Stochastic dispersive PDEs with additive space-time white noise

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or professional qualification.

July 24, 2019, *Leonardo Tolomeo*

Abstract

In this thesis, we will discuss the Cauchy problem for some nonlinear dispersive PDEs with additive space-time white noise forcing. We will focus on two different models: the stochastic nonlinear beam equation (SNLB) with power nonlinearity posed on the three dimensional torus, and the stochastic nonlinear wave equation with cubic nonlinearity in two dimensions, posed both on the torus and on the Euclidean space (SNLW).

For (SNLB), we will present a joint work with R. Mosincat, O. Pocovnicu and Y. Wang [16], which settles local well-posedness for every nonlinearity of the type $|u|^{p-1}u$, and global well-posedness for $p < 11/3$. In the case $p = 3$, we also consider a damped version of the equation, for which we can show invariance of the Gibbs measure. Moreover, we describe the long time-behaviour of the flow, by showing unique ergodicity of the Gibbs measure, and convergence to equilibrium for smooth initial data ([23]).

In the case of (SNLW) with cubic nonlinearity, we consider a renormalised version of the equation, which was introduced by M. Gubinelli, H. Koch and T. Oh. In their work, they established local well-posedness on the two-dimensional torus. We show global existence for these solutions (joint with M. Gubinelli, H. Koch and T. Oh, [10]), and local and global well-posedeness for the same equation posed on the two-dimensional Euclidean space ([22]).

Lay summary

In this thesis, we study some nonlinear dispersive equations, subject to random forcing. Dispersive equations appear ubiquitously in various branches of physics and engineering such as quantum mechanics, nonlinear optics, plasma physics, water waves, and atmospheric sciences.

An important example of these dispersive equations is the *wave* equation, which models many kinds of waves, such as electromagnetic waves, sound waves, and also the recently discovered gravitational waves. In this work, we consider a few of these models, and we consider the way in which the random forcing affects them. From the real world point of view, we introduce this random forcing to model all the microscopic "hidden variables" and "disturbances" in the system, such as the random movement of the air molecules in the case of sound waves.

Even when the amount of randomness is very small, these problems turned out to be substantially more difficult than the ones without randomness, and the techniques to tackle them are very recent, and some of them have been developed only in recent years. In particular, this thesis explores the almost uncharted territory of the *long time behaviour*, which roughly consists in answering the question "what happens to the system after we wait a long time?" In one of the models, we manage to give an almost complete answer (for the first time in this setting), which essentially consists in "the randomness wins, and the information about what was of the system at the beginning is completely scrambled".

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Contents

Chapter 1

Introduction

During my PhD studies, my primary research area has been the study of *stochastic dispersive dynamics*, i.e. the study of disperisve partial differential equations (PDEs) that depend on some random/stochastic objects. A typical examples can be the Cauchy problem stochastic wave equation with additive noise

$$\begin{cases} \partial_t^2 u + \Delta u \pm |u|^{p-1}u = \xi, \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases} \tag{1.0.1}$$

where $\xi = \xi(t,x)$ is some stochastic forcing.

Dispersive PDEs appear ubiquitously in various branches of physics and engineering such as quantum mechanics, nonlinear optics, plasma physics, water waves, and atmospheric sciences. From the modelling point of view, adding randomness to these model is very natural, since they incorporate the uncertainties of the "real world". More specifically, a random initial data would represent the uncertainty in the measurement of the initial condition of our model, while the stochastic forcing models the unpredictable perturbations coming from outside sources. In particular, terms such as ξ are very common in models coming from statistical mechanics, as they represent the effect that a heat bath has on the model.

In many of these models, we can assume that the noise ξ is isotropic in space and time (i.e. it is uniform both in space and time), and that its values are independent in every point of space and time. Hence the so-called *space-time white noise* plays a crucial role. This is the only *distribution*-valued Gaussian random variable ξ such that

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y),$$

or, more rigorously,

$$\mathbb{E}[\langle f, \xi \rangle_{L^2_{t,x}} \langle g, \xi \rangle_{L^2_{t,x}}] = \langle f, g \rangle_{L^2_{t,x}}. \tag{1.0.2}$$

In the following ξ will always denote this particular stochastic object, which is extremely rough. In particular, the space-time white noise on $\mathbb{R}_t \times \mathbb{R}^d_x$ satisfies (glossing over the precise definition of these spaces) $\xi \in C_t^{-\frac{1}{2}-\varepsilon} C_x^{-\frac{d}{2}-\varepsilon}$ for every $\varepsilon > 0$, and it can never be represented as a function.

In this thesis, we study stochastic *dispersive* PDEs of the like of (1.0.1), with a particular focus on the long time behaviour of solutions to these equations.

The ultimate goal is trying to shorten a bit the gap in knowledge with stochastic *parabolic* PDEs. This kind of pursuit seats nicely in the middle of several fields, as it involves tools coming from probability and harmonic analysis, as well as more general PDE techniques.

The main analytical challenges in dealing with these equation come from the roughness of the space-time white noise. Indeed, many physical problems present unexpected complication. Two important examples of these models are given by the Kardar-Parisi-Zhang equation (KPZ)

$$\partial_t h - \partial_x^2 h = \partial_x h^2 - \infty + \xi, \tag{KPZ}$$

which is (conjectured to be) the field theory of many surface growth models and the limiting behaviour of many particle systems; and the Stochastic Quantisation Equation (SQE)

$$\partial_t u - \Delta u + u^3 - \infty \cdot u = \xi, \tag{SQE}$$

which provides a dynamical construction of the quantum field theory Φ^4_d. The presence of these (apparently) nonsensical ∞-s in the formulation of the equations is closely related to the roughness of the forcing ξ, and it is rigorously removed by renormalising the equations.

In the last decade, there have been tremendous developments in the study of singular parabolic PDEs. In 2011, M. Hairer proved *local well posedness* for (KPZ) ([11]), and later on developed the general theory of regularity structure ([12]), which deals with a large class of singular stochastic parabolic PDEs. His work earned him the Fields Medal in 2014.

In 2012, an alternative theory to deal with these stochastic parabolic PDEs has been developed by Gubinelli-Imkeller-Perkowski [7], via the use of *paracontrolled distribution*.

There have been also many developments in the study of the long time behaviour of these singular SPDEs, with global well posedness results (e.g. [15] for 3d (SQE) on the torus), and also the much more precise information about *unique ergodicity* and convergence to equilibrium (e.g. [26] for (SQE) on the 2d torus), which gives a complete statistical description of the long time behaviour of the dynamics starting from any initial data.

On the other end, the available results in the dispersive setting are much further behind. As the only examples of singular dispersive equations, the only results I am aware of (which are not part of this thesis) are [8, 9, 20].

In this thesis, we consider two dispersive models. The first of them is non-singular one, given by the stochastic beam equation on the three dimensional torus

$$u_{tt} + \Delta^2 u + |u|^{p-1}u = \xi,$$

which will be the main focus of Chapter 2, The second one is given by the *renormalised* wave equation in two spatial dimensions, introduced in [8], which instead will be the main focus of Chapter 3.

$$u_{tt} + \Delta^2 u + u^3 - 3 \cdot \infty = \xi.$$

1.1 Stochastic nonlinear beam equations

In Chapter 2, we consider in parallel the Cauchy problem for the defocusing stochastic nonlinear beam equation (SNLB) on $\mathbb{T}^3$

$$\begin{cases} u_{tt} + \Delta^2 u + |u|^{p-1}u = \xi \\ (u_t, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s := H^s \times H^{s-2}, \end{cases} \tag{1.1.1}$$

and the cubic stochastic beam equation with damping (SDNLB),

$$\begin{cases} u_{tt} + u_t + u + \Delta^2 u + u^3 = \xi \\ (u_t, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s \end{cases} \tag{1.1.2}$$

It is convenient to write both equations in first order formulation, by considering $\mathbf{u} := (u, u_t)$. We obtain

$$\begin{cases} \partial_t \mathbf{u} = \begin{pmatrix} 0 & 1 \\ -\Delta^2 & 0 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 0 \\ |u|^{p-1}u \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \in \mathcal{H}^s, \end{cases} \tag{SNLB}$$

and

$$\begin{cases} \partial_t \mathbf{u} = \begin{pmatrix} 0 & 1 \\ -1-\Delta^2 & -1 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \in \mathcal{H}^s. \end{cases} \tag{SDNLB}$$

1.1.1 Equation without damping

In the study of (SNLB), our goal is to prove the following:

Theorem 1.1.1. *Let* $\mathbf{u}_0 \in \mathcal{H}^2$*, let* $p < \frac{11}{3}$*. For every* $0 \leq T < +\infty$*, the equation* (SNLB) *admits a unique solution* $\mathbf{u} = \Psi + \mathbf{v}$*, where* $\Psi = (\psi, \psi_t)$ *is a* $C([0;T]; C^{\frac{1}{2}-} \times C^{-\frac{3}{2}-}(\mathbb{T}^3))$*-valued random variable, and* $\mathbf{v} \in C([0,T]; \mathcal{H}^2)$*. Moreover, this solution satisfies, for some* $\alpha, \beta, \gamma > 0$*,*

$$\|\mathbf{v}(t)\|_{\mathcal{H}^2} \lesssim 1 + \|\mathbf{u}_0\|_{\mathcal{H}^2}^\alpha + \|\Psi\|^\beta_{C([0;T];C^{\frac{1}{2}-}\times C^{-\frac{3}{2}-}(\mathbb{T}^3))} + t^\gamma. \tag{1.1.3}$$

For the local well posedness part of the statement, it is possible to treat the equation for every p, and lower the regularity of the initial data. However, this is the best global well-posedness result available. We refer to the paper [16] for the precise statement.

On $\mathbb{T}^3$, the term ξ is very rough, with it being a random distribution such that $\xi \in C^{-\frac{1}{2}-}H^{-\frac{3}{2}-}$. To deal with (SNLB), we carry on a perturbative analysis, and write the solution as $\mathbf{u} = \mathbf{v} + \Psi$, where Ψ, is the solution to the linear equation $\partial_t^2\Psi + \Delta^2\Psi = \xi$. This is the so-called Da Prato - Debussche trick. We will show in Chapter 2 that $\Psi \in C_t(C^{\frac{1}{2}-} \times C^{-\frac{3}{2}-})$, and this will let us prove local well posedness for the equation for $\mathbf{v}$ via a Banach fixed point argument.

We show global well-posedeness by proving a probabilistic a priori estimate. If one considers the equation without noise, it enjoys conservation of the energy

$$E(\mathbf{u}) = \frac{1}{2}\int u_t^2 + \frac{1}{2}\int (\Delta u)^2 + \frac{1}{p+1}\int |u|^{p+1}.$$

This allows to easily prove global well posedness in $\mathcal{H}^2$ in the energy-subcritical regime, by a simple iteration of local well posedness.

This strategy has clear issues for the equation (SNLB). First of all, because of presence of the stochastic forcing term ξ, one can check that the energy $E(\mathbf{u})$ is *not* formally conserved. Moreover, since one can write $\mathbf{u} = \mathbf{v} + \Psi$, and $\Psi \notin \mathcal{H}^2$, one has that $E(\mathbf{u})(t) = +\infty$ for every $t > 0$. Therefore, we bound $E(\mathbf{v})$ instead. This method was introduced by N. Burq-N. Tzvetkov [2] in the context of cubic nonlinear wave equation with random initial data. We follow this approach to show global well posedness for (SNLB), setting up a Gronwall estimate for $E(\mathbf{v})$. When one considers $\partial_t E(\mathbf{v})$, the worst term in the expansion is given by

$$p\int_{\mathbb{T}^3} u_t |u|^{p-2}\psi(\xi). \tag{1.1.4}$$

This can be bounded using Hölder's inequality by the first and last term of the energy, together with $\psi(\xi) \in C_tC^0(\mathbb{T}^3)$, and we obtain (1.1.4) $\lesssim_\xi E^{\frac{1}{2}+\frac{p-1}{p+1}}$, which is enough to close the Gronwall argument for $p < 3$.

For $3 \leq p < \frac{11}{3}$, we adapt the strategy introduced by T. Oh-O. Pocovnicu in [19] for the study of nonlinear wave equation with random initial data. We rely on integration by parts

$$\int_0^t p\int_{\mathbb{T}^3} u_t|u|^{p-1}\psi = \int_{\mathbb{T}^3} |u|^{p-1}u(t)\psi(t) - \int_0^t\int_{\mathbb{T}^3} |u|^{p-1}u\partial_t\psi$$

and on the fact that $\partial_t\psi(\xi) \in C_tC^{-\frac{3}{2}-}(\mathbb{T}^3)$. We can then estimate $\big\||u|^{p-1}u\big\|_{W^{1,\frac{3}{2}+}}$ by interpolating between $\|u\|_{\dot{H}^2}$ and $\|u\|_{L^{p+1}}$, which in turn can be both bounded using the energy. This allows to conclude the Gronwall - type argument for $p < \frac{11}{3}$.

1.1.2 Equation with damping: invariant measure and unique ergodicity

We consider now the equation (SDNLB). From a formal computation, we expect the measure

$$\mathrm{d}\rho(u, u_t)\text{“}=\text{”}\frac{1}{Z}\exp\Big(-\frac{1}{4}\int u^4 - \frac{1}{2}\int u^2 - \frac{1}{2}\int(\Delta u)^2\Big)\exp\Big(-\frac{1}{2}\int u_t^2\Big)\text{“}\mathrm{d}u\mathrm{d}u_t\text{”}, \tag{1.1.5}$$

to be *invariant*, where "dudu_t" is the non-existent Lebesgue measure on an infinite dimensional vector space (of functions). Heuristically, we expect invariance for this measure by splitting (SDNLB) into

1. $$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-\Delta^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix},$$

   which is a Hamiltonian PDE in the variables u, u_t with Hamiltonian

   $$H(u,u_t) = \frac{1}{4}\int u^4 + \frac{1}{2}\int u^2 + \frac{1}{2}\int (\Delta u)^2 + \frac{1}{2}\int u_t^2,$$

   and so by Liouville's theorem it should preserve the Gibbs measure

   $$\exp\Big(-H(u,u_t)\Big)\text{"d}u\text{d}u_t\text{"} = \rho,$$

   and

2. $$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix},$$

   which is the Ornstein - Uhlenbeck process in the variable u_t, and so it preserves the spatial white noise

   $$\exp\Big(-\frac{1}{2}\int u_t^2\Big)\text{"d}u_t\text{"},$$

   and any measure written in the form $F(u)$"du" $\times \exp\Big(-\frac{1}{2}\int u_t^2\Big)$"d$u_t$".

From a rigorous point of view, we define the measure (1.1.5) to be given by

$$\rho := \frac{1}{Z}\exp\Big(-\frac{1}{4}\int u^4\Big)\text{d}\mu(\mathbf{u}), \tag{1.1.6}$$

where dμ is the law of any Gaussian random variable $\mathbf{X}$ with covariance operator given by

$$\mathbb{E}[\langle \mathbf{f}, \mathbf{X}\rangle_{L^2_x} \langle \mathbf{g}, \mathbf{X}\rangle_{L^2_x}] = \left\langle \begin{pmatrix} 1+\Delta^2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{f}, \mathbf{g} \right\rangle_{L^2_x} \tag{1.1.7}$$

and Z is given by $\int \exp\Big(-\frac{1}{4}\int u^4\Big)\text{d}\mu(\mathbf{u})$, so to make ρ into a probability measure. Notice that formally, by using the formula for the density of a Gaussian given its covariance matrix, we have

$$\begin{aligned} \text{d}\mu(\mathbf{u}) &= \frac{1}{Z_0}\exp\Big(-\frac{1}{2}\int u^2 - \frac{1}{2}\int (\Delta u)^2 - \frac{1}{2}\int u_t^2\Big)\text{d}u\text{d}u_t, \\ \text{d}\rho(\mathbf{u}) &= \frac{1}{Z_1}\exp\Big(-\frac{1}{4}\int u^4 - \frac{1}{2}\int u^2 - \frac{1}{2}\int (\Delta u)^2 - \frac{1}{2}\int u_t^2\Big)\text{d}u\text{d}u_t \end{aligned}$$

where Z_0, Z_1 are renormalisation constants. Therefore, up to a renormalisation constant, ρ is formally given by (1.1.5), and by the previous heuristic, we expect it to be invariant for the flow of (SDNLB). In [23], Indeed, in Chapter 2, we prove

Theorem 1.1.2. *The measure ρ is invariant for the flow $\Phi_t(\cdot;\xi)$ of* (SDNLB)*, in the sense that for every function F measurable and bounded,*

$$\int \mathbb{E}[F(\Phi_r(\mathbf{u}_0,;\xi)]\text{d}\rho(\mathbf{u}_0) = \int F(\mathbf{u}_0)\text{d}\rho(\mathbf{u}_0) \textit{ for every } t > 0.$$

Moreover, there exists a Banach space $X^\alpha \subseteq \mathcal{H}^\alpha$ which contains the Sobolev Space $\mathcal{H}^2$, such that for every $0 < \alpha < \frac{1}{2}$, $\rho(X^\alpha) = 1$, and ρ is the only invariant measure for the flow of (SDNLB) *on X^α.*

To my knowledge, this is the *first* ergodicity result for a stochastic dispersive PDE with additive white-noise forcing.

Notice that in order to have invariance of the measure ρ for the flow of (SDNLB), the space X^α on which we define the flow has to be big enough, in such a way that $\rho(X^\alpha) = 1$. We will define the appropriate space in Chapter 2, but here we notice that this implies that X^α has to contain functions belonging to the space $C^{\frac{1}{2}-} \setminus \mathcal{H}^{\frac{1}{2}}$.

As in the case of (SNLB), we write the solution $\mathbf{u}$ to (SDNLB) as $\mathbf{u} = S(t)\mathbf{u}_0 + \upharpoonleft_t(\xi) + \mathbf{v}$, where $\upharpoonleft_t(\xi)$ solves the linear equation

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 + \Delta^2 & 1 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix},$$

and we carry on a similar analysis to the one for (SNLB) to show local and global well posedness. A difference with the previous argument that is worth pointing out is that we consider the modified energy

$$E(\mathbf{v}) := \frac{1}{2}\int v_t^2 + \frac{1}{2}\int v^2 + \frac{1}{2}\int (\Delta v)^2 + \frac{1}{4}\int v^4 + \frac{1}{8}\int (v + v_t)^2, \tag{1.1.8}$$

so we can make full use of the damping and show a uniform in time estimate for $\|\mathbf{v}\|_{\mathcal{H}^2}$. Once one has global well-posedness, invariance of ρ follows from an approximation argument with a finite-dimensional system of ODEs and an application of the Fokker-Plank equation.

Proving ergodicity and uniqueness of the measure requires some new ideas. One might be tempted to follow the strategy adopted by H.Weber and P. Tsatsoulis in [26] for the Stochastic Quantisation Equation on $\mathbb{T}^2$ and more in general by M. Hairer and J. Mattingly in [13], and show the *strong Feller property* for the flow, i.e.

Definition 1.1.3 (Strong Feller)**.** Let Y be a topological space, and let $\mathcal{A}$ be a σ - algebra on Y. Let $\Phi_t(\cdot;\xi) : Y \to Y$ be a stochastic flow on Y. We say that Φ_t has the *strong Feller* property if, for every $F : Y \to \mathbb{R}$ bounded and $\mathcal{A}$-measurable, and for every $t > 0$, the function $\mathbf{u}_0 \mapsto \mathbb{E}[F(\Phi_t(\mathbf{u}_0;\xi))]$ is a continuous map in $\mathbf{u}_0$.

However, since the linear propagator $S(t)$ does not have any smoothing property (it is actually invertible in the space $\mathcal{H}^s$ for every $s \in \mathbb{R}$), we will prove that the flow of (SDNLB) does *not* satisfy the Strong Feller Property in the X^α topology. In order to recover it, we change the topology of X^α into the one induced by the distance

$$d(\mathbf{u}_0, \mathbf{u}_1) = \min(\|\mathbf{u}_0 - \mathbf{u}_1\|_{\mathcal{H}^2}, 1).$$

This way, we can prove the strong Feller property in this stronger topology. The price to pay is that in this new topology, the space X^α becomes not separable and with infinitely many connected components, so we have to rebuild a few ingredients of the standard theory, since we cannot borrow the usual results in probability theory.

Combining this strong Feller property with a control problem for $\upharpoonleft(\xi)$, we have that for every two different invariant measures ρ_1, ρ_2 for the flow of (SDNLB) s.t. $\rho_1 \perp \rho_2$, then there exists a Borel set S s.t.

$$\rho_1(S + \mathcal{H}^2) = 1, \rho_2(S + \mathcal{H}^2) = 0. \tag{1.1.9}$$

In order to obtain a contradiction from here, we consider the algebraic projection $\pi : X^\alpha \to X^\alpha/\mathcal{H}^2$.

Since the flow can be split in $\Phi(\mathbf{u}_0, \xi)(t) = S(t)\mathbf{u}_0 + \upharpoonleft(\xi)(t) + \mathbf{v}$, with $\mathbf{v} \in \mathcal{H}^2$, and $S(t)$ is a linear operator that maps $\mathcal{H}^2$ into itself, one has that

$$\pi(\Phi_t(\mathbf{u}_0, \xi)) = \pi(S(t)\mathbf{u}_0) + \pi(\upharpoonleft_t(\xi))$$

actually defines a flow on the space $X^\alpha/\mathcal{H}^2$, which is the projection of the flow for the *linear* equation. It is easy to see that the measure μ defined in (1.1.7) is ergodic for the flow of the linear equation, and ergodicity passes to the quotient, so the push-forward measure $\pi_\sharp\mu$ will be ergodic for the the projected flow $\pi(\Phi_t(\mathbf{u}_0, \xi))$. Together with (1.1.9), this implies that if $\tilde{\rho}$ is invariant and that if $\pi_\sharp\tilde{\rho} \ll \pi_\sharp\mu$, then $\tilde{\rho} = \rho$.

We then conclude the argument by showing that every invariant measure $\tilde{\rho}$ satisfies $\pi_\sharp\tilde{\rho} \ll \pi_\sharp\mu$. This essentially follows by the fact that

$$\pi(\Phi_t(\mathbf{u}_0, \xi)) = \pi(S(t)\mathbf{u}_0 + \upharpoonleft_t(\xi)),$$

and $S(t)\mathbf{u}_0 + \upharpoonleft_t(\xi)$ converges in law to μ as $t \to \infty$. However, since the space $X^\alpha/\mathcal{H}^2$ does not have any sensible topology, the map π is not continuous in any meaningful way, so one has to be careful in how to take limits in the argument. The uniformity in time of the energy estimates is crucial in this part.

1.2 Stochastic nonlinear wave equation

Consider the Cauchy problem for the nonlinear wave equation, posed on $\mathbb{T}^2$ or $\mathbb{R}^2$:

$$u_{tt} + \Delta u + u^3 = \xi. \tag{SNLW$_0$}$$

In the works of Albeverio, Haba, Oberguggenberger, and Russo [21, 1, 17, 18], similar stochastic wave equations with a general nonlinearity have been considered, and they have shown that the solutions to (SNLW$_0$) have to be distributions. Moreover, they pointed out the phenomenon of triviality: if u_ε is a solution to (SNLW$_0$) with ξ substituted by a suitable regularisation $\xi_\varepsilon \to \xi$, and $u_\varepsilon \to u$ as a distribution, then u satisfies a *linear* wave equation.

In order to get nontrivial solutions for a nonlinear wave equation, renormalisation is necessary. In [8], M. Gubinelli, H. Koch and T. Oh introduce a time-dependent Wick Renormalisation, and show local well posedness for the equation

$$\begin{cases} u_{tt} - \Delta u + u^3 - \infty \cdot u = \xi, \\ (u, u_t)|_{t=0} = \mathbf{u}_0 \in \mathcal{H}^s := H^s \times H^{s-1}, \end{cases} \tag{∞-SNLW}$$

for $s > \frac{1}{4}$. In Chapter 3, we will analyse this equation give a meaning to the equation, and show global existence of solutions, both on the torus $\mathbb{T}^2$ and the euclidean space $\mathbb{R}^2$. More specifically, after giving a suitable meaning to the equation (∞-SNLW), we will prove the following

Theorem 1.2.1. *Let $M = \mathbb{T}^2$ or $\mathbb{R}^2$, and let $s > \frac{4}{5}$. Then the equation (∞-SNLW) is globally well posed in $\mathcal{H}^s_{\mathrm{loc}}(M) := H^s_{\mathrm{loc}}(M) \times H^{s-1}_{\mathrm{loc}}(M)$. More precisely, for every $\mathbf{u}_0 \in \mathcal{H}^s_{\mathrm{loc}}$, and every $0 \leq T < +\infty$, there exists a unique solution of (∞-SNLW) $u = \psi + v$, where ψ is a random variable belonging to the space $C(\mathbb{R}; C^{0-}_{\mathrm{loc}}(M))$, and $(v, \partial_t v) \in C([-T, T]; \mathcal{H}^s_{\mathrm{loc}}(M))$.*

This result has been proven for $M = \mathbb{T}^2$ in [10], and for $M = \mathbb{R}^2$ in [22].

1.2.1 Global well posedness on $\mathbb{T}^2$

Proceeding similarly to the case of (SNLB), we write the solution $u = \psi + v$, where ψ satisfies the *linear* wave equation $\partial_t^2\psi + \Delta\psi = \xi$. Analogously to the (SNLB) case, we consider the energy functional

$$E(\mathbf{u}) = \frac{1}{2}\int u_t^2 + \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int u^4,$$

which is conserved by solutions to the *deterministic* nonlinear wave equations. However, as pointed out in [8], for every $t > 0$, even for very smooth initial data, we have that $\mathbf{v}(t) \in \mathcal{H}^{1-}$ but $\mathbf{v} \notin \mathcal{H}^1$, so it is impossible to get any meaningful estimate for the energy $E(\mathbf{v})$.

In order to overcome the issue, we make use of the I-method introduced by J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao (see e.g. [3]). We define the smoothing operator I_N associated to the Fourier multiplier

$$m_N(n) := \begin{cases} 1 & \text{for } |n| \leq N \\ \Big(\dfrac{|n|}{N}\Big)^{1-s} & \text{for } |n| \geq 3N, \end{cases}$$

and we estimate the functional $E(I_N\mathbf{v})$ via a Gronwall-type argument. By taking time derivatives of this functional, one obtains

$$\partial_t E(I_N\mathbf{v}) = (\text{commutators}) + \int_{\mathbb{T}^2} I_N v_t (I_N v)^2 I_N \psi + (\text{better terms}).$$

The presence of commutator terms (e.g. $\int I_N v_t[(I_N v)^3 - I_N(v^3)]$) is typical of the I-method, and follows from the fact that the deterministic non linear wave does not conserve the functional $E(I_N\mathbf{u})$. We estimate the commutator terms using Fourier analytic techniques. The main problem in applying a standard I-method argument comes from the extra term, which appears because of the presence of the random forcing.

Indeed, a naive estimate that exploits the smoothing properties of the operator I_N gives $\int_{\mathbb{T}^2} I_N v_t (I_N v)^2 I_N \psi \lesssim N^\delta E$. However, such an estimate, mixed in with the commutator terms, *cannot* rule out finite time blowup of the solution $\mathbf{v}$.

To overcome this issue, we need to use the (sharp) probabilistic bound $\|I_N\psi\|_{L^{\log N}} \lesssim \log N$. This allows us to show that $\partial_t E(I\mathbf{v}(t))) \lesssim E \log N$ as long as $E \lesssim N^D$. By choosing $N = N(\mathbf{u}_0)$ appropriately, this allows to show existence of solutions to (SNLW) up to a random time $\tau \gtrsim 1$ *independent* from the initial data $\mathbf{u}_0$. We conclude the argument by iterating this process, changing the value of N at every step, thus showing existence up to time $k\tau$ after k steps.

As far as I am aware, this is the first application of the I-method that requires to change N in a time-dependent way in order to get the global well posedness result.

1.2.2 Global well posedness in $\mathbb{R}^2$

The main difference between the situation on the torus and the one on the euclidean space is that we do not have a local-well posedness result available. Indeed, if we try to adapt the proof of [8] in this setting, the following happens. Given the initial data $\mathbf{u}_0$ (or even for $\mathbf{u}_0 = 0$), we write the equation solved by $v = u - \psi$, and try to set up a Banach fixed point argument in some Banach space $Y \subseteq C_t\mathcal{H}^s_{\mathrm{loc}}$. Moreover, we can try to exploit the finite speed of propagation, and solve the equation on bounded space-time regions. This can be carried forward, and on a ball $B(x_0, R)$ of center x_0 and radius R,

$$T = T(\|\mathbf{u}_0\|_{\mathcal{H}^s(B)}, \|{:}\psi^l{:}\|_{C^{-\varepsilon}(B)})^1.$$

However, it can be shown that $\sup_{x_0} \|{:}\psi^l{:}\|_{C^{-\varepsilon}(B)} = +\infty$ almost surely (for every l), so

$$\inf_{x_0 \in \mathbb{R}^2} T_{x_0,R} = 0$$

for every $R > 0$. Therefore, this argument *cannot* construct local solutions to (∞-SNLW) on space-time regions of the form $\mathbb{R}^2 \times [0, \varepsilon)$, i.e. it cannot show local well posedness.

Hence, we follow a different strategy. We take a smooth cutoff function ρ_N such that $\rho_N| \equiv 1$ on the ball $\{|x| \le N\}$, and consider the equation for $\mathbf{v}_N = (v_N, (v_N)_t)$ (omitting the subscript N)

$$\begin{cases} v_{tt} - \Delta v + v^3 + 3v^2\rho\psi + 3v\rho :\psi^2: + \rho :\psi^3: = 0^2, \\ \mathbf{v}|_{t=0} = \rho\mathbf{u}_0. \end{cases} \tag{1.2.1}$$

We will discuss this equation more in detail in Chapter 3. For now, we just point out that for $\rho = 1$, this equation formally coincides with the one for $v = u - \psi$. Global well posedness for (1.2.1) can be shown in a similar fashion to Theorem 1.2.1.

We then take a sequence $\mathbf{v}_N$ such that $\rho_n \to 1$. Tailoring the sequence of cutoffs ρ_N appropriately and making use of finite speed of propagation, we will show that v_N is definitely constant on every space-time box $K \times [-T, T]$, so we can define

$$\mathbf{v} := \lim_{N\to\infty} \mathbf{v}_N,$$

which will belong to $\mathcal{H}^s_{\mathrm{loc}}$. Moreover, $u = v + \psi$ will be a global-in-time solution to (∞-SNLW),

[1] The precise meaning of the terms $:\psi^l:$ will be given in Chapter 3.

and by finite speed of propagation again, we can show that every solution $\tilde{u}$ of (∞-SNLW) must satisfy $\tilde{u} = u$ on its domain, hence we get uniqueness of solutions.

Chapter 2

Stochastic beam equation

2.1 Mild formulation

2.1.1 Stochastic beam equation without damping

Consider the *linear* beam equation with forcing $\mathbf{f} = \begin{pmatrix} f \\ f_t \end{pmatrix}$ and initial data $\mathbf{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$,

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ \Delta^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} f \\ f_t \end{pmatrix}.$$

By variation of constants, the solution to this equation is given by

$$\mathbf{u} = S(t)\mathbf{u}_0 + \int_0^t S(t-t')\mathbf{f}(t')\mathrm{d}t', \tag{2.1.1}$$

where $S(t)$ is the operator formally defined as

$$S(t) := \begin{pmatrix} \cos(t\Delta) & \frac{\sin(t\Delta)}{\Delta} \\ -\Delta\sin(t\Delta) & \cos(t\Delta) \end{pmatrix} \tag{2.1.2}$$

with the understanding that

$$\frac{\sin(t0)}{0} := t. \tag{2.1.3}$$

By the formula (SNLB), since we formally have $\mathbf{f} = -\begin{pmatrix} 0 \\ |u|^{p-1}u \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix}$, we expect the solution of (SNLB) to satisfy the *Duhamel* formulation

$$\mathbf{u} = S(t)\mathbf{u}_0 + \int_0^t S(t-t') \begin{pmatrix} 0 \\ \xi(t') \end{pmatrix} \mathrm{d}t' - \int_0^t S(t-t') \begin{pmatrix} 0 \\ |u|^{p-1}u(t') \end{pmatrix} \mathrm{d}t'. \tag{2.1.4}$$

We call the term

$$\Psi(t) := \int_0^t S(t-t') \begin{pmatrix} 0 \\ \xi(t') \end{pmatrix} \mathrm{d}t' \tag{2.1.5}$$

stochastic convolution. Notice that $S(t)$ maps the space $\mathcal{H}^\alpha := H^\alpha \times H^{\alpha-2}$ into itself for every s, and that $\partial_t^k S(t)$ maps $\mathcal{H}^\alpha$ into $\mathcal{H}^{\alpha-4k}$. Therefore, Ψ is well defined as a space-time distribution, through the formula

$$\langle \Psi, \mathbf{f} \rangle = \left\langle \int_{t'}^{+\infty} \pi_2 S(t-t')^* \mathbf{f}(t)\mathrm{d}t, \mathbb{1}_{t'>0}\xi \right\rangle_{t',x}, \tag{2.1.6}$$

which holds for every test function $\mathbf{f}$, where π_2 is the projection on the second component. We will explore more quantitative estimates about Ψ in the next section.

Moreover, instead of considering solutions to (2.1.4), it is more convenient to write (2.1.4) as $\mathbf{u} = \Psi + \mathbf{v}$, and look for solutions of the equation for $\mathbf{v} = \begin{pmatrix} v \\ v_t \end{pmatrix}$, namely

$$\mathbf{v} = S(t)\mathbf{u}_0 - \int_0^t S(t-t') \begin{pmatrix} 0 \\ |\psi + v|^{p-1}(\psi + v)(t') \end{pmatrix} \mathrm{d}t'. \tag{2.1.7}$$

Therefore, in the following, we call $\mathbf{u}$ *solution* of (SNLB) if $\mathbf{v} := \mathbf{u} - \Psi$ solves (2.1.7).

2.1.2 Stochastic beam equation with damping

As for the case without damping, we consider the linear damped beam equation with forcing $\mathbf{f} = \begin{pmatrix} f \\ f_t \end{pmatrix}$ and initial data $\mathbf{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$,

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 + \Delta^2 & 1 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} f \\ f_t \end{pmatrix}. \tag{2.1.8}$$

By variation of constants, the solution to this equation is given by

$$\mathbf{u} = S(t)\mathbf{u}_0 + \int_0^t S(t-t')\mathbf{f}(t')\mathrm{d}t', \tag{2.1.9}$$

where $S(t)$ is the operator formally defined as

$$e^{-\frac{t}{2}} \begin{pmatrix} \cos\left(t\sqrt{\frac{3}{4} + \Delta^2}\right) + \frac{1}{2}\frac{\sin\left(t\sqrt{\frac{3}{4}+\Delta^2}\right)}{\sqrt{\frac{3}{4}+\Delta^2}} & \frac{\sin\left(t\sqrt{\frac{3}{4}+\Delta^2}\right)}{\sqrt{\frac{3}{4}+\Delta^2}} \\ -\left(\sqrt{\frac{3}{4} + \Delta^2} - \frac{1}{4\sqrt{\frac{3}{4}+\Delta^2}}\right)\sin\left(t\sqrt{\frac{3}{4} + \Delta^2}\right) & \cos\left(t\sqrt{\frac{3}{4} + \Delta^2}\right) - \frac{1}{2}\frac{\sin\left(t\sqrt{\frac{3}{4}+\Delta^2}\right)}{\sqrt{\frac{3}{4}+\Delta^2}} \end{pmatrix}. \tag{2.1.10}$$

We note that this operator maps distributions to distributions, and for every $\alpha \in \mathbb{R}$, it maps the Sobolev space $\mathcal{H}^\alpha := H^\alpha \times H^{\alpha-2}$ into itself, with the estimate $\|S(t)\mathbf{u}\|_{\mathcal{H}^\alpha} \lesssim e^{-\frac{t}{2}} \|\mathbf{u}\|_{\mathcal{H}^\alpha}$, and as in the case without damping, $\partial_t^k S(t)$ maps $\mathcal{H}^\alpha$ to $\mathcal{H}^{\alpha-4k}$.

By the formula (2.1.9), we expect the solution of (SDNLB) to satisfy the Duhamel formulation

$$\mathbf{u} = S(t)\mathbf{u}_0 + \int_0^t S(t-t') \begin{pmatrix} 0 \\ \xi(t') \end{pmatrix} \mathrm{d}t' - \int_0^t S(t-t') \begin{pmatrix} 0 \\ u^3(t') \end{pmatrix} \mathrm{d}t'. \tag{2.1.11}$$

From the previous discussion about $S(t)$, we have that

$$\text{¡}_t(\xi) := \int_0^t S(t-t') \begin{pmatrix} 0 \\ \xi(t') \end{pmatrix} \mathrm{d}t' \tag{2.1.12}$$

is a well defined space-time distribution, in the same way as (2.1.6). In the following, when it is not ambiguous, we may omit the argument ξ (i.e. $\text{¡}_t := \text{¡}_t(\xi)$).

Moreover, it is convenient to consider $\text{¡}_t(\cdot)$ as an operator on the appropriate class of space-time distributions, defined by the formula

$$\text{¡}_t(h) := \int_0^t S(t-t') \begin{pmatrix} 0 \\ h(t') \end{pmatrix} \mathrm{d}t' \tag{2.1.13}$$

whenever h is a test function, and its appropriate extension for more general distributions.

Also in this case, instead of looking directly for solutions to (2.1.11), it is more convenient to look at the equation for one of the terms in (2.1.11). Since in this case we want to be able to deal with random initial data as well, it is more convenient to isolate the contribution of $\mathbf{u}_0$ as well as the one of ¡_t. In particular, looking at (2.1.11), we write $\mathbf{u} = S(t)\mathbf{u}_0 + \text{¡}_t(\xi) + \mathbf{v}$, so

$\mathbf{v} = \begin{pmatrix} v \\ v_t \end{pmatrix}$ satisfies

$$\mathbf{v} = -\int_0^t S(t-t') \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \text{¡}_{t'}(\xi) + v(t'))^3 \end{pmatrix} \mathrm{d}t', \tag{2.1.14}$$

where we abused of notation and wrote $S(t)\mathbf{u}_0 + \text{¡}_t(\xi)$ instead of its projection to the first component. Therefore, we call $\mathbf{u}$ *solution* of (SDNLB) if $\mathbf{v} := \mathbf{u} - S(t)\mathbf{u}_0 - \text{¡}_t(\xi)$ satisfies (2.1.14). Since we aim to study a flow for $\mathbf{u}$ which is defined for both smooth initial data and almost every initial data according to the measure ρ, it is important to keep track of the space to which the solution $\mathbf{u}$ is expected to belong. It turns out that a good space is

$$X^\alpha := \left\{ \mathbf{u} \in \mathcal{H}^\alpha | S(t)\mathbf{u} \in C([0,+\infty);\mathscr{C}^\alpha), \|S(t)\mathbf{u}\|_{\mathscr{C}^\alpha} \lesssim e^{-\frac{t}{8}} \right\},$$
$$\|\mathbf{u}\|_{X^\alpha} := \sup_{t>0} e^{\frac{t}{8}} \|S(t)\mathbf{u}\|_{\mathscr{C}^\alpha},$$

for $0 < \alpha < \frac{1}{2}$. Here $\mathscr{C}^\alpha := C^\alpha \times C^{\alpha-2}$. As it is defined, the space X^α might not be separable, which is a helpful hypothesis for some measure theoretical considerations in the following. In order to solve this issue, we will actually denote by X^α the closure of trigonometric polynomials in the X^α norm. Since we have, for $\alpha' > \alpha$,

$$\|\mathbf{u} - P_N\mathbf{u}\|_{X^\alpha} \lesssim N^{-\frac{\alpha'-\alpha}{2}} \|\mathbf{u}\|_{X^{\alpha'}},$$

we have that for every $\alpha' > \alpha$, if $\|\mathbf{u}\|_{X^{\alpha'}} < +\infty$, then $\mathbf{u} \in X^\alpha$.

Lemma 2.1.1. *X^α is a Banach space.*

Proof. $\|\cdot\|_{X^\alpha}$ is clearly a norm, so we just need to show completeness. Let $\mathbf{u}_n$ be a Cauchy sequence in X^α. By definition, for every t, $S(t)\mathbf{u}_n$ is a Cauchy sequence in $\mathscr{C}^\alpha$, so there exists a limit $S(t)\mathbf{u}_n \to \mathbf{u}(t)$ in $\mathscr{C}^\alpha$. Moreover, $S(t)$ is a bounded operator in $\mathcal{H}^\alpha$, so one has that

$$\mathbf{u}(t) = \mathscr{C}^\alpha - \lim S(t)\mathbf{u}_n = \mathcal{H}^\alpha - \lim S(t)\mathbf{u}_n = S(t)(\mathcal{H}^\alpha - \lim \mathbf{u}_n) = S(t)\mathbf{u}(0).$$

Lastly,

$$\begin{aligned}
\lim_n \|\mathbf{u}_n - \mathbf{u}\|_{X^\alpha} &= \lim_n \sup_t e^{\frac{t}{8}} \|S(t)\mathbf{u}_n - S(t)\mathbf{u}(0)\|_{\mathscr{C}^\alpha}, \\
&= \lim_n \sup_t \lim_m e^{\frac{t}{8}} \|S(t)\mathbf{u}_n - S(t)\mathbf{u}_m\|_{\mathscr{C}^\alpha}, \\
&\le \lim_n \lim_m \sup_t e^{\frac{t}{8}} \|S(t)\mathbf{u}_n - S(t)\mathbf{u}_m\|_{\mathscr{C}^\alpha}, \\
&= \lim_n \lim_m e^{\frac{t}{8}} \|S(t)\mathbf{u}_n - S(t)\mathbf{u}_m\|_{X^\alpha}, \\
&= 0.
\end{aligned}$$

□

Since the operator $S(t)$ is *not* bounded on $\mathscr{C}^\alpha$, the space X^α might appear mysterious. However, in the next sections, we will see that the term $\text{¡}_t(\xi)$ belongs to X^α, as well as almost every initial data according to ρ, i.e. $\rho(X^\alpha) = 1$. Moreover, we have the following embedding for smooth functions:

Lemma 2.1.2. *For every $0 < \alpha < \frac{1}{2}$, we have $\mathcal{H}^2 \subset X^\alpha$. Moreover, the identity* $\mathrm{id} : \mathcal{H}^2 \hookrightarrow X^\alpha$ *is a compact operator.*

Proof. Let $\mathbf{u} \in \mathcal{H}^2$. By Sobolev embeddings,

$$\|S(t)\mathbf{u}\|_{\mathscr{C}^\alpha} \lesssim \|S(t)\mathbf{u}\|_{\mathcal{H}^2} \lesssim e^{-\frac{t}{2}} \|\mathbf{u}\|_{\mathcal{H}^2}$$

and given $s \ge 0$, we have

$$\lim_{t\to s} \|S(t)\mathbf{u} - S(s)\mathbf{u}\|_{\mathscr{C}^\alpha} \lesssim \limsup_{t\to s} \|S(t)\mathbf{u} - S(s)\mathbf{u}\|_{\mathcal{H}^2} = 0,$$

hence $\mathbf{u} \in X^\alpha$.

Now let $\mathbf{u}_n$ be a bounded sequence in $\mathcal{H}^2$. By compactness of Sobolev embeddings, up to subsequences, $\mathbf{u}_n \to \mathbf{u}$ in $\mathscr{C}^\alpha$ and $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathcal{H}^s$ for every $s \leq 2$. Therefore, $S(t)\mathbf{u}_n \rightharpoonup S(t)\mathbf{u}$ weakly in $\mathcal{H}^s$ for every $t \geq 0$.

By a diagonal argument, up to subsequences, we have that $S(t)\mathbf{u}_n$ is a converging sequence in $\mathscr{C}^\alpha$ for every $t \in \mathbb{Q}^+$, so by coherence of the limits, $S(t)\mathbf{u}_n \to S(t)\mathbf{u}$ in $\mathscr{C}^\alpha$ for every $t \in \mathbb{Q}^+$. By the property

$$\partial_t S(t) = -\begin{pmatrix} 0 & -1 \\ 1+\Delta^2 & 1 \end{pmatrix} S(t),$$

we have that $\|S(t)\mathbf{u} - S(s)\mathbf{u}\|_{\mathcal{H}^s} \lesssim |t-s|^\varepsilon \|\mathbf{u}\|_{\mathcal{H}^{s+4\varepsilon}}$. Therefore, by taking ε such that $\alpha + 4\varepsilon + \frac{3}{2} < 2$, by the Sobolev embedding $\mathcal{H}^{2-4\varepsilon} \hookrightarrow \mathscr{C}^\alpha$, we have that $S(t)\mathbf{u}_n \to S(t)\mathbf{u}$ in $\mathscr{C}^\alpha$ for every $t \geq 0$ and uniformly on compact sets. Finally, for every T we have

$$e^{\frac{t}{8}} \|S(t)\mathbf{u}_n - S(t)\mathbf{u}\|_{\mathscr{C}^\alpha} \lesssim e^{\frac{T}{8}} \sup_{s\in[0,T]} \|S(s)\mathbf{u}_n - S(s)\mathbf{u}\|_{\mathscr{C}^\alpha} + e^{-\frac{3}{8}T} \sup_n \|\mathbf{u}_n\|_{\mathcal{H}^2}.$$

For $T \gg 1$ big enough and $n \gg 1$ (depending on T), we can make the right hand side arbitrarily small. Therefore, we get $\|\mathbf{u}_n - \mathbf{u}\|_{X^\alpha} \to 0$ as $n \to \infty$, so id is compact. ☐

However, the space X^α is strictly bigger than $\mathcal{H}^2$, and it contains functions at regularity exactly α. Indeed, we have

Lemma 2.1.3. *For every $\alpha_1 > \alpha > 0$, there exists $\mathbf{u}_0 \in X^\alpha$ such that $\mathbf{u}_0 \notin H^{\alpha_1}$.*

Proof. Suppose by contradiction that $X^\alpha \subseteq H^{\alpha_1}$. By the closed graph theorem, this implies that

$$\|\mathbf{u}\|_{\mathcal{H}^{\alpha_1}} \lesssim \|\mathbf{u}\|_{X^\alpha}. \tag{2.1.15}$$

For $n \in \mathbb{Z}^3$, consider $\mathbf{u}_n := \begin{pmatrix} e^{in\cdot x} \\ 0 \end{pmatrix}$. By definition of $S(t)$,

$$S(t)\mathbf{u}_n = e^{-\frac{t}{2}} \begin{pmatrix} \left(\cos\left(t\sqrt{\frac{3}{4}+|n|^4}\right) + \frac{1}{2}\frac{\sin\left(t\sqrt{\frac{3}{4}+|n|^4}\right)}{\sqrt{\frac{3}{4}+|n|^4}} \right) e^{in\cdot x} \\ \frac{\sin\left(t\sqrt{\frac{3}{4}+|n|^4}\right)}{\sqrt{\frac{3}{4}+|n|^4}} e^{in\cdot x} \end{pmatrix}.$$

It is easy to check that $\|S(t)\mathbf{u}_n\|_{\mathscr{C}^\alpha} \sim e^{-\frac{t}{2}} \langle n \rangle^\alpha$, so $\|S(t)\mathbf{u}_n\|_{X^\alpha} \sim \langle n \rangle^\alpha$. On the other hand, $\|\mathbf{u}_n\|_{\mathcal{H}^{\alpha_1}} \sim \langle n \rangle^{\alpha_1}$. By (2.1.15), this implies $\langle n \rangle^\alpha \gtrsim \langle n \rangle^{\alpha_1}$, contradiction. ☐

2.2 Stochastic convolution and invariant measure

2.2.1 Stochastic beam equation without damping

Consider

$$\Psi(t) = \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} := \begin{pmatrix} \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} \xi(t')\mathrm{d}t' \\ \int_0^t \cos((t-t')\Delta)\xi(t')\mathrm{d}t' \end{pmatrix}, \tag{2.2.1}$$

Proposition 2.2.1. *Let $\alpha < \frac{1}{2}$. Then, almost surely:*

$$\Psi \in C_t(\mathbb{R}_+; W^{\alpha,\infty}(\mathbb{T}^3) \times W^{\alpha-2,\infty}(\mathbb{T}^3)). \tag{2.2.2}$$

Proof. This will follow from Proposition (A.3.2), once we check the hypotheses both for ψ and ψ_t for $s = 2-\delta$, $s = -\delta$ respectively and $\theta = \theta(\delta) > 0$, and $0 < \delta < \frac{1}{2} - \alpha$. We have that $\Psi(0) = 0$, so (A.3.4) and (A.3.5) are automatically satisfied. For a test function ϕ, we have for $s \leq t$,

$$
\begin{aligned}
&\mathbb{E}|\left\langle \psi(t)-\psi(s),\phi\right\rangle|^2\\
&=\mathbb{E}\left|\int_0^t\left\langle \frac{\sin((t-t')\Delta)}{\Delta}\phi,\xi(t')\mathrm{d}t'\right\rangle-\int_0^s\left\langle \frac{\sin((s-t')\Delta)}{\Delta}\phi,\xi(t')\right\rangle\right|^2\\
&=\int_0^s\left\|\frac{\sin((t-t')\Delta)-\sin((s-t')\Delta)}{\Delta}\phi\right\|^2_{L^2(\mathbb{T}^3)}+\int_s^t\left\|\frac{\sin((t-t')\Delta)}{\Delta}\phi\right\|^2_{L^2(\mathbb{T}^3)}\\
&\lesssim s\left\||t-s|^{\frac{\delta}{2}}|\Delta|^{-1+\frac{\delta}{2}}\phi\right\|^2_{L^2}+|t-s|\,\|\phi\|^2_{H^{-2}}\\
&\lesssim s|t-s|^{\delta}\,\|\phi\|^2_{H^{-(2-\delta)}}\,,
\end{aligned}
$$

where we used respectively the universal property of white noise (1.0.2) and the inequality $|\sin(a\lambda)-\sin(b\lambda)|\le 2^{1-\eta}|a-b|^{\eta}\lambda^{\eta}$ for every $0\le\eta\le1$, $\lambda\ge0$. Hence, (A.3.4) is satisfied with $s=2-\delta$, $\theta=\frac{\delta}{2}$.

Similarly,

$$
\begin{aligned}
&\mathbb{E}|\left\langle \psi_t(t)-\psi_t(s),\phi\right\rangle|^2\\
&=\mathbb{E}\left|\int_0^t\left\langle \cos((t-t')\Delta)\phi,\xi(t')\mathrm{d}t'\right\rangle-\int_0^s\left\langle \cos((s-t')\Delta\phi,\xi(t')\mathrm{d}t'\right\rangle\right|^2\\
&=\int_0^s\|(\cos((t-t')\Delta)-\cos((s-t')\Delta))\phi\|^2_{L^2(\mathbb{T}^3)}+\int_s^t\|\cos((t-t')\Delta)\phi\|^2_{L^2(\mathbb{T}^3)}\\
&\lesssim s\left\||t-s|^{\frac{\delta}{2}}|\Delta|^{\frac{\delta}{2}}\phi\right\|^2_{L^2}+|t-s|\,\|\phi\|^2_{L^2}\\
&\lesssim s|t-s|^{\delta}\,\|\phi\|^2_{H^{\delta)}}\,,
\end{aligned}
$$

where we used again (1.0.2) and the inequality $|\cos(a\lambda)-\cos(b\lambda)|\le 2^{1-\eta}|a-b|^{\eta}\lambda^{\eta}$ for every $0\le\eta\le1$, $\lambda\ge0$. Hence, (A.3.4) is satisfied also in this case with $s=-\delta$, $\theta=\frac{\delta}{2}$.

Moreover, (A.3.5) is satisfied in both cases since $\langle\psi(t)-\psi(s),\phi\rangle$ and $\langle\psi_t(t)-\psi_t(s),\phi\rangle$ are Gaussians, and for every real-valued Gaussian g, $\mathbb{E}|g|^p=(p-1)!!\mathbb{E}(|g|^2)^{\frac{p}{2}}$ if p is an even integer. Interpolating between consecutive even integers, we get $\mathbb{E}|g|^p\le(p-1)^{\frac{p}{2}}\mathbb{E}(|g|^2)^{\frac{p}{2}}$ for a general $p\ge2$. Notice that this also a special case of (A.2.4). $\square$

2.2.2 Stochastic convolution for the beam equation with damping

Recall the formula for the stochastic convolution,

$$\upharpoonleft_t(\xi):=\int_0^t S(t-t')\begin{pmatrix}0\\ \xi(t')\end{pmatrix}\mathrm{d}t',$$

as in (2.1.12).

Proposition 2.2.2. *For every $\alpha<\frac{1}{2}$,*

$$\mathbb{E}\,\|\upharpoonleft_t(\xi)\|^2_{\mathscr{C}^{\alpha}}<+\infty.$$

Moreover, $\upharpoonleft_t(\xi)\in C([0,+\infty);\mathscr{C}^{\alpha})$ almost surely.

Proof. As in Proposition 2.2.1, the proof follows by applying Proposition A.3.2 to both components of $\upharpoonleft_t(\xi)$. It is easy to check that $\upharpoonleft_0(\xi)=0$, so we just need to check (A.3.6) and (A.3.7).

Let $0 \leq s \leq t$, and let $\mathbf{f} = \begin{pmatrix} f \\ f_t \end{pmatrix}$ be a test function. We have that

$$\begin{aligned}
\mathbb{E}[|\langle \text{ⵑ}_t(\xi) - \text{ⵑ}_s(\xi), \mathbf{f}\rangle|^2] &= \mathbb{E}\left|\int_0^t \langle \pi_2 S(t-t')^*\mathbf{f}, \xi(t')\rangle_{L^2_x} - \int_0^s \langle \pi_2 S(s-t')^*\mathbf{f}, \xi(t')\rangle_{L^2_x}\right|^2 \\
&= \int_0^s \|\pi_2(S(t-t') - S(s-t'))^*\mathbf{f}\|^2_{L^2} + \int_s^t \|\pi_2 S(t-t')^*\mathbf{f}\|^2_{L^2} \qquad (2.2.3) \\
&\lesssim s|t-s|^\delta(\|f\|^2_{H^{-2+2\delta}} + \|f_t\|^2_{H^{2\delta}}) + |t-s|(\|f\|^2_{H^{-2}} + \|f_t\|^2_{L^2}) \\
&\lesssim |t-s|^\delta(\|f\|^2_{H^{-2+2\delta}} + \|f_t\|^2_{H^{2\delta}}),
\end{aligned}$$

where we used the universal property of white noise (1.0.2) in the second equality, and the formula (2.1.10) and the inequality $|\sin(a\lambda) - \sin(b\lambda)|, |\cos(a\lambda) - \cos(b\lambda)| \leq 2^{1-\eta}|a-b|^\eta\lambda^\eta$ for every $0 \leq \eta \leq 1$, $\lambda \geq 0$. Moreover, since $g = \langle \text{ⵑ}_t(\xi) - \text{ⵑ}_s(\xi), \mathbf{f}\rangle$ is Gaussian, we have that $\mathbb{E}|g|^p \leq (p-1)^{\frac{p}{2}}\mathbb{E}(|g|^2)^{\frac{p}{2}}$. Therefore,

$$\begin{aligned}
\mathbb{E}[|\langle \pi_1\text{ⵑ}_t(\xi) - \pi_1\text{ⵑ}_s(\xi), f\rangle|^2] &\lesssim |t-s|^\delta(\|f\|^2_{H^{-2+2\delta}}), \\
\mathbb{E}[|\langle \pi_1\text{ⵑ}_t(\xi) - \pi_1\text{ⵑ}_s(\xi), f\rangle|^p] &\leq (p-1)^{\frac{p}{2}}\mathbb{E}[|\langle \pi_1\text{ⵑ}_t(\xi) - \pi_1\text{ⵑ}_s(\xi), f\rangle|^2]^{\frac{p}{2}},
\end{aligned}$$

so (A.3.6) and (A.3.7) are satisfied for $s = 2 - 2\delta$, $\theta = \frac{\delta}{2}$, and we have

$$\pi_1\text{ⵑ}_t(\xi) \in C([0,+\infty); C^{\frac{1}{2}-2\delta-\varepsilon}).$$

Similarly,

$$\begin{aligned}
\mathbb{E}[|\langle \pi_2\text{ⵑ}_t(\xi) - \pi_2\text{ⵑ}_s(\xi), f_t\rangle|^2] &\lesssim |t-s|^\delta(\|f\|^2_{H^{2\delta}}), \\
\mathbb{E}[|\langle \pi_2\text{ⵑ}_t(\xi) - \pi_2\text{ⵑ}_s(\xi), f_t\rangle|^p] &\leq (p-1)^{\frac{p}{2}}\mathbb{E}[|\langle \pi_2\text{ⵑ}_t(\xi) - \pi_2\text{ⵑ}_s(\xi), f\rangle|^2]^{\frac{p}{2}},
\end{aligned}$$

so (A.3.6) and (A.3.7) are satisfied for $s = -2\delta$, $\theta = \frac{\delta}{2}$, and we have

$$\pi_1\text{ⵑ}_t(\xi) \in C([0,+\infty); C^{-\frac{3}{2}-2\delta-\varepsilon}).$$

□

Proposition 2.2.3. *For every $t \geq 0$, $\|\text{ⵑ}_t(\xi)\|_{X^\alpha} < +\infty$ a.s.. More precisely,*

$$\sup_{s>0} \left\|e^{\frac{s}{8}} S(s)\text{ⵑ}_t(\xi)\right\|_{\mathscr{C}^\alpha} < +\infty \text{ a.s.}$$

for every $\alpha < \frac{1}{2}$.

Proof. Let us first fix $T \geq 1$, and take $s_1, s_2 \in [T-1, T], s_1 \leq s_2$. Let $\mathbf{f} = \begin{pmatrix} f \\ f_t \end{pmatrix}$ be a test function. Proceeding similarly to (2.2.3), we have that

$$\begin{aligned}
&\mathbb{E}[|\langle S(s_2)\text{ⵑ}_t(\xi) - S(s_1)\text{ⵑ}_t(\xi), \mathbf{f}\rangle|^2] \\
&= \mathbb{E}\left|\int_0^t \langle \pi_2 S(t+s_1-t')^*\mathbf{f}, \xi(t')\rangle_{L^2_x} - \int_0^t \langle \pi_2 S(t+s_2-t')^*\mathbf{f}, \xi(t')\rangle_{L^2_x}\right|^2 \\
&= \int_0^t \|\pi_2(S(t+s_1-t') - S(t+s_2-t'))^*\mathbf{f}\|^2_{L^2} \\
&\lesssim e^{-\frac{T}{2}}|s_1-s_2|^\delta(\|f\|^2_{H^{-2+2\delta}} + \|f_t\|^2_{H^{2\delta}}),
\end{aligned}$$

Moreover, by (2.2.3),

$$\begin{aligned}
\mathbb{E}[|\langle S(T-1)\text{ⵑ}_t(\xi), \mathbf{f}\rangle|^2] &= \mathbb{E}[|\langle \text{ⵑ}_t(\xi), S(T-1)^*\mathbf{f}\rangle|^2] \\
&\lesssim (\|\pi_1 S(T-1)^*\mathbf{f}\|^2_{H^{-2+2\delta}} + \|\pi_2 S(T-1)^*\mathbf{f}\|^2_{H^{2\delta}})
\end{aligned}$$

$$\lesssim e^{-\frac{T}{2}}(\|f\|^2_{H^{-2+2\delta}} + \|f_t\|^2_{H^{2\delta}})$$

Therefore, proceeding in the same way as in Proposition 2.2.2, we get that

$$S(s)\upharpoonleft_t(\xi) \in C_s([T\ -\ 1, T]; \mathscr{C}^\alpha),$$

and by the moment bound in Proposition A.3.2, for p big enough we have

$$\mathbb{E}[\|S(s)\upharpoonleft_t(\xi)\|^p_{C_s([T-1,T];\mathscr{C}^\alpha)}]^{\frac{1}{p}} \lesssim e^{-\frac{T}{2}},$$

hence

$$\mathbb{P}(\|S(s)\upharpoonleft_t(\xi)\|_{C_s([T-1,T];\mathscr{C}^\alpha)} > e^{-\frac{T}{8}}) \lesssim e^{-\frac{T}{4}}.$$

Since this is summable for $T \in \mathbb{N}$, by Borel-Cantelli we have that

$$\limsup_{N\in\mathbb{N}} e^{\frac{N}{8}} \ \|S(s)\upharpoonleft_t(\xi)\|_{C_s([N-1,N];\mathscr{C}^\alpha)} \leq 1,$$

and since $S(s)\upharpoonleft_t(\xi) \in C_s([0, +\infty); \mathscr{C}^\alpha)$, we have that

$$\sup_{N\in\mathbb{N}} e^{\frac{N}{8}} \ \|S(s)\upharpoonleft_t(\xi)\|_{C_s([N-1,N];\mathscr{C}^\alpha)} < +\infty,$$

hence $\|\upharpoonleft_t(\xi)\|_{X^\alpha} < +\infty$. □

2.2.3 Invariant measure

By the formula (1.1.6), we want to define the (candidate) invariant measure ρ as a measure which is absolutely continuous with respect to μ, such that its Radon-Nikodym derivative with respect to μ (given by (1.1.7)) is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}\mu}(\mathbf{u}) = \frac{1}{Z}\exp\Big(-\frac{1}{4}\int u^4\Big).$$

In order to make sense of this definition, we need the function $\exp\Big(-\frac{1}{4}\int u^4\Big)$ to belong to the space $L^1(\mu)$. If $u \in L^4(\mathbb{T}^3)$ μ-a.s., this is automatic, since if this is the case $0 \leq \int u^4 < +\infty$ μ-a.s., hence $\exp\Big(-\frac{1}{4}\int u^4\Big) \in L^\infty(\mu) \subset L^1(\mu)$.

Since one has $X^\alpha \subset L^4$ for every $\alpha > 0$, the property $u \in L^4(\mathbb{T}^3)$ follows from the following

Proposition 2.2.4. *Let* $\mathbf{X}$ *be a random variable with law* μ. *Then for every* $0 < \alpha < \frac{1}{2}$, $\|\mathbf{X}\|_{X^\alpha} < +\infty$ *a.s..*

Proof. This proof is very similar to the one of Proposition 2.2.3. Let $T \geq 1$, and let $T-1 \leq s \leq t \leq T$. By (1.1.7) and (2.1.10), for a test function $\mathbf{f} = \begin{pmatrix} f \\ f_t \end{pmatrix}$,

$$\begin{aligned}\mathbb{E}|\langle S(T-1)\mathbf{X}, \mathbf{f}\rangle|^2 &= \mathbb{E}|\langle \mathbf{X}, S(T-1)^*\mathbf{f}\rangle|^2 \\ &\lesssim \|\pi_1 S(T-1)^*\mathbf{f}\|^2_{H^{-2}} + \|\pi_2 S(T-1)^*\mathbf{f}\|^2_{L^2} \\ &\lesssim e^{-\frac{T}{2}}\big(\|f\|^2_{H^{-2+4\delta}} + \|f_t\|^2_{H^{4\delta}}\big),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|\langle (S(t)-S(s))\mathbf{X}, \mathbf{f}\rangle|^2 &= \mathbb{E}|\langle \mathbf{X}, (S(t)-S(s))^*\mathbf{f}\rangle|^2 \\ &\lesssim \|\pi_1(S(t)-S(s))^*\mathbf{f}\|^2_{H^{-2}} + \|\pi_2(S(t)-S(s))^*\mathbf{f}\|^2_{L^2} \\ &\lesssim e^{-\frac{T}{2}}|t-s|^\delta\big(\|f\|^2_{H^{-2+4\delta}} + \|f_t\|^2_{H^{4\delta}}\big).\end{aligned}$$

Therefore, the hypotheses of Proposition (A.3.2) are satisfied for $\pi_1 S(t)\mathbf{X}$ and $\pi_2 S(t)\mathbf{X}$, respectively with $s = 2 - 4\delta, \theta = \frac{\delta}{2}$, and $s = -4\delta$, $\theta = \frac{\delta}{2}$. Therefore, by choosing $0 < 4\delta < \frac{1}{2} - \alpha$,

$$S(t)\mathbf{X} \in C_t([T-1, T]; \mathscr{C}^\alpha),$$

with the moment bound

$$\mathbb{E}[\|S(s)\mathbf{X}\|^p_{C_s([T-1,T];\mathscr{C}^\alpha)}]^{\frac{1}{p}} \lesssim e^{-\frac{T}{2}}.$$

Hence we have

$$\mathbb{P}(\|S(s)\mathbf{X}\|_{C_s([T-1,T];\mathscr{C}^\alpha)} > e^{-\frac{T}{8}}) \lesssim e^{-\frac{T}{4}},$$

and by Borel-Cantelli

$$\limsup_{N\in\mathbb{N}} e^{\frac{N}{8}} \|S(s)\mathbf{X}\|_{C_s([N-1,N];\mathscr{C}^\alpha)} \leq 1,$$

so

$$\sup_{N\in\mathbb{N}} e^{\frac{N}{8}} \|S(s)\mathbf{X}\|_{C_s([N-1,N];\mathscr{C}^\alpha)} < +\infty$$

almost surely. Therefore, $\mathbf{X} \in X^\alpha$. □

2.3 Local in time theory and blowup conditions

This section is dedicated to showing local existence (and uniqueness) of solutions of (2.1.7) and (2.1.14), and describe under which conditions a blowup for solutions of these equations might occur. Given the similarity of two equations, we can present a unified proof. Consider the auxiliary equation

$$\mathbf{v}(t) = F(t) - \int_{t_0}^t S(t-t') \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t') + v(t')) \end{pmatrix} \mathrm{d}t'. \tag{2.3.1}$$

This corresponds to (2.1.7) when we make the choice $F(t) = S(t)\mathbf{u}_0$, $G(t) = \psi$, and in the case $p = 3$, it corresponds to (2.1.14) if we choose $F(t) = 0$, $G(t) = \pi_1(S(t)\mathbf{u}_0 + \mathbf{\dagger}_t(\xi))$. Notice that the meaning of $S(\cdot)$ differs for the two equations, but we will just use the property that $S(t)$ is a uniformly bounded operator from $\mathcal{H}^2$ to $\mathcal{H}^2$.

Remark 2.3.1. Suppose that $\mathbf{v}$ solves (2.3.1) in some interval $I = [t_0, t_0 + \tau]$. Let $t_1 \in I$. Then we have, for every $t \geq t_1$,

$$\begin{aligned}
\mathbf{v}(t) &= F(t) - \int_{t_0}^t S(t-t') \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t') + v(t')) \end{pmatrix} \mathrm{d}t' \\
&= F(t) - \int_{t_0}^{t_1} S(t-t') \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t') + v(t')) \end{pmatrix} \mathrm{d}t' \\
&\quad - \int_{t_1}^t S(t-t') \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t') + v(t')) \end{pmatrix} \mathrm{d}t', \\
&= F(t) - F(t_1) + \mathbf{v}(t_1) - \int_{t_1}^t S(t-t') \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t') + v(t')) \end{pmatrix} \mathrm{d}t'.
\end{aligned}$$

which is an equation in the same form as (2.3.1), with t_1 instead of t_0, and $F(t) - F(t_1) + \mathbf{v}(t_1)$ instead of $F(t)$ (G stays the same).

Remark 2.3.2. Let $p \geq 1$. By the fundamental theorem of calculus,

$$|x|^{p-1}x - |y|^{p-1}y = p(x-y)\int_0^1 |y + s(x-y)|^{p-1}\mathrm{d}s \lesssim (|x|^{p-1} + |y|^{p-1})|x-y|. \tag{2.3.2}$$

Proposition 2.3.3. *Let $p \geq 1$, $0 < \alpha < \frac{1}{2}$, and suppose that $F \in C_t([t_0, t_0+1]; \mathcal{H}^2)$, $G \in C_t([t_0, t_0+1]; \mathscr{C}^\alpha)$, and that $S(t)$ is a bounded operator from $\mathcal{H}^2$ to $\mathcal{H}^2$, uniformly in t. Then (2.3.1) is locally well posed in $\mathcal{H}^2$. More precisely, there exists $T > 0$ such that there exists a unique $\mathbf{v} \in C_t([t_0, t_0+T]; \mathcal{H}^2)$ which solves (2.3.1) $\forall t_0 \leq t \leq t_0 + T$. Moreover, T can* *be chosen as function*

$$T = T\big(\,\|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}\,,\|G\|_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)}\,,\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\,\big)$$

which is nondecreasing in each of its argument.

We also have the following continuity property of the solution $\mathbf{v}$ *in the arguments* F, G *and* S*. Suppose that* $F_n \to F$ *in* $C_t([t_0,t_0+1];\mathcal{H}^2)$*,* $G_n \to G$ *in* $C_t([t_0,t_0+1];\mathscr{C}^\alpha)$*, and* $S_n(t) \to S(t)$ *strongly for every* t*. Let*

$$T_0 = T\big(\sup_n \|F_n\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}\,,\sup_n \|G_n\|_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)}\,,\sup_n\sup_t \|S_n(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\,\big). \tag{2.3.3}$$

Let $\mathbf{v}_n$ *be the solution with data* F_n, G_n *and* S_n*, and let* $\mathbf{v}$ *be the solution with data* F, G, S*. Then we have*

$$\lim_n \|\mathbf{v}-\mathbf{v}_n\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} = 0.$$

Proof. Let $R \geq \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}$. We want to show that there exists T monotone in its argument such that for every τ with

$$\tau \leq T(R, \|G\|_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)}\,,\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}) \leq 1,$$

the map

$$\Gamma(\mathbf{v}) = \Gamma_{F,G,S}(\mathbf{v}) := F(t) - \int_{t_0}^t S(t-t')\begin{pmatrix}0\\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\mathrm{d}t'$$

is a contraction in the set

$$E := \{\mathbf{v}\in C_t([t_0,t_0+\tau]:\mathcal{H}^2) : \|\mathbf{v}\|_{C_t([t_0,t_0+\tau];\mathcal{H}^2)} \leq R+1\}.$$

Existence and uniqueness then follow by Banach contraction mapping theorem, and a straightforward application of Remark 2.3.1 to show uniqueness up to time

$$T\big(\,\|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}\,,\|G\|_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)}\,,\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\,\big).$$

We have that, for $t \leq \tau \leq T \leq 1$, for some constants C_0, by the hypotheses on F, G, S and Sobolev embeddings,

$$\begin{aligned}
&\|\Gamma(\mathbf{v})(t)\|_{\mathcal{H}^2}\\
&\leq \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} + T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}\left\|\begin{pmatrix}0\\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\right\|_{\mathcal{H}^2}\\
&= \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} + T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}\big\||G(t')+v(t')|^{p-1}(G(t')+v(t'))\big\|_{L^2}\\
&\leq \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} + 2^{p-1}T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}(\|G\|_{L^p}^p + \|v\|_{L^p}^p)\\
&\leq \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} + C_1T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}(\|G\|^p_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + \|v\|^p_{C([t_0-T,t_0+T];H^2)})\\
&\leq \|F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} + C_1T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}(\|G\|^p_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + (R+1)^p)
\end{aligned} \tag{2.3.4}$$

and proceeding similarly, by (2.3.2),

$$\begin{aligned}
&\|\Gamma(\mathbf{v})(t)-\Gamma(\mathbf{w})(t)\|_{\mathcal{H}^2}\\
&\leq T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}\left\|\begin{pmatrix}0\\ |G(t')+v(t')|^{p-1}(G(t')+v(t')) - G(t')+w(t')|^{p-1}(G(t')+w(t'))\end{pmatrix}\right\|_{\mathcal{H}^2}\\
&= T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}\big\||G(t')+v(t')|^{p-1}(G(t')+v(t')) - G(t')+w(t')|^{p-1}(G(t')+w(t'))\big\|_{L^2}\\
&\leq C_1T\sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\sup_{t'}(\|G\|^{p-1}_{L^p} + \|v\|^{p-1}_{L^p} + \|w\|^{p-1}_{L^p})\,\|v-w\|_{L^p}
\end{aligned}$$

$$\le C_2 T \sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \left(\|G\|^{p-1}_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + \|v\|^{p-1}_{C([t_0-T,t_0+T];H^2)} + \|w\|^{p-1}_{\mathcal{H}^2}\right) \times \|v-w\|_{C([t_0-T,t_0+T];H^2)}$$

$$\le C_2 T \sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \left(\|G\|^{p-1}_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + 2(R+1)^{p-1}\right) \|\mathbf{v}-\mathbf{w}\|_{C([t_0-T,t_0+T];H^2)}\,. \tag{2.3.5}$$

Therefore, if

$$T = \min\Big\{1, \Big(C_1 \sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \left(\|G\|^{p}_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + (R+1)^p\right)\Big)^{-1}, \frac{1}{2}\Big(C_2 \sup_t \|S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \left(\|G\|^{p-1}_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + 2(R+1)^{p-1}\right)\Big)^{-1}\Big\},$$

by (2.3.4), we get that Γ maps S to S, and by (2.3.5), Γ has Lipschitz constant $\le \frac{1}{2}$, hence Γ is a contraction. Moreover, it is easy to see that this definition of T is monotone in its arguments.

We know move to the continuity part of the statement. We first notice that by the first part of the statement, the solutions $\mathbf{v}_n, \mathbf{v}$ are well defined in the interval, and by choosing $R = \sup_n \sup_n \|F_n\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}$, for every n, Γ_{F_n,G_n,S_n} is a contraction on E with Lipschitz constant $\frac{1}{2}$ with $\tau = T_0$. The same holds for $\Gamma_{F,G,S}$. Therefore, recalling that $\mathbf{v}_n$ is the unique fixed point for Γ_{F_n,G_n,S_n},

$$\begin{aligned}\|\mathbf{v}-\mathbf{v}_n\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} &\le 2\left\|\Gamma_{F_n,G_n,S_n}(\mathbf{v}) - \mathbf{v}\right\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} \\ &= 2\left\|\Gamma_{F_n,G_n,S_n}(\mathbf{v}) - \Gamma_{F,G,S}(\mathbf{v})\right\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)}\,,\end{aligned}$$

so it is enough to show that for every $\mathbf{v} \in C_t([t_0, t_0+T_0];\mathcal{H}^2)$,

$$\left\|\Gamma_{F_n,G_n,S_n}(\mathbf{v}) - \Gamma_{F,G,S}(\mathbf{v})\right\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} \;\to\; 0$$

as $n\to\infty$. We have that

$$\begin{aligned}
&\left\|\Gamma_{F_n,G_n,S_n}(\mathbf{v}) - \Gamma_{F,G,S}(\mathbf{v})\right\|_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} \\
&\le \|F_n - F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} \\
&\quad + \int_{t_0}^{t_0+T_0} \Bigg\| S_n(t-t')\begin{pmatrix} 0 \\ |G_n(t')+v(t')|^{p-1}(G_n(t')+v(t'))\end{pmatrix} \\
&\qquad\qquad - S(t-t')\begin{pmatrix} 0 \\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\Bigg\|_{\mathcal{H}^2} \mathrm{d}t' \\
&\le \|F_n - F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} \\
&\quad + \int_{t_0}^{t_0+T_0} \Bigg\| S_n(t-t')\begin{pmatrix} 0 \\ |G_n(t')+v(t')|^{p-1}(G_n(t')+v(t'))\end{pmatrix} \\
&\qquad\qquad - S_n(t-t')\begin{pmatrix} 0 \\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\Bigg\|_{\mathcal{H}^2} \mathrm{d}t' \\
&\quad + \int_{t_0}^{t_0+T_0} \Bigg\| S_n(t-t')\begin{pmatrix} 0 \\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix} \\
&\qquad\qquad - S(t-t')\begin{pmatrix} 0 \\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\Bigg\|_{\mathcal{H}^2} \mathrm{d}t' \\
&\le \|F_n - F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)} \\
&\quad + \sup_n \sup_t \|S_n(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \int_{t_0}^{t_0+T_0} \Big\| |G_n(t')+v(t')|^{p-1}(G_n(t')+v(t')) \\
&\qquad\qquad - |G(t')+v(t')|^{p-1}(G(t')+v(t'))\Big\|_{L^2} \mathrm{d}t' \\
&\quad + \int_{t_0}^{t_0+T_0} \Bigg\| (S_n(t-t') - S(t-t'))\begin{pmatrix} 0 \\ |G(t')+v(t')|^{p-1}(G(t')+v(t'))\end{pmatrix}\Bigg\|_{\mathcal{H}^2} \mathrm{d}t' \\
&\lesssim \|F_n - F\|_{C_t([t_0,t_0+1];\mathcal{H}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_n \sup_t \|S_n(t)\|_{\mathcal{H}^2\to\mathcal{H}^2} \\
&\quad \times \big(\sup_n \|G_n\|^{p-1}_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} + \|\mathbf{v}\|^{p-1}_{C_t([t_0,t_0+T_0];\mathcal{H}^2)} \big) \|G - G_n\|_{C_t([t_0,t_0+1];\mathscr{C}^\alpha)} \\
&+ \int_{t_0}^{t_0+T_0} \left\| (S_n(t-t') - S(t-t')) \begin{pmatrix} 0 \\ |G(t') + v(t')|^{p-1}(G(t')+v(t')) \end{pmatrix} \right\|_{\mathcal{H}^2} \mathrm{d}t'
\end{aligned}
$$

We have that the first term is converging to 0 by our hypothesis on $F_n \to F$, and the second is converging to 0 by the hypothesis on $G_n \to G$. The third term is converging to 0 by dominated convergence (by $\big\||G(t') + v(t')|^{p-1}(G(t')+v(t'))\big\|_{L^2}$), once we notice that the integrand is pointwise converging to 0 due to the strong convergence of S_n to S for every t. □

Corollary 2.3.4 (Blowup condition for (SNLB))**.** *For $N > 0$, consider the auxiliary equation*

$$
\mathbf{v}_N(t) = P_N S(t)\mathbf{u}_0 - \int_0^t S(t-t') P_N \begin{pmatrix} 0 \\ |P_N\psi + v_N|^{p-1}(P_N\psi + v_N)(t') \end{pmatrix} \mathrm{d}t'. \tag{2.3.6}
$$

Let $\overline{T} > 0$, and suppose that for N big enough, (2.3.6) *admits a solution $\mathbf{v}_N \in C_t([0,\overline{T}];\mathcal{H}^2)$, and that*

$$
\limsup_N \|\mathbf{v}_N\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \le M. \tag{2.3.7}
$$

Then (2.1.7) *admits a (unique) solution $\mathbf{v} \in C_t([0,\overline{T}];\mathcal{H}^2)$, which satisfies*

$$
\|\mathbf{v}\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \le M. \tag{2.3.8}
$$

Moreover, $\|\mathbf{v} - \mathbf{v}_N\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \to 0$ as $N \to \infty$.

Proof. Since $\mathbf{u}_0 \in \mathcal{H}^2$ and $S(t)$ is uniformly bounded in $\mathcal{H}^2$, we have that $P_N S(t)\mathbf{u}_0 \to S(t)\mathbf{u}_0$ in $C_t([0,\overline{T}+1];\mathcal{H}^2)$. Similarly, given $0 < \alpha < \frac{1}{2}$, by (2.2.1) we have that $\psi \in C_t([0,\overline{T}+1];C^{\alpha'})$ for every $\alpha < \alpha' < \frac{1}{2}$, and so $P_N\psi \to \psi$ in $C_t([0,\overline{T}+1];C^\alpha)$. Moreover, since $P_N \to \mathrm{id}$ strongly as operators in $\mathcal{H}^2$, $S(t) \to S(t)P_N$ strongly. Define

$$
T_* := T\big(M + 2\sup_n \|P_N S(t)\mathbf{u}_0\|_{C_t([0,\overline{T}+1];\mathcal{H}^2)} + 1, \sup_n \|P_N\psi\|_{C_t([0,\overline{T}+1];\mathscr{C}^\alpha)}, \sup_n \sup_t \|P_N S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\big).
$$

We have that $T_* \le T_0$ defined in (2.3.3), therefore by Proposition (2.3.3), there exists a unique solution to (SNLB) in $C_t([0,T_*];\mathcal{H}^2)$ and $\lim_N \|\mathbf{v} - \mathbf{v}_N\|_{C_t([0,T_*];\mathcal{H}^2)} = 0$. In particular, we have $\|\mathbf{v}\|_{C_t([0,T_*];\mathcal{H}^2)} \le M$. Define

$$
T^* := \sup \left\{ \tau \,\middle|\, \begin{array}{l} \exists \mathbf{v} \in C_t([0,\tau];\mathcal{H}^2) \text{ solution of (2.1.7)}, \\ \exists \mathbf{v}_N \in C_t([0,\tau];\mathcal{H}^2) \text{ solution of (2.3.6) for every } N \text{ big enough}, \\ \text{and } \lim\limits_N \|\mathbf{v} - \mathbf{v}_N\|_{C_t([0,T^*];\mathcal{H}^2)} = 0. \end{array} \right\}.
$$

In order to conclude the proof, it is enough to show that $T^* > \overline{T}$. By the previous discussion, we know that $T^* \ge T_*$. Suppose by contradiction that $T^* \le \overline{T}$. Let $t_1 := T^* - \frac{T_*}{2}$. By Remark 2.3.1, $\mathbf{v}_N$ solves

$$
\mathbf{v}_N(t) = P_N S(t)\mathbf{u}_0 - P_N S(t_1)\mathbf{u}_0 + \mathbf{v}_N(t_1) - \int_{t_1}^t S(t-t') P_N \begin{pmatrix} 0 \\ |P_N\psi + v_N|^{p-1}(P_N\psi + v_N)(t') \end{pmatrix} \mathrm{d}t',
$$

and $\mathbf{v}$ solves

$$
\mathbf{v}(t) = S(t)\mathbf{u}_0 - S(t_1)\mathbf{u}_0 + \mathbf{v}(t_1) - \int_{t_1}^t S(t-t') \begin{pmatrix} 0 \\ |\psi + v_N|^{p-1}(\psi + v_N)(t') \end{pmatrix} \mathrm{d}t'.
$$

Since $t_1 < T^*$, by definition of T^* one has that $\mathbf{v}_N(t_1) \to \mathbf{v}(t_1)$ in $\mathcal{H}^2$. Recall that we already shown convergence of $P_N S(t)\mathbf{u}_0$ and $P_N\psi$. Therefore, by definition of T_*, by Proposition 2.3.3

we obtain that $\mathbf{v}_N, \mathbf{v}$ admit a unique solution in $C_t([t_1, t_1 + T_*]; \mathcal{H}^2)$ and

$$\lim_N \|\mathbf{v} - \mathbf{v}_N\|_{C_t([t_1,t_1+T_*];\mathcal{H}^2)} = 0.$$

Since the same convergence holds in $[0, t_1]$ and $t_1+T_* = T^*+\frac{T_*}{2} > T^*$, we obtain a contradiction. $\square$

Corollary 2.3.5 (Blowup condition for (SDNLB))**.** *For $N > 0$, consider the auxiliary equation*

$$\mathbf{v}_N(t) = -\int_0^t S(t-t')P_N \begin{pmatrix} 0 \\ (\pi_1(P_N \mathord{\upharpoonleft}_{t'}(\xi) + P_N S(t')\mathbf{u}_0) + v_N(t'))^3 \end{pmatrix} \mathrm{d}t'. \tag{2.3.9}$$

Let $\overline{T} > 0$, and suppose that for N big enough, (2.3.6) *admits a solution $\mathbf{v}_N \in C_t([0, \overline{T}]; \mathcal{H}^2)$, and that*

$$\limsup_N \|\mathbf{v}_N\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \le M. \tag{2.3.10}$$

Then (2.1.14) *admits a (unique) solution $\mathbf{v} \in C_t([0, \overline{T}]; \mathcal{H}^2)$, which satisfies*

$$\|\mathbf{v}\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \le M. \tag{2.3.11}$$

Moreover, $\|\mathbf{v} - \mathbf{v}_N\|_{C_t([0,\overline{T}];\mathcal{H}^2)} \to 0$ as $N \to \infty$.

Proof. The proof is extremely similar to the one of Corollary 2.3.4. By Proposition 2.2.2 and the definition of the space X^α, we have that $\pi_1(P_N \mathord{\upharpoonleft}_{t'}(\xi) + P_N S(t')\mathbf{u}_0) \to \pi_1(\mathord{\upharpoonleft}_{t'}(\xi) + S(t')\mathbf{u}_0)$ in $C_t([0, +\infty); C^{\alpha'})$ for every $0 < \alpha' < \alpha$. Moreover, $P_N S(t) \to S(t)$ strongly as operators in $\mathcal{H}^2$. Hence, we define

$$T_* := T\big(M+1, \sup_n \|\pi_1(P_N \mathord{\upharpoonleft}_{t'}(\xi) + P_N S(t')\mathbf{u}_0)\|_{C_t([0,\overline{T}+1];\mathscr{C}^{\alpha'})}, \sup_n \sup_t \|P_N S(t)\|_{\mathcal{H}^2\to\mathcal{H}^2}\big),$$

and we conclude exactly as in Corollary 2.3.4. $\square$

2.4 Global well posedenss and long-time estimates

2.4.1 Stochastic beam equation without damping

Proposition 2.4.1. *Let $\mathbf{u}_0 \in \mathcal{H}^2$, let $p < \frac{11}{3}$, and let $\mathbf{v}$ be the solution of* (2.1.7) *with initial data $\mathbf{u}_0$. Then there exists $\alpha = \alpha(p), \beta = \beta(p), \gamma = \gamma(p) > 0$ such that we have, for every $0 \le t \le T$,*

$$\|\mathbf{v}(t)\|_{\mathcal{H}^2} \lesssim 1 + \|\mathbf{u}_0\|_{\mathcal{H}^2}^\alpha + \|\psi\|_{C([0,T];\mathcal{H}^2)}^\beta + t^\gamma \tag{2.4.1}$$

Proof. By Proposition 2.3.4, it is enough to show the analogous statement for a solution $\mathbf{v}_N$ of (2.3.6). For ease of notation, we omit the subscript N in the following, and just write $\mathbf{v}$. From the formula (2.3.6) and the fact that

$$\partial_t S(t) P_N = -\begin{pmatrix} 0 & -1 \\ 1+\Delta^2 & 0 \end{pmatrix} P_N$$

is a bounded operator in $\mathcal{H}^2$, we obtain that $\mathbf{v}(t)$ is Fréchet differentiable as a $\mathcal{H}^2$-valued function. We consider the energy functional

$$E(\mathbf{v}) = \frac{1}{2}\int (\Delta v)^2 + \frac{1}{2}\int v_t^2 + \frac{1}{p+1}\int |v|^{p+1}. \tag{2.4.2}$$

Notice that, by the Sobolev embedding $\mathcal{H}^2 \hookrightarrow L^\infty$, $\|\mathbf{v}(t)\|_{\mathcal{H}^2}^2 \lesssim 1 + E(\mathbf{v}) \lesssim 1 + \|\mathbf{v}(t)\|_{\mathcal{H}^2}^{p+1}$.

Therefore, it is enough to show (2.4.1) for the functional E. By (2.3.6), we obtain

$$
\begin{aligned}
\partial_t E(\mathbf{v}) &= \int_{\mathbb{T}^3} v_t\big(\partial_t^2 v + \Delta^2 v + |v|^{p-1}v\big) \\
&= \int_{\mathbb{T}^3} v_t\big(|v|^{p-1}v - |v+P_N\psi|^{p-1}(v+P_N\psi)\big).
\end{aligned}
\tag{2.4.3}
$$

First, we treat the case $1 \le p < 3$. By Cauchy-Schwartz inequality, and (2.3.2), and using that $2p-2 < p+1$, we have for every $t \le T$,

$$
\begin{aligned}
\partial_t E(\mathbf{v}(t)) &\le \left(\int_{\mathbb{T}^3} v_t^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^3} \big(|v+P_N\psi|^{p-1}(v+P_N\psi) - |v|^{p-1}v\big)^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{T}^3} v_t^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^3} |P_N\psi|^2\big(|v|^{2p-2} + |P_N\psi|^{2p-2}\big)\right)^{\frac{1}{2}} \\
&\lesssim \|P_N\psi\|_{L^\infty([0,T]\times\mathbb{T}^3)} \left(\int_{\mathbb{T}^3} v_t^2\right)^{\frac{1}{2}} \left(\int |v|^{p+1}\right)^{\frac{1}{2}\frac{2p-2}{p+1}} \\
&\lesssim \|P_N\psi\|_{L^\infty([0,T]\times\mathbb{T}^3)} E(\mathbf{v}(t))^{\frac{1}{2}\left(1+\frac{2p-2}{p+1}\right)}
\end{aligned}
\tag{2.4.4}
$$

By Gronwall inequality, we obtain

$$
E(\mathbf{v}(t)) \lesssim \left(E(\mathbf{v}(0)) + \|P_N\psi\|_{L^\infty([0,T]\times\mathbb{T}^3)}\, t\right)^{\frac{2p+2}{3-p}}, \tag{2.4.5}
$$

for all $t \in [0,T]$, and recalling $E(\mathbf{v}(0)) \lesssim 1 + \|\mathbf{u}_0(t)\|_{\mathcal{H}^2}^{p+1}$, (2.4.1) follows.

Next, assume $3 \le p < \frac{11}{3}$. By Taylor's reminder theorem, we have

$$
\begin{aligned}
|v+P_N\psi|^{p-1}(v+P_N\psi) - |v|^{p-1}v &= p|v|^{p-1}P_N\psi \\
&\quad + \int_0^1 \frac{p(p-1)}{2}|v+\theta P_N\psi|^{p-3}(v+\theta P_N\psi)(1-\theta)^2 P_N\psi^2 d\theta.
\end{aligned}
\tag{2.4.6}
$$

Therefore, from (2.4.3) we have

$$
\begin{aligned}
\partial_t E(v) &= -p\int_{\mathbb{T}^3} v_t|v|^{p-1}P_N\psi - \frac{p(p-1)}{2}\int_0^1\int_{\mathbb{T}^3} v_t\big(|v+\theta P_N\psi|^{p-3}(v+\theta P_N\psi)\big)(1-\theta)^2 P_N\psi^2 dx d\theta \\
&:= \mathrm{I} + \int_0^1 \mathrm{II}_\theta d\theta.
\end{aligned}
\tag{2.4.7}
$$

We first estimate the term II_θ. By Hölder's inequality, we have, independently of $\theta \in [0,1]$,

$$
\mathrm{II}_\theta \lesssim \left(\int_{\mathbb{T}^3} v_t^2 dx\right)^{\frac{1}{2}} \left(\|P_N\psi(t)\|_{L_x^\infty}^4 \int_{\mathbb{T}^3} |v|^{2(p-2)} + \|P_N\psi(t)\|_{L_x^{2p}}^{2p}\right)^{\frac{1}{2}}
$$

Therefore, noticing that $2(p-2) < p+1$, we have for some $0 < \theta = \theta(p) < 1$ and some $b > 0$, by Young's inequality

$$
\int_0^1 \mathrm{II}_\theta d\theta \lesssim 1 + E^{1-\theta}(\mathbf{v}(t)) + \|P_N\psi\|_{L_x^\infty}^b. \tag{2.4.8}
$$

Now we turn to term I. Since $pv_t|v|^{p-1} = \partial_t\big(v|v|^{p-1}\big)$, integrating by parts we have

$$
\int_0^t \mathrm{I}\, dt' = \int_{\mathbb{T}^3} |v|^{p-1}vP_N\psi dx\Big|_0^t - \int_0^t\int_{\mathbb{T}^3} |v|^{p-1}v(P_N\psi_t)dxdt'
$$

$$= \int_{\mathbb{T}^3} |v(t)|^{p-1} v(t) P_N \psi(t) dx - \int_{\mathbb{T}^3} |v(0)|^{p-1} v(0) P_N \psi(0) dx - \int_0^t \int_{\mathbb{T}^3} |v|^{p-1} v (P_N \psi_t) dx dt'$$
$$\leq \eta \Big(\|v(t)\|_{L_x^{p+1}}^{p+1} + \|v(0)\|_{L_x^{p+1}}^{p+1} \Big) + C_\eta \|P_N \psi(t)\|_{L_x^{p+1}}^{p+1} - \int_0^t \int_{\mathbb{T}^3} |v|^{p-1} v P_N \psi_t dx dt', \tag{2.4.9}$$

for any $\eta > 0$. By Proposition 2.2.1, we have $P_N \psi_t \in C_t([0,T); C^{-\frac{3}{2}-,\infty})$.

From the formula of the functional E given by (2.4.2), we have that $\|v\|_{H^2} \lesssim E^{\frac{1}{2}}$, and $\|v\|_{L^{p+1}} \lesssim E^{\frac{1}{p}}$. Therefore, by Gagliardo-Nirenberg, for every $0 \leq s_0 \leq 2$, and

$$\frac{1}{q} = \frac{s_0}{2} \cdot \frac{1}{2} + \big(1 - \frac{s_0}{2}\big) \frac{1}{p+1},$$

we have $\|v\|_{W^{s_0,q}} \lesssim E^{\frac{s_0}{2} \cdot \frac{1}{2} + (1 - \frac{s_0}{2}) \frac{1}{p+1}} = E^{\frac{1}{q}}$. Hence, by Kato-Ponce's inequality[1], for $\frac{1}{r} = \frac{1}{q} + \frac{p-1}{p+1}$, as long as $\frac{1}{r} < 1$,

$$\big\| |v|^{p-1} v \big\|_{W^{s,r}} \lesssim E^{\frac{1}{q}} \|v\|_{L^{p+1}}^{p-1} \lesssim E^{\frac{1}{q} + \frac{p-1}{p+1}} = E^{\frac{1}{r}}. \tag{2.4.10}$$

The condition $p < \frac{11}{3}$ coincides with the condition $\frac{1}{r} < 1$ for some choice of $s_0 = s_0(p) > \frac{3}{2}$. Therefore, from (2.4.9) and (2.4.10), we get

$$\int_0^t \mathrm{I}\, dt' \lesssim \eta(E(\mathbf{v}(t)) + E(\mathbf{v}(0))) + C_\eta \|P_N \psi\|_{L^\infty([0,T] \times \mathbb{T}^3} + \|P_N \psi_t\|_{C([0,T]; W^{-s_0,\infty})} \int_0^t E(\mathbf{v}(t'))^{\frac{1}{r}} dt'. \tag{2.4.11}$$

Therefore, putting (2.4.7), (2.4.8), (2.4.11) together, and integrating in time, we get

$$(1-\eta)\big(E(\mathbf{v}(t)) - E(\mathbf{v}(0))\big) \lesssim \int_0^t \big(1 + E^{1-\theta}(\mathbf{v}(t')) + \|P_N \psi(t')\|_{L_x^\infty}^b \big) dt' + 2\eta E(0)$$
$$+ C_\eta \|P_N \psi\|_{L^\infty([0,T] \times \mathbb{T}^3)} + \|P_N \psi_t\|_{C([0,T]; C^{-s_0,\infty})} \int_0^t E(\mathbf{v}(t'))^{\frac{1}{r}} dt'. \tag{2.4.12}$$

If we choose $\eta = \frac{1}{2}$, by Gronwall we get (2.4.1) also in this case. □

2.4.2 Stochastic beam equation with damping

In order to make full use of the damping, we consider the modified energy

$$E(\mathbf{v}) := \frac{1}{2} \int v_t^2 + \frac{1}{2} \int v^2 + \frac{1}{2} \int (\Delta v)^2 + \frac{1}{4} \int v^4 + \frac{1}{8} \int (v + v_t)^2.$$

Proposition 2.4.2. *For every $0 < \alpha < \frac{1}{2}$, there exists $c > 0$ such that for every solution $\mathbf{v}_N$ of (2.3.9), we have*

$$E(\mathbf{v}_N(t)) \lesssim_\alpha \Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\uparrow_t\|_{\mathscr{C}^\alpha}^4 + \int_0^t e^{-c(t-t')} \|\uparrow_{t'}\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}} \mathrm{d}t' \Big).$$

Together with Corollary 2.3.5, this implies that

Corollary 2.4.3. *Let $0 < \alpha < \frac{1}{2}$. Given $\mathbf{u}_0 \in X^\alpha$, there exists a solution $\mathbf{v}$ of (2.1.14) in the space $C_t([0,+\infty); \mathcal{H}^2)$ such that*

$$\|\mathbf{v}(t)\|_{\mathcal{H}^2}^2 \lesssim_\alpha \Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\uparrow_t\|_{\mathscr{C}^\alpha}^4 + \int_0^t e^{-c(t-t')} \|\uparrow_{t'}\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}} \mathrm{d}t' \Big) \tag{2.4.13}$$

and for every $T < +\infty$,

$$\|\mathbf{v} - \mathbf{v}_N\|_{C([0,T]; \mathcal{H}^2)} \to 0 \ a.s..$$

[1] see for instance [6]

For the remainder of this subsection, we write $\mathbf{v}$ instead of $\mathbf{v}_N$ for the solution of (2.3.9).

Remark 2.4.4. Any solution $\mathbf{v}$ in $C([0,T^*);\mathcal{H}^2)$ of (2.3.9) actually belongs to $C^1([0,T^*);\mathscr{C}^\infty)$. Indeed, for any $t \le T < T^*$,

$$\begin{aligned}\|\mathbf{v}(t)\|_{\mathcal{H}^{2+s}} &= \left\| \langle\nabla\rangle^s \int_0^t P_N S(t-t') \begin{pmatrix} 0 \\ (S(t)\mathbf{u}_0 + \Psi(\xi) + v)^3 \end{pmatrix} \mathrm{d}t' \right\|_{\mathcal{H}^2} \\ &\lesssim T \sup_{0\le t\le T} \left\| \langle\nabla\rangle^s P_N (S(t)\mathbf{u}_0 + \Psi(\xi) + v)^3 \right\|_{L^2} \\ &\lesssim T N^s \sup_{0\le t\le T} \left\| (S(t)\mathbf{u}_0 + \Psi(\xi) + v)^3 \right\|_{L^2} < +\infty,\end{aligned}$$

where we just used that $\|\langle\nabla\rangle^s P_N\|_{L^2\to L^2} \lesssim N^s$. Similarly,

$$\begin{aligned}\|\partial_t \mathbf{v}(t)\|_{\mathcal{H}^s} &= \left\| \begin{aligned} &\langle\nabla\rangle^s \int_0^t P_N(\partial_t S(t-t')) \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \Psi(\xi) + v)^3 \end{pmatrix} \mathrm{d}t' \\ &\qquad + \langle\nabla\rangle^s \begin{pmatrix} 0 \\ (S(t)\mathbf{u}_0 + P_N \Psi_t(\xi) + v(t))^3 \end{pmatrix} \end{aligned} \right\|_{\mathcal{H}^0} \\ &\lesssim T \sup_{0\le t\le T} \left\| \langle\nabla\rangle^s P_N (S(t)\mathbf{u}_0 + \Psi(\xi) + v)^3 \right\|_{L^2} \\ &\lesssim T N^s \sup_{0\le t\le T} \left\| (S(t)\mathbf{u}_0 + \Psi(\xi) + v)^3 \right\|_{L^2} < +\infty,\end{aligned}$$

where we used that $\|\partial_t S(t)\|_{\mathcal{H}^2\to\mathcal{H}^0} \le 1$.

Actually, proceeding in this way, we can show that $\mathbf{v} \in C_t^\infty([0,T^*);\mathscr{C}^\infty)$, but we never need more than C^1 in the following.

Lemma 2.4.5. *If* $\mathbf{v}$ *solves* (2.3.9)*, then*

$$\partial_t E(\mathbf{v}) = -\frac{1}{4}\Big(3\int v_t^2 + \int v^2 + \int (\Delta v)^2 + \int v^4\Big) \tag{2.4.14}$$

$$+ \partial_t\Big(\frac{1}{8}\int v_t^2\Big) \tag{2.4.15}$$

$$- 3\int v_t v^2 (\Psi_t(\xi) + S(t)\mathbf{u}_0) \tag{2.4.16}$$

$$- \int (v_t + \frac{1}{4}v)[S(t)\mathbf{u}_0) 3v(\Psi_t(\xi) + S(t)\mathbf{u}_0)^2 + (\Psi_t(\xi) + S(t)\mathbf{u}_0)^3] \tag{2.4.17}$$

$$- \frac{3}{4}\int v^3 (\Psi_t(\xi) + S(t)\mathbf{u}_0) \tag{2.4.18}$$

$$+ \int (v_t + \frac{1}{4}v) P_{>N}(v + \Psi_t(\xi) + S(t)\mathbf{u}_0)^3 \tag{2.4.19}$$

Proof. By Remark 2.4.4, $E(\mathbf{v})$ is differentiable, and moreover $\mathbf{v}$ satisfies

$$\partial_t \begin{pmatrix} v \\ v_t \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1+\Delta^2 & 1 \end{pmatrix} \begin{pmatrix} v \\ v_t \end{pmatrix} - \begin{pmatrix} 0 \\ P_N(v + \Psi_t(\xi) + S(t)\mathbf{u}_0)^3 \end{pmatrix}.$$

Therefore, by exchanging time derivatives with integrals and using the equation, we have that

$$\begin{aligned}&\partial_t E(\mathbf{v}) \\ &= \int v_t(v_{tt} + v + \Delta^2 v + v^3) \\ &\quad + \frac{1}{8}\partial_t\Big(\int v^2 + \int v_t^2\Big) + \frac{1}{4}\int v_t^2 + \frac{1}{4}\int v v_{tt}\end{aligned}$$

$$
\begin{aligned}
&= -\int v_t^2 + \int v_t(v^3 - P_N(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3) \\
&\quad + \frac{1}{8}\partial_t\Big(\int v^2 + \int v_t^2\Big) + \frac{1}{4}\int v_t^2 \\
&\quad - \frac{1}{4}\int v v_t - \frac{1}{4}\Big(\int v^2 + \int (\Delta v)^2 + \int v P_N(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3\Big) \\
&= -\int v_t^2 + \int v_t(v^3 - (v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3) \\
&\quad + \int v_t P_{>N}(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3) \\
&\quad + \frac{1}{8}\partial_t\Big(\int v^2 + \int v_t^2\Big) + \frac{1}{4}\int v_t^2 \\
&\quad - \frac{1}{8}\partial_t\Big(\int v_t^2\Big) - \frac{1}{4}\Big(\int v^2 + \int (\Delta v)^2 + \int v(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3\Big) \\
&\quad + \frac{1}{4}\int v P_{>N}(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3. \\
&\overset{\text{rearranging}}{=} -\frac{1}{4}\Big(3\int v_t^2 + \int v^2 + \int (\Delta v)^2 + \int v(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3\Big) \\
&\quad + \partial_t\Big(\frac{1}{8}\int v_t^2\Big) \\
&\quad + \int v_t(v^3 - (v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3) \\
&\quad + \int (v_t + \frac{1}{4}v) P_{>N}(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3,
\end{aligned}
$$

and the claimed identity follows from expanding the cubes. □

Lemma 2.4.6. *If* $\mathbf{v}$ *solves* (2.3.9)*, then*

$$(2.4.19) = 0.$$

Proof. If $\mathbf{v}$ solves (2.3.9), then we can write $\mathbf{v}$ in the form $\mathbf{v} = P_N\mathbf{w}$ for some $\mathbf{w}$, therefore $P_{>N}\mathbf{v} = 0$. Therefore,

$$
\begin{aligned}
(2.4.19) &= \int (v_t + \frac{1}{4}v) P_{>N}(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3 \\
&= \int P_N(v_t + \frac{1}{4}v) P_{>N}(v + \uparrow_t(\xi) + S(t)\mathbf{u}_0)^3 = 0.
\end{aligned}
$$

□

Lemma 2.4.7. *If* $\mathbf{v}$ *solves* (2.3.9)*, then for every* $0 < \alpha < \frac{1}{2}$*,*

$$(2.4.17) \lesssim E^{\frac{3}{4}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\uparrow_t(\xi)\|_{\mathscr{C}^\alpha})^2 + E^{\frac{1}{2}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\uparrow_t(\xi)\|_{\mathscr{C}^\alpha})^3$$

Proof. By Hölder, we have that

$$\int (v_t + \frac{1}{4}v) 3v(\uparrow_t(\xi) + S(t)\mathbf{u}_0)^2 \lesssim (\|v\|_{L^2} + \|v_t\|_{L^2})\, \|v\|_{L^4} \left\|(\uparrow_t(\xi) + S(t)\mathbf{u}_0)^2\right\|_{L^4},$$

so by noticing that $\|v_t\|_{L^2} \lesssim E^{\frac{1}{2}}$, $\|v\|_{L^2} \lesssim E^{\frac{1}{2}}$, $\|v\|_{L^4} \lesssim E^{\frac{1}{4}}$, and $\left\|(\uparrow_t(\xi) + S(t)\mathbf{u}_0)^2\right\|_{L^4} \lesssim \|(\uparrow_t(\xi) + S(t)\mathbf{u}_0)\|_{C^\alpha}^2 \lesssim (\|\mathbf{u}_0\|_{X^\alpha} + \|\uparrow_t(\xi)\|_{\mathscr{C}^\alpha})^2$, we have that

$$\int (v_t + \frac{1}{4}v) 3v(\uparrow + S(t)\mathbf{u}_0)^2 \lesssim E^{\frac{3}{4}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\uparrow_t(\xi)\|_{\mathscr{C}^\alpha})^2.$$

Proceeding similarly,

$$\begin{aligned}\int (v_t + \frac{1}{4}v)(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)^3 &\lesssim (\|v\|_{L^2} + \|v_t\|_{L^2}) \left\|(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)^3\right\|_{L^2} \\ &\lesssim E^{\frac{1}{2}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^3.\end{aligned}$$

□

Lemma 2.4.8. *If* $\mathbf{v}$ *solves* (2.3.9)*, then for every* $0 < \alpha < \frac{1}{2}$*,*

$$(2.4.18) \lesssim E^{\frac{3}{4}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^2$$

Proof. By Hölder,

$$(2.4.18) \lesssim \|v\|_{L^4}^3 \|S(t)\mathbf{u}_0 + \upharpoonleft_t(\xi)\|_{L^4} \lesssim E^{\frac{3}{4}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^2.$$

□

Lemma 2.4.9. *If* $\mathbf{v}$ *solves* (2.3.9)*, then*

$$(2.4.16) = -\partial_t\Big(\int v^3(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)\Big) + \int v^3 \partial_t(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0), \tag{2.4.20}$$

and for every $0 < \alpha < \frac{1}{2}$*,*

$$\int v^3 \partial_t(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0) \lesssim E^{1-\frac{\alpha}{8}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha}).$$

Proof. (2.4.20) follows just from Leibnitz rule. In order to prove the estimate, notice that $\|v\|_{L^4} \lesssim E^{\frac{1}{4}}$, and $\|v\|_{H^2} \lesssim E^{\frac{1}{2}}$. Therefore, by Hölder and fractional Leibnitz respectively,

$$\begin{cases} \left\|v^3\right\|_{L^{\frac{4}{3}}} \lesssim E^{\frac{3}{4}} \\ \left\|v^3\right\|_{W^{2,1}} \lesssim E. \end{cases}$$

Therefore, by interpolation (Gagliardo - Nirenberg), if $\frac{1}{p} = (1 - \frac{\alpha}{2}) + \frac{\alpha}{2} \cdot \frac{3}{4} = 1 - \frac{\alpha}{8}$, then $\left\|v^3\right\|_{W^{2-\alpha,p}} \lesssim E^{(1-\frac{\alpha}{2}) + \frac{\alpha}{2}\cdot\frac{3}{4}} = E^{1-\frac{\alpha}{8}}$. Hence

$$\begin{aligned}\int v^3 \partial_t(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0) &\lesssim \left\|v^3\right\|_{W^{2-\alpha,p}} \|\partial_t(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)\|_{W^{\alpha-2,p'}} \\ &\lesssim \left\|v^3\right\|_{W^{2-\alpha,p}} \|\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0\|_{\mathscr{W}^{\alpha,p'}} \\ &\lesssim E^{1-\frac{\alpha}{8}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha}).\end{aligned}$$

□

Proof of Proposition 2.4.2. Let $F(\mathbf{v}) := E(\mathbf{v}) - \frac{1}{8}\int v_t^2 + \int v^3(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)$. By Hölder and Young's inequalities,

$$\begin{aligned}\Big|\int v^3(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)\Big| &\le \|v\|_{L^4}^3 \|(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)\|_{L^4} \\ &\le E^{\frac{3}{4}}(\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha}) \\ &\le \frac{1}{4}E + \frac{27}{4}(\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha})^4.\end{aligned}$$

Therefore,

$$F \le \frac{5}{4}E + \frac{27}{4}(\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha})^4, \tag{2.4.21}$$

$$E \le 2F + \frac{27}{2}(\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha})^4. \tag{2.4.22}$$

Using Lemma 2.4.5 and (2.4.20), we have that

$$\begin{aligned}\partial_t F =& -\frac{1}{4}\Big(3\int v_t^2 + \int v^2 + \int (\Delta v)^2 + \int v^4\Big)\\ &+ \int v^3 \partial_t(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)\\ &- \int (v_t + \frac{1}{4}v)[3v(\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)^2 + (\upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)^3]\\ &+ \int (v_t + \frac{1}{4}v) P_{>N}(v + \upharpoonleft_t(\xi) + S(t)\mathbf{u}_0)^3.\end{aligned}$$

Therefore, using Lemma 2.4.9, Lemma 2.4.7, Lemma 2.4.8, Lemma 2.4.6, Young's inequality and (2.4.21), for some constant C (that can change line by line) we have

$$\begin{aligned}\partial_t F \leq& -\frac{1}{2}E\\ &+ E^{1-\frac{\alpha}{8}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})\\ &+ E^{\frac{3}{4}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^2 + E^{\frac{1}{2}}(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^3\\ \leq& -\frac{1}{2}E + \frac{1}{4}E\\ &+ C[(\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^{\frac{8}{\alpha}} + (\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^8 + (\|\mathbf{u}_0\|_{X^\alpha} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha})^6]\\ \leq& -\frac{1}{2}E + C\Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\Big)\\ \leq& -\frac{2}{5}F + \frac{27}{10}(\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha})^4 + C\Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\Big)\\ \leq& -\frac{2}{5}F + C\Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\upharpoonleft_t(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\Big).\end{aligned}$$

Therefore, by Gronwall, if $c := \frac{2}{5}$, for some other constant C we have

$$F(\mathbf{v}(t)) \leq e^{-ct}F(\mathbf{v}_0) + C\Big(1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \int_0^t e^{-c(t-t')}\|\upharpoonleft_{t'}(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\,\mathrm{d}t'\Big).$$

Hence, using (2.4.22) and (2.4.21),

$$\begin{aligned}&E(\mathbf{v}(t))\\ \lesssim& F(\mathbf{v}(t)) + (\|\upharpoonleft_t(\xi)\|_{C^\alpha} + \|\mathbf{u}_0\|_{X^\alpha})^4\\ \lesssim& e^{-ct}F(\mathbf{v}(0)) + 1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \int_0^t e^{-c(t-t')}\|\upharpoonleft_{t'}(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\,\mathrm{d}t' + \|\upharpoonleft_t(\xi)\|_{C^\alpha}^4 + \|\mathbf{u}_0\|_{X^\alpha}^4\\ \lesssim& e^{-ct}(\|\mathbf{u}_0\|_{X^\alpha}^4) + 1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\upharpoonleft_t(\xi)\|_{C^\alpha}^4 + \int_0^t e^{-c(t-t')}\|\upharpoonleft_{t'}(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\,\mathrm{d}t'\\ \lesssim& 1 + \|\mathbf{u}_0\|_{X^\alpha}^{\frac{8}{\alpha}} + \|\upharpoonleft\|_{C^\alpha}^4 + \int_0^t e^{-c(t-t')}\|\upharpoonleft_{t'}(\xi)\|_{\mathscr{C}^\alpha}^{\frac{8}{\alpha}}\,\mathrm{d}t'.\end{aligned}$$

□

2.5 Invariance of the measure for the stochastic beam equation with damping

The goal of this section is showing that the flow of (SDNLB) is a stochastic flow which satisfies the semigroup property, and proceed to prove that the measure ρ is invariant for the flow of (SDNLB). Recall that, by the discussion in subsection 2.1.2, if $\mathbf{u}_0 \in X^\alpha$, the flow of (SDNLB) at time t with initial data $\mathbf{u}_0$ is defined as

$$\Phi_t(\mathbf{u}_0;\xi) = S(t)\mathbf{u}_0 + \uparrow_t(\xi) + \mathbf{v}(\mathbf{u}_0,\xi;t), \tag{2.5.1}$$

where $\mathbf{v}$ solves (2.1.14).

Proposition 2.5.1. *The map Φ satisfies the semigroup property, i.e. for every F measurable and bounded,*

$$\mathbb{E}[F(\Phi_{t+s}(\mathbf{u}_0;\xi))] = \mathbb{E}[F(\Phi_s(\Phi_t(\mathbf{u}_0;\xi_1);\xi_2))],$$

where ξ_1, ξ_2 are two independent copies of space-time white noise (defined in (1.0.2)*).*

Proof. Notice that

$$\begin{aligned}
&\Phi_s(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2)\\
&= S(s)\Phi_t(\mathbf{u}_0;\xi_1) + \uparrow_s(\xi_2) + \mathbf{v}(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;s)\\
&= S(t+s)\mathbf{u}_0 + S(s)\uparrow_t(\xi_1) + S(s)\mathbf{v}(\mathbf{u}_0,\xi_1;t) + \uparrow_s(\xi_2) + \mathbf{v}(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;s)\\
&= S(t+s)\mathbf{u}_0 + S(s)\uparrow_t(\xi_1) + \uparrow_s(\xi_2) + S(s)\mathbf{v}(\mathbf{u}_0,\xi_1;t) + \mathbf{v}(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;s).
\end{aligned} \tag{2.5.2}$$

Let $\tilde{\xi}$ defined by

$$\left\langle \tilde{\xi}, \phi \right\rangle = \langle \mathbb{1}_{t'\le t}\xi_1, \phi\rangle + \langle \mathbb{1}_{t'>t}\xi_2(\cdot - t), \phi\rangle .$$

It is easy to see that $\tilde{\xi}$ satisfies (1.0.2), so it is a copy of space-time white noise. Moreover,

$$\begin{aligned}
&S(s)\uparrow_t(\xi_1) + \uparrow_s(\xi_2)\\
&= S(s_0)\int_0^t S(t-t')\begin{pmatrix}0\\ \xi_1(t')\end{pmatrix}\mathrm{d}t' + \int_0^{s_0} S(s_0-t')\begin{pmatrix}0\\ \xi_2(t')\end{pmatrix}\mathrm{d}t'\\
&= \int_0^t S(t+s_0-t_1)\begin{pmatrix}0\\ \xi_1(t')\end{pmatrix}\mathrm{d}t' + \int_t^{t+s_0} S(t+s_0-t')\begin{pmatrix}0\\ \xi_2(t'-t)\end{pmatrix}\mathrm{d}t'\\
&= \int_0^{t+s_0} S(t+s_0-t')\begin{pmatrix}0\\ \tilde{\xi}(t')\end{pmatrix}\mathrm{d}t'\\
&= \uparrow_{t+s_0}(\tilde{\xi}).
\end{aligned} \tag{2.5.3}$$

Lastly, define

$$\mathbf{w}(t_0) = \begin{cases}\mathbf{v}(\mathbf{u}_0,\xi_1;t_0) & \text{if } t_0 \le t,\\ S(t_0-t)\mathbf{v}(\mathbf{u}_0,\xi_1;t) + \mathbf{v}(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;t_0-t) & \text{if } t_0 > t.\end{cases}$$

For $t_0 \le t$, $\mathbf{w}$ solves

$$\begin{aligned}
\mathbf{w}(t_0) &= \mathbf{v}(\mathbf{u}_0,\xi_1;t_0)\\
&= -\int_0^{t_0} S(t_0-t')\begin{pmatrix}0\\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\xi_1) + v(\mathbf{u}_0,\xi_1;t'))^3\end{pmatrix}\mathrm{d}t'\\
&= -\int_0^{t_0} S(t_0-t')\begin{pmatrix}0\\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3\end{pmatrix}\mathrm{d}t',
\end{aligned} \tag{2.5.4}$$

and for $t_0 > t$,

$$\begin{aligned}
\mathbf{w}(t_0) &= S(t_0-t)\mathbf{v}(\mathbf{u}_0,\xi_1;t) + \mathbf{v}(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;t_0-t)\\
&= S(t_0-t)\left(-\int_0^t S(t-t')\begin{pmatrix}0\\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3\end{pmatrix}\mathrm{d}t'\right)\\
&\quad - \int_0^{t_0-t} S(t_0-t-t')\begin{pmatrix}0\\ (S(t')\Phi_t(\mathbf{u}_0;\xi_1) + \uparrow_{t'}(\xi_1) + v(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;t'))^3\end{pmatrix}\mathrm{d}t'\\
&= -\int_0^t S(t_0-t')\begin{pmatrix}0\\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3\end{pmatrix}\mathrm{d}t'
\end{aligned}$$

$$
\begin{aligned}
&- \int_t^{t_0} S(t_0-t') \begin{pmatrix} 0 \\ (S(t'-t)\Phi_t(\mathbf{u}_0;\xi_1) + \uparrow_{t'-t}(\xi_2) + v(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;t'))^3 \end{pmatrix} \mathrm{d}t' \\
&= - \int_0^{t} S(t_0-t') \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3 \end{pmatrix} \mathrm{d}t' \\
&\quad - \int_t^{t_0} S(t_0-t') \\
&\quad \times \begin{pmatrix} 0 \\ (S(t'-t)S(t)\mathbf{u}_0 + S(t'-t)\uparrow_t(\xi_1) + S(t'-t)v(\mathbf{u}_0,\xi_1;t') + \uparrow_{t-t'}(\xi_2) + v(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2;t'))^3 \end{pmatrix} \mathrm{d}t' \\
&= - \int_0^{t} S(t_0-t') \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3 \end{pmatrix} \mathrm{d}t' \\
&\quad - \int_t^{t_0} S(t_0-t') \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3 \end{pmatrix} \mathrm{d}t' \\
&= - \int_0^{t_0} S(t_0-t') \begin{pmatrix} 0 \\ (S(t')\mathbf{u}_0 + \uparrow_{t'}(\tilde{\xi}) + w(t'))^3 \end{pmatrix} \mathrm{d}t',
\end{aligned} \tag{2.5.5}
$$

where we used (2.5.3). Putting (2.5.4) and (2.5.5) together, we have that

$$\mathbf{w}(t_0) = \mathbf{v}(\mathbf{u}_0, \tilde{\xi}; t_0).$$

By (2.5.2) and (2.5.3), this implies that

$$\Phi_s(\Phi_t(\mathbf{u}_0;\xi_1),\xi_2) = \Phi_{t+s}(\mathbf{u}_0;\tilde{\xi}),$$

and so for every F measurable and bounded, $\mathbb{E}[F(\Phi_{t+s}(\mathbf{u}_0;\xi))] = \mathbb{E}[F(\Phi_s(\Phi_t(\mathbf{u}_0;\xi_1);\xi_2))]$. □

Proposition 2.5.2. *Consider the flow given by*

$$\Phi_t^N(\mathbf{u}_0;\xi) := S(t)\mathbf{u}_0 + \uparrow_t(\xi) + \mathbf{v}_N(\mathbf{u}_0;\xi), \tag{2.5.6}$$

where $\mathbf{v}_N$ *solves* (2.3.9). *Then the measure*

$$\mathrm{d}\rho_N(\mathbf{u}) := \frac{1}{Z_N} \exp\Big(-\frac{1}{4}\int (P_N u)^4 \Big) \mathrm{d}\mu(\mathbf{u}) \tag{2.5.7}$$

is invariant for the flow $\Phi_t^N(\cdot;\xi)$*, where* $Z_N = \int \exp\Big(-\frac{1}{4}\int (P_N u)^4\Big)\mathrm{d}\mu(\mathbf{u})$ *(so that* ρ_N *is a probability measure).*

Proof. Let $\mathbf{X}$ be a random variable with law μ, independent from ξ. Invariance of (2.5.7) is equivalent to showing that

$$\mathbb{E}\Big[F(\Phi_t^N(\mathbf{X};\xi)) \exp\Big(-\frac{1}{4}\int (P_N \pi_1 \mathbf{X})^4\Big)\Big] = \mathbb{E}\Big[F(\mathbf{X}) \exp\Big(-\frac{1}{4}\int (P_N \pi_1 \mathbf{X})^4\Big)\Big]$$

for every $F : X^\alpha \to \mathbb{R}$ continuous. Let $M \geq N$. By definition of X^α, we have that $\lim_{M\to\infty} \|\mathbf{u} - P_M \mathbf{u}\|_{X^\alpha} = 0$ for every $u \in X^{\alpha'}$, $\alpha' > \alpha$. Therefore, by Proposition 2.2.3, Proposition 2.2.4, and (2.3.9), one has that for every $t \geq 0$,

$$\lim_{M\to\infty} \big\| \Phi_t^N(P_M \mathbf{X}; P_M \xi) - \Phi^N(\mathbf{X};\xi) \big\|_{X^\alpha} = 0.$$

Therefore, by dominated convergence, it is enough to prove that

$$\mathbb{E}\Big[F(\Phi_t^N(P_M\mathbf{X};P_M\xi)) \exp\Big(-\frac{1}{4}\int (P_N \pi_1 \mathbf{X})^4\Big)\Big] = \mathbb{E}\Big[F(P_N\mathbf{X}) \exp\Big(-\frac{1}{4}\int (\pi_1 P_N \mathbf{X})^4\Big)\Big]. \tag{2.5.8}$$

By (2.5.6), it is easy to check that $\mathbf{Y} = (Y, Y_t)^T := \Phi_t^N(\cdot; P_M\xi)$ solves the SDE

$$\mathrm{d}\mathbf{Y} = \begin{pmatrix} 0 & 1 \\ -(1+\Delta^2) & -1 \end{pmatrix} \mathbf{Y} - P_N \begin{pmatrix} 0 \\ (P_N Y)^3 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}\mathrm{d}W_M \end{pmatrix},$$

where $\mathrm{d}W_M := P_M\xi$ is a space-time white noise on the finite dimensional space given by the image of the map P_M. Therefore, if we show that the measure $\tilde{\rho}$ defined on the rank of P_M,

$$\mathrm{d}\tilde{\rho}(u) := \exp\Big(-\frac{1}{4}\int (P_N u)^4 - \frac{1}{2}\int |u|^2 - \frac{1}{2}\int |\Delta u|^2 - \frac{1}{2}\int |u_t|^2 \Big)\mathrm{d}u\mathrm{d}u_t,$$

is invariant for the flow $\mathbf{Y}$, we get (2.5.8). Since $\mathbf{Y}$ solves an SDE with smooth coefficeints, this is true if and only if $\tilde{\rho}$ solves the Fokker-Planck equation

$$-\operatorname{div}\Big[\Big(\begin{pmatrix} 0 & 1 \\ -(1+\Delta^2) & -1 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix}\Big)\tilde{\rho}(u, u_t)\Big] = 0.$$

We have that

$$\begin{aligned}
&-\operatorname{div}\Big[\Big(\begin{pmatrix} 0 & 1 \\ -(1+\Delta^2) & -1 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix}\Big)\Big]\tilde{\rho}(u, u_t) \\
&= \dim\big(\{P_M u_t\}\big)\tilde{\rho}(u, u_t) \\
&= (2M+1)^3\tilde{\rho}(u, u_t),
\end{aligned} \tag{2.5.9}$$

$$\begin{aligned}
&\Big(-\begin{pmatrix} 0 & 1 \\ -(1+\Delta^2) & -1 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix}\Big)\cdot\nabla\tilde{\rho}(u, u_t) \\
&= -\,\mathrm{d}\tilde{\rho}\Big[\begin{pmatrix} 0 & 1 \\ -(1+\Delta^2) & -1 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix}\Big] \\
&= \Big(\int (P_N u)^3 P_N u_t + \int u u_t + \int \Delta^2 u u_t + \int u_t(-(1+\Delta^2)u - u_t + P_N(P_N u)^3)\Big)\tilde{\rho}(u, u_t) \\
&= -\Big(\int u_t^2\Big)\tilde{\rho}(u, u_t),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_{u_t}\tilde{\rho}(u, u_t) \\
&= \operatorname{tr}\big(\mathrm{d}^2_{u_t}\tilde{\rho}(u, u_t)\big) \\
&= \sum_{\{h_t\}\text{orthonormal basis of }\{P_M u_t\}} \mathrm{d}^2\tilde{\rho}(u, u_t)\Big[\begin{pmatrix} 0 \\ h_t \end{pmatrix}, \begin{pmatrix} 0 \\ h_t \end{pmatrix}\Big] \\
&= \sum_{\{h_t\}\text{orthonormal basis of }\{P_M u_t\}} \Big(-\int h_t^2 + \Big(\int u_t h_t\Big)^2\Big)\tilde{\rho}(u, u_t), \\
&= \Big(-(2M+1)^3 + \int u_t^2\Big)\tilde{\rho}(u, u_t),
\end{aligned}$$

so (2.5) is satisfied. □

Corollary 2.5.3. *The measure ρ is invariant by the flow of* (SDNLB).

Proof. By Corollary 2.4.3, one has that for every $t > 0$ and every $\mathbf{u}_0 \in X^\alpha$, $\Phi_t^N(\mathbf{u}_0; \xi) \to \Phi_t(\mathbf{u}_0; \xi)$ in X^α a.s.. Let $F : X^\alpha \to \mathbb{R}$ be continuous and bounded. By dominated convergence and Proposition 2.5.2, we have

$$\begin{aligned}
\int \mathbb{E}\Big[F(\Phi_t(\mathbf{u}_0; \xi))\Big]\mathrm{d}\rho(\mathbf{u}_0) &= \int \mathbb{E}\Big[F(\Phi_t(\mathbf{u}_0; \xi))\Big]\exp\Big(-\frac{1}{4}\int (u_0)^4\Big)\mathrm{d}\mu(\mathbf{u}_0) \\
&= \lim_{N\to\infty}\int \mathbb{E}\Big[F(\Phi_t^N(\mathbf{u}_0; \xi))\Big]\exp\Big(-\frac{1}{4}\int (P_N\pi_1\mathbf{u}_0)^4\Big)\mathrm{d}\mu(\mathbf{u}_0)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \int F(\mathbf{u}_0) \exp\Big(- \frac{1}{4} \int (P_N \pi_1 \mathbf{u}_0)^4 \Big) \mathrm{d}\mu(\mathbf{u}_0) \\
&= \int F(\mathbf{u}_0) \exp\Big(- \frac{1}{4} \int (\pi_1 \mathbf{u}_0)^4 \Big) \mathrm{d}\mu(\mathbf{u}_0) \\
&= \int F(\mathbf{u}_0) \mathrm{d}\rho(\mathbf{u}_0).
\end{aligned}
$$

□

2.6 Ergodicity for stochastic beam equation with damping

In this section, we proceed to show unique ergodicity for the flow $\Phi_t(\mathbf{u}_0;\xi)$ of (SDNLB). We recall that, as discussed in Section 2.1, the flow is naturally split as $\Phi_t(\mathbf{u}_0;\xi) = \upharpoonleft_t(\xi) + S(t)\mathbf{u}_0 + \mathbf{v}$, where $\mathbf{v} = \mathbf{v}(\mathbf{u}_0, \xi; t)$ solves (2.1.14).

As discussed in the introduction, the flow of (SDNLB) does *not* satisfy the strong Feller property, so more "standard" techniques are not applicable. Indeed, by taking a set $E_t \subset X^\alpha$ of full measure for $\upharpoonleft_t(\xi)$, we can see that

$$
\mathbb{P}(\Phi_t(\mathbf{0};\xi) \in E_t + \mathcal{H}^2) = \mathbb{P}(\upharpoonleft_t + \mathbf{v}(\mathbf{0},\xi;t) \in E_t + \mathcal{H}^2) = \mathbb{P}(\upharpoonleft_t \in E_t + \mathcal{H}^2) = 1.
$$

Taking $0 < \alpha < \alpha_1 < \frac{1}{2}$, let $\overline{\mathbf{u}}_0 \in X^\alpha \setminus \mathcal{H}^{\alpha_1}$, whose existence is guaranteed by Lemma 2.1.3. We have that $S(t)\overline{\mathbf{u}}_0 \notin \mathcal{H}^{\alpha_1}$[2] for every t, and so for every $\lambda \neq 0$,

$$
\mathbb{P}(\Phi(\lambda \overline{\mathbf{u}}_0, \xi)(t) \in E_t + \mathcal{H}^2) = \mathbb{P}(\upharpoonleft_t(\xi) + \lambda S(t)\overline{u}_0 \in E_t + \mathcal{H}^2).
$$

By taking $E_t \subseteq \mathcal{H}^{\alpha_1}$, (as allowed by Proposition 2.2.2), we have that this probability is bounded from above by

$$
\mathbb{P}(\upharpoonleft_t(\xi) + \lambda S(t)\overline{u}_0 \in \mathcal{H}^{\alpha_1}) = \mathbb{P}(S(t)\overline{u}_0 \in \mathcal{H}^{\alpha_1}) = 0.
$$

Therefore, the function

$$
\Psi(\mathbf{u}) := \mathbb{E}[\mathbb{1}_{\{E_t + \mathcal{H}^2\}}(\Phi(\mathbf{u},\xi)(t))]
$$

satisfies $\Psi(\mathbf{0}) = 1$ and $\Psi(\lambda \overline{\mathbf{u}}_0) = 0$ for $\lambda \neq 0$, therefore is not continuous in $\mathbf{0}$. With the same argument, we can see that $\Psi(\mathcal{H}^2) = \{1\}$ and $\Psi(X^\alpha \setminus \mathcal{H}^{\alpha_1}) = \{0\}$, and since both sets are dense in X^α, we have that Ψ is not continuous *anywhere*.

2.6.1 Restricted strong Feller property and irreducibility of the flow

In this subsection, we try to recover some weaker version of the strong Feller property for the flow Φ. The end result will be to prove the following lemma, which will be crucial for the proof of ergodicity:

Lemma 2.6.1. *Let ν_1, ν_2 be two invariant measures for the flow of* (SDNLB) *such that $\nu_1 \perp \nu_2$. Then there exists some $V \subset X^\alpha$ such that $\nu_1(V) = 1$ and $\nu_2(V + \mathcal{H}^2) = 0$.*

In order to prove this, it is convenient to introduce the space $\mathcal{X}^\alpha = X^\alpha$ equipped with the distance

$$
d(\mathbf{u}_0, \mathbf{u}_1) = \|\mathbf{u}_0 - \mathbf{u}_1\|_{\mathcal{H}^2} \wedge 1.
$$

While $\mathcal{X}^\alpha$ is a complete metric space and a vector space, it does not satisfy many of the usual hypotheses on ambient spaces: it is *not* a topological vector space, it is *disconnected*, and it is *not* separable. Moreover, the sigma-algebra $\mathscr{B}$ of the Borel sets on X^α, which is also the sigma-algebra we equip $\mathcal{X}^\alpha$ with, does *not* coincide with the Borel sigma-algebra of $\mathcal{X}^\alpha$ - $\mathscr{B}$ is strictly smaller[3]. However, in this topology, we can prove the strong Feller property.

[2] Since $S(t)$ in invertible in $\mathcal{H}^{\alpha_1}$.

[3] Take $\mathbf{u}_0 \in X^\alpha \setminus \mathcal{H}^2$, and let $E \subseteq \mathbb{R}$ be not Borel. Then it is easy to see that $E\mathbf{u}_0 := \{\lambda \mathbf{u}_0 | \lambda \in E\}$ is *not* in $\mathscr{B}$, but it is *closed* in $\mathcal{X}^\alpha$.

Proposition 2.6.2 (Restricted strong Feller property)**.** *The flow Φ of* (SDNLB) *defined on $\mathcal{X}^\alpha$ has the strong Feller property, i.e. for every $t > 0$, the function*

$$P_t G(\mathbf{u}) := \mathbb{E}[G(\Phi(\mathbf{u}, \xi)(t))]$$

is continuous as a function $\mathcal{X}^\alpha \to \mathbb{R}$ for every $G : \mathcal{X}^\alpha \to \mathbb{R}$ measurable and bounded.

Before being able to prove this Proposition, we need the following (completely deterministic) lemma, which will take the role of support theorem for ξ.

Lemma 2.6.3. *For every $t > 0$, there exists a bounded operator $T_t : \mathcal{H}^2 \to L^2([0,t]; L^2)$ such that for every $\mathbf{w} \in \mathcal{H}^2$,*

$$\mathbf{w} = \int_0^t S(t-t') \begin{pmatrix} 0 \\ (T_t\mathbf{w})(t') \end{pmatrix} \mathrm{d}t' = \upharpoonleft_t (T_t \mathbf{w}).$$

Proof. This lemma is equivalent to proving that the operator $\upharpoonleft_t : L^2([0,t]; L^2) \to \mathcal{H}^2$ has a right inverse. Since $\mathcal{H}^2$ and $L^2([0,t]; L^2)$ are both Hilbert spaces, we have that $\upharpoonleft_t$ has a right inverse if and only if $\upharpoonleft_t^*$ has a left inverse. In Hilbert spaces, this is equivalent to the estimate $\|\mathbf{w}\|_{\mathcal{H}^2} \lesssim \|\upharpoonleft_t^* \mathbf{w}\|_{L^2([0,t];L^2)}$. We have that

$$(\upharpoonleft_t^* \mathbf{w})(s) = \pi_2 S(t-s)^* \mathbf{w},$$

where π_2 is the projection on the second component. Therefore,

$$\|\upharpoonleft_t^* \mathbf{w}\|^2_{L^2([0,t];L^2)} = \int_0^t \|\pi_2 S(s)^* \mathbf{w}\|^2_{L^2}.$$

For convenience of notation, define $L := \sqrt{\frac{3}{4} + \Delta^2}$, and redefine $\|w\|^2_{H^2} := \big(\int |Lw|^2\big)$ [4]. In this space, we have that

$$e^{\frac{t}{2}} \pi_2 S(s)^* \mathbf{w} = L \sin(sL) v + \Big(\cos(sL) - \frac{\sin(sL)}{2L}\Big) v_t.$$

Therefore, if $\lambda_n := \sqrt{\frac{3}{4} + |n|^4}$, by Parseval

$$\|\upharpoonleft_t^* \mathbf{w}\|^2_{L^2([0,t];L^2)} \sim_t \sum_{n \in \mathbb{Z}^3} \int_0^t \Big| \lambda_n \sin(s\lambda_n) \widehat{w}(n) + \Big(\cos(s\lambda_n) - \frac{\sin(s\lambda_n)}{2\lambda_n}\Big) \widehat{w_t}(n) \Big|^2 \mathrm{d}s.$$

Since by Parseval $\|w\|_{H^2} = \|\lambda_n \widehat{w}\|_{l^2}$ and $\|w_t\|_{L^2} = \|\widehat{w}_t\|_{l^2}$, the lemma is proven if we manage to prove that the quadratic form on $\mathbb{R}^2$

$$B_n(x,y) := \int_0^t \Big| \sin(s\lambda_n) x + \Big(\cos(s\lambda_n) - \frac{\sin(s\lambda_n)}{2\lambda_n}\Big) y \Big|^2 \mathrm{d}s$$

satisfies $B_n \geq c_n \,\mathrm{id}$, with $c_n \geq \varepsilon > 0$ for every $n \in \mathbb{Z}^3$. We have that $B_n > 0$, since the integrand cannot be identically 0 for $(x,y) \neq (0,0)$ (if the integrand is 0, by evaluating it in $s = 0$ we get $y = 0$, from which evaluating in almost any other s we get $x = 0$). Therefore, it is enough to prove that $c_n \to c > 0$ as $|n| \to +\infty$. As $|n| \to +\infty$, $\lambda_n \to +\infty$ as well, so

$$\lim_n \int_0^t \sin(s\lambda_n)^2 = \frac{t}{2},$$
$$\lim_n \int_0^t \cos(s\lambda_n)^2 = \frac{t}{2},$$
$$\lim_n \quad \frac{\sin(s\lambda_n)}{2\lambda_n} = 0,$$

[4] It is easy to see that it is equivalent to the usual norm $\int |\sqrt{1+\Delta^2} w|^2$

$$\lim_n \int_0^t \sin(s\lambda_n)\cos(s\lambda_n) = 0.$$

Hence, $B_n \to \frac{t}{2}\,\mathrm{id}$ and so $c_n \to \frac{t}{2} > 0$. □

Proof of Proposition 2.6.2. We recall the decomposition $\Phi(\mathbf{u},\xi)(t) = S(t)\mathbf{u} + \text{\textbullet}_t(\xi) + \mathbf{v}(\mathbf{u},\xi;t)$. For $h \in L^2_{t,x}$, adapted to the natural filtration induced by ξ, let

$$\mathcal{E}(h) := \exp\Big(-\frac{1}{2}\int_0^t \|h(t')\|_{L^2}^2 + \int_0^t \langle h(t'),\xi\rangle_{L^2}\Big).$$

Let $C_1 \gg 1$, $E := \{\xi \mid \|\text{\textbullet}_t(\xi)\|_{C([0,t];\mathscr{C}^\alpha)} \le C_1\}$, and T_t as in Lemma 2.6.3. Let $\mathbf{u}_0 \in \mathcal{X}^\alpha$. By Corollary 2.4.3, as long as $\xi \in E$ and C_2 is big enough (depending on $\mathbf{u}_0, C_1$), then

$$\max(\|\mathbf{v}(\mathbf{u},\xi;t')\|_{C([0;t];\mathcal{H}^2)}, \|S(t')\mathbf{u} + \text{\textbullet}_{t'}(\xi) + v\|^3_{C([0;t];L^2)}) \le C_2$$

in a neighbourhood of $\mathbf{u}_0$. For convenience of notation, we denote

$$(\mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t))(t') = -(\pi_1(S(t')\mathbf{u} + \text{\textbullet}_{t'}(\xi)) + v)^3.$$

Because of (2.1.14), $\mathbf{v}$ satisfies $\mathbf{v}(t) = \text{\textbullet}_t(\mathcal{T}_t\mathbf{v})$, and by the continuity of the flow in the initial data, $\mathcal{T}_t\mathbf{v}$ is continuous in $\mathbf{u}_0$. Moreover, in this way $\mathcal{T}_t\mathbf{v}$ will always be adapted to the natural filtration induced by ξ.

By Girsanov's theorem ([4, Theorem 1]), we have that

$$\begin{aligned}
&\mathbb{E}[G(\Phi(\mathbf{u},\xi)(t))]\\
&= \mathbb{E}[\mathbb{1}_{\xi\in E^c}G(\Phi(\mathbf{u},\xi)(t))] + \mathbb{E}[\mathbb{1}_{\xi\in E}G(S(t)\mathbf{u} + \text{\textbullet}_t(\xi) + \mathbf{v}(\mathbf{u},\xi;t))]\\
&= \mathbb{E}[\mathbb{1}_{\xi\in E^c}G(\Phi(\mathbf{u},\xi)(t))] + \mathbb{E}[\mathbb{1}_{\xi\in E}G(S(t)\mathbf{u} + \text{\textbullet}_t(\xi + \mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t)))]\\
&= \mathbb{E}[\mathbb{1}_{\xi\in E^c}G(\Phi(\mathbf{u},\xi)(t))] + \mathbb{E}[\mathbb{1}_{\xi\in E+\mathcal{T}_t\mathbf{v}}G(S(t)\mathbf{u} + \text{\textbullet}_t(\xi))\mathcal{E}(\mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t))].
\end{aligned}$$

Notice that Novikov condition ((2.1) in ([4, Theorem 1]) is satisfied automatically by the estimate $\|\mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t)\|_{\mathcal{H}^2} \le C_2$, which holds true on $\{\xi \in E\}$[5]. Let $\mathbf{v}_0 \in \mathcal{H}^2$, with $\|\mathbf{v}_0\| \le C_2$.

$$\begin{aligned}
&\mathbb{E}[G(\Phi_t(\mathbf{u}+\mathbf{v}_0,\xi))]\\
&= \mathbb{E}[\mathbb{1}_{\xi\in E^c}G(\Phi_t(\mathbf{u}+\mathbf{v}_0,\xi))]\\
&\quad + \mathbb{E}[\mathbb{1}_{\xi\in E}G(S(t)\mathbf{u} + \text{\textbullet}_t(\xi + T_tS(t)\mathbf{v}_0 + \mathcal{T}_t\mathbf{v}(\mathbf{u}+\mathbf{v}_0,\xi;t)))]\\
&= \mathbb{E}[\mathbb{1}_{\xi\in E^c}G(\Phi(\mathbf{u}+\mathbf{v}_0,\xi)(t))]\\
&\quad + \mathbb{E}[\mathbb{1}_{E+T_tS(t)\mathbf{v}_0+\mathcal{T}_t\mathbf{v}}G(S(t)\mathbf{u} + \text{\textbullet}_t(\xi))\mathcal{E}(T_tS(t)\mathbf{v}_0 + \mathcal{T}_t\mathbf{v})].
\end{aligned}$$

Up to changing $\mathbf{v}$ outside of E, we can assume $\|\mathbf{v}(\mathbf{u},\xi;t)\|_{\mathcal{H}^2} \le C_2$. Therefore, we have (using Girsanov again)

$$\begin{aligned}
&\big|\mathbb{E}[G(\Phi_t(\mathbf{u}+\mathbf{v}_0,\xi))] - \mathbb{E}[G(\Phi_t(\mathbf{u},\xi))]\big|\\
&\le \|G\|_{L^\infty}\Big(2\mathbb{P}(\xi\in E^c) + \mathbb{E}[\mathbb{1}_{(S+\mathcal{T}_t\mathbf{v})^c}\mathcal{E}(\mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t))]\\
&\qquad + \mathbb{E}[\mathbb{1}_{(S+T_tS(t)\mathbf{v}_0+\mathcal{T}_t\mathbf{v})^c}\mathcal{E}(T_tS(t)\mathbf{v}_0 + \mathcal{T}_t\mathbf{v})]\\
&\qquad + \mathbb{E}\big|\mathcal{E}(\mathcal{T}_t\mathbf{v}(\mathbf{u},\xi;t)) - \mathcal{E}(T_tS(t)\mathbf{v}_0 + \mathcal{T}_t\mathbf{v}((\mathbf{u}+\mathbf{v}_0),\xi;t))\big|\Big)\\
&= \|G\|_{L^\infty}(4\mathbb{P}(\xi\in E^c) + \mathbb{E}\big|\mathcal{E}(T_tS(t)\mathbf{v}_0 + \mathcal{T}_t\mathbf{v}(\mathbf{u}+\mathbf{v}_0,\xi;t)) - \mathcal{E}(\mathcal{T}_t\mathbf{v}(\mathbf{u}),\xi;t)\big|)
\end{aligned}$$

Notice that, by Burkholder-Davis-Gundy inequality, for h in the form $h = T_tS(t)\mathbf{w} + \mathcal{T}_t\mathbf{v}$, with both $\|\mathbf{v}\|_{L^2_{t,x}} \le C_2$ and $\|\mathbf{w}\|_{\mathcal{H}^2} \le C_2$,

$$\mathbb{E}[\exp(p\langle h,\xi\rangle_{L^2_{t,x}})] \le \sum_{k\ge0} p^k\frac{1}{k!}\mathbb{E}[|\langle h,\xi\rangle_{L^2_{t,x}}|^k]$$

[5] to define a global adapted process that is equal to $\mathcal{T}_t\mathbf{v}$ on $\{\xi\in E\}$ and bounded by C_2 everywhere, we can for instance stop $\mathcal{T}_t\mathbf{v}$ when its norm reaches C_2

$$
\begin{aligned}
&\leq 1 + \sum_{k \geq 1} p^k C^k \frac{k^{\frac{k}{2}}}{k!} \big(\|T_t S(t) \mathbf{w}\|_{L^2_{t,x}}^k + \mathbb{E}[\|\mathcal{T}_t \mathbf{v}\|^k_{L^2_{t,x}}]\big) \\
&\leq 1 + \Psi_1(\|\mathbf{w}\|_{\mathcal{H}^2}) \|\mathbf{w}\|_{\mathcal{H}^2} + \Psi_2(C_2) \mathbb{E}[\|\mathcal{T}_t \mathbf{v}\|^2_{L^2_{t,x}}]^{\frac{1}{2}}, \\
&\leq \Psi(C_2).
\end{aligned}
$$

where Ψ_1, Ψ_2, Ψ are monotone analytic functions with infinite radius of convergence. With the same computation, we get

$$
\mathbb{E}[\big(\exp(p \langle h, \xi\rangle_{L^2_{t,x}}) - 1\big)^n] \leq \Psi_{3,n}(C_2)(\|\mathbf{w}\|_{\mathcal{H}^2} + \mathbb{E}[\|\mathcal{T}_t \mathbf{v}\|^2_{L^2_{t,x}}]^{\frac{1}{2}}) \lesssim_{n, C_2} \mathbb{E}[\|h\|^2_{L^2_{t,x}}]^{\frac{1}{2}}.
$$

Therefore, by continuity of the flow of (SDNLB) in the initial data, for $\|\mathbf{v}_0\|_{\mathcal{H}^2} \ll 1$, we have that

$$
\begin{aligned}
&\mathbb{E}\big|\mathcal{E}(T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t)) - \mathcal{E}(\mathcal{T}_t \mathbf{v}(\mathbf{u}), \xi; t)\big| \\
=&\mathbb{E}\Big[\exp\Big(- \frac{1}{2} \|\mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}} + \langle \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) \\
&\times \Big(\exp\Big(- \frac{1}{2}(\|T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t)\|^2_{L^2_{t,x}} - \|\mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}}) \\
&\qquad + \langle T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) - 1\Big)\Big] \\
=&\mathbb{E}\Big[\exp\Big(- \frac{1}{2} \|\mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}} + \langle \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) \\
&\times \Big(\exp\Big(- \frac{1}{2}(\|T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t)\|^2_{L^2_{t,x}} - \|\mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}})\Big) - 1\Big) \\
&\qquad \times \exp\Big(\langle T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) \\
&\quad + \exp\Big(\langle T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) - 1\Big)\Big] \\
\leq &\Big[\mathbb{E} \exp\Big(2 \langle \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big)\Big]^{\frac{1}{2}} \\
&\times \Big[\Big(\mathbb{E}\Big(\exp\Big(- \frac{1}{2}(\|T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t)\|^2_{L^2_{t,x}} - \|\mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}})\Big) - 1\Big)^4\Big)^{\frac{1}{4}} \\
&\quad \times \Big(\mathbb{E} \exp\Big(4 \langle T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big)\Big)^{\frac{1}{4}} \\
&\quad + \Big(\mathbb{E}\Big(\exp\Big(\langle T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t), \xi\rangle_{L^2_{t,x}} \Big) - 1\Big)^2\Big)^{\frac{1}{2}}\Big] \\
\lesssim_{C_2} &[\mathbb{E} \, \|T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t) - \mathcal{T}_t \mathbf{v}(\mathbf{u}, \xi; t)\|^2_{L^2_{t,x}}]^{\frac{1}{2}},
\end{aligned}
$$

which is converging to 0 as $\|\mathbf{v}_0\|_{\mathcal{H}^2} \to 0$ because of dominated convergence. Therefore,

$$
\begin{aligned}
&\limsup_{\|v_0\|_{\mathcal{H}^2} \to 0} \big|\mathbb{E}[G(\Phi_t(\mathbf{u} + \mathbf{v}_0, \xi))] - \mathbb{E}[G(\Phi_t(\mathbf{u}, \xi))]\big| \\
\leq &\limsup_{\|v_0\|_{\mathcal{H}^2} \to 0} \|G\|_{L^\infty} (4 \mathbb{P}(\xi \in E^c) + \mathbb{E}\big|\mathcal{E}(T_t S(t) \mathbf{v}_0 + \mathcal{T}_t \mathbf{v}(\mathbf{u} + \mathbf{v}_0, \xi; t)) - \mathcal{E}(\mathcal{T}_t \mathbf{v}(\mathbf{u}), \xi; t)\big|) \\
=&4 \|G\|_{L^\infty} \mathbb{P}(\xi \in E^c).
\end{aligned}
$$

Since the left-hand-side does not depend on C_1, we can send $C_1 \to \infty$, and we obtain that

$$
\lim_{\|v_0\|_{\mathcal{H}^2} \to 0} \big|\mathbb{E}[G(\Phi_t(\mathbf{u} + \mathbf{v}_0, \xi))] - \mathbb{E}[G(\Phi_t(\mathbf{u}, \xi))]\big| = 0,
$$

i.e. $\mathbb{E}[G(\Phi(\mathbf{u}, \xi)(t))]$ is continuous in $\mathbf{u}$ in the $\mathcal{X}^\alpha$ topology. □

While the topology of $\mathcal{X}^\alpha$ does not allow to extend many common consequences of the strong Feller property, we still have the following generalisation of the disjoint supports property.

Corollary 2.6.4. *Let $\nu_1 \perp \nu_2$ be two invariant measures for the flow of* (SDNLB). *Then there exists a measurable open set $V_0 \subseteq \mathcal{X}^\alpha$ such that $\nu_1(V_0) = 1$ and $\nu_2(V_0) = 0$.*

Proof. Let $S_1 \subset \mathcal{X}^\alpha$ be a measurable set with $\nu_1(S_1) = 1$, $\nu_2(S_1) = 0$. Consider the function

$$\Psi(\mathbf{u}) := \mathbb{E}[\mathbb{1}_{S_1}(\Phi(\mathbf{u}, \xi)(t))].$$

By the Proposition 2.6.2, $\Psi : \mathcal{X}^\alpha \to \mathbb{R}$ is continuous. By invariance of ν_j, $\Psi = 1$ ν_1-a.s. and $\Psi = 0$ ν_2-a.s.. Let $V_0 := \{\Psi > \frac{1}{2}\}$. We have that $V_0 \subset \mathcal{X}^\alpha$ is open by continuity of Ψ, it is measurable since Ψ is measurable,

$$\nu_1(V_0) \geq \nu_1(\{\Psi = 1\}) = 1$$

and

$$\nu_2(V_0) \leq \nu_2(\{\Psi \neq 0\}) = 0.$$

□

Lemma 2.6.5 (Irreducibility)**.** *Suppose that ν is invariant for the flow of* (SDNLB)*, and let $E \subset X^\alpha$ such that $\nu(E) = 0$. Then for every $\mathbf{w} \in \mathcal{H}^2$, $\nu(E + \mathbf{w}) = 0$.*

Proof. Since X^α is a Polish space, by inner regularity of ν it is enough to prove the statement when E is compact. Take $C_1 < +\infty$, and let $F := \{\|\upharpoonleft_\cdot(\xi)\|_{C([0,t];\mathscr{C}^\alpha)} \leq C_1\}$. Proceeding in a similar way to Proposition 2.6.2, we have that by the compactness of E, the boundedness of $\upharpoonleft_t(\xi)$ and Proposition 2.4.3, $\mathcal{T}_t\mathbf{v}$ satisfies Novikov's condition on $\{\xi \in F\}$ and

$$\begin{aligned}
0 = \nu(E) &= \int \mathbb{E}[\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v}])\mathrm{d}\nu(\mathbf{u}) \\
&\geq \int \mathbb{E}[\mathbb{1}_F(\xi)\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v})]\mathrm{d}\nu(\mathbf{u}) \\
&= \int \mathbb{E}[\mathbb{1}_{F+\mathcal{T}_t\mathbf{v}}(\xi)\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi))\mathcal{E}(\mathcal{T}_t\mathbf{v})]\mathrm{d}\nu(\mathbf{u}).
\end{aligned}$$

Since $\mathcal{E} > 0$ $\mathbb{P} \times \nu-$a.s., this implies that $\mathbb{1}_{F+\mathcal{T}_t\mathbf{v}}(\xi)\mathbb{1}_E(S(t)\mathbf{u}+\upharpoonleft_t(\xi)) = 0$ $\mathbb{P}\times\nu-$a.s.. By sending $C_1 \to \infty$, by monotone convergence we obtain that $\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi)) = 0$ $\mathbb{P} \times \nu-$a.s..

Let $\mathbf{w} \in \mathcal{H}^2$. Then, proceeding similarly,

$$\begin{aligned}
&\int \mathbb{E}[\mathbb{1}_F(\xi)\mathbb{1}_{E+\mathbf{w}}(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v})]\mathrm{d}\nu(\mathbf{u}) \\
&= \int \mathbb{E}[\mathbb{1}_F(\xi)\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v} - \mathbf{w})]\mathrm{d}\nu(\mathbf{u}) \\
&= \int \mathbb{E}[\mathbb{1}_{F+\mathcal{T}_t\mathbf{v}-T_t\mathbf{w}}(\xi)\mathbb{1}_E(S(t)\mathbf{u} + \upharpoonleft_t(\xi))\mathcal{E}(\mathcal{T}_t\mathbf{v} - T_t\mathbf{w})]\mathrm{d}\nu(\mathbf{u}) = 0,
\end{aligned}$$

since the integrand is 0 $\mathbb{P} \times \nu-$a.s.. By taking $C_1 \to \infty$, by monotone convergence we get

$$\begin{aligned}
0 = &\int \mathbb{E}[\mathbb{1}_{E+\mathbf{w}}(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v})]\mathrm{d}\nu(\mathbf{u}) \\
= &\nu(E + \mathbf{w}).
\end{aligned}$$

□

Proof of Lemma 2.6.1. Let $\nu_1 \perp \nu_2$ be two invariant measures, let $V = V_0$ be the set given by Corollary 2.6.4, and let $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{H}^2$. We have that, by definition, $\nu_1(V) = 1$ and $\nu_2(V) = 0$. By Lemma 2.6.5, $\nu_2(V + \mathbf{w}_n) = 0$ for every $\mathbf{w}_n$. Therefore, $\nu_2(\bigcup_n(V + \mathbf{w}_n)) = 0$. Moreover, since V is open in $\mathcal{X}^\alpha$, we have that $\bigcup_n(V + \mathbf{w}_n) = V + \mathcal{H}^2$. Therefore, $\nu_2(V + \mathcal{H}^2) = 0$. □

2.6.2 Projected flow

In this subsection, we will bootstrap ergodicity of the measure ρ from ergodicity of the flow of the *linear* equation

$$\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1+\Delta^2 & 1 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2}\xi \end{pmatrix}.$$

The measure μ defined in (1.1.7) is invariant for the flow of this equation (which can be seen as a special case of Proposition 2.5.2 for $N=-1$). Let $L(t)\mathbf{u}$ be the flow of (2.6.2), i.e.

$$L(t)\mathbf{u} := S(t)\mathbf{u} + \upharpoonleft_t(\xi).$$

Lemma 2.6.6. *The measure μ is the only invariant measure for* (2.6.2)*. Moreover, for every $\mathbf{u}_0 \in X^\alpha$, the law of $L(t)\mathbf{u}_0$ is weakly converging to μ as $t \to \infty$.*

Proof. Let $\mathbf{u}_0, \mathbf{u}_1 \in X^\alpha$, and let $F : X^\alpha \to \mathbb{R}$ be a Lipschitz function. We have that

$$\begin{aligned}\big|\mathbb{E}[F(L(t)\mathbf{u}_0) - F(L(t)\mathbf{u}_1)]\big| &= \big|\mathbb{E}[F(S(t)\mathbf{u}_0 + \upharpoonleft_t(\xi)) - F(S(t)\mathbf{u}_1 + \upharpoonleft_t(\xi))]\big| \\ &\le \mathbb{E}[\min(\operatorname{Lip}(F)\,\|S(t)\mathbf{u}_0 - S(t)\mathbf{u}_1\|_{X^\alpha}, \|F\|_{L^\infty})] \\ &\le \min(e^{-\frac{t}{8}}\operatorname{Lip}(F)\,\|\mathbf{u}_0 - \mathbf{u}_1\|_{X^\alpha}, \|F\|_{L^\infty})\end{aligned}$$

Therefore by invariance of μ, we have that

$$\begin{aligned}\Big|\mathbb{E}[F(L(t)\mathbf{u}_0)] - \int F(\mathbf{u}_1)\mathrm{d}\mu(\mathbf{u}_1)\Big| &= \Big|\int \big(\mathbb{E}[F(L(t)\mathbf{u}_0) - \mathbb{E}[F(L(t)\mathbf{u}_1)]\big)\mathrm{d}\mu(\mathbf{u}_1)\Big| \\ &\le \int \min(e^{-\frac{t}{8}}\operatorname{Lip}(F)\,\|\mathbf{u}_0 - \mathbf{u}_1\|_{X^\alpha}, \|F\|_{L^\infty})\mathrm{d}\mu(\mathbf{u}_1),\end{aligned}$$

which is converging to 0 by dominated convergence. Since Lipschitz functions are dense in the set of continuous functions, this implies that the law of $L(t)\mathbf{u}_0$ is weakly converging to μ. Similarly, if ν is another invariant measure,

$$\begin{aligned}&\Big|\int F(\mathbf{u}_0)\mathrm{d}\nu(\mathbf{u}_0) - \int F(\mathbf{u}_1)\mathrm{d}\mu(\mathbf{u}_1)\Big| \\ &= \Big|\iint \big(\mathbb{E}[F(L(t)\mathbf{u}_0) - \mathbb{E}[F(L(t)\mathbf{u}_1)]\big)\mathrm{d}\nu(\mathbf{u}_0)\mathrm{d}\mu(\mathbf{u}_1)\Big| \\ &\le \iint \min(e^{-\frac{t}{8}}\operatorname{Lip}(F)\,\|\mathbf{u}_0 - \mathbf{u}_1\|_{X^\alpha}, \|F\|_{L^\infty})\mathrm{d}\nu(\mathbf{u}_0)\mathrm{d}\mu(\mathbf{u}_1),\end{aligned}$$

which is converging to 0 by dominated convergence. Since the left hand side does not depend on t, one gets that $\int F(\mathbf{u}_0)\mathrm{d}\nu(\mathbf{u}_0) - \int F(\mathbf{u}_1)\mathrm{d}\mu(\mathbf{u}_1)$ for every F Lipschitz, so $\mu = \nu$. $\square$

Consider the (algebraic) projection $\pi : X^\alpha \to X^\alpha/\mathcal{H}^2$. While the quotient space does not have a sensible topology, we can define the quotient sigma-algebra,

$$\mathcal{A} := \{F \subseteq X^\alpha/\mathcal{H}^2 \text{ s.t. } \pi^{-1}(F) \subseteq X^\alpha \text{ Borel}\},$$

which makes the map π measurable. While this will not be relevant in the following, we can see that $\mathcal{A}$ is relatively rich: if $E \subset X^\alpha$ is closed and B is the closed unit ball in $\mathcal{H}^2$, since B is compact in X^α, $E + nB$ is closed for every n, so $E + \mathcal{H}^2 = \bigcup_n E + nB$ is Borel. Therefore, $\pi(E) \in \mathcal{A}$.

Since $S(t)$ maps $\mathcal{H}^2$ into itself, is it easy to see that if $\pi(\mathbf{u}) = \pi(\mathbf{v})$, then $\pi(L(t)\mathbf{u}) = \pi(L(t)\mathbf{v})$. Therefore, $\pi(L(t)\mathbf{u})$ is a function of $\pi(u)$, and we define

$$\overline{L}(t)\pi(u) := \pi(L(t)\mathbf{u}).$$

Moreover, if $\Phi_t(\mathbf{u};\xi) = S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v}(\mathbf{u},\xi;t)$ is the flow of (SDNLB), where $\mathbf{v}$ solves (2.1.14), since $\mathbf{v}$ belongs to $\mathcal{H}^2$, we have that

$$\pi(\Phi_t(\mathbf{u};\xi)) = \pi(S(t)\mathbf{u} + \upharpoonleft_t(\xi) + \mathbf{v}(\mathbf{u},\xi;t)) = \pi(S(t)\mathbf{u} + \upharpoonleft_t(\xi)) = \pi(L(t)\mathbf{u}) = \overline{L}(t)\pi(\mathbf{u}).$$

Therefore, also $\pi(\Phi_t(\mathbf{u};\xi))$ is a function of $\pi(\mathbf{u})$, and moreover

$$\pi(\Phi_t(\mathbf{u};\xi)) = \overline{L}(t)\pi(\mathbf{u}), \tag{2.6.1}$$

so the projections of the flows for (2.6.2) and (SDNLB) coincide.

Proposition 2.6.7. *The measure $\pi_\sharp(\mu)$ is ergodic for $\overline{L}(t) : X^\alpha/\mathcal{H}^2 \to X^\alpha/\mathcal{H}^2$.*

Proof. If $G : X^\alpha/\mathcal{H}^2 \to \mathbb{R}$ is a bounded measurable function, then by invariance of μ,

$$\begin{aligned}\int \mathbb{E}[G(\overline{L}(t)x)]\mathrm{d}\pi_\sharp\mu(x) &= \int \mathbb{E}[G(\overline{L}(t)\pi(\mathbf{u}))]\mathrm{d}\mu(\mathbf{u}) \\ &= \int \mathbb{E}[G(\pi(L(t)\mathbf{u}))]\mathrm{d}\mu(\mathbf{u}) \\ &= \int G(\pi(\mathbf{u}))\mathrm{d}\mu(\mathbf{u}) \\ &= \int G(x)\mathrm{d}\pi_\sharp\mu(x),\end{aligned}$$

so $\pi_\sharp\mu$ is invariant.

Let now G be a function such that $\mathbb{E}[G(\overline{L}(t)x)] = G(x)$ for $\pi_\sharp\mu$-a.e. $x \in X^\alpha/\mathcal{H}^2$. Then

$$\mathbb{E}[G \circ \pi(L(t)\mathbf{u})]\mathbb{E}[G(\pi(L(t)\mathbf{u}))] = E[G(\overline{L}(t)\pi(\mathbf{u}))] = G(\pi(u)).$$

Since μ is ergodic (by Proposition 2.6.6), this implies that $G \circ \pi$ is μ-a.s. constant. Therefore, G is $\pi_\sharp\mu$-a.s. constant, so $\pi_\sharp\mu$ is ergodic. ☐

Proposition 2.6.8. *Let ν be an invariant measure for the flow of* (SDNLB) *such that $\pi_\sharp\nu \ll \pi_\sharp\mu$. Then $\nu = \rho$*[6]*.*

Proof. Suppose by contradiction that $\nu \neq \rho$. Let

$$\begin{aligned}\rho_1 &= \frac{1}{(\rho-\nu)_+(X^\alpha)}(\rho-\nu)_+ \\ \rho_2 &= \frac{1}{(\rho-\nu)_+(X^\alpha)}(\nu-\rho)_+.\end{aligned}$$

Since μ, ν are invariant, it is easy to see that ρ_1, ρ_2 are both invariant probabilities[7]. Moreover, $\rho_1 \perp \rho_2$, and $\rho_j \ll \rho + \nu$, so $\pi_\sharp\rho_j \ll \pi_\sharp\rho + \pi_\sharp\nu \ll \pi_\sharp\mu$.

Proceeding as for (2.6.2), and using (2.6.1), we have

$$\begin{aligned}\int \mathbb{E}[G(\overline{L}(t)x)]\mathrm{d}\pi_\sharp\rho_j(x) &= \int \mathbb{E}[G(\overline{L}(t)\pi(\mathbf{u}))]\mathrm{d}\rho_j(\mathbf{u}) \\ &= \int \mathbb{E}[G(\pi(\Phi_t(\mathbf{u};\xi)))]\mathrm{d}\rho_j(\mathbf{u}) \\ &= \int G(\pi(\mathbf{u}))\mathrm{d}\rho_j(\mathbf{u}) \\ &= \int G(x)\mathrm{d}\pi_\sharp\rho_j(x),\end{aligned}$$

therefore $\pi_\sharp\rho_j$ is invariant for $\overline{L}(t)$. Moreover, since $\pi_\sharp\rho_j \ll \pi_\sharp\mu$, by invariance of $\pi_\sharp\rho_j$ and ergodicity of $\pi_\sharp\mu$, we must have $\pi_\sharp\rho_j = \pi_\sharp\mu$. Let now V be the set given by Lemma 2.6.1, i.e. $\rho_1(V) = 1$, $\rho_2(V + \mathcal{H}^2) = 0$. We have

$$0 = \rho_2(V + \mathcal{H}^2) = \pi_\sharp\rho_2(\pi(V + \mathcal{H}^2)) = \pi_\sharp\mu(\pi(V + \mathcal{H}^2))$$

[6] Notice that since $\rho \ll \mu$, then $\pi_\sharp\rho \ll \pi_\sharp\mu$.

[7] Just use the characterisation

$$\text{If } F \geq 0, \int F(x)\mathrm{d}\sigma_+(x) = \sup_{0 \leq G \leq F} \int G(x)\mathrm{d}\sigma(x).$$

$$= \pi_\sharp \rho_1(\pi(V + \mathcal{H}^2)) = \rho_1(V + \mathcal{H}^2) \geq \rho_1(V) = 1,$$

contradiction. $\square$

Corollary 2.6.9. *The measure ρ is ergodic for the flow $\Phi_t(\cdot, \xi) : X^\alpha \to X^\alpha$ of* (SDNLB)*.*

Proof. Let $\nu \ll \rho$, ν invariant. We have that $\pi_\sharp \nu \ll \pi_\sharp \rho \ll \pi_\sharp \mu$. Hence, by Proposition 2.6.8, $\nu = \rho$. Therefore, ρ is ergodic. $\square$

We conclude this section by proving unique ergodicity for the measure ρ. This will be the only part of this argument for which we require the good long-time estimates for the flow given by (2.4.13) (up to this point, whenever we used Corollary 2.4.3, we needed just the qualitative result of global existence and *time-dependent* bounds on the growth of the solution).

In particular, we will prove the following version of Birkhoff's theorem for this process, which in particular implies Theorem 1.1.2.

Proposition 2.6.10. *Let $\Phi_t(\mathbf{u}; \xi)$ be the flow of* (SDNLB)*. For every $u_0 \in X^\alpha$, we have that $\rho_t \rightharpoonup \rho$ as $t \to \infty$, where ρ_t is defined by*

$$\int F(\mathbf{u}) \mathrm{d}\rho_t(\mathbf{u}) := \frac{1}{t} \int_0^t \mathbb{E}[F(\Phi_{t'}(\mathbf{u}_0, \xi))] \mathrm{d}t'.$$

Proof. Consider the usual decomposition

$$\Phi_t(\mathbf{u}_0, \xi) = S(t)\mathbf{u}_0 + \mathord{\upharpoonleft}_t(\xi) + \mathbf{v}(\mathbf{u}_0, \xi; t),$$

where $\mathbf{v}$ solves (2.1.14). We have that the law μ_t of $S(t)\mathbf{u}_0 + \mathord{\upharpoonleft}_t(\xi) = L(\mathbf{u}_0)$ is tight in X^α, since from Proposition 2.6.6, $\mu_t \rightharpoonup \mu$ as $t \to \infty$. Moreover, by this tightness, the estimate (2.4.13) and the compactness of the embedding $\mathcal{H}^2 \hookrightarrow X^\alpha$, we have that also the law of $\mathbf{v}$ is tight. Therefore, the law of $\Phi(\mathbf{u}_0, \xi)(\cdot)$ is tight, so also the sequence ρ_t is. Hence it is enough to prove that every weak limit point $\overline{\rho}$ of ρ_t satisfies $\overline{\rho} = \rho$. Notice that, by definition, $\overline{\rho}$ is invariant. Let $t_n \to \infty$ be a sequence such that $\rho_{t_n} \rightharpoonup \overline{\rho}$. Consider the random variable

$$Y_t := (S(t)\mathbf{u}_0 + \mathord{\upharpoonleft}_t(\xi), \mathbf{v}(\mathbf{u}_0, \xi; t)) \in X^\alpha \times X^\alpha.$$

By the same argument, the law Y_t is tight in $X^\alpha \times X^\alpha$ (with compact sets of the form $K_\varepsilon \times \{y | \, \|y\|_{\mathcal{H}^2} \leq C_\varepsilon\}$). Therefore, if we define ν_t by

$$\int F(\mathbf{u}_1, \mathbf{u}_2) \mathrm{d}\nu_t(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{t} \int_0^t \mathbb{E}[F(Y_t)] \mathrm{d}t,$$

the family will be tight, with the same compact sets. Hence, up to subsequences, $\nu_{t_n} \rightharpoonup \nu$, with ν concentrated on $X^\alpha \times \mathcal{H}^2$. Define the maps $\mathfrak{S}, \pi_1, \pi_2 : X^\alpha \times X^\alpha \to X^\alpha$ by

$$\begin{aligned}
\mathfrak{S}(x, y) &:= x + y, \\
\pi_1(x, y) &:= x, \\
\pi_2(x, y) &:= y.
\end{aligned}$$

Since $\mathfrak{S}(Y_t) = \Phi(\mathbf{u}_0, \xi)(t)$, then $\mathfrak{S}_\sharp \nu = \overline{\rho}$. Moreover, since $\pi_1(Y_t) = S(t)\mathbf{u}_0 + \mathord{\upharpoonleft}_t$, we have that $(\pi_1)_\sharp \nu_t = \mu_t$, so $(\pi_1)_\sharp \nu = \mu$. Recall the projection $\pi : X^\alpha \to X^\alpha / \mathcal{H}^2$. On $X^\alpha \times \mathcal{H}^2$, we have that $\pi \circ \mathfrak{S} = \pi \circ \pi_1$. Therefore, since ν is concentrated on $X^\alpha \times \mathcal{H}^2$,

$$\pi_\sharp \overline{\rho} = \pi_\sharp \mathfrak{S}_\sharp \nu = \pi_\sharp (\pi_1)_\sharp \nu = \pi_\sharp \mu.$$

Hence, by Proposition (2.6.8), we get $\overline{\rho} = \rho$. $\square$

Remark 2.6.11. If we could improve Proposition 2.6.7 to *unique* ergodicity for the measure $\pi_\sharp \mu$, we would automatically improve the result of Corollary 2.6.9 to *unique* ergodicity, without using at all the long time estimates for the growth of $\mathbf{v}$. Indeed, if we knew that the measure $\pi_\sharp \mu$ was uniquely ergodic, then for every invariant measure ν, $\pi_\sharp \nu = \pi_\sharp \mu$ will follow automatically from invariance, and we could apply Proposition 2.6.8 to show $\nu = \rho$.

Chapter 3

Stochastic Wave Equation

3.1 Renormalisation and meaning of the equation

In this chapter, we consider the equation (∞-SNLW),

$$\begin{cases} u_{tt} - \Delta u + u^3 - \infty \cdot u = \xi, \\ (u, u_t)|_{t=0} = \mathbf{u}_0 \in \mathcal{H}^s := H^s \times H^{s-1}, \end{cases} \tag{∞-SNLW}$$

The renormalisation that we apply in (∞-SNLW) is better described as follows. Recall that ξ is defined to be a distribution-valued random variable such that for every test function ϕ, $\langle \phi, \xi \rangle$ is a Gaussian random variable with mean 0 and variance

$$\mathbb{E}[|\langle \phi, \xi \rangle|^2] = \|\phi\|_{L^2}^2 . \tag{3.1.1}$$

Ignoring the term with ∞ in (∞-SNLW), we consider a perturbative expansion $u = v + \psi$, where

$$\psi := \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} \xi(t') \mathrm{d}t' \tag{3.1.2}$$

solves the linear wave equation

$$\psi_{tt} = \Delta \psi + \xi.$$

Formally, the term v would then solve the equation

$$v_{tt} - \Delta v = -(\psi + v)^3 = -\psi^3 - 3\psi^2 v - 3\psi v^2 - v^3.$$

However, because of the roughness of ξ, it can be shown that the terms ψ^3, ψ^2 do not make sense as space-time distributions. Therefore, we introduce the Wick renormalisation

$$\begin{aligned} :\psi^2: &= \psi^2 - \mathbb{E}[|\psi|^2], \\ :\psi^3: &= \psi^3 - 3\mathbb{E}[|\psi|^2]\psi, \end{aligned} \tag{3.1.3}$$

and *define* $v = u - \psi$ to solve the equation

$$\begin{cases} v_{tt} - \Delta v = - :\psi^3: -3 :\psi^2: v^2 - 3\psi v - v^3 \\ v(0, \cdot) = u_0 \in H^s_{\mathrm{loc}}(\mathbb{R}^2), \\ v_t(0, \cdot) = u_1 \in H^{s-1}_{\mathrm{loc}}(\mathbb{R}^2). \end{cases} \tag{3.1.4}$$

While both terms on the right hand side of (3.1.3) diverge (for both definitions), it is actually possible to give a meaning to the renormalised terms $:\psi^2:, :\psi^3:$ by first taking a smooth approximation of the noise ξ and then taking a limit in the space $W^{-\varepsilon,\infty}_{\mathrm{loc}}$. This will be achieved in the next section.

Denoting (formally) $:u^3: = u^3 - 3\mathbb{E}[|\psi|^2]u$, solving the equation (3.1.4) for v corresponds to solving the equation

$$\begin{cases} u_{tt} = \Delta u - :u:^3 + \xi, \\ u(t_0, \cdot) = u_0 \in H^s_{\text{loc}}, \\ u_t(t_0, \cdot) = u_1 \in H^{s-1}_{\text{loc}} \end{cases} \tag{SNLW}$$

for u. Since $\mathbb{E}[|\psi|^2] = +\infty$ for every $t > 0$, by inserting this into (∞-SNLW) we obtain the formula (∞-SNLW). This kind of renormalisation is exactly the same that appears in [8] for the cubic wave equation on the torus.

As in the case for beam equation, we need to define the meaning of solutions to (∞-SNLW). As we already discussed, we write $u = \psi + v$, and we say that u is a solution of (SNLW) in the interval I if $v \in C(I; \mathcal{H}^s_{\text{loc}})$ solves the mild formulation of (3.1.4),

$$\begin{aligned} v(t) = \cos(t(-t_0(|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1 \\ + \int_{t_0}^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}\Big(- :\psi^3: (t') - 3 :\psi^2: (t')v(t') - 3\psi v^2(t') - v^3(t')\Big)\mathrm{d}t'. \end{aligned} \tag{3.1.5}$$

3.2 Stochastic convolution and Wick powers

In this section, we will explore the regularity of the function ψ and the define and study the objects $:\psi^l:$. We provide a unified argument for both cases, which is also more suitable for generalisations for manifolds different from $\mathbb{T}^2$, $\mathbb{R}^2$, or wave equations associated with operators different from the standard Laplacian. See [20] for a situation in which this has be done for the Laplace-Beltrami operator on a compact manifold.

Lemma 3.2.1. *Let f, g be test functions, and let $0 \le s \le t$. Then*

$$\begin{aligned} \mathbb{E}\langle \psi(t), f\rangle \langle \psi(s), g\rangle = \Big\langle \frac{1}{2|\nabla|^2}\Big(s\cos((t-s)|\nabla|) - \frac{\sin(s|\nabla|)\cos(t|\nabla|)}{|\nabla|}\Big) f, g\Big\rangle \\ \iint K_{s,t}(x-y)f(x)g(y)\mathrm{d}x\mathrm{d}y = \langle K_{s,t} * f, g\rangle, \end{aligned} \tag{3.2.1}$$

where $K_{s,t} \in L^p$ for every $p < +\infty$. Moreover, $K_{s,t} \in L^p$ is a continuous function of s, t. Moreover, if $s, t \in \mathbb{R}$, similar formulas hold, with

$$K_{s,t} = \begin{cases} K_{|s|,|t|} & \text{if } s \cdot t \ge 0, |s| \le |t|, \\ K_{|t|,|s|} & \text{if } s \cdot t \ge 0, |s| > |t|, \\ 0 & \text{if } s \cdot t < 0. \end{cases} \tag{3.2.2}$$

Proof. By the definition of ψ (3.1.2), and the universal property of white noise (1.0.2), we have

$$\begin{aligned} \mathbb{E}\langle \psi(t), f\rangle \langle \psi(s), g\rangle &= \mathbb{E}\int_0^t \Big\langle \xi(t'), \frac{\sin((t-t')|\nabla|)}{|\nabla|} f\Big\rangle \mathrm{d}t' \int_0^s \Big\langle \xi(t'), \frac{\sin((s-t')|\nabla|)}{|\nabla|} g\Big\rangle \mathrm{d}t' \\ &= \int_0^s \Big\langle \frac{\sin((t-t')|\nabla|)}{|\nabla|} f, \frac{\sin((s-t')|\nabla|)}{|\nabla|} g\Big\rangle \mathrm{d}t' \\ &= \Big\langle \int_0^s \frac{\sin((t-t')|\nabla|)\sin((s-t')|\nabla|)}{|\nabla|^2} f\mathrm{d}t', g\Big\rangle, \end{aligned}$$

from which we obtain the first equality in (3.2.1) after integration. With the same computation, we can show (3.2.2). By this equality, the Fourier transform of $K_{s,t}$ is given by

$$\widehat{K_{s,t}(\xi)} = \frac{1}{2|\xi|^2}\Big(s\cos((t-s)|\xi|) - \frac{\sin(s|\xi|)\cos(t|\xi|)}{t|\xi|}\Big), \tag{3.2.3}$$

so for every $1 < q < 2$, $\widehat{K_{s,t}}$ is in L^q (and the map is continuous in t, s). Hence, by Hausdorff-Young's inequality, $K_{s,t} \in L^{q'}$ and it is continuous in t, s. □

Lemma 3.2.2. *Let p_τ be the heat kernel. Define $\psi_\tau := \psi * p_\tau$, and let*

$$\begin{cases} :\psi_\tau^1: := \psi_\tau, \\ :\psi_\tau^2: := \psi_\tau^2 - \mathbb{E}[\psi_\tau^2], \\ :\psi_\tau^3: := \psi_\tau^3 - 3\mathbb{E}[\psi_\tau^2]\psi_\tau. \end{cases} \tag{3.2.4}$$

Then for every test function f, g,

$$\mathbb{E}\left\langle :\psi_\tau^l: (t) f, :\psi_{\tau'}^l: (s) g\right\rangle = l! \iint (K_{s,t} * p_{\tau+\tau'})^l (x-y) f(x) g(y) \mathrm{d}x \mathrm{d}y. \tag{3.2.5}$$

Proof. By (3.2.1) and a simple application of Proposition A.3.1, we have that $\psi(s), \psi(t) \in C^{-\varepsilon}$ almost surely. Hence, $\psi_\tau(s), \psi_\tau(t)$ are smooth functions, and satisfy

$$\begin{aligned} \iint f(x)g(y)\mathbb{E}\psi_\tau(s,x)\psi_{\tau'}(t,y)\mathrm{d}x\mathrm{d}y &= \mathbb{E}\left\langle \psi_\tau, f\right\rangle \left\langle \psi_{\tau'}, g\right\rangle \\ &= \mathbb{E}\left\langle \psi, f * p_\tau\right\rangle \left\langle \psi, g * p_{\tau'}\right\rangle \\ &= \left\langle K_{s,t} * f * p_\tau, g * p_{\tau'}\right\rangle \\ &= \left\langle K_{s,t} * p_{\tau+\tau'} * f, g\right\rangle \\ &= \iint (K_{s,t} * p_{\tau+\tau'})(x-y) f(x) g(y) \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Therefore, $\mathbb{E}\psi_\tau(s,x)\psi_\tau(t,y) = (K_{s,t} * p_{\tau+\tau'})(x-y)$ for almost every x, y. Hence we have, using the fundamental property of Wick products (A.2.2)

$$\begin{aligned} \mathbb{E}\left\langle :\psi_\tau^l:, f\right\rangle \left\langle \psi_{\tau'}^l, g\right\rangle &= \iint f(x)g(y)\mathbb{E} :\psi_\tau^l: (s,x) :\psi_{\tau'}^l: (t,y)\mathrm{d}x\mathrm{d}y \\ &= \iint f(x)g(y)\big(\mathbb{E}\psi_\tau(s,x)\psi_{\tau'}(t,y)\big)^l \mathrm{d}x\mathrm{d}y \\ &= \iint (K_{s,t} * p_{\tau+\tau'})^l (x-y) f(x) g(y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

□

Proposition 3.2.3. *Let $1 \le l \le 3$. For every t, the sequence $:\psi_\tau^l: (t)$ is Cauchy in the space $L^p(\mathbb{P}; C^{-\varepsilon}_{\mathrm{loc}})$ for every $p < +\infty$, so it converges to a unique limit $:\psi^l: (t)$. Moreover,*

$$\mathbb{E}\left\langle :\psi^l: (t) f, :\psi^l: (s) g\right\rangle = l! \iint (K_{s,t})^l (x-y) f(x) g(y) \mathrm{d}x\mathrm{d}y, \tag{3.2.6}$$

and $:\psi^l: (t)$ admits a version which belongs to $C(\mathbb{R}; C^{-\varepsilon}_{\mathrm{loc}})$ almost surely.

Proof. We have that $:\psi_\tau^l: (0) = 0$. Moreover, for $\tau, \tau' > 0$, $s \le t$, by the formula (3.2.5), if f is supported on a ball B,

$$\begin{aligned} &\mathbb{E}\left|\left\langle (:\psi_\tau^l: (t) - :\psi_{\tau'}^l: (t)), f\right\rangle\right|^2 \\ &= l! \iint \left[(K_{t,t} * p_{2\tau})^l - 2(K_{t,t} * p_{\tau+\tau'})^l + (K_{t,t} * p_{2\tau'})^l\right](x-y) f(x) f(y) \mathrm{d}x\mathrm{d}y \\ &\lesssim \left(\left\|(K_{t,t} * p_{2\tau})^l - (K_{t,t} * p_{\tau+\tau'})^l\right\|_{L^q(B)} + \left\|(K_{t,t} * p_{\tau+\tau'})^l - (K_{t,t} * p_{2\tau'})^l\right\|_{L^q(B)} \right) \|f\|_{L^{q'}}^2 . \end{aligned}$$

By Lemma 3.2.1 and (3.2.3) (or by (3.2.3) and Hausdorff-Young inequality), $K_{t,t} * p_\delta \to K_{t,t}$ in L^p as $\delta \to 0$. Therefore,

$$\left\|(K_{t,t} * p_{2\tau})^l - (K_{t,t} * p_{\tau+\tau'})^l\right\|_{L^q(B)} + \left\|(K_{t,t} * p_{\tau+\tau'})^l - (K_{t,t} * p_{2\tau'})^l\right\|_{L^q(B)} \to 0,$$

so (A.3.1) holds. Moreover, by (A.2.4), we have (A.3.2). Therefore, by Proposition A.3.3, we have that for p big enough,

$$\lim_{\tau,\tau'\to 0} \mathbb{E}\left\|:\psi_\tau^l:(t) - :\psi_{\tau'}^l:(t)\right\|_{C^{-\varepsilon}(B)}^p = 0.$$

By the same limiting procedure, we get (3.2.6). Finally, for a test function f,

$$\begin{aligned}
&\mathbb{E}|\left\langle :\psi^l:(t) - :\psi^l:(s), f\right\rangle|^2 \\
&= l! \iint (K_{t,t}^l - 2K_{s,t}^l + K_{s,s}^l)(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y \\
&\lesssim \left(\|K_{t,t} - K_{s,t}\|_{L^{lq}} + \|K_{s,t} - K_{s,s}\|_{L^{lq}}\right)\left(\|K_{t,t}\|_{L^{lq}} + \|K_{s,t}\|_{L^{lq}} + \|K_{s,s}\|_{L^{lq}}\right)^{l-1} \|f\|_{L^{q'}}^2 .
\end{aligned}$$

By Lemma 3.2.1, $K \in L^{lq}$ for every $q < +\infty$. Moreover, by (3.2.3),

$$\begin{aligned}
\left|\widehat{K_{t,t}}(\xi) - \widehat{K_{s,t}}(\xi)\right| &= \left|\frac{1}{|\xi|^2}\Big(t - s\cos((t-s)|\xi|) - \frac{(\sin(t|\xi|) - \sin(s|\xi|))\cos(t|\xi|)}{2|\xi|}\Big)\right| \\
&\lesssim \frac{t-s}{|\xi|^2} + \frac{(t-s)^\theta}{|\xi|^{2-\theta}}.
\end{aligned}$$

Therefore, by Hausdorff-Young, if $0 < \theta \ll 1$ satisfies $(2-\theta)(1 + \frac{1}{lq-1}) > 1$, we have that $\|K_{t,t} - K_{s,t}\|_{L^{lq}(B)} \lesssim (t-s)^\theta$. Therefore, by Proposition A.3.4, for every $q < +\infty$, $\varepsilon > 0$, we have $:\psi^l: \in C(\mathbb{R}; C_{\mathrm{loc}}^{-\frac{d}{q}-\varepsilon})$. Since q, ε are arbitrary, we get $:\psi^l: \in C(\mathbb{R}; C_{\mathrm{loc}}^{-\varepsilon})$. ☐

Lemma 3.2.4. *Let $m : \mathbb{R} \to \mathbb{R}$ be a smooth even function such that $m(\lambda) = 1$ for $|\lambda| \le 1$, and $m(\lambda) = \frac{1}{\lambda^\delta}$ for $|\lambda| > 3$. Consider the operator $I_N := m(|\nabla|/N)$. For every ball B, and every $p \ge 2$, we have the estimate*

$$\left(\mathbb{E}[\|I_N\psi\|_{L^p(B)}^p]\right)^{\frac{1}{p}} \lesssim p^{\frac12}|B|^{\frac1p}t^{\frac12}(\log N)^{\frac12}. \tag{3.2.7}$$

Moreover, for every smooth ρ with compact support, the same estimate holds

$$\left(\mathbb{E}[\|I_N\rho\psi\|_{L^p}^p]\right)^{\frac{1}{p}} \lesssim_\rho p^{\frac12}t^{\frac12}(\log N)^{\frac12}. \tag{3.2.8}$$

Proof. From the formulas (3.2.1), (3.2.3),

$$\begin{aligned}
\mathbb{E}[|I_N\psi(x_0)|^2] &= \iint K_{t,t}(x-y)I_N\delta_{x_0}(x)I_N\delta_{x_0}(y)\mathrm{d}x\mathrm{d}y \\
&\sim \int \widehat{K_{t,t}}(\xi)m\left(\frac{|\xi|}{N}\right)^2\mathrm{d}\xi \\
&\lesssim \int \frac{t}{\langle|\xi|\rangle^2}m\left(\frac{|\xi|}{N}\right)^2\mathrm{d}\xi \\
&\lesssim t\int_0^{3N}\frac{1}{\langle|\xi|\rangle^2} + t\int_{3N}^{+\infty}N^\delta\frac{1}{|\xi|^{2+\delta}}\mathrm{d}\xi \\
&\lesssim t\Big(\log N + N^\delta\frac{1}{N^\delta}\Big) \lesssim t\log N,
\end{aligned}$$

where the integrals become sums on $\xi \in \mathbb{Z}^2$ for the case of the torus $\mathbb{T}^2$.
Hence, by (A.2.4),

$$\begin{aligned}
\left(\mathbb{E}[\|I_N\psi\|_{L^p(B)}^p]\right)^{\frac{1}{p}} &= \left(\mathbb{E}\int_B |I_N\psi(x_0)^p|\mathrm{d}x_0\right)^{\frac1p} \\
&= \left(\int_B \mathbb{E}|I_N\psi(x_0)^p|\mathrm{d}x_0\right)^{\frac1p}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim p^{\frac{1}{2}} \left(\int_B (\mathbb{E}|I_N \psi(x_0)^2)^{\frac{p}{2}} \mathrm{d}x_0 \right)^{\frac{1}{p}} \\
&\lesssim p^{\frac{1}{2}} \left(\int_B (t \log N)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&\lesssim p^{\frac{1}{2}} |B|^{\frac{1}{p}} t^{\frac{1}{2}} \log N^{\frac{1}{2}}.
\end{aligned}
$$

Proceeding similarly, we have that

$$
\begin{aligned}
\mathbb{E}[|I_N \rho\psi(x_0)|^2] &\sim \int \widehat{K_{t,t}}(\xi) \left| \int m\left(\frac{|\xi-\eta|}{N}\right) \hat{\rho}(\eta) e^{-i(\xi-\eta)\cdot x_0} \mathrm{d}\eta \right|^2 \mathrm{d}\xi \\
&= \int \widehat{K_{t,t}}(\xi) m\left(\frac{|\xi|}{N}\right)^2 \left| \int \frac{m(|\xi-\eta|)}{m(|\xi|)} \hat{\rho}(\eta) e^{i\eta\cdot x_0} \mathrm{d}\eta \right|^2 \mathrm{d}\xi \\
\text{Integration by parts } &\lesssim \int \widehat{K_{t,t}}(\xi) m\left(\frac{|\xi|}{N}\right)^2 \left\| \frac{m(|\xi-\eta|)}{m(|\xi|)} \right\|_{C^2_\eta}^2 \|\hat{\rho}\|_{W^{2,1}}^2 \frac{1}{\langle x_0 \rangle^4} \mathrm{d}\xi \\
&\lesssim_\rho \frac{1}{\langle x_0 \rangle^4} \int \widehat{K_{t,t}}(\xi) m\left(\frac{|\xi|}{N}\right)^2 \mathrm{d}\xi \\
&\lesssim \frac{1}{\langle x_0 \rangle^4} t \log N,
\end{aligned}
$$

so

$$
\begin{aligned}
\left(\mathbb{E}[\|I_N \rho\psi\|_{L^p}^p]\right)^{\frac{1}{p}} &\lesssim p^{\frac{1}{2}} \left(\int (\mathbb{E}|I_N \rho\psi(x_0)^2)^{\frac{p}{2}} \mathrm{d}x_0 \right)^{\frac{1}{p}} \\
&\lesssim_\rho p^{\frac{1}{2}} \left(\int \frac{1}{\langle x_0 \rangle^{2p}} \mathrm{d}x_0 \right)^{\frac{1}{p}} t^{\frac{1}{2}} (\log N)^{\frac{1}{2}} \\
&\lesssim_\rho p^{\frac{1}{2}} t^{\frac{1}{2}} (\log N)^{\frac{1}{2}}.
\end{aligned}
$$

□

3.3 Local in time theory

3.3.1 SNLW on the torus

In this subsection, we will prove the following, which establishes local well posedness for the equation (∞-SNLW) posed on the torus.

Proposition 3.3.1. *Let $1 > s \geq \frac{2}{3}$, and let $\mathbf{u} = (u_0, u_1)^T \in \mathcal{H}^s(\mathbb{T}^2)$. Consider the equation* (3.1.5)*, starting with initial data $(v(t_0), v_t(t_0))^T = \mathbf{u}_0$. There exists a stopping time*

$$
T = T(\{\|{:}\psi^l{:}\|_{C_t([t_0-1,t_0+1];C^{-\varepsilon})}\}_{l=1,2,3}, \|\mathbf{u}_0\|_{\mathcal{H}^s}) \tag{3.3.1}
$$

such that $T > 0$ a.s., T is nonincreasing in its variables, and such that the equation (3.1.5) *has a unique solution $\mathbf{v} = (v, \partial_t v)^T$ in the space $C([t_0 - T, t_0 + T]; \mathcal{H}^s)$.*

Remark 3.3.2. Since the condition given by the global argument is $s > \frac{4}{5}$, we keep the argument simple and just show this result for $s \geq \frac{2}{3}$. However, with more refined techniques, it is possible to show a similar result for $s > \frac{1}{4}$, as it is shown in [8].

Proof. This proposition will follow by a Banach fixed point argument. Consider the map

$$\begin{aligned}\Gamma_{\mathbf{u}_0} v &:= \cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1 \\ &+ \int_{t_0}^{t} \frac{\sin((t-t')|\nabla|)}{|\nabla|}\left(\sum_{j=0}^{3}\binom{3}{j} v^j :\psi^{3-j}:\right) \mathrm{d}t'.\end{aligned} \tag{3.3.2}$$

defined on functions $v \in C_t([t_0-T, t_0+T]; H^s)$, for some $0 < T \leq 1$. We want to show that for every $R > 2\|\mathbf{u}_0\|_{\mathcal{H}^s}$, there exists some (random) $T > 0$ such that $\Gamma_{\mathbf{u}_0}$ is a contraction on the ball $B(0,R) \subset C_t([t_0-T, t_0+T]; H^s)$.

By the Sobolev embedding $H^s \hookrightarrow H^{\frac{2}{3}} \hookrightarrow W^{0+,6-} \hookrightarrow L^6$, and $s-1<0$, we have that

$$\begin{aligned}\|\Gamma_{\mathbf{u}_0} v\|_{H^s} &\leq \|\mathbf{u}_0\|_{\mathcal{H}^s} + \int_{t_0-T}^{t_0+T} \left\|\sum_{j=0}^{3}\binom{3}{j} v^j :\psi^{3-j}:\right\|_{H^{s-1}} \mathrm{d}t' \\ &\leq \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v^3\|_{L_t^\infty H_x^{s-1}} + \sum_{j=0}^{2}\binom{3}{j}\|v^j :\psi^{3-j}:\|_{L_t^\infty H_x^{s-1}}\Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v\|^3_{L_t^\infty L^6} + \sum_{j=0}^{2}\binom{3}{j}\|v\|^j_{L_t^\infty W^{0+,6-}}\|:\psi^{3-j}:\|_{L_t^\infty W_x^{0-,6+}}\Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v\|^3_{L_t^\infty H_x^s} + \sum_{j=0}^{2}\binom{3}{j}\|v\|^j_{L_t^\infty H_x^s}\|:\psi^{3-j}:\|_{L_t^\infty W_x^{0-,6+}}\Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + T\Big(\|v\|^3_{C_tH^s} + (1+\|v\|^2_{C_tH^s})\max_{1\leq j\leq 3}\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})}\Big).\end{aligned} \tag{3.3.3}$$

By Proposition 3.2.3, $\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})}$ is finite a.s., so given $\eta>0$, for

$$T = T_0(\max_{1\leq j\leq 3}\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})}) \ll 1,$$

we have $T\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})} \leq \eta$. Therefore, for $T \leq T_0$, we have that

$$\|\Gamma_{\mathbf{u}_0} v\|_{H^s} \lesssim R + 2TR^3 + (1+R^2)\eta,$$

so if we choose $\eta \ll \frac{1}{1+R^2}$ and $T \leq T_1(R)$ as well, we get that $\Gamma_{\mathbf{u}_0}$ maps $B(0,R)$ into itself.

Proceeding similarly, we have that

$$\begin{aligned}\|\Gamma_{\mathbf{u}_0} v - \Gamma_{\mathbf{u}_0} w\| &\lesssim T\Big(\|v-w\|_{C_tH^s}(\|v\|^2_{C_tH^s} + \|w\|^2_{C_tH^s}) \\ &\quad + \|v-w\|_{C_tH^s}(1+\|v\|_{C_tH^s}+\|w\|_{C_tH^s})\max_{j=1,2}\|:\psi^j:\|_{L_t^\infty W_x^{0-,6+}}\Big) \\ &\lesssim \|v-w\|_{C_tH^s}(R^2T + (1+R)\eta).\end{aligned}$$

Therefore, taking $T \ll \frac{1}{R^2}$, and η as before, we have that $\Gamma_{\mathbf{u}_0}$ is a contraption on $B(0,R) \subseteq C_tH^s$. Therefore, there exists a unique solution to (3.1.5) in the interval $[[t_0-T, t_0+T]$ with $v \in C_tH^s$. We just need to show that this solution satisfies $\partial_t v \in C_tH^{s-1}$. This solution satisfies $\Gamma_{\mathbf{u}_0} v = v$, so proceeding as in (3.3.3)

$$\begin{aligned}&\|\partial_t v(t)\|_{H^{s-1}} \\ &= \|\partial_t(\Gamma_{\mathbf{u}_0} v)(t)\|_{H^{s-1}} \\ &= \left\| -|\nabla|\sin((t-t_0)|\nabla|)u_0 + \cos((t-t_0))\mathbf{u}_1 + \int_{t_0}^{t}\cos((t-t')|\nabla|)\left(\sum_{j=0}^{3}\binom{3}{j}v^j :\psi^{3-j}:\right)\mathrm{d}t\right\|_{H^{s-1}}\end{aligned}$$

$$\lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_{t_0-T}^{t_0+T} \left\| \sum_{j=0}^{3} \binom{3}{j} v^j :\psi^{3-j}: \right\|_{H^{s-1}} \mathrm{d}t'$$
$$\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + T\Big(\|v\|^3_{C_tH^s} + (1+\|v\|^2_{C_tH^s}) \max_{1\le j\le 3} \|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})}\Big)$$
$$< +\infty.$$

□

Proposition 3.3.3 (Blowup condition for (SNLW)). *Let $s \geq \frac{2}{3}$, let $\mathbf{u}_0 \in \mathcal{H}^s$, and let $v \in C((-T^*,T^*);H^s)$ be a solution of (3.1.5) with $\partial_t v \in C((-T^*,T^*);H^{s-1})$. Suppose that $T^* < +\infty$, and that the solution v cannot be extended to any interval of the form $(-T^*-\varepsilon, T^*+\varepsilon)$. Then*

$$\begin{cases} \limsup_{t\to T^*} \|v(t)\|_{H^s} = +\infty, & or \\ \limsup_{t\to T^*} \|\partial_t v(t)\|_{H^{s-1}} = +\infty, & or \\ \limsup_{t\to -T^*} \|v(t)\|_{H^s} = +\infty, & or \\ \limsup_{t\to -T^*} \|\partial_t v(t)\|_{H^{s-1}} = +\infty. \end{cases} \tag{3.3.4}$$

Proof. Suppose that (3.3.4) is not satisfied, i.e. all the lim sup-s are finite. We want to show that we can extend v to an interval of the form $(-T^*-\varepsilon, T^*+\varepsilon)$.

Call M the maximum of the lim sup-s in (3.3.4), and let

$$T_0 = T(\{\|:\psi^l:\|_{C_t([-T^*,T^*];C^{-\varepsilon}}\}_{l=1,2,3}, 2M)).$$

Let $t_0 = T^* - \frac{T_0}{2}$. By Proposition 3.3.1, we can build a solution of (3.1.5) in the interval $[t_0-T_0, t_0+T_0] = [T^*-\frac{3}{2}T_0, T^*+\frac{1}{2}T_0]$, hence we can extend v to the interval $(-T^*, T^*+\frac{1}{2}T_0)$. Similarly, by choosing $t_0 = -T^*+\frac{1}{2}T_0$, we can extend v to the interval $(-T^*-\frac{1}{2}T_0, T^*+\frac{1}{2}T_0)$. □

3.3.2 SNLW on $\mathbb{R}^2$

In this section, we tackle the local in time theory for the equation (SNLW) posed on $\mathbb{R}^2$. As opposed for the case of the torus, a Banach fixed point argument on some (appropriately crafted) Banach space does not seem to be available. Indeed, the law of the term ψ is space invariant, and because of finite speed of propagation, the values of ψ are independent on far away space-time regions[1]. Therefore for every time t, we expect ψ to have peaks of arbitrary height, which is an obstruction to closing a fixed point argument (since ψ could push v to have arbitrarily high norm on arbitrary short time, somewhere in space). To overcame these issues, we consider the auxiliary equation

$$v(t) = \cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1 + \int_{t_0}^{t} \frac{\sin((t-t')|\nabla|)}{|\nabla|} \times \left(v^3 + \sum_{j=0}^{2}\binom{3}{j} v^j \rho :\psi^{3-j}: \right), \tag{LSNLW}$$

where ρ is a smooth, compactly supported function. Notice that, formally, v satisfies

$$\begin{aligned} v_{tt} - \Delta v + v^3 + 3v^2\rho\psi + 3v\rho :\psi^2: + \rho :\psi^3:= 0, \\ v(t_0,x) = u_0(x) \qquad \partial_t v(t_0,x) = u_1(x). \end{aligned} \tag{3.3.5}$$

[1]From the formula (3.2.3) and a simple application of Paley-Weiner's theorem, we have that the covariance kernel $K_{s,t}$ is supported in the ball $B(0,|t|+|s|) \subset \mathbb{R}^2$. Hence, $\psi(t)|_{B(x_0,R_0)}$ and $\psi(s)|_{B(y_0,R_1)}$ are independent as long as $|x_0-y_0| > R_0+R_1+|t|+|s|$

and

$$v^3 + \sum_{j=0}^{2} \binom{3}{j} v^j \rho :\psi^{3-j}: =: (v+\psi)^3:$$

whenever $\rho = 1$. The effect of the cutoff function ρ is essentially making the forcing terms $\rho :\psi^{3-j}:$ bounded, so we can perform a similar analysis to the one for (SNLW) on the torus. Indeed, we have the following

Proposition 3.3.4. *Let $1 > s \geq \frac{2}{3}$, and let $\mathbf{u} = (u_0, u_1)^T \in \mathcal{H}^s$. Consider the equation* (LSNLW)*, starting with initial data $(v(t_0), v_t(t_0))^T = \mathbf{u}_0$. There exists a stopping time*

$$T = T(\{\|:\psi^l:\|_{C_t([t_0-1,t_0+1];C^{-\varepsilon}(\operatorname{supp}(\rho)))}\}_{l=1,2,3}, \|\mathbf{u}_0\|_{\mathcal{H}^s}) \tag{3.3.6}$$

such that $T > 0$ a.s.,T is nonincreasing in its variables, and such that the equation (3.1.5) *has a unique solution $\mathbf{v} = (v, \partial_t v)^T$ in the space $C([t_0 - T, t_0 + T]; \mathcal{H}^s)$.*

Proof. This proof is essentially the same as the one for the torus case. We consider

$$\begin{aligned} \Gamma_{\mathbf{u}_0} v &:= \cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|} u_1 \\ &\quad + \int_{t_0}^{t} \frac{\sin((t-t')|\nabla|)}{|\nabla|} \left(v^3 + \sum_{j=0}^{2} \binom{3}{j} v^j \rho :\psi^{3-j}: \right) \mathrm{d}t', \end{aligned} \tag{3.3.7}$$

and our goal is to show that for short T, $\Gamma_{\mathbf{u}_0}$ is a contraction on the ball $B(0,R) \subset C_t([t_0 - T, t_0 + T]; H^s)$, for $R > 2\|\mathbf{u}_0\|_{\mathcal{H}^s}$.

By the Sobolev embedding $H^s \hookrightarrow H^{\frac{2}{3}} \hookrightarrow W^{0+,6-} \hookrightarrow L^6$, and $s - 1 < 0$, we have that

$$\begin{aligned} \|\Gamma_{\mathbf{u}_0} v\|_{H^s} &\leq \|\mathbf{u}_0\|_{\mathcal{H}^s} + \int_{t_0-T}^{t_0+T} \left\| \left(v^3 + \sum_{j=0}^{2} \binom{3}{j} v^j \rho :\psi^{3-j}: \right) \right\|_{H^{s-1}} \mathrm{d}t' \\ &\leq \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v^3\|_{L^\infty_t H^{s-1}_x} + \sum_{j=0}^{2} \binom{3}{j} \|v^j \rho :\psi^{3-j}:\|_{L^\infty_t H^{s-1}_x} \Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v\|^3_{L^\infty_t L^6} + \sum_{j=0}^{2} \binom{3}{j} \|v\|^j_{L^\infty_t W^{0+,6-}} \|\rho :\psi^{3-j}:\|_{L^\infty_t W^{0-,6+}_x} \Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + 2T\Big(\|v\|^3_{L^\infty_t H^s_x} + \sum_{j=0}^{2} \binom{3}{j} \|v\|^j_{L^\infty_t H^s_x} \|\rho :\psi^{3-j}:\|_{L^\infty_t W^{0-,6+}_x} \Big) \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + T\Big(\|v\|^3_{C_t H^s} + (1 + \|v\|^2_{C_t H^s}) \max_{1\leq j\leq 3} \|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-}(\operatorname{supp}(\rho)))} \Big). \end{aligned} \tag{3.3.8}$$

Recall that $\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-}(\operatorname{supp}(\rho)))}$ is finite almost surely, by Proposition 3.2.3. Therefore, given $\eta > 0$, for

$$T = T_0(\max_{1\leq j\leq 3} \|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-}(\operatorname{supp}(\rho))}) \ll 1,$$

we have $T\|:\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-}(\operatorname{supp}(\rho)))} \leq \eta$. Therefore, for $T \leq T_0$, we have that

$$\|\Gamma_{\mathbf{u}_0} v\|_{H^s} \lesssim R + 2TR^3 + (1+R^2)\eta,$$

so if we choose $\eta \ll \frac{1}{1+R^2}$ and $T \leq T_1(R)$ as well, we get that $\Gamma_{\mathbf{u}_0}$ maps $B(0,R)$ into itself.

Proceeding similarly, we have that

$$\begin{aligned}\|\Gamma_{\mathbf{u}_0} v - \Gamma_{\mathbf{u}_0} w\|_{H^s} &\lesssim T\Big(\|v-w\|_{C_t H^s} (\|v\|_{C_tH^s}^2 + \|w\|_{C_tH^s}^2) \\ &\qquad + \|v-w\|_{C_tH^s} (1 + \|v\|_{C_tH^s} + \|w\|_{C_tH^s}) \max_{j=1,2} \|\rho :\psi^j:\|_{L_t^\infty W_x^{0-,6+}} \Big) \\ &\lesssim \|v-w\|_{C_tH^s} (R^2 T + (1+R)\eta).\end{aligned} \tag{3.3.9}$$

Therefore, taking $T \ll \frac{1}{R^2}$, and η as before, we have that $\Gamma_{\mathbf{u}_0}$ is a contraption on $B(0,R) \subseteq C_tH^s$. As in Proposition 3.3.1, we just need to show that this solution satisfies $\partial_t v \in C_t H^{s-1}$. This solution solves $\Gamma_{\mathbf{u}_0} v = v$, so proceeding as in (3.3.8)

$$\begin{aligned}&\|\partial_t v(t)\|_{H^{s-1}} \\ &= \|\partial_t(\Gamma_{\mathbf{u}_0} v)(t)\|_{H^{s-1}} \\ &= \Bigg\| -|\nabla| \sin((t-t_0)|\nabla|) u_0 + \cos((t-t_0))\mathbf{u}_1 \\ &\qquad + \int_{t_0}^t \cos((t-t')|\nabla|) \left(\sum_{j=0}^3 \binom{3}{j} v^j :\psi^{3-j}: \right) \mathrm{d}t' \Bigg\|_{H^{s-1}} \\ &\lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_{t_0-T}^{t_0+T} \left\| \left(v^3 + \sum_{j=0}^2 \binom{3}{j} v^j \rho :\psi^{3-j}: \right) \right\|_{H^{s-1}} \mathrm{d}t' \\ &\lesssim \|\mathbf{u}_0\|_{\mathcal{H}^s} + T\Big(\|v\|_{C_tH^s}^3 + (1 + \|v\|_{C_tH^s}^2) \max_{1\le j\le 3} \|\rho :\psi^j:\|_{C_t([t_0-1,t_0+1];C^{0-})} \Big) \\ &< +\infty.\end{aligned}$$

□

Proposition 3.3.5 (Blowup condition for (SNLW))**.** *Let $s \geq \frac{2}{3}$, let $\mathbf{u}_0 \in \mathcal{H}^s$, and let $v \in C((-T^*, T^*); H^s)$ be a solution of* (LSNLW) *with $\partial_t v \in C((-T^*, T^*); H^{s-1})$. Suppose that $T^* < +\infty$, and that the solution v cannot be extended to any interval of the form $(-T^* - \varepsilon, T^* + \varepsilon)$. Then*

$$\begin{cases} \limsup_{t\to T^*} \|v(t)\|_{H^s} = +\infty, & \textit{or} \\ \limsup_{t\to T^*} \|\partial_t v(t)\|_{H^{s-1}} = +\infty, & \textit{or} \\ \limsup_{t\to -T^*} \|v(t)\|_{H^s} = +\infty, & \textit{or} \\ \limsup_{t\to -T^*} \|\partial_t v(t)\|_{H^{s-1}} = +\infty. \end{cases} \tag{3.3.10}$$

Proof. Suppose that (3.3.10) is not satisfied, i.e. all the lim sup-s are finite. We want to show that we can extend v to an interval of the form $(-T^* - \varepsilon, T^* + \varepsilon)$.

Call M the maximum of the lim sup-s in (3.3.10), and let

$$T_0 = T(\{\|:\psi^l:\|_{C_t([-T^*,T^*];C^{-\varepsilon}(\mathrm{supp}(\rho)))}\}_{l=1,2,3}, 2M)),$$

where T is defined in (3.3.6). Let $t_0 = T^* - \frac{T_0}{2}$. By Proposition 3.3.4, we can build a solution of (LSNLW) in the interval $[t_0 - T_0, t_0 + T_0] = [T^* - \frac{3}{2}T_0, T^* + \frac{1}{2}T_0]$, hence we can extend v to the interval $(-T^*, T^* + \frac{1}{2}T_0)$. Similarly, by choosing $t_0 = -T^* + \frac{1}{2}T_0$, we can extend v to the interval $(-T^* - \frac{1}{2}T_0, T^* + \frac{1}{2}T_0)$. □

3.4 Global in time theory

3.4.1 Global estimates

In this section, we show that both (SNLW) posed on $\mathbb{T}^2$ and the auxiliary equation (LSNLW) admit solutions which can be extended for infinite times. More precisely, we will prove the following

Proposition 3.4.1. *Let $s > \frac{4}{5}$, and let $\mathbf{u}_0 \in \mathcal{H}^s(\mathbb{T}^2)$. Then there exists a function*

$$F(t) = F(s, \|\mathbf{u}_0\|_{\mathcal{H}^s}, \{\|{:}\psi^l{:}\|_{C([-t,t];C^{-\varepsilon})}\}_{l=1,2,3}, \{(\log N)^{-\frac{1}{2}} \|I_N\psi\|_{L^{\log N}}\}_{p\geq 2}; t),$$

which is finite almost surely, such that any solution v to (3.1.5) *posed on $\mathbb{T}^2$ with $\mathbf{v} = (v, v_t)^T \in \mathcal{H}^s(\mathbb{T}^2)$ satisfies*

$$\|\mathbf{v}(t)\|_{\mathcal{H}^s} \leq F(t). \tag{3.4.1}$$

Similarly, if $s > \frac{4}{5}$ and $\mathbf{u}_0 \in \mathcal{H}^s(\mathbb{R}^2)$, there exists a function

$$F(t) = F(s, \|\mathbf{u}_0\|_{\mathcal{H}^s}, \{\|\rho :\psi^l{:}\|_{C([-t,t];C^{-\varepsilon})}\}_{l=1,2,3}, \{(\log N)^{-\frac{1}{2}} \|I_N(\rho\psi)\|_{L^{\log N}}\}_{p\geq 2}; t),$$

which is finite almost surely, such that any solution v to (LSNLW) *posed on $\mathbb{R}^2$ with $\mathbf{v} = (v, v_t)^T \in \mathcal{H}^s(\mathbb{R}^2)$ satisfies*

$$\|\mathbf{v}(t)\|_{\mathcal{H}^s} \leq F(t).$$

Together with Proposition 3.3.3, this settles the global well posedness result in the case of the torus $\mathbb{T}^2$, i.e. it proves Theorem 1.2.1 in the case $M = \mathbb{T}^2$. The case $M = \mathbb{R}^2$ requires more ingredients, and the proof will be completed in the next subsection.

As for section 2, we present a unified approach, which is hopefully suitable to extensions to even more general situations[2]. For this, it is convenient to notice that the formulation of (LSNLW) on $\mathbb{R}^2$

$$v(t) = \cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1 + \int_{t_0}^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} \times \left(v^3 + \sum_{j=0}^{2}\binom{3}{j} v^j \rho :\psi^{3-j}: \right),$$

formally coincides with the formulation of (3.1.5), when we take $\rho \equiv 1$ (which is a compactly supported smooth function on the torus). Therefore, we can carry over all the computations and estimates for (LSNLW), as long as we check at the same time that every result holds on $\mathbb{T}^2$ as well. As a byproduct, this argument will show global existence of solutions for the equation (LSNLW) posed on $\mathbb{T}^2$, where ρ is any smooth function (and in particular, for $\rho = 0$, to the deterministic defocusing wave equation[3]).

As discussed in the introduction, we consider the energy $E(\mathbf{w})$, computed on some function $\mathbf{w} = (w, w_t)^T$

$$E(\mathbf{w}) := \frac{1}{2}\int w_t^2 + \frac{1}{2}\int w^2 + \frac{1}{2}\int |\nabla w|^2 + \frac{1}{4}\int w^4. \tag{3.4.2}$$

Clearly, one has that $\|\mathbf{w}\|_{\mathcal{H}^1}^2 \lesssim E(\mathbf{w})$. In order to exploit some kind of "almost conservation" of the functional E for solution of (LSNLW), we consider a smooth Fourier multiplier m such that

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq 1, \\ |\xi|^{-(1-s)} & \text{for } |\xi| \geq 3. \end{cases}$$

By the Mikhlin-Hörmander theorem, this multiplier corresponds to a bounded operator with bounded inverse from $W^{\sigma,p}(\mathbb{R}^2)$ to $W^{\sigma+1-s,p}(\mathbb{R}^2)$ for every $1 < p < +\infty$, and every $\sigma \in \mathbb{R}$.

[2]Even if we will use the Fourier transform in a more crucial way compared to Section 2.

[3]the best known threshold for the deterministic case is $s > \frac{2}{5}$, in [25].

Consider $m_N(\xi) := m(\xi/N)$, and let I_N be the associated operator. By scaling, for every $0 \leq \delta \leq 1-s$, one has the estimates

$$\|I_N f\|_{W^{\sigma+\delta,p}} \lesssim N^{\delta} \|f\|_{W^{\sigma,p}}, \tag{3.4.3}$$

$$\|f\|_{W^{\sigma,p}} \lesssim \|I_N f\|_{W^{\sigma+1-s,p}}. \tag{3.4.4}$$

on the Sobolev spaces defined in $\mathbb{R}^2$, and by transference, the same estimates hold for the $\mathbb{T}^2$ Sobolev spaces.

From (3.4.3),(3.4.4), one has that, for fixed N, $\|\cdot\|_{\mathcal{H}^s}$ is equivalent to $\|I_N \cdot\|_{\mathcal{H}^1}$. Therefore, by the blowup conditions (3.3.4) and (3.3.10), in order to show existence of solutions up to time T, it is enough to show that for some N,

$$\sup_{|s|<T} \|(I_N v, I_N v_t)\|_{\mathcal{H}^1} < +\infty. \tag{3.4.5}$$

As already discussed, taking $\mathbf{v} = (v, v_t)^T$, $E(I_N \mathbf{v})$ controls $\|I_N \mathbf{v}\|_{H^1}$, so (3.4.5) (and more precisely, Proposition 3.4.1) is implied by a priori estimates on the functional $E(I_N\mathbf{v})$. In the remaining of this subscetion, we will abuse of notation and omit the subscript N when is not important in the analysis, writing I instead of I_N. Similarly, we will write E instead of $E(I_N\mathbf{v})$, and $E(s)$ instead of $E(I_N\mathbf{v}(s))$. Moreover, in oder to maintain the same notation, we will write integrals in the Fourier variable, even if in the case of the torus, they are actually sums on the lattice $\mathbb{Z}^2$.

Lemma 3.4.2.

$$\partial_t E(I\mathbf{v})$$

$$= -3\int Iv_t (Iv)^2 I(\rho\psi) \tag{3.4.6}$$

$$-3\int Iv_t Iv I(\rho :\psi^2:) - \int Iv_t I(\rho :\psi^3:) \tag{3.4.7}$$

$$+\int Iv_t\big[\big((Iv)^3 - I(v^3)\big) + 3\big((Iv)^2 I(\rho\psi) - I(v^2\rho\psi)\big) + 3\big((Iv)I(\rho:\psi^2:) - I(v\rho:\psi^2:)\big)\big] \tag{3.4.8}$$

$$+\int Iv_t Iv. \tag{3.4.9}$$

Proof. We will show this proposition by a formal computation using (3.3.5). This computation can be made rigorous by noticing that since $\mathbf{v} \in \mathcal{H}^s, s \geq \frac{2}{3}$, from the formula (LSNLW), one has that (3.3.5) is satisfied in distribution. We omit this part of the argument.

By (3.3.5),

$$\begin{aligned}
&\partial_t E(Iv, Iv_t)\\
&= \int Iv_t\Big(Iv_{tt} - I\Delta v + (Iv)^3 + Iv\Big)\\
&= \int Iv_t\Big(-I\big(v^3 + 3v^2\rho\psi + 3v\rho :\psi^2: + \rho :\psi^3:\big) + (Iv)^3 + Iv\Big).
\end{aligned}$$

The lemma follows by adding and subtracting the terms $3(Iv)^2 I(\rho\psi)$ and $3(Iv)I(\rho:\psi^2:)$. □

We will now proceed to estimate the various terms of the time derivative of $E(Iv, Iv_t)$, with the goal of applying a Gronwall argument. The terms (3.4.7), (3.4.9), are relatively harmless. Estimating the commutator terms in (3.4.8) is the core of the I-method, and will take most of this section. However, from a technical point of view, the hardest term to estimate will be (3.4.6), which is also be the main culprit for the estimate on the growth of E in Proposition 3.4.1 not being explicit. This term is also what makes necessary to iterate the I-method, changing N at every step.

Lemma 3.4.3.

$$(3.4.9) \lesssim E(Iv, Iv_t). \tag{3.4.10}$$

Proof. By Hölder,

$$(3.4.9) \leq \|Iv_t\|_{L^2} \|Iv\|_{L^2} \leq E(Iv, Iv_t).$$

□

Lemma 3.4.4. *For every* $0 < \gamma \leq 1 - s$,

$$(3.4.7) \lesssim N^{\gamma}\Big(E^{\frac{3}{4}} \|\rho :\psi^2:\|_{W^{-\gamma,4}} + E^{\frac{1}{2}} \|\rho :\psi^3:\|_{H^{-\gamma}}\Big). \tag{3.4.11}$$

Proof. By Hölder and (3.4.3),

$$\begin{aligned}
& -3\int Iv_t Iv I(\rho :\psi^2:) - \int Iv_t I(\rho :\psi^3:) \\
\lesssim\; & \|Iv_t\|_{L^2} \|Iv\|_{L^4} \big\|I(\rho :\psi^2:)\big\|_{L^4} + \|v_t\|_{L^2} \big\|I(\rho :\psi^3:)\big\|_{L^2} \\
\lesssim\; & N^{\gamma}\Big(E^{\frac{1}{2}+\frac{1}{4}} \|\rho :\psi^2:\|_{W^{-\gamma,4}} + E^{\frac{1}{2}} \|\rho :\psi^3:\|_{H^{-\gamma}}\Big).
\end{aligned}$$

□

Lemma 3.4.5. *Let* $k \leq 3$. *Then*

$$\big\|(Iv)^k - I(v^k)\big\|_{L^2} \lesssim_s N^{-(1-k(1-s))} \|Iv\|_{H^1}^k. \tag{3.4.12}$$

Proof. Let $v_{\lesssim N} = \int_{|\xi|<N/3} \hat{v}(\xi) e^{i\xi\cdot x}$, and let $v_{\gtrsim N} := v - v_{\lesssim N}$.
Since $\widehat{v_{\lesssim N}}(\xi) \neq 0$ only if $|\xi| < N/3$, by definition of I we have that $Iv_{\lesssim N} = v_{\lesssim N}$. Similarly, $\widehat{v^k_{\lesssim N}}(\xi) \neq 0$ only if $|\xi| < kN/3$, so $I\big(v^k_{\lesssim N}\big) = v^k_{\lesssim N}$. Therefore,

$$\begin{aligned}
& (Iv)^k - I(v^k) \\
=& \big(I(v_{\lesssim N} + v_{\gtrsim N})\big)^k - I\big((v_{\lesssim N} + v_{\gtrsim N})^k\big) \\
=& \big(v_{\lesssim N} + I(v_{\gtrsim N})\big)^k - I\big((v_{\lesssim N} + v_{\gtrsim N})^k\big) \\
=& v^k_{\lesssim N} - I\big(v^k_{\lesssim N}\big) + \sum_{l=0}^{k-1} \binom{k}{l} \Big((Iv_{\gtrsim N}) v^l_{\lesssim N} (Iv_{\gtrsim N})^{k-l-1} - I\big(v_{\gtrsim N} v^l_{\lesssim N} v^{k-l-1}_{\gtrsim N}\big)\Big) \\
=& \sum_{l=0}^{k-1} \binom{k}{l} \Big((Iv_{\gtrsim N}) v^l_{\lesssim N} (Iv_{\gtrsim N})^{k-l-1} - I\big(v_{\gtrsim N} v^l_{\lesssim N} v^{k-l-1}_{\gtrsim N}\big)\Big)
\end{aligned}$$

Therefore, (3.4.12) follows if we prove that for every $l \leq k-1$,

$$\Big\|(Iv_{\gtrsim N}) v^l_{\lesssim N} (Iv_{\gtrsim N})^{k-l-1}\Big\|_{L^2} \lesssim N^{1-k(1-s)} \|Iv\|_{H^1}^k \tag{3.4.13}$$

and

$$\Big\|I\big(v_{\gtrsim N} v^l_{\lesssim N} v^{k-l-1}_{\gtrsim N}\big)\Big\|_{L^2} \lesssim N^{1-k(1-s)} \|Iv\|_{H^1}^k. \tag{3.4.14}$$

Let $\varepsilon := 1 - s$. By Sobolev embeddings, we have that $\|f\|_{L^{\frac{2}{\varepsilon}}} \lesssim \|f\|_{H^{1-\varepsilon}}$. By Hölder, we have

$$\|f\|_{L^{\left(\frac{1}{2} - \frac{(k-1)\varepsilon}{2}\right)^{-1}}} \lesssim \|f\|_{L^2}^{\frac{1-k\varepsilon}{1-\varepsilon}} \|f\|_{L^{\frac{2}{\varepsilon}}}^{\frac{(k-1)\varepsilon}{1-\varepsilon}} \lesssim \|f\|_{L^2}^{\frac{1-k\varepsilon}{1-\varepsilon}} \|f\|_{H^{1-\varepsilon}}^{\frac{(k-1)\varepsilon}{1-\varepsilon}}.$$

Therefore, again by Hölder, we have:

$$\begin{aligned}
& \Big\|(Iv_{\gtrsim N}) v^l_{\lesssim N} (Iv_{\gtrsim N})^{k-l-1}\Big\|_{L^2} \\
\lesssim\; & \big\|(Iv_{\gtrsim N})\big\|_{L^{\left(\frac{1}{2} - \frac{(k-1)\varepsilon}{2}\right)^{-1}}} \big\|Iv_{\lesssim N}\big\|_{L^{\frac{2}{\varepsilon}}}^l \big\|Iv_{\gtrsim N}\big\|_{L^{\frac{2}{\varepsilon}}}^{k-l-1}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\|Iv_{\gtrsim N}\right\|_{L^2}^{\frac{1-k\varepsilon}{1-\varepsilon}} \left\|Iv_{\gtrsim N}\right\|_{H^{1-\varepsilon}}^{\frac{(k-1)\varepsilon}{1-\varepsilon}} \left\|Iv\right\|_{H^{1-\varepsilon}}^{k-1} \\
&\lesssim N^{-(1-\varepsilon)\frac{1-k\varepsilon}{1-\varepsilon}} \left\|Iv\right\|_{H^{1-\varepsilon}}^{\frac{1-k\varepsilon}{1-\varepsilon}} \left\|Iv\right\|_{H^{1-\varepsilon}}^{\frac{(k-1)\varepsilon}{1-\varepsilon}} \left\|Iv\right\|_{H^{1-\varepsilon}}^{k-1} \\
&\lesssim N^{-(1-k\varepsilon)} \left\|Iv\right\|_{H^{1-\varepsilon}}^{k} .
\end{aligned}
$$

Proceeding similarly and using (3.4.4), we have

$$
\begin{aligned}
\left\|I\big(v_{\gtrsim N} v_{\lesssim N}^{l} v_{\gtrsim N}^{k-l-1}\big)\right\|_{L^2} &\lesssim \left\|\big(v_{\gtrsim N} v_{\lesssim N}^{l} v_{\gtrsim N}^{k-l-1}\big)\right\|_{L^2} \\
&\lesssim N^{-(1-k\varepsilon)} \left\|v\right\|_{H^{1-\varepsilon}}^{k} \\
&\lesssim N^{-(1-k\varepsilon)} \left\|Iv\right\|_{H^{1}}^{k} .
\end{aligned}
$$

□

Lemma 3.4.6. *For every $\gamma > 0$, $0 < \tilde{s} < 1$, there exist $p(\gamma) > 1, \eta(\gamma) > 0$ such that*

$$
\left\|(If)(Ig) - I(fg)\right\|_{L^2} \lesssim_{\gamma,\tilde{s}} N^{\gamma - \frac{1-\tilde{s}}{2}} \left\|f\right\|_{H^{1-\tilde{s}}} \left\|g\right\|_{W^{-\eta(\gamma),1} \cap W^{-\eta(\gamma),p(\gamma)}} \tag{3.4.15}
$$

Proof. As in the proof of Lemma 3.4.5, let us define

$$
u_{\lesssim M} := \int_{|\xi|<M/2} \widehat{u}(\xi) e^{i\xi\cdot x}
$$

and $u_{\gtrsim M} := u - u_{\lesssim M}$. Writing $f = f_{\lesssim N^{\frac{1}{2}}} + f_{\gtrsim N^{\frac{1}{2}}}$ and $g = g_{\lesssim N} + g_{\gtrsim N}$, we have that

$$
\begin{aligned}
&(If)(Ig) - I(fg) \\
=\;& \Big(If_{\lesssim N^{\frac{1}{2}}}\Big)\Big(Ig_{\lesssim N}\Big) - I\left(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}\right) && \text{(I)} \\
+\;& \Big(If_{\lesssim N^{\frac{1}{2}}}\Big)\Big(Ig_{\gtrsim N}\Big) - I\left(f_{\lesssim N^{\frac{1}{2}}} g_{\gtrsim N}\right) && \text{(II)} \\
+\;& \Big(If_{\gtrsim N^{\frac{1}{2}}}\Big)\Big(Ig\Big) && \text{(III)} \\
-\;& I\left(f_{\gtrsim N^{\frac{1}{2}}} g\right) && \text{(IV)}
\end{aligned}
$$

We have that

- I = 0, since $If_{\lesssim N^{\frac{1}{2}}} = f_{\lesssim N^{\frac{1}{2}}}$, $Ig_{\lesssim N} = g_{\lesssim N}$ by definition of I, and $(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N})\widehat{\ }(\xi) \neq 0$ only for $|\xi| \leq (N + N^{\frac{1}{2}})/2 < N$, so $I\left(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}\right) = f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}$ as well.
- $\|\text{II}\|_{L^2}$ can be written as $\sup_{\|h\|_{L^2}=1} \int_{\mathbb{R}^2} h \cdot (\text{II})$. Calling $\widehat{f}_{\lesssim N^{\frac{1}{2}}} = a$, $\widehat{g}_{\gtrsim N} = b$, expanding II in Fourier series and using Plancherel, we have to estimate

  $$
  \iint_{\xi_1 < N^{\frac{1}{2}}/3, \xi_2 > N/3} a(\xi_1) b(\xi_2) \Big(m_N(\xi_1) m_N(\xi_2) - m_N(\xi_1+\xi_2)\Big) \hat{h}(\xi_1+\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2.
  $$

  Using the fact that where $a(\xi_1) \neq 0$, $m(\xi_1) \equiv 1$ and that, by the mean value theorem, $|m(\xi_1+\xi_2) - m(\xi_1)| \lesssim_s N^{1-s} |\xi_1| |\xi_2|^{-2+s}$, we have that

  $$
  \begin{aligned}
  &\int_{\mathbb{R}^2} h \cdot (\text{II}) \\
  =\;& \iint_{\xi_1 < N^{\frac{1}{2}}/2, \xi_2 > N/2} a(\xi_1) b(\xi_2) \Big(m_N(\xi_1) m_N(\xi_2) - m_N(\xi_1+\xi_2)\Big) \hat{h}(\xi_1+\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
  \lesssim\;& N^{(1-s)} \iint_{\xi_1 < N^{\frac{1}{2}}/2, \xi_2 \geq N/2} \left|\frac{a(\xi_1)}{|\xi_1|^{\tilde{s}+\delta}}\right| \left|\frac{b(\xi_2)|}{|\xi_2|^{\delta}}\right| \frac{|\xi_1|^{1+\tilde{s}+\delta}}{|\xi_2|^{2-s-\delta}} \left|\hat{h}(\xi_1+\xi_2)\right| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
  \lesssim\;& N^{-\left(\frac{1-\tilde{s}}{2} - \frac{3}{2}\delta\right)} \iint_{\xi_1 < N^{\frac{1}{2}}/2, \xi_2 \geq N/2} \left|\frac{a(\xi_1)}{|\xi_1|^{\tilde{s}+\delta}}\right| \left|\frac{b(\xi_2)|}{|\xi_2|^{\delta}}\right| \left|\hat{h}(\xi_1+\xi_2)\right| \mathrm{d}\xi_1 \mathrm{d}\xi_2
  \end{aligned}
  $$

$$\lesssim N^{-\left(\frac{1-\tilde{s}}{2}-\frac{3}{2}\delta\right)} \left\| \frac{a(\xi_1)}{|\xi_1|^{\tilde{s}+\delta}} \right\|_{L^1} \left\| \frac{b(\xi_2)|}{|\xi_2|^{\delta}} \right\|_{L^2} \left\| \hat{h} \right\|_{L^2}$$
$$\lesssim N^{-\left(\frac{1-\tilde{s}}{2}-\frac{3}{2}\delta\right)} \|f\|_{H^{1-\tilde{s}}} \|g\|_{H^{-\delta}} \|h\|_{L^2},$$

therefore $\|\mathrm{II}\|_{L^2} \lesssim N^{-\left(\frac{1-\tilde{s}}{2}-\frac{3}{2}\delta\right)} \|f\|_{H^{1-\tilde{s}}} \|g\|_{H^{-\delta}}$.

- By Hölder, Sobolev embeddings and (3.4.3),

$$\begin{aligned}\|\mathrm{III}\|_{L^2} &\lesssim \left\| If_{\gtrsim N^{\frac{1}{2}}} \right\|_{L^2} \|Ig\|_{L^\infty} \\ &\lesssim_\delta N^{-\frac{1-\tilde{s}}{2}} \|f\|_{H^{1-\tilde{s}}} \|Ig\|_{W^{3\delta,\delta^{-1}}} \\ &\lesssim_\delta N^{-\frac{1-\tilde{s}}{2}+4\delta} \|f\|_{H^{1-\tilde{s}}} \|g\|_{W^{-\delta,\delta^{-1}}}\end{aligned}$$

- By duality (like for II), self-adjointness of I, fractional Leibnitz inequality, Sobolev embeddings and (3.4.3), we have

$$\begin{aligned}&\int h \cdot (\mathrm{IV}) \\ =& -\int hI(f_{\gtrsim N^{\frac{1}{2}}} g) \\ =& -\int I(h) f_{\gtrsim N^{\frac{1}{2}}} g \\ \lesssim& \left\| I(h) f_{\gtrsim N^{\frac{1}{2}}} \right\|_{W^{2\delta,(1-\delta)^{-1}}} \|g\|_{W^{-2\delta,\delta^{-1}}} \\ \lesssim& \|g\|_{W^{-2\delta,\delta^{-1}}} \Big(\|I(h)\|_{H^{2\delta}} \left\| f_{\gtrsim N^{\frac{1}{2}}} \right\|_{L^{\left(\frac{1}{2}-\delta\right)^{-1}}} \\ &\qquad + \|I(h)\|_{L^{\left(\frac{1}{2}-\delta\right)^{-1}}} \left\| f_{\gtrsim N^{\frac{1}{2}}} \right\|_{H^{2\delta}} \Big) \\ \lesssim& \|g\|_{W^{-2\delta,\delta^{-1}}} \|I(h)\|_{H^{2\delta}} \left\| f_{\gtrsim N^{\frac{1}{2}}} \right\|_{H^{2\delta}} \\ \lesssim& N^{2\delta} N^{-\frac{1}{2}(1-\tilde{s}-2\delta)} \|g\|_{W^{-2\delta,\delta^{-1}}} \|h\|_{L^2} \left\| f_{\gtrsim N^{\frac{1}{2}}} \right\|_{H^{1-\tilde{s}}} \\ \lesssim& N^{-\left(\frac{1-\tilde{s}}{2}-3\delta\right)} \|g\|_{W^{-2\delta,\delta^{-1}}} \|h\|_{L^2} \left\| f_{\gtrsim N^{\frac{1}{2}}} \right\|_{H^{1-\tilde{s}}},\end{aligned}$$

so $\|\mathrm{IV}\|_{L^2} \lesssim N^{-\left(\frac{1-\tilde{s}}{2}-3\delta\right)} \|g\|_{W^{-2\delta,\delta^{-1}}} \|f\|_{H^{1-\tilde{s}}}$.

Therefore, by choosing δ such that $\gamma = 4\delta$, $\eta = \delta$ and $p = \delta^{-1}$, we have (3.4.15). □

Lemma 3.4.7. *Let $k \leq 2$. For every $\gamma > 0$, there exist $p(\gamma) > 1, \eta(\gamma) > 0$ such that*

$$\begin{aligned}&\left\| I\big(v^k \rho : \psi^{3-k} :\big) - (Iv)^k I\big(\rho : \psi^{3-k} :\big) \right\|_{L^2} \\ &\qquad \lesssim_{s,\gamma} N^{-\frac{1-k(1-s)}{2}+\gamma} \|Iv\|_{H^1}^k \left\| \rho : \psi^{3-k} : \right\|_{W^{-\eta(\gamma),1} \cap W^{-\eta(\gamma),p(\gamma)}}. \end{aligned} \tag{3.4.16}$$

Proof. We have that

$$\begin{aligned}&\left\| I\big(v^k \rho : \psi^{3-k} :\big) - (Iv)^k I\big(\rho : \psi^{3-k} :\big) \right\|_{L^2} \\ \lesssim& \left\| I\big(v^k \rho : \psi^{3-k} :\big) - I(v^k) I\big(\rho : \psi^{3-k} :\big) \right\|_{L^2} && \text{(I)} \\ +& \left\| \Big(I(v^k) - (Iv)^k \Big) I\big(\rho : \psi^{3-k} :\big) \right\|_{L^2}. && \text{(II)}\end{aligned}$$

- By Sobolev embeddings and fractional Leibnitz, we have that

$$\left\| v^k \right\|_{H^{1-k(1-s)}} \lesssim \left\| v^k \right\|_{W^{s,\frac{2}{1+(k-1)(1-s)}}} \lesssim \|v\|_{H^s} \|v\|_{L^{\frac{2}{(1-s)}}}^{k-1} \lesssim \|v\|_{H^s}^k.$$

Therefore, by (3.4.15),

$$\|\mathrm{I}\|_{L^2} \lesssim N^{-\frac{1-k(1-s)}{2}+\gamma} \|v\|_{H^s}^k \left\|\rho :\psi^{3-k}:\right\|_{W^{-\eta(\gamma),1} \cap W^{-\eta(\gamma),p(\gamma)}} .$$

From (3.4.4), we have that $\|v\|_{H^s} \lesssim \|Iv\|_{H^1}$, so

$$\|\mathrm{I}\|_{L^2} \lesssim_{(1-s),\gamma} N^{-\frac{1-k(1-s)}{2}+\gamma} \|Iv\|_{H^1}^k \left\|:\psi^{3-k}:\right\|_{W^{-\eta(\gamma),p(\gamma)}} .$$

- From Hölder, (3.4.12) and Sobolev embeddings, we have

$$\begin{aligned}
\|\mathrm{II}\|_{L^2} &\lesssim \left\|I\big(v^k\big) - \big(Iv\big)^k\right\|_{L^2} \left\|I\big(\rho :\psi^{3-k}:\big)\right\|_{L^\infty} \\
&\lesssim_{s,\delta} N^{-(1-k(1-s))} \|Iv\|_{H^1}^k \left\|I\big(\rho :\psi^{3-k}:\big)\right\|_{W^{3\delta,\delta^{-1}}} \\
&\lesssim_{s,\delta} N^{-(1-k(1-s))} N^{4\delta} \|Iv\|_{H^1}^k \left\|\rho :\psi^{3-k}:\right\|_{W^{-\delta,\delta^{-1}}} .
\end{aligned}$$

Choosing δ small enough, we have that $1 - k(1-s) - 4\delta > \frac{1-k(1-s)}{2} - \gamma$, so the main contribution comes from II. We get (3.4.16) by taking $\gamma' = 4\delta$, $p(\gamma') = \max(p(\gamma), \delta^{-1})$, $\eta(\gamma') = \max(\eta(\gamma), \delta)$, and then renaming $\gamma = \gamma'$.

□

Lemma 3.4.8. *There exists $c > 0$ such that for every $0 < \eta < \frac{1}{8}$,*

$$\int_{t_0}^{T} (3.4.6)(s)\mathrm{d}s \lesssim_T \left(|T - t_0| + \int_{t_0}^{T} E^{1+c\eta}\right) \|I\psi\|_{L_{t,x}^{\eta^{-1}}} \tag{3.4.17}$$

Proof. Let $0 < \theta < 1$. Since we have

$$\begin{aligned}
\|Iv\|_{H^1} &\lesssim E^{\frac{1}{2}} \\
\|Iv\|_{L^4} &\lesssim E^{\frac{1}{4}},
\end{aligned}$$

by Gagliardo-Niremberg we have $\|Iv\|_{W^{\theta,\frac{4}{1+\theta}}} \lesssim E^{\frac{1+\theta}{4}}$. Therefore, by Sobolev inequality, we have that $\|Iv\|_{L^{\frac{4}{1-\theta}}} \lesssim E^{\frac{1+\theta}{4}}$, and the implicit constant is uniform in θ as long as $0 \le \theta \le \theta_{\max} < 1$. Take $\theta = 4\eta$. Therefore, by Hölder,

$$\begin{aligned}
\left|\int_{t_0}^{T}\int_{\mathbb{T}^2} Iv_t(Iv)^2 I\psi\right| &\le \int_{t_0}^{T} \|Iv_t\|_{L^2} \|Iv\|_{L^4} \|Iv\|_{L^{\frac{4}{1-4\eta}}} \|I\psi\|_{L_x^{\eta^{-1}}} \\
&\lesssim \int_{t_0}^{T} E^{\frac{1}{2}} E^{\frac{1}{4}} (E^{\frac{1+4\eta}{4}}) \|I\psi\|_{L_x^{\eta^{-1}}} \\
&\lesssim \int_{t_0}^{T} \left(E^{1+\eta}\right) \|I\psi\|_{L_x^{\eta^{-1}}} \\
&\lesssim \left(\int_{t_0}^{T} \left(E^{1+\eta}\right)^{\frac{1}{1-\eta}}\right)^{1-\eta} \|I\psi\|_{L_{x,t}^{\eta^{-1}}} \\
&\lesssim_T \left(\int_{t_0}^{T} E^{\frac{1+\eta}{1-\eta}}\right) \|I\psi\|_{L_{x,t}^{\eta^{-1}}} .
\end{aligned}$$

Therefore, choosing $c = \max_{\eta \in [0,\frac{1}{8}]} \eta^{-1}\left(\frac{1+\eta}{1-\eta} - 1\right)$, we have

$$\left|\int_{t_0}^{T}\int_{\mathbb{T}^2} Iv_t(Iv)^2 I\psi\right| \lesssim_T \left(\int_{t_0}^{T} 1 + E^{1+c\eta}\right) \|I\psi\|_{L_{x,t}^{\eta^{-1}}} ,$$

which gives (3.4.17). □

Lemma 3.4.9. *Let $T > 0$. For every $|t - t_0| \leq T$, for every $0 < \eta \leq \frac{1}{8}$, we have that*

$$\begin{aligned} & E(t) - E(t_0) \\ \lesssim_{s,\gamma,T} & \left(1 + \int_{t_0}^{t} E^{1+c\eta}\right) \|I\psi\|_{L_{t,x}^{\eta^{-1}}} && (3.4.18) \\ + & \int_{t_0}^{t} N^{-(1-3(1-s))} E^2 && (3.4.19) \\ + & \sum_{k=0}^{2} \int_{t_0}^{t} N^{-\frac{1-k(1-s)}{2}+\gamma} E^{\frac{k+1}{2}} \left\|:\psi^{3-k}:\right\|_{L_t^\infty W^{-\eta(\gamma),1} \cap W^{-\eta_k(\gamma),p(\gamma)}} && (3.4.20) \\ + & \int_{t_0}^{t} N^{\gamma} \left(E^{\frac{3}{4}} \left\|\rho :\psi^2:\right\|_{W^{-\gamma,4}} + E^{\frac{1}{2}} \left\|\rho :\psi^3:\right\|_{H^{-\gamma}}\right) && (3.4.21) \\ + & \int_{t_0}^{t} E(s) \mathrm{d}s, && (3.4.22) \end{aligned}$$

where c is the one given by Lemma 3.4.8 and $\eta(\gamma), p(\gamma)$ are the ones given by Lemma 3.4.7.

Proof. By Lemma 3.4.2,

$$E(t) - E(t_0) = \int_{t_0}^{t} (3.4.6)(s) + (3.4.7)(s) + (3.4.8)(s) + (3.4.9)(s) \mathrm{d}s.$$

We have that

- From (3.4.17), $\int_{t_0}^{t} (3.4.6)(s)\mathrm{d}s \lesssim (3.4.18)$,
- From (3.4.11), $\int_{t_0}^{t} (3.4.7)(s)\mathrm{d}s \lesssim (3.4.21)$,
- From (3.4.12) for the fist term and (3.4.16) for the second and third term respectively, and Hölder inequality, $\int_{t_0}^{t} (3.4.8)(s)\mathrm{d}s \lesssim (3.4.19) + (3.4.20)$
- From (3.4.10), $\int_{t_0}^{t} (3.4.9)(s)\mathrm{d}s \lesssim (3.4.22)$.

□

Lemma 3.4.10. *Let $T > 0$, and let*

$$A(N) := \frac{\|I_N \rho\psi\|_{L^{\log N}([-T,T]\times\mathbb{R}^2)}}{\log N}. \tag{3.4.23}$$

For $\gamma > 0$, let M, Λ be such that

$$\max_k \left\|:\rho\psi^{3-k}:\right\|_{L_t^\infty([-T,T];W^{-(\max(\eta_k(\gamma),\gamma)),\max(p_k(\gamma),2)})} = M, \tag{3.4.24}$$

$$A(N) \leq \Lambda, \tag{3.4.25}$$

and suppose that M, Λ are finite. Then, for $\gamma = \gamma(s)$ small enough, $\alpha < 1-3(1-s)$, $\delta < \beta < \alpha$, there exists $\tau = \tau(s, M, \Lambda, \alpha-\beta)$ such that if $E(t_0) \leq N^\beta/2$, $|t_0| \leq T$, $N \geq N_0 = N_0(s, T, M, \Lambda)$, then $E(t) \leq N^\alpha$ for every t such that $|t| \leq T$ and $|t - t_0| \leq \tau$.

Proof. By Lemma 3.4.9, as long as $E \leq N^\alpha$, since $\alpha < 1 - 3(1 - s)$, for N big enough we have that $(3.4.19) + (3.4.20) \leq 1 + \int_{t_0}^{T} E$. Similarly, from Young's inequality, for some universal constant C, we have

$$(3.4.21) \leq \int_{t_0}^{T} E(s)\mathrm{d}s + CN^{4\gamma}M^4.$$

Choosing $\eta = (\log N)^{-1}$ in (3.4.18) we get

$$(3.4.18) \leq A(N) \log N\Big(1 + \int_{t_0}^{T} E^{1+c(\log N)^{-1}}(s)\mathrm{d}s\Big) \leq \Lambda \log N\Big(1 + e^{c\alpha} \int_{t_0}^{T} E(s)\mathrm{d}s\Big).$$

Therefore, as long as $E(t) \le N^\alpha$, for N big enough (depending on s, T, M, Λ),

$$\begin{aligned}
&E(t)\\
&\le E(t_0) + C(s,\gamma,T)\Lambda \log N\Big(1 + \int_{t_0}^T E(s)\mathrm{d}s\Big)\\
&\quad + 1 + \int_{t_0}^T E(s)\mathrm{d}s + CN^{4\gamma}M^4 + C\int_{t_0}^T E(s)\mathrm{d}s\\
&\le \frac{1}{2}N^\beta + C(s,\gamma,T)\Lambda\log N + CN^{4\gamma}M^4\\
&\quad + \Big(1 + C + C(s,\gamma,T)\Lambda\log N\Big)\int_{t_0}^T E(s)\mathrm{d}s.\\
&\le N^\beta + C'(s,\gamma,T,\Lambda)\log N\int_{t_0}^T E(s)\mathrm{d}s.
\end{aligned} \tag{3.4.26}$$

Let $\bar{t} = \max\{s : t_0 \le s \le T, E(s) \le N^\alpha\}$, $\tilde{t} = \min\{s : t_0 \ge s \ge -T, E(s) \le N^\alpha\}$. Then the lemma is proven if we show that if $\bar{t} \ne T$, then $|\bar{t} - t_0| \ge \tau(s,\gamma,T,\Lambda)$ and similarly if $\tilde{t} \ne -T$, then $|\tilde{t} - t_0| \ge \tau(s,\gamma,T,\Lambda)$. For $\tilde{t} \le t \le \bar{t}$, by definition, (3.4.26) holds, so by Gronwall

$$E(t) \le N^\beta \exp\big(|t - t_0|C'(s,\gamma,T,\Lambda)\log N\big). \tag{3.4.27}$$

Suppose that $\bar{t} \ne T$. Then one must have $E(\bar{t}) = N^\alpha$. Therefore, by (3.4.27), $|\bar{t} - t_0| \ge \tau := \frac{(\alpha-\beta)}{C'(s,\gamma,T,\Lambda)}$. The same holds for $\tilde{t}$, and the lemma is proven. □

Proof of Proposition 3.4.1. Let $\varepsilon > 0$, $T > 0$, let $2(1-s) < \beta < \alpha < 1 - 3(1-s)$, and let γ as in Lemma 3.4.10. Moreover, let M as in (3.4.24) and $\Lambda := \sup_{N\in\mathbb{N}} A(N)$. Notice that by Proposition 3.2.3, M is finite almost surely, while by Chebishev and Lemma 3.2.4,

$$\mathbb{P}(A(N) > \Lambda_0) \le \frac{\mathbb{E}\left[A(N)^{\log N}\right]}{\Lambda_0^{\log N}} \le \int_{-T}^T \left(\frac{C(\rho)t}{\Lambda_0^2}\right)^{\frac{\log N}{2}},$$

which is summable in N for $\Lambda_0^2 > e^2 C(\rho)T$. Hence, by Borel-Cantelli,

$$\limsup_N A(N) \le e^2 C(\rho) T,$$

so $\Lambda < +\infty$.

For this choice of M, Λ, γ, α, β, let N_0 and τ be the ones given by Lemma 3.4.10, and take $(u_0, u_1) \in \mathcal{H}^s$. Define a sequence N_k of integers recursively. Take N_1 such that $N_1 \ge \max(N_0, \tilde{N})$ and

$$N_1^{2(1-s)}\|(u_0,u_1)\|^2_{\mathcal{H}^s} + \|u_0\|^4_{H^s} \ll N_1^\beta.$$

By Sobolev embeddings, (3.4.3) and (3.4.4),

$$\begin{aligned}
E((I_{N_1}v(0), I_{N_1}v_t(0))^T) &= E((I_{N_1}u_0, I_{N_1}u_1)^T)\\
&\lesssim \|(I_{N_1}u_0, I_{N_1}u_1)\|^2_{\mathcal{H}^1} + \|I_{N_1}u_0\|^4_{L^4}\\
&\lesssim N_1^{2(1-s)}\|(u_0,u_1)\|^2_{\mathcal{H}^s} + \|I_{N_1}u_0\|^4_{H^s}\\
&\lesssim N_1^{2(1-s)}\|(u_0,u_1)\|^2_{\mathcal{H}^s} + \|u_0\|^4_{H^s},
\end{aligned} \tag{3.4.28}$$

so we will have $E(I_{N_1}\mathbf{v}(0)) \le \frac{1}{2}N_1^\beta$, therefore by Lemma 3.4.10 one has that

$$\|\mathbf{v}(t)\|^2_{\mathcal{H}^s} \lesssim E(I_{N_1}\mathbf{v}) \le N^\alpha \text{ for } t \le T, 0 \le t \le \tau,$$

and similarly backwards in time.

Then take $N_{k+1} \gg N_k$ such that

$$N_{k+1}^{2(1-s)} N_k^{\alpha} + N_k^{2\alpha} \ll N_{k+1}^{\beta}. \tag{3.4.29}$$

If one has

$$E(I_{N_k}\mathbf{v}(\pm(k-1)\tau)) \leq \frac{1}{2} N_k^{\beta} \text{ if } (k-1)\tau \leq T, \tag{3.4.30}$$

by Lemma 3.4.10,

$$\begin{aligned} &\|\mathbf{v}\|^2_{L^\infty([(k-1)\tau,\min(k\tau,T)];\mathcal{H}^s)} \\ \lesssim &\sup_{(k-1)\tau\leq t'\leq\min(k\tau,T)} E(I_{N_k}\mathbf{v}(t')) \leq N_k^{\alpha} \end{aligned} \tag{3.4.31}$$

and similarly backwards in time. Therefore, we get Propostion 3.4.1 with

$$F(t) = N_k^{\alpha} \text{ with } T = t, k = \left\lfloor \frac{t}{\tau} \right\rfloor \tag{3.4.32}$$

as long as we show (3.4.30). Proceeding inductively, we know (3.4.30) for N_1, and proceeding as in (3.4.28),

$$\begin{aligned} &E((I_{N_{k+1}}v(\pm k\tau), I_{N_k}v_t(\pm k\tau))^T) \\ \lesssim\, &N_{k+1}^{2(1-s)} \|(v(\pm k\tau), v_t(\pm k\tau))\|^2_{H^s} + \|v(\pm k\tau)\|^4_{H^s} \\ \lesssim\, &N_{k+1}^{2(1-s)} E((I_N v(\pm k\tau), v_t(\pm k\tau))^T) + E((I_N v(\pm k\tau), v_t(\pm k\tau))^T)^2 \\ \lesssim\, &N_{k+1}^{2(1-s)} N_k^{\alpha} + N_k^{2\alpha} \ll N_{k+1}^{\beta}, \end{aligned}$$

so $E(I_{N_{k+1}}\mathbf{v}(\pm k\tau)) \leq \frac{1}{2} N_{k+1}^{\beta}$ and we have (3.4.30). □

3.4.2 Independence from the cutoff and global well-posedeness for the equation on $\mathbb{R}^2$

In this subsection, we prove that on appropriate space-time regions, the solution to (LSNLW) does not depend on the particular choice of the cutoff ρ, and proceed to prove global well posedness for (SNLW) posed on $\mathbb{R}^2$.

Proposition 3.4.11 (Finite speed of propagation for (SNLW))**.** *Let $R, T > 0$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$. Let u_1, u_2 be solutions to* (SNLW) *on $B(x_0, R)$ for a time T, in the sense that $u_j|_{B(x_0,R)} = \psi|_{B(x_0,R)} + v_j|_{B(x_0,R)}$, $(v_j, \partial_t v_j) \in C([t_0 - T, t_0 + T]; \mathcal{H}^s_{\text{loc}})$, $s \geq \frac{2}{3}$, and*

$$\begin{aligned} v_j(t)|_{B(x_0,R)} = \Bigg(&\cos((t-t_0)|\nabla|)v_j(t_0) + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}\partial_t v_j(t_0) \\ &+ \int_{t_0}^{t} \frac{\sin((t-t')|\nabla|)}{|\nabla|} :(\psi + v_j)^3\colon (t')\mathrm{d}t' \Bigg)\Bigg|_{B(x_0,R)} \end{aligned} \tag{3.4.33}$$

for every $|t - t_0| \leq T$. Suppose moreover that $v_1(t_0) = v_2(t_0)$, $\partial_t v_1(t_0) = \partial_t v_2(t_0)$ on $B(x_0, R)$. Then $v_1(t)|_{B(x_0,R-|t-t_0|)} = v_2(t)|_{B(x_0,R-|t-t_0|)}$ for every $|t - t_0| \leq T$.

Proof. Without loss of generality, assume that $t_0 = 0$, $x_0 = 0$. Let $D_t = B_{R-|t|}$. Recalling that the kernels of $\cos(s|\nabla|)$, $\frac{\sin(s|\nabla|)}{|\nabla|}$ are distributions supported in B_s, from (3.4.33) we have that

$$v_1 - v_2|_{D_t} = \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}\Big((v_1 - v_2)(3 :\psi^2\colon + 3\psi(v_1 + v_2) + v_1^2 + v_2^2 + v_1 v_2)\Big)\Big|_{D_{t'}} \mathrm{d}t'.$$

Proceeding as in (3.3.9), we obtain that

$$\|v_1(t) - v_2(t)\|_{H^s(D_t)}$$

$$\lesssim_R \int_0^t \|v_1 - v_2\|_{H^s(D_t')} (1 + \|v_1\|^2_{C([-R,R];H^s(B_R))} + \|v_2\|^2_{C([-R,R];H^s(B_R))}) \times \max_{1\le j\le 3} \|\rho :\psi^j :\|_{L^1_t W^{0-,6+}_x},$$

therefore by Gronwall,

$$\|v_1(t) - v_2(t)\|_{H^s(D_t)} \le C(R, v_1, v_2, \max_{1\le j\le 3} \|\rho :\psi^j :\|_{L^1_t W^{0-,6+}_x}) \|v_1(0) - v_2(0)\|_{H^s(B_R)} = 0.$$

□

This version of finite speed of propagation immediately implies some kind of consistency for solutions of (LSNLW) with different cutoff functions:

Corollary 3.4.12. *Let $T > 0$, let $t_0 = 0$, and let ρ_1, ρ_2 be two cutoff functions such that $\rho_1(x) = \rho_2(x) = 1$ for every $x \in B_{2T}$. Let $s > \frac{4}{5}$. Let $(u_0, u_1) \in \mathcal{H}^s_{\text{loc}}$, and let v_1, v_2 be respectively the solutions to* (LSNLW) *with cutoff function ρ_1 and ρ_2 and initial data respectively $(\rho_1 u_0, \rho_1 u_1)$ and $(\rho_2 u_0, \rho_2 u_1)$. Then $v_1(t, x) = v_2(t, x)$ for every $|x|, |t| < T$.*

Proof of Theorem 1.2.1 for $M = \mathbb{R}^2$. Let ρ_n be a cutoff function such that $\rho_n(x) = 1$ for every $|x| \le n$. Let $(u_0, u_1) \in \mathcal{H}^s_{\text{loc}}$, and let v_n be the solution of (LSNLW) with cutoff functions ρ_n and initial data $(\rho_n u_0, \rho_n u_1)$. By Proposition 3.4.1 (and Proposition 3.3.5) , we will have $(v_n, \partial_t v_n)^T \in C(\mathbb{R}; \mathcal{H}^s)$. By Corollary 3.4.12,

$$v := \lim_{n\to+\infty} v_n$$

is well defined, and we have that $v|_{[-T,T]\times B_R} = v_{\lceil T\vee R\rceil}|_{[-T,T]\times B_R}$. Therefore the theorem is proven if we show that every solution $\tilde{u} = \psi + \tilde{v}$ of (SNLW) with $v \in C([-T, T]; H^s_{\text{loc}})$ satisfies $\tilde{v}(t) = v(t)$ for every $t \le T$. Let $\phi \in C^\infty_c((-T, T) \times \mathbb{R}^2)$ be a test function. Let $n \in \mathbb{N}$ be such that $\text{supp}(\phi) \subseteq [-n, n] \times B_n$. By Proposition 3.4.11, we have that

$$\tilde{v}|_{[-n,n]\times B_n} = v_n|_{[-n,n]\times B_n} = v|_{[-n,n]\times B_n}.$$

Therefore, $\langle \tilde{v}, \phi\rangle = \langle v, \phi\rangle$, so $\tilde{v} = v$ as space-time distributions. Since they both belong to the space $C([-T, T]; H^s_{\text{loc}})$, the equality must hold in the space $C([-T, T]; H^s_{\text{loc}})$ as well, and so $\tilde{v}(t) = v(t)$ for every $t \le T$. □

Appendix A

Appendix

A.1 Notation

Throughout the thesis, we use the following notation/symbols:

- $$A \lesssim B$$

  denotes that there exists a constant C such that $A \leq CB$. If the constant depends on some parameter p, and we want to highlight this dependence, we will write

  $$A \lesssim_p B.$$

  Similarly,

  $$A \sim B$$

  denotes that there exits constants C_1, C_2 such that $C_1 A \leq B \leq C_2 A$. In most cases, however, we will use the notation only in the situation in which there exists a constant C such that $A = CB$.

- $$\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$$

- If $A \in \mathbb{R}$, we will use $A+$ as a placeholder for a number $B > A$ which can be chosen arbitrarily close to A. Similarly, $A-$ will be a placeholder for a number $B < A$ which can be chosen arbitrarily close to A.

- Given a function f on $M = \mathbb{T}^d, \mathbb{R}^d$,

  $$P_N f(x) := \frac{1}{2\pi} \int_{[-N,N]^d} \hat{f}(\xi) e^{i\xi \cdot x} \mathrm{d}\xi,$$

  where the integral is actually a sum over $\mathbb{Z}^d$ if $M = \mathbb{T}^d$. This can be seen as composition of the Hilbert transform in every of the d variables of $\mathbb{R}^d$ or $\mathbb{T}^d$, so it satisfies

  $$\|P_N f\|_{L^p} \leq C_p^d \|f\|_{L^p}$$

  for every $1 < p < +\infty$. We will also denote

  $$P_{>N} f := f - P_N f.$$

- If $M = \mathbb{T}^2, \mathbb{R}^2$, $\alpha \in \mathbb{R}$, the notation $C^\alpha(M)$ denotes the Besov space $B^\alpha_{\infty,\infty}$. The precise definition and many properties can be found (for instance) in [5, Definition 2.2.1]. Here we just point out that for $0 < \alpha < 1$, they coincide with the spaces of the usual Hölder continuous functions, and that they satisfy the same Sobolev embeddings as the usual Hölder spaces.

- For $s \in \mathbb{R}$, $1 \leq p \leq +\infty$, $W^{s,p}$ will denote the Sobolev space

$$W^{s,p} = \{f | \, \|\langle \nabla \rangle^s f\|_{L^p} < +\infty\}.$$

- H^s denotes the L^2 based Sobolev space

$$H^s = \{f | \, \|\langle \nabla \rangle^s f\|_{L^2} < +\infty\}.$$

- In Chapter 2 and 3, we will sometimes refer to the spaces $\mathscr{C}^\alpha$, $\mathscr{W}^{s,p}$, $\mathcal{H}^s$. In Chapter 2, we denote

$$\mathscr{C}^\alpha = C^\alpha \times C^{\alpha-2}, \mathscr{W}^{s,p} = W^{s,p} \times W^{s-2,p}, \mathcal{H}^s = H^s \times H^{s-2}.$$

  In Chapter 3,

$$\mathscr{C}^\alpha = C^\alpha \times C^{\alpha-1}, \mathscr{W}^{s,p} = W^{s,p} \times W^{s-1,p}, \mathcal{H}^s = H^s \times H^{s-1}.$$

- If X is a Banach space, C_tX will denote continuous functions in the variable t with values in X. If we want to be more specific about the domain I of the variable t, we will write $C(I;X)$. Similarly, for n integer, $C^n(I;X)$ will denote functions from I to X which admits continuous Fréchet derivatives in the variable t up to order n.

A.2 Wiener chaos decomposition and hypercontractivity

Consider the Hermite polynomials generating function

$$F(t,x;\sigma) := e^{tx-\frac{1}{2}\sigma^2t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma). \tag{A.2.1}$$

For simplicity, let $F(t,x) := F(t,x;1)$, and let $H_k(x) := H_k(x;1)$. Fix $d \in \mathbb{N}$. Consider the Hilbert space $L^2(\mathbb{R}^d,\mu)$, where $\mathrm{d}\mu_d := (2\pi)^{-\frac{d}{2}} \exp(-|x|^2/2)$. Then Hermite polynomials satisfy

$$\int_{\mathbb{R}} H_k(x)H_m(x)\mathrm{d}\mu_1(x) = k!\delta_{km},$$

for all $k,m \in \mathbb{N}$. Therefore, by scaling,

$$\int_{\mathbb{R}} H_k(x;\sigma)H_m(x;\sigma)\mathrm{d}\mu_1(x) = k!|\sigma|^k\delta_{km}. \tag{A.2.2}$$

Define the homogeneous Wiener chaos of order k to be an element of the form $\prod_{j=1}^d H_{k_j}(x_j)$, where $k = k_1 + \cdots + k_d$. Denote by $\mathcal{H}_k$ the closure of homogeneous Wiener chaoses of order k in $L^2(\mathbb{R}^d,\mu_d)$. Then, by $L^2(\mathbb{R}^d,\mu_d) = \bigotimes_{j=1}^d L^2(\mathbb{R},\mu_1)$, we have the Ito-Wiener decomposition

$$L^2(\mathbb{R}^d,\mu_d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Consider the operator $L := -(\Delta - x\cdot\nabla)$ (the Ornstein-Uhlenbeck operator). Then any element in $\mathcal{H}_k$ is an eigenvector of L with eigenvalue k, so $\bigoplus_{k=0}^\infty \mathcal{H}_k$ is the spectral decomposition of L^2 associated to L.
Moreover, we have the following hypercontractivity result for the Ornstein-Uhlenbeck semigroup $U(t) := e^{-tL}$, due to Nelson [24].

Lemma A.2.1. *Let $q > 1$ and $p \geq q$. Then, for every $u \in L^q(\mathbb{R}^d,\mu_d)$ and $t \geq \frac{1}{2}\log(\frac{p-1}{q-1})$, we have*

$$\|U(t)u\|_{L^p(\mathbb{R}^d,\mu_d)} \leq \|u\|_{L^q(\mathbb{R}^d,\mu_d)}. \tag{A.2.3}$$

Notice that the constant of the inequality in (A.2.3) (i.e. 1) and the range of p, q, t do not depend on the dimension d. As a consequence, the following holds.

Lemma A.2.2. *Let $F \in \mathcal{H}_k$. Then, for $p \geq 2$, we have*

$$\|F\|_{L^p(\mathbb{R}^d,\mu_d)} \leq (p-1)^{\frac{k}{2}} \|F\|_{L^2(\mathbb{R}^d,\mu_d)} \tag{A.2.4}$$

This estimate follows simply by applying (A.2.3) to F, putting $q = 2$, $t = \frac{1}{2}\log(p-1)$, and recalling that F is an eigenvector of $U(t)$ with eigenvalue e^{-kt}. As a further consequence, we obtain the following lemma.

Lemma A.2.3. *Fix $k \in \mathbb{N}$ and some coefficients $c(n_1, \ldots, n_k)$. Given $d \in \mathbb{N}$, let $\{g_n\}_{n=1}^d$ be a sequence of Gaussian random variables, such that $(g_n)_n$ is a Gaussian vector. Define S_k by*

$$S_k(= \sum_{\Gamma(k,d)} c(n_1, \ldots, n_k) g_{n_1} \cdots g_{n_k},$$

where $\Gamma(k,d)$ is defined by

$$\Gamma(k,d) = \big\{(n_1, \ldots, n_k) \in \{\pm 1, \ldots, \pm d\}^k\big\}.$$

Then, for $p \geq 2$, we have

$$\big(\mathbb{E}\|S_k\|^p\big)^{\frac{1}{p}} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \big(\mathbb{E}\|S_k\|^2\big)^{\frac{1}{2}}. \tag{A.2.5}$$

A.3 Kolmogorov continuity theorems

Proposition A.3.1. *Let $M = \mathbb{T}^k \times \mathbb{R}^{d-k}$, and let X be a distribution-valued random variable such that for every test function ϕ supported in a ball B of radius 1,*

$$\mathbb{E}[|\langle X, \phi\rangle|^2] \leq A_B^2 \|\phi\|_{H^{-s}}^2, \tag{A.3.1}$$

$$\mathbb{E}[|\langle X, \phi\rangle|^p] \leq C_p^p \mathbb{E}[|\langle X, \phi\rangle|^2]^{\frac{p}{2}}. \tag{A.3.2}$$

Then, for every compact K, and for every $\varepsilon > 0$, we have that $X \in C^{s-\frac{d}{2}-\varepsilon}(K)$ almost surely. Moreover, if X is supported in a single ball B, one has for every $p > \frac{2d}{\varepsilon}$,

$$\mathbb{E}[\|X\|_{C^{s-\frac{d}{2}-\varepsilon}}^p]^{\frac{1}{p}} \lesssim A_B C_p. \tag{A.3.3}$$

Proof. Since K is compact, we can cover it with finitely many balls B_j with centre x_j and radius 1, and let ρ_j be a partition of unity subordinated to B_j. By triangle inequality, it is enough to prove that $\rho_j X \in C^{s-\frac{d}{2}-\varepsilon}$. Since this function is compactly supported in a ball of radius 1, we can see it as a function over $\mathbb{R}^d$. Without loss of generality, assume that $x_j = 0$. Let $Y = \rho_j X$. We have that $Y_N(x) := \rho_j X * \varphi_N(x) = \langle X, \rho_j \varphi_N(x - \cdot)\rangle$, so by (A.3.1) and (A.3.2),

$$\mathbb{E}[|Y_N(x)|^p] \leq A_B^p C_p^p \|\rho_j \varphi_N(x-\cdot)\|_{\dot{H}^{-s}}^p = A_B^p C_p^p \|\rho_j(x-\cdot)\varphi_N\|_{H^{-s}}^p \lesssim A_B^p C_p^p \|\varphi_N\|_{H^{-s}(B(x,1))}^p.$$

Since $\varphi_N(y) = N^d \varphi(Ny)$, and φ is a Schwartz function, we have that for every $M \gg 1$,

$$\|\varphi_N\|_{H^{-s}(B(x,1))} = N^{\frac{d}{2}-s} \|\varphi\|_{H^{-s}(B(Nx,N))} \lesssim \frac{N^{\frac{d}{2}-s}}{\langle x\rangle^M}.$$

Therefore, for $p > \frac{2d}{\varepsilon}$, by Sobolev embeddings,

$$\mathbb{E}\|Y\|_{C^{s-\frac{d}{2}-\varepsilon}(\mathbb{R}^d)}^p = \mathbb{E}|\sup_N N^{s-\frac{d}{2}-\varepsilon} \|Y_N\|_{L^\infty}|^p$$

$$\lesssim \mathbb{E}\Big|\sum_N N^{p(s-\frac{d}{2}-\frac{\varepsilon}{2})}\,\|Y_N\|_{L^p}^p\,\Big|$$
$$\lesssim A_B^pC_p^p\sum_N N^{-p\frac{\varepsilon}{2}}\int\frac{1}{\langle x\rangle^{pM}}$$
$$\lesssim A_B^pC_p^p.$$

Moreover, if X is supported on a single ball, we can repeat this proof with $\rho_j\equiv 1$, and we obtain (A.3.3). □

Proposition A.3.2. *Let $M=\mathbb{T}^k\times\mathbb{R}^{d-k}$, and let X be a distribution-valued random variable on $\mathbb{R}_t\times M_x$. Suppose that for every $T>0$, and for every ball B of radius 1, there exist $0<\theta<1$ and constants A_B, C_p such that for every test function $\phi: M\to\mathbb{R}$ supported in B and for every $0\le s\le t\le T$,*

$$\mathbb{E}[|\langle X(0),\phi\rangle|^2]\le A_B^2\,\|\phi\|_{H^{-s}}^2\,, \tag{A.3.4}$$
$$\mathbb{E}[|\langle X(0),\phi\rangle|^p]\le C_p\mathbb{E}[|\langle X,\phi\rangle|^2]^{\frac{p}{2}}, \tag{A.3.5}$$
$$\mathbb{E}[|\langle X(t)-X(s),\phi\rangle|^2]\le A_B^2|t-s|^{2\theta}\,\|\phi\|_{H^{-s}}^2\,, \tag{A.3.6}$$
$$\mathbb{E}[|\langle X(t)-X(s),\phi\rangle|^p]\le C_p\mathbb{E}[|\langle X,\phi\rangle|^2]^{\frac{p}{2}}. \tag{A.3.7}$$

Then for every compact $K\subseteq M$, and every $\varepsilon>0$, we have that $X\in C([0,T];C^{s-\frac{d}{2}-\varepsilon}(K))$ almost surely. Moreover, if X is supported in a singe ball, we have for every $p>\frac{2d}{\varepsilon}$

$$\left(\mathbb{E}[\|X\|^p_{C([0,T];C^{s-\frac{d}{2}\varepsilon}(K))}]\right)^{\frac{1}{p}}\lesssim_p A_B. \tag{A.3.8}$$

Proof. As in the proof of Proposition A.3.1, we cover K with balls B_j of radius 1 and consider a partition of unity ρ_j subordinated to this covering. It is enough to show the result for $Y_j:=\rho_jX$. By Proposition A.3.1, we have that

$$\mathbb{E}[\|Y_j(0)\|^p_{C^{s-\frac{d}{2}-\varepsilon}(M)}]^{\frac{1}{p}}\lesssim A_{B_j}C_p$$
$$\mathbb{E}[\|Y_j(t)-Y_j(s)\|^p_{C^{s-\frac{d}{2}-\varepsilon}(M)}]^{\frac{1}{p}}\lesssim |t-s|^\theta A_{B_j}C_p.$$

for $p>\frac{2d}{\varepsilon}$. By [14, Theorem 1.4.1], this implies that $Y\in C^{\theta-\delta}([0,T];C^{s-\frac{d}{2}-\varepsilon}(M))$ for every $\delta>0$, and that

$$\left(\mathbb{E}[\|Y\|^p_{C^{\theta-\delta}([0,T];C^{s-\frac{d}{2}-\varepsilon}}]\right)^{\frac{1}{p}}\lesssim_p A_B. \tag{A.3.9}$$

so in particular $Y\in C([0,T];C^{s-\frac{d}{2}-\varepsilon}(M))$. Repeating the argument with $Y=X$, we get (A.3.8) □

Proposition A.3.3. *Let $M=\mathbb{T}^k\times\mathbb{R}^{d-k}$, and let X be a distribution-valued random variable such that there exists $1\le q'\le+\infty$ such that for every test function ϕ supported in a ball B of radius 1,*

$$\mathbb{E}[|\langle X,\phi\rangle|^2]\le A_B^2\,\|\phi\|_{L^{q'}}^2\,, \tag{A.3.10}$$
$$\mathbb{E}[|\langle X,\phi\rangle|^p]\le C_p^p\mathbb{E}[|\langle X,\phi\rangle|^2]^{\frac{p}{2}}. \tag{A.3.11}$$

Then, for every compact K, and for every $\varepsilon>0$, we have that $X\in C^{-\frac{d}{q}-\varepsilon}(K)$ almost surely, where $\frac{1}{q}+\frac{1}{q'}=1$. Moreover, if X is supported in a single ball B, one has for every $p>\frac{2d}{\varepsilon}$,

$$\mathbb{E}[\|X\|^p_{C^{-\frac{d}{q}-\varepsilon}}]^{\frac{1}{p}}\lesssim A_BC_p. \tag{A.3.12}$$

Proof. The proof is extremely similar to the one of Proposition A.3.1. We take a compact K, and cover it with finitely many balls B_j with centre x_j and radius 1. Let ρ_j be a partition of unity subordinated to B_j. By triangle inequality, it is enough to prove that $\rho_j X \in C^{-\frac{d}{q}-\varepsilon}$. Since this function is compactly supported in a ball of radius 1, we can see it as a function over $\mathbb{R}^d$. Without loss of generality, we can assume assume that $x_j = 0$. Let $Y = \rho_j X$. We have that $Y_N(x) := \rho_j X * \varphi_N(x) = \langle X, \rho_j \varphi_N(x - \cdot)\rangle$, so by (A.3.4) and (A.3.5),

$$\mathbb{E}[|Y_N(x)|^p] \le A_B^p C_p^p \|\rho_j \varphi_N(x-\cdot)\|_{L^{q'}}^p = A_B^p C_p^p \|\rho_j(x-\cdot)\varphi_N\|_{L^{q'}}^p \lesssim A_B^p C_p^p \|\varphi_N\|_{L^{q'}(B(x,1))}^p .$$

Since $\varphi_N(y) = N^d \varphi(Ny)$, and φ is a Schwartz function, we have that for every $M \gg 1$,

$$\|\varphi_N\|_{L^{q'}(B(x,1))} = N^{\frac{d}{q}} \|\varphi\|_{L^{q'}(B(Nx,N))} \lesssim \frac{N^{\frac{d}{q}}}{\langle x\rangle^M}.$$

By Sobolev embeddings, for $p > \frac{2d}{\varepsilon}$,

$$\begin{aligned}
\mathbb{E}\|Y\|_{C^{-\frac{d}{q}-\varepsilon}(\mathbb{R}^d)}^p &= \mathbb{E}|\sup_N N^{-\frac{d}{q}-\varepsilon}\|Y_N\|_{L^\infty}|^p \\
&\lesssim \mathbb{E}\Big|\sum_N N^{p(-\frac{d}{q}-\frac{\varepsilon}{2})}\|Y_N\|_{L^p}^p\Big| \\
&\lesssim A_B^p C_p^p \sum_N N^{-p\frac{\varepsilon}{2}} \int \frac{1}{\langle x\rangle^{pM}} \\
&\lesssim A_B^p C_p^p.
\end{aligned}$$

Moreover, if X is supported on a single ball, we can repeat this proof with $\rho_j \equiv 1$, and we obtain (A.3.12). ☐

Proposition A.3.4. *Let $M = \mathbb{T}^k \times \mathbb{R}^{d-k}$, and let X be a distribution-valued random variable on $\mathbb{R}_t \times M_x$. Suppose that for every $T > 0$, and for every ball B of radius 1, there exist $0 < \theta < 1$ and constants A_B, C_p such that for every test function $\phi : M \to \mathbb{R}$ supported in B and for every $0 \le s \le t \le T$,*

$$\mathbb{E}[|\langle X(0),\phi\rangle|^2] \le A_B^2 \|\phi\|_{L^{q'}}^2, \tag{A.3.13}$$

$$\mathbb{E}[|\langle X(0),\phi\rangle|^p] \le C_p \mathbb{E}[|\langle X,\phi\rangle|^2]^{\frac{p}{2}}, \tag{A.3.14}$$

$$\mathbb{E}[|\langle X(t)-X(s),\phi\rangle|^2] \le A_B^2 |t-s|^{2\theta} \|\phi\|_{L^{q'}}^2, \tag{A.3.15}$$

$$\mathbb{E}[|\langle X(t)-X(s),\phi\rangle|^p] \le C_p \mathbb{E}[|\langle X,\phi\rangle|^2]^{\frac{p}{2}}. \tag{A.3.16}$$

Then for every compact $K \subseteq M$, and every $\varepsilon > 0$, we have that $X \in C([0,T]; C^{-\frac{d}{q}-\varepsilon}(K))$ almost surely. Moreover, if X is supported in a singe ball, we have for every $p > \frac{2d}{\varepsilon}$

$$\Big(\mathbb{E}[\|X\|_{C([0,T];C^{-\frac{d}{q}-\varepsilon}(K))}^p]\Big)^{\frac{1}{p}} \lesssim_p A_B. \tag{A.3.17}$$

Proof. As in the proof of Proposition A.3.1, we cover K with balls B_j of radius 1 and consider a partition of unity ρ_j subordinated to this covering. It is enough to show the result for $Y_j := \rho_j X$. By Proposition A.3.3, we have that

$$\mathbb{E}[\|Y_j(0)\|_{C^{-\frac{d}{q}-\varepsilon}(M)}^p]^{\frac{1}{p}} \lesssim A_{B_j} C_p$$

$$\mathbb{E}[\|Y_j(t)-Y_j(s)\|_{C^{-\frac{d}{q}-\varepsilon}(M)}^p]^{\frac{1}{p}} \lesssim |t-s|^\theta A_{B_j} C_p.$$

for $p > \frac{2d}{\varepsilon}$. By [14, Theorem 1.4.1], this implies that $Y \in C^{\theta-\delta}([0,T]; C^{-\frac{d}{q}-\varepsilon}(M))$ for every $\delta > 0$, and that

$$\Big(\mathbb{E}[\|Y\|_{C^{\theta-\delta}([0,T];C^{-\frac{d}{q}-\varepsilon}}^p]\Big)^{\frac{1}{p}} \lesssim_p A_B. \tag{A.3.18}$$

so in particular $Y \in C([0,T]; C^{-\frac{d}{q}-\varepsilon}(M))$. Repeating the argument with $Y = X$, we get (A.3.17) $\square$

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