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Convergence problems for singular stochastic dynamics

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or professional qualification.

October 19, 2023, *Younes Zine*

Abstract

In this thesis, we investigate convergence problems for some nonlinear dispersive and parabolic PDEs in the singular stochastic setting. In the first part of the thesis, we study the so-called Smoluchowski-Kramers approximation on convergence of stochastic nonlinear wave equations (SNLW) to stochastic nonlinear heat equations (SNLH), with a polynomial nonlinearity. In particular, we prove that, in the over-damped regime, solutions of SNLW converge to those of the corresponding SNLH. This convergence is established for deterministic initial data. In the second part of the work, we study the inviscid limit for the stochastic complex Ginzburg-Landau equation (SCGL) with the cubic nonlinearity. We prove that, for Gaussian free field initial data, the solution of SCGL converges to that of the cubic nonlinear Schrödinger equation.

Lay summary

In this thesis, we study three equations: the *wave*, *heat* and *Schrödinger* equations. These models appear ubiquitously in various branches of physics and engineering such as quantum mechanics, quantum field theory, nonlinear optics, plasma physics, thermodynamics, financial mathematics and atmospheric sciences. It is therefore essential to rigorously understand the qualitative and quantitative properties of solutions to these equations which are very different in nature. Broadly speaking, the solutions to the wave and Schrödinger equations spread out (disperse) in space, while the solutions to the heat equation converge to a spatially uniform equilibrium over long times.

In this work, we furthermore consider the wave, heat and Schrödinger equations perturbed by a random forcing. From the real-world perspective, such random perturbations model some uncertainty in our knowledge of a system's initial state, due to deficiencies in storing data, or there are random perturbations from outside sources which alter the behaviour of the system itself, such as thermal noise.

In this "random" setting, these systems have been studied independently from one another. In particular, distinct sets of mathematical tools and techniques have been developed to solve these equations, but, so far, their connections remain poorly understood. This thesis aims at filling this gap by developing a *unified framework* to study the random wave, heat and Schrödinger equations at the same time.

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Contents

Chapter 1

Introduction

thesis is concerned with the study of singular stochastic partial differential equations (PDEs), i.e. the study of PDEs that depend on some random stochastic objects. Let $k \in 2\mathbb{N}+1$ and $d \geq 2$ be integers. The models of interest in this thesis are related to the complex-valued Φ_d^{k+1}-measure, as explained below. The Φ_d^{k+1}-measure on $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ is the *Gibbs measure* formally given by

$$d\rho(u) = Z_k^{-1} \exp\Big(- \frac{1}{k+1} \int_{\mathbb{T}^d} |u|^{k+1} dx \Big) d\mu(u). \tag{1.1}$$

Here, Z_k is a normalization factor making ρ a probability measure and μ is the massive *Gaussian free field* with covariance operator $(1-\Delta)^{-1}$ which can be formally written as

$$d\mu(u) = Z^{-1} \exp\Big(- \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{T}^d} |u|^2 dx \Big) du, \tag{1.2}$$

where Z is a normalization factor and du is the non-existent Lebesgue measure on the space of distributions $\mathcal{D}'(\mathbb{T}^d)$. More rigorously, μ is defined as the law of the random variable ϕ such that

$$\phi(x) := \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{in\cdot x}, \tag{1.3}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a family of i.i.d. standard complex Gaussian variables. A simple computation shows that in dimension two and above (i.e. $d \geq 2$), ϕ is merely a distribution and hence, the measure μ is supported on the space of distributions on the torus. Thus, there is a non-trivial obstacle in making sense of the measure ρ as the nonlinearity $|u|^{k+1}$ is a priori ill-defined. We also only expect to be able to construct the measure ρ (1.1) in the "subcritical/critical cases" where ρ is "dominated" by the Gaussian measure μ. By scaling considerations, this gives the following admissibility range for the parameters (d,k):

$$\frac{d}{k+1} \geq \frac{d-2}{2}.$$

The Gibbs measure (1.1) is an important object of study in constructive Euclidean quantum field theory and, despite the difficulties mentioned in the above, has been successfully constructed in the 70's in the subcritical case $\frac{d}{k+1} > \frac{d-2}{2}$: in dimensions two and three; see [52, 69, 29]. On the other hand, in the critical case $(d,k) = (4,3)$, Aizenman and Duminil-Copin [1] proved that the (real-valued version of) Φ_4^4-measure cannot be constructed in any meaningful way.

We associate to ρ the energy functional (Hamiltonian) H given by

$$H(u) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{T}^2} |u|^2 dx + \frac{1}{k+1} \int_{\mathbb{T}^d} |u|^{k+1} dx. \tag{1.4}$$

In the complex-valued setting, the measure ρ (1.1) is formally invariant by several dynamics on $\mathbb{T}^d$:

1. the hyperbolic Φ_d^{k+1}-model, which is given by the stochastic damped nonlinear wave equation:[1]
$$\varepsilon^2\partial_t^2 u_{\varepsilon,\gamma} + \partial_t u_{\varepsilon,\gamma} = (\gamma+i)(\Delta-1)u_{\varepsilon,\gamma} - (\gamma+i)|u_{\varepsilon,\gamma}|^{k-1}u_{\varepsilon,\gamma} + \sqrt{2\gamma}\xi, \qquad (\text{SdNLW}_{\varepsilon,\gamma})$$
for $\varepsilon > 0$ and $\gamma > 0$.

2. the parabolic Φ_d^{k+1}-model, which is given by the stochastic complex nonlinear Ginzburg-Landau equation:
$$\partial_t u_\gamma = (\gamma+i)(\Delta-1)u_\gamma - (\gamma+i)|u_\gamma|^{k-1}u_\gamma + \sqrt{2\gamma}\xi, \qquad (\text{SCGL}_\gamma)$$
for $\gamma > 0$.

3. the dispersive Φ_d^{k+1}-model, which is given by the deterministic nonlinear Schrödinger equation:
$$\partial_t u = i(\Delta-1)u - i|u|^{k-1}u, \qquad (\text{NLS})$$

Here, $\xi(x,t)$ denotes (Gaussian) space-time white noise on $\mathbb{T}^d\times\mathbb{R}$ defined on a probability space $(\Omega,\mathbb{P})$ and with the space-time covariance (formally) given by
$$\mathbb{E}\big[\xi(x_1,t_1)\xi(x_2,t_2)\big] = \delta(x_1-x_2)\delta(t_1-t_2). \qquad (1.5)$$

In the quantum field theory community, the dynamical Φ_d^{k+1}-models $\text{SdNLW}_{\varepsilon,\gamma}$ and SCGL_γ were introduced in order to study properties of the Φ_d^{k+1}-measure. The idea of studying the measure (1.1) through these equations is known as *stochastic quantization* and was streamlined by Parisi and Wu [64] in the context of heat equations; see also [65]. On the other hand, the study of NLS with Gibbs measure initial data (i.e. initial data sampled from the Gibbs measure (1.1)). was originally motivated by statistical mechanics and was initiated by Lebowitz, Rose, and Speer [40].

At the linear level ($|u_{\varepsilon,\gamma}|^{k-1}u_{\varepsilon,\gamma}\equiv 0$), the solution $\Psi_{\varepsilon,\gamma}$ to the stochastic linear damped wave equation is the so-called *stochastic convolution*, satisfying
$$\varepsilon^2\partial_t^2 u_{\varepsilon,\gamma} + \partial_t u_{\varepsilon,\gamma} = (\gamma+i)(\Delta-1)u_{\varepsilon,\gamma} + \sqrt{2\gamma}\xi,$$

One can show that for any $\varepsilon\geq 0$, $\Psi_{\varepsilon,\gamma}(t)\in H^{-\frac{d}{2}+1-\delta}(\mathbb{T}^d)\setminus H^{-\frac{d}{2}+1}(\mathbb{T}^d)$, $\delta>0$, almost surely. Hence, in the two and three-dimensional setting, the stochastic convolution $\Psi_{\varepsilon,\gamma}$ is *only a distribution*, not a function. Similar issues arise in the study of SCGL_γ and NLS with Gibbsian data. For these reasons, SdNLW_ε, SCGL_γ and NLS are called *singular* PDEs. In view of the a priori roughness roughness of the solutions to the equations $\text{SdNLW}_{\varepsilon,\gamma}$, SCGL_γ and NLS, their study is a very challenging problem which has motivated the rapid development of the fields: that of *singular stochastic PDEs* and that of *random dispersive* PDEs.

The field of singular stochastic PDEs originated from the seminal work of Da Prato and Debussche [22] where they proved well-posedness for the two-dimensional parabolic Φ_2^{k+1}-model, $k\in 2\mathbb{N}+1$, with Gibbsian data. Recently, there was a spectacular development in the parabolic setting, led by Hairer, who introduced the theory of *regularity structures*, and proved well-posedness for the three-dimensional parabolic Φ_3^4-model in [34]. Subsequently, Gubinelli, Imkeller, and Perkowski [30] developed the theory of *paracontrolled distributions* to study stochastic parabolic PDEs.

On the other hand, the field of *random dispersive equations* started from Bourgain's breakthrough work [7] on the two-dimensional dispersive Φ_2^4-model with Gibbs measure initial data;

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[1] Strictly speaking, the Φ_d^{k+1}-measure has to be coupled with (up to some rescaling) the white noise measure on $\mathbb{T}^d$ on the coordinate $\partial_t u$. However, for convenience, we omit these technicalities in this discussion.

with later developments by Burq and Tzvetkov [10, 11], and Colliander and Oh [17] in the mid 2000's. Recently, we have seen rapid progress in the resolution of the well-posedness issue for the hyperbolic Φ_2^{k+1}-models as techniques and ideas where imported from the study of singular PDEs. In [32], Gubinelli, Koch and Oh successfully implemented the paracontrolled approach to the study of the three-dimensional stochastic quadratic nonlinear wave equation. This development led to the very recent impressive resolution of the three-dimensional cubic nonlinear wave equation, with Gibbs measure initial data, by Bringmann, Deng, Nahmod, and Yue [9]. As for the study of random Schrödinger equations, Deng, Nahmod and Yue introduced the theory of *random averanging operators* and proved global well-posedness for the super-cubic dispersive Φ_2^{k+1}-models, $k \geq 5$, with Gibbs measure initial data. Later, by introducing the theory of *random tensors*, a refinement of the random averaging operators, they were also able to prove well-posedness for the dispersive $\Phi^4_{3-\delta}$-model[2]; see [25].

Unfortunately, there has been little interaction between these two communities. The main objective of this thesis is to link the fields of singular stochastic parabolic PDEs and random dispersive PDEs through the viewpoint of *convergence problems*.

When $\varepsilon = 0$, $\text{SdNLW}_{\varepsilon,\gamma}$ formally corresponds to SCGL_γ; while for $\gamma = 0$, SCGL_γ corresponds to NLS. Hence, we expect the solution of $\text{SdNLW}_{\varepsilon,\gamma}$ to converge to the solution of SCGL_γ as $\varepsilon \to 0$, which in turn should converge to the solution of NLS as $\gamma \to 0$. The first convergence (from the solution of $\text{SdNLW}_{\varepsilon,\gamma}$ to that of SCGL_γ) is referred to as the *Smoluchowski-Kramers approximation* (or non-relativistic limit) and has been studied extensively in various contexts [12, 13, 14, 15, 16, 18, 19, 20, 28]. The second convergence issue; namely from the solution of SCGL_γ to that of NLS (as $\gamma \to 0$) is called the *inviscid limit* and has been studied in either the deterministic (i.e. $\xi \equiv 0$) or smooth noise settings [4, 43, 54, 75, 38, 68].

In this thesis, we study the Smoluchowski-Kramers approximation and the inviscid limit in the current singular setting of the equations $\text{SdNLW}_{\varepsilon,\gamma}$, SCGL_γ and NLS on the two-dimensional torus $\mathbb{T}^2$. In particular, we settle in Chapter 2 and Chapter 3 below, the question of convergence of Φ^4_2-models, as illustrated below.

$$\begin{tikzcd}[row sep=small]
\text{hyperbolic } \Phi^4_2\text{-model} \\
\varepsilon^2\partial_t^2 u_{\varepsilon,\gamma} + \partial_t u_{\varepsilon,\gamma} = (\gamma+i)(\Delta-1)u_{\varepsilon,\gamma} - (\gamma+i)|u_{\varepsilon,\gamma}|^2 u_{\varepsilon,\gamma} + \sqrt{2\gamma}\xi \arrow[d, squiggly, "\varepsilon \to 0"'] \\
\text{parabolic } \Phi^4_2\text{-model} \\
\partial_t u_\gamma = (\gamma+i)(\Delta-1)u_\gamma - (\gamma+i)|u_\gamma|^2 u_\gamma + \sqrt{2\gamma}\xi \arrow[d, squiggly, "\gamma \to 0"'] \\
\text{dispersive } \Phi^4_2\text{-model} \\
\partial_t u = i(\Delta-1)u - i|u|^2 u
\end{tikzcd}$$

1.1 Smoluchowski-Kramers approximation

In this section, we study the convergence of the solution of $\text{SdNLW}_{\varepsilon,\gamma}$ to the solution of SCGL_γ in the real-valued setting in dimension two. Namely, we consider the following stochastic damped nonlinear wave equation on $\mathbb{T}^2$

$$\begin{cases} \varepsilon^2\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1-\Delta)u_\varepsilon + u_\varepsilon^k = \xi, \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \phi_1), \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+. \tag{1.6}$$

In the above, $\varepsilon \in (0,1]$, $k \geq 2$ is an integer and ξ denotes a real-valued (Gaussian) space-time white noise on $\mathbb{T}^2 \times \mathbb{R}_+$.

As discussed in the above, we aim to show that u_ε converges to the solution u of the stochastic quantization equation:

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[2] Namely, the cubic nonlinear Schrödinger equation with the Gaussian free field initial data with covariance operator $(1-\Delta)^{-1-\delta}$.

$$\begin{cases} \partial_t u + (1-\Delta)u + u^k = \xi, \\ u|_{t=0} = \phi_0, \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{1.7}$$

as $\varepsilon \to 0$.

Let us first discuss informally why the Smoluchowski-Kramers approximation occurs. Consider the following homogeneous linear damped wave equation:

$$\begin{cases} \varepsilon^2 \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1-\Delta)u_\varepsilon = 0 \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \phi_1). \end{cases} \tag{1.8}$$

By taking the spatial Fourier transform, we have

$$\varepsilon^2 \partial_t^2 \widehat{u}_\varepsilon(n) + \partial_t \widehat{u}_\varepsilon(n) + \langle n \rangle^2 \widehat{u}_\varepsilon(n) = 0 \tag{1.9}$$

for $n \in \mathbb{Z}^2$. The roots of the characteristic polynomial $\varepsilon^2 \Lambda^2 + \Lambda + \langle n \rangle^2 = 0$ are given by

$$\Lambda_\varepsilon^{\pm}(n) = \frac{-1 \pm \sqrt{1 - 4\langle n \rangle^2 \varepsilon^2}}{2\varepsilon^2}. \tag{1.10}$$

Note that we have $\Lambda_\varepsilon^{\pm}(n) \in \mathbb{R}$ if and only if $\langle n \rangle \le (2\varepsilon)^{-1}$. In the low frequency regime $\langle n \rangle \le (2\varepsilon)^{-1}$, the solution to (1.9) with

$$(\widehat{u}_\varepsilon(n), \partial_t \widehat{u}_\varepsilon(n))|_{t=0} = (\widehat{\phi}_0(n), \widehat{\phi}_1(n)) \tag{1.11}$$

is given by

$$\widehat{u}_\varepsilon(n,t) = e^{-\frac{t}{2\varepsilon^2}} \cosh(\lambda_\varepsilon(n)t) \widehat{\phi}_0(n) + e^{-\frac{t}{2\varepsilon^2}} \frac{\sinh(\lambda_\varepsilon(n)t)}{\lambda_\varepsilon(n)} \bigg(\frac{1}{2\varepsilon^2} \widehat{\phi}_0(n) + \widehat{\phi}_1(n) \bigg), \tag{1.12}$$

where $\lambda_\varepsilon(n)$ is defined by

$$\lambda_\varepsilon(n) = \frac{\sqrt{1 - 4\langle n \rangle^2 \varepsilon^2}}{2\varepsilon^2}. \tag{1.13}$$

In the high frequency regime $\langle n \rangle > (2\varepsilon)^{-1}$, the solution to (1.9) with initial data (1.11) is given by

$$\widehat{u}_\varepsilon(n,t) = e^{-\frac{t}{2\varepsilon^2}} \cos(\zeta_\varepsilon(n)t) \widehat{\phi}_0(n) + e^{-\frac{t}{2\varepsilon^2}} \frac{\sin(\zeta_\varepsilon(n)t)}{\zeta_\varepsilon(n)} \bigg(\frac{1}{2\varepsilon^2} \widehat{\phi}_0(n) + \widehat{\phi}_1(n) \bigg), \tag{1.14}$$

where $\zeta_\varepsilon(n)$ is defined by

$$\zeta_\varepsilon(n) = \frac{\sqrt{4\langle n \rangle^2 \varepsilon^2 - 1}}{2\varepsilon^2}. \tag{1.15}$$

Let $\mathbf{P}_\varepsilon^{\text{low}}$ and $\mathbf{P}_\varepsilon^{\text{high}}$ be the sharp projections onto the (spatial) frequencies $\{n \in \mathbb{Z}^2 : \langle n \rangle \le (2\varepsilon)^{-1}\}$ and $\{n \in \mathbb{Z}^2 : \langle n \rangle > (2\varepsilon)^{-1}\}$, respectively, defined by

$$\mathbf{P}_\varepsilon^{\text{low}} f = \mathcal{F}^{-1}(\mathbb{1}_{\langle n \rangle \le (2\varepsilon)^{-1}} \widehat{f}(n)) \qquad \text{and} \qquad \mathbf{P}_\varepsilon^{\text{high}} f = \mathcal{F}^{-1}(\mathbb{1}_{\langle n \rangle > (2\varepsilon)^{-1}} \widehat{f}(n)). \tag{1.16}$$

Then, define the operator $S_\varepsilon(t)$ and $\mathcal{D}_\varepsilon(t)$ by setting

$$S_\varepsilon(t) = \frac{\sinh(\lambda_\varepsilon(\nabla)t)}{\lambda_\varepsilon(\nabla)} \mathbf{P}_\varepsilon^{\text{low}} + \frac{\sin(\zeta_\varepsilon(\nabla)t)}{\zeta_\varepsilon(\nabla)} \mathbf{P}_\varepsilon^{\text{high}} \tag{1.17}$$

and

$$\mathcal{D}_\varepsilon(t) = e^{-\frac{t}{2\varepsilon^2}} S_\varepsilon(t). \tag{1.18}$$

Remark 1.1.1. At this point, the operator $\mathcal{D}_\varepsilon$ is only defined for $\varepsilon \in I$ such that

$$I := (0,\infty) \setminus \{\frac{1}{2\langle n\rangle} : n \in \mathbb{Z}^2\}. \tag{1.19}$$

However, we show in Lemma 2.1.5 that the map $\varepsilon \in I \mapsto \mathcal{D}_\varepsilon$ can be extended to $(0,\infty)$. In what follows, since we are interested in the behaviour of the solutions to (1.6) near $\varepsilon = 0$, we will consider that all quantities are defined for $\varepsilon \in [0,1]$.

From (1.12) and (1.14), we see that the solution u_ε to (1.8) is given by

$$u_\varepsilon(t) = \partial_t \mathcal{D}_\varepsilon(t)\phi_0 + \mathcal{D}_\varepsilon(t)(\varepsilon^{-2}\phi_0 + \phi_1). \tag{1.20}$$

Furthermore, by the Duhamel principle, the solution u_ε to the following nonhomogeneous linear damped wave equation:

$$\begin{cases} \varepsilon^2\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1-\Delta)u_\varepsilon = F \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \phi_1). \end{cases} \tag{1.21}$$

is given by

$$u_\varepsilon(t) = \partial_t \mathcal{D}_\varepsilon(t)\phi_0 + \mathcal{D}_\varepsilon(t)(\varepsilon^{-2}\phi_0 + \phi_1) + \int_0^t \varepsilon^{-2}\mathcal{D}_\varepsilon(t-t')F(t')dt' \tag{1.22}$$

From (1.10) with a Taylor expansion, we have

$$\begin{aligned} \Lambda_\varepsilon^+(n) &= \frac{-2\langle n\rangle^2}{1+\sqrt{1-4\langle n\rangle^2\varepsilon^2}} = -\langle n\rangle^2 + O(\langle n\rangle^4\varepsilon^2) \longrightarrow -\langle n\rangle^2, \\ \Lambda_\varepsilon^-(n) &= \frac{-2\langle n\rangle^2}{1-\sqrt{1-4\langle n\rangle^2\varepsilon^2}} \longrightarrow -\infty \end{aligned} \tag{1.23}$$

in the regime $\langle n\rangle = o(\varepsilon^{-\frac{1}{2}})$ as $\varepsilon \to 0$, namely in the regime

$$\sqrt{1-4\langle n\rangle^2\varepsilon^2} = 1 - 2\langle n\rangle^2\varepsilon^2 + O(\langle n\rangle^4\varepsilon^4) \tag{1.24}$$

as $\varepsilon \to 0$. Let χ be a smooth non-negative function such that $\chi \equiv 1$ on $\{x \in \mathbb{R} : |x| \le 1\}$ and $\operatorname{supp}(\chi) \subset \{x \in \mathbb{R} : |x| \le 2\}$. Fix $N \in \mathbb{R}$ and let $\mathbf{P}_{\le N}$ be the sharp projection onto (spatial) frequencies $\{n \in \mathbb{Z}^2 : \langle n\rangle \le N\}$ defined by

$$\mathbf{P}_{\le N} f := \mathcal{F}^{-1}(\mathbb{1}_{\langle n\rangle \le N}\widehat{f}(n)), \tag{1.25}$$

At a formal level, we have

$$\begin{aligned} \varepsilon^{-2}\mathcal{D}_\varepsilon(t)\mathbf{P}_{\le\varepsilon^{-\frac{1}{2}+\theta}} &= \frac{e^{\Lambda_\varepsilon^+(\nabla)t} - e^{\Lambda_\varepsilon^-(\nabla)t}}{\sqrt{1-4\langle\nabla\rangle^2\varepsilon^2}}\mathbf{P}_{\le\varepsilon^{-\frac{1}{2}+\theta}} \longrightarrow P_0(t)\mathbf{P}_{\le\varepsilon^{-\frac{1}{2}+\theta}}, \\ \partial_t\mathcal{D}_\varepsilon(t)\mathbf{P}_{\le\varepsilon^{-\frac{1}{2}+\theta}} &= \Bigg(\Big(1 - \frac{1}{\sqrt{1-4\langle\nabla\rangle^2\varepsilon^2}}\Big)\frac{e^{\Lambda_\varepsilon^+(\nabla)t}}{2} \\ &\quad + \Big(1 + \frac{1}{\sqrt{1-4\langle\nabla\rangle^2\varepsilon^2}}\Big)\frac{e^{\Lambda_\varepsilon^-(\nabla)t}}{2}\Bigg)\mathbf{P}_{\le\varepsilon^{-\frac{1}{2}+\theta}} \\ &\longrightarrow 0 \end{aligned} \tag{1.26}$$

for any $0 < \theta \ll 1$, where

$$P_0(t) = e^{(\Delta-1)t}. \tag{1.27}$$

See Lemma 2.1.7 for a rigorous justification of (1.26).

Remark 1.1.2. By looking more carefully into the kernels involved, one can prove that the convergence (1.26) occurs in the regime $\langle n\rangle = o(\varepsilon^{-1})$. Namely, the convergence in (1.26) is valid with $\mathbf{P}_{\leq \varepsilon^{-\frac{1}{2}+\theta}}$ replaced by $\mathbf{P}_{\leq \varepsilon^{-1+\theta}}$; see Lemma 2.1.7

Remark 1.1.3. Note that (the proof of) Lemma 2.1.7 shows in particular that $\mathbf{P}_{\leq \varepsilon^{-1+\theta}}\big(\varepsilon^{-2}\mathcal{D}_\varepsilon - P_0\big)$ and $\mathbf{P}_{\leq \varepsilon^{-1+\theta}}\partial_t\mathcal{D}_\varepsilon$ - viewed as operators from $H^s(\mathbb{T}^2)$ to itself for any $s \in \mathbb{R}$ - both converge to zero as $\varepsilon \to 0$ only pointwisely in time. In order to get a uniform-in-time convergence, i.e. in $L^\infty\big([0,T]; H^s(\mathbb{T}^2)\big)$ for any $T > 0$ and $s \in \mathbb{R}$, one must work with the operator $\mathbf{P}_{\leq \varepsilon^{-\frac{1}{2}+\theta}}\big(\varepsilon^{-2}\mathcal{D}_\varepsilon + \partial_t\mathcal{D}_\varepsilon\big)$ which enjoys some extra cancellation. See Lemma 2.1.7 and Corollary 2.1.8.

Therefore, we see that u_ε in (1.21) formally converges to

$$u(t) = P_0(t)\phi_0 + \int_0^t P_0(t-t')F(t')dt', \tag{1.28}$$

satisfying the nonhomogeneous linear heat equation:

$$\begin{cases} \partial_t u + (1-\Delta)u = F \\ u|_{t=0} = \phi_0. \end{cases} \tag{1.29}$$

Thus, we expect that the solution to (1.6) converges to the solution of the stochastic quantization equation (1.7) as $\varepsilon \to 0$.

1.1.1 Results and outline of the proof

For any $\varepsilon > 0$, we introduce the stochastic convolutions by

$$\Psi_\varepsilon = \int_0^t \varepsilon^{-2}\mathcal{D}_\varepsilon(t-t')dW(t'), \tag{1.30}$$

$$\Psi_0 = \int_0^t P_0(t-t')dW(t'). \tag{1.31}$$

where W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$, defined on some probability space $(\Omega, \mathbb{P})$:

$$W(t) := \sum_{n\in\mathbb{Z}^2} B_n(t)e_n \tag{1.32}$$

and $\{B_n\}_{n\in\mathbb{Z}^2}$ is defined by $B_n(t) = \langle \xi, \mathbb{1}_{[0,t]}\cdot e_n\rangle_{x,t}$. Here, $\langle\cdot,\cdot\rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^2\times\mathbb{R}$. As a result, we see that $\{B_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued[3] Brownian motions constructed so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^2$. By convention, we normalized B_n such that $\mathrm{Var}(B_n(t)) = t$.

The solutions u_ε and u to (1.6) and (1.7), respectively, are merely distributions, not functions. We first analyze the equations at the linear level.

Given $\varepsilon \in [0,1]$ and $N \in \mathbb{N}$, we define the truncated stochastic convolution $\Psi_{\varepsilon,N} = \mathbf{P}_{\leq N}\Psi_\varepsilon$, solving the truncated linear stochastic wave equation/heat equation (for $\varepsilon = 0$):

$$\varepsilon^2\partial_t^2\Psi_{\varepsilon,N} + \partial_t\Psi_{\varepsilon,N} + (1-\Delta)\Psi_{\varepsilon,N} = \mathbf{P}_{\leq N}\xi, \tag{1.33}$$

with the zero initial data. Here, $\mathbf{P}_{\leq N}$ is as in (1.25). Then, $\Psi_{\varepsilon,N}$ is represented by the following formula:

$$\Psi_{\varepsilon,N} = \int_0^t \varepsilon^{-2}\mathcal{D}_\varepsilon(t-t')d(\mathbf{P}_{\leq N}W)(t'), \tag{1.34}$$

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[3] In particular, B_0 is a standard real-valued Brownian motion.

where W is as in (1.32). For each fixed $\varepsilon \in (0,1]$, $x \in \mathbb{T}^2$ and $t \geq 0$, we see from (1.34) and (1.15) that $\Psi_{\varepsilon,N}(x,t)$ is a mean-zero real-valued Gaussian random variable with variance

$$\begin{aligned}\sigma_{\varepsilon,N}(t) \overset{\text{def}}{=} \mathbb{E}\big[\Psi_{\varepsilon,N}(x,t)^2\big] &= C_\varepsilon(t) + \sum_{\substack{n\in\mathbb{Z}^2\\ \varepsilon^{-1}\ll\langle n\rangle\lesssim N}} \int_0^t \bigg[\frac{\sin(\zeta_\varepsilon(n)(t-t'))}{\zeta_\varepsilon(n)}\bigg]^2 dt' \\ &\sim \log N\end{aligned} \tag{1.35}$$

for some constant $C = C_\varepsilon(t)$ and $N \gg \varepsilon^{-1}$ and ζ_ε as in (1.15). Note that the implicit constant in (1.35) depends on ε and t. We point out that the variance $\sigma_{\varepsilon,N}(t)$ is time-dependent. For any $t > 0$, we see that $\sigma_{\varepsilon,N}(t) \to \infty$ as $N \to \infty$, which can be used to show that $\{\Psi_{\varepsilon,N}(t)\}_{N\in\mathbb{N}}$ is almost surely unbounded in $W^{0,p}(\mathbb{T}^2)$ for any $1 \leq p \leq \infty$. Similar comments apply to $\Psi_{0,N}$ and its variance $\sigma_{0,N}$.

For notational convenience, we denote by $u_{\varepsilon=0}$ the solutions to (1.7) and view it as the solution to (1.6) with $\varepsilon = 0$. For $N \in \mathbb{N}$ and $\varepsilon \in [0,1]$, let $u_{\varepsilon,N}$ denote the solution to (1.6) where the rough noise ξ is replaced by the regularized noise $\mathbf{P}_{\leq N}\xi$. Proceeding with the following decomposition of $u_{\varepsilon,N}$ the solution to (1.6) ([45, 7, 22]):

$$u_{\varepsilon,N} = v_{\varepsilon,N} + \Psi_{\varepsilon,N}. \tag{1.36}$$

This decomposition leads to

$$\varepsilon^2\partial_t^2 v_{\varepsilon,N} + \partial_t v_{\varepsilon,N} + (1-\Delta)v_{\varepsilon,N} + \sum_{\ell=0}^{k}\binom{k}{\ell}\Psi_{\varepsilon,N}^{\ell} v_{\varepsilon,N}^{k-\ell} = 0. \tag{1.37}$$

Due to the deficiency of regularity of $\Psi_{\varepsilon,N}$, the power $\Psi_{\varepsilon,N}^{\ell}$, $\ell \geq 2$, does not converge to any limit as $N \to \infty$. This is where we introduce the Wick renormalization. Namely, we replace $\Psi_{\varepsilon,N}^{\ell}$ by its Wick ordered counterpart:

$$:\!\Psi_{\varepsilon,N}^{\ell}(x,t)\!: \overset{\text{def}}{=} H_\ell(\Psi_{\varepsilon,N}(x,t); \sigma_{\varepsilon,N}(t)), \tag{1.38}$$

where $H_\ell(x;\sigma)$ is the Hermite polynomial of degree ℓ with variance parameter σ. See Appendix A.2. Then, for each $\ell \in \mathbb{N}$, the Wick power $:\!\Psi_{\varepsilon,N}^{\ell}\!:$ converges to a limit, denoted by $:\!\Psi_{\varepsilon}^{\ell}\!:$, in $C([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))$ for any $\varepsilon \geq 0$, $\sigma > 0$, and $T > 0$, almost surely; see Proposition 2.1.10 below.

These renormalizations give rise to the renormalized version of (1.6):

$$\varepsilon^2\partial_t^2 v_{\varepsilon,N} + \partial_t v_{\varepsilon,N} + (1-\Delta)v_{\varepsilon,N} + \sum_{\ell=0}^{k}\binom{k}{\ell} :\!\Psi_{\varepsilon,N}^{\ell}\!:\, v_{\varepsilon,N}^{k-\ell} = 0, \tag{1.39}$$

for $\varepsilon \in [0,1]$. By taking a limit as $N \to \infty$, we then obtain the limiting equations:

$$\varepsilon^2\partial_t^2 v_{\varepsilon} + \partial_t v_{\varepsilon} + (1-\Delta)v_{\varepsilon} + \sum_{\ell=0}^{k}\binom{k}{\ell} :\!\Psi_{\varepsilon}^{\ell}\!:\, v_{\varepsilon}^{k-\ell} = 0, \tag{1.40}$$

Given the almost-sure space-time regularity of the Wick powers $:\!\Psi_{\varepsilon}^{\ell}\!:$, $\ell = 1,\dots,k$, standard deterministic analysis using the product estimates (Lemma A.1.1) yield local well-posedness of (1.39) and (1.40), for each fixed $\varepsilon \in [0,1]$.

Recalling the decomposition (1.36), this argument also shows that the solution $u_{\varepsilon,N} = \Psi_{\varepsilon,N} + v_{\varepsilon,N}$, with $v_{\varepsilon,N}$ solving (1.39), to the renormalized equation with the regularized noise $\mathbf{P}_{\leq N}\xi$

$$\varepsilon^2\partial_t^2 u_{\varepsilon,N} + \partial_t u_{\varepsilon,N} + (1-\Delta)u_{\varepsilon,N} + :\!u_{\varepsilon,N}^{k}\!: = \mathbf{P}_{\leq N}\xi \tag{1.41}$$

where the renormalized nonlinearity $:\!u_{\varepsilon,N}^{k}\!:$ is interpreted as

$$:u_{\varepsilon,N}^k: = :(\Psi_{\varepsilon,N} + v_{\varepsilon,N})^k: = \sum_{\ell=0}^{k} \binom{k}{\ell} :\Psi_{\varepsilon,N}^\ell: v_{\varepsilon,N}^{k-\ell},$$

converge almost surely to the stochastic process $u_\varepsilon = \Psi_\varepsilon + v_\varepsilon$, where v_ε satisfies (1.40). It is in this sense that we say that the following renormalized versions of equations (1.6) and (1.7):

$$\varepsilon^2\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1-\Delta)u_\varepsilon + :u_\varepsilon^k: = \xi, \tag{1.42}$$

$$\partial_t u + (1-\Delta)u + :u^k: = \xi, \tag{1.43}$$

are locally well-posed for $\varepsilon \in (0,1]$ (and for initial data of suitable regularity).

We can now state the main results of the chapter. Firstly, we show the following local existence and Schmoluchowski-Kramers approximation for polynomial nonlinearities. Let $\mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$ for any $s \in \mathbb{R}$.

Theorem 1.1.4. *Fix an integer $k \geq 2$ and let $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for $s > \frac{2k-3}{2k-2}$. Then, the following holds:*

(i) (uniform local well-posedness) *There exists an almost surely postive time $T = T(\omega)$ such that for each $\varepsilon \in (0,1]$, there exists a solution u_ε to (1.42) with initial data (ϕ_0, ϕ_1) and a solution u to (1.43) with initial data ϕ_0 which belong to the class $C\big([0,T]; H^{-\sigma}(\mathbb{T}^2)\big)$, for any $\sigma > 0$.*

(ii) (convergence) *Moreover, u_ε converges almost surely to the solution u in $C\big([0,T]; H^{-\sigma}(\mathbb{T}^2)\big)$ as $\varepsilon \to 0$.*

Remark 1.1.5. We emphasize here that the convergence result in Theorem 1.1.4 means that there exists a set $\Omega_0 \subset \Omega$ (where $(\Omega, \mathbb{P})$ is the underlying probability space on which the noise ξ is defined) such that $\mathbb{P}(\Omega_0) = 1$ and

$$\|u_\varepsilon^\omega - u_0^\omega\|_{C_T H_x^s} \to 0,$$

for any $\omega \in \Omega_0$ as $\varepsilon \to 0$ and with T and s as in Theorem 1.1.4. This is in sharp contrast with the literature where such convergence results are obtained only up to a subsequence. See for instance [13, 28].

The proof of Theorem 1.1.4 follows from two ingredients: the study of the stochastic objects (1.38) as functions in the variable (ε, t) by using a bi-parameter version of the Kolmogorov continuity criterions and a fixed point argument for v_ε the solution to (1.40) in a space of continuous functions of the variable (ε, t). The convergence of u_ε to $u_{\varepsilon=0} = u$ is then a direct consequence of the continuity at $\varepsilon = 0$ of the map $\varepsilon \mapsto v_\varepsilon$. See the discussion in Subsection 2.2.1 for more details.

We note that Fukuizumi, Hoshino and Inui independently studied in [28] the convergence of (1.6) to (1.7) as $\varepsilon \to 0$ for Gibbsian data (1.1). They proved a global in-time convergence similar to Theorem 1.1.4, but only according to the discrete sequence $\varepsilon(j) = \frac{1}{j} \to 0$, $j \in \mathbb{N}$.

In the non-singular case with a polynomial nonlinearity (for instance (1.6) with a colored noise in space or in one space dimension), the result of last theorem can be extended to arbitrary large time intervals since the solutions are known to be global. See for instance [12, 13]. In our singular setting however, the convergence of Theorem 1.1.4 cannot be established over longer times because of a lack of a global well-posedness theory for (1.42) and $k > 3$ (the solutions are known to be global-in-time for $k = 3$; see Remark 1.1.7 below). Since the solution u to (1.43) is known to exist globally in time by an argument of Mourrat and Weber [50] (see also [72]), we can show *asymptotic large time well-posedness* for (1.42) (for $\varepsilon > 0$). More precisely, since u_ε gets closer to u_0 as $\varepsilon \to 0$, we can extend the existence time and the convergence of our local solutions u_ε over larger times as $\varepsilon \to 0$. This is the purpose of the following theorem.

Theorem 1.1.6. *Let $k \geq 2$ be an integer. Fix a (deterministic) target time $T > 0$. Let $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for $s > \frac{2k-3}{2k-2}$. There exists an almost surely positive random variable $\varepsilon_0 = \varepsilon_0(\omega)$ such that for each $\varepsilon \in [0, \varepsilon_0]$ the solutions u_ε and u to (1.42) and (1.43), respectively, constructed in Theorem 1.1.4 exist up to time T.*

Furthermore, $\{u_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ converges to u_0 in $C\big([0,T]; H^{-\sigma}(\mathbb{T}^2)\big)$ as $\varepsilon \to 0$, for any $\sigma > 0$.

We conclude this section with a few remarks.

Remark 1.1.7. Fix $s > \frac{4}{5}$. In [33], the authors proved that (1.42) for $k = 3$ and $\varepsilon = 1$ is globally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. By modifying their argument, one can show global well-posedness in $\mathcal{H}^s(\mathbb{T}^2)$ for $\varepsilon \in (0,1]$. Since the stochastic quantization equation (1.43) is globally well-posed in $H^s(\mathbb{T}^2)$ as well; see [50, 72], the Smoluchowski-Kramers approximation proved in Theorem 1.1.4 holds globally in time, i.e. in $C\big([0,T]; H^{-\sigma}(\mathbb{T}^2)\big)$, $\sigma > 0$, for any $T > 0$ in the cubic case $k = 3$.

Remark 1.1.8. The solutions constructed in Theorems 1.1.4 and 1.1.6 are unique in classes of the form $\Psi_\varepsilon + C([0,T]; H^{1-\sigma}(\mathbb{T}^2))$ for any $\varepsilon \in [0, \varepsilon_0]$ for some appropriate $\varepsilon_0 \in [0,1]$, $T > 0$ and $\sigma > 0$.

1.2 Inviscid limit

In the present chapter, we address the question of the inviscid limit in the singular setting for the equations SCGL$_\gamma$ and NLS with Gibbs measure initial data Φ^4_2 (1.1).

We first aim at defining the Φ^4_2-measure rigorously. Recall the measure μ is given by

$$\mu = \mathrm{Law}(\phi). \tag{1.44}$$

Let $N \in \mathbb{N}$ and define the truncated renormalized interaction potential R_N as follows:

$$R_N(u) = -\frac{1}{4}\int_{\mathbb{T}^2} :|\mathbf{P}_{\le N} u|^4: \, dx. \tag{1.45}$$

Then, we define the truncated renormalized probability measure ρ_N by

$$\rho_N := Z_N^{-1} \exp\big(R_N(u)\big) d\mu(u) \tag{1.46}$$

It turns out that the above definition of the measure ρ_N (1.46) leads to a meaningful object in the sense that ρ_N admits a well-defined limit as $N \to \infty$. Justifying this procedure is the purpose of the next result.

Proposition 1.2.1. *Let $R_N(u)$ and ρ_N be as in (1.45) and (1.46), respectively. Then, the following holds:*

(i) *The truncated renormalized interaction potentials $\{R_N(u)\}_{N\in\mathbb{N}}$ form a Cauchy sequence in $L^p(\mu)$ for any finite $p \ge 1$; thus converging to some random variable $R(u) \in L^p(\mu)$.*

(ii) *Given any finite $p \ge 1$, there exists $C_p > 0$ such that*

$$\sup_{N\in\mathbb{N}} \big\| e^{R_N(u)} \big\|_{L^p(\mu)} \le C_p.$$

Moreover, we have

$$\lim_{N\to\infty} e^{R_N(u)} = e^{R(u)} \qquad \text{in } L^p(\mu). \tag{1.47}$$

As a consequence, the truncated renormalized Gibbs measure ρ_N converge, in the sense of (1.47) to the renormalized Gibbs measure ρ given by

$$d\rho(u) = Z^{-1} e^{R(u)} d\mu. \tag{1.48}$$

Furthermore, the resulting Gibbs measure ρ is equivalent to the Gaussian measure μ.

We now describe here the renormalization procedure that we carry out to make sense of both the dynamics SCGL$_\gamma$ and NLS. Fix $N \in \mathbb{N}$ and consider the truncated Hamiltonian $H_N(u)$ associated to ρ_N given by

$$H_N(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\mathbf{P}_{\le N}\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\mathbf{P}_{\le N} u|^2 dx + \frac{1}{4}\int_{\mathbb{T}^2} :|\mathbf{P}_{\le N} u|^4: dx. \tag{1.49}$$

Here, $:|\mathbf{P}_{\le N} u|^4:$ denotes the following Wick renormalization:[4] and $\mathbf{P}_{\le N}$ is as in (1.25).

$$:|\mathbf{P}_{\le N} u|^4: \overset{\text{def}}{=} |\mathbf{P}_{\le N} u|^4 - 4\sigma_N |\mathbf{P}_{\le N} u|^2 + 2\sigma_N^2, \tag{1.50}$$

with

$$\sigma_N := \mathbb{E}\big[\|\mathbf{P}_{\le N}\phi\|_{L_x^2}^2\big], \tag{1.51}$$

where ϕ is as in (1.3). In particular, we have

$$d\rho_N(u) := Z_N^{-1}\big(e^{-H_N(u)} du_N(u)\big) \otimes d\mu^{>N}(u),$$

where du_N denotes the Lebesgue measure on the space of Fourier modes corresponding to the indices $\{n : \langle n\rangle \le N\}$ and $d\mu^{>N}$ is the push-forward of the measure μ under the map $\mathrm{Id} - \mathbf{P}_{\le N}$.

Let us introduce the following Wick renormalized truncated models induced by H_N:

$$\partial_t u_{\gamma,N} = (\gamma + i)(\Delta - 1)u_{\gamma,N} - (\gamma+i)\mathbf{P}_{\le N}\big((|\mathbf{P}_{\le N} u_{\gamma,N}|^2 - 2\sigma_N)\mathbf{P}_{\le N} u_{\gamma,N}\big) + \sqrt{2\gamma}\xi, \tag{1.52}$$

for $\gamma \in (0,1]$ and

$$\partial_t u_N = i(\Delta - 1)u_N - i\mathbf{P}_{\le N}\big((|\mathbf{P}_{\le N} u_N|^2 - 2\sigma_N)\mathbf{P}_{\le N} u_N\big), \tag{1.53}$$

with initial data ϕ (1.75).

In [22] and [7], Da Prato-Debussche and Bourgain respectively proved the almost sure convergence of $u_{\gamma,N}$ and u_N to some processes u_γ and u in $C\big([0,T]; H^{0-}(\mathbb{T}^2)\big)$ for an almost surely positive time $T > 0$. We formally write that u_γ and u solve the equations

$$\partial_t u_\gamma = (\gamma+i)(\Delta-1)u_\gamma - (\gamma+i)\big((|u_\gamma|^2 - 2\infty)u_\gamma\big) + \sqrt{2\gamma}\xi, \tag{1.54}$$

for $\gamma \in (0,1]$ and

$$\partial_t u = i(\Delta-1)u - i\big((|u|^2 - 2\infty)u\big), \tag{1.55}$$

with initial data sampled from ρ_N (1.46). The equations (1.54) and (1.55) constitute the Wick renormalized counterparts of the models SCGL$_\gamma$ and NLS.

1.2.1 Results

We now state the precise version of our main result regarding the inviscid limit.

Theorem 1.2.2. *The following holds*:

(i) (global well-posedness and invariance) *The renormalized equations* (1.54)*, for any* $\gamma \in (0,1]$*, and* (1.55) *are almost surely globally well-posed with respect to the Gibbs measure* ρ (1.48).

More precisely, for each $\gamma \in [0,1]$*, there exists a non-trivial stochastic process* $u_\gamma \in C\big(\mathbb{R}_+; H^{0-}(\mathbb{T}^2)\big)$ *such that, the solution* $u_{\gamma,N}$ *(resp.* u_N *for* $\gamma = 0$*) to the truncated dynamics* (1.52) *(resp.* (1.53)*) converges to* u_γ *in* $C\big(\mathbb{R}_+; H^{0-}(\mathbb{T}^2)\big)$[5] *as* $N \to \infty$ *in* $\rho \otimes \mathbb{P}$*-probability. Moreover, the law of* $u_\gamma(t)$ *is given by the renormalized Gibbs measure* ρ *for each* $t \ge 0$.

(ii) (convergence) *Let* u_γ *and* u *be the solutions to* (1.54) *and* (1.55) *respectively, constructed in* (i)*. Then,* u_γ *converges to* u *in* $C\big(\mathbb{R}_+; H^{0-}(\mathbb{T}^2)\big)$ *as* $\gamma \to 0$ *in* $\rho \otimes \mathbb{P}$*-probability.*

The main novelty in Theorem 1.2.2 is the convergence (ii). To the best of the author's knowledge, Theorem 1.2.2 (ii) is the first instance of a convergence result of the solution of the Ginzburg-Landau equation to that of the nonlinear Schrödinger equation in a singular setting.

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[4] Note the difference, in the current complex-valued setting with (1.38).

[5] endowed with the compact-open topology.

From an analytical perspective, the difficulty in proving Theorem 1.2.2 comes from the lack of smoothing under the Schrödinger flow (1.55), whereas the heat flow (1.54) essentially gains two derivatives. As such, the well-posedness issue for (1.55) is a much harder problem to consider than that for (1.54) with $\gamma > 0$. Thus, it is natural to choose a space appropriate for the study of NLS (1.55) as a common space where to compare the solutions of (1.54) and (1.55). In what follows, we consider the Fourier restriction norm method (introduced by Bourgain [5] in the context of NLS) utilizing $X^{s,b}$-spaces (defined in Definition 3.1.1 below) adapted to the Schrödinger flow. However, it turns out that these spaces are not well suited for the study of the parabolic equation (1.54) for small $0 < \gamma \ll 1$, which causes issues in our analysis. We elaborate further on this point at the end of Subsection 1.2.2.

Remark 1.2.3. We note that the convergence (ii) in Theorem 1.2.2 above is only a convergence in probability as $\gamma \to 0$, as opposed to the almost sure convergence stated in Theorem 1.1.4. This is essentially because we further replace the "gauged noise" at the level of the gauged version of the equation (1.52) by the usual space-time white noise ξ. We then implement a pathwise approach to this modified equation. See Remark 1.2.8 below for further explanations on this point.

Remark 1.2.4. It would be of interest to study the inviscid limit proved in Theorem 1.2.2 for higher order nonlinearities, i.e; $k \in 2\mathbb{N}+1$ with $k \geq 5$ in SCGL_γ and NLS. With the corresponding Gibbs measure measure initial data, well-posedness for the heat model SCGL_γ was proved by Da Prato and Debussche [22], while well-posedness for NLS was established only recently by Deng, Nahmod, and Yue [24] in a breakthrough work. In order to achieve this recent result, they introduced the theory of random averaging operators. We plan to show the inviscid limit for these models by adapting the random averaging operators theory to parabolic (and stochastic) setting. However, in view of the issue discussed in Remark 1.2.9 below, achieving this convergence might require using the random tensor theory [25], a further extension of the theory of random averaging operators.

1.2.2 Outline of the proof

Here, we outline the proof of Theorem 1.2.2. We fix $N \in \mathbb{N}$. We introduce the following gauge transformation:

$$\mathcal{G}_N(u) = e^{iV_N(u)}u, \tag{1.56}$$

with

$$V_N(u)(t) = 2\int_0^t \Big(\int_{\mathbb{T}^2} |\mathbf{P}_{\leq N} u(t')|^2 dx - \sigma_N \Big) dt' \in \mathbb{R}, \quad t \geq 0, \tag{1.57}$$

for any some smooth $u \in \mathcal{S}(\mathbb{T}^2 \times \mathbb{R})$. Note that the functional $V_N(u)$ does not depend on the spatial variable $x \in \mathbb{T}^2$ and that $V_N(u) = V_N(\mathcal{G}_N(u))$. The gauged variables $\mathfrak{u}_{\gamma,N}$[6] are then given by

$$\mathfrak{u}_{\gamma,N} := \mathcal{G}_N(u_{\gamma,N}), \tag{1.58}$$

for any $\gamma \in [0,1]$. Here, $u_{\gamma,N}$ is the solution to (1.52) with initial data given by $\phi \sim \rho$. The function $\mathfrak{u}_{\gamma,N}$ then solves the following gauged equation:

$$\begin{aligned} \partial_t \mathfrak{u}_{\gamma,N} = (\gamma + i)(\Delta - 1)\mathfrak{u}_{\gamma,N} - \gamma \mathbf{P}_{\leq N}\big(\big(|\mathbf{P}_{\leq N} u_{\gamma,N}|^2 - 2\sigma_N \big) \mathbf{P}_{\leq N} \mathfrak{u}_{\gamma,N} \big) \\ - i\mathbf{P}_{\leq N}\mathfrak{N}(\mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N}) + \sqrt{2\gamma}\,\mathfrak{X}^{\gamma,N}, \end{aligned} \tag{1.59}$$

with initial data $\phi \sim \rho$ and for any $\gamma \in [0,1]$. Here, $\mathfrak{X}^{\gamma,N}$ denotes the *gauged noise* given by

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[6] Here, and throughout this chapter, a quantity indexed by $\gamma = 0$ will always refer to the Schrödinger model. For instance, $u_{\gamma=0,N}$ is the solution to (1.53).

$$\mathfrak{X}^{\gamma,N} := e^{iV_N(u_{\gamma,N})}\xi, \tag{1.60}$$

and $\mathfrak{N}$ denotes the *PDE renormalized* cubic nonlinearity defined by

$$\mathfrak{N}(u) = \Big(|u|^2 - 2\int_{\mathbb{T}^2} |u|^2 dx\Big)u. \tag{1.61}$$

We can then further decompose the nonlinearity $\mathfrak{N}$ into

$$\mathfrak{N}(u) = \mathcal{N}(u) - \mathcal{R}(u), \tag{1.62}$$

where $\mathcal{N}$ and $\mathcal{R}$ are (with a slight abuse of notation) the nonlinearities $\mathcal{N}(u,u,u)$ and $\mathbb{R}(u,u,u)$ associated to the trilinear forms defined by

$$\mathcal{F}_x\big(\mathcal{N}(u_1,u_2,u_3)\big)(n,t) = \sum_{\substack{n_1,n_2,n_3\\ n_2\neq n_1,n_3}} \widehat{u_1}(n_1,t)\overline{\widehat{u_2}(n_2,t)}\widehat{u_3}(n_3,t), \tag{1.63}$$

and

$$\mathcal{F}_x\big(\mathcal{R}(u_1,u_2,u_3)\big)(n,t) = \widehat{u_1}(n,t)\overline{\widehat{u_2}(n,t)}\widehat{u_3}(n,t), \tag{1.64}$$

respectively. We further discuss in Remark 1.2.7 the necessity of introducing the gauge transformation (1.56).

A natural strategy to obtain the convergence of the solution u_γ, the solution of (1.54), to u, the solution of (1.55) is to

(i) solve the gauge equation (1.59). Namely, prove the convergence of $\mathfrak{u}_{\gamma,N}$ to $\mathfrak{u}_\gamma$ as $N\to\infty$,

(ii) show the inviscid limit on the gauged side. Namely, the convergence of $\mathfrak{u}_\gamma$ to $\mathfrak{u}_{\gamma=0}$ as $\gamma\to 0$,

(ii) invert the gauge transformation (1.58).

It turns out that (i) is hard to achieve in view of the dependence in N of the noise $\mathfrak{X}^{\gamma,N}$; see Remark 1.2.8 below. Hence, there is no identifiable pathwise limit $N\to\infty$ of $\mathfrak{u}_{\gamma,N}$ and we cannot pass to the limit at the level of the gauged equation (1.52).

In order to circumvent this issue we fix a function $\gamma\in(0,1]\mapsto N(\gamma)\in\mathbb{N}$ such that $N(\gamma)\to\infty$ as $\gamma\to 0$ to be chosen later and we write

$$u - u_\gamma = (u - u_{\gamma,N(\gamma)}) + (u_{\gamma,N(\gamma)} - u_\gamma) =: \mathrm{I}(\gamma) + \mathrm{I\!I}(\gamma).$$

We divide our argument into two steps.

- **Step 1: $\mathbf{I}(\gamma)\longrightarrow 0$ as $\gamma\to 0$.** We effectively treat (i) and (ii) at the same time. Let us note that the gauged noise $\mathfrak{X}^{\gamma,N}$ has the same law as ξ and is independent of $\phi\sim\rho$; see Proposition 3.1.8. Hence, for each $\gamma\in(0,1]$, the solution $\mathfrak{u}_{\gamma,N(\gamma)}(\phi,\mathfrak{X}^{\gamma,N(\gamma)})$ to (1.59) with initial data $\phi\sim\rho$ has the same law as $\mathfrak{u}_{\gamma,N(\gamma)}(\phi,\xi)$, the solution to

$$\begin{aligned}\partial_t\mathfrak{u}_{\gamma,N} = (\gamma+i)(\Delta-1)\mathfrak{u}_{\gamma,N} - \gamma\mathbf{P}_{\le N}\big(\big(|\mathbf{P}_{\le N}\mathfrak{u}_{\gamma,N}|^2 - 2\sigma_N\big)\mathbf{P}_{\le N}\mathfrak{u}_\gamma\big)\\ - i\mathbf{P}_{\le N}\mathfrak{N}(\mathbf{P}_{\le N}\mathfrak{u}_{\gamma,N}) + \sqrt{2\gamma}\xi,\end{aligned} \tag{1.65}$$

with initial data $\phi\sim\rho$ and where $N = N(\gamma)$. In Section 3.2, we study (1.65) by a pathwise approach[7] and prove the convergence of $\mathfrak{u}_{\gamma,N}(\phi,\xi)$ to some limit $\mathfrak{u}_{\gamma=0}(\phi)$ as $\gamma\to 0$ and $N\to\infty$ in $C\big([0,1];H^{-\varepsilon}(\mathbb{T}^2)\big)$, $\varepsilon>0$. This shows that, by conditioning on $\{\phi=\phi_\star\}$, for some fixed $\phi_\star\in\operatorname{Supp}(\rho)$ we should formally have that

$$\text{“Law}|_{\phi=\phi_\star}\big(\mathfrak{u}_{\gamma,N(\gamma)}(\phi,\mathfrak{X}^{\gamma,N(\gamma)})\big) \longrightarrow \mathfrak{u}_0(\phi_\star)\text{”}, \tag{1.66}$$

---

[7] In Step 1 and Step 2, the analysis is not done at the level of the variables $u_{\gamma,N}$ or $\mathfrak{u}_{\gamma,N}$ but rather at the level of the “nonlinear remainders” (see Section 3.2). We however omit these technicalities in this discussion.

as $\gamma \to 0$. Since $\mathfrak{u}_0(\phi_\star)$ is a constant (in ξ), the convergence in law (1.66) can be upgraded to a convergence in probability with respect to the law of ξ. We then prove the convergence of the gauge transform (iii) which leads to the desired goal: $\mathrm{I}(\gamma) \to 0$.

- **Step 2: $\mathbf{II}(\gamma) \longrightarrow 0$ as $\gamma \to 0$.** We analyze the difference $\mathrm{I\!I}(\gamma) = u_{\gamma,N(\gamma)} - u_\gamma$ by using heat analysis and by implementing Bourgain's invariant measure argument [6, 7] to obtain a (probabilistic) bound on $\mathrm{I\!I}(\gamma)$ with an explicit dependence in γ and $N(\gamma)$. Namely, we prove

$$\|u_\gamma - u_{\gamma,N(\gamma)}\|_{C([0,1];H_x^{-\varepsilon})} \lesssim \exp(\gamma^{-C})N(\gamma)^{-\theta}, \tag{1.67}$$

with high $\rho \otimes \mathbb{P}$-probability for some large $C > 0$. In (1.67), the diverging factor $\exp(\gamma^{-C})$ is due to the use of parabolic smoothing effects (in our local theory) which get weaker as $\gamma \to 0$. Hence, upon choosing $N(\gamma) \sim \exp(\gamma^{-A})$, for $A > 0$ large enough, we deduce that $\mathrm{I\!I}(\gamma) \to 0$. Let us also note here that this step relies heavily on Bourgain's invariant measure argument in the sense that a naive local-in-time version of the convergence (1.67) fails as the local time existence T_γ for u_γ verifies $T_\gamma \to 0$ as $\gamma \to 0$. See Remark 1.2.10 below for more details.

Remark 1.2.5. In Step 1, we implement rigorously the "conditioning" (1.66) by constructing the relevant stochastic objects on full $\rho \otimes \mathbb{P}$-probability sets that "locally look like products of sets of full ρ-probability and $\mathbb{P}$-probability"; i.e. sets of the form

$$\Omega_0 = \bigcup_{\phi_\star \in \Omega_\phi} \{\phi_\star\} \times \Omega_\xi(\phi_\star),$$

such that $\rho(\Omega_\phi) = 1$ and $\mathbb{P}(\Omega_\xi(\phi_\star)) = 1$ for each $\phi_\star \in \Omega_\phi$; see Section 3.4. (Note that we check in Lemma 3.4.1 that such a set Ω_0 is measurable.) The convergence (1.66) then follows from working with any fixed $\phi_\star \in \Omega_\phi$; see Section 3.2.

We now discuss the analysis of (1.65) in Step 1 above as the analysis of (1.65) is, from an analytical perspective, the most challenging part of the proof of Theorem 1.2.2. We first introduce some notations. Let W denote a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:

$$W(t) := \sum_{n \in \mathbb{Z}^2} B_n(t) e^{in\cdot x} \tag{1.68}$$

and $\{B_n\}_{n\in\mathbb{Z}^2}$ is defined by $B_n(t) = \langle \xi, \mathbb{1}_{[0,t]} \cdot e_n \rangle_{x,t}$. Here, $\langle \cdot, \cdot \rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^2 \times \mathbb{R}$. As a result, we see that $\{B_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. By convention, we normalized B_n such that $\mathrm{Var}(B_n(t)) = t$.

For $\gamma \in [0,1]$, we define the operator S_γ by

$$S_\gamma(t,t') = e^{(\gamma|t| + it')(\Delta - 1)}, \quad (t,t') \in \mathbb{R}^2. \tag{1.69}$$

We also set $S_\gamma(t) = S_\gamma(t,t)$ for $t \in \mathbb{R}$. Next, we define the following Duhamel operators:

$$\begin{aligned} \mathcal{I}_\gamma(F)(t) &= \mathbb{1}_{\mathbb{R}_+}(t) \int_0^t S_\gamma(t-t', t-t')F(t')dt' + \mathbb{1}_{\mathbb{R}_-}(t) \int_0^t S_\gamma(t+t', t-t')F(t')dt' \\ \mathcal{I}_0(F)(t) &= \int_0^t S_0(t-t')F(t')dt'. \end{aligned} \tag{1.70}$$

Remark 1.2.6. The above definition of the Duhamel integral (1.70) also appeared in [48, 49]. This naturally extends the "usual" Duhamel integral defined by

$$\mathcal{I}_\gamma^0(F)(t) = \mathbb{1}_{\mathbb{R}_+}(t) \int_0^t e^{(\gamma+i)(t-t')(\Delta-1)}F(t')dt'$$

to the whole real line.

Let $\gamma \in [0,1]$. We denote by Ψ_γ the stochastic convolution

$$\Psi_\gamma(t) = \sqrt{2\gamma} \int_0^t S_\gamma(t - t')dW(t'), \tag{1.71}$$

that is, the solution of the linear equation

$$\partial_t \Psi_\gamma = (\gamma + i)(\Delta - 1)\Psi_\gamma + \sqrt{2\gamma}\xi.$$

For $N \in \mathbb{N}$, we also denote by $\Psi_{\gamma,N}$ the truncated stochastic convolution

$$\Psi_{\gamma,N} = \mathbf{P}_{\leq N}\Psi_\gamma, \tag{1.72}$$

and we denote by $\overset{\bullet}{|}_\gamma$ and $\overset{\bullet}{|}_{\gamma,N}$ the stochastic processes defined by

$$\overset{\bullet}{|}_\gamma(t) = S_\gamma(t)\phi + \Psi_\gamma(t), \tag{1.73}$$

$$\overset{\bullet}{|}_{\gamma,N}(t) = \mathbf{P}_{\leq N}\overset{\bullet}{|}_\gamma(t) = S_\gamma(t)\phi_N + \Psi_{\gamma,N}, \tag{1.74}$$

for $t \geq 0$, where

$$\phi_N = \mathbf{P}_{\leq N}\phi, \tag{1.75}$$

with ϕ is sampled from ρ (1.48). A standard computation shows that $\{(\gamma, t) \mapsto \overset{\bullet}{|}_{\gamma,N}(t)\}_{N\in\mathbb{N}}$ belongs to $C\big([0,1] \times [0,T]; W^{0-,\infty}(\mathbb{T}^2)\big)$, uniformly in $N \in \mathbb{N}$, almost surely for any $T > 0$; see Lemma 3.4.3. We further introduce the following Wick renormalized powers:

$$\begin{aligned} :\!|\overset{\bullet}{|}_{\gamma,N}|^2\!: &= |\overset{\bullet}{|}_{\gamma,N}|^2 - \sigma_N, \\ :\!|\overset{\bullet}{|}_{\gamma,N}|^2\overset{\bullet}{|}_{\gamma,N}\!: &= \big(|\overset{\bullet}{|}_{\gamma,N}|^2 - 2\sigma_N\big)\overset{\bullet}{|}_{\gamma,N}, \end{aligned} \tag{1.76}$$

where σ_N is defined as

$$\sigma_N := \mathbb{E}\big[|\overset{\bullet}{|}_{\gamma,N}(t,x)|^2\big] \sim \log N,$$

and is independent of $(t, x, \gamma) \in \mathbb{R}_+ \times \mathbb{T}^2 \times [0,1]$. The regularities of the stochastic objects (1.76) is also studied in Lemma 3.4.3.

Fix $N \in \mathbb{N}$ and $\gamma \in [0,1]$. Let $\mathfrak{u}_{\gamma,N}$ be the solution to (1.65). We proceed with the following first order expansion ([45, 7, 22]):

$$\mathfrak{u}_{\gamma,N} = \mathfrak{v}_{\gamma,N} + \overset{\bullet}{|}_\gamma = (\mathfrak{v}_{\gamma,N} + \overset{\bullet}{|}_{\gamma,N}) + \mathbf{P}_{>N}\overset{\bullet}{|}_\gamma, \tag{1.77}$$

where $\mathbf{P}_{>N} = \operatorname{Id} - \mathbf{P}_{\leq N}$. We see that the dynamics of the truncated renormalized equation (1.59) decouples into the linear dynamics for the high frequency part given by $\mathbf{P}_{>N}\overset{\bullet}{|}_\gamma^N$ and the nonlinear dynamics for the low frequency part of $\mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N}$:

$$\begin{aligned} \partial_t \mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N} = (\gamma + i)(\Delta - 1)\mathfrak{u}_{\gamma,N} & \\ - \gamma\mathbf{P}_{\leq N}\Big(\big(|\mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N}|^2 - 2\sigma_N\big)\mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N}\Big) & \\ - i\mathbf{P}_{\leq N}\mathfrak{N}(\mathfrak{u}_{\gamma,N}) + \sqrt{2\gamma}\xi, & \end{aligned}$$

Then, the nonlinear remainder $\mathfrak{v}_{\gamma,N} = \mathbf{P}_{\leq N}\mathfrak{u}_{\gamma,N} - \overset{\bullet}{|}_{\gamma,N}$ satisfies the following integral equation:

$$\mathfrak{v}_{\gamma,N} = -\gamma\mathbf{P}_{\leq N}\mathfrak{S}^{\text{Wick}}_{\gamma,N}(\mathfrak{v}_{\gamma,N}) - i\mathbf{P}_{\leq N}\mathfrak{S}^{\text{PDE}}_{\gamma,N}(\mathfrak{v}_{\gamma,N}), \tag{1.78}$$

with the zero initial data and where the nonlinear expressions $\mathfrak{S}^{\text{Wick}}_{\gamma,N}$ and $\mathfrak{S}^{\text{PDE}}_{\gamma,N}$ are given by

$$\begin{aligned}\mathfrak{S}^{\mathrm{Wick}}_{\gamma,N}(\mathfrak{v}) &= \mathcal{I}_\gamma\Big(\big(|\mathfrak{v}+\overset{\bullet}{|}_{\gamma,N}|^2-2\sigma_N\big)(\mathfrak{v}+\overset{\bullet}{|}_{\gamma,N})\Big)\\ &= \mathcal{I}_\gamma\Big(|\mathfrak{v}|^2\mathfrak{v} + :|\overset{\bullet}{|}_{\gamma,N}|^2\overset{\bullet}{|}_{\gamma,N}: +2:|\overset{\bullet}{|}_{\gamma,N}|^2:\mathfrak{v} + (\overset{\bullet}{|}_{\gamma,N})^2\,\mathfrak{v}\\ &\qquad\qquad + 2\overset{\bullet}{|}_{\gamma,N}|\mathfrak{v}|^2 + \overline{\overset{\bullet}{|}_{\gamma,N}}\,\mathfrak{v}^2\Big),\end{aligned}\tag{1.79}$$

and by (2.34), we have that

$$\begin{aligned}\mathfrak{S}^{\mathrm{PDE}}_{\gamma,N}(\mathfrak{v}) &= \mathcal{I}_\gamma\mathfrak{N}\big(\overset{\bullet}{|}_{\gamma,N}+\mathfrak{v}\big)\\ &= \mathcal{I}_\gamma\mathcal{N}(\mathfrak{v}) + \mathcal{I}_\gamma\mathcal{N}(\overset{\bullet}{|}_{\gamma,N})\\ &\qquad + 2\mathcal{I}_\gamma\mathcal{N}\big(\overset{\bullet}{|}_{\gamma,N},\mathfrak{v},\mathfrak{v}\big) + \mathcal{I}_\gamma\mathcal{N}\big(\mathfrak{v},\overset{\bullet}{|}_{\gamma,N},\mathfrak{v}\big)\\ &\qquad + 2\mathcal{I}_\gamma\mathcal{N}\big(\mathfrak{v},\overset{\bullet}{|}_{\gamma,N},\overset{\bullet}{|}_{\gamma,N}\big) + \mathcal{I}_\gamma\mathcal{N}\big(\overset{\bullet}{|}_{\gamma,N},\mathfrak{v},\overset{\bullet}{|}_{\gamma,N}\big)\\ &\qquad - \mathcal{I}_\gamma\mathcal{R}\big(\overset{\bullet}{|}_{\gamma,N}+\mathfrak{v}\big),\end{aligned}\tag{1.80}$$

respectively.

In view of the ill-posedness of NLS below $L^2(\mathbb{T}^2)$ [55, 37], we have to place $\mathfrak{v}_{\gamma,N}$ in $H^s(\mathbb{T}^2)$ for some $s>0$ and uniformly in $N\in\mathbb{N}$. We achieve this by solving a fixed point argument for $\mathfrak{v}_{\gamma,N}$; see Proposition 3.2.2.

The terms in (1.79) will be estimated in the following way: we first construct the stochastic objects $\overset{\bullet}{|}_{\gamma,N}$, $:|\overset{\bullet}{|}_{\gamma,N}|^2:$, $\overset{\bullet}{|}^2_{\gamma,N}$ and $:|\overset{\bullet}{|}_{\gamma,N}|^2\overset{\bullet}{|}_{\gamma,N}:$ in the relevant topologies; and we then exploit the gain of derivatives of the linear operator $\gamma\mathcal{I}_\gamma$ (see Proposition 3.1.14) coming from parabolic smoothing and the product estimates in Appendix A.1 to obtain acceptable bounds for $\gamma\mathfrak{S}^{\mathrm{Wick}}_{\gamma,N}(\mathfrak{v}_{\gamma,N})$ and prove a fixed point argument.

The terms in (1.80) fall into four categories:

(i) a purely deterministic term: $\mathcal{I}_\gamma\mathcal{N}(v_{\gamma,N})$,

(ii) a stochastic term:

$$\Psi_{\gamma,N} := \mathcal{I}_\gamma(\psi_{\gamma,N}) \text{ with } \psi_{\gamma,N} := \mathcal{N}(\overset{\bullet}{|}_{\gamma,N}),\tag{1.81}$$

(iii) two random matrix terms:

$$\begin{aligned}\mathcal{M}^1_{\gamma,N} &: \mathfrak{v}\mapsto\mathcal{I}_\gamma\mathcal{N}\big(\mathfrak{v},\overset{\bullet}{|}_{\gamma,N},\overset{\bullet}{|}_{\gamma,N}\big),\\ \mathcal{M}^2_{\gamma,N} &: \mathfrak{v}\mapsto\mathcal{I}_\gamma\mathcal{N}\big(\overset{\bullet}{|}_{\gamma,N},\mathfrak{v},\overset{\bullet}{|}_{\gamma,N}\big),\end{aligned}\tag{1.82}$$

and two bilinear random operator terms:

$$\begin{aligned}\mathcal{T}^1_{\gamma,N} &: (\mathfrak{u},\mathfrak{v})\mapsto\mathcal{I}_\gamma\mathcal{N}\big(\overset{\bullet}{|}_{\gamma,N},\mathfrak{u},\mathfrak{v}\big),\\ \mathcal{T}^2_{\gamma,N} &: (\mathfrak{u},\mathfrak{v})\mapsto\mathcal{I}_\gamma\mathcal{N}\big(\mathfrak{u},\overset{\bullet}{|}_{\gamma,N},\mathfrak{v}\big),\end{aligned}\tag{1.83}$$

(iv) a resonant term $\mathcal{I}_\gamma\mathcal{R}\big(\overset{\bullet}{|}_{\gamma,N}+\mathfrak{v}\big)$.

All these terms will be considered in Fourier restriction spaces (see Definition 3.1.1). The terms (i) and (iv) are dealt with by using deterministic analysis. In particular, (i) is bounded by using a slight modification of Bourgain's trilinear estimate [5]; see Lemma 3.1.4 in Section 3.1.3. The stochastic terms (ii) and (iii) are estimated in the following way: we first represent them using multiple stochastic integrals, see Appendix A.2.2, and we then use counting arguments (see Section 3.3) combined with the random tensor estimate of Deng, Nahmod, and Yue [25] adapted to our stochastic setting (see Appendix A.4) to close the relevant estimates. Furthermore, we construct the aforementioned terms in spaces of functions that are continuous in $\gamma\in[0,1]$ by using the standard Kolmogorov continuity criterion.

After completing this program, standard PDE analysis proves the convergence of $\mathfrak{v}_{\gamma,N}$ to some non-trivial stochastic process $\mathfrak{v}_\gamma$ for each $\gamma\in[0,1]$ as $N\to\infty$ and allows us to achieve Step 1 above.

Let us now describe the main difficulty in establishing bounds for the objects (ii) and (iii). The key observation is that at the level of the heat Duhamel operators $\mathcal{I}_\gamma$, the dissipative smoothing effects come after a time integration. This smoothing mechanism is needed to handle the roughness in space of the noise ξ. However, in the $X^{s,b}$-spaces analysis, time integration is also used to benefit from multilinear dispersive smoothing effects. Thus, within the framework of $X^{s,b}$-spaces, there is a tradeoff, at the level of the heat model (1.54), between parabolic and multilinear dispersive smoothing effects: they cannot be used simultaneously. We overcome this difficulty by combining both parabolic and dispersive analysis. However, in the weakly dissipative limit $\gamma \to 0$, this results in a loss of derivatives for the remainder $\mathfrak{v}_\gamma$; see Remark 1.2.9 below.

We finish this section by stating a few remarks

Remark 1.2.7 (Wick renormalization for NLS)**.** Let us now consider the Wick renormalized equations (1.52) and (1.53) and describe the main obstruction in proving the convergence in Theorem 1.2.2 (ii) by studying these equations directly (i.e. without introducing the gauge transformation (1.56) and the gauged variable $\mathfrak{u}_{\gamma,N}$ (1.58)).

We proceed with the first oder expansion of $u_{\gamma,N} = \upharpoonleft_\gamma + v_{\gamma,N}$, where $\upharpoonleft_\gamma$ is as in (1.73) and $v_{\gamma,N}$ satisfies the following equation:

$$\begin{aligned} v_{\gamma,N} = -(\gamma+i)\mathcal{I}_\gamma\Big(\, &: |\upharpoonleft_{\gamma,N}|^2 \upharpoonleft_{\gamma,N} : + |v_{\gamma,N}|^2 \overline{v_{\gamma,N}} \\ &+ 2 : |\upharpoonleft_{\gamma,N}|^2 : v_{\gamma,N} + \upharpoonleft_{\gamma,N}^2 \, v_{\gamma,N} \\ &+ 2\upharpoonleft_{\gamma,N} |v_{\gamma,N}|^2 + \overline{\upharpoonleft_{\gamma,N}} \, v_{\gamma,N}^2 \Big). \end{aligned} \tag{1.84}$$

The stochastic terms appearing in (1.84) are as in (1.74) and (1.76) and where $\mathcal{I}_\gamma$ is as in (1.70).

We would like to solve a fixed point problem at the level of $v_{\gamma,N}$. Since the two-dimensional cubic Schrödinger equation is ill-posed below $L^2(\mathbb{T}^2)$ [55, 37] and is "embedded" in (1.84), we wish to prove that $v_{\gamma,N}$ belongs to some space of positive regularity $H^\delta(\mathbb{T}^2)$, for some $\delta > 0$, uniformly in the parameter $N \in \mathbb{N}$ and for every $\gamma \in [0,1]$. This would imply, in particular that $\mathcal{I}_0\big(: |\upharpoonleft_{0,N}|^2 \upharpoonleft_{0,N} : \big)$ belongs to $H^\delta(\mathbb{T}^2)$, uniformly in $N \in \mathbb{N}$. We decompose $: |\upharpoonleft_{0,N}|^2 \upharpoonleft_{0,N} :$ into two components:

$$\begin{aligned} : |\upharpoonleft_{0,N}|^2 \upharpoonleft_{0,N} : (x,t) &= \sum_{\substack{n_1,n_1,n_2 \\ n_2 \neq n_1,n_3 \\ \langle n_j \rangle \leq N}} \frac{g_{n_1}\overline{g_{n_2}}g_{n_3}}{\langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle} e^{i(n_1-n_2+n_3)\cdot x - i(\langle n_1 \rangle^2 - \langle n_2 \rangle^2 + \langle n_3 \rangle^2)t} \\ &\quad + 2\Big(\sum_{\langle n_2 \rangle \leq N} \frac{|g_{n_2}|^2 - 1}{\langle n_2 \rangle^2} \Big) \sum_{\langle n_1 \rangle \leq N} \frac{g_{n_1}}{\langle n_1 \rangle} e^{in_1\cdot x - i\langle n_1 \rangle^2 t} \\ &=: \Psi_{0,N}(x,t) + \upharpoonleft^{(1)}_{0,N}(x,t), \end{aligned} \tag{1.85}$$

where $\Psi_{0,N}$ is as in (1.81). In Lemma 3.4.4 below, we prove that the stochastic term $\Upsilon_{0,N} = \mathcal{I}_0\big(\Psi_{0,N}\big)$ belongs to $C\big([0,1]; H^{\frac{1}{2}-}(\mathbb{T}^2)\big)$. This $\frac{1}{2}$-derivative gain is known as *multilinear smoothing*: by exploiting the interaction of random waves through counting estimates, we can prove a $\frac{1}{2}$-derivatives gain on $\Psi_{0,N}$. See also [32, 24, 25, 8, 63, 73] for examples of multilinear smoothing in other contexts.

The second term $\upharpoonleft^{(1)}_{0,N}$ however verifies $\upharpoonleft^{(1)}_{0,N} = C_N \upharpoonleft_{0,N}$, where C_N is the almost surely converging sequence (as $N \to \infty$) given by $C_N = 2\sum_{\langle n_2 \rangle \leq N} \frac{|g_{n_2}|^2-1}{\langle n_2 \rangle^2}$. Hence, in view of the regularity of $\upharpoonleft_{0,N} \in C\big([0,1]; H^{0-}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)\big)$, we also have $\mathcal{I}_0\big(\upharpoonleft_{0,N}\big) \in C\big([0,1]; H^{0-}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)\big)$[8] and it is not possible to close the fixed point argument (1.84) for $v_{0,N}$ and consequently to study the convergence problem $\gamma \to 0$ at the level of the variables $v_{\gamma,N}$ (1.84).

Note that the term $\upharpoonleft^{(1)}_{0,N}$ comes from the resonant interactions $n_2 \in \{n_1, n_2\}$ in (1.85). The gauge transform (1.56) precisely removes these "bad interactions" from the Schrödinger part of the nonlinearity, making the problem (1.59) amenable to PDE analysis.

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[8] Note that there is no multilinear smoothing at the level of linear objects.

Remark 1.2.8. It would be of interest to develop a pathwise approach to analyze (1.59). This would require to study the convergence (in N) of stochastic object constructed from the noise $\mathfrak{X}^{\gamma,N}$ and that have the form $\{A_N(\mathfrak{X}^{\gamma,N})\}_{N\in\mathbb{N}}$. However, the main obstacle in doing so comes from the dependence of the gauged noise $\mathfrak{X}^{\gamma,N}$ in the parameter N itself. Thus obtaining any difference estimate of the form $A_M(\mathfrak{X}^{\gamma,M}) - A_N(\mathfrak{X}^{\gamma,N})$, $N \geq M$, for these objects is difficult since it involves two correlated space-time white noises.

Remark 1.2.9. Let us discuss the bounds available for the remainders $\mathfrak{v}_\gamma$, $\gamma \in [0,1]$, in (1.78). In the parabolic setting, by a variant of Proposition 3.2.1 in Section 3.2 below, we have that

$$\|\mathfrak{v}_\gamma\|_{C_{T_\gamma}H^1_x} \lesssim 1,$$

for some small time $T_\gamma > 0$ depending on γ. At the level of NLS (1.53), Bourgain's argument [7] gives the bound

$$\|\mathfrak{v}_0\|_{C_T H_x^{\frac{1}{2}-}} < \infty,$$

for some $0 < T \leq 1$. However, surprisingly, because of the issues described in the above, our argument only gives the following uniform (in γ) bound for the remainder $\mathfrak{v}_\gamma$ (at the level of the gauged equation (1.65)):

$$\sup_{\gamma\in[0,1]} \|\mathfrak{v}_\gamma\|_{C_T H_x^{\frac{1}{4}-}} < \infty,$$

for some $0 < T \leq 1$.[9] Hence, there appears to be a $\frac{1}{4}$-derivative loss for the nonlinear remainders $\mathfrak{v}_\gamma$ in the limit $\gamma \to 0$. This is due to the issue mentioned above: in order to study $\mathfrak{v}_\gamma$ in the limit $\gamma \to 0$, we need to combine dissipative and dispersive smoothing effects at the same time by exploiting the (limited budget of) integration in time under the Duhamel integral (1.70) which leads to an effective derivative loss for $\mathfrak{v}_\gamma$ in the limit $\gamma \to 0$.

Remark 1.2.10. By a naive application of Proposition 3.2.1, we basically have that

$$\|u_\gamma - u_{\gamma,N}\|_{C([0,T_\gamma];H_x^{-\varepsilon})} \lesssim N^{-\theta}, \tag{1.86}$$

with high $\rho \otimes \mathbb{P}$-probability and for some small $\varepsilon > 0$ and $\theta > 0$. Here the time $T_\gamma \sim \gamma^{\theta_1}$, $\theta_1 > 0$, and in particular $T_\gamma \to 0$ as $\gamma \to 0$. In Section 3.2, we crucially rely on Bourgain's invariant measure argument to upgrade (1.86) to longer (and independent of γ) time intervals (i.e. effectively upagrade (1.86) to (1.67)).

Remark 1.2.11. We point out that the Fourier restriction norm method had already been applied to heat models in the literature. For instance, in [46, 48, 49], Molinet, Pilod, and Vento used $X^{s,b}$-type spaces to study the well-posedness issue for dispersive perturbations of the Burgers' equation in various contexts. They however do not address the question of the inviscid limit.

1.3 Notations

Before proceeding further, we recall here, for readers' convenience, the notations that are used throughout this thesis.

- **Fourier transforms.** Let $f \in \mathcal{S}'(\mathbb{T}^2)$, $g \in \mathcal{S}'(\mathbb{R})$ and $u \in \mathcal{S}'(\mathbb{T}^2 \times \mathbb{R})$. We define the Fourier transforms $\mathcal{F}_x$, $\mathcal{F}_t$ and $\mathcal{F}_{x,t}$ through the following formulas:

---

[9] The fact the $\mathfrak{v}_0 \in H^{\frac{1}{2}-}(\mathbb{T}^2)$ does not directly come from [7] (where it is proved that $\mathfrak{v}_0 \in H^s(\mathbb{T}^2)$ for some small $s > 0$) but follows however from the arguments in Section 3.4.

$$\begin{aligned}\mathcal{F}_x(f)(n) &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} e^{-in\cdot x} f(x)dx \\ \mathcal{F}_t(g)(\lambda) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t} g(t)dt \\ \mathcal{F}_{x,t}(u)(\lambda,n) &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^2\times\mathbb{R}} e^{-i(n\cdot x+t\lambda)} u(x,t)dxdt\end{aligned} \tag{1.87}$$

for $(n,\lambda)\in\mathbb{Z}^2\times\mathbb{R}$. When there is no confusion, we simple use $\widehat{u}$ to denote the spatial, temporal or space-time Fourier transform of $u\in\mathcal{S}'(\mathbb{T}^2\times\mathbb{R})$. In what follows, we will omit the non-essential dependence in the factor 2π in our estimates for convenience.

We also denote by $\widetilde{u}$ or $\widetilde{\mathcal{F}}(u)$ the twisted space-time Fourier transform of $u\in\mathcal{S}'(\mathbb{T}^2\times\mathbb{R})$ by

$$\widetilde{u}(n,\lambda) = \mathcal{F}_{x,t}(u)\big(n,\lambda-\langle n\rangle^2\big). \tag{1.88}$$

• **Frequency projections.** In addition to the projectors $\mathbf{P}_\varepsilon^{\text{high}}$, $\mathbf{P}_\varepsilon^{\text{low}}$ and $\mathbf{P}_{\le N}$ defined in (1.16) and (1.25) respectively, we also define $\mathbf{P}_\varepsilon^{\text{low},\theta}$ and $\mathbf{P}_\varepsilon^{\text{high},\theta}$ by

$$\begin{aligned}\mathbf{P}_\varepsilon^{\text{low},\theta} f &= \mathcal{F}^{-1}\big(\mathbb{1}_{\langle n\rangle\le(1+\theta)\cdot(2\varepsilon)^{-1}}\widehat{f}(n)\big), \\ \mathbf{P}_\varepsilon^{\text{high},\theta} f &= \mathcal{F}^{-1}\big(\mathbb{1}_{\langle n\rangle>(1+\theta)\cdot(2\varepsilon)^{-1}}\widehat{f}(n)\big),\end{aligned} \tag{1.89}$$

for some $0<\theta\ll 1$, which is chosen to be much smaller than other fixed (i.e. not ε) parameters.

For a number $N\in\mathbb{Z}$, we denote by $\mathbf{P}_N$ the (sharp) projection onto the set $\{n\in\mathbb{Z}^2:\langle n\rangle\sim N\}$. Namely, we have

$$\mathbf{P}_N f(x) = \sum_{\langle n\rangle\sim N}\widehat{f}(n)e^{in\cdot x}, \tag{1.90}$$

for any $N\in\mathbb{N}$. For any set $Q\subset\mathbb{Z}^2$ we denote by $\mathbf{P}_Q$ the Fourier projection onto Q. Also, recall, for $N\in\mathbb{N}$ the projections $\mathbf{P}_{\le N}$ (1.90). We also denote by $\mathbf{P}_{\gg N}$ the sharp Fourier projection onto the set $\{n\in\mathbb{Z}^2:\langle n\rangle\gg N\}$. Note that with our notations, we have

$$\mathbf{P}_{\gg N} = \sum_{N'\gg N}\mathbf{P}_{N'}.$$

• **Basic functions spaces.** For $s\in\mathbb{R}$, the space $H^s(\mathbb{T}^2)$ denotes the usual $L^2(\mathbb{T}^2)$-based Sobolev space and we define $\mathcal{H}^s(\mathbb{T}^2)$ by $\mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2)\times H^{s-1}(\mathbb{T}^2)$. We also use shortcut notations such as $L^\infty_T H^s_x$ and $C_{\varepsilon,T}H^s_x$ (for functions of the form $f=f(\varepsilon,t,x)$) for $L^\infty\big([0,T];H^s(\mathbb{T}^2)\big)$ and $C\big([0,1]\times[0,T];H^s(\mathbb{T}^2)\big)$ respectively, etc.

Let $s\in\mathbb{R}$ and $p\ge 1$. We introduce the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T}^2)$ given by the norm

$$\|f\|_{\mathcal{F}L^{s,p}} := \|\langle n\rangle^s\widehat{f}(n)\|_{\ell^p_n}.$$

• **Probabilistic setting and notations.** In this Chapter 2, we work on a filtered probability space $(\Omega,\mathbb{P},\mathcal{A},\{\mathcal{F}_t\}_{t\ge0})$ and assume that Ω which is rich enough so as to encode all the probabilistic information necessary for our purposes, i.e. the noise ξ (1.5).

In Chapter 3, we work on the filtered probability spaces $(H^{-1}(\mathbb{T}^2)\times\Omega,\rho\otimes\mathbb{P},\mathcal{A},\{\mathcal{F}_t\}_{t\ge0})$ or $(H^{-1}(\mathbb{T}^2)\times\Omega,\rho\otimes\mathbb{P},\mathcal{A},\{\mathcal{F}_t\}_{t\ge0})$, where ρ and μ are the measures (1.44) and (1.48). We assume that the Brownian motions $\{B_n\}_{n\in\mathbb{Z}^2}$ and $\{B_n^{-1}\}_{n\in\mathbb{Z}^2}$ defined in (1.68) and (A.1) in Appendix A.2.2, respectively, are adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$. By a standard procedure, we may assume that the probability space $(H^{-1}(\mathbb{T}^2)\times\Omega,\rho\otimes\mathbb{P},\mathcal{A},\{\mathcal{F}_t\}_{t\ge0})$ (and $(H^{-1}(\mathbb{T}^2)\times\Omega,\mu\otimes\mathbb{P},\mathcal{A},\{\mathcal{F}_t\}_{t\ge0})$) is complete so that every null set is measurable (i.e. belongs to $\mathcal{A}$).

Given a random variable X defined on Ω, we write $\operatorname{Law}(X)$ for its law. Given a measure ρ, we also write $X\sim\rho$ if $\rho=\operatorname{Law}(X)$.

• **Symbols.** We write $A\lesssim B$ to denote an estimate of the form $A\le CB$. Similarly, we write $A\sim B$ to denote $A\lesssim B$ and $B\lesssim A$ and use $A\ll B$ when we have $A\le cB$ for small $c>0$.

We may write $A \lesssim_\theta B$ for $A \leq CB$ with $C = C(\theta)$ if we want to emphasize the dependence of the implicit constant on some parameter θ.

For a complex number z we sometimes use the notations z^1 and z^{-1} for z and $\overline{z}$ respectively. Given a set P, we denote by $|P|$ its cardinality. Given two positive numbers a and b, we denote by $a \vee b$ and $a \wedge b$ the minimum and maximum, respectively, of a and b.

Chapter 2

Smoluchowski-Kramers approximation for singular wave equations

The following chapter is organized as follows. In Section 2.1, we study the convergence of the linear flows and Duhamel integrals defined in (2.2), (2.1) below. In Section 2.2, we turn our attention to the polynomial model (1.42) and present proofs of Theorems 1.1.4 and 1.1.6.

2.1 Convergence of the deterministic and stochastic objects

In this section, we study the convergence as $\varepsilon \to 0$ and prove bounds for various deterministic and stochastic objects depending on $\varepsilon \geq 0$. This is needed to prove Theorems 1.1.4 and 1.1.6. More precisely, we first study the deterministic linear objects (2.1) and (2.2) defined below and we then proceed with the construction of the Wick powers (1.38) uniformly in $\varepsilon \in [0, 1]$ and $N \in \mathbb{N}$.

2.1.1 Linear flows and Duhamel operators

We introduce the following Duhamel operators for $\varepsilon > 0$:

$$\begin{aligned} \mathcal{I}_\varepsilon(F)(t) &= \int_0^t \varepsilon^{-2} \mathcal{D}_\varepsilon(t - t') F(t') dt' \\ \mathcal{I}_0(F)(t) &= \int_0^t P_0(t - t') F(t') dt', \end{aligned} \tag{2.1}$$

for a space-time function F and $t \in \mathbb{R}$, with P_0 and $\mathcal{D}_\varepsilon$ as in (1.27) and (1.18).

In this section, we study the convergence of the linear flow (1.20) and the Duhamel operator (2.1).

Let P_ε be the linear operator associated to the homogeneous linear solution (1.8). For two distributions ϕ_0, ϕ_1, we have

$$P_\varepsilon(t)(\phi_0, \phi_1) := (\varepsilon^{-2} + \partial_t) \mathcal{D}_\varepsilon(t) \phi_0 + \mathcal{D}_\varepsilon(t) \phi_1, \tag{2.2}$$

for $t \geq 0$. Hereafter, we identify the operator P_0 with $(P_0, 0)$ defined by

$$(P_0, 0)(\phi_0, \phi_1)(t) := P_0(t) \phi_0,$$

for $t \geq 0$.

The main result of this section is the following proposition which provides bounds for both P_ε and $\mathcal{I}_\varepsilon$.

Proposition 2.1.1. *Let $s \in \mathbb{R}$ and $T > 0$. For any $\varepsilon \in [0,1]$, we have the following bounds:*

$$\begin{aligned}\sup_{\varepsilon\in[0,1]} \|P_\varepsilon(\phi_0,\phi_1)\|_{C_T H^s_x} &\lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^s_x},\\ \sup_{\varepsilon\in[0,1]} \|\mathcal{I}_\varepsilon(F)\|_{C_T H^s_x} &\lesssim T^{\frac12}\|F\|_{L^\infty_T H^{s-1}_x},\end{aligned} \tag{2.3}$$

for any smooth functions ϕ_0, ϕ_1 and F. Moreover, for any $0 < \theta \ll 1$, we have

$$\begin{aligned}\|(P_\varepsilon - P_0)(\phi_0,\phi_1)\|_{C_T H^s_x} &\lesssim \varepsilon^{\frac{\theta}{2}}\|(\phi_0,\phi_1)\|_{\mathcal{H}^{s+\theta}_x},\\ \|(\mathcal{I}_\varepsilon - \mathcal{I}_0)(F)\|_{C_T H^s_x} &\lesssim T^{\frac12}\varepsilon^{\frac{\theta}{2}}\|F\|_{L^\infty_T H^{s-1+\theta}_x}.\end{aligned} \tag{2.4}$$

for any smooth functions ϕ_0, ϕ_1 and F.

We deduce from Proposition 2.1.1, the following bounds on the operators $\{P_\varepsilon\}_{\varepsilon\in[0,1]}$ and $\{\mathcal{I}_\varepsilon\}_{\varepsilon\in[0,1]}$, viewed as continuous objects in $\varepsilon \in [0,1]$.

Corollary 2.1.2. *Let $s \in \mathbb{R}$, $T > 0$ and $\theta > 0$. We have the following bounds:*

$$\|P_\varepsilon(t)(\phi_0,\phi_1)\|_{C_{\varepsilon,T} H^s_x} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^s_x}, \tag{2.5}$$

$$\|\mathcal{I}_\varepsilon(t)\big(F(\varepsilon,\cdot)\big)\|_{C_{\varepsilon,T} H^s_x} \lesssim T^{\frac12}\|F\|_{C_{\varepsilon,T} H^{s-1}_x}, \tag{2.6}$$

for any smooth ϕ_0, ϕ_1 and F.

Remark 2.1.3. Note that if we replace the $C\big([0,1]\times[0,T]; H^s(\mathbb{T}^2)\big)$ norms in the left-hand side by $C\big((0,1]\times[0,T]; H^s(\mathbb{T}^2)\big)$, then we obtain inequalities without the θ-derivative losses in the right-hand side.

Remark 2.1.4. By combining using the bounds in Proposition 2.1.1 it is easy to prove the convergence of the solution of the nonhomegeneous linear damped wave equation (1.21) to that of linear heat equation (1.29) as $\varepsilon \to 0$, provided the forcing term F is smooth enough. This makes the formal derivation of the Smoluchowski-Kramers approximation in Section 1.1 rigorous.

We prove several useful lemmas first and postpone the proof of Proposition 2.1.1 and Corollary 2.1.2 to the end of the section.

Let $\{\widehat{\mathcal{D}_\varepsilon}(n,t)\}_{n\in\mathbb{Z}^2}$ be the symbol associated to the multiplier $\mathcal{D}_\varepsilon(t)$ defined in (1.18). We define $\eta \in \mathbb{R}^2 \mapsto \widehat{\mathcal{D}_\varepsilon}(\eta,t)$ its natural extension to $\mathbb{R}^2$ given by

$$\widehat{\mathcal{D}_\varepsilon}(\eta,t) := \begin{cases} e^{-\frac{t}{2\varepsilon^2}}\frac{\sinh(\lambda_\varepsilon(\eta)t)}{\lambda_\varepsilon(\eta)} & \text{if } \langle\eta\rangle \le (2\varepsilon)^{-1}\\ e^{-\frac{t}{2\varepsilon^2}}\frac{\sin(\zeta_\varepsilon(\eta)t)}{\zeta_\varepsilon(\eta)} & \text{if } \langle\eta\rangle > (2\varepsilon)^{-1}\end{cases}, \tag{2.7}$$

where $\eta \mapsto \lambda_\varepsilon(\eta)$ and $\eta \mapsto \zeta_\varepsilon(\eta)$ are the obvious extensions to $\mathbb{R}^2$ of the functions λ_ε and ζ_ε defined in (1.13) and (1.15), respectively.

From (2.7), (1.13) and (1.15), it might seem that $(\varepsilon,t,\eta) \mapsto \widehat{\mathcal{D}}_\varepsilon(\eta,t)$ is ill-defined on the hypersurface $\{(\varepsilon,\eta) \in (0,\infty)\times\mathbb{R}^2 : \varepsilon = \frac{1}{2\langle\eta\rangle}\}$. We however prove in the next lemma that $(\varepsilon,t,\eta) \mapsto \widehat{\mathcal{D}}_\varepsilon(\eta,t)$ can actually be extended to a smooth function on $(0,\infty)\times\mathbb{R}_+\times\mathbb{R}^2$ and provide a control on its derivatives in ε and η.

For an integer $p \ge 1$ and a multi-index $\alpha \in \{1,2\}^p$, we denote by $|\alpha| = p$ its length.

Lemma 2.1.5. *Recall the definition of the set I in (1.19). The function $(\varepsilon,t,\eta) \in I\times\mathbb{R}_+\times\mathbb{R}^2 \mapsto \widehat{\mathcal{D}_\varepsilon}(\eta,t)$ can be extended to a C^∞ function on $(0,\infty)\times\mathbb{R}_+\times\mathbb{R}^2$. Moreover, we have the following bound:*

$$|\partial^\alpha_\eta\widehat{\mathcal{D}_\varepsilon}(\eta,t)| \lesssim e^{-\frac{t}{2\varepsilon^2}}t^{|\alpha|+1}\varepsilon^{-|\alpha|}\sum_{p=1}^{|\alpha|}(1+|t\varepsilon^{-1}\eta|^p)\big(\mathbb{1}_{\langle\eta\rangle\le(2\varepsilon)^{-1}}e^{t\lambda_\varepsilon(\eta)} + \mathbb{1}_{\langle\eta\rangle>(2\varepsilon)^{-1}}\big) \tag{2.8}$$

$$|\partial_\varepsilon\widehat{\mathcal{D}_\varepsilon}(\eta,t)| \lesssim \varepsilon^{-5}\langle\eta\rangle^2, \tag{2.9}$$

for any $\varepsilon \in (0,\infty)$, $\eta \in \mathbb{R}^2$, $t \geq 0$ and multi-index α.

Proof. Note that from (2.7) $\widehat{\mathcal{D}_\varepsilon}(\eta,t)$ can be written as a power series:

$$\widehat{\mathcal{D}_\varepsilon}(\eta,t) = e^{-\frac{t}{2\varepsilon^2}} \sum_{j\geq 0} \frac{t^{2j+1}}{(2j+1)!}\big((4\varepsilon^4)^{-1} - \varepsilon^{-2}\langle\eta\rangle^2\big)^j =: e^{-\frac{t}{2\varepsilon^2}} \widehat{S_\varepsilon}(\eta,t). \tag{2.10}$$

This immediately shows that, $(\varepsilon,t,\eta) \mapsto \widehat{\mathcal{D}_\varepsilon}(\eta,t)$ can be extended to a smooth function on $(0,\infty)\times\mathbb{R}_+\times\mathbb{R}^2$. Indeed, the partial derivatives with respect to either ε, t and η or a mix thereof exist and are continuous by standard results on power series. Here, $\widehat{S_\varepsilon}(\eta,t)$ is denotes the extension to $\mathbb{R}^2$ of the symbol of the multiplier $S_\varepsilon(t)$ defined in (1.17).

Let us first show (2.8). Note that from (2.10), we can rewrite $\widehat{S_\varepsilon}(\eta,t)$ as

$$\widehat{S_\varepsilon}(\eta,t) = t g_{\varepsilon,t}(t\varepsilon^{-1}\eta), \quad \eta \in \mathbb{R}^2, \tag{2.11}$$

with $g_{\varepsilon,t}(\eta) := \sum_{j\geq 0} \frac{1}{(2j+1)!}(t^2(4\varepsilon^4)^{-1} - t^2\varepsilon^{-2} - |\eta|^2)^j$ for $\eta \in \mathbb{R}^2$. We can write the derivative $\partial_\eta^\alpha g_{\varepsilon,t}$ as a finite linear combination of functions of the form

$$Q_p(\eta)\sum_{j\geq p} \frac{j!}{(2j+1)!(j-p)!}\big(t^2(4\varepsilon^4)^{-1} - t^2\varepsilon^{-2} - |\eta|^2\big)^{j-p},$$

where $Q_p \in \mathbb{R}[\eta_1,\eta_2]$ (with $\eta = (\eta_1,\eta_2)$) is a polynomial of degree at most p for some integer $1 \leq p \leq |\alpha|$. Hence, we have

$$|\partial_\eta^\alpha g_{\varepsilon,t}(\eta)| \lesssim \sum_{p=1}^{|\alpha|}(1+|\eta|^p)\big|\phi^{(p)}\big(t^2(4\varepsilon^4)^{-1} - t^2\varepsilon^{-2} - |\eta|^2\big)\big|, \tag{2.12}$$

where ϕ is defined by

$$\phi(x) = \sum_{j\geq 0}\frac{x^j}{(2j+1)!}, \tag{2.13}$$

for $x \in \mathbb{R}$ and $\phi^{(p)}$ denotes the pth derivative of ϕ.

We claim the following bound on the derivatives of ϕ:

$$|\phi^{(p)}(x)| \lesssim_p \mathbb{1}_{x\geq 0}\, e^{\sqrt{x}} + \mathbb{1}_{x<0}, \tag{2.14}$$

for any $p \geq 0$. Note that we have the explicit formula $\phi(x) = \frac{\sin(\sqrt{-x})}{\sqrt{-x}}$ for $x < 0$. By induction, it is easy to see that for each $p \geq 0$, $\phi^{(p)}$ can be written as a finite linear combination of terms of the form

$$\frac{a_{k^{(p)}}\cos(\sqrt{-x}) + b_{k^{(p)}}\sin(\sqrt{-x})}{(\sqrt{-x})^{k^{(p)}}}$$

where $k^{(p)}$ is a positive integer and $a_{k^{(p)}}, b_{k^{(p)}}$ are real numbers. We directly deduce the bound $|\phi^{(p)}(x)| \lesssim 1$, for any $x < -1$ and $p \geq 0$. Besides, for $x \in \mathbb{R}$, we have

$$\phi^{(p)}(x) = \sum_{j\geq 0}\frac{(j+p)!}{j!(2j+2p+1)!}x^j.$$

Hence, $|\phi^{(p)}(x)| \lesssim 1$ for $|x| \lesssim 1$. Furthermore, for $x > 0$, we have

$$\sum_{j\geq 0}\frac{(j+p)!}{j!(2j+2p+1)!}x^j \leq \sum_{j\geq 0}\frac{1}{(2j)!}x^j \leq e^{\sqrt{x}},$$

where we used the inequality

$$\frac{(j+p)!(2j)!}{j!(2j+2p+1)!} \leq 1$$

for $j, p \geq 0$. This shows (2.14). We now can estimate the derivative $\partial_\eta^\alpha \widehat{S_\varepsilon}(\eta, t)$ using (2.11), (2.12), (2.14), and (1.13):

$$|\partial_\eta^\alpha \widehat{S_\varepsilon}(\eta, t)| \lesssim t^{|\alpha|+1}\varepsilon^{-|\alpha|}\sum_{p=1}^{|\alpha|}(1+|t\varepsilon^{-1}\eta|^p)\big(\mathbb{1}_{\langle\eta\rangle\leq(2\varepsilon)^{-1}}\, e^{t\lambda_\varepsilon(\eta)} + \mathbb{1}_{\langle\eta\rangle>(2\varepsilon)^{-1}}\big).$$

Combined with (2.10), the above shows (2.8).

Let us now prove (2.9). From (2.10) and the product rule, it suffices to prove that the terms $e^{-\frac{t}{2\varepsilon^2}}\partial_\varepsilon \widehat{S_\varepsilon}(\eta, t)$ and $\varepsilon^{-2}e^{-\frac{t}{2\varepsilon^2}}\widehat{S_\varepsilon}(\eta, t)$, for $\eta \in \mathbb{R}^2$ and $t \geq 0$, verify the bound (2.9). We only prove the former since the bound on $\varepsilon^{-2}e^{-\frac{t}{2\varepsilon^2}}\widehat{S_\varepsilon}(\eta, t)$ follows from similar considerations. By (2.11), we have

$$\partial_\varepsilon \widehat{S_\varepsilon}(\eta, t) = (-\varepsilon^{-5} + 2\varepsilon^{-3}\langle\eta\rangle^2)\, t^2\, \phi'(t^2(4\varepsilon^4)^{-1} - t^2\varepsilon^{-2}\langle\eta\rangle^2),$$

with ϕ as in (2.13). Note that by the inequality $\sqrt{1-x} \leq 1 - \frac{x}{2}$ for $0 \leq x \leq 1$, we get

$$e^{-\frac{t}{2\varepsilon^2}}e^{\lambda_\varepsilon(\eta)t} \leq e^{-\langle\eta\rangle^2 t}, \tag{2.15}$$

for $\langle\eta\rangle \leq (2\varepsilon)^{-1}$. Thus, from (2.14) and (2.15), we deduce

$$e^{-\frac{t}{2\varepsilon^2}}|\partial_\varepsilon \widehat{S_\varepsilon}(\eta, t)| \lesssim \varepsilon^{-5}\langle\eta\rangle^2 \cdot t^2 e^{-\frac{t}{2\varepsilon^2}}(e^{\lambda_\varepsilon(\eta)t}\mathbb{1}_{\langle\eta\rangle\leq(2\varepsilon)^{-1}} + 1) \lesssim \varepsilon^{-5}\langle\eta\rangle^2,$$

where we used the inequality $e^{-y} \lesssim y^{-1}$ for $y > 0$. This proves (2.9). □

In the next lemma, we prove bounds on $\widehat{\mathcal{D}}_\varepsilon(n, t)$ (1.18) (and its time derivative) which are uniform in $\varepsilon > 0$. By Plancherel's identity, this will be sufficient to obtain uniform bounds as (2.3) (at least for $\varepsilon > 0$).

Lemma 2.1.6. *Fix $0 < \theta \ll 1$. We have the following bounds:*

$$\varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n, t)| \lesssim \begin{cases} e^{-\theta t\langle n\rangle^2} & \text{for } \langle n\rangle \leq (1+\theta)(2\varepsilon)^{-1} \\ e^{-\frac{t}{2\varepsilon^2}}\varepsilon^{-1}\langle n\rangle^{-1} & \text{otherwise ,} \end{cases} \tag{2.16}$$

$$|\partial_t \widehat{\mathcal{D}_\varepsilon}(n, t)| \lesssim \begin{cases} e^{-\theta t\langle n\rangle^2} & \text{for } \langle n\rangle \leq (1+\theta)(2\varepsilon)^{-1} \\ e^{-\frac{t}{2\varepsilon^2}} & \text{otherwise ,} \end{cases} \tag{2.17}$$

with implicit constants independent of $\varepsilon > 0$ and $t > 0$.

We infer from (2.16) and (2.17) that both $\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n, t)$ and $\partial_t\widehat{\mathcal{D}_\varepsilon}(n, t)$ behave like the heat propagator P_0 in the low-frequency regime $\langle n\rangle \lesssim \varepsilon^{-1}$. However, in the high-frequency regime $\langle n\rangle \gg \varepsilon^{-1}$, they essentially behave like (a scaled version of) the damped propagator $\mathcal{D}_{\varepsilon=1}(n, t)$ (or its time derivative).

Proof. By the smoothness of the map $(\varepsilon, t, \eta) \mapsto \widehat{\mathcal{D}_\varepsilon}(\eta, t)$ discussed in Lemma 2.1.5 above, it suffices to prove (2.16) and (2.17) with $\langle n\rangle \neq (2\varepsilon)^{-1}$, which we assume in the remaining of the proof.

We first prove (2.16). Fix $t \geq 0$ and $\varepsilon > 0$. By (1.13) and (1.17), we have

$$\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n, t) = \varepsilon^{-2}e^{-\frac{t}{2\varepsilon^2}}\frac{\sinh(\lambda_\varepsilon(n)t)}{\lambda_\varepsilon(n)}, \tag{2.18}$$

in the regime $\langle n\rangle < (2\varepsilon)^{-1}$. Note that from (2.15), we have

$$e^{-\frac{t}{2\varepsilon^2}}\sinh(t\lambda_\varepsilon(n)) \lesssim e^{-\langle n\rangle^2 t}, \tag{2.19}$$

for $\langle n\rangle \le (2\varepsilon)^{-1}$. By using the inequality $\lambda_\varepsilon(n) \gtrsim_\theta \varepsilon^{-2}$ for $\langle n\rangle \le (1-\theta)(2\varepsilon)^{-1}$, we then obtain by (2.19) and (2.18)

$$\varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| \lesssim e^{-t\langle n\rangle^2}. \tag{2.20}$$

for $\langle n\rangle \le (1-\theta)(2\varepsilon)^{-1}$.

We now estimate $\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t)$ for $(1-\theta)(2\varepsilon)^{-1} < \langle n\rangle < (2\varepsilon)^{-1}$.

- **Case 1:** $\lambda_\varepsilon(n)t \le 1$. In this regime, by using the bounds $|\sinh(x)| \lesssim |x|$ for $0 \le x \le 1$ and $e^{-y} \lesssim y^{-1}$ for $y > 0$ and (2.18) with $\langle n\rangle < (2\varepsilon)^{-1}$, we get

$$\begin{aligned} \varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| &\lesssim \varepsilon^{-2}t \cdot e^{-\frac{t}{2\varepsilon^2}} \\ &\lesssim e^{-\frac{t}{4\varepsilon^2}} \lesssim e^{-t\langle n\rangle^2}. \end{aligned} \tag{2.21}$$

- **Case 2:** $\lambda_\varepsilon(n)t > 1$. In this case, we note that we have $\sqrt{1-x} \le 1 - \theta(1+\frac{x}{2})$ for $1-\theta \le x \le 1$ and $0 < \theta \ll 1$ which implies

$$e^{-(1-\theta)\frac{t}{2\varepsilon^2}}\sinh(\lambda_\varepsilon(n)t) \lesssim e^{-\theta t\langle n\rangle^2}, \tag{2.22}$$

for $(1-\theta)(2\varepsilon)^{-1} < \langle n\rangle < (2\varepsilon)^{-1}$. Using (2.18), (2.22) and the inequality $e^{-y} \lesssim y^{-1}$ for $y > 0$, we then get

$$\begin{aligned} \varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| &\lesssim \varepsilon^{-2}t \cdot e^{-\theta\frac{t}{2\varepsilon^2}} \cdot e^{-\theta t\langle n\rangle^2} \\ &\lesssim e^{-\theta t\langle n\rangle^2}. \end{aligned} \tag{2.23}$$

Hence, from (2.21) and (2.23) we deduce

$$\varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| \lesssim e^{-\theta t\langle n\rangle^2} \tag{2.24}$$

for $(1-\theta)(2\varepsilon)^{-1} < \langle n\rangle < (2\varepsilon)^{-1}$.

For $\langle n\rangle > (2\varepsilon)^{-1}$, we have from (1.17) with (1.15) and (1.18):

$$\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t) = \varepsilon^{-2}e^{-\frac{t}{2\varepsilon^2}}\frac{\sin(\zeta_\varepsilon(n)t)}{\zeta_\varepsilon(n)}. \tag{2.25}$$

By using the inequalities $|\sin(x)| \lesssim |x|$ for $x \in \mathbb{R}$ and $e^{-y} \lesssim y^{-1}$ for $y > 0$, (2.25) and $\langle n\rangle \sim \frac{1}{2\varepsilon^2}$, we have

$$\begin{aligned} \varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| &\lesssim \varepsilon^{-2}t \cdot e^{-\frac{t}{2\varepsilon^2}} \\ &\lesssim e^{-\frac{t}{4\varepsilon^2}} \lesssim e^{-\frac{t}{10}\langle n\rangle^2}. \end{aligned} \tag{2.26}$$

for $(2\varepsilon)^{-1} < \langle n\rangle \le (1+\theta)(2\varepsilon)^{-1}$.

For $\langle n\rangle > (1+\theta)\cdot(2\varepsilon)^{-1}$, we have $\zeta_\varepsilon(n) \gtrsim_\theta \varepsilon^{-1}\langle n\rangle$. Thus, we get the following bound from from (2.25),

$$\varepsilon^{-2}|\widehat{\mathcal{D}_\varepsilon}(n,t)| \lesssim e^{-\frac{t}{2\varepsilon^2}}\varepsilon^{-1}\langle n\rangle^{-1} \tag{2.27}$$

Collecting (2.20), (2.24), (2.26) and (2.27) yields (2.16)

We now prove (2.17). Note that from (2.18) and (2.25), we have

$$\partial_t\widehat{\mathcal{D}_\varepsilon}(n,t) = -\frac{1}{2\varepsilon^2}\widehat{\mathcal{D}_\varepsilon}(n,t) + e^{-\frac{t}{2\varepsilon^2}}\partial_t\widehat{S_\varepsilon}(n,t) \tag{2.28}$$

where $\{\widehat{S_\varepsilon}(n,t)\}_{n\in\mathbb{Z}^2}$ is the Fourier symbol associated to $S_\varepsilon(t)$ defined in (1.17).

By (2.16), it suffices to estimate the contribution of $e^{-\frac{t}{2\varepsilon^2}}\partial_t\widehat{S_\varepsilon}(n,t)$. By (1.17), we have

$$\partial_t \widehat{S_\varepsilon}(n,t) = \cosh(\lambda_\varepsilon(n)t)\mathbb{1}_{\langle n\rangle<(2\varepsilon)^{-1}} + \cos(\zeta_\varepsilon(n)t)\mathbb{1}_{\langle n\rangle>(2\varepsilon)^{-1}}. \tag{2.29}$$

Hence, by (2.15), we get

$$e^{-\frac{t}{2\varepsilon^2}}\partial_t \widehat{S_\varepsilon}(n,t) \lesssim \begin{cases} e^{-t\langle n\rangle^2} & \text{if } \langle n\rangle \le (2\varepsilon)^{-1} \\ e^{-\frac{t}{2\varepsilon^2}} & \text{otherwise,}\end{cases} \tag{2.30}$$

which concludes the proof of (2.17). □

In the next lemma, we study the behavior near $\varepsilon = 0$ of the symbols $\widehat{\mathcal{D}}_\varepsilon(n,t)$.

Lemma 2.1.7. *Fix $0<\theta\ll 1$. The following estimates hold:*

$$\big|\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t) - e^{-t\langle n\rangle^2}\big|\mathbb{1}_{\langle n\rangle\lesssim\varepsilon^{-1+\theta}} \lesssim e^{-\frac{t}{2\varepsilon^2}} + \varepsilon^{2\theta}e^{-\frac{t}{2}\langle n\rangle^2}, \tag{2.31}$$

and

$$\big|(\varepsilon^{-2}+\partial_t)\widehat{\mathcal{D}_\varepsilon}(n,t) - e^{-t\langle n\rangle^2}\big|\mathbb{1}_{\langle n\rangle\lesssim\varepsilon^{-1+\theta}} \lesssim \varepsilon^{2\theta}e^{-\frac{t}{2}\langle n\rangle^2}, \tag{2.32}$$

for any $t\ge 0$.

Proof. Fix $t\ge 0$ and $\langle n\rangle \lesssim \varepsilon^{-1+\theta}$. From (2.18) with (1.13), we write

$$\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t) = e^{-\frac{t}{2\varepsilon^2}}\frac{e^{\lambda_\varepsilon(n)t} - e^{-\lambda_\varepsilon(n)t}}{\sqrt{1-4\varepsilon^2\langle n\rangle^2}} =: \mathrm{I} - \mathrm{II}. \tag{2.33}$$

Since $\lambda_\varepsilon(n)$ is non-negative, we have

$$|\mathrm{II}| \lesssim e^{-\frac{t}{2\varepsilon^2}}. \tag{2.34}$$

Furthermore, we get, using (1.10), the inequality $\Lambda_\varepsilon^+(n)+\langle n\rangle^2\le 0$, the asymptotic expansion of $\Lambda_\varepsilon^+(n)$ in (1.23), the mean value theorem with the inequality $e^{-y}\lesssim y^{-1}$ for $y>0$, we deduce

$$\begin{aligned}|\mathrm{I} - e^{-t\langle n\rangle^2}| &\lesssim \Big|\Big(\frac{1}{\sqrt{1-4\varepsilon^2\langle n\rangle^2}} - 1\Big)e^{\Lambda_\varepsilon^+(n)t}\Big| + \big|\big(e^{\Lambda_\varepsilon^+(n)t} - e^{-t\langle n\rangle^2}\big)\big|,\\ &\lesssim \varepsilon^{2\theta}e^{-t\langle n\rangle^2} + t\langle n\rangle^4\varepsilon^2 e^{-t\langle n\rangle^2} \lesssim \varepsilon^{2\theta}e^{-\frac{t}{2}\langle n\rangle^2}.\end{aligned} \tag{2.35}$$

Putting (2.33), (2.34) and (2.35) together gives (2.31).

We now prove (2.32). By using (2.18), (1.10), (1.13) and (2.28), we write

$$(\varepsilon^{-2}+\partial_t)\widehat{\mathcal{D}_\varepsilon}(n,t) = \mathcal{P}_\varepsilon(n,t) + \mathcal{R}_\varepsilon(n,t), \tag{2.36}$$

with

$$\begin{aligned}\mathcal{P}_\varepsilon(n,t) &:= \frac{e^{\Lambda_\varepsilon^+(n)t}}{\sqrt{1-4\langle n\rangle^2\varepsilon^2}}\\ \mathcal{R}_\varepsilon(n,t) &:= \Big(1 - \frac{1}{\sqrt{1-4\langle n\rangle^2\varepsilon^2}}\Big)\frac{e^{\Lambda_\varepsilon^+(n)t}}{2} + \Big(1 - \frac{1}{\sqrt{1-4\langle n\rangle^2\varepsilon^2}}\Big)\frac{e^{\Lambda_\varepsilon^-(n)t}}{2}.\end{aligned}$$

By arguing as in (2.21), we find

$$\begin{aligned}&|\mathcal{P}_\varepsilon(n,t) - e^{-t\langle n\rangle^2}| \lesssim \varepsilon^{2\theta}e^{-\frac{t}{2}\langle n\rangle^2},\\ &|\mathcal{R}_\varepsilon(n,t)| \lesssim \varepsilon^{2\theta}e^{-t\langle n\rangle^2}.\end{aligned} \tag{2.37}$$

Thus, (2.32) follows from (2.36) and (2.37).

□

We deduce from Lemma 2.1.6 and Lemma 2.1.7 and Lemma A.1.2 the following operator bounds. In particular, we prove in (i) and (ii) below that the operator $\mathcal{D}_\varepsilon$ behaves like the heat linear flow on low frequencies $\langle n\rangle \lesssim \frac{1}{2\varepsilon^2}$ and like the wave linear flow on high frequencies $\langle n\rangle \gtrsim \frac{1}{2\varepsilon^2}$.

Lemma 2.1.8. *Fix $0 < \theta \ll 1$. Recall the definitions of $\mathbf{P}_\varepsilon^{\mathrm{low},\theta}$ and $\mathbf{P}_\varepsilon^{\mathrm{high},\theta}$ in (1.89). The following inequalities hold:*
(i) (*parabolic smoothing*) *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$ and $t > 0$. We have*

$$\big\|\varepsilon^{-2}\mathbf{P}_\varepsilon^{\mathrm{low},\theta}\mathcal{D}_\varepsilon(t)f\big\|_{H_x^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|f\|_{H_x^\beta},$$

for any smooth function f and with an implicit constant independent of $\varepsilon > 0$ and $t > 0$.
(ii) (*wave smoothing*) *Let $s \in \mathbb{R}$, $\gamma \in \{0, 1\}$ and $t > 0$. We have*

$$\big\|\varepsilon^{-2}\mathbf{P}_\varepsilon^{\mathrm{high},\theta}\mathcal{D}_\varepsilon(t)f\big\|_{H_x^s} \lesssim e^{-\frac{t}{10\varepsilon^2}}t^{-\frac{\gamma}{2}}\|f\|_{H_x^{s-\gamma}},$$

for any smooth function f and with an implicit constant independent of $\varepsilon > 0$ and $t > 0$.
(iii) *Let $s \in \mathbb{R}$, $0 \leq \gamma \leq 1$ and $t > 0$. We have*

$$\|\mathcal{D}_\varepsilon(t)f\|_{H_x^s} \lesssim \varepsilon^{2-\gamma}\|f\|_{H_x^{s-\gamma}}$$

for any smooth function f and with an implicit constant independent of $\varepsilon > 0$ and $t > 0$.
(iv) *Let $s \in \mathbb{R}$. We have,*

$$\sup_{\varepsilon, t > 0}\big\|\partial_t\mathcal{D}_\varepsilon(t)f\big\|_{H_x^s} \lesssim \|f\|_{H_x^s},$$

for any smooth function f.
(v) *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$, and $t > 0$. We have:*

$$\big\|\mathbf{P}_{\leq \varepsilon^{-1+\theta}}\big((\varepsilon^{-2} + \partial_t)\mathcal{D}_\varepsilon(t) - P_0(t)\big)f\big\|_{H_x^\alpha} \lesssim \varepsilon^{2\theta}t^{-\frac{\alpha-\beta}{2}}\|f\|_{H_x^\beta},$$

for any smooth function f and with an implicit constant independent of $\varepsilon > 0$ and $0 < t \leq T$.

Proof. Items (i), (iv) and (v) are direct consequences of (2.16), (2.17) and (2.32), respectively (along with Lemma A.1.2). We now look at (ii). If $\gamma = 0$, then (ii) comes directly from (2.16). If $\gamma = 1$, then from (2.16), we have

$$\begin{aligned}\big\|\varepsilon^{-2}\mathbf{P}_\varepsilon^{\mathrm{high},\theta}\mathcal{D}_\varepsilon(t)f\big\|_{H_x^\sigma} &\lesssim e^{-\frac{t}{2\varepsilon^2}}\varepsilon^{-1}\|f\|_{H_x^{s-1}}\\ &\lesssim e^{-\frac{t}{10\varepsilon^2}}t^{-\frac{1}{2}}\|f\|_{H_x^{s-1}},\end{aligned}$$

where we used the inequality $e^{-y} \lesssim y^{-\frac{1}{2}}$ for $y > 0$. This proves (ii) for $\gamma = 1$. From (i) and (ii) with $\gamma = 0$, and by interpolation, (iii) follows from the bound

$$\|\mathcal{D}_\varepsilon(t)f\|_{H_x^s} \lesssim \varepsilon\|f\|_{H_x^{s-1}}, \tag{2.38}$$

which we now prove. From (i), (1.89) and the restriction $\langle n\rangle \lesssim \varepsilon^{-1}$, we get

$$\begin{aligned}\|\mathbf{P}_\varepsilon^{\mathrm{low},\theta}\mathcal{D}_\varepsilon(t)f\|_{H_x^s} &\lesssim \varepsilon^2\|\mathbf{P}_\varepsilon^{\mathrm{low},\theta}f\|_{H_x^s}\\ &\lesssim \varepsilon\|f\|_{H_x^{s-1}}.\end{aligned} \tag{2.39}$$

Furthermore, from (2.16) with (1.89), we have

$$\begin{aligned}\|\mathbf{P}_\varepsilon^{\mathrm{high},\theta}\mathcal{D}_\varepsilon(t)f\|_{H_x^s} &\lesssim \varepsilon^2 e^{-\frac{t}{2\varepsilon^2}}\varepsilon^{-1}\|f\|_{H_x^{s-1}}\\ &\lesssim \varepsilon\|f\|_{H_x^{s-1}}.\end{aligned} \tag{2.40}$$

Hence, combining (2.39) and (2.40) gives (2.38). □

Remark 2.1.9. From (2.17), we actually get the following stronger bounds for $\alpha, \beta, s \in \mathbb{R}$ with $\alpha \geq \beta$:

$$\begin{aligned}\|\mathbf{P}_\varepsilon^{\mathrm{low},\theta}\partial_t\mathcal{D}_\varepsilon(t)f\|_{H^\alpha_x} &\lesssim t^{-\frac{\alpha-\beta}{2}}\|f\|_{H^\beta_x},\\ \|\mathbf{P}_\varepsilon^{\mathrm{high},\theta}\partial_t\mathcal{D}_\varepsilon(t)f\|_{H^s_x} &\lesssim e^{-\frac{t}{2\varepsilon^2}}\|f\|_{H^s_x},\end{aligned}$$

for any $\varepsilon, t > 0$ and smooth function f.

We are now ready to prove Proposition 2.1.1.

Proof of Proposition 2.1.1. Let $s \in \mathbb{R}$ and $T > 0$ and fix smooth functions ϕ_0 and ϕ_1. We first prove (2.3). By Lemma A.1.2, P_0 (1.27) cleary satisfies the bound

$$\|P_0(\phi_0,\phi_1)\|_{L^\infty_T H^s_x} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^s_x}. \tag{2.41}$$

Hence, it suffices to prove

$$\sup_{\varepsilon\in(0,1]}\|P_\varepsilon(\phi_0,\phi_1)\|_{L^\infty_T H^s_x} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^s_x}. \tag{2.42}$$

By (2.2), Lemma 2.1.8 (i), (ii), (iii) and (iv) with (1.89), we have

$$\begin{aligned}&\sup_{\varepsilon\in(0,1]}\|(\varepsilon^{-2}+\partial_t)\mathcal{D}_\varepsilon(t)\phi_0\|_{L^\infty_T H^s_x} \lesssim \|\phi_0\|_{H^s_x}\\ &\sup_{\varepsilon\in(0,1]}\|\mathcal{D}_\varepsilon(t)\phi_1\|_{L^\infty_T H^s_x} \lesssim \varepsilon\|\phi_1\|_{H^{s-1}_x}\end{aligned} \tag{2.43}$$

The continuity in time of $P_\varepsilon(\phi_0,\phi_1)$ for some fixed $\varepsilon > 0$ and $(\phi_0,\phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ follows from the dominated convergence theorem and (2.43). Combining (2.43) and (2.2) gives (2.42). This concludes the proof of the first part of (2.3).

By (2.1), Minkowski's inequality and Lemma 2.1.8 (i) and (ii), we obtain the following estimate for $\varepsilon > 0$:

$$\begin{aligned}\|\mathcal{I}_\varepsilon(F)\|_{L^\infty_T H^s_x} &\lesssim \int_0^T \|\varepsilon^{-2}\mathcal{D}_\varepsilon(t)F(t)\|_{H^s_x}dt\\ &\lesssim \int_0^T t^{-\frac{1}{2}}dt\|F\|_{L^\infty_T H^{s-1}_x} \lesssim T^{\frac{1}{2}}\|F\|_{L^\infty_T H^{s-1}_x}.\end{aligned} \tag{2.44}$$

The same inequality holds for $\varepsilon = 0$. The continuity in time of $\mathcal{I}_\varepsilon(F)$ ($\varepsilon \in [0,1]$) follows from a similar computation. This finishes the proof of (2.3).

The estimate on $P_\varepsilon - P_0$ in (2.4) follows from Lemma 2.1.8 (v), Lemma A.1.2 and similar arguments along with the bound $\|\mathcal{F}^{-1}(\widehat{f}\mathbb{1}_{\langle n\rangle>\varepsilon^{-1+\theta}})\|_{H^s_x} \lesssim \varepsilon^{\frac{\theta}{2}}\|f\|_{H^{s+\theta}_x}$ for any $\theta > 0$ small enough. Similarly, the estimate on $\mathcal{I}_\varepsilon - \mathcal{I}_0$ in (2.4) follows from (2.31) in Lemma 2.1.7.

□

Proof of Corollary 2.1.2. The continuity of the map $\varepsilon \in (0,1] \mapsto P_\varepsilon$ is deduced from the smoothness of the map $(\varepsilon,t) \mapsto \widehat{\mathcal{D}_\varepsilon}(n,t)$ for each $n \in \mathbb{Z}^2$ (by Lemma 2.1.5) and the dominated convergence theorem. The continuity at $\varepsilon = 0$ of $\varepsilon \mapsto P_\varepsilon$ follows from (2.4). Hence, (2.117) follows from the above and Proposition 2.1.1. The bound (2.118) follows from similar considerations. □

2.1.2 Wick powers of the stochastic convolution

In this subsection, we construct the Wick powers (1.38).

Proposition 2.1.10. *Let $\ell \in \mathbb{N}$. Fix any finite $p, q \geq 1$, $T > 0$ and $\sigma > 0$. Then, the following holds:*

(i) *Let* $\varepsilon \in [0,1]$*. The sequence* $\{ :\Psi^\ell_{\varepsilon,N}: \}_{N\in\mathbb{N}}$ *defined in* (1.38) *is a Cauchy sequence in* $L^p\big(\Omega; L^q([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))\big)$ *and thus converges, as* $N\to\infty$*, to a limiting stochastic process in* $L^p\big(\Omega; L^q([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))\big)$*, denoted by* $:\Psi^\ell_\varepsilon:$*.*

(ii) *The sequence* $\{(\varepsilon,t) \mapsto :\Psi^\ell_{\varepsilon,N}: \}_{N\in\mathbb{N}}$ *also converges to the process* $(\varepsilon,t)\mapsto :\Psi^\ell_\varepsilon:$ *in* $L^p\big(\Omega; L^q([0,1]\times[0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))\big)$ *and almost surely in* $C\big([0,1]\times[0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)\big)$ *as* $N\to\infty$*.*

In [31] and [22], the processes $\{ :\Psi^\ell_{\varepsilon,N}: \}_{N\in\mathbb{N}}$ were constructed for $\varepsilon = 1$ and $\varepsilon = 0$, respectively. The main novelty in Proposition 2.1.10 lies in (ii), where the stochastic process $(\varepsilon,t)\mapsto :\Psi^\ell_\varepsilon:$ is constructed as a continuous function of both ε and t by using the bi-parameter Kolmogorov continuity criterion (Lemma A.2.3). This implies in particular the convergence of $:\Psi^\ell_\varepsilon:$ to $:\Psi^\ell_0:$ along the continuous parameter $\varepsilon\to 0$.

Proof. Fix $\ell\in\mathbb{N}$, and $\sigma>0$. Our first goal is to bound the variance:

$$\mathbb{E}\Big[\big(\langle\nabla\rangle^{-\sigma} :\Psi^\ell_{\varepsilon,N}(t,\cdot): (x)\big)^2\Big] \lesssim 1, \tag{2.45}$$

uniformly in $N\in\mathbb{N}$, $t\geq 0$, $\varepsilon\in[0,1]$, and $x\in\mathbb{T}^2$.

Fix $N\in\mathbb{N}$, $t\geq 0$, $(x,y)\in(\mathbb{T}^2)^2$ and $\varepsilon\in(0,1]$ (the case $\varepsilon=0$ in (2.45) follows from similar arguments). By (1.34), (1.38) and Lemma A.2.2, we have,

$$\frac{1}{\ell!}\mathbb{E}\big[:\Psi^\ell_{\varepsilon,N}(t,x): :\Psi^\ell_{\varepsilon,N}(t,y): \big] = \mathbb{E}\big[\Psi_{\varepsilon,N}(t,x)\Psi_{\varepsilon,N}(t,y)\big]^\ell$$

Applying the Bessel potentials $\langle\nabla_x\rangle^{-\sigma}$ and $\langle\nabla_y\rangle^{-\sigma}$ and then setting $x=y$, we see from the previous computation that in order to bound the left-hand-side of (2.45), we need to bound terms of the form

$$\sum_{\substack{n_1,\cdots,n_\ell\in\mathbb{Z}^2\\ \langle n_j\rangle\lesssim N}} \langle n_1+\cdots+n_\ell\rangle^{-2\sigma} F_1(n_1,t)\cdots F_\ell(n_\ell,t), \tag{2.46}$$

where we write for $1\leq j\leq \ell$ and $n\in\mathbb{Z}^2$,

$$F_j(n,t) = \mathbb{E}\big[(\mathcal{F}(\Psi_{\varepsilon,N})(n,t))^2\big].$$

Hence, by (1.34) and (2.16), we get

$$\begin{aligned} F_j(n,t) &\lesssim \|\mathbb{1}_{[0,t]}(t')\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t)\|^2_{L^2_{t'}} \\ &\lesssim \langle n\rangle^{-2} \end{aligned} \tag{2.47}$$

for $n\in\mathbb{Z}^2$. This gives

$$(2.46) \lesssim \sum_{\substack{n_1,\cdots,n_\ell\in\mathbb{Z}^2\\ \langle n_j\rangle\lesssim N}} \langle n_1+\cdots+n_\ell\rangle^{-2\sigma} \prod_{j=1}^{\ell}\langle n_j\rangle^{-2} \lesssim 1,$$

and shows (2.45).

Let $r>\frac{4}{\sigma}$ and finite $p,q\geq 1$ with $p\geq q,r$. By Sobolev's and Minkowski's inequalities and Lemma A.2.1 along with (2.45), we have

$$\begin{aligned} \| :\Psi^\ell_{\varepsilon,N}: \|_{L^p(\Omega)L^q_T W^{-\sigma,\infty}_x} &\lesssim \| :\Psi^\ell_{\varepsilon,N}: \|_{L^p(\Omega)L^q_T W^{-\frac{\sigma}{2},r}_x} \\ &\leq \Big\| \| \langle\nabla\rangle^{-\frac{\sigma}{2}} :\Psi^\ell_{\varepsilon,N}: \|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x} \\ &\lesssim p^{\frac{\ell}{2}} \Big\| \| \langle\nabla\rangle^{-\frac{\sigma}{2}} :\Psi^\ell_{\varepsilon,N}: \|_{L^2(\Omega)} \Big\|_{L^q_T L^r_x} \\ &\lesssim T^{\frac{1}{q}} p^{\frac{\ell}{2}} \lesssim_{T,p,\ell} 1. \end{aligned} \tag{2.48}$$

Using the inclusion $L^{p_2}(\Omega) \subset L^{p_1}(\Omega)$ for $p_1 \le p_2$, we obtain a similar bound for any finite $p \ge 1$. Let $p \ge 1$ be finite and $M \ge N$. By similar arguments, we also get

$$\big\| :\Psi^\ell_{\varepsilon,N}: - :\Psi^\ell_{\varepsilon,M}: \big\|_{L^p(\Omega)L^q_T W_x^{-\sigma,\infty}} \lesssim N^{-\gamma}, \tag{2.49}$$

for some small $\gamma > 0$. The bound (2.49) shows that $\{:\Psi^\ell_{\varepsilon,N}:\}_{N\ge1}$ is a Cauchy sequence in $L^p\big(\Omega; L^q\big([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)\big)\big)$, for any $p,q \ge 1$ and $\sigma > 0$. Thus, it converges to some limit denoted by $:\Psi^\ell_\varepsilon:$. This shows the first part of the statement, i.e. item (i).

Before proceeding with the proof of (ii), we note that by arguing as in (2.48) and (2.49), we can construct a process $(\varepsilon,t) \mapsto :\Psi^\ell:(\varepsilon,t)$ as the limit of the sequence of stochastic objects $\{(\varepsilon,t) \mapsto :\Psi^\ell_{\varepsilon,N}:\}_{N\ge1}$ in $L^p\big(\Omega; L^2\big([0,1]\times[0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)\big)\big)$. Furthermore, this construction is coherent with that of $\{:\Psi^\ell_\varepsilon:\}_{\varepsilon\in[0,1]}$ in the sense that $:\Psi^\ell:(\varepsilon,t) = :\Psi^\ell_\varepsilon:(t)$ in $L^p\big(\Omega; L^2\big([0,1]\times[0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)\big)\big)$. This can indeed be observed by using the dominated convergence theorem and the uniformity of the bound (2.49) in $\varepsilon \in [0,1]$. This ensures that the process that we are going to construct below indeed corresponds to $\{:\Psi^\ell_\varepsilon:\}_{\varepsilon\in[0,1]}$. Furthermore, by arguing as in (the proof of) [59, Proposition 3.2], one can prove by using the Borel-Cantelli lemma that the convergence of $\{(\varepsilon,t)\mapsto :\Psi^\ell_{\varepsilon,N}:\}_{N\ge1}$ to $(\varepsilon,t)\mapsto :\Psi^\ell:(\varepsilon,t)$ holds in $L^2\big([0,1]\times[0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)\big)$, almost surely.

We now prove (ii) and investigate the continuity in (ε,t) of our stochastic objects. Let $h_1,h_2 \in \mathbb{R}$. We define the operators δ_{h_1,h_2}, $\delta^1_{h_1}$ and $\delta^2_{h_2}$ by

$$\begin{aligned} \delta_{h_1,h_2}X(\varepsilon,t) &= X(\varepsilon+h_1,t+h_2) - X(\varepsilon,t)\\ \delta^1_{h_1}X(\varepsilon,t) &= X(\varepsilon+h_1,t) - X(\varepsilon,t)\\ \delta^2_{h_2}X(\varepsilon,t) &= X(\varepsilon,t+h_2) - X(\varepsilon,t), \end{aligned} \tag{2.50}$$

Fix $\varepsilon\in[0,1]$ and $t\in[0,T]$. Let $h_1,h_2\in\mathbb{R}$ such that $\varepsilon+h_1\in[0,1]$ and $t+h_2\ge0$. Let $\ell\in\mathbb{N}$, $\sigma>0$. We aim to show the bound

$$\mathbb{E}\big[\big(\langle\nabla\rangle^{-\sigma}\delta_{h_1,h_2} :\Psi^\ell_{\varepsilon,N}(t,):(x)\big)^2\big] \lesssim \|(h_1,h_2)\|_2^\gamma, \tag{2.51}$$

for some $\gamma>0$ and uniformly in all parameters. In (2.51), $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$.

Let $(x,y)\in(\mathbb{T}^2)^2$. We only treat the case $(\varepsilon,\varepsilon+h_1)\in(0,1]$ as the case $\varepsilon=0$ or $\varepsilon+h_1=0$ follows from similar considerations. Expanding the expression

$$\frac{1}{\ell!}\mathbb{E}\big[\delta_{h_1,h_2} :\Psi^\ell_{\varepsilon,N}(t,x): \delta_{h_1,h_2} :\Psi^\ell_{\varepsilon,N}(t,y):\big] \tag{2.52}$$

yields

$$\begin{aligned} (2.52) = \Big(&\mathbb{E}\big[\Psi_{\varepsilon+h_1,N}(t+h_2,x)\Psi_{\varepsilon+h_1,N}(t+h_2,y)\big]^\ell\\ &- \mathbb{E}\big[\Psi_{\varepsilon,N}(t,x)\Psi_{\varepsilon+h_1,N}(t+h_2,y)\big]^\ell\Big)\\ + \Big(&\mathbb{E}\big[\Psi_{\varepsilon,N}(t,x)\Psi_{\varepsilon,N}(t,y)\big]^\ell\\ &- \mathbb{E}\big[\Psi_{\varepsilon+h_1,N}(t+h_2,x)\Psi_{\varepsilon,N}(t,y)\big]^\ell\Big)\\ =:{}& \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.53}$$

We have

$$\begin{aligned} \mathrm{I} = \mathbb{E}\big[&\delta_{h_1,h_2}\Psi_{\varepsilon,N}(t,x)\Psi_{\varepsilon+h_1,N}(t+h_2,y)\big]\\ &\times\sum_{j=0}^{\ell-1}\Big(\mathbb{E}\big[\Psi_{\varepsilon+h_1,N}(t+h_2,x)\Psi_{\varepsilon+h_1,N}(t+h_2,y)\big]^j\\ &\qquad\mathbb{E}\big[\Psi_{\varepsilon,N}(t,x)\Psi_{\varepsilon+h_1,N}(t+h_2,y)\big]^{\ell-1-j}\Big). \end{aligned} \tag{2.54}$$

A similar expression holds for II. Thus, by reasoning as before, in order to estimate $\mathbb{E}\big[\big(\langle\nabla\rangle^{-\sigma}\delta_{h_1,h_2} :\Psi^{\ell}_{\varepsilon,N}(t,):(x)\big)^2\big]$ we are led to bound sums of the form

$$\sum_{\substack{n_1,\cdots,n_\ell\in\mathbb{Z}^2\\ \langle n_j\rangle\lesssim N}} \langle n_1+\cdots+n_\ell\rangle^{-2\sigma} G_1(n_1)\cdots G_\ell(n_\ell), \tag{2.55}$$

with $G_j = G_j(n_j,t,\varepsilon,h_1,h_2)$ $(1\le j\le \ell)$. We have

$$\begin{aligned} G_1(n,t,\varepsilon,h_1,h_2) &= \mathbb{E}\Big[\mathcal{F}(\delta_{h_1,h_2}\Psi_{\varepsilon,N}(n,t))\mathcal{F}(\Psi_{\varepsilon^{(1)},N}(n,t^{(1)}))\Big]\\ G_j(n,t,\varepsilon,h_1,h_2) &= \mathbb{E}\Big[\mathcal{F}(\Psi_{\varepsilon_1^{(j)},N}(n,t_1^{(j)}))\mathcal{F}(\Psi_{\varepsilon_2^{(j)},N}(n,t_2^{(j)}))\Big], \quad 2\le j\le \ell, \end{aligned} \tag{2.56}$$

where $(\varepsilon^{(1)},\varepsilon_1^{(j)},\varepsilon_2^{(j)})\in\{\varepsilon,\varepsilon+h_1\}^3$ and $(t^{(1)},t_1^{(j)},t_2^{(j)})\in\{t,t+h_2\}^3$ for $2\le j\le\ell$. As before, we have

$$G_j(n,t,\varepsilon,h_1,h_2)\lesssim \langle n\rangle^{-2}, \tag{2.57}$$

for $2\le j\le \ell$, uniformly in all parameters. Denoting by $\langle\cdot,\cdot\rangle_{L^2}$ the canonical inner product on $L^2(\mathbb{R})$, we have

$$\begin{aligned} G_1(n,t,\varepsilon,h_1,h_2) &= \big\langle \delta_{h_1,h_2}(\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t-t')), \mathbb{1}_{[0,\min(t,t^{(1)})]}(t')(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t^{(1)}-t')\big\rangle_{L^2_{t'}}\\ &= \big\langle \delta^1_{h_1}(\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t+h_2-t')), \mathbb{1}_{[0,\min(t,t^{(1)})]}(t')(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t^{(1)}-t')\big\rangle_{L^2_{t'}}\\ &\quad + \big\langle \varepsilon^{-2}\delta^2_{h_2}\widehat{\mathcal{D}_\varepsilon}(n,t-t'), \mathbb{1}_{[0,\min(t,t^{(1)})]}(t')(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t^{(1)}-t')\big\rangle_{L^2_{t'}}\\ &= \mathrm{III}+\mathrm{IV}. \end{aligned} \tag{2.58}$$

We now estimate the terms III and IV. By (2.16) we have

$$|\mathrm{III}|, |\mathrm{IV}|\lesssim \langle n\rangle^{-2}. \tag{2.59}$$

Let us assume that $h_1\ge 0$ for convenience. By (2.16), (2.9) and the mean value theorem, we also have the following crude bound:

$$\begin{aligned} |\mathrm{III}| &\lesssim \big\langle \delta^1_{h_1}(\varepsilon^{-2})\,\widehat{\mathcal{D}_\varepsilon}(n,t+h_2-t'), \mathbb{1}_{[0,\min(t,t^{(1)})]}(t')(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t^{(1)}-t')\big\rangle_{L^2_{t'}}\\ &\qquad + \big\langle \varepsilon^{-2}\,\delta^1_{h_1}\big(\widehat{\mathcal{D}_\varepsilon}(n,t+h_2-t')\big), \mathbb{1}_{[0,\min(t,t^{(1)})]}(t')(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t^{(1)}-t')\big\rangle_{L^2_{t'}}\\ &\lesssim h_1\varepsilon^{-3}\,\|\widehat{\mathcal{D}_\varepsilon}(n,t')\|_{L^2_{t'}}\cdot \big\|(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t')\big\|_{L^2_{t'}}\\ &\qquad + \varepsilon^{-2}h_1\|\sup_{\varepsilon_0\in(\varepsilon,\varepsilon+h_1)}\partial_\varepsilon\widehat{\mathcal{D}_{\varepsilon_0}}(n,t')\|_{L^\infty_{t'}}\cdot\big\|(\varepsilon^{(1)})^{-2}\widehat{\mathcal{D}_{\varepsilon^{(1)}}}(n,t')\big\|_{L^1_{t'}}\\ &\lesssim h_1\varepsilon^{-7}\langle n\rangle^2. \end{aligned} \tag{2.60}$$

The bound (2.60) blows up when $\varepsilon>0$ is much smaller than other parameters and we now obtain a bound which is acceptable for small values of ε. We note that

$$\delta^1_{h_1}(\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t')) = \big((\varepsilon+h_1)^{-2}\widehat{\mathcal{D}_{\varepsilon+h_1}}(n,t') - e^{-t'\langle n\rangle^2}\big) + \big(e^{-t'\langle n\rangle^2}-\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t')\big).$$

Hence, by using (2.31), we have

$$|\mathrm{III}|\lesssim_T \varepsilon^{2\theta}+(\varepsilon+h_1)^{2\theta}, \tag{2.61}$$

for $\langle n\rangle\le\min\big(\varepsilon^{-1+\theta},(\varepsilon+h_1)^{-1+\theta}\big)=(\varepsilon+h_1)^{-1+\theta}$. We claim that we have

$$|\mathrm{III}|\lesssim |h_1|^{\gamma_1}\langle n\rangle^{-2+\gamma_2}, \tag{2.62}$$

for some $\gamma_1 > 0$ and some small $0 < \gamma_2 \ll \sigma$. We may assume $h_1\varepsilon^{-7} > h_1^{\gamma}$ for $0 < \gamma \ll 1$, for otherwise, interpolating (2.59) and (2.60) gives (2.62). We then have $\varepsilon \le h_1^{\frac{1-\gamma}{7}}$. If $\langle n\rangle \le (\varepsilon + h_1)^{-1+\theta}$, we have

$$(2.61) \lesssim h_1^{\frac{2\theta(1-\gamma)}{7}}. \tag{2.63}$$

Interpolating (2.63) with (2.59) then yields (2.62). Otherwise $\langle n\rangle > (\varepsilon + h_1)^{-1+\theta}$ and hence

$$(2.59) \lesssim \langle n\rangle^{-2+\gamma}(\varepsilon + h_1)^{\gamma(1-\theta)} \lesssim \langle n\rangle^{-2+\gamma} h_1^{\frac{\gamma(1-\gamma)}{7}(1-\theta)},$$

for $0 < \gamma \ll \sigma$. This concludes the proof of (2.62).

We now estimate IV. By using (2.17), we have

$$|\mathrm{IV}| \lesssim |h_2|\varepsilon^{-2}. \tag{2.64}$$

Since we can write

$$\begin{aligned}\varepsilon^{-2}\delta^2_{h_2}\widehat{\mathcal{D}_\varepsilon}(n,t') = {} & \big(\varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t'+h_2) - e^{-(t'+h_2)\langle n\rangle^2}\big)\\ & + \big(e^{-t'\langle n\rangle^2} - \varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t')\big)\\ & + \big(e^{-(t'+h_2)\langle n\rangle^2} - e^{-t'\langle n\rangle^2}\big),\end{aligned} \tag{2.65}$$

we have, from (2.31), (2.65) and the mean value theorem, the bound

$$|\mathrm{IV}| \lesssim \varepsilon^{2\theta} + |h_2|, \tag{2.66}$$

for $\langle n\rangle \le \varepsilon^{-1+\theta}$. Combining (2.59), (2.64) and (2.66) and arguing as in the estimate of the term III, we deduce

$$|\mathrm{IV}| \lesssim |h_2|^{\gamma_1}\langle n\rangle^{-2+\gamma_2} \tag{2.67}$$

for some $\gamma_1 > 0$ and some small $0 < \gamma_2 \ll \sigma$.

Thus, we deduce from (2.58), (2.62) and (2.67), the estimate

$$|G_1(n,t,\varepsilon,h_1,h_2)| \lesssim \|(h_1,h_2)\|_2^{\gamma}\,\langle n\rangle^{-2+\gamma}, \tag{2.68}$$

for $0 < \gamma \ll \sigma$. Hence, (2.51) follows from (2.55) with (2.56), (2.57) and (2.68).

Arguing as in the computations leading to (2.48) and (2.49), we deduce that for $\ell \in \mathbb{N}$, $M \ge N$, finite $p, q \ge 1$, $\varepsilon \in [0,1]$, $t \in [0,T]$ and $h_1, h_2 \in \mathbb{R}$ such that $\varepsilon + h_1 \in [0,1]$ and $t + h_2 \in [0,T]$, the following bound holds:

$$\begin{aligned}\|\delta_{h_1,h_2} :\Psi^{\ell}_{\varepsilon,N}: (t)\|_{L^p(\Omega)W_x^{-\sigma,\infty}} &\lesssim_{p,\ell} \|(h_1,h_2)\|_2^{\gamma},\\ \|\delta_{h_1,h_2}(:\Psi^{\ell}_{\varepsilon,N}: - :\Psi^{\ell}_{\varepsilon,M}:)(t)\|_{L^p(\Omega)W_x^{-\sigma,\infty}} &\lesssim_{p,\ell} N^{-\gamma}\|(h_1,h_2)\|_2^{\gamma},\end{aligned} \tag{2.69}$$

for $\gamma > 0$ small enough.

Fix $n \in \mathbb{Z}^2$. Given the smoothness of $(\varepsilon,t) \mapsto \widehat{\mathcal{D}_\varepsilon}(n,t')$, the following integration by parts formula holds almost-surely:

$$\int_0^t \varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t')dB_n(t') = \varepsilon^{-2}\widehat{\mathcal{D}_\varepsilon}(n,t)B_n(t) - \int_0^t \varepsilon^{-2}\partial_t\widehat{\mathcal{D}_\varepsilon}(n,t')B_n(t')dt'. \tag{2.70}$$

for any $t \ge 0$ and where the Brownian motion B_n is as in (1.32). Hence, we infer from (2.70), (1.34) and Lemma 2.1.5 that for each $N \in \mathbb{N}$, the map $(\varepsilon,t) \mapsto \Psi_{\varepsilon,N}$ belongs to $C\big((0,1]\times[0,T]; H_x^\infty\big)$; whence so does $(\varepsilon,t) \mapsto :\Psi^{\ell}_{\varepsilon,N}:$ by (1.38). Thus, by applying Lemma A.2.3 on $(0,1]\times[0,T]$, we have the following bounds:

$$\begin{aligned} \sup_{N\in\mathbb{N}} \| :\Psi^{\ell}_{\varepsilon,N}: \|_{L^p(\Omega)C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim 1,\\ \sup_{N\in\mathbb{N}}\sup_{M\geq N} N^{\gamma}\| :\Psi^{\ell}_{\varepsilon,N}: - :\Psi^{\ell}_{\varepsilon,M}: \|_{L^p(\Omega)C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim 1. \end{aligned} \tag{2.71}$$

Note that with the notation of Appendix A.2.2[1], we can write $:\Psi^{\ell}_{\varepsilon,N}:= I_\ell\big[f_N\big]$ for some function $f_N = f_N(n_1,\cdots,n_\ell)$. We then localize f_N as follows:

$$f_N^{N_\star} = f_N\cdot\prod_{j=1}^{\ell}\mathbb{1}_{\langle n_j\rangle\sim N_j}.$$

for dyadic numbers $N_\star = (N_1,\cdots,N_\ell)\in(2^{\mathbb{N}})^\ell$. By arguing as in the proof leading to (2.71), we can then obtain the following estimates:

$$\begin{aligned} \sup_{N\in\mathbb{N}} \|I_\ell\big[f_N^{N_\star}\big]\|_{L^p(\Omega)C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim \max(N_\star)^{-\delta},\\ \sup_{N\in\mathbb{N}}\sup_{M\geq N} N^{\gamma}\|I_\ell\big[f_N^{N_\star}\big] - I_\ell\big[f_M^{N_\star}\big]\|_{L^p(\Omega)C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim \max(N_\star)^{-\delta}. \end{aligned} \tag{2.72}$$

Hence, by using (2.72) and the reductions to frequency localized estimates described in Section 3.4 and Section A.3, we can insert the suprema inside the norms in (2.71). This proves the existence of a set Ω_0 of full $\mathbb{P}$-probability such that on Ω_0, we have

$$\begin{aligned} \sup_{N\in\mathbb{N}} \| :\Psi^{\ell}_{\varepsilon,N}: \|_{C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim 1,\\ \sup_{N\in\mathbb{N}}\sup_{M\geq N} N^{\gamma}\| :\Psi^{\ell}_{\varepsilon,N}: - :\Psi^{\ell}_{\varepsilon,M}: \|_{C((0,1]\times[0,T];W_x^{-\sigma,\infty})} &\lesssim 1. \end{aligned} \tag{2.73}$$

We now verify the continuity at $\varepsilon = 0$ of our stochastic objects. Fix $N\in\mathbb{N}$. By following the proof of (2.51), we can show

$$\mathbb{E}\Big[\big(\langle\nabla\rangle^{-\sigma}\delta^2_{h_2}(:\Psi^{\ell}_{\varepsilon,N}(t): - \Psi^{\ell}_{0,N}(t))(x)\big)^2\Big] \lesssim \varepsilon^{\gamma}|h_2|^{\gamma},$$

for each fixed $\varepsilon\in(0,1]$, $t\in[0,T]$ and $h_2\in\mathbb{R}$ such that $t+h_2\in[0,T]$ and some $0<\gamma,\sigma\ll 1$. From the usual Kolmogorov continuity criterion, we then obtain the bound

$$\Big\| \sup_{\varepsilon\in(0,1]} \varepsilon^{-\gamma}\| :\Psi^{\ell}_{\varepsilon,N}: - :\Psi^{\ell}_{0,N}: \|_{C_T W_x^{-\sigma,\infty}}\Big\|_{L^p(\Omega)} \lesssim_p 1, \tag{2.74}$$

for any $p\geq 1$. Hence, by (2.74) and Chebyshev's inequality, we get a set of full measure Ω_N such that

$$\| :\Psi^{\ell}_{\varepsilon,N}: - :\Psi^{\ell}_{0,N}: \|_{C_T W_x^{-\sigma,\infty}} \lesssim \varepsilon^{\gamma}, \tag{2.75}$$

for any $\varepsilon\in(0,1]$, on Ω_N. We define Ω_1 to be the full probability set

$$\Omega_1 := \Omega_0\cap\bigcap_{N\in\mathbb{N}}\Omega_N.$$

Hence, by (2.75), we get that for each $N\in\mathbb{N}$, $:\Psi^{\ell}_{\varepsilon,N}:$ is continuous at $\varepsilon=0$ on Ω_1. We thus deduce from (2.73) that

$$\begin{aligned} \| :\Psi^{\ell}_{\varepsilon,N}: \|_{C_{\varepsilon,T} W_x^{-\sigma,\infty}} &\lesssim 1\\ \| :\Psi^{\ell}_{\varepsilon,M}: - :\Psi^{\ell}_{\varepsilon,N}: \|_{C_{\varepsilon,T} W_x^{-\sigma,\infty}} &\lesssim N^{-\gamma}, \end{aligned} \tag{2.76}$$

on Ω_1 and for each $M\geq N$. This shows that the sequence $\{:\Psi^{\ell}_{\varepsilon,N}:\}_{N\in\mathbb{N}}$ is almost surely a Cauchy sequence in $C\big([0,1]\times[0,T];W^{-\sigma,\infty}(\mathbb{T}^2)\big)$.

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[1] adapted to the current real-valued setting.

By uniqueness of the almost sure limit in $L^2\big([0,1]\times[0,T];W^{-\sigma,\infty}(\mathbb{T}^2)\big)$, $\sigma>0$, this limit is equal to the map $(\varepsilon,t)\mapsto :\Psi^\ell_\varepsilon(t):$. □

2.2 Proofs of Theorems 1.1.4 and 1.1.6

2.2.1 Local theory

In this subection, we prove Theorem 1.1.4. Fix an integer $k\geq 2$. First, we consider the following *enhanced equation* on functions of the variables (x,ε,t):

$$\begin{cases} \varepsilon^2\partial_t^2 v+\partial_t v+(1-\Delta)v+\sum_{\ell=0}^k\binom{k}{\ell}\,\Xi_\ell\, v^{k-\ell}=0 \\ (v,\mathbb{1}_{\varepsilon>0}\partial_t v))|_{t=0}=(\phi_0,\mathbb{1}_{\varepsilon>0}\phi_1), \end{cases} \qquad (x,\varepsilon,t)\in\mathbb{T}^2\times[0,1]\times\mathbb{R}_+, \tag{2.77}$$

for given initial data (ϕ_0,ϕ_1) and a source $(\Xi_1,\dots,\Xi_k)$ with the understanding that $\Xi_0\equiv 1$.

Here, we present a local well-posedness argument for (2.77) based on Sobolev's inequality as in [33]. More precisely, we prove in Proposition 2.2.1 below, the existence of a solution $v=v(\varepsilon,t)$ to (2.77) which belongs to $C\big([0,1]\times[0,T];H^\kappa(\mathbb{T}^2)\big)$, for some $\kappa>0$. Note that, for each fixed $\varepsilon\in[0,1]$, $v_\varepsilon=v(\varepsilon,\cdot)$ solves (1.40). By using the convergence of the stochastic objects in Proposition 2.1.10, the proof of Theorem 1.1.4 essentially reduces to proving that v_ε converges to $v_{\varepsilon=0}$ as $\varepsilon\to 0$. However, by construction, this immediately follows from the continuity of $\varepsilon\mapsto v(\varepsilon,\cdot)$ at $\varepsilon=0$. Hence, the convergence in Theorem 1.1.4 is a consequence of the existence of the solution v to (2.77). We postpone the proof of Theorem 1.1.4 to the end of this section.

Given $\theta>0$, $T>0$, and $k\geq 2$ define $\mathcal{X}^{k,\theta}_T(\mathbb{T}^2)=\mathcal{X}^\theta_T(\mathbb{T}^2)$ by

$$\mathcal{X}^\theta_T(\mathbb{T}^2):=\big(C\big([0,1]\times[0,T];W^{-\theta,\infty}(\mathbb{T}^2)\big)\big)^{\otimes k},$$

and set

$$\|\Xi\|_{\mathcal{X}^\theta_T}=\sum_{j=1}^k\|\Xi_j\|_{C([0,1]\times[0,T];W_x^{-\theta,\infty})},$$

for $\Xi=(\Xi_1,\Xi_2,\dots,\Xi_k)\in\mathcal{X}^\sigma_T(\mathbb{T}^2)$. In what follows, we use the shorthand notation $\mathcal{X}^\theta(\mathbb{T}^2)$ for $\mathcal{X}^\theta_1(\mathbb{T}^2)$.

We have the following local well-posedness result for (2.77).

Proposition 2.2.1. *Fix an integer $k\geq 2$ and $\delta\leq\frac{1}{2(k-1)}$. Let $0<\theta\ll\delta$. Then, the equation (2.77) is unconditionally locally well-posed in $\mathcal{H}^{1-\delta}\times\mathcal{X}^\theta(\mathbb{T}^2)$. More precisely, given an enhanced data set:*

$$(\phi_0,\phi_1,\Xi)\in\mathcal{H}^{1-\delta}(\mathbb{T}^2)\times\mathcal{X}^\theta(\mathbb{T}^2), \tag{2.78}$$

with $\Xi=(\Xi_1,\Xi_2,\dots,\Xi_k)$, there exist $T=T(\|(\phi_0,\phi_1)\|_{\mathcal{H}^{1-\delta}_x},\|\Xi\|_{\mathcal{X}^\sigma})\in(0,1]$ and a unique solution $v=v(\varepsilon,t)$ to (2.77) in the class

$$C\big([0,1]\times[0,T];H^{1-\delta}(\mathbb{T}^2)\big). \tag{2.79}$$

In particular, the uniqueness of v holds in the entire class (A.17). Furthermore, the solution map

$$(\phi_0,\phi_1,\Xi)\in\mathcal{H}^{1-\delta}(\mathbb{T}^2)\times\mathcal{X}^\theta(\mathbb{T}^2)\mapsto v\in C([0,1]\times[0,T];H^{1-\delta}(\mathbb{T}^2))$$

is locally Lipschitz continuous.

Proof. By writing (2.77) in the Duhamel formulation, we have

$$\begin{aligned} v(\varepsilon,t) = \Gamma(v)(\varepsilon,t) &\overset{\text{def}}{=} P_\varepsilon(t)(\phi_0,\phi_1) \\ &\quad - \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{I}_\varepsilon(t)\big(\Xi_\ell(\varepsilon,t)\, v(\varepsilon,t)^{k-\ell}\big) \end{aligned} \tag{2.80}$$

where the map $\Gamma = \Gamma_{\Xi}$ depends on the enhanced data set Ξ in (2.89) and P_ε and $\mathcal{I}_\varepsilon$ are as in (2.2), (2.1). Fix $0 < T \le 1$. We have from Corollary 2.1.2,

$$\|P_\varepsilon(t)(\phi_0,\phi_1)\|_{C_{\varepsilon,T}H_x^{1-\delta}} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}_x^{1-\delta}} \tag{2.81}$$

We first treat the case $\ell = 0$. From Corollary 2.1.2 and Sobolev's inequality (twice), we obtain

$$\begin{aligned} \|\mathcal{I}_\varepsilon(t)(v^k)\|_{C_{\varepsilon,T}H_x^{1-\delta}} &\lesssim T^{\frac{1}{2}} \|v^k\|_{C_{\varepsilon,T}H_x^{-\delta}} \lesssim T^{\frac{1}{2}} \|v^k\|_{C_{\varepsilon,T}L_x^{\frac{2}{1+\delta}}} \lesssim T^{\frac{1}{2}} \|v\|^k_{C_{\varepsilon,T}L_x^{\frac{2k}{1+\delta}}} \\ &\lesssim T^{\frac{1}{2}} \|v\|^k_{C_{\varepsilon,T}H_x^{1-\delta}}, \end{aligned} \tag{2.82}$$

provided that

$$0 \le \delta \le \frac{1}{k-1}.$$

For $1 \le \ell \le k-1$, it follows from Corollary 2.1.2, Lemma A.1.1 (i) and (ii), and Sobolev's inequality that

$$\begin{aligned} \|\mathcal{I}_\varepsilon(t)(\Xi_\ell\, v^{k-\ell})\|_{C_{\varepsilon,T}H_x^{1-\delta}} &\lesssim T^{\frac{1}{2}} \|\Xi_\ell\, v^{k-\ell}\|_{C_{\varepsilon,T}H_x^{-\delta}} \\ &\lesssim T^{\frac{1}{2}} \|\langle\nabla\rangle^{-\delta}\Xi_\ell\|_{C_{\varepsilon,T}L_x^{\frac{2}{\delta}}} \|\langle\nabla\rangle^{\delta} v^{k-\ell}\|_{C_{\varepsilon,T}L_x^2} \\ &\lesssim T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^\theta} \|\langle\nabla\rangle^{\delta} v\|^{k-\ell}_{C_{\varepsilon,T}L_x^{2(k-\ell)}} \\ &\lesssim T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^\theta} \|v\|^{k-\ell}_{C_{\varepsilon,T}H_x^{1-\delta}}, \end{aligned} \tag{2.83}$$

provided that

$$0 \le \delta \le \frac{1}{2(k-1)}. \tag{2.84}$$

Lastly, again from Corollary 2.1.2, we have

$$\|\mathcal{I}_\varepsilon(t)(\Xi_k)\|_{C_{\varepsilon,T}H_x^{1-\delta}} \lesssim T^{\frac{1}{2}} \|\Xi_k\|_{C_{\varepsilon,T}H_x^{-\delta}} \le T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^\theta}. \tag{2.85}$$

Putting (2.80), (2.81), (2.82), (2.83) and (2.85) together, we have

$$\|\Gamma(v)\|_{C_{\varepsilon,T}H_x^{1-\delta}} \le C_1 \|(\phi_0,\phi_1)\|_{\mathcal{H}_x^{1-\delta}} + C_2 T^{\frac{1}{2}} \big(1 + \|\Xi\|_{\mathcal{X}^\theta}\big)\big(1 + \|v\|_{C_{\varepsilon,T}H_x^{1-\delta}}\big)^k,$$

as long as (2.84) is satisfied.

By similar arguments, the following difference estimate holds:

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{C_{\varepsilon,T}H_x^{1-\delta}} \le C_2 T^{\frac{1}{2}} \|v_1 - v_2\|_{C_{\varepsilon,T}H_x^{1-\delta}} \big(1 + \|\Xi\|_{\mathcal{X}^\theta}\big)\big(1 + \|v\|_{C_{\varepsilon,T}H_x^{1-\delta}}\big)^{k-1},$$

as long as (2.84) is satisfied. Therefore, by choosing $T = T(\|(\phi_0,\phi_1)\|_{\mathcal{H}_x^{1-\delta}}, \|\Xi\|_{\mathcal{X}^\theta}) > 0$ sufficiently small, we conclude that Γ is a contraction in the ball $B_R \subset C\big([0,1]\times[0,T]; H^{1-\delta}(\mathbb{T}^2)\big)$ of radius $R = 2C_1\|(\phi_0,\phi_1)\|_{\mathcal{H}_x^{1-\delta}} + 1$. At this point, the uniqueness holds only in the ball B_R but by a standard continuity argument, we can extend the uniqueness to hold in the entire $C\big([0,1]\times[0,T]; H^{1-\delta}(\mathbb{T}^2)\big)$. The regularity of the map $(\phi_0,\phi_1,\Xi) \in \mathcal{H}^{1-\delta}(\mathbb{T}^2) \times \mathcal{X}^\theta(\mathbb{T}^2) \mapsto v \in C([0,1]\times[0,T]; H^{1-\delta}(\mathbb{T}^2))$ is easily obtained through similar estimates. We omit details. $\square$

We now prove Theorem 1.1.4. We recall that, with a slight abuse of notations, wave equations for which $\varepsilon = 0$ are viewed as heat equations.

Proof of Theorem 1.1.4. Fix $k \geq 2$ and $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for $\frac{2k-3}{2k-2} \leq s < 1$. Let $0 < \theta \ll 1 - s$.

- **Step 1: Construction of solutions.** Let $\Xi = (\Psi_\varepsilon, :\Psi_\varepsilon^2:, \cdots, :\Psi_\varepsilon^k:)_{\varepsilon \in [0,1]}$. On the full probability set Ω_0 constructed in Proposition 2.1.10, we have that $\Xi_N, \Xi \in \mathcal{X}^\theta(\mathbb{T}^2)$ for any $N \in \mathbb{N}$ and $\Xi_N \to \Xi$ in $\mathcal{X}^\theta(\mathbb{T}^2)$ as $N \to \infty$.

By Proposition 2.2.1 we get, for each $N \in \mathbb{N}$, a function $v_N = v_N(\varepsilon, t)$ (resp. $v = v(\varepsilon, T)$) which belongs to $C\big([0,1] \times [0,T]; H^s(\mathbb{T}^2)\big)$ for some almost surely positive time $0 < T \leq 1$ (which is uniform in $N \in \mathbb{N}$ since $\sup_{N \in \mathbb{N}} \|\Xi_N\|_{\mathcal{X}^\theta} < \infty$) and that solves (2.77) with data given by Ξ_N (resp. Ξ). Furthermore, by the continuity of the map $(\phi_0, \phi_1, \Xi) \mapsto v$ proved in Proposition 2.2.1, we deduce that v_N converges to v in $C([0,T]; H^s(\mathbb{T}^2))$ as $N \to \infty$ on Ω_0.

For any $\varepsilon \in [0,1]$ and $N \in \mathbb{N}$, define $v_{\varepsilon,N} = v_N(\varepsilon, \cdot)$ and $v_\varepsilon = v(\varepsilon, \cdot)$. By construction, $v_{\varepsilon,N}$ (resp. v_ε) solves (1.39) (resp. (1.40)) with initial data $(\phi_0, \mathbb{1}_{\varepsilon>0}\phi_1)$[2] and belongs to $C([0,T]; H^s(\mathbb{T}^2))$. For $\varepsilon \in [0,1]$ and $N \in \mathbb{N}$, let $u_{\varepsilon,N} = \Psi_{\varepsilon,N} + v_{\varepsilon,N}$. Then, $u_{\varepsilon,N}$ is the solution to (1.41) with initial data $(\phi_0, \mathbb{1}_{\varepsilon>0}\phi_1)$. Then, by the above and Proposition 2.1.10, $u_{\varepsilon,N}$ converges to the process $u_\varepsilon := \Psi_\varepsilon + v_\varepsilon$ in $C([0,T]; H^{-\sigma}(\mathbb{T}^2))$, $\sigma > 0$, as $N \to \infty$ on Ω_0.

- **Step 2: Convergence.** By the continuity of the map $\varepsilon \mapsto v_\varepsilon$ and Proposition 2.1.10, we have that u_ε converges to $u_{\varepsilon=0}$ in $C([0,T]; H^{-\sigma}(\mathbb{T}^2))$, $\sigma > 0$, as $\varepsilon \to 0$ on Ω_0. This concludes the proof of Theorem 1.1.4. ☐

2.2.2 Asymptotic global well-posedness

The purpose of this section is to prove Theorem 1.1.6. We recall the following global well-posedness result from [50] adapted to our notations. See also [72, Proposition 6.1].

Lemma 2.2.2. *Fix $k \geq 2$. Let $0 < s < 1$ and $\phi_0 \in H^s(\mathbb{T}^2)$. Let $v = v(\varepsilon, t)$ be the solution to (2.80) with data given by $(\phi_0, \phi_1, \Xi) = (\phi_0, \phi_1, \Psi_\varepsilon, :\Psi_\varepsilon^2:, \cdots, :\Psi_\varepsilon^k:)_{\varepsilon \in [0,1]}$ constructed in Theorem 1.1.4. Then, the function $v_0 = v(0, \cdot)$ exists globally in time. Moreover, for any fixed $T > 0$, we have $v_0 \in C\big([0,T]; H^s(\mathbb{T}^2)\big)$.*

In order to prove Theorem 1.1.6, we have to extend the existence time of the solution $\{v_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ to (2.77) restricted to the range $(0, \varepsilon_0]$, for some fixed $\varepsilon > 0$ to be chosen later. The main idea to achieve this goal is to combine two ingredients: the global existence of v_0 provided by Lemma 2.2.2 and the fact that v_0 approximates (locally-in-time) v_ε for $\varepsilon \in (0, \varepsilon_0]$ if ε_0 is small enough. Hence, for any fixed target time $T > 0$, we hope to use the quantity $\|v_0\|_{C_T H_x^s}$, $0 < s < 1$, as an a priori bound on the growth in time of the relevant Sobolev norm of v_ε, $\varepsilon \in (0, \varepsilon_0]$. To do so, we have to iterate the local well-posedness argument of Proposition 2.2.1 starting from a small time $T > 0$ and solve the following fixed point problem:

$$\begin{aligned} v_\varepsilon(t) =& (\varepsilon^{-2} + \partial_t)\mathcal{D}_\varepsilon(t) v_\varepsilon(T) + \mathcal{D}_\varepsilon(t)\partial_t v_\varepsilon(T) \\ &- \sum_{\ell=0}^{k} \binom{k}{\ell} \int_T^t \varepsilon^{-2}\mathcal{D}_\varepsilon(t-t')\big(\Xi_\ell(\varepsilon, t')\, v_\varepsilon(t')^{k-\ell}\big)dt', \quad t \in [T, +\infty). \end{aligned} \tag{2.86}$$

Solving (2.86) requires having a bound on $\partial_t v_\varepsilon(T)$. However, by taking the time-derivative of (2.80) it is easy to see that the expression for $\partial_t v_\varepsilon(T)$ contains a term of the form $\varepsilon^{-4}\widehat{\mathcal{D}_\varepsilon}(T)\phi_0$ which cannot be bounded uniformly in $\varepsilon > 0$ in any Sobolev space. Thus, we have no uniform in $\varepsilon > 0$ control over the $\mathcal{H}^s(\mathbb{T}^2)$ norm of v_ε in Proposition 2.2.1.[3]

Fortunately, it turns out that the term $\mathcal{D}_\varepsilon(t)\partial_t v_\varepsilon(T)$ in (2.86) can be bounded uniformly in $\varepsilon > 0$. It order to capture this effect, we introduce the space $\mathcal{V}_{\varepsilon_0}^s(\mathbb{T}^2)$ defined in (2.88) below which is a modification of the spaces $\mathcal{H}^s(\mathbb{T}^2)$ suitable for the convergence setting at hand.

In what follows, we first state a well-posedness result for an appropriate variation of (2.77). Namely, for fixed $\varepsilon_0 \in (0,1]$, we consider the following problem:

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[2] Here and in what follows, with a slight abouse of notation, we understand $(\phi_0, \mathbb{1}_{\varepsilon>0}\phi_1)$ as (ϕ_0, ϕ_1) for $\varepsilon > 0$ and as ϕ_0 for $\varepsilon = 0$.

[3] This issue is a feature of the convergence problem at hand as the study (i.e. for any *fixed* $\varepsilon > 0$) is usually carried out in the Sobolev spaces $(\mathcal{H}^s(\mathbb{T}^2))_{s \in \mathbb{R}}$; see for instance [31].

$$\begin{cases}\varepsilon^2\partial_t^2 v + \partial_t v + (1-\Delta)v + \sum_{\ell=0}^{k}\binom{k}{\ell}\,\Xi_\ell\, v^{k-\ell} = 0 \\ (v,\partial_t v))|_{t=0} = (\phi_0(\varepsilon),\phi_1(\varepsilon)),\end{cases} \qquad (x,\varepsilon,t)\in\mathbb{T}^2\times(0,\varepsilon_0]\times\mathbb{R}_+, \tag{2.87}$$

for given initial data $\varepsilon\mapsto(\phi_0(\varepsilon),\phi_1(\varepsilon))$ and a source $(\Xi_1,\dots,\Xi_k)$ with the understanding that $\Xi_0\equiv 1$.

Given $\varepsilon\in(0,1]$ and $s\in\mathbb{R}$, we define the space $\mathcal{V}^s_{\varepsilon_0}(\mathbb{T}^2)$ by the norm[4]

$$\|(\phi_0,\phi_1)\|_{\mathcal{V}^s_{\varepsilon_0}} = \|\phi_0(\varepsilon)\|_{C_b((0,\varepsilon_0];H^s_x)} + \|\mathcal{D}_\varepsilon(t)\phi_1(\varepsilon)\|_{C_b((0,\varepsilon_0]\times\mathbb{R}_+;H^s_x)}. \tag{2.88}$$

Proposition 2.2.3. *Fix an integer $k\geq 2$, $\varepsilon_0\in(0,1]$ and $\delta\leq\frac{1}{2(k-1)}$. Let $0<\theta\ll\delta$. Then, the equation (2.87) is unconditionally locally well-posed in $\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\times\mathcal{X}^\theta(\mathbb{T}^2)$. More precisely, given an enhanced data set:*

$$(\phi_0,\phi_1,\Xi)\in\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\times\mathcal{X}^\theta(\mathbb{T}^2), \tag{2.89}$$

with $\Xi=(\Xi_1,\Xi_2,\dots,\Xi_k)$ and any time $T_0\in(0,1)$, there exist a time $T(\|(\phi_0,\phi_1)\|_{\mathcal{V}^{1-\delta}_{\varepsilon_0}},\|\Xi\|_{\mathcal{X}^\theta})\in(0,1]$, independent of ε_0, and a unique solution $\vec{v}:=(v,\partial_t v)$ to (2.87) with initial data (ϕ_0,ϕ_1) at time T_0 in the class

$$C\big([T_0,T];\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\big). \tag{2.90}$$

In particular, the uniqueness of $\vec{v}$ holds in the entire class (2.90). Furthermore, the solution map

$$(\phi_0,\phi_1,\Xi)\in\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\times\mathcal{X}^\theta(\mathbb{T}^2)\mapsto v\in C([T_0,T];\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2))$$

is locally Lipschitz continuous.

Proof. Fix $\varepsilon_0\in(0,1]$. For simplicity, we assume that $T_0=0$. The integral formulation of (2.87) reads

$$\vec{v}=\Gamma(v)=\begin{pmatrix}\Gamma_1(v)\\ \Gamma_2(v)\end{pmatrix}, \tag{2.91}$$

where

$$\begin{aligned}\Gamma_1(v)(\varepsilon,t) &= (\varepsilon^{-2}+\partial_t)\mathcal{D}_\varepsilon(t)\phi_0(\varepsilon)+\mathcal{D}_\varepsilon(t)\phi_1(\varepsilon)\\ &\quad-\sum_{\ell=0}^{k}\binom{k}{\ell}\mathcal{I}_\varepsilon(t)\big(\Xi_\ell(\varepsilon,t)\,v(\varepsilon,t)^{k-\ell}\big),\end{aligned} \tag{2.92}$$

and

$$\begin{aligned}\Gamma_2(v)(\varepsilon,t) &= (\varepsilon^{-2}+\partial_t)\partial_t\mathcal{D}_\varepsilon(t)\phi_0(\varepsilon)+\partial_t\mathcal{D}_\varepsilon(t)\phi_1(\varepsilon)\\ &\quad-\sum_{\ell=0}^{k}\binom{k}{\ell}\int_0^t\varepsilon^{-2}\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'.\end{aligned} \tag{2.93}$$

We fix $T>0$. By arguing as in the proof of Proposition 2.2.1, we obtain the following bound on $\Gamma_1(v)$:

$$\|\Gamma_1(v)\|_{C([0,T]\times(0,\varepsilon_0];H^{1-\delta}_x)}\lesssim\|(\phi_0,\phi_1)\|_{\mathcal{V}^{1-\delta}_{\varepsilon_0}}+T^{\frac{1}{2}}\big(1+\|\Xi\|_{\mathcal{X}^\theta}\big)\big(1+\|v\|_{C_T\mathcal{V}^{1-\delta}_{\varepsilon_0}}\big)^k, \tag{2.94}$$

for any $v\in C\big([0,T];\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\big)$.

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[4] Here, for a metric space X and a Banach space $(Y,\|\cdot\|)$, we denote by $C_b(X,Y)$ the Banach space of bounded and continuous fonctions endowed with the norm $\|f\|_{L^\infty}:=\sup_{x\in X}\|f(x)\|$.

We now turn our attention to Γ_2. Namely, by (2.88), we have to estimate $(t,\varepsilon,t_1)\mapsto \mathcal{D}_\varepsilon(t_1)\Gamma_2(v)(\varepsilon,t)$ in $C\big([0,T]\times(0,\varepsilon_0]\times\mathbb{R}_+;H^{1-\delta}(\mathbb{T}^2)\big)$ for any v in $C\big([0,T];\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\big)$.

Let v in $C\big([0,T];\mathcal{V}^{1-\delta}_{\varepsilon_0}(\mathbb{T}^2)\big)$. We now obtain bounds on $\|\mathcal{D}_\varepsilon(t_1)\Gamma_2(v)(\varepsilon,t)\|_{H^{1-\delta}_x}$ that are uniform in $(\varepsilon,t,t_1)\in(0,\varepsilon_0]\times[0,T]\times\mathbb{R}_+$. We fix $(\varepsilon,t,t_1)\in(0,\varepsilon_0]\times[0,T]\times\mathbb{R}_+$. By Lemma 2.1.8 (iv) and the fact that $\mathcal{D}_\varepsilon$ and $\partial_t\mathcal{D}_\varepsilon$ commute, we have

$$\|\mathcal{D}_\varepsilon(t_1)\partial_t\mathcal{D}_\varepsilon(t)\phi_1(\varepsilon)\|_{H^{1-\delta}_x}\le\|\mathcal{D}_\varepsilon(t_1)\phi_1(\varepsilon)\|_{H^{1-\delta}_x}\lesssim\|(\phi_0,\phi_1)\|_{\mathcal{V}^{1-\delta}_{\varepsilon_0}}. \tag{2.95}$$

We have

$$\begin{aligned}
&\Big\|\mathcal{D}_\varepsilon(t_1)\int_0^t\varepsilon^{-2}\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{1-\delta}_x}\\
&\quad\le\Big\|\varepsilon^{-2}\mathcal{D}_\varepsilon(t_1)\int_0^t\mathbf{P}^{\mathrm{low},\theta}_\varepsilon\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{1-\delta}_x}\\
&\qquad+\Big\|\mathcal{D}_\varepsilon(t_1)\int_0^t\varepsilon^{-2}\mathbf{P}^{\mathrm{high},\theta}_\varepsilon\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{1-\delta}_x}=:\mathrm{I}+\mathrm{II}
\end{aligned}\tag{2.96}$$

By Remark 2.1.9 and Lemma 2.1.8 (i) and (ii) and by arguing as in the proof of Proposition 2.2.1, we have

$$\begin{aligned}
\mathrm{I}&\lesssim\Big\|\int_0^t\mathbf{P}^{\mathrm{low},\theta}_\varepsilon\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{1-\delta}_x}\\
&\lesssim T^{\frac12}\|\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell}\|_{C([0,T]\times(0,\varepsilon_0];H^{-\delta}_x)}\\
&\lesssim T^{\frac12}\big(1+\|\Xi\|_{\mathcal{X}^\theta_T}\big)\big(1+\|v\|_{C_T\mathcal{V}^{1-\delta}_{\varepsilon_0}}\big)^k.
\end{aligned}\tag{2.97}$$

Similarly, using Remark 2.1.9 along with Lemma 2.1.8 (iii), we get

$$\begin{aligned}
\mathrm{II}&\lesssim\varepsilon^{-1}\Big\|\int_0^t\mathbf{P}^{\mathrm{high},\theta}_\varepsilon\partial_t\mathcal{D}_\varepsilon(T-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{-\delta}_x}\\
&\lesssim\varepsilon^{-1}\int_0^t e^{-\frac{t-t'}{2\varepsilon^2}}dt'\,\|\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell}\|_{C([0,T]\times(0,\varepsilon_0];H^{-\delta}_x)}\\
&\lesssim\varepsilon\big(1+\|\Xi\|_{\mathcal{X}^\theta_T}\big)\big(1+\|v\|_{C_T\mathcal{V}^{1-\delta}_{\varepsilon_0}}\big)^k.
\end{aligned}\tag{2.98}$$

Hence, from (2.96), (2.97) and (2.98), we have

$$\begin{aligned}
&\Big\|\mathcal{D}_\varepsilon(t_1)\int_0^t\varepsilon^{-2}\partial_t\mathcal{D}_\varepsilon(t-t')(\Xi_\ell(\varepsilon,t')v(\varepsilon,t')^{k-\ell})dt'\Big\|_{H^{1-\delta}_x}\\
&\qquad\lesssim\big(1+\|\Xi\|_{\mathcal{X}^\theta_T}\big)\big(1+\|v\|_{C_T\mathcal{V}^{1-\delta}_{\varepsilon_0}}\big)^k.
\end{aligned}\tag{2.99}$$

We also have from Lemma 2.1.8 (i), (ii) and (iii),

$$\|\mathcal{D}_\varepsilon(t_1)\varepsilon^{-2}\partial_t\mathcal{D}_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}\le\|\partial_t\mathcal{D}_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}\lesssim\|(\phi_0,\phi_1)\|_{\mathcal{V}^{1-\delta}_{\varepsilon_0}}. \tag{2.100}$$

At last, we look at the term $\mathcal{D}_\varepsilon(t_1)\partial_t^2\mathcal{D}_\varepsilon(t)\phi_0$. By differentiating (2.29) and by combining Lemma 2.1.8 (i), (ii) and (2.30), we get

$$\begin{aligned}
\|\mathcal{D}_\varepsilon(t_1)\partial_t^2\mathcal{D}_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}&\lesssim\varepsilon^{-4}\|\mathcal{D}_\varepsilon(t_1)\mathcal{D}_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}+\|\varepsilon^{-2}\mathcal{D}_\varepsilon(t_1)e^{-\frac{t}{2\varepsilon^2}}\partial_tS_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}\\
&\quad+\|\mathcal{D}_\varepsilon(t_1)e^{-\frac{t}{2\varepsilon^2}}\partial_t^2S_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}\\
&\lesssim\|\phi_0\|_{H^{1-\delta}_x}+\|\mathcal{D}_\varepsilon(t)e^{-\frac{t}{2\varepsilon^2}}\partial_t^2S_\varepsilon(t)\phi_0\|_{H^{1-\delta}_x}
\end{aligned}\tag{2.101}$$

By (2.16) and (1.17) and by analyzing the symbols $\widehat{\mathcal{D}_\varepsilon}(n,t_1)e^{-\frac{t}{2\varepsilon^2}}\partial_t^2\widehat{S_\varepsilon}(n,t)$, $n\in\mathbb{Z}^2$, we easily obtain the following bound:

$$\|\mathcal{D}_\varepsilon(t_1)e^{-\frac{t}{2\varepsilon^2}}\partial_t^2 S_\varepsilon(t)\phi_0\|_{H_x^{1-\delta}} \lesssim \|\phi_0\|_{H_x^{1-\delta}}. \tag{2.102}$$

Hence, from (2.101) and (2.102), we have

$$\|\mathcal{D}_\varepsilon(t_1)\partial_t^2\mathcal{D}_\varepsilon(t)\phi_0\|_{H_x^{1-\delta}} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{V}_{\varepsilon_0}^{1-\delta}}. \tag{2.103}$$

Combining (2.95), (2.99), (2.100) and (2.102) yields

$$\|\mathcal{D}_\varepsilon(t_1)\Gamma_2(v)(\varepsilon,t)\|_{H_x^{1-\delta}} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{V}_{\varepsilon_0}^{1-\delta}} + T^{\frac{1}{2}}\big(1+\|\Xi\|_{\mathcal{X}_T^\theta}\big)\big(1+\|v\|_{C_T\mathcal{V}_{\varepsilon_0}^{1-\delta}}\big)^k, \tag{2.104}$$

uniformly in $(\varepsilon,t,t_1)\in(0,\varepsilon_0]\times[0,T]\times\mathbb{R}_+$. By (2.104), the dominated convergence theorem and since the map $(\varepsilon,t)\in[0,\varepsilon_0]\times[0,T]\mapsto\big(\phi_0(\varepsilon),\phi_1(\varepsilon),\{\Xi_\ell(\varepsilon,t)\}_{1\le\ell\le k},v(\varepsilon,t)\big)$ is continuous and the map $(\varepsilon,t)\mapsto\mathcal{D}_\varepsilon(t)$ is smooth (Lemma 2.1.5), we deduce the estimate

$$\begin{aligned}\|\mathcal{D}_\varepsilon(t_1)&\Gamma_2(v)(\varepsilon,t)\|_{C([0,T]\times(0,\varepsilon_0];H_x^{1-\delta})}\\ &\lesssim \|(\phi_0,\phi_1)\|_{\mathcal{V}_{\varepsilon_0}^{1-\delta}} + T^{\frac{1}{2}}\big(1+\|\Xi\|_{\mathcal{X}_T^\theta}\big)\big(1+\|v\|_{C_T\mathcal{V}_{\varepsilon_0}^{1-\delta}}\big)^k.\end{aligned} \tag{2.105}$$

Hence, by (2.94) and (2.105), we get

$$\|\Gamma(v)\|_{C_T\mathcal{V}_{\varepsilon_0}^{1-\delta}} \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{V}_{\varepsilon_0}^{1-\delta+\theta}} + T^{\frac{1}{2}}\big(1+\|\Xi\|_{\mathcal{X}^\theta}\big)\big(1+\|v\|_{C_T\mathcal{V}_{\varepsilon_0}^{1-\delta}}\big)^k, \tag{2.106}$$

for any $v\in C\big([0,T];\mathcal{V}_{\varepsilon_0}^{1-\delta}(\mathbb{T}^2)\big)$. This proves that Γ maps balls of $C\big([0,T];\mathcal{V}_{\varepsilon_0}^{1-\delta}(\mathbb{T}^2)\big)$ into themselves. We obtain a difference estimate along the same lines. The remaining of the proof follows as in the proof of Proposition 2.2.1. ☐

In the next lemma, we analyze the behavior near $\varepsilon=0$ of the (first coordinate of the) solutions to (2.87) on some (possibly large) time interval $[0,T]$, $T>0$, constructed in Proposition 2.2.3. Namely, we prove that they converge to v_0 given by Lemma 2.2.2.

Lemma 2.2.4 (long time approximation)**.** *Fix an integer $k\ge 2$ and $\frac{2k-3}{2k-2}\le s<1$. Fix $\varepsilon_0\in(0,1]$ and let $0<\theta\ll 1-s$. Consider the data $(\phi_0,\phi_1)\in\mathcal{H}^{s+\theta}(\mathbb{T}^2)$ (and hence $\varepsilon\in(0,\varepsilon_0]\mapsto(\phi_0,\mathbb{1}_{\varepsilon>0}\phi_1)\in\mathcal{V}_{\varepsilon_0}^{s+\theta}$) and $\Xi=(\Xi_1,\Xi_2,\dots,\Xi_k)\in\mathcal{X}_T^\theta(\mathbb{T}^2)$.*

Let $T>0$ and assume that there exists a solution v to (2.87) on $(0,\varepsilon_0]\times[0,T]$ such that

$$v\in C\big((0,\varepsilon_0]\times[0,T];H^s(\mathbb{T}^2)\big)$$

with data $\varepsilon\in(0,\varepsilon_0]\mapsto(\phi_0,\mathbb{1}_{\varepsilon>0}\phi_1)$ and $\Xi=(\Psi_\varepsilon,:\Psi_\varepsilon^2:,\cdots,:\Psi_\varepsilon^k:)_{\varepsilon\in[0,\varepsilon_0]}\in\mathcal{X}_T^\theta(\mathbb{T}^2)$. Let $v_0\in C\big([0,T];H^s(\mathbb{T}^2)\big)$ be the solution to (2.77) on $\{0\}\times[0,T]$ given by Lemma 2.2.2.

Then, there exists a constant $K=K(\|v_0\|_{C_TH_x^s},\|v\|_{C((0,\varepsilon_0]\times[0,T];H_x^{s+\theta})})>0$ and $\gamma>0$ such that

$$\|v(\varepsilon,\cdot)-v_0\|_{C_TH_x^s}\le K\max\big(\varepsilon^\gamma,\|\Xi(\varepsilon)-\Xi(0)\|_{C_T(W_x^{-\theta})^{\otimes k}},\|(\phi_0,\phi_1)\|_{\mathcal{H}_x^{s+\theta}}\big). \tag{2.107}$$

for any $\varepsilon\in(0,\varepsilon_0]$.

Proof. We follow an argument in [57, 58]; by adding an exponential weight to the local well-posedness norm, one is able to globalize directly a bound that only holds locally in time in the original norm. For $T>0$ and let $\lambda>0$ to be chosen later, we define the norm $\|\cdot\|_{S_{\lambda,T}}$ on $C([0,T],H^s(\mathbb{T}^2))$ by

$$\|v\|_{S_{\lambda,T}}\overset{\text{def}}{=}\|e^{-\lambda T}v\|_{C_TH_x^s} \tag{2.108}$$

Note that we have the inequalities

$$\|v\|_{S_{\lambda,T}}\le\|v\|_{C_TH_x^s}\le e^{\lambda T}\|v\|_{S_{\lambda,T}}. \tag{2.109}$$

Fix $\varepsilon\in(0,\varepsilon_0]$. Note that $v=v(\varepsilon,\cdot)$ and v_0 verify (2.80) (with $\varepsilon=0$ for v_0). By (2.109), we have

$$\begin{aligned}\|v(\varepsilon,\cdot)-v_0\|_{S_{\lambda,T}} &\le \|(P_\varepsilon-P_0)(\phi_0,\phi_1)\|_{S_{\lambda,T}} \qquad (2.110)\\ &\qquad+\sum_{\ell=0}^{k}\binom{k}{\ell}\|\mathcal{I}_\varepsilon\big(\Xi_\ell(\varepsilon,\cdot)v(\varepsilon,\cdot)^{k-\ell}\big)-\mathcal{I}_0\big(\Xi_\ell(0,\cdot)v_0^{k-\ell}\big)\|_{S_{\lambda,T}}\\ &\lesssim \|(P_\varepsilon-P_0)(\phi_0,\phi_1)\|_{C_TH^s_x}+\max_{0\le\ell\le k}\|(\mathcal{I}_\varepsilon-\mathcal{I}_0)\big(\Xi_\ell(\varepsilon,\cdot)v(\varepsilon,\cdot)^{k-\ell}\big)\|_{C_TH^s_x}\\ &\qquad+\max_{0\le\ell\le k}\|\mathcal{I}_0\big(\Xi_\ell(\varepsilon,\cdot)v(\varepsilon,\cdot)^{k-\ell}-\Xi_\ell(0,\cdot)v_0^{k-\ell}\big)\|_{S_{\lambda,T}}\\ &=\mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad (2.111)\end{aligned}$$

By (2.4), we get

$$\mathrm{I}\lesssim\varepsilon^{\frac{\theta}{2}}\|(\phi_0,\phi_1)\|_{\mathcal{H}^{s+\theta}_x} \qquad (2.112)$$

We also have from (2.4) and computations similar to those in the proof of Proposition 2.2.1, the following bounds:

$$\begin{aligned}\mathrm{II}&\lesssim T^{\frac{1}{2}}\varepsilon^{\frac{\theta}{2}}\|\Xi_\ell(\varepsilon,\cdot)v(\varepsilon,\cdot)^{k-\ell}\|_{C((0,\varepsilon_0]\times[0,T];H^{s+\theta}_x)}\\ &\lesssim T^{\frac{1}{2}}\varepsilon^{\frac{\theta}{2}}(1+\|\Xi\|_{\mathcal{X}^\theta_T})(1+\|v\|_{C((0,\varepsilon_0]\times[0,T];H^{s+\theta}_x)})^k\end{aligned} \qquad (2.113)$$

We now estimate the term III in the case $\ell=0$ for convenience. From Lemma A.1.2 and proceeding as in (2.82), we have

$$\begin{aligned}\|\mathcal{I}_0\big(v(\varepsilon,\cdot)^k-v_0^k\big)\|_{S_{\lambda,T}}&=\Big\|\int_0^t e^{-\lambda(t-t')}P_0(t-t')\big(e^{-\lambda t'}(v(\varepsilon,t')^k-v_0(t')^k)\big)dt'\Big\|_{C_TH^s_x}\\ &\lesssim\sup_{0\le t\le T}\int_0^t e^{-\lambda(t-t')}(t-t')^{-\frac{1}{2}}dt'\cdot\|e^{-\lambda t}(v(\varepsilon,t)^k-v_0(t)^k)\|_{C_TH^{s-1}_x}\\ &\lesssim\frac{1}{\sqrt{\lambda}}\|v(\varepsilon,\cdot)-v_0\|_{S_{\lambda,T}}(1+\|v_0\|_{C_TH^s_x})^{k-1}(1+\|v\|_{C((0,\varepsilon_0]\times[0,T]H^s_x)})^{k-1}.\end{aligned}$$

By similar arguments, we get the bound

$$\mathrm{III}\lesssim_{v,v_0}\varepsilon^{\frac{\theta}{2}}+\|\Xi(\varepsilon)-\Xi(0)\|_{C_T(W^{-\theta}_x)^{\otimes k}}+\frac{1}{\sqrt{\lambda}}\|v(\varepsilon,\cdot)-v_0\|_{S_{\lambda,T}}. \qquad (2.114)$$

Thus, combining (2.111), (2.112), (2.113) and (2.114), we deduce the existence of $K=K(\|v_0\|_{C_TH^s_x},\|v\|_{C((0,\varepsilon_0]\times[0,T];H^{s+\theta}_x)})>0$ such that

$$\|v(\varepsilon,\cdot)-v_0\|_{S_{\lambda,T}}\le K\varepsilon^{\frac{\theta}{3}}+\|\Xi(\varepsilon)-\Xi(0)\|_{C_T(W^{-\theta}_x)^{\otimes k}}+\frac{K}{\sqrt{\lambda}}\|v_\varepsilon-v_0\|_{S_{\lambda,T}}. \qquad (2.115)$$

This leads to

$$\|v(\varepsilon,\cdot)-v_0\|_{S_{\lambda,T}}\le 2K\max\big(\varepsilon^{\frac{\theta}{3}},\|\Xi(\varepsilon)-\Xi(0)\|_{C_T(W^{-\theta}_x)^{\otimes k}}\big), \qquad (2.116)$$

upon choosing $\lambda=(2K(\varepsilon_0,T))^2$. We hence deduce (2.107) from the second inequality in (2.109) and (2.116).

□

We now prove Theorem 1.1.6.

Proof of Theorem 1.1.6. Fix $k\ge 2$. For convenience, we only prove Theorem 1.1.6 for $T=1$. Let $(\phi_0,\phi_1)\in\mathcal{H}^s(\mathbb{T}^2)$ with $\frac{2k-3}{2k-2}<s<1$. Let $0<\theta\ll 1-s$. We first fix a set of full $\mathbb{P}$-probability Ω_0 such that on Ω_0, the following conditions hold: (i) $\Xi=(\Psi_\varepsilon,:\Psi_\varepsilon^2:,\cdots,:\Psi_\varepsilon^k:)_{\varepsilon\in[0,1]}\in\mathcal{X}^\theta(\mathbb{T}^2)$ (by Proposition 2.1.10) and (ii) $v_0\in C\big([0,1];H^s(\mathbb{T}^2)\big)$ the solution to (2.77) on $\{0\}\times[0,1]$, with data given by (ϕ_0,Ξ), given by Lemma 2.2.2. For the remaining of the proof, we work on Ω_0 without any mention to it.

We fix $\varepsilon_0 = 1$. Consider the problem (2.87) with data $\varepsilon \in (0, \varepsilon_0] \mapsto (\phi_0, \mathbb{1}_{\varepsilon > 0}\phi_1, \Xi(\varepsilon)) \in \mathcal{V}^{s+\theta}_{\varepsilon_0} \times \mathcal{X}^\theta$. By Proposition 2.2.3, there exists a solution $v^{\varepsilon_0} \in C\big([0, T_0]; \mathcal{V}^s_{\varepsilon_0}(\mathbb{T}^2)\big)$ to (2.87) on $(0, \varepsilon_0] \times [0, T_0]$, $T_0 > 0$, with data $\varepsilon \in (0, \varepsilon_0] \mapsto (\phi_0, \mathbb{1}_{\varepsilon > 0}\phi_1, \Xi(\varepsilon)) \in \mathcal{V}^s_{\varepsilon_0} \times \mathcal{X}^\theta$. By Lemma 2.2.4 and the assumptions (i) and (ii), we have

$$\|v^{\varepsilon_0}\|_{C(0,\varepsilon_1] \times [0,T_0]; H^s_x)} \leq \|v_0\|_{C_{T=1} H^s_x} + 1, \tag{2.117}$$

for $\varepsilon_1 > 0$ small enough. Denote by v^{ε_1} the solution to (2.87) on $(0, \varepsilon_1] \times [0, T_0]$ with the same data. Then, we have $v^{\varepsilon_1}(\varepsilon, \cdot) = v^{\varepsilon_0}(\varepsilon, \cdot)$ for $\varepsilon \in (0, \varepsilon_1]$. Hence, by arguing as in the proof of Proposition 2.2.3, we have

$$\begin{aligned}
\|v^{\varepsilon_1}\|_{C([0,T_0]; \mathcal{V}^s_{\varepsilon_1})} &= \|v^{\varepsilon_0}\|_{C([0,T_0]; \mathcal{V}^s_{\varepsilon_1})} \\
&\lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s_x} + \big(1 + \|\Xi\|_{\mathcal{X}^\theta}\big)\big(1 + \|v_0\|_{C_{T=1} H^s_x}\big)^k,
\end{aligned} \tag{2.118}$$

We can now apply the local well-posedness statement of Proposition 2.2.3 to extend v^{ε_1} to a larger time interval $[T_0, T_1]$ whose size $T_1 - T_0$ only depends on the (fixed) constants $\|(\phi_0, \phi_1)\|_{\mathcal{H}^s_x}$, $\|\Xi\|_{\mathcal{X}^\theta}$ and $\|v_0\|_{C_{T=1} H^s_x}$. Thus, iterating this argument allows us to reach the target time $T = 1$ and provides $\varepsilon_\star > 0$ such that $v^{\varepsilon_\star} \in C\big([0, T]; \mathcal{V}^s_{\varepsilon_\star}(\mathbb{T}^2)\big)$ solves (2.87) on $(0, \varepsilon_\star] \times [0, 1]$ with data $\varepsilon \in (0, \varepsilon_\star] \mapsto (\phi_0, \mathbb{1}_{\varepsilon > 0}\phi_1, \Xi(\varepsilon))$. By Proposition 2.2.4, we obtain that $v^{\varepsilon_\star}(\varepsilon, \cdot)$ converges to v_0 on $[0, 1]$ as $\varepsilon \to 0$. Together with Proposition 2.1.10, we have that $\Psi_\varepsilon \to \Psi$ in $C\big([0, 1]; H^{-\sigma}(\mathbb{T}^2)\big)$, $\sigma > 0$ as $\varepsilon \to 0$. This proves the convergence of $u_\varepsilon = \Psi_\varepsilon + v_\varepsilon$ to $u_0 = \Psi_0 + v_0$ in $C\big([0, 1]; H^{-\sigma}(\mathbb{T}^2)\big)$, $\sigma > 0$ as $\varepsilon \to 0$. $\square$

Chapter 3

Inviscid limit for the stochastic complex Ginzburg-Landau equation

3.1 Analytic and probabilistic preliminaries

3.1.1 Fourier restriction norm and Strichartz estimates

In this chapter, we will work with the Fourier restriction norm method introduced by Bourgain [5]. Recall the definition of the twisted space-time Fourier transform $\widetilde{u}$ (1.88) of a function u.

Definition 3.1.1. Let $(s, b) \in \mathbb{R}^2$. We define the $X^{s,b}$ space as the completion of $\mathcal{S}(\mathbb{T}^2 \times \mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}(\mathbb{T}^2\times\mathbb{R})} = \|\langle n\rangle^s \langle\lambda\rangle^b \widetilde{u}(n,\lambda)\|_{\ell^2_n L^2_\lambda(\mathbb{Z}^2\times\mathbb{R})}. \tag{3.1}$$

Given an interval $J \subset \mathbb{R}$, we define the local-in-time version $X^{s,b}(J)$ as a restriction norm:

$$\|u\|_{X^{s,b}(J)} = \inf\{\|v\|_{X^{s,b}(\mathbb{T}^2\times\mathbb{R})} : v|_J = u\}. \tag{3.2}$$

When $J = [0, T]$, we set $X^{s,b}_T = X^{s,b}(J)$.

Let us note that for all $s \in \mathbb{R}$ and $b > \frac{1}{2}$, we have $X^{s,b} \hookrightarrow C(\mathbb{R}; H^s(\mathbb{T}^2))$.

Remark 3.1.2. $X^{s,b}$ spaces are usually defined with the modulation variable $\langle\lambda + |n|^2\rangle$ rather than $\langle\lambda + \langle n\rangle^2\rangle$ as in (1.88) and (3.1). This however does not modify any estimates since these two norms are equivalent.

The following lemma is the so-called L^4-Strichartz estimate and was proved by Bourgain [5].

Lemma 3.1.3. *Let Q be a spatial frequency ball (not necessarily centered at the origin). Then we have the bound*

$$\|\mathbf{P}_Q u\|_{L^4(\mathbb{T}^2\times[0,1])} \lesssim |Q|^\varepsilon \|u\|_{X^{0,\frac{1}{2}}},$$

for any $\varepsilon > 0$.

By interpolating the bound of Lemma 3.1.3 and the following inequality (obtained by Sobolev's inequality in space followed by Minkowski's inequality and Sobolev's inequality in time),

$$\|\mathbf{P}_Q u\|_{L^4(\mathbb{T}^2\times[0,1])} \lesssim |Q|^{\frac{1}{2}} \|u\|_{X^{0,\frac{1}{4}}}, \tag{3.3}$$

gives the next trilinear estimate, which is a slight variation of Bourgain's trilinear estimate [5].

Lemma 3.1.4. *Let $s > 0$, $0 < T \leq 1$, and $0 < \varepsilon \ll 1$. Let $\mathcal{N}$ be as in (1.63). We have*

$$\|\mathcal{N}(u_1, u_2, u_3)\|_{X_T^{s,-\frac{1}{2}+\varepsilon}} \lesssim \max_{\sigma\in S_3} \|u_{\sigma(1)}\|_{X_T^{s,\frac{1}{2}-\varepsilon}} \|u_{\sigma(2)}\|_{X_T^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_{\sigma(3)}\|_{X_T^{0,\frac{1}{2}-\varepsilon}}. \tag{3.4}$$

Here, S_3 denotes the group of permutations on $\{1, 2, 3\}$.

Proof. In view of the identities (1.61) and (2.34) and the fact that (3.4) with $\mathcal{N}$ replaced by either the multilinearities

$$\Big(\int_{\mathbb{T}^2} u_1\overline{u_2}dx\Big)u_3 \quad \text{or} \quad \mathcal{R}(u_1, u_2, u_3)$$

is easy to obtain, it suffices to prove the following estimate

$$\|u_1\overline{u_2}u_3\|_{X_T^{s,-\frac{1}{2}+\varepsilon}} \lesssim \max_{\sigma\in S_3} \|u_{\sigma(1)}\|_{X_T^{s,\frac{1}{2}-\varepsilon}} \|u_{\sigma(2)}\|_{X_T^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_{\sigma(3)}\|_{X_T^{0,\frac{1}{2}-\varepsilon}}. \tag{3.5}$$

Fix $T > 0$. For simplicity, we denote by u_1, u_2 and u_3 some extensions of u_1, u_2 and u_3 onto $\mathbb{R}$ which agree with u_1, u_2 and u_3 on $[0, T]$. Then, by definition of the restricted norms (3.2), (3.6) follows from the bound

$$\|\mathbb{1}_{[0,1]}(t)u_1\overline{u_2}u_3\|_{X^{s,-\frac{1}{2}+\varepsilon}} \lesssim \max_{\sigma\in S_3} \|u_{\sigma(1)}\|_{X^{s,\frac{1}{2}-\varepsilon}} \|u_{\sigma(2)}\|_{X^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_{\sigma(3)}\|_{X^{0,\frac{1}{2}-\varepsilon}}. \tag{3.6}$$

By duality, (3.6) is a consequence of the bound

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)u_1\overline{u_2}u_3 \cdot v \\ &\qquad \lesssim \max_{\sigma\in S_3} \|u_{\sigma(1)}\|_{X^{s,\frac{1}{2}-\varepsilon}} \|u_{\sigma(2)}\|_{X^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_{\sigma(3)}\|_{X^{0,\frac{1}{2}-\varepsilon}} \|v\|_{X^{-s,\frac{1}{2}-\varepsilon}}. \end{aligned} \tag{3.7}$$

Let us denote by n_1, n_2 and n_3, the respective frequencies of u_1, u_2 and u_3, respectively. We may assume that $|n_1| \geq |n_2| \geq |n_3|$. Hence, (3.7) follows from the bounds

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{\gg N_2}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3 \cdot v \\ &\qquad \lesssim N_2^{-\theta}\|u_1\|_{X^{s,\frac{1}{2}-\varepsilon}} \|u_2\|_{X^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_3\|_{X^{0,\frac{1}{2}-\varepsilon}} \|v\|_{X^{-s,\frac{1}{2}-\varepsilon}} \end{aligned} \tag{3.8}$$

for any dyadic numbers $N_2 \geq N_3 \geq 1$ and some small $\theta > 0$, and

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{N_1}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3 \cdot v \\ &\qquad \lesssim N_1^{-\theta}\|u_1\|_{X^{s,\frac{1}{2}-\varepsilon}} \|u_2\|_{X^{100\varepsilon,\frac{1}{2}-\varepsilon}} \|u_3\|_{X^{0,\frac{1}{2}-\varepsilon}} \|v\|_{X^{-s,\frac{1}{2}-\varepsilon}} \end{aligned} \tag{3.9}$$

for any dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ with $N_1 \sim N_2$ and some small $\theta > 0$.

We first prove (3.9). By Hölder's inequality, Lemma 2.45, (3.3), interpolation and the fact that the spatial frequency n of v verifies $n = n_1 - n_2 + n_3$, we have that

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{N_1}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3 \cdot v \\ &\leq \|\mathbf{P}_{N_1}u_1\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{N_2}u_2\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{N_3}u_3\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{\lesssim N_1}v\|_{L^4(\mathbb{T}^2\times[0,1])} \\ &\lesssim N_1^{10\varepsilon}\|\mathbf{P}_{N_1}u_1\|_{X^{s,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{\lesssim N_1}v\|_{X^{-s,\frac{1}{2}-\varepsilon}} \\ &\lesssim N_1^{10\varepsilon}N_2^{-100\varepsilon}\|\mathbf{P}_{N_1}u_1\|_{X^{s,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac{1}{2}-\varepsilon}}\|\mathbf{P}_{\lesssim N_1}v\|_{X^{-s,\frac{1}{2}-\varepsilon}}, \end{aligned}$$

which proves (3.9) since $N_1 \sim N_2$.

We now prove (3.8). Let $\{\Lambda\}_{\Lambda\in\mathfrak{B}}$ be a finitely overlapping family of (countable) balls of radius $\sim N_2$ which covers the set $\{\xi \in \mathbb{R}^2 : |\xi| \gg N_2\}$. Then, since the spatial frequency n of v verifies $n = n_1 - n_2 + n_3$, we have

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{\gg N_2}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3\cdot v\\&=\sum_{\Lambda\in\mathfrak{B}}\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{\gg N_2}\mathbf{P}_{\Lambda}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3\cdot \mathbf{P}_{10\Lambda}v,\end{aligned}\tag{3.10}$$

where 10Λ denotes the ball that has the same center as Λ, but with a radius dilated by a factor 10. Thus, by (3.10), Hölder inequality, Lemma 2.45, (3.3), interpolation and Cauchy-Schwarz's inequality (in Λ), we have that

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{T}^2} \mathbb{1}_{[0,1]}(t)\mathbf{P}_{\gg N_2}u_1\overline{\mathbf{P}_{N_2}u_2}\mathbf{P}_{N_3}u_3\cdot v\\&\le\sum_{\Lambda\in\mathfrak{B}}\|P_\Lambda u_1\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{N_2}u_2\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{N_3}u_3\|_{L^4(\mathbb{T}^2\times[0,1])}\|\mathbf{P}_{10\Lambda}v\|_{L^4(\mathbb{T}^2\times[0,1])}\\&\lesssim N_2^{10\varepsilon}\sum_{\Lambda\in\mathfrak{B}}\|\mathbf{P}_\Lambda u_1\|_{X^{s,\frac12-\varepsilon}}\|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac12-\varepsilon}}\|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac12-\varepsilon}}\|\mathbf{P}_{10\Lambda}v\|_{X^{-s,\frac12-\varepsilon}}\\&\lesssim N_2^{10\varepsilon}\|\mathbf{P}_\Lambda u_1\|_{X^{s,\frac12-\varepsilon}\ell^2_{\Lambda\in\mathfrak{B}}}\|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac12-\varepsilon}}\|\mathbf{P}_{N_3}u_3\|_{X^{0,\frac12-\varepsilon}}\|\mathbf{P}_{10\Lambda}v\|_{X^{-s,\frac12-\varepsilon}\ell^2_{\Lambda\in\mathfrak{B}}}\\&\lesssim N_2^{-\varepsilon}\|u_1\|_{X^{s,\frac12-\varepsilon}}\|u_2\|_{X^{0,\frac12-\varepsilon}}\|u_3\|_{X^{0,\frac12-\varepsilon}}\|v\|_{X^{-s,\frac12-\varepsilon}},\end{aligned}$$

which is acceptable. ☐

Remark 3.1.5. As opposed to Bourgain's trilinear estimate [5], we only require our inputs to be in a space of the form $X^{\alpha,b}$ for $\alpha\in\mathbb{R}$ and $b<\frac12$. This is because, in some cases, the stochastic convolution Ψ_γ (1.71) will be such an input. As Brownian motion is $\big(\frac12-\varepsilon\big)$-Hölder in time for any $\varepsilon>0$, Ψ_γ can only be placed in $X^{\alpha,b}$ for $\alpha<0$ and $b<\frac12$; see Lemma 3.4.3.

Next, we obtain some control on $X^{s,b}$-norms on a large time interval depending on the corresponding $X^{s,b}$-norms on smaller time intervals, for $b=\frac12$. See for instance [8] for a similar statelement when $b\neq\frac12$.

Lemma 3.1.6. *Let $K\in\mathbb{N}$ and $\{J_h\}_{1\le h\le K}$ be a sequence of time subintervals of $[0,1]$ such that $J_h\cap J_{h+1}\neq\emptyset$ for each $1\le h\le K-1$. Fix $s\in\mathbb{R}$. Then, we have the following bound:*

$$\|u\|_{X^{s,\frac12}(\cup_{h=1}^K J_h)}\lesssim\max_{1\le h\le K-1}\log\big(|J_h\cap J_{h+1}|^{-1}\big)\cdot\sum_{h=1}^K\|u\|_{X^{s,\frac12}(J_h)}\tag{3.11}$$

Proof. For the sake of concreteness, we prove (3.11) in the case $K=2$, $J_1=[0,2\delta]$ and $J_2=[\delta,3\delta]$ for some $0<\delta\ll1$. Let $\widetilde u_1$ and $\widetilde u_2$ be two elements in $X^{s,b}(\mathbb{T}^2\times\mathbb{R})$ such that $\widetilde u_1\equiv u$ on $[0,2\delta]$ and $\widetilde u_2\equiv u$ on $[\delta,3\delta]$. Furthermore, let χ_1 and χ_2 be two smooth and compactly supported functions such that $\chi_1\equiv1$ on $[0,2]$, $\chi_2\equiv1$ on $[\delta,3\delta]$ on $[1,3]$ and $\chi_1+\chi_2\equiv1$ on $[1,2]$. We now define $\widetilde u$ by

$$\widetilde u:=\chi_\delta^1\widetilde u_1+\chi_\delta^2\widetilde u_2.\tag{3.12}$$

Here, $\chi_\delta:=\chi(\frac{\cdot}{\delta})$ for a Schwartz function $\chi:\mathbb{R}\to\mathbb{R}$. Note that $\widetilde u\equiv u$ on $[0,3\delta]=J_1\cup J_2$. Hence, (3.11) follows from (3.12), the defintion of $X^{s,b}$-norms and the following estimate:

$$\|\chi_\delta v\|_{X^{s,\frac12}}\lesssim_\chi\log(\delta^{-1})\|v\|_{X^{s,\frac12}},\tag{3.13}$$

for any $v\in X^{s,\frac12}(\mathbb{T}^2\times\mathbb{R})$. It is easy to see that (3.13) follows from

$$\|\chi_\delta f\|_{H_t^{\frac12}}\lesssim_\chi\log(\delta^{-1})\|f\|_{H_t^{\frac12}},\tag{3.14}$$

for any $f\in H^{\frac12}(\mathbb{R})$. We have

$$\|\chi_\delta f\|_{H_t^{\frac{1}{2}}} = \Big\| \int_{\mathbb{R}} K(\lambda, \lambda_1)\widehat{f}(\lambda_1) d\lambda_1 \Big\|_{L^2_\lambda}, \tag{3.15}$$

where $K(\lambda, \lambda_1) = \delta \cdot \langle \lambda \rangle^{\frac{1}{2}} \widehat{\chi}(\delta(\lambda - \lambda_1))$.

- **Case 1:** $|\lambda| \lesssim |\lambda_1|$. The bound (3.14) (without the log-loss) follows from (3.15) and Young's inequality along with $\|\delta\widehat{\chi}(\delta\cdot)\|_{L^1_\lambda} \lesssim 1$.

- **Case 2:** $|\lambda| \gg |\lambda_1|$. If $\delta|\lambda - \lambda_1| \geq |\lambda - \lambda_1|^{\frac{1}{2}}$, then we have that $|K(\lambda, \lambda_1)| \lesssim \langle \lambda - \lambda_1 \rangle^{-10}$ and (3.14) (without the log-loss) follows again from Young's inequality. Otherwise, we have

$$|\lambda_1| \leq |\lambda - \lambda_1| < \delta^{-2}$$

Thus, in that case by applying Young's inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big\| \int_{\mathbb{R}} K(\lambda, \lambda_1)\widehat{f}(\lambda_1) d\lambda_1 \Big\|_{L^2_\lambda} &\lesssim \|\langle \cdot \rangle^{\frac{1}{2}} \delta \widehat{\chi}(\delta \cdot)\|_{L^2} \cdot \|\mathbb{1}_{|\lambda| \lesssim \delta^{-2}} \widehat{f}(\lambda)\|_{L^1} \\ &\lesssim \|\mathbb{1}_{|\lambda| \lesssim \delta^{-2}} \widehat{f}(\lambda)\|_{L^1} \lesssim \log(\delta^{-1}) \|f\|_{L^2}. \end{aligned}$$

This concludes the proof. □

We now introduce the space of multilinear operators used in Subsection 3.4.2 (Section 3.4).

Definition 3.1.7. *Given Banach spaces A_1, A_2 and A_3 we use $\mathcal{L}(A_1; A_3)$ and $\mathcal{B}(A_1 \times A_2, A_3)$ to denote the space of bounded linear and bilinear operators from A_1 to A_3 and $A_1 \times A_2$ to A_3, respectively. We also define the spaces*

$$\begin{aligned} \mathcal{L}^{s_1, s_2, b_1, b_2} &:= \bigcap_{0<T<1} \mathcal{L}\big(X^{s_1, b_1}([0,1]); X^{s_2, b_2}([0,1])\big) \\ \mathcal{B}^{s_1, s_2, b_1, b_2} &:= \bigcap_{0<T<1} \mathcal{L}\big(X^{s_1, b_1}([0,1])^2; X^{s_2, b_2}([0,1])\big) \end{aligned} \tag{3.16}$$

for any $s_1, s_2, b_1, b_2 \in \mathbb{R}$; endowed with the norms

$$\begin{aligned} \|\mathcal{M}\|_{\mathcal{L}^{s_1, s_2, b_1, b_2}} &:= \sup_{0<T<1} \|\mathcal{M}\|_{\mathcal{L}(X^{s_1, b_1}([0,T]); X^{s_2, b_2}([0,T]))} \\ \|\mathcal{T}\|_{\mathcal{B}^{s_1, s_2, b_1, b_2}} &:= \sup_{0<T<1} \|\mathcal{T}\|_{\mathcal{L}(X^{s_1, b_1}([0,T])^2; X^{s_2, b_2}([0,T]))}, \end{aligned} \tag{3.17}$$

respectively and for some small $\theta > 0$.

3.1.2 On the gauged noise

In this subsection, we fix $\gamma \in (0, 1]$ and $N \in \mathbb{N}$ and study the law of the noise $\mathfrak{X}^{\gamma,N} = \mathfrak{X}^{\gamma,N}(\phi, \xi)$ (1.60). We note that we have $\mathfrak{X}^{\gamma,N} = d\mathfrak{W}^{\gamma,N}$ where $\mathfrak{W}^{\gamma,N}$ is the process given by the equation

$$d\mathfrak{W}^{\gamma,N} = e^{iV_N(u_{\gamma,N})} dW. \tag{3.18}$$

Here, V_N is given by (1.57), $u_{\gamma,N}$ is the solution to (1.52) with initial data given by $\phi \sim \rho$ with ρ as in (1.48) and W is the cylindrical Wiener process (1.68). Hence, we may write $\mathfrak{W}^{\gamma,N}$ as

$$\mathfrak{W}^{\gamma,N} = \sum_{n \in \mathbb{Z}^2} \mathfrak{B}_n^{\gamma,N} e^{in\cdot x}, \tag{3.19}$$

where $\mathfrak{B}_n^{\gamma,N}$ is given by

$$d\mathfrak{B}_n^{\gamma,N} = e^{iV_N(u_{\gamma,N})} dB_n, \quad n \in \mathbb{Z}^2,$$

with B_n as in (1.68).

The main goal of this subsection is to prove the following proposition.

Proposition 3.1.8. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Let ϕ be a random variable variable distributed according to the renormalized Gibbs measure ρ (1.48). Then, the noise $\mathfrak{X}^{\gamma,N}(\phi,\xi)$ (1.60) is a space-time white noise independent from ϕ.*

The proposition easily follows from the following lemma.

Lemma 3.1.9. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Let $\phi_\star \in \mathcal{D}'(\mathbb{T}^2)$ be any distribution. The noise $\mathfrak{X}^{\gamma,N}(\phi_\star,\xi)$ is a space-time white noise. Here, $\mathfrak{X}^{\gamma,N}(\phi_\star,\xi)$ is as in (3.18), but where $u_{\gamma,N}$ is the solution to (1.52) with initial data $\phi_\star$.*

Proof of Proposition 3.1.8. Fix $F \in C_b\big(\mathcal{D}'(\mathbb{T}^2)\times\mathcal{D}'(\mathbb{R}\times\mathbb{T}^2);\mathbb{C}\big)$.[1] By Lemma 3.1.9, we have

$$\begin{aligned}\mathbb{E}\big[F\big(\phi,\mathfrak{X}^{\gamma,N}(\phi,\xi)\big)\big] &= \mathbb{E}\Big[\mathbb{E}\big[F\big(\phi,\mathfrak{X}^{\gamma,N}(\phi,\xi)\big)\,\big|\,\phi\big]\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\big[F\big(\phi,\xi\big)\,\big|\,\phi\big]\Big] = \mathbb{E}\big[F\big(\phi,\xi\big)\big].\end{aligned} \tag{3.20}$$

Hence, $\mathrm{Law}\big(\phi,\mathfrak{X}^{\gamma,N}(\phi,\xi)\big) = \mathrm{Law}\big(\phi,\xi\big) = \mathrm{Law}(\phi)\otimes\mathrm{Law}(\xi) = \mathrm{Law}(\phi)\otimes\mathrm{Law}(\mathfrak{X}^{\gamma,N})$, by the independence properties of ϕ and ξ. □

The rest of the subsection is devoted to the proof of Lemma 3.1.9. We first recall some notations from Stochastic Analysis; see [41]. Given two complex-valued stochastic processes $\{X_t\}_t$ and $\{Y_t\}_t$, we define their quadratic covariation $[X,Y]$ as the process

$$[X,Y]_t := \lim_{\|P\|\to 0}\sum_{k=1}^{n}\big(X_{t_k}-X_{t_{k-1}}\big)\big(\overline{Y}_{t_k}-\overline{Y}_{t_{k-1}}\big). \tag{3.21}$$

The sum in (3.21) runs over partitions $P = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, $n \in \mathbb{N}$, of $[0,t]$ whose mesh size[2] $\|P\|$ tends to 0.

We have the following characterization of multivariate complex-valued Brownian motions.

Lemma 3.1.10. *Fix an integer $d \geq 1$. Let $X = (X^1,\cdots,X^d)$ be an adapted process with continuous sample paths defined on a filtered probability space $(\Omega,\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$. The following are equivalent:*

(i) *X is a d-dimensional complex-valued Brownian motion.*

(ii) *The processes $X^1,\cdots,X^d$ are continuous local martingales and $[X^i,X^j] = \mathbb{1}_{i=j}\cdot t$ for every $i,j \in \{1,\cdots,d\}$.*

Proof. The proof is a straightforward adaptation of the proof of [41, Theorem 5.2] to the current complex-valued setting. □

Lemma 3.1.11. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Let $\phi_\star \in \mathcal{D}'(\mathbb{T}^2)$ be any distribution. Let $\{\mathfrak{B}_n^{\gamma,N}(\phi_\star,\xi)\}_{n\in\mathbb{Z}^2}$ be as in (3.19). Then, for each $M \in \mathbb{N}$, the process $\{\mathfrak{B}_n^{\gamma,N}(\phi_\star,\xi) : |n| \leq M\}$ is a $(2M+1)$-dimensional complex-valued Brownian motion. Here, $\{\mathfrak{B}_n^{\gamma,N}(\phi_\star,\xi)\}_{n\in\mathbb{Z}^2}$ is as in (3.18) and (3.19), but where $u_{\gamma,N}$ is the solution to (1.52) with initial data $\phi_\star$.*

Note that by Lemma 3.2.7 below, the function $u_{\gamma,N}$ exists globally in time.

Proof. This is a direct consequence of Ito's formula. □

By passing to the limit $M \to \infty$, Lemma 3.1.11 shows that $\mathfrak{M}^{\gamma,N}(\phi_\star,\xi)$ is a cylindrical Wiener process on $L^2(\mathbb{T}^2)$ and hence proves Proposition 3.1.8.

---

[1] Here, $C_b(X;\mathbb{C})$ denotes the space of continuous and bounded functions from the topological space X to $\mathbb{C}$.

[2] That is, the largest distance between two consecutive elements in P.

3.1.3 Linear estimates

The aim of this subsection is to provide estimates on the linear operators S_γ and $\mathcal{I}_\gamma$ defined in (1.69) and (1.70), respectively.

We first state the following linear homogeneous estimate.

Lemma 3.1.12. *Fix $0 < T \le 1$ and $s, b \in \mathbb{R}$. We have the following bounds:*

$$\|S_\gamma(t) f\|_{H^s_x} \lesssim \|f\|_{H^s_x}, \tag{3.22}$$

and

$$\|S_\gamma(t) f\|_{X^{s,b}_T} \lesssim \langle\gamma\rangle^{b-\frac{1}{2}} \cdot \|f\|_{H^{s+2b-1}_x}, \tag{3.23}$$

$$\|S_0(t) f\|_{X^{s,b}_T} \lesssim \|f\|_{H^s_x}, \tag{3.24}$$

for any $\gamma \in (0,1]$.

Proof. The bound (3.22) follows immediately from the bound $\left|e^{(\gamma|t|+it)\langle n\rangle^2}\right| \le 1$ for any $t \in [0,T]$ and the dominated convergence theorem. The estimate (3.24) is standard and can be found in [70]. Lastly, the proof of (3.23) can essentially be found in [47, Proposition 2.1]. ☐

The following lemma shows that we can gain a small time power at the expense of derivatives in time; see [25, Proposition 2.7] for a proof.

Lemma 3.1.13. *Let $s \in \mathbb{R}$. Fix φ a Schwartz function and $0 < T \le 1$. Then we have*

$$\|\varphi(t/T) u\|_{X^{s,b_1}} \lesssim T^{b_2-b_1} \|u\|_{X^{s,b_2}},$$

for any $b_2 > b_1 > \frac{1}{2}$ and $u \in X^{s,b_1}$ such that $u(0) = 0$.

The next proposition gives bounds on the Duhamel operator $\mathcal{I}_\gamma$, $\gamma \in [0,1]$ (1.70).

Proposition 3.1.14. *Let $s \in \mathbb{R}$. Fix $0 < \varepsilon, \delta \ll 1$ with $\varepsilon \ll \delta^2$ and $0 < T \le 1$. Then, we have the bounds*

$$\left\|\mathcal{I}_\gamma(F)\right\|_{X^{s,\frac{1}{2}+\varepsilon}_T} \lesssim \|F\|_{X^{s,-\frac{1}{2}+\delta}_T}, \tag{3.25}$$

$$\left\|\mathcal{I}_\gamma(F)\right\|_{X^{s,\frac{1}{2}+\varepsilon}_T} \lesssim \gamma^{-(\frac{1}{2}-\delta)} \cdot \|F\|_{L^2_T H^{s-1+2\delta}_x}, \tag{3.26}$$

for any $\gamma \in [0,1]$.

The bound (3.25) is the natural generalization of the standard nonhomogeneous estimate in $X^{s,b}$-spaces (see for instance [70]) to the setting of our convergence problem for which we need to prove bounds uniformly in the parameter $\gamma \in [0,1]$, while (3.26) captures the parabolic smoothing effects of $\mathcal{I}_\gamma$ (1.70) in $X^{s,b}$-spaces. In particular, (3.26) is crucial in the proof of Lemma 3.4.13 and in handling the nonlinear term $\mathfrak{S}^{\mathrm{Wick}}_{\gamma,N}$ (1.79) uniformly in $\gamma \in (0,1]$.

We recall the following technical lemma.

Lemma 3.1.15. *Let $0 \le \alpha, \beta$ such that $\alpha + \beta > 1$. Fix $\mu \in \mathbb{R}$. Then, we have*

$$\int_{\mathbb{R}} \frac{d\lambda}{\langle\lambda\rangle^\alpha \langle\mu-\lambda\rangle^\beta} \lesssim \frac{1}{\langle\mu\rangle^\sigma}$$

with

$$\sigma = \begin{cases} \alpha+\beta-1, & \text{if } \beta < 1 \\ \alpha-\varepsilon, & \text{if } \beta = 1 \\ \alpha, & \text{if } \beta > 1, \end{cases}$$

for any $\varepsilon > 0$.

The next two lemmas essentially reduce the proof of Proposition 3.1.14 to proving bounds on kernels of certain integral operators.

Lemma 3.1.16. *Let T be a linear integral operator with integral kernel K defined on the (time-)frequency side by*

$$\widehat{T(F)}(\lambda) = \int_{\mathbb{R}} \widehat{F}(\mu) K(\lambda, \mu) d\mu,$$

such that the kernel K satisfies the following estimate:

$$|K(\lambda, \mu)| \lesssim \frac{1}{\langle \mu \rangle} \Big(\frac{1}{\langle \lambda \rangle^2} + \frac{1}{\langle \lambda - \mu \rangle^2} \Big), \tag{3.27}$$

for any $(\lambda, \mu) \in \mathbb{R}^2$. Then, the following bound holds:

$$\|T(F)\|_{H^b} \lesssim \|F\|_{H^{-b'}}, \tag{3.28}$$

for any $0 \leq b' < \frac{1}{2}$, $0 \leq b \leq 1$ with $b + b' \leq 1$.

Proof. Fix b and b' as in the above. Let $\widetilde{T}$ be the linear integral operator with kernel given (on the time-Fourier side) by

$$\begin{aligned} \widetilde{K}(\lambda, \mu) &= \frac{\langle \lambda \rangle^b \langle \mu \rangle^{b'}}{\langle \mu \rangle} \Big(\frac{1}{\langle \lambda \rangle^2} + \frac{1}{\langle \lambda - \mu \rangle^2} \Big) \\ &=: \widetilde{K}^1(\lambda, \mu) + \widetilde{K}^2(\lambda, \mu). \end{aligned}$$

Let us note that $H^{-b'} \to H^b$ bounds for T follow from $L^2 \to L^2$ bounds for $\widetilde{T}$, which we prove now.

Next, denote by $\widetilde{T}^1$ and $\widetilde{T}^2$ the linear integral operators whose kernels are given (on the time-Fourier side) by $\widetilde{K}^1$ and $\widetilde{K}^2$ respectively. To conclude the proof, it thus suffices to prove the following bounds:

$$\big\|\widetilde{T}^1(F)\big\|_{L^2} \lesssim \|F\|_{L^2}, \tag{3.29}$$

$$\big\|\widetilde{T}^2(F)\big\|_{L^2} \lesssim \|F\|_{L^2}. \tag{3.30}$$

From the Cauchy-Schwarz inequality and Plancherel's identity , we have

$$\begin{aligned} \big\|\widetilde{T}^1(F)\big\|_{L^2} &\lesssim \|\widetilde{K}_1(\lambda, \mu)\|_{L^2_{\lambda,\mu}} \|F\|_{L^2} \\ &\lesssim_{b,b'} \|F\|_{L^2}. \end{aligned} \tag{3.31}$$

This proves (3.29). We now bound $\|\widetilde{T}^2\|_{L^2 \to L^2}$. We further decompose the kernel $\widetilde{K}^2$ in the following way:

$$\begin{aligned} \widetilde{K}_2(\lambda, \mu) &= \widetilde{K}_2(\lambda, \mu) \mathbb{1}_{|\lambda| \lesssim |\mu|} + \widetilde{K}_2(\lambda, \mu) \mathbb{1}_{|\lambda| \gg |\mu|} \\ &=: \widetilde{K}^{2,<}(\lambda, \mu) + \widetilde{K}^{2,>}(\lambda, \mu). \end{aligned}$$

Let us again denote by $\widetilde{T}^{2,<}$ and $\widetilde{T}^{2,>}$ the associated operators. From the bound $\widetilde{K}^{2,>}(\lambda, \mu) \lesssim \widetilde{K}^1(\lambda, \mu)$, we deduce, as in (3.31), that $\widetilde{T}^{2,>}$ is bounded from L^2_μ to L^2_λ. Next, regarding $T^{2,<}$, by Plancherel's identity, Young's inequality and the condition $|\lambda| \lesssim |\mu|$, we have

$$\begin{aligned} \big\|\widetilde{T}^2(F)\big\|_{L^2} &\lesssim \Big\| \int_{\mathbb{R}} \langle \mu \rangle^{b+b'-1} \langle \lambda - \mu \rangle^{-2} |\widehat{F}(\mu)| d\mu \Big\|_{L^2_\lambda} \\ &\lesssim \Big\| \int_{\mathbb{R}} \langle \lambda - \mu \rangle^{-2} |\widehat{F}(\mu)| d\mu \Big\|_{L^2_\lambda} \lesssim \|F\|_{L^2}, \end{aligned}$$

which proves (3.30) and concludes the proof. □

Lemma 3.1.17. *Let $\{T_a\}_{a\in\mathbb{R}_+}$ be a family of linear integral operators with integral kernels $\{K_a\}_{a\in\mathbb{R}_+}$ defined on the (time-)frequency side by*

$$\widehat{T_a(F)}(\lambda) = \int_{\mathbb{R}} \widehat{F}(\mu) K_a(\lambda,\mu) d\mu.$$

We assume that the kernels $\{K_a\}_{a\in\mathbb{R}_+}$ satisfy the following bounds:

$$|K_a(\lambda,\mu)| \lesssim \frac{1}{\langle a+i\mu\rangle} \min\Big(\frac{1}{\langle\lambda\rangle} + \frac{1}{\langle\lambda-\mu\rangle}, \frac{\langle a\rangle}{\langle\lambda\rangle^2} + \frac{\langle a\rangle}{\langle\lambda-\mu\rangle^2}\Big), \tag{3.32}$$

for any $(\lambda,\mu)\in\mathbb{R}^2$, $a\in\mathbb{R}_+$ and

$$|K_{a_2}(\lambda,\mu) - K_{a_1}(\lambda,\mu)| \lesssim |a_2 - a_1|, \tag{3.33}$$

for any $(\lambda,\mu)\in\mathbb{R}^2$ and $(a_1,a_2)\in(\mathbb{R}_+)^2$. Then, the following bounds hold:

(i) *Let $0\le b<\frac{1}{2}$ and $0\le b'<\frac{1}{2}$. Then, we have*

$$\|T_a(F)\|_{H^b} \lesssim \langle a\rangle^{b'-\frac{1}{2}} \|F\|_{H^{-b'}},$$

for any $\varepsilon>0$ and $a\in\mathbb{R}_+$.

(ii) *Let $0<b<\frac{1}{2}$. Then, we have*

$$\|T_a(F)\|_{H^b} \lesssim \langle a\rangle^{-\frac{1}{2}} \|F\|_{H^\varepsilon},$$

for any $\varepsilon>0$ and $a\in\mathbb{R}_+$.

(iii) *Let $0\le b'<\frac{1}{2}$ and $0\le b\le 1$ with $b+b'\le 1$. Then, we have*

$$\|T_a(F)\|_{H^b} \lesssim \langle a\rangle \|F\|_{H^{-b'}},$$

for any $a\in\mathbb{R}_+$.

(iv) *Let $(a_1,a_2)\in(\mathbb{R}_+)^2$. Then, we have*

$$\|(T_{a_2}-T_{a_1})(F)\|_{H^{-1}} \lesssim |a_2-a_1|\, \|F\|_{H^1},$$

for any $(a_1,a_2)\in(\mathbb{R}_+)^2$.

(v) *Fix any $\varepsilon,\delta>0$ with $\varepsilon\ll\delta^2$. Then, we have*

$$\begin{aligned}
\|T_a(F)\|_{H^{\frac{1}{2}+\varepsilon}} &\lesssim \langle a\rangle^{-\frac{\delta}{2}} \|F\|_{H^{-\frac{1}{2}+\delta}},\\
\|T_a(F)\|_{H^{\frac{1}{2}+\varepsilon}} &\lesssim \langle a\rangle^{-\frac{1}{2}+\delta} \|F\|_{L^2},\\
\|(T_{a_2}-T_{a_1})(F)\|_{H^{\frac{1}{2}+\varepsilon}} &\lesssim |a_2-a_1|^\varepsilon\, \|F\|_{H^{-\frac{1}{2}+\delta}}.
\end{aligned}$$

for any $(a,a_1,a_2)\in(\mathbb{R}_+)^3$.

Proof. The bound (iv) follows from (3.33) and the Cauchy-Schwarz inequality. Note that we have

$$\langle a+i\mu\rangle \ge \max(\langle\mu\rangle, \langle a\rangle). \tag{3.34}$$

Hence, (iii) is a consequence of (3.34), (3.32) and Lemma 3.1.16. Moreover, (v) follows by interpolating (i), (ii), (iii) and (iv).

We now prove (i). Let $0\le b,b'<\frac{1}{2}$. From (3.32), we have the following decomposition:

$$\langle\lambda\rangle^b\langle\mu\rangle^{b'}|K_a(\lambda,\mu)| \lesssim \frac{\langle\lambda\rangle^b\langle\mu\rangle^{b'}}{\langle a+i\mu\rangle\langle\lambda\rangle} + \frac{\langle\lambda\rangle^b\langle\mu\rangle^{b'}}{\langle a+i\mu\rangle\langle\lambda-\mu\rangle} =: \widetilde{K}^1_a(\lambda,\mu) + \widetilde{K}^2_a(\lambda,\mu).$$

Let $\widetilde{T}_a^1$ and $\widetilde{T}_a^2$ be the linear operators with kernels $\widetilde{K}_a^1$ and $\widetilde{K}_a^2$, respectively. As in the proof of Lemma 3.1.16, proving (i) reduces to establishing similar $L^2 \to L^2$ bounds for $\widetilde{T}_a^1$ and $\widetilde{T}_a^2$. Let us first consider $\widetilde{T}_a^1$. We have

$$\|\widetilde{K}_a^1\|^2_{L^2_\lambda L^2_\mu} = \int_{\mathbb{R}^2} \frac{\langle\lambda\rangle^{2b}\langle\mu\rangle^{2b'}}{\langle a+i\mu\rangle^2\langle\lambda\rangle^2} d\lambda d\mu \lesssim \int_{\mathbb{R}} \frac{\langle\mu\rangle^{2b'}}{1+a^2+\mu^2} d\mu. \tag{3.35}$$

If $|a| \le 1$, then we have (3.35) $\lesssim 1$. Otherwise, by using a change of variable, we get

$$(3.35) \lesssim \frac{1}{|a|}\int_{\mathbb{R}} \frac{\langle a\mu\rangle^{2b'}}{\frac{1}{a^2}+1+\mu^2} d\mu \lesssim |a|^{2b'-1}\int_{\mathbb{R}} \frac{1}{\langle\mu\rangle^{2-2b'}} d\mu \lesssim \langle a\rangle^{2b'-1}.$$

Hence, we have

$$\big\|\widetilde{T}^1\big\|_{L^2\to L^2} \lesssim \langle a\rangle^{b'-\frac{1}{2}}, \tag{3.36}$$

by Cauchy-Schwarz inequality.

In bounding $\widetilde{T}_a^2$, we may assume the additional condition $|\lambda| \lesssim |\mu|$ as in the proof of Lemma 3.1.16. By using Plancherel's indentity, (3.34), and the condition $0 \le b, b' < \frac{1}{2}$ along with Young's inequality, we have

$$\begin{aligned}
\big\|\widetilde{T}_a^2(F)\big\|_{L^2} &\lesssim \bigg\| \int_{\mathbb{R}} \frac{\mathbb{1}_{|\lambda|\lesssim|\mu|}}{\langle a+i\mu\rangle^{1-b-b'}\langle\lambda-\mu\rangle}|\widehat{F}(\mu)|d\mu\bigg\|_{L^2_\lambda} \\
&\lesssim \langle a\rangle^{b+b'-1+\varepsilon}\bigg\| \int_{\mathbb{R}} \frac{\mathbb{1}_{|\lambda|\lesssim|\mu|}}{\langle\mu\rangle^{-\varepsilon}\langle\lambda-\mu\rangle}|\widehat{F}(\mu)|d\mu\bigg\|_{L^2_\lambda} \\
&\lesssim \langle a\rangle^{b+b'-1+\varepsilon}\bigg\| \int_{\mathbb{R}} \frac{\mathbb{1}_{|\lambda|\lesssim|\mu|}}{\langle\lambda-\mu\rangle^{1+\varepsilon}}|\widehat{F}(\mu)|d\mu\bigg\|_{L^2_\lambda} \lesssim \langle a\rangle^{b+b'-1+\varepsilon}\|F\|_{L^2},
\end{aligned} \tag{3.37}$$

for any $\varepsilon > 0$. Note that under the assumption $b < \frac{1}{2}$, we have

$$b + b' - 1 + \varepsilon < b' - \frac{1}{2}$$

for $\varepsilon > 0$ small enough. Hence, by (3.37), we have

$$\big\|\widetilde{T}^2\big\|_{L^2\to L^2} \lesssim \langle a\rangle^{b'-\frac{1}{2}} \tag{3.38}$$

Combining (3.36) and (3.38) proves (i). Lastly, from (3.32) and (3.34), we get

$$|K_a(\lambda,\mu)| \lesssim \frac{\langle a\rangle^{-\frac{1}{2}+\varepsilon}}{\langle\mu\rangle^{\frac{1}{2}+\varepsilon}}\Big(\frac{1}{\langle\lambda\rangle} + \frac{1}{\langle\lambda-\mu\rangle}\Big),$$

for any $\varepsilon > 0$ and $a \in \mathbb{R}_+$. Making use of the above inequality and arguing as in the proof of Lemma 3.1.16 yields (ii).

□

We now prove Proposition 3.1.14.

Proof of Proposition 3.1.14. We only prove (3.25) as (3.26) follows from similar arguments. Fix $0 < T \le 1$ and $0 < \varepsilon, \delta \ll 1$ with $\varepsilon^2 \ll \delta$.

Let $\varphi = \varphi(t)$ be a smooth function such that $\varphi \equiv 1$ on $[-1,1]$ and $\varphi \equiv 0$ on $[-2,2]$. Let $F \in X_T^{s,-\frac{1}{2}+\delta}$ and fix G, an extension of F to $\mathbb{R}$, such that $G_{|[0,T]} = F_{|[0,T]}$. Then, $\varphi\mathcal{I}_\gamma(G)$ is an extension of $\varphi\mathcal{I}_\gamma(F)$ to $\mathbb{R}$ which agrees with $\varphi\mathcal{I}_\gamma(F)$ on $[0,T]$. From (1.70), we get

$$\widetilde{\varphi\mathcal{I}_\gamma(G)}(n,\lambda) = \int_{\mathbb{R}} \widetilde{G}(n,\mu)K_\gamma(n,\lambda,\mu)d\mu$$

with

$$K_\gamma(n,\lambda,\mu) := \int_{\mathbb{R}} \varphi(t) e^{-it\lambda} \frac{e^{\gamma(t-|t|)\langle n\rangle^2 + it\mu} - e^{-\gamma|t|\langle n\rangle^2}}{\gamma\langle n\rangle^2 + i\mu} dt$$

Note that we have

$$\|\varphi\mathcal{I}_\gamma(F)\|_{X_T^{s,\frac12+\varepsilon}} \le \|\varphi\mathcal{I}_\gamma(G)\|_{X^{s,\frac12+\varepsilon}} = \Big\|\langle n\rangle^s \big\|\langle\lambda\rangle^b \widetilde{\varphi\mathcal{I}_\gamma(G)}(n,\lambda)\big\|_{L^2_\lambda}\Big\|_{\ell^2_n} \tag{3.39}$$

Thus, (3.25) follows from appropriate bounds on $\big\|\langle\lambda\rangle^b\widetilde{\varphi\mathcal{I}_\gamma(F)}(n,\lambda)\big\|_{L^2_\lambda}$ with $n\in\mathbb{Z}^2$ fixed. From Lemma 3.1.17, it suffices to obtain appropriate bounds on $K_\gamma(n,\lambda,\mu)$. Using integration by parts, we get

$$|K_\gamma(n,\lambda,\mu)| \lesssim \frac{1}{\langle\gamma\langle n\rangle^2 + i\mu\rangle} \min\Big(\frac{1}{\langle\lambda\rangle} + \frac{1}{\langle\lambda-\mu\rangle}, \frac{\langle\gamma\langle n\rangle^2\rangle}{\langle\lambda\rangle^2} + \frac{\langle\gamma\langle n\rangle^2\rangle}{\langle\lambda-\mu\rangle^2}\Big) \tag{3.40}$$

for any $\gamma\in[0,1]$. Furthermore, by the mean value theorem, we have

$$|K_{\gamma_2}(n,\lambda,\mu) - K_{\gamma_1}(n,\lambda,\mu)| \lesssim |\gamma_2-\gamma_1|\langle n\rangle^2, \tag{3.41}$$

for any $(\gamma_1,\gamma_2)\in[0,1]^2$. The conditions (3.40), (3.41) correspond to (3.32) and (3.33) in Lemma 3.1.17. Hence, from Lemma 3.1.17 (iv), we deduce

$$\big\|\langle\lambda\rangle^{\frac12+\varepsilon}\widetilde{\varphi\mathcal{I}_\gamma(G)}(n,\lambda)\big\|_{L^2_\lambda} \lesssim \big\|\langle\mu\rangle^{-\frac12+\delta}\widetilde{G}(n,\mu)\big\|_{L^2_\mu}, \tag{3.42}$$

$$\big\|\langle\lambda\rangle^{\frac12+\varepsilon}\big(\widetilde{\varphi\mathcal{I}_{\gamma_2}} - \widetilde{\varphi\mathcal{I}_{\gamma_2}}\big)(G)(n,\lambda)\big\|_{L^2_\lambda} \lesssim \min\big(1, |\gamma_2-\gamma_1|^\varepsilon\langle n\rangle^{2\varepsilon}\big)\,\big\|\langle\mu\rangle^{-\frac12+\delta}\widetilde{G}(n,\mu)\big\|_{L^2_\mu}. \tag{3.43}$$

Thus, by combining (3.39) and (3.42), we get (3.25) by definition of the $X_T^{s,b}$-restriction norm. □

As a consequence of Lemma 3.1.12, Proposition 3.1.14 and the dominated convergence theorem, we deduce the following result.

Lemma 3.1.18. *Fix $s\in\mathbb{R}$, $0<\varepsilon\ll 1$ and $T>0$. Let $\phi\in H^s(\mathbb{T}^2)$ and $F\in X^{s,\frac12+\varepsilon}([0,T])$. Then, we have the following convergences:*

$$\begin{aligned}\|(S_\gamma - S_0)\phi\|_{C_T H^s_x} &\longrightarrow 0,\\ \|(\mathcal{I}_\gamma - \mathcal{I}_0)F\|_{X_T^{s,\frac12+\varepsilon}} &\longrightarrow 0,\end{aligned}$$

as $\gamma\to 0$.

3.1.4 Resonant estimates

The aim of this subsection is to prove a deterministic estimates to handle the resonant nonlinearity $\mathcal{R}$ (1.64).

Lemma 3.1.19. *Let $s\ge 0$ and $T>0$. We have the following estimate:*

$$\big\|\mathcal{R}(u_1,u_2,u_3)\big\|_{C_T H^s_x} \lesssim \min_{\sigma\in S_3} \|u_{\sigma(1)}\|_{C_T H^{-s}_x} \|u_{\sigma(2)}\|_{C_T \mathcal{F}L^{s,\infty}_x} \|u_{\sigma(3)}\|_{C_T \mathcal{F}L^{s,\infty}_x}.$$

Here, S_3 is the group of permutations on $\{1,2,3\}$.

Proof. Fix $s\ge 0$. By Hölder's inequality, we have

$$\begin{aligned}\|\mathcal{R}(u_1,u_2,u_3)\|_{C_T H^s} &= \Big\|\big\|\langle n\rangle^s \widehat{u_1}(t,n)\overline{\widehat{u_2}(t,n)}\widehat{u_3}(t,n)\big\|_{\ell^2_n}\Big\|_{L^\infty_T}\\ &\le \min_{\sigma\in S_3} \big\|\langle n\rangle^{-s}\widehat{u_{\sigma(1)}}(t,n)\big\|_{L^\infty_T \ell^2_n}\\ &\qquad \times \|\langle n\rangle^s \widehat{u_{\sigma(2)}}(t,n)\|_{L^\infty_T\ell^\infty_n} \|\langle n\rangle^s\widehat{u_{\sigma(3)}}(t,n)\|_{L^\infty_T\ell^\infty_n}.\end{aligned}$$

This concludes the proof.

□

3.2 Proof of Theorem 1.2.2

In this section, we present the proof of Theorem 1.2.2. We provide in 3.2.1 two abstract local-in-time well-posedness arguments. In subsection 3.2.2 below, we discuss globalization and convergence considerations and then proceed with the proof of Theorem 1.2.2.

3.2.1 Local theory

Here, we prove two local well-posedness results for the nonlinear remainders corresponding to the problems (1.52) and (1.65), respectively. See (3.61) and (3.62) below. Throughout this subsection, we introduce the *enhanced data set* $\mathbb{X}$ and $\mathbf{X}$ given by

$$\mathbb{X} = \big(\overset{\bullet}{|},\ :|\overset{\bullet}{|}|^2:,\ \overset{\bullet}{|}^2,\ :|\overset{\bullet}{|}|^2\overset{\bullet}{|}:\ \big), \tag{3.44}$$

$$\mathbf{X} = \big(\overset{\bullet}{|},\ :|\overset{\bullet}{|}|^2:,\ \overset{\bullet}{|}^2,\ :|\overset{\bullet}{|}|^2\overset{\bullet}{|}:,\ \overset{\bullet\bullet\bullet}{Y},\ \mathcal{M}^1,\ \mathcal{M}^2,\ \mathcal{T}^1,\ \mathcal{T}^2\big), \tag{3.45}$$

where, we have

- $\gamma \mapsto \overset{\bullet}{|}_\gamma$ is a distribution-valued family of functions belonging to the space $C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\cap \mathcal{F}L^{1-\delta,\infty}(\mathbb{T}^2)\big) \bigcap C\big([0,1]; X^{-\delta,\frac{1}{2}-\delta}([0,1])\big)$,
- $\gamma \mapsto |\overset{\bullet}{|}_\gamma|^2:$, $\gamma \mapsto \overset{\bullet}{|}^2_\gamma$ and $\gamma \mapsto :|\overset{\bullet}{|}_\gamma|^2\overset{\bullet}{|}_\gamma:$ are distribution-valued families of functions belonging to $C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\big)$,
- $\gamma \mapsto \overset{\bullet\bullet\bullet}{Y}_\gamma$ is a distribution-valued family of function belonging to $C\big([0,1]; X^{s,\frac{1}{2}+\delta}([0,1])\big)$,
- the family of operators $\gamma \mapsto \mathcal{M}^1_\gamma$ and $\gamma \mapsto \mathcal{M}^2_\gamma$ belong to the class $C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ defined in (3.16),
- the family of operators $\gamma \mapsto \mathcal{T}^1_\gamma$ and $\gamma \mapsto \mathcal{T}^2_\gamma$ belong to the class $C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ defined in (3.16),

for some parameters $0 < s < \frac{1}{4}$, $0 < \delta = \delta(s) \ll 1$. Let $\mathcal{Y}^s$ and $\mathcal{Z}^\delta$ be the sets

$$\mathcal{Z}^\delta = C\big([0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\big)^4. \tag{3.46}$$

and

$$\begin{aligned}\mathcal{Y}^s = &\big(C\big([0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\cap \mathcal{F}L^{1-\delta,\infty}(\mathbb{T}^2)\big)\cap X^{-\delta,\frac{1}{2}-\delta}([0,1])\big)^2\\ &\times C\big([0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\big)^3 \times X^{s,\frac{1}{2}+\delta}([0,1])\\ &\times \big(\mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)^2 \times \big(\mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)^2,\end{aligned} \tag{3.47}$$

With this notation, and the assumption on the regularity of the stochastic terms above, we have $\mathbb{X} \in C\big([0,1]; \mathcal{Z}^\delta\big)$ and $\mathbf{X} \in C\big([0,1]; \mathcal{Y}^s\big)$.

Given $\mathbb{X}$ and $\mathbf{X}$ as in (3.44) and (3.45), we define the multilinear expressions $\mathfrak{S}^{\text{Wick}}_{\mathbb{X}} = \mathfrak{S}^{\text{Wick}}_{\mathbf{X}}$ and $\mathfrak{S}^{\text{PDE}} = \mathfrak{S}^{\text{PDE}}_{\mathbf{X}}$ by

$$\mathfrak{S}^{\text{Wick}}(\mathfrak{v}) = \mathcal{I}_\gamma\Big(|\mathfrak{v}|^2\mathfrak{v} + :|\overset{\bullet}{|}|^2\overset{\bullet}{|}: + 2:|\overset{\bullet}{|}|^2:\mathfrak{v} + (\overset{\bullet}{|})^2\,\mathfrak{v} + 2\,\overset{\bullet}{|}|\mathfrak{v}|^2 + \overline{\overset{\bullet}{|}}\,\mathfrak{v}^2\Big), \tag{3.48}$$

and

$$\mathfrak{S}^{\text{PDE}}(\mathfrak{v}) = \mathcal{I}_\gamma\mathcal{N}(\mathfrak{v}) + \mathcal{I}_\gamma\mathcal{R}\big(\overset{\bullet}{|} + \mathfrak{v}\big) + \overset{\bullet\bullet\bullet}{Y} + 2\mathcal{M}^1(\mathfrak{v}) + \mathcal{M}^2(\mathfrak{v}) + 2\mathcal{T}^1(\mathfrak{v},\mathfrak{v}) + \mathcal{T}^2(\mathfrak{v},\mathfrak{v}), \tag{3.49}$$

where the nonlinearities $\mathcal{N}$ and $\mathcal{R}$ are as in (1.63) and (1.64), respectively.

Let $u_{\gamma,N}$ and $\mathfrak{u}_{\gamma,N}$ be the solutions to (1.52) and (1.65), respectively. Then the nonlinear remainders $v_{\gamma,N} = \overset{\bullet}{|}_{\gamma} - u_{\gamma,N}$ and $\mathfrak{v}_{\gamma,N} = \overset{\bullet}{|}_{\gamma,N} - \mathfrak{u}_{\gamma,N}$ essentially solve equations of the form

$$v_\gamma = S_\gamma(t)\phi - (\gamma + i)\mathfrak{S}^{\mathrm{Wick}}_\gamma(v_\gamma), \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{3.50}$$

and

$$\mathfrak{v}_\gamma = S_\gamma(t)\phi - \gamma\mathfrak{S}^{\mathrm{Wick}}_\gamma(\mathfrak{v}_\gamma) - i\mathfrak{S}^{\mathrm{PDE}}_\gamma(\mathfrak{v}_\gamma), \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{3.51}$$

respectively. (See (3.61) and (3.62) for the precise equations that $v_{\gamma,N}$ and $\mathfrak{v}_{\gamma,N}$ solve.)

Next, we study the well-posedness issue for (3.50) and (3.51). We first deal with the problem (3.50) by relying on parabolic smoothing effects (i.e. the gain of derivatives of $\mathcal{I}_\gamma$ (1.70)).

Proposition 3.2.1. *Fix* $\gamma \in (0,1]$ *and let* $0 < \delta \ll 1$. *Then the equation* (3.50) *with initial data* $\phi \in H^1(\mathbb{T}^2)$ *has a unique solution in* $C\big([0,T]; H^1(\mathbb{T}^2)\big)$ *for some positive time* $0 < T \leq 1$ *such that* $T \sim \big(1 + \gamma^{-\theta_1}\|\mathbb{X}_\gamma\|_{C([0,1];\mathcal{Z}^\delta)} + \|\phi\|_{H^1_x}\big)^{-\theta_2}$, *for some* $\theta_1, \theta_2 > 0$. *Here,* $\mathfrak{S}^{\mathrm{Wick}}_\gamma = \mathfrak{S}^{\mathrm{Wick}}$ *is as in* (3.48).

Furthermore, the data-to-solution map $(\phi, \mathbb{X}) \mapsto v_\gamma(\phi, \mathbb{X})$ *is locally Lipschitz-continuous and we have the bound*

$$\|v_\gamma(\phi, \mathbb{X})\|_{C_T H^1_x} \leq C\|\phi\|_{H^1_x}.$$

for some absolute constant $C > 0$.

Proof. The proof easily follows from the estimate (3.69) and by the product estimates in Lemma 3.4.10 with Sobolev's inequality as in Proposition 2.2.1. We omit details. ☐

We now consider the well-posedness issue for (3.51). In view of (3.23) in Lemma 3.1.12, the linear flow $S_\gamma(t)$ is well-behaved in $X^{s,b}$ for $b = \frac{1}{2}$. We hence study (3.51) in $X^{s,\frac{1}{2}}$. To this end, we introduce for each $T > 0$, the space $Y^s([0,T])$ given by

$$Y^s([0,T]) := C\big([0,T]; H^s(\mathbb{T}^2)\big) \cap X^{s,\frac{1}{2}}([0,T]),$$

endowed with norm

$$\|u\|_{Y^s_T} := \|u\|_{C_T H^s_x} + \|u\|_{X^{s,\frac{1}{2}}}.$$

Proposition 3.2.2. *Fix* $\gamma \in [0,1]$ *and let* $0 < s < \frac{1}{4}$, $0 < \delta = \delta(s) \ll 1$. *Then, the equation* (3.51) *with initial data* $\phi \in H^s(\mathbb{T}^2)$, *has a unique solution in the set* $Y^s([0,T])$ *for a positive time* $0 < T \leq 1$ *with* $T \sim \big(1 + \|\mathbf{X}\|_{C([0,1];\mathcal{Y}^s)} + \|\phi\|_{H^s_x}\big)^{-\theta}$, *for some small* $\theta > 0$. *Here,* $\mathfrak{S}^{\mathrm{Wick}}$ *and* $\mathfrak{S}^{\mathrm{PDE}}$ *are as in* (3.48) *and* (3.49), *respectively.*

Furthermore, the data-to-solution map $(\phi, \mathbf{X}) \mapsto \mathfrak{v}_\gamma(\phi, \mathbf{X})$ *is locally Lipschitz continuous and we have the bound*

$$\|\mathfrak{v}_\gamma(\phi, \mathbb{X})\|_{Y^s_T} \leq C\|\phi\|_{H^s_x},$$

for some absolute constant $C > 0$.

Proof. Let $\Gamma = \Gamma_{\mathbf{X}}$ be the mapping defined by

$$\Gamma_{\mathbf{X}}(\mathfrak{v}) = \phi - \gamma\mathfrak{S}^{\mathrm{Wick}}(\mathfrak{v}) - i\mathfrak{S}^{\mathrm{PDE}}(\mathfrak{v}), \tag{3.52}$$

By the contraction mapping principle, it suffices to show that Γ is a contraction from a ball $B \subset Y^s([0,T])$ onto itself, for some time $T > 0$.

We now give a priori bounds on Γ.

• **Estimates on $\gamma\mathfrak{S}_\gamma^{\mathbf{Wick}}$.** Let us, for instance, detail the bound of $\gamma\mathcal{I}_\gamma\big(\upharpoonleft_\gamma|v|^2\big)$. By Lemma 3.1.13, Proposition 3.1.14, the embedding $X^{s,\frac{1}{2}+}([0,T]) \hookrightarrow C\big([0,T];H^s(\mathbb{T}^2)\big)$, the regularity assumption on $\upharpoonleft_\gamma$, Lemma A.1.1 (ii) and (i), Sobolev's inequality and Lemma 3.1.3, we have

$$\begin{aligned}
\big\|\gamma\mathcal{I}_\gamma\big(\upharpoonleft_\gamma|\mathfrak{v}|^2\big)\big\|_{Y_T^s} &\lesssim T^\theta\|\upharpoonleft_\gamma|\mathfrak{v}|^2\|_{C_TH^{s-1+\delta}}\\
&\lesssim T^\theta\|\upharpoonleft_\gamma|\mathfrak{v}|^2\|_{C_TH^{-\delta}}\\
&\lesssim T^\theta\|\upharpoonleft_\gamma\|_{C_TW^{-\delta,\infty-}}\||\mathfrak{v}|^2\|_{C_TW^{\delta,2+}}\\
&\lesssim T^\theta\|\upharpoonleft_\gamma\|_{C_TW^{-\delta,\infty}}\||\mathfrak{v}|^2\|_{C_TH^{2\delta}}\\
&\lesssim T^\theta\|\upharpoonleft_\gamma\|_{C_TW^{-\delta,\infty}}\|\mathfrak{v}\|^2_{C_TW^{2\delta,4}}\\
&\lesssim T^\theta\|\upharpoonleft_\gamma\|_{C_TW^{-\delta,\infty}}\|\mathfrak{v}\|^2_{X_T^{3\delta,\frac{1}{2}}},
\end{aligned}$$

as long as $\delta > 0$ is chosen sufficiently small and for small $\theta > 0$. By Lemma 3.1.13 and (the proof of) Lemma 3.1.4, we also have that

$$\begin{aligned}
\big\|\gamma\mathcal{I}_\gamma\big(|\mathfrak{v}|^2v\big)\big\|_{Y_T^s} &\lesssim T^\theta\||\mathfrak{v}|^2v\|_{X_T^{s,-\frac{1}{2}+\delta}}\\
&\lesssim T^\theta\|\mathfrak{v}\|^3_{X_T^{s,\frac{1}{2}-\delta}}\\
&\lesssim T^\theta\|\mathfrak{v}\|^3_{Y_T^s}.
\end{aligned}$$

The other estimates follow from similar arguments. We hence obtain the following bound:

$$\big\|\gamma\mathfrak{S}_\gamma^{\mathrm{Wick}}(\mathfrak{v})\big\|_{Y_T^s} \lesssim T^\theta\big(1+\|\mathbf{X}\|_{C([0,1];\mathcal{Y}^s)}\big)^3\big(1+\|\mathfrak{v}\|^3_{Y_T^s}\big). \tag{3.53}$$

• **Estimates on $\mathfrak{S}_\gamma^{\mathbf{PDE}}$.** These are direct consequences of Lemma 3.1.12, Proposition 3.1.14, Lemma 3.1.4, Lemma 3.1.19, and the regularity assumption on $\mathbf{X}$. We obtain the bound

$$\big\|\mathfrak{S}_\gamma^{\mathrm{PDE}}(\mathfrak{v})\big\|_{Y_T^s} \lesssim T^\theta\big(1+\|\mathbf{X}\|_{C([0,1];\mathcal{Y}^s)}\big)^3\big(1+\|\mathfrak{v}\|^3_{Y_T^s}\big). \tag{3.54}$$

Combining (3.52), (3.53) and (3.54) with Lemma 3.1.12 gives

$$\|\Gamma(\mathfrak{v})\|_{Y_T^s} \le C_1\|\phi\|_{H_x^s} + C_2T^\theta\big(1+\|\mathbf{X}\|_{C([0,1];\mathcal{Y}^s)}\big)^3\big(1+\|\mathfrak{v}\|^3_{Y_T^s}\big).$$

for some absolute constant $C_1, C_2 > 0$. We obtain difference estimates in a similar fashion and omit details. ☐

3.2.2 Globalization in time and convergence

Here, we present our convergence argument and prove Theorem 1.2.2.[3] We first introduce some notations. Let $s < \frac{1}{4}$ and $0 < \delta = \delta(s) \ll 1$. For each $(\gamma, N) \in [0,1]\times\mathbb{N}$, we define $A_N = A_{\gamma,N}$, $\mathbb{X}_N = \mathbb{X}_{\gamma,N}$ and $\mathbf{X}_N = \mathbf{X}_{\gamma,N}$ by

$$A_{\gamma,N}(t) = \|\upharpoonleft_{\gamma,N}\|^2_{L^2([0,t])L_x^2} - \sigma_N, \tag{3.55}$$

$$\mathbb{X}_N = \big(\upharpoonleft_{\gamma,N},\ :|\upharpoonleft_{\gamma,N}|^2:,\ \upharpoonleft^2_{\gamma,N},\ :|\upharpoonleft_{\gamma,N}|^2\upharpoonleft_{\gamma,N}:\big), \tag{3.56}$$

and

$$\mathbf{X}_N = \big(\upharpoonleft_{\gamma,N},\ :|\upharpoonleft_{\gamma,N}|^2:,\ \upharpoonleft^2_{\gamma,N},\ :|\upharpoonleft_{\gamma,N}|^2\upharpoonleft_{\gamma,N}:,\ \Psi_{\gamma,N},\ \mathcal{M}^1_{\gamma,N},\ \mathcal{M}^2_{\gamma,N},\ \mathcal{T}^1_{\gamma,N},\ \mathcal{T}^2_{\gamma,N}\big), \tag{3.57}$$

By Lemma 3.4.5, Lemma 3.4.3, Lemma 3.4.4, Proposition 3.4.6 and Proposition 3.4.10, A_N, $\mathbb{X}_N$ and $\mathbf{X}_N$ converge to some limits denoted by A_∞, $\mathbb{X}_\infty$ and $\mathbf{X}_\infty$ such that

---

[3] Note that for convenience, we only prove Theorem 1.2.2 on $[0,1]$, i.e. for $T = 1$ in Theorem 1.2.2 (ii).

$$\mathbb{X}_{\gamma,\infty} = \big(\rotatebox[origin=c]{90}{$\multimap$}_\gamma,\ :|\rotatebox[origin=c]{90}{$\multimap$}_\gamma|^2:,\ \rotatebox[origin=c]{90}{$\multimap$}_\gamma^2,\ :|\rotatebox[origin=c]{90}{$\multimap$}_\gamma|^2\rotatebox[origin=c]{90}{$\multimap$}_\gamma:\big), \tag{3.58}$$

and

$$\pmb{\mathbb{X}}_{\gamma,\infty} = \big(\rotatebox[origin=c]{90}{$\multimap$}_\gamma,\ :|\rotatebox[origin=c]{90}{$\multimap$}_\gamma|^2:,\ \rotatebox[origin=c]{90}{$\multimap$}_\gamma^2,\ :|\rotatebox[origin=c]{90}{$\multimap$}_\gamma|^2\rotatebox[origin=c]{90}{$\multimap$}_\gamma:,\ \Psi_\gamma,\ \mathcal{M}_\gamma^1,\ \mathcal{M}_\gamma^2,\ \mathcal{T}_\gamma^1,\ \mathcal{T}_\gamma^2\big), \tag{3.59}$$

in $C\big([0,1]^2;\mathbb{R}\big)$, $C\big([0,1],\mathcal{Z}^{-\delta}\big)$ and $C\big([0,1],\mathcal{Y}^s\big)$ $\rho\otimes\mathbb{P}$-almost surely, respectively. Furthermore, the following tail estimates hold:

$$\begin{aligned} &\mu\otimes\mathbb{P}\big(\|\mathbb{X}_N\|_{C([0,1];\mathcal{Z}^\delta)} > M\big) \lesssim \exp(-cM^\beta),\\ &\mu\otimes\mathbb{P}\big(\|\pmb{\mathbb{X}}_N\|_{C([0,1];\mathcal{Y}^s)} > M\big) \lesssim \exp(-cM^\beta), \end{aligned} \tag{3.60}$$

for any $M \geq 1$ and some constants $c, \beta > 0$.

We also fix a function $\gamma \in [0,1] \mapsto N(\gamma) \in \mathbb{N}$ to be chosen later and further assume that $N(\gamma) \to +\infty$ as $\gamma \to 0$.

Remark 3.2.3. In Section 3.4, we do not prove the tail estimates (3.60) but they follow from the proofs of these results. See also Remark 3.4.9 in Subsection 3.4.3.

For each $N \in \mathbb{N}\cup\{\infty\}$, we let $\mathfrak{S}_N^{\mathrm{Wick}} = \mathfrak{S}_{\mathbb{X}_N}^{\mathrm{Wick}}$ and $\mathfrak{S}_N^{\mathrm{PDE}} = \mathfrak{S}_{\pmb{\mathbb{X}}_N}^{\mathrm{PDE}}$ as in (3.48) and (3.49). We consider the equations[4]

$$v_{\gamma,N} = -(\gamma + i)\mathbf{P}_{\leq N}\mathfrak{S}_{\gamma,N}^{\mathrm{Wick}}(v_{\gamma,N}) \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{3.61}$$

and

$$\mathfrak{v}_{\gamma,N} = -\gamma\mathbf{P}_{\leq N}\mathfrak{S}_{\gamma,N}^{\mathrm{Wick}}(\mathfrak{v}_{\gamma,N}) - i\mathbf{P}_{\leq N}\mathfrak{S}_{\gamma,N}^{\mathrm{PDE}}(\mathfrak{v}_{\gamma,N}), \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{3.62}$$

The equations (3.62) and (3.61) correspond to the equations for the nonlinear remainders at the level of the problems (1.52) and (1.65), respectively. Namely, we have the decompositions

$$\begin{aligned} u_{\gamma,N} &= \rotatebox[origin=c]{90}{$\multimap$}_\gamma + v_{\gamma,N},\\ \mathfrak{u}_{\gamma,N} &= \rotatebox[origin=c]{90}{$\multimap$}_\gamma + \mathfrak{v}_{\gamma,N}, \end{aligned} \tag{3.63}$$

for each $\gamma \in [0,1]$ and $N \in \mathbb{N}$. Here, $u_{\gamma,N}$ and $\mathfrak{u}_{\gamma,N}$ solve (1.52) and (1.65), respectively.

We first study the convergence of $u_{\gamma,N}$, $\gamma > 0$ to its limit u_γ as $N \to \infty$ (see Proposition 3.2.4 below) and obtain a probabilistic estimate on the difference $u_\gamma - u_{\gamma,N}$ which is explicit in the parameters γ and N.

Proposition 3.2.4. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Then, the solution $u_{\gamma,N}$ to (1.52) belongs to $C\big([0,1];H^{-\delta}(\mathbb{T}^2)\big)$, for $\delta > 0$, $\rho\otimes\mathbb{P}$-almost surely and there exists a process $u_\gamma \in C\big([0,1];H^{-\delta}(\mathbb{T}^2)\big)$ such that $u_{\gamma,N}$ converges to u_γ in $C\big([0,1];H^{-\delta}(\mathbb{T}^2)\big)$ as $N \to \infty$ in $\rho\otimes\mathbb{P}$-probability. Furthermore, we have the following tail estimate:*

$$\begin{aligned} &\rho\otimes\mathbb{P}\big(\|u_\gamma - u_{\gamma,N}\|_{C([0,1];H_x^{-\delta})} > \varepsilon\big)\\ &\qquad\qquad \lesssim \exp\big(-c\gamma^{\theta_1}\log(\varepsilon N^{\theta_2})^\beta\big) + \varepsilon^{-2}N^{-2\delta_1} + o_{N\to\infty}(1), \end{aligned} \tag{3.64}$$

for any $\varepsilon > 0$. Here, $c, \theta_1, \theta_1, \beta > 0$ are absolute constants and the Landau symbol $o_{N\to\infty}(1)$ is uniform in all parameters.

In the next lemma, we globalize the solution $v_{\gamma,N}$ to (3.61) by utilizing Bourgain's invariant measure argument.

Lemma 3.2.5. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Recall the definition of the truncated measure ρ_N (1.46). Then, the solution $v_{\gamma,N}$ to (3.61) belongs to $C\big([0,1];H^1(\mathbb{T}^2)\big)$ $\rho_N\otimes\mathbb{P}$-almost surely and there exists a process $v_\gamma \in C\big([0,1];H^1(\mathbb{T}^2)\big)$ which solves (3.61) (with $N = \infty$) such that $v_{\gamma,N}$ converges to v_γ in $C\big([0,1];H^{-\delta}(\mathbb{T}^2)\big)$ as $N \to \infty$ in $\rho\otimes\mathbb{P}$-probability. Furthermore, we have the following tail estimates:*

---

[4] With the convention $\mathbf{P}_{\leq N} = \mathrm{Id}$ when $N = \infty$.

$$\rho_N \otimes \mathbb{P}\big(\|v_{\gamma,N}\|_{C([0,1];H^1_x)} > \gamma^{-A}M\big) \leq \exp\big(-cM^\beta\big), \tag{3.65}$$

and

$$\rho \otimes \mathbb{P}\big(\|v_\gamma\|_{C([0,1];H^1_x)} > \gamma^{-A}M\big) \leq \exp\big(-cM^\beta\big), \tag{3.66}$$

for any $M \geq 1$. Here, $c, \beta, A > 0$ are absolute parameters.

Remark 3.2.6. Note that in view of Proposition 1.2.1 and (3.65), we have

$$\rho \otimes \mathbb{P}\big(\|v_{\gamma,N}\|_{C([0,1];H^1_x)} > \gamma^{-A}M\big) \leq \exp\big(-cM^\beta\big) + o_{N\to\infty}(1).$$

for any $M \geq 1$.

Before moving to the proof to the proof of Lemma 3.2.5, we show a pathwise globalization result for the solutionsto (1.52) for a fixed value of $N \in \mathbb{N}$.

Lemma 3.2.7. *Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Let $\phi_\star \in \mathcal{D}'(\mathbb{T}^2)$ be any distribution and let $u_{\gamma,N}$ be the solution to (1.52) with initial data $\phi_\star$. Then, $u_{\gamma,N}$ exists globally in time and $\mathbf{P}_{\leq N}u_{\gamma,N} \in L^\infty\big(\mathbb{R}_+; L^2(\mathbb{T}^2)\big)$. Furthermore, there exists some absolute constant $C > 0$ such that we have*

$$\big\|\|\mathbf{P}_{\leq N}u_{\gamma,N}(t)\|_{L^2_x}\big\|_{L^{100}(\mathbb{P})} \lesssim_{\phi_\star} \gamma^{-C}\langle t\rangle^C N^C, \tag{3.67}$$

for any $t \geq 0$.

Remark 3.2.8. In the case $\gamma = 0$ (i.e. the Schrödinger equation (1.53)), the statement of Lemma 3.2.7 is also true (without any loss in γ on the right hand-side of (3.67)) and is a simple consequence of the conservation of the L^2_x-norm under the Schrödinger flow, Bernstein's inequality and (for instance) the standard $H^2(\mathbb{T}^2)$-local theory for the Wick renormalized NLS (1.53).

Proof. Let us note that for every $(\gamma, N) \in [0,1] \times \mathbb{N}$, $\mathbf{P}_{>N}u_{\gamma,N}$ solves a linear equation and is hence globally well-posed. We prove that $\widetilde{u}_{\gamma,N} = \mathbf{P}_{\leq N}u_{\gamma,N}$ exists globally in time along with the estimate (3.67). We proceed with a first order expansion $\widetilde{u}_{\gamma,N} = \text{¡}_{\gamma,N}(\phi_\star,\xi) + v_{\gamma,N}$, where $\text{¡}_{\gamma,N}(\phi_\star,\xi)$ is as in (1.74) (but with ϕ_N replaced by $\mathbf{P}_{\leq N}\phi_\star$) and $v_{\gamma,N}$ solves the equation

$$\begin{aligned}
&\partial_t v_{\gamma,N} - (\gamma+i)(\Delta-1)v_{\gamma,N} \\
&\qquad = -(\gamma+i)\mathbf{P}_{\leq N}\Big(:|\text{¡}_{\gamma,N}|^2\text{¡}_{\gamma,N}: + |v_{\gamma,N}|^2 v_{\gamma,N} + 2:|\text{¡}_{\gamma,N}|^2: v_{\gamma,N} \\
&\qquad\qquad + \text{¡}^2_{\gamma,N}\,\overline{v_{\gamma,N}} + 2\text{¡}_{\gamma,N}|v_{\gamma,N}|^2 + \overline{\text{¡}_{\gamma,N}}\,v^2_{\gamma,N}\Big).
\end{aligned} \tag{3.68}$$

There stochastic objects appearing in (3.68) are as in (1.76) (with ϕ_N replaced by $\mathbf{P}_{\leq N}\phi_\star$).

By invoking the bound $e^{-r} \lesssim_\theta r^{-\theta}$ for any $r > 0$ and $\theta > 0$, we deduce the following estimate:

$$\begin{aligned}
\Big\|\int_0^t e^{(\gamma+i)(t-t')(\Delta-1)}F(t',x)dt'\Big\|_{L^\infty_T H^s_x} &= \Big\|\int_0^t e^{-(\gamma+i)(t-t')\langle n\rangle^2}\langle n\rangle^s\widehat{F}(t',n)dt'\Big\|_{L^\infty_T\ell^2_n} \\
&\lesssim \gamma^{-\theta}\Big\|\int_0^t (t-t')^{-\theta}\langle n\rangle^{s-2\theta}\widehat{F}(t',n)dt'\Big\|_{L^\infty_T\ell^2_n} \\
&\lesssim \gamma^{-\theta}T^{1-\theta}\|F\|_{L^\infty_T H^{s-2\theta}_x}.
\end{aligned} \tag{3.69}$$

for any $T > 0$, $s \in \mathbb{R}$ and $0 < \theta < 1$. Hence by (3.69) and standard deterministic estimates (see Lemma A.1.1), it is easy to prove that the equation (3.68) is locally well-posed in $L^2(\mathbb{T}^2)$. Hence, $v_{\gamma,N}$ exists globally in time as long as we have an a priori control of $\|v_{\gamma,N}(t)\|_{L^2_x}$ for $t \geq 0$. We rely on an energy method as in [66, Section 5] and [61] (in the context of stochastic wave equations). Let $\mathcal{E}_\gamma$ be the energy functional defined by

$$\mathcal{E}_\gamma(v)(t) := \frac{1}{2}\int_{\mathbb{T}^2} |v|^2 + \gamma \int_0^t \int_{\mathbb{T}^2} |v(t',x)|^4 dt' dx, \quad t \geq 0. \tag{3.70}$$

We have

$$\frac{d}{dt}\mathcal{E}_\gamma(v_{\gamma,N})(t) = \operatorname{Re}\int_{\mathbb{T}^2} \partial_t v_{\gamma,N}\overline{v_{\gamma,N}} + \gamma\int_{\mathbb{T}^2} |v_{\gamma,N}(t,x)|^4 dx. \tag{3.71}$$

Besides, by (3.68) and integration by parts and noting that $\mathbf{P}_{\leq N} v_{\gamma,N} = v_{\gamma,N}$, we have

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{T}^2} \partial_t v_{\gamma,N}\overline{v_{\gamma,N}} \\
&= -\gamma\int_{\mathbb{T}^2} |\nabla v_{\gamma,N}|^2 + |v_{\gamma,N}|^2 + |v_{\gamma,N}|^4 + \operatorname{Re}(\gamma+i)\int_{\mathbb{T}^2} A_{\gamma,N}\,|v_{\gamma,N}|^2 v_{\gamma,N} \\
&\quad + \operatorname{Re}(\gamma+i)\int_{\mathbb{T}^2} + B_{\gamma,N}\,|v_{\gamma,N}|^2\overline{v_{\gamma,N}} + C_{\gamma,N}\,|v_{\gamma,N}|^2 + D_{\gamma,N}\overline{v_{\gamma,N}}^2 + E_{\gamma,N}\overline{v_{\gamma,N}},
\end{aligned} \tag{3.72}$$

where $A_{\gamma,N}$, $B_{\gamma,N}$ and $C_{\gamma,N}$ are given by linear combinations of the stochastic objects in (3.68). Then, by Young's inequality, we have

$$\begin{aligned}
\Big|\int_{\mathbb{T}^2} A_{\gamma,N}\,|v_{\gamma,N}|^2 v_{\gamma,N}\Big| &\lesssim \|v_{\gamma,N}\|_{L^3_x}^3 \|A_{\gamma,N}\|_{L^\infty_x} \\
&\leq \frac{\gamma}{100}\|v_{\gamma,N}\|_{L^3_x}^4 + C\gamma^{-10}\|A_{\gamma,N}\|_{L^\infty_x}^4,
\end{aligned} \tag{3.73}$$

for some absolute constant $C > 0$. Similarly, we have

$$\Big|\int_{\mathbb{T}^2} B_{\gamma,N}\,|v_{\gamma,N}|^2\overline{v_{\gamma,N}}\Big| \leq \frac{\gamma}{100}\|v_{\gamma,N}\|_{L^3_x}^4 + C\gamma^{-10}\|B_{\gamma,N}\|_{L^\infty_x}^4, \tag{3.74}$$

$$\Big|\int_{\mathbb{T}^2} C_{\gamma,N}\,|v_{\gamma,N}|^2\Big| \leq \frac{\gamma}{100}\|v_{\gamma,N}\|_{L^2_x}^4 + C\gamma^{-10}\|C_{\gamma,N}\|_{L^\infty_x}^2 .,\tag{3.75}$$

$$\Big|\int_{\mathbb{T}^2} D_{\gamma,N}\,\overline{v_{\gamma,N}}^2\Big| \leq \frac{\gamma}{100}\|v_{\gamma,N}\|_{L^2_x}^4 + C\gamma^{-10}\|D_{\gamma,N}\|_{L^\infty_x}^2, \tag{3.76}$$

$$\Big|\int_{\mathbb{T}^2} E_{\gamma,N}\,\overline{v_{\gamma,N}}\Big| \leq \frac{\gamma}{100}\|v_{\gamma,N}\|_{L^2_x}^4 + C\gamma^{-10}\|E_{\gamma,N}\|_{L^\infty_x}^{\frac{4}{3}}. \tag{3.77}$$

Hence, by putting together (3.71), (3.72), (3.73), (3.74), (3.75), (3.76), (3.77) and integrating over the time variable, we deduce

$$\begin{aligned}
\mathcal{E}_\gamma(v_{\gamma,N}) &\leq \|\mathbf{P}_{\leq N}\phi_\star\|_{L^2_x}^2 + \int_0^t \frac{d}{dt}\mathcal{E}_\gamma(v_{\gamma,N})(t')dt' \\
&\leq \|\mathbf{P}_{\leq N}\phi_\star\|_{L^2_x}^2 + \frac{\gamma}{10}\int_0^t \|v_{\gamma,N}\|_{L^4_x}^4 + C\gamma^{-10}F(A_{\gamma,N}, B_{\gamma,N}, C_{\gamma,N}, D_{\gamma,N}, E_{\gamma,N}) \\
&\leq \|\mathbf{P}_{\leq N}\phi_\star\|_{L^2_x}^2 + \frac{1}{10}\mathcal{E}_\gamma(v_{\gamma,N}) + C\gamma^{-10}F(A_{\gamma,N}, B_{\gamma,N}, C_{\gamma,N}, D_{\gamma,N}).
\end{aligned}$$

where

$$\begin{aligned}
F(A_{\gamma,N}, B_{\gamma,N}, C_{\gamma,N}, D_{\gamma,N}, E_{\gamma,N}) &= \|A_{\gamma,N}\|_{L^1([0,t];L^\infty_x)}^4 + \|B_{\gamma,N}\|_{L^1([0,t];L^\infty_x)}^4 \\
&\quad + \|C_{\gamma,N}\|_{L^1([0,t];L^\infty_x)}^2 + \|D_{\gamma,N}\|_{L^1([0,t];L^\infty_x)}^2 + \|E_{\gamma,N}\|_{L^1([0,t];L^2_x)}^{\frac{4}{3}}
\end{aligned}$$

Thus, we have

$$\mathcal{E}_\gamma(v_{\gamma,N})(t) \lesssim \|\mathbf{P}_{\leq N}\phi_\star\|_{L^2_x}^2 + \gamma^{-10}F(A_{\gamma,N}, B_{\gamma,N}, C_{\gamma,N}, D_{\gamma,N}, E_{\gamma,N}). \tag{3.78}$$

for any $t \geq 0$. By (1.76), the spatial Fourier supports of $A_{\gamma,N}$, $B_{\gamma,N}$, $C_{\gamma,N}$ and $D_{\gamma,N}$ are included in a box $\mathcal{B}$ centered at the origin and of size $\sim N$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth radial function whose Fourier support is included in $10\mathcal{B}$ and such that $\widehat{\varphi} \equiv 1$ on $\mathcal{B}$. By Bernstein's inequality, we have

$$\|Z_{\gamma,N}\|_{L^\infty_x} = \|\varphi Z_{\gamma,N}\|_{L^\infty_x} \lesssim N^2 \|Z_{\gamma,N}\|_{L^2_x},$$

for $Z_{\gamma,N} \in \{A_{\gamma,N}, B_{\gamma,N}, C_{\gamma,N}, D_{\gamma,N}\}$. By (3.78), this yields

$$\begin{aligned}\mathcal{E}_\gamma(v_{\gamma,N})(t) &\lesssim \gamma^{-10}N^{10}\big(\|A_{\gamma,N}\|^4_{L^1([0,t];L^2_x)} + \|B_{\gamma,N}\|^4_{L^1([0,t];L^2_x)} + \|C_{\gamma,N}\|^2_{L^1([0,t];L^2_x)} \\ &\qquad + \|D_{\gamma,N}\|^2_{L^1([0,t];L^2_x)} + \|E_{\gamma,N}\|^2_{L^1([0,t];L^2_x)}\big) + \|\mathbf{P}_{\le N}\phi_\star\|^2_{L^2_x} \\ &=: \gamma^{-10}N^{10}G_{\gamma,N} + \|\mathbf{P}_{\le N}\phi_\star\|^2_{L^2_x}\end{aligned} \tag{3.79}$$

By arguing as in the proof of Lemma 3.4.3 below, we deduce that the left hand-side of (3.79) is finite for any $t \ge 0$ $\mathbb{P}$-almost surely. This proves that $v_{\gamma,N}$ (and therefore $u_{\gamma,N}$) exists globally in time. Furthermore, by (3.70) and (3.79), we also have

$$\begin{aligned}\big\|\|v_{\gamma,N}(t)\|_{L^2_x}\big\|_{L^{100}(\mathbb{P})} &\lesssim \gamma^{-10}N^{10}\big\|G_{\gamma,N} + \|\mathbf{P}_{\le N}\phi_\star\|^2_{L^2_x}\big\|_{L^{100}(\mathbb{P})} \\ &\lesssim_{\phi_\star} \gamma^{-10}N^{10}\langle t\rangle^{10},\end{aligned}$$

by arguing as in the proof of Lemma 3.4.3. Moreover, in view of the decomposition $\widetilde{u}_{\gamma,N} = \stackrel{\bullet}{|}_{\gamma,N}(\phi_\star,\xi) + v_{\gamma,N}$ and Bernstein's inequality (inserting a smooth cutoff as in the above), this proves (3.67). ☐

We recall the following invariance result. See [22, 66, 72] for a proof.

Lemma 3.2.9. *Let $\gamma \in [0,1]$ and $N \in \mathbb{N}$. Recall that by Lemma 3.2.7 and Remark 3.2.8, the solution $u_{\gamma,N}(\phi_\star,\xi)$ to (1.52) is globally well-posed in time for any initial data $\phi_\star$. Let $\phi \sim \rho$, where ρ is the truncated renormalized Gibbs measure (1.46). Then, for each time $t \ge 0$, the law of $u_{\gamma,N}(\phi,\xi)(t)$ is given by ρ_N.*

Proof of Lemma 3.2.5. Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Let us recall that, by Lemma 3.2.7 the solution $u_{\gamma,N}$ to (1.52) with initial data ϕ and forcing ξ exists globally in time. We adapt an argument in [27]. By the flow property of the map $t \in \mathbb{R}_+ \mapsto u_{\gamma,N}(\phi,\xi)(t)$ and the decomposition $u_{\gamma,N} = \stackrel{\bullet}{|}_\gamma + v_{\gamma,N}$ (3.63), we have

$$v_{\gamma,N}(\phi,\xi)(t_1+t_2) = S_\gamma(t_2)\big(v_{\gamma,N}(\phi,\xi)(t_1)\big) + v_{\gamma,N}\big(u_{\gamma,N}(\phi,\xi)(t_1), \xi(t_1+\cdot)\big)\big)(t_2). \tag{3.80}$$

for any $t_1, t_2 \ge 0$. Let $k \ge 1$ be an integer to be chosen later. By (3.80) and (3.22) in Lemma 3.1.12, we have the following bound:

$$\begin{aligned}&\|v_{\gamma,N}(\phi,\xi)\|_{L^\infty([0,1];H^1_x)} \\ &\qquad \lesssim k \cdot \sup_{0\le h\le k-1} \big\|v_{\gamma,N}\big(u_{\gamma,N}(\phi,\xi)(\tfrac{h}{k}), \xi(\tfrac{h}{k}+\cdot)\big)\big\|_{L^\infty([0,k^{-1}];H^1_x)}\end{aligned} \tag{3.81}$$

For each $0 \le h \le M-1$, let $I_{\gamma,N}(h)$ be the set given by

$$I_{\gamma,N}(h) := \Big\{\big\|v_{\gamma,N}\big(u_{\gamma,N}(\phi,\xi)(\frac{h}{k}), \xi(\frac{h}{k}+\cdot)\big)\big\|_{L^\infty([0,k^{-1}];H^1_x)} \lesssim 1\Big\}.$$

By Lemma 3.2.9 and the fact that $u_{\gamma,N}(\phi,\xi)\big(\frac{h}{k}\big)$ and $\xi\big(\frac{h}{k}+\cdot\big)$ are independent from each other[5], we have that $v_{\gamma,N}\big(u_{\gamma,N}(\phi,\xi)\big(\frac{h}{k}\big), \xi\big(\frac{h}{k}+\cdot\big)\big)$ and $v_{\gamma,N}\big(\phi,\xi\big)$ are equal in law (with $\phi \sim \rho_N$). Hence, by letting $I_{\gamma,N} := \bigcap_{0\le h\le k-1} I_{\gamma,N}(h)$, we have

$$\rho_N \otimes \mathbb{P}\big(I^c_{\gamma,N}\big) \le k \cdot \rho_N \otimes \mathbb{P}\big(I_{\gamma,N}(0)^c\big). \tag{3.82}$$

Besides, by Proposition 3.2.1 and Proposition 1.2.1 with (3.60), we have that

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[5] This comes from the fact that $t \mapsto u_{\gamma,N}(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$ and that $\xi(t+\cdot)$ is independent from $\{\mathcal{F}_{t'}\}_{t'\le t}$.

$$\begin{aligned}\rho_N \otimes \mathbb{P}\big(I_{\gamma,N}(0)^c\big) &\le \rho_N \otimes \mathbb{P}\big(T(\mathbb{X}_N) \le k^{-1}\big)\\ &\lesssim \rho_N \otimes \mathbb{P}\big(\|\mathbb{X}_N\|_{C([0,1];\mathcal{Z}^\delta)} > \gamma^{\theta_1} k^{\theta_2}\big) \lesssim \exp\big(-c\gamma^{\theta_1}k^{\theta_2}\big).\end{aligned} \tag{3.83}$$

Hence (3.65) follows from (3.81), the definition of the sets $\{I_{\gamma,N}(h)\}_{0\le h\le M-1}$ and $I_{\gamma,N}$ along with (3.82) and (3.83) and by picking $k \sim \gamma^{-C}M$ for some constant $C \gg 1$ and $M \ge 1$.

The tail estimate (3.66) then follows from Proposition 1.2.1 and an approximation argument; see [27] for details. □

We now prove Proposition 3.2.4.

Proof of Proposition 3.2.4. For $\gamma \in (0,1]$, $N \in \mathbb{N} \cup \{\infty\}$ and $M \ge 1$, we define the set $\Omega_{\gamma,N}$ by

$$\Omega_{\gamma,N}(M) = \big\{\|v_{\gamma,N}\|_{C([0,1];H^1_x)} \le \gamma^{-A}M\big\}.$$

Fix $\gamma \in (0,1]$ and $N \in \mathbb{N}$. Then, we claim that on $\Omega_{\gamma,N}(M) \cap \Omega_{\gamma,\infty}(M)$, we have

$$\|v_\gamma - v_{\gamma,N}\|_{C([0,1];H^1_x)} \le \exp\big(c\gamma^{-\theta_1}M^{\theta_2}\big)\big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big), \tag{3.84}$$

for some constants $c, \theta_1, \theta_2 > 0$. We prove (3.84) by a Grönwall-type argument. By (3.48), (3.61) and (3.69), Lemma 3.1.12 and Proposition 3.1.14, we have[6]

$$\begin{aligned}&\|v_\gamma - v_{\gamma,N}\|_{C([0,t];H^1_x)}\\ &\le C\gamma^{-\frac34}t^{\frac14}\big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big)\big(1 + \|v_\gamma\|_{C([0,t];H^1_x)} + \|v_{\gamma,N}\|_{C([0,t];H^1_x)}\big)^2\\ &\quad + C\gamma^{-\frac34}t^{\frac14}\|v_\gamma - v_{\gamma,N}\|_{C([0,t];H^1_x)}\big(1 + \|v_\gamma\|_{C([0,t];H^1_x)} + \|v_{\gamma,N}\|_{C([0,t];H^1_x)}\big)^2,\end{aligned} \tag{3.85}$$

for any $0 \le t \le 1$ and some absolute constant $C > 0$. Hence, if $\tau := \frac{1}{10}C^{-\frac14}\gamma^{3+8A}M^{-8}$ so that $C\gamma^{-\frac34}\tau^{\frac14}(1 + 2\gamma^{-A}M)^2 \le \frac12$, we have (by (3.85))

$$\|v_\gamma - v_{\gamma,N}\|_{C([0,\tau];H^1_x)} \le \big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big)\big(1 + \gamma^{-A}M\big)^2. \tag{3.86}$$

on $\Omega_{\gamma,N}(M) \cap \Omega_{\gamma,\infty}(M)$. Similarly, by (3.48), (3.61) and (3.69), Lemma 3.1.12 and Proposition 3.1.14, we have

$$\begin{aligned}&\|v_\gamma - v_{\gamma,N}\|_{C([\tau,t];H^1_x)}\\ &\le \|v_\gamma(\tau) - v_{\gamma,N}(\tau)\|_{H^1_x} + C\gamma^{-\frac34}(t-\tau)^{\frac14}\|v_\gamma - v_{\gamma,N}\|_{C([\tau,t];H^1_x)}\\ &\qquad \times \big(1 + \|v_\gamma\|_{C([\tau,t];H^1_x)} + \|v_{\gamma,N}\|_{C([\tau,t];H^1_x)}\big)^2\\ &\quad + C\gamma^{-\frac34}(t-\tau)^{\frac14}\big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big)\\ &\qquad \times \big(1 + \|v_\gamma\|_{C([\tau,t];H^1_x)} + \|v_{\gamma,N}\|_{C([\tau,t];H^1_x)}\big)^2,\end{aligned} \tag{3.87}$$

on $\Omega_{\gamma,N}(M) \cap \Omega_{\gamma,\infty}(M)$, for any $\tau \le t \le 1$. Hence, by (3.86), (3.87) and the definition of T_M, we get

$$\|v_\gamma - v_{\gamma,N}\|_{C([\tau,2\tau];H^1_x)} \le 3\big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big)\big(1 + \gamma^{-A}M\big)^2.$$

By iterating this argument $\sim \frac{1}{\tau}$ times, we hence obtain

$$\|v_\gamma - v_{\gamma,N}\|_{C([0,1];H^1_x)} \le 2^{c\frac{1}{\tau}} \cdot \big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} + \|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|^3_{\mathcal{Z}^\delta}\big)\big(1 + \gamma^{-A}M\big)^2,$$

for some $c > 0$. This proves (3.84) by possibly adjusting the constant c in the above.

By (the proof of) Lemma 3.4.3, we have

$$\big\|\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta}\big\|_{L^2(\rho\otimes\mathbb{P})} \lesssim N^{-\delta_1},$$

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[6] In the estimate below, we discard the frequency projection $\mathbf{P}_{\le N}$ in (1.79) as taking it into account only brings out a harmless additive factor $\sim N^{-\theta_1}$ for some small $\theta_1 > 0$ (by giving away a bit of spatial derivatives).

for some $\delta_1 > 0$. Hence, by Chebyshev's inequality, we have the tail estimate:

$$\rho \otimes \mathbb{P}\big(\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} > M\big) \lesssim N^{-2\delta_1} M^{-2}. \tag{3.88}$$

for any $M \geq 1$. Let $F_{\gamma,N}$ be the event $F_{\gamma,N} = \big\{\|\mathbb{X}_\gamma - \mathbb{X}_{\gamma,N}\|_{\mathcal{Z}^\delta} \leq N^{-\frac{\delta_1}{10}}\big\}$. Then, by (3.84), the estimate

$$\|v_\gamma - v_{\gamma,N}\|_{C([0,1];H^1_x)} \leq \exp\big(c\gamma^{-\theta_1} M^{\theta_2}\big) \cdot N^{-\frac{\delta_1}{10}}, \tag{3.89}$$

holds on $\Omega_{\gamma,N}(M) \cap \Omega_{\gamma,\infty}(M) \cap F_{\gamma,N}$.

Thus, by (3.89), (3.88), (3.65) and (3.66) in Lemma 3.2.5 and Remark 3.2.6, we deduce that

$$\begin{aligned}
\rho \otimes \mathbb{P}\Big(&\|v_\gamma - v_{\gamma,N}\|_{C([0,1];H^1_x)} > \exp\big(c\gamma^{-\theta_1} M^{\theta_2}\big) \cdot N^{-\frac{\delta_1}{10}}\Big) \\
&\leq \rho \otimes \mathbb{P}\big(\Omega_{\gamma,N}(M)^c\big) + \rho \otimes \mathbb{P}\big(\Omega_{\gamma,\infty}(M)^c\big) + \rho \otimes \mathbb{P}\big(F^c_{\gamma,N}\big) \\
&\lesssim \exp\big(-cM^\beta\big) + o_{N\to\infty}(1).
\end{aligned} \tag{3.90}$$

Finally, let $u_\gamma := ⫯_\gamma + v_\gamma$. Then, by Lemma 3.2.7, $u_{\gamma,N}$ belongs to $C\big([0,1]; H^{-\delta}(\mathbb{T}^2)\big)$ $\rho \otimes \mathbb{P}$-almost surely. Furthermore, in view of the decomposition $u_{\gamma,N} = ⫯_\gamma + v_{\gamma,N}$ along with (3.88) and (3.90), we have

$$\begin{aligned}
\rho \otimes \mathbb{P}\big(&\|u_\gamma - u_{\gamma,N}\|_{C([0,1];H^{-\delta}_x)} > \varepsilon\big) \\
&\leq \rho \otimes \mathbb{P}\big(\|⫯_\gamma - ⫯_{\gamma,N}\|_{C([0,1];H^{-\delta}_x)} > \frac{\varepsilon}{2}\big) + \rho \otimes \mathbb{P}\Big(\|v_\gamma - v_{\gamma,N}\|_{C([0,1];H^1_x)} > \frac{\varepsilon}{2}\Big) \\
&\leq \varepsilon^{-2} N^{-2\delta_1} + \exp\big(-c\gamma^{\theta_1}\log(\varepsilon N^{\theta_2})^\beta\big) + o_{N\to\infty}(1).
\end{aligned}$$

for any $\varepsilon > 0$ and some absolute constants $c_1, c_2, \theta_1, \delta_1, \beta > 0$. This proves (3.64) and that $u_{\gamma,N}$ converges to u_γ in $C\big([0,1]; H^{-\delta}(\mathbb{T}^2)\big)$ as $N \to \infty$ in $\rho \otimes \mathbb{P}$-probability. ☐

Next, we deal with the globalization and convergence issues on the gauged side, i.e. for (1.59). We first globalize by making use of Bourgain's invariant measure argument the solution to (3.62) for $\gamma = 0$. Our argument follows the proof of Lemma 3.2.5, but is slightly more involved since one needs to proceed with care when combining $X^{s,\frac{1}{2}}$-bounds on different time intervals (Lemma 3.1.6).

Lemma 3.2.10. *Fix $N \in \mathbb{N}$. Recall the definition of the truncated measure ρ_N (1.46). Then, the solution $\mathfrak{v}_{0,N}$ to (3.62) belongs to $Y^s([0,1])$ $\rho_N \otimes \mathbb{P}$-almost surely and there exists a process $\mathfrak{v}_0 \in Y^s([0,1])$ which solves (3.62) (with $N = \infty$) such that $\mathfrak{v}_{0,N}$ converges to $\mathfrak{v}_0$ in $Y^s([0,1])$ as $N \to \infty$ in $\rho \otimes \mathbb{P}$-probability. Furthermore, we have the following tail estimates:*

$$\rho_N \otimes \mathbb{P}\big(\|\mathfrak{v}_{0,N}\|_{Y^s([0,1])} > M^2 \log M\big) \lesssim \exp\big(-cM^\beta\big), \tag{3.91}$$

and

$$\rho \otimes \mathbb{P}\big(\|\mathfrak{v}_0\|_{Y^s([0,1])} > M^2 \log M\big) \lesssim \exp\big(-cM^\beta\big), \tag{3.92}$$

for any $M \gg 1$. Here, $c, \beta > 0$ are absolute parameters.

Proof. Let us recall that, by Remark 3.2.8 the solution $u_{0,N}$ to (1.53) with initial data ϕ exists globally in time. We also recall the definition of $A_{0,N}$ in (3.55). By the flow property of the map $t \in \mathbb{R}_+ \mapsto u_{0,N}(\phi)(t)$, the equality $\mathfrak{u}_{0,N} = e^{2iA_{0,N}(\phi)(t)} u_{0,N}$ (1.58), the conservation of the L^2_x-norm under the Schrödinger flow (1.53) (so that $A_{0,N}(\phi) = A_{0,N}\big(u_{0,N}(\phi)(t_1)\big)$ for any $t_1 \geq 0$), and the decomposition $\mathfrak{u}_{0,N} = ⫯_{0,N} + \mathfrak{v}_{0,N} = S_0(t)\phi + \mathfrak{v}_{0,N}$, we have

$$\mathfrak{v}_{0,N}(\phi)(t_1 + t_2) = S_0(t_2)\big(\mathfrak{v}_{0,N}(\phi)(t_1)\big) + e^{-iA_{0,N}(\phi)(t_1)}\mathfrak{v}\big(u_{0,N}(t_1)\big)(t_2). \tag{3.93}$$

for any $t_1, t_2 \geq 0$. Hence, by (3.93), (3.60), Lemma 3.2.9 and by arguing as in the proof Lemma 3.2.5, we get for any $M \geq 1$,

$$\|\mathfrak{v}_{0,N}(\phi)\|_{C([0,1];H^s_x)} \le M, \tag{3.94}$$

on a set I_N such that $\rho_N \otimes \mathbb{P}(I_N^c) \lesssim \exp(-cM^\beta)$.

Moreover, by Lemma 3.1.6, we have

$$\begin{aligned}
&\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([0,1])} \\
&\quad\lesssim \log(M)\Big(\sum_{h=1}^{M-1}\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([\frac{h}{M},\frac{h+1}{M}])} + \sum_{h=1}^{M-2}\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([\frac{1+2h}{2M},\frac{3+2h}{2M}])}\Big).
\end{aligned} \tag{3.95}$$

Now, fix $1 \le h \le M-1$. Note that $\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([\frac{h}{M},\frac{h+1}{M}])} = \|\mathfrak{v}_{0,N}(\phi)(\cdot + \frac{h}{M})\|_{X^{s,\frac{1}{2}}([0,\frac{1}{M}])}$. Hence, by Lemma 3.1.12 and applying (3.93) iteratively, we obtain

$$\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([\frac{h}{M},\frac{h+1}{M}])} \lesssim M \cdot \sup_{0\le h'\le M-1}\big\|\mathfrak{v}_{0,N}\big(u_{0,N}(\phi)\big(\frac{h'}{M}\big)\big)\big\|_{Y^s([0,\frac{1}{M}])} \tag{3.96}$$

Similarly, we have

$$\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([\frac{1+2h}{2M},\frac{3+2h}{2M}])} \lesssim M \cdot \sup_{0\le h'\le M-2}\big\|\mathfrak{v}_{0,N}\big(u_{0,N}(\phi)\big(\frac{1+2h'}{2M}\big)\big)\big\|_{Y^s([0,\frac{1}{M}])} \tag{3.97}$$

Thus, by (3.60), (3.95), (3.96), (3.97), Lemma 3.2.9 and by arguing as in the proof of Lemma 3.2.5, we have

$$\|\mathfrak{v}_{0,N}(\phi)\|_{X^{s,\frac{1}{2}}([0,1])} \le M^2\log(M), \tag{3.98}$$

on a set J_N such that $\rho_N \otimes \mathbb{P}(J_N^c) \lesssim \exp(-cM^\beta)$. Together with (3.94), the bound (3.98) proves (3.91). The estimate (3.92) then follows from Proposition 3.2.2, a standard approximation argument and Proposition 1.2.1. We omit details. ☐

Lastly, we state a long time approximation result for the sequence $\{\mathfrak{v}_{\gamma,N(\gamma)}\}_{\gamma\in(0,1]}$.

Lemma 3.2.11. *We assume that the following convergence holds:*

$$\|\mathbb{X}_{0,\infty}\|_{\mathcal{Y}^s} + \|\mathbb{X}_{\gamma,N(\gamma)}\|_{\mathcal{Y}^s} \le C, \tag{3.99}$$

for all $\gamma \in (0,1]$ and for some constant $C > 0$, and

$$\|\mathbb{X}_{0,\infty} - \mathbb{X}_{\gamma,N(\gamma)}\|_{\mathcal{Y}^s} \longrightarrow 0, \tag{3.100}$$

as $\gamma \to 0$ and that $\mathfrak{v}_0$, the solution to (3.62) (with $N = \infty$) belongs to $Y^s([0,1])$. Namely, we have

$$\|\mathfrak{v}_0\|_{Y^{s_1}([0,1])} \le C, \tag{3.101}$$

Then, there exists $\gamma_0 > 0$ such that the solution $\mathfrak{v}_{\gamma,N(\gamma)}$ belongs to $Y^{s_1}([0,1])$, for any $0 < s_1 < s$ and all $\gamma \in (0,\gamma_0]$. Furthermore, the following convergence holds:

$$\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}([0,1])} \longrightarrow 0, \tag{3.102}$$

as $\gamma \to 0$.

We first prove a large time approximation result and then proceed with the proof of Lemma 3.2.11.

Lemma 3.2.12. *Fix $0 < T \le 1$ and $\gamma_0 \in (0,1]$. Let us assume that (3.99), (3.100) and (3.101) hold, that $\mathfrak{v}_0$ and $\mathfrak{v}_{\gamma,N(\gamma)}$ both belong to $Y^s([0,T])$, for each $\gamma \in (0,\gamma_0]$, and that there exists a constant $C_1 > 0$ such that*

$$\|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^s([0,T])} \le C_1. \tag{3.103}$$

for every $\gamma \in (0,\gamma_0]$. *Then, the following convergence holds:*

$$\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}([0,T])} \longrightarrow 0, \tag{3.104}$$

as $\gamma \to 0$ *and for any* $0 < s_1 < s$

Proof. Let us first note that by a variant of Lemma 3.2.7, $\mathfrak{v}_{\gamma,N(\gamma)}$ belongs to $Y^s([0,1])$ for each $\gamma \in (0,1]$. Let $\tau > 0$ to be chosen later. By (3.62), (3.49) and (3.48), we have

$$\begin{aligned}
&\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\\
&= \|i\mathfrak{S}^{\text{PDE}}_{0,\infty}(\mathfrak{v}_0) - i\mathbf{P}_{\le N(\gamma)}\mathfrak{S}^{\text{PDE}}_{\gamma,N(\gamma)}(\mathfrak{v}_{\gamma,N(\gamma)}) - \gamma\mathbf{P}_{\le N(\gamma)}\mathfrak{S}^{\text{Wick}}_{\gamma,N(\gamma)}(\mathfrak{v}_{\gamma,N(\gamma)})\|_{Y^{s_1}_\tau}\\
&\le \|(\text{Id} - \mathbf{P}_{\le N(\gamma)})\mathfrak{S}^{\text{PDE}}_{0,\infty}(\mathfrak{v}_0)\|_{Y^{s_1}_\tau} + \gamma\|\mathbf{P}_{\le N(\gamma)}\mathfrak{S}^{\text{Wick}}_{\gamma,N(\gamma)}(\mathfrak{v}_{\gamma,N(\gamma)})\|_{Y^{s_1}_\tau} &(3.105)\\
&\quad + \|(\mathcal{I}_0 - \mathcal{I}_\gamma)(|\mathfrak{v}_0|^2\mathfrak{v}_0)\|_{Y^{s_1}_\tau} + \|\mathcal{I}_\gamma(|\mathfrak{v}_0|^2\mathfrak{v}_0 - |\mathfrak{v}_{\gamma,N(\gamma)}|^2\mathfrak{v}_{\gamma,N(\gamma)})\|_{Y^{s_1}_\tau} &(3.106)\\
&\quad + \sum_{j=1}^{2}\Big(\|\mathcal{M}^j_0(\mathfrak{v}_0) - \mathcal{M}^j_{\gamma,N(\gamma)}(\mathfrak{v}_{\gamma,N(\gamma)})\|_{Y^{s_1}_\tau} + \|\Psi_0 - \Psi_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\\
&\qquad\qquad + \|\mathcal{T}^j_0(\mathfrak{v}_0,\mathfrak{v}_0) - \mathcal{T}^j_{\gamma,N(\gamma)}(\mathfrak{v}_{\gamma,N(\gamma)},\mathfrak{v}_{\gamma,N(\gamma)})\|_{Y^{s_1}_\tau}\Big). &(3.107)
\end{aligned}$$

By the estimates in Proposition 3.2.2, we have for some small $\theta_1 > 0$,

$$\begin{aligned}
(3.105) &\lesssim N(\gamma)^{s_1-s}\big(1 + \|\mathbb{X}_{0,\infty}\|_{\mathcal{Y}^s}\big)^3\big(1 + \|\mathfrak{v}_0\|_{Y^s_\tau}\big)^3\\
&\qquad + \gamma^{\theta_1}\big(1 + \|\mathbb{X}_{\gamma,N(\gamma)}\|_{\mathcal{Z}^\delta}\big)^3\big(1 + \|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^s_\tau}\big)^3\\
&= o_{\gamma\to 0}(1).
\end{aligned}\tag{3.108}$$

Similarly, by Lemma 3.1.18, Lemma 3.1.4 and Proposition 3.1.14, we have that

$$\begin{aligned}
(3.106) &\le o_{\gamma\to 1}(1) + C_0\tau^\theta\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\big(1 + \|\mathfrak{v}_0\|_{Y^{s_1}_\tau} + \|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\big)^2\\
&\le o_{\gamma\to 1}(1) + C_0\big(1 + C + C_1\big)^2\tau^\theta\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau},
\end{aligned}\tag{3.109}$$

for some absolute constant $C_0 > 0$. At last, by the definition of the stochastic objects, we have

$$\begin{aligned}
(3.107) &\le \|\mathbb{X}_{0,\infty} - \mathbb{X}_{\gamma,N(\gamma)}\|_{\mathcal{Y}^s}\big(1 + \|\mathfrak{v}_0\|^2_{Y^{s_1}_\tau}\big)\\
&\qquad + C_0\tau^\theta\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\|\mathbb{X}_{\gamma,N(\gamma)}\|_{\mathcal{Y}^s}\big(\|\mathfrak{v}_0\|_{Y^{s_1}_\tau} + \|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}\big)\\
&\le o_{\gamma\to 0}(1) + C_0C(C + C_1)\tau^\theta\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau}.
\end{aligned}\tag{3.110}$$

We fix $\tau = \big(10C_0(1 + C_0 + C_1)^2\big)^{-\theta}$. Then, by (3.108), (3.109) and (3.110), we have

$$\|\mathfrak{v}_0 - \mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}_\tau} = o_{\gamma\to 0}(1).$$

By repeating this argument $\sim \tau^{-1}$ times on a family of non-disjoint time intervals (so as to be able to apply Lemma 3.1.6), we obtain (3.104). □

We now prove Lemma 3.2.11 by a "bootstrap" argument.

Proof of Lemma 3.2.11. By Proposition 3.2.2 and (3.99), the condition (3.103) in Lemma 3.2.12 holds on some small time interval $[0,T]$ with $T \sim C^{-\theta}$ and $\gamma_0 = 0$. Hence, by (3.104) and (3.101), there exists $\gamma_1 > 0$ such that

$$\|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}([0,T])} \le C + 1,$$

for all $\gamma \in (0,\gamma_1]$ and where $s_1 > 0$ is slightly smaller than s. We may then apply Proposition 3.2.2 again for each $\gamma \in (0,\gamma_1]$ and get

$$\|\mathfrak{v}_{\gamma,N(\gamma)}\|_{Y^{s_1}([0,2T])} \leq 2(C+1),$$

for all $\gamma \in (0,\gamma_1]$. Iterating the last argument $\sim T^{-1}$ times allows us to obtain the bound (3.103) (with $C_1 = C+1$ and s replaced by a slightly smaller exponent s') on $[0,1]$. Hence, the convergence (3.102) then follows from Lemma 3.2.12. $\square$

We now put together the result of the section and prove Theorem 1.2.2.

Proof of Theorem 1.2.2. Let A_N, $\mathbb{X}_N$ and $\mathbf{X}_N$ be as in (3.55), (3.58) and (3.59). By the results of Section 3.4, we have that $\{\gamma \mapsto (A_{\gamma,N}, \mathbb{X}_{\gamma,N}, \mathbf{X}_{\gamma,N})\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]; C([0,1];\mathbb{R}) \times \mathcal{Z}^{-\delta} \times \mathcal{Y}^s\big)$ on a "bundle" Ω_0 of the form

$$\Omega_0 = \bigcup_{\phi_\star \in \Omega_\phi} \{\phi_\star\} \times \Omega_\xi(\phi_\star), \tag{3.111}$$

with $\rho(\Omega_\phi) = 1$ and $\mathbb{P}(\Omega_\xi(\phi_\star)) = 1$ for each $\phi_\star \in \Omega_\phi$. Moreover, denoting as in the above, by $\gamma \mapsto A_{\gamma,\infty}$, $\gamma \mapsto \mathbb{X}_{\gamma,\infty}$ and $\gamma \mapsto \mathbf{X}_{\gamma,\infty}$ and the respective limits of the sequences $\{\gamma \mapsto A_{\gamma,N}\}_{N\in\mathbb{N}}$, $\{\gamma \mapsto \mathbb{X}_{\gamma,N}\}_{N\in\mathbb{N}}$ and $\{\gamma \mapsto \mathbf{X}_{\gamma,N}\}_{N\in\mathbb{N}}$, we have

$$\sup_{N\in\mathbb{N}} N^\theta \|A_{\gamma,\infty}(\phi_\star,\cdot) - A_{\gamma,N}(\phi_\star,\cdot)\|_{C([0,1]^2;\mathbb{R})} \lesssim 1, \tag{3.112}$$

$$\sup_{N\in\mathbb{N}} \|\mathbf{X}_{\gamma,N}(\phi_\star,\cdot)\|_{C([0,1];\mathcal{Y}^s)} + \sup_{N\in\mathbb{N}} N^\theta \|\mathbf{X}_{\gamma,\infty}(\phi_\star,\cdot) - \mathbf{X}_{\gamma,N}(\phi_\star,\cdot)\|_{C([0,1];\mathcal{Y}^s)} \lesssim 1, \tag{3.113}$$

on $\Omega_\xi(\phi_\star)$, for each $\phi_\star \in \Omega_\phi$. Here, $\theta > 0$ is a small absolute number.[7] Note that by Fubini's theorem, we have that $\rho \otimes \mathbb{P}(\Omega_0) = 1$.

In what follows, for the sake of readability, we write $\mathfrak{u}^\star_{\gamma,N}(\phi,\xi)$ and $\mathfrak{u}_{\gamma,N}(\phi,\mathfrak{X}^{\gamma,N})$ for the respective solutions to (1.65) and (1.59) with initial data ϕ. Subsequently, we denote by $\mathfrak{v}^\star_{\gamma,N}(\phi,\xi)$ the nonlinear remainder in the first order expansion (3.63) so that $\mathfrak{u}^\star_{\gamma,N} = \overset{\bullet}{|}_{\gamma,N} + \mathfrak{v}^\star_{\gamma,N}$. Note that with this notation, we have $\mathfrak{u}^\star_{0,N} = \mathfrak{u}_{0,N}$.

• **Step 1: convergence on the gauge side.** We aim to prove that $\mathfrak{u}_{\gamma,N(\gamma)}$ converges to $\mathfrak{u}_0$ as $\gamma \to 0$ in $C\big([0,1]; H^{-\delta}(\mathbb{T}^2)\big)$ in $\rho\otimes\mathbb{P}$-probability. Let $\mathfrak{v}_0$ be the solution to (3.62) (with $N = \infty$). By Lemma 3.2.10, we have that $\|\mathfrak{v}_0\|_{Y^s([0,1])} < \infty$ on a set Ω'_ϕ of full ρ-probability. Since the intersection of sets of the form (3.111) is also a set of the form (3.111), we may assume that $\Omega'_\phi = \Omega_\phi$.

Fix $\phi_\star \in \Omega_\phi$. Next, we note that by (3.113) and by the continuity of $\gamma \mapsto \mathbb{X}_{\gamma,\infty}$ at $\gamma = 0$, we have

$$\begin{aligned} &\|\mathbf{X}_{0,\infty}(\phi_\star) - \mathbf{X}_{\gamma,N(\gamma)}(\phi_\star,\cdot)\|_{\mathcal{Y}^s} \\ &\quad \lesssim \|\mathbf{X}_{0,\infty}(\phi_\star) - \mathbf{X}_{\gamma,\infty}(\phi_\star,\cdot)\|_{\mathcal{Y}^s} + \|\mathbf{X}_{\gamma,\infty}(\phi_\star,\cdot) - \mathbf{X}_{\gamma,N(\gamma)}(\phi_\star,\cdot)\|_{\mathcal{Y}^s} \\ &\quad \lesssim \|\mathbf{X}_{0,\infty}(\phi_\star) - \mathbf{X}_{\gamma,\infty}(\phi_\star,\cdot)\|_{\mathcal{Y}^s} + N(\gamma)^{-\theta} \longrightarrow 0, \end{aligned} \tag{3.114}$$

as $\gamma \to 0$, on $\Omega_\xi(\phi_\star)$. Hence, by (3.114) and (3.113), the conditions (3.99), (3.100) and (3.101) in Lemma 3.2.11 are satisfied. Thus, we have

$$\|\mathfrak{v}_0(\phi_\star) - \mathfrak{v}^\star_{\gamma,N(\gamma)}(\phi_\star,\cdot)\|_{Y^{s_1}([0,1])} \longrightarrow 0, \tag{3.115}$$

as $\gamma \to 0$, on $\Omega_\xi(\phi_\star)$ and for some $s_1 > 0$ which is slightly smaller than s. Hence by (3.63), (3.114) and (3.115), we deduce that

$$\|\mathfrak{u}_0(\phi_\star) - \mathfrak{u}^\star_{\gamma,N(\gamma)}(\phi_\star,\cdot)\|_{C([0,1];H^{-\delta}_x)} \longrightarrow 0, \tag{3.116}$$

as $\gamma \to 0$, on $\Omega_\xi(\phi_\star)$.

Since the map $\xi \mapsto \mathfrak{u}_{\gamma,N(\gamma)}(\phi_\star,\xi)$ is mesurable and $\mathrm{Law}(\xi) = \mathrm{Law}(\mathfrak{X}^{\gamma,N(\gamma)})$ by Proposition 3.1.8. We have that $\mathrm{Law}\big(\mathfrak{u}_{\gamma,N(\gamma)}(\phi_\star,\mathfrak{X}^{\gamma,N(\gamma)})\big) = \mathrm{Law}\big(\mathfrak{u}^\star_{\gamma,N(\gamma)}(\phi_\star,\xi)\big)$ as elements of $C\big([0,1], H^{-\delta}(\mathbb{T}^2)\big)$ and, by (3.116), we have that

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[7] Technically, the bounds (3.112) and (3.113) were not shown in Section 3.4 but they follow from the proofs of the aforementioned results.

$$\mathrm{Law}\big(\mathfrak{u}_{\gamma,N(\gamma)}(\phi_\star,\mathfrak{X}^{\gamma,N(\gamma)})\big)\longrightarrow \mathrm{Law}\big(\mathfrak{u}_0(\phi_\star)\big)\quad \text{in } C\big([0,1],H^{-\delta}(\mathbb{T}^2)\big), \tag{3.117}$$

as $\gamma\to 0$. Since $\mathfrak{u}_0(\phi_\star)$ is a constant (in ξ), the convergence in law (3.117) can be actually be upgraded to a convergence in probability. Thus, in view of the independence of ϕ and $\mathfrak{X}^{\gamma,N(\gamma)}$ (Proposition 3.1.8), Fubini's theorem and the fact that $\rho(\Omega_\phi)=1$, we deduce that $\mathfrak{u}_{\gamma,N(\gamma)}$ converges to $\mathfrak{u}_0$ in $C\big([0,1];H^{-\delta}(\mathbb{T}^2)\big)$ in $\rho\otimes\mathbb{P}$-probability.

• **Step 2: convergence of the gauge transform.** We now consider the gauge transform $\mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})$ (1.56). By (1.56) and (1.58), we have $|u_{\gamma,N}|=|\mathfrak{u}_{\gamma,N}|$, so that

$$\mathcal{G}_N(u_{\gamma,N})=\mathcal{G}_N(\mathfrak{u}_{\gamma,N}), \tag{3.118}$$

for any $N\in\mathbb{N}$.

We fix $\phi_\star\in\Omega_\phi$ and work on $\Omega_\xi(\phi_\star)$. For each $N\in\mathbb{N}$, we have $\mathfrak{u}_{0,N}=\mathfrak{u}^\star_{0,N}$. Since the L^2_x-norm is conserved by the flow of (1.65) with $\gamma=0$ (i.e. the PDE renormalized Schrödinger equation), we have $V_N(\mathfrak{u}_{0,N})=2A_{0,N}$. We denote by $\mathcal{G}_\infty(u_0)$ the limit

$$\mathcal{G}_\infty(u_0):=e^{2iA_{0,\infty}}=\lim_{N\to\infty}\mathcal{G}_N(\mathfrak{u}_{0,N})=\lim_{N\to\infty}e^{iV_N(\mathfrak{u}_{0,N})}\quad\text{on } [0,1]. \tag{3.119}$$

We aim to prove that $\mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1}$ converges to $\mathcal{G}_\infty(u_0)^{-1}$ as $\gamma\to 0$ in $C([0,1];\mathbb{R})$ in $\rho\otimes\mathbb{P}$-probability. Let $(\gamma,N)\in[0,1]\times\mathbb{N}$. In view of (3.55) and the decomposition $\mathfrak{u}^\star_{\gamma,N}=\overset{\bullet}{\mid}_{\gamma,N}+\mathfrak{v}^\star_{0,N}$ (3.63), we have

$$\begin{aligned}
&V_N(\mathfrak{u}^\star_{\gamma,N})(t)\\
&\quad=\|\overset{\bullet}{\mid}_{\gamma,N}+\mathfrak{v}^\star_{\gamma,N}\|^2_{L^2_tL^2_x}-\sigma_N\\
&\quad=A_{\gamma,N}(t)+\|\mathfrak{v}^\star_{\gamma,N}\|^2_{L^2_tL^2_x}+\int_0^t\int_{\mathbb{T}^2}\overset{\bullet}{\mid}_{\gamma,N}\overline{\mathfrak{v}^\star_{\gamma,N}}dt'dx+\int_0^t\int_{\mathbb{T}^2}\overline{\overset{\bullet}{\mid}_{\gamma,N}}\mathfrak{v}^\star_{\gamma,N}dt'dx.
\end{aligned} \tag{3.120}$$

Hence, by (3.119), (3.120) and Step 1, we have

$$\mathcal{G}_\infty(u_0)=e^{iV_\infty(u_0)}, \tag{3.121}$$

with

$$V_\infty(u_0)(t)=A_{0,\infty}(t)+\|\mathfrak{v}^\star_0\|^2_{L^2_tL^2_x}+\int_0^t\int_{\mathbb{T}^2}\overset{\bullet}{\mid}_{0,\infty}\overline{\mathfrak{v}^\star_0}dt'dx+\int_0^t\int_{\mathbb{T}^2}\overline{\overset{\bullet}{\mid}_{0,\infty}}\mathfrak{v}^\star_0dt'dx, \tag{3.122}$$

on $[0,1]$. By (3.112) and arguing as in the proof of the estimate (3.114), we get

$$\|A_{\gamma,N(\gamma)}-A_{0,\infty}\|_{C([0,1];\mathbb{R})}\longrightarrow 0, \tag{3.123}$$

as $\gamma\to 0$. Thus, by (3.120), (3.121), (3.122), the mean value theorem, duality, Step 1, (3.115), (3.114) and (3.123), we have

$$\begin{aligned}
&\|\mathcal{G}_{N(\gamma)}(\mathfrak{u}^\star_{\gamma,N(\gamma)})-\mathcal{G}_\infty(u_0)\|_{C([0,1];\mathbb{R})}=\|e^{iV_{N(\gamma)}(\mathfrak{u}^\star_{\gamma,N})}-e^{iV_\infty(u_0)}\|_{C([0,1];\mathbb{R})}\\
&\lesssim\|A_{\gamma,N(\gamma)}-A_{0,\infty}\|_{C([0,1];\mathbb{R})}+\|\mathfrak{v}^\star_{\gamma,N(\gamma)}-\mathfrak{v}^\star_0\|_{L^2([0,1];L^2_x)}\big(\|\mathfrak{v}^\star_{\gamma,N(\gamma)}\|_{L^2([0,1];L^2_x)}+\|\mathfrak{v}^\star_0\|_{L^2([0,1];L^2_x)}\big)\\
&\qquad+\|\overset{\bullet}{\mid}_{\gamma,N(\gamma)}-\overset{\bullet}{\mid}_{0,\infty}\|_{L^2([0,1];H^{-\delta}_x)}\big(\|\mathfrak{v}^\star_{\gamma,N(\gamma)}\|_{L^2([0,1];H^\delta_x)}+\|\mathfrak{v}^\star_{0,\infty}\|_{L^2([0,1];H^\delta_x)}\big)\longrightarrow 0,
\end{aligned}$$

as $\gamma\to 0$.

Finally, by arguing as in Step 1, this shows that the complex exponential $\mathcal{G}_{N(\gamma)}(\mathfrak{u}_{\gamma,N(\gamma)})^{-1}$ converges to $\mathcal{G}_\infty(u_0)^{-1}$ in $C\big([0,1];\mathbb{R}\big)$ as $\gamma\to 0$ in $\rho\otimes\mathbb{P}$-probability. By (3.118), this proves the convergence

$$\mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1}\longrightarrow\mathcal{G}_\infty(u_0)^{-1}\quad\text{in } C\big([0,1];\mathbb{R}\big),$$

as $\gamma\to 0$, in $\rho\otimes\mathbb{P}$-probability, as desired.

• **Step 3: full convergence.** Let $\gamma \in (0,1]$. By Proposition 3.2.4, Step 1, Step 2 and Lemma 3.4.3, we can construct the distributions $u_\gamma := \overset{\bullet}{\mathrm{l}}_\gamma + v_\gamma$ and $u := \mathcal{G}_\infty(u_0)^{-1}\mathfrak{u}_0$ in $C\big([0,1]; H^{-\delta}(\mathbb{T}^2)\big)$ $\rho \otimes \mathbb{P}$-almost surely. Note that u_γ and u are interpreted as the solutions to (1.54) and (1.55) with initial data $\phi \sim \rho$. We have

$$\begin{aligned} &\rho \otimes \mathbb{P}\Big(\|u - u_\gamma\|_{C([0,1];H_x^{-\delta})} > \varepsilon \Big) \\ &\quad \le \rho \otimes \mathbb{P}\Big(\|u - u_{\gamma,N(\gamma)}\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big) + \rho \otimes \mathbb{P}\Big(\|u_\gamma - u_{\gamma,N(\gamma)}\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big) \\ &\quad =: \mathrm{I}(\gamma) + \mathrm{II}(\gamma). \end{aligned} \tag{3.124}$$

for any $\varepsilon > 0$.

We first prove that $\mathrm{I}(\gamma) \to 0$ as $\gamma \to 0$. Fix $\varepsilon > 0$. By (1.58), we have

$$\begin{aligned} \mathrm{I}(\gamma) &= \rho \otimes \mathbb{P}\Big(\|u - u_{\gamma,N(\gamma)}\|_{C([0,1];H_x^{-\delta})} > \varepsilon \Big) \\ &\le \rho \otimes \mathbb{P}\Big(\big\| \mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1} - \mathcal{G}_\infty(u_0)^{-1} \big\|_{C([0,1];\mathbb{R})} \cdot \|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big) \\ &\qquad + \rho \otimes \mathbb{P}\Big(\|\mathfrak{u}_{\gamma,N(\gamma)} - \mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big) \\ &=: A(\gamma) + B(\gamma). \end{aligned} \tag{3.125}$$

By Step 1, we have that $B(\gamma) \to 0$ as $\gamma \to 0$. Fix $\kappa > 0$ and denote by $E(\gamma, \kappa)$ the event

$$E(\gamma,\kappa) := \Big\{ \big\| \mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1} - \mathcal{G}_\infty(u_0)^{-1} \big\|_{C([0,1];\mathbb{R})} > \kappa \Big\}.$$

Then we have

$$\begin{aligned} A(\gamma) &\le \rho \otimes \mathbb{P}\Big(\Big\{ \big\| \mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1} - \mathcal{G}_\infty(u_0)^{-1} \big\|_{C([0,1];\mathbb{R})} \cdot \|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big\} \cap E(\gamma,\kappa)^c \Big) \\ &\quad + \rho \otimes \mathbb{P}\Big(\Big\{ \big\| \mathcal{G}_{N(\gamma)}(u_{\gamma,N(\gamma)})^{-1} - \mathcal{G}_\infty(u_0)^{-1} \big\|_{C([0,1];\mathbb{R})} \cdot \|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2} \Big\} \cap E(\gamma,\kappa) \Big) \\ &\le \rho \otimes \mathbb{P}\Big(\|\mathfrak{u}_0\|_{C_T H_x^{-\delta}} > \frac{\varepsilon}{2\kappa} \Big) + \rho \otimes \mathbb{P}\big(E(\gamma,\kappa) \big). \end{aligned} \tag{3.126}$$

As $\kappa \to 0$, we have

$$\rho \otimes \mathbb{P}\Big(\|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2\kappa} \Big) \longrightarrow \rho \otimes \mathbb{P}\Big(\|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} = \infty \Big) = 0, \tag{3.127}$$

by Step 1. Furthermore, by Step 2, $\rho \otimes \mathbb{P}\big(E(\gamma,\kappa)\big) \to 0$ as $\gamma \to 0$. Thus, we deduce by the last observation and (3.126) that

$$\limsup_{\gamma \to 0} A(\gamma) \le \rho \otimes \mathbb{P}\Big(\|\mathfrak{u}_0\|_{C([0,1];H_x^{-\delta})} > \frac{\varepsilon}{2\kappa} \Big),$$

and hence, by (3.127), $A(\gamma) \to 0$, as $\gamma \to 0$. By (3.125), this shows that $\mathrm{I}(\gamma) \to 0$ as $\gamma \to 0$.

Next, by (3.64) in Proposition 3.2.4, we have

$$\begin{aligned} \mathrm{II}(\gamma) &= \rho \otimes \mathbb{P}\big(\|u_\gamma - u_{\gamma,N(\gamma)}\|_{C([0,1];H_x^{-\delta})} > \varepsilon \big) \\ &\lesssim \exp\big(-c\gamma^{\theta_1} \log(\varepsilon N(\gamma)^{\theta_2})^\beta \big) + \varepsilon^{-2} N(\gamma)^{-2\delta_1} + o_{\gamma\to 0}(1). \end{aligned} \tag{3.128}$$

Hence, by picking $N(\gamma) = \exp(\gamma^{-A})$ for some large constant $A > 0$, we have, by (3.128), that $\mathrm{II}(\gamma) \to 0$ as $\gamma \to 0$. By (3.124), this proves that

$$\rho \otimes \mathbb{P}\Big(\|u - u_\gamma\|_{C([0,1];H_x^{-\delta})} > \varepsilon \Big) \longrightarrow 0,$$

as $\gamma \to 0$ and for any $\varepsilon > 0$. This finishes the proof of Theorem 1.2.2 $\square$

3.3 Counting estimates

We state here several counting estimates which are used in the construction of the stochastic objects in Section 3.4.

Let $\{(x,\iota_1),(y,\iota_2),(z,\iota_3)\}$ be a set consisting of integers $(x,y,z)\in(\mathbb{Z}^2)^3$ together with some signs $(\iota_1,\iota_2,\iota_3)\in\{\pm1\}^3$. We say that (x,y) is a *pairing* if $x=y$ and $\iota_1=-\iota_2$, and similarly for (y,z), etc. With an abuse of notations, given an affine integral equation of the form

$$\iota_1 x+\iota_2 y+\iota_3 z=d,$$

we say that, for instance, that (x,y) is are paired if $x=y$ and $\iota_1=-\iota_2$.

The following lemma is from [25] but similar counting arguments are already discussed in the work of Bourgain [7].

Lemma 3.3.1. *Given dyadic numbers $N_1\gtrsim N_2\gtrsim N_3$, let $(\iota_1,\iota_2,\iota_3)\in\{\pm1\}^3$ be signs and consider the set S given by*

$$S=\{(x,y,z)\in(\mathbb{Z}^2)^3:\iota_1x+\iota_2y+\iota_3z=d,\ \iota_1\langle x\rangle^2+\iota_2\langle y\rangle^2+\iota_3\langle z\rangle^2=\alpha,\\ |x-a|\lesssim N_1,|y-b|\lesssim N_2,|z-c|\lesssim N_3\}.$$

We assume that there is no pairing in S. Then, the following bound holds:

$$|S|\lesssim N_2^{1+\theta}N_3,$$

for any $\theta>0$ and uniformly in $(a,b,c,d,\alpha)\in(\mathbb{Z}^2)^5$.

Proof. See [25, Lemma 4.3] □

Let us now define the following phase function:

$$\kappa(\bar n):=\langle n\rangle^2-\langle n_1\rangle^2+\langle n_2\rangle^2-\langle n_3\rangle^2, \tag{3.129}$$

with the vectorial notation $\bar n=(n,n_1,n_2,n_3)\in\mathbb{Z}^4$. We use in the remainder of this section the notations of Appendix A.4. Given a tensor $h=h_{nn_1n_2n_3}$, we define the norm

$$\|h\|_1:=\max\big(\|h\|_{n\to n_1n_2n_3},\|h\|_{n_1\to nn_2n_3},\|h\|_{nn_2\to n_1n_3},\|h\|_{nn_3\to n_1n_2}\big). \tag{3.130}$$

and

The following tensor estimates will be useful when handling the random matrix terms in Lemma 3.4.7. In dealing with the different stochastic objects we will need counting estimates that take into account the scenario when the phase $\kappa(\bar n)$ (3.129) is fixed and *dispersionless estimates* (i.e. when there is no condition on $\kappa(\bar n)$).

Lemma 3.3.2 (Tensor bounds I)**.** *Fix $(n_{\star,1},n_{\star,3},n_{\star,3})\in\mathbb{Z}^3$ and $(N_1,N_2,N_3)\in\mathbb{N}^3$. Let $h=h_{nn_1n_2n_3}$ be the tensor given by*

$$h_{nn_1n_2n_3}=\mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}}\prod_{j=1}^{3}\mathbb{1}_{\langle n_j-n_{\star,j}\rangle\sim N_j} \tag{3.131}$$

We also define, for $m\in\mathbb{Z}$, the tensor $h^m=h\mathbb{1}_{\kappa(\bar n)=m}$ with $\kappa=\kappa(\bar n)$ as in (3.129).

(i) *The following bound on h holds:*

$$\|h\|_1\lesssim N_{\max}N_{\mathrm{med}},$$

uniformly in $(n_{\star,1},n_{\star,3},n_{\star,3})\in\mathbb{Z}^3$.

(ii) *The following bound on h^m holds:*

$$\sup_{m\in\mathbb{Z}} \|h^m\|_1 \lesssim N_{\max}^{\frac{1}{2}+\theta} N_{\text{med}}^{\frac{1}{2}}$$

for any $\theta > 0$*, and uniformly in* $(n_{\star,1}, n_{\star,3}, n_{\star,3}) \in \mathbb{Z}^3$*.*

(iii) *If* $N_1 \gtrsim \min(N_2, N_3)$*, then we have the following improvements:*

$$\|h\|_1 \lesssim N_{\max} \min(N_2, N_3),$$
$$\sup_{m\in\mathbb{Z}} \|h^m\|_1 \lesssim N_{\max}^{\frac{1}{2}+\theta} \min(N_2, N_3)^{\frac{1}{2}},$$

for any $\theta > 0$*, and uniformly in* $(n_{\star,1}, n_{\star,3}, n_{\star,3}) \in \mathbb{Z}^3$*.*

Proof. We only prove (ii) as (i) follows from simpler arguments. By Schur's test and Lemma 3.3.1, we have

$$\begin{aligned} \|h^m\|^2_{n\to n_1n_2n_3} &\lesssim \sup_{n_1,n_2,n_3} |\{n : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3\}| && (3.132)\\ &\quad \times \sup_{n} |\{(n_1, n_2, n_3) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| \\ &\lesssim N_{\text{med}}^{1+\theta} N_{\min}, && (3.133)\end{aligned}$$

for any $\theta > 0$. Note that we have used (3.132) $\lesssim 1$ since n is uniquely determined as long as (n_1, n_2, n_3) are fixed.
Similarly and since the only bound available in general for n is $\langle n\rangle \lesssim N_{\max}$, we have

$$\|h^m\|^2_{n_1\to nn_2n_3} \lesssim N_{\max}^{1+\theta} N_{\text{med}}. \tag{3.134}$$

Applying again Schur's test, we get

$$\begin{aligned} \|h^m\|^2_{nn_2\to n_1n_3} &\lesssim \sup_{n,n_2} |\{(n_1, n_3) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| && (3.135)\\ &\quad \times \sup_{n_1,n_3} |\{(n, n_2) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}|. && (3.136)\end{aligned}$$

Note that for $n = n_1 - n_2 + n_3$, we have

$$\begin{aligned} \kappa(\bar{n}) &= |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = 2\langle n_2 - n_1, n_2 - n_3\rangle \\ &= 2\langle n - n_3, n_2 - n_3\rangle = 2\langle n - n_3, n - n_1\rangle, \end{aligned} \tag{3.137}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^2$. This leads to the following formulas:

$$\begin{aligned} \kappa(\bar{n}) &= -2\Big|n_1 - \frac{n+n_2}{2}\Big|^2 + 2\Big|\frac{n-n_2}{2}\Big|^2 \\ &= -2\Big|n_2 - \frac{n_1+n_3}{2}\Big|^2 + 2\Big|\frac{n_1-n_3}{2}\Big|^2. \end{aligned} \tag{3.138}$$

Hence, if n and n_2 are fixed then we deduce from (3.138) that n_1 belongs to a circle of radius at most $\sim N_{\max}$, which leads to the bound (3.135) $\lesssim N_{\max}^{\theta}$, for any $\theta > 0$. Similarly, (3.136) $\lesssim N_{\max}^{\theta}$, for any $\theta > 0$. This yields

$$\|h^m\|^2_{nn_2\to n_1n_3} \lesssim N_{\max}^{\theta}, \tag{3.139}$$

for any $\theta > 0$.

At last, we estimate

$$\|h^m\|^2_{nn_3\to n_1n_2} \lesssim \sup_{n,n_3} |\{(n_1,n_2) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| \tag{3.140}$$

$$\times \sup_{n_1,n_2} |\{(n,n_3) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| \tag{3.141}$$

By (3.137), we deduce that if n and n_3 are fixed, then n_1 and n_2 both belongs to lines. Thus, (3.140) $\lesssim \min(N_1, N_2)$. Similarly, (3.141) $\lesssim N_3$. This proves

$$\|h^m\|^2_{nn_3\to n_1n_2} \lesssim N_{\text{max}} N_{\text{med}}. \tag{3.142}$$

The estimate (ii) follows from (3.133), (3.134), (3.139), and (3.142). The bounds (iii) follow from similar arguments. □

Let $\|\cdot\|_2$ be the norm given by

$$\|h\|_2 := \max\big(\|h\|_{n\to n_1n_2n_3}, \|h\|_{nn_1\to n_2n_3}\big). \tag{3.143}$$

We next state a counting estimate used to handle the bilinear operators in Subsection 3.4.3

Lemma 3.3.3 (Tensor bounds II)**.** *Fix* $(n_{\star,1}, n_{\star,3}, n_{\star,3}) \in \mathbb{Z}^3$ *and* $(N_1, N_2, N_3) \in \mathbb{N}^3$*. Let* $h = h_{nn_1n_2n_3}$ *as in* (3.131)*. We also define, for* $m \in \mathbb{Z}$*, the tensor* $h^m = h\mathbb{1}_{\kappa(\bar{n})=m}$ *with* $\kappa = \kappa(\bar{n})$ *as in* (3.129)*.*

(i) *The following bound on* h *holds:*

$$\|h\|_2 \lesssim N_{\text{max}} N_{\text{min}},$$

uniformly in $(n_{\star,1}, n_{\star,3}, n_{\star,3}) \in \mathbb{Z}^3$*.*

(ii) *The following bound on* h^m *holds:*

$$\sup_{m\in\mathbb{Z}} \|h^m\|_2 \lesssim N_{\text{max}}^{\frac{1}{2}+\theta} N_{\text{min}}^{\frac{1}{2}}$$

for any $\theta > 0$*, and uniformly in* $(n_{\star,1}, n_{\star,3}, n_{\star,3}) \in \mathbb{Z}^3$*.*

Proof. We only prove (ii) as (i) follows from simpler arguments. By Schur's test, we have

$$\|h^m\|^2_{n\to n_1n_2n_3} \lesssim N_{\text{med}}^{1+\theta} N_{\text{min}}. \tag{3.144}$$

for any $\theta > 0$ as in the proof of Lemma 3.3.2. Similarly, in view of (3.140) and (3.141) in the proof of Lemma 3.3.2, we have

$$\begin{aligned}\|h^m\|^2_{nn_1\to n_2n_3} &\lesssim \sup_{n,n_1} |\{(n_2,n_3) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| \\ &\quad \times \sup_{n_2,n_3} |\{(n,n_1) : n = n_1 - n_2 + n_3;\ n_2 \neq n_1, n_3 \text{ and } \kappa(\bar{n}) = m\}| \\ &\lesssim \min(N_2, N_3) \times N_1. \end{aligned} \tag{3.145}$$

□

3.4 Regularities of the stochastic terms

In this section, we establish the almost-sure convergence (in the parameter $N \in \mathbb{N}$) for the stochastic objects introduced in Subsection 1.2.2 in relevant topologies. These stochastic terms are measurable functions in the independent random variables $\phi \sim \rho$ with ρ as in (1.48) and the space-time white noise ξ.

In order to upgrade the convergence in probability described in Subsection 3.2.2, we wish to construct the stochastic terms on a "*bundle*" Ω_0 of the form

$$\Omega_0 = \bigcup_{\phi_\star \in \Omega_\phi} \{\phi_\star\} \times \Omega_\xi(\phi_\star), \tag{3.146}$$

where $\rho(\Omega_\phi) = 1$ and $\mathbb{P}(\Omega_\xi(\phi_\star)) = 1$, for each $\phi_\star \in \Omega_\phi$. We first check in the lemma below that this construction is valid, i.e. that a set of the form (3.146) is indeed measurable.

Lemma 3.4.1. *Let $\Omega_\phi \subset H^{-1}(\mathbb{T}^2)$ be a measurable set of full ρ-measure. For each $\phi_\star \in \Omega_\phi$ let $\Omega_\xi(\phi_\star) \subset \Omega$ be a measurable set of full $\mathbb{P}$-measure. Then, the set Ω_0 defined in (3.146) is a measurable set.*

Proof. It suffices to prove that Ω_0^c is a null set. We have

$$\begin{aligned}
\Omega_0^c &= \bigcap_{\phi_\star \in \Omega_\phi} \big(\{\phi_\star\} \times \Omega_\xi(\phi_\star)\big)^c \\
&= \bigcap_{\phi_\star \in \Omega_\phi} \big(H^{-1}(\mathbb{T}^2) \setminus \{\phi_\star\} \times \Omega\big) \cup \big(H^{-1}(\mathbb{T}^2) \times \Omega \setminus \Omega_\xi(\phi_\star)\big) \cup \big(\, H^{-1}(\mathbb{T}^2) \setminus \{\phi_\star\} \times \Omega \setminus \Omega_\xi(\phi_\star)\big) \\
&\subset \big(H^{-1}(\mathbb{T}^2) \setminus \Omega_\phi \times \Omega\big) \cup \big(H^{-1}(\mathbb{T}^2) \times \Omega \setminus \Omega_\xi(\phi_\star^0)\big) \cup \big(H^{-1}(\mathbb{T}^2) \times \Omega \setminus \Omega_\xi(\phi_\star^0)\big) =: \Omega_1,
\end{aligned}$$

where $\phi_\star^0$ is any fixed element in Ω_ϕ. Note that Ω_1 is a measurable set and that $\rho \otimes \mathbb{P}(\Omega_1) = 0$. This finishes the proof. ☐

It is then easy to see by using the Fubini theorem that $\rho \otimes \mathbb{P}(\Omega_0) = 1$. We further explain how to achieve the construction of the set (3.146) in what follows.

Remark 3.4.2. Let us emphasize the fact that almost-sure convergence of stochastic objects on a set of the form (3.146) is in constrast with the literature, where the stochastic objects are usually constructed on a set Ω_0 of full $\rho \otimes \mathbb{P}$-probability, which cannot be, a priori, written as a set of the form (3.146); see for instance [22, 31, 32, 63, 33, 24, 25].

We first reduce the convergence of the stochastic terms defined in Subsection 1.2.2. Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of stochastic objects of interest which are either distribution-valued or operator-valued. We aim to prove that $A_N = A_N(\phi, \xi)$ converges in a Banach space $(X, \|\cdot\|)$ and on a set of the form (3.146) of full $\rho \otimes \mathbb{P}$-probability.

We assume that the two following estimates hold:

$$\Big\| \sup_{N\in\mathbb{N}} \|A_N\| \Big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim_p 1, \tag{3.147}$$

$$\Big\| \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \|A_M - A_N\| \Big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim_p 1, \tag{3.148}$$

for any $p \geq 1$ and some $\delta > 0$. Then, by Chebyshev's inequality, Proposition 1.2.1 and Fubini's theorem, we have

$$\begin{aligned}
&\rho\Big(\max\Big(\Big\| \sup_{N\in\mathbb{N}} \|A_N\| \Big\|_{L^p(\mathbb{P})}, \Big\| \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \|A_M - A_N\| \Big\|_{L^p(\mathbb{P})}\Big) > \lambda\Big) \\
&\quad\lesssim \frac{\max\Big(\big\| \sup_{N\in\mathbb{N}} \|A_N\| \big\|^p_{L^p(\rho\otimes\mathbb{P})}, \big\| \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \|A_M - A_N\| \big\|^p_{L^p(\rho\otimes\mathbb{P})}\Big)}{\lambda^p} \\
&\quad\lesssim_p \lambda^{-p},
\end{aligned}$$

for any $\lambda > 0$. Hence, there exists a set Ω_ϕ with $\rho(\Omega_\phi) = 1$ such that for each $\phi_\star \in \Omega_\phi$, there exists a positive constant $C(p, \phi_\star) < \infty$ such that

$$\begin{aligned}
&\Big\| \sup_{N\in\mathbb{N}} \|A_N(\phi_\star, \cdot)\| \Big\|_{L^p(\mathbb{P})} \leq C(p, \phi_\star), \\
&\Big\| \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \|A_M(\phi_\star, \cdot) - A_N(\phi_\star, \cdot)\| \Big\|_{L^p(\mathbb{P})} \leq C(p, \phi_\star)
\end{aligned} \tag{3.149}$$

uniformly in positive integers N and M with $N \geq M$. We highlight here that the constant $C(p, \phi_\star)$ is uniform in the parameters N and M. This is why we need to have the suprema *inside* the $L^p(\mu \otimes \mathbb{P})$-norms in the bounds (3.147) and (3.148).

The $\mathbb{P}$-almost sure convergence of $\{A_N(\phi_\star, \cdot)\}_{N\in\mathbb{N}}$ in $(X, \|\cdot\|)$ then follows immediately from (3.149). This finishes the construction of a set Ω_0 as in (3.146) such that $\{A_N\}_{N\in\mathbb{N}}$ converge in $(X, \|\cdot\|)$ on Ω_0.

In practice, the bounds (3.147) and (3.148) follow from certain estimates on appropriate frequency localized versions of A_N; see Lemma A.3.1 in Section A.3 below.

3.4.1 Basic stochastic terms

The purpose of this subsection is to construct the "simple" distribution-valued stochastic objects that appear in the expressions (1.79) and (1.80).

Lemma 3.4.3. *Let $b < \frac{1}{2}$ and $\delta > 0$.*

(i) (linear objects.) *The sequence $\{\gamma \mapsto \overset{\bullet}{|}_{\gamma,N}\}_{N\in\mathbb{N}}$ (1.74) is a Cauchy sequence in $C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\cap \mathcal{F}L^{1-\delta,\infty}(\mathbb{T}^2)\big)\cap C\big([0,1]; X^{-\delta,b}([0,1])\big)$ on a set of the form (3.146). In particular, denoting by $\gamma\mapsto \overset{\bullet}{|}_\gamma$ the limit, we have*

$$\gamma\mapsto \overset{\bullet}{|}_\gamma \in C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\cap \mathcal{F}L^{1-\delta,\infty}(\mathbb{T}^2)\big)\cap C\big([0,1]; X^{-\delta,b}([0,1])\big),$$

$\rho\otimes\mathbb{P}$-almost surely.

(ii) (nonlinear objects.) *Let $\{\gamma\mapsto B_{\gamma,N}\}_{N\in\mathbb{N}}$ be either the sequence $\{\gamma\mapsto :|\overset{\bullet}{|}_{\gamma,N}|^2:\}_{N\in\mathbb{N}}$ or $\{\gamma\mapsto :|\overset{\bullet}{|}_{\gamma,N}|^2\overset{\bullet}{|}_{\gamma,N}:\}_{N\in\mathbb{N}}$ defined in (1.76), or $\{\gamma\mapsto \overset{\bullet}{|}^2_{\gamma,N}\}_{N\in\mathbb{N}}$. Then, $\{\gamma\mapsto B_{\gamma,N}\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\big)$ on a set of the form (3.146). In particular, denoting by $\gamma\mapsto B_\gamma$ the limit, we have*

$$\gamma\mapsto B_\gamma\in C\big([0,1]\times[0,1]; W^{-\delta,\infty}(\mathbb{T}^2)\big),$$

$\rho\otimes\mathbb{P}$-almost surely.

Proof. The proof of the above is similar to that of Proposition 2.1.10 and makes use of the bi-parameter Kolmogorov continuity criterion Lemma A.2.3 and the reductions discussed at the beginning of Section 3.4 and Lemma A.3.1 in Section A.3. We omit details. ☐

We next construct the stochastic term Ψ_γ as the limit of the sequence (1.81) as $N\to\infty$.

Lemma 3.4.4. *For any $0 < s < \frac{1}{2}$ and $0 < \delta = \delta(s) \ll 1$, the sequence $\{\gamma\mapsto \Psi_{\gamma,N}\}_{N\in\mathbb{N}}$ defined in (1.81) is a Cauchy sequence in $C\big([0,1]; X^{s,\frac{1}{2}+\delta}([0,1])\big)$ on a set of the form (3.146). In particular, denoting by $\gamma\mapsto\Psi_\gamma$ the limit, we have*

$$\gamma\mapsto\Psi_\gamma\in C\big([0,1]; X^{s,\frac{1}{2}+\delta}([0,1])\big),$$

$\rho\otimes\mathbb{P}$-almost surely.

Proof. By Proposition 3.1.14, it suffices to show that $\{\gamma\mapsto \psi_{\gamma,N}\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]; X^{s,-\frac{1}{2}+\delta_0}([0,1])\big)$, on a set of the form (3.146) for $\delta_0 > 0$. Let $\gamma\in[0,1]$ and $N\in\mathbb{N}$. From (A.6), we have

$$\mathbb{1}_{[0,1]}\widehat{\psi}_{\gamma,N}(n,t) = I_3[h_{n,t}]. \tag{3.150}$$

with

$$\begin{aligned} h_{\gamma,n,t}(z_1,z_2,z_3) &= \mathbb{1}_{[0,1]}(t)\mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}}\prod_{j=1}^{3}\big(e^{-(\gamma+i)t\langle n_j\rangle^2}\mathbb{1}_{\langle n_j\rangle\leq N}\big)^{\iota_j}\\ &\quad\times\prod_{j=1}^{3}\Big(\mathbb{1}_{\zeta_j=-1}\mathbb{1}_{[0,1]}(t_j)+\mathbb{1}_{\zeta_j=1}\mathbb{1}_{[0,t]}(t_j)\sqrt{2\gamma}e^{(\gamma+i)t_j\langle n_j\rangle^2}\Big)^{\iota_j}, \end{aligned} \tag{3.151}$$

with $z_j = (n_j, t_j, \zeta_j)$ and $\iota_j = 1$ if j is odd and -1 otherwise. In what follows, we will often omit the dependence of the quantities in the variables $(z_j)_{j=1,2,3}$ to ease our notations.

We can further write $h_{\gamma,n,t}$ as a sum of terms of the form

$$\begin{aligned} h^A_{\gamma,n,t}(n_1,t_1,n_2,t_2,n_3,t_3) = \mathbb{1}_{[0,1]}(t)\mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \prod_{j=1}^{3} \big(e^{-(\gamma+i)t\langle n_j\rangle^2}\mathbb{1}_{\langle n_j\rangle\le N}\big)^{\iota_j} \\ \times \prod_{j\in A}\big(\mathbb{1}_{[0,t]}(t_j)\sqrt{2\gamma}e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j} \cdot \prod_{j\in B}\frac{\mathbb{1}_{[0,1]}(t_j)}{\langle n_j\rangle}, \end{aligned}$$

where A and B form a partition of $\{1,2,3\}$ such that $\zeta_j = 1$ for $j \in A$ and $\zeta_j = -1$ for $j \in B$. Then, by using Lemma A.2.5, we compute the twisted space-time Fourier transform (1.88) of (3.150)

$$\widetilde{\mathbb{1}_{[0,1]}\Psi_{\gamma,N}}(n,\lambda) = \sum_{A\subset\{1,2,3\}} I_3\big[\widetilde{h}_{\gamma,n,\lambda}\big], \tag{3.152}$$

where

$$\begin{aligned} \widetilde{h}^A_{\gamma,n,\lambda} = \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \int_{t_{\max}(A)}^{1} e^{-t(i(\lambda-\kappa(\bar{n}))+\gamma\beta_0(\bar{n}))}dt \\ \times \prod_{j\in A}\big(\sqrt{2\gamma}e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j} \cdot \prod_{j\in B}\frac{\mathbb{1}_{[0,1]}(t_j)}{\langle n_j\rangle} \cdot \prod_{j=1}^{3}\mathbb{1}_{[0,T]}(t_j), \end{aligned} \tag{3.153}$$

with

$$\begin{aligned} \kappa(\bar{n}) &:= \langle n\rangle^2 - \langle n_1\rangle^2 + \langle n_2\rangle^2 - \langle n_3\rangle^2 \\ \beta_0(\bar{n}) &:= \langle n_1\rangle^2 + \langle n_2\rangle^2 + \langle n_3\rangle^2, \end{aligned} \tag{3.154}$$

and

$$t_{\max}(A) = \{t_j : j \in A\}. \tag{3.155}$$

We now localize the variables n_j to the regions $\langle n_j\rangle \sim N_j$ for dyadics $N_j \geq 1$ $(j = 1,2,3)$. Let $N_\star = (N_1, N_2, N_3)$ and denote by $\widetilde{h}^{A,N_\star}_{\gamma,n,\lambda}$ the contribution of $\langle n_j\rangle \sim N_j$ to $\widetilde{h}^A_{\gamma,n,\lambda}$. Similarly, we will denote by $\Psi^{N_\star}_{\gamma,N}$ the contribution of $\langle n_j\rangle \sim N_j$ to $\Psi_{\gamma,N}$.

As discussed at the beginning of Section 3.4, we first aim to show the following frequency localized estimate:

$$\big\|\mathbb{1}_{[0,1]}\Psi^{N_\star}_{\gamma,N}\big\|_{L^p(\mu\otimes\mathbb{P})X^{s,-\frac{1}{2}+\delta_0}} \lesssim p^{\frac{3}{2}}N_{\max}^{-\theta}, \tag{3.156}$$

for any dyadics $N_\star = (N_1, N_2, N_3)$, uniformly in $(N,\gamma) \in \mathbb{N}\times[0,1]$, and for some $\theta > 0$. Here, $N_{\max} = \max(N_1, N_2, N_3)$. By Lemma A.2.1, (3.156) follows from the bound

$$\big\|\mathbb{1}_{[0,1]}\Psi^{N_\star}_{\gamma,N}\big\|_{L^2(\mu\otimes\mathbb{P})X^{s,-\frac{1}{2}+\delta_0}} \lesssim N_{\max}^{-\theta}, \tag{3.157}$$

with the same parameters.

By (3.152), and Lemma A.2.4 (iii), we have

$$\begin{aligned} \|\mathbb{1}_{[0,1]}\Psi^{N_\star}_{\gamma,N}\|_{L^2(\mu\otimes\mathbb{P})X^{s,b}} &= \big\|\|\langle n\rangle^s\langle\lambda\rangle^b\widetilde{\mathbb{1}_{[0,1]}\Psi^{N_\star}_{\gamma,N}}(n,\lambda)\|_{\ell^2_nL^2_\lambda}\big\|_{L^2(\mu\otimes\mathbb{P})} \\ &\le \max_{A\subset\{1,2,3\}}\big\|\|\langle n\rangle^s\langle\lambda\rangle^bI_3[\widetilde{h}^{A,N_\star}_{\gamma,n,\lambda}]\|_{\ell^2_nL^2_\lambda}\big\|_{L^2(\mu\otimes\mathbb{P})} \\ &\le \max_{A\subset\{1,2,3\}}\big\|\langle\lambda\rangle^b\|\,\mathtt{Sym}(\langle n\rangle^s\widetilde{h}^{A,N_\star}_{\gamma,n,\lambda})\|_{L^2_{z_1,z_2,z_3}}\big\|_{\ell^2_nL^2_\lambda}, \end{aligned} \tag{3.158}$$

for any $b \in \mathbb{R}$. Hence, from (3.158) and (3.152) with (3.153), we have the crude bound:

$$\|\mathbb{1}_{[0,1]}\overset{\bullet\bullet\bullet}{\vee}{}^{N_\star}_{\gamma,N}\|_{L^2(\mu\otimes\mathbb{P})X^{0,0}} \lesssim N_{\max}^{10},$$

uniformly in $(N,\gamma)\in\mathbb{N}\times[0,1]$. Thus, by interpolation, (3.157) follows from the estimate

$$\big\|\mathbb{1}_{[0,1]}\overset{\bullet\bullet\bullet}{\vee}{}^{N_\star}_{\gamma,N}\big\|_{L^2(\mu\otimes\mathbb{P})X^{s,-\frac{1}{2}-\delta_1}} \lesssim N_{\max}^{-\theta}, \tag{3.159}$$

uniformly in $(N,\gamma)\in\mathbb{N}\times[0,1]$, and for some small $\theta>0$ and small $\delta_1>0$. Next, by Hölder's inequality (using $-\frac{1}{2}-\delta_1<-\frac{1}{2}$) and (3.158), (3.159) reduces to estimating $\big\|\mathtt{Sym}(\langle n\rangle^s\widetilde{h}^{A,N_\star}_{\gamma,n,\lambda})\big\|_{L^2_{z_1,z_2,z_3}}$. Hence, by Jensen's inequality (A.3), (3.151), and relabelling, we only have to show

$$\max_{A\subset\{1,2,3\}}\big\|\langle n\rangle^s\widetilde{h}^{A,N_\star}_{\gamma,n,\lambda}\big\|_{L^2_{z_1,z_2,z_3}} \lesssim N_{\max}^{-\theta}, \tag{3.160}$$

uniformly in $(N,\gamma)\in\mathbb{N}\times[0,1]$ and for some small $\theta>0$.

We now evaluate the t-integration in (the frequency localized version of) (3.153). We have

$$\begin{aligned}
&\int_{t_{\max}(A)}^1 e^{-t(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}dt\cdot\prod_{j\in A}\big(\sqrt{2\gamma}e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j}\\
&=(2\gamma)^{\frac{|A|}{2}}e^{-t_{\max}(A)(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}\,\frac{1-e^{-(1-t_{\max}(A))(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}}{i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n)}\\
&\qquad\times\prod_{j\in A}\big(e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j}
\end{aligned} \tag{3.161}$$

We note that by the definition of $t_{\max}(A)$ (3.155), the following bound holds:

$$\Big|e^{-t_{\max}(A)(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}\cdot\prod_{j\in A}\big(e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j}\Big|\lesssim\prod_{j\in A}e^{-\gamma(t_{\max}(A)-t_j)\langle n_j\rangle^2} \tag{3.162}$$

Further, by the mean value theorem, we have

$$\begin{aligned}
\Big|\frac{1-e^{-(1-t_{\max}(A))(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}}{i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n)}\Big|&\lesssim\frac{1}{\langle i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n)\rangle}\\
&\lesssim\frac{1}{\langle\lambda-\kappa(\bar n))\rangle^{\frac{1}{2}}\cdot\gamma^{\frac{1}{2}}\langle n_{\max}(A)\rangle},
\end{aligned} \tag{3.163}$$

where $n_{\max}(A)$ is the maximum over the set $\{n_j:j\in A\}$. Similarly, we define $n_{\mathrm{med}}(A)$ and $n_{\min}(A)$ as the second largest and smallest elements in the set $\{n_j:j\in A\}$, respectively. Thus, combining (3.161), (3.162) and (3.163) yields

$$\begin{aligned}
&\Big|\int_{t_{\max}(A)}^1 e^{-t(i(\lambda-\kappa(\bar n))+\gamma\beta_0(\bar n))}dt\cdot\prod_{j\in A}\big(\sqrt{2\gamma}e^{(\gamma+i)t_j\langle n_j\rangle^2}\big)^{\iota_j}\Big|\\
&\lesssim\frac{1}{\langle\lambda-\kappa(\bar n))\rangle^{\frac{1}{2}}\cdot\gamma^{\frac{1}{2}}\langle n_{\max}(A)\rangle}\cdot\prod_{j\in A}e^{-\gamma(t_{\max}(A)-t_j)\langle n_j\rangle^2}.
\end{aligned} \tag{3.164}$$

Fix $A\subset\{1,2,3\}$. By (the frequency localized version of) (3.153) and (3.164), we get

$$
\begin{aligned}
\big\| \langle n \rangle^s \widetilde{h}^{A,N_\star}_{\gamma,n,\lambda} \big\|^2_{L^2_{t_1,t_2,t_3}} &\lesssim \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \langle n \rangle^{2s} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j \rangle \sim N_j} \\
&\quad \times \prod_{j\in B} \frac{1}{\langle n_j \rangle^2} \cdot \frac{\gamma^{|A|}}{\langle \lambda - \kappa(\bar{n}) \rangle \cdot \gamma \langle n_{\max}(A) \rangle^2} \\
&\quad \times \Big\| \prod_{j\in A} e^{-\gamma(t_{\max}(A)-t_j)\langle n_j\rangle^2} \mathbb{1}_{[0,T]}(t_j) \Big\|^2_{L^2_{(t_j:j\in A)}} \\
&\lesssim \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \langle n \rangle^{2s} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j \rangle \sim N_j} \prod_{j\in B} \frac{1}{\langle n_j \rangle^2} \\
&\quad \times \frac{\gamma^{|A|}}{\langle \lambda - \kappa(\bar{n}) \rangle \cdot \gamma \langle n_{\max}(A) \rangle^2} \cdot \frac{1}{\gamma^{|A|-1} \langle n_{\mathrm{med}} \rangle^2 \langle n_{\min} \rangle^2} \\
&\lesssim \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \langle n \rangle^{2s} \prod_{j=1}^{3} \frac{\mathbb{1}_{\langle n_j \rangle \sim N_j}}{\langle n_j \rangle^2} \cdot \frac{1}{\langle \lambda - \kappa(\bar{n}) \rangle},
\end{aligned}
\tag{3.165}
$$

From (3.165), we then deduce the following bound:

$$
\begin{aligned}
\max_{A\subset\{1,2,3\}} \big\| \langle n \rangle^s \widetilde{h}^{A,N_\star}_{\gamma,n,\lambda} \big\|^2_{L^2_{z_1,z_2,z_3}} &\lesssim \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N}} \frac{\langle n \rangle^{2s}}{\langle \lambda - \kappa(\bar{n}) \rangle \langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \qquad (3.166) \\
&\lesssim \sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \int_{\mathbb{R}} \frac{d\lambda}{\langle \lambda \rangle^{1+2\delta} \langle \lambda - \kappa(\bar{n}) \rangle} \\
&\lesssim \sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2 \kappa(\bar{n})^{1+2\delta}} \\
&\lesssim \sum_{\kappa\in\mathbb{Z}} \frac{1}{\langle \kappa \rangle^{1+2\delta}} \sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N\\ \kappa(\bar{n})=\kappa}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \\
&\lesssim \sup_{\kappa\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2s} \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N\\ \kappa(\bar{n})=\kappa}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2}, \qquad (3.167)
\end{aligned}
$$

Let us assume $N_1 \geq N_2 \geq N_3$ as the proof is similar in other cases. From Lemma 3.3.1 and by noting that $|n| \lesssim N_{\max}$, we then have

$$
\begin{aligned}
(3.167) &\lesssim N_{\max}^{2s} (N_1 N_2 N_3)^{-2} N_3^2 \sup_{\kappa\in\mathbb{Z}} \sum_{\substack{n_3=n-n_1+n_2\\ n_2\neq n_1,n_3\\ \langle n_j \rangle \sim N_j \leq N\\ \kappa(\bar{n})=\kappa}} 1 \\
&\lesssim N_{\max}^{2s} (N_1 N_2)^{-2} N_1^{1+\varepsilon} N_2 \lesssim T^{\alpha} N_{\max}^{2s-1+\varepsilon},
\end{aligned}
\tag{3.168}
$$

for any $\varepsilon > 0$. This proves (3.160) with $s < \frac{1}{2}$.

We now aim to prove the following bound:

$$
N^{\theta} \Big\| \| \mathbb{1}_{[0,1]} (\Psi^{N_\star}_{\gamma,M} - \Psi^{N_\star}_{\gamma,N}) \|_{X^{s,-\frac{1}{2}+\delta_0}} \Big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim p^{\frac{3}{2}} N_{\max}^{-\theta},
$$

uniformly in integers $M \geq N$, $\gamma \in [0,1]$ and for all dyadics $N_\star = (N_1, N_2, N_3)$ and all $p \geq 1$ and some small $\theta > 0$. Note that $\Psi^{N_\star}_{\gamma,M} - \Psi^{N_\star}_{\gamma,N}$ reads as (the frequency localized versions of)

(3.150) and (3.151) with the additional condition $\max(\langle n_1\rangle, \langle n_2\rangle, \langle n_3\rangle) \geq N$. Hence, by Lemma A.2.1 and by arguing as before, it suffices to prove the estimate

$$\sup_{\gamma\in[0,1]} \big\| \| \mathbb{1}_{[0,1]} (\Psi^{N_\star}_{\gamma,M} - \Psi^{N_\star}_{\gamma,N}) \|_{X^{s,-\frac{1}{2}+\delta_0}} \big\|_{L^2(\mu\otimes\mathbb{P})} \lesssim N_{\max}^{-\theta}, \tag{3.169}$$

for some small $\theta > 0$ and for any dyadic numbers $N_\star = (N_1, N_2, N_3)$ (as, necessarily $N_{\max} \gtrsim N$ so that we can gain a small power of N from the right-hand-side of (3.169)). The proof of the bound (3.169) follows as that of (3.157).

Lastly, we aim at proving the bound:

$$\begin{aligned} N^\theta \big\| \| \mathbb{1}_{[0,1]} \big((\Psi^{N_\star}_{\gamma_2,M} - \Psi_{\gamma_2,N})^{N_\star} - (\Psi^{N_\star}_{\gamma_1,M} - \Psi^{N_\star}_{\gamma_1,N}) \big) \|_{X^{s,-\frac{1}{2}+\delta_0}} \big\|_{L^p(\mu\otimes\mathbb{P})} \\ \lesssim p^{\frac{3}{2}} N_{\max}^{-\theta} (\gamma_2 - \gamma_1)^\theta, \end{aligned} \tag{3.170}$$

for all $p \geq 1$, any integers $M \geq N$, $(\gamma_1, \gamma_2) \in [0,1]^2$ and for all dyadics $N_\star = (N_1, N_2, N_3)$ and some small $\theta > 0$. By interpolation with (3.169), (3.170) follows from

$$\sup_{N\in\mathbb{N}} \big\| \| \mathbb{1}_{[0,1]} (\Psi^{N_\star}_{\gamma_2,N} - \Psi^{N_\star}_{\gamma_1,N}) \|_{X^{s,-\frac{1}{2}+\delta_0}} \big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim p^{\frac{3}{2}} (\gamma_2 - \gamma_1)^\theta N_{\max}^{-\theta}, \tag{3.171}$$

for $(\gamma_1, \gamma_2) \in [0,1]^2$ and for all $p \geq 1$ and all dyadics $N_\star = (N_1, N_2, N_3)$ and some small $\theta > 0$.

We now prove (3.171). Let $(\gamma_1, \gamma_2) \in (0,1]^2$ and $\gamma \in [0,1]$. From (3.151) and the mean value theorem, we have

$$\begin{aligned} |h^{N_\star}_{\gamma_2,n,t} - h^{N_\star}_{\gamma_1,n,t}| &\lesssim N_{\max}^2 \frac{|\gamma_2 - \gamma_1|}{\max(\gamma_1,\gamma_2)^{\frac{1}{2}}} \\ |h^{N_\star}_{\gamma,n,t} - h^{N_\star}_{0,n,t}| &\lesssim N_{\max}^2 \gamma^{\frac{1}{2}}. \end{aligned}$$

Hence, by interpolation, we obtain

$$|h^{N_\star}_{\gamma_2,n,t} - h^{N_\star}_{\gamma_1,n,t}| \lesssim N_{\max}^2 |\gamma_2 - \gamma_1|^\theta, \tag{3.172}$$

for any $(\gamma_1, \gamma_2) \in [0,1]^2$ and for some small $\theta > 0$. Making again use of Lemma A.2.4 (iii) with (3.172)

$$\begin{aligned} \| \mathbb{1}_{[0,1]} (\Psi^{N_\star}_{\gamma_2,N} - \Psi^{N_\star}_{\gamma_1,N}) \|_{L^2(\mu\otimes\mathbb{P}) X^{s,0}} &= \| \Psi^{N_\star}_{\gamma_2,N} - \Psi^{N_\star}_{\gamma_1,N} \|^2_{L^2(\mu\otimes\mathbb{P}) L^2([0,1]; H^s_x)} \\ &\lesssim \| \langle n \rangle^s \| I_3 \big[h_{\gamma_2,n,t} - h_{\gamma_1,n,t} \big] \|_{L^2(\mu\otimes\mathbb{P})} \|_{L^2([0,1];\ell^2_n)} \\ &\lesssim N_{\max}^{10} |\gamma_2 - \gamma_1|^\theta. \end{aligned} \tag{3.173}$$

for some small $\theta > 0$. Hence, interpolating (3.173) and (3.157) gives (3.171).

Finally, by applying the usual Kolmogorov's continuity criterion (see for instance [3]), we obtain the following bounds:

$$\begin{aligned} \sup_{N\in\mathbb{N}} \big\| \| \mathbb{1}_{[0,1]} \Psi^{N_\star}_{\gamma,N} \|_{C_\gamma X^{s,-\frac{1}{2}+\delta_0}} \big\|_{L^p(\mu\otimes\mathbb{P})} &\lesssim p^{\frac{3}{2}} N_{\max}^{-\theta}, \\ \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\theta \big\| \| \mathbb{1}_{[0,1]} \Psi^{N_\star}_{\gamma,M} - \Psi^{N_\star}_{\gamma,N} \|_{C_\gamma X^{s,-\frac{1}{2}+\delta_0}} \big\|_{L^p(\mu\otimes\mathbb{P})} &\lesssim p^{\frac{3}{2}} N_{\max}^{-\theta}, \end{aligned}$$

for all $p \geq 1$ and some small $\theta > 0$. These bounds correspond to (A.11) and (3.149) above and arguing as in the beginning of Section (3.4) finishes the proof of the result. □

At last, we state a lemma that is crucial in handling the behavior in $\gamma \to 0$ of the gauge transform (1.57).

Lemma 3.4.5. *For $\gamma \in [0,1]$ and $N \in \mathbb{N}$, let $\dot{\vert}_{\gamma,N}$ be as in (1.73) and let $A_{\gamma,N}$ be as in (3.55). Then, for any $T > 0$, the sequence $\{A_{\gamma,N}\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]\times[0,1];\mathbb{R}\big)$ on a set of the form (3.146). In particular, denoting the limit by $\gamma \mapsto A_\gamma$, we have*

$$\gamma \mapsto A_\gamma \in C\big([0,1]\times[0,1];\mathbb{R}\big),$$

$\rho \otimes \mathbb{P}$-*almost surely.*

Proof. We only prove the following bound:

$$\sup_{N\in\mathbb{N}} \sup_{0\le t\le 1} \Big\| \|\text{¡}_{\gamma,N}\|^2_{L^2_{t,x}} - \|\text{¡}_{0,N}\|^2_{L^2_{t,x}} \Big\|_{L^2(\Omega)} \lesssim \gamma^{\frac{1}{2}}. \tag{3.174}$$

The rest of the proof follows from that of Proposition 2.1.10, Lemma A.2.1 and the bi-parameter Kolmogorov continuity criterion A.2.3.

Let $0 \le t \le T$ and $N \in \mathbb{N}$. By using the decomposition (1.73), we have

$$\begin{aligned}
&\Big(\|\text{¡}_{\gamma,N}\|^2_{L^2_{t,x}} - \|\text{¡}_{0,N}\|^2_{L^2_{t,x}} \Big)^2 \\
&\quad = \Big(\int_0^t \int_{\mathbb{T}^2} \big(|S_\gamma(t')\phi_N|^2 - |S_0(t')\phi_N|^2\big) dxdt + \int_0^t \int_{\mathbb{T}^2} |\Psi_{\gamma,N}(x,t')|^2 dxdt' &(3.175)\\
&\qquad + \int_0^t \int_{\mathbb{T}^2} \big(S_\gamma(t')\phi_N \overline{\Psi_{\gamma,N}(x,t)} + \overline{S_\gamma(t')\phi_N} \Psi_{\gamma,N}(x,t')\big) dxdt' \Big)^2, &(3.176)
\end{aligned}$$

When computing the square in (3.175) and (3.176), the cross-terms between (3.175) and (3.176) have a null expectation in view of the independence of ϕ_N with $\Psi_{\gamma,N}$. Hence, we have

$$\begin{aligned}
&\mathbb{E}\Big(\|\text{¡}_{\gamma,N}\|^2_{L^2_{t,x}} - \|\text{¡}_{0,N}\|^2_{L^2_{t,x}} \Big)^2 \\
&= \mathbb{E}\Big(\int_0^t \int_{\mathbb{T}^2} \big(|S_\gamma(t')\phi_N|^2 - |S_0(t')\phi_N|^2\big) dxdt \Big)^2 + \mathbb{E}\Big(\int_0^t \int_{\mathbb{T}^2} |\Psi_{\gamma,N}(x,t')|^2 dxdt' \Big)^2 \\
&\quad + \mathbb{E}\Big(\int_0^t \int_{\mathbb{T}^2} \big(S_\gamma(t')\phi_N \overline{\Psi_{\gamma,N}(x,t')} + \overline{S_\gamma(t')\phi_N} \Psi_{\gamma,N}(x,t')\big) dxdt' \Big)^2 &(3.177)\\
&\quad + 2\mathbb{E}\Big(\int_0^t \int_{\mathbb{T}^2} \big(|S_\gamma(t')\phi_N|^2 - |S_0(t')\phi_N|^2\big) dxdt' \times \int_0^t \int_{\mathbb{T}^2} |\Psi_{\gamma,N}(x,t')|^2 dxdt' \Big) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By Plancherel's formula, Lemma A.2.7, Lemma A.2.4, (1.69) and (1.71), we have

$$\begin{aligned}
\mathrm{I} &= \mathbb{E} \sum_{\langle n_1\rangle, \langle n_2\rangle \le N} \int_0^t \int_0^t dt_1 dt_2 \frac{\big(e^{-2\gamma t\langle n_1\rangle^2} - 1\big)\big(e^{-2\gamma t\langle n_2\rangle^2} - 1\big)}{\langle n_1\rangle^2 \langle n_2\rangle^2} |g_{n_1}|^2 |g_{n_2}|^2 \\
&\quad + \mathbb{E} \sum_{\langle n_1\rangle, \langle n_2\rangle \le N} \int_0^t \int_0^t dt_1 dt_2 \big|\widehat{\Psi_{\gamma,N}}(n_1,t_1)\big|^2 \big|\widehat{\Psi_{\gamma,N}}(n_2,t_2)\big|^2 \\
&= 2 \sum_{\langle n\rangle \le N} \int_0^t \int_0^t dt_1 dt_2 \frac{\big(e^{-2\gamma t_1\langle n\rangle^2} - 1\big)\big(e^{-2\gamma t_2\langle n\rangle^2} - 1\big)}{\langle n\rangle^4} &(3.178)\\
&\quad - 2 \sum_{\langle n_1\rangle, \langle n_2\rangle \le N} \int_0^t \int_0^t dt_1 dt_2 \frac{\big(e^{-2\gamma t_1\langle n_1\rangle^2} - 1\big)\big(e^{-2\gamma t_2\langle n_2\rangle^2} - 1\big)}{\langle n_1\rangle^2 \langle n_2\rangle^2} &(3.179)
\end{aligned}$$

Similarly, by independence, we have

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^t \int_{\mathbb{T}^2} \big(S_\gamma(t')\phi_N \overline{\Psi_{\gamma,N}(x,t')} + \overline{S_\gamma(t')\phi_N} \Psi_{\gamma,N}(x,t')\big) dxdt' \Big)^2 \\
&= \mathbb{E} \sum_{\langle n_1\rangle, \langle n_2\rangle \le N} \int_0^t \int_0^t \widehat{S_\gamma(t_1)\phi_M}(n_1) \overline{\widehat{\Psi_{\gamma,N}}(n_1,t_1)} \cdot \overline{\widehat{S_\gamma(t_2)\phi_N}(n_2)} \widehat{\Psi_{\gamma,N}}(n_2,t_2) dt_1 dt_2 \\
&\lesssim \sum_{\langle n\rangle \le N} \int_0^t \int_0^t \frac{e^{-\gamma(t_1+t_2)\langle n\rangle^2}}{\langle n\rangle^2} \cdot \frac{|e^{-\gamma|t_2-t_1|\langle n\rangle^2} - e^{-\gamma(t_2+t_1)\langle n\rangle^2}|}{\langle n\rangle^2}. &(3.180)
\end{aligned}$$

At last, by independence, we have

$$
\begin{aligned}
\mathbb{III} &= 2\mathbb{E}\Big(\int_0^t\int_{\mathbb{T}^2}\big(|S_\gamma(t')\phi_N|^2-|S_0(t')\phi_N|^2\big)dxdt'\Big)\times\mathbb{E}\Big(\int_0^t\int_{\mathbb{T}^2}|\Psi_{\gamma,N}(x,t')|^2dxdt\Big)\\
&= 2\sum_{\langle n_1\rangle,\langle n_2\rangle\le M}\int_0^t\int_0^t\frac{\big(e^{-2\gamma t_1\langle n_1\rangle^2}-1\big)\big(e^{-2\gamma t_2\langle n_2\rangle^2}-1\big)}{\langle n_1\rangle^2\langle n_2\rangle^2}=-(3.179).
\end{aligned}
$$

By combining the above with the bound (obtained via the mean value theorem),

$$
(3.178),(3.179)\lesssim\gamma^{\frac{1}{2}}, \tag{3.181}
$$

we immediately deduce (3.174).

□

3.4.2 Linear random operators

In this section, we deal with the linear random operators[8] $\mathcal{M}^1_\gamma$ and $\mathcal{M}^2_\gamma$ defined in (1.82). In particular, we prove the following proposition.

Proposition 3.4.6. *Let* $0<s<\frac{1}{4}$, $0<\delta=\delta(s)\ll1$. *Then, the sequences* $\{\gamma\mapsto\mathcal{M}^1_{\gamma,N}\}_{N\in\mathbb{N}}$ *and* $\{\gamma\mapsto\mathcal{M}^2_{\gamma,N}\}_{N\in\mathbb{N}}$ *defined in* (1.82) *are Cauchy sequences in the class* $C\big([0,1];\mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ *on a set of the form* (3.146). *In particular, denoting the respective limits by* $\gamma\mapsto\mathcal{M}^1_\gamma$ *and* $\gamma\mapsto\mathcal{M}^2_\gamma$, *we have*

$$
(\gamma\mapsto\mathcal{M}^1_\gamma,\gamma\mapsto\mathcal{M}^2_\gamma)\in C\big([0,1];\mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)^2.
$$

$\rho\otimes\mathbb{P}$*-almost surely.*

We now introduce the following trilinear forms:

$$
\begin{aligned}
\mathcal{N}^{1,>}(u,v,w) &= \sum_{N_2,N_3}\mathcal{N}\big(\mathbf{P}_{\gg N_2\vee N_3}u,\mathbf{P}_{N_2}v,\mathbf{P}_{N_3}w\big)\\
\mathcal{N}^{2,>}(u,v,w) &= \sum_{N_1,N_3}\mathcal{N}\big(\mathbf{P}_{N_1}u,\mathbf{P}_{\gg N_1\vee N_3}v,\mathbf{P}_{N_3}w\big),
\end{aligned}
\tag{3.182}
$$

In (3.182), the summations are over dyadic numbers $N_1,N_2,N_3\ge1$. Similarly, we denote by $\mathcal{N}^{j,<}$, the trilinear form

$$
\mathcal{N}^{j,<}(u,v,w):=\mathcal{N}^j(u,v,w)-\mathcal{N}^{j,>}(u,v,w), \tag{3.183}
$$

for $1\le j\le2$. We then decompose the random operators as follows

$$
\begin{aligned}
\mathcal{M}^1_{\gamma,N} &=: \mathcal{M}^{1,>}_{\gamma,N}+\mathcal{M}^{1,<}_{\gamma,N}\\
\mathcal{M}^2_{\gamma,N} &=: \mathcal{M}^{2,>}_{\gamma,N}+\mathcal{M}^{2,<}_{\gamma,N},
\end{aligned}
\tag{3.184}
$$

with

$$
\begin{aligned}
\mathcal{M}^{1,\dagger}_{\gamma,N}(v) &= \mathcal{I}_\gamma\mathcal{N}^{1,\dagger}\big(v,\overset{\bullet}{|}_{\gamma,N},\overset{\bullet}{|}_{\gamma,N}\big)\\
\mathcal{M}^{2,\dagger}_{\gamma,N}(v) &= \mathcal{I}_\gamma\mathcal{N}^{2,\dagger}\big(\overset{\bullet}{|}_{\gamma,N},v,\overset{\bullet}{|}_{\gamma,N}\big),
\end{aligned}
\tag{3.185}
$$

for $\dagger\in\{<,>\}$.

Proposition 3.4.6 is a direct consequence of the following two lemmas.

Lemma 3.4.7. *Let* $0<s<\frac{1}{2}$, $0<\delta=\delta(s)\ll1$. *Then, the sequences* $\{\mathcal{M}^{1,>}_{\gamma,N}\}_{N\in\mathbb{N}}$ *and* $\{\mathcal{M}^{2,>}_{\gamma,N}\}_{N\in\mathbb{N}}$ *defined in* (1.82) *are Cauchy sequences in the class* $C\big([0,1];\mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ *on a set*

---

[8] Such operators are also called random matrices in the literature on random dispersive equations; see for instance [7, 24, 25, 63, 9]

of the form (3.146). In particular, denoting the respective limits by $\gamma \mapsto \mathcal{M}^{1,>}_{\gamma}$ and $\gamma \mapsto \mathcal{M}^{2,>}_{\gamma}$, we have

$$(\mathcal{M}^{1,>}_{\gamma}, \mathcal{M}^{2,>}_{\gamma}) \in C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)^2.$$

$\rho \otimes \mathbb{P}$-almost surely.

Lemma 3.4.8. *Let $0 < s < \frac{1}{4}$, $0 < \delta = \delta(s)$. Then, the sequences $\{\mathcal{M}^{1,<}_{\gamma,N}\}_{N\in\mathbb{N}}$ and $\{\mathcal{M}^{2,<}_{\gamma,N}\}_{N\in\mathbb{N}}$ defined in (1.82) are Cauchy sequences in the class $C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\gamma \mapsto \mathcal{M}^{1,<}_{\gamma}$ and $\gamma \mapsto \mathcal{M}^{2}_{\gamma}$, we have*

$$(\mathcal{M}^{1,>}_{\gamma}, \mathcal{M}^{2,>}_{\gamma}) \in C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)^2.$$

$\rho \otimes \mathbb{P}$-almost surely.

We first start with the proof of Lemma 3.4.7.

Proof of Lemma 3.4.7. We focus on the operator $\{\mathcal{I}_\gamma \mathcal{N}^{1,>}(\cdot, \upharpoonleft_{\gamma,N}, \upharpoonleft_{\gamma,N})\}_{N\in\mathbb{N}}$ since the case $j = 2$ is similar; namely, we fix $j = 1$. By Proposition 3.1.14, (3.184), and (3.185) it suffices to prove that $\{\mathcal{N}^{1,>}(\cdot, \upharpoonleft_{\gamma,N}, \upharpoonleft_{\gamma,N})\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},-\frac{1}{2}+\delta_1}\big)$ for some small $\delta_1 > 0$. (We only prove the relevant uniform in $N \in \mathbb{N}$ bound as convergence follow these and considerations as in Lemma 3.4.4.)

By definition of the $X^{s,b}_T$ norms, and (3.17) and the reductions at the beginning Section 3.4 and Section A.3, it suffices to prove

$$\sup_{\|v\|_{X^{s,\frac{1}{2}}} \leq 1} \big\| \mathbb{1}_{[0,1]}(t) \mathcal{N}\big(\mathbf{P}_{\gg N_2 \vee N_3} v, \mathbf{P}_{N_2} \upharpoonleft_{\gamma,N}, \mathbf{P}_{N_3} \upharpoonleft_{\gamma,N}\big)\big\|_{C_\gamma X^{s,-\frac{1}{2}+\delta_1}} \lesssim (N_2 \vee N_3)^{-\delta_0}, \tag{3.186}$$

on a set of the form (3.146) and for some small $\delta_0 > 0$, uniformly in $N \in \mathbb{N}$, and for dyadic numbers N_2, N_3.

Let us fix $N \in \mathbb{N}$, $\gamma \in [0,1]$, some dyadic numbers N_2, N_3, and $v \in X^{s,\frac{1}{2}}$. Let $\{\Lambda\}_{\Lambda\in\mathfrak{B}}$ be a finitely overlapping family of (countable) balls of radius $\sim N_2 \vee N_3$ which covers the set $\{\xi \in \mathbb{R}^2 : |\xi| \gg N_2 \vee N_3\}$. By orthogonality and the fact that sets of the form (3.146) are stable under intersections, we deduce that (3.186) follows from the bound:

$$\begin{aligned} \sup_{\|v\|_{X^{s,\frac{1}{2}}} \leq 1} &\big\| \mathbb{1}_{[0,1]}(t) \mathcal{N}\big(\mathbf{P}_\Lambda \mathbf{P}_{\gg N_2 \vee N_3} v, \mathbf{P}_{N_2} \upharpoonleft_{\gamma,N}, \mathbf{P}_{N_3} \upharpoonleft_{\gamma,N}\big)\big\|_{C_\gamma X^{s,-\frac{1}{2}+\delta_1}} \\ &\lesssim (N_2 \vee N_3)^{-\delta_0}, \end{aligned} \tag{3.187}$$

on a set of the form (3.146) and uniformly in $\Lambda \in \mathfrak{B}$. Next, we observe that by the Kolmogorov continuity criterion (in the variable γ) and by proving and difference estimate that gains a small power of the difference of two γ's as in (3.171), the bound (3.187) essentially follows from the following estimate:

$$\begin{aligned} \sup_{\gamma\in[0,1]} &\Big\| \big\| \mathbb{1}_{[0,1]}(t) \mathcal{N}\big(\mathbf{P}_\Lambda \mathbf{P}_{\gg N_2 \vee N_3} v, \mathbf{P}_{N_2} \upharpoonleft_{\gamma,N}, \mathbf{P}_{N_3} \upharpoonleft_{\gamma,N}\big)\big\|_{\mathcal{L}(X^{s,\frac{1}{2}}; X^{s,-\frac{1}{2}+\delta_1})} \Big\|_{L^p(\mu\otimes\mathbb{P})} \\ &\lesssim (N_2 \vee N_3)^{-\delta_0}, \end{aligned} \tag{3.188}$$

for $p \geq 1$, uniformly in $\Lambda \in \mathfrak{B}$, and $N \in \mathbb{N}$. Namely, we dropped the dependence of γ in the $C_\gamma X^{s,-\frac{1}{2}+\delta_1}$-norm in (3.187). In the remainder of the proof, we show (3.188).

Fix $\gamma \in [0,1]$, $N \in \mathbb{N}$, $\Lambda \in \mathfrak{B}$ and two dyadic numbers N_2 and N_3. In what follows, we will omit the dependence of the objects in γ and N for convenience. Let us note however that the bounds we obtain are uniform in these parameters. Let $N_\star = (\Lambda, N_2, N_3)$. Applying Lemma A.2.7 gives

$$\begin{aligned}\mathbb{1}_{[0,1]}(t)\mathcal{N}(\mathbf{P}_\Lambda \mathbf{P}_{\gg N_2\vee N_3} v, \mathbf{P}_{N_2} \overset{\bullet}{|}_{\gamma,N}, \mathbf{P}_{N_3} \overset{\bullet}{|}_{\gamma,N})(n,t)& \\ = \int_{\mathbb{R}} d\mu \sum_{n_1\in\mathbb{Z}^2} \widetilde{\mathbf{P}_\Lambda v}(n_1,\mu) I_2\big[h^{N_\star}_{n_1,n,\mu,t}\big],&\end{aligned} \tag{3.189}$$

where $h^{N_\star}_{n_1,n,\mu,t} = h^{N_\star}_{n_1,n,\mu,t}(z_2,z_3)$ (with $z_j = (n_j,t_j,\zeta_j)$, for $2\le j\le 3$, is given by

$$\begin{aligned} h^{N_\star}_{n_1,n,\mu,t}(z_2,z_3) = e^{it(\mu-\langle n_1\rangle^2)} \mathbb{1}_{[0,1]}(t) \mathbb{1}_{n_1\in\Lambda} \mathbb{1}_{\langle n_1\rangle \gg N_2\vee N_3} \prod_{j=2}^{3} \mathbb{1}_{\langle n_j\rangle\sim N_j}& \\ \times \mathbb{1}_{\substack{n_1-n_2+n_3=n \\ \langle n_2\rangle,\langle n_3\rangle\le N}} \mathbb{1}_{n_2\neq n_1,n_3}\, f_t\otimes \overline{f_t}(z_2,z_3),& \end{aligned} \tag{3.190}$$

with f_t as in (A.5). We may then write $h^{N_\star}_{n_1,n,\mu,t}$ as a sum of terms of the form

$$\begin{aligned} h^{A,N_\star}_{n_1,n,\mu,t}(z_2,z_3) = e^{-t(i(\mu+\langle n\rangle^2-\kappa(\bar n))+\gamma\beta_1(\bar n))} \mathbb{1}_{[t_{\max}(A),1]}(t) \mathbb{1}_{n_1\in\Lambda} \mathbb{1}_{\langle n_1\rangle\gg N_2\vee N_3} \prod_{j=2}^{3} \mathbb{1}_{\langle n_j\rangle\sim N_j}& \\ \times \mathbb{1}_{\substack{n_1-n_2+n_3=n \\ \langle n_2\rangle,\langle n_3\rangle\le N}} \mathbb{1}_{n_2\neq n_1,n_3} \cdot \prod_{j\in B} \frac{\mathbb{1}_{[0,1]}(t_j)}{\langle n_j\rangle}& \\ \times \prod_{j\in A} \Big(\sqrt{2\gamma} e^{(\gamma+i)t_j\langle n_j\rangle^2} \mathbb{1}_{[0,1]}(t_j)\Big)^{\iota_j},& \end{aligned} \tag{3.191}$$

where A and B form a partition of $\{2,3\}$ such that $\zeta_j = 1$ for $j\in A$ and $\zeta_j = -1$ for $j\in B$. In the above, $\kappa(\bar n)$ is as in (3.154) and $t_{\max}(A)$ and $\beta_1(\bar n)$ are defined by

$$\begin{aligned} t_{\max}(A) &:= \max\{t_j : j\in A\} \\ \beta_1(\bar n) &:= \langle n_2\rangle^2 + \langle n_3\rangle^2. \end{aligned} \tag{3.192}$$

Similarly, we define $n_{\min}(A) = \min\{n_j : j\in A\}$ and $n_{\max}(A) = \max\{n_j : j\in A\}$.

Taking the twisted space-time Fourier transform in (3.189) and applying Lemma A.2.5 gives

$$\begin{aligned} \widetilde{\mathcal{F}}\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_\Lambda \mathbf{P}_{\gg N_2\vee N_3} v, \mathbf{P}_{N_2} \overset{\bullet}{|}_{\gamma,N}, \mathbf{P}_{N_3} \overset{\bullet}{|}_{\gamma,N}\big)(n,\lambda)& \\ = \sum_{A\subset\{1,2\}} \int_{\mathbb{R}} d\mu \sum_{n_1\in\mathbb{Z}^2} \widetilde{\mathbf{P}_\Lambda v}(n_1,\mu) H^{A,N_\star}(n_1,n,\mu,\lambda),& \end{aligned} \tag{3.193}$$

where $H^{A,N_\star} = I_2\big[\widetilde{h}^{A,N_\star}_{n_1,n,\mu,\lambda}\big]$, and $\widetilde{h}^{A,N_\star}_{n_1,n,\mu,\lambda} = \widetilde{h}^{A,N_\star}_{n_1,n,\mu,\lambda}(z_2,z_3)$ is given by

$$\begin{aligned} \widetilde{h}^{A,N_\star}_{n_1,n,\mu,\lambda}(z_2,z_3) = \frac{e^{-\Phi_1(\bar n)} - e^{-t_{\max}(A)\Phi_1(\bar n)}}{\Phi_1(\bar n)} \mathbb{1}_{n_1\in\Lambda} \mathbb{1}_{\langle n_1\rangle\gg N_2\vee N_3} \prod_{j=2}^{3} \mathbb{1}_{\langle n_j\rangle\sim N_j}& \\ \times \mathbb{1}_{\substack{n_1-n_2+n_3=n \\ \langle n_2\rangle,\langle n_3\rangle\le N}} \mathbb{1}_{n_2\neq n_1,n_3} \cdot \prod_{j\in B} \frac{\mathbb{1}_{[0,1]}(t_j)}{\langle n_j\rangle}& \\ \times \prod_{j\in A} \Big(\sqrt{2\gamma} e^{(\gamma+i)t_j\langle n_j\rangle^2} \mathbb{1}_{[0,1]}(t_j)\Big)^{\iota_j},& \end{aligned} \tag{3.194}$$

with

$$\Phi_1(\bar n) := i(\lambda-\mu-\kappa(\bar n)) + \gamma\beta_1(\bar n). \tag{3.195}$$

We now address the proof of (3.188) with $-\frac{1}{2}+\delta_1$ replaced by $-\frac{1}{2}-\delta_1$ (arguing by interpolation with the trivial $X^{0,0}$ estimate as in Lemma 3.4.4). Similarly, by interpolation with the trilinear estimate in Lemma 3.1.4, we may assume that the input v belongs to $X^{s,\frac{1}{2}+\delta_1}$. By the Minkowski and Cauchy-Schwarz inequalities (in μ), we estimate

$$
\begin{aligned}
&\Big\| \sup_{\|v\|_{X^{s,\frac12+\delta_1}}\le 1} \big\| \mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_\Lambda \mathbf{P}_{\gg N_2\vee N_3} v, \mathbf{P}_{N_2} \overset{\bullet}{\upharpoonleft}_{\gamma,N}, \mathbf{P}_{N_3} \overset{\bullet}{\upharpoonleft}_{\gamma,N}\big)\big\|_{X^{s,-\frac12-\delta_1}} \Big\|_{L^p(\mu\otimes\mathbb{P})} \\
&\quad\lesssim \max_{A\subset\{1,2\}} \Big\| \sup_{\|v\|_{X^{s,\frac12+\delta_1}}\le 1} \big\| \langle n\rangle^s \langle\lambda\rangle^{-\frac12-\delta_1} \int_{\mathbb{R}} d\mu \sum_{n_1\in\mathbb{Z}^2} \widetilde{\mathbf{P}_\Lambda v}(\gamma_0,n_1,\mu) H^{A,N_\star}(n_1,n,\mu,\lambda)\big\|_{\ell^2_n L^2_\lambda} \Big\|_{L^p(\mu\otimes\mathbb{P})} \\
&\quad\lesssim \max_{A\subset\{2,3\}} \sup_{\mu,\lambda\in\mathbb{R}} \Big\| \big\| \mathfrak{H}^{A,N_\star}(n_1,n,\mu,\lambda)\big\|_{\ell^2_{n_1}\to\ell^2_n} \Big\|_{L^p(\mu\otimes\mathbb{P})}, \qquad (3.196)
\end{aligned}
$$

with

$$
\mathfrak{H}^{A,N_\star}(n_1,n,\mu,\lambda) = \langle n\rangle^s \langle n_1\rangle^{-s} H^{A,N_\star}(n_1,n,\mu,\lambda).
$$

Hence, (3.188) reduces to the bound

$$
\max_{A\subset\{2,3\}} \sup_{\mu,\lambda\in\mathbb{R}} \Big\| \big\| \mathfrak{H}^{A,N_\star}(n_1,n,\mu,\lambda)\big\|_{\ell^2_{n_1}\to\ell^2_n} \Big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim p(N_2\vee N_3)^{-\delta_0}, \qquad (3.197)
$$

for $p\ge 1$, and some small $\delta_0>0$. From (3.194), we have

$$
\mathfrak{H}^{A,N_\star}(n_1,n,\mu,\lambda) = I_2\big[\mathfrak{h}^{A,N_\star}_{n,n_1}(n_2,n_3)\mathfrak{f}^A_{n,n_1,\mu,\lambda}(z_2,z_3)\big], \qquad (3.198)
$$

with

$$
\begin{aligned}
\mathfrak{h}^{A,N_\star}_{n,n_1}(n_2,n_3) := \mathbb{1}_{n_1\in\Lambda} \mathbb{1}_{\langle n_1\rangle\gg N_2\vee N_3} \prod_{j=2}^3 \mathbb{1}_{\langle n_j\rangle\sim N_j} \mathbb{1}_{\substack{n_1-n_2+n_3=n \\ \langle n_2\rangle,\langle n_3\rangle\le N}} \mathbb{1}_{n_2\ne n_1,n_3} \\
\times \langle n\rangle^s \langle n_1\rangle^{-s} \prod_{j\in B} \frac{1}{\langle n_j\rangle},
\end{aligned}
\qquad (3.199)
$$

and

$$
\begin{aligned}
\mathfrak{f}^A_{m,n,n_1,\mu,\lambda}(z_2,z_3) := &\frac{e^{-\Phi_1(\bar n)} - e^{-t_{\max}(A)\Phi_1(\bar n)}}{\Phi_1(\bar n)} \\
&\times \prod_{j=1}^3 \mathbb{1}_{[0,1]}(t_j) \cdot \prod_{j\in A} \Big(\sqrt{2\gamma} e^{(\gamma+i)t_j\langle n_j\rangle^2}\Big)^{\iota_j}.
\end{aligned}
\qquad (3.200)
$$

We then have from (3.192), (3.195) and the mean value theorem

$$
\Big| \frac{e^{-\Phi_1(\bar n)} - e^{-t_{\max}(A)\Phi_1(\bar n)}}{\Phi_1(\bar n)} \Big| \lesssim \frac{e^{-\gamma t_{\max}(A)\beta_1(\bar n)}}{\langle \Phi_1(\bar n)\rangle} \qquad (3.201)
$$

Hence, putting together (3.200) and (3.201) yields (as in (3.165))

$$
\|\mathfrak{f}^A_{m,n,n_1,\mu,\lambda}\|_{L^2_{t_2,t_3}([0,1]^2)} \lesssim \frac{\gamma^{\frac{|A|}{2}}}{\langle\Phi_1(\bar n)\rangle} \min\Big(1, \frac{1}{\gamma^{\frac12} r(A)}\Big) =: d_\gamma(A), \qquad (3.202)
$$

where

$$
r(A) = \frac{\mathbb{1}_{A=\{2,3\}}}{\langle n_{\min}(A)\rangle} + \frac{\mathbb{1}_{A=\{2\}}}{\langle n_3\rangle} + \frac{\mathbb{1}_{A=\{3\}}}{\langle n_2\rangle} + \mathbb{1}_{A=\emptyset}. \qquad (3.203)
$$

Note that the factor $\langle n_{\min}(A)\rangle$ in (3.203) comes from the case where $n_{\max}(A)$ and $t_{\max}(A)$ correspond to the same index (say $n_{\max}(A)=n_2$ and $t_{\max}(A)=t_2$, for instance). In other cases, $\langle n_{\min}(A)\rangle$ may be replaced by the better factor $\langle n_{\max}(A)\rangle$.

Let $\mathfrak{L}^{A,N_\star} = \mathfrak{L}^{A,N_\star}_{nn_1n_2n_3}$ be the tensor (which also depends on μ and λ) defined by

$$\begin{aligned}\mathfrak{L}^{A,N_\star}_{nn_1n_2n_3} = \mathbb{1}_{n_1\in\Lambda}\mathbb{1}_{\langle n_1\rangle\gg N_2\vee N_3}\prod_{j=2}^{3}\mathbb{1}_{\langle n_j\rangle\sim N_j}\cdot\mathbb{1}_{\substack{n_1-n_2+n_3=n\\ \langle n_2\rangle,\langle n_3\rangle\le N}}\mathbb{1}_{n_2\neq n_1,n_3}\\ \times\prod_{j\in B}\frac{1}{\langle n_j\rangle}\cdot d_\gamma(A).\end{aligned}\tag{3.204}$$

From Lemma A.4.2, (3.198) and (3.202), we have

$$(3.197)\lesssim p(N_2\vee N_3)^{\varepsilon}\max_{A\subset\{2,3\}}\sup_{\mu,\lambda\in\mathbb{R}}\|\mathfrak{L}^{A,N_\star}\|_1,\tag{3.205}$$

for any $\varepsilon>0$, and with $\|\cdot\|_1$ as in (3.130). Thus, to obtain (3.197), it suffices to show

$$\|\mathfrak{L}^{A,N_\star}\|_1\lesssim\gamma^{\theta|A|}(N_2\vee N_3)^{-\theta},\tag{3.206}$$

for any $A\subset\{2,3\}$, $\gamma\in[0,1]$, some small $\theta>0$, and uniformly in $(\mu,\lambda)\in\mathbb{R}^2$. In the rest of the proof we hence prove (3.206).

We divide our analysis into three cases: (i) $A=\{2,3\}$ and $B=\emptyset$, (ii) $A=\{2\}$ and $B=\{3\}$ or $A=\{3\}$ and $A=\{2\}$, and (iii) $A=\emptyset$ and $B=\{2,3\}$.

- **Case (i):** $A=\{2,3\}$ and $B=\emptyset$. In this case, we have

$$d_\gamma(\{2,3\})=\frac{\gamma}{\langle\Phi_1(\bar{n})\rangle}\min\Big(1,\frac{1}{\gamma^{\frac12}\langle n_{\min}(\{2,3\})\rangle}\Big).$$

We obtain several bounds depending on the size of $d_\gamma(\{2,3\})$. Namely, by (3.192) and (3.195), we have the following three bounds:

$$d_\gamma(\{2,3\})\lesssim\frac{1}{\langle n_{\max}(\{2,3\})\rangle^2},\tag{3.207}$$

$$d_\gamma(\{2,3\})\lesssim\frac{\gamma^{\frac12}}{\langle\lambda-\mu-\kappa(\bar{n})\rangle\langle n_{\min}(\{2,3\})\rangle},\tag{3.208}$$

$$d_\gamma(\{2,3\})\lesssim\frac{\gamma^{-\frac12}}{\langle n_{\max}(\{2,3\})\rangle^2\langle n_{\min}(\{2,3\})\rangle}.\tag{3.209}$$

Plugging (3.207) into (3.204) gives, by Lemma 3.3.2 (i),

$$\|\mathfrak{L}^{\{2,3\},N_\star}\|_1\lesssim(N_2\vee N_3)^{-2}N_2N_3=(N_2\vee N_3)^{-1}(N_2\wedge N_3).\tag{3.210}$$

Next, we have the following decomposition:

$$\|\mathfrak{L}^{\{2,3\},N_\star}\|_1\le\big\|\mathfrak{L}^{\{2,3\},N_\star}\mathbb{1}_{\langle\lambda-\mu-\kappa(\bar{n})\rangle\le(N_2\vee N_3)^{10}}\big\|_1\tag{3.211}$$

$$+\big\|\mathfrak{L}^{\{2,3\},N_\star}\mathbb{1}_{\langle\lambda-\mu-\kappa(\bar{n})\rangle>(N_2\vee N_3)^{10}}\big\|_1\tag{3.212}$$

By Lemma 3.3.2 (i), (3.204) and (3.208), we have

$$(3.212)\lesssim\gamma^{\frac12}(N_2\vee N_3)^{-10}(N_2\wedge N_3)^{-1}N_2N_3\lesssim\gamma^{\frac12}(N_2\vee N_3)^{-9}.\tag{3.213}$$

Furthermore, we may estimate (3.211) by fixing the value of the phase function $\kappa(\bar{n})$. More precisely, we have from Lemma 3.3.2 (ii), along with (3.204) and (3.208),

$$\begin{aligned}(3.211)&\lesssim\gamma^{\frac12}\sum_{\substack{m\in\mathbb{Z}\\ \langle\lambda-\mu-m\rangle\le(N_2\vee N_3)^{10}}}\frac{1}{\langle\lambda-\mu-m\rangle}\big\|\mathfrak{L}^{\Lambda,\{2,3\}}\mathbb{1}_{\kappa(\bar{n})=m}\big\|_1\\ &\lesssim\gamma^{\frac12}\log\big(1+N_2\vee N_3\big)\sup_{m\in\mathbb{Z}}\big\|\mathfrak{L}^{\Lambda,\{2,3\}}\mathbb{1}_{\kappa(\bar{n})=m}\big\|_1\\ &\lesssim\gamma^{\frac12}(N_2\vee N_3)^{\frac12+\theta}(N_2\wedge N_3)^{-\frac12},\end{aligned}\tag{3.214}$$

for any $\theta > 0$. Hence, (3.213) and (3.214) yield

$$\|\mathfrak{L}^{\{2,3\},N_\star}\|_1 \lesssim \gamma^{\frac{1}{2}}(N_2 \vee N_3)^{\frac{1}{2}+\theta}(N_2 \wedge N_3)^{-\frac{1}{2}}, \tag{3.215}$$

for any $\theta > 0$.

Lastly, from (3.209), (3.204) and Lemma 3.3.2 (i), we have

$$\|\mathfrak{L}^{\{2,3\},N_\star}\|_1 \lesssim \gamma^{-\frac{1}{2}}(N_2 \vee N_3)^{-1}. \tag{3.216}$$

Finally, interpolating (3.210), (3.215), and (3.216) gives

$$\|\mathfrak{L}^{\{2,3\},N_\star}\|_1 \lesssim \gamma^{\theta}(N_2 \vee N_3)^{-\theta},$$

for some small $\theta > 0$; which is acceptable in view of (3.206).

• **Case (ii):** $A = \{2\}$ and $B = \{3\}$ or $A = \{3\}$ and $B = \{2\}$. We assume $A = \{2\}$ and $B = \{3\}$ as the other case is similar. In this case, we have

$$d_\gamma(\{2\}) = \frac{\gamma^{\frac{1}{2}}}{\langle \Phi_1(\bar{n})\rangle} \min\Big(1, \frac{1}{\gamma^{\frac{1}{2}}\langle n_3\rangle}\Big).$$

As in case (i), we hence have the bounds

$$d_\gamma(\{2\}) \lesssim \frac{1}{\langle \lambda - \mu - \kappa(\bar{n})\rangle\langle n_3\rangle}, \tag{3.217}$$

$$d_\gamma(\{2\}) \lesssim \frac{\gamma^{\frac{1}{2}}}{\langle \lambda - \mu - \kappa(\bar{n})\rangle}, \tag{3.218}$$

$$d_\gamma(\{2\}) \lesssim \frac{\gamma^{-\frac{1}{2}}}{\big(\langle n_2\rangle \vee \langle n_3\rangle\big)^2}. \tag{3.219}$$

From (3.217) with (3.204), and arguing as in the case (3.208) to fix values of the phase $\kappa(\bar{n})$, we have by Lemma 3.3.2 (ii),

$$\begin{aligned}\|\mathfrak{L}^{\{2\},N_\star}\|_1 &\lesssim N_3^{-2}(N_2 \vee N_3)^{\frac{1}{2}+\theta}(N_2 \wedge N_3)^{\frac{1}{2}} \\ &\lesssim (N_2 \vee N_3)^{\frac{1}{2}+\theta}N_3^{-\frac{3}{2}},\end{aligned} \tag{3.220}$$

for any $\theta > 0$.

Similarly, we have from (3.204), (3.218), and Lemma 3.3.2 (ii),

$$\|\mathfrak{L}^{\{2\},N_\star}\|_1 \lesssim \gamma^{\frac{1}{2}}(N_2 \vee N_3)^{\frac{1}{2}+\theta}N_3^{-\frac{1}{2}}. \tag{3.221}$$

for any $\theta > 0$.

Lastly, by (3.204), (3.219) and Lemma 3.3.2 (i),

$$\|\mathfrak{L}^{\{2\},N_\star}\|_1 \lesssim \gamma^{-\frac{1}{2}}(N_2 \vee N_3)^{-1}, \tag{3.222}$$

Interpolating (3.220), (3.221), and (3.222) gives (3.206) in this case.

• **Case (iii):** $A = \emptyset$ and $B = \{2,3\}$. In this case, we have

$$d_\gamma(\emptyset) = \frac{1}{\langle \Phi_1(\bar{n})\rangle} \lesssim \frac{1}{\langle \lambda - \mu - \kappa(\bar{n})\rangle}$$

Hence, from the above, (3.204), we have by arguing as in (3.208),

$$\|\mathfrak{L}^{\emptyset,N_\star}\|_1 \lesssim N_2^{-1}N_3^{-1}(N_2 \vee N_3)^{\frac{1}{2}+\theta}(N_2 \wedge N_3)^{\frac{1}{2}}, \tag{3.223}$$

for any $\theta > 0$, which is acceptable in view of (3.206). This concludes the proof of (3.186). □

Remark 3.4.9. Note that technically, the proof of Lemma 3.4.7 in the above does not provide the tail estimates (3.60) since we used a (pathwise) orthogonality argument in going from (3.186) to (3.187). It is however possible to recover the desired $L^p(\mu\otimes\mathbb{P})$ by proving a variant of Lemma A.4.2 adapted to the current "high × low × low → high" setting; see [25, Proposition 4.15].

We now prove Lemma 3.4.8.

Proof of Lemma 3.4.8. We focus on the operator $\{\gamma \mapsto \mathcal{I}_\gamma \mathcal{N}^{1,<}(\cdot, \overset{\bullet}{|}_{\gamma,N}, \overset{\bullet}{|}_{\gamma,N})\}_{N\in\mathbb{N}}$ since the case $j = 2$ is similar; namely, we fix $j = 1$. By Proposition 3.1.14, (3.184), and (3.185) it suffices to prove that $\{\gamma \mapsto \mathcal{N}^{1,<}(\cdot, \overset{\bullet}{|}_{\gamma,N}, \overset{\bullet}{|}_{\gamma,N})\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1]; \mathcal{L}^{s,s,\frac{1}{2},-\frac{1}{2}+\delta_1}\big)$ for some small $\delta_1 > 0$. (We only prove the relevant uniform in $N \in \mathbb{N}$ bound as convergence follow these and considerations as in Lemma 3.4.4.)

By arguing as in Lemma 3.4.7 and by using the reductions at the beginning of Section 3.4 and Section A.3, it suffices to prove the following bound

$$\sup_{\gamma\in[0,1]} \Big\| \big\| \mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1} v, \mathbf{P}_{N_2} \overset{\bullet}{|}_{\gamma,N}, \mathbf{P}_{N_3} \overset{\bullet}{|}_{\gamma,N}\big) \big\|_{\mathcal{L}(X^{s,\frac{1}{2}}; X^{s,-\frac{1}{2}+\delta_1})} \Big\|_{L^p(\mu\otimes\mathbb{P})} \lesssim N_{\max}^{-\delta_0}, \tag{3.224}$$

for some small $\delta_0 > 0$, uniformly in $N \in \mathbb{N}$, and for dyadic numbers N_1, N_2, N_3. Here, $N_{\max} = \max(N_1, N_2, N_3)$.

We now prove (3.224) with $-\frac{1}{2} + \delta_1$ replaced by $-\frac{1}{2} - \delta_1$ arguing as in Lemma 3.4.4). We also assume that the input v belongs to $X^{s,\frac{1}{2}+\delta_1}$ by interpolation with the trilinear estimate in Lemma 3.1.4. Let us fix dyadic numbers N_1, N_2, N_3 and let $N_\star = (N_1, N_2, N_3)$. By similar considerations as in (the proof of) Lemma 3.4.7 and Lemma A.4.2,

$$\begin{aligned}
&\Big\| \sup_{\|v\|_{X^{s,\frac{1}{2}+\delta_1}} \le 1} \big\| \mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1} v, \mathbf{P}_{N_2} \overset{\bullet}{|}_{\gamma,N}, \mathbf{P}_{N_3} \overset{\bullet}{|}_{\gamma,N}\big) \big\|_{X^{s,-\frac{1}{2}-\delta_1}} \Big\|_{L^p(\mu\otimes\mathbb{P})} \\
&\quad \lesssim p N_{\max}^{\varepsilon} \max_{A\subset\{2,3\}} \sup_{\mu,\lambda\in\mathbb{R}} \|\mathfrak{L}^{A,N_\star}\|_1,
\end{aligned} \tag{3.225}$$

for any $\varepsilon > 0$, and with $\|\cdot\|_1$ as in (3.130), and where $\mathfrak{L}^{A,N_\star} = \mathfrak{L}^{A,N_\star}_{nn_1n_2n_3}$ is the tensor (which also depends on μ and λ) given by

$$\begin{aligned}
\mathfrak{L}^{A,N_\star}_{nn_1n_2n_3} &= \langle n\rangle^s \langle n_1\rangle^{-s} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j\rangle \sim N_j} \cdot \mathbb{1}_{\substack{n_1-n_2+n_3=n \\ \langle n_2\rangle, \langle n_3\rangle \le N}} \mathbb{1}_{n_2\ne n_1,n_3} \\
&\quad \times \prod_{j\in B} \frac{1}{\langle n_j\rangle} \cdot d_\gamma(A),
\end{aligned} \tag{3.226}$$

with $d_\gamma(A)$ as in (3.202), (3.203), and (3.195). Thus, to obtain (3.224), it suffices to show

$$\|\mathfrak{L}^{A,N_\star}\|_1 \lesssim \gamma^{\theta|A|} N_{\max}^{-\theta}, \tag{3.227}$$

for any $A \subset \{2,3\}$, $\gamma \in [0,1]$, some small $\theta > 0$, and uniformly in $(\mu,\lambda) \in \mathbb{R}^2$. We assume that $N_1 \ll N_2 \wedge N_3$ (hence, $N_{\max} = N_2 \vee N_3$) for now and divide our analysis into three cases: (i) $A = \{2,3\}$ and $B = \emptyset$, (ii) $A = \{2\}$ and $B = \{3\}$ or $A = \{3\}$ and $A = \{2\}$, and (iii) $A = \emptyset$ and $B = \{2,3\}$.

- **Case (i):** $A = \{2,3\}$ and $B = \emptyset$. In this case, we have

$$d_\gamma(\{2,3\}) = \frac{\gamma}{\langle \Phi_1(\bar{n})\rangle} \min\Big(1, \frac{1}{\gamma^{\frac{1}{2}} \langle n_{\min}(\{2,3\})\rangle}\Big).$$

As in the proof of Lemma 3.4.7, we obtain by using the different bounds on $d_\gamma(A)$ (3.207), (3.208) and (3.209), along with Lemma 3.3.2 (i) and (ii),

$$\|\mathfrak{L}^{\{2,3\}}\|_1 \lesssim N_1^{-s}(N_2 \vee N_3)^{s-1}(N_2 \wedge N_3), \tag{3.228}$$
$$\|\mathfrak{L}^{\{2,3\}}\|_1 \lesssim \gamma^{\frac{1}{2}} N_1^{-s}(N_2 \vee N_3)^{s+\frac{1}{2}+\theta}(N_2 \wedge N_3)^{-\frac{1}{2}}, \tag{3.229}$$
$$\|\mathfrak{L}^{\{2,3\}}\|_1 \lesssim \gamma^{-\frac{1}{2}} N_1^{-s}(N_2 \vee N_3)^{s-1}, \tag{3.230}$$

for any $\theta > 0$. By interpolation with (3.229), (3.227) follows from the bound

$$\|\mathfrak{L}^{A}\|_1 \lesssim (N_2 \vee N_3)^{-\theta}, \tag{3.231}$$

for some small $\theta > 0$; which we now prove. Fix any $0 < \theta_0 \ll 1$. By (3.230), if the following bound holds

$$\gamma^{-\frac{1}{2}}(N_2 \vee N_3)^{s-1} \lesssim (N_2 \vee N_3)^{-\theta_0},$$

then (3.231) holds. Otherwise, we have

$$\gamma^{\frac{1}{2}} \lesssim (N_2 \vee N_3)^{s-1+\theta_0}, \tag{3.232}$$

Plugging (3.232) into (3.229) gives

$$(3.229) \lesssim N_1^{-s}(N_2 \vee N_3)^{2s-\frac{1}{2}+\theta+\theta_0}(N_2 \wedge N_3)^{-\frac{1}{2}}. \tag{3.233}$$

We now note that if we have

$$(N_2 \vee N_3)^{s-1}(N_2 \wedge N_3) \lesssim (N_2 \vee N_3)^{-\theta_0},$$

then (3.231) holds. Otherwise, we have

$$(N_2 \wedge N_3) \gtrsim (N_2 \vee N_3)^{1-s-\theta_0}, \tag{3.234}$$

Putting (3.234) into (3.233) yields

$$(3.229) \lesssim N_1^{-s}(N_2 \vee N_3)^{\frac{5}{2}s-1+\theta+2\theta_0}. \tag{3.235}$$

Hence by choosing $s < \frac{2}{5}$ and $\theta, \theta_0 > 0$ small enough, (3.200) ensures that (3.231) holds.

- **Case (ii):** $A = \{2\}$ and $B = \{3\}$ or $A = \{3\}$ and $B = \{2\}$. We assume $A = \{2\}$ and $B = \{3\}$ as the other case is similar. In this case, we have

$$d_\gamma(\{2\}) = \frac{\gamma^{\frac{1}{2}}}{\langle \Phi_1(\bar{n})\rangle} \min\Big(1, \frac{1}{\gamma^{\frac{1}{2}}\langle n_3\rangle}\Big).$$

As in the proof of Lemma 3.4.7, we obtain by using the different bounds on $d_\gamma(A)$, (3.226), (3.217), (3.218) and (3.219), along with Lemma 3.3.2 (i) and (ii),

$$\|\mathfrak{L}^{\{2\}}\|_1 \lesssim N_1^{-s}(N_2 \vee N_3)^{s+\frac{1}{2}+\theta} N_3^{-\frac{3}{2}}, \tag{3.236}$$
$$\|\mathfrak{L}^{\{2\}}\|_1 \lesssim \gamma^{\frac{1}{2}} N_1^{-s}(N_2 \vee N_3)^{s+\frac{1}{2}+\theta} N_3^{-\frac{1}{2}}, \tag{3.237}$$
$$\|\mathfrak{L}^{\{2\}}\|_1 \lesssim \gamma^{-\frac{1}{2}} N_1^{-s}(N_2 \vee N_3)^{s-1}, \tag{3.238}$$

As before, by interpolation with (3.237), it suffices to prove

$$\|\mathfrak{L}^{\{2\}}\|_1 \lesssim N_{\max}^{-\theta}, \tag{3.239}$$

for some small $\theta > 0$. Fix any $0 < \theta_0 \ll 1$. If we have

$$\gamma^{-\frac{1}{2}}(N_2 \vee N_3)^{s-1} \lesssim (N_2 \vee N_3)^{-\theta_0}, \tag{3.240}$$

Plugging (3.240) into (3.237) gives

$$(3.237) \lesssim N_1^{-s}(N_2 \vee N_3)^{2s-\frac{1}{2}+\theta} N_3^{-\frac{1}{2}}. \tag{3.241}$$

Thus, from (3.241), we can conclude that (3.239) holds under the condition $s < \frac{1}{4}$.

- **Case (iii):** $A = \emptyset$ and $B = \{2, 3\}$. In this case, we have

$$d_\gamma(\emptyset) = \frac{1}{\langle \Phi_1(\bar{n}) \rangle} \lesssim \frac{1}{\langle \lambda - \mu - \kappa(\bar{n}) \rangle}$$

Hence, from the above, (3.226), we have by arguing as in (3.208),

$$\begin{aligned} \|\mathfrak{L}^\emptyset\|_1 &\lesssim N_1^{-s} N_2^{-1} N_3^{-1} (N_2 \vee N_3)^{s+\frac{1}{2}+\theta} (N_2 \wedge N_3)^{\frac{1}{2}} \\ &\lesssim N_1^{-s} (N_2 \vee N_3)^{s-\frac{1}{2}+\theta} (N_2 \wedge N_3)^{-\frac{1}{2}} \end{aligned}$$

for any $\theta > 0$, which is acceptable in view of (3.227). This proves (3.227) for $N_1 \gtrsim N_2 \wedge N_3$.

The case $N_1 \gtrsim N_2 \wedge N_3$ is a direct consequence of the above computations for the case $N_1 \ll N_2 \wedge N_3$ and Lemma 3.3.2 (iii) (since if $N_1 \gtrsim N_2 \wedge N_3$, we can exploit the extra N_1^{-s}-factor to obtain better estimates). This concludes the proof of (3.224). □

3.4.3 Bilinear random operators

The purpose of this subsection is to treat the bilinear random operator terms $\mathcal{T}_\gamma^1$ and $\mathcal{T}_\gamma^2$ in (1.83).

Proposition 3.4.10. *Fix $0 < s < \frac{1}{4}$ and $0 < \delta = \delta(s) \ll 1$. Let $j \in \{1, 2\}$. The sequence $\{\mathcal{T}_{\gamma,N}^j\}_{N\in\mathbb{N}}$ defined in (1.83) is a Cauchy sequence in the class $C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\mathcal{T}_\gamma^j$, we have*

$$\mathcal{T}_\gamma^j \in C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big).$$

$\rho \otimes \mathbb{P}$-almost surely.

Let $j \in \{1, 2\}$. By (1.74), we can decompose $\mathcal{T}_{\gamma,N}^j$ as

$$\mathcal{T}_{\gamma,N}^j = \mathcal{T}_{\gamma,N}^{j,\ominus} + \mathcal{T}_{\gamma,N}^{j,\oplus} \tag{3.242}$$

with

$$\begin{aligned} \mathcal{T}_{\gamma,N}^{j,\ominus}(u,v) &:= \mathcal{I}_\gamma \mathcal{N}(\Psi_{\gamma,N}, u, v) \\ \mathcal{T}_{\gamma,N}^{j,\oplus}(u,v) &:= \mathcal{I}_\gamma \mathcal{N}(S_\gamma(t)\phi_N, u, v) \end{aligned} \tag{3.243}$$

We reduce the proof of Proposition 3.4.10 to the construction of the sequence of operators $\{\mathcal{T}_{\gamma,N}^{j,\ominus}\}_{N\in N}$ and $\{\mathcal{T}_{\gamma,N}^{j,\oplus}\}_{N\in N}$ in appropriate spaces.

Proposition 3.4.11. *Fix $0 < s < \frac{1}{4}$ and $0 < \delta = \delta(s) \ll 1$. Let $j \in \{1, 2\}$. The sequence $\{\mathcal{T}_{\gamma,N}^{j,\ominus}\}_{N\in\mathbb{N}}$ defined in (3.243) is a Cauchy sequence in the class $C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\mathcal{T}_\gamma^{j,\ominus}$, we have*

$$\mathcal{T}_\gamma^{j,\ominus} \in C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big).$$

$\rho \otimes \mathbb{P}$-almost surely.

Proposition 3.4.12. *Fix $0 < s < \frac{1}{2}$ and $0 < \delta = \delta(s) \ll 1$. Let $j \in \{1, 2\}$. The sequence $\{\mathcal{T}_{\gamma,N}^{j,\oplus}\}_{N\in\mathbb{N}}$ defined in (3.243) is a Cauchy sequence in the class $C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\mathcal{T}_\gamma^{j,\oplus}$, we have*

$$\mathcal{T}_\gamma^{j,\oplus} \in C\big([0,1]; \mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big).$$

$\rho \otimes \mathbb{P}$-almost surely.

In what follows, we prove Proposition 3.4.11 as Proposition 3.4.12 follows from similar (and in fact simpler) arguments. To this end, we further decompose $\mathcal{T}^{j,\ominus}_{\gamma,N}$, $j \in \{1,2\}$. First, we introduce the following frequency localized multilinear forms:

$$\begin{aligned} \mathcal{N}^{1,\ominus,>}(u,v,w) &= \sum_{N_1 \gg N_2 \vee N_3} \mathcal{N}\big(\mathbf{P}_{N_1}u, \mathbf{P}_{N_2}v, \mathbf{P}_{N_3}w\big), \\ \mathcal{N}^{2,\ominus,>}(u,v,w) &= \sum_{N_2 \gg N_1 \vee N_3} \mathcal{N}\big(\mathbf{P}_{N_1}u, \mathbf{P}_{N_2}v, \mathbf{P}_{N_3}w\big), \end{aligned} \tag{3.244}$$

and

$$\mathcal{N}^{j,\ominus,<}(u,v,w) := \mathcal{N}^{j}(u,v,w) - \mathcal{N}^{j,\ominus,>}(u,v,w), \tag{3.245}$$

for $j \in \{1,2\}$. We now introduce the operators

$$\begin{aligned} \mathcal{T}^{1,\ominus,\dagger}_{\gamma,N}(u,v) &= \mathcal{I}_\gamma \mathcal{N}^{1,\ominus,\dagger}\big(\Psi_{\gamma,N}, u, v\big), \\ \mathcal{T}^{2,\ominus,\dagger}_{\gamma,N}(u,v) &= \mathcal{I}_\gamma \mathcal{N}^{2,\ominus,\dagger}\big(u, \Psi_{\gamma,N}, v\big), \end{aligned} \tag{3.246}$$

for $\dagger \in \{<,>\}$.

With these notations, Proposition 3.4.11 follows from the two next lemmas.

Lemma 3.4.13. *Fix $s > 0$ and $0 < \delta = \delta(s) \ll 1$. Let $j \in \{1,2\}$. The sequence $\{\mathcal{T}^{j,\ominus,<}_{\gamma,N}\}_{N\in\mathbb{N}}$ defined in (3.246) is a Cauchy sequence in the class $C\big([0,1];\mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\mathcal{T}^{j,\ominus,<}_{\gamma}$, we have*

$$\mathcal{T}^{j,\ominus,<}_{\gamma} \in C\big([0,1];\mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big).$$

$\rho \otimes \mathbb{P}$-almost surely.

Lemma 3.4.14. *Fix $0 < s < \frac{1}{4}$ and $0 < \delta = \delta(s) \ll 1$. Let $j \in \{1,2\}$. The sequence $\{\mathcal{T}^{j,\ominus,>}_{\gamma,N}\}_{N\in\mathbb{N}}$ defined in (3.246) is a Cauchy sequence in the class $C\big([0,1];\mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big)$ on a set of the form (3.146). In particular, denoting the respective limits by $\mathcal{T}^{j,\ominus,>}_{\gamma}$, we have*

$$\mathcal{T}^{j,\ominus,>}_{\gamma} \in C\big([0,1];\mathcal{B}^{s,s,\frac{1}{2},\frac{1}{2}+\delta}\big).$$

$\rho \otimes \mathbb{P}$-almost surely.

The rest of this subsection is devoted to the proof of Lemmas 3.4.13 and 3.4.14.

Proof of Lemma 3.4.13. We focus on the case $j = 1$, i.e. on the operator $\{\mathcal{I}_\gamma \mathcal{N}^{1,\ominus,<}(\Psi_{\gamma,N},\cdot,\cdot)\}_{N\in\mathbb{N}}$ since the case $j = 2$ is similar; namely, we fix $j = 1$. By Proposition 3.1.14, (3.245), (3.246), and it suffices to prove that $\{\mathcal{N}^{1,\ominus,<}(\Psi_{\gamma,N},\cdot,\cdot)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,1];\mathcal{B}^{s,s,\frac{1}{2},-\frac{1}{2}+\delta_1}\big)$ for some small $\delta_1 > 0$. (We only prove the relevant uniform in $N \in \mathbb{N}$ bound as convergence follow these and considerations as in Lemma 3.4.4.)

By arguing as in Lemma 3.4.7 and by using the reductions at the beginning of Section 3.4 and Section A.3, it suffices to prove the following frequency localized estimate:

$$\begin{aligned} \sup_{\gamma\in[0,1]} \sup_{N\in\mathbb{N}} \Big\| \sup_{\substack{\|u\|_{X^{s,\frac{1}{2}}}\le 1 \\ \|v\|_{X^{s,\frac{1}{2}}}\le 1}} \big\| \mathbb{1}_{[0,1]}(t) \mathcal{N}(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v) \big\|_{X^{s,-\frac{1}{2}+\delta_1}} \Big\|_{L^p(\mu\otimes\mathbb{P})} & \\ \lesssim p (N_2 \vee N_3)^{-\delta_0}, & \end{aligned} \tag{3.247}$$

for some small $\delta_0 > 0$, and for dyadic numbers N_1, N_2, N_3, with $N_1 \lesssim N_2 \vee N_3$. Let us note that by Lemma 3.1.4, if $N_2 \wedge N_3 \gtrsim N_1^{\frac{1}{100}}$, then (3.247) follows directly by transferring some derivatives from $\Psi_{\gamma,N}$ to u and v if needed. Hence, in the remainder of the proof, we assume that $N_2 \wedge N_3 \ll N_1^{\frac{1}{100}}$ and prove (3.247) in that case.

Let $(\gamma, N) \in [0,1] \times \mathbb{N}$ and N_1, N_2, N_3 be dyadic numbers. Let us assume that we have $N_2 \geq N_3$ and hence we have $N_3 \ll N_1^{\frac{1}{100}}$. We also write $N_\star = (N_1, N_2, N_3)$. Next, by further decomposing u into balls of radius $\sim N_2$ (as in the proof of Lemma 3.4.7) if need be, we may assume that $N_2 \sim N_1$. By (A.6), we have

$$\begin{aligned} \mathcal{F}_x\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v\big)\big)(n,t) &= \int_{\mathbb{R}^2} d\mu_2 d\mu_3 \\ \times \sum_{n_2, n_3 \in \mathbb{Z}^2} &\overline{\widetilde{u}(n_2,\mu_2)}\widetilde{v}(n_3,\mu_3) I_1\big[h^{\ominus,<,N_\star}_{nn_2n_3\mu_2\mu_3 t}\big] \end{aligned} \tag{3.248}$$

with

$$\begin{aligned} h^{\ominus,<,N_\star}_{nn_2n_3\mu_2\mu_3 t}(n_1,t_1) = \sqrt{2\gamma}\, &\mathbb{1}_{\substack{n=n_1-n_2+n_3 \\ n_2\neq n_1,n_3}} \mathbb{1}_{\langle n_1\rangle \leq N} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j\rangle \sim N_j} \\ &\times e^{-t(i(-\mu_2+\mu_3+\langle n_1\rangle^2-\langle n_2\rangle^2+\langle n_3\rangle^2)+\gamma\langle n_1\rangle^2)} \\ &\times e^{t_1(\gamma+i)\langle n_1\rangle^2}\mathbb{1}_{[0,1]}(t)\mathbb{1}_{[0,t]}(t_1). \end{aligned} \tag{3.249}$$

Taking the Fourier transform with Lemma A.2.5 gives

$$\begin{aligned} \mathcal{F}_{x,t}\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v\big)\big)(n,\lambda) &= \int_{\mathbb{R}^2} d\mu_2 d\mu_3 \\ \times \sum_{n_2, n_3 \in \mathbb{Z}^2} &\overline{\widetilde{u}(n_2,\mu_2)}\widetilde{v}(n_3,\mu_3) I_1\Big[\widetilde{h}^{\ominus}_{nn_2n_3\mu_2\mu_3 t}\Big], \end{aligned} \tag{3.250}$$

with

$$\begin{aligned} \widetilde{h}^{\ominus,<,N_\star}_{nn_2n_3\mu_2\mu_3\lambda}(n_1,t_1) = \sqrt{2\gamma}\, &\mathbb{1}_{\substack{n=n_1-n_2+n_3 \\ n_2\neq n_1,n_3}} \mathbb{1}_{\langle n_1\rangle \leq N} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j\rangle \sim N_j} \\ &\times \frac{e^{-\Phi_2(\bar{n})} - e^{-t_1\Phi_2(\bar{n})}}{\Phi_2(\bar{n})} e^{t_1(\gamma+i)\langle n_1\rangle^2}\mathbb{1}_{[0,1]}(t_1). \end{aligned} \tag{3.251}$$

In the above, we denoted by $\Phi_2(\bar{n})$, the phase function

$$\Phi_2(\bar{n}) := i(\lambda - \mu_2 - \mu_3 - \kappa(\bar{n})) + \gamma\langle n_1\rangle^2, \tag{3.252}$$

where $\kappa(\bar{n})$ is as in (3.129). By the mean value theorem, we have

$$\Big|\frac{e^{-\Phi_2(\bar{n})} - e^{-t_1\Phi_2(\bar{n})}}{\Phi_2(\bar{n})}\Big| \lesssim \frac{e^{-t_1\Phi_1(\bar{n})}}{\langle\Phi_2(\bar{n})\rangle} \tag{3.253}$$

We may replace $-\frac{1}{2}+\delta_1$ replaced by $-\frac{1}{2}-\delta_1$ in (3.247) (arguing as in Lemma 3.4.4) and we may assume that the inputs u and v belong to $X^{s,\frac{1}{2}+\delta_1}$. By similar considerations as in (the proof of) Lemma 3.4.7, we have

$$\begin{aligned} \Big\| \sup_{\substack{\|u\|_{X^{s,\frac{1}{2}+\delta_1}} \leq 1 \\ \|v\|_{X^{s,\frac{1}{2}+\delta_1}} \leq 1}} \big\|\mathbb{1}_{[0,1]}(t)\mathcal{N}(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v)\big\|_{X^{s,-\frac{1}{2}-\delta_1}} \Big\|_{L^p(\mu\otimes\mathbb{P})} \\ \sup_{\lambda,\mu_2,\mu_3\in\mathbb{R}} \lesssim \Big\| \big\| \langle n\rangle^s\langle n_2\rangle^{-s}\langle n_3\rangle^{-s}\widetilde{h}^{\ominus,<,N_\star}_{nn_2n_3\mu_2\mu_3\lambda}\big\|_{\ell^2_n\to\ell^2_{n_2n_3}}\Big\|_{L^p(\mu\otimes\mathbb{P})}, \end{aligned} \tag{3.254}$$

Furthermore, applying Lemma A.4.2 gives

$$\begin{aligned} \Big\| \big\| \langle n\rangle^s\langle n_2\rangle^{-s}\langle n_3\rangle^{-s}\widetilde{h}^{\ominus,<,N_\star}_{nn_2n_3\mu_2\mu_3\lambda}\big\|_{n\to n_2n_3}\Big\|_{L^p(\mu\otimes\mathbb{P})} \\ \lesssim p(N_2\vee N_3)^\theta \|\mathfrak{h}^{\ominus,<,N_\star}\|_2, \end{aligned} \tag{3.255}$$

for any $\theta > 0$, and uniformly in $\lambda, \mu_2, \mu_3 \in \mathbb{R}$, where $\mathfrak{h}^{\ominus,<,N_\star} = \mathfrak{h}^{\ominus,<,N_\star}_{nn_1n_2n_3}$ (which also depends on λ, μ_2, μ_3) denotes the tensor

$$\mathfrak{h}^{\ominus,<,N_\star}_{nn_1n_2n_3} = \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \mathbb{1}_{\langle n_1\rangle \le N} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j\rangle \sim N_j} \cdot \frac{\langle n\rangle^s}{\langle n_2\rangle^s \langle n_3\rangle^s} \cdot \frac{\sqrt{2\gamma}}{\langle \Phi_2(\bar{n})\rangle}. \tag{3.256}$$

with $\|\cdot\|_2$ as in (3.143). From (3.251), (3.252), (3.253), and Lemma 3.3.3 (i) and (ii) (fixing the values of $\kappa(\bar{n})$ as in Lemma 3.4.8), we obtain the bounds

$$\|\mathfrak{h}^{\ominus,<,N_\star}\|_2 \lesssim \gamma^{-\frac12}(N_2N_3)^{-s}N_2^{s-1}N_3, \tag{3.257}$$

$$\|\mathfrak{h}^{\ominus,<,N_\star}\|_2 \lesssim \gamma^{\frac12}(N_2N_3)^{-s}N_2^{s+\frac12+\theta}N_3^{\frac12}, \tag{3.258}$$

for any $\theta > 0$. Interpolating (3.257) and (3.258) gives

$$(3.255) \lesssim p\gamma^{\theta}N_1^{-\theta},$$

for some small $\theta > 0$. This concludes the proof. $\square$

Next, we deal with the proof of Lemma 3.4.14. It turns out that a straightforward adaptation of the proof of Lemma 3.4.13 fails: for some range of frequencies, the bounds corresponding to (3.257) and (3.258) are not sufficient to close the relevant estimates. In order to overcome this issue, we use directly use the heat smoothing coming from the Duhamel operator $\mathcal{I}_\gamma$, $\gamma > 0$, in Lemma 3.1.16 (3.26), along with the spatial integrability of the stochastic convolution $\Psi_{\gamma,N}$; see Remark 3.4.15 below.

Proof of Lemma 3.4.14. We focus on the operator $\{\mathcal{I}_\gamma\mathcal{N}^{1,\ominus,>}(\Psi_{\gamma,N},\cdot,\cdot)\}_{N\in\mathbb{N}}$ since the case $j = 2$ is similar; namely, we fix $j = 1$. By Lemma 3.1.13, (3.244), (3.246), and it suffices to prove that $\{\mathcal{I}_\gamma\mathcal{N}(\Psi_{\gamma,N},\cdot,\cdot)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}^{s,s,\frac12,\frac12+\delta}$ and without the T^θ-factor gain in the $\mathcal{B}^{s,s,\frac12,\frac12+\delta}$-norm (3.17) for some small $\delta > \theta$. (We only prove the relevant uniform in $N \in \mathbb{N}$ bound as convergence follow these and considerations as in Lemma 3.4.4.)

By arguing as in Lemma 3.4.7 and by using the reductions at the beginning of Section 3.4 it is enough to prove the following frequency localized

$$\Bigg\| \sup_{\substack{\|u\|_{X^{s,\frac12+\delta_1}}\le 1\\ \|v\|_{X^{s,\frac12+\delta_1}}\le 1}} \big\|\mathcal{I}_\gamma\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v\big)\big)\big\|_{X^{s,\frac12+\delta}} \Bigg\|_{L^p(\mu\otimes\mathbb{P})} \lesssim pN_1^{-\delta_0}, \tag{3.259}$$

for some small $\delta_0, \delta_1 > 0$, uniformly in $N \in \mathbb{N}$ and $\gamma \in [0,1]$, and for dyadic numbers N_1, N_2, N_3, with $N_1 \gg N_2, N_3$.

Next, fix $0 < \eta \ll 1$. Let N_1, N_2, N_3 be dyadic numbers, and write $N_\star = (N_1, N_2, N_3)$. If we have $N_2 \wedge N_3 \gtrsim N_1^{\frac12+\eta}$. Then, by using Proposition 3.1.14 and Lemma 3.1.4, we obtain (3.259) by observing the inequality $N_1^s \lesssim (N_2N_3)^{s-10\eta}$. (Hence, u and v can absorb the derivatives coming from $\Psi_{\gamma,N}$.)

Thus, in the following, we assume the condition $N_2 \wedge N_3 \ll N_1^{\frac12+\eta}$. By using Proposition 3.1.14 and arguing as in the proof of Lemma 3.4.13, we have

$$\begin{aligned}
&\Bigg\| \sup_{\substack{\|u\|_{X^{s,\frac12+\delta_1}}\le 1\\ \|v\|_{X^{s,\frac12+\delta_1}}\le 1}} \big\|\mathcal{I}_\gamma\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N}, \mathbf{P}_{N_2}u, \mathbf{P}_{N_3}v\big)\big)\big\|_{X^{s,\frac12+\delta}} \Bigg\|_{L^p(\mu\otimes\mathbb{P})} \\
&\quad \lesssim \sup_{\lambda,\mu_2,\mu_3} pN_1^{\theta}\|\mathfrak{h}^{\ominus,>,N_\star}\|_2,
\end{aligned} \tag{3.260}$$

uniformly in $\gamma \in [0,1]$ and $N \in \mathbb{N}$, where $\mathfrak{h}^{\ominus,>,N_\star} = \mathfrak{h}^{\ominus,>,N_\star}_{nn_1n_2n_3}$ (which also depends on λ, μ_2, μ_3) denotes the tensor

$$\mathfrak{h}^{\ominus,>,N_\star}_{nn_1n_2n_3} = \mathbb{1}_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \mathbb{1}_{\langle n_1\rangle\le N} \prod_{j=1}^{3} \mathbb{1}_{\langle n_j\rangle\sim N_j}\cdot \frac{\langle n\rangle^s}{\langle n_2\rangle^s\langle n_3\rangle^s}\cdot\frac{\sqrt{2\gamma}}{\langle\Phi_2(\bar n)\rangle}. \tag{3.261}$$

with $\|\cdot\|_2$ as in (3.143) and $\Phi_2(\bar n)$ as in (3.252). From (3.261), and Lemma 3.3.3 (i) and (ii) (fixing the values of $\kappa(\bar n)$ as in Lemma 3.4.8), we obtain the bounds

$$\|\mathfrak{h}^{\ominus,>,N_\star}\|_2 \lesssim \gamma^{-\frac12}(N_2N_3)^{-s}N_1^{s-1}(N_2\wedge N_3), \tag{3.262}$$

$$\|\mathfrak{h}^{\ominus,>,N_\star}\|_2 \lesssim \gamma^{\frac12}(N_2N_3)^{-s}N_1^{s+\frac12+\theta}(N_2\wedge N_3)^{\frac12}, \tag{3.263}$$

for any $\theta>0$. Unfortunately, in the regime $N_2\wedge N_3 = cN_1^{\frac12+\eta}$ for some $0<c\ll 1$, interpolating (3.262) and (3.263) does not suffice to obtain acceptable bounds on (3.260).

Instead, we rely on the derivative gain from the Duhamel operator $\mathcal{I}_\gamma$ for $\gamma>0$ in Proposition 3.1.14. Let us fix $\gamma>0$ and u,v with $\|u\|_{X^{s,\frac12+\delta_1}}\le 1$ and $\|v\|_{X^{s,\frac12+\delta_1}}\le 1$. We have with (3.26) in Proposition 3.1.14, the following estimate:

$$\begin{aligned}
&\big\|\mathcal{I}_\gamma\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N},\mathbf{P}_{N_2}u,\mathbf{P}_{N_3}v\big)\big)\big\|_{X^{s,\frac12+\delta}}\\
&\quad\lesssim \gamma^{-\frac12+\delta}\big\|\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N},\mathbf{P}_{N_2}u,\mathbf{P}_{N_3}v\big)\big\|_{L^2_t([0,1])H^{s-1+2\delta}_x}\\
&\quad\lesssim \gamma^{-\frac12+\delta}N_1^{s-1+2\delta}\big\|\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N},\mathbf{P}_{N_2}u,\mathbf{P}_{N_3}v\big)\big\|_{L^2([0,1];L^2_x)},
\end{aligned}\tag{3.264}$$

where we used $N_1\gg N_2,N_3$. Furthermore, we have

$$\begin{aligned}
&\mathcal{F}_x\big(\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N},\mathbf{P}_{N_2}u,\mathbf{P}_{N_3}v\big)\big)(n,t)\\
&\quad=\sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}}\widehat{\mathbf{P}_{N_1}\Psi_{\gamma,N}}(n_1,t)\overline{\widehat{\mathbf{P}_{N_2}u}}(n_2,t)\widehat{\mathbf{P}_{N_3}v}(n_3,t)\\
&\quad=\sum_{n=n_1-n_2+n_3}\widehat{\mathbf{P}_{N_1}\Psi_{\gamma,N}}(n_1,t)\overline{\widehat{\mathbf{P}_{N_2}u}}(n_2,t)\widehat{\mathbf{P}_{N_3}v}(n_3,t)\\
&\qquad-\widehat{\mathbf{P}_{N_1}\Psi_{\gamma,N}}(n,t)\sum_{n_2\in\mathbb{Z}^2}\overline{\widehat{\mathbf{P}_{N_2}u}}(n_2,t)\widehat{\mathbf{P}_{N_3}v}(n_2,t)\\
&\quad=\mathrm{I}-\mathrm{II}.
\end{aligned}\tag{3.265}$$

Note that in the last computation, we have removed the condition $n_2\neq n_1$ under the assumption $N_1\gg N_2,N_3$. Given the fact that $\mathrm{I}=\mathcal{F}_x\big(\mathbf{P}_{N_1}\Psi_{\gamma,N}\overline{\mathbf{P}_{N_2}u}\,\mathbf{P}_{N_3}v\big)(n,t)$, we obtain by using Hölder and Cauchy-Schwarz inequalities along with Lemma 3.1.3, the following bound:

$$\begin{aligned}
\|\mathrm{I}\|_{L^2([0,1];L^2_x)}&\lesssim\|\mathbf{P}_{N_1}\Psi_{\gamma,N}\|_{L^\infty([0,1],L^\infty_x)}\big\|\overline{\mathbf{P}_{N_2}u}\,\mathbf{P}_{N_3}v\big\|_{L^2([0,1];L^2_x)}\\
&\lesssim\|\mathbf{P}_{N_1}\Psi_{\gamma,N}\|_{L^\infty([0,1],L^\infty_x)}\|\mathbf{P}_{N_2}u\|_{L^4([0,1]\times\mathbb{T}^2)}\big\|\mathbf{P}_{N_3}v\big\|_{L^4([0,1]\times\mathbb{T}^2)}\\
&\lesssim N_1^\theta\|\mathbf{P}_{N_1}\Psi_{\gamma,N}\|_{L^\infty([0,1];W^{-\theta,\infty}_x)}\|\mathbf{P}_{N_2}u\|_{C_\gamma X^{\theta,\frac12+\delta_1}}\big\|\mathbf{P}_{N_3}v\big\|_{X^{\theta,\frac12+\delta_1}}\\
&\lesssim N_1^\theta(N_2N_3)^{-s+\theta}\|\Psi_{\gamma,N}\|_{L^\infty([0,1];W^{-\theta,\infty}_x)},
\end{aligned}\tag{3.266}$$

for small $\theta>0$. Similarly, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\mathrm{II}\|_{L^2([0,1],L^2_x)}&\lesssim\|\mathbf{P}_{N_1}\Psi_{\gamma,N}\|_{L^\infty([0,1],L^2_x)}\sup_{t\in[0,1]}\Big|\sum_{n_2\in\mathbb{Z}^2}\overline{\widehat{\mathbf{P}_{N_2}u}}(n_2,t)\widehat{\mathbf{P}_{N_3}v}(n_2,t)\Big|\\
&\lesssim N_1^\theta\|\Psi_{\gamma,N}\|_{L^\infty([0,1],H^{-\theta}_x)}\|\mathbf{P}_{N_2}u\|_{L^\infty([0,1],L^2_x)}\|\mathbf{P}_{N_3}v\|_{L^\infty([0,1],L^2_x)}\\
&\lesssim N_1^\theta(N_2N_3)^{-s}\|\Psi_{\gamma,N}\|_{L^\infty([0,1],H^{-\theta}_x)}\|\mathbf{P}_{N_2}u\|_{X^{s,\frac12+\delta_1}}\|\mathbf{P}_{N_3}v\|_{X^{s,\frac12+\delta_1}}\\
&\lesssim N_1^\theta(N_2N_3)^{-s}\|\Psi_{\gamma,N}\|_{L^\infty([0,1],H^{-\theta}_x)},
\end{aligned}\tag{3.267}$$

for small $\theta>0$. Note that we used the continuous embedding $C_TH^s\hookrightarrow X^{s,\frac12+\delta_1}$ in the above. Collecting (3.264), (3.265), (3.266), and (3.267) gives (with $0<T\le 1$)

$$\begin{aligned}\Big\| \sup_{\substack{\|u\|_{X^{s,\frac12+\delta_1}}\le 1\\ \|v\|_{X^{s,\frac12+\delta_1}}\le 1}} \big\|\mathcal{I}_\gamma\big(\mathbb{1}_{[0,1]}(t)\mathcal{N}\big(\mathbf{P}_{N_1}\Psi_{\gamma,N},\mathbf{P}_{N_2}u,\mathbf{P}_{N_3}v\big)\big)\big\|_{C_\gamma X^{s,\frac12+\delta}}\Big\|_{L^p(\mu\otimes\mathbb{P})} \\ \lesssim \gamma^{-\frac12+\delta} N_1^{s-1+2\delta+\theta}(N_2N_3)^{-s+\theta}\big\|\|\Psi_{\gamma,N}\|_{L^\infty([0,1],H_x^{-\theta})}\big\|_{L^p(\mu\otimes\mathbb{P})} \\ \lesssim \gamma^{-\frac12+\delta} N_1^{s-1+2\delta+\theta}(N_2N_3)^{-s+\theta}p. \qquad (3.268)\end{aligned}$$

Hence, interpolating (3.268) and (3.260) with (3.263) gives (3.259) and concludes the proof. ☐

Remark 3.4.15. Let us note that (3.268) improves on (3.262) since it essentially saves one derivative in the smallest frequency $N_2 \wedge N_3$.

Appendix A

Appendix

A.1 Deterministic estimates

We recall the following product estimates. See [31] for a proof.

Lemma A.1.1. *Let* $0 \leq s \leq 1$.
(i) *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have*

$$\|\langle\nabla\rangle^s (fg)\|_{L^r(\mathbb{T}^2)} \lesssim \Big(\|f\|_{L^{p_1}(\mathbb{T}^2)} \|\langle\nabla\rangle^s g\|_{L^{q_1}(\mathbb{T}^2)} + \|\langle\nabla\rangle^s f\|_{L^{p_2}(\mathbb{T}^2)} \|g\|_{L^{q_2}(\mathbb{T}^2)} \Big).$$

(ii) *Suppose that* $1 < p, q, r < \infty$ *satisfy the condition:* $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{2}$. *Then, we have*

$$\big\|\langle\nabla\rangle^{-s} (fg)\big\|_{L^r(\mathbb{T}^2)} \lesssim \big\|\langle\nabla\rangle^{-s} f\big\|_{L^p(\mathbb{T}^2)} \big\|\langle\nabla\rangle^s g\big\|_{L^q(\mathbb{T}^2)}.$$

Note that while Lemma A.1.1 (ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{2}$ in [31], the general case $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{2}$ follows from the inclusion $L^{r_1}(\mathbb{T}^2) \subset L^{r_2}(\mathbb{T}^2)$ for $r_1 \geq r_2$.

We record the following elementary result on the regularizing effect of the heat semi-group in Sobolev spaces whose proof is a simple consequence of Plancherel's identity.

Lemma A.1.2. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$ *and* $t > 0$. *We have*

$$\|P_0(t) f\|_{H^\alpha(\mathbb{T}^2)} \lesssim t^{-\frac{\alpha-\beta}{2}} \|f\|_{H^\beta(\mathbb{T}^2)},$$

where P_0 *is as in* (1.27).

Proof. By Plancherel's identity, we have the following bound:

$$\begin{aligned} \|P_0(t) f\|_{H^\alpha(\mathbb{T}^2)} &= \|\langle n\rangle^\alpha e^{-t\langle n\rangle^2} \widehat{f}(n)\|_{\ell^2_n} \\ &\lesssim t^{-\frac{\alpha-\beta}{2}} \|\langle n\rangle^\alpha \langle n\rangle^{\beta-\alpha} \widehat{f}(n)\|_{\ell^2_n} = t^{-\frac{\alpha-\beta}{2}} \|f\|_{H^\beta(\mathbb{T}^2)}, \end{aligned}$$

where we used the bound $e^{-x} \lesssim x^{-\frac{\alpha-\beta}{2}}$ for any $x > 0$. □

A.2 Stochastic tools

In this appendix, we recall some basic tools from probability theory and Euclidean quantum field theory ([39, 53, 67, 69]) and construct the multiple stochastic integrals used throughout this thesis.

A.2.1 Wiener chaoses and bi-parameter Kolmogorov continuity criterion

We now state several useful probabilistic estimates used in this thesis. This subsection is written in the real-valued setting but the adaptation to the complex-valued setting, relevant to Chapter (3), is straighforward.

First, recall the Hermite polynomials $H_k(x;\sigma)$ defined through the generating function:

$$F(t,x;\sigma) \overset{\text{def}}{=} e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma).$$

For readers' convenience, we write out the first few Hermite polynomials:

$$H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x, \quad H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x.$$

Note that the Hermite polynomials verify the following standard identity:

$$H_k(x+y;\sigma) = \sum_{\ell=0}^{k} x^{k-\ell} H_\ell(y;\sigma).$$

Next, we recall the Wiener chaos estimate. Let (H, B, μ) be an abstract Wiener space. Namely, μ is a Gaussian measure on a separable Banach space B with $H \subset B$ as its Cameron-Martin space. Given a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ of $H^* = H$, we define a polynomial chaos of order k to be an element of the form $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j\rangle)$, where $x \in B$, $k_j \neq 0$ for only finitely many j's, $k = \sum_{j=1}^{\infty} k_j$, H_{k_j} is the Hermite polynomial of degree k_j, and $\langle\cdot,\cdot\rangle = {}_B\langle\cdot,\cdot\rangle_{B^*}$ denotes the B-B^* duality pairing. We then denote the closure of the span of polynomial chaoses of order k under $L^2(B,\mu)$ by $\mathcal{H}_k$. The elements in $\mathcal{H}_k$ are called homogeneous Wiener chaoses of order k. We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$$

for $k \in \mathbb{N}$.

Let $L = \Delta - x \cdot \nabla$ be the Ornstein-Uhlenbeck operator.[1] Then, it is known that any element in $\mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$. Then, as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [52], we have the following Wiener chaos estimate [69, Theorem I.22]. See also [71, Proposition 2.4].

Lemma A.2.1. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

We recall the following property of Wick products; see [69, Theorem I.3] for a proof.

Lemma A.2.2. *Let f and g be Gaussian random variables with variances σ_f and σ_g. Then, we have*

$$\mathbb{E}\big[H_k(f;\sigma_f)H_m(g;\sigma_g)\big] = \delta_{km} k! \big\{\mathbb{E}[fg]\big\}^k.$$

We now state a two-dimensional version of the usual Kolmogorov continuity criterion for readers' convenience. Before doing so, let us recall that given an index set T, we say that a stochastic process $\{\widetilde{X}_t\}_{t\in T}$ is a modification of another process $\{X_t\}_{t\in T}$ (both defined on the same probability space $(\Omega, \mathbb{P})$) if for any $t \in T$, we have

$$\mathbb{P}\big(X_t = \widetilde{X}_t\big) = 1.$$

---

[1] For simplicity, we write the definition of the Ornstein-Uhlenbeck operator L when $B = \mathbb{R}^d$.

Lemma A.2.3 (bi-parameter Kolmogorov continuity criterion)**.** *Let $I \times J$ be two intervals of $\mathbb{R}$. Let $\{X(\varepsilon,t)\}_{(\varepsilon,t)\in I\times J}$ be a stochastic process defined on a probability space $(\Omega,\mathbb{P})$ and taking values in a complete metric space (E,d). Assume that there exist real numbers $q,\alpha>0$ and $A>0$ such that*

$$\mathbb{E}\big[d\big(X(\varepsilon_1,t_1),X(\varepsilon_2,t_2)\big)^q\big]\le A\|(\varepsilon_1-\varepsilon_2,t_1-t_2)\|_2^{2+\alpha},$$

for any $(\varepsilon_1,\varepsilon_2)\in I^2$ and $(t_1,t_2)\in J^2$ and where $\|\cdot\|_2$ denotes the canonical Euclidean norm on $\mathbb{R}^2$. Then, there exists a modification $\widetilde{X}$ of X whose sample paths are Hölder continuous on $I\times J$ with exponent δ for any $0<\delta=\delta(\alpha,q)$. Namely, we have

$$d\big(X^\omega(\varepsilon_1,t_1),X^\omega(\varepsilon_2,t_2)\big)\lesssim_\omega A^\gamma\|(\varepsilon_1-\varepsilon_2,t_1-t_2)\|_2^\delta,$$

for some $\gamma>0$ and for any $\omega\in\Omega_0$, where $\Omega_0\subset\Omega$ is a full $\mathbb{P}$-probability set. Furthermore, we have the following tail estimate:

$$\mathbb{P}\bigg(\frac{d\big(X^\omega(\varepsilon_1,t_1),X^\omega(\varepsilon_2,t_2)\big)}{A^\gamma\|(\varepsilon_1-\varepsilon_2,t_1-t_2)\|_2^\delta}>M\bigg)\lesssim M^{-q}.$$

The proof of Lemma A.2.3 follows from a slight modification of the argument in the proof of [2, Theorem 2.1]. See [3] for the proof of the tail estimate.

A.2.2 Multiple stochastic integrals

In this section, we go over the basic definitions and properties of multiple stochastic integrals. See [53] and also [8, Section 4] for further discussion.

Let λ be the measure on $Z:=\mathbb{Z}^2\times\mathbb{R}_+\times\{-1,1\}$ defined by

$$d\lambda=dndtd\zeta,$$

where dn and $d\zeta$ are the counting measures on $\mathbb{Z}^2$ and $\{-1,1\}$. Given $k\in\mathbb{N}$, we set $\lambda_k=\bigotimes_{j=1}^k\lambda$ and $L^2(Z^k)=L^2\big((\mathbb{Z}^2\times\mathbb{R}_+\times\{-1,1\})^k,\lambda_k\big)$.

Recall that $\{B_n\}_{n\in\mathbb{Z}^2}$ is an i.i.d. family of complex Brownian motions defined in (1.68). Let $\{B_n^{-1}\}_{n\in\mathbb{Z}^2}$ be an i.i.d. family of standard complex Brownian motions, independent from $\{B_n\}_{n\in\mathbb{Z}^2}$ and which verifies

$$g_n=\int_0^1 dB_n^{-1}(s) \tag{A.1}$$

for any $n\in\mathbb{Z}^2$ with $\{g_n\}_{n\in\mathbb{Z}^2}$ as in (1.3).

Given a function $f\in L^2(Z^k)$, we can adapt the discussion in [53, Section 1.1] (in particular, [53, Example 1.1.2]) to the complex-valued setting and define the multiple stochastic integral $I_k[f]$ by

$$I_k[f]=\sum_{n_1,\dots,n_k\in\mathbb{Z}^2}\int_{[0,\infty)^k}\sum_{\zeta_1,\cdots,\zeta_k\in\{-1,1\}}f(n_1,t_1,\dots,n_k,t_k)dB_{n_1}^{\zeta_1}(t_1)\dots dB_{n_k}^{\zeta_2}(t_k).$$

Let $f\in L^2(Z^k)$ be a function. We define its symmetrization $\texttt{Sym}(f)$ by

$$\texttt{Sym}(f)(z_1,\dots,z_k)=\frac{1}{k!}\sum_{\sigma\in S_k}f(z_{\sigma(1)},\dots,z_{\sigma(k)}), \tag{A.2}$$

where $z_j=(n_j,t_j,\zeta_j)$ and S_k denotes the symmetric group on $\{1,\dots,k\}$. Note that by Jensen's equality, we have

$$|\texttt{Sym}(f)(z_1,\dots,z_k)|^p\le\frac{1}{k!}\sum_{\sigma\in S_k}|f(z_{\sigma(1)},\dots,z_{\sigma(k)})|^p \tag{A.3}$$

for any $p \geq 1$. We say that f is symmetric if $\mathtt{Sym}(f) = f$. We now recall some basic properties of multiple stochastic integrals.

Lemma A.2.4. *Let $k, \ell \in \mathbb{N}$. The following statements hold for any $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$:*

(i) *$I_k : L^2(Z^k) \to \mathcal{H}_k \subset L^2(\Omega)$ is a linear operator, where $\mathcal{H}_k$ denotes the kth Wiener chaos defined in Subsection 3.1.2.*[2]

(ii) $I_k[\mathtt{Sym}(f)] = I_k[f]$.

(iii) *Ito isometry*:

$$\mathbb{E}\Big[I_k[f] \cdot \overline{I_\ell[g]}\Big] = \mathbb{1}_{k=\ell} \cdot k! \int_{(\mathbb{Z}^2 \times \mathbb{R} \times \{-1,1\})^k} \mathtt{Sym}(f)\overline{\mathtt{Sym}(g)} d\lambda_k.$$

(iv) *Furthermore, suppose that f is symmetric. Then, we have*

$$I_k[f] = k! \sum_{n_1,\cdots,n_k \in \mathbb{Z}^2} \sum_{\zeta_1,\ldots,\zeta_k \in \{-1,1\}} \int_0^\infty \int_0^{t_1} \int_0^{t_{k-1}} f(n_1, t_1, \zeta_1, \ldots, n_k, t_k, \zeta_k) dB_{n_k}^{\zeta_k}(t_k) \cdots dB_{n_1}^{\zeta_1}(t_1),$$

where the iterated integral on the right-hand side is understood as an iterated Ito integral.

We state a version of Fubini's theorem for multiple stochastic integrals that is convenient for our purpose. See, for example, [63, Lemma B.2] for a proof and [23, Theorem 4.33] for a general version of the stochastic Fubini theorem.

Lemma A.2.5. *Let $k \geq 1$. Given finite $T > 0$, let $f \in L^2((\mathbb{Z}^2 \times [0,T] \times \{-1,1\})^k \times [0,T], d\lambda_k \otimes dt\big)$. (In particular, we assume that the temporal support (for the variables $t_1, \ldots, t_k, t$) of f is contained in $[0,T]^{k+1}$ for any $(n_1, \zeta_1, \ldots, n_k, \zeta_k)$.) Then, we have*

$$\int_0^T I_k[f(\cdot, t)]dt = I_k\bigg[\int_0^T f(\cdot, t)dt\bigg] \tag{A.4}$$

in $L^2(\Omega)$.

We conclude this section by stating the product formula (Lemma A.2.7). Before doing so, we first recall the contraction of two functions.

Definition A.2.6. Let $k, \ell \in \mathbb{N}$. Given an integer $0 \leq r \leq \min(k, l)$, we define the contraction $f \otimes_r g$ of r indices of $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$ by

$$\begin{aligned}(f \otimes_r g)(z_1, \ldots, z_{k+\ell-2r}) = \sum_{m_1,\ldots,m_r \in \mathbb{Z}^2} \int_{\mathbb{R}_+^r} \sum_{\iota_1,\ldots,\iota_j \in \{-1,1\}} & f(z_1, \ldots, z_{k-r}, \alpha_1, \ldots, \alpha_r) \\ & \times g(z_{k+1-r}, \ldots, z_{k+\ell-2r}, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_r) ds_1 \cdots ds_r,\end{aligned}$$

where $\alpha_j = (m_j, s_j, \iota_j)$ and $\widetilde{\alpha}_j = (m_j, s_j, \iota_j)$. For convenience, we may simply write $\otimes$ instead of $\otimes_0$.

Note that even if f and g are symmetric, their contraction $f \otimes_r \overline{g}$ is not symmetric in general. We now state the product formula. See [53, Proposition 1.1.3].

Lemma A.2.7 (product formula)**.** *Let $k, \ell \in \mathbb{N}$ with $k \geq \ell$. Let $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$ be symmetric functions. Then, we have*

$$I_k[f] \cdot \overline{I_\ell[g]} = \sum_{r=0}^{\ell} r! \binom{k}{r} \binom{\ell}{r} I_{k+\ell-2r}[f \otimes_r \overline{g}].$$

and

$$I_k[f] \cdot I_\ell[g] = I_{k+\ell}[f \otimes g].$$

---

[2] Strictly speaking, $\mathcal{H}_k$ is a space of real-valued random variables, but we mean here that the real and imaginary parts of I_k have $\mathcal{H}_k$ as a target space.

Let $f_t = f_t(n', t', \zeta)$ and $g_{n,t} = g_{n,t}(z_1, z_2, z_3)$ (for $z_j = (n_j, t_j, \zeta_j)$, $1 \leq j \leq 3$) be the functions defined by

$$
\begin{aligned}
f_t(n', t', \zeta) = \mathbb{1}_{\mathbb{R}_+}(t)\Big(&\mathbb{1}_{\zeta=-1}\mathbb{1}_{[0,1]}(t')\frac{e^{-(\gamma+i)t\langle n'\rangle^2}}{\langle n'\rangle} \\
&+ \sqrt{2\gamma}\mathbb{1}_{\zeta=1}\mathbb{1}_{[0,t]}(t')e^{-(\gamma+i)(t-t')\langle n'\rangle^2}\Big)
\end{aligned}
\tag{A.5}
$$

and

$$
g_{n,t}(z_1, z_2, z_3) = \mathbb{1}_{\substack{n=n_1-n_2+n_3 \\ n_2\neq n_1,n_3}} f_t \otimes \overline{f_t} \otimes_0 f_t(z_1, z_2, z_3),
$$

Then, from Lemma A.2.7, (1.71), (1.73), and (1.81), we have the formulae

$$
\begin{aligned}
\mathcal{F}_x(\overset{\bullet}{\mid}_\gamma)(n, t) &= I_1\big[\mathbb{1}_{n=n'}f_t\big] \\
\mathcal{F}_x(\overset{\bullet\bullet\bullet}{\vee}_\gamma)(n, t) &= I_3\big[g_{n,t}\big]
\end{aligned}
\tag{A.6}
$$

A.3 Construction of stochastic objects: reduction to frequency localized estimates

In this section, we demonstrate how to construct stochastic objects by proving some frequency localized estimates (on the truncated versions) thereof. Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of stochastic objects of interest which are either distribution-valued or operator-valued and defined on a probability space $(\Omega, \mathbb{P})$. We aim to prove the following estimates:

$$
\Big\| \sup_{N\in\mathbb{N}} \|A_N\| \Big\|_{L^p(\Omega)} \lesssim_p 1, \tag{A.7}
$$

$$
\Big\| \sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \|A_M - A_N\| \Big\|_{L^p(\Omega)} \lesssim_p 1, \tag{A.8}
$$

for any $p \geq 1$. Here, $\|\cdot\|$ is a norm on a Banach space X.

In this work, we need the strong bounds (A.7) and (A.8) to prove the convergence of the stochastic objects $\{A_N\}_{N\in\mathbb{N}}$ on sets of the form (3.146) in Chapter 3 (Section 3.4). In particular, this argument differs from the usual construction relying on estimates similar to (A.7) and (A.8) (but with the suprema *outside* the $L^p(\Omega)$-norms) and the Borel-Cantelli lemma; see [59, Proposition 3.2].

Next, we show how to derive (A.7) and (A.8) from "simpler" frequency localized estimates; see Lemma A.3.1 below. In the current setting, A_N can be written (with the notations of Appendix A.2.2) as:

$$
A_N = I_k[f_N],
$$

for some $k \in \mathbb{N}$ and where $f_N \in L^2(Z^k)$ is given by

$$
f_N = g \cdot \prod_{j=1}^{k} \mathbb{1}_{\langle n_j\rangle \leq N},
$$

for some $g \in L^2(Z^k)$.

Let $N_\star = (N_1, \dots, N_k)$ be a vector of dyadic numbers. We introduce the following frequency localized objects:

$$
A_N^{N_\star} = I_k[f_N^{N_\star}],
$$

where

$$f_N^{N_\star} = f_N \cdot \prod_{j=1}^{k} \mathbb{1}_{\langle n_j \rangle \sim N_j}. \tag{A.9}$$

By construction, we have

$$A_N = \sum_{N_\star \in (2^{\mathbb{N}})^k} A_N^{N_\star}. \tag{A.10}$$

We prove the following result.

Lemma A.3.1. *We consider a sequence as in the above such that* (A.9) *and* (A.10) *hold. Let us assume that the following bounds are satisfied:*

$$\sup_{N\in\mathbb{N}} \big\| \|A_N^{N_\star}\| \big\|_{L^p(\Omega)} \lesssim_p \max(N_\star)^{-\delta}, \tag{A.11}$$

$$\sup_{N\in\mathbb{N}} \sup_{M\geq N} N^\delta \big\| \|A_M^{N_\star} - A_N^{N_\star}\| \big\|_{L^p(\Omega)} \lesssim_p \max(N_\star)^{-\delta}, \tag{A.12}$$

for some $\delta > 0$, any $p \geq 1$ and any dyadic numbers $N_\star = (N_1, \dots, N_k)$.

Then, the bounds (A.7) *and* (A.8) *hold.*

Proof. We argue that (A.11) implies (A.7). Fix $N_\star = (N_1, \dots, N_k) \in (2^{\mathbb{N}})^k$. For $N \gg \max(N_\star)$, we can freely replace the frequency localizations $\mathbb{1}_{\langle n_j \rangle \leq N}$ in (A.9) by $\mathbb{1}_{\langle n_j \rangle \leq 10\cdot\max(N_\star)}$, i.e. $f_N^{N_\star} = f_{10\cdot\max(N_\star)}^{N_\star}$. This leads to

$$A_N^{N_\star} = I_k[f_{10\cdot\max(N_\star)}^{N_\star}],$$

and hence, $\sup_{N\gg\max(N_\star)} \|A_N^{N_\star}\| = \|A_{10\cdot\max(N_\star)}^{N_\star}\|$. Thus, by (A.10) and Minkowski's inequality, we have

$$\begin{aligned}
&\big\| \sup_{N\in\mathbb{N}} \|A_N\| \big\|_{L^p(\Omega)} \\
&\quad\lesssim \sum_{N_\star\in(2^{\mathbb{N}})^k} \Big(\big\| \sup_{N\lesssim\max(N_\star)} \|A_N^{N_\star}\| \big\|_{L^p(\Omega)} + \big\| \sup_{N\gg\max(N_\star)} \|A_N^{N_\star}\| \big\|_{L^p(\Omega)} \Big) \\
&\quad\lesssim \sum_{N_\star\in(2^{\mathbb{N}})^k} \Big(\big\| \|A_N^{N_\star}\| \big\|_{\ell^q(1\leq N\leq 10\cdot\max(N_\star)) L^p(\Omega)} + \|A_{10\cdot\max(N_\star)}^{N_\star}\|_{L^p(\Omega)} \Big) \\
&\quad\lesssim \sum_{N_\star\in(2^{\mathbb{N}})^k} \Big(\max(N_\star)^{\frac{2}{q}-\delta} + \max(N_\star)^{-\delta} \Big) \lesssim 1,
\end{aligned}$$

provided that $q > \frac{2}{\delta}$ and $p \geq q$.[3] This proves (A.7). By similar arguments, the bound (A.8) can be deduced from (A.12). ☐

A.4 Random tensors

In this section, we provide the basic definition and some lemmas on (random) tensors from [25, 8]. See [25, Sections 2 and 4] and [8, Section 4] for further discussion.

Definition A.4.1. Let A be a finite index set. We denote by n_A the tuple $(n_j : j \in A)$. A tensor $h = h_{n_A}$ is a function: $(\mathbb{Z}^2)^A \to \mathbb{C}$ with the input variables n_A. Note that the tensor h may also depend on $\omega \in \Omega$. The support of a tensor h is the set of n_A such that $h_{n_A} \neq 0$.

Given a finite index set A, let (B, C) be a partition of A. We define the norms $\|\cdot\|_{n_A}$ and $\|\cdot\|_{n_B \to n_C}$ by

$$\|h\|_{n_A} = \|h\|_{\ell^2_{n_A}} = \Big(\sum_{n_A} |h_{n_A}|^2 \Big)^{\frac{1}{2}}$$

---

[3] The same bound then holds for every $p \geq 1$ in view of the nestedness of the spaces $(L^p(\Omega))_{1\leq p<\infty}$

and

$$\|h\|^2_{n_B\to n_C} = \sup\bigg\{\sum_{n_C}\Big|\sum_{n_B} h_{n_A} f_{n_B}\Big|^2 : \|f\|_{\ell^2_{n_B}} = 1\bigg\}, \tag{A.13}$$

where we used the short-hand notation $\sum_{n_Z}$ for $\sum_{n_Z\in(\mathbb{Z}^2)^Z}$ for a finite index set Z. Note that, by duality, we have $\|h\|_{n_B\to n_C} = \|h\|_{n_C\to n_B} = \|\overline{h}\|_{n_B\to n_C}$ for any tensor $h = h_{n_A}$. If $B = \varnothing$ or $C = \varnothing$, then we have $\|h\|_{n_B\to n_C} = \|h\|_{n_A}$.

For example, when $A = \{1,2\}$, the norm $\|h\|_{n_1\to n_2}$ denotes the usual operator norm $\|h\|_{\ell^2_{n_1}\to\ell^2_{n_2}}$ for an infinite dimensional matrix operator $\{h_{n_1n_2}\}_{n_1,n_2\in\mathbb{Z}^3}$. By bounding the matrix operator norm by the Hilbert-Schmidt norm (= the Frobenius norm), we have

$$\|h\|_{\ell^2_{n_1}\to\ell^2_{n_2}} \le \|h\|_{\ell^2_{n_1,n_2}} \tag{A.14}$$

Let (B,C) be a partition of A. Then, by duality, we can write (A.13) as

$$\|h\|_{n_B\to n_C} = \sup\bigg\{\Big|\sum_{n_B,n_C} h_{n_A} f_{n_B} g_{n_C}\Big| : \|f\|_{\ell^2_{n_B}} = \|g\|_{\ell^2_{n_C}} = 1\bigg\},$$

from which we obtain

$$\sup_{n_A}|h_{n_A}| = \sup_{n_B,n_C}|h_{n_Bn_C}| \le \|h\|_{n_B\to n_C}. \tag{A.15}$$

Next, we state the following random matrix estimate. This lemma is essentially the content of Propositions 2.8 and 4.14 in [25]; see also Proposition 4.50 in [8].

Let A be a finite index set. We set $z_A = (k_A, t_A, \zeta_A)$ for $(k_A, t_A, \zeta_A) \in (\mathbb{Z}^2)^A\times\mathbb{R}^A\times\{-1,1\}^A$ and write $f_{z_A} = f(z_A) = f(n_A, t_A, \zeta_A)$.

Lemma A.4.2. *Let A be a finite index set with $k = |A| \ge 1$. Let $h = h_{bcn_A}$ be a tensor such that $n_j \in \mathbb{Z}^2$ for each $j \in A$ and $(b,c) \in (\mathbb{Z}^2)^d$ for some integer $d \ge 2$. Given $N \ge 1$, assume that*

$$\operatorname{supp} h \subset \big\{|b - b_\star|, |c - c_\star|, |n_j - n_{j,\star}| \lesssim N \text{ for each } j \in A\big\}, \tag{A.16}$$

for some $(a_\star, b_\star, (n_{j,\star})_{j\in A}) \in (\mathbb{Z}^2)^2\times(\mathbb{Z}^2)^k$. Given a (deterministic) tensor $h_{bcn_A} \in \ell^2_{bcn_A}$, define the tensor $H = H_{bc}$ by

$$H_{bc} = I_k\big[h_{bcn_A} f_{z_A}\big] \tag{A.17}$$

for $f \in \ell^\infty_{n_A}\big((\mathbb{Z}^2)^A; L^2_{t_A}(\mathbb{R}^A_+)\times\ell^2_{\zeta_A}(\{-1,1\})\big)$, where I_k denotes the multiple stochastic integral defined in Appendix A.2.2. Then, for any $\theta > 0$, we have

$$\Big\|\|H_{bc}\|_{b\to c}\Big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^\theta \Big(\max_{(B,C)} \Big\|\|h f(n_A,t_A)\|_{L^2_{t_A}\ell^2_{\zeta_A}}\Big\|_{bn_B\to cn_C}\Big), \tag{A.18}$$

where the maximum is taken over all partitions (B,C) of A.

Proof. The proof is a slight modification of the proof of [63, Lemma C.3]. □

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