$$

This is motivated by the fact that

$$\langle n_1+\cdots+n_{k+\ell-1}\rangle \lesssim \max(\langle n_1+\cdots+n_{k+\ell}\rangle, \langle n_{k+\ell}\rangle).$$

With this splitting we have,

$$\sum_{|n_{k+\ell}|\le N} \frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{2\varepsilon}} \frac{1}{\langle n_{k+\ell}\rangle^2} \lesssim \tag{4.4.5}$$

$$\frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}} \sum_{|n_{k+\ell}|\le N} \frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{2\varepsilon}\langle n_{k+\ell}\rangle^{2-\varepsilon}}$$

$$+\frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}} \sum_{|n_{k+\ell}|\le N} \frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{\varepsilon}\langle n_{k+\ell}\rangle^{2}}$$

$$\lesssim \frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}}$$

independently of N. Hence,

$$\sum_{|n_1|,\ldots,|n_{k+\ell}|\le N} \frac{1}{\langle n_1\cdots+n_{k+\ell}\rangle^{2\varepsilon}} \prod_{j=1}^{k+\ell}\frac{1}{\langle n_j\rangle^2} \lesssim \sum_{|n_1|,\ldots,|n_{k+\ell-1}|\le N} \frac{1}{\langle n_1\cdots+n_{k+\ell-1}\rangle^{\varepsilon}} \prod_{j=1}^{k+\ell-1}\frac{1}{\langle n_j\rangle^2}$$

and so the desired bound follows by induction.

We have shown,

$$\mathbb{E}\left[|\langle\nabla_x\rangle^{-\varepsilon} :\Psi_N k(x,t)^k\overline{\Psi_N(x,t)}^{\ell}: |^2\right] < \infty$$

independently of N. Using the Propositions A.1.2 and A.2.1 in similar way to (4.4.3),

$$\begin{aligned}
\mathbb{E}[\| :\Psi_N(\cdot,t)^k\overline{\Psi_N(\cdot,t)}^{\ell}: \|^p_{W^{-\varepsilon,\infty}}] &\lesssim_{p,\varepsilon} \mathbb{E}[\| :\Psi_N(\cdot,t)^k\overline{\Psi_N(\cdot,t)}^{\ell}: \|^p_{W^{-\varepsilon/2,p}}] \\
&= \int_{\mathbb{T}^2} \mathbb{E}[|\langle\nabla\rangle^{-\varepsilon/2} :\Psi_N(\cdot,t)^k\overline{\Psi_N(\cdot,t)}^{\ell}: |^p]dx \\
&\lesssim \int_{\mathbb{T}^2} \mathbb{E}[|\langle\nabla\rangle^{-\varepsilon/2} :\Psi_N(\cdot,t)^k\overline{\Psi_N(\cdot,t)}^{\ell}: (\cdot,t)|^2]dx \\
&\lesssim 1 < \infty
\end{aligned}$$

which shows $:\Psi_N^k\overline{\Psi_N}^{\ell}:\in L^p(\Omega;L^\infty([0,T];W^{-\varepsilon,\infty}))$ uniformly in N.

Now we show that $:\Psi_N^k\overline{\Psi_N}^{\ell}:$ is Cauchy in $L^p(\Omega,L^\infty([0,T],W^{-\varepsilon,\infty}))$. For $N\ge M\ge 1$, similar to (4.4.4) we have,

$$\begin{aligned}
\mathbb{E}\Big[&(:\Psi_N(x,t)^k\overline{\Psi_N(x,t)}^{\ell}:- :\Psi_M(x,t)^k\overline{\Psi_M(x,t)}^{\ell}:) \\
&\times\overline{(:\Psi_N(y,t)^k\overline{\Psi_N(y,t)}^{\ell}: - :\Psi_M(y,t)^k\overline{\Psi_M(y,t)}^{\ell}:)}\Big] \\
&= C_{k,l}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]^k\overline{\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]}^{\ell}
\end{aligned}$$

$$
\begin{aligned}
&- C_{k,l}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_M(y,t)}\big]^k\overline{\mathbb{E}[\Psi_N(x,t)\overline{\Psi_M(y,t)}]}^\ell \\
&- C_{k,l}\mathbb{E}\big[\Psi_M(x,t)\overline{\Psi_N(y,t)}\big]^k\overline{\mathbb{E}\big[\Psi_M(x,t)\overline{\Psi_N(y,t)}\big]}^\ell \\
&+ C_{k,l}\mathbb{E}\big[\Psi_M(x,t)\overline{\Psi_M(y,t)}\big]^k\overline{\mathbb{E}\big[\Psi_M(x,t)\overline{\Psi_M(y,t)}\big]}^\ell \\
=\ & C_{k,l}\Big(\sum_{|n|\le N} e_n(x-y)\zeta(n)\Big)^k\Big(\overline{\sum_{|n|\le N} e_n(x-y)\zeta(n)}\Big)^\ell \\
&- C_{k,\ell}\Big(\sum_{|n|\le M} e_n(x-y)\zeta(n)\Big)^k\Big(\overline{\sum_{|n|\le M} e_n(x-y)\zeta(n)}\Big)^\ell \\
=\ & C_{k,l}\sum_{|n_1|,\ldots,|n_{k+\ell}|\le N} e_{n_1\cdots+n_{k+\ell}}(x-y)\prod_{j=1}^{k+\ell}\zeta(n_j) \\
&- C_{k,l}\sum_{|n_1|,\ldots,|n_{k+\ell}|\le M} e_{n_1\cdots+n_{k+\ell}}(x-y)\prod_{j=1}^{k+\ell}\zeta(n_j). \qquad (4.4.6)
\end{aligned}
$$

Using the notation

$$
\Gamma_{N,M}(\overline{n}) = \{|n_1|,\ldots,|n_{k+\ell}| \le N : |n_j| > M \text{ for some } j\}
$$

we have

$$
\text{LHS of (4.4.6)} = C_{k,l}\sum_{\Gamma_{N,M}(\overline{n})} e_{n_1\cdots+n_{k+\ell}}(x-y)\prod_{j=1}^{k+\ell}\zeta(n_j).
$$

Applying the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ and then setting $x=y$ we get

$$
\begin{aligned}
&\mathbb{E}\big[\big|\langle\nabla_x\rangle^{-\varepsilon}\big(:\Psi_N(x,t)^k\overline{\Psi_N(x,t)}^\ell:-:\Psi_M(x,t)^k\overline{\Psi_M(x,t)}^\ell:\big)\big|^2\big] \\
&\qquad\qquad = C_{k,l}\sum_{\Gamma_{N,M}(\overline{n})}\frac{1}{\langle n_1\cdots+n_{k+\ell}\rangle^{2\varepsilon}}\prod_{j=1}^{k+\ell}\zeta(n_j).
\end{aligned}
$$

We can estimate this sum in a way similar to (4.4.5). Indeed, without loss of generality we can assume $|n_{k+\ell}| > M$. Then, adapting the estimate in (4.4.5), we have,

$$
\begin{aligned}
&\sum_{N<|n_{k+\ell}|\le M}\frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{2\varepsilon}}\frac{1}{\langle n_{k+\ell}\rangle^2} \lesssim \qquad (4.4.7)\\
&\qquad \frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}}\sum_{N<|n_{k+\ell}|\le M}\frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{2\varepsilon}\langle n_{k+\ell}\rangle^{2-\varepsilon}} \\
&\qquad + \frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}}\sum_{N<|n_{k+\ell}|\le M}\frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{\varepsilon}\langle n_{k+\ell}\rangle^{2}} \\
&\qquad \lesssim \frac{1}{\langle n_1+\cdots+n_{k+\ell-1}\rangle^{\varepsilon}}\frac{1}{M^{\frac{\varepsilon}{2}}}.
\end{aligned}
$$

This shows,

$$\sum_{\Gamma_{N,M}(\overline{n})} \frac{1}{\langle n_1 \cdots + n_{k+\ell}\rangle^{2\varepsilon}} \prod_{j=1}^{k+\ell} \zeta(n_j) \lesssim M^{-\frac{\varepsilon}{2}}.$$

Using the Propositions A.1.2 and A.2.1 in a similar way to (4.4.3) we have,

$$\mathbb{E}[\| :\Psi_N(\cdot,t)^k \overline{\Psi_N(\cdot,t)}^\ell: - :\Psi_M(\cdot,t)^k \overline{\Psi_M(\cdot,t)}^\ell: \|^p_{W^{-\varepsilon,\infty}}] \lesssim_{p,\varepsilon} M^{-\frac{\varepsilon}{2}}. \tag{4.4.8}$$

We now show a time difference estimate for $:\Psi_N^k \overline{\Psi}_N^\ell:$. This will show that $:\Psi_N^k \overline{\Psi}_N^\ell:$ is almost surely continuous in time and hence, combined with the previous part of this proof, $:\Psi_N^k \overline{\Psi}_N^\ell:$ is Cauchy in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}))$.

We define the time deference operator

$$\Delta_h :\Psi_N^k \overline{\Psi_N}^\ell(x,t): \stackrel{\text{def}}{=} :\Psi_N^k \overline{\Psi_N}^\ell(x,t+h): - :\Psi_N^k \overline{\Psi_N}^\ell(x,t):$$

for $|h| < 1$. In the following we will assume $h > 0$ for simplicity. Simple modifications are needed for the $h < 0$ case. Expanding and then using Proposition 4.3.3 we have,

$$\begin{aligned}
&\mathbb{E}\Big[\big(\Delta_h :\Psi_N^k \overline{\Psi_N}^\ell(x,t):\big)\overline{\big(\Delta_h :\Psi_N^k \overline{\Psi_N}^\ell(y,t):\big)}\Big] \qquad (4.4.9)\\
&\quad= \mathbb{E}\Big[:\Psi_N^k \overline{\Psi_N}^\ell(x,t+h): \overline{:\Psi_N^k \overline{\Psi_N}^\ell(y,t+h):}\Big]\\
&\qquad - \mathbb{E}\Big[:\Psi_N^k \overline{\Psi_N}^\ell(x,t): \overline{:\Psi_N^k \overline{\Psi_N}^\ell(y,t+h):}\Big]\\
&\qquad + \mathbb{E}\Big[:\Psi_N^k \overline{\Psi_N}^\ell(x,t): \overline{:\Psi_N^k \overline{\Psi_N}^\ell(y,t):}\Big]\\
&\qquad - \mathbb{E}\Big[:\Psi_N^k \overline{\Psi_N}^\ell(x,t+h): \overline{:\Psi_N^k \overline{\Psi_N}^\ell(y,t):}\Big]\\
&\quad= C_{k,\ell}\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t+h)}\big]^k \overline{\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t+h)}\big]}^\ell\\
&\qquad - C_{k,\ell}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t+h)}\big]^k \overline{\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t+h)}\big]}^\ell\\
&\qquad + C_{k,\ell}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]^k \overline{\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]}^\ell\\
&\qquad - C_{k,\ell}\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t)}\big]^k \overline{\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t)}\big]}^\ell\\
&\quad= (\mathrm{I}) + (\mathrm{II})
\end{aligned}$$

where in (I), we group the first and second terms on the right hand side of (4.4.9) and in (II) we group the third and fourth terms on the right hand side of (4.4.9). Using the purely algebraic formula

$$a^k\overline{a}^\ell - b^k\overline{b}^\ell = (a-b)\overline{a}^\ell \sum_{i=0}^{k-1} b^i a^{k-1-i} + \overline{(a-b)} b^k \sum_{i=0}^{\ell-1} \overline{b}^i \overline{a}^{\ell-1-i} \tag{4.4.10}$$

we can write (I) as,

$$
\begin{aligned}
(\mathrm{I}) = C_{k,\ell}&\mathbb{E}\big[\Delta_h\Psi_N(x,t)\overline{\Psi_N(y,t+h)}\big]\overline{\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t+h)}\big]}^{\ell} \\
&\times\sum_{i=0}^{k-1}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t+h)}\big]^{i}\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t+h)}\big]^{k-1-i} \\
&- C_{k,\ell}\overline{\mathbb{E}\big[\Delta_h\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]}\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]^{\ell} \\
&\times\sum_{i=0}^{\ell-1}\overline{\mathbb{E}\big[\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]}^{i}\,\overline{\mathbb{E}\big[\Psi_N(x,t+h)\overline{\Psi_N(y,t)}\big]}^{\ell-1-i} \\
=& (\mathrm{Ia}) + (\mathrm{Ib}).
\end{aligned}
$$

Using equation (4.4.10) again we have a similar decomposition for (II),

$$
(\mathrm{II}) = (\mathrm{IIa}) + (\mathrm{IIb}).
$$

From (4.4.1) we have,

$$
\mathbb{E}\big[\Delta_h\Psi_N(x,t)\overline{\Psi_N(y,t+h)}\big] = \sum_{|n|\le N} e_n(x-y)\big(\zeta(n)-\zeta(n,t,t+h)\big)
$$

A similar equality holds for $\mathbb{E}\big[\Delta_h\Psi_N(x,t)\overline{\Psi_N(y,t)}\big]$. Using the mean value theorem,

$$
\big|\zeta(n)-\zeta(n,t,t+h)\big| \lesssim \min\big(|h|,\langle n\rangle^{-2}\big).
$$

Hence by interpolation with (4.4.2),

$$
\big|\zeta(n)-\zeta(n,t+t+h)\big| \lesssim |h|^{\alpha}\langle n\rangle^{2-2\alpha}. \tag{4.4.11}
$$

Taking the $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ Bessel potentials of (4.4.9), setting $x=y$ and then using the estimates (4.4.2) and (4.4.11) we have,

$$
\mathbb{E}\Big[\big|\Delta_h\big(\langle\nabla\rangle^{-\varepsilon}:\Psi_N^k\overline{\Psi_N}^{\ell}(\cdot,t):\big)\big|^2\Big] \lesssim |h|^{\alpha}\sum_{|n_1|,\dots,|n_{k+\ell}|\le N}\frac{1}{\langle n_1+\cdots+n_{k+\ell}\rangle^{2\varepsilon}}\frac{1}{\langle n_1\rangle^{2-2\alpha}}\prod_{j=2}^{k+\ell}\frac{1}{\langle n_j\rangle^2}.
$$

If $\alpha<\varepsilon$ the summation in the above equation can be summed using a method similar to (4.4.7). Using Propositions A.1.2 and A.2.1,

$$
\mathbb{E}\Big[\|\Delta_h :\Psi_N^k\overline{\Psi_N}^{\ell}(x,t):\|^p_{W^{-\varepsilon,\infty}}\Big] \lesssim |h|^{\alpha p}. \tag{4.4.12}
$$

Choosing p large enough so that $\alpha p>1$, the Kolmogorov continuity criterion, see [6, Propostion 8.2], implies that $:\Psi_N^k\overline{\Psi_N}^{\ell}:\in C([0,T];W^{-\varepsilon,\infty})$ almost surely. For the convergence of $:\Psi_N^k\overline{\Psi_N}^\ell:$ in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}))$, in a manner similar to (4.4.12) and (4.4.8), for $N \geq M \geq 1$ we can show

$$\mathbb{E}\left[\|\Delta_h\big(:\Psi_N^k\overline{\Psi_N}^\ell(\cdot,t): - :\Psi_M^k\overline{\Psi_M}^\ell(\cdot,t): \big)\|^p_{W^{-\varepsilon,\infty}}\right] \lesssim |h|^{\alpha p} M^{-\varepsilon/2}.$$

Choosing p large enough so that $\alpha p > 1$, from the Kolmogorov continuity criterion we have that $:\Psi_N^k\overline{\Psi_N}^\ell:$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}))$ and so denoting it's limit by $:\Psi^k\overline{\Psi}^\ell:$ we have that $:\Psi^k\overline{\Psi}^\ell:\in C([0,T]; W^{-\varepsilon,\infty}))$ almost surely. □

Remark 4.4.1. The above argument can be easily adapted to show the paths of $:\Psi^k\overline{\Psi}^\ell:$ are in $C^\alpha([0,T]; C^{-\varepsilon})$ almost surely for $\alpha < \varepsilon$. See for example [38].

4.5 Local well-posedness of the WSCGL

4.5.1 Statement of results

In this section, we present the proof of Theorem 4.1.2. To do, this we reformulate Theorem 4.1.2 in a way slightly more amenable to PDE techniques.

For $\varepsilon > 0$ to be fixed later, we consider the space $\widehat{C}_T^{-\varepsilon}$ of $(m+1)\times m$-tuples of functions in $C([0,T]; C^{-\varepsilon})$. That is $\vec{z} = \{z_{i,j}\} \in \widehat{C}_T^{-\varepsilon}$ if

$$z_{i,j} \in C([0,T]; C^{-\varepsilon}) \text{ for all } 0 \leq i \leq m \text{ and } 0 \leq j \leq m-1.$$

We define a norm on $\widehat{C}_T^{-\varepsilon}$ as follows:

$$\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}} = \max_{i,j} \|z_{i,j}\|_{C([0,T];C^{-\varepsilon})}.$$

Instead of studying the equation for v (4.1.5) directly, for $\vec{z} \in \widehat{C}_T^{-\varepsilon}$ we study the equation

$$\begin{cases} \partial_t v = (a_1 + ia_2)[\Delta - 1]v - F(v, \vec{z}) \\ v|_{t=0} = v_0 \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+ \tag{4.5.1}$$

where

$$F(v,\vec{z}) = (c_1 + ic_2) \sum_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1}} \binom{m}{i}\binom{m-1}{j} z_{m-i,m-1-j} v^i \overline{v}^j.$$

The local well-posedness argument in this section will work for any choice of $\vec{z} \in \widehat{C}_T^{-\varepsilon}$. Proposition 4.1.1 shows that, $\{: \Psi^\ell\overline{\Psi}^k :\}_{i,j} \in \widehat{C}_T^{-\varepsilon}$. Hence if we show that (4.5.1) is locally well-posed, Theorem 4.1.2 will follow. The point of proving local well-posedness this way is that it draws a clear line between the probabilistic techniques used in the construction of the stochastic objects in Section 4.4 and the PDE techniques used in this section.

As usual, we interpret (4.5.1) in the mild sense. That is we say v solves (4.5.1) on $[0,T]$ if for all $t \in [0,T]$,

$$v(t) = S(t)v_0 - \int_0^t S(t-t')F(v,\vec{z})dt'. \tag{4.5.2}$$

In the following we look for solutions in the Banach space $X_T^{s_1,s_2}$ defined through the norm

$$\|v\|_{X_T^{s_1,s_2}} = \|v\|_{\frac{s_2-s_1}{2},s_2,T} + \|v\|_{L^\infty([0,T];C^{s_0})}$$

where

$$\|v\|_{\alpha,\beta,T} = \sup_{t\in[0,T]} t^\alpha \|v(t)\|_{C^\beta}.$$

The goal in this section is to prove the following.

Proposition 4.5.1. *Suppose $m \geq 2$, an integer, and $s_0 < 0$ are such that*

$$-(2m-1)\frac{s_0}{2} < 1. \tag{4.5.3}$$

Then (4.5.1) is locally well-posed for initial data in C^{s_0}. More precisely for $\varepsilon > 0$ small enough there exists $\theta > 0$ such that for $R > 1$, given $v_0 \in C^{s_0}$ and $\vec{z} \in \widehat{C}_T^{-\varepsilon}$ such that

$$\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}}, \|v_0\|_{C^{-s_0}} \leq R$$

there exists a unique solution, $v \in C([0,T];C^{s_0}) \cap C((0,T];C^{2\varepsilon})$ where $T \sim R^{-\theta}$. Moreover, if $v_0, u_0 \in C^{-s_0}$ and $\vec{z}, \vec{x} \in \widehat{C}_T^{-\varepsilon}$ satisfy

$$\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}}, \|v_0\|_{C^{s_0}}, \|\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}, \|u_0\|_{C^{s_0}} \leq R$$

then the respective solutions $v_1, v_2 \in C((0,T];C^{-s_0})$ to (4.5.2) with initial data and forcing $v_0, \vec{z}$ and $u_0, \vec{x}$ satisfy

$$\|v_1 - v_2\|_{X_T^{s_0,2\varepsilon}} \lesssim \|u_0 - v_0\|_{C^{s_0}} + \|\vec{z} - \vec{x}\|_{\widehat{C}_T^{-\varepsilon}}.$$

A similar local well-posedness result holds for $C^{2\varepsilon}$-initial data, but with time of existence depending on the L^p-norm of the initial data. See [63] for a similar result for SQE.

Proposition 4.5.2. *Suppose $m \geq 2$ is an integer, $p > 2m-1$ and $\varepsilon > 0$ is sufficiently small but fixed. Then for initial data in $C^{2\varepsilon}$, there exists a unique solution to (4.5.1) in $C([0,T],C^{2\varepsilon})$. Moreover, this solution depends continuously on $\vec{z}$ and v_0. More precisely there exists $\theta > 0$ such that for $R > 1$, given $v_0 \in C^{2\varepsilon}$ and $\vec{z} \in \widehat{C}_T^{-\varepsilon}$ such that*

$$\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}}, \|v_0\|_{L^p} \leq R$$

there exists a unique solution, $v \in C([0,T]; C^{2\varepsilon})$ *where* $T \sim R^{-\theta}$, *to* (4.5.2). *Moreover, if* $v_0, u_0 \in C^{2\varepsilon}$ *and* $\vec{z}, \vec{x} \in \widehat{C}_T^{-\varepsilon}$ *satisfy*

$$\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}}, \|v_0\|_{C^{2\varepsilon}}, \|\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}, \|u_0\|_{C^{2\varepsilon}} \leq R$$

then the respective solutions $v_1, v_2 \in C([0,T]; C^{s_0})$ *to* (4.5.2) *with initial data and forcing* $v_0, \vec{z}$ *and* $u_0, \vec{x}$ *satisfy*

$$\|v_1 - v_2\|_{C([0,T];C^{2\varepsilon})} \lesssim_R \|u_0 - v_0\|_{C^{2\varepsilon}} + \|\vec{z} - \vec{x}\|_{\widehat{C}_T^{-\varepsilon}}.$$

In light of the instantaneous smoothing from regularity s_0 to 2ε in Proposition 4.5.1, this proposition will allow us to prove global well-posedness by demonstrating an a priori estimate on the growth of the L^p-norm of solutions to (4.5.1). We do this later.

4.5.2 Proof of local well-posedness results

Before we prove the above propositions we first state and prove a useful elementary lemma.

Lemma 4.5.3. *Suppose* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\alpha < 1$ *and* $\beta < 1$. *Then,*

$$\int_0^t (t-s)^{-\alpha} s^{-\beta}\, ds \sim t^{1-\alpha-\beta}.$$

Proof. We split the integral into two parts and estimate each piece separately:

$$\begin{aligned}\int_0^t (t-s)^{-\alpha} s^{-\beta}\, ds &= \int_0^{t/2} (t-s)^{-\alpha} s^{-\beta}\, ds + \int_{t/2}^t (t-s)^{-\alpha} s^{-\beta}\, ds \\ &\sim t^{-\alpha} \int_0^{t/2} s^{-\beta}\, ds + t^{-\beta} \int_{t/2}^t (t-s)^{-\alpha}\, ds \\ &\sim t^{1-\alpha-\beta}\end{aligned}$$

where in the last line we simply evaluated the two integrals. □

The following local well-posedness proof, using the Da Prato-Debussche trick, the linear heat smoothing estimate Proposition4.2.1 and the product estimate (A.1.9) is standard, see for example, [22,63,93]. For completeness we go through the argument here.

Proof of Proposition 4.5.1. We will first show the existence of the solution in $X_T^{s_0,2\varepsilon}$.

Suppose $\|\vec{z}\|_{\widehat{C}_T^{-\varepsilon}}, \|v_0\|_{C^{s_0}} \leq R$. For $R_0 > R$ yet to be chosen, let B_{R_0} be the ball of radius R_0 and center 0 in $X_T^{s_0,2\varepsilon}$. We aim to show the map

$$\Gamma v(t) = S(t)v_0 - \int_0^t S(t-t')F(v,\vec{z})(t')\, dt'$$

is a contraction mapping on B_{R_0}. The linear heat smoothing estimate, Proposition 4.2.1, gives,

$$\|S(t)v_0\|_{C^{s_0}} \lesssim t^{\frac{s_0-2\varepsilon}{2}}\|v_0\|_{C^{2\varepsilon}}.$$

From the multiplicative estimate (A.1.9) we have,

$$\|F(v,\vec{z})(t')\|_{C^{-\varepsilon}} \lesssim \sum_{\substack{0\leq i\leq m\\ 0\leq j\leq m-1}} \|z_{i,j}\|_{C^{-\varepsilon}}\|v\|^{i+j}_{C^{2\varepsilon}} \lesssim R_0^{2m-1}(t')^{(2m-1)\frac{s_0-2\varepsilon}{2}} \tag{4.5.4}$$

where we used the fact that $t'^{\frac{2\varepsilon-s_0}{2}}\|v(t')\|_{C^{2\varepsilon}} \leq R_0$ for $v \in B_{R_0}$ and the fact that $\|z_{i,j}\|_{C^{-\varepsilon}} \leq \|\vec{z}\|_{\widehat{C}^{-\varepsilon}} \leq R \leq R_0$. We then have,

$$\begin{aligned} t^{\frac{2\varepsilon-s_0}{2}}\|\Gamma v\|_{C^{2\varepsilon}} &\lesssim \|v_0\|_{C^{s_0}} + t^{\frac{2\varepsilon-s_0}{2}}\int_0^t (t-t')^{-\frac{3\varepsilon}{2}}\|F(v,\vec{z})(t')\|_{C^{-\varepsilon}}\,dt' \\ &\lesssim \|v_0\|_{C^{s_0}} + R_0^{2m-1}t^{\frac{2\varepsilon-s_0}{2}}\int_0^t (t-t')^{-\frac{3\varepsilon}{2}}(t')^{(2m-1)\frac{s_0-2\varepsilon}{2}}\,dt' \\ &\lesssim \|v_0\|_{C^{s_0}} + R_0^{2m-1}t^{1-\frac{3\varepsilon}{2}+(2m-2)\left(\frac{s_0-2\varepsilon}{2}\right)} \end{aligned} \tag{4.5.5}$$

where in the final inequality we used Lemma 4.5.3 and the condition (4.5.3). Taking a supremum we have

$$\|\Gamma v\|_{\frac{2\varepsilon-s_0}{2},2\varepsilon,T} \leq C\|v_0\|_{C^{s_0}} + CR_0^{2m-1}T^{1-\frac{3\varepsilon}{2}+(2m-2)\left(\frac{s_0-2\varepsilon}{2}\right)}$$

where C is the implicit constant (4.5.5).

Similarly, choosing ε small enough so that $s_0 < -\varepsilon$ and using (4.5.4),

$$\begin{aligned} \|\Gamma v\|_{C^{s_0}} &\lesssim \|v_0\|_{C^{s_0}} + \int_0^t \|F(v,\vec{z})(t')\|_{C^{-\varepsilon}}\,dt' \\ &\lesssim \|v_0\|_{C^{s_0}} + R_0^{2m-1}t^{1+(2m-1)\frac{s_0-2\varepsilon}{2}}. \end{aligned} \tag{4.5.6}$$

Taking a supremum we have,

$$\|\Gamma v\|_{L^\infty([0,T];C^{s_0})} \leq C\|v_0\|_{C^{s_0}} + CR_0^{2m-1}T^{1+(2m-1)\frac{s_0-2\varepsilon}{2}}. \tag{4.5.7}$$

Adding (4.5.7) and (4.5.6) and choosing ε small so that $\frac{s_0-2\varepsilon}{2} < -\frac{3\varepsilon}{2}$,

$$\|\Gamma v\|_{X_T^{s_0,2\varepsilon}} \leq C\|v_0\|_{C^{s_0}} + CR_0^{2m-1}T^{1+(2m-1)\frac{s_0-2\varepsilon}{2}}.$$

By the condition (4.5.3) we can choose $\varepsilon > 0$ small enough so that

$$\theta := 1 + (2m-1)\left(\frac{s_0-2\varepsilon}{2}\right) > 0$$

and so the power of T is positive. Hence choosing $R_0 = 2CR$ and T satisfying

$$CR^{2m}T^{1-\frac{3\varepsilon}{2}+(2m-2)\frac{s_0-2\varepsilon}{2}} \leq \frac{R}{2}$$

we find that Γ maps B_{R_0} to B_{R_0}. Now we verify the contraction property. It follows form the algebra property (A.1.4) and the polynomial difference formula (4.4.10) that for $v_1, v_2 \in B_{R_0}$,

$$\|v_1^i\overline{v_1}^j - v_2^i\overline{v_2}^j\|_{C^{2\varepsilon}} \lesssim (t')^{(i+j-1)\frac{s_0-2\varepsilon}{2}} R^{i+j-1}\|v_1 - v_2\|_{C^{2\varepsilon}}.$$

Hence we have the difference estimate

$$\|F(v_1, \vec{z})(t') - F(v_2, \vec{z})(t')\|_{C^{-\varepsilon}} \lesssim (t')^{(2m-1)\frac{s_0-2\varepsilon}{2}} R^{2m-1}\|v_1 - v_2\|_{C^{2\varepsilon}}. \tag{4.5.8}$$

Using this estimate and estimates similar to those in (4.5.5) and (4.5.6) we can show that $\Gamma : B_{R_0} \to B_{R_0}$ is a contraction mapping. By the contraction mapping theorem it follows that Γ has a unique fixed point and hence, (4.5.1) has a solution in $X_T^{s_0,2\varepsilon}$.

Using Grönwall and standard PDE techniques the uniqueness of the solution on B_{R_0} can be extended to all of $X_T^{s_0,2\varepsilon}$. Using Proposition 4.2.2 and standard PDE techniques it can be shown that the solution we constructed above is in fact in $C((0,T]; C^{2\varepsilon}) \cap C([0,T]; C^{s_0})$. Further using standard PDE techniques it can be shown that the solution depends continuously on the noise and initial data.

The proofs of these three statements are quite standard. We will just prove the continuous dependence. Let

$$v_1(t) = S(t)v_0 - \int_0^t S(t-t')F(v_1, \vec{z})(t')\,dt'$$

and

$$v_2(t) = S(t)u_0 - \int_0^t S(t-t')F(v_2, \vec{x})(t')\,dt'$$

be the solutions on $[0,T]$ for $T \sim R^{-\theta}$ constructed by the above contraction mapping argument. Adding and subtracting $F(v_1, \vec{x})$ we have

$$\begin{aligned} F(v_1,\vec{z}) - F(v_2, \vec{x}) &= \\ &+ (c_1 + ic_2) \sum_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1}} \binom{m}{i}\binom{m-1}{j} (z_{m-i,m-1-j} - x_{m-i,m-1-j}) v_1^i\overline{v_1}^j \\ &+ (c_1 + ic_2) \sum_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1}} \binom{m}{i}\binom{m-1}{j} x_{m-i,m-1-j}(v_1^i\overline{v_1}^j - v_2^i\overline{v_2}^j) \\ &= \text{(I)} + \text{(II)} \end{aligned}$$

Using estimates similar to (4.5.4) to estimate (I) and (4.5.8) to estimate (II),

$$\|F(v_1,\vec{z})(t') - F(v_2,\vec{x})(t')\|_{C^{-\varepsilon}} \lesssim R_0^{2m-2}(t')^{(2m-2)\frac{s_0-2\varepsilon}{2}}\|v_1 - v_2\|_{\frac{2\varepsilon-s_0}{2},2\varepsilon,T} + R_0^{2m-1}(t')^{(2m-1)\frac{s_0-2\varepsilon}{2}}\|\vec{z}-\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}.$$

So,

$$\begin{aligned}\|v_1 - v_2\|_{\frac{2\varepsilon-s_0}{2},2\varepsilon,T} &\lesssim \|v_0 - u_0\|_{C^{s_0}} + T^{\frac{2\varepsilon-s_0}{2}}\int_0^T (t')^{-\frac{3\varepsilon}{2}}\|F(v_1,\vec{z}) - F(v_2,\vec{x})\|_{C^{-\varepsilon}}\,dt' \\ &\lesssim \|v_0 - u_0\|_{C^{s_0}} \\ &\quad + T^{1-\frac{3\varepsilon}{2}-(2m-3)\frac{2\varepsilon-s_0}{2}} R_0^{2m-1}\left(\|\vec{z}-\vec{x}\|_{\widehat{C}_T^{-\varepsilon}} + \|v_1 - v_2\|_{X_T^{s_0,2\varepsilon}}\right).\end{aligned}$$

Similarly we have

$$\|v_1-v_2\|_{L^\infty([0,T];C^{s_0})} \lesssim \|v_0-u_0\|_{C^{s_0}}+T^{1-(2m-2)\frac{2\varepsilon-s_0}{2}}R_0^{2m-1}\big(\|\vec{z}-\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}+\|v_1-v_2\|_{X_T^{s_0,2\varepsilon}}\big).$$

Adding the above estimates gives,

$$\|v_1-v_2\|_{X_T^{s_0,2\varepsilon}} \le C\|v_0-u_0\|_{C^{s_0}}+CT^{1-(2m-2)\frac{2\varepsilon-s_0}{2}}R_0^{2m-1}\big(\|\vec{z}-\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}+\|v_1-v_2\|_{X_T^{s_0,2\varepsilon}}\big).$$

Choosing T small enough, we can bring $\frac{1}{2}\|v_1 - v_2\|_{X_T^{s_0,2\varepsilon}}$ to the left hand side of the above inequality giving,

$$\|v_1 - v_2\|_{X_T^{s_0,2\varepsilon}} \lesssim \|v_0 - u_0\|_{C^{s_0}} + \|\vec{z}-\vec{x}\|_{\widehat{C}_T^{-\varepsilon}}.$$

□

We now outline the proof of Proposition 4.5.2. For more details we refer the reader to [63] where a similar result is proven for the 2-dimensional SQE.

Proof of Proposition 4.5.2. Suppose $\|\vec{z}\|_{\widehat{C}^{-\varepsilon}}, \|v_0\|_{L^p} \le R$. We let B denote the Banach space defined through the norm $\|\cdot\|_{\varepsilon+\frac{1}{p},2\varepsilon,T}$. Following [63, Theorem 6.2] we will first show that there exists a solution in B. Then we will show that the solution constructed is in fact in $C([0,T];C^{2\varepsilon})$. For $R_0 > R$ yet to be chosen let B_{R_0} be the ball of radius R_0 and center 0 measured in the norm $\|\cdot\|_{\varepsilon+\frac{1}{p},2\varepsilon,T}$. From the mild formulation we have,

$$\|\Gamma v(t)\|_{C^{2\varepsilon}} \le \|S(t)v_0\|_{C^{2\varepsilon}} + \int_0^t (t-t')^{-\frac{3\varepsilon}{2}}\|F(v,\vec{z})(t')\|_{C^{-\varepsilon}}\,dt'.$$

Using the embedding (A.1.5) and the smoothing estimate Proposition 4.2.1,

$$\|S(t)v_0\|_{C^{2\varepsilon}} \lesssim \|S(t)v_0\|_{B_{p,\infty}^{2\varepsilon+\frac{2}{p}}} \lesssim t^{-\varepsilon-\frac{1}{p}}\|v_0\|_{L^p}.$$

Using the above estimate and an estimate similar to (4.5.4) in the proof of Proposition 4.5.1 we have

$$t^{\varepsilon+\frac{1}{p}}\|\Gamma v(t)\|_{C^{2\varepsilon}} \lesssim \|v_0\|_{L^p} + t^{\varepsilon+\frac{1}{p}}\int_0^t (t-t')^{-\frac{3\varepsilon}{2}}(t')^{-(2m-1)(\varepsilon+\frac{1}{p})}R_0^{2m-1}\,dt'.$$

If $2m-1<p$ and $\varepsilon>0$ is small enough then the integral in the above equation can be evaluated using Lemma 4.5.3 and taking a supremum,

$$\|\Gamma v\|_{\varepsilon+\frac{1}{p},2\varepsilon,T} \le C\|v_0\|_{L^p} + CR_0^{2m-1}T^{1-\frac{3\varepsilon}{2}-(2m-2)(\varepsilon+\frac{1}{p})}.$$

Choosing $R_0=2CR$ and T so that

$$CR_0^{2m-1}T^{1-\frac{3\varepsilon}{2}-(2m-2)(\varepsilon+\frac{1}{p})} \le \frac{1}{2}R$$

it follows that Γ maps B_{R_0} to itself. Using arguments similar to those in the proof of Proposition 4.5.1 one can verify a difference estimate for Γ. Hence by the Contraction Mapping Theorem, Γ has a fixed point.

Using Proposition 4.2.2 and arguments in [63, Proposition 6.2] one can show the solution constructed above is in fact in $C([0,T];C^{2\varepsilon})$ and is unique in this space.

The proof of the continuous dependence on $v_0\in C^s$ and $\vec{z}$ is similar to the proof of continuous dependence in the proof of Proposition 4.5.1. □

4.6 Global well-posedness of WSCGL

In this section we place the additional assumption that $c_1<0$. This means that the nonlinearity is defocusing with respect to the heat part of SCGL.

In this section we will prove Theorem 4.1.3. To do this we will prove the following global well-posedness result for (4.5.1).

Proposition 4.6.1. *Let $m\ge 2$ be an integer and suppose $c_1>0$ and $s_0>-\frac{2}{2m-1}$. Set $r=\left|\frac{a_2}{a_1}\right|$. Suppose*

$$2m-1<2+2(r^2+2r\sqrt{1+r^2})$$

and suppose $\varepsilon=\varepsilon(m,r)>0$ is sufficiently small. Then for any $T>0$, $v_0\in C^{s_0}$ and $\vec{z}\in\widehat{C}_T^{-\varepsilon}$ there exists a unique solution v to (4.5.1) with $v\in C((0,T];C^{2\varepsilon})\cap C([0,T];C^{s_0})$.

From Proposition 4.1.1 $\{:\Psi^k\overline{\Psi^\ell}:\}_{k,\ell}\in\widehat{C}_T^{-\varepsilon}$. Thus, if we can prove the above proposition, Theorem 4.1.3 will follow

To prove this, we will establish an a priori L^p bound coming from a "Testing against v^{p-1}" identity. This is similar to the method in [46, 63, 64, 93]. However, as in [46] our situation is more delicate than the situation in [63, 64, 93]. Due to some extra terms appearing in our "Testing against v^{p-1}" identity, we are only be able to establish a suitable a priori L^p bound for small p. However, for the L^p-norm to control the time of existence in Proposition 4.5.2 we need $p > 2m-1$. Hence we only get global well-posedness when these two ranges overlap.

We now state and prove a "Testing against v^{p-1}" inequality. We became aware of inequalities of this type through the work of Hoshino [46]. Inequalities of this type first appeared in the works [26, 34].

Proposition 4.6.2. *Let $T > 0$ be fixed and $m \geq 2$ be an integer. Set $r = \left|\frac{a_2}{a_1}\right|$. Suppose*

$$2m - 1 < 2 + 2(r^2 + 2r\sqrt{1+r^2})$$

and suppose $\varepsilon = \varepsilon(m,r) > 0$ is sufficiently small. Further, suppose $v_0 \in C_x^\infty$, $z_{i,j} \in C_t^\infty C_x^\infty$ for all i, j and $v \in C_t^\infty C_x^\infty$ solves (4.5.1). Then, for $\eta > 0$ small enough, v satisfies the following inequality

$$\begin{aligned} \frac{1}{p}\left(\|v(t)\|_{L^p}^p - \|v(t_0)\|_{L^p}^p\right) + \int_{t_0}^t \|v^{p+2m-2}(t')\|_{L^1}\, dt' + 4\eta a_1 \int_{t_0}^t \|v^{p-2}|\nabla v|^2(t')\|_{L^1}\, dt' \\ \leq \int_{t_0}^t |\langle F_0(v,\vec{z}), |v|^{p-2}v\rangle|(t')\, dt' \end{aligned} \tag{4.6.1}$$

where

$$F_0(v,\vec{z}) = (c_1 + ic_2) \sum_{\substack{0\leq i\leq m \\ 0\leq j\leq m-1 \\ (i,j)\neq(m,m-1)}} \binom{m}{i}\binom{m-1}{j} z_{m-i,m-1-j} v^i \overline{v}^j.$$

Proof. We will assume $a_2 \geq 0$ as if $a_2 < 0$ we can take the conjugate of (4.5.2) so that $\overline{v}$ solves (4.5.2) with the sign of a_2 switched. As v is sufficiently smooth we can compute,

$$\begin{aligned} \frac{1}{p}\partial_t \|v(t)\|_{L^p}^p &= \frac{1}{p}\partial_t \int_{\mathbb{T}^2} (v\overline{v})^{p/2}\, dx. \\ &= \frac{1}{2}\int_{\mathbb{T}^2} (v\overline{v})^{p/2-1}\left(v\partial_t\overline{v} + \overline{v}\partial_t v\right) dx \\ &= \frac{1}{2}\int_{\mathbb{T}^2} (v\overline{v})^{p/2-1}\left((a_1 - ia_2)v\Delta\overline{v} + (a_1 + ia_2)\overline{v}\Delta v\right) dx \\ &\quad + \int_{\mathbb{T}^2} (v\overline{v})^{p/2-1}\,\mathrm{Re}\left(\overline{v}F(v,\vec{z}\right) dx. \end{aligned} \tag{4.6.2}$$

Integrating by parts and then applying the product rule gives,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^2}(v\overline{v})^{p/2-1}\left((a_1-ia_2)v\Delta\overline{v}\right)\,dx &= -\frac{1}{2}(a_1-ia_2)\int_{\mathbb{T}^2}\nabla\left[(v\overline{v})^{p/2-1}v\right]\cdot\nabla\overline{v}\,dx\\ &= -\frac{p}{4}(a_1-ia_2)\int_{\mathbb{T}^2}|v|^{p-2}|\nabla v|^2\,dx\\ &\quad -\frac{p-2}{4}(a_1-ia_2)\int_{\mathbb{T}^2}|v|^{p-4}v^2(\nabla\overline{v})^2\,dx.\end{aligned}$$

Here we are using the notation $v^2 = v_1^2 + v_2^2$ for $v \in \mathbb{C}^2$. Note that this is distinct from $|v|^2$. Using this expression we can write the first line in the third equality in (4.6.2) as

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^2}(v\overline{v})^{p/2-1}\big((a_1-ia_2)v\Delta\overline{v}+(a_1+ia_2)\overline{v}\Delta v\big)\,dx &=\\ &-\frac{p}{2}a_1\int_{\mathbb{T}^2}|v|^{p-2}|\nabla v|^2\,dx\\ &-\frac{p-2}{4}a_1\int_{\mathbb{T}^2}|v|^{p-4}\left[v^2(\nabla\overline{v})^2+\overline{v}^2(\nabla v)^2\right]\,dx\\ &-i\frac{p-2}{4}a_2\int_{\mathbb{T}^2}|v|^{p-4}\left[\overline{v}^2(\nabla v)^2-v^2(\nabla\overline{v})^2\right]\,dx.\end{aligned} \tag{4.6.3}$$

Making use of the elementary algebraic calculus identities

$$\begin{aligned}v^2(\nabla\overline{v})^2+\overline{v}^2(\nabla v)^2 &= (v\nabla\overline{v}-\overline{v}\nabla v)^2+2|v|^2|\nabla v|^2,\\ \nabla|v|^2 &= \overline{v}\nabla v+v\nabla\overline{v},\\ 4|v|^2|\nabla v|^2 &= (\nabla|v|^2)^2-(v\nabla\overline{v}-\overline{v}\nabla v)^2,\end{aligned}$$

for $\eta > 0$ we can write (4.6.3) as

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^2}(v\overline{v})^{p/2-1}\big((a_1-ia_2)v\Delta\overline{v}+(a_1+ia_2)\overline{v}\Delta v\big)\,dx &=\\ -(p-1)a_1\int_{\mathbb{T}^2}|v|^{p-2}|\nabla v|^2\,dx-\frac{p-2}{4}a_1\int_{\mathbb{T}^2}|v|^{p-4}(v\nabla\overline{v}-\overline{v}\nabla v)^2\,dx&\\ -\quad i\frac{p-2}{4}a_2\int_{\mathbb{T}^2}|v|^{p-4}\nabla|v|^2\left(\overline{v}\nabla v-v\nabla\overline{v}\right)\,dx&\\ =-4\eta a_1\int_{\mathbb{T}^2}|v|^{p-2}|\nabla v|^2\,dx-\big(\frac{p-1}{4}-\eta\big)a_1\int_{\mathbb{T}^2}|v|^{p-4}(\nabla|v|^2)^2\,dx&\\ +\big(\frac{1}{4}-\eta\big)a_1\int_{\mathbb{T}^2}|v|^{p-4}(v\nabla\overline{v}-\overline{v}\nabla v)^2\,dx&\\ -\frac{p-2}{4}a_2\int_{\mathbb{T}^2}|v|^{p-4}\nabla|v|^2 i\left(\overline{v}\nabla v-v\nabla\overline{v}\right)\,dx&\\ =-4\eta a_1\int_{\mathbb{T}^2}|v|^{p-2}|\nabla v|^2\,dx-\int_{\mathbb{T}^2}|v|^{p-4}A_{p,\eta}(f,g)\,dx&\end{aligned}$$

where

$$f = i\left(\overline{v}\nabla v - v\nabla\overline{v}\right), \quad g = \nabla|v|^2$$

and $A_{p,\eta}(f,g)$ is the quadratic form

$$A_{p,\eta}(f,g) = \big(\frac{1}{4} - \eta\big)a_1 f^2 + \frac{p-2}{4}a_2 fg + \big(\frac{p-1}{4} - \eta\big)a_1 g^2.$$

Note that both f and g are real valued and so the quadratic form $A_{p,\eta}$ takes real arguments. If

$$2 < p < 2 + 2r(r + \sqrt{1+r^2})$$

then for small enough $\eta = \eta(p,r)$ the matrix

$$a_2\begin{pmatrix} (\frac{1}{4}-\eta)r & \frac{p-2}{8} \\ \frac{p-2}{8} & (\frac{p-1}{4}-\eta)r \end{pmatrix}$$

is non-negative definite, has non-negative trace and non-negative determinant, and so $A_{p,\eta}(f,g) \geq 0$. In this case

$$\frac{1}{2}\int_{\mathbb{T}^2}(v\overline{v})^{p/2-1}\left((a_1 - ia_2)v\Delta\overline{v} + (a_1+ia_2)\overline{v}\eta v\right)\,dx \leq -4\eta a_1\int_{\mathbb{T}^2}|v|^{p-2}|\Delta v|^2\,dx. \tag{4.6.4}$$

Note that the left hand side of the above inequality is real valued and so the inequality makes sense. We now consider the term on the second line of (4.6.2). As

$$F(v,\vec{z}) = (c_1 + ic_2)|v|^{2m-2}v + F_0(v,\vec{z})$$

we have

$$\int_{\mathbb{T}^2}(v\overline{v})^{p/2-1}\operatorname{Re}\left(\overline{v}F(v,\vec{z})\right)\,dx = c_1\int_{\mathbb{T}^2}|v|^{p+2m-2}\,dx + \operatorname{Re}\int_{\mathbb{T}^2}|v|^{p-2}\overline{v}F_0(v,\vec{z}). \tag{4.6.5}$$

Putting (4.6.2), (4.6.4) and (4.6.5) together after integrating from t_0 to t gives the desired L^p inequality

$$\frac{1}{p}\left(\|v(t)\|_{L^p}^p - \|v(t_0)\|_{L^p}^p\right) + \int_{t_0}^t\|v^{p+2m-2}(t')\|_{L^1}\,dt' + 4\eta a_1\int_{t_0}^t\|v^{p-2}|\nabla v|^2(t')\|_{L^1}\,dt'$$
$$\leq \int_{t_0}^t|\langle F_0(v,\vec{z}), |v|^{p-2}v\rangle|(t')\,dt'.$$

□

It is not immediately clear how this proposition helps prove global well-posedness. Proposition 4.6.2 only holds for smooth initial data, noise and solutions to (4.5.1). Without knowing the time continuity properties of v it is not even clear (4.6.1) even makes sense for rough solutions of (4.5.1). In [63] this problem was solved by proving a certain amount of time continuity of v and then proving an a priori bound of type

(4.6.1) for rough v. In this chapter we take an alternative PDE approach which we outline here.

Consider the solution v_N of (4.5.2) with smoothed forcing and initial data. That is the equation,

$$\begin{cases} \partial_t v_N = [(a_1+a_2)\Delta - 1]v_N + F(v_N, \vec{z}_N) \\ v|_{t=0} = v_0^N \end{cases} \tag{4.6.6}$$

where $\vec{z}_N \in (C_t^\infty C_x^\infty)^{m\times(m-1)}$ converges to $\vec{z}$ in $\widehat{C}_T^{-\varepsilon}$ and $v_0^N \in C_x^\infty$ converges to v_0 in C^{s_0}. We denote $\vec{z}_N = \{z_{i,j}^N\}$.

Such sequences exist as $C_t^\infty C_x^\infty$ is a dense subset of $C([0,T]; C^{-\varepsilon})$ and C_x^∞ is a dense subset of C^{s_0}. Alternatively, consider suitably mollified $\vec{z}$ and v_0. It can be shown that $v_N \in C_t^\infty C_x^\infty$ and hence v_N is sufficiently regular for the hypothesis of Proposition 4.6.2 to hold. We then prove an a priori L^p bound on v_N that is independent of N, globalising the solution v_N. Using the fact that $\vec{z}_N \to \vec{z}$ in $\widehat{C}_T^{-\varepsilon}$ and $v_0^N \to v_0$ in C^{s_0} one can can use the continuous dependence of the solution in v_0 and $\vec{z}$, from Proposition 4.5.2 in addition to the L^p bound to show that v is also a global solution.

With this in mind, to prove Proposition 4.6.1, it suffices to prove the following bound.

Proposition 4.6.3. *Suppose* $2 < p < 2 + 2(r^2 + 2r\sqrt{1+r^2})$ *and* $\varepsilon > 0$ *is sufficiently small. Let* $T > 0$ *and* $0 < t_0 < T$. *Then there exists* $C = C(m, p, \varepsilon, \vec{z}) > 0$ *such that if* v_N *is a solution to* (4.6.6) *on* $[0, T]$ *then for all* $t \in [t_0, T]$,

$$\|v_N(t)\|_{L^p} \leq \|v_N(t_0)\|_{L^p} + Ct.$$

The $t_0 > 0$ is present in this bound as for $s_0 < 0$, a function $v_0 \in C^{s_0}$ is not always in L^p. However, in light of the instantaneous smoothing from the local well-posedness theory of Proposition 4.5.1, $v(t_0) \in C^{2\varepsilon} \subset L^p$. To prove Proposition 4.6.3 we use an almost identical proof to that in [63, 93]. We present the details here.

Proof of Proposition 4.6.3. In the following we write v instead of v_N for simplicity. Set

$$A_t = 4\eta c_1 \|v^{p-2}(t)|\nabla v(t)|^2\|_{L^1} \quad \text{and} \quad B_t = \|v^{p+2m-2}(t)\|_{L^1}.$$

Recall

$$F_0(v, \vec{z}_N) = (c_1 + ic_2) \sum_{\substack{0\leq i<m \\ 0\leq j\leq m-1 \\ (i,j)\neq(m,m-1)}} \binom{m}{i}\binom{m-1}{j} z_{m-i,m-1-j}^N v^i \overline{v}^j.$$

From (4.6.1) it suffices to show,

$$|\langle |v|^{p-2}\overline{v}, F_0(v, \vec{z}_N)\rangle(t)| \leq A_t + B_t + C$$

for some constant C. To do this it suffices to prove

$$|\langle z^N_{m-i,m-1-j} v^i \overline{v}^j, |v|^{p-2}v\rangle| \le \delta(A_s + B_s) + C(\delta) \tag{4.6.7}$$

for some small $\delta > 0$, for each i and j.

We will just prove (4.6.7) for the case $(i,j) = (m-1, m-1)$, the other cases are similar, and in fact slightly easier as the homogeneity in v is lower. Using (A.1.6) and the convergence and hence is boundedness of $z^N_{1,0}$ in $C^{-\varepsilon}$, we have,

$$\begin{aligned} |\langle z^N_{1,0} v^{m-1} \overline{v}^{m-1}, |v|^{p-2}v\rangle| &= |\langle z^N_{1,0}, |v|^{p-2} v \overline{v}^{m-1} v^{m-1}\rangle| \\ &\lesssim \||v|^{2m-4+p} v\|_{B^{\varepsilon}_{1,1}} \|z^N_{1,0}\|_{C^{-\varepsilon}} \\ &\lesssim \||v|^{2m-4+p} v\|_{B^{\varepsilon}_{1,1}}. \end{aligned}$$

Applying (A.1.10), Cauchy-Schwarz and then Jensen's inequality gives,

$$\begin{aligned} \||v|^{2m-4+p} v\|_{B^{\varepsilon}_{1,1}} &\lesssim \|v^{2m-3+p}\|^{1-\varepsilon}_{L^1} \|v^{2m-4+p} \nabla v\|^{\varepsilon}_{L^1} + \|v^{2m-3+p}\|_{L^1} \\ &\lesssim \|v^{p-2}|\nabla v|^2\|^{\varepsilon/2}_{L^1} \|v^{p+4m-6}\|^{\varepsilon/2}_{L^1} \|v^{2m-3+p}\|^{1-\varepsilon}_{L^1} + \|v^{2m-3+p}\|_{L^1} \qquad (4.6.8) \\ &\lesssim A_t^{\varepsilon/2} B_t^{\frac{2m-3+p}{2m-2+p}(1-\varepsilon)} \|v^{p+4m-6}\|^{\varepsilon/2}_{L^1} + B_t^{\frac{2m-3+p}{p+2m-2}}. \end{aligned}$$

Note that, as $x \mapsto x^{\frac{p+4m-6}{p+2m-2}}$ is not concave for $m \ge 2$, we cannot use Jensen's inequality to control $\|v^{p+4m-6}\|_{L^1}$ by a power of B_t. To get around this problem we use a trick in [93]. Using Proposition A.1.2 in the form

$$\|f\|_{L^q} \lesssim \|f\|_{L^2} + \|\nabla f\|_{L^2}$$

with $f = v^{p/2}$ implies

$$\|v^{pq/2}\|^{1/2}_{L^1} \lesssim \|v^p\|^{q/4}_{L^1} + \|v^{p-2}|\nabla v|^2\|^{q/4}_{L^1}.$$

In particular with $q = \frac{2(p+4m-6)}{p}$ we have,

$$\|v^{p+4m-6}\|^{1/2}_{L^1} \lesssim \|v^p\|^{\frac{p+4m-6}{2p}}_{L^1} + \|v^{p-2}|\nabla v|^2\|^{\frac{p+4m-6}{2p}}_{L^1} \lesssim B_t^{\frac{p+4m-6}{2(p+2m-2)}} + A_t^{\frac{p+4m-6}{2p}} \tag{4.6.9}$$

where the second inequality follows from Jensen's inequality which is now applicable. Putting (4.6.8) and (4.6.9) together gives

$$\begin{aligned} |\langle z^N_{1,0} v^{m-1} \overline{v}^{m-1}, |v|^{p-2}\rangle| &\lesssim A_t^{\varepsilon/2 + \frac{p+4m-6}{2p}\varepsilon} B_t^{\frac{2m-3+p}{2m-2+p}(1-\varepsilon)} \\ &\quad + A_t^{\varepsilon/2} B_t^{\frac{(2m-3+p)}{2m-2+p}(1-\varepsilon) + \frac{p+4m-6)}{2(p+2m-2)}\varepsilon} + B_t^{\frac{2m-3+p}{p+2m-2}}. \end{aligned}$$

Choosing ε small enough so that

$$\varepsilon/2 + \frac{p+4m-6}{2p}\varepsilon + \frac{2m-3+p}{2m-2+p}(1-\varepsilon) < 1$$

and

$$\varepsilon/2 + \frac{(2m-3+p)}{2m-2+p}(1-\varepsilon) + \frac{p+4m-6}{2(p+2m-2)}\varepsilon < 1$$

we can use Young's inequality to get,

$$|\langle z_{1,0}^N v^{m-1}\overline{v}^{m-1}, |v|^{p-2}v\rangle| \leq \delta(A_t + B_t) + C(\delta).$$

Here we are choosing a preliminary $\delta' = \frac{\delta}{C}$ to absorb the implicit constants in the preceding inequalities. This completes the proof. $\square$

Chapter 5

Long term behaviour of Gaussian measures transported by partial differential equations: a numerical study

In this chapter we perform numerical simulations, studying the long term transport of Gaussian measures supported on periodic functions under the flow of the fractional Schrödinger equation (FNLS,) (1.3.1) and the dispersion generalised Benjamin-Bona-Mahony equation (gBBM), (1.3.6). In the first section of this chapter, we explain how we generate the data used to approximate the flow of the Gaussian measures being studied. In the second section, we explain how we reorganise the data into an amenable form. In the third and fourth sections, we display visualisations of the flow of the measures studied and explain their significance.

The work in this chapter is exploratory in nature. We aim to get a heuristic idea of the behaviour of the objects studied and to illustrate concepts previously studied in this thesis without being overly burdened with error analysis. For this reason, any conclusion reached in this chapter should be viewed skeptically.

5.1 Data generation method

In this section we give an overview of how we will simulate Gaussian measures under the flow of PDEs numerically. There are two steps needed to do this.

- **Step 1:** Sample from the initial Gaussian measure.

- **Step 2:** Solve the PDE numerically and quickly with given initial data.

In our case, Step 1 is trivial. This is because the measures we study for

FNLS and gBBM both arise as the law of a function valued random variable of the form[1]

$$\omega \mapsto \sum_{n\in\mathbb{Z}} c_n g_n(\omega) e^{2\pi i n x} \tag{5.1.1}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of Gaussian random variables, not necessarily independent, and $\{c_n\}_{n\in\mathbb{Z}}$ is some, non random, sequence of constants. By generating a sample of the sequence $\{g_n\}_{n\in\mathbb{Z}}$, which is easily done numerically up to a truncation, we can generate a sample from the measure being studied.

Step 2 is more challenging. In this chapter, we use spectral methods to solve FNLS (1.3.1) and gBBM (1.3.6). We will explain this method in detail by example, focusing on FNLS and the Gaussian measure induced by the map (1.3.2). From our discussion for FNLS (1.3.1), it will be obvious how we derive the spectral formulation we use for gBBM (1.3.6). In Section 5.4.1 we state the numerical formulation we use for gBBM without explaining its derivation in detail.

We note that spectral methods have previously been used to study many different dispersive PDEs, including FNLS, see for example [52].

To start, we take the Fourier transform of FNLS (1.3.1), giving the infinite dimensional system of ODEs,

$$\begin{aligned} i\frac{du_n}{dt} &= -|2\pi n|^{2\alpha} u_n + \lambda \mathcal{F}[|u|^2 u](n) \\ &= -|2\pi n|^{2\alpha} u_n + \lambda \sum_{n=n_1-n_2+n_3} u_{n_1}\overline{u_{n_2}} u_{n_3} \end{aligned} \tag{5.1.2}$$

In (5.1.2) we used the product-to-convolution property of the Fourier transform to write the nonlinearity as a sum. We then truncate the infinite dimensional system of ODEs (5.1.2), to only include the N frequencies, $-N/2 \le n \le N/2-1$. This gives us the finite dimensional system of ODEs

$$i\frac{du_n}{dt} = -|2\pi n|^{2\alpha} u_n + \lambda \sum_{\substack{n=n_1-n_2+n_3 \\ -N/2\le n_i \le N/2-1}} u_{n_1}\overline{u_{n_2}} u_{n_3}, \quad -N/2 \le n \le N/2-1. \tag{5.1.3}$$

[1] We recall here that, as stated in the introduction Section 1.3, we are breaking with a convention established previously in this thesis. We are defining the 1-dimensional torus $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ instead of $\mathbb{T} = \mathbb{R}\backslash(2\pi\mathbb{Z})$. This means that several Fourier analytical concepts need to be defined differently. For example, the nth Fourier coefficient of a function $f : \mathbb{T} \to \mathbb{C}$ is now defined as

$$\mathcal{F}[f](n) = f_n = \int_{\mathbb{T}} f(x) e^{-2\pi i x n} dx.$$

The trigonometric polynomial orthonormal basis for $L^2(\mathbb{T})$ is now $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$, as in (5.1.1). We define the torus in this way because it is convenient for some numerical algorithms we will use, in particular [32].

Our initial data, being sampled according to (1.3.2) is now just a finite dimensional vector sampled according to

$$\omega \mapsto \left(\frac{g_{-N/2}(\omega)}{\langle N/2 \rangle^s}, \ldots, \frac{g_{N/2-1}(\omega)}{\langle N/2-1 \rangle^s} \right).$$

Unlike the infinite dimensional system of ODEs (5.1.2), it is possible to numerically implement (5.1.3). However, the nonlinearity of (5.1.3) is numerically expensive to evaluate, one needs to do a summation of N^2 terms, N-times and so the computational complexity is therefore $O(N^3)$. We need N to be large, $N \sim 10^3$, so the finite dimensional system (5.1.3) is a good approximation of the infinite dimensional system (5.1.2). Further, from the law of large numbers, the more samples generated, the more accurate our Monte-Carlo approximation will be. We want to solve (5.1.3) around 10^6 times. Putting this together, the formulation (5.1.3), the most obvious approximation of (5.1.2), is too inefficient to be of use.

To get around this issue, we use properties of the Fourier transform. Instead of simply truncating (5.1.2), we can write the nonlinearity of this equation as

$$\lambda \mathcal{F}[|u|^2 u](n) = \lambda \mathcal{F}\big[|\mathcal{F}^{-1}[\{u_n\}_{n\in\mathbb{Z}}]|^2 \mathcal{F}^{-1}[\{u_n\}_{n\in\mathbb{Z}}]\big]$$

Here $\mathcal{F}^{-1}$ is the inverse Fourier transform of a sequence,

$$\mathcal{F}^{-1}[\{f_n\}_{n\in\mathbb{Z}}] = \sum_{n\in\mathbb{Z}} f_n e^{2\pi i n x}.$$

Taking another truncation, replacing the Fourier transform with the discrete Fourier transform (DFT), defined by,

$$\mathcal{F}_{\text{DFT}}[\{f(\tfrac{k}{N})\}_{k\in\mathbb{Z}}](n) := \sum_{k=0}^{N} f(\tfrac{k}{N}) e^{-2\pi i \frac{k}{N} n} \sim f_n \tag{5.1.4}$$

and replacing the inverse Fourier Transform by the inverse DFT (IDFT),

$$\mathcal{F}^{-1}_{\text{DFT}}[\{f_n\}_{n\in\mathbb{Z}}](\tfrac{k}{N}) := \frac{1}{N} \sum_{k=0}^{N} f_n e^{2\pi i \frac{k}{N} n} \sim f(\tfrac{k}{N}) \tag{5.1.5}$$

we can approximate the nonlinearity of (5.1.2) as

$$\lambda \mathcal{F}\big[|u|^2 u\big](n) \sim \lambda \mathcal{F}_{\text{DFT}}\big[|\mathcal{F}^{-1}_{\text{DFT}}[\{u_n\}]|^2 \mathcal{F}^{-1}_{\text{DFT}}[\{u_n\}]\big](n).$$

The DFT (5.1.4), is ubiquitous in applied mathematics and signal processing. It is a linear map, mapping a vector of length N, which one can think of as representing the values of some function taken from evenly spaced points over the interval $[0, 1]$, to another vector of length N, which one can think of as being the Fourier coefficients $-N/2 \leq n \leq N/2 - 1$ of the function. The DFT (5.1.4) essentially amounts to using a Riemann sum to calculate Fourier coefficients. Similarly, the IDFT (5.1.5) can be thought of as using N terms of a Fourier series to evaluate the value of a function at a point. For more information on the DFT and IDFT, we refer the reader to [87].

To summarise, using the DFT and IDFT, we approximate (5.1.2) by the finite dimensional system of ODEs,

$$\begin{cases} i\frac{du_n}{dt} &= -|2\pi n|^{2\alpha}u_n + \lambda\mathcal{F}_{\text{DFT}}[|\mathcal{F}^{-1}_{\text{DFT}}[\{u_n\}]|^2\mathcal{F}^{-1}_{\text{DFT}}[\{u_n\}]](n) \\ u_n(0) &= \left(\frac{g_{-N/2}(\omega)}{\langle N/2\rangle^s}, \dots, \frac{g_{N/2-1}(\omega)}{\langle N/2-1\rangle^s}\right) \end{cases} \quad -N/2 \leq n \leq N/2-1.$$

At first glance, it seems that the computational complexity of DFT (5.1.4) is $O(N^2)$, one needs to do a sum of N terms for each of the N Fourier coefficients, and hence the complexity of the nonlinearity above is $O(N^2)$. This would be an improvement compared to (5.1.3). However, there is a remarkable algorithm for calculating the discrete Fourier transform, discovered by Cooley and Tukey in [19], which has computational complexity $O(N\log N)$. This algorithm, aptly named the fast Fourier transform in the literature, is pervasive in applied mathematics. Using the FFT, we can evaluate the nonlinearity above in $O(N\log N)$. With the FFT, it takes approximately one week to complete the simulations for this chapter, without the FFT it would take months or years.

This gives the mathematical formulation we use to solve FNLS (1.3.1) and gBBM (1.3.6) efficiently. Another important aspect for numerically solving these PDEs is the software we use to implement the spectral numerical method described above.

We code the numerical methods in this chapter in the Julia programming language, [7]. The Julia language was first released in 2012. It is a dynamically typed language, similar to programming languages such as Python. In layman's terms, *dynamically typed* means one writes

$$x = 3 \qquad \text{instead of} \qquad \text{int } x = 3.$$

As a consequence of this, Julia code tends to be easy to write and read. However, unlike most dynamically typed languages, Julia code compiles directly into machine code, making it significantly faster than most dynamically typed languages, running at speeds close to the low level C programming language. We refer the reader to [7] for more information on the Julia language. We note that we use version 1.2 of Julia.

To solve the finite dimensional systems of ODEs occurring in this chapter, we use the DifferentialEquations.jl Julia module, see [69]. This module has a large number of ODE solvers, which is of help in choosing an appropriate solver for the system being solved. We use the solver described in [2] for FNLS (1.3.1) and the solver [94] for gBBM (1.3.6).

As for the FFT, we use an implementation of this algorithm called 'The Fastest Fourier Transform in the West' (FFTW). See [32] for details of this implementation. The FFTW claims to be the fastest ever implementation of the FFT algorithm. FFTW is written in the C programming language and so we use the Julia module FFTW.jl, a Julia wrapper for FFTW.

5.2 Data sorting method

To simulate Gaussian measures under the flow of PDEs by Monte-Carlo, we do the following a large number, around 10^6, of times.

- **1)** Generate a sample from the Gaussian measure being studied.
- **2)** Numerically solve the PDE using the method described in Section 5.1.
- **3)** Save the solution in a file as a 2-dimensional array, where one axis is the Fourier coefficients, and the other axis is the time slices the Fourier coefficients are saved at.

Doing this, we are left with a folder containing around 10^6 files, each file approximately 500kb in size. As there is a computational cost to opening files, we want to open files as infrequently as possible. Hence the data stored in this form is not amenable to analysis. To get around this issue, and for the sake of simplicity, we reorganise the data into a smaller number of files, in a simpler form.

We sort the data into N (the number of Fourier coefficients) files. Each file stores a 2-dimensional array, corresponding to a single Fourier coefficient. One axis of the 2-dimensional array corresponds to the number of samples gathered the other axis corresponds to the time slices the Fourier coefficient sample is saved at.

This approach allows for faster investigation of the data. For example, to plot the variance of the first Fourier coefficient, we only have to open a single file, instead of opening 10^6 files. The code used to perform this data transformation is shown in Appendix D.7. This code took approximately 5 hours to run per 10^6 files, but was worth the upfront cost as it greatly speed up the data exploration process.

For legacy reasons the code we use to reorganise the files is written in Python instead of Julia, like the PDE solvers in Appendix D.1 and Appendix D.4. This is because an earlier version of the project this chapter is based on was written entirely in Python. The data sorting code was not a bottleneck in terms of speed and so we stuck with the working code instead of rewriting it solely for consistency reasons.

5.3 Fractional nonlinear Schrödinger equation

5.3.1 Problem and numerical formulation

We performed a Monte-Carlo simulation for the equation FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2). Following the discussion in Section 5.1 we generated samples by solving the finite dimensional system of ODEs

$$\begin{cases} i\frac{du_n}{dt} &= -|2\pi n|^{2\alpha}u_n + \lambda\mathcal{F}_{\text{DFT}}\big[|\mathcal{F}_{\text{DFT}}^{-1}[\{u_n\}]|^2\mathcal{F}_{\text{DFT}}^{-1}[\{u_n\}]\big](n) \\ u_n(0) &= \left(\frac{g_{-N/2}(\omega)}{\langle N/2\rangle^s},\dots,\frac{g_{N/2-1}(\omega)}{\langle N/2-1\rangle^s}\right) \end{cases} \quad -N/2 \le n \le N/2-1. \tag{5.3.1}$$

We chose $\alpha = 0.55$ in (5.3.1). During preliminary testing, we found that a lower value of α meant individual samples of (5.3.1) could be generated faster. This is for stiffness reasons. When α is large, the value $i|2\pi n|^{2\alpha}$ is large. This means that for large n, we expect the magnitude of $\frac{du_n}{dt}$ to be large, and for small n we expect the magnitude of $\frac{du_n}{dt}$ to be small. Following this heuristic, the stiffness of the system (5.3.1), and hence computation time, increases with α. It would then make sense to choose α close to 0. But as mentioned in the introduction, Section 1.3, one of the motivations of this project is to visualise how quasi-invariance manifests in terms of statistical properties of Fourier coefficients. In Chapter 2 of this thesis we proved that FNLS is quasi-invariant for $\alpha > 0.5$. These two reasons motivate our choice of $\alpha = 0.55$. We chose $\lambda = 100$ in (5.3.1) to make nonlinear behaviour noticeable and $s = 4.2$ so the initial data sampled from (1.3.2) is of high regularity and (5.3.1) can be solved quickly.

We used a second order accurate, adaptive backwards differentiation formula (ABDF2) method to solve (5.3.1). See [2] For more information on this algorithm. We used this method because ABDF2 is a stiff ODE solver and is suitable for large systems of equations.

We chose $N = 2^9$ for the size of the the system of ODEs (5.3.1). We chose this value because the FFT algorithm is most efficient with a power of 2 (see [19] for more information) and to find a balance ensuring samples could be generated sufficiently fast (N needs to be small), but are also sufficiently accurate (N needs to be large). With this choice of N and the other parameters explained above, we could generate samples at a rate of a sample every 4 seconds. We chose to generate 10^6 samples. There was no particular motivation for choosing this number other than it is large and at the limit of how much computer time we were willing to use. With this number of samples, this simulation for FNLS took approximately 4 days of computer time to run.

We refer the reader to Appendices D.1, D.2 and D.3 for the code used to implement the simulation for FNLS.

5.3.2 Results

In this subsection we present the results of the simulation described in the previous subsection. In particular, we look at the evolution of the variance of the Fourier coefficients,

$$\mathrm{Var}(u_n(t)) = \mathbb{E}[|u_n(t)|^2] - |\mathbb{E}[u_n(t)]|^2.$$

We do not present plots for negative Fourier coefficients, $n < 0$ because, due to the rotational symmetry of the measure (1.3.2),

$$u^\omega(0) \overset{\text{law}}{=} e^{i\theta} u^\omega(0)$$

and the rotational symmetry of FNLS (1.3.1), we have the following equality in law,

$$u^\omega_n(t) \overset{\text{law}}{=} u^\omega_{-n}(t) \tag{5.3.2}$$

and so the variance graphs for $n > 0$ and $-n$ would be identical. Moreover, for this reason we chose to not save the negative Fourier coefficients in this simulation, as they can mostly be recovered, saving storage space.

Due to accuracy and floating point arithmetic considerations (see the relative tolerance and absolute tolerance choices of 1×10^{-5} and 1×10^{-7} respectively in Appendix D.1), we do not trust the accuracy of the simulation for FNLS when the Fourier coefficients are of a very small size. This is the case for frequencies $n > 4$. Hence we only present information for frequencies $n = 0, 1, 2, 3, 4$.

Figures 5.1, 5.2, 5.3, 5.4 and 5.5 illustrate the evolution of the variance for these frequencies.

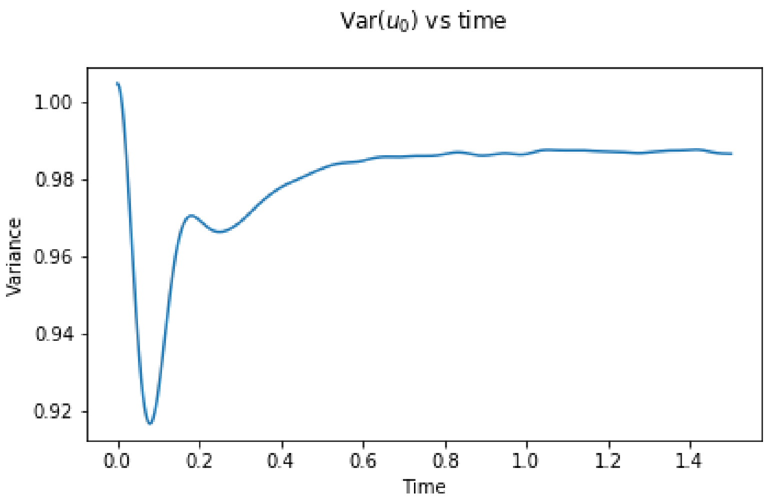

Figure 5.1: Evolution of $\mathrm{Var}(u_0(t))$ for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

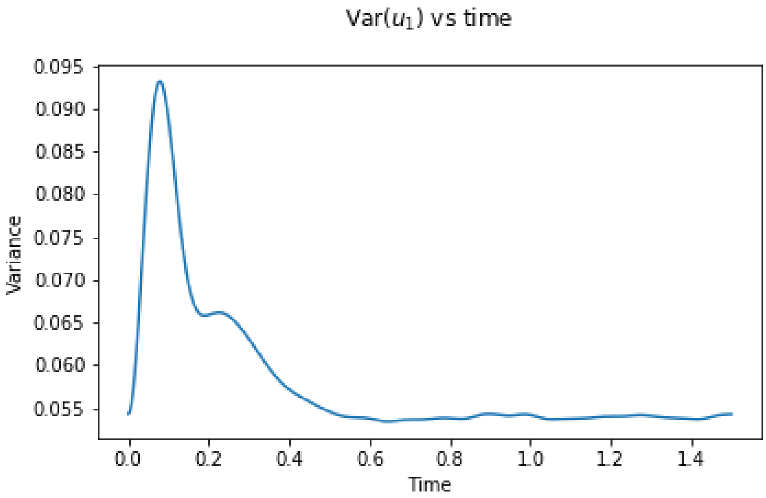

Figure 5.2: Evolution of Var($u_1(t)$) for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

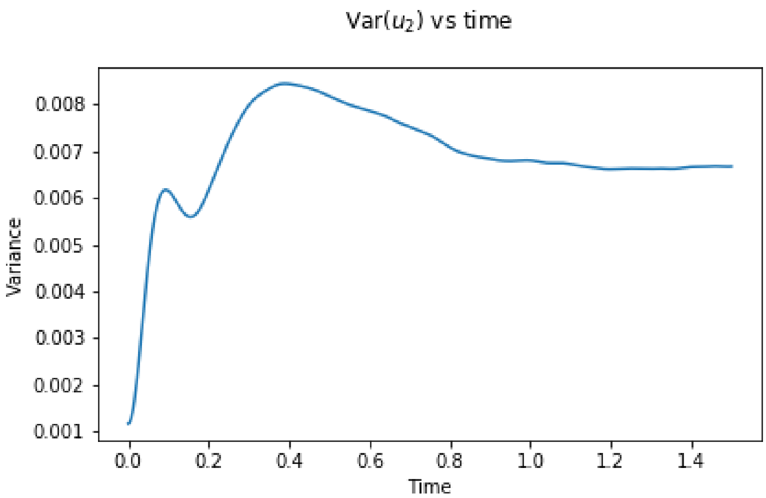

Figure 5.3: Evolution of Var($u_2(t)$) for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

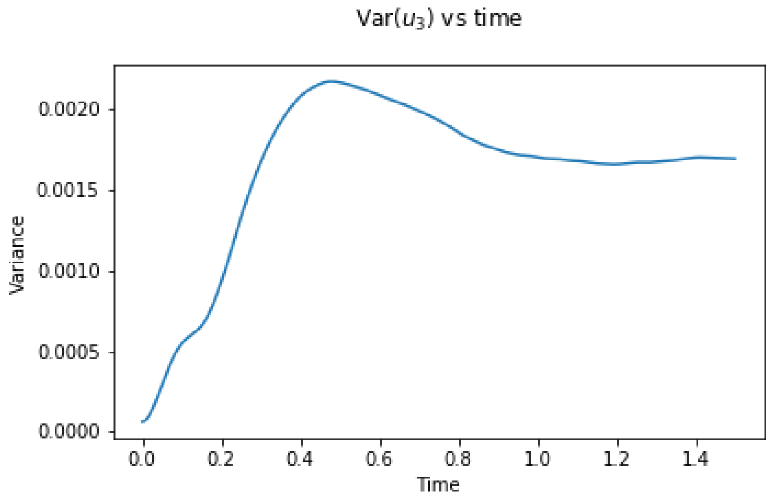

Figure 5.4: Evolution of Var($u_3(t)$) for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

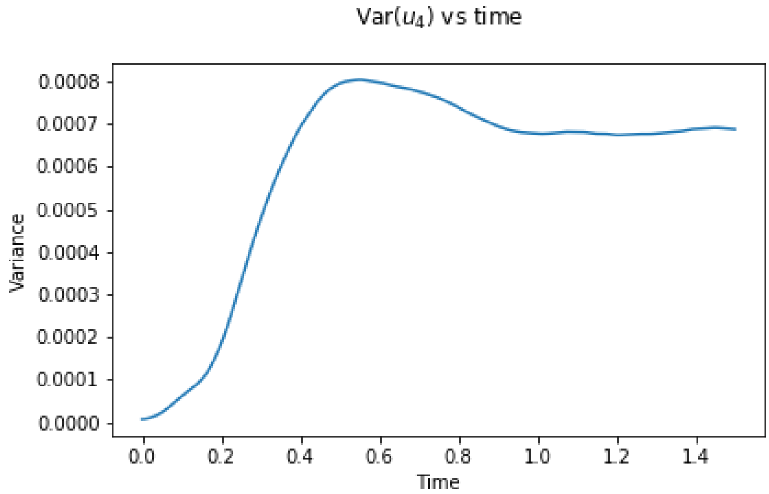

Figure 5.5: Evolution of Var($u_4(t)$) for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

We also plot the covariance

$$\mathrm{Covar}(u_n(t), u_m(t)) = \mathbb{E}[u_n(t)\overline{u_m(t)}] - \mathbb{E}[u_n(t)]\overline{\mathbb{E}[u_m(t)]}.$$

We only do this for $n = 1$ and $n = 2$ as the behaviour observed is representative for other pairs.

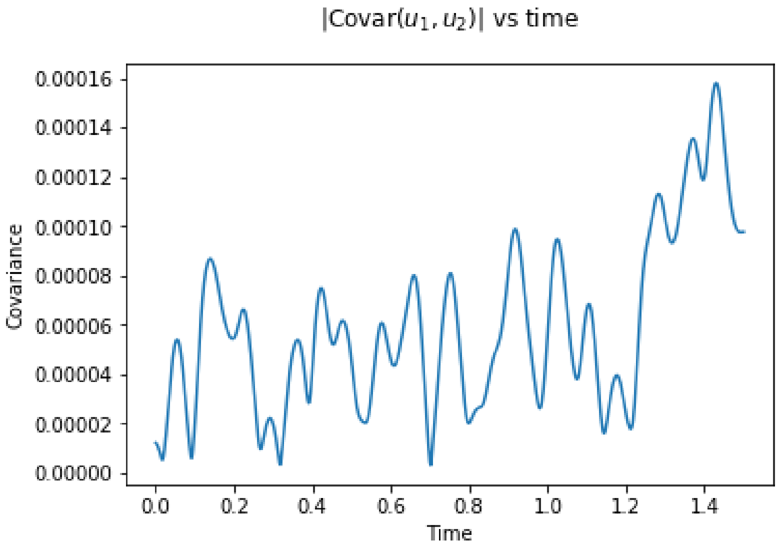

Figure 5.6: Evolution of $|\mathrm{Covar}(u_1(t), u_2(t))|$ for FNLS (1.3.1) with initial data sampled from the Gaussian measure (1.3.2).

5.3.3 Discussion

Before we discuss the implications of the above figures, we briefly discuss the accuracy of the Monte-Carlo simulation performed. Note that FNLS (1.3.1) converses the L^2-norm,

$$\|u(t)\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} |u_n(t)|^2. \tag{5.3.3}$$

We want to check that this quantity is being conserved. However, as mentioned in Section 5.3.2, we chose not to save the negative Fourier coefficients to lower the amount of data we needed to store.

To get around this issue, we note that the expected value of the L^2-norm (5.3.3) is also conserved under the flow of FNLS (1.3.1). From (5.3.2) and using that the Fourier coefficients u_n for $|n| > 4$ are very small, we expect the following quantity to be approximately conserved in our Monte-Carlo simulation,

$$F(t) = \mathbb{E}\big[|u_0(t)|^2 + 2\sum_{n=1}^{4} |u_n(t)|^2\big].$$

We have saved all the required Fourier coefficients to calculate this quantity. We plot $F(t)$ in Figure 5.7.

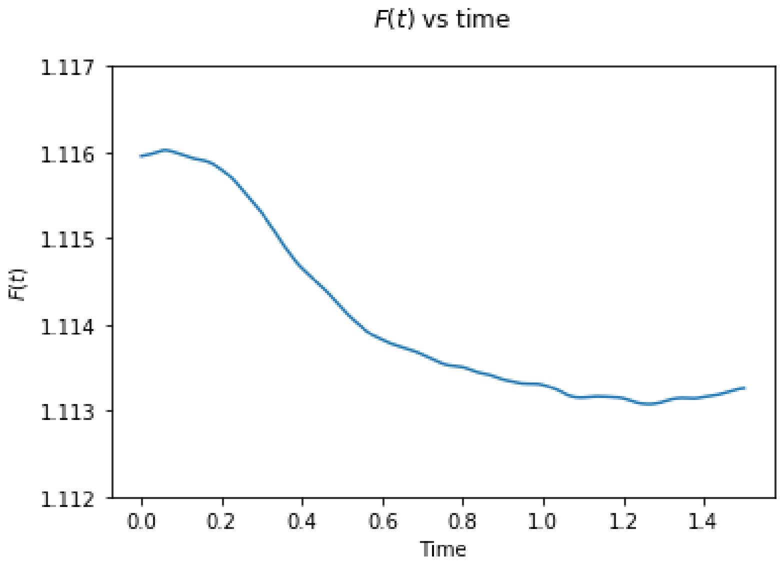

Figure 5.7: Plot of $F(t)$, demonstrating a quantity expected to be conserved by the simulation for FNLS is in fact almost conserved.

Keeping in mind the scale of Figure 5.7, this quantity is close to being conserved giving credence to the Monte-Carlo simulation we performed for FNLS (1.3.1).

We observe in our simulation that $\mathrm{Var}(u_n)$ for large n seems to increase more as a percentage than for small n. This connects to the growth of higher Sobolev norms and would be interesting to numerically investigate further. However, we chose not to pursue this.

In Figure 5.6, we see that the covariance of the first and second Fourier coefficients starts close to zero and remains near zero. This agrees with the theoretical observation in [78, Footnote 5] that if an equation and $u(0)$ are translation invariant in space, then we expect $\mathrm{Covar}(u_n(t), u_m(t)) = 0$ for all t unless $n = m$.

Figures 5.1, 5.2, 5.3, 5.4 and 5.5 appear to show that each Fourier coefficient converges to some value. Based on this weak evidence, we make the hypothesis that the Gaussian measure (1.3.2) under the flow of FNLS (1.3.1) converges in some suitable sense. We are not sure what limit measure this could be, or in what sense this convergence would take place. We expect that the limiting measure would be an invariant measure for FNLS (1.3.1). One candidate could be the invariant Gibbs measure possessed by FNLS (1.3.1) formally given by

$$Z^{-1} e^{-\frac{1}{2}\int |u|^2\,dx - \frac{1}{2}\int |(-\partial_x^2)^{\frac{\alpha}{2}} u|^2\,dx - \frac{\lambda}{4}\int |u|^4\,dx} du.$$

However, we are not sure if convergence to the Gibbs measure is possible as the Gibbs measures is supported on $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$ for $\varepsilon > 0$, while our initial data is of regularity $H^{3.7-\varepsilon}(\mathbb{T})$.

5.4 gBBM equation

In the previous section, we provided weak evidence that the Gaussian measure (1.3.2) transported under the flow of FNLS (1.3.1) converges in some sense. In this section we perform a simulation for gBBM (1.3.6) where we believe there is a candidate for convergence.

As explained in the introduction, in particular (1.3.8) and (1.3.9) of Section 1.3.8, the following Gibbs measures are invariant under the flow of gBBM (1.3.6)

$$d\mu_{\frac{\gamma}{2}} = Z^{-1} e^{-\beta^2 \frac{1}{2} \|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}} du.$$

This is the measure induced by the following random Fourier series,

$$\omega \mapsto \beta \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n(\omega)}{|2\pi n|^{\frac{\gamma}{2}}} e^{2\pi i n x},$$

where g_n are mutually independent, except for $\overline{g_n} = g_{-n}$, standard Gaussian random variables.

We perform a Monte-Carlo simulation for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4), which is a small perturbation of the Gibbs measure (1.3.8), to test our hypothesis of convergence.

5.4.1 Problem and numerical formulation

Following the discussion in Section 5.1 for FNLS, we study the following numerical formulation of gBBM.

$$\begin{cases} \frac{du_n}{dt} &= \lambda \frac{-2\pi n i}{1+|2\pi n|^5} \mathcal{F}_{\text{RDFT}}[(\mathcal{F}^{-1}_{\text{RDFT}}[\{u_n\}])^2] \\ u_n(0) &= \left(g_0(\omega), \dots, \frac{g_N(\omega)}{\sqrt{1+(2\pi N)^\gamma}} \right) \end{cases} \quad 0 \le n \le N. \tag{5.4.1}$$

where $g_0, g_1, g_2, \dots, g_N$ are the random variables given by (1.3.5).

The only difference in the derivation of (5.3.1) for FNLS and (5.4.1) for gBBM above, is that for the latter we use the real discrete Fourier transform (RDFT), $\mathcal{F}_{\text{RDFT}}$, instead of the DFT. This is because solutions of gBBM are real valued. The RDFT, is a version of the DFT for real valued functions. The RDFT is essentially given by the same formula as the DFT (5.1.4) but from the reality condition $\overline{f_n} = f_{-n}$, one does not need to do redundant calculations for the negative Fourier coefficients. This makes the RDFT slightly quicker than the DFT. The RDFT is a linear map taking a vector of length 2^N to a vector of length $2^{N-1} + 1$. For more information on the RDFT, see [87]. See [32] for information on the algorithm we use to calculate the RDFT and Appendix D.4 for the Julia code we use to solve gBBM.

We study gBBM (1.3.6) with $\gamma = 5$ as we found this value to be approachable computationally. From the presence of ,

$$\frac{-2\pi ni}{1 + |2\pi n|^\gamma}$$

we can see that large values of γ means smaller magnitude oscillations on high frequencies. This stands in contrast to the situation for FNLS and α in the previous section. From this, we heuristically expect that (5.4.1) is a non-stiff system of ODEs. We use the solver Tsit5, described in [94], to solve the system. We chose this solver as it is the solver-of-first-choice for non-stiff systems of ODEs in the module DifferentialEquations.jl.

In this simulation, we chose N, the dimension of the finite dimension system (5.4.1) to be $N = 2^{12}$. This value is much larger than for FNLS in the previous section. We chose a larger value of N because we found the system (5.4.1) could be solved much faster than (5.3.1) and so we had time to spare for accuracy. We ran this Monte-Carlo simulation with 10^6 samples. With these parameters, we could generate random samples at a rate of approximately a sample every 2 seconds. The Monte-Carlo simulation took approximately two and a half days to complete. For more details on the simulation for gBBM (1.3.6), see the code in Appendices D.4, D.5 and D.6.

5.4.2 Results

In this subsection, we present the results of the simulation described in the previous subsection. Instead of simply plotting the variances of the Fourier coefficients, like in Section 5.3.2 we plot the scaled variances

$$|2\pi n|^5 \mathrm{Var}(u_n(t)). \tag{5.4.2}$$

We do this because if our hypothesis of convergence to the Gibbs measure (1.3.8) is correct, for all n we expect this scaled variance to converge to the same value, β^2, for some $\beta > 0$.

We do not show the plots for negative Fourier coefficients because, as gBBM is real valued, these would be identical to plots for the positive Fourier coefficients. Further, we do not trust the accuracy of the simulation when the Fourier coefficents are very small in magnitude. See the error tolerances in Appendix D.4. This is the case for $n > 4$. Hence we only show the plots for $n = 1, 2, 3, 4$.

We plot the scaled variances (5.4.2) in Figures 5.8, 5.9, 5.10 and 5.11

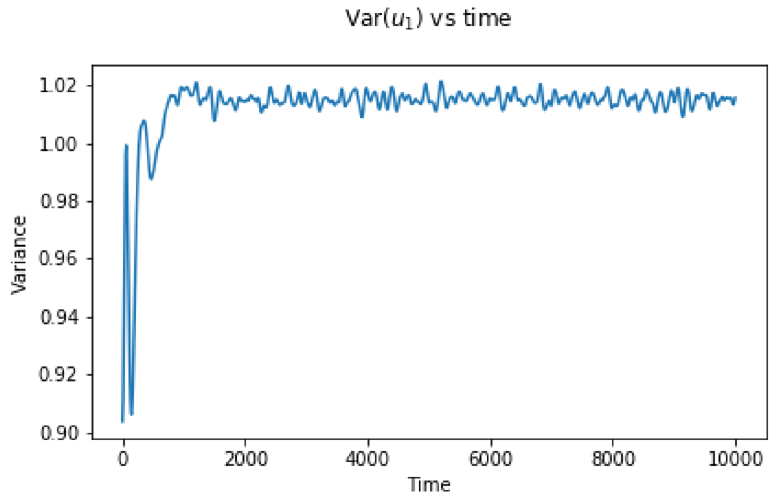

Figure 5.8: Evolution of $(2\pi)^5\text{Var}(u_1(t))$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

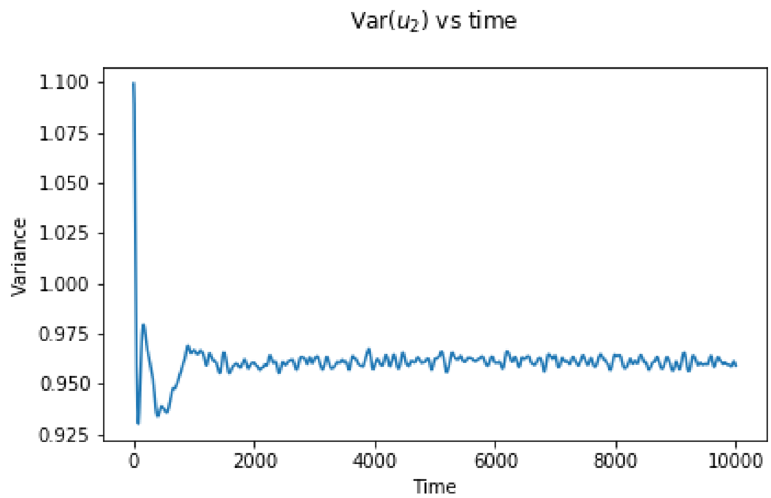

Figure 5.9: Evolution of $(4\pi)^5\text{Var}(u_2(t))$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

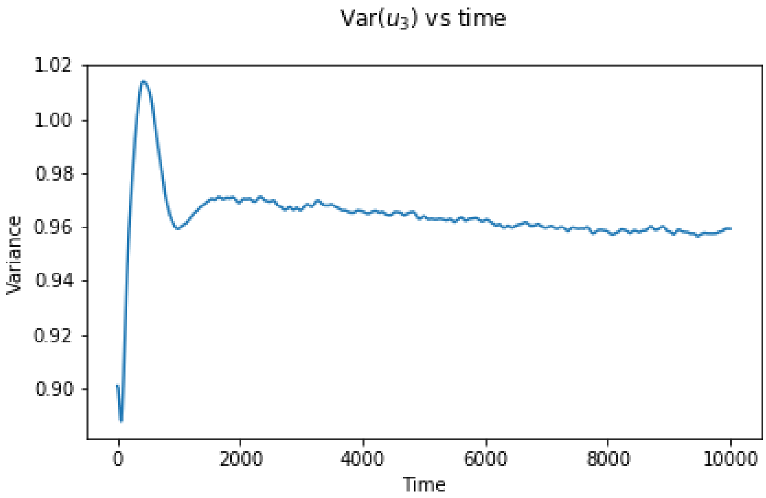

Figure 5.10: Evolution of $(6\pi)^5\text{Var}(u_3(t))$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

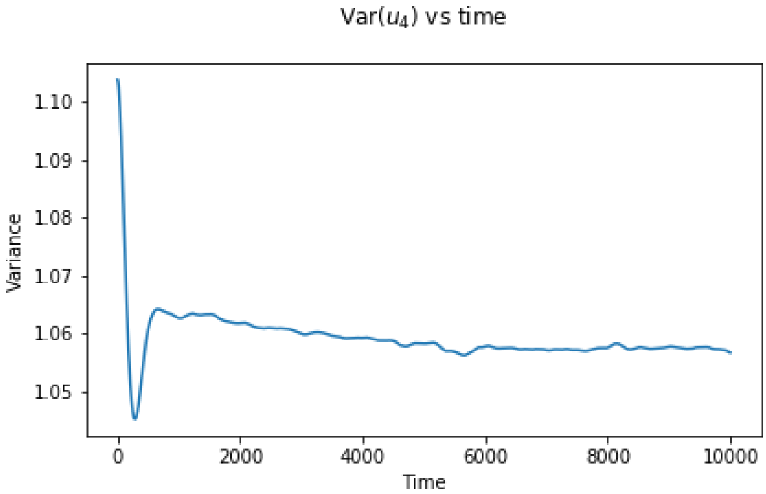

Figure 5.11: Evolution of $(8\pi)^5\text{Var}(u_4(t))$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

We also plot the covariances,

$$\text{Covar}(u_n(t), u_m(t)) = \mathbb{E}[u_n(t)\overline{u_m(t)}] - \mathbb{E}[u_n(t)]\overline{\mathbb{E}[u_m(t)]}$$

in Figures 5.12 and 5.13. We only plot the covariances for two pairs as the plots below are representative of the behaviour observed for other pairs.

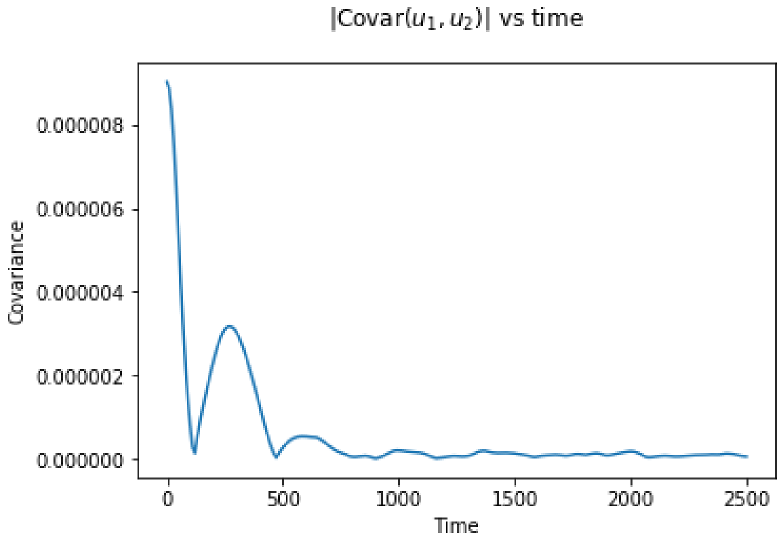

Figure 5.12: Evolution of $|\mathrm{Covar}(u_1(t), u_2(t))|$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

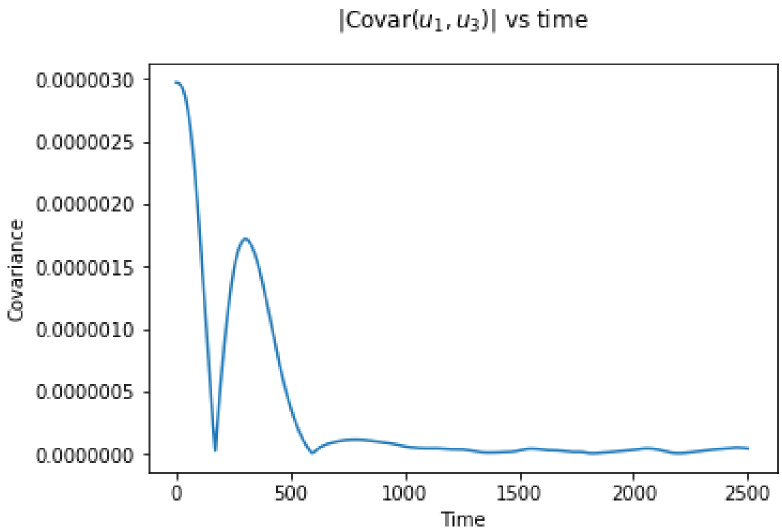

Figure 5.13: Evolution of $|\mathrm{Covar}(u_1(t), u_3(t))|$ for gBBM (1.3.6) with initial data sampled from the Gaussian measure (1.3.4).

5.4.3 Discussion

We recall that gBBM (1.3.6) conserves that mean, which is also the $0th$ Fourier coefficient,

$$u_0 = \int_{\mathbb{T}} u(x,t)\,dx$$

Hence, we expect that our simulation conserves the expected value of this. We would like to plot this quantity to test the accuracy of the simulation. However, the conservation of this quantity is built into the solver we use for gBBM (1.3.6), see Appendix D.4. The equation gBBM (1.3.6) has another conserved quantity, the energy (1.3.7), this quantity is almost surely infinite in the support of the initial measure (1.3.4). Hence we can not use this quantity for testing purposes. We continue in our discussion as the accuracy of our simulation for FNLS (1.3.1) was acceptable and we follow the same methodology in this simulation for gBBM (1.3.6).

From the covariances, Figures 5.12 and 5.13, the Fourier coefficients start correlated but then become mutually uncorrelated, a property enjoyed by the Gibbs measure (1.3.8). This is weak evidence of our convergence to the measure (1.3.8) hypothesis. We note that we did not expect the covariances to be conserved, like we did for FNLS as the assumptions in [78, Footnote 5] are not met.

However, from Figures 5.8, 5.9, 5.10 and 5.11, the scaled variance quantity (5.4.2) converges to different values, for different n. If our hypothesis of convergence to a Gibbs measure (1.3.8) was correct, we would expect these to converge to the same value. Indeed, the variance of the nth coefficient of (1.3.9) is

$$\beta^2 \frac{1}{(2\pi n)^\gamma}$$

so we expect these scaled quantities to converge to β^2, whatever this may be.

This would appear to disprove our hypothesis of convergence to the Gibbs measure (1.3.8). However, we do not make a conclusive claim ruling out convergence to the Gibbs measure (1.3.8). The quantities (5.4.2) in Figures 5.8, 5.9, 5.10 and 5.11 do seem to approach a common value, but stop short of reaching it. It is possible that this is due to not having enough samples in our Monte-Carlo simulation. We would like to preform a more comprehensive error analysis concerning how likely our Monte-Carlo simulation was to give the results just presented. Presently this is beyond our expertise. We aim to attempt to address this in the future.

From the apparent convergence of the variances (albeit perhaps to different values than hypothesized) in Figures 5.8, 5.9, 5.10 and 5.11 and the convergence of the covariances in Figures 5.12 and 5.13, it appears that the measure (1.3.4) transported by the flow gBBM (1.3.6) still converges, in some sense, to some other, presently unknown, measure.

Based on the results of this Monte-Carlo simulation for gBBM (1.3.6), and of the Monte-Carlo simulation for FNLS (1.3.1), we are lead to make Conjecture 1.3.1.

We note that we only have weak numerical support for this conjecture. It would be interesting to redo the simulations in this chapter, thoroughly taking into account sources of error. One useful tool would be confidence intervals for variances. We aim to purse this in the future. For the moment, Conjecture 1.3.1 should be viewed skeptically.

Appendix A

Basic estimates

A.1 Function spaces and Basic estimates

In this section we define the Function spaces used in this thesis, and state some useful basic estimates on these spaces. For proofs and a more detailed approach to the results in this section, we refer the reader to [3].

First we briefly recall some basic definitions in Littlewood-Paley theory. For a more detailed treatment, see [3]. We denote the Fourier transform of a function $u \in L^1(\mathbb{T}^d)$ by

$$\widehat{u}(n) = \frac{1}{(2\pi)^d} \int u(x) e^{-in\cdot x} \, \mathrm{d}x.$$

The Fourier transform of $u \in \mathcal{D}'(\mathbb{T}^d)$, the space of distributions on $\mathbb{T}^d$ is defined in the usual fashion by duality. We denote by $\mathbf{P}_{\leq N}$ the Dirichlet truncation of a distribution in Fourier space:

$$\mathbf{P}_{\leq N} u = \sum_{|n| \leq N} \widehat{u}(n) e^{in\cdot x}. \tag{A.1.1}$$

Consider functions χ, $\tilde{\chi}$ such that

$$\begin{aligned} &\mathrm{supp}\tilde{\chi} \subset B\big(0, \tfrac{4}{3}\big), \\ &\mathrm{supp}\chi \subset B\big(0, \tfrac{4}{3}\big) \setminus B\big(0, \tfrac{3}{8}\big). \end{aligned}$$

We define $\chi_0 = \tilde{\chi}$ and

$$\chi_j(\cdot) = \chi(2^{-j}\cdot), \quad j \geq 1.$$

We also define δ_j, the *Littlewood-Paley projector*, associated to symbol χ_j by

$$\delta_j u(x) = \big(\chi_j(\nabla) u\big)(x) = \sum_{n \in \mathbb{Z}^3} \chi_j(n) \widehat{u}(n) e^{in\cdot x}.$$

We now define the Besov spaces we use in this thesis through Littlewood-Paley decomposition.

For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$,we define the Besov space $B^s_{p,q}(\mathbb{T}^d)$ as the completion of $C^\infty(\mathbb{T}^d)$, the space of smooth functions, under the norm

$$\|u\|_{B^s_{p,q}(\mathbb{T}^d)} = \left\| \left(2^{sj} \|\delta_j u\|_{L^p_x(\mathbb{T}^d)} \right)_{j \geq 0} \right\|_{\ell^q_j}.$$

For $s \in \mathbb{R}$, we define the Hölder spaces $C^s(\mathbb{T}^d) := B^s_{\infty,\infty}(\mathbb{T}^d)$. These spaces are used extensively in Chapter 4.

We now state some important estimates used in this thesis.

Lemma A.1.1. *The following estimates hold*

(i) Let $s_0, s_1, s \in \mathbb{R}$ and $\nu \in [0,1]$ be such that $s = (1-\nu)s_0 + \nu s_1$. Then,

$$\|u\|_{H^s(\mathbb{T}^d)} \lesssim \|u\|^{1-\nu}_{H^{s_0}(\mathbb{T}^d)} \|u\|^{\nu}_{H^{s_1}(\mathbb{T}^d)}. \tag{A.1.2}$$

(ii) Let $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in [1, \infty]$. Then,

$$\begin{aligned}
&\|u\|_{B^{s_0}_{p_0,q_0}(\mathbb{T}^d)} \lesssim \|u\|_{B^{s_1}_{p_1,q_1}(\mathbb{T}^d)} \quad \text{for } s_0 \leq s_1, p_0 \leq p_1 \text{ and } q_0 \geq q_1, \\
&\|u\|_{B^{s_0}_{p_0,q_0}(\mathbb{T}^d)} \lesssim \|u\|_{B^{s_1}_{p_0,\infty}(\mathbb{T}^d)} \quad \text{for } s_0 < s_1, \\
&\|u\|_{B^{0}_{p_0,\infty}(\mathbb{T}^d)} \lesssim \|u\|_{L^p(\mathbb{T}^d)} \lesssim \|u\|_{B^{0}_{p_0,1}(\mathbb{T}^d)}.
\end{aligned} \tag{A.1.3}$$

(iii) Let $s > 0$. Then,

$$\|uv\|_{C^s(\mathbb{T}^d)} \lesssim \|u\|_{C^s(\mathbb{T}^d)} \|v\|_{C^s(\mathbb{T}^d)}. \tag{A.1.4}$$

(iv) Let $1 \leq p_1 \leq p_0 \leq \infty$, $q \in [1, \infty]$ and $s_1 = s_0 + d\left(\frac{1}{p_1} - \frac{1}{p_0}\right)$. Then,

$$\|u\|_{B^{s_0}_{p_0,q}(\mathbb{T}^d)} \lesssim \|u\|_{B^{s_1}_{p_1,q}(\mathbb{T}^d)}. \tag{A.1.5}$$

(v) Let $s \in \mathbb{R}$ and $p, p', q, q' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then,

$$\left| \int_{\mathbb{T}^d} uv \, dx \right| \leq \|u\|_{B^s_{p,q}(\mathbb{T}^d)} \|u\|_{B^{-s}_{p',q'}(\mathbb{T}^d)}. \tag{A.1.6}$$

(vi) Let $p, p_0, p_1, p_2, p_3 \in [1, \infty]$ be such that $\frac{1}{p_0} + \frac{1}{p_1} = 1$ and $\frac{1}{p_2} + \frac{1}{p_3} = 1$. Then for $s > 0$,

$$\|uv\|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \|u\|_{B^s_{p_0,q}(\mathbb{T}^d)} \|v\|_{L^{p_1}(\mathbb{T}^d)} + \|v\|_{B^s_{p_2,q}(\mathbb{T}^d)} \|u\|_{L^{p_3}(\mathbb{T}^d)}. \tag{A.1.7}$$

(vii) Let $m > 0$ be an integer, $s > 0$ and $q, p, p_0, p_1 \in [0, \infty]$ satisfy $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p}$. Then

$$\|u^{m+1}\|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \|u^m\|_{L^{p_0}(\mathbb{T}^d)} \|u\|_{B^s_{p_1,q}(\mathbb{T}^d)}. \tag{A.1.8}$$

(viii) Let $s_0 < 0 < s_1$ be such that $s_0 + s_1 > 0$. Then,

$$\|uv\|_{C^{s_0}(\mathbb{T}^d)} \lesssim \|u\|_{C^{s_0}(\mathbb{T}^d)} \|v\|_{C^{s_1}(\mathbb{T}^d)}. \tag{A.1.9}$$

(ix) Let $s \in (0,1)$. Then,

$$\|f\|_{B^s_{1,1}(\mathbb{T}^d)} \lesssim \|f\|^{1-\sigma}_{L^1(\mathbb{T}^d)} \|\nabla f\|^{\sigma}_{L^1(\mathbb{T}^d)} + \|f\|_{L^1(\mathbb{T}^d)} \tag{A.1.10}$$

We refer to [3,65] for proofs. We also note the Bernstein inequality:

$$\|\delta_j u\|_{L^p(\mathbb{T}^d)} \lesssim 2^{dj(\frac{1}{p}-\frac{1}{q})} \|\delta_j u\|_{L^q(\mathbb{T}^d)} \tag{A.1.11}$$

for $p \leq q \leq \infty$.

We also use the $L^p(\mathbb{T}^d)$ based Sobolev spaces, $W^{s,p}(\mathbb{T}^d)$, which we define for $s > 0$ and $p \in [1, \infty]$, as the completion of smooth functions under the norm

$$\|f\|_{W^{s,p}(\mathbb{T}^d)} = \|\langle \nabla \rangle^s f\|_{L^p(\mathbb{T}^d)}.$$

When $p = 2$, we write $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

We have the following Sobolev embedding result for these spaces.

Proposition A.1.2. *Suppose $s_0 \leq s_1$ and $1 \leq q \leq p \leq \infty$ satisfy $s_1 = s_0 + \frac{d}{2}(\frac{1}{p} - \frac{1}{q})$. Then,*

$$\|f\|_{W^{s_0,p}(\mathbb{T}^d)} \lesssim \|f\|_{W^{s_1,q}(\mathbb{T}^d)}.$$

We are interested in the Sobolev space corresponding to $p = \infty$. The following proposition shows that in the 2-dimensional setting, up to a ε loss in regularity we can transfer estimates between $W^{s,\infty}(\mathbb{T}^d)$ and $C^s(\mathbb{T}^d)$.

Proposition A.1.3. *For all $s \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|f\|_{C^s(\mathbb{T}^2)} \lesssim \|f\|_{W^{s,\infty}(\mathbb{T}^2)} \lesssim \|f\|_{C^{s+\varepsilon}(\mathbb{T}^2)}.$$

A.2 Wiener chaos and hypercontractivity

The hypercontractive estimate is a key tool to estimate nonlinear functions of Gaussian random variables. We recall this estimate here. See S. Janson's book [48,67] for more information on hypercontractivity and Wiener chaos spaces.

Let X_n, $n \geq 1$ be a sequence of i.i.d. standard Gaussian random variables. We define $\mathcal{H}_k$, the homogeneous Wiener chaos of order k to be the closed span of the polynomials

$$\prod_{n=1}^{\infty} H_{k_n}(X_n),$$

where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^{\infty} k_n$. We have

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \oplus_{k=0}^{\infty} \mathcal{H}_k.$$

Here $\mathcal{F}$ is the σ-algebra generated by X_n, $n \geq 1$. The next lemma gives the crucial hypercontractivity estimate. See [48] for a proof.

Proposition A.2.1. *Let $p \geq 2$ be finite. If $X \in \mathcal{H}_k$, then*

$$\left(E[|X|^p]\right)^{\frac{1}{p}} \leq (p-1)^{\frac{k}{2}} \left(E[|X|^2]\right)^{\frac{1}{2}}. \tag{A.2.1}$$

Appendix B

Proof of Lemma 2.1.3 for $\alpha > 1$

Setting $k = n_1 - n$ and $j = n_3 - n$, it is equivalent to prove

$$g(j,k,n) := ||n+k|^{2\alpha} - |n+k+j|^{2\alpha} + |n+j|^{2\alpha} - |n|^{2\alpha}| \gtrsim |k||j|(|k|+|j|+|n|)^{2\alpha-2}.$$

Since $2\alpha > 2$, the function $f(x) = |x|^{2\alpha} \in C^2(\mathbb{R})$ and satisfies

$$f'(x) = 2\alpha|x|^{2\alpha-2}x, \quad f''(x) = 2\alpha(2\alpha-1)|x|^{2\alpha-2}.$$

We follow a similar argument to the case $\frac{1}{2} < \alpha < 1$ in [25]. Without loss of generality we can assume that $\max(|j|,|k|) = |j|$ and $j \neq 0$. For any $c \in \mathbb{R}$, define $f_c(x) := |x+c|^{2\alpha} - |x-c|^{2\alpha}$. Then, we have

$$g(j,k,n) = |f_{j/2}(n+j/2) - f_{j/2}(n+k+j/2)|.$$

The mean value theorem implies

$$g(j,k,n) \gtrsim |k| \min_{x\in I} |f'_{j/2}(x)|,$$

where I is either the interval $(n+j/2, n+j/2+k)$ or the interval $(n+j/2+k, n+j/2)$. It suffices to show

$$|f'_c(x)| \gtrsim |c| \max(|x|,|c|)^{2\alpha-2} \text{ for } |c| \geq \frac{1}{2}. \tag{B.0.1}$$

To see this, we first suppose $|n| \lesssim |j|$. Then, for any $x \in I$, we have

$$|f'_{j/2}(x)| \gtrsim |j|^{2\alpha-1} \gtrsim |j|(|k|+|j|+|n|)^{2\alpha-2}.$$

Now suppose $|n| \gg |j|$. Then $x \in I$ implies $|x| \sim |n|$ and hence

$$\min_{x\in I} |f'_{j/2}(x)| \gtrsim |j||n|^{2\alpha-2} \gtrsim |j|(|k|+|j|+|n|)^{2\alpha-2}.$$

In order to verify (B.0.1), we may assume that $x \geq 0$ as f_c is odd and similarly, we assume $c \geq \frac{1}{2}$ as f'_c is odd in c. We have

$$f'_c(x) = 2\alpha|x+c|^{2\alpha-2}(x+c) - 2\alpha|x-c|^{2\alpha-2}(x-c),$$

and we consider three cases.

•**Subcase 2.1:** $0 \le x \le c$

Here we have

$$f_c'(x) = 2\alpha \left[|x+c|^{2\alpha-1} + |x-c|^{2\alpha-1}\right] \gtrsim c^{2\alpha-1}.$$

•**Subcase 2.2:** $c < x \le 2c$

We have

$$\begin{aligned} f_c'(x) &= 2\alpha \left[(x+c)^{2\alpha-1} - (x-c)^{2\alpha-1}\right] \\ &= 2\alpha c^{2\alpha-1} \left[\left(\frac{x}{c}+1\right)^{2\alpha-1} - \left(\frac{x}{c}-1\right)^{2\alpha-1}\right] \gtrsim c^{2\alpha-1}\left(\frac{x}{c}\right)^{2\alpha-2} \sim cx^{2\alpha-2}. \end{aligned}$$

• **Subcase 2.3:** $x > 2c$

Using the mean value theorem, we have

$$f_c'(x) = 2\alpha x^{2\alpha-1} \left[\left(1+\frac{c}{x}\right)^{2\alpha-1} - \left(1-\frac{c}{x}\right)^{2\alpha-1}\right] \gtrsim x^{2\alpha-1}\frac{c}{x} \sim cx^{2\alpha-2}.$$

This completes the proof of (B.0.1).

Appendix C

Bound for the growth of the H^σ-norm for NLW in the critical case

In this Appendix, we derive the growth bound (3.1.9) in the case $d = 3$, $k = 5$. This is the energy-critical nonlinearity. In this case, it is known solutions to NLW exist globally (see [90, Chapter 5], [88, Chapter V]) and scatter.

We begin by establishing a bound for the solutions of the equation on $\mathbb{R}^3$ instead of $\mathbb{T}^3$. We work with the homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^3)$ defined for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ as the completion of Schwartz functions under the norm

$$\|u\|_{\dot{W}^{s,p}(\mathbb{R}^3)} = \||\nabla|^s u\|_{L^p(\mathbb{R}^3)}$$

where $|\nabla|^s$ is the Fourier multiplier $\mathcal{F}[|\nabla|^s u](\xi) = |\xi|^s \mathcal{F}[u](\xi)$. When $p = 2$, we use the notation $\dot{H}^s(\mathbb{R}^3)$ instead of $\dot{W}^{s,p}(\mathbb{R}^3)$.

The following global space time bound is well known in the literature (see for example [91, Theorem 1.1, Footnote 2])

$$\|u\|_{L^5_t L^{10}_x} \leq C(H_0),$$

where H_0 is the initial energy. Let $\eta > 0$, and divide $\mathbb{R}_+$ into a finite number of interval I_j, $j = 1, \dots, J$ such that

$$\|u\|_{L^5_t(I_j; L^{10}_x)} \leq \eta.$$

Denote by t_j^- the left endpoint of the interval I_j, so that

$$\begin{aligned} I_j &= [t_j^-, t_j^+), \\ t_{j+1}^- &= t_j^+. \end{aligned}$$

We now recall the Strichartz estimate for the linear wave equation. A pair of exponents (q, r), $q \geq 2$, $2 \leq r < \infty$, is called s-admissible (in dimension 3) if

$$\begin{aligned} \frac{1}{q} + \frac{1}{r} &\leq \frac{1}{2}, \\ \frac{1}{q} + \frac{3}{r} &= \frac{3}{2} - s. \end{aligned}$$

Let $s > 0$. If (q,r) and $(\tilde{q},\tilde{r})$ are s–admissible (respectively, $1-s$–admissible pairs, then if $(\tilde{u},\tilde{v})$ solves the linear wave equation with right hand side given by F, we have [90, Theorem 2.6]

$$\|(\tilde{u},\tilde{v})\|_{L_t^\infty(I;\vec{H}^s(\mathbb{R}^3))} + \|\tilde{u}\|_{L_t^q(I;L_x^r)} \lesssim \|(\tilde{u}_0,\tilde{v}_0)\|_{\vec{H}^s(\mathbb{R}^3)} + \|F\|_{L_t^{\tilde{q}'}(I;L_x^{\tilde{r}'})}.$$

Here $\dot{\vec{H}}^s = \dot{H}^s \times \dot{H}^{s-1}$. We choose $p = 5$, $r = 10$, $\tilde{q}' = 1$, $\tilde{r}' = 2$. We choose $\tilde{u} = |\nabla|^{\sigma-1}u$ and obtain

$$\|(u,v)\|_{C_t(I;\vec{H}^\sigma)} + \|u\|_{L_t^5(I;\dot{W}_x^{\sigma-1,10})} \leq C\|(u_0,v_0)\|_{\vec{H}^\sigma} + C\||\nabla|^{\sigma-1}(u^5)\|_{L_t^1(I;L_x^2)}.$$

Applying the Leibniz rule (A.1.8) with $s = \sigma - 1$, $p_0 = 10$, $p_1 = 5/4$ and $p = q = 2$ to the final term, we obtain, on each interval I_j, the estimate

$$\begin{aligned}\|(u,v)\|_{C_t(I_j;\vec{H}^\sigma)} + \|u\|_{L_t^5(I_j;\dot{W}_x^{\sigma-1,10})} &\leq C\big(\|(u,v)(t_j^-)\|_{\vec{H}^\sigma} + \|u\|_{L_t^5(I_j;\dot{W}_x^{\sigma-1,10})}\|u\|^4_{L_t^5(I_j;L_x^{10})}\big)\\ &\leq C\big(\|(u,v)(t_j^-)\|_{\vec{H}^\sigma} + \eta^4\|u\|_{L_t^5(I_j;\dot{W}_x^{\sigma-1,10})}\big).\end{aligned}$$

For η small enough, we obtain

$$\|(u,v)\|_{C_t(I_j;\vec{H}^\sigma)} + \|u\|_{L^5(I_j;\dot{W}_x^{\sigma-1,10})} \leq C'\|(u,v)(t_j^-)\|_{\vec{H}^\sigma(\mathbb{R}^3)}.$$

Repeating this over each of the J intervals, we obtain the bound

$$\|(u,v)\|_{C_t(\mathbb{R}_+;\vec{H}^\sigma)} + \|u\|_{L^5(\mathbb{R}_+;\dot{W}_x^{\sigma-1,10}(\mathbb{R}^3))} \leq (C')^J.$$

The same argument applied to the negative time direction gives

$$\|(u,v)\|_{C_t(\mathbb{R};\vec{H}^\sigma)} + \|u\|_{L^5(\mathbb{R};\dot{W}_x^{\sigma-1,10}(\mathbb{R}^3))} \leq (C')^J. \tag{C.0.1}$$

To transfer the estimate (C.0.1) to $\mathbb{T}^3 \approx [-\frac{1}{2},\frac{1}{2})^3$, we consider initial data $(u_0,v_0) \in \dot{\vec{H}}^\sigma(\mathbb{T}^3)$. We extend (u_0,v_0) to a periodic function $(\bar{u}_0,\bar{v}_0)$ on $\mathbb{R}^3$. Let $\eta \in C_c^\infty(\mathbb{R}^3)$ such that $\eta = 1$ on $[-1,1]^3$, and define

$$\eta_T(x) = \eta(x/\langle T\rangle).$$

Consider the initial data problem on $\mathbb{R}^3$:

$$\begin{cases}\partial_t w - \Delta w + w^5 = 0\\ (w,\partial_t w)|_{t=0} := (\eta_T\bar{u}_0, \eta_T\bar{v}_0).\end{cases}$$

By (C.0.1), we have the estimate

$$\|(w,\partial_t w)\|_{C_t([0,T],\vec{H}^\sigma)} \leq C(\|(w,\partial_t w)|_{t=0}\|_{\vec{H}^\sigma(\mathbb{R})}).$$

Then, by [72, Eqn. (3.11)], we have, for $f \in \dot{H}^s(\mathbb{T}^3)$

$$(1/C)\langle T\rangle^{3/2}\|f\|_{\dot{H}^s(\mathbb{T}^3)} \leq \|\eta_T f\|_{\dot{H}^s(\mathbb{R}^3)} \leq C\langle T\rangle^{3/2}\|f\|_{\dot{H}^s(\mathbb{T}^3)}.$$

It follows that

$$\|(w, \partial_t w)\|_{C_t([0,T],\vec{H}^\sigma)} \lesssim T^{3/2}.$$

By finite speed of propagation, for $|t| \leq T$, the restriction of $(w(t), \partial_t w(t))$, to $[-\frac{1}{2}, \frac{1}{2})^3 \approx \mathbb{T}^3$ coincides with the solution $(u(t), v(t))$ of the quintic nonlinear wave equation on $\mathbb{T}^3$ with initial data (u_0, v_0).

In the periodic setting, we can control the L^2-norm of u and $\partial_t u$ by the Hamiltonian, finally giving

$$\|(u, \partial_t u)\|_{C_t([0,T],\vec{H}^\sigma)} \lesssim T^{3/2}.$$

Appendix D

Code excerpts

D.1 FNLS solver: Julia code

```
using FFTW
using DifferentialEquations

function FNLS_solver(Fu0::Array{ComplexF64}, t::Real, dt::Real)
    alpha = 0.55
    lambda = 10
    N = length(Fu0)
    k = convert(Array{ComplexF64,1}, -div(N,2):(div(N,2)-1))
    k = fftshift(k)
    disp_rel = -im*abs2.(2*pi*k).^alpha

    function G(dFu, Fu::Array{ComplexF64}, p, t)
        u = ifft(Fu)
        dFu[:] = disp_rel.*Fu - lambda*im*N^2*fft(abs2.(u).*u)
    end
    prob = ODEProblem(G, Fu0, (0,t))
    sol = solve(prob, ABDF2(autodiff = false), saveat = 0.0:dt:t,
                abstol = 1e-7, reltol = 1e-5)
    return sol.u
end
```

D.2 Random initital data for FNLS: Julia code

```
using Random
using FFTW

function random_data(N::Int, s::Real)::Array{ComplexF64}
    k = convert(Array{ComplexF64,1}, -div(N,2):(div(N,2)-1))
    k = fftshift(k)
```

```
    out = randn(ComplexF64, N)./ (k.^2 .+ 1).^(s/2)
end
```

D.3 Sampling evoloution of measure for FNLS: Julia code

```
using Random
include("initial_data.jl")
include("FNLS_solver.jl")
using NPZ

function sample(n)
    s = 4.2
    t = 1.5
    NUM_FRAMES = 750
    N = 2^9
    dt = t/NUM_FRAMES
    save_loc = "path"

    println("Starting sample " * string(n) * "...")
    Fu0 = random_data(N, s)
    Fu = FNLS_solver(Fu0, t, dt)
    save_freqs = cat(1:30,  N :-1 : N-30, dims = 1)
    save_path = save_loc*string(n)*".npy"
    npzwrite(save_path, (hcat(Fu...)')[:,save_freqs])
    println("Sample "*string(n)*" done.")
end

Random.seed!(82354692435761234031468)

NUM_TRIALS = 100000
map(sample, 1 : NUM_TRIALS)
```

D.4 gBBM solver: Julia code

```
using FFTW
using DifferentialEquations

function BBM_solver(Fu0::Array{ComplexF64}, t::Real, dt::Real)
    gamma = 5
    lambda = 10000
    N = length(Fu0)
    k = convert(Array, 0:(N-1))
    disp_rel = (-2*pi*im*k)./(1 .+ (2*pi*k).^gamma)
```

```julia
    function G(dFu, Fu::Array{ComplexF64}, p, t)
        dFu[:] = disp_rel.*(lambda*N*rfft(irfft(Fu, 2*(N-1)).^2))
        dFu[1] = 0
    end

    prob = ODEProblem(G, Fu0, (0,t))
    sol = solve(prob, Tsit5(), saveat = 0.0:dt:t,
                abstol = 1e-12, reltol = 1e-9)
    return sol.u
end
```

D.5 Random initial data for gBBM: Julia code

```julia
using Random

function random_data2(N::Int, s::Real)::Array{ComplexF64}
    out = Array{ComplexF64}(undef, N)
    out[1] = 0
    #Make first 4 coefficents correlated
    h = randn(ComplexF64, 4)
    out[2] = sqrt(0.225)
                * (randn(ComplexF64) + h[1] + h[2] + h[3])
                / ((1 + (2*pi*1)^(2*s))^(1/2))
    out[3] = sqrt(0.275)
                * (randn(ComplexF64) + h[2] + h[3] + h[4])
                / ((1 + (2*pi*2)^(2*s))^(1/2))
    out[4] = sqrt(0.225)
                * (randn(ComplexF64) + h[3] + h[4] + h[1])
                / ((1 + (2*pi*3)^(2*s))^(1/2))
    out[5] = sqrt(0.275)
                * (randn(ComplexF64) + h[4] + h[1] + h[2])
                / ((1 + (2*pi*4)^(2*s))^(1/2))
    #The other coefficents
    for k in 5:(N-1)
        out[k+1] = randn(ComplexF64)
                      / ((1 + (2*pi*k)^(2*s))^(1/2))
    end
    return out
end
```

D.6 Sampling evoloution of measure for gBBM: Julia code

```julia
using Distributed
include("initial_data.jl")
```

```
include("bbm_solver.jl")
using NPZ

function sample(n)
    s = 2.5
    t = 10000.0
    NUM_FRAMES = 1000
    N = 2^12 + 1
    dt = t/NUM_FRAMES
    save_loc = "path"

    println("Starting sample "*string(n))
    Fu0 = random_data2(N, s)
    Fu = BBM_solver(Fu0, t, dt)
    save_freqs = 1:10
    save_path = save_loc*string(n)*".npy"
    npzwrite(save_path, (hcat(Fu...)')[:,save_freqs])
    println("Sample "*string(n)*" done.")
end

Random.seed!(32084628435673757489)

NUM_TRIALS = 100000
time map(sample, 1:NUM_TRIALS)
```

D.7 Samples to frequencies sorter: Python code

```
import numpy as np

NUM_SAMPLES = 100000
NUM_FRAMES = 1001    #This is T/dt
TOTAL_FREQS = 8
BUFFER = 4    #RAM limits, only do few frequencies at once.
SAMPLES_PATH = "samples_path"
FREQS_SAVE_PATH = "freqs_save_path"

NUM_LOOPS = TOTAL_FREQS//BUFFER    #Loops through sample data

def samples_to_frequencies(START_FREQ, NUM_FREQS):
    Futs = np.zeros((NUM_FRAMES, NUM_SAMPLES, NUM_FREQS)) + 0j

    for sample_num in range(0,NUM_SAMPLES):

        ld_path = SAMPLES_PATH+str(sample_num+1)+r".npy"
        Fsample = np.load(ld_path)
```

```python
        END_FREQ = START_FREQ + NUM_FREQS
        Futs[:,sample_num,:]=Fsample[:,START_FREQ:END_FREQ]

        if sample_num*100 % NUM_SAMPLES == 0 and sample_num != 0:
            print(f"{sample_num*100//NUM_SAMPLES}%")

    for freq_num in range(START_FREQ, START_FREQ + NUM_FREQS):
        np.save(FREQS_SAVE_PATH+"freq"+str(freq_num + 1)+".npy",
                Futs[:, :, freq_num - START_FREQ])

for k in range(NUM_LOOPS):
    print(f"Converting freqs {k*BUFFER} through {(k+1)*BUFFER-1}")
    samples_to_frequencies(k*BUFFER, BUFFER)
    print("Done")
```

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