Multijoints and Multilinear Duality

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Michael Chi Yung Tang

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For Aemilia

Lay Summary

The joints problem is related to geometric questions at the heart of harmonic analysis. In three dimensions, a joint is a point of intersection of three lines that do not lie within a common plane. Given a collection of lines, one can ask how many joints those lines can form – this is the joints problem.

"Duality" describes the relationship between two complementary problems that are distinct but logically equivalent. Any two such problems are said to be dual to one another. The problem that is dual to the joints problem looks to understand geometric properties which abstract on the conventional notion of volume. Understanding this geometric problem is the aim of this thesis.

Abstract

This thesis studies the dual formulation of the multijoint problem – the discrete analogue of the multilinear Kakeya problem.

Let $\mathbb{F}$ be a field. Given a set of lines in $\mathbb{F}^d$, each coloured by one of d distinct colours, a multijoint is a point formed by the intersection of d distinctly coloured, linearly independent lines. The multijoint problem looks to control the number of multijoints by the numbers of lines and can be formulated as a geometric multilinear inequality. Inequalities of this form have a dual formulation, whereby solving the problem is equivalent to proving the existence of so-called factorisations, however this only applies if the geometry of the problem is transversal.

The multilinear Kakeya theorem in Euclidean space was proven in full generality *via* the existence of factorisations. The multijoint problem was recently solved, and while it is analogous to the multilinear Kakeya theorem, the question of the existence of factorisations for this discrete problem remained unanswered.

In this work, we resolve the dual multijoint problem and prove the existence of factorisations in general. Moreover, we deduce the analogue of a theorem due to Bourgain and Guth, where the Euclidean wedge product is replaced by the discrete one. More precisely, we show that there is a constant $C = C(d)$ so that for any function $S : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$, there is a “factorising” function $s : \mathbb{F}^d \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ such that

$$(\omega_1 \wedge \cdots \wedge \omega_d) S(p)^d \leq \prod_{j=1}^{d} s(p, \omega_j),$$

for every $p \in \mathbb{F}^d$ and every tuple of directions $(\omega_j)_{j=1}^d \in (\mathbb{S}^{d-1})^d$, and

$$\sum_{p \in l} s(p, e(l)) \leq C \|S\|_d$$

for every line $l \subset \mathbb{F}^d$, where $e(l) \in \mathbb{S}^{d-1}$ denotes the direction of l and $\wedge$ denotes the discrete wedge product. This property is common to both the discrete wedge product in $\mathbb{F}^d$ and the continuous wedge product in $\mathbb{R}^d$, and establishes a perspective from which these two geometries are “the same”.

Until now, there was no duality theorem for the multijoint problem in general, and there was no formal duality which applied directly to the joints problem. We conclude with a proof that there is, indeed, a duality for the joints problem.

Contents

Notation

We write $A \lesssim B$ to mean that there is a non-negative constant C, depending only on dimension, so that $A \leq CB$. We write $A \lesssim_{k_1,\dots,k_d} B$ to mean that there is a constant C, depending only on $k_1, \dots, k_d$ and the dimension, so that $A \leq CB$. We write $B \gtrsim A$ and $B \gtrsim_{k_1,\dots,k_d} A$ to mean $A \lesssim B$ and $A \lesssim_{k_1,\dots,k_d} B$, respectively. Moreover, by $A \sim B$, we mean $A \lesssim B$ and $B \lesssim A$, and finally, by $A \sim_{k_1,\dots,k_d} B$, we mean $A \lesssim_{k_1,\dots,k_d} B$ and $B \lesssim_{k_1,\dots,k_d} A$.

Given d sets $Y_1, \dots, Y_d$ and $\boldsymbol{y} \in Y_1 \times \cdots \times Y_d$ we write y_j to mean the j-th entry of $\boldsymbol{y}$ for each $1 \leq j \leq d$, so $\boldsymbol{y} = (y_j)_{j=1}^d$. We write $y \in \boldsymbol{y}$ to mean $y \in \{y_j\}_{j=1}^d$. We will use standard multi-index notation.

We write χ_A to denote the indicator function of a set A.

We will use boldface text to indicate that a word or phrase is being defined.

Outline

Chapter 1 and Chapter 2 contain introductory material. The substance of this thesis can be found in Chapter 3, Chapter 4 and Chapter 5.

Chapter 1

Introduction to Joints and Multijoints

Let $d \in \mathbb{N}$, let $\mathcal{L}$ be a finite set of lines in $\mathbb{R}^d$ and let $p \in \mathbb{R}^d$. Suppose there are d lines, $l_1, \dots, l_d \in \mathcal{L}$, that all contain p and are such that their directions, $e(l_1), \dots, e(l_d) \in \mathbb{S}^{d-1}$, span $\mathbb{R}^d$. Then we say that p is a **joint** and that the d-tuple $\boldsymbol{l}$ **forms a joint at** p.

Given a d-tuple, $\boldsymbol{l} \in \mathcal{L}^d$, let us say that $\delta(p, \boldsymbol{l}) = 1$ if the lines $\boldsymbol{l} = (l_j)_j$ form a joint at p, and $\delta(p, \boldsymbol{l}) = 0$ otherwise. Let $J = \{p : \exists \boldsymbol{l} \in \mathcal{L}^d \text{ so that } \delta(p, \boldsymbol{l}) = 1\}$ be the set of joints formed by tuples $\boldsymbol{l} \in \mathcal{L}^d$. The joints problem seeks to understand how many joints J there can be, given some set of lines $\mathcal{L}$. Trivially, $|J| \leq |\mathcal{L}|^d$ since $|J|$ is bounded by the number of d-tuples of lines. Hence, we would like to learn for which $q \geq 1$,

$$|J| \lesssim |\mathcal{L}|^{\frac{d}{q}}. \tag{1.1}$$

This problem was conceived by Chazelle *et al.*, [CEG$^+$92], where inequality (1.1) was proved for $q = 12/7$ in $\mathbb{R}^3$. Here, the joints problem was studied in the context of computer science, with the aim of showing that an algorithm which enumerates collections of rods in $\mathbb{R}^3$ so that the i-th element obstructs the $(i+1)$-st element, as seen from a fixed viewpoint, will always terminate. A joint was then formulated by interpreting a cycle of obstructing lines in $\mathbb{R}^3$, depicted in Figure 1.1, as a point of intersection of three non-coplanar lines.

Consider the configuration of lines in $\mathbb{R}^3$ described by Figure 1.2. Generalising so that for each colour there are N^2 copies of each line, parallel to each coordinate axis, we see that this forms a lattice of N^3 joints using $3N^2$ lines. Therefore, $|J| = N^3 = (N^2)^{\frac{3}{2}} \sim |\mathcal{L}|^{\frac{3}{2}}$. Hence, it was conjectured that (1.1) should hold for $q = d - 1$. Further improvements in $\mathbb{R}^3$ were made in [EKS11, BCT06, SW04, Sha94] until Guth and Katz proved the endpoint case, $q = 2$ in 2008, [GK10]. The endpoint result was extended to $\mathbb{R}^d$ in [KSS10, Qui10]. Thereafter, Carbery and Iliopoulou, Dvir and Tao generalised these results from $\mathbb{R}^d$ to $\mathbb{F}^d$ for an arbitrary field $\mathbb{F}$, [CI14, Tao14, Dvi12]. We continue our discussion with respect to an arbitrary field, which we denote by $\mathbb{F}$, unless stated otherwise.

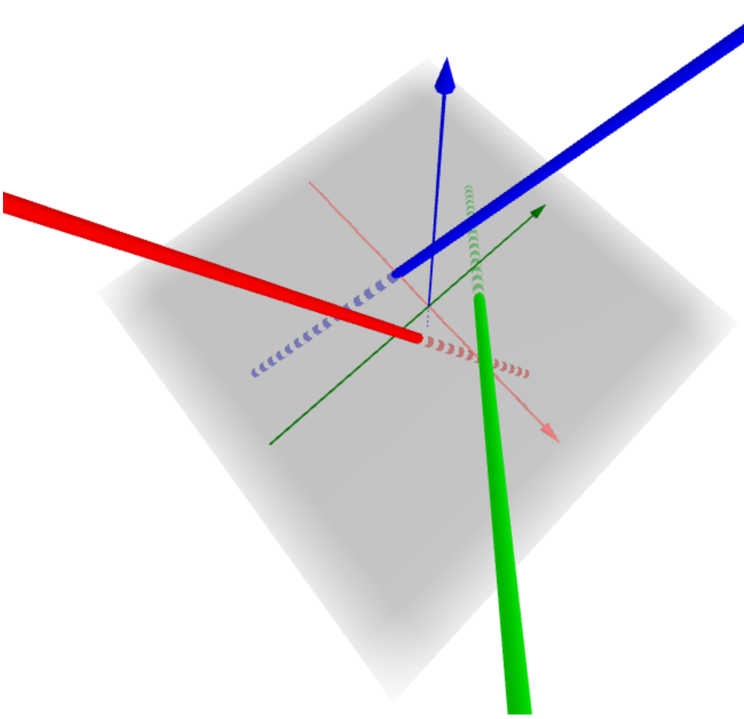

Figure 1.1: Cycle of three pairwise obstructing cylinders in $\mathbb{R}^3$. Figure generated using [HBA$^+$18].

1.1 Multiplicity

To refine our question, let us write the trivial estimate as $\sum_{p\in J} 1 \lesssim |\mathcal{L}|^d$. The **multiplicity** of each joint $p \in J$ is the number of tuples $\boldsymbol{l}$ which form a joint at p. This can be expressed as $\sum_{\boldsymbol{l}\in\mathcal{L}^d} \delta(p, \boldsymbol{l})$, and the trivial estimate is

$$\sum_{p\in J} \sum_{\boldsymbol{l}\in\mathcal{L}^d} \delta(p, \boldsymbol{l}) \lesssim \left(\sum_{l\in\mathcal{L}} 1 \right)^d .$$

Considering an arbitrary function $f : \mathcal{L} \to \mathbb{R}_{\geq 0}$, another trivial estimate is

$$\sum_{p\in J} \left(\sum_{\boldsymbol{l}\in\mathcal{L}^d} \delta(p, \boldsymbol{l}) \prod_{l\in\boldsymbol{l}} f(l) \right) \lesssim \left(\sum_{l\in\mathcal{L}} f(l) \right)^d . \tag{1.2}$$

In particular, we regard (1.2) as a geometric inequality. Accordingly, the joints problem becomes a question of understanding the operator

$$f \mapsto \sum_{\boldsymbol{l}} \delta(\,\cdot\,, \boldsymbol{l}) \prod_{l\in\boldsymbol{l}} f(l),$$

and hence, a question of understanding the joints kernel, δ.

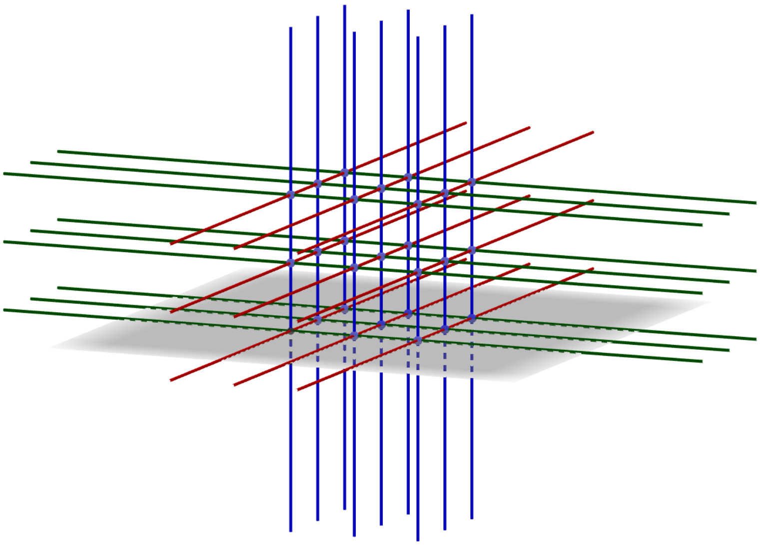

Figure 1.2: An example configuration of axis-parallel lines in $\mathbb{R}^3$. Figure generated using [HBA$^+$18].

Remark. In other references (*e.g.* [Zha20, TYZ20]), instead of formulating the joints problem as a geometric inequality, the set $\mathcal{L}$ is generalised to a multiset. Arguments in the multiset-setting are formally equivalent to considering f-weighted multiplicity for $\mathbb{Z}_{\geq 0}$-valued f only. ◀

Since the ℓ^q-norm decreases in q, it is well defined to ask: which is the least $q \in (0, 1]$ such that

$$\left(\sum_{p \in \mathbb{F}^d} \left(\sum_{\boldsymbol{l} \in \mathcal{L}^d} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^q \right)^{\frac{1}{q}} \lesssim \left(\sum_{l \in \mathcal{L}} f(l) \right)^d$$

holds for arbitrary non-negative f? By the change of variables $q \mapsto q/d$, the question becomes: which is the least $q \in (0, d]$ for which

$$\left(\sum_{p \in \mathbb{F}^d} \left(\sum_{\boldsymbol{l} \in \mathcal{L}^d} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^{\frac{q}{d}} \right)^{\frac{1}{q}} \lesssim \left(\sum_{l \in \mathcal{L}} f(l) \right) \tag{1.3}$$

holds for arbitrary non-negative f? The case of $q = d$ corresponds to the trivial estimate, (1.2). Examples such as the axis-parallel configuration in Figure 1.2 show that any such exponent must satisfy $q \geq d/(d-1)$. Zhang proved in [Zha20] that (1.3) holds at the endpoint $q = d/(d-1)$ for an arbitrary field, and established a multilinear generalisation of (1.3). This is a strictly stronger result than inequality (1.1), which does not account for multiplicity.

1.2 Multilinearity

Consider d finite sets of lines, $\mathcal{L}_1, \ldots, \mathcal{L}_d \subset \mathbb{F}^d$. Let $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ and suppose p lies within the intersection of those lines. If the directions of those lines span $\mathbb{F}^d$, then we call p a **multijoint**. Moreover, we say that **$\boldsymbol{l}$ forms a multijoint at** p. We extend the definition of δ so that $\delta(p, \boldsymbol{l}) = 1$ if $\boldsymbol{l}$ forms a multijoint at p, and 0 otherwise. The prefix “multi” acknowledges that multijoints are the multilinear generalisation of joints. We can now ask: which is the least $q \in (0, d]$ such that

$$\left(\sum_{p \in \mathbb{F}^d} \left(\sum_{\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d} \delta(p, \boldsymbol{l}) \prod_{j=1}^{d} f_j(l_j) \right)^{\frac{q}{d}} \right)^{\frac{1}{q}} \lesssim \prod_{j=1}^{d} \left(\sum_{l_j \in \mathcal{L}_j} f_j(l_j) \right)^{\frac{1}{d}} \tag{1.4}$$

holds for arbitrary non-negative functions f_j? Considering Figure 1.2, we see that any exponent q for which inequality (1.4) holds must satisfy $q \geq d/(d-1)$. When $d = 3$, the endpoint case of $q = 2$ was proved with $\mathbb{F} = \mathbb{R}$ by Iliopoulou in [Ili15] with f_j identically equal to 1. For arbitrary dimension d, the endpoint case $q = d/(d-1)$ of this multijoint inequality is precisely the multilinear generalisation that was proven by Zhang in [Zha20], where he proved the analogous result for multisets. That is, he proved (1.4) for $\mathbb{Z}_{\geq 0}$-valued f_j. By homogeneity, this establishes the result for $\mathbb{Q}_{\geq 0}$-valued f_j and by density this is sufficient for inequality (1.4) to hold for arbitrary $\mathbb{R}_{\geq 0}$-valued f_j. In principle, this result appears more general than the joints inequality (1.3) since the weights f_j may be chosen independently. However in [Zha20], Zhang also observed that the joint problem with multiplicity and the multijoint problem with multiplicity are equivalent.

1.3 Joints of Varieties

Having considered how to count joints and multijoints formed by lines, we turn to multijoints formed by more general algebraic objects. Let Γ_j be a set of k_j-dimensional varieties in $\mathbb{F}^n$ for each $1 \leq j \leq d$, where $k_1 + \ldots + k_d = n$. We do not require the k_j to be distinct. Since the ambient dimension is not necessarily equal to d, we retain d as the level of multilinearity and use n to denote the dimension of the ambient vector space. Examples of such sets Γ_j include sets of affine k_j-planes, which are varieties of degree 1.

Let $\gamma \subset \mathbb{F}^n$ be a variety. Roughly speaking, a point $p \in \gamma$ is **smooth** if $T_p\gamma$, the tangent space of γ at p, is well-defined. Suppose $p \in \mathbb{F}^n$ is such that for each $1 \leq j \leq d$, there is variety $\gamma_j \in \Gamma_j$ for which p is a smooth point and $T_p\gamma_1, \ldots, T_p\gamma_d$, the tangent spaces of each γ_j at p, span $\mathbb{F}^n$. Then we say that p is a **multijoint** and that the tuple **$\boldsymbol{\gamma}$ forms a multijoint at** p. As before, we let $\delta(p, \boldsymbol{\gamma}) = 1$ if $\boldsymbol{\gamma}$ forms a multijoint at p, and 0 otherwise. In this setting, the correct generalisation of (1.4) is

$$\left(\sum_{p\in\mathbb{F}^n}\left(\sum_{\gamma\in\Gamma_1\times\cdots\times\Gamma_d}\delta(p,\gamma)\prod_{j=1}^{d}f_j(\gamma_j)\right)^{\frac{q}{d}}\right)^{\frac{1}{q}}\lesssim_{k_1,\dots,k_d}\prod_{j=1}^{d}\left(\sum_{\gamma_j\in\Gamma_j}f_j(\gamma_j)\deg\gamma_j\right)^{\frac{1}{d}}.$$

Note that while the ambient dimension n implicitly contributes to the implied constant, it is only the degree of multilinearity d that features in the exponents of this inequality. Iliopoulou proved results on joints and multijoints formed by real algebraic curves of uniformly bounded degree in $\mathbb{R}^3$, [Ili13, Ili15]. A result for planes (*i.e.* degree 1 varieties) of arbitrary dimensions in $\mathbb{R}^n$ was established away from the endpoint by Yang, [Yan16].

Yang's proof is among the minority of works on multijoint problems which uses polynomial partitioning, tying his argument to the topological properties of the Euclidean space, $\mathbb{R}^n$. Earlier arguments for counting multijoints using polynomial partitioning were developed by Iliopoulou in [Ili13, Ili15]. However, Yang's work is the only proof to date in the development of the joints problem which uses polynomial partitioning to perform an induction using polynomials of bounded degree, which in turn necessitates loss in the exponent. Later, Carbery and Iliopulou, [CI20], and Yu and Zhao, [YZ19] independently proved estimates on multijoints formed by $(d-k)$ sets of lines and a single set of k-planes in $\mathbb{F}^n$. Eventually, the general problem for multijoints formed by algebraic varieties of any dimension and degree at the endpoint,

$$\sum_{p\in\mathbb{F}^n}\left(\sum_{\gamma\in\Gamma_1\times\cdots\times\Gamma_d}\delta(p,\gamma)\prod_{j=1}^{d}f_j(\gamma_j)\right)^{\frac{1}{d-1}}\lesssim_{k_1,\dots,k_d}\prod_{j=1}^{d}\left(\sum_{\gamma_j\in\Gamma_j}f_j(\gamma_j)\deg\gamma_j\right)^{\frac{1}{d-1}}\tag{1.5}$$

was solved by Tidor, Yu and Zhao, [TYZ20]. Similar to Zhang's work, [Zha20], their proof established the analogous result for multisets of varieties, which is sufficient to establish (1.5) for arbitrary non-negative functions f_j.

Chapter 2

Multilinear Duality

Almost all of the arguments and techniques used to study joints and multijoints can be understood from a dual perspective. This chapter introduces and discusses multilinear duality, which was conceived through the study of the multilinear Kakeya problem.

2.1 Multilinear Kakeya

Let $\mathcal{T}_1, \ldots, \mathcal{T}_d$ be sets of 1-neighbourhoods of doubly-infinite lines, which we call tubes, in $\mathbb{R}^d$. For any tube T, let $e(T) \in \mathbb{S}^{d-1}$ be a unit vector parallel to the central line of T, and for $\omega_1, \ldots, \omega_d \in \mathbb{S}^{d-1}$, we adopt a non-standard definition of the Euclidean wedge product whereby $\omega_1 \wedge \cdots \wedge \omega_d$ is equal to the non-negative volume of the parallelepiped with edges $\omega_1, \ldots, \omega_d$. The endpoint multilinear Kakeya theorem, first proved by Bourgain and Guth in [BG11], states that

$$\int_{\mathbb{R}^d} \left(\sum_{\boldsymbol{T} \in \mathcal{T}_1 \times \cdots \times \mathcal{T}_d} \left(\prod_{j=1}^{d} f_j(T_j) \chi_{T_j}(x) \right) (e(T_1) \wedge \cdots \wedge e(T_d)) \right)^{\frac{1}{d-1}} \mathrm{d}x$$
$$\lesssim \prod_{j=1}^{d} \left(\sum_{T_j \in \mathcal{T}_j} f_j(T_j) \right)^{\frac{1}{d-1}} . \qquad (2.1)$$

Earlier results were proved when the sets of tubes were assumed to be transversal in [BCT06], with near-optimal exponents, and in [Gut10], at the endpoint.

The geometry defined by this wedge product $\wedge$ in $\mathbb{R}^d$ is understood to be the **continuous setting**. From the left-hand side of inequality (2.1), we can extract an integral kernel of the form

$$(x, \boldsymbol{T}) \mapsto \left(\prod_{j=1}^{d} \chi_{T_j}(x) \right) \big(e(T_1) \wedge \cdots \wedge e(T_d) \big) .$$

The discrete analogue of a tube is a line. Replacing instances of tubes in this kernel with lines results in a candidate formula for the multijoint kernel δ,

$$(p, \boldsymbol{l}) \mapsto \left(\prod_{j=1}^{d} \chi_{l_j}(p) \right) \big(e(l_1) \wedge \cdots \wedge e(l_d) \big) .$$

By replacing the wedge product with the “discrete wedge product”, which we will define prior to Theorem 2.1.3, below, we see that this candidate for the discrete analogue of the multilinear Kakeya kernel gives a valid formula for δ.

Suppose in (2.1) that we substitute continuous objects and operators with their discrete counterparts. That is, suppose we substitute the multilinear Kakeya kernel with the multijoint kernel δ, tubes with lines and the Lebesgue measure on $\mathbb{R}^d$ with the counting measure on $\mathbb{R}^d$. Then the multilinear Kakeya inequality, (2.1), corresponds precisely to (1.4) at the endpoint $q = d/(d-1)$ with $\mathbb{F} = \mathbb{R}$.

Guth’s proof of the transversal multilinear Kakeya Theorem, [Gut10], was subsequently interpreted using techniques accessible to the harmonic analyst by Carbery and Valdimarsson in [CV13]. The cornerstone of the Carbery–Valdimarsson argument, also implicit in [BG11], is the following Theorem 2.1.1.

Theorem 2.1.1 (Carbery–Valdimarsson, [CV13])**.** *Let $S : \mathcal{Q} \to [1, \infty)$ be finitely supported, where $\mathcal{Q}$ is the lattice of unit cubes in $\mathbb{R}^d$. There exists a function $s : \mathcal{Q} \times (\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_d) \to \mathbb{R}_{\geq 0}$ so that*

$$e(T_1) \wedge \cdots \wedge e(T_d) S(Q)^d \leq \prod_{j=1}^{d} s(Q, T_j),$$

for all $Q \in \mathcal{Q}$ and $T_j \in \mathcal{T}_j$ such that $T_j \cap Q \neq \emptyset$ for all $1 \leq j \leq d$, and so that

$$\sum_{Q : T \cap Q \neq \emptyset} s(Q, T) \lesssim \|S\|_d$$

for all $T \in \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_d$.

We highlight two properties of the function s. Firstly, we note that despite the multilinearity of the problem, the function s does not have have a subscript. In other multilinear situations, we might expect that instead of a single function s, we may need to consider d functions $s_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}_{\geq 0}$. Secondly, the arguments of s comprise of a cube Q and a tube T_j. This indicates that for two parallel tubes T and T', we may have that $s(\cdot, T)$ and $s(\cdot, T')$ are distinct. Specifying a tube T is equivalent to specifying a cube and direction, hence we may interpret the function s in this theorem as having three arguments – two cubes and a direction. These two observations describe additional degrees of freedom that we might expect to be unnecessary. Indeed, a more universal dual statement, which does not refer to any fixed sets of tubes, was found by Bourgain and Guth and is described by the following theorem.

Theorem 2.1.2 (Bourgain–Guth, [BG11])**.** *Let* $S : \mathcal{Q} \to [1, \infty)$ *be finitely supported. Then there exists a function* $s : \mathcal{Q} \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ *so that*

$$(\wedge_{\omega \in \boldsymbol{\omega}} \omega) S(Q)^d \leq \prod_{\omega \in \boldsymbol{\omega}} s(Q, \omega),$$

for all $Q \in \mathcal{Q}$ *and* $\boldsymbol{\omega} \in (\mathbb{S}^{d-1})^d$*, and so that*

$$\sum_{Q : T \cap Q \neq \emptyset} s(Q, e(T)) \lesssim \|S\|_d$$

for any tube $T \subset \mathbb{R}^d$*.*

Remark. For any given S, the factorising function s from Theorem 2.1.2 is explicitly constructed in terms of geometric quantities associated to a particular polynomial hypersurface. Specifically, the function $s(Q, e(T))$ is defined as the directional surface area of a polynomial hypersurface within Q in the direction $e(T)$. Moreover, the quantity $\prod_j s(Q, e(T_j))$ is realised by the "visibility" of the polynomial hypersurface in Q. However, for any S, it is only the existence of the polynomial hypersurface that is known, and it is not constructively identified. The existence is a consequence of the Borsuk–Ulam Theorem. ◀

This theorem removes the subscript j from the factorising functions and reduces the second argument of such functions to a direction only. While the Bourgain–Guth theorem was used to prove the endpoint multilinear Kakeya theorem, and the analogous multijoint problem has been resolved by Zhang, [Zha20], the discrete analogue of the above Bourgain–Guth theorem has remained open until now.

The primary aim of this thesis is to prove the discrete analogue of the Bourgain–Guth theorem, Theorem 2.1.3, below.

For any field $\mathbb{F}$, let $\mathbb{S}^{d-1}$ denote the projective space of lines through the origin in $\mathbb{F}^d$. Given any d-tuple $\boldsymbol{\omega} \in (\mathbb{S}^{d-1})^d$, we define the **discrete wedge product** by $\wedge_{\omega \in \boldsymbol{\omega}} \omega = 1$ if the entries of $\boldsymbol{\omega}$ are linearly independent, and 0 otherwise.

Theorem 2.1.3 (Discrete Bourgian–Guth Theorem)**.** *For all finitely supported* $S : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$ *with* $\|S\|_d = 1$*, there exists a function* $s : \mathbb{F}^d \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ *so that*

$$(\omega_1 \wedge \cdots \wedge \omega_d) S(p)^d \lesssim \prod_{j=1}^{d} s(p, \omega_j),$$

for all $p \in \mathbb{F}^d$ *and every* $\boldsymbol{\omega} \in (\mathbb{S}^{d-1})^d$*, and so that,*

$$\sum_{p \in l} s(p, e(l)) \leq 1$$

for all lines $l \subset \mathbb{F}^d$*.*

2.2 Multilinear Duality

In the context of multilinear inequalities involving geometric means, and in light of Theorem 2.1.2, we wish to formalise the Bourgain–Guth proof strategy for multilinear Kakeya in the abstract setting. This was achieved by Carbery, Hänninen and Valdimarsson, [CHV20a], who proved under appropriate assumptions, that such inequalities are indeed equivalent to factorisation statements, thus establishing the theory of multilinear duality.

Theorem 2.2.1 (Multilinear Duality, [Coc18, CHV20a])**.** *Let X be a σ-finite measure space and let $Y_1, \dots, Y_d$ be measure spaces. For each $1 \leq j \leq d$, let T_j be a linear operator which maps functions on Y_j to functions on X. Let $\alpha_1 + \dots + \alpha_d = 1$, where $\alpha_j > 0$ for each $1 \leq j \leq d$, and let $A > 0$. The following are equivalent:*

(i)

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{L^p(X)} \leq A \prod_{j=1}^{d} \|f_j\|_{L^1(Y_j)}^{\alpha_j} \tag{2.2}$$

for all non-negative $f_j \in L^1(Y_j)$ for each $1 \leq j \leq d$.

(ii) For every non-negative $S \in L^{p'}(X)$, there are functions $S_j : X \to \mathbb{R}_{\geq 0}$ for each $1 \leq j \leq d$ so that

$$S(x) \leq \prod_{j=1}^{d} S_j(x)^{\alpha_j}$$

for a.e. $x \in X$, and so that

$$\int_X T_j f_j(x) S_j(x) \,\mathrm{d}x \leq A \|S\|_{p'} \|f_j\|_1$$

for all non-negative $f_j \in L^1(Y_j)$ and all $1 \leq j \leq d$.

(iii) For every non-negative $S \in L^{p'}(X)$, there are functions $S_j : X \to \mathbb{R}_{\geq 0}$ for each $1 \leq j \leq d$ so that

$$S(x) \leq \prod_{j=1}^{d} S_j(x)^{\alpha_j}$$

for a.e. $x \in X$, and so that

$$\sum_{j=1}^{d} \alpha_j \int_X T_j f_j(x) S_j(x) \,\mathrm{d}x \leq A \|S\|_{p'} \|f_j\|_1$$

for all non-negative $f_1 \in L^1(Y_1), \dots, f_d \in L^1(Y_d)$.

(iv) For every non-negative $S \in L^{p'}(X)$, there are functions $S_j : X \to \mathbb{R}_{\geq 0}$ for

each $1 \leq j \leq d$ *so that*

$$S(x) \leq \prod_{j=1}^{d} S_j(x)^{\alpha_j}$$

for a.e. $x \in X$*, and so that*

$$\prod_{j=1}^{d} \left(\int_X T_j f_j(x) S_j(x) \, \mathrm{d}x \right)^{\alpha_j} \leq A \|S\|_{p'} \left\|f_j\right\|_1$$

for all non-negative $f_j \in L^1(Y_1), \ldots, f_d \in L^1(Y_d)$*.*

We say that a tuple of functions $(S_j)_j$ which satisfies any one of the above items *(ii)*, *(iii)* or *(iv)* from Theorem 2.2.1, **factorises** the test function S.

Inequality (1.4) is not of the form (2.2) since $\delta(p, \boldsymbol{l})$ is not expressed as a tensor product of kernels of linear operators. Therefore, we cannot apply Theorem 2.2.1 unless we assume that the sets of lines are **transversal**. We say that the sets of lines $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are transversal if every tuple $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ so that there exists $p \in J$ such that $p \in l_j$ for all $1 \leq j \leq d$ also satisfies $\delta(p, \boldsymbol{l}) = 1$. In this case, we can write

$$\delta(p, \boldsymbol{l}) = \prod_{j=1}^{d} \chi_{l_j}(p).$$

2.2.1 Counting Multijoints by Optimisation

For problems, such as the multijoint problem, the theory surrounding Theorem 2.2.1, as described by Carbery, Hänninen and Valdimarsson in [CHV20a], suggests analysis by convex optimisation. Specifically, their work shows that, under the assumption that (2.2) holds for any $S : J \to \mathbb{R}_{\geq 0}$, it is possible to demonstrate the existence of a tuple $(S_j)_j$ which factorises S by convex optimisation. In this subsection, we explore this argument in the setting of multijoints.

The multijoints problem seeks to prove the inequality

$$\left\| M^{\frac{1}{d}} \right\|_{\frac{d}{d-1}} \lesssim \prod_{j=1}^{d} \left(\sum_{l_j \in \mathcal{L}_j} f_j(l_j) \right)^{\frac{1}{d}}, \tag{2.3}$$

where $M(p) = M[f_1, \ldots, f_d](p) = \sum_{\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d} \delta(p, \boldsymbol{l}) \prod_{j=1}^{d} f_j(l_j)$ and $f_1, \ldots, f_d$ are arbitrary non-negative functions on $\mathcal{L}_1, \ldots, \mathcal{L}_d$, respectively. For each $1 \leq j \leq d$, the operator T_j from Theorem 2.2.1 is given by $T_j f_j(p) = \sum_{l_j \in \mathcal{L}_j} f_j(l_j) \chi_{l_j}(p)$ for $p \in J$.

Let us examine the left-hand side of (2.3). To evaluate the $L^{d/(d-1)}(J)$-norm, we sum against a test function $S \in L^d(J)$ with unit norm. Suppose the problem is transversal. Then,

$$\sum_{p\in J} S(p)M(p)^{\frac{1}{d}} = \sum_{p\in J} S(p)\prod_{j=1}^{d}(\sum_{l_j\ni p} f_j(l_j))^{\frac{1}{d}} = \sum_{p\in J}\frac{1}{d}\sum_{j=1}^{d} S_j(p)\sum_{l_j\ni p} f_j(l_j),$$

where

$$S_j(p) := S(p)\left(\sum_{l_j\ni p} f_j(l_j)\right)^{-1}\prod_{j'=1}^{d}\left(\sum_{l_{j'}\ni p} f_{j'}(l_{j'})\right)^{\frac{1}{d}}.$$

By realising that this equality is an extremiser for the arithmetic mean–geometric mean (AM–GM) inequality, we deduce that

$$\begin{aligned}\sum_{p\in J} S(p)M(p)^{\frac{1}{d}} &= \inf_{(S_j)_j\in\mathcal{C}_S}\frac{1}{d}\sum_{p\in J}\sum_{j=1}^{d} S_j(p)\sum_{l_j\ni p} f_j(l_j)\\ &= \inf_{(S_j)_j\in\mathcal{C}_S}\frac{1}{d}\sum_{j=1}^{d}\sum_{l_j\in\mathcal{L}_j} f_j(l_j)\sum_{p\in l_j\cap J} S_j(p),\end{aligned}$$

where $\mathcal{C}_S$ is the set of all tuples of functions $(S_j)_j$ so that $S(p) \le \prod_j S_j(p)^{\frac{1}{d}}$ for all $p \in J$. The set $\mathcal{C}_S$ is convex, and we are infimising a linear, and hence convex, function of $(S_j)_j$. Therefore, the above infimum describes a convex optimisation problem. We summarise our discussion as follows.

Lemma 2.2.2. *Suppose the sets of lines $\mathcal{L}_1, \dots, \mathcal{L}_d$ are transversal with associated joints J. For any non-negative $S : J \to \mathbb{R}_{\ge 0}$ with $\|S\|_d = 1$,*

$$\inf_{(S_j)_j\in\mathcal{C}_S}\frac{1}{d}\sum_{j=1}^{d}\sum_{l_j\in\mathcal{L}_j} f_j(l_j)\sum_{p\in l_j\cap J} S_j(p) \le \left\|M^{\frac{1}{d}}\right\|_{L^{\frac{d}{d-1}}(J)}.$$

Suppose, in addition, that (2.3) *holds for any non-negative functions $f_1, \dots, f_d$ such that $\sum_{l_j\in\mathcal{L}_j} f_j(l_j) = 1$ for each $1 \le j \le d$. Then*

$$\inf_{(S_j)_j\in\mathcal{C}_S}\frac{1}{d}\sum_{j=1}^{d}\sum_{l_j\in\mathcal{L}_j} f_j(l_j)\sum_{p\in l_j\cap J} S_j(p) \lesssim 1. \tag{2.4}$$

In particular, if $(S_j)_j \in \mathcal{C}_S$ realises the infimum in (2.4)*, then $\sum_{p\in l_j\cap J} S_j(p) \lesssim 1$ for every $l_j \in \mathcal{L}_j$ and $1 \le j \le d$.*

Thus, for any fixed transversal configuration of lines, we may compute the factorisation of S, by finding a tuple $(S_j)_j$ which realises the infimum on the left-hand side of (2.4). However, there are two important observations to note. Firstly, if we compute $(S_j)_j$ in this way then the tuple will, in general, depend on the functions $f_1, \dots, f_d$. Therefore, this approach does not guarantee the existence of the tuple $(S_j)_j$ as described in Theorem 2.2.1. Secondly, we deduced the uniform upper bound described by inequality (2.4) under the assumption that inequality

(2.3) is true. Without making that assumption we cannot deduce any such upper bound without further work.

2.2.2 Minimax

To find examples of functions as described in Theorem 2.2.1, which are independent of $f_1, \dots, f_d$, consider the case where $d = 2$, so that $d/(d-1) = 2$ and follow arguments that are outlined in [Coc18]. In this case, (2.3) reduces to $\sum_{p \in J} M(p) \lesssim (\sum_{l_1 \in \mathcal{L}_1} f_1(l_1))(\sum_{l_2 \in \mathcal{L}_2} f_2(l_2))$. However, by changing the order of summations, we deduce that this inequality is true with implied constant equal to 1, uniformly over all f_1 and f_2. Indeed,

$$\begin{aligned}
\sum_{p \in J} M(p) &= \sum_{p \in J} \prod_{j=1}^{2} \sum_{l_j \in \mathcal{L}_j} f_j(l_j) \chi_{l_j}(p) \\
&= \sum_{l_1 \in \mathcal{L}_1} \sum_{l_2 \in \mathcal{L}_2} \sum_{p \in J} f_1(l_1) f_2(l_2) \chi_{l_1 \cap l_2}(p) \le \sum_{l_1 \in \mathcal{L}_1} \sum_{l_2 \in \mathcal{L}_2} f_1(l_1) f_2(l_2).
\end{aligned}$$

Hence, we may apply Lemma 2.2.2 with normalised f_1, f_2, for any non-negative $S \in L^2(J)$ of unit size, to deduce that

$$\inf_{S^2 \le S_1 S_2} \left[\frac{1}{2} \sum_{l_1 \in \mathcal{L}_1} f_1(l_1) \sum_{p \in l_1 \cap J} S_1(p) + \frac{1}{2} \sum_{l_2 \in \mathcal{L}_2} f_2(l_2) \sum_{p \in l_2 \cap J} S_2(p) \right] \le 1. \tag{2.5}$$

Therefore

$$\sup_{\|f_j\|_1 \le 1} \inf_{(S_1, S_2) \in \mathcal{C}_S} \left[\frac{1}{2} \sum_{l_1 \in \mathcal{L}_1} f_1(l_1) \sum_{p \in l_1 \cap J} S_1(p) + \frac{1}{2} \sum_{l_2 \in \mathcal{L}_2} f_2(l_2) \sum_{p \in l_2 \cap J} S_2(p) \right] \le 1. \tag{2.6}$$

We wish to assert that the "sup inf" in (2.6) can be replaced by a "min sup". This is valid following an application of a minimax theorem.

Theorem 2.2.3 (Lopsided Minimax, [AE84]). *Let A and B be convex subsets of vector spaces, where B additionally has a topology. Suppose that*

- $\exists a_0 \in A$ *such that* $b \mapsto \Psi(a_0, b)$ *is* $\inf$*-compact,*
- $\forall a \in A$*, the mapping* $b \mapsto \Psi(a, b)$ *is lower semi-continuous,*

for a map $\Psi : A \times B \to \mathbb{R}$ *which is convex in* b *for all* a*, and concave in* a *for all* b*. Then* $\exists b_0 \in B$ *so that* $\sup_{a \in A} \Psi(a, b_0)$ *attains the common value*

$$\sup_{a \in A} \inf_{b \in B} \Psi(a, b) = \min_{b \in B} \sup_{a \in A} \Psi(a, b).$$

Remark. This minimax theorem is widely applicable, and its use here may be considered heavy-handed. It is likely that there is a simpler minimax theorem that would suffice for our purpose, however, this Lopsided Minimax Theorem adequately suits our purpose. ◀

To apply Theorem 2.2.3 to the left-hand side of (2.6), with $A = \{(f_1, f_2) : \|f_1\|_1, \|f_2\|_1 \leq 1\}$ and $B = \mathcal{C}_S$, it remains to verify that the hypotheses are satisfied. The map

$$\Psi((f_1, f_2), (S_1, S_2)) := \frac{1}{2}\sum_{l_1\in\mathcal{L}_1} f_1(l_1) \sum_{p\in l_1\cap J} S_1(p) + \frac{1}{2}\sum_{l_2\in\mathcal{L}_2} f_2(l_2) \sum_{p\in l_2\cap J} S_2(p)$$

is linear in both arguments – so the lower semi-continuity condition is verified. For inf-compactness, let f_1 and f_2 be the constant functions on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and consider the sublevel set

$$\Big\{(S_1, S_2) \in \mathcal{C}_S : \Psi\big((f_1, f_2), (S_1, S_2)\big) \leq \lambda\Big\}.$$

We may insist that the functions S_1 and S_2 are non-negative. Since (2.6) holds, both S_1 and S_2 are bounded above pointwise by $r = 2\max_{j=1,2}\max_{l_j\in\mathcal{L}_j} f_j(l_j)^{-1}$. As Ψ is continuous, for any $\lambda \in \mathbb{R}$, these sublevel sets are closed subsets of $[0, r]^{|J|} \times [0, r]^{|J|}$, which is compact. Hence, the sublevel sets are compact and Theorem 2.2.3 applies. Therefore

$$\min_{(S_1,S_2)\in\mathcal{C}_S} \sup_{\|f_j\|_1\leq 1} \left[\frac{1}{2}\sum_{l_1\in\mathcal{L}_1} f_1(l_1) \sum_{p\in l_1\cap J} S_1(p) + \frac{1}{2}\sum_{l_2\in\mathcal{L}_2} f_2(l_2) \sum_{p\in l_2\cap J} S_2(p)\right] \leq 1. \tag{2.7}$$

Let (S_1, S_2) be a minimiser. The choice of minimiser is independent of the functions f_1, f_2 and hence (S_1, S_2) satisfies part *(ii)* of Theorem 2.2.1. Therefore the pair (S_1, S_2) factorises S, as desired. In conclusion, for the given sets of lines $\mathcal{L}_1, \mathcal{L}_2$, computing a minimiser (S_1, S_2) for the min sup on the left-hand side of (2.7) will yield a factorising pair.

2.2.3 Example Factorisations

Suppose we do not know *a priori* whether (2.3) holds, and let us reconsider the discussion preceding Lemma 2.2.2. In this subsection we present some numerical experiments which demonstrate that it is, in principle, possible to establish geometric multilinear inequalities, such as inequality (2.3), by brute force computation alone and without considering any minimax theorem.

Let $\mathcal{L}_1$ be the set of thin lines and $\mathcal{L}_2$ the set of thick lines depicted in Figure 2.1. Consider the normalised test functions $(S^{(1)}(p_i))_{1\leq i\leq 5} = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$, and $(S^{(2)}(p_i))_{1\leq i\leq 5} = \frac{1}{\sqrt{20}}(1, 1, 4, 1, 1)$. Since we are not using a minimax theorem, we will not introduce a supremum over the positive functions f_j as in Subsection 2.2.2. Thus, for the sake of this example, we let both f_1 and f_2 be constant and normalised on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. This choice is arbitrary. For both of the example functions S, in order to find factorisations, we compute (using

MATLAB) a pair (S_1, S_2) which minimises

$$\min_{(S_1,S_2)\in\mathcal{C}_S} \frac{1}{6}\sum_{l_1\in\mathcal{L}_1}\sum_{p\in l_1\cap J} S_1(p) + \frac{1}{6}\sum_{l_2\in\mathcal{L}_2}\sum_{p\in l_2\cap J} S_2(p). \tag{2.8}$$

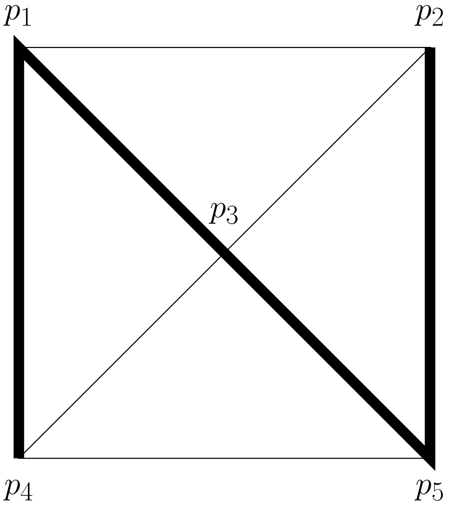

Figure 2.1: Example multijoint configuration with two sets of lines given by thin and thick, respectively.

Minimisers for this problem with test functions $S = S^{(1)}, S^{(2)}$ – computed with the aid of MATLAB – are described in Table 2.1 and Table 2.2, respectively. We provide graphical representations of the respective minimisers in Figure 2.2. In this representation, for each $1 \leq i \leq 5$, the values of the factorising functions S_1, S_2 at p_i are proportional to the length of the thin and thick line segments, respectively, which originate from p_i. Further examples can be found in Appendix A.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)}S_2^{(1)}}$	$S^{(1)}$
p_1	0.723	0.276	0.457	0.457
p_2	0.276	0.723	0.457	0.457
p_3	0.447	0.457	0.457	0.457
p_4	0.276	0.723	0.457	0.457
p_5	0.723	0.276	0.457	0.457

Table 2.1: Minimisers of (2.8) with $S = S^{(1)}$. Values rounded to 3 decimal places.

Despite the fact that we made an arbitrary choice and fixed f_1 and f_2, the values in Table 2.1 and Table 2.2 allow us to verify that the resulting minima (S_1, S_2) satisfy

$$\sum_{p\in l_j} S_j(p) \leq 1 \tag{2.9}$$

for every $l_j \in \mathcal{L}_j$ and $j = 1, 2$. For each $j = 1, 2$, let $g_j : \mathcal{L}_j \to \mathbb{R}_{\geq 0}$ be such that $\sum_{l_j} g_j(l_j) = 1$. We deduce from (2.9) that each pair (S_1, S_2) also satisfies

$$\sum_{l_j \in \mathcal{L}_j} g_j(l_j) \sum_{p \in l_j} S_j(p) \leq \|S\|_2.$$

	$S_1^{(2)}$	$S_2^{(2)}$	$\sqrt{S_1^{(2)} S_2^{(2)}}$	$S^{(2)}$
p_1	0.947	0.053	0.224	0.224
p_2	0.053	0.947	0.224	0.224
p_3	0.894	0.894	0.894	0.894
p_4	0.053	0.947	0.224	0.224
p_5	0.947	0.053	0.224	0.224

Table 2.2: Minimisers of (2.8) with $S = S^{(2)}$. Values rounded to 3 decimal places.

Hence, the pairs (S_1, S_2) satisfy part *(ii)* of Theorem 2.2.1, thus factorising the test functions $S^{(1)}$ and $S^{(2)}$.

In summary, this subsection demonstrates that it is possible, in principle, to prove inequality (2.3) by brute force computation only. This discussion remains valid for $d \geq 2$.

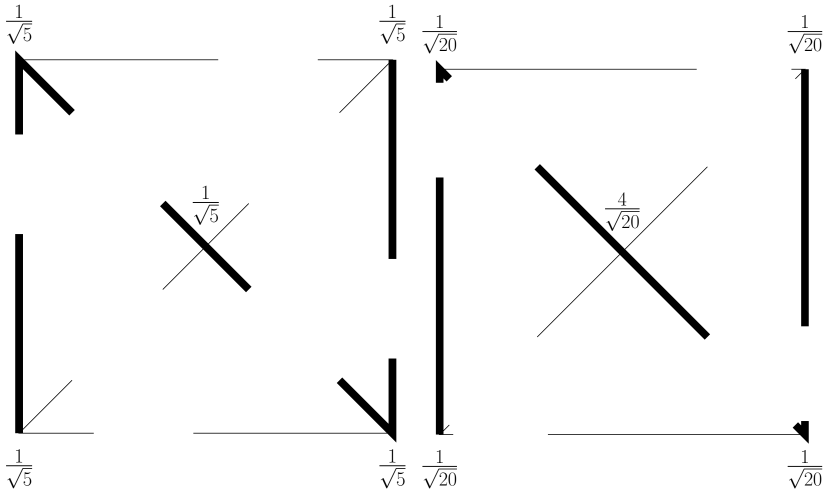

Figure 2.2: Representation of solution with $(S(p_i))_{1 \leq i \leq 5} = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$ (left) and $(S(p_i))_{1 \leq i \leq 5} = \frac{1}{\sqrt{20}}(1, 1, 4, 1, 1)$ (right).

2.3 The Dual Multijoint Problem

Having briefly considered problems of transversal lines, we return to a more general setting. Abstractly, for any geometric multilinear inequality with some integral kernel K, we can derive the sufficiency statement Proposition 2.3.1, below.

Proposition 2.3.1. *Let $X, Y_1, \dots, Y_d$ be measure spaces and let $K : X \times Y_1 \times \dots \times Y_d \to \mathbb{R}_{\geq 0}$. Suppose there is a constant $A > 0$ such that for every non-negative $S \in L^{p'}(X)$, there are functions $S_j : X \times Y_j \to \mathbb{R}_{\geq 0}$ so that*

$$K(x, \boldsymbol{y})S(x) \leq \prod_{j=1}^{d} S_j(x, y_j)^{\alpha_j} \tag{2.10}$$

for a.e. $x \in X$ and $\boldsymbol{y} \in Y_1 \times \dots \times Y_d$, and so that

$$\int_X S_j(x, y_j)\, \mathrm{d}x \leq A\|S\|_{p'} \tag{2.11}$$

for a.e. $y_j \in Y_j$ and every $1 \leq j \leq d$. Then

$$\left\| \left(\int_{Y_1 \times \dots \times Y_d} K(\cdot, \boldsymbol{y}) \prod_{j=1}^{d} f_j(y_j)^{\alpha_j}\, \mathrm{d}\boldsymbol{y} \right) \right\|_{L^p(X)} \leq A \prod_{j=1}^{d} \|f_j\|_{L^1(Y_j)}^{\alpha_j} \tag{2.12}$$

for all non-negative $f_j \in L^1(Y_j)$ for each $1 \leq j \leq d$.

If the hypotheses of Proposition 2.3.1 are satisfied, then we again say that the functions $(S_j)_j$ factorise the test function S.

Remark. The hypotheses for Proposition 2.3.1 contain a single constant, A. It is precisely the same constant as is in (2.12). ◀

Proof. We establish inequality (2.12) by integrating against an arbitrary non-negative test function $S \in L^{p'}(X)$. By hypothesis, we can find functions S_j satisfying (2.10) and (2.11). Hence

$$\begin{aligned}
&\int_X \left(\int_{Y_1 \times \dots \times Y_d} K(x, \boldsymbol{y}) \prod_{j=1}^{d} f_j(y_j)^{\alpha_j}\, \mathrm{d}\boldsymbol{y} \right) S(x)\, \mathrm{d}x \\
= &\int_X \int_{Y_1 \times \dots \times Y_d} K(x, \boldsymbol{y}) S(x) \prod_{j=1}^{d} f_j(y_j)^{\alpha_j}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}x \\
\leq &\int_X \int_{Y_1 \times \dots \times Y_d} \prod_{j=1}^{d} \big(S_j(x, y_j) f_j(y_j)\big)^{\alpha_j}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}x
\end{aligned}$$

by (2.10). By Hölder's inequality, this is at most

$$\begin{aligned}
&\prod_{j=1}^{d}\left(\int_X\int_{Y_j} S_j(x,y_j)f_j(y_j)\,\mathrm{d}y_j\,\mathrm{d}x\right)^{\alpha_j}\\
=\ &\prod_{j=1}^{d}\left(\int_{Y_j} f_j(y_j)\int_X S_j(x,y_j)\,\mathrm{d}x\,\mathrm{d}y_j\right)^{\alpha_j}\\
\leq\ &\prod_{j=1}^{d}\left(A\|S\|_{p'}\int_{Y_j} f_j(y_j)\,\mathrm{d}y_j\right)^{\alpha_j}\\
=\ &A\|S\|_{p'}\prod_{j=1}^{d}\|f_j\|_1^{\alpha_j},
\end{aligned}$$

where we have applied (2.11). This concludes the proof. □

Remark. Observe that the functions f_j sit idly throughout the manipulations in this proof. This motivates a heuristic whereby the functions $S_1,\dots,S_d$ describe intrinsic properties of the kernel K. ◀

Stated in this generality, the converse to Proposition 2.3.1 is false. Indeed, a counterexample is provided in [CHV20a, Proposition 8.1]. Nevertheless, we show in Chapter 5 that there is a duality for the (non-transversal) joints problem which encompasses Proposition 2.3.1. Specifically, we will prove Theorem 2.3.2, below.

It is of great interest that there is a duality for the joints problem, which we will prove in Chapter 5, given that the joints problem does not express the boundedness of a linear operator, nor a multilinear operator.

Theorem 2.3.2 (Duality for Joints). *Let $\mathcal{L}$ be a set of lines in $\mathbb{F}^d$ and let J be the associated set of joints. Then, there are constants $C_1, C_2 > 0$, depending only on d, so that the following are equivalent:*

(i) For all $f:\mathcal{L}\to\mathbb{R}_{\geq 0}$,

$$\sum_{p\in J}\left(\sum_{\boldsymbol{l}\in\mathcal{L}^d}\delta(p,\boldsymbol{l})\prod_{l\in\boldsymbol{l}}f(l)\right)^{\frac{1}{d-1}}\leq C_1\left(\sum_{l\in\mathcal{L}}f(l)\right)^{\frac{d}{d-1}}.$$

(ii) For all $S:J\to\mathbb{R}_{\geq 0}$ there exists a function $s:J\times\mathcal{L}\to\mathbb{R}_{\geq 0}$ so that

$$\delta(p,\boldsymbol{l})S(p)^d\leq C_2\prod_{l\in\boldsymbol{l}}s(p,l)$$

for all $p\in J$ and $\boldsymbol{l}\in\mathcal{L}^d$, and

$$\sum_{p\in l\cap J}s(p,l)\leq\|S\|_d$$

for all $l\in\mathcal{L}$.

In the next section, we place this notion of duality in a historical context. Moving forward, we will reserve the prefix "multi", for multilinear problems which account for d sets of lines, $\mathcal{L}_1, \ldots, \mathcal{L}_d$.

Remark. It follows that whether or not duality of this type holds, depends on the spaces $X, Y_1, \ldots, Y_d$, and especially the integral kernel K. In Chapter 4, we will show that both the joints problem and the multijoint problem follow from the existence of the factorisations detailed by Theorem 2.3.2, item *(ii)*. Hence, although combinatorially distinct, the joints problem and the multijoint problem both reduce to the same dual statement which elucidates properties of the kernel δ, or more specifically, the discrete wedge product. ◀

Therefore, the (multi)joint problem, with sets of lines $\mathcal{L}_1, \ldots, \mathcal{L}_d \subset \mathbb{F}^d$ and multijoints J, is completely equivalent to the following question, which we solve (independently of Theorem 2.3.2) in Chapter 3.

> **Dual Multijoint Problem.** *Let* $S : J \to \mathbb{R}_{\geq 0}$*. For each* $1 \leq j \leq d$*, does there exist a function* $S_j : J \times \mathcal{L}_j \to \mathbb{R}_{\geq 0}$ *so that:*
>
> - $\delta(p, \boldsymbol{l})S(p)^d \leq \prod_{j=1}^{d} S_j(p, l_j)$ *for all* $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$*, and*
> - $\sum_{p \in l_j \cap J} S_j(p, l_j) \lesssim \|S\|_d$ *for all* $1 \leq j \leq d$ *and* $l_j \in \mathcal{L}_j$*?*

2.4 Remarks on the History of Joints – A Dual Perspective

Dvir's elegant proof of the finite field Kakeya conjecture, [Dvi09], is a celebrated application of the polynomial method, which we discuss in more detail below. If $\mathbb{F}$ is a finite field, then a set $E \subset \mathbb{F}^d$ is a **Kakeya set** if for all $b \in \mathbb{F}^d \setminus \{0\}$, there exists $a \in \mathbb{F}^d$ so that the line $a + b\mathbb{F} \subset E$.

Theorem 2.4.1 (Dvir, [Dvi09])**.** *Let* $\mathbb{F}$ *be a finite field and let* $E \subset \mathbb{F}^d$ *be a Kakeya set. Then*

$$|E| \geq C|\mathbb{F}|^d$$

where C *depends only on* d*.*

The proof of Theorem 2.4.1 is so short that its essence can be summarised in a few sentences. Suppose for a contradiction that the conclusion does not hold. By parameter counting, we can find a non-zero polynomial $f \in \mathbb{F}[x_1, \ldots, x_d]$ of degree smaller than $|\mathbb{F}|$ that vanishes on E. However, since E is a Kakeya set, we deduce that f has a large number of zeros. Since f has low degree, we deduce that it must be the zero polynomial – a contradiction.

Almost every contribution to the joints problem involves some modification of Dvir's method. Although not explicit in the literature, these polynomial-based arguments can be seen to approach problems from the perspective of their respective duals. Indeed, they prove the existence of functions with properties that are very similar to those described by Theorem 2.2.1 and Theorem 2.3.2.

Let g be an "averaging" operator of one or more arguments, such as an arithmetic mean, geometric mean or maximum. Suppose, for any test function S, that there exist functions $(S_j)_j$ which satisfy

- $S(x) \leq Ag[S_1, \ldots, S_d](x)$ for a.e. $x \in X$, and
- $\int_X T_j f_j(x) S_j(x, y_j)\, \mathrm{d}x \leq \|S\|_{p'} \|f_j\|_1$ for all $f_j \in L^1(Y_j)$, a.e. $y_j \in Y_j$ and $1 \leq j \leq d$.

We say that the functions $(S_j)_j$ **factorise** the test function S with respect to g. Crucially, the factorisation $(S_j)_j$ is independent of the functions $(f_j)_j$.

Consider the multijoint problem where each set $Y_1 = \mathcal{L}_1, \ldots, Y_d = \mathcal{L}_d$ is finite. Since the second bullet, above, holds for any f_j supported at a singleton in $\mathcal{L}_j$, it is equivalent that there exist functions $(S_j)_j$ which satisfy

- $S(p) \leq Ag[S_1, \ldots, S_d](p)$ for all $p \in J$, and
- $\sum_{p \in l_j \cap J} S_j(p, l_j) \leq \|S\|_d$ for all $l_j \in \mathcal{L}_j$ and $1 \leq j \leq d$.

In the remainder of this section, we revisit landmark contributions to the joints problem. We phrase key results in a dual setting to understand those contributions from a new perspective. In particular, we continue with the following variants of the questions that we have in mind:

- What if the test function S is constant?
- How does the "size" of the averaging operator g affect the question?
- What if g is non-uniformly weighted? What do the weights depend on?

However, the hierarchy of dual statements is primarily determined by the "size" of the averaging operator g, where small averages, such as the geometric mean, are more challenging than large averages, such as the arithmetic mean or maximum. Until now, a dual result for an averaging operator, given by the geometric mean with weights depending only on dimension – Theorem 3.0.1, which we prove in Chapter 3 – was not known.

2.4(a) Quilodrán's Lemma

A major early contribution to the joints problem was due to Quilodrán in [Qui10]. He proved that (1.1) holds with $q = d-1$, thereby establishing the endpoint joints theorem in $\mathbb{R}^d$. Lemma 2.4.2, below, is precisely Quilodrán's original result, [Qui10, Proposition 1], written in the language of the dual problem.

Lemma 2.4.2. *Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^d$. There is a function $s : J \times \mathcal{L} \to [0, 1]$ so that*

$$|J|^{-\frac{1}{d}} \lesssim \max_{l : l \ni p} s(p, l),$$

for every $p \in J$*, and so that*

$$\sum_{p \in l \cap J} s(p,l) \leq 1$$

for all $l \in \mathcal{L}$*.*

Lemma 2.4.2 is, indeed, sufficient to establish inequality (1.1) with $q = d-1$, since

$$\begin{aligned} |J|^{\frac{d-1}{d}} &= \sum_{p \in J} |J|^{-\frac{1}{d}} \\ &\lesssim \sum_{p \in J} \max_{l \ni p} s(p,l) \\ &\leq \sum_{p \in J} \sum_{l \in \mathcal{L}: l \ni p} s(p,l) \\ &= \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} s(p,l) \\ &\leq |\mathcal{L}|. \end{aligned}$$

Remark. With respect to the dual problem, if $S(p) = |J|^{-\frac{1}{d}}$ for all p, then $S : J \to \mathbb{R}_{\geq 0}$ satisfies $\|S\|_d = 1$. Hence, to show that Lemma 2.4.2 is sufficient to prove (1.1), as above, we sum the characteristic function χ_J against the test function $S(p) = |J|^{-\frac{1}{d}}$. That is, to deduce (1.1), which does not account for multiplicity nor multilinearity, it suffices to factorise the constant test function only, with respect to an averaging operator with uniform weights. ◀

Lemma 2.4.2 follows from the cornerstone result of Quilodrán's original proof, which is stated as follows in Lemma 2.4.3.

Lemma 2.4.3 (Quilodrán, [Qui10, Proposition 1])**.** *Let* $\mathcal{L}$ *be a set of lines in* $\mathbb{R}^d$*. For each* $p \in J$*, choose a line* $\tilde{l}(p) \in \mathcal{L}$ *which contains* p*. Of all choices* $\tilde{l} : J \to \mathcal{L}$*, there is one such that for every* $l \in \mathcal{L}$*,*

$$\left| p \in l \cap J : \tilde{l}(p) = l \right| \leq C|J|^{\frac{1}{d}}, \tag{2.13}$$

for a constant C*, depending only on* d*.*

For $p \in J$ and $l \in \mathcal{L}$, let us set

$$s(p,l) := \begin{cases} C^{-1}|J|^{-\frac{1}{d}} & \tilde{l}(p) = l \\ 0 & \text{otherwise.} \end{cases}$$

Since every $p \in J$ has some line l so that $\tilde{l}(p) = l$,

$$C^{-1}|J|^{-1/d} = s(p, \tilde{l}(p)) = \max_{l : l \ni p} s(p,l).$$

Inequality (2.13) divided by $C|J|^{1/d}$ becomes $\sum_{p \in l \cap J} s(p,l) \leq 1$. Hence, s satisfies the conclusion of Lemma 2.4.2, thus establishing the joints inequality $|J|^{\frac{d-1}{d}} \leq C|\mathcal{L}|$.

2.4(b) The Lemma of Carbery and Valdimarsson

Following Quilodrán's success, it was observed by Carbery and Valdimarsson that Quilodrán's method operated within this dual framework, [CV14]. The averaging operator in Quilodrán's result, Lemma 2.4.2, is a maximum over at most $|\mathcal{L}|$ numbers. Carbery and Valdimarsson improve on this by establishing a result for an average over $\lesssim 1$ arguments. More importantly, they reduced the "size" of the averaging operator from a maximum to an arithmetic mean.

Lemma 2.4.4 (Carbery–Valdimarsson, [CV14])**.** *Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be transversal sets of lines in $\mathbb{F}^d$ with multijoints J. Then there exists a constant $C = C(d) > 0$, and a function $\kappa : J \to \{1, \ldots, d\}$ such that*

$$\left|\{p \in l_j \cap J : \kappa(p) = j\}\right| \leq C|J|^{\frac{1}{d}}$$

for all $l_j \in \mathcal{L}_j$ and $1 \leq j \leq d$.

Remark. Interestingly, the Carbery–Valdimarsson argument did not use the polynomial method. ◀

Written from the dual perspective, Lemma 2.4.4 reads as follows.

Lemma 2.4.5. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be transversal sets of lines in $\mathbb{F}^d$ with multijoints J. Then for each $1 \leq j \leq d$, there is a function $S_j : J \times \mathcal{L}_j \to [0,1]$ so that*

$$|J|^{-\frac{1}{d}} \lesssim \sum_{j=1}^{d} S_j(p, l_j)$$

for all $p \in J$ and $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ which form a multijoint at p, and

$$\sum_{p \in l_j \cap J} S_j(p, l_j) \leq 1$$

for all $l_j \in \mathcal{L}_j$ and $1 \leq j \leq d$.

Provided that the transversal sets of lines satisfy $|\mathcal{L}_1|, \ldots, |\mathcal{L}_d| = L$, Lemma 2.4.5 is sufficient to deduce that $|J| \lesssim L^{\frac{d}{d-1}}$. Under the constraint $|\mathcal{L}_1|, \ldots, |\mathcal{L}_d| = L$, this inequality is equivalent to the result of Quilodrán, [Qui10], despite the improved dual statement. Similar to Lemma 2.4.2, the Carbery–Valdimarsson factorisation lemma factorises the uniform test function with respect to an averaging operator with uniform weights.

2.4(c) Zhang's Lemma

In 2017, Zhang proved the multijoint conjecture in [Zha20]. That is, he established (1.4) with $q = d/(d-1)$. His proof used the polynomial method, and his arguments can also be seen from a dual perspective. His proof of the full multilinear problem with multiplicity reduces the problem to one where each $\mathcal{L}_j = \mathcal{L}$ is the same set. In [Zha20], to demonstrate his new technique, Zhang proved the weaker inequality, $|J| \lesssim \prod_{j=1}^{d} \|f_j\|_{\ell^1(\mathcal{L}_j)}^{1/(d-1)}$ for arbitrary functions $f_j : \mathcal{L}_j \to \mathbb{Z}_{\geq 0}$. The corresponding factorisation lemma is as follows in Lemma 2.4.6 and summarises the critical aspects from Zhang's proof of [Zha20, Theorem 1.2].

Lemma 2.4.6 (Zhang, [Zha20])**.** *Let $\mathbb{F}$ be a field. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d \subset \mathbb{F}^d$ be sets of lines with multijoints J. There is a function $s : J \times (\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d) \to \mathbb{R}_{\geq 0}$ such that for every $p \in J$ there is an index $1 \leq j(p) \leq d$ and a line $l_{j(p)}(p) \in \mathcal{L}_{j(p)}$ containing p, so that*

$$\frac{1}{\left(|J| \prod_{j=1}^{d} |\mathcal{L}_j|\right)^{\frac{1}{d}}} \lesssim \frac{1}{|\mathcal{L}_{j(p)}|} s(p, l_{j(p)}(p)),$$

and moreover,

$$\sum_{p \in l \cap J} s(p, l) \leq 1$$

for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$.

Compare Lemma 2.4.6 to Lemma 2.4.5. Both lemmas factorise the constant test function. However, where Lemma 2.4.5 uses an averaging operator with uniform weights, in contrast, Lemma 2.4.6 uses an "averaging" operator with weights $|\mathcal{L}_j|^{-1}$.

Lemma 2.4.6 is a further multilinear generalisation of Lemma 2.4.2. Equipped with this result, for each $1 \leq j \leq d$, let $J_j = \{p : j(p) = j\}$. By the pigeonhole principle, there is an index $1 \leq j_0 \leq d$ so that $|J| \lesssim \left|J_{j_0}\right|$. It follows that

$$\begin{aligned}
|J| \frac{1}{\left(|J| \prod_{j=1}^{d} |\mathcal{L}_j|\right)^{\frac{1}{d}}} &\lesssim \left|J_{j_0}\right| \frac{1}{\left(|J| \prod_{j=1}^{d} |\mathcal{L}_j|\right)^{\frac{1}{d}}} \\
&= \sum_{p \in J_{j_0}} \frac{1}{\left(|J| \prod_{j=1}^{d} |\mathcal{L}_j|\right)^{\frac{1}{d}}} \\
&\lesssim \sum_{p \in J_{j_0}} \frac{1}{\left|\mathcal{L}_{j_0}\right|} s(p, l_{j_0}(p)) \\
&\leq \sum_{p \in J} \sum_{l_{j_0} \in \mathcal{L}_{j_0} : p \in l_{j_0}} \frac{1}{\left|\mathcal{L}_{j_0}\right|} s(p, l_{j_0}) \\
&= \sum_{l_{j_0} \in \mathcal{L}_{j_0}} \frac{1}{\left|\mathcal{L}_{j_0}\right|} \sum_{p \in l_{j_0} \cap J} s(p, l_{j_0}) \\
&\leq \sum_{l_{j_0} \in \mathcal{L}_{j_0}} \frac{1}{\left|\mathcal{L}_{j_0}\right|} \\
&= 1.
\end{aligned}$$

It follows that,

$$|J| \lesssim \prod_{j=1}^{d} |\mathcal{L}_j|^{\frac{1}{d-1}}.$$

While not stated explicitly, the dual-type statement, Lemma 2.4.7 below, may be extracted from [Zha20]. The displayed inequalities in Lemma 2.4.7 follow from properties of a particular polynomial, as described in the discussion around [Zha20, Inequality (5.6)].

Lemma 2.4.7 (Zhang, [Zha20])**.** *Let $\mathcal{L}$ be a set of lines in $\mathbb{F}^d$, and let $f : \mathcal{L} \to \mathbb{R}_{\geq 0}$. Let J be the set of joints and let $M[f](p) := \sum_{\boldsymbol{l}} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l)$. There exists a function $s : J \times \mathcal{L} \to \mathbb{R}_{\geq 0}$ so that*

$$\left(\sum_{p \in J} M[f](p)^{\frac{1}{d-1}}\right)^{-\frac{1}{d}} M[f](p)^{\frac{1}{d-1}} \leq C \sum_{l \in \mathcal{L} : l \ni p} f(l) s(p, l)$$

for all $p \in J$, and

$$\sum_{p \in l \cap J} s(p, l) \leq 1$$

for all $l \in \mathcal{L}$.

Lemma 2.4.7, which solved the joints problem, can now be seen as a dual statement for a test function with non-constant weight and includes an f-weighted averaging operator. In particular, the first inequality in Lemma 2.4.7 is averaged over subsets of $\mathcal{L}$, whereas the first inequality in Lemma 2.4.5 holds pointwise over $\mathcal{L}_1 \times \cdots \times \mathcal{L}_d$.

Remark. Compare Lemma 2.4.7 to item *(ii)* of Theorem 2.3.2. Both results consider the joints problem so that only one arbitrary function f needs to be considered, rather than a d-tuple of arbitrary functions as in the multilinear multijoint problem. In this context, one may describe the joints problem as "pseudo-linear". This apparent simplification allows for the factorisation of non-uniform test functions. ◀

Lemma 2.4.7 proves a stronger result than Lemma 2.4.2. The first inequality in Lemma 2.4.2 bounds the constant function, and consequently we can only hope to bound $\sum_{p\in J} 1$. However, the first inequality in Lemma 2.4.7 places a bound on the multiplicity function M, which allows us to prove the stronger statement (1.4) with $q = d/(d-1)$.

Indeed, Lemma 2.4.7 is sufficient to establish (1.4) with $q = d/(d-1)$. Applying the first inequality yields

$$\left\| M[f]^{\frac{1}{d}} \right\|_{d/d-1} = \left(\sum_{q\in J} M[f](q)^{\frac{1}{d-1}} \right)^{-\frac{1}{d}} \sum_{p\in J} M[f](p)^{\frac{1}{d-1}} \le C \sum_{p\in J} \sum_{l:l\ni p} f(l) s(p,l).$$

Changing the order of summation and applying the second inequality, this is at most

$$C \sum_{l\in\mathcal{L}} f(l) \sum_{p\in l\cap J} s(p,l) \le C \sum_{l\in\mathcal{L}} f(l) = C\|f\|_{L^1(\mathcal{L})},$$

as desired.

2.4(d) Yang's Proof

In 2016, Yang established higher-dimensional analogues of the multijoint problem with ε-loss, [Yan16]. Namely, he proved the following.

Theorem 2.4.8 (Yang, [Yan16])**.** *Let $\varepsilon > 0$. Let $\Gamma_1, \ldots, \Gamma_d$ be sets of k_1-$, \ldots, k_d$-planes, respectively, in $\mathbb{R}^n$ so that $k_1 + \ldots + k_d = n$, and let J be the set of multijoints formed by tuples $\gamma \in \Gamma_1 \times \cdots \times \Gamma_d$. Then*

$$|J| \lesssim_{\varepsilon,n,d,k_1,\ldots,k_d} \prod_{j=1}^{d} \left|\Gamma_j\right|^{\frac{1}{d-1}+\varepsilon}.$$

Yang's work is specific to $\mathbb{R}^d$ as it invokes polynomial partitioning, and even implicitly, does not relate to multilinear duality, Theorem 2.2.1. In the special case of transversal lines, he established (1.4) with $q = (d-1) - \varepsilon$. However, his argument can be adapted (without applying Theorem 2.2.1 directly) to work in the dual setting to prove the following. We will not include a proof.

Theorem 2.4.9. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_d \subset \mathbb{R}^d$ be sets of transverse lines with multijoints J, and $\varepsilon > 0$. Let $S : J \to \mathbb{R}_{\ge 0}$. Then, for all $1 \le j \le d$, there are functions S_j*

so that

$$S(p) \leq \prod_{j=1}^{d} S_j(p)^{\frac{1}{d}}$$

for all $p \in J$*, and*

$$\sum_{p \in l_j \cap J} S_j(p) \leq C \left(\sum_{p \in J} S(p)^{d-\varepsilon} \right)^{\frac{1}{d-\varepsilon}}$$

for all $l_j \in \mathcal{L}_j$ *and* $1 \leq j \leq d$*, where* C *depends only on* d *and* ε*.*

Remark. This dual statement accounts for an arbitrary test function S, and the averaging operator is the desired geometric mean. However, the inductive nature of Yang's arguments necessitates ε-loss in the second displayed inequality of Theorem 2.4.9, above. ◀

Hablicsek provided an earlier proof of (1.4) for transversal lines in [Hab14]. This work does not obviously present itself as relating to the dual multijoint problem.

2.4(e) The Lemma of Tidor, Yu And Zhao

In 2020, the work of Tidor, Yu and Zhao, [TYZ20], further generalised the multijoint problem to prove (1.5), the multijoint inequality for collections of algebraic varieties of any degree and any dimension. A key achievement in this work is to upgrade the arithmetic means from Lemma 2.4.2, Lemma 2.4.5 and Lemma 2.4.7 to a weighted geometric mean. While not stated explicitly in their work, the following result, Theorem 2.4.10, may be deduced as a consequence of [TYZ20, Lemma 5.10], stated in the special case of lines.

Theorem 2.4.10. *Let* $\mathcal{L}_1, \dots, \mathcal{L}_d$ *be finite sets of lines in* $\mathbb{F}^d$ *with associated multijoints* J*, and for each* $1 \leq j \leq d$*, let* $f_j : \mathcal{L}_j \to \mathbb{R}_{\geq 0}$*. Then, there is a constant* $C = C(d)$ *so that for all* $S : J \to \mathbb{R}_{\geq 0}$*, there exists a function* $s : J \times (\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d) \to \mathbb{R}_{\geq 0}$ *so that*

$$S(p)^d \leq C \prod_{\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d} \left(\prod_{j=1}^{d} s(p, l_j) \right)^{\omega(\boldsymbol{l})}$$

for all $p \in J$*, where* $\omega(\boldsymbol{l}) = \frac{\delta(p,\boldsymbol{l}) \prod_{j=1}^{d} f_j(l_j)}{\sum_{\boldsymbol{l}'} \delta(p,\boldsymbol{l}') \prod_{j'=1}^{d} f_{j'}(l'_{j'})}$*, and so that*

$$\sum_{p \in l \cap J} s(p, l) = \|S\|_d$$

for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$*.*

Theorem 2.4.10 features a geometric mean as desired. However, the number of factors is too large and depends on the sets $\mathcal{L}_j$ and functions f_j. Moreover, the weights $\omega(\boldsymbol{l})$ depend on the functions $f_1, \ldots, f_d$.

Remark. Consider the averages in Lemma 2.4.2, Lemma 2.4.5, Lemma 2.4.7, Theorem 2.4.10 and Theorem 2.1.3. We can order the sizes of these averages as follows,

$$\text{Geometric Mean} \leq \text{Arithmetic Mean} \leq \text{Maximum},$$

which is the reverse of the chronological order of their respective proofs. This agrees with the heuristic that these dual problems first look to find functions with large averages, since this is harder to achieve for smaller averages. The earliest result, Lemma 2.4.2, features a maximum. This was followed by Lemma 2.4.7 and Lemma 2.4.5, which feature arithmetic means and thereafter, Theorem 2.4.10 which features an f-dependent geometric mean. Finally, Theorem 2.1.3, proved in this thesis, involves the classical geometric mean, and is independent of the weights f. ◀

2.5 The Polynomial Method

It is truly remarkable that, aside from elementary functional analysis, we only require elementary facts about polynomials to handle the multijoint problem. This section introduces the main tools we will employ, and motivates the arguments that follow.

2.5.1 Toolkit

Moving forward, we will work in an arbitrary field $\mathbb{F}$ and any derivative will be the **Hasse** derivative. All arguments remain valid when $\mathbb{F} = \mathbb{R}$ with the usual derivative operator.

Let $f \in \mathbb{F}[x_1, \ldots, x_d]$ and $\alpha \in \mathbb{N}^d$ be a multi-index. The α-th Hasse derivative of f is the coefficient of the monomial $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ in the polynomial $p(x + z) \in \big(\mathbb{F}[x_1, \ldots, x_d]\big)[z_1, \ldots, z_d]$. This is denoted by $D^\alpha f$. For further details see [DKSS13] or [CI20].

Let $\lambda \in \mathbb{N}$ and define

$$\mathbb{F}_\lambda[x_1, \ldots, x_d] := \{f \in \mathbb{F}[x_1, \ldots, x_d] : \deg f \leq \lambda\}.$$

Moreover, if X is a vector space, let X^* denote its dual vector space. The following fact about spaces of polynomials, Proposition 2.5.1, follows from the so-called "stars and bars" argument which we include for completeness.

Proposition 2.5.1. *For any field* $\mathbb{F}$ *and* $\lambda \in \mathbb{N}$, $\dim \mathbb{F}_\lambda[x_1, \ldots, x_d] = \binom{\lambda+d}{d}$. *Hence* $\dim \Big(\big(\mathbb{F}_\lambda[x_1, \ldots, x_d]\big)^*\Big) = \binom{\lambda+d}{d}$.

Proof. The dimension of $\mathbb{F}_\lambda[x_1, \ldots, x_d]$ is equal to the number of tuples, $\alpha = (\alpha_1, \ldots, \alpha_d)$, of non-negative integers so that $\alpha_1 + \ldots + \alpha_d \leq \lambda$. Such tuples are in one-to-one correspondence with $(\lambda+d)$-tuples which contain precisely λ stars and d bars. For each $1 \leq j < d$, let α_j be equal to the number of stars between the j-th and $(j+1)$-st bars, and let α_d be the number of stars after the d-th line. For example,

$$\star\star\,||\,\star\,|\,\star\star \quad \leftrightarrow \quad x_2x_3^2 \in \mathbb{F}_5[x_1,x_2,x_3].$$

The number of such $(\lambda+d)$-tuples is precisely $\binom{\lambda+d}{d}$. □

We will also make use of the following property of finite dimensional vector spaces. For a vector space X, let $x \in X$ and $\phi \in X^*$. We adopt the notation $\langle \phi, x\rangle := \phi(x)$ to emphasise the role of vector space duality throughout our discussion.

Proposition 2.5.2. *Let X be a finite dimensional vector space and $\phi_1,\dots,\phi_M \in X^*$ be a basis. Then $x \in X$ is zero if and only if $\langle \phi_m, x\rangle = 0$ for all $1 \leq m \leq M$.*

Proof. By definition, $\phi \in Y^*$ is zero if and only if $\psi(\phi) = 0$ for all $\psi \in Y^{**}$. Since X is finite dimensional, X is isomorphic to X^{**}. Hence, we may apply this definition with $Y^* = X$ and $Y^{**} = X^*$ to deduce that $x \in X$ is zero if and only if $\langle \phi, x\rangle = 0$ for all $f \in X^*$. Since X^* is finite dimensional, this holds if and only if $\langle \phi_m, x\rangle = 0$ for all $1 \leq m \leq M$. □

Observe that checking whether a (Hasse) derivative of a low degree polynomial at a point is zero, is equivalent to checking whether $\langle \phi, f\rangle = 0$ for an appropriately chosen $\phi \in \big(\mathbb{F}_\lambda[x_1,\dots,x_d]\big)^*$. By viewing $\mathbb{F}_\lambda[x_1,\dots,x_d]$ as a vector space and applying Proposition 2.5.2, we arrive at Proposition 2.5.3.

Proposition 2.5.3. *Let $\mathbb{F}$ be a field, $\lambda \in \mathbb{N}$ and $X = \mathbb{F}_\lambda[x_1,\dots,x_d]$ and suppose $\phi_1,\dots,\phi_M \in X^*$ are linearly independent. If there exists a non-zero $f \in X$ that satisfies $\langle \phi_m, f\rangle = 0$ for all $1 \leq m \leq M$ then*

$$M < \binom{\lambda+d}{d}.$$

The following result, also commonly used in the polynomial method, is the converse of Proposition 2.5.3.

Lemma 2.5.4 (Parameter Counting)**.** *Let $\mathbb{F}$ be a field and $X = \mathbb{F}_\lambda[x_1,\dots,x_d]$ for $\lambda \in \mathbb{N}$, and suppose $\phi_1,\dots,\phi_M \in X^*$. If $M < \binom{d+\lambda}{d}$, then there is a non-zero $f \in X$ so that $\langle \phi_m, f\rangle = 0$, for each $1 \leq m \leq M$.*

Note that linear dependence is not required in this statement.

Proof of Lemma 2.5.4. The conditions $\langle \phi_m, f\rangle = 0$ for each $1 \leq m \leq M$ describe precisely M homogeneous linear constraints on the coefficients of f. By Proposition 2.5.1, there are $\binom{\lambda+d}{d}$ coefficients. By the Rank-Nullity Theorem, the null space has dimension at least $\binom{\lambda+d}{d} - M$ and by hypothesis, this is strictly positive. Hence, the vector space of f satisfying $\langle \phi_m, f\rangle = 0$ for each $1 \leq m \leq M$ is at least 1-dimensional. In particular, there is a non-zero $f \in \mathbb{F}_\lambda[x_1,\dots,x_d]$ so that $\langle \phi_m, f\rangle = 0$ for all $1 \leq m \leq M$, as desired. □

Combining Proposition 2.5.3 with Lemma 2.5.4, we deduce the following.

Lemma 2.5.5. *Let* $\mathbb{F}$ *be a field,* $\lambda \in \mathbb{N}$ *and* $X = \mathbb{F}_\lambda[x_1, \dots, x_d]$ *and suppose* $\phi_1, \dots, \phi_M \in X^*$. *Then there exists a non-zero* $f \in X$ *that satisfies* $\langle \phi_m, f \rangle = 0$ *for all* $1 \leq m \leq M$ *if and only if* $\{\phi_1, \dots, \phi_M\}$ *does not span* X^*.

Equivalently, suppose in addition that the functionals $\phi_1, \dots, \phi_M$ *are linearly independent. Then there exists a non-zero* $f \in X$ *that satisfies* $\langle \phi_m, f \rangle = 0$ *for all* $1 \leq m \leq M$ *if and only if*

$$M < \binom{\lambda + d}{d}.$$

2.5.2 Quilodrán is a Special Case of Tidor–Yu–Zhao

While the recent Tidor–Yu–Zhao refinement of the polynomial method is novel, [TYZ20, YZ19], we can see their ideas pre-empted in Quilodrán's proof, [Qui10]. Lemma 2.5.6, below, is the result at the heart of Quilodrán's paper, [Qui10], from which the aforementioned Lemma 2.4.2 follows. In this subsection, we re-examine Quilodrán's method and reinterpret his original proof of Lemma 2.5.6 to emphasise how it relates to the new Tidor–Yu–Zhao method.

Lemma 2.5.6 (Quilodrán, [Qui10, Lemma 1])**.** *Let* $\mathcal{L}$ *be a set of lines in* $\mathbb{R}^d$ *with associated joints* J, *and let* $\lambda \in \mathbb{N}$. *Let* $J' \subseteq J$ *be such that for all* $l \in \mathcal{L}$, *if* $l \cap J' \neq \emptyset$ *then* $|l \cap J'| \geq \lambda$. *Then* $|J'| \geq C\lambda^d$ *for a constant* C, *depending only on* d.

Lemma 2.5.6 follows from the following proposition and corollary.

Proposition 2.5.7. *Suppose that* J' *is as described in Lemma 2.5.6, and* $f \in \mathbb{R}_{\lambda-1}[x_1, \dots, x_d]$. *If* $f(p) = 0$ *for all* $p \in J'$ *then* f *is the zero polynomial.*

Equivalently, for every non-zero $f \in \mathbb{R}_{\lambda-1}[x_1, \dots, x_d]$, there is some $p \in J'$ so that $f(p) \neq 0$.

Corollary 2.5.7a. *If* J' *is as described in Lemma 2.5.6 then* $|J'| \geq \binom{(\lambda-1)+d}{d}$.

Proof. For a contradiction, suppose otherwise. By parameter counting, we can find a non-zero polynomial f of degree at most $(\lambda - 1)$, such that $f(p) = 0$ for all $p \in J'$. By Proposition 2.5.7, f must be the zero polynomial, a contradiction. □

There is a constant C depending only on d such that $\binom{(\lambda-1)+d}{d} \geq C\lambda^d$, and hence Lemma 2.5.6 follows.

Remark. Quilodrán's paper, [Qui10], includes a proof that Lemma 2.5.6 implies Lemma 2.4.3. However, we want to highlight that the proof of Lemma 2.5.6 can be summarised by the following statement: the dual space $(\mathbb{R}_{\lambda-1}[x_1, \dots, x_d])^*$ is spanned by the maps $\{f \mapsto f(p) : p \in J'\}$. Hence, Lemma 2.4.3, a dual joints statement, follows from a covering statement. ◀

Proof of Proposition 2.5.7. Suppose $f \in \mathbb{R}[x_1, \ldots, x_d]$ is a polynomial of degree at most $(\lambda - 1)$ which vanishes on J'. For any $p \in J'$ and any line $l \in \mathcal{L}$ through p, we have that l contains at least λ points in J'. Since $f|_l(p') = 0$ for all $p' \in l \cap J'$, the polynomial $f|_l$ has at least λ zeros. Since $\deg f \leq (\lambda - 1)$, we have that $f|_l$ must be the zero polynomial. Hence $\big((\nabla f) \cdot e(l)\big)|_l$ is the zero polynomial, where $e(l)$ denotes the direction of l. However, there are d lines, $l_1(p), \ldots, l_d(p) \in \mathcal{L}$, which contain p such that their directions are linearly independent. Hence, $\nabla f(p)$ is orthogonal to d linearly independent vectors, and therefore $\nabla f(p) = 0$.

Since $p \in J'$ was arbitrary, it follows that $\nabla f(p) = 0$ for all $p \in J'$. In particular, all of the components of ∇f are zero on J'. Thus, by repeating this argument on each component of ∇f, we deduce that $\partial_1^{r_1} \cdots \partial_d^{r_d} f(p) = 0$ for all $p \in J'$, and all multi-indices $r = (r_i)_i \in \mathbb{N}^d$. Hence, f is the zero polynomial, as desired. ☐

By assuming that each line $l \in \mathcal{L}$ contains at least λ points, we ensure that the evaluation maps $\{f|_l \mapsto f(p)\}_{p \in l \cap J'}$ span the dual space

$$\big\{f|_l : f \in \mathbb{R}_{\lambda-1}[x_1, \ldots, x_d]\big\}^*$$

for each line $l \in \mathcal{L}$. This spanning property lies very much at the heart of the TIdor–Yu–Zhao method.

2.6 Overview of the Thesis

Having described multilinear duality and introduced the polynomial method, we begin our work in earnest.

In Chapter 3, we adapt the recent contributions to the polynomial method by [YZ19, TYZ20] to establish new multijoint results in the dual domain. We simplify the technical details from [TYZ20] to emphasise the relationship between the dual multijoint problem and the polynomial method. Specifically, the introduction of so-called “handicaps” allows us to choose polynomial vanishing conditions which respect the geometry of the multijoint problem.

The structure of Chapter 4 mimics that of Chapter 3. It adapts the higher-dimensional Tidor–Yu–Zhao generalisations, [TYZ20], to solve the analogous dual problem in greater generality. These two chapters form the basis for a publication, which is currently in preparation.

The results of Chapter 5 were proved jointly with Anthony Carbery. In Chapter 5, we prove Theorem 2.3.2 – a more general duality result, specifically adapted to the joints problem. Of notable interest is Theorem 5.1.1, which describes a property of the discrete wedge product. This theorem is simple to state, and motivates wider questions in multilinear duality. This chapter will also form the basis of a publication, which is currently in preparation.

Chapter 3

The Discrete Bourgain–Guth Theorem

In this chapter we prove our discrete Bourgain–Guth theorem and discuss its analogy to the Bourgain–Guth theorem, [BG11]. Some of the materials in the remaining chapters of this thesis have subsequently been prepared for publication in the preprints [CT22] and [Tan22].

Let us fix the field $\mathbb{F}$, finite or otherwise. The discussion that follows is completely independent of the cardinality of $\mathbb{F}$. We define an equivalence relation $\sim$ on $\mathbb{F}^d \setminus \{0\}$ whereby $x, x' \in \mathbb{F}^d$ satisfy $x \sim x'$, if and only if there is some non-zero $t \in \mathbb{F}$ so that $x = tx'$. We define the **sphere** in $\mathbb{F}^d$, written as $\mathbb{S}^{d-1} := (\mathbb{F}^d \setminus \{0\})/\sim$, to be the set of $\sim$-equivalence classes of $\mathbb{F}^d$. We refer to any element of $\mathbb{S}^{d-1}$ as a **direction**. Technically, $\mathbb{S}^{d-1}$ is the projective space of lines in $\mathbb{F}^d$, however it is convenient to interpret this space as a sphere. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in (\mathbb{S}^{d-1})^d$. Recall that the discrete wedge product is defined by $\wedge_{j=1}^d \omega_j = \omega_1 \wedge \cdots \wedge \omega_d = 1$ if representatives of $\omega_1, \dots, \omega_d$ span $\mathbb{F}^d$, and 0 otherwise.

Let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be sets of lines in $\mathbb{F}^d$. We can now define the multijoint kernel $\delta : \mathbb{F}^d \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_d \to \{0, 1\}$. Let $\boldsymbol{l} = (l_1, \dots, l_d) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ and $p \in \mathbb{F}^d$. Then

$$\delta(p, \boldsymbol{l}) := \left(\prod_{j=1}^d \chi_{l_j}(p) \right) \wedge_{j=1}^d e(l_j),$$

where $e(l_j) \in \mathbb{S}^{d-1}$ is the direction parallel to l_j. Then $p \in \mathbb{F}^d$ is a multijoint if and only if there exists a tuple of lines $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ so that $\delta(p, \boldsymbol{l}) = 1$, and we define $J = \{p : \exists \boldsymbol{l} \text{ s.t. } \delta(p, \boldsymbol{l}) = 1\}$ to be the set of all multijoints. The multijoint multiplicity function $M : J \to \mathbb{N}$ is given by $M(p) = \big|\{\boldsymbol{l} : \delta(p, \boldsymbol{l}) = 1\}\big|$. The multijoint problem asks if there is a constant $C = C(d)$ so that

$$\sum_{p \in J} M(p)^{\frac{1}{d-1}} \leq C \prod_{j=1}^d \big|\mathcal{L}_j\big|^{\frac{1}{d-1}}.$$

By a modification of the Tidor–Yu–Zhao perturbation argument, [TYZ20], we will establish a result that strongly relates to the key result of Carbery and Valdimarsson, which they used to prove the multilinear Kakeya theorem, [CV13].

We will prove the following results in this chapter.

Theorem 3.0.1 (Multijoint Factorisation)**.** *Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be finite sets of lines in $\mathbb{F}^d$ with associated multijoints J. There is a constant $C = C(d)$ so that for all $S : J \to \mathbb{R}_{\geq 0}$ with $\|S\|_d = 1$, there exists a function $s : J \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ so that*

$$\delta(p, \boldsymbol{l})S(p)^d \leq C \prod_{j=1}^{d} s(p, e(l_j)),$$

for all $p \in J$ and $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$, and so that

$$\sum_{p \in l \cap J} s(p, l) = 1$$

for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$.

Remark. The exact equality in the second displayed conclusion is manifest from the proof, and not an *a posteriori* modification. ◀

Thereafter, we will deduce the following, Theorem 3.0.2.

Theorem 3.0.2 (Discrete Bourgain–Guth)**.** *For all finitely supported $S : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$ with $\|S\|_d = 1$, there exists a function $s : \mathbb{F}^d \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ so that*

$$(\omega_1 \wedge \cdots \wedge \omega_d)S(p)^d \lesssim \prod_{j=1}^{d} s(p, \omega_j),$$

for all $p \in \mathbb{F}^d$ and every $\boldsymbol{\omega} \in (\mathbb{S}^{d-1})^d$, and so that,

$$\sum_{p \in l} s(p, e(l)) \leq 1$$

for all lines $l \subset \mathbb{F}^d$.

Theorem 3.0.2 is a freestanding statement. That is, unlike Theorem 3.0.1, it does not refer to any sets of lines $\mathcal{L}_j$ or multijoints. It may be interpreted as describing a property of affine space and the discrete wedge product. One such interpretation is that the product $\prod_j s(\cdot, u_j)$ is a good pointwise approximation, with respect to a weight S, for the discrete wedge product. Alternatively, the finite sets $\mathcal{L}_j$ referred to in Theorem 3.0.2 may be interpreted as finitely supported functions on $\mathbb{S}^{d-1}$.

3.1 Multilinear Restriction in Finite Fields

Before we begin proving the main result, Theorem 3.0.1, we revisit the analogy between the multijoint problem and the multilinear Kakeya theorem. Carbery, Hänninen and Valdimarsson observed that the existence of transversal factorisations for the multilinear Kakeya theorem implies that multilinear restriction estimates in $\mathbb{R}^d$ hold, conditional on the Mizohata–Takeuchi conjecture, [CHV20b, Section 1.1]. Accordingly, in this section, we prove that multilinear restriction estimates in $\mathbb{F}^d$, for any finite field $\mathbb{F}$, hold conditional on a finite field version of the Mizohata–Takeuchi conjecture.

Let $\mathbb{F}$ be a finite field. For polynomials $g_1, \ldots, g_d$, let $E_j \subset \{p : g_j(p) = 0\}$ for each $1 \leq j \leq d$, and let $\sigma_j : E_j \to \mathbb{R}_{\geq 0}$ be the normalised surface measure on each E_j; that is, let σ_j be constant on E_j and unit-normalised. Suppose that

$$\nabla g_1(p_1) \wedge \cdots \wedge \nabla g_d(p_d) = 1$$

for all $p_1 \in E_1, \ldots, p_d \in E_d$, where $\wedge$ denotes the discrete wedge product. Then we say that the sets $E_1, \ldots, E_d$ are **transversal**.

Let $\gamma = \{x_1 = x_2^2 + \ldots + x_d^2\}$ be the paraboloid in $\mathbb{F}^d$ and suppose that the sets $E_1, \ldots, E_d$ are transversal. For each $1 \leq j \leq d$, suppose we can express $E_j = \Gamma_j(U_j)$ for some map Γ_j whose components are polynomials and $U_j \subset \mathbb{F}^{d-1}$.

Remark. The sets E_j may be thought of as "local" neighbourhoods of the paraboloid, in the sense that each E_j is covered by a single chart U_j. ◀

We denote the Fourier extension (dual restriction) operator, acting on a function $f_j : U_j \to \mathbb{C}$ by

$$\mathcal{E}_j f_j(q) := \sum_{p \in U_j} e(2\pi i q \cdot \Gamma_j(p)) f_j(p) \sigma_j(p),$$

where σ_j is the normalised uniform measure on U_j, and e is a non-principal character. In analogy with the Euclidean case, [CHV20b, Section 1.1], we can therefore ask whether or not it is true that

$$\left\| \prod_{j=1}^{d} \left| \mathcal{E}_j f_j \right|^{\frac{2}{d}} \right\|_{d/(d-1)} \lesssim \prod_{j=1}^{d} \left\| f_j \right\|_{L^2(U_j, \mathrm{d}\sigma_j)}^{\frac{2}{d}}$$

for functions $f_j : U_j \to \mathbb{C}$. This is the transversal multilinear restriction conjecture for the paraboloid in finite fields, which remains open.

From Theorem 3.0.2, taking $s_j(p) := \min_{l \in \mathcal{L}_j : l \ni p} s(p, e(l))$, we deduce Theorem 3.1.1.

Theorem 3.1.1 (Transversal Multijoint Factorisation). *Let $\mathbb{F}$ be an arbitrary field. For each $1 \leq j \leq d$, let $\mathcal{L}_j$ be transversal sets of lines. For every $S : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$, there are positive functions $s_j : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$ so that*

$$S(p) \leq \prod_{j=1}^{d} s_j(p)^{\frac{1}{d}}$$

for every $p \in \mathbb{F}^d$, and so that

$$\sum_{p \in l_j} s_j(p) \lesssim \|S\|_d$$

for every $1 \leq j \leq d$ and $l_j \in \mathcal{L}_j$.

Remark. Prior to Theorem 3.0.2, Theorem 3.1.1 was already known for arbitrary fields. Indeed, one could apply multilinear duality, Theorem 2.2.1, to the multijoint estimates proved by Zhang, [Zha20]. By proving Theorem 3.1.1 *via* Theorem 3.0.2, we offer a new and constructive proof without any application of duality. ◀

Conjecture (Finite Field Mizohata–Takeuchi Conjecture, [Tak80])**.** *Let $U \subset \mathbb{F}^{d-1}$. Suppose that there exist $g \in \mathbb{F}[x_1, \dots, x_d]$ and $\Gamma : U \to \mathbb{F}^d$ such that $Z_g = \Gamma(U)$. Let $w : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$ be a non-negative weight. For all $p \in U$, let $l(p) \subset \mathbb{F}^d$ be the line through $\Gamma(p)$ with direction $\nabla g(p)$. Then*

$$\sum_{q \in \mathbb{F}^d} \left|\mathcal{E}f(q)\right|^2 w(q) \lesssim \left(\sup_{p' \in U} \left(\sum_{q \in l(p')} w(q) \right) \right) \sum_{p \in U} \left|f(p)\right|^2$$

for all $f : U \to \mathbb{C}$, where $\mathcal{E}f(q) := \sum_{p \in U} e(2\pi i q \cdot \Gamma(p)) f(p) \sigma(p)$, and σ is the normalised surface measure on U.

Proposition 3.1.2. *The finite field Mizohata–Takeuchi Conjecture implies the finite field multilinear restriction conjecture.*

Proof. Using a duality argument, we sum $\prod_{j=1}^d \left|\mathcal{E}_j f_j\right|^{\frac{2}{d}}$ against a positive test function S with $\|S\|_d = 1$. Since the sets $E_1, \dots, E_d$ are transversal, the vectors $\nabla g_1(p_1), \dots, \nabla g_d(p_d)$ are linearly independent for $p_1 \in U_1, \dots, p_d \in U_d$. For each $1 \leq j \leq d$, let $l_j(p_j) := p_j + \nabla g_j(p_j) \cdot \mathbb{F}$. Let $\mathcal{L}_j = \{l_j(p_j) : p_j \in U_j\}$. By construction, the sets of lines $\mathcal{L}_1, \dots, \mathcal{L}_d$ are transversal. Hence, Theorem 3.1.1 applies and there are functions $s_1, \dots, s_d$ so that

$$S(q)^d \lesssim \prod_{j=1}^d s_j(q),$$

for all $q \in \mathbb{F}^d$, and so that

$$\sum_{q \in l_j} s_j(q) \leq 1$$

for all $l_j \in \mathcal{L}_j$, and for all $1 \leq j \leq d$. Hence

$$\sum_{q \in \mathbb{F}^d} S(q) \prod_{j=1}^d \left|\mathcal{E}_j f_j(q)\right|^{\frac{2}{d}} \lesssim \sum_{q \in \mathbb{F}^d} \prod_{j=1}^d \left(\left|\mathcal{E}_j f_j(q)\right|^2 s_j(q) \right)^{\frac{1}{d}} \leq \prod_{j=1}^d \left(\sum_{q \in \mathbb{F}^d} \left|\mathcal{E}_j f_j(q)\right|^2 s_j(q) \right)^{\frac{1}{d}}.$$

For each $1 \leq j \leq d$, we apply the finite field Mizohata–Takeuchi conjecture to each hypersurface E_j with respective weights s_j to deduce

$$\sum_{q \in \mathbb{F}^d} \left|\mathcal{E}_j f_j(q)\right|^2 s_j(q) \lesssim \left(\sup_{p' \in U_j} \left(\sum_{q \in l_j(p')} s_j(q)\right)\right) \sum_{p \in U_j} \left|f_j(p)\right|^2 \leq \left\|f_j\right\|^2_{L^2(U_j, \mathrm{d}\sigma_j)}.$$

Hence

$$\sum_{q \in \mathbb{F}^d} S(q) \prod_{j=1}^{d} \left|\mathcal{E}_j f_j(q)\right|^{\frac{2}{d}} \lesssim \prod_{j=1}^{d} \left\|f_j\right\|^2_{L^2(U_j, \mathrm{d}\sigma_j)}.$$

Since $S \in L^d(\mathbb{F}^d)$ was arbitrary with $\|S\|_d = 1$, we have deduced that

$$\left\|\prod_{j=1}^{d} \left|\mathcal{E}_j f_j\right|^{\frac{2}{d}}\right\|_{L^{d/(d-1)}(\mathbb{F}^d)} \lesssim \prod_{j=1}^{d} \left\|f_j\right\|^{\frac{2}{d}}_{L^2(U_j, \mathrm{d}\sigma_j)}$$

as desired. □

3.2 The Polynomial Method

This section marks the beginning of our proof of the discrete Bourgain–Guth theorem, and we start by revisiting the polynomial method.

Many applications of the polynomial method would find a non-zero polynomial $f \in \mathbb{F}[x_1, \ldots, x_d]$ of low degree which vanishes at every point of J. More generally, we may ask that f satisfies certain **vanishing conditions**. A vanishing condition is an equation of the form $\langle \phi, f \rangle = 0$ where ϕ is a linear form on $\mathbb{F}[x_1, \ldots, x_d]$. Such conditions include evaluation at a point, or evaluation of derivatives at a point.

It was observed in [TYZ20] that choosing vanishing conditions in the traditional way is naïve, and unwittingly imposes more vanishing conditions than necessary. Following [TYZ20], we will begin by choosing vanishing conditions, with some degrees of freedom, in an optimal way such that a vanishing lemma remains valid. Once the vanishing lemma is established, the use of polynomials is concluded. We then show that these vanishing conditions were chosen in a way that satisfies desirable uniform boundedness, monotonicity and continuity properties with respect to these degrees of freedom. This allows for a heuristically simple perturbation argument and we conclude the argument by choosing vanishing conditions in a way that respects the geometry of the particular multijoint configuration with which we are working. In Section 3.3, our results diverge from [TYZ20] to establish the discrete Bourgain–Guth theorem.

The advantages of using the new Tidor–Yu–Zhao approach come in two parts. Firstly, the careful choice of vanishing conditions allows for some degrees of freedom. Secondly, interpreting vanishing as elements of a dual space allows us to establish an analogue of Bézout's Theorem which generalises from counting points on lines, to points on varieties. In this section, we capitalise on the former.

3.2.1 Choosing Vanishing Conditions

To choose vanishing conditions, we will cycle through J, the set of multijoints, and accumulate conditions as we do. The number of multijoints is at most $|J| \leq \prod_j |\mathcal{L}_j|$. Indeed, the total number of d-tuples of lines is given by $\prod_j |\mathcal{L}_j|$. Since this is finite, we may enumerate $J = \{p_1, \dots, p_{|J|}\}$. We equip J with a total order according to this enumeration. That is, for $p_i, p_{i'} \in J$, we say $p_i \leq p_{i'}$ if and only if $i \leq i'$, where equality corresponds to the case of $i = i'$. Let $\alpha : J \to \mathbb{Z}$. We say that such a function is a **handicap**. For handicaps, we will denote evaluation at a point $p \in J$ by α_p.

Remark. This function describes the aforementioned degrees of freedom. More precisely, given a fixed $p_0 \in J$, the degrees of freedom are the $(|J| - 1)$ entries of $(\alpha_{p_0} - \alpha_p)_{p \in J \setminus \{p_0\}}$. ◀

We define a total order (called the **priority order** in [TYZ20]), denoted by $\prec$, on $J \times \mathbb{Z}_{\geq 0}$ as follows. We say $(p, r) \prec (p', r')$ if

- $r - \alpha_p < r' - \alpha_{p'}$, or
- $r - \alpha_p = r' - \alpha_{p'}$ and $p < p'$.

Moreover, $(p, r) = (p', r')$ if and only if $p = p'$ and $r = r'$. We write $\preceq$ to allow for this equality case.

Consider the case where α is identically 0. In this case, we may list the elements of $J \times \mathbb{Z}_{\geq 0}$ in $\prec$-order as follows:

$$\begin{aligned} &(p_1, 0) \prec (p_2, 0) \prec \cdots \prec (p_{|J|}, 0) \prec \\ &\qquad \prec (p_1, 1) \prec (p_2, 1) \prec \cdots \prec (p_{|J|}, 1) \prec \\ &\qquad\qquad \prec (p_1, 2) \prec (p_2, 2) \prec \cdots \prec (p_{|J|}, 2) \prec \cdots . \end{aligned}$$

The most convenient way to visualise handicaps is to think of $\mathbb{Z}_{\geq 0}$ as a column vector. We may depict $J \times \mathbb{Z}_{\geq 0}$ as $|J|$ adjacent columns, each indexed by some $p \in J$ and ordered according to the total order on J. Suppose $(p, r) \prec (p', r')$. The first bulleted condition stated above, says that (p, r) is on a lower row than (p', r'). The second says that if both (p, r) and (p', r') are on the same row, then

p is to the left of p'. This is illustrated below in the case of the zero handicap:

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & & \vdots & & \\
\curlyvee & & \curlyvee & & \curlyvee & & \rotatebox{45}{$\prec$} \\
(p_1,2) & \prec & (p_2,2) & \prec\cdots\prec & (p_{|J|},2) & & \\
\curlyvee & & \curlyvee & & \curlyvee & & \rotatebox{45}{$\prec$} \\
(p_1,1) & \prec & (p_2,1) & \prec\cdots\prec & (p_{|J|},1) & & \\
\curlyvee & & \curlyvee & & \curlyvee & & \rotatebox{45}{$\prec$} \\
(p_1,0) & \prec & (p_2,0) & \prec\cdots\prec & (p_{|J|},0), & &
\end{array}$$

where we use $\rotatebox{45}{$\prec$}$ to indicate that the pairs $(p_{|J|},r)$ immediately precedes $(p_1,r+1)$ for all $r\in\mathbb{Z}_{\geq 0}$, in this example.

Passing to a general handicap α shifts the p_i-th column vertically by α_{p_i} – downwards for $\alpha_{p_i}>0$, and upwards otherwise. Therefore, if $\alpha_{p_i}>\alpha_{p_{i'}}$, then the p_i-th column sits lower than the $p_{i'}$-th column. That is, columns which have a greater handicap sit lower in this column-representation.

For a non-trivial example, let $|J|=4$ and let $\alpha=(0,-1,2,0)$. We list the first terms of $J\times\mathbb{Z}_{\geq 0}$ under $\prec_\alpha$, the total order on $J\times\mathbb{Z}_{\geq 0}$ that is determined by α, as

$$\begin{aligned}
&(p_3,0)\prec(p_3,1)\prec(p_1,0)\prec(p_3,2)\prec \\
&\qquad\prec(p_4,0)\prec(p_1,1)\prec(p_2,0)\prec(p_3,3)\prec \\
&\qquad\qquad\prec(p_4,1)\prec(p_1,2)\prec(p_2,1)\prec(p_3,4)\prec\cdots.
\end{aligned}\tag{3.1}$$

Remark. Writing the elements out in this way better intuits $\prec$ as a total order. ◀

Alternatively, writing $J\times\mathbb{Z}_{\geq 0}$ in columns where lower rows precede higher rows gives:

$$\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1,3) & \prec & (p_2,2) & \prec & (p_3,5) & \prec & (p_4,3) & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1,2) & \prec & (p_2,1) & \prec & (p_3,4) & \prec & (p_4,2) & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1,1) & \prec & (p_2,0) & \prec & (p_3,3) & \prec & (p_4,1) & \\
\curlyvee & & & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1,0) & \prec & & & (p_3,2) & \prec & (p_4,0) & \\
 & & & & \curlyvee & & & \\
 & & & & (p_3,1) & & & \\
 & & & & \curlyvee & & & \\
 & & & & (p_3,0). & & &
\end{array}\tag{3.2}$$

Similar to before, the symbol $\rotatebox{45}{$\prec$}$ indicates that the pair at the end of a row immediately precedes the first element of the row above.

As an example, we can now easily see that $(p_1,0)\prec(p_3,3)$ since $(p_1,0)$ is on a row below $(p_3, 3)$. Similarly, $(p_1, 1) \prec (p_3, 3)$ since they are on the same row, but $(p_1, 1)$ lies to the left of $(p_3, 3)$. From this column-representation, we can also easily reconstruct the linear order of $J \times \mathbb{Z}_+$, as in (3.1). Each row of the diagram contains $\prec$-adjacent pairs. We start by writing the elements in the lowest row as a block. This is directly succeeded by the elements from the row directly above as a block. We continue to add the next row above as a block, *ad infinitum.*

More formally, for an arbitrary handicap α, each row is a level set of the form $\{(p, r) : r - \alpha_p = t\}$ for some $t \in \mathbb{N}$. It follows from the definition of $\prec$ that if $t' < t$ then every element of the level set $\{(p, r) : r - \alpha_p = t'\}$ precedes every element of the t-level set. Hence any pair (p, r) belonging to the t'-level set comes $\prec$-before every element of the t-level sets for every $t > t'$.

Remark. Writing the elements of $J \times \mathbb{Z}_{\geq 0}$ in this way better intuits the construction of the sets $B(p, l, \alpha, \lambda)$ which we will introduce in equation (3.3). ◀

Remark. The enumeration of $J = \{p_1, \ldots, p_{|J|}\}$ at the beginning of Subsection 3.2.1 is used solely to equip J with a total order, so that for any $p, q \in J$, precisely one of $p < q$, $p = q$, or $p > q$ is true. This total order on J is now fixed for the entirety of this argument, and we may now erase this enumeration. Later, in the proof of Lemma 3.3.1, we will again enumerate J for the purpose of labelling only. In particular, this will be completely independent of our fixed total order on J. ◀

To help gain intuition, we state a property of $\prec$.

Proposition 3.2.1. *For any $(p, r) \in J \times \mathbb{Z}_{\geq 0}$, we have $(p, 0), \ldots, (p, r - 1) \prec (p, r)$.*

To suggest how polynomials will enter into our discussion, observe that Proposition 3.2.1 is an analogue of the fact that if a polynomial $f \in \mathbb{F}[t]$ vanishes to order r at p, then it also vanishes to order r' at p for every $r' \leq r$.

We now describe how to choose vanishing conditions on some fixed line $l \subset \mathbb{F}^d$. Recall that $\mathbb{F}_\lambda[x_1, \ldots, x_d]$ denotes the space of d-variate polynomials of degree less than or equal to $\lambda \in \mathbb{N}$. Given $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$, we denote the restriction of f to a line l by $f|_l$, so that $f|_l(p) = f(p)$ for all $p \in l$. By identifying

$$\big\{f|_l : f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]\big\} = \mathbb{F}_\lambda[t],$$

we may assume that l is a coordinate axis. Let $\mathbb{B}^\lambda$ be the vector space of linear forms on $\mathbb{F}_\lambda[t]$. Then $\dim \mathbb{B}^\lambda = \dim \mathbb{F}_\lambda[t] = (\lambda + 1)$. Suppose $(p_0, r_0) = (p_0(\alpha, \lambda), r_0(\alpha, \lambda)) \in (l \cap J) \times \mathbb{Z}_{\geq 0}$ is such that[1]

$$\Big|\big\{(p, r) \preceq (p_0, r_0) : p \in l \cap J\big\}\Big| = \lambda + 1.$$

That is, (p_0, r_0) is the $(\lambda + 1)$-st member of $(l \cap J) \times \mathbb{Z}_{\geq 0}$.

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[1] This is what is meant when we say that we choose vanishing conditions optimally. We choose them so that any given line is not associated with too many – a substitute for Bézout's theorem.

Each pair, (p,r), indexes a vanishing condition. Given (p,r), let $\langle \phi_p^r, f\rangle := f^{(r)}(p)$, where $f^{(r)}$ denotes the (directional) Hasse derivative of order r (with direction $e(l)$) of f. Recall that f is a d-variate polynomial restricted to l.

Proposition 3.2.2. *With* (p_0, r_0) *as above,*

$$\operatorname{span}\{\phi_p^r : (p,r) \preceq (p_0,r_0) : p \in l \cap J\} = \mathbb{B}^\lambda.$$

Remark. Unlike the higher-dimensional case, simply accumulating vanishing conditions, ϕ_p^r, as we increment along $(l \cap J) \times \mathbb{Z}_{\geq 0}$ is sufficient, as we are only handling the set of linear forms on $\mathbb{F}_\lambda[t]$, and not a higher-dimensional space. As such, we can omit much of the linear algebra which features in the general argument to demonstrate which elements of the argument are strictly due to handicaps. ◀

Proof. For a contradiction, suppose that the vanishing conditions, ϕ_p^r, are not spanning. Then by Lemma 2.5.5 there exists a non-zero $f \in \mathbb{F}_\lambda[t]$ so that $\langle \phi_p^r, f\rangle = 0$ for all $(p,r) \preceq (p_0,r_0)$. Let $\mathcal{P} := \{p \in l \cap J : \exists r \text{ s.t. } (p,r) \preceq (p_0,r_0)\}$. For each $p \in \mathcal{P}$, let

$$m_p = 1 + \max\{r : (p,r) \preceq (p_0,r_0)\}.$$

By Proposition 3.2.1, $\langle \phi_{p,r'}, f\rangle = 0$ for all $0 \leq r' < m_p$. That is, f vanishes to order at least m_p at p. Hence f vanishes to order at least m_p for all $p \in \mathcal{P}$. Let $\operatorname{mult}(f,p)$ denote the multiplicity of the zero of f at p. Then

$$\lambda \geq \sum_{p \in \mathcal{P}} \operatorname{mult}(f,p) \geq \sum_{p \in \mathcal{P}} m_p = \lambda + 1,$$

a contradiction. □

For each $p \in l \cap J$, let us define

$$B(p,l,\alpha,\lambda) := \left\{(p,r) \in (l \cap J) \times \mathbb{Z}_{\geq 0} : (p,r) \preceq (p_0,r_0)\right\}, \tag{3.3}$$

where (p_0,r_0) is the $(\lambda+1)$-st element of $(l \cap J) \times \mathbb{Z}_{\geq 0}$ under $\prec$. We write $B(p,l,\alpha,\lambda) = \{(p,r) \preceq (p_0,r_0)\}$ for short. The cardinalities of these sets will be central to our discussion. In particular, they are associated to sets of vanishing conditions at p, with direction $e(l)$. Hence, we define

$$\tilde{S}(p,l,\alpha,\lambda) := \left|B(p,l,\alpha,\lambda)\right|,$$

for all $p \in l \cap J$. With fixed l, α and λ, the sets $B(p,l,\alpha,\lambda)$ for $p \in l \cap J$ are distinct, and

$$\sum_{p \in l \cap J} \left|B(p,l,\alpha,\lambda)\right| = \lambda + 1. \tag{3.4}$$

This is a fundamental property of the sets $B(p,l,\alpha,\lambda)$, and serves to replace Bézout's Theorem in our polynomial method.

Remark. Let (p_0, r_0) be the $(\lambda+1)$-st element of $((l \cap J) \times \mathbb{Z}_{\geq 0})$ under $\prec$. Observe that we can write equation (3.3) as

$$B(p, l, \alpha, \lambda) = \left\{(p, r) : p \in (p + \mathbb{F} \cdot e(l)) \text{ and } (p, r) \preceq (p_0, r_0)\right\}.$$

In particular, since the first argument of B is a point in l, we can identify $l = p + \mathbb{F} \cdot e(l)$. Hence, for the definition of $B(p, l, \alpha, \lambda)$, it suffices to reduce its second argument to $e(l)$, the direction of l. For notational convenience, we will continue to write the second argument as a line. ◀

Corollary 3.2.2a. *For any α and λ,*

$$\operatorname{span}_{p \in l \cap J}\{\phi_p^r : (p, r) \in B(p, l, \alpha, \lambda)\} = \mathbb{B}^\lambda.$$

Corollary 3.2.2b. *If $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$ is such that $f|_l$ is non-zero, then there exists $(p, r) \preceq (p_0, r_0)$ so that $\langle \phi_p^r, f|_l \rangle \neq 0$.*

Proposition 3.2.3 (Translation Invariance)**.** *Let l be a line. Let α be a handicap and $c \in \mathbb{Z}$. Let $(\alpha + c)_p := \alpha_p + c$ for all $p \in J$. Then*

$$B(p, l, \alpha, \lambda) = B(p, l, \alpha + c, \lambda)$$

for all $p \in (l \cap J)$. Furthermore, $\tilde{S}(p, l, \alpha, \lambda) = \tilde{S}(p, l, \alpha + c, \lambda)$.

3.2.2 Vanishing Lemma

Having chosen sets of vanishing conditions $B(p, l, \alpha, \lambda)$, we show that these sets satisfy a vanishing lemma, Lemma 3.2.4, below.

Lemma 3.2.4 (Vanishing Lemma)**.** *Fix α and λ. For each $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ and $p \in J$, build the sets $B(p, l) = B(p, l, \alpha, \lambda)$ as in Subsection 3.2.1. For each $p \in J$, choose lines $l_1(p) \in \mathcal{L}_1, \ldots, l_d(p) \in \mathcal{L}_d$ so that $\delta(p, \boldsymbol{l}(p)) = 1$. If $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$ is non-zero, then there exists $p \in J$ and $(p, r_j) \in B(p, l_j(p))$ for each $1 \leq j \leq d$ so that*

$$\left(\nabla \cdot e(l_1(p))\right)^{r_1} \cdots \left(\nabla \cdot e(l_d(p))\right)^{r_d} f(p) \neq 0.$$

Remark. Note the absence of a j-subscript on these sets B. Indeed, the construction depends only on $e(l)$, and not the set which contains l. ◀

Proof. Suppose for a contradiction that there is a non-zero $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$, so that for all $p \in J$ and all $(p, r_j) \in B(p, l_j(p))$ for $1 \leq j \leq d$,

$$\left(\nabla \cdot e(l_1(p))\right)^{r_1} \cdots \left(\nabla \cdot e(l_d(p))\right)^{r_d} f(p) = 0.$$

Let $\operatorname{mult}(f, p)$ denote the multiplicity of the zero of f at p; that is, the least positive integer m so that there is an m-th derivative of f that does not vanish at p. Fix a choice of p that minimises $(p, \operatorname{mult}(f, p))$ with respect to $\prec$. By the definition of multiplicity, there is a derivative D of order $\operatorname{mult}(f, p)$ so that

$Df(p) \neq 0$. Since $l_1(p), \ldots, l_d(p)$ have directions spanning $\mathbb{F}^d$, we can find non-zero $r_1, \ldots, r_d$ such that $r_1 + \ldots + r_d = \mathrm{mult}(f, p)$ and

$$(\nabla \cdot e(l_1(p)))^{r_1} \cdots (\nabla \cdot e(l_d(p)))^{r_d} f(p) \neq 0. \tag{3.5}$$

By hypothesis, there exists j so that $(p, r_j) \notin B(p, l_j(p))$. Fix such j, and by relabelling, we may assume that $j = 1$ without loss of generality.

Consider $(p', r') \prec (p, r_1)$ for any $p' \in l_1(p) \cap J$. Then

$$(p', r' + r_2 + \ldots + r_d) \prec (p, r_1 + \ldots + r_d) = (p, \mathrm{mult}(f, p)).$$

The multijoint p was chosen so that $(p, \mathrm{mult}(f, p)) \preceq (p', \mathrm{mult}(f, p'))$. Hence

$$(p', r' + r_2 + \ldots + r_d) \prec (p', \mathrm{mult}(f, p')).$$

By the definition of $\prec$, we deduce that

$$r' + r_2 + \ldots + r_d < \mathrm{mult}(f, p').$$

Therefore,

$$\left(\nabla \cdot e(l_1(p))\right)^{r'} \left(\nabla \cdot e(l_2(p))\right)^{r_2} \cdots \left(\nabla \cdot e(l_d(p))\right)^{r_d} f(p') = 0.$$

Let

$$g := \left(\nabla \cdot e(l_2(p))\right)^{r_2} \cdots \left(\nabla \cdot e(l_d(p))\right)^{r_d} f.$$

We have established that $(\nabla \cdot e(l_1(p')))^{r'} g(p') = 0$ for all $(p', r') \prec (p, r_1)$. On the other hand, by (3.5), $g|_{l_1(p)}$ is not the zero polynomial. Therefore, the set

$$\left\{\phi_{p', r'} : (p', r') \prec (p, r_1),\ p' \in l_1(p) \cap J\right\}$$

does not span $\mathbb{B}^\lambda$.

By Proposition 3.2.2, we deduce that (p, r_1) is in the first $(\lambda + 1)$ elements of $l_1(p) \cap J$ under $\prec$, and so our construction requires $(p, r_1) \in B(p, l_1(p))$. This contradicts our choice of $r_1, \ldots, r_d$, concluding the proof. $\square$

A general form of this lemma can be found in [TYZ20]. However, our proof in the special case of lines is more simple as it does not need to account for linearly independent vanishing conditions, and counting alone is sufficient.

Corollary 3.2.4a. *Fix α and λ. For each $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ and $p \in J$, build the sets $B(p, l, \alpha, \lambda)$, as in Subsection 3.2.1. For each $p \in J$, choose lines $l_1(p) \in \mathcal{L}_1, \ldots, l_d(p) \in \mathcal{L}_d$ so that $\delta(p, \boldsymbol{l}(p)) = 1$. Then*

$$\sum_{p \in J} \prod_{j=1}^{d} \tilde{S}(p, l_j(p), \alpha, \lambda) \geq \binom{\lambda + d}{d}.$$

Proof. Recall that $\tilde{S}(p, l, \alpha, \lambda) = \left|B(p, l, \alpha, \lambda)\right|$ and suppose that the conclusion is false. By parameter counting, there exists a non-zero $f \in \mathbb{F}_\lambda[x_1, \dots, x_d]$ so that

$$(\nabla \cdot e(l_1(p)))^{r_1} \cdots (\nabla \cdot e(l_d(p)))^{r_d} f(p) = 0$$

for all $(p, r_j) \in B(p, l_j(p), \alpha, \lambda)$ and all $1 \leq j \leq d$, contrary to Lemma 3.2.4. $\square$

Remark. We follow the convention, in this proof of Corollary 3.2.4a, and hence in Lemma 3.2.4, that if $B(p, l_j(p), \alpha, \lambda) = \emptyset$ for some p and j, then the set of conditions imposed on f at p is empty. In particular, we deduce that choosing vanishing conditions in this way prohibits situations such that for all $p \in J$, there exists j so that $B(p, l_j(p), \alpha, \lambda) = \emptyset$.

The constant C in Theorem 3.0.1 is exactly the implicit constant in the inequality $\binom{\lambda+d}{d} \gtrsim \lambda^d$. Within this proof, for the case of lines, this inequality is the only place where a dimensional constant occurs. Moreover, this use of parameter counting concludes our use of the polynomial method.

Corollary 3.2.4a states that the products $\prod_j \tilde{S}(p, l_j(p))$ are large, on average. In Subsection 3.3.1, we will choose a handicap α so that the products $\prod_j \tilde{S}(p, l_j(p))$ are all approximately equal as p varies, with respect to a weight given by $S(p)^{-d}$. This allows us to lift the “on average” bound to a pointwise bound over $p \in J$.

3.2.3 Vanishing Conditions as Handicap Varies

In [TYZ20], some technical linear algebra on vector spaces of polynomials is required to precisely quantify how $\tilde{S}(p, l, \alpha, \lambda)$ changes with α, so that the numerology of dimensions and degrees works out in the full generality of their result. In the one dimensional case of lines discussed here, this is not necessary. Indeed, Corollary 3.2.2a states that we just need to consider the first $(\lambda + 1)$ terms of $(l \cap J) \times \mathbb{Z}_{\geq 0}$ under $\prec$ to construct our sets $B(p, l, \alpha, \lambda)$ and there are no subtleties to be considered.

This changes in the higher-dimensional case. Consider a 2-plane π. There is no longer a unique choice of direction in $e(\pi)$. Moreover, if there are many points in $(\pi \cap J)$ that are collinear, then we may encounter issues of linear dependence before we have accumulated $\binom{\lambda+2}{2}$ vanishing conditions on π.[2] Since linear independence is easy to understand in the 1-dimensional case of lines, we may proceed by handling the handicap α directly without a linear algebra detour.

To understand why these results are intuitive, consider how the diagram in display 3.2 changes as we increase or decrease α at some fixed $p \in J$. Indeed, consider the effect of replacing a handicap by one which is identical, except at one point p, where it is precisely one less. This shifts the p-th column up by one. With reflection, we see that either $\tilde{S}$ is unchanged, or, $\tilde{S}$ reduces at p by exactly one, increases at some other p' by exactly one, and remains unchanged otherwise.

Lemma 3.2.5 (Uniform Boundedness). *Let α be a handicap on J, $\lambda \in \mathbb{N}$ and*

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[2] The choice of $\binom{\lambda+2}{2}$ happens to be the correct generalisation of $(\lambda + 1)$ in our discussion.

let l be a line. Suppose there are $p, q \in (l \cap J)$ so that $\alpha_p < \alpha_q - \lambda$. Then $\tilde{S}(p, l, \alpha, \lambda) = 0$.

Lemma 3.2.5 is so-called since it states that there is a uniform upper bound on α_q for those $q \in l \cap J$ such that $q \neq p$ and $\tilde{S}(p, l, \alpha, \lambda) \neq 0$.

Proof. Recall that

$$\tilde{S}(p, l, \alpha, \lambda) = \big|B(p, l, \alpha, \lambda)\big|.$$

By definition $(q, \lambda) \prec (p, 0)$. By Proposition 3.2.1,

$$(q, 0) \prec (q, 1) \prec \cdots \prec (q, \lambda - 1) \prec (q, \lambda) \prec (p, 0)$$

and so there are at least $(\lambda + 1)$ pairs preceding $(p, 0)$ under $\prec$. By definition $B(p, l, \alpha, \lambda) = \emptyset$ and hence $S(p, l, \alpha, \lambda) = 0$. □

Lemma 3.2.6 (Monotonicity). *Let $\lambda \in \mathbb{N}$, and let l be a line. Suppose $\alpha^{(1)}$ and $\alpha^{(2)}$ are handicaps so that $\exists p \in l \cap J$ such that $\alpha^{(1)}_p - \alpha^{(1)}_{p'} \leq \alpha^{(2)}_p - \alpha^{(2)}_{p'}$ for all $p' \in l \cap J$. Then*

$$\tilde{S}(p, l, \alpha^{(1)}, \lambda) \leq \tilde{S}(p, l, \alpha^{(2)}, \lambda).$$

Proof. By Proposition 3.2.3 we can assume that $\alpha^{(i)}_p = 0$ for $i = 1, 2$, so that $\alpha^{(2)}_{p'} \leq \alpha^{(1)}_{p'}$ for all $p' \in l \cap J$ and let $\prec_i = \prec_{\alpha^{(i)}}$. It suffices to show that if (p, r) is among the $\prec_1$-first $(\lambda + 1)$ elements of $(l \cap J) \times \mathbb{Z}_{\geq 0}$, then it is among the $\prec_2$-first $(\lambda + 1)$ elements of $(l \cap J) \times \mathbb{Z}_{\geq 0}$. This follows if we can show that if $(p, r) \prec_1 (p', r')$, then $(p, r) \prec_2 (p', r')$. Equivalently, the handicaps $\alpha^{(1)}$ and $\alpha^{(2)}$ are such that that there is no pair (p', r') which occurs $\prec_1$-after (p, r) which also occurs $\prec_2$-before (p, r).

Let (p', r') be such that $(p, r) \prec_1 (p', r')$. Let us assume that $r - 0 < r' - \alpha^{(1)}_{p'}$. Applying our hypothesis, we have that $r - 0 < r' - \alpha^{(2)}_{p'}$ and therefore $(p, r) \prec_2 (p', r')$, as desired. Otherwise, we have the equality case $r = r' - \alpha^{(1)}_{p'}$ and $p < p'$. Applying our hypothesis, we deduce $r \leq r' - \alpha^{(2)}_{p'}$. If this inequality is strict then the result holds. Otherwise, it is an equality and $p < p'$ so the result holds, concluding the proof. □

Lemma 3.2.7 (Continuity). *Let $\lambda \in \mathbb{N}$, l be a line, $p \in l \cap J$, and let $\alpha^{(1)}, \alpha^{(2)}$ be handicaps. Then*

$$\Big|\tilde{S}(p, l, \alpha^{(1)}, \lambda) - \tilde{S}(p, l, \alpha^{(2)}, \lambda)\Big| \leq \sum_{p' \in l \cap J} \Big|(\alpha^{(1)}_p - \alpha^{(1)}_{p'}) - (\alpha^{(2)}_p - \alpha^{(2)}_{p'})\Big|.$$

Remark. By Proposition 3.2.3, we may translate $\alpha^{(2)}$ so that $\alpha^{(2)}_p = \alpha^{(1)}_p$. The conclusion of Lemma 3.2.7 thus reads as:

$$\Big|\tilde{S}(p, l, \alpha^{(1)}, \lambda) - \tilde{S}(p, l, \alpha^{(2)}, \lambda)\Big| \leq \sum_{p' \in l \cap J} \Big|\alpha^{(1)}_{p'} - \alpha^{(2)}_{p'}\Big|.$$

This is more easily recognisable as a Lipschitz continuity result. In fact, if desired, we could redefine handicaps and identify α by its equivalence class, $\alpha+\mathbb{Z}^{|J|}$, under equivalence by translation invariance. ◀

Proof. By translation invariance, Proposition 3.2.3, the left-hand side of the target inequality remains unchanged if we translate $\alpha^{(1)}$ and $\alpha^{(2)}$ by $\alpha_p^{(1)}$ and $\alpha_p^{(2)}$, respectively.

Let $p \in l \cap J$ and consider $(\alpha_{p'}^{(1)} - \alpha_p^{(1)})_{p' \in J}$. For $p' \neq p$ we will incrementally modify $(\alpha_{p'}^{(1)} - \alpha_p)$ by ± 1 until we get to $(\alpha_{p'}^{(2)} - \alpha_p^{(2)})$. This can be done in $\left|(\alpha_{p'}^{(1)} - \alpha_p^{(1)}) - (\alpha_{p'}^{(2)} - \alpha_p^{(2)})\right|$ increments.

It therefore suffices to consider how $\tilde{S}(p, l, \alpha, \lambda)$ changes on any one increment. However, by the paragraph preceding Lemma 3.2.5, for any handicap α, if we increment $\alpha_{p'}$ by ± 1 then $\tilde{S}(p, l, \alpha, \lambda)$ either remains unchanged, or changes by ± 1. Let $e_{p'} : J \to \mathbb{Z}$ be given by $(e_{p'})_q = 1$ if $p' = q$ and $(e_{p'})_q = 0$ if $p' \neq q$. This may be thought of as the p'-th basis vector for the space of maps $J \to \mathbb{Z}$. Then, for an increment $\alpha \mapsto \alpha' = \alpha + e_{p'}$

$$0 \leq \tilde{S}(p, l, \alpha, \lambda) - \tilde{S}(p, l, \alpha', \lambda) \leq 1. \tag{3.6}$$

Moreover, we can reverse the roles of α and α' for an increment of the form $\alpha \mapsto \alpha - e_{p'}$. Thereafter, we sum these inequalities over all increments to yield the result. □

3.3 Multijoints of Lines

We are now equipped to begin counting multijoints.

3.3.1 Choosing A Handicap

We have constructed numbers $\tilde{S}(p, l, \alpha, \lambda)$ according to the Tidor–Yu–Zhao argument in [TYZ20] but using more elementary arguments to establish “good” properties with respect to α. We now deviate from their argument to establish a new result.

For technical reasons, we introduce a further definition. Let $p \in J$. For each $1 \leq j \leq d$, we say that a line $l \in \mathcal{L}_j$ **contributes** to p if there is a tuple $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ so that $l = l_j$ and $\delta(p, \boldsymbol{l}) = 1$. In particular, if $\boldsymbol{l}$ is a tuple so that $\delta(p, \boldsymbol{l}) = 1$, then for each $1 \leq j \leq d$, l_j contributes to p.

Let $E \subset J$. We may assume that every $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ contributes to some $p \in E$. Indeed, for the purposes of Theorem 3.0.1, if l is a line which does not contribute to any point then we may set $s(\cdot, l) \equiv 0$ and so l may reasonably be ignored.

We say $p, q \in J$ are **adjacent** if there is some $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ which contributes to p and contributes to q. We say that p and q are **connected** if there is a sequence of points $p = p_1, \ldots, p_N = q \in E$ so that p_i and p_{i+1} are adjacent for all $1 \leq i < N$. Connectedness defines an equivalence relation on E. Moreover, this induces a partition of each $\mathcal{L}_1, \ldots, \mathcal{L}_d$ given by those lines which contribute to any element of any one equivalence class of E.

The first result that we will prove is Theorem 3.0.1 and to do so we may assume that $\operatorname{supp} S$ is connected and that every line contributes to some $p \in \operatorname{supp} S$. Indeed, let E' be a connected component of $\operatorname{supp} S$ and let $\mathcal{L}_j(E')$ be the subset of lines in $\mathcal{L}_j$ which contribute to some $p \in E'$. For each component E', assume that for any $S : E' \to \mathbb{R}_{\geq 0}$, there is a function $s_{E'} : E' \times (\mathcal{L}_1(E') \cup \cdots \cup \mathcal{L}_d(E')) \to \mathbb{R}_{\geq 0}$ satisfying the first display of Theorem 3.0.1 for all $p \in E'$ and $\boldsymbol{l} \in \mathcal{L}_1(E') \times \cdots \times \mathcal{L}_d(E')$, and satisfying the second display for all $l \in \mathcal{L}_1(E') \cup \cdots \cup \mathcal{L}_d(E')$. Now, for each $p \in \operatorname{supp} S$ there is a unique set E' so that $p \in E'$. Hence, we can define $s(p, l) := s_{E'}(p, l)$ for every $l \in \mathcal{L}_1, \ldots, \mathcal{L}_d$ which contributes to p. This defines s for every pair (p, l) for which $p \in \operatorname{supp} S$ and l contributes to p. Extending the function s by zero, so that it is defined on $J \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$. By the definition of connectedness, such s is well defined.

The first condition of Theorem 3.0.1 is satisfied by s, automatically. Furthermore, consider the condition $\sum_{p \in l_j \cap E} s(p, l_j) = 1$. Since all elements of the sum are contained within a single line, they belong to the same connected component. Therefore, in computing this sum, we need consider only one of the original functions $s_{E'}$ defined on $E' \times \mathcal{L}(E') \times \cdots \times \mathcal{L}(E')$. Hence, s satisfies conclusions of Theorem 3.0.1.

It therefore suffices to assume that $\operatorname{supp} S$ contains a single connected component.

Lemma 3.3.1 (Handicap). *Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be sets of lines in $\mathbb{F}^d$ with associated multijoints J. Let $S : J \to \mathbb{R}_{\geq 0}$ and suppose that any two multijoints in $\operatorname{supp} S$ are connected via a path of multijoints in $\operatorname{supp} S$, and that $(\lambda + 1) \geq |\operatorname{supp} S|$. There is a handicap $\alpha : J \to \mathbb{Z}$ so that for all $p \in \operatorname{supp} S$, the numbers*

$$\min_{\boldsymbol{l} : \delta(p, \boldsymbol{l}) = 1} \frac{1}{S(p)^d} \left(\prod_{j=1}^{d} \frac{\tilde{S}(p, l_j, \alpha, \lambda)}{\lambda + 1} \right) \tag{3.7}$$

all lie in a common interval with length $\leq h'/\lambda$ for some $h' = h'(|J|, S)$ depending on S and $|J|$, but not λ. Furthermore, we may choose α so that $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ for all $p \notin \operatorname{supp} S$.

Remark. The definition of "contributes" is due to the fact that at each $p \in \operatorname{supp} S$, the quantity (3.7) is equal to zero if and only if $\tilde{S}(p, l, \alpha, \lambda) = 0$ for some $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ which contributes to p. ◀

Remark. The analogous result, proven by Tidor Yu and Zhao, [TYZ20, Lemma

5.10], for any fixed tuple of functions $f_j : \mathcal{L}_j \to \mathbb{Z}_{>0}$, considers the quantity

$$\frac{1}{S(p)^{dM(p)}} \prod_{\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d} \left(\prod_{j=1}^{d} \frac{\tilde{S}(p, l_j, \alpha, \lambda)}{\lambda + 1} \right)^{\omega(\boldsymbol{l})}$$

where $M(p) = \sum_{\boldsymbol{l}} \delta(p, \boldsymbol{l}) \prod_j f_j(l_j)$ and $\omega(\boldsymbol{l}) = \frac{\delta(p,\boldsymbol{l}) \prod_{j=1}^{d} f_j(l_j)}{\sum_{\boldsymbol{l}'} \delta(p,\boldsymbol{l}') \prod_{j'=1}^{d} f_{j'}(l_{j'})}$. This is sufficient to resolve the multijoints problem, but not the conjectured dual result.

Despite this difference, we prove Lemma 3.3.1 by appropriate modifications of methods from [TYZ20]. ◀

We regard the interval length h'/λ, as given by Lemma 3.3.1, as an error bound. In the context of the overarching argument in this chapter, the size of h', large or small, is completely irrelevant. However, the constant h' can be expressed as a function which grows polynomially in both $|J|$ and the quantities $S(p)^{-d}$ over all $p \in J$. Importantly, h' is independent of the parameter λ, which is an input to Lemma 3.3.1. In Subsection 3.3.2, we will let $\lambda \to \infty$ so that the interval has arbitrarily small width. That is, for sufficiently large λ, all the values in (3.7) become as close to equal as we desire.

Lemma 3.3.1 makes no assertion as to the location of the given interval. We will examine the location of this interval in Subsection 3.3.2, using our newly adapted polynomial method. Indeed, Lemma 3.3.1 will be paired with Corollary 3.2.4a to deduce that the interval is bounded away from 0. Lemma 3.3.1, however, should be considered as separate from the polynomial method section of the argument.

To prove Lemma 3.3.1 we require some notation and useful facts. Recall that if $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$, and $p \in l \cap J$, then

$$\tilde{S}(p, l, \alpha, \lambda) = \left| B(p, l, \alpha, \lambda) \right|,$$

where the sets $B(p, l, \alpha, \lambda)$ are defined by (3.3), and satisfy

$$\sum_{p \in l} \tilde{S}(p, l, \alpha, \lambda) = \lambda + 1$$

for every $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$, by construction. Let α be a handicap. For any $p \in \operatorname{supp} S$, let $\boldsymbol{l} = (l_j)_j \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ be such that $p = \cap_j l_j$. We define

$$W(\boldsymbol{l}, \alpha) := \frac{1}{S(p)^d} \left(\prod_{j=1}^{d} \frac{\tilde{S}(p, l_j, \alpha, \lambda)}{\lambda + 1} \right).$$

Note that W additionally depends on λ and p. However, λ is fixed for Lemma 3.3.1, and the dependence on p is implicit through the tuple $\boldsymbol{l}$ which must satisfy $p \in \cap_j l_j$. Therefore, we suppress p and λ from our definition of W. Now, for each $p \in J$,

$$w_p(\alpha) := \min_{\boldsymbol{l} : \delta(p, \boldsymbol{l}) = 1} W(\boldsymbol{l}, \alpha).$$

Proposition 3.3.2. *Let* $(\lambda+1) \geq |\operatorname{supp} S|$*. Let* $w : \{\alpha : J \to \mathbb{Z}\} \to \mathbb{R}^{|J|}_{\geq 0}$ *be the map such that* $w(\alpha) = (w_p(\alpha))_p$*. Define* $A = A(\lambda) \subset \{\alpha : J \to \mathbb{Z}\}$*, to be the set of* α *such that*

- $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ *for all* $p \notin \operatorname{supp} S$*, and*
- $\tilde{S}(p, l, \alpha, \lambda) \neq 0$ *for all* $p \in \operatorname{supp} S$ *and all* $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ *that contribute to* p*.*

The image $w(A)$ *is finite and non-empty.*

Remark. The first bullet of Proposition 3.3.2, above, which prescribes a condition on α, is necessary for (3.17) in a later part of the argument, since we wish to allow $\operatorname{supp} S$ to be a strict subset of J. This condition was not required in [TYZ20].

The second bulleted condition, however, is not strictly necessary. Hence, it can reasonably be ignored. However, recall the emptyset convention, remarked upon in Subsection 3.2.1. In the case where we allow the sets $B(p, l, \alpha, \lambda)$ to be possibly empty, we will eventually be able to deduce that α, as given by Lemma 3.3.1, is such that every $B(p, l, \alpha, \lambda) \neq \emptyset$ whenever $p \in \operatorname{supp} S$, and l contributes to p, *via* Corollary 3.2.4a. We include this technicality so as to guarantee non-emptiness *a priori* with the intention of sharing a more complete understanding of the properties of handicaps, and how to derive them. ◀

Proposition 3.3.3. *With* A *as defined in Proposition 3.3.2, suppose* $w_0 \in w(A)$*. Then* $\exists \alpha \in A$ *such that* $w(\alpha) = w_0$ *and for all* $q \notin \operatorname{supp} S$ *and lines* $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ *so that* $l \ni q$*, and all* $p \in l \cap \operatorname{supp} S$*, we have* $\alpha_q < \alpha_p - \lambda$*.*

Remark. Proposition 3.3.3 is a technical result to guarantee that we may perturb α to α' such that the conditions $\tilde{S}(q, \cdot, \alpha', \lambda) = 0$ for all $q \notin \operatorname{supp} S$ are preserved. ◀

Proof of Proposition 3.3.2. Let $p \in \operatorname{supp} S$. By translation invariance (Proposition 3.2.3),

$$w(A) = w(\{\alpha \in A : \alpha_p = 0\}).$$

Hence, we may further insist that $\alpha_p = 0$ for all $\alpha \in A$. We begin by proving that A is non-empty. Consider

$$\alpha = -(\lambda+1) \sum_{q \notin \operatorname{supp} S} e_q.$$

Then α satisfies the two bulleted conditions specified in Proposition 3.3.2 and thus, A is non-empty. Indeed, to establish the first bullet, note that every line in $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ intersects $\operatorname{supp} S$. Hence, for any $q \notin \operatorname{supp} S$ and every line $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ such that $l \ni q$, there exists $p \in l \cap \operatorname{supp} S$. By the definition of α,

$$\alpha_q = -(\lambda+1) = 0 - (\lambda+1) = \alpha_p - (\lambda+1) < \alpha_p - \lambda.$$

By uniform boundedness (Lemma 3.2.5), $\tilde{S}(q,l,\alpha,\lambda)=0$. Since l is arbitrary, $\tilde{S}(p,\cdot,\alpha,\lambda)=0$, as desired.

For the second bullet, since $\alpha_p=\alpha_q$ for all $p,q\in\operatorname{supp}S$, the $\prec$-first $|\operatorname{supp}S|$ elements of $J\times\mathbb{Z}_{\geq 0}$ are precisely $\{(p,0):p\in\operatorname{supp}S\}$. By assumption, $(\lambda+1)\geq|\operatorname{supp}S|$. Hence $(p,0)\in B(p,l,\alpha,\lambda)$ for all $p\in\operatorname{supp}S$, as per (3.3), the definition of B, establishing the second bullet.

We now show that $w(A)$ is finite. Recall we have chosen $p\in\operatorname{supp}S$ so that $\alpha_p=0$. It suffices to show that if $\alpha\in A$ with $\alpha_p=0$, and $q\in\operatorname{supp}S$ is such that $q\neq p$, then

$$\alpha_q\in[-\lambda|J|,\lambda|J|]. \tag{3.8}$$

Indeed, this implies that $\{\alpha|_{\operatorname{supp}S}:\alpha\in A\}\subseteq[-\lambda|J|,\lambda|J|]^{\operatorname{supp}S}$. Since $\operatorname{supp}S$ is finite, there are finitely many possible values of $(w_p(\alpha))_{p\in\operatorname{supp}S}$. Since $w_q(\alpha)=0$ for $q\notin\operatorname{supp}S$, $w(A)$ is finite, as desired.

To prove (3.8), let $\alpha\in A$ with $\alpha_p=0$, and let $q\in\operatorname{supp}S\setminus\{p\}$. Since $\operatorname{supp}S$ is connected, there is some sequence of multijoints $p=p_1,\dots,p_N=q$ so that for every $1\leq i<N$, multijoints p_i and p_{i+1} are adjacent, and each $p_1,\dots,p_N\in\operatorname{supp}S$. For each $1\leq i\leq N$, let $\alpha_i:=\alpha_{p_i}$. We claim that

$$\alpha_i-\lambda\leq\alpha_{i+1}\leq\alpha_i+\lambda, \tag{3.9}$$

for all $1\leq i<N$. Indeed, by uniform boundedness (Lemma 3.2.5), if either the first or the second inequality is violated, and l is any line contributing to both p_i and p_{i+1}, then $\tilde{S}(p_{i+1},l,\alpha,\lambda)=0$ or $\tilde{S}(p_i,l,\alpha,\lambda)=0$, respectively. This contradicts the second bullet of Proposition 3.3.2, in the definition of A.

Since (3.9) holds for all $1\leq i<N$ and $N\leq|J|$, this implies (3.8), concluding the proof. □

Recall the representation of $(J\times\mathbb{Z}_{\geq 0},\prec)$ as adjacent columns. To intuit why Proposition 3.3.3 is true, draw such a representation and highlight the $\prec$-first $(\lambda+1)$ pairs $(p,r)\in J\times\mathbb{Z}_{\geq 0}$. Since $\tilde{S}(q,l,\alpha,\lambda)=0$ for $q\notin\operatorname{supp}S$, we deduce that none of the highlighted pairs (p,r) satisfy $p=q$. Hence, there are no highlighted pairs in the quadrant whose bottom left corner is $(q,0)$. Furthermore, if we reduce the handicap at q by one, we shift the q-th column vertically up and still there are no highlighted pairs in the quadrant whose bottom left corner is $(q,0)$. Hence, the first $(\lambda+1)$ elements under the handicap $(\alpha-e_q)$ remain the same. We can repeat the argument to deduce that the first $(\lambda+1)$ elements remain unchanged when we change the handicap from α to $(\alpha-ce_q)$, for any $c\in\mathbb{N}$.

Proof of Proposition 3.3.3. Since $w_0\in w(A)$, we can find some $\beta\in A$ so that $w(\beta)=w_0$. Let $p\in\operatorname{supp}S$. By translation invariance (Proposition 3.2.3), we may assume that $\beta_p=0$.

Let $c\in\mathbb{N}$ and define $\alpha=\beta-c\sum_{q\notin\operatorname{supp}S}e_q$. By the definition of A, for all $q\notin\operatorname{supp}S$, and $l\ni q$, we have $\tilde{S}(q,l,\beta,\lambda)=0$. Since $\tilde{S}(\cdot,\cdot,\beta,\lambda)=\left|B(\cdot,\cdot,\beta,\lambda)\right|$, by (3.3), the definition of B, we deduce that

$$\left|(p', r') : p' \in l \cap J \text{ and } (p', r') \prec (q, 0)\right| \geq \lambda + 1, \tag{3.10}$$

for all $q \notin \operatorname{supp} S$, and $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ such that $l \ni q$. For any $c > 0$, inequality (3.10) remains valid if we replace $\prec = \prec_\beta$ with $\prec_\alpha$. Furthermore, for every $l \in \mathcal{L}_1, \ldots, \mathcal{L}_d$, the $\prec_\beta$-first $(\lambda + 1)$ elements of $(l \cap J) \times \mathbb{Z}_{\geq 0}$ are also the $\prec_\alpha$-first elements of $(l \cap J) \times \mathbb{Z}_{\geq 0}$. Hence,

$$\tilde{S}(\cdot, \cdot, \beta, \lambda) = \tilde{S}(\cdot, \cdot, \alpha, \lambda),$$

and therefore $(w_p(\alpha))_p = (w_p(\beta))_p$. Therefore, we may take c sufficiently large to ensure that for all $q \notin \operatorname{supp} S$, $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ such that $l \ni q$, and for all $p \in l \cap \operatorname{supp} S$,

$$\alpha_q < \alpha_p - \lambda.$$

This is possible since there are finitely many such choices of q, l and p. This gives the desired α. □

Let us label each $p \in J$ so that $J = \{p_1, \ldots, p_{|J|}\}$. For any α, there is a permutation $\sigma = \sigma_\alpha \in S_{|J|}$ so that $w_{p_{\sigma(1)}}(\alpha) \geq \cdots \geq w_{p_{\sigma(|J|)}}(\alpha)$. By Proposition 3.3.2, the set $w(A)$ is finite. Therefore, of all $w(\alpha) \in w(A)$, we can choose one so that $(w_{p_{\sigma(i)}}(\alpha))_{1 \leq i \leq |J|} \in \mathbb{R}^{|J|}_{\geq 0}$ is minimal with respect to lexicographical order on $\mathbb{R}^{|J|}$.[3] Let

$$w(\alpha) = (w_{p_{\sigma(i)}}(\alpha))_{1 \leq i \leq |J|} \tag{3.11}$$

be such a minimum with corresponding minimiser α. By relabelling the indices of each $p \in J$, we may assume that σ is the identity permutation so that

$$w_{p_1}(\alpha) \geq \cdots \geq w_{p_{|J|}}(\alpha) \geq 0.$$

For ease of notation, let $w_i := w_{p_i}(\alpha)$ for each $1 \leq i \leq |J|$.

Remark. Uniform boundedness has allowed us to choose a minimal α, for an appropriate notion of minimum, *via* a compactness-type argument. This α is the candidate we will use to conclude the proof of Lemma 3.3.1. ◄

Proposition 3.3.4 (Continuity of Perturbations)**.** *Let* $1 \leq t \leq |J|$. *Let*

$$v = \sum_{\substack{1 \leq i \leq t: \\ p_i \in \operatorname{supp} S}} e_i + \sum_{\substack{i: \\ p_i \notin \operatorname{supp} S}} e_i$$

and $\alpha' = \alpha - v$. *There is a constant* $h = h(|J|, S)$, *which does not depend on* λ, *so that*

$$\left|w_i(\alpha) - w_i(\alpha')\right| \leq \frac{h}{2\lambda}$$

---

[3] Consider tuples $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$. If $a_i = b_i$ for all $1 \leq i \leq N$, then $a = b$ under lexicographical order. Otherwise, there is some i so that $a_i \neq b_i$. Let i_0 be the least such index. If $a_{i_0} < b_{i_0}$ then $a < b$ under lexicographical order. Otherwise $b > a$. Lexicographical order is a total order on $\mathbb{R}^N$.

for all $1 \le i \le N$, *where* $N = |\operatorname{supp} S|$.

The first summation in the definition of v, in Proposition 3.3.4, is of most importance. The second summation is included only for technical reasons and one may assume $\operatorname{supp} S = J$ on first reading. The constant h described here is related, but not equal to h' in Lemma 3.3.1.

Proof of Proposition 3.3.4. We construct h directly. Fix $1 \le i \le N$, fix a d-tuple of lines $\boldsymbol{l}$ which realises $w_i(\alpha)$, and fix $\boldsymbol{l}'$ which realises $w_i(\alpha')$. Consider

$$\big|w_i(\alpha) - w_i(\alpha')\big| = \big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}', \alpha')\big|.$$

If $\boldsymbol{l} = \boldsymbol{l}'$ then $\big|w_i(\alpha) - w_i(\alpha')\big| = \big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}, \alpha')\big|$. Otherwise, $\boldsymbol{l} \neq \boldsymbol{l}'$, in which case

$$\big|w_i(\alpha) - w_i(\alpha)\big| = \big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}', \alpha')\big| \le \max_{\tilde{\boldsymbol{l}}} \Big|W(\tilde{\boldsymbol{l}}, \alpha) - W(\tilde{\boldsymbol{l}}, \alpha')\Big|,$$

where the maximum is over all $\tilde{\boldsymbol{l}} \in (\arg\min W(\cdot, \alpha) \cup \arg\min W(\cdot, \alpha'))$.[4] This is an application of the following elementary fact, Proposition 3.3.5.

Proposition 3.3.5. *Let* $\phi, \psi : X \to \mathbb{R}_{\ge 0}$ *be functions for some finite set* X*. Then*

$$\left|\min_x \phi(x) - \min_x \psi(x)\right| \le \max_{x \in (\arg\min \phi \cup \arg\min \psi)} \big|\phi(x) - \psi(x)\big|.$$

Proof. Without loss of generality, we may assume $\min \phi \ge \min \psi$. Let $\phi(a) = \min \phi$ and let $\psi(b) = \min \psi$. Then

$$\begin{aligned} |\min \phi - \min \psi| &= \phi(a) - \psi(b) \\ &\le \phi(b) - \psi(b) \\ &\le \max_{x \in (\arg\min \phi \cup \arg\min \psi)} \big|\phi(x) - \psi(x)\big|. \end{aligned}$$

□

Whether or not $\boldsymbol{l} = \boldsymbol{l}'$, we may assume

$$\big|w_i(\alpha) - w_i(\alpha')\big| \le \Big|W(\tilde{\boldsymbol{l}}, \alpha) - W(\tilde{\boldsymbol{l}}, \alpha')\Big|,$$

for some $\tilde{\boldsymbol{l}}$ which minimises $W(\cdot, \alpha)$ or $W(\cdot, \alpha')$. However, for any $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$ which forms a multijoint at $p = p_i$,

$$\begin{aligned} &S(p)^d(\lambda+1)^d \big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}, \alpha')\big| \\ &= \left|\prod_{j=1}^d \Big(\tilde{S}(p, l_j, \alpha', \lambda) - (\tilde{S}(p, l_j, \alpha', \lambda) - \tilde{S}(p, l_j, \alpha, \lambda))\Big) - \prod_{j=1}^d \tilde{S}(p, l_j, \alpha', \lambda)\right|. \end{aligned} \tag{3.12}$$

---

[4] For a function $\phi : X \to \mathbb{R}$ we let $\arg\min \phi = \{x \in X : \phi(x) = \min \phi\}$.

Taking the right-hand side of equality (3.12), we expand the first product. In this expansion, there is precisely one occurrence of the term $\prod_j \tilde{S}(p, l_j, \alpha', \lambda)$, which now cancels the second product of (3.12). We are left with terms of the form

$$\left(\prod_{j\in A} \tilde{S}(p, l_j, \alpha', \lambda)\right)\left(\prod_{j'\in B} (\tilde{S}(p, l_j, \alpha', \lambda) - \tilde{S}(p, l_j, \alpha, \lambda))\right) \tag{3.13}$$

where $A \sqcup B = \{1, \ldots, d\}$ and $B \neq \emptyset$. To establish an upper bound on (3.12), by the triangle inequality, it suffices to bound each such term of the form (3.13) separately. By continuity (Lemma 3.2.7), for any $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ and $p \in l \cap J$,

$$\left|\tilde{S}(p, l, \alpha, \lambda) - \tilde{S}(p, l, \alpha', \lambda)\right| \leq |J|,$$

since $\|\alpha - \alpha'\|_{\ell^1(J)} = \|v\|_{\ell^1(J)} \leq |J|$. Combining this with the fact that each $\tilde{S}(p, l, \alpha', \lambda) \leq (\lambda+1)$ by construction, each term (3.13) is bounded by $(\lambda+1)^a|J|^b$ for non-negative integers $a + b = d$. Hence, (3.12) is bounded by a polynomial in $(\lambda + 1)$ and $|J|$, and can be dominated by the term which contains $(\lambda + 1)^{(d-1)}$. Hence

$$S(p)^d(\lambda + 1)^d\big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}, \alpha')\big| \leq h_{d-1}(\lambda + 1)^{d-1} + \ldots + h_1(\lambda + 1) + h_0.$$

This in turn is at most $\frac{h}{2}(\lambda + 1)^{d-1}$ provided that h is sufficiently large. We can express h as a polynomial in $|J|$ that does not depend on λ. Dividing by $(\lambda + 1)^d$ and updating h to be additionally depend on $S(p)^{-d}$, we deduce

$$\big|w_i(\alpha) - w_i(\alpha')\big| \leq \big|W(\boldsymbol{l}, \alpha) - W(\boldsymbol{l}, \alpha')\big| \leq \frac{h}{2\lambda}$$

for some h which can be expressed as a function in $|J|$ and the quantities $\{S(p)^{-d} : p \in J\}$, but does not depend on λ. This defines the constant h. □

The remainder of this subsection is dedicated to proving Lemma 3.3.1. Since $w_1(\alpha) \geq \ldots \geq w_{|J|}(\alpha) \geq 0$, it suffices to show that all the differences $w_i(\alpha) - w_{i+1}(\alpha)$ are small. We will prove this by contradiction. To begin, we assume there is some index $1 \leq t < N$ so that $w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda$, with h as given by Proposition 3.3.4. We construct a perturbation of the handicap α. Let α' be the perturbed handicap. The following three claims are established for the perturbation:

1. The large entries of the tuple $w(\alpha)$ remain large, and the small entries remain small.
2. The perturbed handicap α' is such that $w(\alpha') \in w(A)$, so $w(\alpha) \leq w(\alpha')$ with respect to lexicographical order.
3. If $w(\alpha) \neq w(\alpha')$ then $w(\alpha') < w(\alpha)$. Hence, $w(\alpha') = w(\alpha)$.

Thereafter, we realise that the perturbation can be applied to α', the already perturbed handicap. However, by connectedness, there is a pair $p, q \in \operatorname{supp} S$

so that $w_p(\alpha)$ is large, $w_q(\alpha)$ is small and there is a line which contributes to both p and q. If we perturb sufficiently many times, what results is a perturbed handicap α' so that $w(\alpha) = w(\alpha')$, and

$$\alpha'_p < \alpha'_q - \lambda.$$

By uniform boundedness (Lemma 3.2.5), $\tilde{S}(p, l, \alpha', \lambda) = 0$ and hence $w_p(\alpha) = 0$. Therefore, $0 = w_p(\alpha) \geq w_q(\alpha) \geq 0$, and hence $w_q(\alpha) = 0$. However, we assumed, for a contradiction, that $w_p(\alpha)$ is large and that $w_q(\alpha)$ is small so that $w_p(\alpha) > w_q(\alpha)$, which is a contradiction.

Proof of Lemma 3.3.1. Let $\operatorname{supp} S = \{p_1, \ldots, p_N\}$ for some $N \leq |J|$ be connected, and let $h' = |J|h$, where h is as given by Proposition 3.3.4. Since the tuple $(w_i)_i$ has at most $|J|$ non-zero entries, it suffices to show that

$$w_i - w_{i+1} \leq \frac{h}{\lambda} \tag{3.14}$$

for all $1 \leq i < N$. Indeed, summing this inequality over all $1 \leq i \leq N$, we deduce $\max_i w_i - \min_i w_i \leq |J|h/\lambda$. That is, the interval containing all $(w_i)_{1 \leq i \leq N}$ has width $\sim_J 1/\lambda$, where the implied constant does not depend on λ.

We now prove that (3.14) holds for all $1 \leq i < N$. Suppose for a contradiction that there is some index $1 \leq i < N$ so that (3.14) is false. Let t be the least such index. For this choice of t, define

$$v = \sum_{\substack{1 \leq i \leq t: \\ p_i \in \operatorname{supp} S}} e_i + \sum_{\substack{i: \\ p_i \notin \operatorname{supp} S}} e_i$$

and let $\alpha' = \alpha - v$.

Let $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$. Recall that $\sum_{p \in l} \tilde{S}(p, l, \alpha, \lambda) = (\lambda + 1)$ for any α. It follows that, if $\tilde{S}(p, l, \alpha', \lambda) > \tilde{S}(p, l, \alpha, \lambda)$ for some $p \in l \cap J$, then there exists some $p' \in l \cap J$ so that $p' \neq p$ and $\tilde{S}(p', l, \alpha', \lambda) < \tilde{S}(p', l, \alpha, \lambda)$. For those i such that $p_i \notin \operatorname{supp} S$, we have that $w_i(\alpha) = 0 = w_i(\alpha')$. Moreover, by monotonicity (Lemma 3.2.6), for i such that $p_i \in \operatorname{supp} S$, we have that if $i \leq t$ then $w_i(\alpha') \leq w_i(\alpha)$ (where the handicap is decreased) and if $i > t$ then $w_i(\alpha') \geq w_i(\alpha)$ (where the handicap is unchanged). Of all the inequalities indexed by $i \leq t$, at least one is strict if and only if another inequality indexed by $i > t$ is also strict. By Proposition 3.3.4, the difference $\left|w_i(\alpha) - w_i(\alpha')\right|$ is bounded above by $h/(2\lambda)$ for all $1 \leq i \leq N$, and by our definition of t,

$$w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda. \tag{3.15}$$

Let $\sigma' \in S_{|J|}$ be a permutation such that[5]

$$w_{\sigma'(1)}(\alpha') \geq \cdots \geq w_{\sigma'(|J|)}(\alpha'), \tag{3.16}$$

---

[5] A convenient way to understand what σ' describes is as follows: the i-th largest multijoint, p, of the tuple $(w_p(\alpha'))_{p \in J}$ has index $\sigma'(i)$.

where $\sigma'(i) = i$ for $N < i \leq |J|$.

Claim 3.3.6. If $1 \leq i \leq t$, then $1 \leq \sigma'(i) \leq t$, and if $t < i' \leq N$, then $t < \sigma'(i') \leq |J|$.

Proof. Suppose $p_i, p_{i'} \in \operatorname{supp} S$ are such that $i \leq t < i'$. By Proposition 3.3.4,

$$\begin{aligned} w_i(\alpha') - w_{i'}(\alpha') &= \big(w_i(\alpha') - w_i(\alpha)\big) + \big(w_i(\alpha) - w_{i'}(\alpha)\big) + \big(w_{i'}(\alpha) - w_{i'}(\alpha')\big) \\ &\geq -\frac{h}{2\lambda} + \big(w_i(\alpha) - w_{i'}(\alpha)\big) - \frac{h}{2\lambda}. \end{aligned}$$

Using inequalities (3.15) and the fact that $w_i(\alpha)$ is decreasing in i, this is at least

$$\big(w_t(\alpha) - w_{t+1}(\alpha)\big) - \frac{h}{\lambda} > -\frac{h}{\lambda} + \frac{h}{\lambda} = 0.$$

Hence, if $1 \leq i \leq t$, then $w_i(\alpha') \geq w_{i'}(\alpha')$ for all $i' > t$. Therefore $1 \leq \sigma'(i) \leq t$. Similarly, if $t < i \leq N$, then $w_{i'}(\alpha') \geq w_i(\alpha')$ for all $1 \leq i' \leq t$. Hence, $t < \sigma'(i) \leq N$, as desired. □

Claim 3.3.7. Let the set A be as in Proposition 3.3.2. Then $w(\alpha') \in w(A)$. Hence $w(\alpha')$ was among those considered when the minimiser $w(\alpha)$ was chosen.

Proof. Since $w(\alpha) \in W(A)$, we have that $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ for all $p \notin \operatorname{supp} S$. Hence the assumptions of Proposition 3.3.3 are satisfied, and we may assume that α is such that for all $q \notin \operatorname{supp} S$, and all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ so that $l \ni q$, if $p \in l \cap \operatorname{supp} S$, then

$$\alpha_q < \alpha_p - \lambda.$$

We will show that $\tilde{S}(q, \cdot, \alpha', \lambda) = 0$ for all $q \notin \operatorname{supp} S$. Let $q \notin \operatorname{supp} S$. If $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ so that $l \ni q$, and $p \in l \cap \operatorname{supp} S$, then

$$\alpha'_q = \alpha_q - 1 < (\alpha_p - \lambda) - 1 = (\alpha_p - 1) - \lambda \leq \alpha'_p - \lambda.$$

We assumed, at the beginning of Subsection 3.3.1, that every line in $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ intersects $\operatorname{supp} S$, so such p exists. Hence, by uniform boundedness (Lemma 3.2.5), $\tilde{S}(q, l, \alpha', \lambda) = 0$.

We now show that $\tilde{S}(p, \cdot, \alpha', \lambda) \geq 1$ for all $p \in \operatorname{supp} S$. Since $w(\alpha) \in W(A)$, we have that $w_p(\alpha) > 0$ for all $p \in \operatorname{supp} S$. We have shown that if $t < i \leq |J|$, then $w_i(\alpha') \geq w_i(\alpha)$. Hence $w_i(\alpha') > 0$ for all i such that $t < i \leq N$. For $1 \leq i \leq t$, we have that $w_i(\alpha') > w_N(\alpha')$. Hence $w_i(\alpha') > 0$ for all $1 \leq i \leq N$. Consequently, $\tilde{S}(p, l, \alpha', \lambda) \geq 1$ for all $p \in \operatorname{supp} S$ and all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ which contribute to p, and $w(\alpha') \in w(A)$.[6]

That is, $(w_p(\alpha'))_p$ was among those tuples considered when $(w_p(\alpha))_p$ was chosen. □

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[6] Note that we cannot deduce the positivity of $\tilde{S}(p, l, \alpha', \lambda)$ for those l which do not contribute to p.

Claim 3.3.8. If $w(\alpha') \neq w(\alpha)$ then $w(\alpha') < w(\alpha)$ with respect to lexicographical order. Hence $w(\alpha) = w(\alpha')$.

Proof. Since $w_1(\alpha) \geq \cdots \geq w_{|J|}(\alpha)$, we have that $(w_i(\alpha))_{1\leq i\leq t} \geq (w_{\sigma'(i)}(\alpha))_{1\leq i\leq t}$ with respect to lexicographical order.

Suppose that there is some $1 \leq i \leq |J|$ so that $w_i(\alpha') \neq w_i(\alpha)$. Then there is an $i \leq t$ so that $w_i(\alpha') < w_i(\alpha)$ and $1 \leq \sigma'(i) \leq t$. That is, $w_i(\alpha')$ is strictly smaller than $w_i(\alpha)$ and $w_{\sigma'(i)}(\alpha')$ is among the t largest values of $(w_i(\alpha'))_{1\leq i\leq |J|}$ by Claim 3.3.6. Hence

$$(w_{\sigma'(i)}(\alpha))_{1\leq i\leq t} > (w_{\sigma'(i)}(\alpha'))_{1\leq i\leq t},$$

and therefore $w(\alpha')$ is of strictly lower lexicographical order than $w(\alpha)$. Moreover, by Claim 3.3.7, $w(\alpha') \in w(A)$. This contradicts that $w(\alpha) \in w(A)$ was chosen to be minimising. ☐

Thus, $w_i(\alpha') = w_i(\alpha - v) = w_i(\alpha)$ for all $1 \leq i \leq |J|$. We have not yet contradicted our assumption that (3.14) is false, however, we have deduced that a perturbation $\alpha \mapsto (\alpha - v)$ must leave $(w_i)_i$ unchanged. To conclude, we observe that we may return to (3.11), use $\alpha' = \alpha - v$ as our minimiser and repeat the application of Claim 3.3.6, Claim 3.3.7 and Claim 3.3.8. We may repeat this process *ad infinitum* to deduce that

$$w_i(\alpha) = w_i(\alpha - v) = w_i(\alpha - 2v) = \cdots = w_i(\alpha - cv)$$

for all $1 \leq i \leq |J|$ and any $c \in \mathbb{N}$. By the connectedness of $\operatorname{supp} S$, we can find a line $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ contributing to distinct multijoints $p_i, p_j \in \operatorname{supp} S$ for some i and j satisfying $i \leq t < j \leq N$. Taking c sufficiently large (depending on λ) forces $w_i(\alpha') = 0$ by uniform boundedness (Lemma 3.2.5) and hence $w_i(\alpha) = 0$. That is, one of the large entries of the tuple $(w_i(\alpha))_i$ is zero. Since $(w_i)_i$ is decreasing in i and each w_i is non-negative, $w_{i'}(\alpha) = 0$ for all $i \leq i' \leq |J|$. This contradicts our assumption that $w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda$, and hence (3.14) holds for all $1 \leq i < N$. ☐

3.3.2 Good Vanishing Conditions Yield a Factorisation

We now deduce Theorem 3.0.1 as a corollary of Lemma 3.3.1. Let $S : J \to \mathbb{R}_{\geq 0}$ be a test function with $\|S\|_d = 1$ and let λ be sufficiently large. By Lemma 3.3.1, we may choose a handicap, $\alpha = \alpha(\lambda)$, so that the numbers $\tilde{S}(p, l_j, \alpha, \lambda)$, as constructed in Subsection 3.2.1, are such that there is an interval containing

$$w_p(\alpha) = \min_{\delta(p,\mathbf{l})=1} \frac{1}{S(p)^d} \left(\prod_{j=1}^{d} \frac{\tilde{S}(p, l_j, \alpha, \lambda)}{\lambda + 1} \right)$$

for all $p \in \operatorname{supp} S$, with length proportional to $1/\lambda$. Moreover, $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ for all p such that $S(p) = 0$. We now show that the upper endpoint w of the interval is $\gtrsim_d 1$. Thereafter, since we can choose λ so that the interval has arbitrarily small width, we deduce that everything contained in that interval is also $\gtrsim_d 1$.

Let $\lambda \in \mathbb{N}$ and let $\varepsilon := h/\lambda$, where h is given by Lemma 3.3.1. We can find ω so that

$$w - \varepsilon \leq w_p(\alpha) \leq w$$

for all $p \in \operatorname{supp} S$ and so that $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ if $S(p) = 0$. For each $p \in J$, let $(l_j(p))_j$ be a choice of lines which minimises $\prod_{j=1}^{d} \tilde{S}(p, l_j(p), \alpha, \lambda)$ over all tuples $\boldsymbol{l}$ so that $\delta(p, \boldsymbol{l}) = 1$. Corollary 3.2.4a applies and hence,

$$w = \sum_{p\in J} S(p)^d w \geq \sum_{p\in J} \prod_{j=1}^{d} \frac{\tilde{S}(p, l_j(p), \alpha, \lambda)}{\lambda+1} \geq \frac{1}{(\lambda+1)^d}\binom{\lambda+d}{d} \gtrsim_d 1. \qquad (3.17)$$

This implies that $w \gtrsim_d 1$.

Remark. Our argument is such that the lower endpoint, $(w - \varepsilon)$ will increase with λ. In contrast, the lower bound on the upper endpoint, w, is uniform in λ. Hence, we use the upper endpoint (and not the lower one) in order to establish the first inequality of Theorem 3.0.1. ◀

Therefore, each $B(p, l_j, \alpha, \lambda) \neq \emptyset$ as long as $\varepsilon = h/\lambda$ is sufficiently small, or equivalently, for λ sufficiently large. Set

$$s_\lambda(p, l_j) := \tilde{S}(p, l_j, \alpha, \lambda)/(\lambda + 1).$$

For each $p \in J$, if $l_j \in \mathcal{L}_j$ is such that $p \notin l_j$ then set $s_\lambda(p, l_j) = 0$. Then for any $p \in \operatorname{supp} S$, we have

$$1 \lesssim_d w \leq \frac{h}{\lambda} + \frac{1}{S(p)^d} \prod_{j=1}^{d} s_\lambda(p, l_j(p)).$$

Remark. As suggested previously, from the inequality above, we could deduce that every $s_\lambda(p, l_j) \neq 0$, and hence $s_\lambda(p, l) \neq 0$ for all $p \in \operatorname{supp} S$ and all l that contribute to p. ◀

Since $\|S\|_d = 1$, $S(p) \leq 1$ for all $p \in J$. Hence

$$S(p)^d \lesssim_d \frac{h}{\lambda} + \prod_{j=1}^{d} s_\lambda(p, l_j)$$

for a tuple of lines $\boldsymbol{l}$ that minimises the right-hand side over all tuples such that $\delta(p, \boldsymbol{l}) = 1$. Hence

$$\delta(p, \boldsymbol{l}) S(p)^d \lesssim_d \frac{h}{\lambda} + \prod_{j=1}^{d} s_\lambda(p, l_j) \qquad (3.18)$$

for all tuples of lines $\boldsymbol{l}$. By (3.4), for all $1 \leq j \leq d$, $l_j \in \mathcal{L}_j$,

$$\sum_{p \in l_j \cap J} s_\lambda(p, l_j) = \frac{1}{\lambda+1} \sum_{p \in l_j \cap J} \tilde{S}(p, l_j, \alpha, \lambda) = 1. \tag{3.19}$$

Inequality (3.18) and equation (3.19) are uniform in λ. For any λ, the function

$$\frac{1}{S(p)} s_\lambda(p, l) = \frac{1}{S(p)} \frac{\tilde{S}(p, l, \alpha, \lambda)}{\lambda+1}$$

can be realised as an $\mathbb{R}$-valued vector in $[0,1]^{|J| \times |\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d|}$. Hence, passing to a subsequence if necessary, we may let $s = \lim_{\lambda \to \infty} s_\lambda$. Letting $\lambda \to \infty$ in (3.18) and (3.19) shows that this s satisfies the properties stated in Theorem 3.0.1.

Remark. We could fix a sufficiently large λ to conclude the proof at (3.19). However, taking $\lambda \to \infty$ yields the best dimensional constant this argument can offer. ◀

3.3.3 The Discrete Bourgain–Guth Theorem

We now conclude the proof of Theorem 3.0.2. Let $S : \mathbb{F}^d \to \mathbb{R}_{\geq 0}$ with $\|S\|_d = 1$ be finitely supported. If the field $\mathbb{F}$ is finite, then we may set each $\mathcal{L}_j$ to be the set of all lines, apply Theorem 3.0.1, and we are done. Hence, we may assume that $\mathbb{F}$ is infinite.

Let $\mathcal{L}$ be the set of lines l which pass through at least two elements of $\operatorname{supp} S$. Since S is finitely supported, there are at most finitely many lines in $\mathcal{L}$. Consider the joints J formed by $\mathcal{L}$. If $\operatorname{supp} S$ is not a subset of J then we supplement $\mathcal{L}$ with additional lines as follows. For each $p \in J$, consider $\mathcal{L}_p = \{l \in \mathcal{L} : p \in l\}$. For each $1 \leq k \leq d$, let $l_1, \ldots, l_k \in \mathcal{L}_p$ have linearly independent directions. Since the field is infinite, we can find $(d-k)$ lines, $l_{k+1}, \ldots, l_d$ which contain p and satisfy $e(l_1) \wedge \cdots \wedge e(l_d) = 1$. We repeat this procedure for all of the k-tuples of linearly independent lines in $\mathcal{L}_p$ for all $1 \leq k \leq d$, and for all $p \in \operatorname{supp} S \setminus J$. In doing so, we add at most finitely many lines to $\mathcal{L}$.

Now, let $\mathcal{L}_j = \mathcal{L}$ for each $1 \leq j \leq d$. Then each $\mathcal{L}_j$ is a finite set of lines and the associated set of joints J is such that $\operatorname{supp} S \subseteq J$ and $\|S\|_d = 1$. Hence Theorem 3.0.1 applies, so there exists a function $s : J \times \mathbb{S}^{d-1} \to \mathbb{R}_{\geq 0}$ so that

$$\delta(p, \boldsymbol{l}) S(p)^d \leq C \prod_{j=1}^{d} s(p, e(l_j)), \tag{3.20}$$

for all $p \in J$ and $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$, and so that

$$\sum_{p \in l \cap J} s(p, e(l)) = 1 \tag{3.21}$$

for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$.

We now extend s so that it is defined on $\mathbb{F}^d \times \mathbb{S}^{d-1}$. Observe from Subsection 3.2.1 that if $l \cap J = \{p\}$, then $\tilde{S}(p, l, \alpha, \lambda) = \lambda + 1$ for any handicap α and any $\lambda \in \mathbb{N}$. Therefore, s is such that for any line $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ so that $l \cap J = \{p\}$, $s(p, e(l)) = 1$. Every line which contains at least two elements of $\operatorname{supp} S$ is an element of $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$. Hence, any line not in $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ contains at most 1 element of $\operatorname{supp} S$. If $l \cap \operatorname{supp} S = \{p\}$ then we define $s(p, e(l)) = 1$, and if $l \cap \operatorname{supp} S = \emptyset$, then we define $s(p, e(l)) = 0$ for all $p \in l$. Finally, for any $p \notin \operatorname{supp} S$, we set $s(p, \cdot) = 0$. Thus s is defined on $\mathbb{F}^d \times \mathbb{S}^{d-1}$. For any $l \notin \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$, we have that

$$\sum_{p \in \mathbb{F}} s(p, e(l)) \leq 1.$$

Since we already have that (3.21) holds for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$, the second display of Theorem 3.0.2 follows. Turning to the first display, let $\omega_1, \ldots, \omega_d \in \mathbb{S}^{d-1}$ be such that $\omega_1 \wedge \cdots \wedge \omega_d = 1$ and let $p \in \mathbb{F}^d$. For each $1 \leq j \leq d$, let $l_j = p + \mathbb{F} \cdot \omega_j$. Suppose that $l_j \in \mathcal{L}_j$ for each $1 \leq j \leq k$ and $l_j \notin \mathcal{L}_j$ for $k < j \leq d$. By construction, there are lines $l'_j \in \mathcal{L}_j$ so that $l'_j = l_j$ for each $1 \leq j \leq k$ and so that $\delta(p, \boldsymbol{l}') = 1$. Since (3.21) holds for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ we have that $s(p, e(l)) \leq 1$ for all such lines. If $k < j \leq d$, then $l_j \notin \mathcal{L}_j$ and so $s(p, \omega_j) = 1$. Thus $s(p, e(l'_j)) \leq s(p, \omega_j)$ for $j < k \leq d$. Hence,

$$(\omega_1 \wedge \cdots \wedge \omega_d) S(p)^d = \delta(p, \boldsymbol{l}') S(p)^d \lesssim \prod_{j=1}^{d} s(p, e(l')) \leq \prod_{j=1}^{d} s(p, \omega_j).$$

This verifies the first display, as desired.

3.4 Abstracting the Argument

Recall that the multijoint problem is the discrete analogue of the multilinear Kakeya theorem. Now that we have a proof of the discrete Bourgain–Guth theorem, further to being analogous to the multilinear Kakeya theorem, we can see that the proof of the discrete Bourgain–Guth theorem is also analogous to the Carbery–Valdimarsson proof of the Bourgain–Guth theorem, [CV13]. Indeed, in both proofs, the implied constants are derived from covering lemmas. In the continuous case, we use the Borsuk–Ulam theorem *via* the equivalent Lusternik–Schnirelmann lemma, and the analogous covering statement for the discrete case is parameter counting. This suggests that to prove other multilinear problems, for example counting multijoints where the ambient space $\mathbb{F}^d$ is replaced with a d-dimensional algebraic variety or multilinear Kakeya on a manifold, it should suffice to find an appropriate covering lemma. However, the analogous Bourgain–Guth-type results are interesting in their own right.

Continuous	Discrete	Other $\wedge$
Lusternik–Schnirelmann	Parameter Counting	Covering Lemma
$\Downarrow$	$\Downarrow$	$\Downarrow$
Bourgain–Guth	Discrete Bourgain–Guth	Analogous Bourgain–Guth
$\Downarrow$	$\Downarrow$	$\Downarrow$
Multilinear Kakeya Theorem	Multijoint Problem	Geometric Multilinear Inequality

3.4.1 Parameter Counting as a Covering Lemma

Parameter counting is traditionally viewed as an existence statement. However, it can be easily interpreted as a covering lemma. Consider the key lemmas, Lemma 3.4.1 and Lemma 3.4.2, for the multilinear Kakeya and discrete Bourgain–Guth theorems, respectively.

Lemma 3.4.1 (Lusternik–Schnirelmann)**.** *Suppose that* $A_i \subset \mathbb{S}^N$ *for* $1 \leq i \leq m$, *and suppose that for each* i, $A_i \cap (\overline{-A_i}) = \emptyset$. *If* $m \leq N$, *then there is some* $x \in \mathbb{S}^N$ *not contained in the union of the* $2m$ *sets* A_i *and* $-A_i$.

Lemma 3.4.2 (Parameter Counting)**.** *Let* $\phi_1, \ldots, \phi_m$ *be homogeneous linear functionals which act on* $\mathbb{F}_\lambda[x_1, \ldots, x_d]$. *If* $m < \binom{d+\lambda}{d}$ *then there is a non-zero* $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$ *so that* $\langle \phi_i, f \rangle = 0$ *for each* $1 \leq i \leq m$.

Recall that one definition of the sphere $\mathbb{S}^N$, with respect to $\mathbb{R}$, is as a quotient of $\mathbb{R}^{N+1} \setminus \{0\}$ under the equivalence relation $x \sim y$ if and only if $x = ty$ for some $t \in \mathbb{R}$. Suppose that $f \in \mathbb{F}_\lambda[x_1, \ldots, x_d]$, and ϕ is a linear functional acting on $\mathbb{F}_\lambda[x_1, \ldots, x_d]$. Similarly, if $\langle \phi, f \rangle = 0$ then $\langle \phi, tf \rangle = 0$. That is, when we apply parameter counting, we are choosing a 1-dimensional vector space of polynomials, all of which may be thought of as being "the same" from the perspective of $\phi_1, \ldots, \phi_m$. In particular, we may regard $\mathbb{F}_\lambda[x_1, \ldots, x_d]$ as $\mathrm{Gr}(1, \mathbb{F}^d)$, or a $\left(\binom{\lambda+d}{d} - 1\right)$-sphere. Under this analogy, the Lusternik–Shnirelmann and parameter counting lemmas are different statements of the same covering lemma when $\mathbb{F} = \mathbb{R}$. Indeed, the set $\{f : \langle \phi_i, f \rangle = 0\}$ describes a great circle in $\mathbb{F}_\lambda[x_1, \ldots, x_d]$. Hence the sets $\{f : \langle \phi_i, f \rangle < 0\}$ replace A_i, where $A_i \cap (\overline{-A_i}) = \emptyset$, as in the hypothesis of the Lusternik–Schnirelmann Lemma, Lemma 3.4.1. Therefore, the statement that the sets A_i and $-A_i$ do not cover $\mathbb{S}^N$ is equivalent to saying that there is a point in the intersection of all the great circles $(\{f : \langle \phi_i, f \rangle = 0\})_{i=1}^m$.

For the multilinear Kakeya theorem and multijoints problem, the sets to which the covering lemma is applied are defined by sets of bisecting polynomials and polynomials vanishing to high order, respectively.

The Lusternik–Schnirelmann lemma is equivalent to the Borsuk–Ulam Theorem, which is a profoundly topological result that is specific to the real numbers. In contrast, the parameter counting lemma relies only on linear algebra and is valid for any field.

Chapter 4

The Generalised Discrete Bourgain–Guth Theorem

Regarding lines as degree 1 varieties of dimension 1, Theorem 3.0.1 and Theorem 3.0.2 can be generalised in both dimension and degree. Given a k-plane π, let us define the **direction** of π to be the k-dimensional vector space that is parallel to π, and denote it by $e(\pi)$. Suppose $\pi_1, \ldots, \pi_d$ are k_1-, $\ldots, k_d$-planes in $\mathbb{F}^n$, respectively, where $k_1 + \ldots + k_d = n$. We define the discrete wedge product on k_j-planes by

$$\wedge_{j=1}^{d} e(\pi_j) := \wedge_{j=1}^{d} \wedge_{k=1}^{k_j} \omega_{j,k},$$

where $\omega_{j,1}, \ldots, \omega_{j,k_j} \in \mathbb{S}^{n-1}$ is a choice of k_j vectors which form a basis for $e(\pi_j)$ for each $1 \leq j \leq d$. We define δ at $p \in \mathbb{F}^n$ by

$$\delta(p, \boldsymbol{\pi}) := \left(\prod_{j=1}^{d} \chi_{\pi_j}(p) \right) \wedge_{j=1}^{d} e(\pi_j).$$

For $p \in \mathbb{F}^n$, if there is a tuple $\boldsymbol{\pi}$ so that $\delta(p, \boldsymbol{\pi}) = 1$, we say that p is a ***k*-multijoint**, or multijoint in short.

Theorem 4.0.1 (***k***-Multijoint Factorisation)**.** *Let $\Pi_1, \ldots, \Pi_d$ be finite sets of k_1-$, \ldots, k_d$-planes in $\mathbb{F}^n$, respectively, so that $k_1 + \ldots + k_d = n$, and define $J = \{p : \exists \boldsymbol{\pi}$ so that $\delta(p, \boldsymbol{\pi}) = 1\}$. There is a constant $C = C(k_1, \ldots, k_d)$ so that for all $S : J \to \mathbb{R}_{\geq 0}$ with $\|S\|_d = 1$ there exists a function $s_k : J \times \mathrm{Gr}(k_j, \mathbb{F}^n) \to \mathbb{R}_{\geq 0}$ for each $k \in \{k_1, \ldots, k_d\}$ so that*

$$\delta(p, \boldsymbol{\pi}) S(p)^d \leq C \prod_{j=1}^{d} s_{k_j}(p, e(\pi_j)),$$

for all $p \in J$ and $\boldsymbol{\pi} \in \Pi_1 \times \cdots \times \Pi_d$, and so that

$$\sum_{p \in \pi_j \cap J} s_{k_j}(p, \pi_j) = 1$$

for all $\pi_j \in \Pi_j$ *and all* $1 \leq j \leq d$.

As in the case for lines, we will deduce the stronger freestanding result, Theorem 4.0.2, below:

Theorem 4.0.2 (Discrete Bourgain–Guth for k_j-Planes)**.** *Let* $k_1 + \ldots + k_d = n$. *For all finitely supported* $S : \mathbb{F}^n \to \mathbb{R}_{\geq 0}$ *with* $\|S\|_d = 1$, *for each* $1 \leq j \leq d$ *there exists a function* $s_{k_j} : \mathbb{F}^n \times \mathrm{Gr}(k_j, \mathbb{F}^n) \to \mathbb{R}_{\geq 0}$ *for each* $1 \leq j \leq d$ *so that*

$$(V_1 \wedge \cdots \wedge V_d) S(p)^d \lesssim_{k_1,\ldots,k_d} \prod_{j=1}^{d} s_{k_j}(p, V_j),$$

for all $p \in \mathbb{F}^n$ *and* $V_j \in \mathrm{Gr}(k_j, \mathbb{F}^n)$ *for* $1 \leq j \leq d$, *and so that for any* $1 \leq j \leq d$,

$$\sum_{p \in \pi_j} s_{k_j}(p, e(\pi_j)) \leq 1$$

for all affine k_j*-subspaces* $\pi_j \subset \mathbb{F}^n$.

The factorising functions s_{k_j} have a subscript, k_j since the domain of the second argument of the function, $\mathrm{Gr}(k_j, \mathbb{F}^n)$, depends on j in contrast to Theorem 3.0.2 where each of these Grassmannian spaces is $\mathbb{S}^{d-1} = \mathrm{Gr}(1, \mathbb{F}^d)$.

Remark. Let $d = 2$. Then every multijoint problem with two sets of lines (1-dimensional (over $\mathbb{C}$) affine subspaces) in $\mathbb{C}^2$ is equivalent to a multijoint problem with two sets of 2-dimensional (with respect to $\mathbb{R}$) planes in $\mathbb{R}^4$. In each setting, we can construct two sets of N lines or planes in $\mathbb{C}^2$ or $\mathbb{R}^4$, respectively, such that every pair of distinct lines or planes intersect and form a multijoint. Such configurations therefore have $\binom{N}{2}$ multijoints. As N becomes large, this is approximately $N^2/2$. Hence, the optimal constant for both of these problems is 1/2. However, the implied constant from Theorem 4.0.2 is a function of $k_1, \ldots, k_d$ and therefore, our argument would yield different constants for these two problems. This difference is due to varying dimensions of polynomial vector spaces and not properties of the field we are working with. ◀

We can use this to establish the combinatorially more general result, also discussed by Tidor, Yu and Zhao in [TYZ20].

Let $\boldsymbol{m} = (m_1, \ldots, m_r)$ and $\boldsymbol{k} = (k_1, \ldots, k_r)$ be tuples of positive integers and let $m_1 k_1 + \ldots + m_r k_r = n$ be the ambient dimension and $m_1 + \ldots + m_r = d$ be the level of multilinearity.

Remark. To retain the integer d as the level of multilinearity, we use $r \in \mathbb{N}$ to denote the number of sets of planes. ◀

Let Π_j be a set of k_j-planes in $\mathbb{F}^n$ for each $1 \leq j \leq r$. Let $p \in \mathbb{F}^n$ and $\boldsymbol{\pi} = (\pi_{j,m})_{1 \leq j \leq r, 1 \leq m \leq m_j}$ be such that $\pi_{j,m} \in \Pi_j$ for each $1 \leq m \leq m_j$ and $1 \leq j \leq r$. Define

$$\delta(p, \boldsymbol{\pi}) := \left(\prod_{j=1}^{r} \prod_{m=1}^{m_j} \chi_{\pi_{j,m}}(p) \right) \wedge_{j=1}^{r} \wedge_{m=1}^{m_j} e(\pi_{j,m}).$$

If $\delta(p,\boldsymbol{\pi})=1$, we say that the planes $\boldsymbol{\pi}$ **form an $(\boldsymbol{m},\boldsymbol{k})$-multijoint** at p. That is to say, $p\in\mathbb{F}^n$ is an $(\boldsymbol{m},\boldsymbol{k})$-multijoint formed by $\boldsymbol{\pi}$ if $\pi_{j,m}$ contains p for each $1\le j\le r$ and $1\le m\le m_j$, and the directions $(e(\pi_{j,m}))_{j,m}$ span $\mathbb{F}^n$. The definition of $(\boldsymbol{m},\boldsymbol{k})$-multijoints is a generalisation of both joints and multijoints.

Let J be the set of associated $(\boldsymbol{m},\boldsymbol{k})$-multijoints. For each $1\le j\le r$, let $\tilde{\Pi}_{j,m}=\Pi_j$ for each $1\le m\le m_j$. The collections of planes $(\tilde{\Pi}_{j,m})_{j,m}$ defines a multijoint problem. Hence, by Theorem 4.0.2, for any $S:J\to\mathbb{R}_{\ge 0}$ with $\|S\|_d=1$, there are functions $s_{k_j}:J\times\mathrm{Gr}(k_j,\mathbb{F}^n)\to\mathbb{R}_{\ge 0}$, for each $1\le j\le d$, so that

$$\delta(p,\boldsymbol{\pi})S(p)^d\le C\prod_{j=1}^{d}s_{k_j}(p,e(\pi_j)),$$

for all $p\in J$ and $\boldsymbol{\pi}\in\Pi_1^{m_1}\times\cdots\times\Pi_r^{m_r}$, and so that

$$\sum_{p\in\pi_j\cap J}s_{k_j}(p,e(\pi_j))=1 \tag{4.1}$$

for all $\pi_j\in\Pi_j$ and all $1\le j\le d$. We define $M_{(\boldsymbol{m},\boldsymbol{k})}(p)$, the associated $(\boldsymbol{m},\boldsymbol{k})$-multijoint operator by

$$M_{(\boldsymbol{m},\boldsymbol{k})}[f_1,\dots,f_r](p)=\sum_{(\pi_{1,t_1})_{t_1=1}^{m_1}\in\Pi_1^{m_1}}\cdots\sum_{(\pi_{r,t_r})_{t_r=1}^{m_r}\in\Pi_r^{m_r}}\delta(p,\boldsymbol{\pi})\prod_{j=1}^{r}\frac{1}{m_j!}\prod_{t_j=1}^{m_j}f_j(\pi_{j,t_j})$$

for arbitrary $f_j:\Pi_j\to\mathbb{R}_{\ge 0}$, and where we temporarily use $\boldsymbol{\pi}$ to denote the d-tuple of planes $\big((\pi_{j,t_j})_{1\le t_j\le m_j}\big)_{1\le j\le r}$. We include the factors of $m_j!$ so that (unordered) m_j-tuples are not double counted. Then

$$\begin{aligned}&\sum_{p\in J}\left(\left(\prod_{j=1}^{r}m_j!\right)M_{(\boldsymbol{m},\boldsymbol{k})}[f_1,\dots,f_r](p)\right)^{\frac{1}{d}}S(p)\\&=\sum_{p\in J}\left(\sum_{(\pi_{1,t_1})_{t_1=1}^{m_1}\in\Pi_1^{m_1}}\cdots\sum_{(\pi_{r,t_r})_{t_r=1}^{m_r}\in\Pi_r^{m_r}}\delta(p,\boldsymbol{\pi})\prod_{j=1}^{r}\prod_{t_j=1}^{m_j}f_j(\pi_{j,t_j})S(p)^d\right)^{\frac{1}{d}}\end{aligned}$$

By the properties of s_{k_j}, up to a constant $C=C(k_1,m_1,\dots,k_r,m_r)$ given by Theorem 4.0.2, this is at most

$$\sum_{p\in J}\left(\sum_{(\pi_{1,t_1})_{t_1=1}^{m_1}\in\Pi_1^{m_1}}\cdots\sum_{(\pi_{r,t_r})_{t_r=1}^{m_r}\in\Pi_r^{m_r}}\prod_{j=1}^{r}\prod_{t_j=1}^{m_j}f_j(\pi_{j,t_j})s_{k_j}(p,e(\pi_{j,t_j}))\right)^{\frac{1}{d}}.$$

Applying Hölder's inequality, we bound this from above by

$$\prod_{j=1}^{r}\prod_{t_j=1}^{m_j}\left(\sum_{p\in J}\sum_{\substack{\pi_{j,t_j}\in\Pi_j:\\ p\in\pi_{j,t_j}}} f_j(\pi_{j,t_j})s_{k_j}(p,e(\pi_{j,t_j}))\right)^{\frac{1}{d}}.$$

Note that each multiplicand of the inner product no longer depends on t_j. Hence, this is equal to

$$\prod_{j=1}^{r}\left(\sum_{p\in J}\sum_{\substack{\pi_j\in\Pi_j:\\ p\in\pi_j}} f_j(\pi_j)s_{k_j}(p,e(\pi_j))\right)^{\frac{m_j}{d}}.$$

Additionally changing the order of summation and then applying (4.1), this is equal to

$$\prod_{j=1}^{r}\left(\sum_{\pi_j\in\Pi_j} f_j(\pi_j)\sum_{p\in\pi_j\cap J} s_{k_j}(p,e(\pi_j))\right)^{\frac{m_j}{d}}=\prod_{j=1}^{r}\left(\sum_{\pi_j\in\Pi_j} f_j(\pi_j)\right)^{\frac{m_j}{d}}=\prod_{j=1}^{r}\|f_j\|_1^{\frac{m_j}{d}}.$$

Since S was arbitrary, we have shown that

$$\left(\sum_{p\in J}\left(\left(\prod_{j=1}^{r}m_j!\right)M_{(m,k)}[f_1,\dots,f_r](p)\right)^{\frac{1}{d}\frac{d}{d-1}}\right)^{\frac{d-1}{d}}\le C(k_1,\dots,k_r)\prod_{j=1}^{r}\|f_j\|_1^{\frac{m_j}{d}}.$$

Hence, we have established the $(\boldsymbol{m},\boldsymbol{k})$-multijoint inequality,

$$\sum_{p\in J}M_{(\boldsymbol{m},\boldsymbol{k})}[f_1,\dots,f_r](p)^{\frac{1}{d-1}}\le\left(\frac{C(k_1,\dots,k_r)}{\prod_{j=1}^{r}m_j!}\right)^{\frac{1}{d-1}}\prod_{j=1}^{r}\|f_j\|_1^{\frac{m_j}{d-1}}.$$

By establishing the $(\boldsymbol{m},\boldsymbol{k})$-multijoint inequality in this way we have explicitly identified the $\boldsymbol{m}$-dependence and $\boldsymbol{k}$-dependence in this combinatorial variant of the multijoints problem. In particular, we account for these combinatorial differences in the calculations above, without modifying the multilinear dual statement, Theorem 4.0.2. This generalises Zhang's observation, in [Zha20], that the joint and multijoint problems are equivalent.

More generally, the new Tidor–Yu–Zhao polynomial method used in Chapter 3 applies equally well to varieties, [TYZ20]. Hence, we expect that a discrete Bourgain–Guth result should hold for k_j-dimensional varieties and in this chapter we prove that this is indeed the case.

Theorem 4.0.3. *Given a k-dimensional variety $\gamma\subset\mathbb{F}^n$, let p be a non-singular point in γ and let $T_p\gamma$ denote the tangent plane to γ at p. Let $\Gamma_1,\dots,\Gamma_d$ be sets* *of* k_1-$, \dots,$ k_d*-dimensional varieties in* $\mathbb{F}^n$*, respectively, with* $k_1 + \ldots + k_d = n$*. We define* δ *by*

$$\delta(p, \boldsymbol{\gamma}) := \left(\prod_{j=1}^{d} \chi_{\gamma_j}(p) \right) \wedge_{j=1}^{d} e(T_p \gamma_j).$$

Let $J = \{p : \exists \boldsymbol{\gamma} \text{ so that } \delta(p, \boldsymbol{\gamma}) = 1\}$*. There is a constant* $C = C(k_1, \dots, k_d)$ *so that for all* $S : J \to \mathbb{R}_{\geq 0}$ *with* $\|S\|_d = 1$ *there exist functions* $s_k : J \times \left(\cup_{j : k_j = k} \Gamma_j \right) \to \mathbb{R}_{\geq 0}$ *for each* $k \in \{k_1, \dots, k_d\}$*, so that*

$$\delta(p, \boldsymbol{\gamma}) S(p)^d \leq C \prod_{j=1}^{d} s_{k_j}(p, \gamma_j),$$

for all $p \in J$ *and* $\boldsymbol{\gamma} \in \Gamma_1 \times \cdots \times \Gamma_d$*, and so that*

$$\sum_{p \in \gamma_j \cap J} s_{k_j}(p, \gamma_j) = \deg \gamma_j$$

for all $\gamma_j \in \Gamma_j$ *and all* $1 \leq j \leq d$*.*

Remark. The subscript on each function s_k denotes the dimension of the elements of the set of objects it acts on. Additionally, Theorem 4.0.3 allows for objects of differing degree within the same set Γ_j. ◀

Lemma 3.2.5, Lemma 3.2.6 and Lemma 3.2.7, for any line-multijoint problem, were simple to prove without any reference to polynomials. However, for the generalisations considered in Theorem 4.0.3, Tidor, Yu and Zhao realised that the correct numerology could be achieved by deriving preliminary linear algebra results. In fact, we are also required to be more precise about how the numbers $\tilde{S}(p, l, \alpha, \lambda)$ are chosen.

Before proceeding, we make a comment on the field, $\mathbb{F}$; specifically, regarding our freedom to consider a field extension without loss of generality and without changing the problem in any meaningful way. A k-plane $\pi \subset \mathbb{F}^n$ can be identified by a k-tuple of vectors $(\omega_1, \dots, \omega_k)$ which span $e(\pi)$ and a point $p \in \pi$. If we extend our field to a larger one, $\tilde{\mathbb{F}}$, we can embed $\mathbb{F}^n \subset \tilde{\mathbb{F}}^n$, and hence we can realise the vectors $\omega_1, \dots, \omega_k$ and the point p as elements of $\tilde{\mathbb{F}}^n$. Hence, we can embed any multijoint problem of k_j-planes in $\mathbb{F}^n$ as a multijoint problem of k_j-planes in $\tilde{\mathbb{F}}^n$. In particular, the set of multijoints is preserved since this embedding preserves linear independence. However, we may find additional points for which there is a tuple $\boldsymbol{\pi}$ such that $\delta(p, \boldsymbol{\pi}) = 1$. Therefore, there may be more joints in $\tilde{\mathbb{F}}^n$ than in $\mathbb{F}^n$, and we may find more multijoints on each $\pi \in \Pi_1 \cup \cdots \cup \Pi_d$. Hence, by restricting our attention to test functions S such that $\operatorname{supp} S$ is a subset of the $\mathbb{F}$-multijoints, any tuple of functions s_k that satisfies the conclusion of Theorem 4.0.1, defined over $\tilde{\mathbb{F}}$, will satisfy the same conditions restricted to multijoints defined over $\mathbb{F}$. Therefore, at any point, we can extend the field $\mathbb{F}$ where necessary.

4.1 The Polynomial Method

In this section, we discuss the results required to extend the polynomial method to apply to higher-dimensional affine planes.

4.1.1 Preliminary Results

The proof of Theorem 4.0.1 generalises that of Theorem 3.0.1. In particular, we establish uniform boundedness, monotonicity and continuity properties, similar to Subsection 3.2.3, for numbers $\tilde{S}_k$, which we will define in Subsection 4.1.2. In order to generalise our work in Chapter 3 with the correct numerology, we follow [TYZ20] to establish preliminary facts on vector spaces of polynomials. This subsection, Subsection 4.1.1, is independent from any properties of joints, and as such, may be omitted on first reading.

Let U be a vector space and let $V \leq U$. The **codimension** of V in U is given by

$$\operatorname{codim}_U V := \dim U - \dim V.$$

An elementary fact about codimension is that it decreases as we restrict to subspaces. That is, if $U, W \leq V$ then

$$\operatorname{codim}_U (W \cap U) \leq \operatorname{codim}_V W. \tag{4.2}$$

Let $\mathcal{P} \subset \mathbb{F}^k$ be a finite set of points and $\lambda \in \mathbb{N}$. Let $v = (v_p)_{p \in \mathcal{P}} \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$. We say that f vanishes to order at least v_p at $p \in \mathcal{P}$ if $Df(p) = 0$, for all derivative operators D of order r for each $0 \leq r < v_p$. Define

$$\mathbb{T}(v, \lambda) := \left\{ f \in \mathbb{F}_\lambda[x_1, \dots, x_k] : f \text{ vanishes to order } \geq v_p \text{ at } p, \forall p \in \mathcal{P} \right\}.$$

Remark. In Chapter 3 we considered the case where $k = 1$. We did not need to introduce the set $\mathbb{T}(v, \lambda)$ in that chapter, but our observations there would show that $\dim \mathbb{T}(v, \lambda)$ changes by $-1, 0$ or $+1$ when we replace v with $(v \pm e_q)$ for some $q \in \mathcal{P}$, where $e_q \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$ has entry 1 at q, and 0 for all $p \neq q$. ◀

For $p \in \mathcal{P}$, the vector space $\mathbb{T}(v + e_p, \lambda)$ is a subspace of $\mathbb{T}(v, \lambda)$. Hence, we may define

$$b_p(v, \lambda) := \operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v + e_p, \lambda).$$

Lemma 4.1.1 (Preliminary Uniform Boundedness)**.** *Let $\lambda \in \mathbb{N}$. If $v \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$ is such that $v_p > \lambda$ for some $p \in \mathcal{P}$ then* $\dim \mathbb{T}(v, \lambda) = 0$.

Proof. This is the fact that the only polynomial of degree at most λ, which vanishes to order more than λ at a point, is the zero polynomial. □

Lemma 4.1.2 (Preliminary Monotonicity)**.** *Let $p \in \mathcal{P}$. If $v^{(1)}, v^{(2)} \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$ satisfy $v^{(1)} \geq v^{(2)}$ pointwise with equality at p, then $b_p(v^{(1)}, \lambda) \leq b_p(v^{(2)}, \lambda)$ for all $\lambda \in \mathbb{N}$.*

Proof. Let $1 \leq i \leq 2$ and recall that p is given. Let $f \in \mathbb{T}(v^{(i)}, \lambda)$ and consider the map $f \mapsto (Df(p))_D$ where D varies over all derivative operators of order $v_p^{(1)} = v_p^{(2)}$ at p. This map sends f to all its $v_p^{(1)} = v_p^{(2)}$-th order derivatives at p. By the definition of vanishing order, the kernel of this map is precisely the vector space of polynomials in $\mathbb{T}(v, \lambda)$ which vanish to order at least $v_p^{(1)} + 1 = v_p^{(2)} + 1$ at p, which we can write as $\mathbb{T}(v^{(i)} + e_p, \lambda)$. Therefore, this map has rank

$$\operatorname{codim}_{\mathbb{T}(v^{(i)},\lambda)} \mathbb{T}(v^{(i)} + e_p, \lambda) = b_p(v^{(i)}, \lambda).$$

Since $v^{(1)} \geq v^{(2)}$, if $f \in \mathbb{T}(v^{(1)}, \lambda)$, then $f \in \mathbb{T}(v^{(2)}, \lambda)$ and hence $\mathbb{T}(v^{(1)}, \lambda)$ is a subspace of $\mathbb{T}(v^{(2)}, \lambda)$. So

$$\mathbb{T}(v^{(1)} + e_p, \lambda) \leq \mathbb{T}(v^{(1)}, \lambda) \leq \mathbb{T}(v^{(2)}, \lambda).$$

Hence,

$$\begin{aligned} b_p(v^{(1)}, \lambda) &= \operatorname{codim}_{\mathbb{T}(v^{(1)},\lambda)} \mathbb{T}(v^{(1)} + e_p, \lambda) \\ &= \operatorname{codim}_{\mathbb{T}(v^{(1)},\lambda) \cap \mathbb{T}(v^{(2)},\lambda)} \Big(\mathbb{T}(v^{(1)} + e_p, \lambda) \cap \mathbb{T}(v^{(2)}, \lambda) \Big) \\ &\leq \operatorname{codim}_{\mathbb{T}(v^{(2)},\lambda)} \mathbb{T}(v^{(2)}, \lambda) \\ &= b_p(v^{(2)}, \lambda), \end{aligned}$$

where the inequality follows from (4.2). □

Lemma 4.1.3 (Preliminary Continuity)**.** *Let $p, q \in \mathcal{P}$ be distinct. Suppose the sequence of $\mathcal{P}$-tuples $v^{(0)}, v^{(1)}, \ldots \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$ is increasing pointwise and does so strictly at p. Then*

$$0 \leq \sum_{r \geq 0} b_p(v^{(r)}, \lambda) - \sum_{r \geq 0} b_p(v^{(r)} + e_q, \lambda) \leq \binom{\lambda + k - 1}{k - 1}.$$

Remark. Crucially, $\binom{\lambda+k-1}{k-1} \sim_k \lambda^{k-1}$ so that $\lambda^{-k}\binom{\lambda+(k-1)}{k-1} \to 0$ as $\lambda \to \infty$. ◀

Proof of Lower Bound for Lemma 4.1.3. For every $r \geq 0$, apply Lemma 4.1.2 with $v^{(r)}$ and $v^{(r)} + e_q$. Since $q \neq p$, we have that $v^{(r)} + e_q \geq v^{(r)}$ with equality at p and therefore $b_p(v^{(r)} + e_q, \lambda) \leq b_p(v^{(r)}, \lambda)$. Hence

$$0 \leq \sum_{r \geq 0} b_p(v^{(r)}, \lambda) - b_p(v^{(r)} + e_q, \lambda) = \sum_{r \geq 0} b_p(v^{(r)}, \lambda) - \sum_{r \geq 0} b_p(v^{(r)} + e_q, \lambda).$$

□

To establish the upper bound for Lemma 4.1.3, we consider the following useful fact, Proposition 4.1.4.

Proposition 4.1.4 (Useful Fact)**.** *For any $v \in \mathbb{Z}_{\geq 0}^{\mathcal{P}}$, $b_p(v + e_q, \lambda) \geq b_p(v, \lambda - 1)$.*

Proof. Suppose the field $\mathbb{F}$ is large enough so that there is a linear polynomial $g \in \mathbb{F}[x_1, \ldots, x_k]$ which vanishes at q, and not at any other element of $\mathcal{P}$. The space of polynomials which are in $\mathbb{T}(v+e_q,\lambda)$ and divisible by g is isomorphic to $g\cdot\mathbb{T}(v,\lambda-1)$ for any $v\in\mathbb{Z}^{\mathcal{P}}_{\geq 0}$. Since $g\cdot\mathbb{T}(v,\lambda-1)\leq\mathbb{T}(v+e_q,\lambda)$, we may apply (4.2) so that

$$\begin{aligned} b_p(v+e_q,\lambda) &= \operatorname{codim}_{\mathbb{T}(v+e_q,\lambda)}\mathbb{T}(v+e_q+e_p,\lambda) \\ &\geq \operatorname{codim}_{g\cdot\mathbb{T}(v,\lambda-1)}\big(g\cdot\mathbb{T}(v+e_p,\lambda-1)\big) \\ &= \operatorname{codim}_{\mathbb{T}(v,\lambda-1)}\big(\mathbb{T}(v+e_p,\lambda-1)\big) \\ &= b_p(v,\lambda-1), \end{aligned}$$

where we use that $g\cdot\mathbb{T}(v,\lambda-1)$ is isomorphic to $\mathbb{T}(v,\lambda-1)$, and similarly for $g\cdot\mathbb{T}(v+e_q,\lambda-1)$. □

Proof of Upper Bound for Lemma 4.1.3. By Proposition 4.1.4,

$$\sum_{r\geq 0}b_p(v^{(r)},\lambda)-\sum_{r\geq 0}b_p(v^{(r)}+e_q,\lambda)\leq\sum_{r\geq 0}b_p(v^{(r)},\lambda)-\sum_{r\geq 0}b_p(v^{(r)},\lambda-1).$$

So for the upper bound, it suffices to show that

$$\sum_{r\geq 0}b_p(v^{(r)},\lambda)-\sum_{r\geq 0}b_p(v^{(r)},\lambda-1)\leq\binom{\lambda+k-1}{k-1}.$$

Let $r\geq 0$. Then

$$\begin{aligned} & b_p(v^{(r)},\lambda)-b_p(v^{(r)},\lambda-1) \\ &= \operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda)}\mathbb{T}(v^{(r)}+e_p,\lambda)-\operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda-1)}\mathbb{T}(v^{(r)}+e_p,\lambda-1) \\ &= \Big(\dim\mathbb{T}(v^{(r)},\lambda)-\dim\mathbb{T}(v^{(r)}+e_p,\lambda)\Big) \\ &\qquad -\Big(\dim\mathbb{T}(v^{(r)},\lambda-1)-\dim\mathbb{T}(v^{(r)}+e_p,\lambda-1)\Big) \\ &= \Big(\dim\mathbb{T}(v^{(r)},\lambda)-\dim\mathbb{T}(v^{(r)},\lambda-1)\Big) \\ &\qquad -\Big(\dim\mathbb{T}(v^{(r)}+e_p,\lambda)-\dim\mathbb{T}(v^{(r)}+e_p,\lambda-1)\Big) \\ &= \operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda)}\mathbb{T}(v^{(r)},\lambda-1)-\operatorname{codim}_{\mathbb{T}(v^{(r)}+e_p,\lambda)}\mathbb{T}(v^{(r)}+e_p,\lambda-1). \end{aligned}$$

Inequality (4.2) gives

$$\operatorname{codim}_{\mathbb{T}(v^{(r)}+e_p,\lambda)}\mathbb{T}(v^{(r)}+e_p,\lambda-1)\geq\operatorname{codim}_{\mathbb{T}(v^{(r+1)},\lambda)}\mathbb{T}(v^{(r+1)},\lambda-1)$$

since $v^{(r)}+e_p\leq v^{(r+1)}$, by hypothesis. Hence

$$\begin{aligned} & b_p(v^{(r)},\lambda)-b_p(v^{(r)},\lambda-1) \\ &\qquad \leq \operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda)}\mathbb{T}(v^{(r)},\lambda-1)-\operatorname{codim}_{\mathbb{T}(v^{(r+1)},\lambda)}\mathbb{T}(v^{(r+1)},\lambda-1). \end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{r\geq 0} b_p(v^{(r)},\lambda) - \sum_{r\geq 0} b_p(v^{(r)},\lambda-1)\\
\leq\ & \sum_{r\geq 0}\Big(\operatorname{codim}_{\mathbb{T}(v^{(r)},\lambda)}\mathbb{T}(v^{(r)},\lambda-1) - \operatorname{codim}_{\mathbb{T}(v^{(r+1)},\lambda)}\mathbb{T}(v^{(r+1)},\lambda-1)\Big)\\
=\ & \operatorname{codim}_{\mathbb{T}(v^{(0)},\lambda)}\mathbb{T}(v^{(0)},\lambda-1)\\
\leq\ & \operatorname{codim}_{\mathbb{T}(0,\lambda)}\mathbb{T}(0,\lambda-1)\\
=\ & \dim \mathbb{F}_\lambda[x_1,\ldots,x_k] - \dim \mathbb{F}_{(\lambda-1)}[x_1,\ldots,x_k].
\end{aligned}$$

This is equal to the number of monomials in k variables of degree precisely λ, which is equal to $\binom{\lambda+k-1}{k-1}$. □

Remark. This proof is the only instance where a field extension may be required. ◀

4.1.2 Choosing Vanishing Conditions

Let $\Pi_1,\ldots,\Pi_d$ be sets of k_1-$,\ldots,k_d$-dimensional planes in $\mathbb{F}^n$, respectively, where $k_1+\ldots+k_d=n$. Let $J=\{p : \exists\boldsymbol{\pi} \text{ so that } \delta(p,\boldsymbol{\pi})=1\}$ be the associated set of $\boldsymbol{k}$-multijoints. Let $\lambda\in\mathbb{N}$ be a parameter and let $\alpha : J\to\mathbb{Z}$ be a handicap. Choose an ordering of the set J and thereafter, define the total order $\prec=\prec_\alpha$ on $J\times\mathbb{Z}_{\geq 0}$, as in Subsection 3.2.1.

Let $\pi\subset\mathbb{F}^n$ be a k-plane which we fix for the remainder of Chapter 4. Without loss of generality, we assume that π is spanned by the coordinate vectors $e_1,\ldots,e_k$, so we identify

$$\big\{f|_\pi : f\in\mathbb{F}[x_1,\ldots,x_n]\big\} = \mathbb{F}[x_1,\ldots,x_k]. \tag{4.3}$$

For each (p,r), we define $\mathbb{B}_r(p,\pi,\lambda)$ to be the vector space of linear functionals which act on $\mathbb{F}_\lambda[x_1,\ldots,x_k]$ of the form $f\mapsto Df(p)$ for some differential operator D, of order r, acting on k-variate polynomials.

To define the sets $B(p,\pi,\alpha,\lambda)$ in this general setting, we perform an iterative procedure starting with the $\prec$-least element of $(\pi\cap J)\times\mathbb{Z}_{\geq 0}$ and proceeding to the $\prec$-next element on each iteration. Starting with $(p,0)$, the least element of $(\pi\cap J)\times\mathbb{Z}_{\geq 0}$, we choose a set $B_0(p,\pi,\alpha,\lambda)\subset\mathbb{B}_0(p,\pi,\lambda)$ that is a basis for $\mathbb{B}_0(p,\pi,\lambda)$.

Assume for each $(p',r')\prec(p,r)$, we have chosen sets $B_{r'}(p',\pi,\alpha,\lambda)$ so that the disjoint union, $\cup_{(p',r')\prec(p,r)}B_{r'}(p',\pi,\alpha,\lambda)$ is a basis for $\operatorname{span}_{(p',r')\prec(p,r)}\mathbb{B}_{r'}(p',\pi,,\lambda)$. Thereafter, choose $B_r(p,\pi,\alpha,\lambda)\subset\mathbb{B}_r(p,\pi,\lambda)$ so that the disjoint union

$$\bigcup_{(p',r')\preceq(p,r)} B_{r'}(p',\pi,\alpha,\lambda)$$

is a basis for $\operatorname{span}_{(p',r')\preceq(p,r)}\mathbb{B}_{r'}(p',\pi,\lambda)$.

Remark. Writing these unions out with more complete notation yields

$$\bigcup_{(p',r')\preceq(p,r)} B_{r'}(p',\pi,\alpha,\lambda) = \bigcup_{\substack{(p',r')\in(\pi\cap J)\times\mathbb{Z}_{\geq 0}:\\ (p',r')\preceq(p,r)}} B_{r'}(p',\pi,\alpha,\lambda).$$

Similar to the case for lines, observe that we can write these unions as

$$\bigcup_{\substack{p'\in(p+e(\pi))\cap J,\, r'\in\mathbb{Z}_{\geq 0}:\\ (p',r')\preceq(p,r)}} B_{r'}(p',\pi,\alpha,\lambda),$$

since $\pi = p + e(\pi)$. Since the first argument of B is p, the only information this construction requires from the second argument of B is which vector space is parallel to π, namely, $e(\pi) \in \mathrm{Gr}(k,\mathbb{F}^n)$. It follows that for this k-plane construction, we may also reduce the second argument to $e(\pi)$, although for notational convenience, we again continue to simply write $B(p,\pi,\alpha,\lambda)$. ◀

Let

$$B(p,\pi,\alpha,\lambda) = \bigcup_{r\geq 0} B_r(p,\pi,\alpha,\lambda).$$

By construction, the disjoint union

$$\bigcup_{p\in\pi\cap J} B(p,\pi,\alpha,\lambda)$$

is a basis for the space of linear functionals on $\mathbb{F}_\lambda[x_1,\dots,x_k]$. Hence

$$\sum_{p\in\pi\cap J} \left|B(p,\pi,\alpha,\lambda)\right| = \dim \mathbb{F}_\lambda[x_1,\dots,x_k] = \binom{\lambda+k}{k}. \tag{4.4}$$

As in Subsection 3.2.3, let

$$\tilde{S}_k(p,\pi,\alpha,\lambda) = \left|B(p,\pi,\alpha,\lambda)\right|$$

for $\pi \in \cup_{j:k_j=k}\Pi_j$ and $p \in \pi\cap J$. Note that if $\pi' \subset \mathbb{F}^n$ is a k-plane so that $\pi'\cap J = \pi\cap J$ and $\pi' \neq \pi$, then

$$\tilde{S}_k(\cdot,\pi',\alpha,\lambda) = \tilde{S}_k(\cdot,\pi,\alpha,\lambda). \tag{4.5}$$

Our analysis will only consider pairs (p,π) so that $p\in\pi$. Therefore, it is not necessary to define $\tilde{S}_k$ for pairs such that $p\notin\pi$. These quantities remain central to our analysis. Moreover, it follows immediately from (4.4) that

$$\sum_{p\in\pi\cap J} \tilde{S}_k(p,\pi,\alpha,\lambda) = \dim \mathbb{F}_\lambda[x_1,\dots,x_k] = \binom{\lambda+k}{k}.$$

Given a polynomial $f \in \mathbb{F}_\lambda[x_1,\dots x_k]$, each set $B(p,\pi,\alpha,\lambda)$ indexes a set of vanishing conditions, $\langle\phi,f\rangle = 0$ for $\phi\in\mathbb{B}(p,\pi,\lambda)$. We will relate the preliminary lemmas from Subsection 4.1.1 to the numbers

$$\tilde{S}(p,\pi,\alpha,\lambda) = \big|B(p,\pi,\alpha,\lambda)\big| = \sum_{r\geq 0} \big|B_r(p,\pi,\alpha,\lambda)\big|$$

by considering a vector $v = v^{(p,r)}(\alpha)$, chosen so that $b_p(v,\lambda) = \big|B_r(p,\pi,\alpha,\lambda)\big|$.

Recall that we fixed the k-plane π, and assumed that is it spanned by the first k standard basis vectors so that (4.3) is valid.

Let $\alpha : J \to \mathbb{Z}$ be a handicap, and let $(p,r) \in (\pi \cap J) \times \mathbb{Z}_{\geq 0}$. Define the vector $v^{(p,r)}(\alpha) \in \mathbb{Z}^J_{\geq 0}$ by choosing its p'-th entry

$$v^{(p,r)}_{p'}(\alpha) := \min\{r' : (p,r) \preceq (p',r')\}.$$

In particular, for this vector v, we have the following.

Proposition 4.1.5. *Let* $(p,r) \in (\pi \cap J) \times \mathbb{Z}_{\geq 0}$ *and* $v := v^{(p,r)}(\alpha)$. *Then*

$$\big\{f \in \mathbb{F}_\lambda[x_1,\ldots,x_k] : \langle \phi, f\rangle = 0,\, \forall \phi \in \cup_{(p',r')\prec(p,r)} B_{r'}(p',\pi,\alpha,\lambda)\big\} = \mathbb{T}(v,\lambda).$$

Remark. The set $\mathbb{T}(v,\lambda)$ contains precisely those polynomials which satisfy the vanishing conditions that we have chosen for each pair, $(p',r') \prec (p,r)$. ◀

Proof. Denote the set on the left-hand side by X. By construction, for any $p' \in \pi \cap J$,

$$v^{(p,r)}_{p'}(\alpha) = \min\{r' : (p,r) \preceq (p',r')\} = \big|\{r' : (p',r') \prec (p,r)\}\big| + 1.$$

By the spanning property of the sets $B_{r'}(p',\pi,\alpha,\lambda)$, $f \in X$ if and only if $Df(p') = 0$ for all derivatives D, of order smaller than $v_{p'}$ acting on polynomials of degree at most λ restricted to π for all $p' \in \pi \cap J$. That is, $f \in X$ if and only if f vanishes to order $v_{p'}$ at p' for every $p' \in \pi \cap J$. So $X = \mathbb{T}(v,\lambda)$. ☐

It follows that

$$\big|B_r(p,\pi,\alpha,\lambda)\big| = \operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v+e_p,\lambda) = b_p(v,\lambda). \tag{4.6}$$

Indeed, by modifying the argument for Proposition 4.1.5,

$$\big\{f : \langle \phi, f\rangle = 0,\, \forall \phi \in \cup_{(p',r')\preceq(p,r)} B_{r'}(p',\pi,\alpha,\lambda)\big\} = \mathbb{T}(v+e_p,\lambda),$$

where the union on the left-hand side now includes (p,r). Hence, the set of linear functionals $B_r(p,\pi,\alpha,\lambda)$ is chosen to comprise of linearly independent dual maps so that if $f \in \mathbb{T}(v,\lambda)$ and $\langle\phi, f\rangle = 0$ for all $\phi \in B_r(p,\pi,\alpha,\lambda)$, then $f \in \mathbb{T}(v+e_p,\lambda)$. The least number of such dual maps required is precisely $\operatorname{codim}_{\mathbb{T}(v,\lambda)} \mathbb{T}(v+e_p,\lambda)$.

4.1.3 Example

Recall the example directly preceding (3.1), where $|J| = 4$ and $\alpha = (0, -1, 2, 0)$. The first terms of $J \times \mathbb{Z}_{\geq 0}$ under $\prec_\alpha$ are

$$\begin{aligned} &(p_3, 0) \prec (p_3, 1) \prec (p_1, 0) \prec (p_3, 2) \prec \\ &\qquad \prec (p_4, 0) \prec (p_1, 1) \prec (p_2, 0) \prec (p_3, 3) \prec \\ &\qquad\qquad \prec (p_4, 1) \prec (p_1, 2) \prec (p_2, 1) \prec (p_3, 4) \prec \cdots , \end{aligned}$$

which can also be written as

$$\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1, 3) & \prec & (p_2, 2) & \prec & (p_3, 5) & \prec & (p_4, 3) & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1, 2) & \prec & (p_2, 1) & \prec & (p_3, 4) & \prec & (p_4, 2) & \\
\curlyvee & & \curlyvee & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1, 1) & \prec & (p_2, 0) & \prec & (p_3, 3) & \prec & (p_4, 1) & \\
\curlyvee & & & & \curlyvee & & \curlyvee & \rotatebox{45}{$\prec$} \\
(p_1, 0) & \prec & & & (p_3, 2) & \prec & (p_4, 0) & \\
 & & & & \curlyvee & & & \\
 & & & & (p_3, 1) & & & \\
 & & & & \curlyvee & & & \\
 & & & & (p_3, 0). & & &
\end{array}$$

Let us retain this column layout and replace each pair (p, r) by the integer $b_p(v^{(p,r)}) = b_p(v^{(p,r)}(\alpha), \lambda)$.

$$\begin{array}{c||c||c||c}
\vdots & \vdots & \vdots & \vdots \\
+ & + & + & + \\
b_{p_1}(v^{(p_1,3)}) & b_{p_2}(v^{(p_2,2)}) & b_{p_3}(v^{(p_3,5)}) & b_{p_4}(v^{(p_4,3)}) \\
+ & + & + & + \\
b_{p_1}(v^{(p_1,2)}) & b_{p_2}(v^{(p_2,1)}) & b_{p_3}(v^{(p_3,4)}) & b_{p_4}(v^{(p_4,2)}) \\
+ & + & + & + \\
b_{p_1}(v^{(p_1,1)}) & b_{p_2}(v^{(p_2,0)}) & b_{p_3}(v^{(p_3,3)}) & b_{p_4}(v^{(p_4,1)} \\
+ & & + & + \\
b_{p_1}(v^{(p_1,0)}) & & b_{p_3}(v^{(p_3,2)}) & b_{p_4}(v^{(p_4,0)}) \\
 & & + & \\
 & & b_{p_3}(v^{(p_3,1)}) & \\
 & & + & \\
 & & b_{p_3}(v^{(p_3,0)}) &
\end{array}$$

Since $\tilde{S}(p, \pi, \alpha, \lambda) = \big|B(p, \pi, \alpha, \lambda)\big| = \sum_{r \geq 0} \big|B_r(p, \pi, \alpha, \lambda)\big|$, we may compute $\tilde{S}(p, \pi, \alpha, \lambda)$ by summing all entries in the p-th column of this diagram. Given that the quantity $b_p(v, \lambda)$ satisfies uniform boundedness, monotonicity and continuity properties, it is reasonable to expect that $\tilde{S}(p, \pi, \alpha, \lambda)$ does too.

In the case where $k = 1$, as in Subsection 3.2.1, each $b_p(v^{(p,r)}) = b_p(v^{(p,r)}(\alpha, \lambda))$ was either 1 or 0 depending on whether or not (p, r) was within the $\prec_\alpha$-first $(\lambda + 1)$ elements of $J \times \mathbb{Z}_{\geq 0}$. Hence, for $k = 1$, this diagram contains precisely $(\lambda + 1)$ 1's, and all other entries are 0.

4.1.4 Uniform Boundedness, Monotonicity and Continuity

This subsection utilises the preliminary results from Subsection 4.1.1 to establish the uniform boundedness, monotonicity and continuity results for higher-dimensional planes. Upon first reading, we may take these properties as given and omit this subsection.

For the remainder of this subsection, let $\pi \subset \mathbb{F}^n$ be a k-plane.

Lemma 4.1.6 (k-plane Uniform Boundedness). *Let $\lambda \in \mathbb{N}$ and $\alpha : J \to \mathbb{Z}$ be a handicap. If α is such that $\alpha_p < \alpha_q - \lambda$ for some $p, q \in \pi \cap J$, then $\tilde{S}_k(p, \pi, \alpha, \lambda) = 0$.*

Proof. Let $r \in \mathbb{Z}_{\geq 0}$ and $v = v^{(p,r)}(\alpha)$. Then $v_q > \lambda$ and $b_p(v, \lambda) = 0$ by preliminary uniform boundedness (Lemma 4.1.1). The result follows from (4.6). □

Lemma 4.1.7 (k-plane Monotonicity). *Let $\lambda \in \mathbb{N}$, and $\alpha^{(1)}, \alpha^{(2)} \in \mathbb{Z}^J$ be two handicaps. Suppose $\exists p \in \pi \cap J$ so that $\alpha^{(1)}_p - \alpha^{(1)}_{p'} \leq \alpha^{(2)}_p - \alpha^{(2)}_{p'}$ for all $p' \in \pi \cap J$. Then*

$$\tilde{S}_k(p, \pi, \alpha^{(1)}, \lambda) \leq \tilde{S}_k(p, \pi, \alpha^{(2)}, \lambda).$$

Proof. For $1 \leq i \leq 2$ and $r \geq 0$, let $v^{(i)}_r = v^{(p,r)}(\alpha^{(i)})$. We write $v^{(i)}_r$ explicitly as

$$v^{(p,r)}_{p'}(\alpha^{(i)}) = \begin{cases} \max(r - \alpha^{(i)}_p + \alpha^{(i)}_{p'} + 1, 0) & \text{if } p' < p, \\ \max(r - \alpha^{(i)}_p + \alpha^{(i)}_{p'}, 0) & \text{otherwise.} \end{cases}$$

The conditions on $\alpha^{(i)}$ give that $v^{(1)}_r \geq v^{(2)}_r$, coordinate-wise, with equality at p. By (4.6), $\tilde{S}_k(p, \pi, \alpha^{(i)}, \lambda) = \sum_{r \geq 0} \left| B_r(p, \pi, \alpha^{(i)}, \lambda) \right|$. Hence, it suffices to show that

$$b_p(v^{(1)}_r, \lambda) \leq b_p(v^{(2)}_r, \lambda),$$

which follows from Lemma 4.1.2. □

Lemma 4.1.8 (k-plane Continuity). *Let $p \in \pi \cap J$, let $\alpha^{(1)}, \alpha^{(2)} \in \mathbb{Z}^J$ be handicaps and $\lambda \in \mathbb{N}$. Then*

$$\begin{aligned} &\left| \tilde{S}(p, \pi, \alpha^{(1)}, \lambda) - \tilde{S}(p, \pi, \alpha^{(2)}, \lambda) \right| \\ &\qquad \leq \binom{\lambda + k - 1}{k - 1} \sum_{p' \in J} \left| (\alpha^{(1)}_p - \alpha^{(1)}_{p'}) - (\alpha^{(2)}_p - \alpha^{(2)}_{p'}) \right|. \end{aligned}$$

Remark. By translation invariance, Proposition 3.2.3, we may choose $\alpha_p^{(i)} = 0$ for $i = 1, 2$. The resulting inequality reads as

$$\left|\tilde{S}(p, \pi, \alpha^{(1)}, \lambda) - \tilde{S}(p, \pi, \alpha^{(2)}, \lambda)\right| \leq \binom{\lambda + k - 1}{k - 1} \sum_{p' \in J} \left|\alpha_{p'}^{(1)} - \alpha_{p'}^{(2)}\right|.$$

◀

Proof. By translation invariance, Proposition 3.2.3, the left-hand side of the target inequality remains unchanged if we translate $\alpha^{(1)}$ and $\alpha^{(2)}$ by $\alpha_p^{(1)}$ and $\alpha_p^{(2)}$, respectively.

Let $p \in \pi \cap J$ and consider $(\alpha_{p'} - \alpha_p)_{p' \in J}$. For any $p' \neq p$, can perform a sequence of precisely $\left|(\alpha_p^{(1)} - \alpha_{p'}^{(1)}) - (\alpha_p^{(2)} - \alpha_{p'}^{(2)})\right|$ pointwise modifications of ±1 in order to change $(\alpha_p^{(1)} - \alpha_{p'}^{(1)})$ into $(\alpha_{p'}^{(2)} - \alpha_p^{(2)})$. Suppose at some increment we modify $\alpha \mapsto \alpha + e_{p'}$ for $p' \neq p$, and have that

$$0 \leq \tilde{S}_k(p, \pi, \alpha, \lambda) - \tilde{S}_k(p, \pi, \alpha + e_{p'}, \lambda) \leq \binom{\lambda + k - 1}{k - 1}. \tag{4.7}$$

Summing (4.7) over all modifications, the sum collapses to the desired left-hand side. This yields an upper bound given by the number of modifications, the desired right-hand side. Hence, if suffices to establish (4.7) for $p' \neq p$.

The lower bound in (4.7) follows from Lemma 4.1.7 with α and $\alpha + e_{p'}$, since for $p' \neq p$ and any $q \in J$,

$$(\alpha_p + (e_{p'})_p) - (\alpha_q + (e_{p'})_q) = (\alpha_p - \alpha_q) + (0 - (e_{p'})_q) \leq (\alpha_p - \alpha_q).$$

For the upper bound, recall that $\tilde{S}_k(p, \pi, \alpha^{(i)}, \lambda) = \sum_{r \geq 0} \left|B_r(p, \pi, \alpha^{(i)}, \lambda)\right|$. By (4.6), it remains to show that for all $q \neq p$

$$\sum_{r \geq 0} b_p(v^{(p,r)}(\alpha), \lambda) - \sum_{r \geq 0} b_p(v^{(p,r)}(\alpha + e_q), \lambda) \leq \binom{\lambda + k - 1}{k - 1}.$$

Recall the formula

$$v_{p'}^{(p,r)}(\alpha) = \begin{cases} \max(r - \alpha_p + \alpha_{p'} + 1, 0) & \text{if } p' < p, \\ \max(r - \alpha_p + \alpha_{p'}, 0) & \text{otherwise.} \end{cases}$$

Hence, if r is sufficiently large, for example $r - \alpha_p + \alpha_q \geq 0$, then $v^{(p,r)}(\alpha + e_q) = v^{(p,r)}(\alpha) + e_q$. Let r_0 be the least such r. For $r < r_0$, we have that

$v^{(p,r)}(\alpha+e_q)=v^{(p,r)}(\alpha)$, and hence

$$\begin{aligned}\sum_{r\geq 0} b_p(v^{(p,r)}(\alpha),\lambda)-\sum_{r\geq 0} b_p(v^{(p,r)}(\alpha+e_q),\lambda)&\\ \leq \sum_{r\geq 0} b_p(v^{(p,r)}(\alpha),\lambda)-\sum_{r\geq 0} b_p(v^{(p,r)}(\alpha)+e_q,\lambda).&\end{aligned}$$

The result follows as Lemma 4.1.3 gives

$$\sum_{r\geq 0} b_p(v^{(p,r)}(\alpha),\lambda)-\sum_{r\geq 0} b_p(v^{(p,r)}(\alpha)+e_q,\lambda)\leq \binom{\lambda+k-1}{k-1}.$$

□

4.2 Multijoints of k_j-Planes

We are now equipped to count multijoints formed by k_j-planes.

4.2.1 Choosing A Handicap

In this subsection, we state and prove the generalisation of Lemma 3.3.1. We previously remarked, in Subsection 3.3.1, that it is not necessary to choose α so that $\tilde{S}_{k_j}(p,l,\alpha,\lambda)>0$, *a priori*, for all l that contribute to p. We have already determined in Chapter 3 how to choose α to satisfy these conditions before applying Corollary 4.2.9a. In this subsection we relax these *a priori* assumptions and deduce that that these conditions are satisfied for the chosen α, after the fact. This allows us to remove some technicalities from the argument.

We generalise the definition of connectedness to apply to k_j-planes. We say $\pi\in\Pi_1\cup\cdots\cup\Pi_d$ **contributes** to a multijoint p if there is some tuple $\boldsymbol{\pi}\in\Pi_1\times\cdots\times\Pi_d$ so that $\boldsymbol{\pi}\ni\pi$ and $\delta(p,\boldsymbol{\pi})=1$. We say that $p,q\in J$ are **adjacent** if there is some $\pi\in\Pi_1\cup\cdots\cup\Pi_d$ that contributes to p and contributes to q. We say a set $E\subseteq J$ is **connected** if given any $p,q\in E$, there is a sequence of points $p=p^{(1)},\ldots,p^{(N)}=q\in E$ so that $p^{(i)}$ and $p^{(i+1)}$ are adjacent for all $1\leq i<N$. This defines an equivalence relation on any $E\subseteq J$. As for Theorem 3.0.1, we may assume that $\operatorname{supp} S$ is itself connected.

Lemma 4.2.1 (k-Plane Handicap)**.** *Let $\lambda\in\mathbb{N}$, and $\Pi_1,\ldots,\Pi_d$ be sets of k_1-,$\ldots,k_d$-planes in $\mathbb{F}^n$ where $k_1+\ldots+k_d=n$, with associated multijoints J. Let $S:J\to\mathbb{R}_{\geq 0}$ and suppose that any two multijoints in $\operatorname{supp} S$ are connected via a path of multijoints in $\operatorname{supp} S$. Then there is a handicap $\alpha:J\to\mathbb{Z}$ so that for all $p\in\operatorname{supp} S$,*

$$\min_{\boldsymbol{\pi}:\delta(p,\boldsymbol{\pi})=1}\frac{1}{S(p)^d}\left(\prod_{j=1}^{d}\frac{\tilde{S}_{k_j}(p,\pi_j,\alpha,\lambda)}{\binom{\lambda+k_j}{k_j}}\right) \tag{4.8}$$

lies in a common interval with length $\leq h'/\lambda$ *for some* $h' = h'(S, J, k_1, \dots, k_d)$*, which does not depend on* λ*. Furthermore, we may choose* α *so that* $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ *for all* $p \notin \operatorname{supp} S$*.*

Remark. The structure of this proof is identical to that of Lemma 3.3.1 except for the numerology in (4.8), which is a consequence of generalising Lemmas 3.2.5-3.2.7 to Lemmas 4.1.6-4.1.8. ◀

Let α be a handicap. For any $p \in J$ and any tuple of planes $\boldsymbol{\pi} \in \Pi_1 \times \cdots \times \Pi_d$ so that $\delta(p, \boldsymbol{\pi}) = 1$, let

$$W(\boldsymbol{\pi}, \alpha) := \frac{1}{S(p)^d} \left(\prod_{j=1}^{d} \frac{\tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda)}{\binom{\lambda + k_j}{k_j}} \right).$$

Note that W additionally depends on λ and p. However, λ is fixed for this Lemma 4.2.1 and the dependence on p is implicit through the tuples $\boldsymbol{\pi}$ which must form a multijoint at p. Now, for each $p \in J$, we define

$$w_p(\alpha) := \min_{\boldsymbol{\pi} : \delta(p, \boldsymbol{\pi}) = 1} W(\boldsymbol{\pi}, \alpha).$$

Proposition 4.2.2. *Let* $\lambda \in \mathbb{N}$*. Let* $w : \{\alpha : J \to \mathbb{Z}\} \to \mathbb{R}_{\geq 0}^{|J|}$ *be the map such that* $w(\alpha) = (w_p(\alpha))_p$*. Define* $A = A(\lambda) \subset \{\alpha : J \to \mathbb{Z}\}$ *to be the set of* α *such that* $\tilde{S}_{k_j}(p, \cdot, \alpha, \lambda) = 0$ *for all* $p \notin \operatorname{supp} S$ *and* $1 \leq j \leq d$*. The image* $w(A)$ *is finite and non-empty.*

Remark. Proposition 4.2.2 contains fewer conditions on α than its precursor, Proposition 3.3.2. ◀

Proof. Let $p \in \operatorname{supp} S$. By translation invariance (Proposition 3.2.3),

$$w(A) = w(\{\alpha \in A : \alpha_p = 0\}).$$

Hence we may further insist that $\alpha_p = 0$ for all $\alpha \in A$. We begin by proving that A is non-empty. Consider

$$\alpha = -(\lambda + 1) \sum_{q \notin \operatorname{supp} S} e_q.$$

Then $\alpha \in A$. Indeed, every plane in $\Pi_1 \cup \cdots \cup \Pi_d$ intersects $\operatorname{supp} S$. Hence, for any $q \notin \operatorname{supp} S$, every plane $\pi \in \Pi_1 \cup \cdots \cup \Pi_d$ such that $\pi \ni q$, there exists $p \in \pi \cap \operatorname{supp} S$. By the definition of α,

$$\alpha_q = -(\lambda + 1) = 0 - (\lambda + 1) = \alpha_p - (\lambda + 1) < \alpha_p - \lambda.$$

For each $1 \leq j \leq d$, by uniform boundedness (Lemma 4.1.6), $\tilde{S}_{k_j}(q, \pi_j, \alpha, \lambda) = 0$. Since π_j is arbitrary, $\tilde{S}_{k_j}(p, \cdot, \alpha, \lambda) = 0$, as desired.

We now prove that $w(A)$ is finite. Let $q, q' \in \operatorname{supp} S$ and suppose that π contributes to both q and q'. By uniform boundedness (Lemma 4.1.6), there are only finitely many differences $(\alpha_q - \alpha_{q'})$ such that $\tilde{S}_{k_j}(q', \pi, \alpha, \lambda) \neq 0$. Given any $q \in \operatorname{supp} S \setminus \{p\}$, there is a finite sequence of adjacent multijoints $p = p_0, p_1, \dots, p_i = q$. The number of choices for α_q such that $\tilde{S}_{k_j}(q, \pi_j, \alpha, \lambda) \neq 0$ for some $\pi_j \in \Pi_j$ contributing to q and p_{i-1} is a product of finitely many numbers (an i-fold product to be precise), and hence, it is finite. Since $\operatorname{supp} S$ is finite, we conclude that there are at most finitely many possible values of $(w_p(\alpha))_{p \in \operatorname{supp} S}$, as desired. ☐

Proposition 4.2.3. *With A as in Proposition 4.2.2, suppose $w_0 \in w(A)$. Then $\exists \alpha \in A$ such that $w(\alpha) = w_0$ and for all $q \notin \operatorname{supp} S$, all planes $\pi \in \Pi_1 \cup \dots \cup \Pi_d$ so that $\pi \ni q$, and all $p \in \Pi \cap \operatorname{supp} S$, we have $\alpha_q < \alpha_p - \lambda$.*

Proof. The proof is identical to that of Proposition 3.3.3. ☐

Let us label each $p \in J$ so that $J = \{p_1, \dots, p_{|J|}\}$. For any α, there exists a permutation $\sigma = \sigma_\alpha \in S_{|J|}$ so that $w_{p_{\sigma(1)}}(\alpha) \geq \dots \geq w_{p_{\sigma(|J|)}}(\alpha)$. By Proposition 4.2.2, the set $w(A)$ is finite. Therefore, of all $w(\alpha) \in w(A)$, we can choose one so that $(w_{p_{\sigma(i)}}(\alpha))_{1 \leq i \leq |J|} \in \mathbb{R}^{|J|}_{\geq 0}$ is minimal with respect to lexicographical order on $\mathbb{R}^{|J|}$.

Let

$$w(\alpha) = (w_{p_{\sigma(i)}}(\alpha))_{1 \leq i \leq |J|} \tag{4.9}$$

be such a minimum and let α be a minimiser. By relabelling the indices of each $p \in J$, we may assume that σ is the identity permutation so that

$$w_{p_1}(\alpha) \geq \dots \geq w_{p_{|J|}}(\alpha) \geq 0.$$

For ease of notation, let $w_i := w_{p_i}(\alpha)$ for each $1 \leq i \leq |J|$.

Proposition 4.2.4 (Continuity of Perturbations)**.** *Let $1 \leq t \leq |J|$. Let*

$$v = \sum_{\substack{1 \leq i \leq t: \\ p_i \in \operatorname{supp} S}} e_i + \sum_{\substack{i: \\ p_i \notin \operatorname{supp} S}} e_i$$

and $\alpha' = \alpha - v$. There is a constant h which depends on $S, k_1, \dots, k_d$ and $|J|$, but not on λ, so that

$$\left| w_i(\alpha) - w_i(\alpha') \right| \leq \frac{h}{2\lambda}$$

for all $1 \leq i \leq N$, where $N = |\operatorname{supp} S|$.

Remark. The conclusion of the generalised numerology in the preliminary lemmas is that the upper bound in Proposition 4.2.4 is still proportional to λ^{-1}. Note that the constant of proportionality depends on S and $|J|$. ◀

Proof. We construct h directly. Fix $1 \leq i \leq N$, fix a d-tuple of planes $\boldsymbol{\pi}$, which realises $w_i(\alpha)$, and fix $\boldsymbol{\pi}'$, which realises $w_i(\alpha')$. Consider

$$\left| w_i(\alpha) - w_i(\alpha') \right| = \left| W(\boldsymbol{\pi}, \alpha) - W(\boldsymbol{\pi}', \alpha') \right|.$$

If $\boldsymbol{\pi} = \boldsymbol{\pi}'$ then $\big|w_i(\alpha) - w_i(\alpha')\big| = \big|W(\boldsymbol{\pi}, \alpha) - W(\boldsymbol{\pi}, \alpha')\big|$. Otherwise, $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$, in which case

$$\big|w_i(\alpha) - w_i(\alpha)\big| = \big|W(\boldsymbol{\pi}, \alpha) - W(\boldsymbol{\pi}', \alpha')\big| \leq \max_{\tilde{\boldsymbol{\pi}}} \big|W(\tilde{\boldsymbol{\pi}}, \alpha) - W(\tilde{\boldsymbol{\pi}}, \alpha')\big|,$$

where the maximum is over all $\tilde{\boldsymbol{\pi}} \in (\arg\min W(\cdot, \alpha) \cup \arg\min W(\cdot, \alpha'))$. Hence, we may assume

$$\big|w_i(\alpha) - w_i(\alpha')\big| \leq \big|W(\tilde{\boldsymbol{\pi}}, \alpha) - W(\tilde{\boldsymbol{\pi}}, \alpha')\big|,$$

for some $\tilde{\boldsymbol{\pi}}$, which minimises either $W(\cdot, \alpha)$ or $W(\cdot, \alpha')$. However, for any $\boldsymbol{\pi} \in \Pi_1 \cup \cdots \cup \Pi_d$ which forms a multijoint at $p = p_i$,

$$\begin{aligned} &S(p)^d \left(\prod_{j=1}^{d} \binom{\lambda + k_j}{k_j} \right) \big|W(\boldsymbol{\pi}, \alpha) - W(\boldsymbol{\pi}, \alpha')\big| \\ =&\left| \prod_{j=1}^{d} \left(\tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) - \left(\tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) - \tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda) \right) \right) - \prod_{j=1}^{d} \tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) \right|. \end{aligned} \tag{4.10}$$

Taking (4.10), we expand the first product so that we can cancel both occurrences of $\prod_j \tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda)$. We are then left with terms of the following form:

$$\left(\prod_{j \in A} \tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) \right) \left(\prod_{j' \in B} (\tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) - \tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda)) \right), \tag{4.11}$$

where $A \sqcup B = \{1, \ldots, d\}$ and $B \neq \emptyset$. To establish an upper bound on (4.10), by the triangle inequality, it suffices to bound each such term of the form (4.11) separately. By continuity (Lemma 4.1.8), for any $\pi_j \in \Pi_j$ and $p \in \pi_j \cap J$,

$$\left| \tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda) - \tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) \right| \leq \binom{\lambda + k_j - 1}{k_j - 1} |J|,$$

since $\|\alpha - \alpha'\|_{\ell^1(J)} = \|v\|_{\ell^1(J)} \leq |J|$. Combining this with the fact that each $\tilde{S}_{k_j}(p, \pi, \alpha', \lambda) = \binom{\lambda + k_j}{k_j}$ by construction, each term of the form (4.11) is bounded by

$$\prod_{j \in A} \binom{\lambda + k_j}{k_j} \prod_{j \in B} \binom{\lambda + k_j - 1}{k_j - 1} |J| \sim_{k_1, \ldots, k_d, J} \lambda^{\sum_{j \in A} k_j + \sum_{j' \in B} (k_{j'} - 1)},$$

for sets A, B so that $A \sqcup B = \{1, \ldots, d\}$ and $B \neq \emptyset$. Hence, (4.10) can be dominated by

$$S(p)^d \left(\prod_{j=1}^{d} \binom{\lambda + k_j}{k_j} \right) \big|W(\boldsymbol{\pi}, \alpha) - W(\boldsymbol{\pi}, \alpha')\big| \leq h_{n-1} \lambda^{n-1} + \ldots + h_1 \lambda + h_0$$

for $h_{n-1}, \ldots, h_0$ sufficiently large, depending on S, $k_1, \ldots, k_d$ and $|J|$. This is in turn at most $\frac{h}{2}\lambda^{n-1}$ for some h which can be expressed as a function of S, $|J|$ and $k_1, \ldots, k_d$, that does not depend on λ. Dividing by

$$\left(\prod_{j=1}^{d}\binom{\lambda+k_j}{k_j}\right) \sim_{k_1,\ldots,k_d} \lambda^n$$

and possibly updating h further, depending on $S(p)^{-d}$, we deduce

$$\left|w_i(\alpha) - w_i(\alpha')\right| \le \left|W(\boldsymbol{\pi},\alpha) - W(\boldsymbol{\pi},\alpha')\right| \le \frac{h}{2\lambda}$$

for some h which can be expressed as a function in $|J|, k_1, \ldots, k_d$ and the quantities $\{S(p)^{-d} : p \in J\}$, but does not depend on λ. This defines h. □

The remainder of this subsection is dedicated to proving Lemma 4.2.1. The proof is analogous to that of Lemma 3.3.1. Since $w_1(\alpha) \ge \ldots \ge w_{|J|}(\alpha) \ge 0$, it suffices to show that all the differences $w_i(\alpha) - w_{i+1}(\alpha)$ are small. We will prove this by contradiction. To begin, we assume there is some index $1 \le t < N$ so that $w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda$, with h as given by Proposition 4.2.4. We construct a perturbation of the handicap α. Let α' be the perturbed handicap. The following three claims are established for the perturbation:

1. The large entries of the tuple $w(\alpha)$ remain large, and the small entries remain small.
2. The perturbed handicap α' is such that $w(\alpha') \in w(A)$, so $w(\alpha) \le w(\alpha')$.
3. If $w(\alpha) \ne w(\alpha')$ then $w(\alpha') < w(\alpha)$. Hence, $w(\alpha') = w(\alpha)$.

Thereafter, we realise that the perturbation can be applied to α', the already perturbed handicap. Moreover, by the connectedness of $\operatorname{supp} S$, there is a pair $p, q \in \operatorname{supp} S$ so that $w_p(\alpha)$ is large, $w_q(\alpha)$ is small and there is a plane which contributes to both p and q. If we perturb sufficiently many times, what results is a perturbed handicap α' so that $w(\alpha) = w(\alpha')$, and

$$\alpha'_p < \alpha'_q - \lambda.$$

By uniform boundedness (Lemma 4.1.6), $\tilde{S}_{k_j}(p, \pi_j, \alpha', \lambda) = 0$, and hence $w_p(\alpha) = 0$. So $0 = w_p(\alpha) \ge w_q(\alpha) \ge 0$ and hence $w_q(\alpha) = 0$. However, we assumed for a contradiction that $w_p(\alpha)$ was large and that $w_q(\alpha)$ was small so that $w_p(\alpha) > w_q(\alpha)$, which is a contradiction.

Proof of Lemma 4.2.1. Let $\operatorname{supp} S = \{p_1, \ldots, p_N\}$ for some $N \le |J|$ be connected, and let $h' = |J|h$, where h is given by Proposition 4.2.4. Since the tuple $(w_i)_i$ has at most $|J|$ non-zero entries, it suffices to show that

$$w_i - w_{i+1} \le \frac{h}{\lambda} \tag{4.12}$$

for all $1 \le i < N$. Indeed, summing this inequality over all $1 \le i \le N$, we deduce $\max_i w_i - \min_i w_i \le |J| h/\lambda$. That is, the interval containing all $(w_i)_{1 \le i \le N}$ has width $\sim_{k_j, J, S} 1/\lambda$, where the implied constant does not depend on λ.

We now prove that (4.12) holds for all $1 \le i < N$. Suppose for a contradiction that there is some index $1 \le i < N$ so that (4.12) does not hold. Let t be the least such index, and for this choice of t, define

$$v = \sum_{\substack{1 \le i \le t: \\ p_i \in \operatorname{supp} S}} e_i + \sum_{\substack{i: \\ p_i \notin \operatorname{supp} S}} e_i$$

and let $\alpha' = \alpha - v$.

Let $\pi \in \Pi_j$ for some $1 \le j \le d$. Recall that $\sum_{p \in \pi} \tilde{S}_{k_j}(p, \pi, \alpha, \lambda) = \binom{\lambda + k_j}{k_j}$ for any α. It follows that, if $\tilde{S}_{k_j}(p, \pi, \alpha', \lambda) > \tilde{S}_{k_j}(p, \pi, \alpha, \lambda)$ for some $p \in \pi \cap J$, then there exists some $p' \in \pi \cap J$ so that $p' \neq p$ and $\tilde{S}_{k_j}(p', \pi, \alpha', \lambda) < \tilde{S}_{k_j}(p', \pi, \alpha, \lambda)$. For those i such that $p_i \notin \operatorname{supp} S$, we have that $w_i(\alpha) = 0 = w_i(\alpha')$. Moreover, by monotonicity (Lemma 4.1.7), for i such that $p_i \in \operatorname{supp} S$, we have that if $i \le t$ then $w_i(\alpha') \le w_i(\alpha)$ (where the handicap is decreased) and if $i > t$ then $w_i(\alpha') \ge w_i(\alpha)$ (where the handicap is unchanged). One inequality is strict if and only if the other is too. By Proposition 4.2.4, the difference $\big|w_i(\alpha) - w_i(\alpha')\big|$ is bounded above by $h/(2\lambda)$ for all $1 \le i \le N$, and by the definition of t,

$$w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda. \tag{4.13}$$

Let $\sigma' \in S_{|J|}$ be a permutation such that

$$w_{\sigma'(1)}(\alpha') \ge \cdots \ge w_{\sigma'(|J|)}(\alpha'), \tag{4.14}$$

where $\sigma'(i) = i$ for $N < i \le |J|$.

Claim 4.2.5. If $1 \le i \le t$, then $1 \le \sigma'(i) \le t$, and if $t < i' \le N$, then $t < \sigma'(i') \le |J|$.

Proof. Suppose $p_i, p_{i'} \in \operatorname{supp} S$ are such that $i \le t < i'$. By Proposition 4.2.4,

$$\begin{aligned} w_i(\alpha') - w_{i'}(\alpha') &= \big(w_i(\alpha') - w_i(\alpha)\big) + \big(w_i(\alpha) - w_{i'}(\alpha)\big) + \big(w_{i'}(\alpha) - w_{i'}(\alpha')\big) \\ &\ge -\frac{h}{2\lambda} + \big(w_i(\alpha) - w_{i'}(\alpha)\big) - \frac{h}{2\lambda}. \end{aligned}$$

Using inequalities (4.13) and the fact that $w_i(\alpha)$ is decreasing in i, this is at least

$$\big(w_t(\alpha) - w_{t+1}(\alpha)\big) - \frac{h}{\lambda} > \frac{h}{\lambda} - \frac{h}{\lambda} = 0.$$

Hence, if $1 \le i \le t$, then $w_i(\alpha') \ge w_{i'}(\alpha')$ for all $i' > t$. Therefore $1 \le \sigma'(i) \le t$. Similarly, if $t < i \le N$, then $w_{i'}(\alpha') \ge w_i(\alpha')$ for all $1 \le i' \le t$. Hence, $t < \sigma'(i) \le N$, as desired. □

Claim 4.2.6. Let the set A be as in Proposition 4.2.2. Then $w(\alpha') \in w(A)$. Hence $w(\alpha')$ was among those considered when the minimiser $w(\alpha)$ was chosen.

Proof. Since $w(\alpha) \in w(A)$, we have that $\tilde{S}_{k_j}(p, \cdot, \alpha, \lambda) = 0$ for all $1 \leq j \leq d$ and $p \notin \operatorname{supp} S$. Since Proposition 4.2.3 applies, we may assume that α is such that for all $q \notin \operatorname{supp} S$, and all $\pi \in \Pi_1 \cup \cdots \cup \Pi_d$ so that $\pi \ni q$, if $p \in \pi \cap \operatorname{supp} S$, then

$$\alpha_q < \alpha_p - \lambda.$$

We will show that $\tilde{S}_{k_j}(q, \cdot, \alpha', \lambda) = 0$ for all $1 \leq j \leq d$ and $q \notin \operatorname{supp} S$. Let $q \notin \operatorname{supp} S$. If $\pi \in \Pi_1 \cup \cdots \cup \Pi_d$ is a k_j-plane such that $\pi \ni q$, and $p \in \pi \cap \operatorname{supp} S$, then

$$\alpha'_q = \alpha_q - 1 < (\alpha_p - \lambda) - 1 = (\alpha_p - 1) - \lambda \leq \alpha'_p - \lambda.$$

We may assume that every plane in $\Pi_1 \cup \cdots \cup \Pi_d$ intersects $\operatorname{supp} S$, so such p exists. Hence, by uniform boundedness (Lemma 4.1.6), $\tilde{S}_{k_j}(q, \pi, \alpha', \lambda) = 0$. Hence $w(\alpha') \in w(A)$ and so $(w_p(\alpha'))_p$ was among those tuples considered when $(w_p(\alpha))_p$ was chosen. ☐

Claim 4.2.7. If $w(\alpha') \neq w(\alpha)$, then $w(\alpha') < w(\alpha)$. Hence $w(\alpha) = w(\alpha')$.

Proof. Suppose that there is some $1 \leq i \leq |J|$ so that $w_i(\alpha') \neq w_i(\alpha)$. Then there is an $i \leq t$ so that $w_i(\alpha') < w_i(\alpha)$ and $1 \leq \sigma'(i) \leq t$. That is, $w_i(\alpha')$ is strictly smaller than $w_i(\alpha)$ and $w_{\sigma'(i)}(\alpha')$ is among the t largest values of $(w_i(\alpha'))_{1 \leq i \leq |J|}$ by Claim 4.2.5. Hence

$$(w_{\sigma'(i)}(\alpha'))_{1 \leq i \leq t} <_{\text{lex}} (w_i(\alpha))_{1 \leq i \leq t},$$

and therefore, $w(\alpha')$ is of strictly lower lexicographical order than $w(\alpha)$. Moreover, by Claim 4.2.6, $w(\alpha') \in w(A)$. This contradicts that $w(\alpha) \in w(A)$ was chosen to be minimising. ☐

Thus, $w_i(\alpha') = w_i(\alpha - v) = w_i(\alpha)$ for all $1 \leq i \leq |J|$. We have not yet contradicted our assumption that (4.12) is false, however we have deduced that the perturbation $\alpha \mapsto (\alpha - v)$ must leave $(w_i)_i$ unchanged. To conclude, we observe that we may return to (4.9), use $\alpha' = \alpha - v$, and repeat the application of Claim 4.2.5, Claim 4.2.6 and Claim 4.2.7. We may repeat this process *ad infinitum* to deduce that

$$w_i(\alpha) = w_i(\alpha - v) = w_i(\alpha - 2v) = \cdots = w_i(\alpha - cv)$$

for all $1 \leq i \leq |J|$. By connectedness of $\operatorname{supp} S$, we can find a plane $\pi \in \Pi_j$ for some $1 \leq j \leq d$ contributing to distinct multijoints $p_i, p_j \in \operatorname{supp} S$ for some i and j satisfying $i \leq t < j \leq N$. Taking c sufficiently large (depending on λ) forces $w_i(\alpha') = 0$ by uniform boundedness (Lemma 4.1.6) and hence $w_i(\alpha) = 0$. That is, one of the large entries of the tuple $(w_i(\alpha))_i$ is zero. Since $(w_i)_i$ is decreasing in i and each w_i is non-negative, $w_{i'}(\alpha) = 0$ for all $i \leq i' \leq |J|$. This contradicts our assumption that $w_t(\alpha) - w_{t+1}(\alpha) > h/\lambda$, and hence (4.12) holds for all $1 \leq i < N$. ☐

4.2.2 Vanishing Lemma

To conclude, we lift Lemma 3.2.4 to a k_j-plane generalisation. Recall (Subsection 4.1.2) that the sets $B(p,\pi,\alpha,\lambda)$ are sets of dual maps of the form $Df(p)$ for some derivative operators which act on polynomials restricted to π. Let us abuse notation and write $D \in B(p,\pi,\alpha,\lambda)$ to mean a derivative operator D, so that $Df(p) = \langle \phi, f \rangle$, for some $\phi \in B(p,\pi,\alpha,\lambda)$.

To prove the higher-dimensional vanishing lemma, we first require an intermediate result on compositions of Hasse derivatives.

Lemma 4.2.8. *Suppose that* $p \in \mathbb{F}^n$ *is a joint formed by* $\boldsymbol{\pi}$ *where* $\dim \pi_j = k_j$ *for each* $1 \le j \le d$ *and* $k_1 + \ldots + k_d = n$. *Suppose that* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *is non-zero and vanishes to order precisely* r *at* p. *Then there exist* $r_1, \ldots, r_d \in \mathbb{Z}_{\ge 0}$ *such that* $r_1 + \ldots + r_d = r$, *and there exist* $D_j \in \mathbb{B}_{r_j}(p, \pi_j(p))$ *for each* $1 \le j \le d$ *so that*

$$(D_1 \cdots D_d) f(p) \neq 0.$$

Proof. Without loss of generality, assume that each π_j contains the coordinate vectors $e_{k_1+\ldots+k_{j-1}+1}, \ldots, e_{k_1+\ldots+k_j}$, and that p is at the origin. Let μx^γ be a monomial of f of lowest degree so that the coefficient $\mu \neq 0$. This is possible since f is non-zero. Since f vanishes to order r at p, then $\gamma_1 + \ldots + \gamma_d = r$. Let $r_1 = \gamma_1 + \ldots + \gamma_{k_1}$, let $r_2 = \gamma_{k_1+1} + \ldots + \gamma_{k_1+k_2}$ and so on, until $r_d = \gamma_{k_1+\ldots+k_{d-1}+1} + \ldots + \gamma_n$. For each $1 \le j \le d$, let D_j be the $(0, \ldots, 0, \gamma_{k_1+\ldots+k_{j-1}+1}, \ldots, \gamma_{k_1+\ldots+k_j}, 0, \ldots, 0)$-th Hasse derivative. Then each D_j is of order r_j and $r_1 + \ldots + r_d = r$. Moreover,

$$(D_1 \cdots D_d) f(p) = \mu \neq 0,$$

as desired. □

Lemma 4.2.9 (Vanishing Lemma)**.** *Fix* α *and* λ. *For each* $1 \le j \le d$, $\pi_j \in \Pi_j$ *and* $p \in J$, *build the sets* $B(p,\pi_j) = B(p,\pi_j,\alpha,\lambda)$ *and* $B_r(p,\pi_j) = B_r(p,\pi_j,\alpha,\lambda)$ *as above, in Subsection 4.1.2. For each* $p \in J$, *choose planes* $\pi_j(p) \in \Pi_j$ *for* $1 \le j \le d$ *so that* $\delta(p, \boldsymbol{\pi}(p)) = 1$. *If* $f \in \mathbb{F}_\lambda[x_1, \ldots, x_n]$ *is non-zero then there exists* $p \in J$ *and* $D_j \in B(p, \pi_j(p))$ *for* $1 \le j \le d$ *so that*

$$D_1 \cdots D_d f(p) \neq 0.$$

Proof. Suppose for a contradiction that there is a non-zero $f \in \mathbb{F}_\lambda[x_1, \ldots, x_n]$, so that for all $p \in J$ and all $D_j \in B(p, \pi_j(p))$ for $1 \le j \le d$,

$$D_1 \cdots D_d f(p) = 0.$$

Let $\operatorname{mult}(f,p)$ denote the multiplicity of the zero of f at p; that is, the least positive integer m, so that there is an m-th derivative of f that does not vanish at p. Fix a choice of p that minimises $(p, \operatorname{mult}(f,p))$ with respect to $\prec$. By Lemma 4.2.8 there are $D_j \in \mathbb{B}_{r_j}(p, \pi_j(p))$ so that $r_1 + \ldots + r_d = \operatorname{mult}(f,p)$, so that

$$D_1 \cdots D_d f(p) \neq 0. \tag{4.15}$$

Of all possible choices of $r_1 + \ldots + r_d = \operatorname{mult}(f, p)$, choose one that maximises

$$\Big|\{j' \in \{1, \ldots, d\} : D_j \in B_{r_j}(p, \pi_{j'}(p))\}\Big|. \tag{4.16}$$

By hypothesis, there exists j so that $D_j \notin B(p, \pi_j(p))$, which we fix. By relabelling, we can assume that $j = 1$, without loss of generality.

Consider $(p', r') \prec (p, r_1)$ for some $p' \in \pi_1(p) \cap J$. Then

$$(p', r' + r_2 + \ldots + r_d) \prec (p, r_1 + \ldots + r_d) = (p, \operatorname{mult}(f, p)).$$

The multijoint p was chosen so that $(p, \operatorname{mult}(f, p)) \prec (p', \operatorname{mult}(f, p'))$. Hence

$$(p', r' + r_2 + \ldots + r_d) \prec (p', \operatorname{mult}(f, p')).$$

By the definition of $\prec$, we deduce that

$$r' + r_2 + \ldots + r_d < \operatorname{mult}(f, p').$$

Therefore,

$$DD_2 \cdots D_d f(p) = 0$$

for all $D \in B_{r'}(p, \pi_1)$ and all $1 \leq r' \leq r_1$.

Let

$$g := D_2 \cdots D_d f.$$

We have established that $Dg(p') = 0$ for all $D \in B_{r'}(p', \pi_1)$ for $(p', r') \prec (p, r_1)$, and by (4.15), $g|_{\pi_1(p)}$ is not the zero polynomial and moreover, $D_1 g|_{\pi_1(p)}$ is non-zero, where D_1 has order r_1. Therefore, the set

$$\bigcup_{(p', r') \prec (p, r_1)} B_{r'}(p', \pi_1(p'))$$

does not span $\left(f|_{\pi_1(p)} : f \in \mathbb{F}_\lambda[x_1, \ldots, x_n]\right)^*$. Hence, by construction, $B_{r_1}(p, \pi_1(p))$ is non-empty and

$$\operatorname{span} \cup_{(p', r') \preceq (p, r_1)} B_{r'}(p', \pi_1(p), \alpha, \lambda) = \operatorname{span} \cup_{(p', r') \preceq (p, r_1)} \mathbb{B}_{r'}(p', \pi_1(p), \lambda).$$

Hence, we can find a linear combination of elements in $B_{r_1}(p, \pi_1(p))$ that realises D. In particular, there must exist $D_1 \in B_{r_1}(p, \pi_1(p))$ so that $D_1 g(p) \neq 0$. This contradicts our choice of $r_1, \ldots, r_d$, concluding the proof. ☐

Corollary 4.2.9a. *Fix α and λ. For each $1 \leq j \leq d$, $\pi_j \in \Pi_j$ and $p \in J$, build the sets $B(p, \pi_j, \alpha, \lambda)$ as above, in Subsection 4.1.2. For each $p \in J$, choose planes $\pi_1(p) \in \Pi_1, \ldots, \pi_d(p) \in \Pi_d$ so that $\delta(p, \boldsymbol{\pi}(p)) = 1$. Let $\tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda) := \big|B(p, \pi_j, \alpha, \lambda)\big|$. Then*

$$\sum_{p \in J} \prod_{j=1}^{d} \tilde{S}_{k_j}(p, \pi_j(p), \alpha, \lambda) \geq \binom{\lambda + n}{n}.$$

Proof. For a contradiction, suppose that the conclusion is false. By parameter counting, there exists a non-zero $f \in \mathbb{F}_\lambda[x_1, \ldots, x_n]$ so that

$$D_1 \cdots D_d f(p) = 0$$

for all $D_j \in B(p, \pi_j(p), \alpha, \lambda)$, and all $1 \leq j \leq d$. This contradicts Lemma 4.2.9. $\square$

4.2.3 Good Vanishing Conditions Yield A Factorisation

We now deduce Theorem 4.0.1 as a corollary of Lemma 4.2.1. Let $S : J \to \mathbb{R}_{\geq 0}$ be non-negative with $\|S\|_d = 1$ and let λ be sufficiently large. By Lemma 4.2.1, we may choose a handicap $\alpha = \alpha(\lambda)$, so that the numbers $\tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda)$, as constructed in Subsection 4.1.2, are such that there is an interval containing

$$w_p(\alpha) = \min_{\delta(p,\boldsymbol{\pi})=1} \frac{1}{S(p)^d} \left(\prod_{j=1}^{d} \frac{\tilde{S}(p, \pi_j, \alpha, \lambda)}{\binom{\lambda+k_j}{k_j}} \right)$$

for all $p \in \operatorname{supp} S$ with length at most h'/λ. Moreover, $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ for all p such that $S(p) = 0$. We conclude as in the case for lines.

Let $\lambda \in \mathbb{N}$ and let $\varepsilon = h'/\lambda$, where h' is the constant given in Lemma 4.2.1. Then we can find ω so that

$$w - \varepsilon \leq w_p(\alpha) \leq w$$

for all $p \in \operatorname{supp} S$ and so that $\tilde{S}(p, \cdot, \alpha, \lambda) = 0$ if $S(p) = 0$. The sets $B(p, \pi_j, \alpha, \lambda)$ satisfy the hypothesis of Lemma 4.2.9 and therefore Corollary 4.2.9a applies. For each $p \in J$, let $(\pi_j(p))_j$ be a choice of planes which minimises $\prod_{j=1}^{d} \tilde{S}(p, \pi_j(p), \alpha, \lambda)$ over all tuples $\boldsymbol{\pi}$ so that $\delta(p, \boldsymbol{\pi}) = 1$. Then,

$$w = \sum_{p \in J} S(p)^d w \geq \sum_{p \in J} \prod_{j=1}^{d} \frac{\tilde{S}(p, \pi_j(p), \alpha, \lambda)}{\binom{\lambda+k_j}{k_j}} \geq \frac{1}{\prod_{j=1}^{d} \binom{\lambda+k_j}{k_j}} \binom{\lambda+n}{n} \gtrsim_{k_1,\ldots,k_d} 1, \tag{4.17}$$

where we have used the fact that $k_1 + \ldots + k_d = n$. This implies that $w \gtrsim_{k_1,\ldots,k_d} 1$.

Remark. The implied constant in (4.17), depending only on $k_1, \ldots, k_d$, is precisely the one that appears in Theorem 4.0.1. ◀

If λ is large enough, then the length of the interval $\varepsilon = h'/\lambda < w$ so that the left endpoint $w - \varepsilon$ will be strictly positive. Hence, each $B(p, \pi, \alpha, \lambda) \neq \emptyset$ for all $p \in J$ and every π that contributes to p.

Define

$$s_{k_j,\lambda}(p, \pi_j) := \frac{\tilde{S}(p, \pi_j, \alpha, \lambda)}{\binom{\lambda+k_j}{k_j}}.$$

For each $p \in J$, if $\pi_j \in \Pi_j$ is such that $p \notin \pi_j$, then set $s_{k_j,\lambda}(p, \pi_j) = 0$. Thus, for any $p \in \operatorname{supp} S$ we have

$$1 \lesssim_{k_1,\dots,k_d} w \leq \frac{h}{\lambda} + \frac{1}{S(p)^d} \prod_{j=1}^{d} s_{k_j,\lambda}(p, \pi_j(p)).$$

Since $\|S\|_d = 1$, $S(p) \leq 1$ for all $p \in J$. Hence

$$S^d(p) \lesssim_{k_1,\dots,k_d} \frac{h}{\lambda} + \prod_{j=1}^{d} s_{k_j,\lambda}(p, \pi_j(p))$$

for a tuple of planes $\boldsymbol{\pi}(p)$ that minimises the right-hand side over all tuples such that $\delta(p, \boldsymbol{\pi}) = 1$. Hence

$$\delta(p, \boldsymbol{\pi}) S^d(p) \lesssim_{k_1,\dots,k_d} \frac{h}{\lambda} + \prod_{j=1}^{d} s_{k_j,\lambda}(p, \pi_j) \tag{4.18}$$

for all tuples of planes $\boldsymbol{\pi}$. By construction, for all $1 \leq j \leq d$, $\pi_j \in \Pi_j$,

$$\sum_{p \in \pi_j \cap J} s_{k_j,\lambda}(p, \pi_j) = \frac{1}{\binom{\lambda+k_j}{k_j}} \sum_{p \in \pi_j \cap J} \tilde{S}_{k_j}(p, \pi_j(p), \alpha(\lambda), \lambda) = 1. \tag{4.19}$$

Both inequality (4.18) and equation (4.19) are uniform in ε and each function

$$\frac{1}{S(p)} s_{k_j,\lambda}(p, \pi_j) = \frac{1}{S(p)} \frac{\tilde{S}_{k_j}(p, \pi_j, \alpha, \lambda)}{\binom{\lambda+k_j}{k_j}}$$

can be realised as an $\mathbb{R}$-valued vector in $[0,1]^{|J| \times |\Pi_j|}$. Hence, passing to a subsequence if necessary, we may let $s_{k_j} = \lim_{\lambda \to \infty} s_{k_j,\lambda}$ for each $1 \leq j \leq d$. Letting $\lambda \to \infty$ in (4.18) and (4.19) shows that this s satisfies the properties stated in Theorem 4.0.1. □

We will not include a proof of Theorem 4.0.3, but we will highlight the necessary modifications of the arguments to follow to establish the result. The proof differs from the arguments in this chapter in only three places. Firstly, the definition of the sets $\mathbb{B}_r(p, \pi, \lambda)$, above, change so as to be defined more generally for a k-dimensional variety γ, rather than for a k-plane π. Secondly, we generalise equation (4.4), above, to

$$\sum_{p \in \gamma \cap J} \left| B(p, \gamma, \alpha, \lambda) \right| = \dim \gamma \binom{\lambda}{k} + O_\gamma(\lambda^{\dim \gamma - 1}).$$

Finally, a generalisation of Lemma 4.2.9 and Corollary 4.2.9a, to follow, with k_j-dimensional varieties replacing k_j-planes, is required. This updated definition and two updated results are detailed in Section 4 and Section 5 of [TYZ20], and the remaining argument does not require any modification.

4.2.4 The Discrete Bourgain–Guth Theorem for k_j-Planes

With Theorem 4.0.1 proved, we turn to Theorem 4.0.2. If the field $\mathbb{F}$ is finite, then we can apply Theorem 4.0.1 with each $\pi_j = \mathrm{Gr}(k_j, \mathbb{F}^n)$. Hence, we may assume that the field is infinite.

Let $S : \mathbb{F}^n \to \mathbb{R}_{\geq 0}$ be finitely supported. Consider

$$\mathcal{O} := \left\{ \left(\pi_j \cap \operatorname{supp} S\right)_j : \delta(p, \boldsymbol{\pi}) = 1,\, p \in \mathbb{F}^n,\, \pi_j \subset \mathbb{F}^n \right\},$$

where in the definition of $\mathcal{O}$, we consider any tuple of k_j-planes of which there may be infinitely many and any $p \in \operatorname{supp} S$. However, since $\operatorname{supp} S$ is finite, so too is $\mathcal{O}$. For every $(E_j)_j \in \mathcal{O}$, choose a tuple of k_j-planes $\boldsymbol{\pi}$ so that $\delta(p, \boldsymbol{\pi}) = 1$ and $\pi_j \cap \operatorname{supp} S = E_j$ for every $1 \leq j \leq d$. Hence, we define the finite sets Π_j to consist of all such planes π_j. In particular, since the field is infinite, for each $p \in \operatorname{supp} S$, the tuple $(\{p\})_j$ belongs to $\mathcal{O}$ and hence $\operatorname{supp} S$ is a subset of the multijoints formed by the finite sets $\Pi_1, \ldots, \Pi_d$. Hence, there exist factorising functions $\tilde{s}_{k_j}$ that satisfy the displays described in Theorem 4.0.2.

Recall from (4.5) that if two k_j-planes, π and π', are distinct and satisfy $\pi \cap \operatorname{supp} S = \pi' \cap \operatorname{supp} S$, then $\tilde{S}_{k_j}(\cdot, \pi, \alpha, \lambda) = \tilde{S}_{k_j}(\cdot, \pi', \alpha, \lambda)$. Hence, the functions $\tilde{s}_{k_j}$ satisfy $\tilde{s}_{k_j}(\cdot, e(\pi)) = \tilde{s}_{k_j}(\cdot, e(\pi'))$.

Now, consider the pairs $(p, V_j) \in \operatorname{supp} S \times \mathrm{Gr}(k_j, \mathbb{F}^n)$. By construction, there exists $\pi_j \in \Pi_j$ so that $\pi_j \cap \operatorname{supp} S = (p + V_j) \cap \operatorname{supp} S$ and it is well-defined to set $s_{k_j}(p, V_j) := \tilde{s}_{k_j}(p, e(\pi_j))$. We additionally set $s_{k_j}(p, \cdot) = 0$ for any $p \notin \operatorname{supp} S$, whereby each s_{k_j} is finitely supported and defined on $\mathbb{F}^n \times \mathrm{Gr}(k_j, \mathbb{F}^n)$. Each s_{k_j} automatically satisfies

$$\sum_{p \in \pi_j} s_{k_j}(p, e(\pi_j)) = \sum_{p \in \pi_j \cap J} s_{k_j}(p, e(\pi_j)) = 1$$

for any k_j-plane $\pi_j \subset \mathbb{F}^n$, establishing the second display of Theorem 4.0.2.

Turning to the first display, let $p \in \operatorname{supp} S$ and let $V_j \in \mathrm{Gr}(k_j, \mathbb{F}^n)$ be such that $V_1 \wedge \cdots \wedge V_d = 1$. For each $1 \leq j \leq d$ there exists $\pi_j \in \Pi_j$ so that $(p + V_j) \cap \operatorname{supp} S = \pi_j \cap \operatorname{supp} S$ and $\delta(p, \pi_1, \ldots, \pi_d) = 1$, by construction. Hence

$$\begin{aligned} (V_1 \wedge \cdots \wedge V_d) S(p)^d &= \delta(p, \boldsymbol{\pi}) S(p)^d \\ &\lesssim_{k_1, \ldots, k_d} \prod_{j=1}^{d} \tilde{s}_{k_j}(p, e(\pi_j)) \\ &= \prod_{j=1}^{d} s_{k_j}(p, V_j). \end{aligned}$$

This concludes the proof of Theorem 4.0.2.

Chapter 5

The (Multi)Joint Duality Theorem

The results in this chapter were developed in collaboration with Anthony Carbery.

In Chapter 3, we proved the dual multijoint theorem, Theorem 3.0.1, for arbitrary sets of lines $\mathcal{L}_1, \ldots, \mathcal{L}_d \subset \mathbb{F}^d$. Considering the joints problem, let each $\mathcal{L}_j = \mathcal{L}$ for $1 \leq j \leq d$. We can use Theorem 3.0.1 to conclude, *via* Proposition 2.3.1, that

$$\sum_{p \in J} \left(\sum_{\boldsymbol{l}} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^{\frac{1}{d-1}} \lesssim \left(\sum_{l \in \mathcal{L}} f(l) \right)^{\frac{d}{d-1}}. \tag{5.1}$$

In this chapter, we prove Theorem 2.3.2 – that there is a duality for the (non-transversal) joints problem. By Proposition 2.3.1, it suffices to prove Theorem 5.0.1, which states that the joints inequality (5.1) implies the dual joints theorem.

Theorem 5.0.1 (Joints Factorisations are Necessary)**.** *Let $\mathcal{L}$ be a finite set of lines in $\mathbb{F}^d$ with joints J. Suppose that*

$$\sum_{p \in J} \left(\sum_{\boldsymbol{l}} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^{\frac{1}{d-1}} \lesssim \left(\sum_{l \in \mathcal{L}} f(l) \right)^{\frac{d}{d-1}}$$

for every $f : \mathcal{L} \to \mathbb{R}_{\geq 0}$. Then for any $S : J \to \mathbb{R}_{\geq 0}$, there is a function $s : J \times \mathcal{L} \to \mathbb{R}_{\geq 0}$ such that

$$\delta(p, \boldsymbol{l}) S(p)^d \leq \prod_{l \in \boldsymbol{l}} s(p, l)$$

for all $p \in J$ and $\boldsymbol{l} \in \mathcal{L}^d$, and

$$\sum_{p \in l \cap J} s(p, l) \lesssim \|S\|_d$$

for all $l \in \mathcal{L}$.

Theorem 5.0.1 is stated with respect to a single set of lines $\mathcal{L}$, and hence, it describes duality for the joints problem. This is somewhat surprising, given that the corresponding inequality for joints (5.1) describes the boundedness of an operator which is neither linear nor multilinear and so we are, *a priori*, less inclined to expect a duality result to hold in this setting.

In contrast, the multijoints problem is described by multilinear inequality (1.4), and so we do expect that some form of duality should hold. We deduce this multijoint duality as a consequence of Theorem 5.0.1. Indeed, for any sets of lines $\mathcal{L}_1, \dots, \mathcal{L}_d \subset \mathbb{F}^d$ with multijoints J, the joints problem with lines $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ also has joints J. Therefore, since these duality statements are independent of the functions f and f_j, which feature in the inequalities, the analogous multilinear statement, Corollary 5.0.1a below, holds.

Corollary 5.0.1a. *Suppose that for any finite sets of lines, $\mathcal{L}_1, \dots, \mathcal{L}_d \subset \mathbb{F}^d$, with multijoints J,*

$$\sum_{p \in J} \left(\sum_{\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d} \delta(p, \boldsymbol{l}) \prod_{j=1}^{d} f_j(l_j) \right)^{\frac{1}{d-1}} \lesssim \prod_{j=1}^{d} \left(\sum_{l_j \in \mathcal{L}_j} f_j(l_j) \right)^{\frac{1}{d-1}},$$

for every $f_j : \mathcal{L} \to \mathbb{R}_{\geq 0}$ for $1 \leq j \leq d$. Then for any sets of lines $\mathcal{L}_1, \dots, \mathcal{L}_d \subset \mathbb{F}^d$ with multijoints J, and any $S : J \to \mathbb{R}_{\geq 0}$, there is a function $s : J \times (\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d) \to \mathbb{R}_{\geq 0}$ such that

$$\delta(p, \boldsymbol{l}) S(p)^d \leq \prod_{j=1}^{d} s(p, l_j)$$

for all $p \in J$ and $\boldsymbol{l} \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d$, and

$$\sum_{p \in l \cap J} s(p, l) \lesssim \|S\|_d$$

for all $l \in \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$.

Remark. Zhang proved that the multijoint problem is equivalent to the joints problem, [Zha20]. Therefore, the hypotheses of Theorem 5.0.1 and Corollary 5.0.1a are logically equivalent. Similarly, any theorem that is dual to the multijoint problem is also dual to the joints problem, and *vice versa.* In particular, Theorem 5.0.1 establishes duality for both the joints problem and the multijoint problem.

◀

5.1 Reduction to a Geometric Problem

In this section, we reduce the proof of Theorem 5.0.1 to a geometric result on the properties of the kernel δ. This reductive discussion invokes methods from convex optimisation.

Let $\mathbb{F}$ be a field, let $\mathcal{L} \subset \mathbb{F}^d$ be a set of lines with associated joints J and let $\mathbb{S}^{d-1}$ be the sphere in $\mathbb{F}^d$. For $\boldsymbol{l} \in \mathcal{L}^d$ recall that the joints kernel δ is given by $\delta(p, \boldsymbol{l}) = 1$ if the tuple $\boldsymbol{l}$ forms a joint at p, and 0 otherwise. Moreover, δ can be written as

$$\delta(p, \boldsymbol{l}) = \left(\prod_{l \in \boldsymbol{l}} \chi_l(p) \right) \wedge_{l \in \boldsymbol{l}} e(l),$$

where we use $\wedge$ to denote the discrete wedge product and, given any line l, $e(l) \in \mathbb{S}^{d-1}$ is its direction. For $p \in \mathbb{F}^d$, let $\mathcal{L}(p)$ be the set of all lines $l \in \mathcal{L}$ such that $p \in l$. Without loss of generality, we may assume that $0 \in J$ and our proof of Theorem 5.0.1 will reduce to the following.

Theorem 5.1.1 (Discrete Wedge Product Theorem)**.** *For each* $f : \mathcal{L}(0) \to \mathbb{R}_{\geq 0}$ *there is a function* $s : \mathcal{L}(0) \to \mathbb{R}_{\geq 0}$ *such that*

$$\wedge_{l \in \boldsymbol{l}} e(l) \leq \prod_{l \in \boldsymbol{l}} s(l)$$

for all $\boldsymbol{l} \in \mathcal{L}(0)^d$*, and*

$$\sum_{l \in \mathcal{L}(0)} s(l) f(l) \lesssim \left(\sum_{\boldsymbol{l} \in \mathcal{L}(0)^d} \left(\wedge_{l \in \boldsymbol{l}} e(l) \right) \prod_{l \in \boldsymbol{l}} f(l) \right)^{1/d}.$$

In summary, Theorem 5.1.1 states that the discrete wedge product can be well-approximated by a map $\boldsymbol{l} \mapsto \prod_{l \in \boldsymbol{l}} s(l)$. Recall that the discrete wedge product is a discrete analogue of the Euclidean wedge product. Therefore, we may interpret the discrete wedge product as a measure of angle or volume and so we may understand Theorem 5.1.1 as a geometric result which describes a property of the pair $(\mathbb{F}^d, \wedge)$.

Remark. We expect that appropriate modifications of the arguments in this chapter will be sufficient to prove a generalisation of Theorem 5.1.1 for sets of k_j-planes in $\mathbb{F}^n$, where $k_1 + \ldots + k_d = n$. ◀

Our main purpose is to derive Theorem 5.0.1. However, in contrast to the purely abstract formulation of multilinear duality principles (as in [CHV20a]), which applies to situations where the geometry is essentially transversal – for joints, this would mean that $\delta(p, \boldsymbol{l}) = 1$ for every $\boldsymbol{l} \in \mathcal{L}(p)^d$ – we will require use of the explicit form of the joint kernel δ, or more specifically, the discrete wedge product, in our analysis.

Fix $S : J \to \mathbb{R}_{\geq 0}$. Let

$$\mathcal{C} = \left\{ s : J \times \mathcal{L} \to \mathbb{R}_{\geq 0} \, : \, \delta(p, \boldsymbol{l}) S(p)^d \leq \prod_{l \in \boldsymbol{l}} s(p, l), \forall p \in J, \forall \boldsymbol{l} \in \mathcal{L}^d \right\}$$

denote our set of candidate functions s. We seek an $s \in \mathcal{C}$ such that for all $l \in \mathcal{L}$,

$$\sum_{p \in l \cap J} s(p,l) \lesssim \|S\|_d. \tag{5.2}$$

Remark. In contrast to the displayed inequalities in Theorem 5.0.1, we work with a dimensional constant in inequality (5.2) (which bounds a sum from above), and constant 1 in the definition of $\mathcal{C}$ (which bounds a product from below). By scaling the function s, these conditions are equivalent. However, for our analysis, it is convenient to define $\mathcal{C}$ in this way. ◀

Inequality (5.2) is equivalent to

$$\sum_{l \in \mathcal{L}} f(l) \sum_{p \in l \cap J} s(p,l) \lesssim \|S\|_d$$

for all non-negative f defined on $\mathcal{L}$, such that $\sum_{l \in \mathcal{L}} f(l) \leq 1$. Moreover, this equivalence is preserved if we restrict our attention to strictly positive f.

Let

$$\mathcal{F} = \left\{ f : \mathcal{L} \to \mathbb{R}_{>0} \,:\, \sum_{l \in \mathcal{L}} f(l) \leq 1 \right\}.$$

Therefore, it suffices to prove that

$$\min_{s \in \mathcal{C}} \sup_{f \in \mathcal{F}} \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l) s(p,l) \lesssim \|S\|_d.$$

Note that the mappings

$$f \mapsto \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l) s(p,l)$$

for $s \in \mathcal{C}$ fixed and

$$s \mapsto \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l) s(p,l)$$

for $f \in \mathcal{F}$ fixed are linear. In particular they are concave, convex and continuous with respect to the standard topology on finite dimensional vector spaces.

Recall the lopsided minimax theorem, below.

Theorem 5.1.2 (Lopsided Minimax, [AE84])**.** *Let A and B be convex subsets of vector spaces, where B additionally has a topology. Suppose that*

- $\exists a_0 \in A$ *such that* $b \mapsto \Psi(a_0, b)$ *is* inf*-compact,*
- $\forall a \in A$ *the mapping* $b \mapsto \Psi(a,b)$ *is lower semi-continuous,*

for a map $\Psi : A \times B \to \mathbb{R}$, which is convex in b for all a, and concave in a for all b. Then $\exists b_0 \in B$ so that $\sup_{a \in A} \Psi(a, b_0)$ attains the common value

$$\sup_{a \in A} \inf_{b \in B} \Psi(a,b) = \min_{b \in B} \sup_{a \in A} \Psi(a,b).$$

The set $\mathcal{F}$ is convex, and likewise, $\mathcal{C}$ is convex by the AM–GM inequality, $a^{1-\theta}b^{\theta} \leq (1-\theta)a + \theta b$. Moreover, we endow $\mathcal{C}$ with the standard finite dimensional vector space topology. Hence, to apply Theorem 5.1.2 to the map $\Psi : \mathcal{F} \times \mathcal{C} \to \mathbb{R}$, with

$$\Psi(f, s) = \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l)s(p, l),$$

it remains to verify that there exists $f_0 : \mathcal{L} \to \mathbb{R}_{\geq 0}$ so that $\Psi(f_0, \cdot) : \mathcal{C} \to \mathbb{R}$ is inf-compact. To do so, let $f_0 : \mathcal{L} \to \mathbb{R}_{\geq 0}$ be the constant function with $\sum_l f_0(l) = 1$ and let $\lambda \in \mathbb{R}$. Consider the sublevel set

$$\mathcal{C}(\lambda) := \left\{ s \in \mathcal{C} : \Psi(f_0, s) \leq \lambda \right\}.$$

If $s \in \mathcal{C}(\lambda)$ then $0 \leq s(p, l) \leq \lambda$ for all $p \in J$ and $l \in \mathcal{L}$. Hence, $\mathcal{C}(\lambda) \subseteq [0, \lambda]^{|J| \times |\mathcal{L}|}$ and therefore $\mathcal{C}(\lambda)$ is a subset of a compact set. Since Ψ is continuous, and $\mathcal{C}(\lambda)$ is the $\Psi(f_0, \cdot)$-pre-image of the closed set $(-\infty, \lambda]$, we deduce that $\mathcal{C}(\lambda)$ is a closed subset of a compact set in a finite-dimensional $\mathbb{R}$-vector space. Hence, $\mathcal{C}(\lambda)$ is compact, and the hypotheses of Theorem 5.1.2 are verified.

Hence, it follows that

$$\min_{s \in \mathcal{C}} \sup_{f \in \mathcal{F}} \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l)s(p, l) = \sup_{f \in \mathcal{F}} \inf_{s \in \mathcal{C}} \sum_{l \in \mathcal{L}} \sum_{p \in l \cap J} f(l)s(p, l).$$

Matters have thus reduced to proving that

$$\inf_{s \in \mathcal{C}} \sum_{p \in J} \sum_{l \in \mathcal{L}(p)} f(l)s(p, l) \lesssim \|S\|_d, \tag{5.3}$$

for any $f \in \mathcal{F}$.

Fix $f \in \mathcal{F}$. The constraints that define $\mathcal{C}$ for each $p \in J$ are independent of each other. Therefore, with

$$\mathcal{C}_p = \left\{ s(p, \cdot) : \mathcal{L}(p) \to \mathbb{R}_{\geq 0} \ : \ \delta(p, \boldsymbol{l}) \leq \prod_{l \in \boldsymbol{l}} s(p, \boldsymbol{l}) \text{ for all } \boldsymbol{l} \in \mathcal{L}(p)^d \right\},$$

we have

$$\inf_{s \in \mathcal{C}} \sum_{p \in J} \sum_{l \in \mathcal{L}_p} f(l)s(p, l) = \sum_{p \in J} S(p) \inf_{s \in \mathcal{C}_p} \sum_{l \in \mathcal{L}(p)} f(l)s(p, l).$$

To complete our reduction, suppose that Theorem 5.1.1 holds, and apply it with p in the role of 0 so that

$$\inf_{s \in \mathcal{C}} \sum_{p \in J} S(p) \sum_{l \in \mathcal{L}(p)} f(l)s(p, l) \lesssim \sum_{p \in J} S(p) \left(\sum_{\boldsymbol{l} \in \mathcal{L}^d} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^{1/d}.$$

By Hölder's inequality, this is at most

$$\|S\|_d \left(\sum_p \left(\sum_{\boldsymbol{l} \in \mathcal{L}^d} \delta(p, \boldsymbol{l}) \prod_{l \in \boldsymbol{l}} f(l) \right)^{\frac{1}{d-1}} \right)^{\frac{d-1}{d}}.$$

Finally, we apply (5.1) to bound the above display by

$$\|S\|_d \sum_{l \in \mathcal{L}} f(l) \leq \|S\|_d,$$

where we have used the fact that f satisfies $\sum_{l \in \mathcal{L}} f(l) \leq 1$. Thus (5.3) is established, subject to Theorem 5.1.1.

5.2 The Discrete Wedge Product ($d = 3$)

The proof of Theorem 5.1.1 is notationally dense. Thus, we begin by detailing the proof when $d = 3$ so that calculations are more explicit.

Let $\mathcal{L} \subset \mathbb{F}^3$ be a set of lines with joints J. Suppose that $0 \in J$, assume that $\mathcal{L} = \mathcal{L}(0)$ and fix $f : \mathcal{L}(0) \to \mathbb{R}_{\geq 0}$. Let $s : \mathcal{L} \to \mathbb{R}_{\geq 0}$. We say that s is **admissible** if

$$\wedge_{l \in \boldsymbol{l}} e(l) \leq \prod_{l \in \boldsymbol{l}} s(l)$$

for all $\boldsymbol{l} \in \mathcal{L}^3$. Then we will show that there is an admissible s so that

$$\left(\sum_{l \in \mathcal{L}} s(l) f(l) \right)^3 \lesssim \sum_{\boldsymbol{l} \in \mathcal{L}^3} \left(\wedge_{l \in \boldsymbol{l}} e(l) \right) \prod_{l \in \boldsymbol{l}} f(l). \tag{5.4}$$

We proceed in a case-by-case analysis and implicitly assume that every line in this section, which appears in a summation index, is an element of $\mathcal{L} = \mathcal{L}(0)$. The cases we consider, indexed by subsection and with notions of heaviness to be defined, are as follows:

5.2.1 There is a heavy line.

- **5.2.1(a)** There is a heavy plane.
- **5.2.1(b)** Every plane is not heavy.

5.2.2 Every line is not heavy.

- **5.2.2(a)** There is a heavy plane.
- **5.2.2(b)** Every plane is not heavy.

In each case, we explicitly construct an admissible s such that the conclusion holds.

5.2.1 There is a heavy line.

Suppose there is a line $l_1 \in \mathcal{L}$ so that

$$f(l_1) > \sum_{l \neq l_1} f(l).$$

The line l_1 is said to be **heavy**.

5.2.1(a) There is a heavy plane.

Suppose there is a plane π such that $l_1 \subset \pi$ and

$$\sum_{l \subset \pi, l \neq l_1} f(l) > 4 \sum_{l \not\subset \pi} f(l).$$

Such π is said to be **heavy**. Let $F_1 = f(l_1)$, $F_2 = \sum_{l \subset \pi, l \neq l_1} f(l)$ and let $F_3 = \sum_{l \not\subset \pi} f(l)$. Since f is positive, each $F_j > 0$. We define s by

$$s(l) = \begin{cases} \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_1} & l = l_1 \\ \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_2} & l \subset \pi, l \neq l_1 \\ \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_3} & l \not\subset \pi. \end{cases}$$

By the definition of s,

$$\begin{aligned} \sum_{l \in \mathcal{L}} s(l) f(l) &= \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_1} f(l_1) + \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_2} \sum_{l \subset \pi, l \neq l_1} f(l) + \frac{\prod_{j=1}^3 F_j^{\frac{1}{3}}}{F_3} \sum_{l \not\subset \pi} f(l) \\ &\lesssim \prod_{j=1}^3 F_j^{\frac{1}{3}} \\ &= f(l_1)^{\frac{1}{3}} \left(\sum_{l \subset \pi, l \neq l_1} f(l) \right)^{\frac{1}{3}} \left(\sum_{l \not\subset \pi} f(l) \right)^{\frac{1}{3}}. \end{aligned}$$

The tuples $\boldsymbol{l}$ featuring in this expression form a subset of those such that $\wedge_{l \in \boldsymbol{l}} e(l) = 1$ and hence (5.4) is satisfied, as desired. It remains to verify that s is admissible.

Since l_1 is heavy, $F_1 > F_2$ and since π is heavy, $F_2 > 4F_3$. Let $\boldsymbol{l} \in \mathcal{L}^3$ satisfy $\wedge_{l \in \boldsymbol{l}} e(l) = 1$. Let $m_1 = \left|\{l \in \boldsymbol{l} : l = l_1\}\right|$, $m_2 = \left|\{l \in \boldsymbol{l} : l \subset \pi \text{ and } l \neq l_1\}\right|$ and $m_3 = \left|\{l \in \boldsymbol{l} : l \not\subset \pi\}\right|$. Then

$$\prod_{l \in \boldsymbol{l}} s(l) = \frac{\prod_{j=1}^3 F_j}{F_1^{m_1} F_2^{m_2} F_3^{m_3}}.$$

Since $\wedge_{l\in\boldsymbol{l}} e(l) = 1$, each $m_j \leq j$. If each $m_j = 1$, then we are done. Otherwise, $(m_1, m_2, m_3) \in \{(0,2,1),(0,1,2),(0,0,3)\}$. If $(m_1, m_2, m_3) = (0,2,1)$ then

$$\prod_{l\in\boldsymbol{l}} s(l) = \frac{F_1}{F_2} > 1.$$

If $(m_1, m_2, m_3) = (0,1,2)$ then

$$\prod_{l\in\boldsymbol{l}} s(l) = \frac{F_1}{F_3} > \frac{F_2}{F_3} > 4.$$

Finally, if $(m_1, m_2, m_3) = (0,0,3)$ then

$$\prod_{l\in\boldsymbol{l}} s(l) = \frac{F_1}{F_3}\frac{F_2}{F_3} > \frac{F_2^2}{F_3^2} > 4^2.$$

Hence s is admissible.

5.2.1(b) Every plane is not heavy.

Suppose that every plane π, such that $l_1 \subset \pi$, satisfies

$$\sum_{l\subset\pi, l\neq l_1} f(l) \leq 4 \sum_{l\not\subset\pi} f(l).$$

Let

$$\rho = \left(\frac{\sum_{l\neq l_1} f(l)}{f(l_1)} \right)^{\frac{2}{3}}.$$

Then $\rho < 1$ since l_1 is heavy, and $\rho > 0$ since f is positive. Let

$$s(l) = \begin{cases} \rho & l = l_1 \\ \rho^{-\frac{1}{2}} & l \neq l_1. \end{cases}$$

Then

$$\sum_{l\in\mathcal{L}} s(l)f(l) = \rho f(l_1) + \rho^{-\frac{1}{2}} \sum_{l\neq l_1} f(l) \lesssim f(l_1)^{\frac{1}{3}} \left(\sum_{l\neq l_1} f(l) \right)^{\frac{2}{3}}.$$

It remains to show that s is admissible, and that

$$\left(\sum_{l\neq l_1} f(l) \right)^2 \lesssim \sum_{l_2, l_3 \neq l_1} (\wedge_{j=1}^3 e(l_j)) \prod_{j=2}^3 f(l_j). \tag{5.5}$$

Checking s, if $\boldsymbol{l}$ satisfies $\wedge_{l\in\boldsymbol{l}} e(l) = 1$, then $\left|\{l \in \boldsymbol{l} : l = l_1\}\right| \leq 1$. If $l_1 \in \boldsymbol{l}$, then $\prod_{l\in\boldsymbol{l}} s(l) = \rho\rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}} = 1$. Otherwise, since $\rho < 1$, $\prod_{l\in\boldsymbol{l}} s(l) = \rho^{-\frac{3}{2}} > 1$, as desired.

In this subcase, it remains to prove (5.5). Given lines $l_2 \neq l_1$, let $\pi(l_1, l_2)$ be the unique plane that contains l_1 and l_2. Then

$$\left(\sum_{l\neq l_1} f(l)\right)^2 = \sum_{l_2,l_3} (\wedge_{j=1}^3 e(l_j)) f(l_2) f(l_3) + \sum_{l_2\neq l_1} \sum_{l_3\subset \pi(l_1,l_2)} f(l_2) f(l_3).$$

However, since every plane containing l_1 is not heavy,

$$\begin{aligned}\sum_{l_2\neq l_1} \sum_{l_3\subset \pi(l_1,l_2)} f(l_2) f(l_3) &\leq 4 \sum_{l_2\neq l_1} \sum_{l_3\not\subset \pi(l_1,l_2)} f(l_2) f(l_3) \\ &= \sum_{l_2,l_3} (\wedge_{j=1}^3 e(l_j)) f(l_2) f(l_3).\end{aligned}$$

Hence (5.5) holds, concluding the case of heavy lines.

5.2.2 Every line is not heavy.

Suppose that there are no heavy lines. That is, for every $l_1 \in \mathcal{L}$,

$$f(l_1) \leq \sum_{l\neq l_1} f(l_1).$$

5.2.2(a) There is a heavy plane.

Suppose there is a plane π so that

$$\sum_{l\subset\pi} f(l) > 4 \sum_{l\not\subset\pi} f(l).$$

Such π is said to be a **heavy** plane. Let

$$\rho = \left(\frac{\sum_{l\not\subset\pi} f(l)}{\sum_{l\subset\pi} f(l)}\right)^{\frac{1}{3}},$$

and define

$$s(l) = \begin{cases} \rho & l \subset \pi \\ \rho^{-2} & l \not\subset \pi. \end{cases}$$

Then

$$\sum_{l\in\mathcal{L}} s(l) f(l) = \rho \sum_{l\subset\pi} f(l) + \rho^{-2} \sum_{l\not\subset\pi} f(l) \lesssim \left(\sum_{l\subset\pi} f(l)\right)^{\frac{2}{3}} \left(\sum_{l\not\subset\pi} f(l)\right)^{\frac{1}{3}}.$$

To check that s is admissible, we note that $\rho^3 < 1/4$ since π is heavy. If $\boldsymbol{l}$ satisfies $\wedge_{l\in\boldsymbol{l}} e(l) = 1$ then $\left|\{l \in \boldsymbol{l} : l \subset \pi\}\right| \leq 2$. If there are no $l \in \boldsymbol{l}$ such that $l \subset \pi$

then $\prod_{l \in \boldsymbol{l}} s(l) = \rho^{-6} > 1$. If there is a unique $l \in \boldsymbol{l}$ such that $l \subset \pi$, then $\prod_{l \in \boldsymbol{l}} = \rho\rho^{-2}\rho^{-2} = \rho^{-3} > 1$. Finally, if there are precisely two lines $l \in \boldsymbol{l}$ so that $l \subset \pi$, then $\prod_{l \in \boldsymbol{l}} s(l) = \rho\rho\rho^{-2} = 1$. To conclude this subcase, it remains to check

$$\left(\sum_{l \subset \pi} f(l)\right)^2 \lesssim \sum_{l_1, l_2 \subset \pi} (e(l_1) \wedge e(l_2)) f(l_1) f(l_2), \tag{5.6}$$

where $e(l) \wedge e(l') = 1$ if the lines l and l' are distinct, and 0 otherwise. To prove (5.6), we first show that,

$$f(l_1) \leq 4 \sum_{l \subset \pi, l \neq l_1} f(l) \tag{5.7}$$

for all $l_1 \subset \pi$. Let $l_1 \subset \pi$. Since l_1 is not heavy

$$f(l_1) \leq \sum_{l \neq l_1} f(l) = \sum_{l \subset \pi, l \neq l_1} f(l) + \sum_{l \not\subset \pi} f(l).$$

Since π is heavy, this is bounded above by

$$\sum_{l \subset \pi, l \neq l_1} f(l) + \frac{1}{4} \sum_{l \subset \pi} f(l) = \sum_{l \subset \pi, l \neq l_1} f(l) + \frac{1}{4}\left(f(l_1) + \sum_{l \subset \pi, l \neq l_1} f(l)\right).$$

Rearranging, we deduce that

$$f(l_1) \leq \frac{5}{3} \sum_{l \subset \pi, l \neq l_1} f(l),$$

thus proving (5.7).

To conclude, we prove (5.6) directly, as follows:

$$\begin{aligned} \left(\sum_{l \subset \pi} f(l)\right)^2 &= \sum_{l_1, l_2 \subset \pi} (e(l_1) \wedge e(l_2)) f(l_1) f(l_2) + \sum_{l \subset \pi} f(l)^2 \\ &\lesssim \sum_{l_1, l_2 \subset \pi} (e(l_1) \wedge e(l_2)) f(l_1) f(l_2) + \sum_{l \subset \pi} f(l) \sum_{l \subset \pi, l \neq l_1} f(l) \\ &\lesssim \sum_{l_1, l_2 \subset \pi} (e(l_1) \wedge e(l_2)) f(l_1) f(l_2), \end{aligned}$$

hence concluding this subcase.

5.2.2(b) Every plane is not heavy.

Suppose that every plane π is such that

$$\sum_{l \subset \pi} f(l) \leq 4 \sum_{l \not\subset \pi} f(l).$$

Then the result holds with the admissible choice of $s(l) = 1$ for all $l \in \mathcal{L}$. Indeed,

$$\left(\sum_{l \in \mathcal{L}} f(l)\right)^3 = \sum_{\boldsymbol{l} \in \mathcal{L}^3} (\wedge_{l \in \boldsymbol{l}} e(l)) \prod_{l \in \boldsymbol{l}} f(l) + \sum_{t=1}^{2} \sum_{\substack{\boldsymbol{l} \in \mathcal{L}^3: \\ \dim \operatorname{span}\{e(l) : l \in \boldsymbol{l}\} = t}} \prod_{l \in \boldsymbol{l}} f(l).$$

The result follows from the inequality,

$$\sum_{\substack{\boldsymbol{l} \in \mathcal{L}^3: \\ \dim \operatorname{span}\{e(l) : l \in \boldsymbol{l}\} = t}} \prod_{l \in \boldsymbol{l}} f(l) \lesssim \sum_{\substack{\boldsymbol{l} \in \mathcal{L}^3: \\ \dim \operatorname{span}\{e(l) : l \in \boldsymbol{l}\} = t+1}} \prod_{l \in \boldsymbol{l}} f(l),$$

for $t = 1, 2$. This is a special case of Lemma 5.3.2. Proving this concrete case is not more informative than the general case, and so we will not prove this special case separately.

This concludes our analysis of the problem for $d = 3$, subject to a proof of Lemma 5.3.2, which we reserve until the end of this chapter.

Remark. At the heart of this $d = 3$ argument is a sequence of nested subspaces. If there is a heavy line and a heavy plane, then the sequence is given by $\{0\} \subset l_1 \subset \pi \subset \mathbb{F}^3$. If there is a heavy line, but no heavy planes, then the sequence is $\{0\} \subset l_1 \subset \mathbb{F}^3$. These sequences are used to identify a subset $\{\boldsymbol{l} \in \mathcal{L}^3 : \wedge_{l \in \boldsymbol{l}} e(l) = 1\}$, on which $\prod_{l \in \boldsymbol{l}} f(l)$ is large, which we can exploit. This structure is, perhaps, more illuminating when $d = 4$ as demonstrated, below. In general, there are 2^{d-1} subcases to consider. We will choose admissible weights s to partition $\mathcal{L}$ into "layers", each of which are easy to handle, as a result of the forthcoming Lemma 5.3.2. ◀

5.2.3 Brief Overview when $d = 4$

Before moving to the general case, we give a schematic overview of the case-structure for the argument when $d = 4$. On the one hand, we see the nested subspace-structure emerge in a further concrete setting. On the other, it is informative to see how the structure manifests within the proof.

The nested lists below detail the subcase-structure and the expression which dominates $(\sum_l s(l)f(l))^4$ for to each subcase, when $d = 4$.

- There is a heavy line l_1.
    - There is a heavy 2-plane π containing l_1.
        - There is a heavy 3-plane H containing π.
            - $\displaystyle f(l_1)\left(\sum_{l_2\subset\pi, l_2\neq l_1} f(l_2)\right)\left(\sum_{l_3\subset H, l_3\not\subset\pi} f(l_3)\right)\left(\sum_{l_4\not\subset H} f(l_4)\right)$
        - Every 3-plane containing π is not heavy.
            - $\displaystyle f(l_1)\left(\sum_{l_2\subset\pi, l_2\neq l_1} f(l_2)\right)\left(\sum_{l_3, l_4\not\subset\pi} \big(e(l_3)\wedge e(l_4)\big) f(l_3)f(l_4)\right)$
    - Every 2-plane containing l_1 is not heavy.
        - There is a heavy 3-plane H containing l_1.
            - $\displaystyle f(l_1)\left(\sum_{l_2, l_3\subset H, l_2, l_3\neq l_1} \big(\wedge_{j=1}^{3} e(l_j)\big) f(l_2)f(l_3)\right)\left(\sum_{l_4\not\subset H} f(l_4)\right)$
        - Every 3-plane containing l_1 is not heavy.
            - $\displaystyle f(l_1)\left(\sum_{l_2, l_3, l_4\neq l_1} (\wedge_{j=1}^{4} e(l_j)) f(l_2)f(l_3)f(l_4)\right)$
- Every line is not heavy.
    - There is a heavy 2-plane π.
        - There is a heavy 3-plane H containing π.
            - $\displaystyle \left(\sum_{l_1, l_2\subset\pi} \big(e(l_1)\wedge e(l_2)\big) f(l_1)f(l_2)\right)\left(\sum_{l_3\subset H, l_3\not\subset\pi} f(l_3)\right)\left(\sum_{l_4\not\subset H} f(l_4)\right)$
        - Every 3-plane containing π is not heavy.
            - $\displaystyle \left(\sum_{l_1, l_2\subset\pi} \big(e(l_1)\wedge e(l_2)\big) f(l_1)f(l_2)\right)\left(\sum_{l_3, l_4\not\subset\pi} \big(e(l_3)\wedge e(l_4)\big) f(l_3)f(l_4)\right)$
    - Every 2-plane is not heavy.
        - There is a heavy 3-plane H.
            - $\displaystyle \left(\sum_{l_1, l_2, l_3\subset H} (\wedge_{j=1}^{3} e(l_j)) f(l_1)f(l_2)f(l_3)\right)\left(\sum_{l_4\not\subset H} f(l_4)\right)$
        - Every 3-plane is not heavy.
            - $\displaystyle \sum_{\boldsymbol{l}} (\wedge_{j=1}^{4} e(l_j)) \prod_{l\in\boldsymbol{l}} f(l)$

5.3 The Discrete Wedge Product ($d \geq 3$)

Let $\mathcal{L} \subset \mathbb{F}^d$ be a set of lines with joints J. Suppose that $0 \in J$ and assume that every $l \in \mathcal{L}$ is such that $0 \in l$. Let $f : \mathcal{L} \to \mathbb{R}_{>0}$. We say that a function $s : \mathcal{L} \to \mathbb{R}_{\geq 0}$ is **admissible** if

$$\wedge_{l \in \boldsymbol{l}} e(l) \leq \prod_{l \in \boldsymbol{l}} s(l)$$

for all $\boldsymbol{l} \in \mathcal{L}^d$. We will show that there is an admissible s so that

$$\left(\sum_{l \in \mathcal{L}} s(l) f(l) \right)^d \lesssim \sum_{\boldsymbol{l} \in \mathcal{L}^d} (\wedge_{l \in \boldsymbol{l}} e(l)) \prod_{l \in \boldsymbol{l}} f(l). \tag{5.8}$$

5.3.1 Constructing a Flag with Good Properties

Let $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_{d-1}$ be defined by $\alpha_k = 2^{k-1}$. Choose k_1 to be the least integer, such that $1 \leq k_1 \leq d-1$, for which there is an α_{k_1}**-heavy** k_1**-plane** π_1; that is, for which

$$\sum_{l \subset \pi_1} f(l) > \alpha_{k_1} \sum_{l \not\subset \pi_1} f(l).$$

There may be no such π_1. However, if one exists, then we continue by choosing the smallest k_2, with $k_1 < k_2 \leq d-1$, for which there is an α_{k_2}**-heavy** k_2**-plane** π_2 **which contains** π_1; that is, for which

$$\sum_{l \subset \pi_2, l \not\subset \pi_1} f(l) > \alpha_{k_2} \sum_{l \not\subset \pi_2} f(l).$$

Again, there may be no such π_2. However, if one exists, then we continue by choosing the smallest k_3, with $k_2 < k_3 \leq d-1$, for which there is an α_{k_3}**-heavy** k_3**-plane** π_3 **which contains** π_2. That is, for which

$$\sum_{l \subset \pi_3, l \not\subset \pi_2} f(l) > \alpha_{k_3} \sum_{l \not\subset \pi_3} f(l).$$

We continue this process until we are forced to stop and one of the following occurs:

1. There are no α_k-heavy planes of any dimension $k < d$;
2. We have arrived at some π_N of dimension $k_N = d-1$; or,
3. We have a π_N of dimension $k_N < d-1$, and there are no α_k-heavy k-planes for any $k_N < k \leq d-1$.

The result of this process is a (possibly empty) flag $(\pi_n)_{1 \leq n \leq N}$ of subspaces. We will deduce a useful property, Lemma 5.3.1, which improves the above inequalities that we have introduced to define the flag.

The case in which the sequence of subspaces is empty, will be accounted for later. Therefore, we assume that we have $\pi_1 \subset \pi_2 \subset \cdots \subset \pi_N$ (with $N \geq 1$), a non-empty maximal[1] increasing sequence of subspaces of dimensions $1 \leq k_1 < k_2 < \cdots < k_N \leq d-1$, respectively, such that for each $1 \leq n \leq N$,[2]

$$\sum_{l \subset \pi_n, l \not\subset \pi_{n-1}} f(l) > \alpha_{k_n} \sum_{l \not\subset \pi_n} f(l), \tag{5.9}$$

and such that for all subspaces π, which contain π_{n-1} and which satisfy $k_{n-1} < \dim \pi < k_n$,

$$\sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) \leq \alpha_{\dim \pi} \sum_{l \not\subset \pi} f(l),$$

and thus

$$\sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) \leq \alpha_{k_n - 1} \sum_{l \not\subset \pi} f(l). \tag{5.10}$$

We can qualitatively improve the right-hand side of inequality (5.10) to include the additional constraint that $l \subset \pi_n$ at the expense of a multiplicative constant:

Lemma 5.3.1. *For every $1 \leq n \leq N$, and every subspace π such that $k_{n-1} < \dim \pi < k_n$ and $\pi_{n-1} \subset \pi \subset \pi_n$,*

$$\sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) \leq 4\alpha_{k_n - 1} \sum_{l \subset \pi_n, l \not\subset \pi} f(l). \tag{5.11}$$

Proof. For each $1 \leq n \leq N$ and for each subspace π such that $\pi_{n-1} \subsetneq \pi \subsetneq \pi_n$ and $k_{n-1} < \dim \pi < k_n$, we have, by (5.10) and (5.9),

$$\begin{aligned}
\sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) &\leq \alpha_{k_n-1} \sum_{l \not\subset \pi} f(l) \\
&= \alpha_{k_n-1} \sum_{l \not\subset \pi, l \subset \pi_n} f(l) + \alpha_{k_n-1} \sum_{l \not\subset \pi_n} f(l) \\
&< \alpha_{k_n-1} \sum_{l \not\subset \pi, l \subset \pi_n} f(l) + \frac{\alpha_{k_n-1}}{\alpha_{k_n}} \sum_{l \subset \pi_n, l \not\subset \pi_{n-1}} f(l) \\
&= \alpha_{k_n-1} \sum_{l \not\subset \pi, l \subset \pi_n} f(l) + \frac{\alpha_{k_n-1}}{\alpha_{k_n}} \left(\sum_{l \subset \pi_n, l \not\subset \pi} f(l) + \sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) \right).
\end{aligned}$$

Rearranging the resulting inequality gives

$$\left(1 - \frac{\alpha_{k_n-1}}{\alpha_{k_n}}\right) \sum_{l \subset \pi, l \not\subset \pi_{n-1}} f(l) \leq \alpha_{k_n-1} \left(1 + \frac{1}{\alpha_{k_n}}\right) \sum_{l \subset \pi_n, l \not\subset \pi} f(l),$$

---

[1] Maximal in the sense that there is no flag, which contains this one, and has length $> N$.

[2] We interpret the condition $l \not\subset \pi_0$ to be the void condition.

or equivalently,

$$\sum_{l\subset\pi, l\not\subset\pi_{n-1}} f(l) \leq \alpha_{k_n-1}\left(\frac{\alpha_{k_n}+1}{\alpha_{k_n}-\alpha_{k_n-1}}\right)\sum_{l\subset\pi_n, l\not\subset\pi} f(l).$$

With $\alpha_k = 2^{k-1}$ we have

$$1 < \frac{\alpha_{k_n}+1}{\alpha_{k_n}-\alpha_{k_n-1}} \leq 4,$$

as required. □

5.3.2 Arranging the f-Mass According to the Flag

For $1 \leq n \leq N+1$, let

$$F_n = \sum_{l\subset\pi_n, l\not\subset\pi_{n-1}} f(l),$$

where we interpret the conditions $l \not\subset \pi_0$ and $l \subset \pi_{N+1}$ as being void. Therefore, (5.9) reads as

$$F_n > \alpha_{k_n} F_{n+1}. \tag{5.12}$$

Moreover, since $f > 0$, each $F_n > 0$.

Suppose we have $1 < r \leq d$ directions $\omega_1, \ldots, \omega_r \in \mathbb{S}^{d-1}$. We define the discrete wedge product on r-tuples of directions by $\omega_1 \wedge \cdots \wedge \omega_r = 1$ if the r vectors are linearly independent, and 0 otherwise.

The following lemma, Lemma 5.3.2, handles the subspaces between adjacent heavy subspaces.

Lemma 5.3.2. *There are constants* $\beta_n = \beta_n(d)$ *such that for* $1 \leq n \leq N+1$*,*

$$F_n^{k_n-k_{n-1}} \leq \beta_n \sum_{\substack{l_{k_{n-1}+1},\ldots,l_{k_n}\subset\pi_n,\\ l_{k_{n-1}+1},\ldots,l_{k_n}\not\subset\pi_{n-1}}} (\wedge_{j=k_{n-1}+1}^{k_n} e(l_j)) \prod_{j=k_{n-1}+1}^{k_n} f(l_j),$$

where we define $k_0 = 0$ *and* $k_{N+1} = d$*.*

In particular, Lemma 5.3.2 applies in the exceptional case that the sequence of subspaces is empty, in which case its conclusion reads as follows:

$$\left(\sum_l f(l)\right)^d \lesssim \sum_{\boldsymbol{l}\in\mathcal{L}^d} (\wedge_{l\in\boldsymbol{l}} e(l)) \prod_{l\in\boldsymbol{l}} f(l).$$

That is, the choice $s(l) = 1$ for all $l \in \mathcal{L}$ is an admissible choice of weight s that satisfies (5.8), verifying the desired condition in the exceptional case. Hence, we assume that the flag $(\pi_n)_{1\leq n\leq N}$ is non-empty until the proof of Lemma 5.3.2.

We momentarily postpone the proof of Lemma 5.3.2 to conclude the overarching structure in the proof of Theorem 5.1.1, and hence Theorem 5.0.1.

For parameters $\rho_1, \ldots, \rho_{N+1}$ to be defined, we define s by

$$s(l) = \begin{cases} \rho_1 & l \subset \pi_1, \\ \rho_2 & l \subset \pi_2, l \not\subset \pi_1, \\ \vdots & \\ \rho_n & l \subset \pi_n, l \not\subset \pi_{n-1}, \\ \vdots & \\ \rho_N & l \subset \pi_N, l \not\subset \pi_{N-1}, \\ \rho_{N+1} & l \not\subset \pi_N. \end{cases}$$

Using the arrangement of the mass of f into layers, we therefore have

$$\sum_l s(l) f(l) = \rho_1 F_1 + \cdots + \rho_n F_n + \cdots + \rho_{N+1} F_{N+1}.$$

A direct consequence of Lemma 5.3.2 is that

$$F_1^{k_1-k_0} \cdots F_n^{k_n-k_{n-1}} \cdots F_{N+1}^{d-k_N} \leq \left(\prod_{n=1}^{N+1} \beta_n \right) \sum_{\boldsymbol{l} \in \mathcal{L}^d} (\wedge_{l \in \boldsymbol{l}} e(l)) \prod_{l \in \boldsymbol{l}} f(l). \tag{5.13}$$

In light of inequality (5.13), we consider

$$\rho_n = F_n^{-1} \prod_{n'=1}^{N+1} F_{n'}^{\frac{k_{n'}-k_{n'-1}}{d}} \tag{5.14}$$

where $k_0 = 0$ and $k_{N+1} = d$, which verifies (5.8), as desired.

5.3.3 Concluding Detail

Our proof of Theorem 5.1.1 is now complete, up to a proof of Lemma 5.3.2 and showing that s is admissible. We first prove that s is admissible. Recall that s is admissible if it satisfies

$$1 \leq \prod_{l \in \boldsymbol{l}} s(l)$$

for all $\boldsymbol{l} \in \mathcal{L}^d$ such that $\wedge_{l \in \boldsymbol{l}} e(l) = 1$.

Let $\boldsymbol{l} \in \mathcal{L}^d$ satisfy $\wedge_{l \in \boldsymbol{l}} e(l) = 1$. For each $1 \leq n \leq N$, let

$$m_n = \left| \{ l \in \boldsymbol{l} : l \subset \pi_n \text{ and } l \not\subset \pi_{n-1} \} \right|.$$

Since $\wedge_{l \in \boldsymbol{l}} e(l) = 1$, we have that $m_n \leq k_n$ for each $1 \leq n \leq N$. Indeed, suppose that $m_n > k_n$ for some n. Then the number of lines contained in π_n exceeds $\dim \pi_n$. Hence such m_n lines are linearly dependent, and consequently the whole tuple $\boldsymbol{l}$ must be linearly dependent. Admissibility of the function s, therefore, follows from Lemma 5.3.3, below.

Lemma 5.3.3 (Admissibility)**.** *Suppose that* $m_1 + \cdots + m_{N+1} = d$ *and* $m_n \leq k_n$ *for all* $1 \leq n \leq N+1$. *Then*

$$\rho_1^{m_1} \cdots \rho_n^{m_n} \cdots \rho_{N+1}^{m_{N+1}} \geq 1.$$

Proof. Rearranging the exponents on each ρ_n gives

$$\begin{aligned} & \rho_1^{m_1} \cdots \rho_n^{m_n} \cdots \rho_{N+1}^{m_{N+1}} \\ = \;& \left(\rho_1^{k_1} \cdots \rho_n^{k_n - k_{n-1}} \cdots \rho_{N+1}^{d-k_N}\right) \left(\rho_1^{m_1-k_1} \cdots \rho_n^{m_n-(k_n-k_{n-1})} \cdots \rho_{N+1}^{m_{N+1}-(d-k_N)}\right) \\ = \;& \rho_1^{m_1-k_1} \cdots \rho_n^{m_n-(k_n-k_{n-1})} \cdots \rho_{N+1}^{m_{N+1}-(d-k_N)}. \end{aligned}$$

By equation (5.14), the definition of each ρ_n, this is equal to

$$\begin{aligned} = \;& F_1^{-(m_1-k_1)} \cdots F_n^{-(m_n-(k_n-k_{n-1}))} \cdots F_{N+1}^{-(m_{N+1}-(d-k_N))} \\ & \times \left(\prod_{n=1}^{N+1} F_n^{\frac{k_n-k_{n-1}}{d}}\right)^{m_1-k_1+\ldots+m_n-(k_n-k_{n-1})+\ldots+m_{N+1}-(d-k_N)} \\ = \;& F_1^{-(m_1-k_1)} \cdots F_n^{-(m_n-(k_n-k_{n-1}))} \cdots F_{N+1}^{-(m_{N+1}-(d-k_N))} \\ & \times \left(\prod_{n=1}^{N+1} F_n^{\frac{k_n-k_{n-1}}{d}}\right)^{m_1+\ldots+m_{N+1}-d} \\ = \;& F_1^{k_1-m_1} \cdots F_n^{k_n-k_{n-1}-m_n} \cdots F_{N+1}^{d-k_N-m_{N+1}}. \end{aligned}$$

We claim that $\rho_1^{m_1} \cdots \rho_{N+1}^{m_{N+1}}$ is minimised when $(m_n)_{n=N+1}^1 = (k_n - k_{n-1})_{n=N+1}^1$, whereby

$$\rho_1^{m_1} \cdots \rho_n^{m_n} \cdots \rho_{N+1}^{m_N} = F_1^{k_1-(k_1-0)} \cdots F_{N+1}^{d-k_N-(k_{N+1}-k_N)} = 1.$$

Indeed, if $(m_n)_{n=N+1}^1 \neq (k_n - k_{n-1})_{n=N+1}^1$ then there exists n such that $m_n > (k_n - k_{n-1})$. Of all such $n \leq N+1$, let us choose the maximal one and call it n_0. Since $m_1 + \ldots + m_{N+1} = d$, there exists $n \neq n_0$ so that $m_n < (k_n - k_{n-1})$. Suppose that $n > n_0$ (this is void in the case where $n_0 = N+1$). By the maximal choice of n_0, we have that $m_{n'} \leq (k_{n'} - k_{n'-1})$ for all $n' > n_0$, and since $n > n_0$, one of these inequalities is strict. Hence,

$$\left|\{l \in \boldsymbol{l} : l \not\subset \pi_{n_0}\}\right| = \sum_{n'>n_0} m_n < \sum_{n>n_0} (k_n - k_{n-1}) = d - k_{n_0}.$$

Hence, $\left|\{l \in \boldsymbol{l} : l \subset \pi_{n_0}\}\right| > k_{n_0}$, and so $\wedge_{l \in \boldsymbol{l}} e(l) = 0$, contrary to the choice of $\boldsymbol{l}$.

Therefore n is such that, $n < n_0$ and hence every $n' > n_0$ satisfies $m_{n'} \geq (k_{n'} - k_{n'-1})$. However, it follows from the maximality property of n_0 that these are all satisfied with equality. Hence, $(m_n)_{n=N+1}^1$ is of a greater lexicographical order than $(k_n - k_{n-1})_n$. Considering (5.12), $\rho_1^{m_1} \cdots \rho_{N+1}^{m_{N+1}}$ is reduced if the tuple $(m_{N+1},\ldots,m_1)$ is replaced by another tuple $(m'_{N+1},\ldots m'_1)$ of a lower lexicographical order, and satisfies $m'_1+\ldots+m'_{N+1}=d$, and $m'_n\le k_n$ for each $1\le n\le N+1$. Hence,

$$\rho^{m_1}\cdots\rho^{m_{N+1}}\ge\rho^{k_1-k_0}\cdots\rho^{k_{N+1}-k_N},$$

completing this proof. □

It only remains to prove Lemma 5.3.2, and we do so now.

Proof of Lemma 5.3.2. It is here that we finally use Lemma 5.3.1, the fact that for all subspaces π which contain π_{n-1} and which are strictly contained in π_n, inequality (5.11) holds.

For convenience, fix n, let $k_{n-1}=k$ and let $k_n=k+r$. Relabel π_{n-1} as Π_0 and π_n as Π_1. Let Δ be the set of all lines which are contained in Π_1, but which are not contained in Π_0.

Thus, inequality (5.11) of Lemma 5.3.1 reads as follows: for all subspaces π with $\Pi_0\subset\pi\subset\Pi_1$

$$\sum_{l\subset\pi,l\not\subset\Pi_0}f(l)\le 4\alpha_{k_{n-1}}\sum_{l\not\subset\pi,l\subseteq\Pi_1}f(l). \tag{5.15}$$

Similarly, the conclusion of Lemma 5.3.2 reads as

$$\left(\sum_{l\in\Delta}f(l)\right)^r\le\beta_n\sum_{l_1,\ldots,l_r\in\Delta}(\wedge_{j=1}^r e(l_j))f(l_1)\cdots f(l_r). \tag{5.16}$$

Let $(l_1,\ldots,l_r)$ be an r-tuple of lines, each of which lies in Δ. Then, together with Π_0, they span a subspace of Π_1, with dimension $k+t$, for some $t\in\{1,\ldots,r\}$. For $1\le t\le r$ let

$$\Gamma_t=\{(l_1,\ldots,l_r)\in\Delta^r\ :\ \dim\operatorname{span}\{e(l_1),\ldots,e(l_r)\}=t\}.$$

Note that $e(l_1)\wedge\cdots\wedge e(l_r)=1$ for lines $l_j\subset\Pi_1$ such that $l_j\not\subset\Pi_0$ if and only if $(l_1,\ldots,l_r)\in\Gamma_r$.

We expand the left-hand side of (5.16) as follows:

$$\sum_{t=1}^r\sum_{(l_1,\ldots,l_r)\in\Gamma_t}f(l_1)\cdots f(l_r).$$

Therefore, to prove (5.16) it suffices to show that for $1\le t\le r$,

$$\sum_{(l_1,\ldots,l_r)\in\Gamma_t}f(l_1)\cdots f(l_r)\lesssim\sum_{(l_1,\ldots,l_r)\in\Gamma_r}f(l_1)\cdots f(l_r).$$

Similarly, the above inequality follows if we can show that for $1 \leq t < r$,

$$\sum_{(l_1,\ldots,l_r)\in\Gamma_t} f(l_1)\cdots f(l_r) \lesssim \sum_{(l_1,\ldots,l_r)\in\Gamma_{t+1}} f(l_1)\cdots f(l_r).$$

If $(l_1,\ldots,l_r) \in \Gamma_t$, then for some t-tuple – which is without loss of generality $(l_1,\ldots,l_t)$ – we have that $\{\Pi_0, l_1,\ldots,l_t\}$ spans a $(k+t)$-plane $H(l_1,\ldots,l_t)$ satisfying $\Pi_0 \subset H(l_1,\ldots,l_t) \subset \Pi_1$. Therefore

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_r)\in\Gamma_t} f(l_1)\cdots f(l_r)\\
\leq\ &\binom{r}{t} \sum_{\substack{l_1,\ldots,l_t\in\Delta^t:\\ e(l_1)\wedge\cdots\wedge e(l_t)=1}} f(l_1)\cdots f(l_t) \sum_{\substack{l_{t+1},\ldots,l_r\subset H(l_1,\ldots,l_t):\\ l_{t+1},\ldots,l_r\not\subset\Pi_0}} f(l_{t+1})\cdots f(l_r)\\
=\ &\binom{r}{t} \sum_{\substack{l_1,\ldots,l_t\in\Delta^t:\\ e(l_1)\wedge\cdots\wedge e(l_t)=1}} f(l_1)\cdots f(l_t) \left(\sum_{\substack{l\subset H(l_1,\ldots,l_t):\\ l_{t+1},\ldots,l_r\not\subset\Pi_0}} f(l) \right)^{r-t}.
\end{aligned}$$

Since $\Pi_0 \subset H(l_1,\ldots,l_t) \subset \Pi_1$, we use Lemma 5.3.1, *via* (5.15), to estimate the bracketed expression above by

$$4\alpha_k \left(\sum_{\substack{l\subset H(l_1,\ldots,l_t):\\ l\not\subset\Pi_0}} f(l) \right)^{r-t-1} \left(\sum_{l\not\subset H(l_1,\ldots,l_t),\, l\subset\Pi_1} f(l) \right).$$

This shows that

$$\sum_{(l_1,\ldots,l_r)\in\Gamma_t} f(l_1)\cdots f(l_r) \lesssim \sum_{(l_1,\ldots,l_r)\in\Gamma_{t+1}} f(l_1)\cdots f(l_r),$$

as desired. □

5.4 Topics for Further Investigation

Let $X, Y_1, \ldots, Y_d$ be measure spaces, subject to appropriate hypotheses. Let $K : X \times Y_1 \times \cdots \times Y_d \to \mathbb{R}_{\geq 0}$. The results in this chapter suggest that we should expect some form of duality to hold for **nice** kernels K. More specifically, and in light of Theorem 5.0.1, we say that K is nice if

$$\inf_{(\omega_j)_j\in\mathcal{C}(x)} \prod_{j=1}^{d} \left(\int_{Y_j} \omega_j(y_j) f_j(y_j)\, \mathrm{d}y_j \right)^{\alpha_j} \lesssim \int_{Y_1\times\cdots\times Y_d} K(x,\boldsymbol{y}) \prod_{j=1}^{d} f_j(y_j)^{\alpha_j}\, \mathrm{d}\boldsymbol{y},$$

for all non-negative $f_j : Y_j \to \mathbb{R}_{\geq 0}$, where

$$\mathcal{C}(x) = \left\{ \omega_j : Y_j \to \mathbb{R}_{\geq 0} : K(x, \boldsymbol{y}) \leq \prod_j \omega_j(y_j)^{\alpha_j} \right\},$$

and $\alpha_1 + \ldots + \alpha_d = 1$. Suppose that K is a nice kernel. Following technical adaptations of arguments in [Coc18] and [CHV20a], we expect the following to be equivalent:

(i) For all non-negative $f_j \in L^1(Y_j)$, for each $1 \leq j \leq d$,

$$\left\| \left(\int_{Y_1 \times \cdots \times Y_d} K(\cdot, \boldsymbol{y}) \prod_{j=1}^{d} f_j(y_j)^{\alpha_j} \, \mathrm{d}\boldsymbol{y} \right) \right\|_{L^p(X)} \lesssim \prod_{j=1}^{d} \|f_j\|_{L^1(Y_j)}^{\alpha_j}.$$

(ii) For every non-negative $S \in L^{p'}(X)$, there are functions $S_j : X \times Y_j \to \mathbb{R}_{\geq 0}$ so that

$$K(x, \boldsymbol{y}) S(x) \leq \prod_{j=1}^{d} S_j(x, y_j)^{\alpha_j}$$

for a.e. $x \in X$ and a.e. $\boldsymbol{y} \in Y_1 \times \cdots \times Y_d$, and so that

$$\int_X S_j(x, y_j) \, \mathrm{d}x \lesssim \|S\|_{p'}$$

for a.e. $y_j \in Y_j$ and every $1 \leq j \leq d$.

However, of greater interest would be to understand which kernels K are nice. Perhaps the most immediate example to consider, given our work here on the multijoint problem, is the multilinear Kakeya kernel. In particular, denoting the Euclidean wedge product by $\wedge$, we do not currently know whether the kernel associated to the operator

$$\boldsymbol{T} \mapsto \left(\prod_{j=1}^{d} \chi_{T_j}(\cdot) \right) (e(T_1) \wedge \cdots \wedge e(T_d))$$

is nice, where each T_j is a tube in $\mathbb{R}^d$.

More generally, it may be interesting to examine the entire class of nice kernels. For example, are there any nice kernels which are not geometric in nature? Alternatively, it would also be of interest to find examples of kernels which are not nice, as suggested in the discussion following [CHV20a, Proposition 8.1].

Both the multilinear Kakeya kernel and the multijoints kernel have symmetry in common. In contrast, the kernel for the higher-dimensional multijoint problem of k_1-, …, k_d-planes will not be symmetric, in general. We expect that our arguments from this chapter will generalise to establish a duality theorem for

$\boldsymbol{k}$-multijoints, and hopefully this will be informative as to the role of symmetry within this type of duality.

With regards to questions raised by Chapter 3 and Chapter 4, in Section 3.4 we described a framework in which our proof of the discrete Bourgain–Guth theorem ran parallel to the Carbery–Valdimarsson proof of the Bourgain–Guth theorem. As suggested in Section 3.4, it would be of interest to discover what further variants of the multijoint problem could be proven by considering covering lemmas for polynomials defined on algebraic varieties. For example, we may consider counting joints formed by lines, all contained in a particular ruled surface – a joints problem where the ambient space, usually $\mathbb{F}^d$, is generalised to a d-dimensional variety.

We have not discussed in any detail the notion of "visibility" which is central to the Carbery–Valdimarsson and Bourgain–Guth arguments, [CV14, BG11]. A salient point of visibility is that the factorising function s is constructed *via* a particular algebraic hypersurface Z with large visibility. For any cube Q and tube T, the value of s is given by the directional surface area of Z with respect to Q and $e(T)$, and the first inequality in item *(ii)*, above, manifests as an estimate on visibility. No such "universal" hypersurface arose in Chapter 3 or Chapter 4, and this warrants further research to search for a corresponding geometric object in the discrete setting.

References

[AE84] Jean Pierre Aubin and Ivar Ekeland. *Applied nonlinear analysis.* Wiley, 1984.

[BCT06] Jonathan Bennett, Anthony Carbery, and Terence Tao. On the multilinear restriction and Kakeya conjectures. *Acta Mathematica*, 196(2):261–302, 2006.

[BG11] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geometric and Functional Analysis*, 21:1239–1295, 2011.

[CEG$^+$92] B. Chazelle, H. Edelsbrunner, L.J. Guibas, R. Pollack, R. Seidel, M. Sharir, and J. Snoeyink. Counting and cutting cycles of lines and rods in space. *Computational Geometry. Theory and Applications*, 1(6):305–323, 1992.

[CHV20a] Anthony Carbery, Timo S. Hänninen, and Stéfan Ingi Valdimarsson. Multilinear duality and factorisation for Brascamp–Lieb-type inequalities with applications (to appear in *Journal of the European Mathematical Society*). arXiv:1809.02449, 2020.

[CHV20b] Anthony Carbery, Timo S. Hänninen, and Stefán Ingi Valdimarsson. Disentanglement, multilinear duality and factorisation for non-positive operators (to appear in *Analysis & PDE*). arXiv:2003.03326, 2020.

[CI14] Anthony Carbery and Marina Iliopoulou. Counting joints in vector spaces over arbitrary fields. arXiv:1403.6438, 2014.

[CI20] Anthony Carbery and Marina Iliopoulou. Joints formed by lines and a k-plane, and a discrete estimate of Kakeya type. *Discrete Analysis*, (18):1–45, 2020.

[Coc18] Anthony Carbery and other contributors. Lecture notes for Brascamp–Lieb and related inequalities. *Maxwell Institute Graduate School In Analysis and its Applications*, Autumn 2018.

[CT22] Anthony Carbery and Michael Tang. Non-transversal multilinear duality and joints. arXiv:2203.02320, 2022.

[CV13] Anthony Carbery and Stéfan Ingi Valdimarsson. The endpoint multilinear Kakeya theorem *via* the Borsuk–Ulam theorem. *Journal of Functional Analysis*, 264(7):1643–1663, 2013.

[CV14] Anthony Carbery and Stefán Ingi Valdimarsson. Colouring multijoints. *Discrete & Computational Geometry*, 52(4):730–742, 2014.

[DKSS13] Zeev Dvir, Swastik Kopparty, Shubhangi Saraf, and Madhu Sudan. Extensions to the method of multiplicities, with applications to Kakeya sets and mergers. *SIAM Journal on Computing*, 42(6):2305–2328, 2013.

[Dvi09] Zeev Dvir. On the size of Kakeya sets in finite fields. *Journal of the American Mathematical Society*, 22(4):1093–1097, 2009.

[Dvi12] Zeev Dvir. Incidence theorems and their applications. *Foundations and Trends in Theoretical Computer Science*, 6(4):257–393, 2012.

[EKS11] György Elekes, Haim Kaplan, and Micha Sharir. On lines, joints, and incidences in three dimensions. *Journal of Combinatorial Theory, Series A*, 118(3):962–977, 2011.

[GK10] Larry Guth and Nets Hawk Katz. Algebraic methods in discrete analogs of the Kakeya problem. *Advances in Mathematics*, 225(5):2828–2839, 2010.

[Gut10] Larry Guth. The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture. *Acta Mathematica*, 205(2):263–286, 2010.

[Hab14] Márton Hablicsek. On the joints problem with multiplicities. arXiv:1408.5791, 2014.

[HBA$^+$18] M. Hohenwarter, M. Borcherds, G. Ancsin, B. Bencze, M. Blossier, J. Éliás, K. Frank, L. Gál, A. Hofstätter, F. Jordan, Z. Konečný, Z. Kovács, E. Lettner, S. Lizelfelner, B. Parisse, C. Solyom-Gecse, C. Stadlbauer, and M. Tomaschko. GeoGebra 5.0.507.0, 2018. `http://www.geogebra.org`.

[Ili13] Marina Iliopoulou. Counting joints with multiplicities. *Proceedings of the London Mathematical Society*, 106(3):675–702, 2013.

[Ili15] Marina Iliopoulou. Counting multijoints. *Journal of Combinatorial Theory, Series A*, 136:143–163, 2015.

[KSS10] Haim Kaplan, Micha Sharir, and Eugenii Shustin. On lines and joints. *Discrete & Computational Geometry*, 44(4):838–843, 2010.

[Qui10] René Quilodrán. The joints problem in $\mathbb{R}^n$. *SIAM Journal on Discrete Mathematics*, 23(4):2211–2213, 2010.

[Sha94] Micha Sharir. On joints in arrangements of lines in space and related problems. *Journal of Combinatorial Theory, Series A*, 67(1):89–99, 1994.

[SW04] Micha Sharir and Emo Welzl. Point-line incidences in space. *Combinatorics, Probability and Computing*, 13(1):203–220, 2004.

[Tak80] Jiro Takeuchi. On the Cauchy problem for some non-Kowalewskian equations with distinct characteristic roots. *Journal of Mathematics of Kyoto University*, 20(1):105–124, 1980.

[Tan22] Michael Tang. Multijoints and factorisation. arXiv:2203.02328, 2022.

[Tao14] Terence Tao. Algebraic combinatorial geometry: the polynomial method in arithmetic combinatorics, incidence combinatorics, and number theory. *EMS Surveys in Mathematical Sciences*, 1(1):1–46, 2014.

[TYZ20] Jonathan Tidor, Hung-Hsun Hans Yu, and Yufei Zhao. Joints of varieties. arXiv:2008.01610, 2020.

[Yan16] Ben Yang. Generalizations of joints problem. arXiv:1606.08525, 2016.

[YZ19] Hung-Hsun Hans Yu and Yufei Zhao. Joints tightened. arXiv:1911.08605, 2019.

[Zha20] Ruixiang Zhang. A proof of the multijoints conjecture and Carbery's generalization. *Journal of the European Mathematical Society*, 22(8):2405–2417, 2020.

Appendix A

Additional Numerical Examples

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)} S_2^{(1)}}$	$S^{(1)}$
p_1	0.946	0.846	0.0.894	0.894
p_2	0.054	0.926	0.223	0.223
p_3	0.622	0.080	0.223	0.223
p_4	0.324	0.154	0.223	0.223
p_5	0.676	0.074	0.223	0.223

Table A.1: Minimisers of (2.8) with $S = \frac{1}{\sqrt{20}}(4, 1, 1, 1, 1)$. Values rounded to 3 decimal places.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)} S_2^{(1)}}$	$S^{(1)}$
p_1	0.525	0.870	0.676	0.676
p_2	0.475	0.963	0.676	0.676
p_3	0.306	0.093	0.169	0.169
p_4	0.220	0.130	0.169	0.169
p_5	0.780	0.034	0.169	0.169

Table A.2: Minimisers of (2.8) with $S = \frac{1}{\sqrt{35}}(4, 4, 1, 1, 1)$. Values rounded to 3 decimal places.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)}S_2^{(1)}}$	$S^{(1)}$
p_1	0.154	0.324	0.224	0.224
p_2	0.846	0.946	0.894	0.894
p_3	0.080	0.622	0.224	0.224
p_4	0.074	0.676	0.224	0.224
p_5	0.926	0.054	0.224	0.224

Table A.3: Minimisers of (2.8) with $S = \frac{1}{\sqrt{20}}(1,4,1,1,1)$. Values rounded to 3 decimal places.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)}S_2^{(1)}}$	$S^{(1)}$
p_1	0.963	0.475	0.676	0.676
p_2	0.037	0.780	0.169	0.169
p_3	0.093	0.306	0.169	0.169
p_4	0.870	0.525	0.676	0.676
p_5	0.130	0.220	0.169	0.169

Table A.4: Minimisers of (2.8) with $S = \frac{1}{\sqrt{35}}(4,1,1,4,1)$. Values rounded to 3 decimal places.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)}S_2^{(1)}}$	$S^{(1)}$
p_1	0.971	0.471	0.676	0.676
p_2	0.029	0.970	0.169	0.169
p_3	0.917	0.499	0.676	0.676
p_4	0.054	0.529	0.169	0.169
p_5	0.946	0.030	0.169	0.169

Table A.5: Minimisers of (2.8) with $S = \frac{1}{\sqrt{35}}(4,1,4,1,1)$. Values rounded to 3 decimal places.

	$S_1^{(1)}$	$S_2^{(1)}$	$\sqrt{S_1^{(1)}S_2^{(1)}}$	$S^{(1)}$
p_1	0.529	0.054	0.169	0.169
p_2	0.471	0.971	0.676	0.676
p_3	0.499	0.917	0.676	0.676
p_4	0.030	0.946	0.169	0.169
p_5	0.970	0.029	0.169	0.169

Table A.6: Minimisers of (2.8) with $S = \frac{1}{\sqrt{35}}(1,4,4,1,1)$. Values rounded to 3 decimal places.

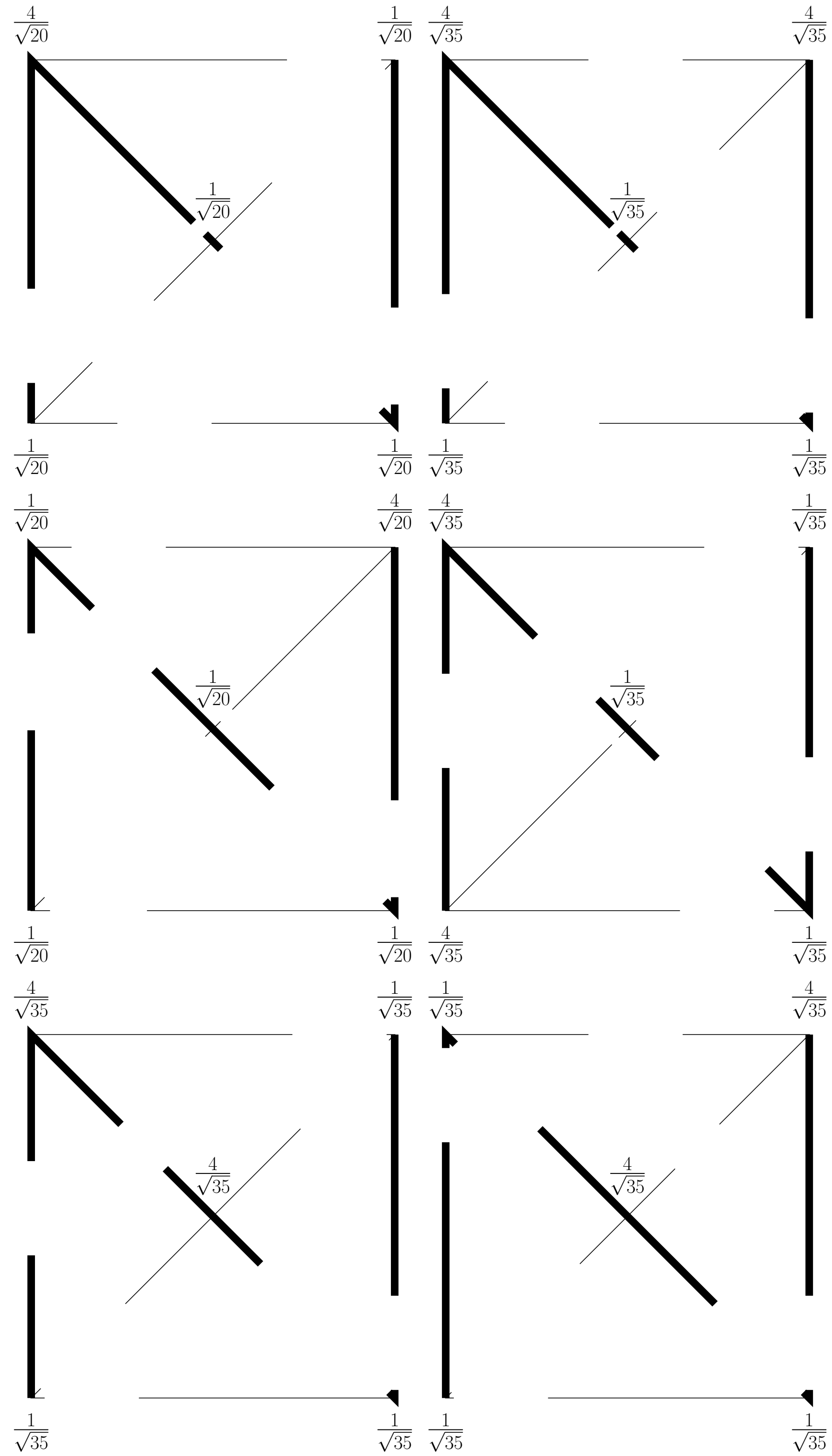

Figure A.1: Further graphical representations of example minimisers for various test functions whose values are shown for each minimiser.