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# Classical Physics from Quantum Fields

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# Lay summary

Two pinnacles of modern physics are Quantum Field Theory (QFT) and Einstein’s theory of General Relativity. The former is used to describe particles at the subatomic level, and has done so with great precision and success at the Large Hadron Collider at CERN. The latter describes gravity on the largest scales. In particular it allows us to study the motion of black holes, the most massive and intriguing objects in the universe. These theories however are wildly different. One is applicable to the smallest scales and follows the probabilistic rules of quantum mechanics, while the other is a deterministic classical theory. It must be possible to extract classical information from quantum physics: this is the main theme of this thesis.

With the recent observation of gravitational waves – ripples in space time itself caused by the motion of black holes – there has been a surge of interest in their study. Despite these being some of the most violent interactions in the universe, the waves detected on Earth are incredibly weak and require highly advanced instrumentation to detect. From a theoretical perspective, this means we need to make extremely precise predictions.

Though these theories of particle physics and gravity appear unrelated at first sight, recent discoveries show that they are linked in a surprisingly simple way through the so-called Double Copy. By treating black holes like fundamental particles, we can compute observable quantities — things experimentalists can measure from data — using the tools of QFT. This is able to bypass much of the computational difficulty that is present in traditional methods, and utilise the vast armoury of techniques developed by particle physicists. By computing *scattering amplitudes* (used to compute probabilities for two particles to interact), we can deduce the angle by which the particles are deflected in a scattering process. We can do this for a variety of models, such as electric charges interacting in electromagnetism, or in a more sophisticated setup for black holes interacting via gravity.

# Abstract

Gravitational wave physics is now well established as an experimental science, no longer existing only in the minds and blackboards of the theorist. There is an abundance of events being detected with a very high accuracy by the LIGO/Virgo/KAGRA collaboration. This necessitates the development of innovative techniques for producing very high precision calculations. Motivated by the Double Copy, a relation between scattering amplitudes in gauge theory and gravity, an amplitudes based approach seems to be a promising avenue to pursue. Yet amplitudes are quantum objects and gravitational waves are classical; there must be a bridge between these two regimes. This leads us to the topic of this thesis - how can we extract classical observable quantities directly from quantum scattering amplitudes.

We begin by reviewing the KMOC formalism for computing classical observables from amplitudes. We investigate how, by imposing classically sensible minimal uncertainty constraints on observables, we can learn about amplitudes in the classical limit. Black holes in general will spin, and so including spin effects into the calculations of gravitational waveforms is important. Motivated by this we apply the KMOC formalism to a theory of scalars carrying an  $SU(N)$  colour charge. In this theory we compute changes in and radiation of momentum and also colour charge. This requires a detour into the study of coherent states which are another key tool to study classical from quantum.

An older approach to semi-classical physics is the eikonal method which has recently received a surge of interest. It allows one to easily compute the scattering angle and impulse during a scattering event. We review the eikonal resummation and use to describe the final state after such a (conservative) scattering event. Returning to coherent states we propose an ansatz for a quantum final state, parametrised only by the eikonal phase and a coherent state parameter, which captures all the necessary data to compute the radiation in the classical limit.

# Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have appeared in collaboration in the following papers.

## Chapter 4

[1]: L. de la Cruz, B. Maybee, D. O’Connell and A. Ross, *Classical Yang-Mills observables from amplitudes*, *JHEP* **12** (2020) 076 [ 2009.03842].

## Chapters 3 and 5

[2]: A. Cristofoli, R. Gonzo, N. Moynihan, D. O’Connell, A. Ross, M. Sergola et al., *The Uncertainty Principle and Classical Amplitudes*, 2112.07556.

## Chapter 6

A. Luna, D. O’Connell, N. Moynihan, A. Ross. *One Loop Impulse from the Spinning Eikonal*, (Work in progress) .

(Alasdair Ross, July 2023)

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# Chapter 1

## Introduction - Background and Motivation

The goal of this thesis is to investigate novel approaches to computing observables in classical scattering events using methods from quantum field theory (QFT). By consistency it should be possible to recover classical physics from quantum mechanics in a suitable limit. This is often referred to as the ‘correspondence principle’. Why should we be interested in this? The motivation comes from an apparently unrelated area of physics – Gravitational Waves (GW) and a relation called the Double Copy (DC).

Gravitational waves have been theoretically understood for many years, however the first detection was made by the LIGO/ Virgo collaboration (now Ligo-Virgo-Kagra) in 2015 [3, 4]. Since then the number of merger observations has grown substantially and as of time of writing (after three observing runs in the detectors) there have been 90 black hole merger event detections[5]. With the advent of new ground and space base detectors, as well as improvements in current instrumentation, the data output will increase dramatically. Not only will more events be detected but the precision will also improve by an estimated two orders of magnitude [6–8]. The reason it took so long to make a detection is that the signals are incredibly weak. The strain<sup>1</sup> measured in the interferometers is of the order of  $10^{-22} - 10^{-23}$ , and there were many challenges to overcome in building a detector capable of resolving such minute scales. These signals, despite being very weak when we detect them, come from incredibly violent and energetic

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<sup>1</sup>Relative change in length of the detector arms as gravitational waves pass by

interactions in space – the binary inspiral of two black holes (or neutron stars) orbiting and merging into one. In the process they emit gravitational radiation with energy equivalent of many solar masses which we later detect here on earth. See [9] for a recent review of the current status of GW observations.

The merger process can be split into 3 phases- inspiral, coalescence/merger, and ringdown. The majority of the data comes from the inspiral phase where perturbation theory can be applied. The merger and ringdown phases occur in a much shorter time period, and are highly non linear and thus are typically approached numerically. There are many approaches used to study the inspiral problem including EOB [10, 11], Worldline EFTs [12–14], Self force [15–17] and Numerical Relativity [18–21]. These all have their merits and domains where they perform best, depending on relative masses, speeds etc. See [22–25] for reviews of these approaches.

Analytical approaches to the two body problem in GR typically come in two flavours. First is the Post-Minkowskian (PM) approximation where we expand in the gravitational constant  $G$  but allow velocities to be relativistic (formally an expansion in  $G$ ). Second is the Post-Newtonian (PN) expansion, where again one expands for small coupling and also small velocities. Both are weak field perturbative expansions and can be viewed as corrections to flat space special relativity and Newtonian gravity respectively. The PM scheme is most natural for unbound scattering situations, while the PN lends itself to bound systems. Because of this, efforts studying the two body dynamics classically are based in the PN scheme. Current state of the art PN results are pushing 5PN order [26–31].

Remarkably, ideas and methods from quantum field theory offer a promising avenue of investigation [32–50]. Powerful techniques have been developed to compute scattering amplitudes, and we have learned how to extract classical physics efficiently from amplitudes [51–93]. Why would it be sensible to model black holes using methods from quantum field theory? One reason it is not completely crazy is the “No Hair Theorem” [94–96] which says that a black hole is uniquely described by only 3 parameters - Mass, Charge and Spin (angular momentum). Far from the black hole, finite size effects can be ignored and so, if we squint, it looks rather a lot like a fundamental point particle with possibly a charge and spin, which is exactly what QFT is designed for. Another, more formal, viewpoint is that we can use Effective Field Theory and the first approximation is that of point particles. Finite size effects are then included with higher dimension

operators.

We obviously also need a way to extract the classical limit of this description, seeing as black holes are classical not quantum objects, and this will be studied largely in the KMOC framework [45].

The key link we need for computations is the double copy: the observation that scattering amplitudes in gravitational theories can be computed from amplitudes in gauge theories. Since perturbation theory is far simpler in Yang-Mills (YM) theory than in standard approaches to gravity, the double copy has revolutionary potential.

A wide arsenal of techniques has been developed for computing scattering amplitudes, initially for precision computations at particle colliders, but now finding a new application. This includes unitarity based methods and recursion relations for building up amplitudes from smaller simpler ones, either for higher multiplicities or higher loop orders [97–102]. Then there are methods for computing loop integrals which is often the bottleneck in the state of the art calculations, both from amplitudes and classical approaches. This includes IBP relations [103–106], differential equations techniques [107–110], and method of regions [111]. See [112] for a recent review of Feynman integrals.

Amplitudes based approaches have already found great success, computing the  $4PM$  observables [46, 63, 113–115], in full agreement with other traditional approaches [116, 117] as well as EFT [118–121]. In fact in [46, 63], the amplitudes approach arrived at a  $3PM$  result before the more traditional approaches. This was the first time when genuine state of the art was performed using amplitudes. Amplitudes methods are typically applied to either computing potentials, via an EFT matching [44, 57, 61, 75, 122, 123], or observables like impulse in scattering events [45, 74, 88, 124–127]. Classical Hamiltonians allow bound state dynamics to be studied, and bound observables directly from amplitudes can be done by an analytic continuations to map unbounded to bound observables [65, 67, 120]. For a review of the PM expansion via amplitudes, see [93].

It is also worth emphasising the importance of spin. The spin of the individual bodies in a compact binary coalescence event influences the details of the outgoing gravitational radiation. Measurement of spin is therefore one of the primary physics outputs of gravitational wave observations. It is fortunate, then, that spin can be incorporated rather naturally in the formalism of quantum field theory [75, 88, 89, 128–147], even for very large (classical) spins. However the details of spin

dynamics quickly becomes complex. Spin effects in the impulse have been studied at 3/4PM order via QFT methods [89, 123]

This provides the motivation and background for the investigations in this thesis and can be neatly summarised in the following figure<sup>2</sup>.

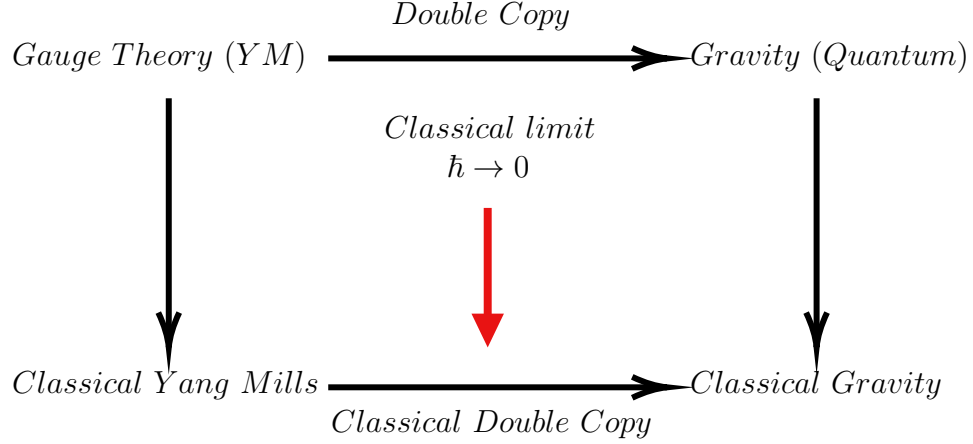


**Figure 1.1** *The big picture: Double Copy and classical limits as tools to study classical gravitational observables relevant to gravitational wave experiments.*

The focus in this thesis is moving down in the diagram, not across. Indeed we will not be making explicit use of the double copy, and generally will stick to the gauge theory side as an arena for testing the validity and practicality of the classical limit procedures. Toy models like these have been applied by others, testing the waters for higher order calculations [148–152].

There are many incarnations of the double copy, all of which come with the theme “gravity is the square of gauge theory” in some form. Of the precise statements of a double copy, the first was by Kawai, Lewellen and Tye (KLT) [153] coming from studying string amplitudes. They noticed that closed string amplitudes (reducing to gravitons) could be written as the product of open string amplitudes (reducing to gauge theory). Then more recently was the BCJ double copy and the concept of ‘colour kinematics duality’ [154, 155]. These are both statements in terms of quantum scattering amplitudes. Finally there are notions of a “classical double copy”, such as the Kerr-Schild double copy [38, 156], relating purely classical solutions to the vacuum field equations for gauge theory or GR. This was extensively reviewed recently [157, 158], and I refer the interested reader

<sup>2</sup>Note however that this is not saying that the classical double copy, as it is usually defined, is the same as the classical limit of the quantum double copy.

there for more details and references. As for a deeper reason for this squaring relation between gauge theories and gravity theories, nobody knows.

## Thesis outline

To summarise, the outline of this thesis is as follows. We begin by reviewing the background material that is applied throughout this thesis. The aim is for this to be pedagogical. We start with the KMOC formalism, then coherent states and finally eikonal resummation.

Since we are ultimately interested in classical physics, in chapter 3 we will look at how we can use minimal uncertainty conditions to constrain and derive relationships between loop and tree amplitudes. Next in chapter 4 we apply this to the study of observables in Yang Mills theory. We compute momentum impulse, colour impulse as well as the associated radiation.

In chapter 5 we take the relations of chapter 3, which suggest a generalised exponential structure like that of the eikonal, and we will provide an ansatz for the final state of a system after conservative scattering that is a hybrid of KMOC and eikonal. We will then extend this to include radiation in the final state with the use of coherent states. This is the pinnacle idea in this thesis – we combine KMOC, coherent states and eikonal into one.

In the final chapter 6 we apply this to the case of particles with spin. We will compute the impulse in EM and GR using the KMOC formalism, and then using the hybrid KMOC eikonal. This reveals some subtleties where one must be very careful using the KMOC-Eikonal state, particular regarding the spin. We find these results to agree mutually and with the literature. Finally we end with conclusions in chapter 7.

## Chapter 2

# Background Material - KMOC Eikonal and Coherent States

In this chapter I will review the material that forms the basis of this thesis. I will begin with the KMOC formalism, then the basics of coherent states and their relevance to classical limits, and finally discuss eikonal exponentiation. The aim of this chapter is pedagogy as well as setting conventions and notations for the later chapters. Some of the material is reviewed elsewhere, for example references [\[159–161\]](#) are useful.

### 2.1 KMOC Review

We begin by reviewing the KMOC formalism [\[45\]](#) which is the main framework used throughout this thesis. This is a quantum-first approach for computing observable quantities and taking a classical limit, starting from a quantum field theory basis. The basic idea is to take two stable incoming particles, described by a quantum state, which are very distant in the far past. We scatter these particles off one another gently, so that they do not approach too closely (and so that we can describe the scattering, if we choose, using classical field theory). Particles will nevertheless radiate. We assume that the particles remain unbound throughout the process. Ultimately we are interested in describing the scattering of two classical bodies (think of this as a model for black holes) and so we shall first need to set up sensible quantum states to describe these.

### 2.1.1 Particle States

We would like a state that describes a real particle with a well defined momentum and also position, since we are interested in classical objects. These requirements are naturally opposing each other in quantum mechanics as a result of the uncertainty principle, so we need a best compromise. Quite generally the way forward is to use coherent states, though in simple cases this is not necessary. Indeed for pure scalars, where we only need to describe kinematics, we will not need to make use of them. Coherent states will be introduced in the next subsection.

In the original paper [45], it was shown how to construct states explicitly in scalar QED for simplicity, however it is possible to apply the formalism much more generally to other fields with other degrees of freedom. Indeed in a follow up article by two of the original authors and Vines [133], they generalise the setup to include particles with spin. It has also been applied to gravity with great success, up to 4(3) PM order [123, 149, 162] in the conservative (radiative) sectors. In the next chapter we shall apply this to scalar particles with a non abelian colour charge, so I will keep the discussion here as general as possible for now.

As we will need to take a classical limit, amounting to sending  $\hbar \rightarrow 0$ , we will keep factors of  $\hbar$  explicit. This makes equations and dimensional analysis arguments slightly less trivial than the standard conventions where  $\hbar = c = 1$ . We are not completely crazy though, and still work with  $c = 1$ . Where necessary we will use dimensions of mass  $[M]$  and length  $[L]$ , as the two dimensionful scales<sup>1</sup>.

First let us construct a single particle state, and later generalise to two particles. These states are all defined in the far past, that is they are ‘in’ states. Later in chapter 5 we will discuss outgoing final states too. We use a basis of momentum eigenstates,

$$|p_i\rangle = a_i^\dagger(p) |0\rangle, \quad (2.1)$$

where the  $i$  index corresponds to the representation index of some symmetry group(s). This could be, for example, a colour gauge group or describe the spin of the particle. For a scalar particle in QED this index is absent. The

---

<sup>1</sup>We use the notation  $[x]$  for the dimensions of the quantity  $x$ .

creation/annihilation operators are chosen to be normalised such that

$$[a_i(p), a^{\dagger j}(q)] = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \delta_i^j \equiv \hat{\delta}_{\Phi}^3(p - q) \delta_i^j, \quad (2.2)$$

and likewise for the antiparticle operators. Each ladder operator therefore has dimensions of  $M^{-1}$ . In order to avoid a proliferation of factors of  $2\pi$  we will introduce a ‘hat’ notation

$$\hat{d}^n p \equiv \frac{d^n p}{(2\pi)^n}, \quad \hat{\delta}^{(n)}(x) \equiv (2\pi)^n \delta^{(n)}(x). \quad (2.3)$$

Thus the momentum eigenstates are normalised as

$$\langle p'_i | p^j \rangle = 2E_p \hat{\delta}^{(3)}(\underline{p}' - \underline{p}) = \hat{\delta}_{\Phi}^3(p - p') \delta_i^j. \quad (2.4)$$

Finally a single particle state will be written

$$|\psi\rangle = \sum_i \int d\Phi(p) \varphi_i(p) |p^i\rangle, \quad (2.5)$$

where we defined an on shell phase space measure

$$d\Phi(k) = \frac{d^4 k}{(2\pi)^4} 2\pi \Theta(k^0) \delta(k^2 - m^2) \equiv \hat{d}^4 k \hat{\delta}^{(+)}(k^2 - m^2), \quad (2.6)$$

and wave packet  $\varphi_i(p)$ . We will work with the mostly minus metric signature throughout this thesis. The phase space measure is restricted to positive energy states as we are considering real particles. In many cases, in particular for colour as we will come to later, the wavepacket can be split into a kinematic part and a group part

$$\varphi_i(p) = \varphi(p) \chi_i(p), \quad (2.7)$$

and we will refer to these as the momentum and colour wavepackets respectively. Each are normalised independently as

$$\int d\Phi(p) |\varphi(p)|^2 = \sum_i \chi_i \chi^i = 1. \quad (2.8)$$

Equivalently we could phrase this by saying state decomposes into a tensor product of a kinematic part, associated with space time symmetries, and ‘colour’ part, associated with other symmetry groups,

$$|\psi\rangle = \sum |\psi_{\text{kin}}\rangle \otimes |\psi_{\text{colour}}\rangle. \quad (2.9)$$

I will discuss the details of the colours states and wavepackets later in chapter 4 on colourful observables, after introducing coherent states. For now let us look at the key features required of the momentum wavepackets, to give a reasonable classical limit, keeping the discussion general without specifying any specific form for these. Specific example wavepackets and details of them are given in the original work [45].

The key property these must enforce is the localisation of the particle. More precisely the wavepacket should be a sharply peaked function centred about the classical momentum  $p^\mu = mu^\mu$ . There will be some intrinsic spread or width  $\ell_w$  associated with the wavepacket, which we can roughly think of as the size of the particle, and there will also be a Compton wavelength associated with the momentum  $p^\mu$ . Note that we are only considering massive particles here – the application to massless particles was studied in [163] and we refer the interested reader there for more details. To be a good, classical, well-localised description, the Compton wavelength must be much less than the spread of the wave packet

$$\frac{\hbar}{m} = \ell_c \ll \ell_w. \quad (2.10)$$

Intuitively this means the quantum fuzzyness ( $\ell_c$ ) is much smaller than the ‘size’ of the particle ( $\ell_w$ ). The precise form of the wavepacket is not important for us, only that this localisation property is good.

Now that we have a way to describe a single particle, we can look at describing a pair of particles. After all, we are going to be studying observables in 2 body scattering events. The initial state should describe two classical particles, initially separated by an impact parameter  $b$ , and is given by <sup>2</sup>

$$|\Psi\rangle = \sum_{i,j} \int d\Phi(p_1, p_2) \varphi(p_1, p_2) e^{ip_1 \cdot b / \hbar} \chi_i \chi_j |p^i, p^j\rangle. \quad (2.11)$$

Here I’ve introduced a condensed notation

$$\begin{aligned} d\Phi(p_1, p_2, \dots, p_n) &= d\Phi(p_1) d\Phi(p_2) \dots d\Phi(p_n), \\ \varphi(p_1, p_2, \dots, p_n) &= \varphi(p_1) \varphi(p_2) \dots \varphi(p_n). \end{aligned} \quad (2.12)$$

It is sometimes convenient to absorb the translation in the two-particle momentum-

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<sup>2</sup>We will use  $|\Psi\rangle$  for two particle states and  $|\psi\rangle$  for single particle ones.

space wavefunction and write

$$\phi_b(p_1, p_2) \equiv \varphi(p_1, p_2) e^{ib \cdot p_1 / \hbar}. \quad (2.13)$$

I've chosen to use a single impact parameter  $b^\mu$  with origin of the coordinates being at particle 2, however one could instead have kept an arbitrary origin and had impact parameters  $b_i^\mu$ . In this case the phase representing the translation of the states would be

$$e^{ip_1 \cdot b} \rightarrow e^{ip_1 \cdot b_1} e^{ip_2 \cdot b_2}. \quad (2.14)$$

Here we encounter a similar inequality condition, that for the point particles to be well separated classically the impact parameter must be much more than the size of the particles. That is  $\ell_w \ll |b|$ . Combined with the earlier inequality, we have a pair which is sometimes referred to as the ‘Goldilocks inequalities’

$$\ell_c \ll \ell_w \ll \sqrt{-b^2}. \quad (2.15)$$

More generally, if  $\ell_s$  is the distance of closest approach of the particles during scattering, we require that

$$\ell_c \ll \ell_w \ll \ell_s. \quad (2.16)$$

Small-angle scattering has the property that  $\ell_s \simeq b$ . This has schematic representation as depicted in fig. 2.1

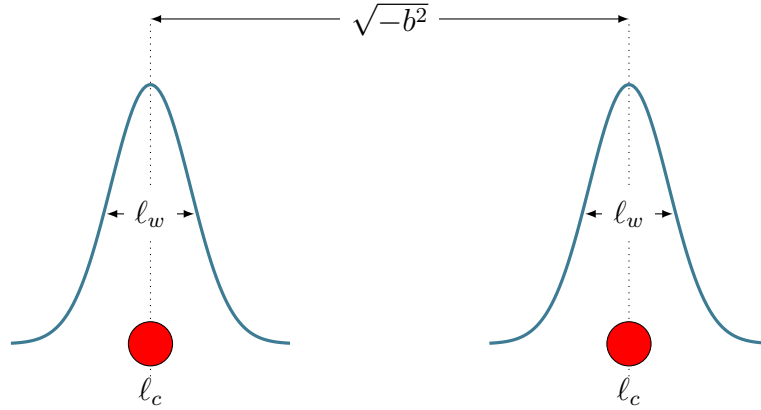


**Figure 2.1** *Schematic representation of the Goldilocks inequalities*

This final condition in eq. (2.16) is also required for perturbation theory to be valid. In particular in gravity, which is the motivating goal, this is the requirement that gravity is weak. The separation between the black holes is large compared to the schwarzschild radius  $b \gg R_s$ , (equivalent to  $\ell_w$ ). This is also the physical

regime for the inspiral phase of the binary BH merger.

### 2.1.2 Observables

We will now discuss the construction of various observables in this formalism. The discussion here will be quite general, and later we will study specific cases in more detail (scalar QCD in chapter 4).

Given some Hermitian quantum operator we can compute its expectation value in the far past with the ‘in’ states constructed in section 2.1.1. We can then allow the particles to evolve forward in time, interact in some way. In the far future then measure the same expectation value again and determining the change as a result of the scattering. This is the sort of observable one could imagine actually measuring in an experiment. To compute this we evolve the in-states with the  $\hat{S}$  matrix

$$|\Psi\rangle \rightarrow \hat{S} |\Psi\rangle. \quad (2.17)$$

Then for some operator  $\hat{\mathcal{O}}$  the change will be given by <sup>3</sup>

$$\begin{aligned} \Delta\mathcal{O} &= \langle\Psi| \hat{S}^\dagger \hat{\mathcal{O}} \hat{S} |\Psi\rangle - \langle\Psi| \hat{\mathcal{O}} |\Psi\rangle \\ &= \langle\Psi| \hat{S}^\dagger [\hat{\mathcal{O}}, \hat{S}] |\Psi\rangle. \end{aligned} \quad (2.18)$$

Writing  $\hat{S} = \hat{1} + i\hat{T}$  and using unitarity we can write this as

$$\Delta\mathcal{O} = i \langle\Psi| [\hat{\mathcal{O}}, \hat{T}] |\Psi\rangle + \langle\Psi| \hat{T}^\dagger [\hat{\mathcal{O}}, \hat{T}] |\Psi\rangle. \quad (2.19)$$

This expression is a non perturbative statement, though to use in practice we will have to resort to perturbation theory. In this thesis we will mainly be interested in the cases of momentum and colour as well as the radiation associated with each of these

$$\hat{\mathcal{O}} = \{\hat{\mathbb{P}}_1^\mu, \hat{\mathbb{C}}_1^a, \dots\}, \quad (2.20)$$

though one could also consider other observables such as spin or angular momentum [115, 133]. The next step is to insert our in-states into this equation and follow through the algebra. In anticipation of this, notice we will encounter

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<sup>3</sup>Here the hat represents it being an operator, not a factor of  $2\pi$  and the distinction should be clear from context.

terms in the form of scattering amplitudes, so I will choose notation for this now

$$\langle p'_1, \dots, p'_m | \hat{T} | p_1, \dots, p_n \rangle = \mathcal{A}(p_1, \dots, p_n \rightarrow p'_1, \dots, p'_m) \delta^4 \left( \sum p_i - \sum p'_j \right). \quad (2.21)$$

Including the ‘colour’ states too we can write

$$\begin{aligned} \langle \chi'_1, \dots, \chi'_m; p'_1, \dots, p'_m | \hat{T} | p_1, \dots, p_n; \chi_1, \dots, \chi_n \rangle \delta^4(p_{in} - p_{out}) \\ = \chi_1^{i_1} \dots \chi_n^{i_n} \mathcal{A}_{i_1, \dots, i_n}^{j_1, \dots, j_m}(p_1, \dots, p_n \rightarrow p'_1, \dots, p'_m) \chi'_{1j_1} \dots \chi'_{mj_m} \\ \equiv \langle \vec{\chi}' | \mathcal{A}(p_1, \dots, p_n \rightarrow p'_1, \dots, p'_m) | \vec{\chi} \rangle \delta^4(p_{in} - p_{out}) \end{aligned} \quad (2.22)$$

where we’ve introduced the short hand  $|\vec{\chi}\rangle = |\chi_1, \dots, \chi_n\rangle$ .

To tackle the quadratic term in eq. (2.19) we use the resolution of the identity which for our purposes can be written

$$\mathbb{1} = \sum_X \int d\Phi(r_1, r_2, r_X) d\mu(\xi_1, \xi_2) |r_1, r_2, X; \xi_2, \xi_2\rangle \langle r_1, r_2, X; \xi_2, \xi_2|. \quad (2.23)$$

Here  $X$  represents states with arbitrary number of particles (usually gauge particles), though we will mostly be sticking to the case where  $X = \emptyset$  as we will work only up to one loop. The measure  $d\mu(\xi)$  is an appropriate measure for the other degrees of freedom associated with the  $\xi$ , and could be discrete or continuous. In the case of scalars with no extra charges we can simply remove any  $d\mu(\xi) |\xi\rangle \langle \xi|$  etc. in this expression.

Using this resolution of the identity we can write the observable of interest as

$$\begin{aligned} \Delta \mathcal{O} &= \Delta \mathcal{O}_1 + \Delta \mathcal{O}_2 \\ \Delta \mathcal{O}_1 &= i \int d\Phi(p'_1, p'_2, p_1, p_2) \varphi^*(p'_1, p'_2) \varphi(p_1, p_2) e^{i(p_1 - p'_1) \cdot b/\hbar} \\ &\quad \times \delta \mathcal{O}_{p'p} \langle \chi' | \mathcal{A}(p_1, p_2 \rightarrow p'_1, p'_2) | \chi \rangle \delta^4(p_1 + p_2 - p'_1 - p'_2) \\ \Delta \mathcal{O}_2 &= \sum_X \int d\Phi(p'_1, p'_2, p_1, p_2) d\Phi(r_1, r_2, r_X) d\mu(\xi_1, \xi_2) \varphi^*(p'_1, p'_2) \varphi(p_1, p_2) e^{i(p_1 - p'_1) \cdot b/\hbar} \\ &\quad \times \langle \chi' | \mathcal{A}^*(p'_1, p'_2 \rightarrow r_1, r_2, r_X) | \xi \rangle \delta \mathcal{O}_{rp} \langle \xi | \mathcal{A}(p_1, p_2 \rightarrow r_1, r_2, r_X) | \chi \rangle \\ &\quad \times \delta^4(r_1 + r_2 + r_X - p'_1 - p'_2) \delta^4(p_1 + p_2 - r_1 - r_2 - r_X). \end{aligned} \quad (2.24)$$

Here  $r_X$  is the total momentum carried by the messenger(s)  $X$ . The expression  $\delta \mathcal{O}_{ab}$  represents the difference of the operator acting to the left on a state  $\langle a|$  and acting on the right on  $|b\rangle$ , i.e.  $\delta \mathcal{O}_{p'p} = \mathcal{O}(p') - \mathcal{O}(p)$ . In the quadratic

part, this is associated with the variables in  $\mathcal{A}$  not  $\mathcal{A}^*$ . The precise form of  $\delta\mathcal{O}$  will vary depending on the operator, for example for momentum of particle 1  $\delta\mathcal{O}_{p'p} = (p' - p)$ .

We can manipulate this to a slightly more compact form that will also be helpful when we come to the classical limit procedure. First, define momentum shifts/mismatches by  $p'_i = p_i + q_i$  and  $r_i = p_i + w_i$ . It is also worth noting that both the incoming and outgoing particles are on shell. In terms of the momentum mismatches the on-shell conditions are

$$p_i^2 = p_i'^2 = (p_i + q_i)^2 = p_i^2 + 2p_i \cdot q_i + q_i^2 = m_i^2. \quad (2.25)$$

With this we can write the on shell phase space measures as

$$d\Phi(p'_i) = \hat{d}^4 q_i \hat{\delta}(2p_i \cdot q_i + q_i^2) \quad (2.26)$$

and similarly for  $r_i$ . These mismatches are not the difference between ingoing and outgoing momenta in the scattering event, but rather the difference between the momenta in the wavepacket and the conjugate, so are actually both initial state momenta, and importantly both on shell. However in the amplitudes inside these expressions they show up as momentum transfers.

Using this and the overall momentum conserving delta function to set  $q_1 = -q_2 = q$  we find <sup>4</sup>

$$\begin{aligned} \Delta\mathcal{O} = & \int d\Phi(p_1, p_2) \varphi^*(p_1+q, p_2-q) \varphi(p_1, p_2) \int \hat{d}^4 q e^{iq \cdot b/\hbar} \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \\ & \times \left[ i\delta\mathcal{O}_q \langle \chi' | \mathcal{A}(p_1, p_2 \rightarrow p_1+q, p_2-q) | \chi \rangle \right. \\ & + \sum_X \int \hat{d}^4 w_1 \hat{d}^4 w_2 d\mu(\xi_1, \xi_2) \hat{\delta}(2p_1 \cdot w_1 + w_1^2) \hat{\delta}(2p_2 \cdot w_2 + w_2^2) \delta^4(w_1 + w_2 + r_X) \\ & \left. \times \langle \chi' | \mathcal{A}^*(p'_1, p'_2 \rightarrow r_1, r_2, r_X) | \xi \rangle \delta\mathcal{O}_{rp} \langle \xi | \mathcal{A}(p_1, p_2 \rightarrow p_1+w_1, p_1+w_2, r_X) | \chi \rangle \right]. \end{aligned} \quad (2.27)$$

The term in square brackets we shall refer to as a “generalised impulse kernel”, and will denote it  $\mathcal{I}_{\mathcal{O}}$ . Most of our efforts will be directed to computing these.

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<sup>4</sup>In some cases, such as when there is an external emitted messenger particle, it is better to leave  $q_1$  and  $q_2$  separate. We will encounter these when we look at some radiative observables later.

This is a fully quantum expression written in terms of scattering amplitudes. We will mostly refer to the two terms as ‘linear’ and ‘quadratic’ however they are also sometimes referred to as virtual and real respectively, borrowing terminology from the study of cross section computations. Indeed the integral making up the virtual kernel resembles the kind of integral that shows up in the phase space integral of a cross section. Real versus virtual simply refers to the state of the ‘exchanged’ particles. The linear integrand involves a loop amplitude with off shell (virtual) exchanged particles, while in the quadratic part the exchanged particles are on shell. Technically the quadratic integrand is a product of amplitudes, but this can be viewed as a cut of a higher loop amplitude, which can in turn be viewed as a loop amplitude with on shell internal lines. For this reason we may also refer to this contribution as a cut contribution. Incidentally this distinction between real and virtual kernels turns out often not to be that useful from a computational perspective. Rather it can be more helpful to split and mix these and re-express in terms of real and imaginary parts of the amplitudes, often removing the need to explicitly compute cut phase space integrals <sup>5</sup>.

These kernels in eq. (2.27) have a diagrammatic depiction below.

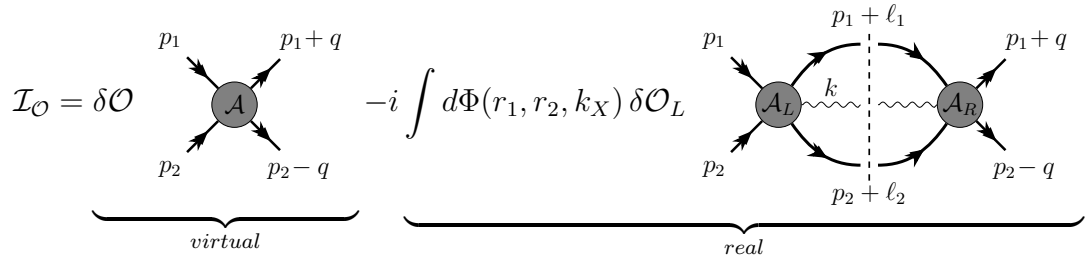


**Figure 2.2** *Generalised impulse kernel diagrammatically*

### 2.1.3 Taking the Classical limit

The final task is to extract the classical limit. Naively this is simply the limit of  $\hbar \rightarrow 0$ , however it requires a careful consideration of the scales and  $\hbar$  dependence of different variables. When implementing this limit in calculations there are subtle cancellations between terms which appear to be singular in the  $\hbar \rightarrow 0$  limit.

<sup>5</sup>The two uses of the terminology ‘Real’ are distinct and unrelated. Real in this context is the complex number real not on/off shell real as used earlier.

First consider the coupling  $g$ . In electrodynamics this is  $e$ , while in gravity it is  $\kappa = \sqrt{32\pi G}$ . The dimensionless coupling is  $g/\sqrt{\hbar}$ , so in all expressions with amplitudes we must replace  $g \rightarrow g/\sqrt{\hbar}$ . Next for all momentum mismatches, loop momenta and massless momenta are re-scaled to wavenumbers:  $q \rightarrow \hbar \bar{q}, \ell \rightarrow \hbar \bar{\ell}, k \rightarrow \hbar \bar{k}$  [35]. The intuition here is that classically, massless particles are described as waves rather than particles, and thus a wavenumber rather than a momentum. Removing this factor of  $\hbar$  turns the momentum to a wavenumber. We keep these barred quantities order 1 while we send  $\hbar \rightarrow 0$ .

In computations of kernels we must Laurent expand in powers of  $\hbar$ . This means that first we must make use of the on-shell conditions by replacing, for example,  $2p_1 \cdot \bar{q} \rightarrow -\hbar \bar{q}^2$ , and then do the expansion. In the end the classical observable will be uniform in  $\hbar$ , and at this point we drop any  $\hbar$  suppressed terms. In particular inside the on-shell delta functions in the observable, integrated against the kernel, we can drop the  $\hbar \bar{q}^2$  term.

The loop momentum rescaling can be seen as a consequence of the  $q$  shift scaling,  $q \rightarrow \hbar \bar{q}$ , arising in the Fourier transform. This can be interpreted in the context of the ‘Method of Regions’ [111, 162, 164] used for computing loop integrals. The idea of this is as follows; first note that the integration region for the loop integrals is only up to a cutoff at the scale of the mass of the massive particles. This is because we’re working in the classical regime below the particle creation threshold – in the gravitational context we are not trying to pair produce black holes. Then we split the integration domain into a hard and a soft region, where the loop momentum scales with mass, or is much smaller. This will change how we should expand the integrands; in the hard region we expand some massless propagators which will give terms with positive powers of  $q^2$ , while in the soft region we expand massive propagators which will give some negative powers of  $q^2$ . It is only the latter that survive in our long range perturbative regime <sup>6</sup>. Thus the only relevant part of the integral comes from when  $\ell \sim q$ , and thus we should also re-scale  $\ell \rightarrow \hbar \bar{\ell}$  as well as for  $q$ . The soft region can be further subdivided into potential and radiation regions, however we won’t be making use of this separation. See [162] for details.

This procedure of rescaling momenta associated with the gauge bosons is related to a soft limit. That is where momenta of emitted/exchanged gauge bosons is much smaller (softer) than the massive particle momenta (hard). The difference

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<sup>6</sup>See section 2.1.4 for a discussion

is that we take all massless momenta to be small, rather than just one. In our set up  $\hbar$  serves as a way of counting the softness. There will be apparent divergences which will often appear in intermediate steps and we must ensure they cancel in the final observables. We will refer to these as “super classical”<sup>7</sup> We will see another incarnation of the soft limit expansion when we discuss the eikonal approximation shortly.

With these scalings the conjugate wavepackets associated with the momentum  $p' = p \pm q$  can be approximated by the original ones, since the momenta  $q_i$  are negligible compared to the width of the wavefunctions,

$$\varphi^*(p \pm \hbar \vec{q}) \simeq \varphi^*(p). \quad (2.28)$$

This simplifies the expression for the observables (2.27) by making the replacement

$$\varphi^*(p_1 + q, p_2 - q) \varphi(p_1, p_2) \simeq |\varphi(p_1, p_2)|^2. \quad (2.29)$$

Then integrating over these sharply peaked wavepackets, which are very sharply peaked around the classical momenta, simply has the effect of replacing integration variables by their classical values. Similarly, as the observable has the structure of a convolution of the sharply-peaked wavefunctions multiplied by delta functions and otherwise smooth functions, we may neglect the  $q^2$  shifts inside the delta functions<sup>8</sup>. With this we introduce the notation

$$\left\langle\!\left\langle f(p_1, \dots) \right\rangle\!\right\rangle \equiv \int d\Phi(p_1, \dots) |\varphi(p_1, \dots)|^2 \langle \vec{\chi} | f(p_1, \dots) | \vec{\chi} \rangle. \quad (2.30)$$

The states  $|\chi\rangle$  are as before in eq. (2.22), however for a proper classical limit, there are extra conditions that must be satisfied. It is important that the  $|\chi\rangle$  are the appropriate “classical states” which we will discuss shortly. This notation instructs us to perform the  $\hbar$  scaling’s just described, neglect the  $q^2$  shifts, and evaluate the phase space integrals by replacing momenta and other external variables by their classical values, i.e.  $p_i \rightarrow m_i u_i$ . With this we can write the observable (2.27) as

$$\Delta\mathcal{O} = i \left\langle\!\left\langle \int \hat{d}^4 q \delta(2p_1 \cdot q) \delta(2p_2 \cdot q) e^{iq \cdot b/\hbar} \mathcal{I}_{\mathcal{O}} \right\rangle\!\right\rangle. \quad (2.31)$$

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<sup>7</sup>Some authors use ‘hyperclassical’ or simply ‘classically singular’, but we will stick with superclassical.

<sup>8</sup>The  $q^2$  shifts are still important and we impose this on shell condition in the integrand, allowing us to neglect the shift in the delta function.

It's helpful to remember the kernel is schematically

$$\mathcal{I}_{\mathcal{O}} = \delta\mathcal{O}\mathcal{A} - i \int \delta\mathcal{O}\mathcal{A}\mathcal{A}^* = \mathcal{I}_v + \mathcal{I}_r, \quad (2.32)$$

and diagrammatically was shown in section 2.1.2.

Removing factors of  $\hbar$  according to the rules above, the classical kernel and observable are

$$\Delta\mathcal{O} = i \left\langle\left\langle \hbar^2 \int \hat{d}^4 \bar{q} \delta(2p_1 \cdot \bar{q}) \delta(2p_2 \cdot \bar{q}) e^{i\bar{q} \cdot b} \mathcal{I}_{\mathcal{O}}|_{class} \right\rangle\right\rangle. \quad (2.33)$$

### 2.1.4 Simplifications and comments

There are a variety of important simplifications that can drastically reduce the size of the expressions that show up in these KMOc calculations. Firstly when performing the  $\hbar$  expansion, we can immediately drop any terms which are analytic in  $q$ , that is any term in the kernel of the form positive powers of  $q^2$ . These, when Fourier transformed, will produce terms in the observables involving delta functions of the impact parameter, or derivatives thereof. Since at the very start we made the assumption that the impact parameter was large, these delta functions can be ignored. They correspond to short range dynamics, but we are only interested in the long range data.

At one loop, which is the highest order we will go to in this thesis, similar statements hold. There are a selection of integrals which will vanish or are known to give quantum corrections. These occur when a massless propagator is cancelled by a numerator term, reducing the integral to a scaleless integral which vanishes in dim reg, or corresponds to a 'contact term' which is quantum.

In many calculations it will be necessary to use the following identity to combine propagators into delta functions

$$i\hat{\delta}(x) = 2\pi i\delta(x) = \frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon}. \quad (2.34)$$

This is sometimes referred to as “reverse unitarity”, since it can be viewed as combining two regular propagators into a cut propagator. It's particularly helpful and necessary when massaging expressions to show that superclassical terms cancel.

Finally it can be helpful to rewrite the kernels in terms of real and imaginary parts of amplitudes. We will make use of this in chapter 6. We will see in more detail how these work when they are used in action throughout the thesis.

### 2.1.5 Summary - The KMOC recipe

To summarise this section I will recap and provide a “step by step KMOC recipe” for computing classical observables.

#### KMOC recipe

1. Set up in-states of the form in equation (2.11) and choose an observable.
2. Compute observable as in (2.19) and express in terms of scattering amplitudes (2.24).
3. Define momentum shifts  $q_i = p'_i - p_i$  and convert final state  $p'$  integrals to  $q$  integrals (2.27).
4. Re-scale all momentum shifts and massless loop momenta.  $q \rightarrow \hbar \bar{q}, \ell \rightarrow \hbar \bar{\ell}$ . The barred quantities are wave numbers and remain finite in the classical limit, while we take  $\hbar \rightarrow 0$ .
5. Compute the necessary amplitudes and expand as a Laurent series in  $\hbar$ . Take care with massive propagators, making use of the on-shell conditions. The  $\mathcal{O}(\hbar^0)$  is the classical part. Be careful with ‘superclassical’ parts with negative powers of  $\hbar$  - these should either cancel or vanish. If they don’t then something has gone wrong.
6. Evaluate wavepacket averaging double brackets by replacing external momenta and any other degrees of freedom by their classical values. e.g.  $p_i \rightarrow m_i u_i$ . (2.30)
7. Perform loop and Fourier integrals.

## 2.2 Coherent states

In this section I will review the basics of coherent states. These have a long history in applications to semi-classical physics [165, 166] as well as more recent interest [1, 75, 144, 163, 167]. They are states which quite generally [165, 168] give a best possible description of the classical dynamics of a quantum state — they are the most classical quantum states — and thus will form a vital part of this thesis. These were first studied by Schrödinger looking at solutions for the simple harmonic oscillator (SHO) [169], and later studied and generalised by many others. These states can be viewed as the vacuum solution that is displaced by a fixed distance and have two key features,

1. they minimise the uncertainty conditions, for example position and momentum in case of SHO,
2. the expectation values evolve according to the classical equations of motion.

Later, states of this type were studied and generalised by Glauber [170, 171], particularly in the context of optics. It was Glauber who introduced the term coherent, motivated by this optical application. This notion of coherent state was further generalised by Perelomov who defined coherent states for a general Lie group [172, 173]. Right now I will describe the Glauber coherent states, then the Perelomov generalisation and also discuss Schwinger boson formalism. The Perelomov generalisation, along with Schwinger formalism, will be used when constructing  $SU(3)$  coherent states in chapter 4.

### 2.2.1 Coherent states: the basics

Consider a pair of creation and annihilation operators<sup>9</sup> with standard commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (2.35)$$

Define a ‘displacement operator’  $D(\alpha)$  by

$$D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \quad (2.36)$$

---

<sup>9</sup>Now these are standard quantum mechanics operators rather than field theory ones. We also set  $\hbar = 1$  in this section.

where  $\alpha$  is a complex number. This operator is unitary and satisfies some simple relations <sup>10</sup>

$$\begin{aligned} D(\alpha)^\dagger \hat{a}^{(\dagger)} D(\alpha) &= \hat{a} + \alpha^{(*)} \\ D(\alpha) D(\beta) &= D(\alpha + \beta) e^{i \operatorname{Im} \alpha \beta^*}. \end{aligned} \quad (2.37)$$

The first of these relations provides the origin of the name displacement operator. Then we define the coherent state  $|\alpha\rangle$  by acting on the vacuum with this operator,

$$|\alpha\rangle = D(\alpha) |0\rangle. \quad (2.38)$$

This can be recast using the Baker-Campbell-Hausdorff (BCH) formula, which in this situation takes the form

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \quad (2.39)$$

This version relies on the fact that the commutator commutes with each operator  $[[A, B], A] = [[A, B], B] = 0$ . A second form of BCH formula is

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]]. \quad (2.40)$$

Using the first of these we find

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle. \quad (2.41)$$

The first key feature these states have (with either normalisation) is that they are eigenstates of the annihilation operator with eigenvalue  $\alpha$

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle. \quad (2.42)$$

In some texts this eigenstate property is taken as the starting definition of a coherent state and then it is shown that it can be written in the form above. With this definition, coherent states are often introduced simply via the exponential of the creation operator, without the  $e^{-|\alpha|^2/2}$  prefactor. This is an equally good definition but just has a different normalisation; the states defined via the displacement operator are normalised to unity.

With the state in exponential form we can see that the lowering operator acts as a derivative with respect to the raising one  $\hat{a} \rightarrow d/d\hat{a}^\dagger$ . I will now list and describe some of the other features:

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<sup>10</sup>Ref [160] provides an introduction, in similar style to what is presented here.

- They can be expressed in a basis of number operator states by expanding the exponential

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \text{with} \quad |n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle. \quad (2.43)$$

This is a key point. We can view a coherent state as an infinite superposition of states with arbitrary occupation numbers. It is these high occupation number states which are responsible for the classical behaviour.

- They are not orthogonal. With the normalisation used here we have

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{|\alpha - \beta|^2}{2}\right). \quad (2.44)$$

- They form an over-complete set. This can be seen from the two features above and means that it is possible to express a coherent state as a sum of other coherent states

$$|\beta\rangle = \sum c_n |\alpha_n\rangle. \quad (2.45)$$

They still however have a completeness relation which is written

$$\mathbb{1} = \frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle\alpha|. \quad (2.46)$$

- The number operator expectation and variance is given by  $|\alpha|^2$ , and furthermore the distribution of number eigenstates in the coherent state is Poisson distributed, that is

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2} \langle n\rangle^n}{n!}. \quad (2.47)$$

The final key feature is the minimal uncertainty - it was this that motivates using coherent states in this thesis. However to demonstrate this property we need to have specific operator(s) in mind. Later in the thesis we will look at specific examples, but for now if we go back to the quantum SHO then we can choose  $\alpha = x_0/d$  where  $d = 1/\sqrt{m\omega}$  is the typical length scale of the oscillator. Then the coherent state becomes simply the translation of the vacuum by distance  $x_0$

$$|x_0\rangle = \exp\{-ipx_0\} |0\rangle, \quad (2.48)$$

which satisfies  $\Delta X \Delta P = \hbar/2$ , thus saturating the minimum possible uncertainty.

This was so far a very basic introduction to the simplest coherent states, but we will need more general ones in this thesis. A generalisation was provided by Perelomov [172], which generalise the notion to a generic Lie group. This discussion can become very mathematical in nature so I will refer to the references above for more details. The essence of the construction is replacing  $D(\alpha) \rightarrow T(g)$ , where  $T(g)$  is some representation of the group element  $g$  in the Lie group.

### 2.2.2 Schwinger boson formalism

Schwinger provided a way to construct the algebra of angular momentum using a pair of ladder operators [174]<sup>11</sup>. Construction of states using this is relatively simple, even for other  $SU(N)$  groups. The  $SU(2)$  case is simplest because of the symmetry of indices in product representations, and the associated coherent states are often referred to as spin coherent states. Using the Schwinger boson formalism to construct coherent states for angular momentum was done in [176]. There are constructions for more general  $SU(N)$  irreducible representation states [177, 178] as well as coherent states [179–181]. More recently similar constructions have found application in amplitudes based approaches to classical observables in gravity [144]. I will describe the  $SU(3)$  coherent state construction explicitly in chapter 4. This is sufficiently complex to highlight the subtleties of larger groups but also still demonstrate the key features of coherent states that we want - namely the classical behaviour.

Consider a pair of uncoupled operators,  $\hat{a}_\pm, \hat{a}_\pm^\dagger$  which obey

$$[\hat{a}_+, \hat{a}_-] = [\hat{a}_+^\dagger, \hat{a}_-^\dagger] = 0, \quad [\hat{a}_+, \hat{a}_+^\dagger] = [\hat{a}_-, \hat{a}_-^\dagger] = 1. \quad (2.49)$$

We can now construct operators  $J_\pm, J_z$

$$J_+ = \hbar \hat{a}_+^\dagger \hat{a}_-, \quad J_- = \hbar \hat{a}_-^\dagger \hat{a}_+, \quad J_z = \frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-). \quad (2.50)$$

Equivalently we can write

$$J_i = \hat{a}_a^\dagger \sigma_{ab}^i \hat{a}_b \quad (2.51)$$

where we defined  $\hat{a}_{+(-)}^\dagger = \hat{a}_{1(2)}^\dagger$ . It is easy to verify that these obey the standard

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<sup>11</sup>See [175] for nice introduction to this.

commutation relations of angular momentum

$$[J_z, J_\pm] = \pm \hbar J_\pm \quad [J_+, J_-] = 2\hbar J_z. \quad (2.52)$$

If we define the number operators for each oscillator type  $N_+, N_-$  as

$$\hat{N}_+ = \hat{a}_+^\dagger \hat{a}_+, \quad \hat{N}_- = \hat{a}_-^\dagger \hat{a}_-, \quad \hat{N} = \hat{N}_+ + \hat{N}_-, \quad (2.53)$$

then the total square angular momentum operator takes the form

$$\hat{J}^2 = \hbar^2 \left( \frac{\hat{N}}{2} \right) \left( \frac{\hat{N}}{2} + 1 \right). \quad (2.54)$$

This is very reminiscent of the expression  $\langle J^2 \rangle = \hbar^2 j(j+1)$  and suggests  $j \sim \langle N \rangle / 2$ . More precisely, define tensor states

$$|n_+, n_-\rangle = |n_+\rangle |n_-\rangle = \frac{(\hat{a}_+^\dagger)^{n_+} (\hat{a}_-^\dagger)^{n_-}}{\sqrt{n_+!} \sqrt{n_-!}} |0\rangle \quad (2.55)$$

which are eigenstates of each  $\hat{N}_\pm$  individually. These transform in the spin  $n = 2j$  representation. This can be seen by acting on these states with  $\hat{J}^2$  and  $\hat{J}_z$ , finding that the usual  $J^2, J_z$  basis states  $|j, m\rangle$  correspond to  $n_+ = j + m, n_- = j - m$  <sup>12</sup>

$$|j, m\rangle = \frac{(\hat{a}_+^\dagger)^{j+m} (\hat{a}_-^\dagger)^{j-m}}{\sqrt{(j+m)!} \sqrt{(j-m)!}} |0\rangle. \quad (2.56)$$

The intuition here is that the plus type operators create and destroy spin up states, while the minus type create/destroy spin down states.

More generally speaking note that say the irreducible representations of  $SU(2)$  are all symmetrised tensor powers of the fundamental, so the state

$$a^{\dagger i_1} a^{\dagger i_2} \dots a^{\dagger i_{2j}} |0\rangle, \quad (2.57)$$

which is automatically symmetric in all its indices, transforms in the spin  $j$  representation. For groups larger than  $SU(2)$ , the construction of a general irreducible representation requires both symmetrisation and antisymmetrisation over appropriate sets of indices. This leads to expressions which are involved already for  $SU(3)$  [179, 180].

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<sup>12</sup>or equivalently  $2j = n_+ + n_-$ ,  $2m = n_+ - n_-$ .

Here we constructed the usual angular momentum states one works with, but we can also define coherent states in this formalism too. Since we've built these from creation/annihilation operators it is easy to apply the same construction for a coherent state. Since we have two sets of operators we need two complex numbers to define the state, we will use  $\alpha_1$  and  $\alpha_2$ . It's useful also to use a more compact notation writing as vectors  $\vec{\alpha} = (\alpha_1, \alpha_2)$ . Then a coherent state, which is an eigenstate of each lowering operator individually, can be defined very analogously to eq. (2.41)

$$\begin{aligned} |\vec{\alpha}\rangle &= \exp\{-(\alpha_1^2 + \alpha_2^2)/2\} \exp\{\alpha_1 \hat{a}_+^\dagger + \alpha_2 \hat{a}_-^\dagger\} |0\rangle \\ &= \exp\{-|\vec{\alpha}|^2/2\} \exp\{\vec{\alpha} \cdot \vec{\hat{a}}^\dagger\} |0\rangle. \end{aligned} \quad (2.58)$$

This construction can be generalised quite easily to  $SU(N)$ . We content ourselves with a brief discussion of the  $SU(3)$  case which captures all of the interesting features of the general case, which we present in section 4.1.3. There we will see that it is the large occupation number states dominate this classical behaviour. In some sense it is only these states that matter, and using coherent states is just a neat way to describe it and work with in calculations.

## 2.3 Eikonal exponentiation

We've established that the classical limit comes from essentially a soft expansion in the exchanged momenta. This soft limit is very closely related to the eikonal approximation. Eikonal methods have long been used to extract classical physics from quantum mechanics. Recent years have seen a renewed surge of interest in this approach, especially in the context of gravitational scattering [48, 57, 59, 71, 74, 76, 85, 87, 91, 124, 126, 161, 162, 182–194], though this has roots in earlier work [195–200]. Originally born out of the study of high energy/Regge scattering [201–208] where the Feynman diagrammatics dramatically simplify, eikonal physics now have much wider application. The simplification in this regime allows diagrams, expressed in impact parameter space rather than momentum space, to be summed exactly to an exponential form. This exponential depends on the  $2 \rightarrow 2$  scattering amplitude, and contains information about classical quantities such as the deflection angle. There are rich connections to soft/IR physics and Wilson lines [209–214] which lead to a formal

proof of the exponentiation quite generally [210]. Nowadays the exponentiation is taken as a starting point and applied to various scattering regimes. For a recent extensive review see [161]

### 2.3.1 Resummation of Amplitudes

The idea of eikonal resummation, or any resummation scheme for that matter, is that there are two scales or parameters in the problem. One being the coupling and one being something else, usually kinematical in nature, which in this case is the softness. It can in general however be any other scale that becomes large in some region. What one finds is that at the lowest order in the softness the amplitude can be resummed to all loops, i.e. powers of coupling.

So how does eikonal resummation work?

In the soft limit, if one is interested only in the leading soft divergence one can use effective so called eikonal Feynman rules. This amounts to taking the standard rules, and expanding in powers of the soft momenta and keeping only the leading term. This dramatically simplifies computations and allows one to show how the amplitudes resum into an exponential. Alternatively one can compute the full amplitude and then expand, but for computing leading divergences the former is simpler.

Let us now illustrate how the exponentiation comes about. Here I will demonstrate it using effective eikonal Feynman rules in scalar QED for simplicity. The 3 point vertex is

$$\begin{array}{c} \text{---} \overrightarrow{p} \text{---} \\ \uparrow q \\ \text{---} \overrightarrow{p-q} \text{---} \end{array} = 2p^\mu - q^\mu \rightarrow 2p^\mu \quad (2.59)$$

and massive propagators of the form become

$$\overrightarrow{p + \sum k} = \frac{i}{(p + \sum k)^2 - m^2 + i\epsilon} \rightarrow \frac{i}{2p \cdot \sum k + i\epsilon}. \quad (2.60)$$

Here  $\sum k$  is a sum of internal or external photon momenta that enter internal massive propagators.

We are interested in the 4-point amplitude  $\mathcal{A}(p_1, p_2 \rightarrow p_1 + q, p_2 - q)$ . Using these rules the leading tree amplitude, including factors of  $\hbar$  is<sup>13</sup>,

$$i\mathcal{A}_{(4,0)} = \text{diagram} = ie^2 \frac{4p_1 \cdot p_2}{\hbar^2 \bar{q}^2}. \quad (2.61)$$

Now we Fourier transform this in the plane perpendicular<sup>14</sup> to  $p_1$  and  $p_2$ , and call this  $\chi_0$

$$\begin{aligned} \chi_0(b) &\equiv \mathcal{FT}[\mathcal{A}_4] = \int \hat{d}^{d-2} q_\perp e^{iq \cdot b / \hbar} \mathcal{A}_{(4,0)} \\ &\equiv \int \hat{d}^d q \delta(2p_1 \cdot \bar{q}) \delta(2p_2 \cdot \bar{q}) e^{iq \cdot b / \hbar} \mathcal{A}_{(4,0)}(q) \\ &= 4e^2 p_1 \cdot p_2 \int \hat{d}^d q \delta(2p_1 \cdot \bar{q}) \delta(2p_2 \cdot \bar{q}) e^{iq \cdot b / \hbar} \frac{1}{q^2}. \end{aligned} \quad (2.62)$$

Now consider the one loop amplitude. The leading in soft limit parts come from the box and cross box diagrams. The triangles, involving the four point vertex, are suppressed by one power of  $\hbar$  relative to the box and cross box. Taking these two amplitudes, and making aggressive use of the on shell conditions and dropping anything that is beyond the leading eikonal — we drop anything that generates an extra factor of  $\hbar$ , for example in expanding massive propagators — we find

$$\begin{aligned} i\mathcal{A}_{(4,1)}|_{sc} &= \text{diagram} + \text{diagram} \\ &= e^4 \frac{(4p_1 \cdot p_2)^2}{\hbar^4} \int \hat{d}^4 \ell \frac{1}{\ell^2 (q - \ell)^2} \\ &\quad \times \left\{ \frac{1}{(2p_1 \cdot \ell + i\epsilon)(-2p_2 \cdot \ell + i\epsilon)} + \frac{1}{(2p_1 \cdot \ell + i\epsilon)(2p_2 \cdot (\ell - q) + i\epsilon)} \right\} \\ &= e^4 \frac{(4p_1 \cdot p_2)^2}{\hbar^4} \int \hat{d}^4 \ell \frac{1}{\ell^2 (q - \ell)^2} \frac{1}{(2p_1 \cdot \ell + i\epsilon)} \left\{ \frac{1}{(-2p_2 \cdot \ell + i\epsilon)} + \frac{1}{(2p_2 \cdot \ell + i\epsilon)} \right\} \\ &= e^4 \frac{(4p_1 \cdot p_2)^2}{\hbar^4} \int \hat{d}^4 \ell \frac{1}{\ell^2 (q - \ell)^2} \frac{1}{(2p_1 \cdot \ell + i\epsilon)} i\delta(2p_2 \cdot \ell). \end{aligned} \quad (2.63)$$

We used the on shell condition to replace  $2p_1 \cdot \bar{q} \rightarrow -\hbar \bar{q}^2$  inside the massive propagator, then dropped this term as it produces a higher order correction and

<sup>13</sup>Here we are just counting the soft scaling, so only send  $q \rightarrow \hbar \bar{q}$ , and leave the coupling alone.

<sup>14</sup>Many papers simply write  $d^{d-2} q_\perp$  without explaining the precise meaning and normalisation. In particular the factors of 2, masses (or energies in the massless case) in the delta functions are often scaled outside of the integral.

used the delta function relation (2.34) to produce the delta function. Now shifting  $\ell \rightarrow q - \ell$  and averaging over these equivalent integration variables, making the same on shell simplification again, we get another delta function

$$i\mathcal{A}_{(4,1)} = -e^4 \frac{(4p_1 \cdot p_2)^2}{\hbar^4} \frac{1}{2} \int \hat{d}^4 \ell \frac{1}{\ell^2(q - \ell)^2} \hat{\delta}(2p_1 \cdot \ell) \hat{\delta}(2p_2 \cdot \ell). \quad (2.64)$$

This is already starting to resemble the square of the tree amplitude, but it is a convolution rather than product. Performing a Fourier transform as above (2.62), turns the convolution into a product. Noticing the appearance of the 2-d integral in the loop momenta, we have

$$\begin{aligned} \mathcal{FT}[i\mathcal{A}_{(4,1)}] &= -\frac{(4p_1 \cdot p_2)^2}{2} \int \hat{d}^{d-2} \ell_\perp \hat{d}^{d-2} \bar{q}_\perp \frac{1}{\ell^2(q - \ell)^2} e^{iq \cdot b} \\ &= -\frac{(4p_1 \cdot p_2)^2}{2} \int \hat{d}^{d-2} \ell_\perp \frac{1}{\ell^2} e^{i\ell \cdot b} \int \hat{d}^{d-2} \bar{q}_\perp \frac{1}{q^2} e^{iq \cdot b} \\ &= -\frac{1}{2} \chi_0^2(b), \end{aligned} \quad (2.65)$$

where we relabelled  $q \rightarrow q' = q - l$  and then dropped the prime. Note we have dropped many terms that contribute at the next order in softness; we are only interested in the most singular parts for now. We see that this is exactly the product of the tree amplitude, up to the factor of  $1/2$  and the sign. This is important. If we were to look at 2 loops, the story is very similar. The leading divergence comes from the double box and associated crossings, all built from three point vertices only. Then Fourier transforming these can be massaged to find it is the cube of the tree Fourier transform, with a factor of  $1/3!$ . More generally these diagrams giving the leading contribution are referred to as ladder and crossed ladder diagrams, as shown in figure 2.3.

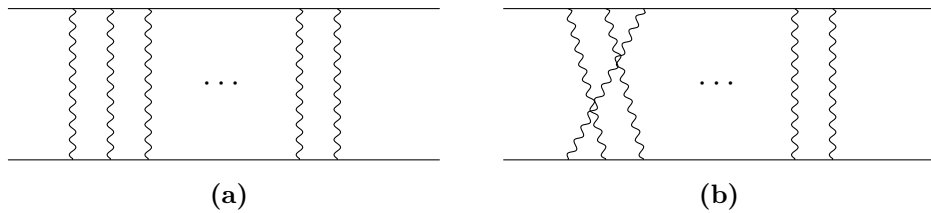


**Figure 2.3** *Example ladder and crossed ladder diagrams contributing to the leading eikonal.*

Computing these we find

$$\mathcal{FT}[i\mathcal{A}_{4,L}(q)] = i\tilde{\mathcal{A}}_{4,L}(b) = \frac{i^{L+1}}{(L+1)!} \mathcal{FT}[\mathcal{A}_{4,0}(q)]^{L+1}. \quad (2.66)$$

These are exactly the terms in an exponential expansion, so we can write the position space amplitude as an exponential of the leading eikonal phase.

$$1 + i\tilde{\mathcal{A}}_4(b)|_{eik} = e^{i\chi_0(b)} \quad (2.67)$$

with  $\chi_0$  defined in eq. (2.62). This is called the leading eikonal exponentiation, and  $\chi_0$  is the leading eikonal phase. We will find it helpful to remove one factor of  $\hbar$  from the eikonal phase, so that (2.67) becomes  $1 + i\tilde{\mathcal{A}} = e^{i\chi/\hbar}$ .

What about the subleading one-loop corrections from triangle type diagrams? These are suppressed by one power of  $\hbar$  relative to the dominant terms. It would be nice if they also form an exponential. They do in fact, though only partially. It is known that some terms do, in particular the triangle diagrams, but there are also remainder terms which do not exponentiate. These are quantum [210, 211]. The subleading eikonal,  $\chi_1$  is obtained from triangle type diagrams [75, 183],

$$\begin{aligned} \chi_0(b) &= \mathcal{FT} \left[ \text{triangle diagram} \right] \\ \chi_1(b) &= \mathcal{FT} \left[ \text{triangle diagram} + \text{triangle diagram} \right]. \end{aligned} \quad (2.68)$$

If we look at the schematic form of the amplitudes and their impact parameter space cousins, we see how this pattern is quite reasonable. Writing the amplitudes in  $d = 4 - 2\epsilon$  dimensions <sup>15</sup>,

$$\begin{aligned} \mathcal{A}_0 &= \frac{a_{(0,cl)}}{\hbar^3 q^2} \\ \mathcal{A}_1 &= \frac{a_{(1,sc)}}{\hbar^4 (q^2)^{1+\epsilon}} + \frac{a_{(1,cl)}}{\hbar^3 (q^2)^{\frac{1}{2}+\epsilon}} + \frac{a_{(1,qm)}}{\hbar^2 (q^2)^\epsilon} \\ \mathcal{A}_2 &= \frac{a_{(2,ssc)}}{\hbar^5 (q^2)^{1+2\epsilon}} + \frac{a_{(2,sc)}}{\hbar^4 (q^2)^{\frac{1}{2}+2\epsilon}} + \frac{a_{(2,cl)}}{\hbar^3 (q^2)^{2\epsilon}} + \frac{a_{(2,qm)}}{\hbar^3 (q^2)^{-\frac{1}{2}+2\epsilon}} + \dots \end{aligned} \quad (2.69)$$

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<sup>15</sup>There are also powers of  $\epsilon$  associated with  $\hbar$  but these have been omitted for simplicity. They vanish when we set  $d = 4$ .

and their Fourier transforms

$$\begin{aligned}
\tilde{\mathcal{A}}_0 &= \frac{\tilde{a}_{(0,cl)}}{\hbar(b^2)^{-\epsilon}} \\
\mathcal{A}_1 &= \frac{\tilde{a}_{(1,cl)}}{\hbar^2(b^2)^{-2\epsilon}} + \frac{\tilde{a}_{(1,cl)}}{\hbar(b^2)^{\frac{1}{2}-2\epsilon}} + \frac{\tilde{a}_{(1,qm)}}{(b^2)^{1-2\epsilon}} \\
\tilde{\mathcal{A}}_2 &= \frac{\tilde{a}_{(2,ssc)}}{\hbar^3(b^2)^{-3\epsilon}} + \frac{\tilde{a}_{(2,sc)}}{\hbar^2(b^2)^{\frac{1}{2}-3\epsilon}} + \frac{\tilde{a}_{(2,cl)}}{\hbar(b^2)^{1-3\epsilon}} + \frac{\tilde{a}_{(2,qm)}}{(b^2)^{\frac{3}{2}-3\epsilon}} + \dots
\end{aligned} \tag{2.70}$$

In this picture, the first column is the leading soft limit, coming from the ladder type diagrams and these all arise as iterations of  $\chi_0$ . Similarly the second column involves iterations of the  $\chi_1$  and  $\chi_0$ , and so on. Each row represents a given loop order, while each column represents a different order of soft divergence. This sort of picture is common to other resummation ideas, where computing loops (rows) becomes a bad approximation but instead we can resum entire columns (at all loops) into compact expressions. The classical terms all come with a factor of  $1/\hbar$  as we expect, and each higher perturbative order is suppressed by a factor<sup>16</sup> of  $1/b$ . By counting powers of  $\hbar$  and  $b$ , the iterative type structure becomes clear, at least on dimensional grounds. It is also reasonable to expect iterations of this type to occur by considering unitarity cuts [76, 124].

These observations suggest a general exponential structure for the eikonal

$$1 + i\tilde{\mathcal{A}}_4(b) = (1 + \Delta)e^{i\chi/\hbar} \tag{2.71}$$

where  $\chi = \chi_0 + \chi_1 + \dots$  and  $\Delta = \Delta_1 + \dots$  contains all the terms which do not exponentiate and are quantum corrections. Since these are quantum we will generally ignore these terms. If we expand both sides of eq. (2.71) and compare to eq. (2.70), we'd see a set of relations between the parts of amplitudes and their relation to  $\chi$  and  $\Delta$  [124, 161]

While it is not known if this formula eq. (2.71) holds to all orders generally, it has been formally proven at leading order (see for example [183, 210]), and has been successfully applied in a variety of calculations [52, 74, 91, 124, 185, 187, 188, 215]. At two loops and beyond, the eikonal, as defined here, acquires an imaginary part. This causes issues which we will address in chapter 5, with a proposed remedy to this situation.

Note that we've been working here with two massive particles. The same analysis

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<sup>16</sup>Remember there is power of  $g$  implicit inside the coefficients  $\tilde{a}_{(L)}$ , so is really  $g/b$  suppressed.

can be done for massless particles too, indeed this is more often studied, or at least the massless limit (ultra relativistic so essentially massless), for example [52, 85, 185].

### 2.3.2 Observables and stationary phase

We've argued that the leading eikonal four point amplitude can be written in terms of an exponential of a phase in position space. We generally prefer momentum space for explicit calculations, so we need to Fourier transform back to momentum space. We might also want to do this if we are computing some object integrating against some momentum states – indeed we will be doing this later in chapter 5. Suppose for now we simply want to Fourier transform the amplitude itself. The inverse transform must be 2 dimensional and happen in the same plane as before, so we write <sup>17</sup>

$$\begin{aligned}\mathcal{A}(q) &= \int d^{d-2}b e^{iq \cdot b} \tilde{\mathcal{A}}(b) \\ &= \int d^{d-2}b e^{iq \cdot b} \{e^{i\chi(b)} - 1\}.\end{aligned}\tag{2.72}$$

This integral in general will not be simple to do, and so we must resort to applying a stationary phase approximation [216]. While it may be tempting to expand the exponential and integrate order by order, this would only be undoing all of the work that was done to avoid individually divergent terms by resumming to an exponential.

The stationary phase condition is

$$\begin{aligned}0 &= \frac{\partial}{\partial b_\mu} (q \cdot b + \chi(b)) \\ \implies q_{stationary} &\equiv Q_\mu = -\frac{\partial \chi}{\partial b^\mu}.\end{aligned}\tag{2.73}$$

I've chosen the notation  $Q$  to signify that it is the value which satisfies the stationary phase condition. This is the value of momentum transfer in the large phase (small  $\hbar$ ) limit, which is precisely the impulse in the classical limit, thus we identify  $Q^\mu$  with the impulse. From this one can also extract the scattering angle

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<sup>17</sup>Now ignoring the quantum remainder term

too. Referring to fig. 2.4 we see that the angle is given by

$$\begin{aligned} \Delta p_1 &= |p_1| \sin(\theta/2) \\ \implies \theta &= \frac{1}{|p_1|} |\Delta p_1| = \frac{1}{p_1} \left| \frac{\partial \chi}{\partial b} \right|. \end{aligned} \quad (2.74)$$

In the last line we used the small angle approximation for simplicity.

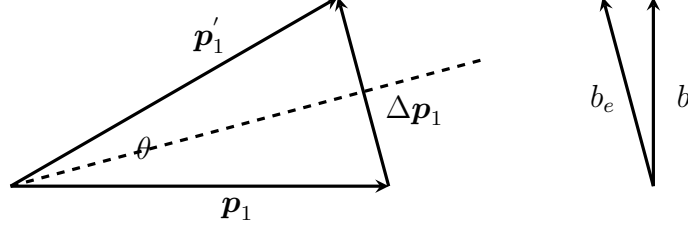


**Figure 2.4** *Relation of scattering angle, impulse and impact parameters in the eikonal setup.*

The usual perturbative expansion in the coupling  $g$  is a small angle expansion and by resumming to all orders in the coupling, we gain access to larger angles. This is the benefit of the eikonal resummation — it allows us to extend the range of validity from small to larger angles.

Computing the derivatives for the stationary phase conditions can be confusing and subtle, and great care must be taken, particularly if going beyond leading order. We will discuss this in detail in chapters 5 and 6. One must also be careful about the impact parameter. In the literature there is also a variety of notation used. The impact parameter in the eikonal, as defined above, lies naturally in the direction of  $q^\mu$ , and is often called the “eikonal impact parameter”. This is sometimes written  $b = b_e$ . This is not however the impact parameter associated with the initial kinematic setup which we’ve called  $b$ . Often this is referred to as  $b_J$  since it is associated with the initial angular momentum  $J = p_1 b_J$ . These differ by the  $\cos(\theta/2)$  as can be seen in figure 2.4. The differences are only important beyond leading order and this is discussed in chapter 5. There I will try to emphasise all the subtleties and provide a systematic guide to understanding it.

The fact that the eikonal phase seems to capture all the necessary classical information should not be surprising, indeed when considering the path integral formulation of quantum field theory, the stationary phase approximation to the path integral gives Hamilton’s action principle and thus all of classical mechanics. This also gives confidence that the classical information should exponentiate in

some form, whether that be eikonal or some other formulation.

### 2.3.3 Subtleties and other exponential schemes

Let us conclude with some remarks and comments on this exponentiation. We’ve already mentioned that there are terms in an amplitude which do not exponentiate. The form given in (2.71), is typically taken as the starting point by most people in the literature now.

There are also other exponentiation schemes that are similar but distinct from the eikonal. In the work of [192] they write the S-Matrix as an exponential of an operator  $N$ , rather than the usual  $1+iT$ ,  $\hat{S} = 1+i\hat{T} \rightarrow \exp(i\hat{N})$ . Matrix elements of  $\hat{N}$  relate to the radial action which is another related structure often studied in classical mechanics, but has shown up in other amplitudes based approaches through an “amplitude action relation” [113, 217, 218]. In a selection of works [64, 219–221], a so called HEFT<sup>18</sup> phase is defined. This is related to amplitudes in the heavy mass limit, rather than small momentum transfer as in the case of the eikonal. All these are similar and likely related, however how so is still not well understood. We shall focus on the eikonal in the rest of this thesis.

## Chapter Summary

In this chapter we introduced the KMOC formalism, explaining how to compute observables in terms of scattering amplitudes for a generic theory, and then a recipe for extracting the classical limit. Next we discussed coherent states, another key ingredient for ensuring sensible classical limits. These can be used along side KMOC for describing non kinematical degrees of freedom. Finally the notion of eikonal exponentiation was discussed, as an alternative approach for obtaining classical impulses and scattering angles from amplitudes. In the following chapters we will make use of all of these topics together, applying them to observables in Yang Mills theory, then combining KMOC and eikonal into a hybrid framework to include radiation. Finally we apply this to particles with spin and discuss the subtleties that arise there.

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<sup>18</sup>HEFT stands for Heavy Mass Effective Theory

# Chapter 3

## Uncertainty constraints

The quintessential difference between classical and quantum physics is uncertainty. Quantum observables naturally have a variance which is absent in classical physics. Standard textbook lore says that classical physics comes from tree level and that loops are quantum mechanical corrections [222, 223]. This is however not true, and can in fact be the reverse situation – classical loops and quantum trees. This has been studied before, for example [35]. To investigate this we take a look at uncertainty, or variance, in important classical observables using scattering amplitudes, following the methods of KMOC. In the classical limit the variance must be negligible. We find that this condition of negligible variance or uncertainty<sup>1</sup> generates an infinite set of relations which involve different amplitudes. More specifically, in the correspondence regime we can expand our amplitudes in powers of momentum transfer over centre-of-mass energy. This is a semi-classical expansion because the momentum transfers in KMOC are order  $\hbar^2$ . The zero-variance conditions relate certain terms in this “transfer” expansion between amplitudes with different numbers of loops and legs. For example, five-point amplitudes in the transfer expansion are related to lower loop five-point and four-point amplitudes.

At four points, the zero-variance conditions are very familiar: they are the relations required for eikonal exponentiation. Our work therefore shows that there is a generalisation of the eikonal formula beyond four points, which we will

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<sup>1</sup>Strictly speaking many of the objects we consider are not variances in the sense they are not of the form  $\langle X^2 \rangle - \langle X \rangle^2$ . More generally they are just classical uncertainty conditions.

<sup>2</sup>Alternatively one can think of the expansion as a large-mass expansion because the centre-of-mass energy is dominated by the particle masses.

come to in chapter 5. Where we outline the structure of the generalisation, which involves a coherent radiative state entangled with the four-point dynamics.

We begin this chapter in section 3.1 with a discussion of a basic requirement for a successful quantum description of such a classical event: negligible variance in a measurement of the field strength. As we will see, this requirement becomes a non-trivial constraint on scattering amplitudes. The leading obstruction is given by the six-point tree amplitude; this amplitude must be suppressed relative to the corresponding six-point one-loop amplitude. As a check we compute the tree amplitude, demonstrating explicitly that it has the required suppression. We build on this observation in section 3.2 to find an infinite series of constraints on multi-loop, multi-leg scattering amplitudes, verifying the first non-trivial constraint on the five-point one-loop amplitude in section 3.2.2.

### 3.1 Negligible Uncertainty

In classical electrodynamics, a key role is played by the field strength  $F_{\mu\nu}(x)$ . This object is a complete gauge-invariant characterisation of the field; once it is known, quantities such as the energy-momentum radiated to infinity and the field angular momentum are easily determined. In a quantum description, the field becomes an operator  $\mathbb{F}_{\mu\nu}(x)$ . In a semiclassical situation, the expectation value of this operator on a state  $|\psi\rangle$  should equal the classical field, up to negligible quantum corrections:

$$\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle = F_{\mu\nu}(x) + \mathcal{O}(\hbar). \quad (3.1)$$

Note that we have schematically indicated the presence of small, order  $\hbar$ , quantum corrections. More precisely, these corrections must be suppressed by dimensionless ratios involving Planck's constant; the precise ratios depend on the actual physical context.

Since in the quantum theory a single-valued field is replaced by the expectation value of an operator, we must address the quintessentially quantum mechanical issue of uncertainty. The uncertainty can be characterised by the variance

$$\langle\psi|\mathbb{F}_{\mu\nu}(x)\mathbb{F}_{\rho\sigma}(y)|\psi\rangle - \langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle\langle\psi|\mathbb{F}_{\rho\sigma}(y)|\psi\rangle. \quad (3.2)$$

In the domain of validity of the classical approximation, this variance must be

negligible.

This section is devoted to an investigation of this condition of negligible uncertainty. Working in the KMOC formalism at lowest order in perturbation theory, we will see that the expectations  $\langle \psi | \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) | \psi \rangle$  and  $\langle \psi | \mathbb{R}_{\mu\nu\rho\sigma}(x) \mathbb{R}_{\alpha\beta\gamma\delta}(y) | \psi \rangle$  are determined by tree-level *six*-point amplitudes while  $\langle \psi | \mathbb{F}_{\mu\nu}(x) | \psi \rangle$  and  $\langle \psi | \mathbb{R}_{\mu\nu\rho\sigma}(x) | \psi \rangle$  are determined by *five*-point tree amplitudes. We must then face the question of how it can be that the variance is negligible.

### 3.1.1 Field strength expectations

We begin by reviewing the evaluation of single field-strength observables in KMOC. Although this material appears elsewhere in the literature, the calculation will give us an opportunity to introduce notation which we will use in the remainder of our paper. We will discuss the electromagnetic case in some detail. The gravitational case is completely analogous to the electromagnetic case, so we only quote key results.

The electromagnetic field operator is

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_{\eta=\pm} \int d\Phi(k) \left[ a_\eta(k) \varepsilon_\mu^{(\eta)*}(k) e^{-i\vec{k}\cdot x} + \text{h.c.} \right]. \quad (3.3)$$

This is a sum over helicities  $\eta$ ; the notation h.c. indicates that the Hermitian conjugate of the explicit term should be included. Notice that the polarisation vectors  $\varepsilon^{(\eta)}(k)$  are homogeneous in the momentum, so they can equally be taken to be functions of the momentum or of the wavenumber.

Although we could choose to discuss the variance of the field  $\mathbb{A}_\mu$ , we focus on the gauge-invariant variance in the field strength operator  $\mathbb{F}_{\mu\nu}$ . The explicit expression for this operator follows from eq. (3.3) and can be written as

$$\mathbb{F}_{\mu\nu}(x) = \frac{1}{\sqrt{\hbar}} \sum_{\eta=\pm} \int d\Phi(k) \left[ -i a_\eta(k) \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*}(k) e^{-i\vec{k}\cdot x} + \text{h.c.} \right]. \quad (3.4)$$

Because we frequently encounter antisymmetric operators like this, we have defined the antisymmetrisation symbol

$$a_{[\mu} b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu \quad (3.5)$$

without any factor of  $1/2$ .

At this point we employ the KMOC setup to compute expectations. The incoming state is the 2 particle asymptotic state described in chapter 2, but a pure scalar i.e. no extra ‘colour’ type degrees of freedom

$$\begin{aligned} |\psi\rangle &= \int d\Phi(p_1, p_2) \phi(p_1, p_2) e^{ib \cdot p_1/\hbar} |p_1, p_2\rangle \\ &= \int d\Phi(p_1, p_2) \phi_b(p_1, p_2) |p_1, p_2\rangle . \end{aligned} \quad (3.6)$$

Now that we have an incoming state, we may write the outgoing state in terms of the time evolution operator  $U(+\infty, -\infty)$ , which is equal to the  $S$  matrix:

$$\begin{aligned} |\psi\rangle_{\text{out}} &= S |\psi\rangle \\ &= \int d\Phi(p_1, p_2) \phi_b(p_1, p_2) S |p_1, p_2\rangle . \end{aligned} \quad (3.7)$$

Since the  $S$  matrix can be written in terms of scattering amplitudes, the outgoing state itself can be written in terms of integrals over amplitudes and the incoming state.

Let us now consider the leading-order field strength in our situation. In the far past, we have

$$\begin{aligned} \langle\psi| \mathbb{F}_{\mu\nu}(x) |\psi\rangle &= \frac{1}{\sqrt{\hbar}} \sum_{\eta} \int d\Phi(k) \left[ \langle\psi| -ia_{\eta}(k) |\psi\rangle \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*}(k) e^{-i\bar{k}\cdot x} + \text{h.c.} \right] \\ &= 0 . \end{aligned} \quad (3.8)$$

The expectation value vanishes because there are no photons in the initial state:  $a_{\eta}(k) |\psi\rangle = 0$ . Classically, the interpretation is that the initial state contains no incoming radiation. Notice that the expectation value is not sensitive to the Coulomb fields of the incoming particles; instead, we are computing the asymptotic value of the field at infinity, namely the coefficient of the  $1/\text{distance}$  piece of the field strength.

In the far future, the expectation value is

$$\begin{aligned} \langle\psi| S^{\dagger} \mathbb{F}_{\mu\nu}(x) S |\psi\rangle &= \frac{1}{\sqrt{\hbar}} \sum_{\eta} \int d\Phi(k) \left[ -i \langle\psi| S^{\dagger} a_{\eta}(k) S |\psi\rangle \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*}(k) e^{-i\bar{k}\cdot x} + \text{h.c.} \right] . \end{aligned} \quad (3.9)$$

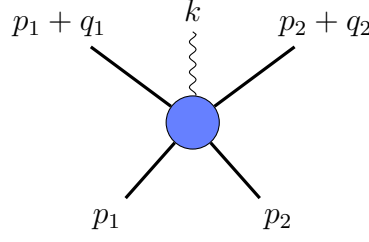


**Figure 3.1** *The kinematic configuration we choose for the five-point amplitude which determines the leading-order radiation field.*

This no longer vanishes. We may evaluate it at lowest perturbative order by writing the  $S$  matrix in terms of the transition matrix  $T$  as  $S = 1 + iT$ , and then expressing in terms of amplitudes. Generically we will denote the perturbative coupling as  $g$ . Thus we have

$$\begin{aligned} \langle \psi | S^\dagger a_\eta(k) S | \psi \rangle &= i \langle \psi | (a_\eta(k)T - T^\dagger a_\eta(k)) | \psi \rangle + \langle \psi | T^\dagger a_\eta(k) T | \psi \rangle \\ &= i \langle \psi | a_\eta(k) T | \psi \rangle + \langle \psi | T^\dagger a_\eta(k) T | \psi \rangle \\ &\simeq i \langle \psi | a_\eta(k) T | \psi \rangle . \end{aligned} \quad (3.10)$$

In the middle line above, we used the fact that  $a_\eta(k) | \psi \rangle = 0$ ; in the last line we neglected the term involving two  $T$  matrices which does not contribute at lowest order by counting powers of  $g$ .

Further expanding the state using eq. (3.6) we may write

$$\begin{aligned} \langle \psi | S^\dagger \mathbb{F}_{\mu\nu}(x) S | \psi \rangle &= 2 \operatorname{Re} \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(p'_1, p'_2, p_1, p_2, k) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) \\ &\quad \times \langle k^\eta, p'_1, p'_2 | T | p_1, p_2 \rangle \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*} e^{-i\bar{k} \cdot x} . \end{aligned} \quad (3.11)$$

The matrix element  $\langle k^\eta, p'_1, p'_2 | T | p_1, p_2 \rangle$  is, at lowest order, a five-point tree amplitude so it is proportional to  $g^3$ . This is consistent with a classical analysis of the outgoing radiation field.

It will be useful for us to simplify these expressions further, following again the discussion of reference [45] at leading order. The matrix element  $\langle k^\eta, p'_1, p'_2 | T | p_1, p_2 \rangle$  is the amplitude times a momentum-conserving delta function; our expectation value instructs us to integrate over all momenta in the amplitude. We may relabel these external momenta as shown in fig. 3.1.

Now we employ the classical limit procedure, described in section 2.1.3. Writing  $p'_1 = p_1 + q_1$ ,  $p'_2 = p_2 + q_2$ , the measure and wavepackets have shifts involving the

mismatches  $q_1$  and  $q_2$ . Both these are negligible and so we neglect them. Then performing the phase space integrals and using the double-angle bracket notation, reminding us to drop these  $q$  shifts and replace momenta by their classical values  $p_i \rightarrow m_i u_i$ . We thus write the field expectation value as

$$\begin{aligned} \langle \psi | S^\dagger F_{\mu\nu}(x) S | \psi \rangle &= 2 \operatorname{Re} \hbar^{7/2} \sum_{\eta} \left\langle\left\langle \int d\Phi(\bar{k}) \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p_1 \cdot \bar{q}_1) \hat{\delta}(2p_2 \cdot \bar{q}_2) \right. \right. \\ &\quad \times \mathcal{A}_{5,0}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^\eta) \hat{\delta}^4(\bar{k} + \bar{q}_1 + \bar{q}_2) \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*} e^{-i(\bar{k} \cdot x + \bar{q}_1 \cdot b)} \left. \right\rangle\right\rangle. \end{aligned} \quad (3.12)$$

We have scaled out all the  $\hbar$  dependence except that of the scattering amplitude itself. The amplitude  $\mathcal{A}_{5,0}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^\eta)$  is a five-point, tree amplitude. More generally, we will denote  $n$  point  $L$  loop amplitudes as  $\mathcal{A}_{n,L}$  (in gauge theory) or  $\mathcal{M}_{n,L}$  (in gravity).

For these expressions to make sense classically, it better be that the overall  $\hbar$  dependence of the amplitudes cancels that of the observable. Indeed we may write

$$\mathcal{A}_{5,0}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^\eta) = \hbar^{-7/2} \mathcal{A}_{5,0}^{(0)}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^\eta) + \mathcal{O}(\hbar) \quad (3.13)$$

where the quantity  $\mathcal{A}_{5,0}^{(0)}$  is independent of  $\hbar$ , as was noticed in [45]. We will return to this structure below.

The physical interpretation of these expectation values is that they compute the radiative part of the field at large distances. To see this explicitly, the  $\bar{k}$  integral needs to be performed taking advantage of the large distance between the point of measurement  $x$  and the particles. The integration can be performed using textbook methods and was recently reviewed in detail in ref. [163]. The question of central interest to us in this section, however, is to compute the uncertainty in the field strength; to do so, we turn to computing the expectation of two field strengths.

### 3.1.2 Expectation of two field strengths

Having computed a single expectation, we now need the second part of eq. (3.2) which is to compute the expectation of two operators. It will be quite straightforward for us to compute expectations of products of operators using

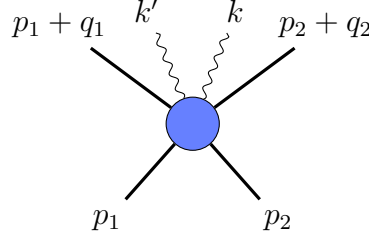


**Figure 3.2** *The kinematic configuration we choose for the six-point amplitude appearing at leading-order expectation of a pair of field strength operators.*

precisely the methods of the previous subsection. In electrodynamics, we need to compute

$$\begin{aligned} \langle \psi | S^\dagger \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) S | \psi \rangle &= -\frac{1}{\hbar} \sum_{\eta, \eta'} \int d\Phi(k', k) \langle \psi | S^\dagger \left[ -ia_\eta(k) \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*} e^{-i\bar{k} \cdot x} + \text{h.c.} \right] \\ &\quad \times \left[ -ia_{\eta'}(k') \bar{k}'_{[\rho} \varepsilon_{\sigma]}^{(\eta')*} e^{-i\bar{k}' \cdot y} + \text{h.c.} \right] S | \psi \rangle . \end{aligned} \quad (3.14)$$

Working at lowest order, and taking advantage of the fact that  $a_\eta(k) |\psi\rangle = 0$ , the expectation simplifies to

$$\begin{aligned} \langle \psi | S^\dagger \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) S | \psi \rangle &= -\frac{2}{\hbar} \text{Re} \sum_{\eta, \eta'} \int d\Phi(k', k) \langle \psi | a_\eta(k) a_{\eta'}(k') iT | \psi \rangle \\ &\quad \times \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*} \bar{k}'_{[\rho} \varepsilon_{\sigma]}^{(\eta')*} e^{-i(\bar{k} \cdot x + \bar{k}' \cdot y)} , \end{aligned} \quad (3.15)$$

up to a purely quantum single-photon effect [163]. Expanding the wavefunctions, we encounter the matrix element  $\langle p'_1, p'_2 | a_\eta(k) a_{\eta'}(k') T | p_1, p_2 \rangle$ : a six-point tree amplitude. The classical limit is determined precisely as in the previous section with the result

$$\begin{aligned} \langle \psi | S^\dagger \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) S | \psi \rangle &= -2\hbar^5 \text{Re} \sum_{\eta, \eta'} \left\langle \int d\Phi(\bar{k}', \bar{k}) \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \delta(2p_1 \cdot \bar{q}_1) \delta(2p_2 \cdot \bar{q}_2) \right. \\ &\quad \times i\mathcal{A}_{6,0} \hat{\delta}^4(\bar{k} + \bar{k}' + \bar{q}_1 + \bar{q}_2) \bar{k}_{[\mu} \varepsilon_{\nu]}^{(\eta)*} \bar{k}'_{[\rho} \varepsilon_{\sigma]}^{(\eta')*} e^{-i(\bar{k} \cdot x + \bar{k}' \cdot y + \bar{q}_1 \cdot b)} \left. \right\rangle . \end{aligned} \quad (3.16)$$

The amplitude is shown in figure 3.2.

The expectation of two field strengths is given to leading order in  $g$  by a tree-level six-point amplitude.

### 3.1.3 Negligible variance?

We have now seen explicitly that the expectation of a single field strength is determined by a five-point amplitude, while the expectation of two field strengths is a six-point tree amplitude at lowest order in the coupling  $g$ . But for the uncertainty in the field strength to be negligible, we need the variance to be negligible. How can this happen?

Let us count powers of the coupling in the electromagnetic variance. The product of two field-strength expectations is

$$\langle \psi | \mathbb{F}_{\mu\nu}(x) | \psi \rangle \langle \psi | \mathbb{F}_{\rho\sigma}(y) | \psi \rangle \sim (\mathcal{A}_{5,0})^2 \sim (g^3)^2. \quad (3.17)$$

while the expectation of two field strengths is

$$\langle \psi | \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) | \psi \rangle \sim \mathcal{A}_{6,0} \sim g^4. \quad (3.18)$$

Thus the situation seems to be very bad: the variance is dominated by the expectation of two field-strength operators! For the classical limit to emerge as expected, we need the six-point tree amplitude to be suppressed somehow.

One possibility is that it is suppressed by powers of  $\hbar$ , but naively that is not the case. Consider, for example, the 6-point Feynman diagram shown in figure 3.3. We can count the powers of  $\hbar$  associated to the diagram as follows. Each vertex contributes a factor  $\hbar^{-\frac{1}{2}}$ ; this is because the dimensionless coupling is  $e/\sqrt{\hbar}$  or  $\kappa/\sqrt{\hbar}$ . Each messenger propagator contributes a factor  $\hbar^{-2}$  because messenger momenta are of order  $\hbar$ . Meanwhile each massive propagator contributes a factor  $\hbar^{-1}$ ; this arises since the momenta flowing through these propagators are a sum of a massive on-shell momentum  $p$  and a messenger momentum  $k$ , so that the propagator denominator is  $(p+k)^2 - m^2 \simeq 2\hbar p \cdot \bar{k}$ . We conclude that the six-point tree amplitude contains terms of order  $\hbar^{-6}$ . Referring back to eq. (3.16) for the expectation of two field strengths, we see that the observables contribute a total of  $\hbar^{+5}$ . Based on this counting, the observable seems to scale as  $\hbar^{-1}$ , which would be a serious obstruction to the emergence of a classical limit. Evidently there is more to understand here: the powers of  $\hbar$  don't make sense.

It is a familiar story that power counting Feynman diagrams can be misleading: upon combining diagrams to evaluate an amplitude, there can be cancellations. In fact this already happens in the case of the five-point tree; there, naive power

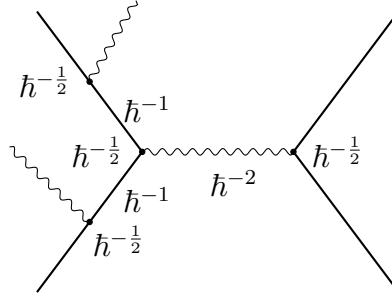


**Figure 3.3** *A sample Feynman diagram contributing to the six-point tree amplitude, indicating powers of  $\hbar$  assigned by naive power counting to the propagators and vertices.*

counting suggests that the amplitude scales as  $\hbar^{-9/2}$  but in fact the leading term in the amplitude is of order  $\hbar^{-7/2}$  [54]<sup>3</sup>. The question, then, of the fate of the six-point tree amplitude in the expectation value of two field strengths becomes a question of the overall  $\hbar$  scaling of six-point tree amplitudes in QED and gravity. We will shortly demonstrate explicitly that the QED amplitude in fact scale as  $\hbar^{-4}$ ; the gravitational case is discussed in reference [224]. Two powers of  $\hbar$  cancel; consequently the contribution of the six-point tree to the variance is entirely at the quantum level.

It is amusing that at next-to-leading order in the perturbative coupling  $g$ , namely order  $g^6$ , the expectation value of two field strengths is sensitive to one-loop six-point amplitudes and to products of two tree five-point amplitudes<sup>4</sup>. These products of tree-level five-point amplitudes can be viewed as the cut of a six-point one-loop amplitude. It is easy to check that these scale as  $\hbar^{-5}$ , so in this sense they are *enhanced* relative to the six-point tree amplitude. This is as desired for negligible uncertainty:

$$\begin{aligned} \langle \psi | \mathbb{F}_{\mu\nu}(x) | \psi \rangle \langle \psi | \mathbb{F}_{\rho\sigma}(y) | \psi \rangle &\sim (\mathcal{A}_{5,0})^2 \sim (g^3)^2; \\ \langle \psi | \mathbb{F}_{\mu\nu}(x) \mathbb{F}_{\rho\sigma}(y) | \psi \rangle &\sim \mathcal{A}_{6,1} \sim (\mathcal{A}_{5,0})^2 \sim (g^3)^2. \end{aligned} \quad (3.19)$$

Here we see, as promised, an explicit example of how a one loop amplitude dominates in the classical limit, and the tree level is quantum suppressed.

<sup>3</sup>In reference [54], the analysis of the five-point tree amplitude was performed in gravity using the large mass expansion, which is equivalent to expanding in small  $\hbar$ .

<sup>4</sup>We learned from G. Veneziano that the use of a cut 6-point function to compute observables which are quadratic in the fields (such as number or energy densities in inclusive cross-sections) goes back to Mueller’s generalized optical theorem from the seventies (see e.g. [225] and references therein).

### 3.1.4 Explicit six-point tree amplitudes

We now compute the leading contribution in  $\hbar$  of the six-point tree amplitude, in electrodynamics. The cancellation of excess  $\hbar$  is similar in gravity (see eg [224]).

Suppose particle 1 has charge  $Q_1$  while particle 2 has charge  $Q_2$ . Then there are three gauge-invariant six-point tree partial amplitudes:

$$\mathcal{A}_{6,0}(p_1 + q_1, p_2 + q_2 \rightarrow p_1, p_2, k) = Q_1^3 Q_2 A_{(3,1)} + Q_1^2 Q_2^2 A_{(2,2)} + Q_1 Q_2^3 A_{(1,3)}. \quad (3.20)$$

The “charge-ordered” partial amplitudes  $A_{(3,1)}$ ,  $A_{(2,2)}$ , and  $A_{(1,3)}$  are analogues of color-ordered amplitudes in gauge theory, which motivates our choice of notation.

Evidently there can be no cancellation of powers of  $\hbar$  between these partial amplitudes because of the different powers of the charges. Thus the problem reduces to computing the leading-in- $\hbar$  terms in these amplitudes. There are two partial amplitudes to consider, since  $A_{(1,3)}$  can be obtained from  $A_{(3,1)}$  by trivially swapping the labels 1 and 2.

Let us compute the partial amplitudes  $A_{(3,1)}$  and  $A_{(2,2)}$ . We do so making use of a convenient gauge which greatly reduces the labour necessary to see that two powers of  $\hbar$  do indeed cancel. The gauge we chose (referring to the momentum routing in equation (3.20)) is

$$p_1 \cdot \varepsilon(k_i) = 0 \text{ for } i = 1, 2, \quad (3.21)$$

where  $p_1$  is the momentum of particle 1 while  $k_i$  is the outgoing momentum of photon  $i$ .

The effect of this choice is twofold; it removes many diagrams from the calculation and those that remain get an  $\hbar$  enhancement from each emission vertex. For example, consider  $A_{(3,1)}$ : in this case, the emitted photon is radiated from particle 1. With our momentum labelling convention the emission vertices will produce factors of the form  $(2p_1 + \hbar \bar{Q}) \cdot \varepsilon(k_i)$ , where  $\bar{Q}$  is some combination of wavenumbers. The first part vanishes leaving only the  $\hbar$  enhanced  $\varepsilon(k_i) \cdot \bar{Q}$  term. Any diagram with a photon emitted from the outgoing line of particle 1 vanishes, as it is proportional to  $(2p_1 + \hbar \bar{k}_i) \cdot \varepsilon(k_i) = 0$ . In what follows we will write  $\varepsilon(k_i) = \varepsilon_i$ . We will also suppress the  $i\epsilon$  factors in the propagators<sup>5</sup>; they all

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<sup>5</sup>The  $i\epsilon$  factors are often important — and will play an important role in section 3.2.2 — but in this computation they are spectators.

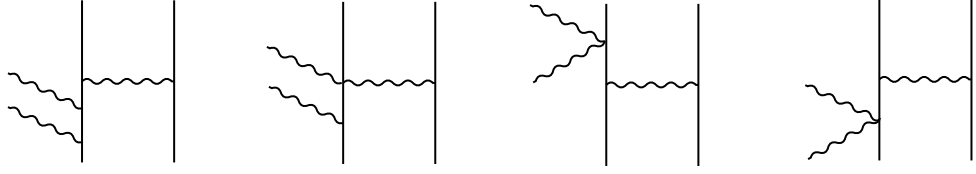


**Figure 3.4** *Diagrams in  $Q_1^3 Q_2$  sector.*

implicitly come with  $+i\epsilon$ . Finally we will refer to each diagram contributing to the amplitude by, for example,  $D_{(3,1)}$  so that

$$iA_{(3,1)} = iD_{(3,1)cubic} + iD_{(3,1)quartic}. \quad (3.22)$$

Some of the sub-amplitudes are given by a single diagram, whereas others are made up of multiple diagrams.

Our gauge choice is most powerful in the case of  $A_{(3,1)}$ , so we discuss this case in most detail. The Feynman diagrams that constitute this amplitude can be split into 3 classes: the first involves single photon emissions coming from cubic vertices, the second has precisely one photon emitted into the final state from a quartic vertex, while the third class has two photons emitted from the same quartic vertex. These classes are shown in figure 3.4, after removing diagrams which vanish by gauge choice. The first diagram is an example of the first class, the second an example of the second class and the last two diagrams are in the third class.

We choose a particular ordering of  $k_1$  and  $k_2$  for the calculation, and include the permuted case by swapping  $k_1 \leftrightarrow k_2$ . For the first class there is a single diagram to compute after gauge fixing, and it is trivial to write down the leading term in the amplitude and see that it has the desired scaling. This diagram is

$$\begin{aligned} D_{(3,1)}|_{cubic} &= \frac{1}{\hbar^4 \bar{q}_2^2} \left[ \frac{2(\varepsilon_2 \cdot \bar{q}_1)(\varepsilon_1 \cdot (2\bar{q}_1 - \bar{k}_2))(2p_2 + \hbar\bar{q}_2) \cdot (2p_1 - \hbar\bar{q}_2)}{(-2p_1 \cdot \bar{q}_2 + \hbar\bar{q}_2^2)(2p_1 \cdot (\bar{q}_1 - \bar{k}_1) + \hbar(\bar{q}_1 - \bar{k}_1)^2)} \right] \\ &= \frac{1}{\hbar^4 \bar{q}_2^2} \left[ \frac{4p_1 \cdot p_2 (\varepsilon_2 \cdot \bar{q}_1)(\varepsilon_1 \cdot (2\bar{q}_1 - \bar{k}_2))}{2(-p_1 \cdot \bar{q}_2)(p_1 \cdot (\bar{q}_1 - \bar{k}_1))} + \mathcal{O}(\hbar) \right] \\ &= \frac{2p_1 \cdot p_2}{\hbar^4 \bar{q}_2^2} \left[ \frac{(\varepsilon_2 \cdot \bar{q}_1)(\varepsilon_1 \cdot (\bar{q}_1 - \bar{q}_2))}{(p_1 \cdot (\bar{k}_1 + \bar{k}_2))(p_1 \cdot \bar{k}_1)} \right] + \mathcal{O}(\hbar^{-3}). \end{aligned} \quad (3.23)$$

We used momentum conservation  $q_1 + q_2 = k_1 + k_2$  to write our expressions in terms of just the  $k_i$  or the  $q_i$ . The choice here is most natural for obtaining a

similarly simple expression for the amplitude with  $k_1$  and  $k_2$  swapped.

The second class is actually tractable without fixing a gauge, as there is only a single cancellation to show. However with our gauge fixing this class becomes trivial, and there is only a single diagram to compute. This is

$$D_{(3,1)}|_{cubic/quartic} = \frac{4(p_2 \cdot \varepsilon_2)(\bar{q}_1 \cdot \varepsilon_1)}{\hbar^4 \bar{q}_2^2 p_1 \cdot \bar{k}_1}. \quad (3.24)$$

The final class is unaffected by our choice of gauge as it is proportional to  $\varepsilon_1 \cdot \varepsilon_2$ . After gauge fixing this is naively  $\hbar^{-5}$ , so we must find a single cancellation. The mechanism of the cancellation is identical to the case of the five-point tree amplitude. This is done in [45, 54], but we shall review it here for completeness. The key step is to make use of the on-shell conditions

$$(p_i + q_i)^2 = p_i^2 = m_i^2, \quad i = 1, 2, \quad (3.25)$$

which allows us, after the  $\hbar$  rescaling, to replace  $2p_i \cdot \bar{q}_i \rightarrow -\hbar \bar{q}_i^2$  in the propagators. Lastly, we Taylor expand. There are two diagrams to compute which are,

$$\begin{aligned} D_{(3,1)}|_{quartic,1} &= -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2} \left[ \frac{(2p_1 + \hbar(2\bar{q}_1 + \bar{q}_2)) \cdot (2p_2 + \hbar\bar{q}_2)}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2) + \hbar(\bar{k}_1 + \bar{k}_2)^2} \right] \\ &= -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2} \left[ \frac{4p_1 \cdot p_2}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} \right. \\ &\quad \left. + \hbar \left( \frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} - \frac{4p_1 \cdot p_2 (\bar{k}_1 + \bar{k}_2)^2}{(2p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} \right) \right] \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} D_{(3,1)}|_{quartic,2} &= -\frac{2\varepsilon_1 \cdot \varepsilon_2}{\hbar^5 \bar{q}_2^2} \left[ \frac{4p_1 \cdot p_2}{-2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} \right. \\ &\quad \left. + \hbar \left( \frac{4p_1 \cdot \bar{q}_2}{-2p_1 \cdot (\bar{k}_1 + \bar{k}_2)} + \frac{4p_1 \cdot p_2 ((\bar{k}_1 + \bar{k}_2)^2 - \bar{q}_1^2)}{(2p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} \right) \right]. \end{aligned} \quad (3.27)$$

Notice the most singular terms are equal up to a sign, and so cancel. Combining the remaining terms we obtain

$$D_{(3,1)}|_{quartic} = -\frac{\varepsilon_1 \cdot \varepsilon_2}{\hbar^4 \bar{q}_2^2} \left[ \frac{4(p_1 \cdot p_2)(\bar{q}_2 \cdot (\bar{k}_1 + \bar{k}_2))}{(p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} + \frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{p_1 \cdot (\bar{k}_1 + \bar{k}_2)} \right]. \quad (3.28)$$

These can be combined as  $A_{(3,1)} = D_{(3,1)cubic} + D_{(3,1)quartic}$  yielding,

$$\begin{aligned}
A_{(3,1)} = & \frac{1}{\hbar^4 \bar{q}_2^2} \left[ \frac{4p_1 \cdot p_2 (\varepsilon_2 \cdot \bar{q}_1) (\varepsilon_1 \cdot (\bar{q}_1 - \bar{q}_2))}{2(p_1 \cdot (\bar{k}_1 + \bar{k}_2)) (p_1 \cdot \bar{k}_1)} + \frac{4p_1 \cdot p_2 (p_2 \cdot \varepsilon_2) (\bar{q}_1 \cdot \varepsilon_1)}{p_1 \cdot \bar{k}_1} \right. \\
& - (\varepsilon_1 \cdot \varepsilon_2) \left( \frac{4(p_1 \cdot p_2) (\bar{q}_2 \cdot (\bar{k}_1 + \bar{k}_2))}{(p_1 \cdot (\bar{k}_1 + \bar{k}_2))^2} + \frac{4p_2 \cdot \bar{q}_1 + 2p_1 \cdot \bar{q}_2}{p_1 \cdot (\bar{k}_1 + \bar{k}_2)} \right) \Big] \\
& + (k_1 \leftrightarrow k_2).
\end{aligned} \tag{3.29}$$

The story is very similar for  $A_{(2,2)}$ . We can split into the same 3 classes, use gauge fixing to get rid of one factor of  $\hbar$  and then massage using the on-shell constraints to show the final cancellation. Here we just quote the result and give details in appendix B.1. The result is

$$\begin{aligned}
A_{(2,2)} = & \frac{4}{\hbar^4 (\bar{q}_2 - \bar{k}_2)^2} \left[ 4\varepsilon_1 \cdot \varepsilon_2 + \frac{(\varepsilon_1 \cdot p_2) (\varepsilon_2 \cdot \bar{q}_2)}{p_2 \cdot \bar{k}_2} - \frac{(\varepsilon_2 \cdot p_2) (\varepsilon_1 \cdot (\bar{q}_2 + \bar{q}_1))}{2p_2 \cdot \bar{k}_2} \right. \\
& \frac{p_1 \cdot p_2 (\varepsilon_1 \cdot \bar{q}_1) (\varepsilon_2 \cdot \bar{q}_2)}{(p_2 \cdot \bar{k}_2) (p_1 \cdot \bar{k}_1)} - \frac{(p_1 \cdot \bar{k}_2) (\varepsilon_1 \cdot \bar{q}_1) (\varepsilon_2 \cdot p_2)}{(p_2 \cdot \bar{k}_2) (p_1 \cdot \bar{k}_1)} \\
& \left. - \frac{(\varepsilon_1 \cdot p_2) (\varepsilon_2 \cdot p_2) (\bar{q}_2 \cdot \bar{k}_2)}{(p_2 \cdot \bar{k}_2)^2} - \frac{p_1 \cdot p_2 (\varepsilon_1 \cdot \bar{q}_1) (\varepsilon_2 \cdot p_2) \bar{q}_2 \cdot \bar{k}_2}{(p_2 \cdot \bar{k}_2)^2 (p_1 \cdot \bar{k}_1)} \right] \\
& + (k_1 \leftrightarrow k_2).
\end{aligned} \tag{3.30}$$

Finally  $A_{(1,3)}$  can be obtained by swapping the labels  $1 \leftrightarrow 2$  in the expression for  $A_{(3,1)}$  (written in the gauge where  $p_2 \cdot \varepsilon_i = 0$ . Of course the partial amplitudes themselves are gauge-invariant.) In all cases, the  $\hbar$  scaling is as required from negligible variance.

## 3.2 Transfer relations

Our discussion so far reveals that scattering amplitudes, viewed as Laurent series in  $\hbar$ , obey certain properties which permit the emergence of a classical limit through negligible uncertainty. This Laurent expansion can also be viewed as an expansion in small momentum transfers divided by the centre-of-mass energy  $\sqrt{s}$ . In fact, the emergence of the classical limit imposes an *infinite* set of these relationships, which we will call “transfer relations” on scattering amplitudes. In this section we will describe the origin of these relations, and explicitly demonstrate a non-trivial example at one loop and five points.

### 3.2.1 Mixed variances

To see where these relationships are coming from, recall that the double field-strength expectation (3.16) depends on a six-point amplitude. We have seen that the dominant term is actually the six-point one-loop amplitude, occurring at next-to-leading order in the expansion in  $g$ . At this order an additional term contributes to the double field-strength expectation; this term is the product of two five-point amplitudes. Now, negligible uncertainty demands that the complete double field-strength expectation must be the product of two single field-strength expectations. At leading order in the coupling, and leading non-trivial order in  $\hbar$ , we conclude that there must exist a relationship between the leading-in- $\hbar$  six-point one-loop amplitude and the product of two five-point trees.

Further examples of relationships between amplitudes can be obtained by considering expectations of three (or more) field strengths, leading to relationships between seven- (or higher-) point loop amplitudes and products of three (or more) five-point amplitudes.

Yet more relationships occur by considering expectations of products of operators including field strengths and momenta. For example, consider the variance

$$V_{\mu\nu\rho} \equiv \langle \psi | S^\dagger F_{\nu\rho}(x) S P_\mu | \psi \rangle - \langle \psi | S^\dagger F_{\nu\rho}(x) S | \psi \rangle \langle \psi | P_\mu | \psi \rangle . \quad (3.31)$$

This is the variance in a measurement of the initial momentum and the future field strength; it must be negligible in the classical regime. In a quantum-first approach, however, this variance will not vanish. Indeed it need not even be real:

$$V_{\mu\nu\rho}^* = \langle \psi | P_\mu S^\dagger F_{\nu\rho}(x) S | \psi \rangle - \langle \psi | S^\dagger F_{\nu\rho}(x) S | \psi \rangle \langle \psi | P_\mu | \psi \rangle \neq V_{\mu\nu\rho} . \quad (3.32)$$

We can derive an interesting constraint on the five-point one-loop amplitude by demanding that imaginary part of this variance vanishes in the classical approximation. We therefore define

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho} &= i(V_{\mu\nu\rho}^* - V_{\mu\nu\rho}) \\ &= i \langle \psi | P_\mu S^\dagger F_{\nu\rho}(x) S - S^\dagger F_{\nu\rho}(x) S P_\mu | \psi \rangle . \end{aligned} \quad (3.33)$$

Expanding the states as usual, we easily find

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho} = \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) i(p'_{1\mu} - p_{1\mu}) \\ \times \langle p'_1 p'_2 | i(\mathbb{F}_{\nu\rho}(x)T - T^\dagger \mathbb{F}_{\nu\rho}(x)) + T^\dagger \mathbb{F}_{\nu\rho}(x)T | p_1 p_2 \rangle . \end{aligned} \quad (3.34)$$

The factor  $i(p'_{1\mu} - p_{1\mu})$  is important here: working at leading perturbative order, this factor is of order  $\hbar$ . It is also worth noting that we may write the expectation of the field strength itself as

$$\begin{aligned} \langle \psi | \hat{S}^\dagger \mathbb{F}_{\nu\rho} \hat{S} | \psi \rangle = \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) \\ \times \langle p'_1 p'_2 | i(\mathbb{F}_{\nu\rho}(x)T - T^\dagger \mathbb{F}_{\nu\rho}(x)) + T^\dagger \mathbb{F}_{\nu\rho}(x)T | p_1 p_2 \rangle . \end{aligned} \quad (3.35)$$

Thus the  $i(p'_{1\mu} - p_{1\mu}) \sim \hbar$  factor in the variance is the key distinction between the variance, which vanishes classically, and the field strength which of course should not vanish classically. As we have already seen that the field strength is related to five-point amplitudes, it is now clear that the condition of vanishing  $\mathcal{O}_{\mu\nu\rho}$  will become a condition on five-point amplitudes.

It is useful to break the variance  $\mathcal{O}_{\mu\nu\rho}$  up into two structures:

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(1)} = \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) i(p'_{1\mu} - p_{1\mu}) \\ \times \langle p'_1 p'_2 | i(\mathbb{F}_{\nu\rho}(x)T - T^\dagger \mathbb{F}_{\nu\rho}(x)) | p_1 p_2 \rangle , \end{aligned} \quad (3.36)$$

and

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(2)} = \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) i(p'_{1\mu} - p_{1\mu}) \\ \times \langle p'_1 p'_2 | T^\dagger \mathbb{F}_{\nu\rho}(x)T | p_1 p_2 \rangle . \end{aligned} \quad (3.37)$$

Both of these objects are real, which is convenient in terms of keeping the expressions simple.

We may simplify these structures using the explicit expression for the field strength given in equation (3.4). For  $\mathcal{O}^{(1)}$  we find

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(1)} = 2 \operatorname{Re} \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(p'_1, p'_2, p_1, p_2, k) \phi_b^*(p'_1, p'_2) \phi_b(p_1, p_2) \times \\ \times i(p'_{1\mu} - p_{1\mu}) \langle k^\eta, p'_1, p'_2 | T | p_1, p_2 \rangle \bar{k}_{[\nu} \varepsilon_{\rho]}^{(\eta)*} e^{-i\bar{k} \cdot x} , \end{aligned} \quad (3.38)$$

which should be compared to equation (3.11). Again we see that the crucial new

ingredient is a factor  $i(p'_{1\mu} - p_{1\mu})$ . In the classical regime, we may write this term as

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(1)} = 2 \operatorname{Re} \hbar^{9/2} \sum_{\eta} \left\langle\!\left\langle \int d\Phi(\bar{k}) \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p_1 \cdot \bar{q}_1) \hat{\delta}(2p_2 \cdot \bar{q}_2) \right. \right. \\ \left. \left. \times i \bar{q}_{\mu} \mathcal{A}_5(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^{\eta}) \hat{\delta}^4(\bar{k} + \bar{q}_1 + \bar{q}_2) \bar{k}_{[\nu} \varepsilon_{\rho]}^{(\eta)*} e^{-i(\bar{k} \cdot x + \bar{q}_1 \cdot b)} \right\rangle\!\right\rangle. \end{aligned} \quad (3.39)$$

Referring back once more to equation (3.12), the additional  $\hbar$  suppression is now manifest.

In order to control the  $\hbar$  expansion of scattering amplitudes, it is useful to introduce some further notation. Let us write the amplitudes as explicit Laurent series in  $\hbar$  building on equation (3.13). For example, at five-points we may write

$$\begin{aligned} \mathcal{A}_{5,0}(i \rightarrow f) &= \hbar^{-7/2} \left( \mathcal{A}_{5,0}^{(0)}(i \rightarrow f) + \hbar \mathcal{A}_{5,0}^{(1)}(i \rightarrow f) + \dots \right), \\ \mathcal{A}_{5,1}(i \rightarrow f) &= \hbar^{-9/2} \left( \mathcal{A}_{5,1}^{(0)}(i \rightarrow f) + \hbar \mathcal{A}_{5,1}^{(1)}(i \rightarrow f) + \dots \right). \end{aligned} \quad (3.40)$$

We have scaled out the dominant (inverse) power of  $\hbar$ ; the quantities  $\mathcal{A}_{n,L}^{(p)}$  are  $\hbar$ -independent gauge-invariant sub-amplitudes; it is precisely these quantities that are related by our reasoning. This expansion defines an infinite set of objects  $\mathcal{A}_{n,L}^{(p)}$  which could in principle be reassembled into the full amplitude. They are a kind of partial amplitude, but distinct from the usual use of this term. We will therefore refer to them as “fragmentary amplitudes”, or simply as “fragments.”

Since  $\hbar$  is dimensionful, it is useful to view these fragmentary amplitudes in a slightly different way. Amplitudes are functions of Mandelstam invariants; in the semi-classical region, we are expanding in powers of momentum transfers, such as  $q^2 = \hbar^2 \bar{q}^2$  at four points, divided by Mandelstam  $s = (p_1 + p_2)^2$ . The semiclassical expansion is an expansion in powers of  $\hbar \sqrt{-\bar{q}^2/s}$ . More general amplitudes involve a richer set of momentum transfers  $\bar{q}_{ij}^2$ ; our expansion is in powers of  $\hbar \sqrt{-\bar{q}_{ij}^2/s}$ . We only consider amplitudes with two incoming massive particles.

We may also view the expansion as being in (inverse) powers of the large mass of the scattering particles [54, 64, 70, 77]. This makes contact with effective field theory, especially heavy quark effective theory or, more generally, heavy particle effective theories as has been emphasised in references [64, 70, 77]. Our fragmentary amplitudes correspond in this context to the standard HQET

expansion in inverse powers of heavy masses. It seems likely that a study of the properties of amplitudes in these theories would illuminate the structure of the fragmentary amplitudes.

Notice that this expansion is analogous, but different, to a soft expansion. In the soft expansion we take the momentum of an individual particle soft. In this transfer expansion we take the momenta in all messenger lines to be of the same order, and small compared to the incoming centre of mass energy. It is possible to perform the transfer expansion and then, in a second stage, to single out some line, say an outgoing photon, and take its momentum to be softer than all other messenger lines. This yields the soft limit of the transfer expansion. It corresponds to the low-frequency limit in the classical approximation.

Now at classical order ( $\hbar^0$ ) the tree level amplitude  $\mathcal{A}_{5,0}$  does not appear in  $\mathcal{O}_{\mu\nu\rho}^{(1)}$  on account of the explicit factor  $\hbar^{9/2}$  in equation (3.39). The leading in  $g$ , non-trivial, classical contribution arises from the fragment  $\mathcal{A}_{5,1}^{(0)}$ . We conclude then that

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(1)} = 2 \operatorname{Re} \sum_{\eta} \left\langle\!\left\langle \int d\Phi(\bar{k}) \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p_1 \cdot \bar{q}_1) \hat{\delta}(2p_2 \cdot \bar{q}_2) \times \right. \right. \\ \left. \left. \times i \bar{q}_{\mu} \mathcal{A}_{5,1}^{(0)}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^{\eta}) \hat{\delta}^4(\bar{k} + \bar{q}_1 + \bar{q}_2) \bar{k}_{[\nu} \varepsilon_{\rho]}^{(\eta)*} e^{-i(\bar{k} \cdot x + \bar{q}_1 \cdot b)} \right\rangle\!\right\rangle. \end{aligned} \quad (3.41)$$

The relevant fragmentary amplitude is the leading-in- $\hbar$  five-point one-loop amplitude, sometimes known as the “superclassical” part of the one-loop amplitude. Of course in this context this fragment of the amplitude is contributing precisely at classical order.

Now the full  $\mathcal{O}_{\mu\nu\rho}$  should vanish at classical order. Since  $\mathcal{O}_{\mu\nu\rho}^{(1)} \neq 0$ , it must be that the second structure  $\mathcal{O}_{\mu\nu\rho}^{(2)}$  cancels the contribution of equation (3.41). We find that

$$\begin{aligned} \mathcal{O}_{\mu\nu\rho}^{(2)} = 2 \operatorname{Re} \sum_{\eta} \left\langle\!\left\langle \int d\Phi(\bar{k}) \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{d}^4 \bar{w}_1 \hat{d}^4 \bar{w}_2 \hat{\delta}(2p_1 \cdot \bar{q}_1) \hat{\delta}(2p_2 \cdot \bar{q}_2) \hat{\delta}(2p_1 \cdot \bar{w}_1) \hat{\delta}(2p_2 \cdot \bar{w}_2) \right. \right. \\ \left. \left. \times \bar{q}_{\mu} \hat{\delta}^4(\bar{k} + \bar{q}_1 + \bar{q}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{w}_1 - \bar{w}_2) \bar{k}_{[\nu} \varepsilon_{\rho]}^{(\eta)*} e^{-i(\bar{k} \cdot x + \bar{q}_1 \cdot b)} \right. \right. \\ \left. \left. \times \mathcal{A}_{5,0}^{(0)}(p_1 p_2 \rightarrow p_1 + w_1, p_2 + w_2, k^{\eta}) \mathcal{A}_{4,0}^{(0)}(p_1 + w_1, p_2 + w_2 \rightarrow p_1 + q_1, p_2 + q_2) \right\rangle\!\right\rangle. \end{aligned} \quad (3.42)$$

Comparing equations (3.41) and (3.42), the condition for vanishing  $\mathcal{O}$  is

$$\begin{aligned}
& i\mathcal{A}_{5,1}^{(0)}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k^\eta) \\
&= - \int \hat{d}^4 \bar{w}_1 \hat{d}^4 \bar{w}_2 \hat{\delta}(2p_1 \cdot \bar{w}_1) \hat{\delta}(2p_2 \cdot \bar{w}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{w}_1 - \bar{w}_2) \\
&\quad \times \mathcal{A}_{5,0}^{(0)}(p_1 p_2 \rightarrow p_1 + w_1, p_2 + w_2, k^\eta) \mathcal{A}_{4,0}^{(0)}(p_1 + w_1, p_2 + w_2 \rightarrow p_1 + q_1, p_2 + q_2).
\end{aligned} \tag{3.43}$$

Thus the dominant part of the five-point one-loop amplitude is given by the tree five-point and tree four-point amplitudes; we will check this relation explicitly in the next subsection.

Clearly this explicit example is one among an infinite set of relationships. Variances involving one field strength operator and two momenta will lead to relationships among two-loop five-point amplitudes and the product of one five-point tree and two four-point trees. We can continue, in principle, as far as we wish generating similar relations. These negligible uncertainty relations generalise the well-known relations between multiloop four-point amplitudes required for eikonal exponentiation. Indeed consideration of expectations such as

$$\langle \psi | S^\dagger \mathbb{P}_{\mu_1} \mathbb{P}_{\mu_2} \cdots \mathbb{P}_{\mu_n} S | \psi \rangle \simeq \langle \psi | S^\dagger \mathbb{P}_{\mu_1} | \psi \rangle \langle \psi | S^\dagger \mathbb{P}_{\mu_2} | \psi \rangle \cdots \langle \psi | S^\dagger \mathbb{P}_{\mu_n} | \psi \rangle, \tag{3.44}$$

shows that there must be a relationship between the  $n - 1$  loop four-point amplitude and the product of  $n$  tree amplitudes.

Thus we find a remarkable abundance of relationships between multiloop, multileg amplitudes, considered as Laurent series in  $\hbar$ , forced on us by the *absence* of uncertainty in the classical regime. In the next section of this article we will interpret these relationships in terms of a radiative generalisation of the eikonal exponentiation.

As well as finding explicit relations between different fragmentary amplitudes, we can use similar ideas to determine the  $\hbar$  scaling associated with fragments in the transfer expansion. The kinds of multiple cancellations of  $\hbar$  powers we saw at six points must continue to occur at higher points. The reason again follows from considering expectations of products of more than two field-strength operators.

The arguments are based simply on counting powers of coupling and  $\hbar$ . We know that for the single expectation, at leading order, we have

$$\langle \mathbb{F}_{\mu\nu} \rangle \sim g^3. \tag{3.45}$$

This means that we must also have  $\langle \mathbb{F}^n \rangle \sim (g^3)^n$ . Now we perform the KMOC analysis of  $\langle \mathbb{F}^n \rangle$ . Following the steps of the calculation earlier in section 3.1.2 we find, schematically, that

$$\langle \mathbb{F}^n \rangle \sim \hbar^{3n/2+2} \int \mathcal{A}_{4+n}. \quad (3.46)$$

These relations allow us to deduce two things. Firstly the relevant fragment of the complete amplitude  $\mathcal{A}_{4+n}$  must scale as  $\hbar^{-3n/2-2}$ . Secondly this fragment must have  $3n$  powers of the coupling  $g$  — this corresponds to having  $n-1$  loops. From this we can also infer the scaling of all other loop and tree amplitudes, in the classical limit, since each loop contributes an extra factor of  $\hbar^{-1}$ . In particular the tree scaling will be

$$\mathcal{A}_{4+n,0} \sim \hbar^{-n/2-3}. \quad (3.47)$$

This is consistent with the scaling we computed above for six points ( $n=2$ ), and we have also checked explicitly at seven points. It is interesting to see how these two very simple power counting arguments have completely constrained the  $\hbar$  scaling of all  $2 \rightarrow 2+n$  amplitudes, as well as telling us which number of loops should contribute classically.

### 3.2.2 One loop factorisation

We now turn to verifying equation (3.43). For simplicity we focus on the case of scalar QED, though the general nature of our arguments indicates that the result should also hold in gravity and in Yang-Mills theory. In order to keep the computational labour to the minimum, we take advantage of lessons we learned in the context of the six-point tree amplitude in section 3.1.4. We use that the scalar QED five-point amplitudes can be reduced to gauge-invariant partial amplitudes, and make use of the same gauge choice. In particular we write the amplitudes as

$$\begin{aligned} \mathcal{A}_{4,0}^{(0)}(p_1 p_2 \rightarrow p_1 + w_1, p_2 + w_2) &= Q_1 Q_2 A_{(1,1)}, \\ \mathcal{A}_{5,0}^{(0)}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k) &= Q_1 Q_2^2 A_{(1,2)} + Q_1^2 Q_2 A_{(2,1)}, \\ \mathcal{A}_{5,1}^{(0)}(p_1 p_2 \rightarrow p_1 + q_1, p_2 + q_2, k) &= Q_1^2 Q_2^3 A_{(2,3)} + Q_1^3 Q_2^2 A_{(3,2)}, \end{aligned} \quad (3.48)$$

In view of the symmetry between  $A_{2,3}$  and  $A_{3,2}$  we may compute just one choice: we choose to focus on the charge sector  $Q_1^2 Q_2^3$ . This means we instead choose an

gauge

$$\varepsilon_\eta(\bar{k}) \cdot p_2 = 0. \quad (3.49)$$

The leading-in- $\hbar$  parts of the tree partial amplitudes in this gauge are

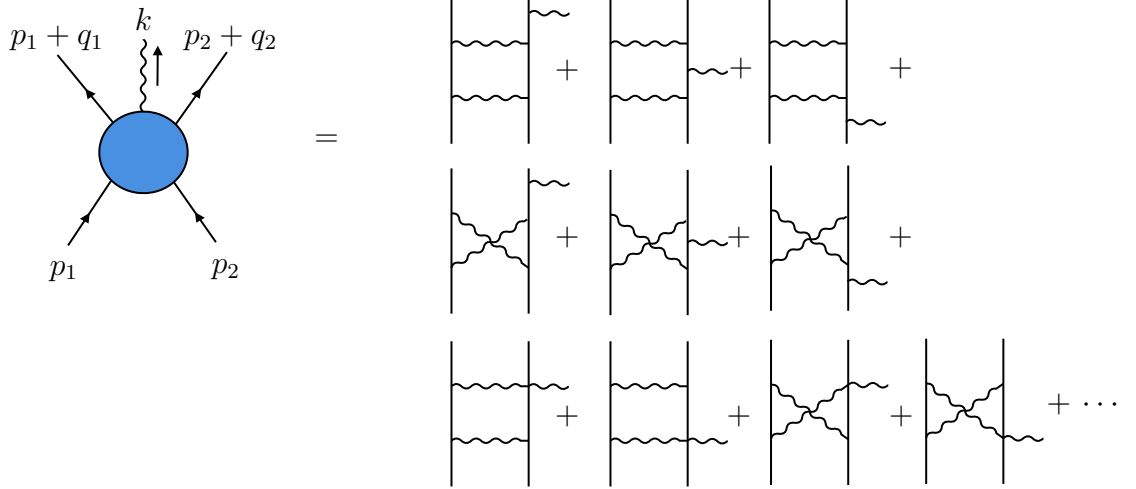
$$A_{(1,1)} = e^2 \frac{4p_1 \cdot p_2}{\bar{w}_1^2}, \quad (3.50)$$

and

$$A_{(1,2)} = \frac{4e^3}{\bar{q}_1^2} \left( p_1 \cdot \varepsilon_\eta + \frac{p_1 \cdot p_2 \varepsilon_\eta \cdot \bar{q}_1}{p_2 \cdot \bar{k}} \right) \quad (3.51)$$

at 4 and 5 points respectively.

The anatomy of (3.43) is that the leading-in- $\hbar$  fragment of the one-loop five-point amplitude  $A_{(2,3)}$  will organise itself into a product of  $A_{(1,1)}$  times the four terms that constitute  $A_{(1,2)}$ . Our strategy will be to isolate these terms one by one. Let us start gathering the relevant diagrams of  $A_{(2,3)}$ . At one loop, and in the classical limit, we need to consider the transfer expansion of the following five-point diagrams with a photon emitted in the final state:



The ellipsis indicate purely quantum diagrams which are not relevant for us.

As in 3.1.4 we tidy our expressions up by making use of the on-shell conditions. It is helpful to compute the first six diagrams in the figure above, which involve only three-point vertices, separately from the rest of the diagrams (which involve

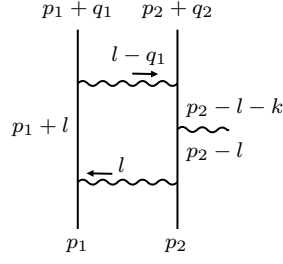


**Figure 3.5** *Momentum routing for the pentagon contribution to the five-point one-loop amplitude.*

contact four-point vertices). We therefore write

$$A_{(2,3)} = A_{(2,3)}^{3\text{pt}} + A_{(2,3)}^{4\text{pt}}. \quad (3.52)$$

We focus first on the six diagrams which constitute  $A_{(2,3)}^{3\text{pt}}$ . For clarity, let us begin by describing the contribution from the pentagon diagram, with the emitted photon coming from an internal loop leg rather than external leg, in our gauge (3.49) in detail. We choose the momentum routing shown in figure 3.5. On the support of the momentum-conserving delta function  $\hat{\delta}^4(q_1 + q_2 + k)$ , its contribution to  $A_{(2,3)}$  is

$$ie^5 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2 (\bar{\ell} - \bar{q}_1)^2} \frac{(4p_1 \cdot p_2)^2 (-2\varepsilon_\eta \cdot \bar{\ell})}{(2p_1 \cdot \bar{\ell})(-2p_2 \cdot \bar{\ell})(-2p_2 \cdot (\bar{k} + \bar{\ell}))}. \quad (3.53)$$

The sum of the cubic diagrams yields

$$\begin{aligned} A_{(2,3)}^{3\text{pt}} = ie^5 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2 (\bar{\ell} - \bar{q}_1)^2} (4p_1 \cdot p_2)^2 \times \\ \left( \frac{-2\varepsilon_\eta \cdot \bar{\ell}}{(2p_1 \cdot \bar{\ell})(-2p_2 \cdot \bar{\ell})(-2p_2 \cdot (\bar{k} + \bar{\ell}))} - \frac{2\varepsilon_\eta \cdot \bar{q}_1}{(2p_1 \cdot \bar{\ell})(-2p_2 \cdot \bar{\ell})(2p_2 \cdot \bar{k})} \right. \\ \left. + \frac{2\varepsilon_\eta \cdot (\bar{\ell} - \bar{q}_1)}{(2p_1 \cdot \bar{\ell})(2p_2 \cdot (\bar{k} + \bar{\ell}))(2p_2 \cdot \bar{\ell})} - \frac{2\varepsilon_\eta \cdot \bar{q}_1}{(2p_1 \cdot \bar{\ell})(2p_2 \cdot (\bar{k} + \bar{\ell}))(2p_2 \cdot \bar{k})} \right). \end{aligned} \quad (3.54)$$

Note that now the  $i\epsilon$ 's in linear propagators are important. We include them implicitly with the keeping track of the signs using the notation that

$$\frac{1}{\pm p \cdot \bar{\ell}} \equiv \frac{1}{\pm p \cdot \bar{\ell} + i\epsilon} \neq \pm \frac{1}{p \cdot \bar{\ell}} = \frac{1}{\pm p \cdot \bar{\ell} \pm i\epsilon}. \quad (3.55)$$

which allows us to make use of eq. (2.34) as

$$-i\hat{\delta}(p \cdot \bar{\ell}) = \frac{1}{p \cdot \bar{\ell}} + \frac{1}{-p \cdot \bar{\ell}}. \quad (3.56)$$

In order to make use of this identity we apply a change of variables  $l \rightarrow \bar{q}_1 - l$  in the last two terms in (3.54). This, along with the on-shell conditions, allows pairs of terms to take an almost identical form — denominators differ only by a sign in the  $p_1 \cdot l$  term which allows us to turn the propagators into a delta function. The amplitude then becomes

$$A_{(2,3)}^{3\text{pt}} = 4e^5(p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q}_1)^2} \frac{\hat{\delta}(p_1 \cdot \bar{\ell})}{-p_2 \cdot \bar{\ell}} \left( -\frac{\varepsilon_\eta \cdot \bar{q}_1}{p_2 \cdot \bar{k}} - \frac{\varepsilon_\eta \cdot \bar{\ell}}{-p_2 \cdot (\bar{k} + \bar{\ell})} \right). \quad (3.57)$$

It is possible to expose a second delta function in this expression by writing  $-\varepsilon_\eta \cdot \bar{q}_1 = \varepsilon_\eta \cdot (\bar{\ell} - \bar{q}_1) - \varepsilon_\eta \cdot \bar{\ell}$ . The two terms involving  $\varepsilon_\eta \cdot \bar{\ell}$  under the integral sign can be simplified by a partial fraction, yielding

$$\begin{aligned} A_{(2,3)}^{3\text{pt}} &= 4e^5(p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q}_1)^2} \frac{\hat{\delta}(p_1 \cdot \bar{\ell})}{p_2 \cdot \bar{k}} \left( \frac{\varepsilon_\eta \cdot (\bar{\ell} - \bar{q}_1)}{-p_2 \cdot \bar{\ell}} - \frac{\varepsilon_\eta \cdot \bar{\ell}}{-p_2 \cdot (\bar{\ell} - \bar{q}_1)} \right) \\ &= 4ie^5(p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q}_1)^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot (\bar{\ell} - \bar{q}_1)) \frac{\varepsilon_\eta \cdot \bar{\ell}}{p_2 \cdot \bar{k}}. \end{aligned} \quad (3.58)$$

To obtain the second of these equalities, we redefined the variable of integration to  $\bar{\ell}' = -(\bar{\ell} - \bar{q}_1)$  in the first term, and dropped the prime.

Next, we address the remaining diagrams contributing to  $A_{(2,3)}$  which now involve four-point vertices. After a straightforward computation, we find

$$A_{(2,3)}^{4\text{pt}} = 4ie^5 p_1 \cdot p_2 \varepsilon_\eta \cdot p_1 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q}_1)^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot (\bar{\ell} - \bar{q}_1)). \quad (3.59)$$

At this stage we can see the structure of the required factorisation — we have exposed the delta functions present in equation (3.43).

Combining the contact terms of equation (3.59) with the rest of the diagrams, equation (3.58), we find that the total expression for the amplitude fragment is

$$A_{(2,3)} = 4ie^5 p_1 \cdot p_2 \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2(\bar{\ell} - \bar{q}_1)^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot (\bar{\ell} - \bar{q}_1)) \left( \varepsilon_\eta \cdot p_1 + p_1 \cdot p_2 \frac{\varepsilon_\eta \cdot \bar{\ell}}{p_2 \cdot \bar{k}} \right). \quad (3.60)$$

The final step is to compare this result with the prediction for  $A_{(2,3)}$  from

equation (3.43). Using the amplitudes of equation (3.48), it is easy to see that the prediction is

$$A_{(2,3)} = \int \hat{d}^4 \bar{w}_1 \hat{d}^4 \bar{w}_2 \hat{\delta}(2p_1 \cdot \bar{w}_1) \hat{\delta}(2p_2 \cdot \bar{w}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{w}_1 - \bar{w}_2) \times \frac{4e^3}{\bar{w}_1^2} \left( p_1 \cdot \varepsilon_\eta + \frac{p_1 \cdot p_2 \varepsilon_\eta \cdot \bar{w}_1}{p_2 \cdot \bar{k}} \right) \frac{4e^2}{(\bar{q}_1 - \bar{w}_1)^2} p_1 \cdot p_2. \quad (3.61)$$

We first perform the the integral over  $\bar{w}_2$  using the four-fold delta function, relabelling  $\bar{w}_1 = \bar{\ell}$ ,

$$A_{(2,3)} = \int \hat{d}^4 \bar{\ell} \hat{\delta}(2p_1 \cdot \bar{\ell}) \hat{\delta}(2p_2 \cdot (\bar{q}_1 + \bar{q}_2)) \times \frac{4e^3}{\bar{\ell}^2} \left( p_1 \cdot \varepsilon_\eta + \frac{p_1 \cdot p_2 \varepsilon_\eta \cdot \bar{w}_1}{p_2 \cdot \bar{k}} \right) \frac{4e^2}{(\bar{q}_1 - \bar{\ell})^2} p_1 \cdot p_2 \quad (3.62)$$

Recognising that  $\bar{k} = -\bar{q}_1 - \bar{q}_2$  we recover equation (3.60).

We have thus verified the non trivial condition that the superclassical part of the one loop five point amplitude can indeed be written as a product of the tree level four and five point amplitudes. This is the five point analogue of first check for the eikonal, that the one loop box and cross box amplitudes can be written in terms of a product of two tree amplitudes, as in eq. (2.65). We will encounter this same condition again later in chapter 5, where it serves as the first non trivial check to allow for the inclusion of radiation in an eikonal like picture.

## Summary

In this chapter we studied the uncertainty, or variance, of quantum operators in the classical limit. By imposing sensible classical constraints, namely that the variance must vanish, we find relations between amplitude fragments. These can lead to conclusions that loops are more important than trees in the classical limit. Studying more general variance type objects, built from multiple potentially different operators, yields an infinite set of possible relations.

We saw how the 6 point tree, with two external photons, is quantum suppressed and demonstrated the desired  $\hbar$  scaling explicitly. We also saw how the leading in  $\hbar$  fragment of the one loop 5 point amplitude,  $\mathcal{A}_{5,1}$  is built from the integrated product of  $\mathcal{A}_{4,0}$  and  $\mathcal{A}_{5,0}$ . This is reminiscent of the first non trivial condition that allowed for the resummation of the 4 point amplitude to the leading eikonal.

We will revisit this 5 point version later in chapter 5, where it serves as the first non trivial check for a generalised eikonal formula to include a coherent radiative state.

# Chapter 4

## Colourful observables

Yang-Mills theory, treated as a classical field theory, shares many of the important physical features of gravity, including non-linearity and a subtle gauge structure. In this respect the YM case has always served as an excellent toy model for gravitational dynamics. But our developing understanding of the double copy has taught us that the connection between Yang-Mills theory and gravity is deeper than this. Detailed aspects of the perturbative dynamics of gravity, including gravitational radiation, can be deduced from Yang-Mills theory and the double copy [41, 54, 226, 227].

Another reason for interest in YM theory is as it provides a toy model for studying spin. Both spin and colour are non abelian charges which adds to the complexity compared to pure scalars. Colour however is simpler than spin because the latter depends on momentum – it is intimately related to the structure of the Poincaré group since spin relates to the little group. YM does not suffer from this momentum interference and thus provides a good stepping stone for studying spin. A last motivation is because of the double copy. There are known formulations, both quantum mechanical and classical, so studying the classical limit of YM could shed some light on the relation between these two aspects of the DC. In this chapter however we won't be commenting any further on this double copy aspect.

This chapter is organised as follows. In section 4.1 we review the theoretical treatment of colour in quantum Yang-Mills theory, emphasising the importance of coherent states in the classical limit. We will see that point-like particles emerging from an underlying matter field in a representation  $R$  of the gauge

group have an adjoint-valued colour charge which naturally has dimensions of angular momentum. In section 4.2 we will set up expressions for the classical and momentum impulse when scattering states have colour, and provide an explicit computation of these observables at NLO. In section 4.3 we turn to radiation, constructing the total radiated colour charge in terms of amplitudes and explicitly recovering the classical LO radiation current derived in [41]. In section 4.4 we reproduce our earlier results using purely classical methods.

## 4.1 Review of the theory of colour

The classical equations of central interest in this chapter describe the Yang-Mills field with some gauge group coupled to several classical point-like particles. These particles each carry colour charges  $c^a$  which are time-dependent vectors in the adjoint representation of the gauge group. Often known as Wong's equations [228], the equations of motion for  $N$  of these particles following worldlines  $x_\alpha(\tau_\alpha)$  and with velocities  $v_\alpha$  and momenta  $p_\alpha$  are

$$\frac{dp_\alpha^\mu}{d\tau_\alpha} = g c_\alpha^a(\tau_\alpha) F^{a\mu\nu}(x_\alpha(\tau_\alpha)) v_{\alpha\nu}(\tau_\alpha), \quad (4.1a)$$

$$\frac{dc_\alpha^a}{d\tau_\alpha} = g f^{abc} v_\alpha^\mu(\tau_\alpha) A_\mu^b(x_\alpha(\tau_\alpha)) c_\alpha^c(\tau_\alpha), \quad (4.1b)$$

$$D^\mu F_{\mu\nu}^a(x) = J_\nu^a(x) = g \sum_{\alpha=1}^N \int d\tau_\alpha c_\alpha^a(\tau_\alpha) v_{\alpha\nu}(\tau_\alpha) \delta^{(4)}(x - x_\alpha(\tau_\alpha)), \quad (4.1c)$$

where the Yang-Mills field is  $A_\mu = A_\mu^a T^a$  and  $F_{\mu\nu}^a$  is the associated field strength tensor.

The Wong equations describe classical point particles arising from a worldline action. Now let us review the emergence of non-Abelian colour charges  $c_\alpha^a$  — a vector in the *adjoint* representation for each particle species  $\alpha$  — from a quantum field theory with scalars  $\varphi_\alpha$  in *any* representation  $R_\alpha$  of the gauge group, coupled to the Yang-Mills field. We specialise to the case of two different scalar fields, so that  $\alpha = 1, 2$ . Our action is

$$S = \int d^4x \left( \sum_{\alpha=1}^2 \left[ (D_\mu \varphi_\alpha^\dagger) D^\mu \varphi_\alpha - \frac{m_\alpha^2}{\hbar^2} \varphi_\alpha^\dagger \varphi_\alpha \right] - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right), \quad (4.2)$$

where  $D_\mu = \partial_\mu + ig A_\mu^a T_R^a$ . We have restored factors of  $\hbar$  for convenience below. The generator matrices (in a representation  $R$ ) are  $T_R^a = (T_R^a)_i^j$  and satisfy the

Lie algebra  $[T_R^a, T_R^b]_i{}^j = if^{abc}(T_R^c)_i{}^j$ .

### 4.1.1 Colour in classical field theory

For simplicity we begin with the case of just a single massive scalar. At the classical level, the colour charge can be obtained from the Noether current  $j_\mu^a$  associated with the global part of the gauge symmetry. The colour charge is explicitly given by

$$\int d^3x j_0^a(t, \vec{x}) = i \int d^3x \left( \varphi^\dagger T_R^a \partial_0 \varphi - (\partial_0 \varphi^\dagger) T_R^a \varphi \right). \quad (4.3)$$

Notice that a direct application of the Noether procedure has led to a colour charge with dimensions of action, or equivalently, of angular momentum.

It's worth dwelling on dimensional analysis in the context of the Wong equations (4.1) since they motivate us to make certain choices which may, at first, seem surprising. The Yang-Mills field strength

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc} A_\mu^b A_\nu^c \quad (4.4)$$

is obviously an important actor in these classical equations. Classical equations should contain no factors<sup>1</sup> of  $\hbar$ , so we choose to maintain this precise expression for the field strength when  $\hbar \neq 1$ . By inspection it follows that  $[gA_\mu^a] = L^{-1}$ . We can develop this further; since the action of equation (4.2) has dimensions of angular momentum, the Yang-Mills field strength must have dimensions of  $\sqrt{M/L^3}$ . Thus, from equation (4.4),

$$[A_\mu^a] = \sqrt{\frac{M}{L}}, \quad [g] = \frac{1}{\sqrt{ML}}. \quad (4.5)$$

This conclusion about the dimensions of  $g$  is in contrast to the situation in electrodynamics, where  $[e] = \sqrt{ML}$ . Put another way, in electrodynamics the dimensionless fine structure constant is  $e^2/4\pi\hbar$  while in our conventions the analogue is  $\hbar g^2/4\pi$ ! It is possible to arrange matters such that the YM and EM cases are more similar, but we find the present conventions to be convenient in perturbative calculations.

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<sup>1</sup>An equivalent point of view is that any factors of  $\hbar$  appearing in an equation which has classical meaning should be absorbed into parameters of the classical theory.

Continuing with our discussion of dimensions, note that the Yang-Mills version of the Lorentz force, equation (4.1a), demonstrates that the quantity  $gc^a$  must have the same dimension as the electric charge. This is consistent with our observation above that the colour has dimensions of angular momentum.

At first our assignment of dimensions of  $g$  may seem troubling from the perspective of the KMOC classical limit recipe. The fact that  $g$  has dimensions of  $1/\sqrt{ML}$  implies that the dimensionless coupling at each vertex is  $g\sqrt{\hbar}$ , so factors of  $\hbar$  associated with the coupling appear with the opposite power to the case of electrodynamics (and gravity). However, because the colour charges are dimensionful the net power of  $\hbar$  turns out to be the same, and thus the KMOC approach is ultimately unaltered. We will also see below that the dimensionful nature of the colour clarifies the classical limit of this aspect of the theory. To see how this works we must quantise.

### 4.1.2 Colour of single-particle quantum states

Dimensional analysis demonstrates that  $\varphi$  has dimensions of  $\sqrt{M/L}$ , so its mode expansion is

$$\varphi_i(x) = \frac{1}{\sqrt{\hbar}} \int d\Phi(p) \left( a_i(p) e^{-ip \cdot x/\hbar} + b_i^\dagger(p) e^{ip \cdot x/\hbar} \right). \quad (4.6)$$

The index  $i$  labels the representation  $R$ . For example, in the fundamental representation of  $SU(N)$ ,  $i = 1, \dots, N$ .

After quantisation, the colour charge of equation (4.3) becomes a Hilbert space operator which will be important below. To emphasise that this is an operator we write the quantised form as  $\mathbb{C}^a$ :

$$\begin{aligned} \mathbb{C}^a &= i \int d^3x \left( \varphi^\dagger T_R^a \partial_0 \varphi - (\partial_0 \varphi^\dagger) T_R^a \varphi \right) \\ &= \hbar \int d\Phi(p) \left( a^\dagger(p) T_R^a a(p) + b^\dagger(p) T_R^a b(p) \right), \end{aligned} \quad (4.7)$$

using the generators of the conjugate representation  $\bar{R}$ ,  $T_{\bar{R}}^a = -T_R^a$ . The overall  $\hbar$  factor guarantees that the colour has dimensions of angular momentum, as we require. It is important to note that these global colour operators inherit the

usual Lie algebra of the generators, modified by factors of  $\hbar$ , so that

$$[\mathbb{C}^a, \mathbb{C}^b] = i\hbar f^{abc} \mathbb{C}^c. \quad (4.8)$$

Now we turn to the action of the colour operator on the single-particle states

$$|p^i\rangle = a^{\dagger i}(p)|0\rangle. \quad (4.9)$$

Acting with the colour charge operator of equation (4.7) we immediately see that

$$\mathbb{C}^a |p^i\rangle = \hbar (T_R^a)_j^i |p^j\rangle, \quad \langle p_i | \mathbb{C}^a = \hbar \langle p_j | (T_R^a)_i^j. \quad (4.10)$$

Thus inner products yield generators scaled by  $\hbar$ :

$$\langle p_i | \mathbb{C}^a | p^j \rangle \equiv (C^a)_i^j = \hbar (T_R^a)_i^j. \quad (4.11)$$

The  $(C^a)_i^j$  are simply rescalings of the usual generators  $T_R^a$  by a factor of  $\hbar$ , and thus satisfy the rescaled Lie algebra in equation (4.8); since this rescaling is important for us, it is useful to make the distinction between the two.

$$\mathbb{C}^a |\psi\rangle = \int d\Phi(p) (C^a)_i^j \phi(p) \chi_j |p^i\rangle. \quad (4.12)$$

Furthermore, we define the colour charge of the particle as

$$\langle \psi | \mathbb{C}^a | \psi \rangle = \chi^{i*} (C^a)_i^j \chi_j. \quad (4.13)$$

Computing this charge and extracting its classical limit is the topic of the next section 4.1.3.

As a final remark on these rescaled generators, let us write out the covariant derivative in the representation  $R$ . In terms of  $C^a$ , the  $\hbar$  scaling of interactions is precisely the same as in QED (and in perturbative gravity):

$$D_\mu = \partial_\mu + i g A_\mu^a T^a = \partial_\mu + \frac{ig}{\hbar} A_\mu^a C^a; \quad (4.14)$$

for comparison, the covariant derivative used by KMOC in QED was  $\partial_\mu + ie A_\mu / \hbar$  [45]. Thus we have arranged that factors of  $\hbar$  appear in the same place in YM theory as in electrodynamics, provided that the colour is measured by  $C^a$ . This ensures that the basic rules for obtaining the classical limits of amplitudes

will be as described in section 2.1.3. In practical calculations one can thus restore  $\hbar$ 's in colour factors and work using  $C^a$ 's everywhere. However, it is worth emphasising that unlike classical colour charges, the factors  $C^a$  do not commute.

### 4.1.3 Colour of point-like particles in the classical regime

The classical point-particle picture emerges from sharply peaked quantum wavepackets. In [45], linear exponential generalisations of Gaussian wavepackets were chosen for relativistic momentum space wavepackets. The essential property that must be satisfied was for the particle to have a sharply-defined position *and* a sharply-defined momentum whenever the classical limit was taken. To understand colour, governed by the Yang-Mills-Wong equations in the classical arena, a similar picture should emerge for our quantum colour operator in equation (4.7). Following the KMOC procedure we will first consider the full quantum states and then the classical limit. We define the classical limit of the colour charge to be

$$c^a \equiv \langle \psi | \mathbb{C}^a | \psi \rangle. \quad (4.15)$$

Our focus in this section is on the colour structure of our particle. The full state is a tensor product of colour and kinematics:

$$|\psi\rangle = \sum |\psi_{\text{colour}}\rangle \otimes |\psi_{\text{kin}}\rangle, \quad (4.16)$$

but as we ignore kinematics for now, we simply write the colour part of the state  $|\psi_{\text{colour}}\rangle \rightarrow |\psi\rangle$  in the remainder of this section. Then, in the classical limit, the critical requirements on the colour part of the state are that

$$\begin{aligned} \langle \psi | \mathbb{C}^a | \psi \rangle &= \text{finite}, \\ \langle \psi | \mathbb{C}^a \mathbb{C}^b | \psi \rangle &= \langle \psi | \mathbb{C}^a | \psi \rangle \langle \psi | \mathbb{C}^b | \psi \rangle + \text{negligible}. \end{aligned} \quad (4.17)$$

Since the colour operator explicitly involves a factor of  $\hbar$ , another parameter must be large so that the colour expectation  $\langle \psi | \mathbb{C}^a | \psi \rangle$  is much bigger than  $\hbar$  in the classical region. This situation is basically the same as for the usual classical limit of angular momentum: in that case we take the spin quantum number  $j$  large so that  $\hbar j$  is a classical angular momentum. For colour, we similarly need the size of the representation  $R$  to be large. (We will see this explicitly in the case of  $SU(3)$  in a moment.) For the second requirement we must select appropriate

colour wavefunctions  $|\psi\rangle$  which are analogous to the kinematic wavepacket in KMOC [45].

Here we will use the Perelomov formulation since it is applicable to a generic lie group and thus can be utilised for both colour and kinematic parts. For the explicit construction of the appropriate colour states we will use the Schwinger boson formalism.

One can construct all irreducible representations from tensor products only of fundamentals [178, 181]; however, for our treatment of  $SU(3)$  it is helpful to instead make use of the fundamental and antifundamental, and tensor these together to generate representations. Following [179], we introduce two sets of ladder operators  $a_i$  and  $b^i$ ,  $i = 1, 2, 3$ , which transform in the  $\mathbf{3}$  and  $\mathbf{3}^*$  respectively. The colour operator can then be written as

$$\mathbb{C}^e = \hbar \left( a^\dagger \frac{\lambda^e}{2} a - b^\dagger \frac{\bar{\lambda}^e}{2} b \right), \quad e = 1, \dots, 8, \quad (4.18)$$

where  $\lambda^e$  are the Gell-Mann matrices and  $\bar{\lambda}^e$  are their conjugates. The operators  $a$  and  $b$  satisfy the usual commutation relations

$$[a_i, a^{\dagger j}] = \delta_i^j, \quad [b^i, b_j^\dagger] = \delta^i_j, \quad [a_i, b^j] = 0, \quad [a^{\dagger i}, b_j^\dagger] = 0. \quad (4.19)$$

By virtue of these commutators, the colour operator (4.18) obeys the commutation relation (4.8).

There are two Casimir operators given by the number operators<sup>2</sup>

$$\mathcal{N}_1 \equiv a^\dagger \cdot a, \quad \mathcal{N}_2 \equiv b^\dagger \cdot b, \quad (4.20)$$

with eigenvalues  $n_1$  and  $n_2$  respectively, so we label irreducible representations by  $[n_1, n_2]$ . Naïvely, the states we are looking for are constructed by acting on the vacuum state as follows:

$$(a^{\dagger i_1} \dots a^{\dagger i_{n_1}}) (b_{j_1}^\dagger \dots b_{j_{n_2}}^\dagger) |0\rangle. \quad (4.21)$$

However, these states are  $SU(3)$  reducible and thus cannot be used in our construction of coherent states. We write the irreducible states schematically by acting with a Young projector  $\mathcal{P}$  which appropriately (anti-) symmetrises upper

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<sup>2</sup>Here we define  $a^\dagger \cdot a \equiv \sum_{i=1}^3 a^{\dagger i} a_i$  and  $|\xi|^2 \equiv \sum_{i=1}^3 |\xi_i|^2$ .

|       |       |         |           |       |       |         |           |
|-------|-------|---------|-----------|-------|-------|---------|-----------|
| $j_1$ | $j_2$ | $\dots$ | $j_{n_2}$ | $i_1$ | $i_2$ | $\dots$ | $i_{n_1}$ |
|       |       |         |           |       |       |         |           |

**Figure 4.1** Young tableau of  $SU(3)$

and lower indices, thereby subtracting traces:

$$|\psi\rangle_{[n_1, n_2]} \equiv \mathcal{P} \left( (a^{\dagger i_1} \dots a^{\dagger i_{n_1}}) (b_{j_1}^{\dagger} \dots b_{j_{n_2}}^{\dagger}) |0\rangle \right). \quad (4.22)$$

In general these operations will lead to involved expressions for the states, but we can understand them from their associated Young tableaux (Fig. 4.1). Each double box column represents an operator  $b_i^{\dagger}$  and each single column box represents the operator  $a^{\dagger i}$ , and thus for a mixed representation we have  $n_2$  double columns and  $n_1$  single columns.

Having constructed the irreducible states, one can define a coherent state parametrised by two triplets of complex numbers  $\xi_i$  and  $\zeta^i$ ,  $i = 1, 2, 3$ . These are normalised according to

$$|\xi|^2 = |\zeta|^2 = 1, \quad \xi \cdot \zeta = 0. \quad (4.23)$$

We won't require fully general coherent states, but instead their projections onto the  $[n_1, n_2]$  representation, which are

$$|\xi \zeta\rangle_{[n_1, n_2]} \equiv \frac{1}{\sqrt{(n_1! n_2!)}} (\zeta \cdot b^{\dagger})^{n_2} (\xi \cdot a^{\dagger})^{n_1} |0\rangle. \quad (4.24)$$

The square roots ensure that the states are normalised to unity<sup>3</sup>. With this normalisation we can write the identity operator as

$$\mathbb{1}_{[n_1, n_2]} = \int d\mu(\xi, \zeta) \left( |\xi \zeta\rangle \langle \xi \zeta| \right)_{[n_1, n_2]}, \quad (4.25)$$

where  $\int d\mu(\xi, \zeta)$  is the  $SU(3)$  Haar measure, normalised such that  $\int d\mu(\xi, \zeta) = 1$ . Its precise form is irrelevant for our purposes.

With the states in hand, we can return to the expectation value of the colour operator  $\mathbb{C}^a$  in equation (4.7). The size of the representation, that is  $n_1$  and  $n_2$ ,

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<sup>3</sup>Note that the Young projector in equation (4.22) is no longer necessary since the constraint  $\xi \cdot \zeta = 0$  removes all the unwanted traces.

must be large compared to  $\hbar$  in the classical regime so that the final result is finite. To see this let us compute this expectation value explicitly. By definition we have

$$\langle \xi \zeta | \mathbb{C}^e | \xi \zeta \rangle_{[n_1, n_2]} = \frac{\hbar}{2} \left( \langle \xi \zeta | a^\dagger \lambda^e a | \xi \zeta \rangle_{[n_1, n_2]} - \langle \xi \zeta | b^\dagger \bar{\lambda}^e b | \xi \zeta \rangle_{[n_1, n_2]} \right). \quad (4.26)$$

After a little algebra we find that

$$\langle \xi \zeta | \mathbb{C}^e | \xi \zeta \rangle = \frac{\hbar}{2} (n_1 \xi^* \lambda^e \xi - n_2 \zeta^* \bar{\lambda}^e \zeta). \quad (4.27)$$

We see that a finite charge requires a scaling limit in which we take  $n_1, n_2$  large as  $\hbar \rightarrow 0$ , keeping the product  $\hbar n_i$  fixed for at least one value of  $i$ . The classical charge is therefore the finite c-number

$$c^a = \langle \xi \zeta | \mathbb{C}^a | \xi \zeta \rangle_{[n_1, n_2]} = \frac{\hbar}{2} (n_1 \xi^* \lambda^a \xi - n_2 \zeta^* \bar{\lambda}^a \zeta). \quad (4.28)$$

The other feature we must check is the expectation value of products. Considering the expectation of two colour charge operators in these states, we have

$$\begin{aligned} \langle \xi \zeta | \mathbb{C}^a \mathbb{C}^b | \xi \zeta \rangle_{[n_1, n_2]} &= \frac{1}{(n_1! n_2!)} \langle 0 | (\zeta^* \cdot b)^{n_2} (\xi^* \cdot a)^{n_1} (a^\dagger \lambda^a a - b^\dagger \bar{\lambda}^a b) \\ &\quad \times (a^\dagger \lambda^b a - b^\dagger \bar{\lambda}^b b) (\zeta \cdot b^\dagger)^{n_2} (\xi \cdot a^\dagger)^{n_1} | 0 \rangle \\ &= \frac{1}{(n_1! n_2!)} \langle 0 | (\zeta^* \cdot b)^{n_2} (\xi^* \cdot a)^{n_1} \left( a^\dagger \lambda^a a a^\dagger \bar{\lambda}^b b - b^\dagger \bar{\lambda}^a b a^\dagger \lambda^b a \right. \\ &\quad \left. - a^\dagger \lambda^a a b^\dagger \bar{\lambda}^b b - b^\dagger \bar{\lambda}^a b b^\dagger \bar{\lambda}^b b \right) (\zeta \cdot b^\dagger)^{n_2} (\xi \cdot a^\dagger)^{n_1} | 0 \rangle. \end{aligned} \quad (4.29)$$

Note that we can consider the  $a$  and  $b$  terms separately since they commute. The terms with only two  $a$  operators (or  $b$ 's) reduce to products of the form

$$\begin{aligned} \langle \xi \zeta | a^{\dagger i} a_j | \xi \zeta \rangle_{[n_1, n_2]} &= \frac{1}{(n_1! n_2!)} \langle 0 | (\zeta^* \cdot b)^{n_2} (\xi^* \cdot a)^{n_1} a^{\dagger i} a_j (\zeta \cdot b^\dagger)^{n_2} (\xi \cdot a^\dagger)^{n_1} | 0 \rangle \\ &= n_1 \xi^{*i} \xi_j, \end{aligned} \quad (4.30)$$

and similarly for expressions involving  $b^\dagger b_j^\dagger$ . We made use of the fact that the states are normalised. Next we have the term involving four  $a$ 's (or  $b$ 's), which yields

$$\langle \xi \zeta | a^{\dagger i} a_j a^{\dagger k} a_l | \xi \zeta \rangle_{[n_1, n_2]} = n_1 \xi^{*i} \xi_l \delta_j^k + n_1 (n_1 - 1) \xi^{*i} \xi_l \xi^{*k} \xi_j. \quad (4.31)$$

In the limit where  $n_1$  is large we can replace<sup>4</sup> the  $n(n-1)$  factor with  $n^2$ . Using this and returning all  $\lambda$  and  $\hbar$  factors we find

$$\hbar^2(\lambda^a)_i{}^j(\lambda^b)_k{}^l\langle\xi\zeta|a_j^\dagger a_i a_k^\dagger a_l|\xi\zeta\rangle_{[n_1,n_2]} = \hbar^2 n_1^2 \xi^* \lambda^a \xi \xi^* \lambda^b \xi + \hbar^2 n_1 \xi^* \lambda^a \cdot \lambda^b \xi. \quad (4.32)$$

Now, gathering all the terms with a pair of  $a$ 's and a pair of  $b$ 's, which are simply products of the expressions in equation (4.30) contracted with Gell-Mann matrices, we have that

$$\begin{aligned} \langle\xi\zeta|\mathbb{C}^a\mathbb{C}^b|\xi\zeta\rangle_{[n_1,n_2]} &= \hbar^2 \left( n_1^2 \xi^* \lambda^a \xi \xi^* \lambda^b \xi + n_2^2 \zeta^* \bar{\lambda}^a \zeta \zeta^* \bar{\lambda}^b \zeta - n_1 n_2 \xi^* \lambda^a \xi \zeta^* \bar{\lambda}^b \zeta \right. \\ &\quad \left. - n_1 n_2 \xi^* \lambda^b \xi \zeta^* \bar{\lambda}^a \zeta \right) + \hbar \left( \hbar n_1 \xi^* \lambda^a \cdot \lambda^b \xi - \hbar n_2 \zeta^* \bar{\lambda}^a \cdot \bar{\lambda}^b \zeta \right). \end{aligned} \quad (4.33)$$

Recognising the charge expectation values  $\langle\xi\zeta|\mathbb{C}^a\mathbb{C}^b|\xi\zeta\rangle_{[n_1,n_2]}$  from equation (4.28), this can be written as

$$\begin{aligned} \langle\xi\zeta|\mathbb{C}^a\mathbb{C}^b|\xi\zeta\rangle_{[n_1,n_2]} &= \langle\xi\zeta|\mathbb{C}^a|\xi\zeta\rangle_{[n_1,n_2]} \langle\xi\zeta|\mathbb{C}^b|\xi\zeta\rangle_{[n_1,n_2]} \\ &\quad + \hbar \left( \hbar n_1 \xi^* \lambda^a \cdot \lambda^b \xi - \hbar n_2 \zeta^* \bar{\lambda}^a \cdot \bar{\lambda}^b \zeta \right) \\ &= c^a c^b + \mathcal{O}(\hbar). \end{aligned} \quad (4.34)$$

The product  $\hbar n_i$  is finite in the classical limit  $\hbar \rightarrow 0$ ,  $n_i \rightarrow \infty$ . The term inside the brackets on the second line is itself finite, but comes with a lone  $\hbar$  coefficient, and thus vanishes in the classical limit. Thus we've demonstrated explicitly that the expectation factorises up to terms which are explicitly  $\hbar$  suppressed.

This is in fact a special case of a more general construction discussed in detail by Yaffe [229]. The same argument can be used to demonstrate an important property of the coherent states in the classical limit, which is that the overlap  $\langle\chi'|\chi\rangle$  is very strongly peaked about  $\chi = \chi'$  [229]. We have thus constructed explicit colour states which ensure the correct classical behaviour of the colour charges.

In the calculation of the colour impulse and radiated colour in sections 4.2 and 4.3, we will only need to make use of the finiteness and factorisation properties, so we will avoid further use of the explicit form of the states. Henceforth we write  $\chi_i$  for the parameters of a general colour state  $|\chi\rangle$  with these properties, and  $d\mu(\chi)$  for the Haar measure of the colour group, whatever it may be.

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<sup>4</sup>The correction term vanishes in the classical  $\hbar \rightarrow 0$  limit regardless.

#### 4.1.4 Colour of several particles

Now that we have reviewed the theory of colour for a single particle, it's time to consider what happens when more than one particle is present. We will shortly discuss the dynamics of colour in detail; here, we set up initial states describing more than one point-like particle.

We take our particles to be distinguishable, so they are associated with distinct quantum fields  $\varphi_\alpha$  with  $\alpha = 1, 2$ . We only consider two different particles explicitly, though it is no more difficult to consider the many-particle case. We also restrict to scalar fields, again for simplicity. The action is therefore as given in equation (4.2). Both fields  $\varphi_\alpha$  must be in representations  $R_\alpha$  which are large, so that a classical limit is available for the individual colours.

We discussed the two or multiparticle states that we use for scattering in section 2.1.1. These are states that classically have well defined position, momentum and colour and we also assume no radiation in the far past. In other words, particle  $\alpha$  has a wavepacket  $\phi_\alpha(p_\alpha)$  describing its momentum-space distribution, and a colour wavepacket  $\chi_\alpha$  which both minimise uncertainty in the classical limit. The phase space and momentum wavepackets simply go along for the ride when looking at colour operators, so we can ignore these for now.

We now measure the colour of multi-particle states by acting with a colour operator which is simply the sum of the individual colour operators (4.7) for each of the scalar fields. For example, acting on the state  $|p_1 \chi_1; p_2 \chi_2\rangle$  we have

$$\begin{aligned} \mathbb{C}^a |p_1 \chi_1; p_2 \chi_2\rangle &= |p_1^{i'} p_2^{j'}\rangle \left( (C_1^a)_{i'}^i \delta_{j'}^j + \delta_{i'}^i (C_2^a)_{j'}^j \right) \chi_{1i} \chi_{2j} \\ &= \int d\mu(\chi'_1) d\mu(\chi'_2) |p_1 \chi'_1; p_2 \chi'_2\rangle \langle \chi'_1 \chi'_2 | C_1^a \otimes 1 + 1 \otimes C_2^a | \chi_1 \chi_2 \rangle \quad (4.35) \\ &= \int d\mu(\chi'_1) d\mu(\chi'_2) |p_1 \chi'_1; p_2 \chi'_2\rangle \langle \chi'_1 \chi'_2 | C_{1+2}^a | \chi_1 \chi_2 \rangle, \end{aligned}$$

where  $C_\alpha^a$  is the colour in representation  $R_\alpha$  and we have written  $C_{1+2}^a$  for the colour operator on the tensor product of representations  $R_1$  and  $R_2$ . In the classical regime, using the property that the overlap between states sets  $\chi'_i = \chi_i$  in the classical limit, it follows that

$$\langle p_1 \chi_1; p_2 \chi_2 | C_{1+2}^a | p_1 \chi_1; p_2 \chi_2 \rangle = c_1^a + c_2^a, \quad (4.36)$$

so the colours simply add.

## 4.2 Impulse

Now we move on to the dynamics of colour. Our focus in this section will be on the colour impulse — that is, the total change in colour during a scattering event — leaving radiation of colour to the next section. We begin by setting up the colour impulse observable in the vein of [45, 133] before turning to explicit examples at LO and NLO.

### 4.2.1 Colourful scattering observables in quantum field theory

A natural observable in Yang-Mills theory is the total change in the colour charge of one of the massive scattering particles. Taking eq. (2.19) and choosing  $\mathcal{O} = \mathbb{C}_1^a$  we have

$$\begin{aligned}\langle \Delta c_1^a \rangle &= \langle \Psi | S^\dagger \mathbb{C}_1^a S | \Psi \rangle - \langle \Psi | \mathbb{C}_1^a | \Psi \rangle \\ &= i \langle \Psi | [\mathbb{C}_1^a, T] | \Psi \rangle + \langle \Psi | T^\dagger [\mathbb{C}_1^a, T] | \Psi \rangle,\end{aligned}\tag{4.37}$$

We call this observable the *colour impulse*, as it mirrors the structure of the momentum impulse  $\Delta p_1^\mu$  of [45] and angular impulse  $\Delta s_1^\mu$  of [133]. An immediate novelty for this impulse is that it is a Lorentz scalar, instead transforming in the adjoint representation of the gauge group.

Substituting the 2-particle wavepackets in equation (2.11) yields

$$\begin{aligned}\langle \Delta c_1^a \rangle &= \prod_{i=1,2} \int d\Phi(p_i) d\Phi(p'_i) \phi_i(p_i) \phi_i^*(p'_i) e^{ib \cdot (p_1 - p'_1)/\hbar} \\ &\quad \times \langle p'_1 \chi_1; p'_2 \chi_2 | i[\mathbb{C}_1^a, T] + T^\dagger [\mathbb{C}_1^a, T] | p_1 \chi_1; p_2 \chi_2 \rangle,\end{aligned}\tag{4.38}$$

which we expand in terms of amplitudes by inserting complete sets of states,

$$\mathbb{1} = \sum_X \int d\Phi(r_1) d\Phi(r_2) d\mu(\zeta_1) d\mu(\zeta_2) |r_1 \zeta_1; r_2 \zeta_2; X\rangle \langle r_1 \zeta_1; r_2 \zeta_2; X|. \tag{4.39}$$

The set  $X$  could contain any number of extra gluon or scalar states, whose phase space measures and sums over any other quantum numbers are left implicit in the summation over  $X$ .

It is frequently convenient to write amplitudes in Yang-Mills theory in colour-ordered form; for example, see [230] for an application to amplitudes with multiple

different external particles. The full amplitude  $\mathcal{A}$  is decomposed onto a basis of colour factors times partial amplitudes  $A$ . The colour factors are associated with some set of Feynman topologies. Once a basis of independent colour structures is chosen, the corresponding partial amplitudes must be gauge invariant. Thus,

$$\begin{aligned} \langle p'_1 \chi_1; p'_2 \chi_2 | T | p_1 \chi_1; p_2 \chi_2 \rangle &= \langle \chi_1 \chi_2 | \mathcal{A}(p_1, p_2 \rightarrow p'_1, p'_2) | \chi_1 \chi_2 \rangle \hat{\delta}^{(4)}(p_1 + p_2 - p'_1 - p'_2) \\ &= \sum_D \langle \mathcal{C}(D) \rangle A_D(p_1, p_2 \rightarrow p'_1, p'_2) \hat{\delta}^{(4)}(p_1 + p_2 - p'_1 - p'_2), \end{aligned} \quad (4.40)$$

where  $\mathcal{C}(D)$  is the colour factor of diagram  $D$  and  $A_D$  is the associated partial amplitude. The remaining expectation value is over the colour states  $\chi_i$ . Using this notation and the action of the colour operator in equation (4.35), we can write the colour impulse as

$$\begin{aligned} \langle \Delta c_1^a \rangle &= \prod_{i=1,2} \sum_D \int d\Phi(p_i) d\Phi(p'_i) \phi_i(p_i) \phi_i^*(p'_i) e^{ib \cdot (p_1 - p'_1)/\hbar} \\ &\quad \times \left[ i \langle [C_1^a, \mathcal{C}(D)] \rangle A_D(p_1, p_2 \rightarrow p'_1, p'_2) \hat{\delta}^{(4)}(p_1 + p_2 - p'_1 - p'_2) \right. \\ &\quad + \sum_{D'} \sum_X \int d\Phi(r_i) \langle \mathcal{C}(D')^\dagger [C_1^a, \mathcal{C}(D)] \rangle A_{D'}^*(p'_1, p'_2 \rightarrow r_1, r_2, r_X) \\ &\quad \left. \times A_D(p_1, p_2 \rightarrow r_1, r_2, r_X) \hat{\delta}^{(4)}(p_1 + p_2 - p'_1 - p'_2) \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \right]. \end{aligned} \quad (4.41)$$

Finally, let us introduce the momentum mismatch  $q_i = p'_i - p_i$ , and transfer  $w_i = r_i - p_i$ . After integrating over the delta functions, we arrive at

$$\begin{aligned} \langle \Delta c_1^a \rangle &= i \int d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \\ &\quad \times \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0) \phi_1(p_1) \phi_2(p_2) \phi_1^*(p_1 + q) \phi_2^*(p_2 - q) e^{-ib \cdot q/\hbar} \\ &\quad \times \left\{ \sum_D \langle [C_1^a, \mathcal{C}(D)] \rangle A_D(p_1, p_2 \rightarrow p_1 + q, p_2 - q) \right. \\ &\quad - i \prod_{i=1,2} \sum_X \int \hat{d}^4 w_i \hat{\delta}(2p_1 \cdot w_i + w_i^2) \hat{\delta}^{(4)}(w_1 + w_2 - r_X) \Theta(p_i^0 + w_i^0) \\ &\quad \times \sum_{D, D'} \langle \mathcal{C}(D')^\dagger [C_1^a, \mathcal{C}(D)] \rangle A_D(p_1, p_2 \rightarrow p_1 + w_1, p_2 + w_2, r_X) \\ &\quad \left. \times A_{D'}^*(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X) \right\}. \end{aligned} \quad (4.42)$$

It is interesting to note that factors of expectation values of colour commutators,

such as  $\langle [C_1^a, \mathcal{C}(D)] \rangle$ , in the colour impulse play a similar role to that of the momentum mismatch  $q^\mu$  in the momentum impulse  $\Delta p_1^\mu$  [45].

The momentum impulse in QED and gravity was discussed in detail in [45]. In Yang-Mills theory, the presence of colour leads to slight modifications of those KMOC expressions. The basic difference is the colour structure of the amplitude. The observable itself is built from the (colour singlet) momentum operator  $\mathbb{P}_1^\mu$ , so factors of  $C_1^a$  appearing in the colour impulse, equation (4.42), do not arise in the momentum case. We proceed by writing the full amplitude as a sum over colour structures, finding

$$\begin{aligned}
\langle \Delta p_1^\mu \rangle = & i \int d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \\
& \times \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0) \phi(p_1) \phi(p_2) \phi^*(p_1 + q) \phi^*(p_2 - q) e^{-ib \cdot q} \\
& \times \left\{ \sum_D q^\mu \langle \mathcal{C}(D) \rangle A_D(p_1, p_2 \rightarrow p_1 + q, p_2 - q) \right. \\
& - i \prod_{i=1,2} \sum_X \int \hat{d}^4 w_i \hat{\delta}(2p_1 \cdot w_i + w_i^2) \hat{\delta}^{(4)}(w_1 + w_2 - r_X) \Theta(p_i^0 + w_i^0) \\
& \times \sum_{D, D'} \langle \mathcal{C}(D')^\dagger \mathcal{C}(D) \rangle A_D(p_1, p_2 \rightarrow p_1 + w_1, p_2 + w_2, r_X) \\
& \left. \times A_{D'}^*(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X) \right\}.
\end{aligned} \tag{4.43}$$

By construction, both impulse observables are well defined in the classical regime. Once wavefunctions of the types described in section 4.1 are used, the details of the wavefunctions will not be important. However, to extract expressions which are valid in the classical approximation, it is important to be aware that the commutators of the  $C^a$  contain powers of  $\hbar$ . In particular one must take care to expand all commutators of colour factors.

All other powers of  $\hbar$  appear as described previously. In brief, the rescaled covariant derivative of equation (4.14) ensures that each factor of the coupling  $g$  is accompanied by a factor  $\hbar^{-1/2}$  as in QED. The rest is as usual— massless external and loop momenta are products of a factor of  $\hbar$  and a wavenumber; care must be taken with squares of massless momenta  $q^2$  in delta functions. Finally, small shifts of order  $\hbar \bar{q}$  to the dominant momenta of order  $m$  in wavefunctions can be neglected; this is an example of a general property of coherent states in

the classical limit [229]. The colour impulse thus becomes

$$\langle \Delta c_1^a \rangle \rightarrow \Delta c_1^a = i \left\langle\left\langle \int \hat{d}^4 \bar{q} \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}} \mathcal{G}^a \right\rangle\right\rangle, \quad (4.44)$$

where we define the colour kernel  $\mathcal{G}^a$  to be

$$\begin{aligned} \mathcal{G}^a &= \hbar^2 \sum_D [C_1^a, \mathcal{C}(D)] A_D(p_1, p_2 \rightarrow p_1 + q, p_2 - q) \\ &\quad - i \hbar^4 \sum_X \prod_{i=1,2} \int \hat{d}^4 \bar{w}_i \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) \hat{\delta}^{(4)}(\bar{w}_1 + \bar{w}_2 - \bar{r}_X) \sum_{D, D'} \mathcal{C}(D')^\dagger [C_1^a, \mathcal{C}(D)] \\ &\quad \times A_{D'}^*(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X) A_D(p_1, p_2 \rightarrow p_1 + w_1, p_2 - w_2, r_X). \end{aligned} \quad (4.45)$$

We designed this kernel so that it is of order  $\hbar^0$  in the classical approximation. Clearly at LO only the first term, linear in the amplitude, contributes; the second integral contributes from NLO, where it is an integral over tree level diagrams, while the first term involves one-loop amplitudes.

In the same notation, the momentum impulse is

$$\langle \Delta p_1^\mu \rangle \rightarrow \Delta p_1^\mu = i \left\langle\left\langle \int \hat{d}^4 \bar{q} \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}} \mathcal{I}^\mu \right\rangle\right\rangle, \quad (4.46)$$

with momentum kernel

$$\begin{aligned} \mathcal{I}^\mu &= \hbar^3 \bar{q}^\mu \sum_D \mathcal{C}(D) A_D(p_1, p_2 \rightarrow p_1 + q, p_2 - q) \\ &\quad - i \hbar^5 \sum_X \prod_{i=1,2} \int \hat{d}^4 \bar{w}_i \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) \hat{\delta}^{(4)}(\bar{w}_1 + \bar{w}_2 - \bar{r}_X) \bar{w}_1^\mu \sum_{D, D'} \mathcal{C}(D')^\dagger \mathcal{C}(D) \\ &\quad \times A_{D'}^*(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X) A_D(p_1, p_2 \rightarrow p_1 + w_1, p_2 - w_2, r_X) \end{aligned} \quad (4.47)$$

when the scattering particles carry colour.

## 4.2.2 Leading order

We may now compute the colour impulse explicitly. We begin at leading order (LO) for the scalar YM theory defined by equation (4.2), moving to next-to-leading order (NLO) in the next subsection. We will strip coupling constants from amplitudes, writing  $\bar{A}_D^{(n)}$  for the charge stripped partial amplitudes at  $\mathcal{O}(g^{2n+2})$ .

At LO the colour kernel is

$$\mathcal{G}^{a,(0)} = \hbar g^2 \sum_D [C_1^a, \mathcal{C}(D)] \bar{A}_D^{(0)}(p_1, p_2 \rightarrow p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q}). \quad (4.48)$$

Here only the  $t$ -channel tree topology contributes, so the sum between colour and kinematics is trivial; we simply have<sup>5</sup>

$$\bar{A}_{\searrow\swarrow} = \frac{4p_1 \cdot p_2 + \hbar \bar{q}^2}{\hbar^2 \bar{q}^2}, \quad \mathcal{C} \left( \searrow \swarrow \right) = C_1 \cdot C_2 = C_1^a C_2^a, \quad (4.49)$$

and therefore the colour impulse factor is

$$[C_1^a, \mathcal{C}(\searrow\swarrow)] = [C_1^a, C_1^b] C_2^b = i\hbar f^{abc} C_1^c C_2^b. \quad (4.50)$$

Note in eq. (4.49) we have introduced the dot to indicate the summation of adjoint indices. This however is not a matrix product; indeed there are four fundamental indices corresponding to the four external scalar legs which have been suppressed.

Inserting these expressions into the colour kernel, all factors of  $\hbar$  cancel as expected for a classical observable. The classical limit is

$$\begin{aligned} \Delta_{C_1}^{a,(0)} &= -g^2 \left\langle\!\left\langle f^{abc} C_1^c C_2^b \int \hat{d}^4 \bar{q} \delta(p_1 \cdot \bar{q}) \delta(p_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}} \frac{p_1 \cdot p_2}{\bar{q}^2} \right\rangle\!\right\rangle \\ &= g^2 f^{abc} c_1^b c_2^c u_1 \cdot u_2 \int \hat{d}^4 \bar{q} \delta(u_1 \cdot \bar{q}) \delta(u_2 \cdot \bar{q}) \frac{e^{-ib \cdot \bar{q}}}{\bar{q}^2}. \end{aligned} \quad (4.51)$$

Notice that while evaluating the large double angle brackets we obtained classical colour charges as expectation values of the  $C_\alpha$ .

The remaining integral is straightforward but divergent. While we use dimensional regulation throughout the remainder of the chapter to define divergent integrals, in this case it is convenient to take a different approach.

The logarithmic divergence in the colour may seem surprising at first. However, the spacetime position of the particle is also logarithmically divergent in four dimensions; this is simply the familiar divergence due to the long-range nature of  $1/r^2$  forces in four dimensions. We therefore introduce a cutoff regulator  $L$  of

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<sup>5</sup>We adopt the convention that time runs vertically in Feynman diagrams.

dimensions length as follows. Consider the following quantity

$$2b^2 \frac{\partial \Delta c_1^{a,(0)}}{\partial b^2} = b_\mu \frac{\partial \Delta c_1^{a,(0)}}{\partial b_\mu} = -ig^2 f^{abc} c_1^b c_2^c \gamma b_\mu \int \hat{d}^4 \bar{q} \hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q}) \frac{e^{-ib \cdot \bar{q}} \bar{q}^\mu}{\bar{q}^2}, \quad (4.52)$$

where  $\gamma = u_1 \cdot u_2$ . The integral on the RHS was evaluated in [45]. Using that result, it is easy to show that the solution of the differential equation is

$$\Delta c_1^{a,(0)} = \frac{\gamma g^2 f^{abc} c_1^b c_2^c}{4\pi \sqrt{\gamma^2 - 1}} \log \left( \frac{b^2}{L^2} \right), \quad (4.53)$$

where we have included the regulator explicitly.

### 4.2.3 Next to leading order

At NLO the classical colour kernel, with  $\hbar$ 's from couplings removed, is

$$\begin{aligned} \mathcal{G}^{a,(1)} = & g^4 \sum_{\Gamma} [C_1^a, \mathcal{C}(\Gamma)] \bar{A}_{\Gamma}^{(1)}(p_1, p_2 \rightarrow p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q}) \\ & - ig^4 \hbar^2 \int \hat{d}^4 \bar{\ell} \hat{\delta}(2p_1 \cdot \bar{\ell} + \hbar \bar{\ell}^2) \hat{\delta}(2p_2 \cdot \bar{\ell} - \hbar \bar{\ell}^2) \mathcal{C}(\succ \prec)^\dagger [C_1^a, \mathcal{C}(\succ \prec)] \\ & \times \bar{A}_{\succ \prec}^*(p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q} \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}) \bar{A}_{\succ \prec}(p_1, p_2 \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}), \end{aligned} \quad (4.54)$$

where  $\Gamma$  is a set of one-loop topologies which span the independent colour factors. By the analysis of [45], the topologies relevant in the classical regime are

(4.55)

We will refer to these as the box  $B$ , cross box  $C$ , triangles  $T_{ij}$  and non-Abelian diagrams  $Y_{ij}$  respectively. The latter, involving the 3-gluon interaction vertex, are new to the Yang-Mills calculation. These diagrams are evaluated in appendix B.2. Of course these are not the only diagrams we must calculate; there is also the product of trees, which we will view as a cut box  $\hat{B}$ , in the non-linear part of the colour kernel. Note that we now have two distinct colour structures to calculate, one for the loops and one for the cut box. We will investigate in detail how these structures affect the cancellation of classically singular terms, but first let us work with the 1-loop, linear piece.

## 1-loop amplitude

At NLO we need to calculate the 1-loop scalar amplitude

$$\begin{aligned} \mathcal{A}^{(1)} = & \mathcal{C}(\text{box}) B + \mathcal{C}(\text{cross}) C + \mathcal{C}(\text{triangle}) T_{12} \\ & + \mathcal{C}(\text{triangle}) T_{21} + \mathcal{C}(\text{triangle}) Y_{12} + \mathcal{C}(\text{triangle}) Y_{21} . \end{aligned} \quad (4.56)$$

A first task is to choose a basis of independent colour structures. The complete set of colour factors can easily be calculated:

$$\begin{aligned} \mathcal{C}(\text{box}) &= C_1^a C_2^a C_1^b C_2^b , \\ \mathcal{C}(\text{cross}) &= C_1^a C_2^b C_1^b C_2^a , \\ \mathcal{C}(\text{triangle}) &= \frac{1}{2} \mathcal{C}(\text{box}) + \frac{1}{2} \mathcal{C}(\text{cross}) = \mathcal{C}(\text{triangle}) , \\ \mathcal{C}(\text{triangle}) &= \hbar C_1^a f^{abc} C_2^b C_2^c , \\ \mathcal{C}(\text{triangle}) &= \hbar C_1^a C_1^b f^{abc} C_2^c . \end{aligned} \quad (4.57)$$

Again here we have suppressed fundamental indices, but now we encounter matrix products. For example, reinstating these for the box colour factor we have

$$\mathcal{C}(\text{box}) = (C_1^a)_{im} (C_2^a)_{kn} (C_1^b)_{mj} (C_2^b)_{nl} = (C_1^a C_1^b)_{ij} (C_2^a C_2^b)_{kl} . \quad (4.58)$$

At first sight, we appear to have four independent colour factors: the box, cross box and the two non-Abelian triangles. However, it is very simple to see that the latter are in fact both proportional to the tree colour factor of equation (4.49); for example,

$$\begin{aligned} \mathcal{C}(\text{triangle}) &= \frac{\hbar}{2} C_1^a f^{abc} [C_2^b, C_2^c] = \frac{i\hbar^2}{2} f^{abc} f^{bcd} C_1^a C_2^d \\ &= \frac{i\hbar^2}{2} \mathcal{C}(\text{triangle}) , \end{aligned} \quad (4.59)$$

where we have used equation (4.8). Moreover, similar manipulations demonstrate that the cross-box colour factor is not in fact linearly independent:

$$\begin{aligned}
\mathcal{C}(\text{cross-box}) &= C_1^a C_1^b (C_2^a C_2^b - i\hbar f^{abc} C_2^c) \\
&= (C_1 \cdot C_2)(C_1 \cdot C_2) - \frac{i\hbar}{2} [C_1^a, C_2^b] f^{abc} C_2^c \\
&= \mathcal{C}(\text{box}) + \frac{\hbar^2}{2} \mathcal{C}(\text{tree}).
\end{aligned} \tag{4.60}$$

Thus at 1-loop the classically significant part of the amplitude has a basis of two colour structures: the box and tree. Hence the decomposition of the 1-loop amplitude into partial amplitudes and colour structures is

$$\begin{aligned}
\mathcal{A}^{(1)} &= \mathcal{C}(\text{box}) \left[ B + C + T_{12} + T_{21} \right] \\
&\quad + \frac{\hbar^2}{2} \mathcal{C}(\text{tree}) \left[ C + \frac{T_{12}}{2} + \frac{T_{21}}{2} + iY_{12} + iY_{21} \right].
\end{aligned} \tag{4.61}$$

This expression for the amplitude is particularly useful when taking the classical limit. The second term is proportional to two powers of  $\hbar$  while the only possible singularity in  $\hbar$  at one loop order is a factor  $1/\hbar$  in the evaluation of the kinematic parts of the diagrams. Thus, it is clear that the second line of the expression must be a quantum correction, and can be dropped in calculating the classical colour impulse. Perhaps surprisingly, these terms include the sole contribution from the non-Abelian triangles  $Y_{ij}$ , and thus we will not need to calculate these diagrams. We learn that classically, the 1-loop scalar YM amplitude has a basis of only one colour factor, and moreover depends on the same topologies as in electrodynamics, so we have

$$\mathcal{A}^{(1)} = \mathcal{C}(\text{box}) \mathcal{A}^{(1, \text{QED})} + \mathcal{O}(\hbar), \tag{4.62}$$

in terms of the one-loop QED amplitude  $\mathcal{A}^{(1, \text{QED})}$ .

The colour impulse factor in equation (4.54) therefore reduces to a single commutator. To calculate this we need to repeatedly apply the commutation

relation in equation (4.8), which yields

$$\begin{aligned}
[C_1^a, \mathcal{C}(\text{box})] &= [C_1^a, C_1^b C_1^c] C_2^b C_2^c \\
&= i\hbar f^{acd} (C_1^d C_1^b C_2^c C_2^b + C_1^b C_1^d C_2^b C_2^c) \\
&= i\hbar f^{acd} (C_1^d C_2^c (C_1 \cdot C_2) + (C_1^d C_1^b + i\hbar f^{bde} C_1^e) (C_2^c C_2^b + i\hbar f^{bce} C_2^e)) \\
&= i\hbar f^{acd} (2C_1^d C_2^c (C_1 \cdot C_2) - i\hbar f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) + \mathcal{O}(\hbar^2)) .
\end{aligned} \tag{4.63}$$

The colour impulse factor is itself a series in  $\hbar$ . The partial amplitude is also a Laurent series in  $\hbar$ , which is presented in appendix B.2. In brief, the leading term in this expansion is the apparently singular (enhanced by one inverse power of  $\hbar$ ) part  $\mathcal{A}_{-1}^{(1, \text{QED})} \sim \mathcal{O}(\hbar^{-2})$ , and the classical term  $\mathcal{A}_0^{(1, \text{QED})} \sim \mathcal{O}(\hbar^{-1})$ . This has a very important consequence for the impulse kernel — unlike in the QED case, the apparently singular term  $\mathcal{A}_{-1}^{(1, \text{QED})}$  in the partial amplitude now contributes classically, because of the second term in the colour impulse factor:

$$\begin{aligned}
\mathcal{G}_{1\text{-loop}}^{a,(1)} &= \hbar g^4 \left\{ 2i f^{acd} C_1^d C_2^c (C_1 \cdot C_2) (\mathcal{A}_{-1}^{(1, \text{QED})} + \mathcal{A}_0^{(1, \text{QED})}) \right. \\
&\quad \left. + \hbar f^{acd} f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) \mathcal{A}_{-1}^{(1, \text{QED})} \right\} . \tag{4.64}
\end{aligned}$$

However, there are still singular terms in the first line; their cancellation requires including the quadratic part of the colour kernel.

### Cut box

Rather than viewing the second term in equation (4.54) as a product of trees, we will treat the quadratic piece as a weighted cut of the box diagram, and define

$$\begin{aligned}
\mathcal{B} &= -i\hbar^2 \int \hat{d}^4 \bar{\ell} \hat{\delta}(2p_1 \cdot \bar{\ell} + \hbar \bar{\ell}^2) \hat{\delta}(2p_2 \cdot \bar{\ell} - \hbar \bar{\ell}^2) \\
&\quad \times \bar{A}_{\succ\prec}(p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q} \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}) \bar{A}_{\succ\prec}(p_1, p_2 \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}) .
\end{aligned} \tag{4.65}$$

Using the tree in equation (4.49) we can Laurent expand this expression in  $\hbar$ , as discussed in appendix B.2, which yields the leading terms

$$\begin{aligned}\mathcal{B}_{-1} &= -i \frac{4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{\hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \\ \mathcal{B}_0 &= -i \frac{2(p_1 \cdot p_2)^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\bar{\ell} \cdot \bar{q}}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left\{ \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}(p_2 \cdot \bar{\ell}) \hat{\delta}'(p_1 \cdot \bar{\ell}) \right\} .\end{aligned}\tag{4.66}$$

To determine the classical contributions we must calculate the associated colour impulse factor,

$$\begin{aligned}\mathcal{C}(\text{><})^\dagger [\mathbb{C}_1^a, \mathcal{C}(\text{><})] &= i\hbar (C_1 \cdot C_2) f^{abc} C_1^c C_2^b \\ &= i\hbar f^{abc} (C_1^c C_1^d + i\hbar f^{dce} C_1^e) (C_2^b C_2^d + i\hbar f^{dbe} C_2^e) \\ &= i\hbar f^{acd} C_1^d C_2^c (C_1 \cdot C_2) + \hbar^2 f^{acd} f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) + \mathcal{O}(\hbar^3) .\end{aligned}\tag{4.67}$$

Clearly we have a similar situation to equation (4.64): the colour impulse factor is again an expansion in  $\hbar$ , and its leading term yields a classical contributions from  $\mathcal{B}_0$ . Meanwhile  $\mathcal{B}_{-1}$  also contributes classically from the correction to the colour structure — however, there is still a singular term:

$$\begin{aligned}\mathcal{G}_{\text{cut box}}^{a,(1)} &= i\hbar g^4 f^{acd} \left( C_1^d C_2^c (C_1 \cdot C_2) (\mathcal{B}_{-1} + \mathcal{B}_0) \right. \\ &\quad \left. - i\hbar f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) \mathcal{B}_{-1} \right) \\ &= 2g^4 f^{acd} (p_1 \cdot p_2)^2 \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left\{ C_1^d C_2^c (C_1 \cdot C_2) \right. \\ &\quad \times \left[ \frac{2\hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell})}{\hbar} + \bar{\ell} \cdot \bar{q} \left( \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}(p_2 \cdot \bar{\ell}) \hat{\delta}'(p_1 \cdot \bar{\ell}) \right) \right] \\ &\quad \left. - 2i f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right\} .\end{aligned}\tag{4.68}$$

## Combining terms

It now remains to calculate the full colour kernel,

$$\mathcal{G}^{a,(1)} = \mathcal{G}_{1\text{-loop}}^{a,(1)} + \mathcal{G}_{\text{cut box}}^{a,(1)} .\tag{4.69}$$

The first priority is to study the classically singular terms which sit in both parts of the kernel. Recall that each part of equation (4.54) came with a different colour

kernel. The upshot of this fact is that, after explicit calculation, the singular terms now involve one common colour structure:

$$\begin{aligned} \mathcal{G}_{-1}^{a,(1)} = & i\hbar g^4 f^{acd} C_1^d C_2^c (C_1 \cdot C_2) \left( 2\mathcal{A}_{-1}^{(1,\text{QED})} + \mathcal{B}_{-1} \right) \\ & + \hbar^2 g^4 f^{acd} f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) \left( \mathcal{A}_{-1}^{(1,\text{QED})} + \mathcal{B}_{-1} \right). \end{aligned} \quad (4.70)$$

As detailed in appendix B.2, the singular term in the expansion of the QED amplitude originates entirely from box diagrams, and takes a neat form in terms of delta functions:

$$\mathcal{A}_{-1}^{(1,\text{QED})} = i \frac{2(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}). \quad (4.71)$$

This expression is the same as the  $\mathcal{B}_{-1}$  term in equation (4.66). The coefficients are such that the terms in the first line of equation (4.70) cancel, ensuring the apparently singular part of the colour kernel vanishes.

However, an interesting new feature of the colour impulse is that the colour structure in the second line of equation (4.70) combines with the sum of the 1-loop singular terms to give a non-zero classical contribution:

$$\left[ \mathcal{G}_{-1}^{a,(1)} \right]_{\mathcal{O}(\hbar^0)} = -2ig^4 f^{acd} f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) (p_1 \cdot p_2)^2 \int \hat{d}^4 \bar{\ell} \frac{\hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2}. \quad (4.72)$$

With all possible singular terms safely dealt with, it remains to combine the  $\mathcal{O}(\hbar^0)$  terms in equation (4.64) and equation (4.68). Conveniently, these all have the same colour factor:

$$\mathcal{G}^{a,(1)} = ig^4 \hbar f^{acd} C_1^d C_2^c (C_1 \cdot C_2) \left( 2\mathcal{A}_0^{(1,\text{QED})} + \mathcal{B}_0 \right) + \left[ \mathcal{G}_{-1}^{a,(1)} \right]_{\mathcal{O}(\hbar^0)}. \quad (4.73)$$

Now we can sum the diagrams in the partial amplitude, the explicit expressions

for which are given in appendix B.2. The result is

$$\begin{aligned}
\mathcal{G}^{a,(1)} = g^4 \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} & \left\{ 4i f^{acd} C_2^c C_1^d (C_1 \cdot C_2) \right. \\
& \times \left[ \hat{\delta}(p_1 \cdot \bar{\ell}) \left[ m_1^2 + (p_1 \cdot p_2)^2 \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{1}{(p_2 \cdot \bar{\ell} - i\epsilon)^2} + i\hat{\delta}'(p_2 \cdot \bar{\ell}) \right) \right] \right. \\
& + \hat{\delta}(p_2 \cdot \bar{\ell}) \left[ m_2^2 + (p_1 \cdot p_2)^2 \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{1}{(p_1 \cdot \bar{\ell} + i\epsilon)^2} - i\hat{\delta}'(p_1 \cdot \bar{\ell}) \right) \right] \\
& \left. \left. - 2i f^{acd} f^{dbe} (C_1^e C_2^b C_2^c - C_2^e C_1^b C_1^c) (p_1 \cdot p_2)^2 \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right] \right\}. \tag{4.74}
\end{aligned}$$

### Final result

Finally the observable, the colour impulse, is given by

$$\Delta c_1^{a,(1)} = \left\langle \left\langle i \int \hat{d}^4 \bar{q} \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}} \mathcal{G}^{a,(1)} \right\rangle \right\rangle. \tag{4.75}$$

Upon substituting the kernel in equation (4.74), we can average over the momentum and colour wavefunctions implicit in the expectation value. For sharply peaked momentum wavefunctions and large  $SU(N)$  representations, this merely has the effect of sending  $p_i \mapsto m_i u_i$ , and replacing quantum colour factors with products of commuting classical charges. Hence we finally obtain the NLO colour impulse

$$\begin{aligned}
\Delta c_1^{a,(1)} = g^4 \int \hat{d}^4 \bar{q} \hat{d}^4 \bar{\ell} \hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q}) e^{-i\bar{q} \cdot b} \frac{1}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} \\
\times \left\{ \hat{\delta}(u_1 \cdot \bar{\ell}) \left[ \frac{f^{acd} c_1^c c_2^d (c_1 \cdot c_2)}{m_2} \left[ 1 + (u_1 \cdot u_2)^2 \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{1}{(u_2 \cdot \bar{\ell} - i\epsilon)^2} \right. \right. \right. \right. \\
\left. \left. \left. + i\hat{\delta}'(u_2 \cdot \bar{\ell}) \right) \right] - f^{acd} f^{dbe} c_1^b c_1^c c_2^e \frac{(u_1 \cdot u_2)^2}{2} \hat{\delta}(u_2 \cdot \bar{\ell}) \right] \\
+ \hat{\delta}(u_2 \cdot \bar{\ell}) \left[ \frac{f^{acd} c_1^c c_2^d (c_1 \cdot c_2)}{m_1} \left[ 1 + (u_1 \cdot u_2)^2 \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{1}{(u_1 \cdot \bar{\ell} + i\epsilon)^2} \right. \right. \right. \\
\left. \left. \left. - i\hat{\delta}'(u_1 \cdot \bar{\ell}) \right) \right] + f^{acd} f^{dbe} c_1^e c_2^b c_2^c \frac{(u_1 \cdot u_2)^2}{2} \hat{\delta}(u_1 \cdot \bar{\ell}) \right] \right\}. \tag{4.76}
\end{aligned}$$

This is found to agree with the result obtained by solving the classical equations of motion, which is discussed in section 4.4.

#### 4.2.4 Momentum

It is evident that the usual (momentum) impulse in YM theory should be similar to the QED case discussed in [45]. But it is also natural to expect some new terms in the YM impulse in view of the self-coupling of the YM field. Diagrams involving this self-coupling are present at NLO. In this subsection, we investigate the impulse in the YM case with this thought in mind. We begin with equation (4.47) for the impulse kernel  $\mathcal{I}^\mu$ , which now only involves colour factors of the partial amplitudes themselves.

At leading order we can just reuse the expressions in equation (4.49), finding

$$\mathcal{I}^{\mu,(0)} = 4g^2 \frac{(p_1 \cdot p_2)}{\bar{q}^2} \bar{q}^\mu C_1 \cdot C_2. \quad (4.77)$$

Then, substituting into equation (4.46) and taking the classical limit as before we have the LO momentum impulse

$$\Delta p_1^{\mu,(0)} = ig^2 c_1 \cdot c_2 \int \hat{d}^4 \bar{q} \frac{\hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}}}{\bar{q}^2} (u_1 \cdot u_2) \bar{q}^\mu. \quad (4.78)$$

This expression is closely related to the NLO impulse in QED, which can be obtained from the YM case by replacing  $c_1 \cdot c_2$  with the product of the electric charges of the two particles. This relationship is natural, since at leading order the gluons do not self-interact.

Just as in the colour case, at NLO the momentum kernel has linear, 1-loop and quadratic, cut box components:

$$\begin{aligned} \mathcal{I}^{\mu,(1)} = & \hbar g^4 \bar{q}^\mu \sum_{\Gamma} \mathcal{C}(\Gamma) \bar{A}_{\Gamma}^{(1)}(p_1, p_2 \rightarrow p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q}) \\ & - ig^4 \hbar^3 \int \hat{d}^4 \bar{\ell} \hat{\delta}(2p_1 \cdot \bar{\ell} + \hbar \bar{\ell}^2) \hat{\delta}(2p_2 \cdot \bar{\ell} - \hbar \bar{\ell}^2) \bar{\ell}^\mu \mathcal{C}(\text{box})^\dagger \mathcal{C}(\text{box}) \\ & \times \bar{A}_{\text{box}}(p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q} \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}) \bar{A}_{\text{box}}(p_1, p_2 \rightarrow p_1 + \hbar \bar{\ell}, p_2 - \hbar \bar{\ell}). \end{aligned} \quad (4.79)$$

The decomposition of the 1-loop amplitude onto the colour basis in equation (4.61) makes computing the first term (linear in the one-loop amplitude)

in the impulse kernel trivial; we have

$$\mathcal{I}_{1\text{-loop}}^{\mu,(1)} = \hbar g^4 \bar{q}^\mu \mathcal{C} \left( \text{triangle} \right) \left( \mathcal{A}_{-1}^{(1,\text{QED})} + \mathcal{A}_0^{(1,\text{QED})} \right). \quad (4.80)$$

This means that the non-Abelian triangle Feynman diagrams do not contribute to the impulse: a somewhat surprising result, since it is only in these diagrams that the self-interaction of the gluons appears.

Meanwhile we will denote the kinematic terms in the quadratic piece of the momentum kernel  $\mathcal{B}^\mu$ , which has the same definition as equation (4.65) but dressed with an extra loop momentum — explicit expressions are given in appendix B.2. Its colour factor is simply

$$\mathcal{C} \left( \text{box} \right)^\dagger \mathcal{C} \left( \text{box} \right) = (C_2 \cdot C_1)(C_1 \cdot C_2) = \mathcal{C} \left( \text{triangle} \right). \quad (4.81)$$

Thus there is only one relevant colour structure in the NLO momentum impulse, that of the box. The momentum kernel factorises accordingly:

$$\mathcal{I}^{\mu,(1)} = g^4 (C_1 \cdot C_2)^2 \left[ \hbar \bar{q}^\mu \left( \mathcal{A}_{-1}^{(1,\text{QED})} + \mathcal{A}_0^{(1,\text{QED})} \right) + \mathcal{B}_{-1}^\mu + \mathcal{B}_0^\mu \right]. \quad (4.82)$$

This is just a colour dressing of the NLO momentum impulse in QED — in particular, the cancellation of singular terms between the cut box and 1-loop diagrams is guaranteed [45]. Gathering all the terms from triangles, boxes and the cut box in appendix B.2 and inserting into equation (4.46), upon taking the classical limit in the now familiar way we find

$$\begin{aligned} \Delta p_1^{\mu,(1)} = & \frac{g^4 (c_1 \cdot c_2)^2}{2} \int \hat{d}^4 \bar{\ell} \hat{d}^4 \bar{q} \frac{\hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q})}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} \left[ \bar{q}^\mu \left\{ \frac{\hat{\delta}(u_2 \cdot \bar{\ell})}{m_1} + \frac{\hat{\delta}(u_1 \cdot \bar{\ell})}{m_2} \right. \right. \\ & + (u_1 \cdot u_2)^2 \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{\hat{\delta}(u_1 \cdot \bar{\ell})}{m_2 (u_2 \cdot \bar{\ell} - i\epsilon)^2} + \frac{\hat{\delta}(u_2 \cdot \bar{\ell})}{m_1 (u_1 \cdot \bar{\ell} + i\epsilon)^2} \right) \Big\} \\ & \left. \left. - i \bar{\ell}^\mu \bar{\ell} \cdot (\bar{\ell} - \bar{q}) \left( \frac{\hat{\delta}'(u_1 \cdot \bar{\ell}) \hat{\delta}(u_2 \cdot \bar{\ell})}{m_2} - \frac{\hat{\delta}(u_1 \cdot \bar{\ell}) \hat{\delta}'(u_2 \cdot \bar{\ell})}{m_1} \right) \right] \right]. \end{aligned} \quad (4.83)$$

We have found that in the non-Abelian theory the final result for the impulse is identical to QED [45] with the charge to colour replacement  $Q_1 Q_2 \rightarrow c_1 \cdot c_2$ . In fact this result follows from the colour basis decomposition in equation (4.61) and in particular the fact that the non-Abelian triangle diagrams only contribute to the  $\hbar^2$  suppressed second colour structure. At 2 loops and higher, there are more

singular terms in the kinematic parts of the amplitudes, so these  $\hbar^2$  terms in the colour factors will generate new classical pieces rather than being quantum.

## 4.3 Radiation

One of the strengths of studying impulse-like observables is that radiative phenomena are naturally included, as explored in depth in [45]. Moreover, the double copy makes radiation in Yang-Mills theory a powerful tool for studying its gravitational counterpart [41, 54, 231]. The general discussion of radiation of momentum in KMOC [45] applies directly to the Yang-Mills case, but weakly-coupled YM theory contains another interesting observable: the total colour radiated to infinity. In this section, we study this radiation of colour in the quantum formalism. As an explicit example we compute the leading order classical colour current found in [41] from scattering amplitudes.

### 4.3.1 Total radiated colour

The construction and calculation of the colour impulse relied on the adjoint-valued colour operator in equation (4.7) for a scalar field in representation  $R$ . To study the total radiated colour we need a similar operator for the gluon radiation field. This can easily be obtained by restricting to the adjoint representation, namely by taking  $(T_{\text{adj}}^a)_b^c = if^{bac}$ . Since the gluon field is real and has two helicity eigenstates, its colour operator is

$$\mathbb{F}^a = i\hbar f^{bac} \sum_{\sigma=\pm} \int d\Phi(k) a_{\sigma}^{b\dagger}(k) a_{\sigma}^c(k), \quad (4.84)$$

where  $\sigma$  labels the helicity. Of course, it is also possible to derive this expression directly from the Noether charge for vector fields in the adjoint representation, given in equation (4.117), in close analogy to our discussion in sections 4.1.1 and 4.1.2.

This adjoint colour charge is of interest elsewhere in the literature since it plays a role in the physics of YM theory at asymptotic infinity [232–238]. In this connection, the nature of the final state of the radiation is relevant [237]. Here, we do not compute this final state explicitly. Instead, we compute expectation values of operators on the final state. In chapter chapters 5 and 6 however we

will address the nature of the final state, including radiation for pure scalars as well as particles with extra charges such as spin or colour.

Following the KMOC procedure to obtain an expression for the total colour charge radiated from a scattering event leads to

$$\langle R_{\text{col}}^a \rangle = \langle \Psi | T^\dagger \mathbb{F}^a T | \Psi \rangle, \quad (4.85)$$

where we made use of the fact that there are no gauge bosons in the incoming state of equation (2.11). Before expanding in terms of on-shell scattering amplitudes, it is worth demonstrating colour conservation in our formalism. At the operator level, assuming only the quantum fields corresponding to particles 1 and 2 are present in addition to the Yang-Mills field, the statement that colour is conserved is

$$[\mathbb{C}_1^a + \mathbb{C}_2^a + \mathbb{F}^a, T] = 0. \quad (4.86)$$

It then immediately follows that

$$\begin{aligned} \langle \Delta c_1^a \rangle + \langle \Delta c_2^a \rangle &= \langle \Psi | T^\dagger [\mathbb{C}_1^a, T] | \Psi \rangle + \langle \Psi | T^\dagger [\mathbb{C}_2^a, T] | \Psi \rangle \\ &= -\langle \Psi | T^\dagger [\mathbb{F}^a, T] | \Psi \rangle = -\langle \Psi | T^\dagger \mathbb{F}^a T | \Psi \rangle = -\langle R_{\text{col}}^a \rangle, \end{aligned} \quad (4.87)$$

where the second line holds from the absence of gluon radiation in the incoming state. Total colour is therefore conserved in the quantum theory, as it must be given the associated global symmetry.

Let us proceed in expanding equation (4.85) by inserting complete sets of states; for leading order radiation we need to consider an extra explicit gluon with momentum  $k$  and a colour index, so we will take the set  $X$  in the resolution of the identity in equation (4.39) to just include the contribution

$$\sum_{b,\sigma} \int d\Phi(k) |k^b, \sigma\rangle \langle k^b, \sigma|. \quad (4.88)$$

Note that higher order corrections could also be obtained by adding further states, but we will just be interested in the lowest order case here. Using equation (4.84)

and integrating over intermediate delta functions we find

$$\begin{aligned}
\langle R_{\text{col}}^a \rangle &= \sum_{b,c,\sigma} \int d\Phi(k) d\Phi(\tilde{k}) d\Phi(r_1) d\Phi(r_2) d\mu(\zeta_1) d\mu(\zeta_2) \\
&\quad \times \langle \Psi | T^\dagger | r_1 r_2 k^b, \sigma; \zeta_1 \zeta_2 \rangle \langle k^b, \sigma | \mathbb{F}^a | \tilde{k}^c, \sigma \rangle \langle r_1 r_2 \tilde{k}^c, \sigma; \zeta_1 \zeta_2 | T | \Psi \rangle \\
&= -i\hbar \sum_{b,c,\sigma} \int d\Phi(k) d\Phi(r_1) d\Phi(r_2) d\mu(\zeta_1) d\mu(\zeta_2) f^{abc} \mathcal{R}^{*b}(r_1, r_2; k, \sigma) \mathcal{R}^c(r_1, r_2; k, \sigma),
\end{aligned} \tag{4.89}$$

where

$$\begin{aligned}
\mathcal{R}^a(r_1, r_2; k, \sigma) &= \int d\Phi(p_1) d\Phi(p_2) \phi_1(p_1) \phi_2(p_2) e^{ib \cdot p_1 / \hbar} \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - k) \\
&\quad \times \sum_D \langle \zeta_1 \zeta_2 | \mathcal{C}^a(D) | \chi_1 \chi_2 \rangle A_D(p_1, p_2 \rightarrow r_1, r_2; k, \sigma).
\end{aligned} \tag{4.90}$$

We have factorised the amplitude into colour structures  $\mathcal{C}(D)$  and partial amplitudes  $A_D$ , as in equation (4.40). However, here the colour factor gains a free index from the external gluon state.

To take the classical limit of equation (4.89), we introduce radiation kernels as in [45], finding

$$R_{\text{col}}^a = -i f^{abc} \sum_{\sigma} \left\langle \left\langle \hbar^{-2} \int d\Phi(k) \mathcal{R}^{*b}(k, \sigma) \mathcal{R}^c(k, \sigma) \right\rangle \right\rangle. \tag{4.91}$$

The large angle brackets, defined in eq. (2.30), are the expectation value over the incoming scalar wavepackets, and thus include the colour states. The radiation kernels inherit the colour index of the external gluon, and take the form

$$\begin{aligned}
\mathcal{R}^a(k, \sigma) &= \hbar^{\frac{3}{2}} \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p_1 \cdot q_1 + q_1^2) \hat{\delta}(2p_2 \cdot q_2 + q_2^2) \hat{\delta}^{(4)}(k - q_1 - q_2) e^{ib \cdot q_1 / \hbar} \\
&\quad \times \sum_D \mathcal{C}^a(D) A_D(p_1 + q_1, p_2 + q_2 \rightarrow p_1, p_2; k, \sigma).
\end{aligned} \tag{4.92}$$

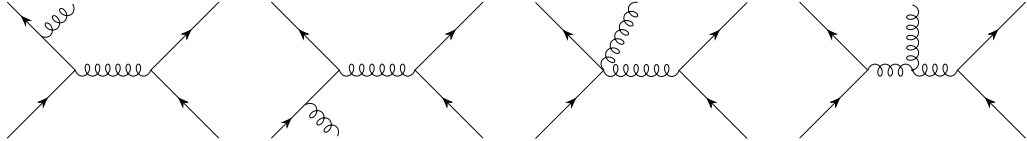
The powers of  $\hbar$  are organised such that the radiation kernel will be  $\mathcal{O}(\hbar^0)$  and therefore classical in the limit. Note that because the colour charge has dimensions of angular momentum, the  $\hbar$  scaling here works out the same way as in the total radiated momentum.

### 4.3.2 Leading order evaluation

Let us explicitly compute the leading order radiation kernel for the scattering of two massive scalar particles, described by the action in equation (4.2). In the classical limit the LO kernel is given in terms of coupling constant stripped amplitudes by

$$\begin{aligned} \mathcal{R}^{a,(0)}(\bar{k}) &= \hbar^2 g^3 \int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \hat{\delta}(2p_2 \cdot \bar{q}_2 + \hbar \bar{q}_2^2) \\ &\times \hat{\delta}^{(4)}(\bar{k} - \bar{q}_1 - \bar{q}_2) e^{ib \cdot \bar{q}_1} \sum_D \mathcal{C}^a(D) \bar{A}_D^{(0)}(p_1 + q_1, p_2 + q_2 \rightarrow p_1, p_2; k, \sigma). \end{aligned} \quad (4.93)$$

Clearly the terms in the amplitude which contribute to the classical radiation are those at  $\mathcal{O}(\hbar^{-2})$ . The shifts in the delta functions are important for obtaining this accurately. The relevant amplitude is the non-Abelian extension of the 5-point tree studied in [45]; this was used in [54] to take the double copy and calculate radiation in Einstein gravity. The relevant Feynman topologies for emission from particle 1 are<sup>6</sup>



(4.94)

so we need to calculate the classical terms in the 5-point amplitude

$$\begin{aligned} \mathcal{A}^{(0)}(\bar{k}^a) &= \sum_D \mathcal{C}^a(D) \bar{A}_D^{(0)}(p_1 + q_1, p_2 + q_2 \rightarrow p_1, p_2; k, \sigma) \\ &= [\mathcal{C}^a(\text{diagram 1}) A_{\times\leftarrow} + \mathcal{C}^a(\text{diagram 2}) A_{\leftarrow\times} + \mathcal{C}^a(\text{diagram 3}) A_{\times\leftarrow} + (1 \leftrightarrow 2)] + \mathcal{C}^a(\text{diagram 4}) A_{\leftarrow\times}. \end{aligned} \quad (4.95)$$

Explicitly, the colour factors are given by

$$\begin{aligned} \mathcal{C}^a(\text{diagram 1}) &= (C_1^a \cdot C_1^b) C_2^b, & \mathcal{C}^a(\text{diagram 2}) &= (C_1^b \cdot C_1^a) C_2^b, \\ \mathcal{C}^a(\text{diagram 3}) &= \frac{1}{2} \mathcal{C}^a(\text{diagram 1}) + \frac{1}{2} \mathcal{C}^a(\text{diagram 2}), & \mathcal{C}^a(\text{diagram 4}) &= \hbar f^{abc} C_1^b C_2^c, \end{aligned} \quad (4.96)$$

with the replacement  $1 \leftrightarrow 2$  for diagrams with gluon emission from particle 2. Just as in the 4-point case at 1-loop, this is an overcomplete set for specifying a basis, because

$$\mathcal{C}^a(\text{diagram 2}) = (C_1^a \cdot C_1^b) C_2^b + i \hbar f^{bac} C_1^c C_2^b = \mathcal{C}^a(\text{diagram 1}) + i \mathcal{C}^a(\text{diagram 4}). \quad (4.97)$$

<sup>6</sup>The momentum routing is as indicated in equation (4.93).

Hence the full basis of colour factors is only 3 dimensional, and the colour decomposition of the 5-point tree is

$$\begin{aligned} \mathcal{A}^{(0)}(\bar{k}^a) = \mathcal{C}^a \left( \text{triangle diagram} \right) & \left( A_{\text{triangle}} + A_{\text{triangle}} + A_{\text{triangle}} \right) \\ & + \frac{1}{2} \mathcal{C}^a \left( \text{box diagram} \right) \left( A_{\text{box}} + 2iA_{\text{box}} + iA_{\text{box}} \right) + (1 \leftrightarrow 2). \end{aligned} \quad (4.98)$$

Given that the second structure is  $\mathcal{O}(\hbar)$ , it would appear that we could again neglect the second term as a quantum correction. However, this intuition is not quite correct, as calculating the associated partial amplitude shows:

$$\begin{aligned} A_{\text{triangle}} + 2iA_{\text{triangle}} + iA_{\text{triangle}} = -\frac{4i \varepsilon_\mu^h(\bar{k})}{\hbar^2} & \left[ \frac{2p_1 \cdot p_2}{\bar{q}_2^2} \frac{p_1^\mu}{p_1 \cdot \bar{k}} + \frac{1}{\hbar \bar{q}_1^2 \bar{q}_2^2} \left( 2p_2 \cdot \bar{k} p_1^\mu \right. \right. \\ & \left. \left. - p_1 \cdot p_2 (\bar{q}_1^\mu - \bar{q}_2^\mu) - 2p_1 \cdot \bar{k} p_2^\mu \right) + \mathcal{O}(\hbar^0) \right], \end{aligned} \quad (4.99)$$

where we have used  $p_1 \cdot \bar{q}_2 = p_1 \cdot \bar{k} - \hbar \bar{q}_1^2/2$  on the support of the on-shell delta functions in the kernel equation (4.93). The partial amplitude appears to be singular, as there is an extra power of  $\hbar$  downstairs. However, this will cancel against the extra power in the colour structure, yielding a classical contribution. Meanwhile in the other partial amplitude the potentially singular terms cancel trivially, as in QED, and the contribution is classical:

$$\begin{aligned} A_{\text{triangle}} + A_{\text{triangle}} + A_{\text{triangle}} = \frac{2}{\hbar^2} \frac{\varepsilon_\mu^h(\bar{k})}{\bar{q}_2^2 p_1 \cdot \bar{k}} & \left[ 2p_1 \cdot p_2 \bar{q}_2^\mu + \frac{p_1 \cdot p_2}{p_1 \cdot \bar{k}} p_1^\mu (\bar{q}_1^2 - \bar{q}_2^2) \right. \\ & \left. - 2p_1 \cdot \bar{k} p_2^\mu + 2p_2 \cdot \bar{k} p_1^\mu + \mathcal{O}(\hbar) \right]. \end{aligned} \quad (4.100)$$

Summing all colour factors and partial amplitudes, the classically significant part of the 5-point amplitude is

$$\begin{aligned} \bar{\mathcal{A}}^{(0)}(\bar{k}^a) = \sum_D \mathcal{C}^a(D) \bar{A}_D^{(0)}(\bar{k}) \\ = -\frac{4\varepsilon_\mu^h(\bar{k})}{\hbar^2} \left\{ \frac{C_1^a(C_1 \cdot C_2)}{\bar{q}_2^2 \bar{k} \cdot p_1} \left[ -(p_1 \cdot p_2) \left( \bar{q}_2^\mu - \frac{\bar{k} \cdot \bar{q}_2}{\bar{k} \cdot p_1} p_1^\mu \right) + \bar{k} \cdot p_1 p_2^\mu - \bar{k} \cdot p_2 p_1^\mu \right] \right. \\ \left. + \frac{if^{abc} C_1^b C_2^c}{\bar{q}_1^2 \bar{q}_2^2} \left[ 2\bar{k} \cdot p_2 p_1^\mu - p_1 \cdot p_2 \bar{q}_1^\mu + p_1 \cdot p_2 \frac{\bar{q}_1^2}{\bar{k} \cdot p_1} p_1^\mu \right] + (1 \leftrightarrow 2) \right\}, \end{aligned} \quad (4.101)$$

where we have used that  $\bar{q}_1^2 - \bar{q}_2^2 = -2\bar{k} \cdot \bar{q}_2$  since the outgoing radiation is on-shell. Finally, we can substitute into the radiation kernel in equation (4.93) and take the classical limit. Averaging over the wavepackets sets  $p_i = m_i u_i$  and replacing

quantum colour charges with their classical counterparts, yielding

$$\begin{aligned} \mathcal{R}^{a,(0)}(\bar{k}) = & -g^3 \int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}^{(4)}(\bar{k} - \bar{q}_1 - \bar{q}_2) \hat{\delta}(u_1 \cdot \bar{q}_1) \hat{\delta}(u_2 \cdot \bar{q}_2) e^{ib \cdot \bar{q}_1} \varepsilon_\mu^h \\ & \times \left\{ \frac{c_1 \cdot c_2}{m_1} \frac{c_1^a}{\bar{q}_2^2 \bar{k} \cdot u_1} \left[ -(u_1 \cdot u_2) \left( \bar{q}_2^\mu - \frac{\bar{k} \cdot \bar{q}_2}{\bar{k} \cdot u_1} u_1^\mu \right) + \bar{k} \cdot u_1 u_2^\mu - \bar{k} \cdot u_2 u_1^\mu \right] \right. \\ & \left. + \frac{if^{abc} c_1^b c_2^c}{\bar{q}_1^2 \bar{q}_2^2} \left[ 2\bar{k} \cdot u_2 u_1^\mu - u_1 \cdot u_2 \bar{q}_1^\mu + u_1 \cdot u_2 \frac{\bar{q}_1^2}{\bar{k} \cdot u_1} u_1^\mu \right] + (1 \leftrightarrow 2) \right\}. \end{aligned} \quad (4.102)$$

Our result is equal to the leading order current  $\tilde{K}^{a,(0)}$  obtained in [41] by iteratively solving the Wong equations in equation (4.1a) and equation (4.1b) for timelike particle worldlines. We will show this explicitly in the next section.

## 4.4 Classical perspectives

In this section we compute the same classical observables, impulse and radiation, using purely classical techniques. These calculations are not too complex, and serve to verify the results we obtained using scattering amplitudes. This gives confidence in applying these quantum methods to gravity, for example, where the classical calculations can become significantly more involved. We start with the colour and momentum impulses before moving to the total radiated colour charge, discussing its relation to asymptotic symmetries.

### Impulses from equations of motion

We start with the NLO colour impulse, initially in the more general case of a system of  $N$  interacting particles, and later restrict to the  $N = 2$  case for comparison with earlier sections.

As discussed previously, the appropriate equations of motion for each particle's worldline are the Yang-Mills-Wong equations in equation (4.1a) and equation (4.1b). We are seeking perturbative solutions, and therefore expand worldline quantities in the coupling:

$$\begin{aligned} x_\alpha^\mu(\tau_\alpha) &= b_\alpha^\mu + u_\alpha \tau_\alpha + \Delta^{(1)} x_\alpha^\mu(\tau_\alpha) + \Delta^{(2)} x_\alpha^\mu(\tau_\alpha) + \dots, \\ v_\alpha^\mu(\tau_\alpha) &= u_\alpha + \Delta^{(1)} v_\alpha^\mu(\tau_\alpha) + \Delta^{(2)} v_\alpha^\mu(\tau_\alpha) + \dots, \\ c_\alpha^a(\tau_\alpha) &= c_\alpha^a + \Delta^{(1)} c_\alpha^a(\tau_\alpha) + \Delta^{(2)} c_\alpha^a(\tau_\alpha) + \dots. \end{aligned} \quad (4.103)$$

Here  $\Delta^{(i)}x_\alpha^\mu$  indicates quantities entering at  $\mathcal{O}(g^{2i})$ . Calculating higher order corrections requires solving for the gauge field  $A_\mu^a(x)$ , using equation (4.1c). Provided the particle worldlines remain well separated, we can find perturbative solutions using the field equation in the form [41]

$$\begin{aligned}\partial^2 A_\mu^a(x) &= K_\mu^a(x), \\ K_\mu^a(x) &\equiv J_\mu^a(x) + gf^{abc}A^{b\nu}(x)(\partial_\nu A_\mu^c(x) - F_{\mu\nu}^c(x)).\end{aligned}\tag{4.104}$$

The current  $K_\mu^a$  is conserved but gauge dependent; for simplicity we have chosen Lorenz gauge. Writing  $\Delta^{(i)}A_\mu^a(\bar{\ell})$  for perturbative corrections to gauge-field quantities at order  $\mathcal{O}(g^{2i-1})$ , the LO and NLO gauge fields are solutions to the equations

$$\begin{aligned}\partial^2[\Delta^{(1)}A_\mu^a(x)] &= \Delta^{(1)}J_\mu^a(x), \\ \partial^2[\Delta^{(2)}A_\mu^a(x)] &= \Delta^{(2)}J_\mu^a(x) + gf^{abc}[\Delta^{(1)}A^{b\nu}(x)(\partial_\nu \Delta^{(1)}A_\mu^c(x) - \Delta^{(1)}F_{\mu\nu}^c(x))],\end{aligned}\tag{4.105}$$

respectively — note that the LO equation is the same as for Abelian electrodynamics. We solve by Fourier transforming using tildes to represent the Fourier transformed quantity. Solving the LO equation, it is easy to show that the leading order field is

$$\Delta^{(1)}\tilde{A}^{a\mu}(\bar{\ell}) = -g \sum_\alpha \hat{\delta}(u_\alpha \cdot \bar{\ell}) e^{i\bar{\ell} \cdot b_\alpha} \frac{c_\alpha^a u_\alpha^\mu}{\bar{\ell}^2}.\tag{4.106}$$

It will be useful to define the straight line trajectory  $y_\alpha^\mu \equiv b_\alpha^\mu + u_\alpha^\mu \tau_\alpha$  corresponding to the initial unperturbed worldlines. With this definition the LO equations of motion become

$$\begin{aligned}m_\alpha \frac{d^2 \Delta^{(1)}x_\alpha^\mu}{d\tau_\alpha^2} &= g c_\alpha^a \int \hat{d}^4 \bar{\ell} e^{-i\bar{\ell} \cdot y_\alpha} \Delta^{(1)}\tilde{F}^{a\mu\nu}(\bar{\ell}) u_{\alpha\nu}, \\ \frac{d\Delta^{(1)}c_\alpha^a}{d\tau_\alpha} &= gf^{abc} \int \hat{d}^4 \bar{\ell} e^{-i\bar{\ell} \cdot y_\alpha} u_\alpha^\mu \Delta^{(1)}\tilde{A}_\mu^b(\bar{\ell}) c_\alpha^c,\end{aligned}\tag{4.107}$$

while at NLO,  $\mathcal{O}(g^4)$ , we have

$$\begin{aligned}
\frac{d^2 \Delta^{(2)} x_\alpha^\mu}{d\tau_\alpha^2} &= \frac{g}{m_\alpha} \int \hat{d}^4 \bar{\ell} e^{-i\bar{\ell} \cdot y_\alpha} \left[ \Delta^{(1)} \tilde{F}^{a\mu\nu}(\bar{\ell}) \Delta^{(1)} v_{\alpha\nu} c_\alpha^a + \Delta^{(1)} \tilde{F}^{a\mu\nu}(\bar{\ell}) \right. \\
&\quad \times u_{\alpha\nu} \Delta^{(1)} c_\alpha^a + \Delta^{(2)} \tilde{F}^{a\mu\nu}(\bar{\ell}) u_{\alpha\nu} c_\alpha^a - i\bar{\ell} \cdot \Delta^{(1)} x_\alpha \Delta^{(1)} \tilde{F}^{a\mu\nu}(\bar{\ell}) u_{\alpha\nu} c_\alpha^a \left. \right], \\
\frac{d\Delta^{(2)} c_\alpha^a}{d\tau_\alpha} &= g f^{abc} \int \hat{d}^4 \bar{\ell} e^{-i\bar{\ell} \cdot y_\alpha} \left[ u_\alpha \cdot \Delta^{(1)} \tilde{A}^b(\bar{\ell}) \Delta^{(1)} c_\alpha^c + \Delta^{(1)} v_\alpha \cdot \Delta^{(1)} A^b(\bar{\ell}) c_\alpha^c \right. \\
&\quad \left. + u_\alpha \cdot \Delta^{(2)} \tilde{A}^b c_\alpha^c - i\bar{\ell} \cdot \Delta^{(1)} x_\alpha u_\alpha \cdot \Delta^{(1)} \tilde{A}^b(\bar{\ell}) c_\alpha^c \right].
\end{aligned} \tag{4.108}$$

These NLO equations involve the LO corrections to the fields and the particles' colours, positions and velocities, so although they are not the main quantities of interest we will need to integrate the expressions in equation (4.107); for example,

$$\Delta c_\alpha^a(\tau_\alpha) = \int_{-\infty}^{\tau_\alpha} d\tau'_\alpha \frac{dc_\alpha^a}{d\tau'_\alpha}. \tag{4.109}$$

In performing these integrals one must include an  $i\epsilon$  convergence factor, so the definition of  $y_\alpha^\mu$  is modified such that  $\bar{\ell} \cdot y_\alpha = \bar{\ell} \cdot b_\alpha + (u_\alpha \cdot \bar{\ell} + i\epsilon)\tau_\alpha$ . This yields

$$\begin{aligned}
\Delta^{(1)} x_\alpha^\mu(\tau_\alpha) &= ig^2 \sum_{\beta \neq \alpha} \frac{c_\alpha \cdot c_\beta}{m_\alpha} \int \hat{d}^4 \bar{\ell} e^{i(\bar{\ell} \cdot b_\beta - \bar{\ell} \cdot y_\alpha)} \hat{\delta}(\bar{\ell} \cdot u_\beta) \frac{\bar{\ell} \cdot u_\alpha u_\beta^\mu - \bar{\ell}^\mu u_\alpha \cdot u_\beta}{\bar{\ell}^2 (\bar{\ell} \cdot u_\alpha + i\epsilon)^2}, \\
\Delta^{(1)} c_\alpha^a(\tau_\alpha) &= ig^2 \sum_{\beta \neq \alpha} f^{abc} c_\alpha^b c_\beta^c u_\alpha \cdot u_\beta \int \hat{d}^4 \bar{\ell} e^{i(\bar{\ell} \cdot b_\beta - \bar{\ell} \cdot y_\alpha)} \frac{\hat{\delta}(\bar{\ell} \cdot u_\beta)}{\bar{\ell}^2} \frac{1}{(\bar{\ell} \cdot u_\alpha + i\epsilon)}.
\end{aligned} \tag{4.110}$$

We now have the information to determine the NLO field, which is

$$\begin{aligned}
\Delta^{(2)} \tilde{A}_\mu^a(\bar{\ell}_1) &= -g^3 \frac{1}{\bar{\ell}_1^2} \sum_{\beta \neq \alpha} \int \hat{d}^4 \bar{\ell}_2 e^{i(\bar{\ell}_1 - \bar{\ell}_2) \cdot b_\alpha} e^{i\bar{\ell}_2 \cdot b_\beta} \hat{\delta}((\bar{\ell}_1 - \bar{\ell}_2) \cdot u_\alpha) \hat{\delta}(\bar{\ell}_2 \cdot u_\beta) \\
&\quad \times \left\{ \frac{c_\alpha^a c_\alpha \cdot c_\beta}{m_\alpha \bar{\ell}_2^2} \left( -\frac{\bar{\ell}_2^\mu u_\alpha \cdot u_\beta}{\bar{\ell}_2 \cdot u_\alpha} + \frac{\bar{\ell}_1 \cdot \bar{\ell}_2 u_\alpha \cdot u_\beta u_\alpha^\mu}{(\bar{\ell}_2 \cdot u_\alpha)^2} - \frac{\bar{\ell}_1 \cdot u_\beta u_\alpha^\mu}{\bar{\ell}_2 \cdot u_\alpha} + u_\beta^\mu \right) \right. \\
&\quad \left. + \frac{if^{abc} c_\alpha^b c_\beta^c}{\bar{\ell}_2^2} \left( \frac{\bar{\ell}_2^\mu u_\alpha \cdot u_\beta}{(\bar{\ell}_1 - \bar{\ell}_2)^2} - \frac{2\bar{\ell}_2 \cdot u_\alpha u_\beta^\mu}{(\bar{\ell}_1 - \bar{\ell}_2)^2} + \frac{u_\alpha \cdot u_\beta u_\alpha^\mu}{\bar{\ell}_2 \cdot u_\alpha} \right) \right\}.
\end{aligned} \tag{4.111}$$

It is now very simple to use the Fourier transform of the Yang-Mills equation in equation (4.104) to calculate the LO momentum space current. In this context it is useful to rename the momentum of the field  $\bar{k}$ , relabel  $\bar{\ell}_2 = \bar{q}_2$ , and introduce

$\bar{q}_1 = \bar{\ell}_1 - \bar{\ell}_2$ . Then we find that

$$\begin{aligned} \Delta^{(2)} \tilde{K}^{a\mu}(\bar{k}) &= g^3 \sum_{\beta \neq \alpha} \int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}^{(4)}(\bar{k} - \bar{q}_1 - \bar{q}_2) \hat{\delta}(u_\alpha \cdot \bar{q}_1) \hat{\delta}(u_\beta \cdot \bar{q}_2) e^{ib_\alpha \cdot \bar{q}_1} e^{ib_\beta \cdot \bar{q}_2} \\ &\times \left\{ \frac{c_\alpha \cdot c_\beta}{\bar{q}_2^2 \bar{k} \cdot u_\alpha} \frac{c_\alpha^a}{m_\alpha} \left[ - (u_\alpha \cdot u_\beta) \left( \bar{q}_2^\mu - \frac{\bar{k} \cdot \bar{q}_2}{\bar{k} \cdot u_\alpha} u_\alpha^\mu \right) + \bar{k} \cdot u_\alpha u_\beta^\mu - \bar{k} \cdot u_\beta u_\alpha^\mu \right] \right. \\ &\quad \left. + \frac{if^{abc} c_\alpha^b c_\beta^c}{\bar{q}_1^2 \bar{q}_2^2} \left[ 2\bar{k} \cdot u_\beta u_\alpha^\mu - u_\alpha \cdot u_\beta \bar{q}_1^\mu + u_\alpha \cdot u_\beta \frac{\bar{q}_1^2}{\bar{k} \cdot u_\alpha} u_\alpha^\mu \right] \right\}. \end{aligned} \quad (4.112)$$

This result was first obtained in Ref. [41]. Comparing against equation (4.102), we can see that (up to an irrelevant overall sign) the two particle restriction of the current is equal to the LO radiation kernel calculated using amplitudes.<sup>7</sup>

Returning to the impulse, we can skip over the current and substitute the NLO field into equation (4.108). The observable quantities, the impulses, are defined by

$$\Delta c_\alpha^a \equiv \int_{-\infty}^{\infty} d\tau_\alpha \frac{dc_\alpha^a}{d\tau_\alpha}, \quad \Delta p_\alpha^\mu \equiv m_\alpha \int_{-\infty}^{\infty} d\tau_\alpha \frac{dv_\alpha^\mu}{d\tau_\alpha}. \quad (4.113)$$

Using the results for  $\Delta^{(2)} c_\alpha^a$  it is straightforward to show, after a redefinition of the integration variables, that the NLO colour impulse takes the form

$$\begin{aligned} \Delta c_\alpha^{a,(2)} &= f^{abc} \int \hat{d}^4 \bar{q} e^{ib_2 \cdot \bar{q}} e^{-ib_1 \cdot \bar{q}} \hat{\delta}(\bar{q} \cdot u_1) \hat{\delta}(\bar{q} \cdot u_2) \int \frac{\hat{d}^4 \bar{\ell}}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} \\ &\times \left\{ \hat{\delta}(\bar{\ell} \cdot u_2) \left[ \frac{c_1 \cdot c_2 c_1^b c_2^c}{m_1} \left( 1 + \frac{(u_1 \cdot u_2)^2 (\bar{\ell}^2 - \bar{\ell} \cdot \bar{q})}{(\bar{\ell} \cdot u_1 - i\epsilon)^2} \right) + if^{bde} c_2^c c_2^d c_1^e \frac{(u_1 \cdot u_2)^2}{\bar{\ell} \cdot u_1 - i\epsilon} \right] \right. \\ &\quad + \hat{\delta}(\bar{\ell} \cdot u_1) \left[ \frac{c_1 \cdot c_2 c_1^b c_2^c}{m_2} \left( 1 + \frac{(u_1 \cdot u_2)^2 (\bar{\ell}^2 - \bar{\ell} \cdot \bar{q})}{(\bar{\ell} \cdot u_2 + i\epsilon)^2} \right) + if^{bde} c_1^c c_1^d c_2^e \frac{(u_1 \cdot u_2)^2}{\bar{\ell} \cdot u_2 + i\epsilon} \right] \\ &\quad \left. - 2if^{bde} c_1^c c_1^d c_2^e \hat{\delta}(\bar{\ell} \cdot u_1) \frac{\bar{\ell} \cdot u_2}{\bar{q}^2} \right\}. \end{aligned} \quad (4.114)$$

Notice that the signs of the  $i\epsilon$  on the second and third lines in the above equation are different. This is a simple consequence of a change of variables required to associate the loop momenta of the classical calculation to that derived from amplitudes. To see that this indeed is the same as our earlier result we must manipulate the  $i\epsilon$  factors in the denominators further. For the quadratic

<sup>7</sup>Notice that we set  $b_2 = 0$  in sections 4.2 and 4.3 using translation symmetry.

denominator we can replace<sup>8</sup>

$$\frac{1}{(\bar{\ell} \cdot u_\alpha + i\epsilon)^2} = i\hat{\delta}'(\bar{\ell} \cdot u_\alpha) + \frac{1}{(\bar{\ell} \cdot u_\alpha - i\epsilon)^2}, \quad (4.115)$$

and for the linear denominator we make the shift  $\bar{\ell} \rightarrow \bar{q} - \bar{\ell}$ , which simply has the effect of changing the sign of the  $i\epsilon$ <sup>9</sup>. Then these terms can be averaged and combined to form a delta function. These procedures are the same as for the momentum impulse [45] and are briefly reviewed in appendix B.2. After these manipulations, the result from amplitudes in equation (4.83) matches the colour impulse computed classically in equation (4.114), up to the term on the last line.

This final term is spurious, and can be traced back to the non-Abelian correction to the gauge field at NLO, shown in the bottom line of equation (4.111). We observe that the term is proportional to the integral

$$u_2 \cdot I = \int \hat{d}^4\bar{\ell} \hat{\delta}(\bar{\ell} \cdot u_1) \frac{\bar{\ell} \cdot u_2}{\bar{\ell}^2(\bar{\ell} - \bar{q})^2}, \quad (4.116)$$

which vanishes on the support of  $\hat{\delta}(\bar{q} \cdot u_1)$  and  $\hat{\delta}(\bar{q} \cdot u_2)$ . This can easily be seen by writing<sup>10</sup>  $I^\mu = Au_1^\mu + B\bar{q}^\mu$ ; then we have that  $A = 0$ , and hence  $u_2 \cdot I = 0$ .

The momentum impulse follows through similarly, and for the case of two particles we find that the result is the same as in Abelian electrodynamics, calculated by KMOC [45], but with the replacement  $Q_1 Q_2 \rightarrow c_1 \cdot c_2$ . This is in agreement with the quantum calculation of section 4.2.4.

## Total radiated colour and asymptotic symmetries

Our classical calculation of the NLO impulse relied on the order  $g^3$  gauge field in equation (4.111). This gauge field in isolation is also of interest, because it is the leading radiation field generated by the scattering of the two particles. It therefore describes the transport of momentum and colour by the classical YM field itself. This classical transport of colour deserves more discussion.

The Noether current for a vector field transforming in the adjoint of the colour

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<sup>8</sup>This is a generalisation of eq. (2.34), and can be obtained by differentiation.

<sup>9</sup>The  $\bar{q} \cdot u_\alpha$  term can be ignored due to the delta function  $\delta(\bar{q} \cdot u_\alpha)$ .

<sup>10</sup>Any additional vectorial dependence arising by regulating the divergent integral will have vanishing dot product with  $u_2$ .

group is

$$j_\mu^a(x) = -f^{abc} A^{b\nu}(x) (\partial_\mu A_\nu^c(x) - \partial_\nu A_\mu^c(x)). \quad (4.117)$$

To measure the instantaneous rate of colour radiation at a time  $t$  during a scattering event, we surround the particles by a large sphere and measure the flux of  $j_\mu^a$  across its surface, taking the limit that the radius of the sphere goes to infinity. More specifically, we are interested in outgoing radiation from our scattering event, so we take this large radius limit at fixed retarded time  $u = t - r$ . We then integrate over all retarded times. Thus the surface of integration — namely the null future boundary  $\mathcal{I}^+$  of Minkowski space — is three-dimensional, parameterised by  $u$  and the coordinates on the two-sphere. The colour radiated to  $\mathcal{I}^+$  is<sup>11</sup>

$$\begin{aligned} R_{\text{col}}^a &= \int_{\mathcal{I}^+} *j^a \\ &= - \int_{-\infty}^{\infty} du \lim_{r \rightarrow \infty} \int d\Omega_2 r^2 j_r^a, \end{aligned} \quad (4.118)$$

where  $j^a = j_\mu^a dx^\mu$  and  $*j^a$  is its Hodge dual.

To evaluate this integral, we need an expression for the asymptotic field. This satisfies the Yang-Mills equation, given in linearised form in equation (4.104). Since there is no incoming radiation in our situation, we impose retarded boundary conditions. Using the standard large-distance expansion of the retarded Green's function we readily find

$$A_\mu^a(x) = \frac{1}{4\pi r} \int \frac{d\omega}{2\pi} e^{-i\omega u} \tilde{K}_\mu^a(\bar{k}) \Big|_{\bar{k}^\nu = (\omega, \omega \hat{x})} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (4.119)$$

where  $r = |\vec{x}|$  and  $t = x^0$ . It may also help the reader to record the derivative of the field, which is

$$\partial_\mu A_\nu^a(x) = -\frac{i}{4\pi r} \int \frac{d\omega}{2\pi} \bar{k}_\mu \tilde{K}_\nu^a(\bar{k}) e^{-i\omega u} \Big|_{\bar{k}^\rho = (\omega, \omega \hat{x})} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (4.120)$$

Hence upon integrating over delta functions the total radiated charge is

$$R_{\text{col}}^a = \frac{i}{(4\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d\Omega_2 f^{abc} \tilde{K}^{b\nu}(-\bar{k}) \left( k_\nu \tilde{K}_r^c(k) - \bar{k}_r \tilde{K}_\nu^c(\bar{k}) \right) \Big|_{\bar{k} = (\omega, \omega \hat{x})}. \quad (4.121)$$

This expression can be considerably simplified. Current conservation is  $k^\nu \tilde{K}_\nu^a(k) = 0$ , so the first term in parentheses vanishes. In the second term, we have  $\bar{k}_r = -\omega$ .

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<sup>11</sup>We use Bondi coordinates  $u, r, \theta$  and  $\phi$ .

We can further exploit the symmetry of the integral, and reality of the current  $K_\mu^a(x)$  to show

$$\begin{aligned}
R_{\text{col}}^a &= \frac{i}{(2\pi)^3} \int_0^\infty d\omega \omega^2 \int d\Omega_2 \frac{1}{2\omega} f^{abc} \tilde{K}_\nu^b(-\bar{k}) \tilde{K}^{c\nu}(\bar{k}) \Big|_{\bar{k}=(\omega, \omega \hat{x})} \\
&= \frac{i}{(2\pi)^3} \int d^3\bar{k} d\bar{k}^0 \frac{\delta(\bar{k}^0 - |\vec{k}|)}{2|\vec{k}|} f^{abc} \tilde{K}_\nu^b(-\bar{k}) \tilde{K}^{c\nu}(\bar{k}) \\
&= \int d\Phi(\bar{k}) i f^{abc} \tilde{K}_\nu^{b*}(\bar{k}) \tilde{K}^{c\nu}(\bar{k}) .
\end{aligned} \tag{4.122}$$

Finally, we use completeness of the polarisation vectors to write the classical total radiated colour in precisely the same form as the quantum expression:

$$R_{\text{col}}^a = -i f^{abc} \sum_{\sigma=\pm} \int d\Phi(\bar{k}) \left( \varepsilon_\sigma^* \cdot \tilde{K}^*(\bar{k}) \right)^b \left( \varepsilon_\sigma \cdot \tilde{K}(\bar{k}) \right)^c . \tag{4.123}$$

Comparing with equation (4.91) confirms that, in the classical limit, the radiation kernel coincides (up to a possible sign) with  $\varepsilon_\sigma \cdot \tilde{K}^a(\bar{k})$  at large distances.

Let us finish with a few additional remarks on this radiated colour. In ordinary electrodynamics it is elementary that charge is connected with the current appearing in the equation of motion. Although we made use of the Noether current in our discussion above, it remains the case that the radiated charge is connected to the current  $K^a$  in the linearised form of equation (4.104). It is easy to check, using the explicit asymptotic field of equation (4.119), that the radiated charge is

$$R_{\text{col}}^a = \frac{1}{g} \int_{\mathcal{I}^+} * K^a . \tag{4.124}$$

We may now make use of equation (4.104) in the form

$$d * F^a = - * K^a \tag{4.125}$$

to write

$$R_{\text{col}}^a = -\frac{1}{g} \int_{\mathcal{I}^+} d * F^a , \tag{4.126}$$

where  $F^a$  is the linearised field strength. The radiated charge may therefore also be reconstructed by integration over the boundaries  $\mathcal{I}_\pm^+$  of  $\mathcal{I}^+$  as

$$g R_{\text{col}}^a = \int_{\mathcal{I}_-^+} * F^a - \int_{\mathcal{I}_+^+} * F^a = g c_{\text{initial}} - g c_{\text{final}} . \tag{4.127}$$

In other words, the radiated charge is the difference between initial and final charges, as measured by integrating the electric fields over large spheres in the far past and the far future: total colour charge is conserved, as we also saw using quantum mechanical methods in equation (4.87).

## Summary

In this chapter we applied the KMOC framework to a theory of scalars coupled to a non abelian Yang Mills field. The motivation for this was two-fold; one is the double copy and the other that studying colour provides a toy model for spin.

We saw explicitly how coherent states can be utilised to ensure the correct classical dynamics of these colour charged particles. We used this to compute momentum impulse and radiation, with the surprising finding that the result is the same as for abelian scalar QED, up to the replacement of charge by colours. The non-linearity did, or non abelian diagrams, did not show up at this order as a result of the  $\hbar$ 's inside the colour factors. This is a little peculiar because it is natural to expect the non-linearity of the Yang-Mills field to enter at this order (and it does so in the quantum theory).

In addition to the observables familiar from electrodynamics and gravity, namely the momentum impulse and the total radiated momentum, we constructed two new observables: the colour impulse and total radiated colour charge. This can be viewed analogously to the change in spin and radiation of angular momentum for spinning particles, which is of particular physical interest. Our expression for the colour impulse is in many ways similar to the KMOC expression for the ordinary impulse. In essence the impulse describes a transfer of a small amount of momentum  $\hbar\bar{q}$ , weighted by an amplitude of order  $1/\hbar$ . Thus the momentum transferred by many gluons leads to a macroscopic impulse. In the colour case, the small momentum transferred is replaced by a colour commutator. At one loop, the colour impulse, though built from the same amplitudes as the momentum impulse, has a variety of colour structures as a result of the colour commutators. These commutators coming with factors of  $\hbar$  were responsible for rendering the non abelian terms quantum.

We verified these results by solving the classical equations of motion – the Wong equations – perturbatively. We did not comment on whether it is easier to do

this directly or to use the quantum first approach. In the examples here both calculations were tractable, however this is not the case for gravity. There these quantum amplitude based methods acquire a great deal more power compared to any known classical methods.

# Chapter 5

## KMOC meets Eikonal

We have seen how classical physics, in particular the impulse and scattering angle, can be obtained via KMOC and eikonal methods. These two approaches share many features, in particular that they involve a soft/small momentum transfer limit of scattering amplitudes. The aim of this chapter is to bring these two approaches together.

In this chapter we interpret these zero-variance conditions of chapter 3 in eikonal terms, arguing that radiation exponentiates in a manner analogous to the conservative terms. We propose a formula for the  $S$  matrix acting on our state which is a product of a coherent radiative state entangled with the more traditional eikonal (conservative) dynamics.

We have seen that the uncertainty, or the variance, in the measurement of a scattering observable can be computed in terms of amplitudes and, moreover, that the classical absence of uncertainty leads to an infinite set of relationships among fragments of amplitudes expanded in powers of momentum transfer, which is a Laurent series in  $\hbar$ . In a purely conservative limit, these relationships can be understood in terms of eikonal exponentiation. Our goal now is to utilise the eikonal formula, emphasising its connection to final state dynamics. We will then build on this eikonal state to incorporate radiative dynamics as a kind of coherent state so that the variance is naturally small.

## 5.1 Eikonal final state

In this section our goal is to explain the link between eikonal methods and the KMOC approach. Firstly recall that the small  $\hbar$  expansion in the KMOC formalism is essentially the same as the soft expansion in the eikonal literature; the  $\hbar$  scaling counts the order of softness. The key connection is to compute the final state using the methods of KMOC instead of computing observables directly. We will see that this final, outgoing, state is controlled by the usual eikonal function. In this section we restrict to a purely conservative scattering scenario: then eikonal exponentiation is exact. We take two incoming particles and (since the scattering is conservative) assume that the outgoing state is also an element of the two-particle Hilbert space.

We begin with the standard definition of the eikonal as the transverse Fourier transform of the four-point amplitude

$$e^{i\chi(x; s)/\hbar}(1 + i\Delta(x; s)) - 1 = i \int \hat{d}^4 q \, \hat{\delta}(2p_1 \cdot q) \hat{\delta}(2p_2 \cdot q) e^{-iq \cdot x/\hbar} \mathcal{A}_4(s, q^2) , \quad (5.1)$$

where  $\chi(x; s)$  is the eikonal function and  $\Delta(x; s)$  is the so-called quantum remainder which takes into account contributions that do not exponentiate (see for example [124]). This remainder is important for computing the eikonal function — but it will play no role in the remainder of our article, so we will omit it. Meanwhile the two Dirac delta functions appearing in equation (5.1) ensure that we integrate only over the components of  $q$  transverse to the momenta. This is often just written instead as  $\hat{d}^{D-2} q_\perp$  (times a Jacobian factor). Indeed, the parameter  $x$  should be thought of as an element of the  $D - 2$  dimensional spatial slice perpendicular to  $p_1$  and  $p_2$ : this is most evident on the right-hand-side of the equation, where the Dirac delta functions project away any components of  $q$  in the (timelike)  $p_1$  and  $p_2$  directions. Consequently no components of  $x$  in the space spanned by  $p_1$  and  $p_2$  enter the dot product  $q \cdot x$ .

The eikonal can be written as an expansion in powers of the generic coupling  $g$

$$\chi(x; s) = \sum_{n=0}^{\infty} \chi_n(x; s) \quad , \quad \chi_n(x; s) \sim g^{2n} , \quad (5.2)$$

and we will, as others have (see for example [59, 71, 76, 124, 188, 239]), assume that this holds to all orders.

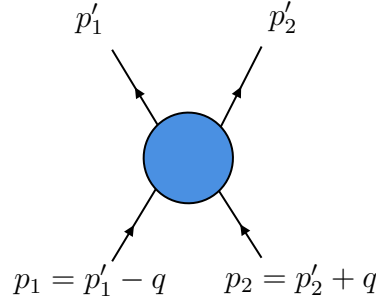


**Figure 5.1** *Momentum labelling at four points*

Starting with the in state (3.6) from section 3.1.2, we obtain the final state by acting with the  $S$  matrix. Writing  $S = 1 + iT$  and inserting a complete set of states, we have

$$\begin{aligned} S|\psi\rangle &= |\psi\rangle + i \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b(p_1, p_2) |p'_1, p'_2\rangle \langle p'_1, p'_2| T |p_1, p_2\rangle \\ &= |\psi\rangle + i \int d\Phi(p'_1, p'_2, p_1, p_2) \phi_b(p_1, p_2) \mathcal{A}_4(s, q^2) \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2) |p'_1, p'_2\rangle . \end{aligned} \quad (5.3)$$

Notice that we made explicit use of our assumption of conservative scattering by restricting the complete set of states to the two-particle Hilbert space. We also ignore any extra degrees of freedom, such as spin or colour, and address these in the next chapter.

With the momentum labelling in figure 5.1 we can convert the  $p_1$  and  $p_2$  phase space integrals to integrals over  $q$ <sup>1</sup>. Doing so, we may write

$$\begin{aligned} S|\psi\rangle &= |\psi\rangle \\ &+ i \int d\Phi(p'_1, p'_2) \hat{d}^4 q \hat{\delta}(2p'_1 \cdot q - q^2) \hat{\delta}(2p'_2 \cdot q + q^2) \phi_b(p'_1 - q, p'_2 + q) \mathcal{A}_4(s, q^2) |p'_1, p'_2\rangle . \end{aligned} \quad (5.4)$$

At this point, the  $q$  integral is tantalising similar to the  $q$  integral in the eikonal formula (5.1). However there is a key difference in the nature of the delta functions: those in equation (5.4) involve  $q^2$  terms which are absent in equation (5.1). This issue appeared recently in refs. [219, 220]: there the authors proceeded using a “HEFT” phase, which is analogous to yet distinct from the eikonal phase. We will instead continue with the eikonal phase.

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<sup>1</sup>Notice that here we convert the  $p$  rather than  $p'$  integrals into  $q$  integrals as we have done before, because now we’re working with final state variables so it is really  $p'$  external data that we have to work with.

It may be worth remarking that the  $q^2$  terms in these delta functions are suppressed in specific examples. One such example is the leading order impulse [45]. Nevertheless the impulse at NLO does indeed involve the full delta functions [45].

To incorporate the full delta functions, we follow the route described in [76]. We introduce new momentum variables  $\tilde{p}$  as

$$\tilde{p}_1 = p'_1 - \frac{q}{2} \quad \tilde{p}_2 = p'_2 + \frac{q}{2}. \quad (5.5)$$

Now, rather than using the eikonal equation (5.1) directly we can take advantage of its inverse Fourier transform in the following form<sup>2</sup>

$$i\hat{\delta}(2\tilde{p}_1 \cdot q)\hat{\delta}(2\tilde{p}_2 \cdot q)\mathcal{A}_4(s, q^2) = \frac{1}{\hbar^4} \int d^4x e^{iq \cdot x/\hbar} \{e^{i\chi(x_\perp; s)/\hbar} - 1\}, \quad (5.6)$$

where we have written  $x_\perp$  as one of the arguments of the eikonal function  $\chi(x_\perp; s)$  to emphasise that  $\chi(x_\perp; s)$  only depends on components of  $x$  which are orthogonal to the space spanned by  $\tilde{p}_1$  and  $\tilde{p}_2$ . Indeed, integrating over the two components of  $x$  which *are* in the space spanned by  $\tilde{p}_1$  and  $\tilde{p}_2$ , one recovers the two Dirac delta functions on the left-hand-side of equation (5.6). In this way, we find that the final state is

$$S|\psi\rangle = \int d\Phi(p'_1, p'_2) |p'_1, p'_2\rangle \frac{1}{\hbar^4} \int d^4q d^4x e^{iq \cdot x/\hbar} e^{i\chi(x_\perp; s)/\hbar} \phi_b(p'_1 - q, p'_2 + q). \quad (5.7)$$

It is worth emphasising once again that  $x_\perp$  is perpendicular to  $\tilde{p}_i$ , rather than to  $p_i$ , so that

$$\tilde{p}_1 \cdot x_\perp = 0 = \tilde{p}_2 \cdot x_\perp. \quad (5.8)$$

In particular,  $x_\perp$  depends on  $q$ . This will become important and we will see how exactly it depends later on.

## 5.2 The impulse from the eikonal

In this section, we will recover one beautiful result from the literature on the eikonal function: the scattering angle can be extracted from the eikonal function using a stationary phase argument. As our interest is not so much in this

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<sup>2</sup>As noted above, we dropped the quantum remainder  $\Delta$  which plays no role in our analysis.

conservative case but rather in its radiative generalisation (which we discuss below), we wish to emphasise that it is, in fact, possible to extract the *full* final momentum from the eikonal using stationary phase. Later, in section 5.5, we will use the same ideas to extract the final momentum in the case of radiation — with radiation, of course, knowledge of the direction of the final momentum is insufficient to recover the full momentum.

The impulse is the observable

$$\begin{aligned}\Delta p_1^\mu &\equiv \langle \psi | S^\dagger \mathbb{P}_1^\mu S | \psi \rangle - \langle \psi | \mathbb{P}_1^\mu | \psi \rangle \\ &= \langle \psi | S^\dagger [\mathbb{P}_1^\mu, S] | \psi \rangle .\end{aligned}\tag{5.9}$$

It is convenient to focus on

$$[\mathbb{P}_1^\mu, S] | \psi \rangle .\tag{5.10}$$

As we shall see, in essence the operator  $[\mathbb{P}_1^\mu, S]$  pulls out a factor of the momentum transfer multiplying  $S | \psi \rangle$ . We will evaluate  $[\mathbb{P}_1^\mu, S] | \psi \rangle$  by stationary phase; it is then trivial to determine  $\langle \psi | S^\dagger$  in the same way.

We begin our expression for the final-state wavepacket, equation (5.7), quickly finding

$$\begin{aligned}[\mathbb{P}_1^\mu, S] | \psi \rangle &= \int d\Phi(p'_1, p'_2) e^{ib \cdot p'_1 / \hbar} | p'_1, p'_2 \rangle \frac{1}{\hbar^4} \int \hat{d}^4 q \, d^4 x \, e^{iq \cdot x / \hbar} e^{-ib \cdot q / \hbar} \\ &\quad \times e^{i\chi(x_\perp; s) / \hbar} \phi(p'_1 - q, p'_2 + q) q^\mu .\end{aligned}\tag{5.11}$$

To obtain formulae for the impulse, we apply the stationary phase approximation to the  $x$  and  $q$  integrals. (Our approach is very similar to that of Ciafaloni and Colferai [240] who previously discussed wavepacket dynamics, the eikonal, and stationary phase.)

The stationary phase condition for  $x$  is<sup>3</sup>

$$q_\mu = -\frac{\partial}{\partial x^\mu} \chi(x_\perp, s) .\tag{5.12}$$

One thing to note immediately is that this  $q^\mu$  is *not* of order  $\hbar$ : it is a classical momentum, of order  $g^2$ . Further, it is useful to note that for scalars  $\chi$  is actually a function of  $x_\perp^2$  (since the four-point function is a function of  $s$  and  $q^2$ ). For

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<sup>3</sup>One can find this expression also by promoting the eikonal phase to an operator  $\hat{\chi}(x_\perp, s)$ : the impulse on particle  $i$  in the transverse plane  $\hat{Q}_{\perp i}^\mu$  is then related to the standard commutator  $[\hat{Q}_{\perp i}^\mu, e^{i\hat{\chi}(\hat{x}_\perp; s) / \hbar}] = e^{i\hat{\chi}(\hat{x}_\perp; s) / \hbar} \partial_{\perp i}^\mu \hat{\chi}(\hat{x}_\perp, s)$  and the result follows from unitarity.

particles with spin this is no longer true and instead depends on the full vector not its norm. This will be addressed in the next chapter and for now we will continue with this  $\chi(x_\perp^2)$ .

$$q^\mu = -2\chi'(x_\perp^2, s)x_\perp^\mu, \quad (5.13)$$

where  $\chi'(x_\perp^2, s)$  is the derivative of  $\chi$  with respect to its first argument. Since  $\tilde{p}_1 \cdot x_\perp = 0 = \tilde{p}_2 \cdot x_\perp$ , it now follows that  $\tilde{p}_1 \cdot q = 0 = \tilde{p}_2 \cdot q$ : thus the on-shell delta functions in equation (5.6) have reappeared, now as “equations of motion” following from the stationary phase conditions<sup>4</sup>.

The second stationary phase condition, associated with  $q$ , is

$$x^\mu - b^\mu + \frac{\partial}{\partial q_\mu} \chi(x_\perp^2, s) = 0. \quad (5.14)$$

The  $q$  derivative is non-vanishing because  $x_\perp$  depends on  $q$ . It is often useful to introduce a particular notation for the variables  $q$  and  $x$  when they satisfy the stationary phase conditions: we will denote these by  $q_*$  and  $x_*$ . Armed with this notation, we may use equation (5.12) we may write equation (5.14) as

$$x^\mu = b^\mu + q_{*\nu} \frac{\partial}{\partial q_\mu} x_\perp^\nu. \quad (5.15)$$

Performing the derivative is straightforward, but requires some notation which we relegate to appendix C. The result may be expressed in the form

$$x_{*\perp}^\mu = b^\mu - \tilde{N}_q(p_1^\mu - p_2^\mu) - \tilde{N}_{0q}(p_1^\mu + p_2^\mu), \quad (5.16)$$

where  $\tilde{N}_q$  and  $\tilde{N}_{0q}$  can be interpreted as Lagrange multipliers ensuring that  $x_\perp \cdot \tilde{p}_1 = 0 = x_\perp \cdot \tilde{p}_2$ .

Our result for  $x_{*\perp}$  takes a familiar form in the CM frame. Then  $x_{*\perp}^0 = 0$  and, using,  $\mathbf{p}_2 = -\mathbf{p}_1$ , we have

$$\begin{aligned} \mathbf{x}_{*\perp} &= \mathbf{b} - 2N_q^2(\mathbf{b} \cdot \mathbf{q}_*)\mathbf{p} \\ \Rightarrow \mathbf{b} \cdot \mathbf{x}_{*\perp} &= \mathbf{b}^2. \end{aligned} \quad (5.17)$$

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<sup>4</sup>Since some of the  $x$  integrations may be performed exactly to yield delta functions, we are slightly abusing terminology by referring to all of the conditions on  $x$  and  $q$  as “stationary phase conditions”. Some of the conditions arise approximately by stationary phase, and some are exact conditions due to the delta functions. Nevertheless it is most convenient to treat all the conditions as one set.

Denoting the scattering angle by  $\Psi$ , we may write this result<sup>5</sup> as

$$|\mathbf{b}| = |\mathbf{x}_{*\perp}| \cos(\Psi/2), \quad (5.18)$$

where  $\Psi$  is the scattering angle.

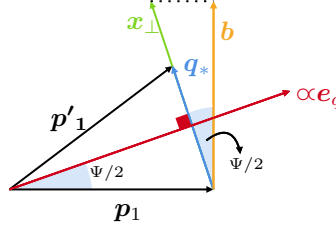


**Figure 5.2** *Geometry of eikonal scattering.*

In this way, we have performed the integrals over  $q$  and  $x$ . The result has been to evaluate the factor  $q^\mu$  as a derivative of the eikonal function. To obtain the full expectation, we simply evaluate  $\langle \psi | S^\dagger$  using stationary phase in precisely the same way: the only difference (apart from the obvious Hermitian conjugation) is the absence of the  $q^\mu$  factor. We then exploit unitarity to conclude that<sup>6</sup>

$$\Delta p_1^\mu = q^\mu = -\partial^\mu \chi(x_\perp, s), \quad (5.19)$$

where all quantities are defined on using the solution of the stationary phase conditions.

Notice that we have determined the *complete* impulse four-vector, not just the scattering angle. The distinction between these quantities is obviously unimportant at the level of conservative dynamics, but it is important when radiation occurs. The key aspect of our argument which leads to the full impulse rather than the scattering angle is the presence of a perpendicular projector. To see how this works, let us discuss an explicit example: the impulse at next-to-leading order in gravitational fast scattering.

Focusing on the scattering between two massive bodies in general relativity, the

<sup>5</sup>In this context, the quantity  $x_\perp$  is sometimes referred to as the “eikonal” impact parameter. It is often thus written as  $b_e$ .

<sup>6</sup>See appendix C of [163], performing all the remaining integrals.

next-to-leading amplitude is

$$i\mathcal{M}_4 = -\frac{16\pi G m_1 m_2 (2\gamma^2 - 1)}{q^2} + \frac{6\pi^2 G^2 m_1^2 m_2^2 (5\gamma^2 - 1) (m_1 + m_2)}{\sqrt{-q^2}} \quad (5.20)$$

which gives the NLO eikonal phase<sup>7</sup>

$$\chi = -2Gm_1m_2 \left( \frac{(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \log |x_\perp| - \frac{3\pi}{8} \frac{(5\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{G(m_1 + m_2)}{|x_\perp|} \right), \quad (5.21)$$

where we have defined  $\gamma = u_1 \cdot u_2$  as the scalar product between the four-velocities of the particles. Using equation (5.19) and straightforward differentiation, we obtain the following expression in terms of  $x_\perp^\mu$ ,

$$\Delta p_1^\mu = -\frac{2Gm_1m_2x_\perp^\mu}{|x_\perp^2|} \left( \frac{(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} + \frac{3\pi}{8} \frac{(5\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{G(m_1 + m_2)}{|x_\perp|} \right), \quad (5.22)$$

where the four-velocities in  $\gamma$  can now be identified with the incoming four-velocities of the particles due to the integrals over the wavepackets that took place to arrive at (5.19). At this point, it is important to remember that  $x_\perp^\mu$  is not quite  $b^\mu$ . It is trivial to show that  $x_\perp^\mu$  coincides with  $b^\mu$  at leading order in the gravitational coupling, but this is no longer the case at next-to-leading order; then instead

$$x_\perp^\mu = b^\mu + \frac{G(2\gamma^2 - 1)}{(\gamma^2 - 1)^{3/2}} (u_1^\mu(m_2 + \gamma m_1) - u_2^\mu(m_1 + \gamma m_2)), \quad (5.23)$$

where

$$|x_\perp^2| = |b^2| + \mathcal{O}(G^2). \quad (5.24)$$

We can now express the impulse in terms of the impact parameter  $b^\mu$ . At the order we are interested in we find,

$$\begin{aligned} \Delta p_1^\mu = & -\frac{2Gm_1m_2b^\mu}{|b^2|} \left( \frac{(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} + \frac{3\pi}{8} \frac{(5\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{G(m_1 + m_2)}{|b|} \right) \\ & - \frac{G^2m_1m_2(2\gamma^2 - 1)^2((\gamma m_1 + m_2)u_1^\mu - (\gamma m_2 + m_1)u_2^\mu)}{(\gamma^2 - 1)^2|b^2|}, \end{aligned} \quad (5.25)$$

in perfect agreement with the literature [162].

From the perspective of this thesis, the key achievement of the eikonal resum-

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<sup>7</sup>Fourier transform formulae are in the appendix.

mation is that negligible variance becomes automatic in the stationary phase argument. Indeed since the stationary phase condition (5.12) sets the momentum transfer to a specific, classical, value it is clear that the expectation value of any polynomial in the momentum operator will evaluate to the classical expectation.

### 5.3 Coherent radiative state

In the context of radiation, another exponentiation is well known to lead to small or negligible variance. This occurs when the radiative state is a coherent state with large occupation number. Indeed, in this context the requirement of minimal uncertainty is sometimes referred to as complete coherence [170, 171]. In this subsection, we describe how these states naturally describe the radiation in the final state, working in electrodynamics for simplicity.

Consider the state

$$|\alpha^\eta\rangle = \mathcal{N}_\alpha \exp\left(\int d\Phi(k) \alpha^\eta(k) a_\eta^\dagger(k)\right) |0\rangle. \quad (5.26)$$

This is a field theory generalisation of the coherent state defined in section 2.2, in eq. (2.41). It is not difficult to verify that  $|\alpha^\eta\rangle$  is indeed a coherent state satisfying the usual properties: in particular, the minimal uncertainty relation (3.2) is obeyed explicitly if the occupation number is large [163]. Normalising these states to unity requires

$$\mathcal{N}_\alpha = \exp\left(-\frac{1}{2} \int d\Phi(k) |\alpha^\eta(k)|^2\right). \quad (5.27)$$

Following [163] and computing the expectation value of the photon field  $\mathbb{A}_\mu$  in this state we have,

$$\begin{aligned} \langle \alpha^\eta | \mathbb{A}_\mu(x) | \alpha^\eta \rangle &= \frac{1}{\sqrt{\hbar}} \sum_{\eta=\pm} \int d\Phi(k) \left[ \langle \alpha^\eta | \hat{a}_\eta(k) | \alpha^\eta \rangle \varepsilon_\mu^{(\eta)*}(k) e^{-i\bar{k}\cdot x} + \text{h.c.} \right] \\ &= \frac{1}{\sqrt{\hbar}} \sum_{\eta=\pm} \int d\Phi(k) \left[ \alpha_\eta(k) \varepsilon_\mu^{(\eta)*}(k) e^{-i\bar{k}\cdot x} + \text{h.c.} \right] \\ &= \hbar^{3/2} \sum_{\eta=\pm} \int d\Phi(\bar{k}) \left[ \alpha_\eta(k) \varepsilon_\mu^{(\eta)*}(k) e^{-i\bar{k}\cdot x} + \text{h.c.} \right], \end{aligned} \quad (5.28)$$

where in the last line we rescaled  $k \rightarrow \hbar \bar{k}$ . Thus we see that in the classical limit

we expect the coherent state parameter  $\alpha(k)$  to scale as  $\hbar^{-3/2}$ .

This state only describes the gauge particles (photons in this case), but we would like to use this to describe a state with the two massive scattering particles as well as the radiation emitted during the interaction. This is the situation one would generally expect after some scattering event. We will do exactly this later and determine the form of  $\alpha$  at leading order. We will find that  $\alpha$  depends on the 5-point amplitude, much as the eikonal depends on the 4-point.

## 5.4 Extension with coherent radiation

The exponentiated eikonal final state of equation (5.7) beautifully describes semiclassical conservative dynamics, leading to a transparent method for extracting the impulse (or scattering angle) from amplitudes in a manner which automatically enforces minimal uncertainty. We have also seen that coherent states naturally enforce minimal uncertainty for radiation. Now let us put these two ideas together to form a proposal for an eikonal-type final state in the fully dynamical, radiative, case.

It is very natural to consider a modification of the eikonal formula which includes radiation, and indeed this idea has received attention [74] in the literature. Given that our motivation is to extend the eikonal while maintaining its minimal uncertainty property, an obvious way to proceed is to include an additional factor in the eikonal formula which has the structure of a coherent state like (5.26). If this radiative part of the state has large occupation number, expectations of products of field-strength operators will naturally factorise into products of expectations of the operators.

We will simply propose one possibility for the structure of this final state, depending on a coherent waveshape parameter  $\alpha^{(\eta)}$  (of helicity  $\eta$ ) in addition to an eikonal function  $\chi$ . We believe there is strong evidence in favour of the basic structure of our proposal, and in particular in the idea that two objects  $\chi$  and  $\alpha^{(\eta)}$  suffice to define it; however, it seems possible to implement the idea in somewhat different ways. We will discuss the basic virtues of our proposal in the remainder of this section, leaving it to future work to determine further details. Since we are primarily interested in classical effects, we will continue to neglect

the quantum remainder  $\Delta$  in this discussion<sup>8</sup>.

To describe our proposal, we begin with the eikonal final state (5.7). With an eye towards a situation where momentum is lost to radiation, we need a description in which the sum of the momenta of the two final particles differs from the initial momenta. A first step, then, is to Fourier transform the wavepacket to position space:

$$S|\psi\rangle|_{\text{conservative}} = \int d\Phi(p'_1, p'_2) \int \hat{d}^4\bar{q} d^4x d^4x_1 d^4x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ \times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} |p'_1, p'_2\rangle. \quad (5.29)$$

Our proposal is now very straightforward: we simply incorporate a coherent state by assuming that

$$S|\psi\rangle = \int d\Phi(p'_1, p'_2) \int \hat{d}^4\bar{q} d^4x d^4x_1 d^4x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ \times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp \left[ \sum_\eta \int d\Phi(k) \alpha^{(\eta)}(k, x_1, x_2) a_\eta^\dagger(k) \right] |p'_1, p'_2\rangle. \quad (5.30)$$

This is a minimal proposal: more generally, one could imagine that the coherent waveshape parameter  $\alpha^{(\eta)}$  depends on other variables, for example  $x$  or  $q$  which appear in the eikonal dynamics. We will nevertheless restrict ourselves to this minimal proposal. However, it is important that the state is not merely an outer product of a conservative eikonal state with a radiative factor. Some entanglement is necessary so that the radiation can backreact on the motion. In the case of the present proposal, the integrals over the variables  $x_1$ ,  $x_2$  and  $x$  perform this role.

There is a connection between our proposal here and recent work [192] on the exponential structure of the  $S$  matrix.

Following the discussion in section 5.3, we know that the waveshape should be proportional to  $\hbar^{-3/2}$  so we may also write the state as

$$S|\psi\rangle = \int d\Phi(p'_1, p'_2) \int \hat{d}^4\bar{q} d^4x d^4x_1 d^4x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ \times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp \left[ \frac{1}{\hbar^{3/2}} \sum_\eta \int d\Phi(k) \bar{\alpha}^{(\eta)}(k, x_1, x_2) a_\eta^\dagger(k) \right] |p'_1, p'_2\rangle. \quad (5.31)$$

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<sup>8</sup>Indeed radiative quantum effects will require  $\Delta$  to be upgraded to an operator.

In this expression, the classical waveshape  $\bar{\alpha}^{(\eta)}$  is *independent* of  $\hbar$ , just as the eikonal function  $\chi$  is independent of  $\hbar$ .

In order to determine  $\alpha^{(\eta)}$ , we follow the same steps as in sections 3.1.2 and 5.1; we act on the incoming state with the  $S$  matrix, and then expand in terms of integrals of amplitudes. To isolate the waveshape, we consider the overlap of our proposed final state with the bra  $\langle p'_1 p'_2 k^\eta |$ :

$$\begin{aligned} \langle p'_1 p'_2 k^\eta | S | \psi \rangle &= \int \hat{d}^4 \bar{q} d^4 x d^4 x_1 d^4 x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ &\quad \times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \alpha^{(\eta)}(k, x_1, x_2). \end{aligned} \quad (5.32)$$

Next we expand the  $S$  matrix as

$$\begin{aligned} \langle p'_1 p'_2 k^\eta | S | \psi \rangle &= \int d\Phi(p_1, p_2) \phi_b(p_1, p_2) \langle p'_1 p'_2 k^\eta | S | p_1 p_2 \rangle \\ &= \int d\Phi(p_1, p_2) \int d^4 x_1 d^4 x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)/\hbar} \\ &\quad \times i\mathcal{A}_5(p_1 p_2 \rightarrow p'_1 p'_2 k^\eta) \hat{\delta}^4(p_1 + p_2 - p'_1 - p'_2 - k). \end{aligned} \quad (5.33)$$

We note that the five-point amplitude appearing here could in principle include disconnected components beginning at order  $g$ . This order  $g$  disconnected term would involve exactly zero-energy photons, and does not contribute to observables such as the radiated momentum. We therefore omit this term in the remainder of this chapter.

To continue, it is useful to perform a change of variable in the phase space measures, taking  $q_1 \equiv p_1 - p'_1$  and  $q_2 \equiv p_2 - p'_2$  as variables of integration. Neglecting Heaviside theta functions (which will always be unity in the domain of validity of our calculation) we find

$$\begin{aligned} \langle p'_1 p'_2 k^\eta | S | \psi \rangle &= \int d^4 x_1 d^4 x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ &\quad \times \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p'_1 \cdot q_1 + q_1^2) \hat{\delta}(2p'_2 \cdot q_2 + q_2^2) e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)/\hbar} \\ &\quad \times i\mathcal{A}_5(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \hat{\delta}^4(q_1 + q_2 - k). \end{aligned} \quad (5.34)$$

Requiring equations (5.32) and (5.34) to be equal for any (appropriately classical)

initial wavepacket  $\tilde{\phi}_b(x_1, x_2)$  we deduce that

$$\begin{aligned} \alpha^{(\eta)}(k, x_1, x_2) &= i \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p'_1 \cdot q_1 + q_1^2) \hat{\delta}(2p'_2 \cdot q_2 + q_2^2) \hat{\delta}^4(q_1 + q_2 - k) e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \\ &\quad \times \mathcal{A}_5(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \\ &\quad \times \left[ \int \hat{d}^4 q \hat{d}^4 x e^{iq \cdot x} e^{iq \cdot (x_2 - x_1)/\hbar} e^{i\chi(x_\perp; s)/\hbar} \right]^{-1}. \end{aligned} \quad (5.35)$$

It is easy to use the eikonal equation (5.1) to show that equivalently we may write the waveshape as

$$\begin{aligned} \alpha^{(\eta)}(k, x_1, x_2) &= i \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p'_1 \cdot q_1 + q_1^2) \hat{\delta}(2p'_2 \cdot q_2 + q_2^2) \hat{\delta}^4(q_1 + q_2 - k) e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \\ &\quad \times \mathcal{A}_5(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \\ &\quad \times \left[ 1 + \int \hat{d}^4 q \hat{\delta}(2\tilde{p}_1 \cdot q) \hat{\delta}(2\tilde{p}_2 \cdot q) e^{iq \cdot (x_2 - x_1)/\hbar} i \mathcal{A}_4(s, q^2) \right]^{-1}. \end{aligned} \quad (5.36)$$

This last expression makes the physical meaning transparent: the waveshape is obtained by removing iterated contributions of four-point amplitudes from the five-point amplitude.

To see this in more detail, it is instructive to expand the  $\alpha^{(\eta)}$  order-by-order in perturbation theory. We again consider a generic coupling  $g$  and expand the waveshape as

$$\alpha^{(\eta)} = \alpha_0^{(\eta)} + \alpha_1^{(\eta)} + \dots. \quad (5.37)$$

The leading order term,  $\alpha_0^{(\eta)}$ , follows immediately from equation (5.36):

$$\begin{aligned} \alpha_0^{(\eta)}(k, x_1, x_2) &= i \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p'_1 \cdot q_1 + q_1^2) \hat{\delta}(2p'_2 \cdot q_2 + q_2^2) \hat{\delta}^4(q_1 + q_2 - k) \\ &\quad \times e^{i(q_1 \cdot (x_1 + b) + q_2 \cdot x_2)/\hbar} \mathcal{A}_{5,0}(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta). \end{aligned} \quad (5.38)$$

It is determined by the tree-level five-point amplitude  $\mathcal{A}_{5,0}$ , so it is of order  $g^3$  in gauge theory and gravity. The fact that the leading-order classical radiation field is intimately related to five-point amplitudes was already discussed in [54, 132, 163, 241].

More precisely,  $\alpha_0^{(\eta)}$  is really determined by the fragment  $\mathcal{A}_{5,0}^{(0)}$ . This follows by counting powers of  $\hbar$ . Indeed extracting dominant powers of  $\hbar$  using

equation (3.40), we find

$$\begin{aligned} \alpha_0^{(\eta)}(k, x_1, x_2) &= \frac{i}{\hbar^{3/2}} \int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p'_1 \cdot \bar{q}_1) \hat{\delta}(2p'_2 \cdot \bar{q}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{k}) \\ &\quad \times e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \mathcal{A}_{5,0}^{(0)}(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta). \end{aligned} \quad (5.39)$$

Note that the factor  $\hbar^{-3/2}$  arises as expected on general grounds. The conclusion is that the leading-in- $\hbar$  part of the five-point tree amplitude determines the radiation. The amplitude itself contains higher order terms in  $\hbar$ ; rather than arising from the radiative factor in our proposal (5.31), these terms arise from a generalised quantum remainder.

The next-to-leading order correction to the waveshape following from equation (5.36) is

$$\begin{aligned} \alpha_1^{(\eta)}(k, x_1, x_2) &= i \int \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p'_1 \cdot q_1 + q_1^2) \hat{\delta}(2p'_2 \cdot q_2 + q_2^2) \hat{\delta}^4(q_1 + q_2 - k) \\ &\quad \times e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)/\hbar} \mathcal{A}_{5,1}(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \\ &\quad - \alpha_0^{(\eta)}(k, x_1, x_2) \int \hat{d}^4 q \hat{\delta}(2\tilde{p}_1 \cdot q) \hat{\delta}(2\tilde{p}_2 \cdot q) e^{iq \cdot (x_2 - x_1)/\hbar} i \mathcal{A}_{4,0}(s, q^2). \end{aligned} \quad (5.40)$$

This correction involves the five-point one-loop amplitude, after subtracting an iteration term. To understand the role of the subtraction, it is instructive to extract the leading-in- $\hbar$  part of  $\alpha_1^{(\eta)}$ :

$$\begin{aligned} \alpha_1^{(\eta)}(k, x_1, x_2) &= \frac{i}{\hbar^{5/2}} \int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p'_1 \cdot \bar{q}_1) \hat{\delta}(2p'_2 \cdot \bar{q}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{k}) \\ &\quad \times e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \mathcal{A}_{5,1}^{(0)}(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \\ &\quad - \frac{1}{\hbar^{5/2}} \bar{\alpha}_0^{(\eta)}(k, x_1, x_2) \int \hat{d}^4 \bar{q} \hat{\delta}(2p'_1 \cdot \bar{q}) \hat{\delta}(2p'_2 \cdot \bar{q}) e^{i\bar{q} \cdot (x_2 - x_1)} i \mathcal{A}_{4,0}^{(0)}(s, q^2) \\ &\quad + \mathcal{O}(\hbar^{-3/2}). \end{aligned} \quad (5.41)$$

At this stage it seems that there is an unwanted order  $\hbar^{-5/2}$  term in the NLO waveshape! Consistency with our proposal therefore demands

$$\begin{aligned} &\int \hat{d}^4 \bar{q}_1 \hat{d}^4 \bar{q}_2 \hat{\delta}(2p'_1 \cdot \bar{q}_1) \hat{\delta}(2p'_2 \cdot \bar{q}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - \bar{k}) e^{i(\bar{q}_1 \cdot x_1 + \bar{q}_2 \cdot x_2)} \\ &\quad \mathcal{A}_{5,1}^{(0)}(p'_1 + q_1, p'_2 + q_2 \rightarrow p'_1, p'_2, k^\eta) \\ &= -\bar{\alpha}_0^{(\eta)}(k, x_1, x_2) \int \hat{d}^4 \bar{q} \hat{\delta}(2p'_1 \cdot \bar{q}) \hat{\delta}(2p'_2 \cdot \bar{q}) e^{i\bar{q} \cdot (x_2 - x_1)} \mathcal{A}_{4,0}^{(0)}(s, q^2). \end{aligned} \quad (5.42)$$

Since  $\bar{\alpha}_0^{(\eta)}$  is determined by  $\mathcal{A}_{5,0}^{(0)}$ , this requirement relates  $\mathcal{A}_{5,1}^{(0)}$  to  $\mathcal{A}_{5,0}^{(0)}$  and  $\mathcal{A}_{4,0}^{(0)}$ . The requirement is nothing but a Fourier transform of the zero-variance relation (3.43) which we encountered in section 3.2.1. So we see that the zero-variance relations retain their importance in the context of this eikonal/coherent resummation: their validity admits the possibility of exponentiation.

In the same vein, it is interesting to project our proposal onto a two-photon final state:

$$\begin{aligned} \langle p'_1, p'_2, k_1^{\eta_1}, k_2^{\eta_2} | S | \psi \rangle &= \int \hat{d}^4 \bar{q} d^4 x d^4 x_1 d^4 x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ &\times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \alpha^{(\eta_1)}(k_1, x_1, x_2) \alpha^{(\eta_2)}(k_2, x_1, x_2). \end{aligned} \quad (5.43)$$

Since the waveshape is at least of order  $g^3$ , it follows that this overlap begins at order  $g^6$ . However by expanding the  $S$  matrix out directly, we encounter a six-point amplitude. The conclusion is that our proposal does not populate the (order  $g^4$ ) tree-level six-point amplitude. Of course this is as it should be: we saw that the six-point tree is suppressed in the classical region in section 3.1.4. Similarly the seven-point tree and one-loop amplitudes are suppressed, etc.

As a final remark, note that we did not introduce any normalisation factor in our proposal. This may be surprising in view of the necessity to normalise the coherent state as in equation (5.26). The origin of the difference is simply that unitarity must already guarantee the normalisation of the final state. As is by now well understood, at two loops the eikonal function  $\chi$  ceases to be real in the radiative case. Instead the imaginary part of  $\chi$  is related to  $\sum_\eta |\alpha^{(\eta)}|^2$ ; this supplies the necessary normalisation.

## 5.5 Radiation reaction

Once there is radiation, there must also be radiation reaction: the particle's motion must change in the radiative case relative to the conservative case to account for the loss of momentum to radiation. In this section we will see that the waveshape indeed contributes to the impulse of a particle in the manner required.

We begin by acting on our conjectural final state, equation (5.30), with the

momentum operator of the field corresponding to particle 1:

$$\begin{aligned} \mathbb{P}_1^\mu S |\psi\rangle &= \int d\Phi(p'_1, p'_2) \int \hat{d}^4 \bar{q} d^4 x d^4 x_1 d^4 x_2 \tilde{\phi}_b(x_1, x_2) e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} \\ &\times e^{i[q \cdot (x - x_1 + x_2) + \chi(x_\perp; s)]/\hbar} \exp \left[ \sum_\eta \int d\Phi(k) \alpha^{(\eta)}(k, x_1, x_2) a_\eta^\dagger(k) \right] p_1'^\mu |p'_1, p'_2\rangle . \end{aligned} \quad (5.44)$$

The operator simply inserts a factor  $p_1'^\mu$ . We proceed by rewriting this factor in terms of a derivative  $-i\hbar\partial/\partial x_{1\mu}$  acting on the exponential factor in the first line of equation (5.44), and then integrating by parts. Neglecting the boundary term, the result is

$$\begin{aligned} \mathbb{P}_1^\mu S |\psi\rangle &= \int d\Phi(p'_1, p'_2) \int \hat{d}^4 \bar{q} d^4 x d^4 x_1 d^4 x_2 e^{i(p'_1 \cdot x_1 + p'_2 \cdot x_2)/\hbar} e^{i(q \cdot x + \chi(x_\perp; s))/\hbar} \\ &\times i\hbar\partial_1^\mu \left( \tilde{\phi}_b(x_1, x_2) e^{iq \cdot (x_2 - x_1)/\hbar} \exp \left[ \sum_\eta \int d\Phi(k) \alpha^{(\eta)}(k, x_1, x_2) a_\eta^\dagger(k) \right] \right) |p'_1, p'_2\rangle . \end{aligned} \quad (5.45)$$

Expanding out the derivative, we encounter three terms. In the first, the derivative acts on the spatial wavefunction: as usual in quantum mechanics, this term will evaluate (in the expectation value of the final momentum) to the contribution of the initial momentum. The second term arises when the derivative operator acts on  $e^{iq \cdot (x_2 - x_1)/\hbar}$ , which inserts a factor of  $q^\mu$ . This term is familiar from equation (5.11) in section 5.2, and contributes to the impulse as a (suitably projected) derivative of the eikonal function. Only the final term is new: it involves the waveshape, and must then be the origin of radiation reaction in our approach.

Since we have discussed the conservative impulse in detail in section 5.2, we focus on the final (new) term here. In this term, the derivative brings down a factor of

$$\sum_\eta \int d\Phi(k) \partial_1^\mu \alpha^{(\eta)}(k, x_1, x_2) a_\eta^\dagger(k). \quad (5.46)$$

Now to extract the momentum observable we multiply by  $\langle\psi|S^\dagger$ . Since this will introduce yet more integrals, it is helpful to define a modified KMOC style

‘classical average angle brackets’  $\langle\langle \dots \rangle\rangle$ , defined by

$$\begin{aligned} \langle\langle \dots \rangle\rangle &= \int d\Phi(p'_1, p'_2) \int \hat{d}^4 \bar{q} d^4 x \hat{d}^4 \bar{Q} d^4 y d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 \tilde{\phi}_b^*(y_1, y_2) \tilde{\phi}_b(x_1, x_2) \\ &\times e^{i(p'_1 \cdot (x_1 - y_1) + p'_2 \cdot (x_2 - y_2))/\hbar} e^{i(q \cdot (x_2 - x_1) - Q \cdot (y_2 - y_1))/\hbar} e^{i(q \cdot x + \chi(x_\perp; s) - Q \cdot y - \chi^*(y_\perp; s))/\hbar} \\ &\times \exp \left[ -\frac{1}{2} \sum_\eta \int d\Phi(k) |\alpha^{(\eta)}(k, x_1, x_2) - (\alpha^{(\eta)})^*(k, y_1, y_2)|^2 \right] (\dots). \end{aligned} \quad (5.47)$$

The  $a^\dagger(k)$  will act on this left state will produce a delta function and  $\alpha^*$ . After using the delta function it gives just a factor  $\alpha^*$ . Putting this together, and using the angle brackets shorthand we get

$$\langle \mathbb{P}_1^\mu \rangle_{\text{reaction}} = \langle\langle i \sum_\eta \int d\Phi(k) \alpha^{(\eta)*}(k, x_1, x_2) \partial_1^\mu \alpha^{(\eta)}(k, x_1, x_2) \rangle\rangle. \quad (5.48)$$

Notice that the manipulations leading to equation (5.48) were exact (under the assumption of equation (5.30).) However equation (5.48) involves a number of integrals which would need to be performed to arrive at a concrete expression for the impulse. In section 5.2, we performed these integrals by stationary phase. In the present (radiative) case a similar approach would be possible, but the stationary phase conditions are significantly more complicated. For example, demanding the phase of the  $q$  integral to be stationary leads to a condition involving the variables  $x_2$  and  $x_1$ ; further demanding that the phases of these  $x_i$  integrals should be stationary leads to an equation involving the integral of a quadratic function of the waveshape. In this way the stationary phase conditions involve an intricate interplay of the eikonal and the waveshape. Of course this is as it should be: the complexity of radiation reaction must be captured by the final state.

As a simpler sanity check of our machinery, we evaluate the radiation reaction contribution to the impulse at lowest non-trivial perturbative order. In [45] the leading order radiation reaction term was written as

$$I_{rad}^\mu = e^6 \langle\langle \int d\Phi(\bar{k}) \prod_{i=1,2} \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{q}'_i \bar{q}_1^\mu \mathcal{Y}(\bar{q}_1, \bar{q}_2, \bar{k}) \mathcal{Y}^*(\bar{q}'_1, \bar{q}'_2, \bar{k}) \rangle\rangle, \quad (5.49)$$

with

$$\mathcal{V}(\bar{q}_1, \bar{q}_2, \bar{k}) = \delta(p_1 \cdot \bar{q}_1) \delta(p_2 \cdot \bar{q}_2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2 - k) e^{ib \cdot \bar{q}_1} \mathcal{A}_{5,0}^{(0)}(\bar{q}_1, \bar{k}^\eta). \quad (5.50)$$

It is straightforward to verify that this is equivalent to the second term in equation (5.48). In fact, recalling the leading order waveshape formula (5.39), the match between the two expressions (5.49) and (5.48) is immediate, once our definition of average over wave packets (5.47) is taken into account.

We note in passing that the expectation  $\langle \psi | S^\dagger \mathbb{F}_{\mu\nu} S | \psi \rangle$  (or  $\langle \psi | S^\dagger \mathbb{R}_{\mu\nu\rho\sigma} S | \psi \rangle$ ) can be determined in a similar way. The annihilation operators in the field strength operator immediately bring down a single power of the waveshape. At leading non-trivial order in  $g$ , it is then straightforward to see that the field strength is determined by the five-point tree amplitude (specifically the leading fragment in  $\hbar$ ) consistent with reference [163]. Similarly, the momentum radiated into messengers can be computed as the expectation

$$\langle \psi | S^\dagger \sum_\eta \int d\Phi(k) k^\mu a_\eta^\dagger(k) a_\eta(k) S | \psi \rangle. \quad (5.51)$$

In this case, the creation and annihilation operators bring down the waveshape times its conjugate. At leading perturbative order, the momentum radiated is the square of the five-point tree, as observed in [45]. The result is also consistent with classical field theory: in that context, radiation is described by the energy-momentum tensor, which is quadratic in the field strength. Finally, we note that conservation of momentum holds as discussed in [45].

## Summary

In this chapter we took the KMOC formalism and incorporated the eikonal formula. This allowed us to write a form for the final semi-classical state  $\hat{S} |\psi\rangle$ , containing all the classical data, in terms of the eikonal phase. Then including radiation can be achieved by introducing a coherent state for the outgoing radiation of the photon (or gluon/graviton). It was important that this was not merely a tensor product of the conservative and coherent states, but rather included some mixing that allows for momentum and radiation to talk to each other. In this picture we are able to compute the full impulse knowing only the eikonal  $\chi$  and waveshape  $\alpha$ , which depend on the all order amplitudes  $\mathcal{A}_4$  and  $\mathcal{A}_5$

respectively. It is known that at the leading level the radiation field is described by the five-point, but our claim here is that this is an all order statement: all information relevant to classical physics is contained in these amplitudes. As the four- and five-point amplitudes are enough to describe the classical dynamics, it makes sense that higher-point amplitudes are classically suppressed.

The exponential structure of this state yields relations between the four and five point amplitudes, order by order, in a similar vein to the minimal variance conditions of chapter 3. Indeed the first non trivial condition for the exponential formula, relating the leading fragment of the one loop 5-point amplitude to the 4 and 5-point tree amplitudes, was identical to the condition in section 3.2.2.

While we provided evidence and justification for why this final state ansatz given in equation (5.30) is reasonable, it should be emphasised that we made no attempt to prove it. It certainly seems possible that the waveshape  $\alpha$  could depend, in general, on more variables. There is still no known general proof of the eikonal exponentiation in the conservative case, let alone the radiative one too. It would be interesting if future work could shed light on this, and even refine our ansatz state. Likewise exploring the higher loop structure, and relations between loops and cuts is also an avenue worth exploring that we have not done in this thesis.

In the next chapter we take the ideas explored in this chapter and extend it to include other degrees of freedom - in particular colour and spin.

## Chapter 6

# Spinning impulse in Gravity: KMOC vs Eikonal

Our discussion of the eikonal thus far has been for scalar particles in EM or GR, with no colour or spin degrees of freedom. However, the application of scattering amplitudes to classical systems involving colour and spin is also an important topic [70, 75, 130, 131, 133–135, 138, 143, 144, 191, 242–244], motivated in large part by the dynamics of inspiraling black holes with angular momentum. In this chapter, we expand this discussion to include minimum uncertainty spin and colour states. For simplicity, however, we revert to the purely conservative case throughout this section, leaving radiation of spin to future work.

We begin by reviewing the eikonal formula again, this time including other degrees of freedom such as spin or colour. We consider a generic Lie group valued charges, but have in mind the application to spin ultimately. We follow the same steps as in the last chapter to obtain a semi-classical final state in terms of the eikonal, this time only working in the conservative case. Next we construct observables in this setup which can include new observables, like the colour impulse discussed in chapter 4, and this reveals new structures in the observables. We discover an infinite set of nested commutators which, in the classical limit, are replaced by Poisson brackets. We still make use of stationary phase and evaluate these Poisson brackets on the solution to the stationary phase conditions  $x_*$ ,  $q_*$ . As a first application of this we reproduce the colour impulse results of chapter 4.

We then move to looking at the impulse for a particle with spin. We consider two massive particles, one scalar and one spin one vector, and work to NLO in

the coupling (1 loop) and linear order in spin. For the impulse we can ignore the commutator structure and simply evaluate the stationary phase momentum transfer. Nevertheless there are still intricacies that one must be aware of that are not present without spin. We compute the impulse in the limit where the spinning particle is much heavier than the scalar which can be viewed as a scalar particle moving in a fixed Kerr background. This limit, rather than the converse of a light spinning particle, has been considered in other works [245]. We also perform the same computation using the KMOC procedure.

## 6.1 Eikonal state with spin degrees of freedom

In section 5, we derived a formula for the final state involving an eikonal operator acting on the two-particle state. In general, states are labelled by an assortment of quantum numbers which can also evolve during a scattering event, and so in this section we will generalise the eikonal  $S$ -matrix operator to include Lie-algebra valued quantum numbers, focussing on spin for clarity.

We take equation (2.11) as the initial state, and now use  $a_1$  and  $a_2$  for the spin indices. Then the amplitude acquires indices becoming a matrix in (2.22). For the specific case of  $2 \rightarrow 2$  scattering this becomes

$$\langle p'_1, a'_1; p'_2, a'_2 | T | p_1, a_1; p_2, a_2 \rangle = \mathcal{A}_4(p_1, p_2 \rightarrow p'_1, p'_2)_{a_1 a_2}^{a'_1 a'_2} \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2). \quad (6.1)$$

Now that the scattering amplitudes are matrices we follow [78, 143, 246] and assume that eikonal exponentiation takes a matrix form<sup>1</sup> so that eq. (5.1) becomes

$$\exp \left( i\chi(x_\perp, s)/\hbar \right)_{a_1, a_2}^{a'_1, a'_2} = \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} + i \int \hat{d}^4 q \, \hat{\delta}(2p_1 \cdot q) \hat{\delta}(2p_2 \cdot q) e^{-iq \cdot x/\hbar} \mathcal{A}_4(s, q^2)_{a_1, a_2}^{a'_1, a'_2}. \quad (6.2)$$

and its inverse (5.6) becomes

$$i\hat{\delta}(2\tilde{p}_1 \cdot q) \hat{\delta}(2\tilde{p}_2 \cdot q) \mathcal{A}_4(s, q^2)_{a_1, a_2}^{a'_1, a'_2} = \frac{1}{\hbar^4} \int d^4 x \, e^{iq \cdot x/\hbar} \left\{ \exp \left( i\chi(x_\perp, s)/\hbar \right)_{a_1, a_2}^{a'_1, a'_2} - \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \right\}. \quad (6.3)$$

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<sup>1</sup>We continue to ignore the quantum remainder in this discussion.

Acting with the S-matrix on the initial state, and using this eikonal we can write the final state with spin indices as

$$S|\psi\rangle = \frac{1}{\hbar^4} \int d\Phi(p'_1, p'_2) |p'_1, a'_1; p'_2, a'_2\rangle \int \hat{d}^4 q d^4 x \phi_b(p'_1 - q, p'_2 + q) e^{iq \cdot x / \hbar} \times \exp\left(i\chi(x_\perp, s)/\hbar\right)_{a_1, a_2}^{a'_1, a'_2} \xi_1^{a_1} \xi_2^{a_2}. \quad (6.4)$$

Here the vectors  $\xi_q^{a_1}, \xi_2^{a_2}$  describe the spin or colour degrees of freedom, as in eq. (2.11). While we have constructed this with spin in mind, it is applicable to any operator with quantum numbers  $a_i$ , and so we will now consider the expectation value of a generic Lie-algebra valued operator  $\hat{\mathcal{O}}$  that has a sensible classical limit. In particular the momentum operator  $\mathbb{P}^\mu$ , colour  $\mathbb{C}^a$ . We can also look at the operator for spin which is the given by the Pauli-Lubanski operator  $\mathbb{W}^\mu$  defined as

$$\mathbb{W}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbb{P}_\nu \mathbb{S}_{\rho\sigma}, \quad (6.5)$$

Such operators act on one particle states, are linear in the momentum and obey momentum conservation, meaning their matrix elements can be written

$$\langle p, a | \mathcal{O} | k, b \rangle = \hat{\delta}_\Phi(p - k) \mathcal{O}_b^a(p). \quad (6.6)$$

We will focus on operators that act on one particle at a time, say particle 1. In order to compute the change in an observable, we need to find the action of the commutator in (2.18). We will compute each half of the commutator acting on  $\hat{S}|\psi\rangle$  separately. Starting with  $\mathcal{O}_1 S|\psi\rangle$  we have

$$\mathcal{O}_1 S|\psi\rangle = \frac{1}{\hbar^4} \int d\Phi(p'_1, p'_2) |p'_1, b_1; p'_2, a'_2\rangle \int \hat{d}^4 q d^4 x \phi_b(p'_1 - q, p'_2 + q) e^{iq \cdot x / \hbar} \times \mathcal{O}_1(p'_1)_{a'_1}^{b_1} \exp\left(i\chi(x_\perp, s)/\hbar\right)_{a_1, a_2}^{a'_1, a'_2} \xi_1^{a_1} \xi_2^{a_2}, \quad (6.7)$$

and then similarly  $S\mathcal{O}_1|\psi\rangle$  gives

$$S\mathcal{O}_1|\psi\rangle = \frac{1}{\hbar^4} \int d\Phi(p'_1, p'_2) |p'_1, b_1; p'_2, a'_2\rangle \int \hat{d}^4 q d^4 x \phi_b(p'_1 - q, p'_2 + q) e^{iq \cdot x / \hbar} \times \exp\left(i\chi(x_\perp, s)/\hbar\right)_{a'_1, a_2}^{b_1, a'_2} \mathcal{O}_1(p'_1 - q)_{a_1}^{a'_1} \xi_1^{a_1} \xi_2^{a_2} \quad (6.8)$$

We inserted a complete set of states, as in (2.23), which in the specific case of

two particles takes the form

$$\mathbb{1} = \sum_{b_1, b_2} \int d\Phi(k_1, k_2) |k_1, b_1; k_2, b_2\rangle \langle k_1, b_1; k_2, b_2|. \quad (6.9)$$

Notice that we encounter the operator  $\mathcal{O}_1(p'_1 - q)$  at a shifted value of the momentum, arising from  $\mathcal{O}(p_1)$  being expressed in terms of  $p'_1$  and  $q$ . Since the operators we are considering are linear in the momentum, we can write

$$\mathcal{O}_1(p'_1 - q) = \mathcal{O}_1(p'_1) - \mathcal{O}_1(q). \quad (6.10)$$

The notation  $\mathcal{O}_1(q)$  indicates that momentum factors in the operator are evaluated at momentum  $q$ . Using this notation, we can eventually write

$$\begin{aligned} [\mathcal{O}_1(p'_1), S] |\psi\rangle &= \frac{1}{\hbar^4} \int d\Phi(p'_1, p'_2) |p'_1, b_1; p'_2, a'_2\rangle \int \hat{d}^4 q d^4 x \phi_b(p'_1 - q, p'_2 + q) e^{iq \cdot x / \hbar} \\ &\times \left\{ [\mathcal{O}_1(p'_1), \exp(i\chi(x_\perp, s)/\hbar)]_{a'_1, a'_2}^{b_1, a'_2} + \exp\left(i\chi(x_\perp, s)/\hbar\right)_{a'_1, a'_2}^{b_1, a'_2} \mathcal{O}_1(q)^{a'_1}_{a_1} \right\} \xi_1^{a_1} \xi_2^{a_2}. \end{aligned} \quad (6.11)$$

There are two matrix structures inside the integral: one is a commutator, while the second is a more simple matrix product. This structure is a generalisation of the structure discussed in [133], where the second term was called the “direct” term in contrast to the “commutator” term.

In computing the observable, we’re going to multiply this by the the quantity  $\langle\psi| S^\dagger$ , and then evaluate integrals by stationary phase. It will be helpful to rewrite (6.11) in such a way as to bring the matrix  $\exp(i\chi(x_\perp, s)/\hbar)$  to the left. Then in the evaluation of the overlap  $\langle\psi| S^\dagger [\mathcal{O}_1(p'_1), S] |\psi\rangle$ , the matrix and (on the solution of the stationary phase conditions) its inverse will simplify.

We only have to move the eikonal operator though the commutator term in equation (6.11); we can do so using the Baker-Campbell-Hausdorff lemma in the form [247]

$$[\mathcal{O}, e^{i\chi/\hbar}] = e^{i\chi/\hbar} \left( -\frac{i}{\hbar} [\chi, \mathcal{O}] + \frac{1}{2\hbar^2} [\chi, [\chi, \mathcal{O}]] + \cdots + \frac{(-i)^n}{n!\hbar^n} [\chi, [\chi, [\chi, \dots [\chi, \mathcal{O}]]] \dots] \right), \quad (6.12)$$

where the last term contains  $n$  nested commutators involving  $\chi$ .

So far, we have not said anything about the states  $|\xi\rangle$ , be it spin or colour. As discussed in chapter 2 and chapter 4, in order to achieve the desired classical

properties, namely that expectations factorise up to negligible corrections, the states must be coherent or at least have a large representation size, in the sense that the spin quantum number  $S$  times  $\hbar$  is a classical angular momentum  $S\hbar$ . For the spin operator this means

$$\langle \mathbb{S}_{\mu\nu} \mathbb{S}_{\rho\sigma} \rangle = \langle \mathbb{S}_{\mu\nu} \rangle \langle \mathbb{S}_{\rho\sigma} \rangle + \text{small}, \quad (6.13)$$

where  $\mathbb{S}_{\mu\nu}$  is a spin Lorentz generator and  $\langle \mathbb{S}_{\mu\nu} \rangle$  is its expectation value on the spin state. The key point is that the small correction term is of order  $S\hbar^2$  compared to the explicit term on the right-hand side, which is of order  $S^2\hbar^2$ . For the case of  $SU(3)$  colour this is shown explicitly in section 4.1.3.

We saw in chapter 4, for example in eqs. (4.61) and (4.63), how factors of  $\hbar$  in commutators were crucial for the correct classical limits, and it's the same story in the eikonal. Much like with colour, the commutator of the Pauli-Lubanski vector contains an  $\hbar$  and is given by

$$[\mathbb{W}_\mu, \mathbb{W}_\nu] = -i\hbar \epsilon_{\mu\nu\rho\sigma} \mathbb{W}^\rho \mathbb{P}^\sigma. \quad (6.14)$$

When we take the classical limit (large representation states and  $\hbar \rightarrow 0$ ), we can replace operators by their classical expectations. We still need to retain the commutator structure however. Since each nested commutator in eq. (6.12) comes with a factor of  $-i/\hbar$ , we can introduce Poisson brackets<sup>2</sup>, for example

$$\{\langle \mathbb{W}_\mu \rangle, \langle \mathbb{W}_\nu \rangle\} = -\epsilon_{\mu\nu\rho\sigma} \langle \mathbb{W}^\rho \rangle \langle \mathbb{P}^\sigma \rangle. \quad (6.15)$$

These Poisson brackets are directly inherited from the underlying algebraic structure of the quantum field theory, replacing operators by expectations and scaling out appropriate factors of  $i\hbar$ .

We can now make use of this inside the infinite series of commutators on the right hand side of eq. (6.12) – the  $\hbar$  factors cancel and we replace the commutator by Poisson brackets. Doing so and introducing some shorthand notation we write

$$\{\mathcal{O}_1(p'_1), e^{i\chi(x_\perp, s; \langle w \rangle)/\hbar}\}_{\text{B.H.}} \equiv e^{i\chi/\hbar} \left( -\{\mathcal{O}, \chi\} - \frac{1}{2}\{\chi, \{\mathcal{O}, \chi\}\} + \dots \right). \quad (6.16)$$

We emphasise that  $\{\cdot, \cdot\}_{\text{B.H.}}$  is *not* a Poisson bracket: it is simply convenient

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<sup>2</sup>This is the reverse of canonical quantisation, replacing commutators by Poisson bracket rather than vice versa.

notation for the nested commutator structure in equation eq. (6.16).

It was important to first use BCH and then make the classical replacement, since the ‘small’ term in eq. (6.13) is intimately related to the right hand side of the commutator. We need to use the commutation relations in order to cancel the  $\hbar$  factors in the right hand side of the BCH expansion. This is reminiscent of the subtle combinations of  $\hbar$  that occurred for colour observables in chapter 4

Finally we can write the commutator  $[\mathcal{O}, S]$  acting on  $|\psi\rangle$  as

$$\begin{aligned} [\mathcal{O}_1(p'_1), S] |\psi\rangle &= \frac{1}{\hbar^4} \int d\Phi(p'_1, p'_2) \int \hat{d}^4 q d^4 x \phi_b(p'_1 - q, p'_2 + q) e^{iq \cdot x / \hbar} \\ &\times \left( \{ \mathcal{O}_1(p'_1), e^{i\chi(x_\perp, s; \langle w \rangle) / \hbar} \}_{\text{B.H.}} + \mathcal{O}_1(q) e^{i\chi(x_\perp, s; \langle w \rangle) / \hbar} \right) |p'_1, \xi_1; p'_2, \xi_2\rangle, \end{aligned} \quad (6.17)$$

where all operators have been replaced with scalar functions, at the expense of introducing Poisson brackets.

In this form, the eikonal has also become a scalar function  $\chi(x_\perp, s; \langle w \rangle)$  which now depends on the expectation value of operators, eg Pauli-Lubanski  $\langle w \rangle$  (and/or in the Yang-Mills case, the colour). We can therefore perform the  $x$  and  $q$  integrals by stationary phase, following precisely the methods of section 5.2. The stationary phase conditions are

$$q_\mu = -\frac{\partial}{\partial x^\mu} \chi(x_\perp, s; \langle w \rangle). \quad (6.18)$$

We note that now  $\chi$  is a linear function of  $x_\perp^\mu$ , rather than  $x_\perp^2$  in the scalar case, along with the spin vector  $s^\mu = \langle w \rangle$ , and so  $q_\mu^*$  need not point only along  $x_\perp^\mu$ . It will in general point along some other direction in the plane perpendicular to  $\tilde{p}_1$  and  $\tilde{p}_2$ . The final result for the out state is then

$$\begin{aligned} [\mathcal{O}_1(p'_1), S] |\psi\rangle &= \int d\Phi(p'_1, p'_2) \phi_b(p'_1 - q_*, p'_2 + q_*) e^{iq_* \cdot (x_*) / \hbar} \\ &\times \left( \{ \mathcal{O}_1(p'_1), e^{i\chi(x_{\perp*}, s) / \hbar} \}_{\text{B.H.}} + \mathcal{O}_1(q_*) e^{i\chi(x_{\perp*}, s) / \hbar} \right) |p'_1, \xi_1; p'_2, \xi_2\rangle. \end{aligned} \quad (6.19)$$

We can similarly evaluate  $\langle \psi | S^\dagger$  on the stationary phase too<sup>3</sup>, which results in

$$\langle \psi | S^\dagger = \int d\Phi(p_1, p_2) \phi_b^\dagger(p_1 - q_*, p_2 + q_*) e^{-iq_* \cdot (x_*) / \hbar} e^{-i\chi^\dagger(x_{\perp*}, s)} \langle p_1, \xi_1; p_2, \xi_2 |. \quad (6.20)$$

---

<sup>3</sup>We’re working in conservative sector so  $\chi$  is real.

Combining all this we obtain an expression for the observable

$$\begin{aligned}
\Delta \mathcal{O}_1 &= \langle \psi | S^\dagger [\mathcal{O}_1, S] | \psi \rangle \\
&= \int d\Phi(p_1, p_2) |\phi(p_1 - q_*, p_2 + q_*)|^2 \left( e^{-i\chi^\dagger(x_{\perp*}, s)/\hbar} \{ \mathcal{O}_1(p_1), e^{i\chi(x_{\perp*}, s)/\hbar} \}_{\text{B.H.}} + \mathcal{O}_1(q_*) \right) \\
&= \left\langle\left\langle e^{-i\chi^\dagger(x_{\perp*}, s)/\hbar} \{ \mathcal{O}_1(p_1), e^{i\chi(x_{\perp*}, s)/\hbar} \}_{\text{B.H.}} + \mathcal{O}_1(q_*) \right\rangle\right\rangle.
\end{aligned} \tag{6.21}$$

The Baker-Hausdorff lemma in the form of equation (6.16) immediately tells us how to expand the Poisson bracket, leading to a neat expression for the change in  $\mathcal{O}_1$

$$\Delta \mathcal{O}_1 = \mathcal{O}_1(q_*) - \{ \mathcal{O}_1(p_1), \chi \} - \frac{1}{2} \{ \chi, \{ \mathcal{O}_1(p_1), \chi \} \} + \frac{1}{6} \{ \chi, \{ \chi, \{ \mathcal{O}_1(p_1), \chi \} \} \} + \dots \tag{6.22}$$

where we recall that  $\chi$  depends on  $x_{\perp*}^\mu$  along with spin, colour, and momentum; all evaluated at their classical expectations. It is useful to note that for operators without momentum dependence, the term involving  $q_*$  vanishes, since it was induced by a shift in momentum and the linearity of the operator.

In order to evaluate these Poisson brackets we use a fundamental property, namely that

$$\{ A, f(B) \} = \{ A, B \} \frac{\partial f}{\partial B}, \tag{6.23}$$

to convert commutators with  $\chi$  into derivatives of  $\chi$  and fundamental Poisson brackets, such as eq. (6.15).

We shall shortly apply this to the computation of the impulse in section 6.3. Assuming  $\mathbb{P}_1^\mu$  to commute with  $\chi$ , we simply need to evaluate the stationary phase value for  $q_*$ . The  $x_{\perp}^\mu$  dependence of  $\chi$  will be different in the spinning case, which will lead to additional spin contributions to the impulse. These new dependencies makes evaluating the stationary phase conditions and computing the impulse much more intricate than the scalar case.

## 6.2 Classical Colour from the Eikonal

Let us begin by reproducing the results of chapter 4 in this eikonal framework. The change in the classical colour is then given by eq. (6.21)

$$\Delta c_1^a = \left\langle\left\langle e^{-i\chi^\dagger(x_{\perp*}, c^i)/\hbar} \left\{ c_1^a, e^{i\chi(x_{\perp*}, c^i)/\hbar} \right\}_{\text{B.H.}} \right\rangle\right\rangle, \quad (6.24)$$

where  $c^a = \langle \psi | \mathbb{C} | \psi \rangle$  and  $\mathcal{O}(q_*)$  term has dropped out since the colour does not depend on momentum. The first few terms in the BCH expansion are given by

$$\Delta c_1^a = -\{c_1^a, \chi\} - \frac{1}{2}\{\chi, \{c_1^a, \chi\}\} + \dots, \quad (6.25)$$

Using eq. (6.23) along with the relation  $\{c^a, c^b\} = f^{abc}c^c$ , we can write the first bracket as

$$\{c_1^a, \chi\} = \{c_1^a, c_1^b\} \frac{\partial \chi}{\partial c_1^b} = f^{abc} \frac{\partial \chi}{\partial c_1^b} c_1^c. \quad (6.26)$$

At order  $\mathcal{O}(g^4)$ , we expect contributions from the tree-level and one-loop in the first bracket, as well as the tree-level terms in the second bracket eq. (6.25). The tree-level and one-loop triangle eikonals are given by [1, 248]

$$\begin{aligned} \chi_0(x_{\perp}, c^i) &= g^2(c_1 \cdot c_2) \frac{\gamma}{4\pi\sqrt{\gamma^2 - 1}} \log \left| \frac{x_{\perp}^2}{L^2} \right|, \\ \chi_1(x_{\perp}, c^i) &= \frac{g^4(c_1 \cdot c_2)^2(m_1 + m_2)}{32\pi m_1 m_2 \sqrt{\gamma^2 - 1} |x_{\perp}|}, \end{aligned} \quad (6.27)$$

where  $L$  is an infrared cutoff. We can plug the tree-level eikonal into the bracket to find, at first order,

$$\Delta c_1^{a,0} = -\{c_1^a, \chi_1\} = \frac{\gamma g^2 f^{abc} c_1^b c_2^c}{4\pi\sqrt{\gamma^2 - 1}} \log \left| \frac{x_{\perp}^2}{L^2} \right|. \quad (6.28)$$

The second bracket is given by

$$\Delta c_1^{a,1} = -\frac{1}{2}\{\chi, \{c_1^a, \chi\}\} \quad (6.29)$$

$$= -\frac{1}{2} \frac{g^4 \gamma^2}{(4\pi)^2 (\gamma^2 - 1)} \log^2 \left| \frac{x_{\perp}^2}{L^2} \right| f^{abc} [\{\chi, c_2^b\} c_1^c + c_2^b \{\chi, c_1^c\}] \quad (6.30)$$

$$= -\frac{1}{2} \frac{g^4 \gamma^2}{(4\pi)^2 (\gamma^2 - 1)} \log^2 \left| \frac{x_{\perp}^2}{L^2} \right| [f^{abc} f^{bde} c_1^c c_1^d c_2^e - f^{abc} f^{bde} c_2^c c_2^d c_1^e]. \quad (6.31)$$

The contribution from the triangle eikonal is evaluated in the first bracket to give

$$\begin{aligned}\{c_1^a, \chi_2\} &= \frac{g^4(m_1 + m_2)}{32\pi m_1 m_2 \sqrt{\gamma^2 - 1} |x_\perp|} \{c_1^a, (c_1 \cdot c_2)^2\} \\ &= -\frac{g^4(c_1 \cdot c_2) f^{abc} c_1^b c_2^c}{16\pi \sqrt{\gamma^2 - 1} |x_\perp|} \left( \frac{1}{m_1} + \frac{1}{m_2} \right).\end{aligned}\tag{6.32}$$

Putting this together gives the full  $\mathcal{O}(g^4)$  colour impulse

$$\begin{aligned}\Delta c_1^a &= \frac{\gamma g^2 f^{abc} c_1^b c_2^c}{4\pi \sqrt{\gamma^2 - 1}} \log \left| \frac{x_\perp^2}{L^2} \right| + \frac{g^4(c_1 \cdot c_2) f^{abc} c_1^b c_2^c}{16\pi \sqrt{\gamma^2 - 1} |x_\perp|} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \\ &\quad - \frac{1}{2(4\pi)^2(\gamma^2 - 1)} \log^2 \left| \frac{x_\perp^2}{L^2} \right| [f^{abc} f^{bde} c_1^c c_1^d c_2^e - f^{abc} f^{bde} c_2^c c_2^d c_1^e].\end{aligned}\tag{6.33}$$

In chapter 4 we did not compute the integrals explicitly, however they were done in [248] and the results here match.

## 6.3 NLO Spinning impulse - Eikonal

Let us use the eikonal framework to compute the impulse with spin. The impulse does not involve any of the commutator / Poisson bracket terms, so we only need to evaluate the stationary phase momentum shift  $\Delta p = q_*$ . We do this iteratively solving the equations order by order. We must be careful to keep track of the variables in the eikonal, namely the full vector  $x_\perp^\mu$ , the spin  $s^\mu$  and the shifted momenta  $\tilde{p}_i^\mu$ , since these all contain hidden  $q$  dependence.

We will work two massive particles, one particle spinning which we take to be particle one, and the other scalar. Given the complexity of the stationary phase calculation, in particular the iteration structure of  $x_\perp$ , we will simplify things by working in the limit where  $m_1 \gg m_2$ . This means the spinning background is essentially fixed, and we only need to worry about the dynamics of the scalar particle. This limit has also been investigated in other work, for example [245], where there the authors take this limit in order to obtain an all-orders-in-spin result.

Recall we have two stationary phase equations to solve in eqs. (5.12) and (5.14), one for  $x$  and one for  $q$ . Before diving into the computation, let us just outline the procedure and philosophy. The eikonal, as defined in eq. (5.1), is a function of  $p_1, p_2, s_1, x_\perp$ . The  $q$  dependence in the amplitude is converted into  $x_\perp$  dependence

by the Fourier transform. However we evaluate this eikonal function on the *tilde* variables  $\tilde{p}_1, \tilde{p}_2$  and  $\tilde{s}_1 \equiv s(\tilde{p}_1)$  which reintroduces some explicit  $q$  dependence. We do this because we had to introduce shifted variables in our state and amplitude in order to make use of the definition eq. (5.1). The other source of  $q$  dependence is via  $x_\perp$ , which implicitly depends on  $q$  (recall it is defined by  $\tilde{p}_i \cdot x_\perp = 0$ ). With all this together we can evaluate the stationary phase condition for  $x$ , then project it to  $x_\perp$ . The derivatives of the eikonal with respect to  $x$  are much more straightforward. The final step is to substitute in the expression for  $x_\perp$  into  $\partial\chi/\partial x$ . At NLO there will be contributions arising from corrections to  $x_\perp$  plugged into the leading order impulse expression.

Let us now solve the stationary phase equations, which I will repeat here for clarity making explicit the variables on which  $\chi$  depends,

$$q_\mu = -\frac{\partial}{\partial x^\mu} \chi(x_\perp, s_1, \tilde{p}) \quad (6.34)$$

and

$$x^\mu - b^\mu + \frac{\partial}{\partial q_\mu} \chi(x_\perp, s_1, \tilde{p}) = 0. \quad (6.35)$$

Firstly we need the eikonal phases, which we get by Fourier transforming the momentum space amplitudes,

$$\mathcal{M}_{4,0} = -\frac{16\pi G m_1^2 m_2^2 ((2\gamma^2 - 1) - 2i\gamma\epsilon(p_1, p_2, q, s_1))}{q^2}, \quad (6.36)$$

$$\begin{aligned} \mathcal{M}_{4,1}|_{sc} &= -\frac{16\pi i G^2 m_1^3 m_2^3 \left( (2\gamma^2 - 1)^2 - 2i\gamma(2\gamma^2 - 1)\epsilon(p_1, p_2, q, s_1) \right)}{\epsilon(q^2)^{1+\epsilon}} \\ \mathcal{M}_{4,1}|_{cl} &= \frac{6\pi^2 G^2 m_1^2 m_2^2 (5\gamma^2 - 1) (m_1 + m_2)}{(-q^2)^{1/2+\epsilon}} \\ &\quad - \frac{2i\pi^2 G^2 m_1^2 m_2^2 \gamma (5\gamma^2 - 3) (4m_1 + 3m_2) \epsilon(p_1, p_2, q, s_1)}{(\gamma^2 - 1) (-q^2)^{1/2+\epsilon}}. \end{aligned} \quad (6.37)$$

Taking the tree and classical part of the one loop amplitudes in eqs. (6.36) and (6.37) and performing the Fourier transforms (given in the appendix A.2.1)

we find

$$\begin{aligned}\chi_0(x_\perp, s_1, \tilde{p}) &= -\frac{4Gm_1m_2}{\sqrt{\gamma^2-1}} \left[ \frac{(2\gamma^2-1)}{4} \log(-x_\perp^2) - \frac{\gamma\epsilon(\tilde{a}_1, \tilde{u}_1, \tilde{u}_2, x_\perp)}{x_\perp^2} \right], \\ \chi_1(x_\perp, s_1, \tilde{p}) &= \frac{\pi G^2 m_1 m_2}{4\sqrt{\gamma^2-1}} \left[ \frac{3(5\gamma^2-1)(m_1+m_2)}{\sqrt{-x_\perp^2}} \right. \\ &\quad \left. + \frac{\gamma(5\gamma^2-3)(4m_1+3m_2)\epsilon(\tilde{a}_1, \tilde{u}_1, \tilde{u}_2, x_\perp)}{x_\perp^2(\gamma^2-1)} \right].\end{aligned}\tag{6.38}$$

These are functions of  $x_\perp$  not  $x$ , and note that both have the form of

$$\chi = \chi_{scal}(x_\perp^2) + \chi_{spin}(x_\perp^2)\epsilon(x_\perp, \tilde{p}_1, \tilde{p}_2, \tilde{s}_1),\tag{6.39}$$

where  $\chi_I(x_\perp^2)$  is a function of the norm of  $x_\perp$ , just like the scalar case. The fact that it is  $\tilde{p}_1$  and  $\tilde{p}_2$  that are contracted with the Levi-Civita is important when it comes to evaluating the stationary phase conditions later and in appendix C. Writing  $x_\perp^\mu = \Pi^{\mu\nu}x_\nu$  and using the chain rule we can write the first stationary phase condition as

$$q_*^\mu = -\Pi^{\mu\nu} \frac{\partial}{\partial x_\perp^\nu} \chi(x_\perp, s_1, \tilde{p}).\tag{6.40}$$

This projector often acts simply as a metric - the orthogonal subtracted terms will vanish. Computing the derivatives with respect to  $x_\perp$  gives

$$\begin{aligned}\frac{\partial \chi_0}{\partial x_{\perp\mu}} &= \frac{G(x_\perp^\mu(2[2\gamma^2-1]m_1m_2x_\perp^2 + 4\gamma\epsilon(\tilde{p}_1, \tilde{p}_2, \tilde{s}_1, x_\perp)) + 2\gamma x_\perp^2 \epsilon^\mu(\tilde{p}_1, \tilde{p}_2, \tilde{s}_1))}{\sqrt{\gamma^2-1}x_\perp^4} \\ \frac{\partial \chi_1}{\partial x_{\perp\mu}} &= -\frac{3\pi G^2 m_1 m_2 (5\gamma^2-1)(m_1+m_2)x_\perp^\mu}{4(\gamma^2-1)^{1/2}(-x_\perp^2)^{3/2}} \\ &\quad + \frac{\pi G^2 m_1 m_2 (5\gamma^2-3)\gamma(4m_1+3m_2)(3x_\perp^\mu \epsilon(\tilde{p}_1, \tilde{p}_2, \tilde{s}_1, x_\perp) + x_\perp^2 \epsilon^\mu(\tilde{p}_1, \tilde{p}_2, \tilde{s}_1))}{4(\gamma^2-1)^{3/2}(-x_\perp^2)^{5/2}}.\end{aligned}\tag{6.41}$$

Notice that these are either proportional to  $x_\perp^\mu$  or  $\epsilon^\mu(\tilde{p}_1, \tilde{p}_2, \tilde{s}_1)$ . Both of these terms are already orthogonal so the projector acts trivially<sup>4</sup> and so we can ignore the projector in (6.40).

The next task is to compute  $x_\perp^\mu$  via the second of the two stationary phase conditions, which involves derivatives with respect to  $q^\mu$ . We do this order by order as well, noting that we only need the  $O(G)$  correction to  $x_\perp^\mu$  to get an  $O(G)$

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<sup>4</sup>This would not have been true if we had  $\epsilon^\mu(p_1, p_2, s_1)$ . Since the projector is built from the  $e_0^\mu$  and  $e_q^\mu$  which in turn are built from the  $\tilde{p}'s$ , these projector terms acting on the levi civita gives zero.

impulse contribution from  $\chi_0$ . In the  $O(G)$  correction to  $x_\perp^\mu$  there will be terms coming from  $x_*^\mu$  and also from the projection procedure.

Since there is  $q$  dependence in  $x_\perp, \tilde{p}$  and  $\tilde{s}$ , we can write the second stationary phase condition, using the chain rule, as

$$x_*^\mu = b^\mu - \frac{\partial \chi}{\partial x_\perp^\nu} \frac{\partial x_\perp^\nu}{\partial q_\mu} - \sum_i \frac{\partial \chi}{\partial \tilde{p}_i^\nu} \frac{\partial \tilde{p}_i^\nu}{\partial q_\mu} - \frac{\partial \chi}{\partial \tilde{s}_1^\nu} \frac{\partial \tilde{s}_1^\nu}{\partial q_\mu}. \quad (6.42)$$

Using the results given in appendix C, this can be written as

$$x^\mu = b^\mu - N_q(x \cdot e_q) q^\mu|_{q=q_*} - \frac{1}{2} \left( \frac{\partial \chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu}. \quad (6.43)$$

In all these expressions we compute derivatives and only at the very end evaluate at  $q = q_* = \Delta p$ .

At lowest order we can set  $x^\mu = x_\perp^\mu = b^\mu$ , and set  $\tilde{p} \rightarrow p$ , since all extra derivatives in eq. (6.43) contain at least one extra power of  $G$ . Substituting this into the first of eq. (6.41), we get the leading order (1PM) impulse

$$\begin{aligned} \Delta p_1^\mu &= q_*^\mu = \frac{2Gm_1m_2}{\sqrt{\gamma^2 - 1}} \left[ \frac{(2\gamma^2 - 1)}{b^2} b^\mu + 2\gamma\epsilon^\nu(a_1, u_1, u_2) \left( \frac{\Pi_\nu^\mu}{b^2} - \frac{2b^\mu b_\nu}{b^4} \right) \right] \\ &= \frac{2Gm_1m_2}{\sqrt{\gamma^2 - 1}} \left[ \frac{(2\gamma^2 - 1)}{b^2} b^\mu + \frac{2\gamma\epsilon^\mu(a_1, u_1, u_2)}{b^2} - \frac{4\gamma b^\mu \epsilon(b, a_1, u_1, u_2)}{b^4} \right]. \end{aligned} \quad (6.44)$$

Now let us move onto the next to leading order impulse. To compute this we need the  $O(G)$  correction to  $x_\perp^\mu$  which we get by first finding the correction to  $x_*^\mu$  and then projecting. This requires inputting  $\chi_0$ , and  $x_\perp^\mu = b^\mu$  on the right hand side of eqs. (6.38) and (6.43). We should now do some  $G$  power counting. Firstly note that all these corrections involve  $\chi$  which is  $O(G)$  at least. Secondly we should be treating  $q^\mu = O(G)$  since at the end we will evaluate on the stationary phase condition  $q^\mu = q_*^\mu$ . This means that any terms inside or multiplying  $\chi$  that contain a  $q^\mu$  or  $q^2$  can be dropped as they will contribute to  $x_\perp^\mu$  at  $O(G^2)$  and thus  $O(G^3)$  in the impulse. In particular we can replace  $\tilde{p}^\mu \rightarrow p^\mu$ , and set  $q^2 = 0$  inside the normalisation factors  $N_q, N_{0q}$  in all our expressions from now on.

The final term in eq. (6.43) requires a form for  $\tilde{s}_1^\mu$ . If we assume it to be linear in momentum then we can write

$$\tilde{s}^\mu = s^\mu(p) + \frac{1}{2} s^\mu(q). \quad (6.45)$$

The spin vector can be written in terms of the spin tensor

$$s_1^\mu(q) = \frac{1}{2m_1} \epsilon^{\mu\nu\rho\sigma} q_\nu S_{\rho\sigma}(p_1),$$

$$\text{where } S_{\rho\sigma}(p_1) = \frac{1}{m_1} \epsilon_{\rho\sigma\mu\nu} p_1^\mu s_1^\nu(p_1). \quad (6.46)$$

This gives a spin vector, evaluated at  $q$  to be

$$s_1^\mu(q) = \frac{1}{2m_1^2} \epsilon^{\mu\nu\rho\sigma} q_\nu \epsilon_{\rho\sigma\lambda\tau} p_1^\lambda s_1^\tau(p_1) = \frac{1}{m_1^2} ((q \cdot p_1) s_1^\mu(p_1) - (s_1 \cdot q) p_1^\mu). \quad (6.47)$$

As a sanity check we can check that the spin supplementary condition (SSC) holds for  $\tilde{s}$ , that is  $p_1 \cdot s_1 = \tilde{p}_1 \cdot \tilde{s}_1 = 0$ .

Differentiating we find

$$\frac{\partial \tilde{s}^\mu}{\partial q^\nu} = \frac{1}{2} (s^\mu(p_1) p_1^\nu - s^\nu(p_1) p_1^\mu). \quad (6.48)$$

Taking the form of  $\chi_0$  and using the spin derivative above we can write the last term as

$$-\frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu} = -\frac{2Gm_1 m_2 \gamma \epsilon(s, u_1, u_2, x_\perp)}{(\gamma^2 - 1) x_\perp^2} p_1^\mu$$

$$\propto G p_1^\mu. \quad (6.49)$$

The middle terms are

$$\frac{1}{2} \left( \frac{\partial \chi_0}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi_0}{\partial \tilde{p}_{2\mu}} \right) = \frac{2G\gamma}{b^2} (\epsilon^\mu(b, s_1, p_1) + \epsilon^\mu(b, s_1, p_1))$$

$$= \frac{2G\gamma}{N_0 b^2} \epsilon^\mu(b, s_1, e_o). \quad (6.50)$$

Taking these, along with the leading order expression for  $q_*^\mu$  we get<sup>5</sup>

$$x_*^\mu = b^\mu - \frac{2\gamma G m_2 m_2 p_1^\mu \epsilon(s_1, u_1, u_2, b)}{\sqrt{\gamma^2 - 1} b^2} - \frac{2\gamma G (\epsilon^\mu(p_1, s_1, b) + \epsilon^\mu(p_2, s_1, b))}{\sqrt{\gamma^2 - 1} b^2} + O(G^2, s_1^2)$$

$$\equiv b^\mu + \delta^{(1)} x_*^\mu. \quad (6.51)$$

The  $N_q(x \cdot e_q) q^\mu$  term in eq. (6.43), is actually order  $G^2$ . This is true even without spin.

Then we can project this using the procedure outlined in the appendix, which

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<sup>5</sup>now dropping terms suppressed by powers of  $m_2/m_1$ .

yields, after a little algebra,

$$x_{\perp}^{\mu} = b^{\mu} + \frac{G(m_1(\gamma u_1^{\mu} - u_2^{\mu}))((2\gamma^2 - 1)b^2 - 4\gamma\epsilon(b, s_1, u_1, u_2))}{(\gamma^2 - 1)^{3/2}b^2} - \frac{2G\gamma m_1\epsilon^{\mu}(b, s_1, u_1)}{(\gamma^2 - 1)^{1/2}b^2}. \quad (6.52)$$

Note that  $x_{\perp}^2 = b^2 + O(G^2)$ , so we can safely replace  $x_{\perp} \rightarrow b$  in all denominators when we come to evaluate the impulse. It is interesting to note the spin derivative terms in eq. (6.49), do not contribute at this order to  $x_{\perp}$ . The projection imparts another power of  $G$ , and thus this contributes to the impulse at  $O(G^3)$ .

Before we take  $x_{\perp}^{\mu}$  and substitute into the derivatives of  $\chi_0$ , note that there is another NLO contribution to the impulse coming from evaluating  $\chi_0$  on the  $\tilde{p}$  variables, as in eqs. (6.38) and (6.41). Using the definitions eqs. (5.5) and (6.47) we have

$$\begin{aligned} \epsilon^{\mu}(\tilde{s}_1, \tilde{p}_1, \tilde{p}_2) &= \epsilon^{\mu}(s_1, p_1, p_2) + \frac{1}{2}(\epsilon^{\mu}(s_1, p_1, q) + \epsilon^{\mu}(s_1, p_2, q)) \\ &\rightarrow \epsilon^{\mu}(s_1, p_1, p_2) + \frac{1}{2}\epsilon^{\mu}(s_1, p_1, q) \end{aligned} \quad (6.53)$$

here the second term is dropped in the last line since it is suppressed in the  $m_1 \gg m_2$  limit. Note the spin shift has no effect here because it is proportional to  $s_1$  or  $p_1$  and vanishes when contracted with the Levi-Civita. Now inserting the leading order  $q_*$ , and writing  $s_1 = m_1 a_1$ , we get

$$\epsilon^{\mu}(\tilde{s}_1, \tilde{p}_1, \tilde{p}_2) = \epsilon^{\mu}(s_1, p_1, p_2) + \frac{2Gm_1^2 m_2 \gamma}{\sqrt{\gamma^2 - 1}b^2} [\epsilon^{\mu}(b, s_1, u_1)]. \quad (6.54)$$

Substituting  $x_{\perp} \rightarrow \delta^{(1)}x_{\perp}$  and using eq. (6.54) in the leading order  $\chi_0$  derivative we get the iterated NLO impulse terms from  $\chi_0$ ,

$$\begin{aligned} \Delta^{(1)}p_1^{\mu} \Big|_{\chi_0} &= \frac{2G^2 m_1^2 m_2 (2\gamma^2 - 1)(\gamma u_1^{\mu} - u_2^{\mu})((2\gamma^2 - 1)b^2 - 8\gamma\epsilon(b, a_1, u_1, u_2))}{(\gamma^2 - 1)^2 (-b^2)^2} \\ &\quad - \frac{8G^2 m_1^2 m_2 \gamma (2\gamma^2 - 1)\epsilon^{\mu}(b, a_1, u_1)}{(\gamma^2 - 1)(-b^2)^2}. \end{aligned} \quad (6.55)$$

Now taking the derivative of  $\chi_1$  and evaluating at  $x_{\perp} = b$  and  $\tilde{p}_i = p_i$ , we get the

$\chi_1$  contribution to the impulse

$$\Delta^{(1)} p_1^\mu \Big|_{\chi_1} = -\frac{\pi \gamma G^2 m_1^2 m_2 (5\gamma^2 - 3)}{(\gamma^2 - 1)^{3/2} (-b^2)^{3/2}} \left( \epsilon^\mu(a_1, u_1, u_2) + \frac{3b^\mu \epsilon(b, a_1, u_1, u_2)}{b^2} \right) - \frac{3\pi G^2 m_1^2 m_2 b^\mu (5\gamma^2 - 1)}{4(\gamma^2 - 1)^{1/2} (-b^2)^{3/2}}. \quad (6.56)$$

Combining these and regrouping this into the different vectorial directions we get the NLO impulse

$$\begin{aligned} \Delta^{(1)} p_1^\mu = & \frac{2G^2 m_1^2 m_2 \left[ 2(2\gamma^2 - 1)^2 b^2 - 8\gamma(2\gamma^2 - 1) \epsilon(b, a_1, u_1, u_2) \right] (\gamma u_1^\mu - u_2^\mu)}{(\gamma^2 - 1)^2 b^4} \\ & - \frac{8\gamma G^2 m_1^2 m_2 (2\gamma^2 - 1) \epsilon^\mu(b, a_1, u_1)}{(\gamma^2 - 1) b^4} \\ & + \frac{\gamma G^2 m_1^2 m_2 \pi (5\gamma^2 - 3) \epsilon^\mu(s_1, u_1, u_2)}{(\gamma^2 - 1)^{3/2} (-b^2)^{3/2}} \\ & - \frac{3\pi G^2 m_1^2 m_2 b^\mu}{4\sqrt{\gamma^2 - 1} (-b^2)^{3/2}} \left( (5\gamma^2 - 1) + \frac{4\gamma(5\gamma^2 - 3) \epsilon(b, s_1, u_1, u_2)}{(\gamma^2 - 1) (-b^2)} \right). \end{aligned} \quad (6.57)$$

It can be helpful to write this on a different basis, as here we have 6 vectorial directions, so clearly this is a redundant description<sup>6</sup>. We will choose our directions to be

$$\{u_1^\mu, u_2^\mu, b^\mu, \epsilon^\mu(b, u_1, u_2)\}, \quad (6.58)$$

for which we have the following relations

$$\begin{aligned} \epsilon^\mu(b, u_1, s_1) &= -\frac{\epsilon(b, s_1, u_1, u_2)}{\gamma^2 - 1} (\gamma u_1^\mu - u_2^\mu) + \frac{s \cdot u_2}{\gamma^2 - 1} \epsilon^\mu(b, u_1, u_2) \\ \epsilon^\mu(b, u_2, s_1) &= -\frac{\epsilon(b, s_1, u_1, u_2)}{\gamma^2 - 1} (\gamma u_2^\mu - u_1^\mu) + \frac{\gamma s \cdot u_2}{\gamma^2 - 1} \epsilon^\mu(b, u_1, u_2) \\ \epsilon^\mu(s_1, u_1, u_2) &= \frac{b^\mu \epsilon(b, s_1, u_1, u_2)}{b^2} + \frac{b \cdot s_1}{b^2} \epsilon^\mu(b, u_1, u_2). \end{aligned} \quad (6.59)$$

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<sup>6</sup>Note however that other authors often present the result in this redundant set of vectors.

Substituting these into eq. (6.57) above we get

$$\begin{aligned}
\Delta^{(1)} p^\mu = & -\frac{3G^2 m_1^2 m_2 \pi (5\gamma^2 - 1) \sqrt{-b^2} b^\mu}{4\sqrt{\gamma^2 - 1} b^4} \\
& - \frac{2\pi\gamma G^2 m_1^2 m_2 \epsilon(b, a_1, u_1, u_2) (5\gamma^2 - 3) b^\mu}{(\gamma^2 - 1)^{3/2} (-b^2)^{5/2}} \\
& + \frac{2G^2 m_1^2 m_2 (\gamma u_1^\mu - u_2^\mu)}{(\gamma^2 - 1)^2 b^4} \\
& \times \left( (2\gamma^2 - 1)^2 b^2 - 4\gamma (2\gamma^2 - 1) \epsilon(b, a_1, u_1, u_2) \right) \\
& - \gamma G^2 m_1^2 m_2 \epsilon^\mu(b, u_1, u_2) \left( \frac{8(2\gamma^2 - 1)(a_1 \cdot u_2)}{(\gamma^2 - 1)^2 b^4} \right. \\
& \left. - \frac{\pi\gamma (5\gamma^2 - 3)(b \cdot a_1)}{(\gamma^2 - 1)^{3/2} (-b^2)^{5/2}} \right). \tag{6.60}
\end{aligned}$$

As a sanity check, note that setting  $s_1 \rightarrow 0$  we recover the scalar impulse in eq. (5.25). This is now in agreement with the literature [88, 245, 249].

Our procedure is firmly rooted in QFT and also has the potential to apply to all orders. This computation, of the impulse at 1 loop (2PM) in gravity for particles with spin has been performed via a variety of approaches. One such work [88] makes use of the WQFT formalism to compute an eikonal phase and then obtain the impulse. There the authors compute the full impulse for two spinning particles and to second order in spin. The setup and variables used in their calculation is slightly different to the one presented here, but still based on an eikonal phase. One difference is that they impose total angular momentum conservation to produce some of the terms in their result. Clearly angular momentum should be conserved, but we should be able to recover it as a result of the theory rather than impose it as an extra constraint. Seeing how angular momentum conservation comes about, either in KMOC or via the eikonal, is something we have not explored in this thesis but is certainly a useful and interesting avenue to pursue.

## 6.4 NLO Spinning impulse - Kosower:2018adc

Let us finish by investigating the spinning impulse, this time using KMOC. We use the standard setup: incoming momenta  $p_1, p_2$  separated by an impact parameter  $b$ , and with particle one spinning. We will again work in the probe limit where

$m_1 \gg m_2$ .

The impulse kernel is the same as discussed before, though the relevant amplitude depends on spin. Since at one loop superclassical pieces arise we must take care to ensure they cancel in the observable. However the algebra is already involved for the scalar case and will only be more complicated with spin added in to the problem, so we follow the inspiration of [162] and decompose the kernels into real and imaginary parts. This makes it possible to completely bypass the computation to show the superclassical cancellations for the scalar case, and does much of the work for us when we have spin too.

Recall the full impulse kernel, at one loop, is

$$\mathcal{I}^\mu = iq^\mu \mathcal{A}_4^{(1)}(q) + \int \hat{d}^4 \ell \hat{\delta}(2p_1 \cdot \ell + \ell^2) \hat{\delta}(2p_2 \cdot \ell - \ell^2) \ell^\mu \mathcal{A}_4^{(0)}(\ell) \mathcal{A}_4^{*(0)}(q - \ell). \quad (6.61)$$

Much of the complexity in computing this arises in the cut term because of the weighting factor  $\ell^\mu$ . If this were absent then the cut would simply be the imaginary part of the amplitude, by the optical theorem. Written in terms of the  $T$  matrix, unitarity of the S-matrix is  $2 \text{Im } T = TT^\dagger = T^\dagger T$ . Taking the matrix element of this between two in / out states and making use of the resolution of the identity we have, at 1 loop,

$$2 \text{Im } \mathcal{A}_4^{(1)}(p_I \rightarrow p'_I) = \int \hat{d}^4 \ell \hat{\delta}(2p_1 \cdot \ell + \ell^2) \hat{\delta}(2p_2 \cdot \ell - \ell^2) \mathcal{A}^{(0)}(p \rightarrow p \pm \ell) \mathcal{A}^{*(0)}(p \pm \ell \rightarrow p'). \quad (6.62)$$

We will follow the approach by [162] which is to decompose  $\ell^\mu$  onto a basis in terms of the other vectors in the problem. By making a clever choice for these, in particular modified velocity variables, the cut integral can be recast in terms of the imaginary part of the one loop amplitude. We could take the tilde variables used before, however these are not orthogonal or normalised, so instead we define dual vectors  $\tilde{u}_i^\mu$  which are orthogonal to the  $\tilde{u}_i$

$$\begin{aligned} \tilde{u}_1^\mu &= \frac{y\tilde{u}_2^\mu - \tilde{u}_1^\mu}{y^2 - 1}, & \tilde{u}_2^\mu &= \frac{y\tilde{u}_1^\mu - \tilde{u}_2^\mu}{y^2 - 1} & \tilde{u}_i \cdot \tilde{u}_j &= \delta_{ij} \\ y &= \tilde{u}_1 \cdot \tilde{u}_2 = \gamma + \frac{q^2}{2m_1 m_2}. \end{aligned} \quad (6.63)$$

At one loop we will typically be able to drop the  $q$  shift terms so we have

$$\tilde{u}_1^\mu \sim \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2 - 1}, \quad \tilde{u}_2^\mu \sim \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2 - 1}. \quad (6.64)$$

Now we can decompose  $\ell^\mu$  as

$$\ell^\mu = \frac{\ell \cdot q}{q^2} q^\mu + \sum_{i=1,2} (\ell \cdot \tilde{u}_i) \tilde{u}_i^\mu - \frac{\epsilon(l, q, \tilde{u}_1, \tilde{u}_2)}{q^2} \epsilon^\mu(q, \tilde{u}_1, \tilde{u}_2). \quad (6.65)$$

Computing the dot products, using the definitions above, we find

$$\ell \cdot \tilde{u}_1 = \frac{q^2}{4m_1} + \dots, \quad \ell \cdot \tilde{u}_2 = -\frac{q^2}{4m_2} + \dots \quad (6.66)$$

where the dots involve higher powers of  $q$  and thus yield quantum corrections, or terms proportional to  $\ell^2$  which we can drop at one loop (we will discuss why shortly). The final direction  $\epsilon^\mu(q, \tilde{u}_2, \tilde{u}_2)$  is orthogonal to the other directions by construction. This decomposition is not unique and we could have used different vectors, and obtained different coefficients for the last term. In particular we haven't used the spin vector yet.

At one loop we encounter a variety of integrals, in particular ones of the form

$$I = \int \hat{d}^4 \ell \frac{(q - \ell)^2}{\ell^2 (q - \ell)^2 (u_i \cdot \ell) (u_j \cdot \ell)}. \quad (6.67)$$

This integral vanishes in dimensional regularisation since it is a scaleless integral. Alternatively this can be viewed as pinching one of the photon propagators which would lead to a contact type interaction which is irrelevant for the long range physics we are interested in. Similarly the integral with  $\ell^2$  rather than  $(q - \ell)^2$  in the numerator can also be ignored. Performing a  $\ell \rightarrow q - \ell$  shift will yield  $q^2$  corrections however these are quantum/contact terms which can be safely dropped. Similarly if we consider instead  $\ell \cdot q$  in the numerator we have

$$\begin{aligned} \int \hat{d}^4 \ell \frac{\ell \cdot q}{\ell^2 (q - \ell)^2 (u_i \cdot \ell) (u_j \cdot \ell)} &= \frac{1}{2} \int \hat{d}^4 \ell \frac{\ell^2 - (\ell - q)^2 + q^2}{\ell^2 (q - \ell)^2 (u_i \cdot \ell) (u_j \cdot \ell)} \\ &\rightarrow \frac{q^2}{2} \int \hat{d}^4 \ell \frac{1}{\ell^2 (q - \ell)^2 (u_i \cdot \ell) (u_j \cdot \ell)}. \end{aligned} \quad (6.68)$$

The upshot of this is that we can drop many terms in the computation and replace  $\ell \cdot q \rightarrow q^2/2$  inside the integrals. This allows many vectorial  $\ell$  integrals to be turned into scalar ones, which become standard cuts and then imaginary part of the corresponding loop amplitude, by eq. (6.62). Putting this together we can

write the kernel as

$$\begin{aligned}
\mathcal{I}^\mu &= q^\mu \left[ \mathcal{A}_4^{(1)}(q) - i \operatorname{Im} \mathcal{A}_4^{(1)}(q) \right] - i \frac{q^2}{2} \left( \frac{\tilde{u}_1^\mu}{m_1} + \frac{\tilde{u}_2^\mu}{m_2} \right) \operatorname{Im} \mathcal{A}_4^{(1)}(q) \\
&\quad + i \epsilon^\mu(q, \tilde{u}_1, \tilde{u}_2) \int \hat{d}^4 \ell \delta(2p_1 \cdot \ell + \ell^2) \delta(2p_2 \cdot \ell - \ell^2) \frac{\epsilon(l, q, \tilde{u}_1, \tilde{u}_2)}{q^2} \mathcal{A}_4^{(0)}(l) \mathcal{A}_4^{(0)}(q-l) \\
&= q^\mu \operatorname{Re} \mathcal{A}_4^{(1)}(q) - i \frac{q^2}{2} \left( \frac{\tilde{u}_1^\mu}{m_1} - \frac{\tilde{u}_2^\mu}{m_2} \right) \operatorname{Im} \mathcal{A}_4^{(1)}(q) \\
&\quad + i \epsilon^\mu(q, \tilde{u}_1, \tilde{u}_2) \int \hat{d}^4 \ell \frac{\epsilon(l, q, \tilde{u}_1, \tilde{u}_2)}{q^2} \delta(2p_1 \cdot \ell + \ell^2) \delta(2p_2 \cdot \ell - \ell^2) \mathcal{A}_4^{(0)}(l) \mathcal{A}_4^{(0)}(q-l),
\end{aligned} \tag{6.69}$$

where we noticed that the combination in square brackets is simply the real part of the amplitude  $i\mathcal{A} + \operatorname{Im} \mathcal{A} = \operatorname{Re} \mathcal{A}$ . This will allow us to completely bypass determining the cancellations of superclassical terms between the loop and the cut. The remaining terms are manifestly classical, with the possible exception of the last term which we will discuss shortly.

Since the real and imaginary terms are already familiar for scalars, we will focus our efforts here on evaluating the final residual cut term in eq. (6.69), leaving the evaluation of the real/imaginary parts to appendix B.3. This cut term is the remaining iteration contribution from KMOC, and we saw how the iterations in the eikonal setup were important to recover the new contributions to the impulse.

As well as dealing with super classical terms, this splitting of the kernel also simplifies the computation of the real/cut integrations. It is however still non trivial and we will need to make use of the one loop simplifications in eq. (6.67) again here. In the case of only scalars interacting this final term will not contribute to the impulse since the  $\ell$  integral must be proportional to  $q^\mu$  and so the kernel vanishes as this is contracted with the Levi Civita which already has a  $q$  contracted. From a physics perspective the scalar scattering must lie in a plane defined by  $q^\mu, u_1^\mu$  and  $u_2^\mu$  and this direction is perpendicular so no impulse in this direction. For particles with spin however this is no longer true and this provides important contributions.

We need the tree amplitude, as given in eq. (6.36). Keeping only the linear in spin terms<sup>7</sup> we have

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<sup>7</sup>We only need to focus on the linear in spin terms from now on, as the scalar terms will vanish as discussed.

$$\mathcal{M}_4(l)\mathcal{M}_4(q-l) \rightarrow i \frac{(16\pi G)^2 4m_1^4 m_2^4 \gamma (2\gamma^2 - 1)}{\ell^2(q - \ell^2)} [\epsilon(l, s_1, u_1, u_2) + \epsilon(q - l, s_1, u_2, u_2)]. \quad (6.70)$$

Next in the kernel we multiply by  $\epsilon(q, l, \tilde{u}_1, \tilde{u}_2)$ . This, along with the other Levi-Civita term can be turned into a series of scalar products<sup>8</sup>. Many of these vanish by virtue of the choice of variables or physical conditions such as  $q \cdot \tilde{u}_i = 0$  or  $s_1 \cdot u_1 = 0$ . Performing the algebra yields

$$\begin{aligned} \epsilon(l, q, u_1, u_2)\epsilon(q, s, u_1, u_2) &= \frac{(q^2)^2(\gamma m_2 + m_1)s_1 \cdot u_2}{4m_1 m_2} \\ &\rightarrow \frac{m_1(q^2)^2 s_1 \cdot u_2}{4m_1 m_2} \end{aligned} \quad (6.71)$$

where we made use of the replacement  $q \cdot \ell \rightarrow q^2/2$ . It is reassuring that the superclassical pieces here vanishes since the super classical terms from the loop amplitudes have already been dealt with in the impulse. This gives us a kernel

$$\begin{aligned} \mathcal{I}_\epsilon^\mu &= -i\epsilon^\mu(q, \tilde{u}_1, \tilde{u}_2)q^2 \int \hat{d}^4 \ell \frac{\hat{\delta}(2p_1 \cdot \ell)\hat{\delta}(2p_2 \cdot \ell)}{\ell^2(q - \ell)^2} \\ &\quad \times (16\pi G)^2 m_1^4 m_2^3 \gamma (2\gamma^2 - 1) s_1 \cdot u_2. \end{aligned} \quad (6.72)$$

Performing the integral, as in appendix A.2.2, again dropping the  $q$  terms in  $\tilde{u}_i$ , we get a final kernel of

$$\mathcal{I}_\epsilon^\mu = -\frac{\epsilon^\mu(q, u_1, u_2)}{\gamma^2 - 1} \frac{32i\pi\gamma(2\gamma^2 - 1)G^2 m_1^3 m_2^2 ((a_1 \cdot u_2))}{\varepsilon\sqrt{\gamma^2 - 1}(-q^2)^\epsilon} \quad (6.73)$$

We can now perform the Fourier integrals to get the impulse and we find

$$\Delta p_1^\mu|_\epsilon = -\frac{8G^2 m_1^2 m_2 (2\gamma^2 - 1)\gamma(a_1 \cdot u_2)\epsilon^\mu(b, u_1, u_2)}{(\gamma^2 - 1)^2(-b^2)^2} \quad (6.74)$$

Comparing the two calculations of the impulse, by eikonal and KMOC, we can see some similarities. We see how the iteration terms of the eikonal, coming from  $\chi_0$  and  $x_\perp$  map over to the quadratic part of the impulse kernel. In the eikonal calculation  $x_\perp$  is where the  $u_1^\mu$  and  $u_2^\mu$  directions appear, while in KMOC it is in the imaginary part of the amplitude. These both start out life as the product of two tree amplitudes.

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<sup>8</sup>Using eq. (A.11)

## 6.5 Summary

This final chapter was a culmination of all the previous work in this thesis. We applied KMOC and the eikonal to computing observables with colour and spin. We began by describing the modifications necessary to include spin degrees of freedom to the semi-classical final state of eq. (5.30). This allows for the computation of new observables which depend explicitly on these spin degrees of freedom, and doing so reveals new structures in the observable; namely a bracket structure of commutators (quantum theory) or Poisson brackets (classical). We used this to reproduce the NLO colour impulse results of chapter 4.

We then moved on to a new computation - impulse with spin in gravity for two massive particles, one scalar and one vector. We did this two ways: one via KMOC where we employed a variety of simplifications in the spirit of [162]. Splitting the impulse kernel into real and imaginary parts removes much of, but not all, of the intricacies of ensuring that super classical terms do indeed cancel. In the case of scalars this becomes trivial, while with spin there is still a little work to do. In particular there remains a quadratic ‘cut’ type term, which is crucial for giving the new direction to the impulse when spin is involved and scattering no longer takes place in the plane defined by initial momenta. It also highlighted how these new terms come from iterations of the tree amplitude.

We performed this same computation using the eikonal semi-classical final state. Here evaluating the stationary phase conditions required a careful consideration of the implicit dependence on  $q$  and on which variables the eikonal depends. Doing so allowed us to compute the probe limit impulse for a light scalar moving in a heavy spinning background.

# Chapter 7

## Conclusions

The central theme of this thesis has been the study of methods to compute classical observable quantities starting from a quantum field theory basis, in particular via scattering amplitudes. The motivation for this comes in two parts. Firstly is the need for precision calculations at current and future gravitational wave observatories [3, 4]. Second is the double copy [154, 155, 157, 158], allowing the wealth of knowledge of gauge theory scattering amplitudes to be applied to gravity. The final ingredient is to understand how to obtain classical information from this quantum amplitude data.

We explored a variety of approaches for this, mostly working with toy models of scalars interacting in electrodynamics or chromodynamics to illustrate and test ideas. We employed three main approaches which we discussed in chapter 2: KMOC, coherent states and the eikonal exponentiation. Coherent states allowed us to ensure the appropriate classical aspects of the non kinematical degrees of freedom, such as spin or colour. The kinematics is tackled via KMOC and eikonal. Both of these naturally lend themselves to scattering observables and have amplitudes at the core. We described how to construct observables in the quantum theory and then the recipe for taking the classical limit. In the eikonal exponentiation amplitudes are able to be resummed into an exponential of the eikonal phase. This naturally lives in position space. Fourier transforming back to momentum space, making use of the stationary phase approximation, allows us to compute the classical impulse and scattering angle.

In chapter 3 we investigated how imposing reasonable classical conditions — that the variance of some quantum operator should vanish in the classical limit — we

can derive constraints on the fragment structure of scattering amplitudes. We saw how tree amplitudes, such as the 6 point with 2 outgoing messenger particles, can in fact be quantum suppressed relative to the loop level amplitude. We only considered some simple variances, however by considering more elaborate combinations of operators an, in principle, infinite number of relations can be derived. For example that all  $n$ -point trees are suppressed for  $n > 5$  in Yang-Mills theory and in gravity. One of the simplest of these variance relations yields precisely the conditions required for eikonal exponentiation. Studying more of these in more depth could shed light on higher order structures in the classical limit, allowing for more efficient computations, or perhaps provide guidance towards a general proof or understanding of the eikonal exponentiation. This would certainly be of benefit for applying eikonal type methods to higher loop orders.

In chapter 4 we applied the KMOC procedure to scalar QCD. The non linear structure of the gluons of QCD compared to photons in QED as well as the non-Abelian charge provide a stepping stone to the study of spinning particles in gravity. Colour charge is simpler than spin because of its lack of momentum dependence, however the use of large representation states, or equivalently spin coherent states, give rise to the correct classical limits for both colour and spin. Surprisingly we discovered that the non-Abelian diagrams did not contribute to the impulse at the order we worked to. Similarly some amplitude fragments which were superclassical in QED do now contribute to the colour impulse. Studying this classical limit of Yang Mills theory, via KMOC, could provide insights into the relation between the classical and quantum double copy.

Finally, in chapters 5 and 6 we merged the KMOC and eikonal frameworks. We used this to describe the final semi-classical state after scattering. In the conservative sector we generalised this to particles with spin or colour degrees of freedom too. This allowed us, via a stationary phase approximation, to compute observables such as the impulse or scattering angle as well as other operators only present for non scalars. For these we encountered an infinite set of commutators which, after using coherent spin states, become Poisson brackets in the classical limit. In the pure scalar case we also included radiation, by including a coherent radiative state, allowing radiation reaction to be described. This requires only two functions: the eikonal  $\chi$  and waveshape  $\alpha$  which depend only on the four and five point amplitudes respectively. Our claim is that these amplitudes contain all of the classical physics. The first non trivial condition for this generalised eikonal

exponential structure, that the superclassical fragment of the five-point one loop amplitude is formed from the tree level four and five point amplitudes, was verified and identical to one of the uncertainty conditions in chapter 3. An interesting question following this relates to the higher loop structure of the eikonal function  $\chi$ . In particular with the suppression of higher-point trees, what class of cuts are relevant for a unitarity-based construction of  $\chi$ ? For example, the two-loop amplitude built from five-point trees is classically relevant, while power counting shows that the three-loop contribution from six-point trees does not contribute to  $\chi$  at this order.

These two chapters leave open many questions and avenues for future work. Firstly is the validity and generality of the eikonal formula eq. (5.1). We, along with other authors, take it to be true beyond the leading eikonal, however being able to prove or disprove this could well become necessary if we are to push it to higher loops. Perhaps the methods of chapter 3 could provide an approach in this direction. Similarly is the eikonal-coherent final state. We provided it as a reasonable ansatz, and refining and testing to higher orders is necessary. If the semi-classical final state is determined by exponentiating only four- and five-point amplitudes, it must follow that *every* higher-point amplitude, in the classical regime, is determined by four- and five-point amplitudes. When we included spin, we only worked in the conservative case — is it possible to include the full radiative effects in this case? While we focused on the eikonal approach to classical dynamics, it is worth emphasising that we could have phrased our discussion in terms of other, closely related, quantities such as radial action or HEFT phases. Recently the radial action has received particular attention in the literature [113]; this radial action is a classical limit of the usual quantum phase shifts [218], and is a close cousin of the eikonal function [146]. Understanding how they are related or not would be an interesting avenue to pursue. We focused on unbound (scattering) events here but what about the bounded case? Indeed this is most relevant to BH inspirals. It is possible to determine the physics of bound states from our observables. This can be done concretely using effective theories [44]. Can we modify the formalisms in this thesis to include bound states perhaps using analytic continuation, in a manner similar to the work of Kälın and Porto [65, 67, 120]

As I hope the work in this thesis has demonstrated, amplitudes based methods can be a useful and powerful approach to classical gravitational wave physics, and this boundary field is certainly a rich and exciting one to work in.

# Appendix A

## Conventions and Identities

Here we collect the various conventions and useful identities used in the main text for general reference.

### A.1 Conventions

We make use of ‘hats’ to absorb factors of  $2\pi$ ’s.

$$\hat{\mathrm{d}}^d p = \frac{\mathrm{d}^d p}{(2\pi)^d}, \quad \hat{\delta}(x) = 2\pi\delta(x). \quad (\text{A.1})$$

Fourier transform conventions are

$$\begin{aligned} f(x) &= \int \hat{\mathrm{d}}q \, \tilde{f}(q) e^{-iq \cdot x} \\ \tilde{f}(q) &= \int \mathrm{d}x \, f(x) e^{iq \cdot x}. \end{aligned} \quad (\text{A.2})$$

The Dirac delta function can be written as a Fourier transform as

$$\hat{\delta}(\bar{q}) = 2\pi\delta(\bar{q}) = \int \mathrm{d}x e^{i\bar{q}x}. \quad (\text{A.3})$$

Another useful representation of the delta function, and its derivative, is

$$i\delta(x) = \frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} \quad (\text{A.4})$$

$$\delta'(x) = \frac{i}{(x - i\epsilon)^2} - \frac{i}{(x + i\epsilon)^2} \quad (\text{A.5})$$

In the classical calculation solving equations of motion iteratively, we encounter integrals of the form

$$\int_{-\infty}^{\tau} d\tau e^{\pm ip \cdot u \tau}. \quad (\text{A.6})$$

To ensure the convergence of these integrals we make the replacement  $p \cdot u \rightarrow p \cdot u \pm i\epsilon$ . The sign of the  $i\epsilon$  depends on whether the integrand involves  $e^{+ip \cdot u}$  or  $e^{-ip \cdot u}$ . The sign of the  $i\epsilon$  is important as we will encounter terms with positive and negative signs which can be combined using eq. (A.5) above. The replacements are as follows

$$\int_{-\infty}^{\tau} d\tau e^{ip \cdot u \tau} \rightarrow \int_{-\infty}^{\tau} d\tau e^{i(p \cdot u - i\epsilon)\tau} = \frac{1}{i(p \cdot u - i\epsilon)} \quad (\text{A.7})$$

$$\int_{-\infty}^{\tau} d\tau e^{-ip \cdot u \tau} \rightarrow \int_{-\infty}^{\tau} d\tau e^{-i(p \cdot u + i\epsilon)\tau} = \frac{-1}{i(p \cdot u + i\epsilon)}. \quad (\text{A.8})$$

With the Levi-Civita tensor we use the following convention and notations

$$\epsilon_{0123} = +1 \quad (\text{A.9})$$

and

$$\epsilon^{\mu\nu\rho\sigma} a_{\mu} b_{\nu} c_{\rho} d_{\sigma} = \epsilon(a, b, c, d), \quad \epsilon^{\mu\nu\rho\sigma} a^{\nu} b^{\rho} c^{\sigma} = \epsilon^{\mu}(a, b, c), \quad \text{etc.} \quad (\text{A.10})$$

It satisfies the following decomposition for products

$$\epsilon_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon^{\beta_1 \beta_2 \beta_3 \beta_4} = \delta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\beta_1 \beta_2 \beta_3 \beta_4} \equiv 4! \delta_{[\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} \delta_{\alpha_3}^{\beta_3} \delta_{\alpha_4]}^{\beta_4}. \quad (\text{A.11})$$

The spin vector for a spin-1 particle can be obtained via its polarisation tensor as

$$s^{\mu}(p) = \frac{i}{m} \epsilon^{\mu\nu\rho\sigma} p_{\nu} \varepsilon(p)_{\rho} \varepsilon^{*}(p)_{\sigma}. \quad (\text{A.12})$$

The spin vector and spin tensor are related by

$$\begin{aligned} s^\mu(p) &= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} p_\nu S_{\rho\sigma} \\ S^{\mu\nu} &= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} p_\rho s_\sigma \end{aligned} \tag{A.13}$$

## A.2 Useful integrals

Here we summarise some useful integrals that show up in the main text. In particular the  $d$  dimensional Fourier integrals, regulated in dimensional regularisation. We also present the evaluation of the 2 particle cut of the one loop box integral, used in section 6.3

### A.2.1 Fourier transform integrals

Moving on to Fourier transform integrals, the basic ones required for computing the impulse from the impulse kernel have been reviewed elsewhere, for example [162, 249]. They are

$$\begin{aligned} J(\alpha, b^2) &= \int \hat{d}^d q \hat{\delta}(q \cdot u_1) \hat{\delta}(q \cdot u_2) e^{iq \cdot b} \frac{1}{(-q^2)^\alpha} \\ &= \frac{1}{\sqrt{\gamma^2 - 1}} \frac{\Gamma(d/2 - 1 - \alpha)}{2^{2\alpha} \pi^{d/2-1} \Gamma(\alpha)} \frac{1}{|b|^{d-2-2\alpha}} \end{aligned} \tag{A.14}$$

The impact parameter in the expression is always space-like so  $\underline{b}_\perp^2 = -b^2 \equiv |b|^2$ , or equivalently  $|b| = \sqrt{-b^2}$ . Integrals with  $q^\mu$  in the numerator can be obtained by differentiation, being careful to bear in mind that this picks up projectors. The first vector integral is

$$\begin{aligned} J^\mu(\alpha, b^2) &= -i \frac{\partial}{\partial b_\mu} J(\alpha, b^2) \\ &= \int \hat{d}^d q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) \frac{e^{iq \cdot b}}{(-q^2)^\alpha} q^\mu \\ &= \frac{i}{\sqrt{\gamma^2 - 1}} \frac{\Gamma(d/2 - \alpha)}{2^{2\alpha} \pi^{(d-2)/2} \Gamma(\alpha)} \frac{2b^\mu}{|b|^{d-2-2\alpha}}. \end{aligned} \tag{A.15}$$

The tensor integral can similarly be found

$$\begin{aligned}
J^{\mu\nu}(\alpha, b^2) &= \int \hat{d}^d q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) \frac{e^{iq \cdot b}}{(-q^2)^\alpha} q^\mu q^\nu \\
&= -\frac{1}{\sqrt{\gamma^2 - 1}} \frac{\Gamma(d/2 - \alpha)}{2^{2\alpha-1} \pi^{(d-2)/2} \Gamma(\alpha)} \frac{1}{|b|^{d+2-2\alpha}} ((d-2\alpha)b^\mu b^\nu + |b|^2 \Pi^{\mu\nu}).
\end{aligned} \tag{A.16}$$

## A.2.2 Cut box integral

An integral we encounter is

$$C^\mu = \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l) \frac{l^\mu}{l^2(q-l)^2} \tag{A.17}$$

Performing a PV type tensor reduction we know it must be in the directions of  $u_1^\mu, u_2^\mu$  and  $q^\mu$

$$C^\mu = au_1^\mu + bu_2^\mu + cq^\mu. \tag{A.18}$$

Contracting with each vector in turn and making use of the delta functions we find  $a = b = 0$  and  $c \neq 0$

$$C^\mu = q^\mu \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l) \frac{l \cdot q}{l^2(q-l)^2} \rightarrow \frac{1}{2} q^\mu \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l) \frac{1}{l^2(q-l)^2} \tag{A.19}$$

after making use of the replacements discussed in eq. (6.67). Alternatively we can arrive at the same result by shifting and averaging the loop momenta. Since  $u_i \cdot q = \mathcal{O}(q^2)$ , as imposed by delta functions that will show up when performing the  $q$  integral, we can shift  $l \rightarrow q - l$  in  $C^\mu$  and the integral does not change (up to quantum corrections). Performing this and averaging we have

$$\begin{aligned}
C^\mu &= \frac{1}{2} \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l) \frac{l^\mu + (q-l)^\mu}{l^2(q-l)^2} \\
&= \frac{q^\mu}{2} \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l) \frac{1}{l^2(q-l)^2} \\
&\equiv \frac{q^\mu}{2} C.
\end{aligned} \tag{A.20}$$

where we define  $C$  to be the loop integral, corresponding to the cut of a scalar box. This integral is

$$\tilde{C} = \int \hat{d}^4 l \frac{\hat{\delta}(u_1 \cdot l) \hat{\delta}(u_2 \cdot l)}{l^2(q-l)^2}. \tag{A.21}$$

can be evaluated using standard textbook methods. First one can choose a frame and perform two of the integrals using the delta functions. Then the resulting integral is a standard loop type integral in  $d - 2$  dimensions, with euclidean signature. Setting  $d = 4 - 2\epsilon$  we find

$$\tilde{C} = \frac{(4\pi)^{\epsilon-1} \Gamma(-\epsilon)^2 \Gamma(\epsilon+1)}{\beta\gamma \Gamma(-2\epsilon)} \frac{1}{(q^2)^{1+\epsilon}} \quad (\text{A.22})$$

Expanding this in the limit  $\epsilon \rightarrow 0$  gives

$$\tilde{C} = -\frac{1}{2\epsilon(\pi\beta\gamma q^2)} + \frac{-\log \bar{q}^2 + \gamma - \log(4\pi)}{2\pi\beta\gamma q^2} + \mathcal{O}(\epsilon). \quad (\text{A.23})$$

### A.3 Feynman rules

Here we list the Feynman rules used in calculations involving colour, following the conventions for dimensionful quantities discussed in section 4.1.3. For the action in equation (4.2), these are

$$\begin{aligned}
& \begin{array}{c} \mu, a \\ \text{---} \text{---} \text{---} \\ j \text{---} \text{---} \text{---} i \\ \text{---} \text{---} \text{---} \\ p_1 \quad p_2 \end{array} = -ig(C^a)_i^j (p_1^\mu + p_2^\mu), \\
& \begin{array}{c} \mu, a \quad \nu, b \\ \text{---} \text{---} \text{---} \text{---} \\ k \text{---} \text{---} \text{---} i \\ \text{---} \text{---} \text{---} \end{array} = ig^2 \eta^{\mu\nu} \left[ (C^a)_i^j (C^b)_j^k + (C^b)_i^j (C^a)_j^k \right], \\
& \begin{array}{c} \rho, c \\ \text{---} \text{---} \text{---} \\ p_3 \\ \text{---} \text{---} \text{---} \\ p_1 \quad p_2 \\ \text{---} \text{---} \text{---} \\ \mu, a \quad \nu, b \end{array} = \hbar g f^{abc} (\eta^{\mu\nu} (p_1^\rho - p_2^\rho) + \eta^{\nu\rho} (p_2^\mu - p_3^\mu) + \eta^{\mu\rho} (p_3^\nu - p_1^\nu)), \\
& \begin{array}{c} \rho, c \quad \sigma, d \\ \text{---} \text{---} \text{---} \text{---} \\ \mu, a \quad \nu, b \end{array} = -i\hbar^2 g^2 \left[ f^{abe} f^{cde} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ace} f^{bde} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ade} f^{bce} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) \right].
\end{aligned} \quad (\text{A.24})$$

The  $\hbar$  factors in the gluon self interactions are imposed on us by choosing our conventions to match electrodynamics. This relies on the covariant derivative in equation (4.14), which includes the dimensional colour factors. The Yang-Mills coupling alone has the dimensions in equation (4.5), so we must include an extra  $\hbar$  in pure YM interactions if always treating couplings in the same manner as electrodynamics or gravity.

In scalar QED we can simply set the colour charges  $C^a$  to be kronecker deltas. The gluon interaction vertices are not present in the abelian case.

# Appendix B

## Supplementary calculations

In this appendix we provide details of some extra computations that supplement those in the main text.

### B.1 Further details at six points

In this appendix we provide the full details of the gauge fixed calculation which were not included in [3.1.4](#).

For  $A_{(2,2)}$  we have diagrams in the same classes as in the main text. There is only one location for the cubic emission on particle 1 with our gauge choice, and we will label the diagrams by 1 (2) for emission on the outgoing (incoming) particle 2 line. We find

$$\begin{aligned}
 D_6(Q_1^2 Q_2^2)|_{cubic,1} &= \frac{1}{\hbar^5} \frac{\varepsilon_1 \cdot (2\bar{q}_1) \varepsilon_2 \cdot (2p_2) (2p_2 + \hbar\bar{q}_2 + \hbar\bar{k}_2) \cdot (2p_1 + \hbar\bar{q}_1 - \hbar\bar{k}_1)}{(\bar{q}_2 - \bar{k}_2)^2 (2p_2 \cdot \bar{k}_2) (2p_1 \cdot (\bar{q}_1 - \bar{k}_1) + \hbar(\bar{q}_1 - \bar{k}_1)^2)} \\
 &\quad + 2 \left( p_1 \cdot (\bar{k}_2 + \bar{q}_2) + p_2 \cdot (\bar{q}_1 - \bar{k}_1) \right) \Bigg] \\
 &= - \frac{(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2)}{\hbar^4 (\bar{q}_2 - \bar{k}_2)^2 (p_2 \cdot \bar{k}_2) (p_1 \cdot \bar{k}_1)} \left[ \frac{4p_1 \cdot p_2}{\hbar} - \frac{4p_1 \cdot p_2 \bar{q}_1 \cdot \bar{k}_1}{p_1 \cdot \bar{k}_1} \right. \\
 &\quad \left. + (4p_1 \cdot \bar{k}_2 + 2p_1 \cdot \bar{k}_1 + 2p_2 \cdot \bar{k}_2) \right].
 \end{aligned} \tag{B.1}$$

To get the last line we used momentum conservation to replace, for example,

$k_1 - q_1 \rightarrow q_2 - k_2$  and then dropped the  $p_i \cdot \bar{q}_i$  as this gets an extra  $\hbar$  from the on shell conditions. The second is

$$\begin{aligned}
D_6(Q_1^2 Q_2^2)|_{cubic,2} &= \frac{1}{\hbar^5} \frac{[\varepsilon_1 \cdot (2\bar{q}_1)\varepsilon_2 \cdot (2p_2 + \hbar\bar{q}_2)][(2p_2 + \hbar\bar{q}_2 - \hbar\bar{k}_2) \cdot (2p_1 + \hbar\bar{q}_1 - \hbar\bar{k}_1)]}{(\bar{q}_2 - \bar{k}_2)^2 (2p_1 \cdot (\bar{q}_1 - \bar{k}_1) + \hbar(\bar{q}_1 - \bar{k}_1)^2) (2p_2 \cdot (\bar{q}_2 - \bar{k}_2) + \hbar(\bar{q}_2 - \bar{k}_2)^2)} \\
&= \frac{1}{\hbar^4(\bar{q}_2 - \bar{k}_2)^2(p_2 \cdot \bar{k}_2)(p_1 \cdot \bar{k}_1)} \left[ \frac{4p_1 \cdot p_2(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2)}{\hbar} + 4p_1 \cdot p_2(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot \bar{q}_2) \right. \\
&\quad \left. + 2(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2)(p_1 \cdot \bar{k}_1 + p_2 \cdot \bar{k}_2) - 4p_1 \cdot p_2(\varepsilon_1 \cdot \bar{q}_1)(\varepsilon_2 \cdot p_2) \left( \frac{\bar{q}_1 \cdot \bar{k}_1}{p_1 \cdot \bar{k}_1} + \frac{\bar{q}_2 \cdot \bar{k}_2}{p_2 \cdot \bar{k}_2} \right) \right]. \tag{B.2}
\end{aligned}$$

Now for the diagrams with cubic and quartic vertices. They both have the quartic vertex on particle 1 and two locations for cubic emission on particle 2. These, again labelled by 1 (2), are,

$$\begin{aligned}
D_6(Q_1^2 Q_2^2)_{cubic/quartic,1} &= -\frac{2}{\hbar^5} \frac{\varepsilon_1 \cdot (2p_2)\varepsilon_2 \cdot (2p_2 + \hbar\bar{q}_2)}{(\bar{q}_2 - \bar{k}_1)^2(2p_2 \cdot \bar{k}_2)} \\
&= -\frac{1}{\hbar^4(\bar{q}_2 - \bar{k}_1)^2(p_2 \cdot \bar{k}_2)} \left[ \frac{4(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot p_2)}{\hbar} + 2(\varepsilon_1 \cdot \bar{q}_2)(\varepsilon_2 \cdot p_2) \right] \tag{B.3}
\end{aligned}$$

and the second

$$\begin{aligned}
D_6(Q_1^2 Q_2^2)_{cubic/quartic,2} &= -\frac{2}{\hbar^5} \frac{\varepsilon_1 \cdot (2p_2 + \hbar\bar{q}_2 - \hbar\bar{k}_2)\varepsilon_2 \cdot (2p_2 + 2\hbar\bar{q}_2)}{(\bar{q}_1 - \bar{k}_1)^2(2p_2 \cdot (\bar{q}_2 - \bar{k}_2) + \hbar(\bar{q}_2 - \bar{k}_2)^2)} \\
&= +\frac{1}{\hbar^4(\bar{q}_1 - \bar{k}_1)^2(p_2 \cdot \bar{k}_2)} \left[ \frac{4(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot p_2)}{\hbar} + 2(\varepsilon_2 \cdot p_2)(\varepsilon_1 \cdot (\bar{q}_2 - \bar{k}_2)) \right. \\
&\quad \left. 4(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot \bar{q}_2) - \frac{4(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot p_2)(\bar{q}_2 \cdot \bar{k}_2)}{p_2 \cdot \bar{k}_2} \right]. \tag{B.4}
\end{aligned}$$

Where we used the on-shell condition  $2p_2 \cdot \bar{q}_2 = -\hbar\bar{q}_2^2$ , and then Taylor expanded the propagator denominator. Note how the two most singular terms cancel when we combine these diagrams.

The final diagram doesn't require the gauge fixing to simplify anything as it is already the correct order in a general gauge. This is

$$D_6(Q_1^2 Q_2^2)_{quartic} = \frac{4\varepsilon_1 \cdot \varepsilon_2}{\hbar^4(\bar{q}_2 - \bar{k}_2)^2}. \tag{B.5}$$

Combining these gives the result (3.30) in the main text.

## B.2 Diagrams and amplitude expressions

In this appendix we gather the expressions necessary for calculating the 1-loop partial amplitudes introduced in section 4.2.3. We are only interested in the leading classical terms of the relevant topologies, and will not list quantum corrections.

From equation (4.61) we know that the relevant 1-loop topologies for NLO observables in YM theory are those which contributed to the analogous QED calculation in [45]. The QED amplitude

$$\mathcal{A}^{(1,\text{QED})} = B + C + T_{12} + T_{21} \quad (\text{B.6})$$

is constructed from the triangle  $T_{ij}$ , box  $B$ , and cross box  $C$  diagrams. Beginning with the triangles, the leading classical terms are

$$\begin{aligned}
 iT_{12} &= \text{Diagram 1} = i \frac{2m_1^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\hat{\delta}(p_1 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} + \mathcal{O}(\hbar^0), \\
 iT_{21} &= \text{Diagram 2} = i \frac{2m_2^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\hat{\delta}(p_2 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{\ell} - \bar{q})^2} + \mathcal{O}(\hbar^0).
 \end{aligned} \quad (\text{B.7})$$

We refer the curious reader to Ref. [45] for the detailed calculations: heuristically, the  $\hbar$  expansion of the diagrams is conducted by rescaling  $\ell \rightarrow \hbar \bar{\ell}$  and  $q \rightarrow \hbar \bar{q}$  on the support of the delta functions in equation (4.44). Propagator denominators are expanded as a series in  $\hbar$ . Noting that the loop integrals are symmetric under the replacement  $\bar{\ell} \rightarrow \bar{q} - \bar{\ell}$ , this change of variables can be exploited to change the sign of the (Feynman)  $i\epsilon$  in massive propagators. Then, averaging over the

two expressions for the integral and applying the identities

$$\begin{aligned} i\hat{\delta}(x) &= \frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} \\ -i\hat{\delta}'(x) &= \frac{1}{(x - i\epsilon)^2} - \frac{1}{(x + i\epsilon)^2} \end{aligned} \quad (\text{B.8})$$

leads to the expressions in equation (B.7). This symmetrisation trick is unnecessary for calculating the individual terms from box topologies, for which we choose the following momentum routing:

$$iB = \quad iC = \quad i\cancel{B} = \quad (\text{B.9})$$

The cut box appears outside of  $\mathcal{A}^{(1,\text{QED})}$ , forming the quadratic part of the NLO colour kernel, and is defined in equation (4.66). However, these diagrams all have kinematic coefficients of the form

$$D = D_{-1} + D_0 + \mathcal{O}(\hbar^0), \quad (\text{B.10})$$

where  $D_{-1} \sim \mathcal{O}(\hbar^{-2})$  and  $D_0 \sim \mathcal{O}(\hbar^{-1})$ . We choose to label the terms like this as the  $\mathcal{O}(\hbar^{-1})$  terms are those which generally contribute classically, and we ignore the  $\mathcal{O}(\hbar^0)$  terms as they always act as quantum corrections. The  $\mathcal{O}(\hbar^{-2})$  terms would give rise to contributions to the impulse which are classically singular, and as shown in section 4.2.3 it is necessary to consider all three diagrams in order to see that they cancel. We have

$$\begin{aligned} B_{-1} &= i \frac{4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2 (p_2 \cdot \bar{\ell} - i\epsilon)(p_1 \cdot \bar{\ell} + i\epsilon)}, \\ C_{-1} &= -i \frac{4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \frac{1}{(p_2 \cdot \bar{\ell} + i\epsilon)(p_1 \cdot \bar{\ell} + i\epsilon)}, \\ \cancel{B}_{-1} &= -i \frac{4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{\hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2}, \end{aligned} \quad (\text{B.11})$$

and

$$\begin{aligned}
B_0 &= i \frac{2p_1 \cdot p_2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2 (p_1 \cdot \bar{\ell} + i\epsilon)(p_2 \cdot \bar{\ell} - i\epsilon)} \\
&\quad \times \left\{ 2(p_2 - p_1) \cdot \bar{\ell} + (p_1 \cdot p_2) \bar{\ell}^2 \left( \frac{1}{(p_2 \cdot \bar{\ell} - i\epsilon)} - \frac{1}{(p_1 \cdot \bar{\ell} + i\epsilon)} \right) \right\}, \\
C_0 &= -i \frac{2p_1 \cdot p_2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2 (p_1 \cdot \bar{\ell} + i\epsilon)(p_2 \cdot \bar{\ell} + i\epsilon)} \\
&\quad \times \left\{ 2(p_1 + p_2) \cdot \bar{\ell} - (p_1 \cdot p_2) \left( \frac{\bar{\ell}^2}{(p_1 \cdot \bar{\ell} + i\epsilon)} + \frac{\bar{\ell}^2 - 2\bar{q} \cdot \bar{\ell}}{(p_2 \cdot \bar{\ell} + i\epsilon)} \right) \right\}, \\
\mathbb{B}_0 &= i \frac{2(p_1 \cdot p_2)^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \bar{\ell}^2 \left\{ \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}'(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right\}.
\end{aligned} \tag{B.12}$$

Here the delta functions in  $\mathbb{B}$  originate in its definition as the quadratic part of the colour kernel in equation (4.54). Applying the symmetrisation trick and equation (B.8) ensures that the sums of the box and cross box contributions can also be recast in terms of delta functions — hence, upon including the triangle contributions, the leading terms in the expansion of the QED amplitude in equation (B.6) are

$$\begin{aligned}
\mathcal{A}_{-1}^{(1,\text{QED})} &= i \frac{2(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \\
\mathcal{A}_0^{(1,\text{QED})} &= \frac{2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{1}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left\{ i(p_1 \cdot p_2)^2 \left[ \frac{1}{2} (\bar{\ell}^2 - 2\bar{\ell} \cdot \bar{q}) \left( \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) \right. \right. \right. \\
&\quad \left. \left. - \hat{\delta}'(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right) - (\bar{\ell}^2 - \bar{\ell} \cdot \bar{q}) \left( \frac{i\hat{\delta}(p_1 \cdot \bar{\ell})}{(p_2 \cdot \bar{\ell} - i\epsilon)^2} + \frac{i\hat{\delta}(p_2 \cdot \bar{\ell})}{(p_1 \cdot \bar{\ell} + i\epsilon)^2} \right) \right] \\
&\quad \left. + m_1^2 \hat{\delta}(p_1 \cdot \bar{\ell}) + m_2^2 \hat{\delta}(p_2 \cdot \bar{\ell}) \right\}.
\end{aligned} \tag{B.13}$$

A similar averaging procedure can also be applied to the cut box, yielding

$$\mathbb{B}_0 = -i \frac{2(p_1 \cdot p_2)^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\bar{\ell} \cdot \bar{q}}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left\{ \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}'(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right\}, \tag{B.14}$$

which is the result listed in equation (4.66).

Finally, for the momentum impulse the cut box  $\mathbb{B}^\mu$  is dressed by a power of the

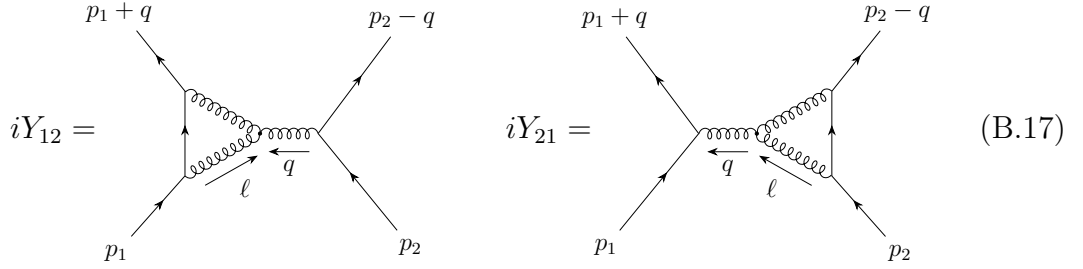
loop momentum, and thus

$$\begin{aligned}\mathbb{B}_{-1}^\mu &= -i \frac{4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{\bar{\ell}^\mu \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2}, \\ \mathbb{B}_0^\mu &= i \frac{2(p_1 \cdot p_2)^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\bar{\ell}^\mu}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left\{ \bar{\ell}^2 \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \bar{\ell}^2 \hat{\delta}(p_2 \cdot \bar{\ell}) \hat{\delta}'(p_1 \cdot \bar{\ell}) \right\}.\end{aligned}\tag{B.15}$$

Shifting with  $\bar{\ell} \rightarrow \bar{q} - \bar{\ell}$  and averaging the two expressions, these can be written equivalently as

$$\begin{aligned}\mathbb{B}_{-1}^\mu &= -i \frac{2g^4(p_1 \cdot p_2)^2}{\hbar^2} \int \hat{d}^4 \bar{\ell} \frac{\bar{q}^\mu}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}), \\ \mathbb{B}_0^\mu &= i \frac{2(p_1 \cdot p_2)^2}{\hbar} \int \hat{d}^4 \bar{\ell} \frac{\bar{\ell} \cdot (\bar{\ell} - \bar{q})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \bar{\ell}^\mu \left( \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}'(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right) \\ &\quad - i \frac{(p_1 \cdot p_2)^2}{\hbar} \bar{q}^\mu \int \hat{d}^4 \bar{\ell} \frac{(\bar{\ell}^2 - 2\bar{q} \cdot \bar{\ell})}{\bar{\ell}^2 (\bar{q} - \bar{\ell})^2} \left( \hat{\delta}(p_1 \cdot \bar{\ell}) \hat{\delta}'(p_2 \cdot \bar{\ell}) - \hat{\delta}'(p_1 \cdot \bar{\ell}) \hat{\delta}(p_2 \cdot \bar{\ell}) \right).\end{aligned}\tag{B.16}$$

Applying the analysis of [45], one would expect the non-Abelian triangles



$$iY_{12} = \quad \quad \quad iY_{21} = \tag{B.17}$$

to contribute to the NLO observables. The series in  $\hbar$  for these partial amplitudes contain terms at  $\mathcal{O}(\hbar^{-2})$ ; however, the decomposition of the full amplitude onto the colour basis in equation (4.61) shows that these always act as quantum corrections, and thus we need not calculate these diagrams.

## B.3 Real and Imaginary Kernel

In this appendix we compute the contribution to the one loop impulse from real and imaginary parts of the one loop amplitude in eq. (6.69).

The real and imaginary parts of eq. (6.69) are familiar from the scalar case, they nevertheless produce new spin dependent terms in the impulse, when the amplitudes. Firstly note that the imaginary amplitude term in eq. (6.69) comes

with an extra  $q^2$ , which brings with it an extra power of  $\hbar$  implicitly. This means that the superclassical parts of the amplitude contribute classically to the impulse in this term. These superclassical terms are equivalent to being iterations of the tree amplitudes in the real/quadratic kernel.

One must be careful taking the amplitudes in eqs. (6.36) and (6.37) and blindly plugging them into the kernel. At first sight it appears as though there are both real and imaginary parts in these amplitudes. This is not completely true though, and we should treat the quantity  $i\epsilon(q, s, u_1, u_2)$  as *real*. Indeed this property has been applied by other authors <sup>1</sup> [130, 134, 135, 250]. One reason this should seem plausible is that it means the tree amplitude is purely real, as it should be by the optical theorem. The upshot of all this is that the superclassical amplitude is purely imaginary and the classical part is purely real, just as with scalars. This means the superclassical amplitude goes in the imaginary part and the classical amplitude goes in the real part of the kernel. With this in mind we can plug the amplitudes into the first line of eq. (6.69). At this order we can drop the  $q$  terms in  $\check{u}_i$ , as they'll become quantum, so

$$\begin{aligned} \frac{\check{u}_1^\mu}{m_1} - \frac{\check{u}_2^\mu}{m_2} &= -\frac{(\gamma m_1 + m_2)u_1^\mu - (\gamma m_2 + m_1)u_2^\mu}{m_1 m_2 (\gamma^2 - 1)} + O(q) \\ &\rightarrow -\frac{m_1(\gamma u_1^\mu - u_2^\mu)}{m_1 m_2 (\gamma^2 - 1)} + O(q). \end{aligned} \quad (\text{B.18})$$

Using this we obtain impulse kernels

$$\begin{aligned} \mathcal{I}^\mu|_{real} &= q^\mu \left( \frac{6\pi^2 G^2 m_1^2 m_2^2 (5\gamma^2 - 1) (m_1 + m_2)}{(-q^2)^{1/2+\varepsilon}} \right. \\ &\quad \left. - \frac{2i\pi^2 G^2 m_1^2 m_2^2 \gamma (5\gamma^2 - 3) (4m_1 + 3m_2) \epsilon(p_1, p_2, q, s_1)}{(\gamma^2 - 1) (-q^2)^{1/2+\varepsilon}} \right), \\ \mathcal{I}^\mu|_{imaginary} &= -\frac{16\pi i G^2 m_1^3 m_2^2 \left( (2\gamma^2 - 1)^2 - 2i\gamma(2\gamma^2 - 1)\epsilon(p_1, p_2, q, s_1) \right)}{\varepsilon (\gamma^2 - 1) (q^2)^\varepsilon} (\gamma u_1^\mu - u_2^\mu). \end{aligned} \quad (\text{B.19})$$

Performing the Fourier integrals using appendix A.2.1, the basis transformation

---

<sup>1</sup>They use it on shell where it can be written as

$$i\epsilon^\mu(q, u_1, u_2) = \sinh(w)q^\mu$$

where  $w$  is the rapidity.

eq. (6.59) and taking the  $m_1 \gg m_2$  limit, we get

$$\begin{aligned}
\Delta p_1^\mu|_{real} &= -\frac{3\pi(5\gamma^2-1)G^2m_1^2m_2b^\mu}{4(b^2)^{3/2}} \\
&\quad -\frac{\pi\gamma(5\gamma^2-3)G^2m_1^2m_2(2b^\mu\epsilon(b,s_1,u_1,u_2)-(b\cdot s_1)\epsilon^\mu(b,u_1,u_2))}{b^5(\gamma^2-1)} \\
\Delta p_1^\mu|_{imaginary} &= -\frac{2G^2m_1^2m_2(2\gamma^2-1)^2}{(\gamma^2-1)^2b^4}(\gamma u_1^\mu-u_2^\mu) \\
&\quad +\frac{8G^2m_1^2m_2\gamma(2\gamma^2-1)\epsilon(b,s_1,u_1,u_2)}{(\gamma^2-1)^2b^4}(\gamma u_1^\mu-u_2^\mu).
\end{aligned} \tag{B.20}$$

The real and imaginary splitting immediately removes the superclassical terms so that the impulse is purely classical. Combining this with the result for the cut term eq. (6.74), we recover the full probe limit impulse eq. (6.57).

# Appendix C

## Projection in the plane of scattering

The relation between the eikonal impact parameter  $x_\perp^\mu$  and  $b^\mu$  can be stated clearly by first introducing some notation. Let us define the following four-vectors in momentum space

$$\begin{cases} e_0^\mu & \equiv N_0(\tilde{p}_1^\mu + \tilde{p}_2^\mu) = N_0(p_1^\mu + p_2^\mu) \\ e_q^\mu & \equiv N_q(\tilde{p}_1^\mu - \tilde{p}_2^\mu) - N_{0q}(\tilde{p}_1^\mu + \tilde{p}_2^\mu) \\ & = N_q(p_1^\mu - p_2^\mu + q^\mu) - N_{0q}(p_1^\mu + p_2^\mu) , \end{cases} \quad (\text{C.1})$$

where the normalization factors  $N_0$ ,  $N_q$  and  $N_{0q}$  are fixed by requiring  $e_0^2 = 1$ ,  $e_q^2 = -1$  and  $e_0 \cdot e_q = 0$ .

$$\left. \begin{aligned} e_0^2 &= +1 \\ e_q^2 &= -1 \\ e_0 \cdot e_q &= 0 \end{aligned} \right\} \implies \begin{cases} N_0 = \frac{1}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}} \\ N_q = \pm \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}}{\sqrt{q^2(m_1^2 + m_2^2 + 2m_1 m_2 \gamma) + 4m_1^2 m_2^2 (\gamma^2 - 1)}} \\ N_{0q} = \mp \frac{(m_1^2 - m_2^2)}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma} \sqrt{q^2(m_1^2 + m_2^2 + 2m_1 m_2 \gamma) + 4m_1^2 m_2^2 (\gamma^2 - 1)}} \end{cases} \quad (\text{C.2})$$

Note the relative sign between the last two terms. For concreteness we will take the positive sign of  $N_q$ . Note also that these are functions of  $(m_1, m_2, \gamma)$  and  $q^2$ , except for  $N_0$  which has no  $q$  dependence. In the CM frame and for the special case of equal masses these have a simpler form and interpretation -  $e_0 \propto (2E, 0, 0, 0)$ . This explains the subscript 0 as it reduces to a vector in the time direction only. Note that it also has no  $q$  dependence, whereas  $e_q$  does.

By definition of  $x_\perp^\mu$ , that is  $x_\perp \cdot \tilde{p}_i = 0$ , the following identities hold:

$$e_0 \cdot x_\perp = 0 \quad , \quad e_q \cdot x_\perp = 0 . \quad (\text{C.3})$$

As a consequence, we can write the projection of  $x^\mu$  on the plane orthogonal to  $\tilde{p}_1^\mu$  and  $\tilde{p}_2^\mu$  as

$$\begin{aligned} x_\perp^\mu &= x^\mu - (x \cdot e_0)e_0^\mu + (x \cdot e_q)e_q^\mu \\ x_\perp^\mu &= (\eta^{\mu\nu} - e_0^\mu e_0^\nu + e_q^\mu e_q^\nu)x_\nu \\ &\equiv \Pi^{\mu\nu}x_\nu \end{aligned} \quad (\text{C.4})$$

where the different signs in the the last two terms are a consequence of  $e_0^\mu$  being time-like while  $e_q^\mu$  space-like.

With the eikonal phase taking the form in eq. (6.39), then we can write  $q_*^\mu = -\partial\chi/\partial x_{\perp\mu}$  and thus the other stationary phase condition eq. (6.42) can be written

$$x^\mu = b^\mu + q_{\nu,*} \frac{\partial x_\perp^\nu}{\partial q_\mu} \Big|_{q=q_*} - \frac{1}{2} \left( \frac{\partial\chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial\chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial\chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu}, \quad (\text{C.5})$$

where we used the definitions of  $\tilde{p}$  get the factor half in the middle term. We can now differentiate eq. (C.4) with respect to  $q$ , noting that  $e_0^\mu$  is independent of  $q^\mu$ , to find

$$x^\mu = b^\mu + q_{\nu,*} \frac{\partial}{\partial q_\mu} \left[ (x \cdot e_q) e_q^\nu \right] \Big|_{q=q_*} - \frac{1}{2} \left( \frac{\partial\chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial\chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial\chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu}. \quad (\text{C.6})$$

The vector  $q_*^\mu$  is parallel to  $x_\perp^\mu$ , since it explicitly involves a projector, and thus we know that  $q_* \cdot e_q \propto x_\perp \cdot e_q = 0$ . This means the term in equation (C.6) where the derivative acts inside the dot product will vanish and we have

$$x^\mu = b^\mu + q_{\nu,*} (x \cdot e_q) \frac{\partial e_q^\nu}{\partial q_\mu} \Big|_{q=q_*} - \frac{1}{2} \left( \frac{\partial\chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial\chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial\chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu}. \quad (\text{C.7})$$

The remaining derivative can be easily performed using the definition of  $e_q^\mu$ . The normalisations do depend on  $q$ , and will have non vanishing derivatives. However these terms, when contracted with  $q_{*\nu}$  as in eq. (C.7), will vanish as they are

proportional to  $q_* \cdot \tilde{p} = 0$ . Only the second term survives so we can write

$$q_\nu \frac{\partial e_q^\nu}{\partial q_\mu} \Big|_{q=q_*} = - (N_q q^\mu) \Big|_{q=q_*} \quad (\text{C.8})$$

leading to

$$x^\mu = b^\mu - N_q (x \cdot e_q) q^\mu \Big|_{q=q_*} - \frac{1}{2} \left( \frac{\partial \chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu}. \quad (\text{C.9})$$

This is the value of  $x^\mu$  satisfying the stationary phase conditions, so the last thing to do is to project it to  $x_\perp^\mu$  using eq. (C.4).

For the scalar case the term involving spin clearly is not present and similarly the  $\partial \chi / \partial \tilde{p}$  terms also vanish, which greatly simplifies things. In that case we can write the eikonal impact parameter as

$$x_\perp^\mu = b^\mu - (x \cdot e_0) e_0^\mu - [(x \cdot e_q) N_q q^\mu] \Big|_{q=q_*} + [(x \cdot e_q) e_q^\mu] \Big|_{q=q_*}. \quad (\text{C.10})$$

The scalar products  $x \cdot e_0$  and  $x \cdot e_q$  can be evaluated using the definitions of  $e_0^\mu$  and  $e_q^\mu$ , to give  $x \cdot e_0 = 0$  and  $x \cdot e_q = b \cdot e_q$ . A straightforward calculation gives

$$x_\perp^\mu = b^\mu - [(e_q \cdot b) (N_q q^\mu - e_q^\mu)] \Big|_{q=q_*}. \quad (\text{C.11})$$

Expressing  $e_q^\mu$  in terms of  $p_1^\mu$ ,  $p_2^\mu$  and  $q^\mu$  we obtain

$$x_\perp^\mu = b^\mu - \left[ (e_q \cdot b) [N_{0q} (p_1^\mu + p_2^\mu) - N_q (p_1^\mu - p_2^\mu)] \right] \Big|_{q=q_*}, \quad (\text{C.12})$$

which agrees — when evaluated in the center of mass frame — with the expression for the eikonal impact parameter in (5.17), where  $\tilde{N}_q = -(e_q \cdot b) N_q$  and  $\tilde{N}_{0q} = (e_q \cdot b) N_{0q}$ .

For the spinning case the expressions don't simplify so dramatically, however given the form of  $\chi$  then

$$\frac{\partial \chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi}{\partial \tilde{p}_{2\mu}} \propto \epsilon^\mu(x_\perp, \tilde{s}, e_0), \quad (\text{C.13})$$

and so

$$x \cdot e_0 = - \frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu} \cdot e_0. \quad (\text{C.14})$$

The expression for  $x \cdot e_q$  becomes

$$x \cdot e_q = b \cdot e_q - \frac{1}{2} \left( \frac{\partial \chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi}{\partial \tilde{p}_{2\mu}} \right) \cdot e_q - \frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu} \cdot e_q. \quad (\text{C.15})$$

Putting this all together we have

$$\begin{aligned} x_\perp^\mu = & b^\mu - (N_q q^\mu)|_{q=q_*} - \frac{1}{2} \left( \frac{\partial \chi}{\partial \tilde{p}_{1\mu}} - \frac{\partial \chi}{\partial \tilde{p}_{2\mu}} \right) - \frac{\partial \chi}{\partial \tilde{s}_1^\lambda} \frac{\partial \tilde{s}_1^\lambda}{\partial q_\mu} \\ & - (x \cdot e_0) e_0^\mu + (x \cdot e_q) e_q^\mu |_{q=q_*}. \end{aligned} \quad (\text{C.16})$$

These are fully general equations for  $x_\perp$ , which we can solve iteratively order by order in  $G$ . This is done in the main text.

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