

THEOREMS IN THE TENSOR CALCULUS,
with applications to
Relativity.

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2. Taylor's Theorem in the Tensor Calculus.
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3. Normal Covariant Derivatives.
Proc. London Math. Soc. 33 (1931) 66 to 76.
4. On the "elementary" solution of Laplace's Equation.
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6. The potential of an electron in a space-time of constant curvature.
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9. On the definition of spatial distance in general Relativity.
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10. Note on refraction and reflection in general Relativity.
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CHAPTER I.

Introduction.

§1. The present work deals with certain aspects of tensor analysis which have hitherto gained little attention. The subject is usually treated from a definitely geometrical point of view, the geometrical significance of the theorems taking precedence over the analytical. The reason for this is obvious enough, being indeed partly explained by Einstein's use of tensors in formulating his essentially geometrical theory of general Relativity, and by Levi-Civita's discovery of the close connexion between covariant differentiation and the concept of parallelism. Moreover it is necessary to recognise that tensor analysis owes much of its development to the ideas suggested by geometrical reasoning, for it might otherwise have remained a simple theory of the quadratic differential

form.

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Nevertheless it must not be forgotten that the theory of tensors, at any rate in so far as its associated geometry is riemannian, forms a wide and beautiful generalisation of the ordinary differential calculus. With this in mind I have therefore made some small attempt to develop the subject for its own sake, keeping the geometrical aspect of the theory somewhat in the background. It is true that it is convenient constantly to use geometrical ideas and terminology, but for the purpose of the present work they are to be regarded for the most part as unessential.

§ 2. Elementary text-books divide the infinitesimal calculus into three main parts, dealing respectively with the differentiation of functions of one variable,

partial differentiation, and integration. From the point of view of the following work, the Absolute Differential Calculus of Ricci and Levi-Civita is the analogue or generalisation of the first of these. What then corresponds to the other two? I have endeavoured in Chapter III to show that, in spite of the fact that the covariant derivative necessarily involves partial derivatives (except in the trivial case when the associated geometry is one-dimensional), it is possible to develop a tensor analogy to the ordinary theory of partial differentiation. The work in that chapter is not pursued to greater lengths because it is the point of view rather than the essential subject-matter which is new.

Some reference to the possibility of establishing an integral calculus

of tensors is made in Chapter IV,
page 74. It is there suggested that
the central problem of such a theory
must be the defining of an operation
inverse to that of covariant differentiation.
I have hitherto had no real success
in my attempts to do this. It is in
any case obvious enough that no
unique operation of this kind can
exist, on account of the arbitrary
functions which must necessarily
appear. I have therefore made
tentative efforts in the direction of
beginning an integral theory by
obtaining solutions of tensor differential
equations involving Beltrami's second
differential parameter, equations, that
is, in which the differential operator
is the tensor generalisation of the
Laplacian operator ∇^2 . This particular
type of equation has been chosen

because it is perhaps the simplest to begin with, and because solutions of such equations may be of use in applied mathematics.

In particular, the third section of Chapter IV contains a solution of a set of partial differential equations which arise in Relativity. An earlier attempt to solve the same equations also appears in Chapter I, paper 1.

§3. From another point of view almost the whole of this dissertation may be regarded as an investigation of the properties of a characteristic scalar function in an n -dimensional Riemannian space. The function, which I denote by Ω , is defined to be one-half* of the square of the geodesic distance between two points of the space, so that it is a function of $2n$ variables, the

* In Chapter I, paper 1 the coefficient $\frac{1}{2}$ is not inserted. I changed the notation in the later papers at the suggestion of Prof. E.A. Milne, to whom I am greatly indebted.

coordinates of the two points.

An effort is made to demonstrate the fundamental importance of this function* in the analytical theory of tensors. One reason for its importance is easy to see: namely that to be given the function is to be given, *inter alia*, the metric of the riemannian space and therefore all its intrinsic properties. For the square of the geodesic distance between two points P, P' becomes, when P, P' are close to one another, the fundamental quadratic form which determines the metric properties of the space. Hence a complete description of the invariant properties of Ω and its derivatives must include the whole theory of invariants of quadratic

* My attention was first drawn to the function by reading Hadamard's "Lectures on Cauchy's Problem", in which it is employed in the formation of the "elementary solution" of the general linear partial differential equation of the second order. It has also been used by J. L. Synge [Proc. London Math. Soc., 32 (1931) 241] in discussing the relations between the sides and angles of geodesic triangles in riemannian space. Synge's work was done independently of, but almost contemporaneously with my own earlier papers.

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differential forms.

§4. There is another fairly obvious reason for the appearance of the function in problems of the nature discussed in the following pages. It is well known that, in a three-dimensional euclidean space, the rectangular cartesian coordinates (x, y, z) transform like the components of a vector for rotations of the axes. What does this vector become under more general transformations of the variables? The answer is easily found. Since the vector (x, y, z) is the gradient of the scalar $\frac{1}{2}(x^2 + y^2 + z^2)$, which is an absolute invariant since it is proportional to the square of the distance of the point (x, y, z) from the origin, its components relative to the new set of variables u, v, w are $\frac{\partial \Omega}{\partial u}$, $\frac{\partial \Omega}{\partial v}$, $\frac{\partial \Omega}{\partial w}$, where Ω is the function into which $\frac{1}{2}(x^2 + y^2 + z^2)$ transforms; that is, Ω is one-half of the square of

the geodesic distance between the point (u, v, w) and the point into which the cartesian origin is transformed.

A similar thing is true of a euclidean space of any number of dimensions.

Consequently, if $(x^k) \equiv (x^1, x^2, \dots, x^n)$ and $(\bar{x}^k) \equiv (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ are the coordinates of an n -dimensional flat riemannian space, and if $\Omega = \frac{1}{2} s^2$, where s is the geodesic distance between them, then the derivatives $\frac{\partial \Omega}{\partial x^1}, \frac{\partial \Omega}{\partial x^2}, \dots, \frac{\partial \Omega}{\partial x^n}$ are the components of the "position-vector" of the point (x^k) with respect to the point (\bar{x}^k) . In more general riemannian spaces the derivatives are, as might be expected, closely associated with normal coordinates, so that it is hardly surprising that they appear in the tensor generalisation of many of the theorems of the differential calculus.

CHAPTER II.

8A.

Theorems in the Absolute Differential Calculus.

Paper 1: "Some Theorems in the Tensor Calculus"*
(Page 9)

Paper 2: "Taylor's Theorem in the Tensor
Calculus". (Page 15)

Paper 3: "Normal Covariant Derivatives."
(Page 23)

* The contents of Paper 1 formed part of a thesis submitted to the examiners for the University of Oxford Senior Mathematical Scholarship, 1929.

SOME THEOREMS IN THE TENSOR CALCULUS

By H. S. RUSE.

[Received 17 July, 1929.—Read 14 November, 1929.]

[*Extracted from the Proceedings of the London Mathematical Society, Ser. 2, Vol. 31, Part 3.*]*Notation and definitions.*

1. Let (x^1, x^2, \dots, x^p) be generalized coordinates in a p -dimensional Euclidean manifold* specified by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

due regard being had for the summation convention.

For brevity, we denote the point whose coordinates are (x^1, x^2, \dots, x^p) by (x^i) .

Let $P, (x^i)$, be a variable point and $Q, (q^i)$, be a fixed point of the space. We exclude from our discussion points P, Q which are singular, *e.g.* points at which any of the functions $g_{\mu\nu}$ become infinite. We shall call Q the base-point of the manifold.

Let Ω denote the square of the geodesic distance from Q to P . Thus we define Ω by the relation

$$\sqrt{\Omega} = \int_{QP} ds, \quad (2)$$

the integral being taken along the arc of the geodesic (assumed unique) from Q to P .

Then it is evident from the definition that Ω is a scalar function of x^1, x^2, \dots, x^p , and it is a well known fact that it satisfies the partial differential equation

$$g^{\mu\nu} \Omega_\mu \Omega_\nu = 4\Omega^\dagger, \quad (3)$$

where Ω_μ, Ω_ν are covariant derivatives of Ω . This is indeed easily proved by the methods of the Calculus of Variations.

We notice that when the coordinates are rectangular Cartesian, we have

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^p)^2,$$

* *I.e.* one for which the Riemann tensor is everywhere zero.

† See, *e.g.*, Hadamard, *Lectures on Cauchy's problem* (1923), 90, § 59.

so that $g_{\mu\nu} = \delta_{\nu}^{\mu}$ (where $\delta_{\nu}^{\mu} = 0$ or 1 according as $\mu \neq \nu$ or $\mu = \nu$) and the function Ω is simply

$$\Omega = \sum_{r=1}^p (x^r - q^r)^2 = \sum_{\mu, \nu} \delta_{\nu}^{\mu} (x^{\mu} - q^{\mu})(x^{\nu} - q^{\nu}). \quad (4)$$

It is the properties of the function Ω , defined by equation (2), that we propose to investigate.

2. THEOREM. *The scalar Ω satisfies the partial differential equations*

$$\Omega_{\mu\nu} = 2g_{\mu\nu}, \quad (5)$$

where $\Omega_{\mu\nu}$ is the second covariant derivative of Ω .

In other words, the fundamental tensor $g_{\mu\nu}$ is, in a Euclidean space, the second covariant derivative of the scalar $\frac{1}{2}\Omega$; which statement is really meaningless, since we cannot form covariant derivatives without an *a priori* knowledge of the value of $g_{\mu\nu}$.*

By way of proof, we demonstrate the truth of the theorem for two particular cases, and infer its general truth from the tensor character of equation (5). That the theorem does in fact hold for all Euclidean spaces may, however, easily be proved by a transference to normal coordinates.

Consider then the case in which the coordinates are rectangular Cartesian. Since the Christoffel symbols are then all zero, we have simply

$$\begin{aligned} \Omega_{\mu\nu} &= \frac{\partial^2 \Omega}{\partial x^{\mu} \partial x^{\nu}} = 2\delta_{\nu}^{\mu} \quad \{\text{by equation (4)}\} \\ &= 2g_{\mu\nu}. \end{aligned}$$

Again, if the metric is given by

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ &\quad (p = 3, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi), \end{aligned}$$

then $g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu),$

and the only non-vanishing Christoffel symbols are

$$\{22, 1\} = -r, \quad \{33, 1\} = -r \sin^2 \theta, \quad \{33, 2\} = -\sin \theta \cos \theta,$$

$$\{12, 2\} = \{21, 2\} = \{13, 3\} = \{31, 3\} = \frac{1}{r},$$

$$\{23, 3\} = \{32, 3\} = \cot \theta.$$

* I owe this remark to Dr. E. T. Copson.

Also, the square of the geodesic distance between the points (r, θ, ϕ) and (a, α, β) is given by

$$\Omega = r^2 + a^2 - 2ar \{ \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos(\phi - \beta) \}.$$

Substituting these values in the formula

$$\Omega_{\mu\nu} = \frac{\partial^2 \Omega}{\partial x^\mu \partial x^\nu} - \{ \mu\nu, \alpha \} \frac{\partial \Omega}{\partial x^\alpha}$$

we at once get $\Omega_{11} = 2 = 2g_{11}$, $\Omega_{22} = 2r^2 = 2g_{22}$, and so on.

COROLLARY. In equation (5), raising the suffix ν , we get

$$\Omega_\mu^\nu = 2g_\mu^\nu = 2\delta_{\mu\nu}^{\nu}, \quad (6)$$

and, putting $\nu = \mu$ (and summing),

$$\Omega_\mu^\mu = 2p. \quad (7)$$

Taylor's series; invariant form.

3. We can now establish the following theorem:—

Let $V(x^i)$ be any scalar which possesses covariant derivatives $V_\lambda, V_{\lambda\mu}, V_{\lambda\mu\nu}, \dots$, in a given region \mathfrak{K} of the continuum, and let $V(q^i)$ be its value at the point (q^i) in \mathfrak{K} . Then, if (x^i) be also in \mathfrak{K} and Ω be the square of the geodesic distance between (q^i) and (x^i) ,

$$V(q^i) = V(x^i) - \frac{\Omega^\lambda V_\lambda}{2 \cdot 1!} + \frac{\Omega^\lambda \Omega^\mu V_{\lambda\mu}}{2^2 \cdot 2!} - \frac{\Omega^\lambda \Omega^\mu \Omega^\nu V_{\lambda\mu\nu}}{2^3 \cdot 3!} + \dots, \quad (8)$$

where $\Omega^\lambda, \Omega^\mu, \dots$, are contravariant derivatives of Ω , and where the summation convention is observed throughout.

We remark that, although we shall not concern ourselves with questions of rigour, the series on the right-hand side is assumed to be either terminating, convergent, or to possess a suitable "remainder after n terms."

To prove the theorem, we observe that the series certainly represents some scalar function of x^1, x^2, \dots, x^p , say F . Forming the first covariant derivative $\partial F / \partial x^\kappa$ by term-by-term covariant differentiation (which we assume legitimate), it at once becomes clear that $\partial F / \partial x^\kappa = 0$; provided that we utilize equation (6) and remember that covariant differentiations are commutative in a Euclidean space. Hence it follows that F is independent of the x 's, and is unaltered if in the series it represents we put $(x^i) = (q^i)$. We at once get

$$F = V(q^i),$$

since a consideration of the Cartesian case renders evident the fact that the first contravariant derivative of Ω is zero when $(x^i) = (q^i)$. And this proves the theorem.

We notice that when the coordinates are orthogonal Cartesian, we have, by equation (4),

$$\Omega^\lambda = g^{\lambda\alpha} \frac{\partial \Omega}{\partial x^\alpha} = 2(x^\lambda - q^\lambda), \quad (9)$$

while $V_\lambda, V_{\lambda\mu}, \dots$, are simply $\partial V / \partial x^\lambda, \partial^2 V / \partial x^\lambda \partial x^\mu, \dots$. Substitution in equation (8) gives

$$V(q^i) = V(x^i) + \frac{1}{1!} \sum_{\lambda=1}^p (q^\lambda - x^\lambda) \frac{\partial V}{\partial x^\lambda} + \frac{1}{2!} \sum_{\mu, \nu} \delta_\nu^\mu (q^\mu - x^\mu)(q^\nu - x^\nu) \frac{\partial^2 V}{\partial x^\mu \partial x^\nu} + \dots$$

which is the classical Taylor's theorem.

The theorem may readily be verified to give a true result in a few simple cases; for example, the case in which

$$ds^2 = dr^2 + r^2 d\theta^2, \quad \Omega = r^2 + a^2 - 2ar \cos(\theta - \alpha),$$

and

$$V = r^2 - 2kr \cos \theta + k^2.$$

Homogeneous tensors.

4. Let $T_{\beta_1 \beta_2 \dots \beta_s}^{\alpha_1 \alpha_2 \dots \alpha_r}$ be a tensor with r contravariant and s covariant suffixes, which we denote briefly by T_β^α . In the following the word "tensor" is taken to include scalars as a particular case.

Then we define T_β^α to be a homogeneous tensor of degree n with respect to the base-point (q^i) if it satisfies the partial differential equations

$$\Omega^\mu T_{\beta\mu}^\alpha = 2n T_\beta^\alpha, \quad (10)$$

where $\Omega^\mu = g^{\mu\lambda} (\partial \Omega / \partial x^\lambda)$ and $T_{\beta\mu}^\alpha$ is the first covariant derivative of T_β^α .

We notice from equation (9) that in the orthogonal Cartesian case, with the base-point (q^i) taken at the origin, equation (10) reduces to

$$\sum_{\mu=1}^p x^\mu \frac{\partial}{\partial x^\mu} T_\beta^\alpha = n T_\beta^\alpha,$$

so that each component of the tensor is homogeneous in the ordinary sense.

THEOREM I. *If T_β^α is a homogeneous tensor of degree n , then*

$$\Omega^\mu \Omega^\nu T_{\beta\mu\nu}^\alpha = 2n(2n-2) T_\beta^\alpha, \quad (11)$$

the suffixes μ, ν denoting covariant or contravariant differentiations.

$$\begin{aligned}
 \text{For } \Omega^\mu \Omega^\nu T_{\beta\mu\nu}^\alpha &= \Omega^\nu (\Omega^\mu T_{\beta\mu\nu}^\alpha) - \Omega^\nu \Omega_\nu^\mu T_{\beta\mu}^\alpha \\
 &= \Omega^\nu (\Omega^\mu T_{\beta\mu\nu}^\alpha) - \frac{1}{2} (\Omega^\nu \Omega_\nu)^\mu T_{\beta\mu}^\alpha \\
 &= \Omega^\nu \cdot 2n T_{\beta\nu}^\alpha - 2\Omega^\mu T_{\beta\mu}^\alpha, \quad \text{by equations (10) and (3),} \\
 &= 2n(2n-2) T_{\beta\nu}^\alpha, \quad \text{by equation (10).}
 \end{aligned}$$

Proceeding thus, we could prove that

$$\Omega^{\mu_1} \Omega^{\mu_2} \dots \Omega^{\mu_r} T_{\beta\mu_1\mu_2\dots\mu_r}^\alpha = 2n(2n-2)\dots(2n-2r+2) T_{\beta\nu}^\alpha,$$

which, in the orthogonal Cartesian case, reduces to Euler's theorem on homogeneous functions, namely,

$$\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + \dots + x^p \frac{\partial}{\partial x^p} \right)^r T_{\beta}^\alpha = n(n-1)\dots(n-r+1) T_{\beta}^\alpha,$$

the base-point being taken at the origin.

THEOREM II. *The first covariant derivative $T_{\beta\gamma}^\alpha$ of a homogeneous tensor T_{β}^α of degree n is a homogeneous tensor of degree $n-1$.*

For since, in a Euclidean space, covariant differentiations are commutative, we have

$$\begin{aligned}
 \Omega^\mu T_{\beta\gamma\mu}^\alpha &= \Omega^\mu T_{\beta\mu\gamma}^\alpha = (\Omega^\mu T_{\beta\mu}^\alpha)_\gamma - \Omega_\gamma^\mu T_{\beta\mu}^\alpha \\
 &= (2n T_{\beta}^\alpha)_\gamma - 2\Omega_\gamma^\mu T_{\beta\mu}^\alpha, \quad \text{by equations (10) and (6),} \\
 &= 2(n-1) T_{\beta\gamma}^\alpha,
 \end{aligned}$$

and the theorem follows from the definition.

Additional theorems on homogeneous tensors.

The following theorems result at once from the definition.

A. *If T_{β}^α , U_{β}^α are two homogeneous tensors of the same kind, both of degree n , then $T_{\beta}^\alpha + U_{\beta}^\alpha$ is homogeneous of degree n .*

B. *If T_{β}^α is a homogeneous tensor of degree n , and V_{κ}^{ϵ} a homogeneous tensor of degree m , then the product $T_{\beta}^\alpha V_{\kappa}^{\epsilon}$ is a homogeneous tensor of degree $(n+m)$.*

C. Ω is itself a homogeneous scalar of degree 2. {For, by equation (3), $\Omega^\lambda \Omega_\lambda = 2n\Omega$, where $n = 2$.}

D. *The fundamental tensors $g_{\mu\nu}$, $g^{\mu\nu}$ are homogeneous of degree zero.*

From A, B, and D it therefore follows that the operations of raising and lowering suffixes, and of "contraction", may be performed without

altering the homogeneity or degree of a homogeneous tensor. For these operations really consist of multiplications (by the fundamental tensor), and of summations of like tensors.

An application to the solution of a set of partial differential equations.

Let Δ denote the operation of forming the second covariant derivative of a tensor, of raising the second suffix, and contracting. Thus

$$\Delta T_{\beta}^{\alpha} = g^{\mu\nu} T_{\beta\mu\nu}^{\alpha} = T_{\beta\mu}^{\alpha\mu}.$$

Let $\Delta^2, \Delta^3, \dots$ denote repetitions of this operation.

Then, if $\Phi_{\beta}^{\alpha} \equiv \Phi_{\beta_1 \beta_2 \dots \beta_n}^{\alpha_1 \alpha_2 \dots \alpha_n}$ be a tensor which satisfies the partial differential equations

$$\Delta \Phi_{\beta}^{\alpha} = 0, \quad (12)$$

we have the theorem:

A solution of the partial differential equations (12) is given by

$$\begin{aligned} \Phi_{\beta}^{\alpha} = & \left[1 - \frac{\Omega \Delta N_{\beta}^{\alpha}}{2(2n+p-4)} + \frac{\Omega^2 \Delta^2 N_{\beta}^{\alpha}}{2 \cdot 4 \cdot (2n+p-4)(2n+p-6)} - \dots \right. \\ & \left. + \frac{(-1)^m \Omega^m \Delta^m N_{\beta}^{\alpha}}{2 \cdot 4 \dots (2m) \cdot (2n+p-4) \dots (2n+p-2m-2)} + \dots \right], \quad (13) \end{aligned}$$

where $\Omega^2, \Omega^3, \dots, \Omega^m, \dots$ denote the 2nd, 3rd, ..., m -th, ... powers of Ω , and N_{β}^{α} is a homogeneous tensor of degree n of the same kind as Φ_{β}^{α} . The series on the right-hand side is assumed to be either terminating or convergent.

The theorem is capable of immediate proof by direct substitution of the series in equation (12), term-by-term differentiation being assumed legitimate. We omit the proof on account of its length, but remark that the theorem is an obvious generalization of one due to E. W. Hobson on the solution of Laplace's equation*.

In conclusion, we note that if $p = 4$, and Φ_{β}^{α} is a four-vector ϕ_{μ} , we obtain a solution of the electromagnetic equations $\Delta \phi_{\mu} = 0$ in the Theory of Relativity†, provided that the tensor N_{β}^{α} be so chosen that we have in addition $g^{\mu\nu} \phi_{\mu\nu} = 0$.

* *Proc. London Math. Soc.* (1), 24 (1892), 55, 66.

† Eddington, *Mathematical theory of relativity* (1924), § 74.

Chapter II,

Paper 2. TAYLOR'S THEOREM IN THE TENSOR CALCULUS

By H. S. RUSE.

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[*Extracted from the Proceedings of the London Mathematical Society, Ser. 2, Vol. 32, Part 2.*]

In a previous paper[†] an invariant formula was found which expressed the value of a scalar at any point of a Euclidean space in terms of its value at another point; the formula being, in fact, a tensor form of Taylor's series. We propose now to extend the theorem so that it shall be applicable to tensors in the most general Riemannian space. In so doing we have occasion to employ the normal coordinates of Lipschitz[‡], for which, incidentally, an explicit definition is obtained in place of the implicit definition usually given.

1. *Notation.*

The summation convention is observed throughout. Let

$$(x^i) \equiv (x^1, x^2, \dots, x^n)$$

be generalized coordinates in a Riemannian space of n dimensions characterized by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

Let (x^i) and (\bar{x}^i) be two non-singular points of the space. To avoid an unduly complicated notation, we sometimes employ x^1, x^2, \dots, x^n as variables (current coordinates), and sometimes as the coordinates of a fixed point. It will be seen that no ambiguity results.

The components of any tensor $T_{q\dots}^p\dots$ will, of course, be functions of the variables (x^i) ; their value at (\bar{x}^i) will be denoted by $\bar{T}_{q\dots}^p\dots$.

[†] *Proc. London Math. Soc.*, 31 (1930), 225. (Paper 1 of this Chapter).

[‡] An account of these is given by Veblen, "Invariants of quadratic differential forms", *Camb. Math. Tracts*, No. 24 (1927), Ch. 6.

2. *Normal coordinates.*

It has been shown[†] that the equations of the geodesics which pass through the point (\bar{x}^i) may be written in the form

$$x^i = \bar{x}^i + a^i s - \frac{1}{2!} \gamma_{\lambda\mu}^i a^\lambda a^\mu s^2 - \frac{1}{3!} \gamma_{\lambda\mu\nu}^i a^\lambda a^\mu a^\nu s^3 - \dots \quad (i = 1, 2, \dots, n), \quad (2)$$

where s denotes the length measured from (\bar{x}^i) to (x^i) of the arc of the geodesic; where $\gamma_{\lambda\mu}^i, \gamma_{\lambda\mu\nu}^i, \dots$ are the values at (\bar{x}^i) of certain functions of the Christoffel symbols and their derivatives; and where the parameters a^i specify the geodesic under consideration. In fact,

$$a^i = \frac{d\bar{x}^i}{ds}, \quad (3)$$

the right-hand side being the value at (\bar{x}^i) of dx^i/ds for the geodesic in question.

If now we introduce the normal coordinates (y^i) defined by the equation

$$x^i = \bar{x}^i + y^i - \frac{1}{2!} \gamma_{\lambda\mu}^i y^\lambda y^\mu - \frac{1}{3!} \gamma_{\lambda\mu\nu}^i y^\lambda y^\mu y^\nu - \dots \quad (i = 1, 2, \dots, n), \quad (4)$$

then the equations of the geodesics passing through the point (\bar{x}^i) , which is the origin of the new coordinates (y^i) , are simply

$$y^i = a^i s. \quad (5)$$

From (4) it at once follows that

$$\left(\frac{\partial x^i}{\partial y^j} \right)_0 = \delta_j^i, \quad (6)$$

the suffix 0 implying (as always hereafter) evaluation at the point $(y^i) = 0$ [*i.e.* at the point (\bar{x}^i)], and δ_j^i being equal to zero or unity according as $i \neq j$ or $i = j$.

3. *The function Ω .*

Let Ω denote one half[‡] of the square of the geodesic distance between the points (\bar{x}^i) and (x^i) . Thus Ω is a scalar defined by the equation

$$(2\Omega)^{\frac{1}{2}} = \int ds, \quad (7)$$

[†] Veblen, *loc. cit.*, § 2.

[‡] The function here denoted by Ω is one half of that denoted by Ω in the earlier paper referred to above.

the integral being taken along the arc of the geodesic from (\bar{x}^i) to (x^i) . That is,

$$\Omega = \frac{1}{2}s^2, \quad (7 a)$$

where s is as defined above.

It should be noticed that Ω is a function of both the x^i and the \bar{x}^i . Thus, for the three-dimensional Euclidean case, with rectangular Cartesian coordinates x^1, x^2, x^3 , we have

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad (8)$$

$$\text{and} \quad \Omega = \frac{1}{2} \{ (x^1 - \bar{x}^1)^2 + (x^2 - \bar{x}^2)^2 + (x^3 - \bar{x}^3)^2 \}. \quad (9)$$

To find the derivatives of Ω with respect to the x^i and the \bar{x}^i , consider the variation of the integral in (7) obtained by integrating along a neighbouring geodesic with end-points $(x^i + \delta x^i)$, $(\bar{x}^i + \delta \bar{x}^i)$. We get†, after integrating by parts,

$$\begin{aligned} \delta(2\Omega)^{\frac{1}{2}} &= \left[g_{\mu\sigma} \frac{dx^\mu}{ds} \delta x^\sigma \right] - \int g_{\mu\sigma} \left(\frac{d^2 x^\mu}{ds^2} + \{ \alpha\beta, \mu \} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right) \delta x^\sigma ds \\ &= g_{\mu\sigma} \frac{dx^\mu}{ds} \delta x^\sigma - \bar{g}_{\mu\sigma} \frac{d\bar{x}^\mu}{ds} \delta \bar{x}^\sigma, \end{aligned}$$

the integral vanishing since the integrand is zero along a geodesic.

Hence, using equation (3), we get

$$\frac{\partial}{\partial x^\sigma} (2\Omega)^{\frac{1}{2}} = g_{\mu\sigma} \frac{dx^\mu}{ds}, \quad (10)$$

$$\text{and} \quad \frac{\partial}{\partial \bar{x}^\sigma} (2\Omega)^{\frac{1}{2}} = -\bar{g}_{\mu\sigma} \frac{d\bar{x}^\mu}{ds}. \quad (11)$$

It follows at once from equations (5), (7 a), and (11), that

$$\frac{\partial \Omega}{\partial \bar{x}^\sigma} = -\bar{g}_{\mu\sigma} y^\mu. \quad (12)$$

As is customary, we shall write Ω_σ , Ω^σ respectively for $\frac{\partial \Omega}{\partial x^\sigma}$ and $g^{\sigma\alpha} \frac{\partial \Omega}{\partial x^\alpha}$.

Further, we shall denote $\frac{\partial \Omega}{\partial \bar{x}^\sigma}$ by $\Omega_{(\sigma)}$, and $\bar{g}^{\sigma\alpha} \frac{\partial \Omega}{\partial \bar{x}^\alpha}$ by $\Omega^{(\sigma)}$. And later we shall use $\Omega_\mu^{(\lambda)}$ to represent $\bar{g}^{\lambda\alpha} \frac{\partial^2 \Omega}{\partial x^\mu \partial \bar{x}^\alpha}$, and $\Omega_{(\mu)}^\lambda$ to represent $g^{\lambda\alpha} \frac{\partial^2 \Omega}{\partial x^\alpha \partial \bar{x}^\mu}$.

† Cf. Eddington, *Mathematical theory of relativity* (1924), § 28.

In other words, we shall use simple suffixes to denote covariant and contravariant differentiations with respect to the x^i , and bracketed suffixes to denote what may analogously be called "covariant and contravariant differentiations" with respect to the \bar{x}^i .

Multiplying both sides of equation (12) by $-\bar{g}^{i\sigma}$ (and summing for σ), we get

$$y^i = -\Omega^{(i)}. \tag{13}$$

We thus have the result: *The normal coordinates (y^i) may be obtained as functions of the original coordinates (x^i) by the contravariant differentiation of the scalar Ω with respect to the \bar{x}^i .*

This gives an explicit formula for the normal coordinates, in place of the implicit definition (4).

4. Taylor's theorem.

Consider first an arbitrary covariant vector T_p , each of whose components is a function of the x^i .

Transform to the normal coordinates (y^i), and let $*T_p$ denote the vector in the transformed system. Then, by the ordinary transformation law for tensors,

$$*T_p = \frac{\partial x^\lambda}{\partial y^p} T_\lambda. \tag{14}$$

Further,
$$T_p = \frac{\partial y^\lambda}{\partial x^p} *T_\lambda,$$

and hence, by (13),
$$T_p = -\Omega_p^{(\lambda)} *T_\lambda, \tag{15}$$

where the repetition of the suffix λ implies a summation in spite of the fact that in one case the suffix is bracketed.

The point (\bar{x}^i) is now the origin ($y^i = 0$). Hence, by (6) and (14),

$$\begin{aligned} (*T_p)_0 &= \left(\frac{\partial x^\lambda}{\partial y^p} \right)_0 \bar{T}_\lambda, \\ &= \delta_p^\lambda \bar{T}_\lambda, \end{aligned}$$

i.e.,
$$(*T_p)_0 = \bar{T}_p. \tag{16}$$

Now each component of the vector $*T_p$ is a function of the y^i . Hence, by the ordinary Taylor's theorem, assuming the necessary conditions of convergence to be satisfied,

$$*T_p = (*T_p)_0 + \frac{1}{1!} y^\alpha \left(\frac{\partial *T_p}{\partial y^\alpha} \right)_0 + \frac{1}{2!} y^\alpha y^\beta \left(\frac{\partial^2 *T_p}{\partial y^\alpha \partial y^\beta} \right)_0 + \dots \tag{17}$$

But the derivatives $\left(\frac{\partial^* T_p}{\partial y^a}\right)_0, \left(\frac{\partial^2 T_p}{\partial y^a \partial y^b}\right)_0, \dots$ evaluated at the y -origin are equal to the values at (\bar{x}^i) of the "affine extensions"† $T_{p, a}, T_{p, a\beta}, \dots$ of the vector T_p .

Substituting in (17), and using equations (13) and (16), we get

$$*T_p = \bar{T}_p - \frac{1}{1!} \Omega^{(\alpha)} \bar{T}_{p, a} + \frac{1}{2!} \Omega^{(\alpha)} \Omega^{(\beta)} \bar{T}_{p, a\beta} - \frac{1}{3!} \Omega^{(\alpha)} \Omega^{(\beta)} \Omega^{(\gamma)} \bar{T}_{p, a\beta\gamma} + \dots$$

$\alpha, \beta, \gamma, \dots$ being summation suffixes.

Hence, by (15),

$$T_p = -\Omega_p^{(\lambda)} \left[\bar{T}_\lambda - \frac{1}{1!} \Omega^{(\alpha)} \bar{T}_{\lambda, a} + \frac{1}{2!} \Omega^{(\alpha)} \Omega^{(\beta)} \bar{T}_{\lambda, a\beta} - \dots \right], \tag{18}$$

which is a tensor formula expressing the vector T_p at (x^i) in terms of its value at (\bar{x}^i) .

Interchanging the x 's and the \bar{x} 's,

$$\bar{T}_p = -\Omega_{(p)}^\lambda \left[T_\lambda - \frac{1}{1!} \Omega^\alpha T_{\lambda, a} + \frac{1}{2!} \Omega^\alpha \Omega^\beta T_{\lambda, a\beta} - \dots \right], \tag{19}$$

which gives the value of T_p at (\bar{x}^i) in terms of its value at (x^i) . And this is the formula which we sought to establish.

Multiplying both sides of (19) by \bar{g}^{pq} , and summing for p , we get for the *contravariant* vector T^q the equation‡

$$\bar{T}^q = -\Omega_\lambda^{(q)} \left[T^\lambda - \frac{1}{1!} \Omega^\alpha T^\lambda_{, a} + \frac{1}{2!} \Omega^\alpha \Omega^\beta T^\lambda_{, a\beta} - \dots \right], \tag{20}$$

since $\bar{g}^{pq} \Omega_{(p)}^\lambda T_\lambda = \Omega_\lambda^{(q)} T^\lambda$, and so on.

A similar result holds for a tensor of any rank. It was purely for the sake of simplicity that a *vector* was chosen for the purpose of establishing the formula. By an exactly similar method, therefore, the following general theorem may be proved:

Let $T_p^{q_1 \dots q_s}$ be a mixed tensor of rank $r+s$ (p denoting the group of suffixes $p_1 p_2 \dots p_r$ and q the group $q_1 q_2 \dots q_s$). Then, if the series on the right is convergent,

$$\bar{T}_p^{q_1 \dots q_s} = (-1)^{r+s} \prod_{l=1}^r \Omega_{p_l}^{(\lambda_l)} \prod_{m=1}^s \Omega_{(q_m)}^{\mu_m} \left[T_\lambda^{\mu_1 \dots \mu_s} - \frac{1}{1!} \Omega^\alpha T_\lambda^{\mu_1 \dots \mu_s}{}_{, \alpha} + \frac{1}{2!} \Omega^\alpha \Omega^\beta T_\lambda^{\mu_1 \dots \mu_s}{}_{, \alpha\beta} - \dots \right], \tag{21}$$

where λ, μ denote respectively the groups of suffixes $\lambda_1 \lambda_2 \dots \lambda_r, \mu_1 \mu_2 \dots \mu_s$.

† The m -th affine extension of a tensor is equal to its m -th covariant derivative plus certain terms involving the Riemann tensor; Veblen, *loc. cit.*, § 5, *et seq.*

‡ In this equation, $T^\lambda_{, a\beta}$ stands for $g^{\lambda\alpha} T_{\alpha, a\beta}$ not $(g^{\lambda\alpha} T_{, \alpha})_{, a\beta}$, these being in general different. A similar remark applies to formula (21).

In particular, for an arbitrary scalar T , the theorem becomes

$$\bar{T} = T - \frac{1}{1!} \Omega^a T_{,a} + \frac{1}{2!} \Omega^a \Omega^b T_{,ab} - \frac{1}{3!} \Omega^a \Omega^b \Omega^c T_{,abc} + \dots, \quad (22)$$

a result obtained for the special case of a Euclidean space in the earlier paper mentioned above.

5. Conclusion.

It is interesting to notice that the generalizations of ordinary partial derivatives which appear in the above theorem are not covariant derivatives but affine extensions; these generalizations being the same when the Riemann tensor is everywhere zero (*i.e.*, when the space is Euclidean).

The formula (21) contains the classical Taylor's theorem as a particular case. For, if the space is Euclidean and the coordinates rectangular Cartesian, we have [by equations (8) and (9) if we take three dimensions only]

$$\begin{aligned} \Omega_{\lambda}^{(\gamma)} &= \Omega'_{(\lambda)} = -\delta_{\lambda}^{\gamma}, \\ \Omega^a &= x^a - \bar{x}^a, \end{aligned}$$

and it is immediately evident that (21) reduces to the ordinary Taylor's theorem applied separately to each component of the tensor.

In practice the actual calculation of the function Ω usually leads to algebra which rapidly becomes intractable. It is therefore difficult to give examples of the application of the theorems of this paper. But it may be remarked that the theorem of § 3 is readily illustrated in the case of the Euclidean metric

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (x^1 = r, x^2 = \theta),$$

for which, of course,

$$\Omega = \frac{1}{2} \{r^2 + \bar{r}^2 - 2r\bar{r} \cos(\theta - \bar{\theta})\}.$$

And it is easily shown that the formula (19) reduces to a simple identity if applied to the vector whose components are $T_1 = r - \cos \theta$, $T_2 = r \sin \theta$. All covariant derivatives of this vector of orders higher than the first are zero.

It is further to be noticed that formula (13), namely $y^i = -\Omega^{(i)}$, provides a method of obtaining cartesian coordinates when the metric of a flat space is given; for in such a space the normal coordinates are of course ordinary cartesians, though not necessarily rectangular. In particular, the formula may be used to find a galilean frame of reference in a space-time of zero curvature. For example, consider

the "Quasi-Uniform Gravitational Field" * specified by

$$ds^2 = (1 + \frac{2gx}{c^2}) dt^2 - \frac{1}{c^2} (\frac{dx^2}{1 + \frac{2gx}{c^2}} + dy^2 + dz^2) \dots (23)$$

where g, c are constants. By solving the equations of the geodesics, it is readily found that

$$\Omega = \frac{c^2}{g^2} \sqrt{(1 + \frac{2gx}{c^2})(1 + \frac{2g\bar{x}}{c^2})} \cosh \frac{g}{c}(t - \bar{t}) - \frac{1}{g}(x + \bar{x}) - \frac{c^2}{g} - \frac{1}{2c^2} [(y - \bar{y})^2 + (z - \bar{z})^2] \dots (24).$$

Consequently

$$\left. \begin{aligned} \Omega_{(0)} &= \frac{\partial \Omega}{\partial t} = -\frac{c}{g} \sqrt{(1 + \frac{2gx}{c^2})(1 + \frac{2g\bar{x}}{c^2})} \sinh \frac{g}{c}(t - \bar{t}), \\ \Omega_{(1)} &= \frac{\partial \Omega}{\partial x} = \frac{1}{g} \sqrt{\frac{1 + \frac{2g\bar{x}}{c^2}}{1 + \frac{2gx}{c^2}}} \cosh \frac{g}{c}(t - \bar{t}) - \frac{1}{g}, \\ \Omega_{(2)} &= \frac{\partial \Omega}{\partial y} = \frac{1}{c^2}(y - \bar{y}), \\ \Omega_{(3)} &= \frac{\partial \Omega}{\partial z} = \frac{1}{c^2}(z - \bar{z}). \end{aligned} \right\} \dots (25)$$

Also $\bar{g}^{00} = \frac{1}{1 + \frac{2g\bar{x}}{c^2}}$, $\bar{g}'' = -c^2(1 + \frac{2g\bar{x}}{c^2})$, $\bar{g}^{22} = -c^2 = \bar{g}^{33}$,
 $\bar{g}^{\mu\nu} = 0$ if $\mu \neq \nu$.

So, taking the base-point $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to be $(0, 0, 0, 0)$, the normal coordinates are

$$\left. \begin{aligned} y^0 &= -\bar{g}^{00} \Omega_{(0)} = \frac{c}{g} \sqrt{1 + \frac{2g\bar{x}}{c^2}} \sinh \frac{gt}{c}, \\ y^1 &= -\bar{g}'' \Omega_{(1)} = \frac{c^2}{g} \sqrt{1 + \frac{2g\bar{x}}{c^2}} \cosh \frac{gt}{c} - \frac{c^2}{g}, \\ y^2 &= -\bar{g}^{22} \Omega_{(2)} = y, \\ y^3 &= -\bar{g}^{33} \Omega_{(3)} = z. \end{aligned} \right\} \dots (26)$$

It is now readily proved that

* First introduced by E.T. Whittaker, Proc. Roy. Soc. A, 116 (1927) 722.

$$(dy^0)^2 - \frac{1}{c^2} [(dy^1)^2 + (dy^2)^2 + (dy^3)^2]$$

$$= \left(1 + \frac{2gx}{c^2}\right) dt^2 - \frac{1}{c^2} \left[\frac{dx^2}{\left(1 + \frac{2gx}{c^2}\right)} + dy^2 + dz^2 \right]$$

$$= ds^2,$$

so that (y^0, y^1, y^2, y^3) is a galilean coordinate-system.

(Note: the transformation (26) was found in 1927 by Professor E.T. Whittaker, and given by him in a lecture to the Edinburgh University Research Class. The method by which he obtained it is unknown to me).

Chapter II, NORMAL COVARIANT DERIVATIVES
 Paper 3.

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1. *Introduction.*

The geometry associated with the analysis appearing in the present paper is Riemannian. The problem attacked is that of finding a natural tensor generalization of the partial derivative of the elementary differential calculus. The generalization arrived at occupies a position midway between the ordinary covariant derivative (upon which Levi-Civita based his theory of parallelism) and that which arises from the more recent theories of teleparallelism. For it actually reduces in certain special cases to the ordinary covariant derivative, and yet can be used to define a teleparallelism.

The next paragraph contains an outline of the theory of normal coordinates (upon which the ideas of this paper are based). In §§ 3, 4, 5 the new covariant derivative is defined and its properties investigated. In §§ 6, 7 appears the consequent theory of parallelism, while the last paragraph contains an extension to general Riemannian space of the idea of homogeneous tensors introduced for the case of a Euclidean space in a previous paper†.

Finally, it should be stated that the chief disadvantage of the present theory is that it gives a special importance to a particular point of space, namely the origin of normal coordinates.

2. *Preliminary definitions.*

Consider a Riemannian space of n dimensions specified by the metric‡

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

† *Proc. London Math. Soc.* (2), 31 (1930), 225. (Paper 1 of this Chapter).

‡ The summation convention is adopted throughout.

Let $(\bar{x}^i) \equiv (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ be a fixed point of the space, which we shall call the *base point*. Then, as is well known, the differential equations of the geodesics are

$$\frac{d^2 x^\mu}{ds^2} + \{ \alpha\beta, \mu \} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \quad (2)$$

The solution of these equations which is such that $(x^i) = (\bar{x}^i)$ and $dx^i/ds = a^i$ when $s = 0$ is†

$$x^i = \bar{x}^i + a^i s - \frac{1}{2!} c_{jk}^i a^j a^k s^2 - \frac{1}{3!} c_{jkl}^i a^j a^k a^l s^3 - \dots, \quad (3)$$

where $c_{jk}^i, c_{jkl}^i, \dots$ are certain determinate constants.

In other words, equations (3) are those of the geodesics passing through the point (\bar{x}^i) , any particular geodesic being specified by the values chosen for the arbitrary constants a^i . s is the length of the arc of the geodesic measured from (\bar{x}^i) to (x^i) .

If then we introduce new variables (y^i) defined by the relations

$$x^i = \bar{x}^i + y^i - \frac{1}{2!} c_{jk}^i y^j y^k - \dots, \quad (4)$$

the equations of the geodesics through (\bar{x}^i) , which is the origin $(y^i) = 0$ of the new variables, become simply

$$y^i = a^i s. \quad (5)$$

The coordinates (y^i) are said to be *normal*. The components of a tensor in these normal coordinates will be denoted by the affixing of an asterisk. Thus, for example,

$$*g_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}.$$

From the definition of a^i we get, by (1),

$$\bar{g}_{\mu\nu} a^\mu a^\nu = 1,$$

the superposed bar indicating (as always hereafter) evaluation at the point (\bar{x}^i) , that is, at $(y^i) = 0$.

Hence, from (5),

$$s^2 = \bar{g}_{\mu\nu} y^\mu y^\nu. \quad (6)$$

† Veblen, "Invariants of quadratic differential forms", *Camb. Mathematical Tract*, No. 24 (1927), 84. Our notation differs slightly from that of Veblen.

Thus, defining the scalar Ω to be one-half of the square of the geodesic distance between the points (\bar{x}^i) and (x^i) in the x -coordinate system, and between the origin and the point (y^i) in the y system, we have

$$\Omega = \frac{1}{2} \bar{g}_{\mu\nu} y^\mu y^\nu. \quad (7)$$

It should be noticed that the $\bar{g}_{\mu\nu}$ are functions of the \bar{x}^i only.

From the definition it is evident that Ω is a function of both the x^i and \bar{x}^i ; now it has been proved elsewhere† that

$$y^i = -\bar{g}^{ia} \frac{\partial \Omega}{\partial \bar{x}^a}.$$

Following the notation of the paper referred to, $\partial \Omega / \partial \bar{x}^a$ will be denoted by $\Omega_{(a)}$, and $\bar{g}^{ia} \Omega_{(a)}$ by $\Omega^{(i)}$. Unbracketed suffixes will be used to denote covariant differentiations with respect to the (x^i) . Thus

$$y^i = -\Omega^{(i)}. \quad (8)$$

Hence
$$\frac{\partial y^i}{\partial x^j} = -\frac{\partial \Omega^{(i)}}{\partial x^j} = -\Omega_j^{(i)}, \quad \text{say.} \quad (9)$$

Again, from (7), we get

$$\frac{\partial \Omega}{\partial y^\mu} = \bar{g}_{\mu\nu} y^\nu,$$

that is

$$\begin{aligned} * \Omega_\mu &= \bar{g}_{\mu\nu} y^\nu \\ &= * g_{\mu\nu} y^\nu, \end{aligned} \quad (10)$$

since†

$$\bar{g}_{\mu\nu} y^\nu = * g_{\mu\nu} y^\nu.$$

Raising the suffix μ in (10), we get at once

$$* \Omega^\mu = y^\mu. \quad (11)$$

This equation differs from (8) in that it gives a relation between the y^i and the components in normal coordinates of the contravariant vector Ω^i , whereas (8) defines the y^i as functions of the x^i 's.

Finally, if the coordinates x^i are transformed by the arbitrary relations

$$x^i = x^i(x'^\mu), \quad (12)$$

then the normal coordinates (y'^μ) corresponding to the new variables

† *Proc. London Math. Soc.* (2), 32 (1931), 87. (Page 18 above).

‡ Veblen, *loc. cit.*, 96 (14.8).

x'^{μ} are connected with the (y^{μ}) by the linear relations†

$$y'^i = \left(\frac{\partial x'^i}{\partial x^j} \right)_0 y^j, \quad (13)$$

the suffix 0 indicating evaluation at the common y origin, that is at (\bar{x}^i) .

3. Normal covariant derivatives.

In any given coordinate system (x^i) , let the functions $\Gamma_{\mu\nu}^{\lambda}$ be defined by the relations

$$\Gamma_{\mu\nu}^{\lambda} = \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}, \quad (14)$$

where the (y^{μ}) are the corresponding normal coordinates defined by either of the equations (4), (8). To find the law of transformation of these functions, change the coordinates to another system (x'^i) , letting (y'^i) be the corresponding normal coordinates having the same origin as the (y^i) . Then

$$\begin{aligned} \Gamma'_{\mu\nu}{}^{\lambda} &= \frac{\partial x'^{\lambda}}{\partial y'^{\alpha}} \frac{\partial^2 y'^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial y'^{\tau}} \frac{\partial y'^{\tau}}{\partial y'^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial y'^{\alpha}}{\partial y^{\tau}} \frac{\partial y^{\tau}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \right) \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial y'^{\sigma}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\alpha}} \right)_0 \frac{\partial}{\partial x'^{\mu}} \left[\left(\frac{\partial x'^{\alpha}}{\partial x^{\tau}} \right)_0 \frac{\partial y^{\tau}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \right], \end{aligned}$$

by (13). Remembering that $(\partial x'^{\alpha}/\partial x^{\tau})_0$ is a constant so far as differentiation with respect to x'^{μ} is concerned, we quickly get

$$\Gamma'_{\mu\nu}{}^{\lambda} = \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \Gamma_{\kappa\rho}^{\beta} + \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial^2 x^{\beta}}{\partial x'^{\mu} \partial x'^{\nu}}. \quad (15)$$

From this it at once follows that, if T^{λ} be an arbitrary contravariant vector, then

$$T^{\lambda}{}_{;\mu} \equiv \frac{\partial T^{\lambda}}{\partial x^{\mu}} + \Gamma_{\alpha\mu}^{\lambda} T^{\alpha} \quad (16)$$

is a tensor‡.

This we shall call the *normal covariant derivative of the vector T^{λ} with (\bar{x}^{λ}) as base point*. And in general the normal covariant derivative

† Veblen, *loc. cit.*, 86 (3. 5).

‡ Veblen, *loc. cit.*, ch. 3, §§ 10, 11.

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of any tensor $T_{\mu_1 \mu_2 \dots}^{\lambda_1 \lambda_2 \dots}$ will be defined to be

$$T_{\mu_1 \mu_2 \dots}^{\lambda_1 \lambda_2 \dots} = \frac{\partial T_{\mu_1 \mu_2 \dots}^{\lambda_1 \lambda_2 \dots}}{\partial x^v} + \Gamma_{\alpha v}^{\lambda_1} T_{\mu_1 \mu_2 \dots}^{\alpha \lambda_2 \dots} + \dots - \Gamma_{\mu_1 v}^{\alpha} T_{\alpha \mu_2 \dots}^{\lambda_1 \lambda_2 \dots} - \dots \quad (17)$$

The normal covariant derivative of a scalar is defined to be the ordinary partial derivative of the scalar.

It should be noticed that, by (14), it follows at once that $\Gamma_{\mu\nu}^{\lambda}$ is in general symmetrical in μ and ν ; *i.e.*

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}. \quad (18)$$

To find an explicit formula for the functions Γ in terms of the x 's, we notice that, since

$$\frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial y^{\alpha}}{\partial x^{\mu}} = \delta_{\mu}^{\lambda},$$

δ_{μ}^{λ} being the Kronecker symbol, it follows that

$$\frac{\partial x^{\lambda}}{\partial y^{\alpha}} = \left(\text{cofactor of } \frac{\partial y^{\alpha}}{\partial x^{\lambda}} \text{ in the Jacobian } \frac{\partial(y^i)}{\partial(x^j)} \right) \div \frac{\partial(y^i)}{\partial(x^i)}.$$

By (9),

$$\begin{aligned} \frac{\partial(y^i)}{\partial(x^j)} &= (-)^n \|\Omega_j^{(i)}\| \\ &= (-)^n J, \quad \text{say,} \end{aligned}$$

where $\|\Omega_j^{(i)}\|$ denotes the determinant of the $\Omega_j^{(i)}$.

Hence $\frac{\partial x^{\lambda}}{\partial y^{\alpha}} = -J^{-1} \times (\text{cofactor of } \Omega_{\lambda}^{(\alpha)} \text{ in } J)$,

so that, by (9) and (14),

$$\Gamma_{\mu\nu}^{\lambda} = J^{-1} \times (\text{cofactor of } \Omega_{\lambda}^{(\alpha)} \text{ in } J) \times \frac{\partial^2 \Omega^{(\alpha)}}{\partial x^{\mu} \partial x^{\nu}}. \quad (19)$$

And since Ω can be calculated for a given metric directly from the differential equations (2), this gives an explicit formula for the Γ 's in terms of the x 's.

4. THEOREM I. *In the y coordinate system the normal covariant derivative of a tensor is the ordinary partial derivative.*

For, by the law of transformation (15),

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda} &= \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial y^{\beta}}{\partial x^{\mu}} \frac{\partial y^{\gamma}}{\partial x^{\nu}} * \Gamma_{\beta\gamma}^{\alpha} + \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \\ &= \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial y^{\beta}}{\partial x^{\mu}} \frac{\partial y^{\gamma}}{\partial x^{\nu}} * \Gamma_{\beta\gamma}^{\alpha} + \Gamma_{\mu\nu}^{\lambda}, \end{aligned}$$

whence

$$* \Gamma_{\beta\gamma}^{\alpha} = 0. \quad (20)$$

Hence, by (16),
$$*T^{\lambda}_{;\mu} = \frac{\partial *T^{\lambda}}{\partial y^{\mu}}, \tag{21}$$
 and so for any tensor†.

THEOREM II. The "normal Riemann tensor", defined to be

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^b_{jk} \Gamma^i_{bl} - \Gamma^b_{jl} \Gamma^i_{bk},$$

is everywhere identically zero‡.

For, by (20), it at once follows that

$$*B^i_{jkl} = 0,$$

and this, being a tensor equation, must be true in all systems of coordinates. Hence

$$B^i_{jkl} = 0. \tag{22}$$

From this it follows that $T^{\mu}_{;\lambda;\nu} = T^{\mu}_{;\nu;\lambda}$, and so for any tensor.

THEOREM III. In a flat space $\Gamma^{\lambda}_{\mu\nu} = \{\mu\nu, \lambda\}$.

By a "flat" space is meant one for which the ordinary Riemann tensor, namely

$$K^i_{jkl} = \frac{\partial}{\partial x^j} \{jk, i\} - \frac{\partial}{\partial x^k} \{jl, i\} + \{jk, b\} \{bl, i\} - \{jl, b\} \{bk, i\},$$

is everywhere identically zero.

For in a flat space the normal coordinates are ordinary Cartesian coordinates (not necessarily rectangular). Hence $*\{\mu\nu, \lambda\} = 0$. It follows that if T^{μ} is an arbitrary vector and $T^{\nu}_{;\mu} = \frac{\partial T^{\nu}}{\partial x^{\mu}} + \{a\nu, \mu\} T^a$, then

$$\begin{aligned} *T^{\nu}_{;\mu} &= \frac{\partial *T^{\nu}}{\partial y^{\mu}} \\ &= *T^{\nu}_{;\mu}. \end{aligned}$$

This is a tensor formula, so

$$T^{\nu}_{;\mu} = T^{\nu}_{;\mu}. \tag{23}$$

The theorem follows on account of the arbitrariness of T^{μ} .

† It is worth remarking that the ordinary covariant derivative $T^{\lambda}_{;\mu} = \partial T^{\lambda} / \partial x^{\mu} + \{a\lambda, \mu\} T^a$ possesses the property that $*T^{\lambda}_{;\mu} = \partial *T^{\lambda} / \partial y^{\mu}$ at the origin $y^i = 0$, but this equation, unlike (20), is not in general true at all points of space.

‡ That B^i_{jkl} is indeed a tensor follows from Veblen, *loc. cit.*, 41, where a proof appears for the case of any set of functions Γ^i_{jk} , symmetrical in the lower suffixes, which satisfy the law of transformation (15).

In a flat space, therefore, the normal covariant derivative is the ordinary covariant derivative, and the functions $\Gamma_{\mu\nu}^\lambda$ are independent of the base point (\bar{x}^λ). In a general Riemannian space, however, the Γ 's are functions of the \bar{x}^λ as well as of the x^λ . Hence the base point is of special importance in the present theory, a different set of Γ 's being obtained by a different choice of base point.

5. *The normal covariant derivatives of the fundamental tensor and of the scalar Ω .*

Since, in general, $\partial^* g_{\mu\nu} / \partial y^\lambda \neq 0$ unless the $*g_{\mu\nu}$ are constants (which is the case only when the space is flat), it follows that

$$g_{\mu\nu;\lambda} \neq 0 \tag{24}$$

in general. Similarly

$$g^{\mu\nu}{}_{;\lambda} \neq 0. \tag{25}$$

But since $*g_\nu^\mu = \delta_\nu^\mu$, we have $\partial^* g_\nu^\mu / \partial y^\lambda = 0$, whence

$$g_{\nu;\lambda}^\mu = 0. \tag{26}$$

In forming normal covariant derivatives we may therefore treat the mixed tensor g_ν^μ as a constant, but not the covariant and contravariant tensors $g_{\mu\nu}$, $g^{\mu\nu}$.

Now consider the scalar Ω . By (11),

$$*\Omega^\mu = y^\mu,$$

whence

$$\frac{\partial^* \Omega^\mu}{\partial y^\nu} = \delta_\nu^\mu = *g_\nu^\mu.$$

From this we deduce at once that

$$\Omega^\mu{}_{;\nu} = g_\nu^\mu, \tag{27}$$

where

$$\Omega^\mu = g^{\mu\alpha} \frac{\partial \Omega}{\partial x^\alpha}. \tag{28}$$

Again, by (10),

$$\begin{aligned} \frac{\partial^* \Omega_\mu}{\partial y^\nu} &= *g_{\mu\nu} + y^\lambda \frac{\partial^* g_{\mu\lambda}}{\partial y^\nu} \\ &= *g_{\mu\nu} + *(\Omega)^\lambda \frac{\partial^* g_{\mu\lambda}}{\partial y^\nu} \end{aligned}$$

by (11). Hence

$$\Omega_{\mu;\nu} = g_{\mu\nu} + \Omega^\lambda g_{\mu\lambda;\nu}. \tag{29}$$

If the space is flat, this becomes

$$\Omega_{\mu\nu} = g_{\mu\nu},$$

where $\Omega_{\mu\nu}$ is the ordinary second covariant derivative of Ω . This result was found in a previous paper†.

6. Parallelism.

Let T^μ be any contravariant vector. Then we shall say that T^μ has been carried by *normal parallel transport over the displacement* (dx^μ) if the increment in T^μ is given by

$$dT^\mu = -\Gamma_{\alpha\beta}^\mu T^\alpha dx^\beta. \quad (30)$$

The conditions of integrability of these equations are $B_{jkl}^i = 0$, which are satisfied (§ 4, Theorem II). Equations (30) therefore define a teleparallelism. In fact, by (14) and (30),

$$\frac{\partial T^\mu}{\partial x^\beta} + \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^\alpha \partial x^\beta} T^\alpha = 0.$$

Multiplying by $\hat{\partial}y^\sigma/\hat{\partial}x^\mu$ (and summing for μ) we get

$$\frac{\hat{\partial}y^\sigma}{\hat{\partial}x^\mu} \frac{\partial T^\mu}{\partial x^\beta} + \frac{\partial^2 y^\sigma}{\partial x^\alpha \partial x^\beta} T^\alpha = 0,$$

or
$$\frac{\partial}{\partial x^\beta} \left[\frac{\hat{\partial}y^\sigma}{\hat{\partial}x^\alpha} T^\alpha \right] = 0;$$

whence
$$\frac{\hat{\partial}y^\sigma}{\hat{\partial}x^\alpha} T^\alpha = \text{constant},$$

or, by (9),
$$\Omega_a^{(\sigma)} T^a = \text{constant}. \quad (31)$$

These are therefore the integrated equations of parallel transport.

7. The autoparallels.

Putting $T^\mu = dx^\mu/ds$ in (30), we get the differential equations of the autoparallels, namely

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \quad (32)$$

† *Proc. London Math. Soc.* (2), 31 (1930), 225. (Page 10, above)

In normal coordinates (y^i) these are

$$\frac{d^2 y^\mu}{ds^2} = 0,$$

which on integration give

$$y^\mu = b^\mu + a^\mu s, \quad (33)$$

where the b^μ and a^μ are arbitrary constants. In particular, the autoparallels through the base point are given by

$$y^\mu = a^\mu s,$$

and thus, by (5), coincide with the geodesics through that point. But in general the autoparallels *not* passing through the base point will not coincide with the geodesics.

Substituting from (8) in (33), we get

$$\Omega^{(\omega)} = -(b^\mu + a^\mu s),$$

which are thus the equations of the autoparallels in the x coordinates, that is, the solutions of equations (32).

8. Homogeneous tensors.

We now extend to general Riemannian space the idea of homogeneous tensors outlined in the case of a flat space in the paper quoted above (§ 5).

Consider an arbitrary tensor T_{β}^{α} , which, without loss of generality, we may take to be of the second rank only. We shall call T_{β}^{α} a *homogeneous tensor of degree m , with respect to the base point (\bar{x}^i)* , if it satisfies the differential equations

$$\Omega^{\mu} T_{\beta;\mu}^{\alpha} = m T_{\beta}^{\alpha}, \quad (34)$$

where Ω^{μ} is defined by equation (28).

A homogeneous scalar V of degree m is similarly defined to be one satisfying the equation

$$\Omega^{\mu} V_{,\mu} = m V. \quad (35)$$

In the y coordinates (34) becomes, by (11) and (21),

$$y^{\mu} \frac{\partial^* T_{\beta}^{\alpha}}{\partial y^{\mu}} = m^* T_{\beta}^{\alpha}, \quad (36)$$

so that each component of the tensor is homogeneous in the ordinary sense.

THEOREM A. *If T_{β}^{α} is homogeneous of degree m , its first normal covariant derivative is homogeneous of degree $m-1$.*

For, by (36),

$$y^{\mu} \frac{\partial^2 * T_{\beta}^{\alpha}}{\partial y^{\nu} \partial y^{\mu}} = (m-1) \frac{\partial * T_{\beta}^{\alpha}}{\partial y^{\nu}} \quad (37)$$

from which we deduce at once that

$$\Omega^{\mu} T_{\beta:r;\mu}^{\alpha} = (m-1) T_{\beta:r}^{\alpha}, \quad (38)$$

which proves the theorem.

THEOREM B. *Euler's general theorem on homogeneous functions.*

Multiplying (37) by y^{ν} and summing for ν , we get, using (36),

$$y^{\mu} y^{\nu} \frac{\partial^2 * T_{\beta}^{\alpha}}{\partial y^{\mu} \partial y^{\nu}} = m(m-1) * T_{\beta}^{\alpha},$$

which gives

$$\Omega^{\mu} \Omega^{\nu} T_{\beta;\mu;\nu}^{\alpha} = m(m-1) T_{\beta}^{\alpha}.$$

Proceeding in this way we could obtain the tensor form of Euler's general theorem on homogeneous functions, namely

$$\Omega^{\mu_1} \Omega^{\mu_2} \dots \Omega^{\mu_r} T_{\beta;\mu_1;\mu_2;\dots;\mu_r}^{\alpha} = m(m-1)\dots(m-r+1) T_{\beta}^{\alpha}.$$

THEOREM C. *The scalar Ω is itself homogeneous of degree 2.*

For, as is well known, Ω satisfies the differential equation†

$$\Omega^{\mu} \Omega_{\mu} = 2\Omega,$$

and the theorem follows from the definition (35).

THEOREM D. *If T_{β}^{α} , U_{β}^{α} are two homogeneous tensors of degree m of the same kind, so also is $T_{\beta}^{\alpha} + U_{\beta}^{\alpha}$.*

This results at once from the definition (34).

THEOREM E. *If T_{β}^{α} , H_{κ}^{ϵ} are homogeneous tensors of degree m and m' respectively, then the product $T_{\beta}^{\alpha} H_{\kappa}^{\epsilon}$ is homogeneous of degree $m+m'$.*

For it is easily proved that

$$\begin{aligned} (T_{\beta}^{\alpha} H_{\kappa}^{\epsilon})_{;\mu} &= T_{\beta;\mu}^{\alpha} H_{\kappa}^{\epsilon} + T_{\beta}^{\alpha} H_{\kappa;\mu}^{\epsilon} \\ &= (m+m') T_{\beta}^{\alpha} H_{\kappa}^{\epsilon}. \end{aligned}$$

† This is indeed an immediate consequence of equations (7), (10), and (11).

THEOREM F. *The fundamental tensors $g_{\mu\nu}$, $g^{\mu\nu}$ are in general not homogeneous.*

This is obvious, since it is not in general true that there exists a constant m such that

$$y^\lambda \frac{\partial *g_{\mu\nu}}{\partial y^\lambda} = m *g_{\mu\nu} \quad \text{or} \quad y^\lambda \frac{\partial *g^{\mu\nu}}{\partial y^\lambda} = m *g^{\mu\nu}.$$

In the case when the space is flat we have, however,

$$\frac{\partial *g_{\mu\nu}}{\partial y^\lambda} = 0 = \frac{\partial *g^{\mu\nu}}{\partial y^\lambda},$$

since all the $*g$'s are constants. Hence in a flat space the $g_{\mu\nu}$ and $g^{\mu\nu}$ are homogeneous of degree zero, as was found in the earlier paper quoted above.

THEOREM G. *The operation of contracting a covariant with a contravariant tensor-suffix may be performed without affecting the degree or homogeneity of a homogeneous tensor.*

This is an immediate consequence of the fact that a "contraction" is a summation.

THEOREM H. *The operations of raising and lowering the suffixes of a tensor may be performed without altering the degree or homogeneity of the tensor only in the case when the space is flat.*

For these operations consist of multiplying the tensor by $g_{\mu\nu}$ or $g^{\mu\nu}$ and contracting. And the lack of homogeneity of the latter tensors except in the case of flat space is proved above.

CHAPTER III.An Absolute Partial Differential
Calculus.

AN ABSOLUTE PARTIAL DIFFERENTIAL CALCULUS

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1. Introduction

IN the Absolute Differential Calculus* of Ricci and Levi-Civita, a tensor is a set of functions, obeying certain laws of transformation, of a *single* set of variables. The present paper outlines a theory of tensor-functions of *two* independent sets of variables. In a certain sense the ordinary tensor theory is analogous to the elementary differential calculus of functions of a single variable, while the following work is comparable with the theory of the partial differentiation of functions of two independent variables. It is true that the formation of a covariant derivative in the Absolute Differential Calculus involves partial differentiation with respect to each one of the given set of variables, but it is the peculiar nature of the subject that the process is thought of as a single operation and is usually symbolized by the addition of one subscript to the operand, just as the derivative of a function $f(x)$ in the elementary calculus is often represented by the same symbol with an added accent: that is, by $f'(x)$.

Now although on the one hand the suffix notation for covariant derivatives is adhered to in this paper, yet, on the other, there are two kinds of subscripts (bracketed and unbracketed), corresponding to covariant differentiation with respect to one or other of the two given sets of variables. This necessity of introducing two notations is, from the point of view here adopted, analogous to the similar need in the elementary differential calculus, in which the partial derivatives of a function $f(x, y)$ of two independent variables are often represented by $f_x(x, y)$ and $f_y(x, y)$.

It was these considerations which led to the choice of the title of this paper.

There is no theoretical reason why the ideas presented below should not be extended to tensor-functions of any number of sets of independent variables, instead of two only. There is, however, a practical reason, for such an extension would involve algebra of an unsatis-

* This is referred to below as 'the ordinary tensor theory'.

factorily elaborate character. The notation and terminology of this paper, based as they are on those of the ordinary tensor theory, are already complicated enough without further generalization.

The subject is treated from an analytical standpoint, although § 5 contains references to geometry. The earlier paragraphs are devoted mainly to definitions, which in themselves contain the essence of the paper. In the concluding paragraph appear a few remarks concerning the general aspect of the theory.

2. Preliminary definitions

Let $(x^\mu) \equiv (x^1, x^2, \dots, x^n)$ and $(\bar{x}^i) \equiv (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m)$ be two sets of independent variables. It may be stated at once that Greek suffixes will always refer to the x 's (and will therefore take the values 1, 2, ..., n) and Latin to the \bar{x} 's, taking the values 1, 2, ..., m .

Suppose that we have n independent functions $f_\mu(x^\nu)$ of the (x^μ) , but *not* of the (\bar{x}^i) , and m independent functions $\phi_i(\bar{x}^j)$ of the (\bar{x}^i) , but *not* of the (x^μ) . We can then define two new sets of variables $(x'^\mu) \equiv (x'^1, x'^2, \dots, x'^n)$ and $(\bar{x}'^i) \equiv (\bar{x}'^1, \bar{x}'^2, \dots, \bar{x}'^m)$ by the equations

$$x'^\mu = f_\mu(x^\nu), \quad (\mu = 1, 2, \dots, n), \quad (1)$$

and
$$\bar{x}'^i = \phi_i(\bar{x}^j), \quad (i = 1, 2, \dots, m). \quad (2)$$

Now let $T^{\mu(p)}$ denote the array

$$\begin{pmatrix} T^{1(1)} & T^{1(2)} & \dots & T^{1(m)} \\ T^{2(1)} & T^{2(2)} & \dots & T^{2(m)} \\ \dots & \dots & \dots & \dots \\ T^{n(1)} & T^{n(2)} & \dots & T^{n(m)} \end{pmatrix}$$

where $T^{1(1)}, T^{1(2)}, \dots$ are each functions, mn in number, of both the (x^μ) and the (\bar{x}^i) .

Further, let

- (i) $T'^{\mu(p)}$ be a set of mn functions of the (x'^μ) and (\bar{x}'^i) ;
- (ii) $'T^{\mu(p)}$ be a set of mn functions of the (x^μ) and (\bar{x}'^i) ;
- (iii) $'T'^{\mu(p)}$ be a set of mn functions of the (x'^μ) and (\bar{x}^i) .

Then, if these sets of functions are related by the equations

$$T'^{\mu(p)} = \frac{\partial x'^\mu}{\partial x^\alpha} T^{\alpha(p)} \quad (p = 1, 2, \dots, m), \quad (3)$$

$$'T^{\mu(p)} = \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} T^{\mu(a)} \quad (\mu = 1, 2, \dots, n), \quad (4)$$

$$'T'^{\mu(p)} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} T^{\alpha(a)} \quad \begin{pmatrix} p = 1, 2, \dots, m \\ \mu = 1, 2, \dots, n \end{pmatrix}, \quad (5)$$

the repetitions of the suffixes α and a implying summations from 1 to n and from 1 to m respectively, we shall say that $T^{\mu(p)}$ are the components of a contravariant double-vector in the systems (x^μ) and (\bar{x}^i) , and that it has components $T'^{\mu(p)}$ in the systems (x'^μ) and (\bar{x}^i) , components $T^{\mu(p)}$ in the systems (x^μ) and (\bar{x}'^i) , and components $T'^{\mu(p)}$ in the systems (x'^μ) and (\bar{x}'^i) ; or alternatively, $T^{\mu(p)}$ will be called a contravariant x -vector and a contravariant \bar{x} -vector.*

From this definition it is evident that, if the (\bar{x}^i) are kept constant, then $T^{\mu(1)}, T^{\mu(2)}, \dots, T^{\mu(m)}$ are m contravariant vectors in the sense of the ordinary tensor theory. A similar remark applies if the (x^μ) are kept constant and the (\bar{x}^i) allowed to vary. The indices μ and p are therefore ordinary tensor-suffixes, μ belonging to the x 's and p to the \bar{x} 's. For greater clearness the tensor-suffixes belonging to the \bar{x} 's will be placed in brackets in addition to being written in Latin characters.

More generally, a set of $m^{r+s}n^{\rho+\sigma}$ functions $T^{\lambda_1\lambda_2\dots\lambda_r(p_1p_2\dots p_r)}_{\mu_1\mu_2\dots\mu_\sigma(a_1a_2\dots a_s)}$ of the x 's and \bar{x} 's will be said to constitute a double-tensor if they transform according to the laws:

$$\text{I. } T^{\lambda_1\lambda_2\dots\lambda_r(p_1p_2\dots p_r)}_{\mu_1\mu_2\dots\mu_\sigma(a_1a_2\dots a_s)} = \frac{\partial x'^{\lambda_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\lambda_2}}{\partial x^{\alpha_2}} \dots \frac{\partial x'^{\lambda_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}} \frac{\partial x^{\beta_2}}{\partial x'^{\mu_2}} \dots \frac{\partial x^{\beta_\sigma}}{\partial x'^{\mu_\sigma}} T^{\alpha_1\alpha_2\dots\alpha_r(p_1\dots p_r)}_{\beta_1\beta_2\dots\beta_\sigma(a_1\dots a_s)}$$

for a transformation of the (x^μ) to the (x'^μ) with the (\bar{x}^i) unaltered;

$$\text{II. } T^{\lambda_1\lambda_2\dots\lambda_r(p_1p_2\dots p_r)}_{\mu_1\mu_2\dots\mu_\sigma(a_1a_2\dots a_s)} = \frac{\partial \bar{x}'^{\nu_1}}{\partial \bar{x}^{a_1}} \frac{\partial \bar{x}'^{\nu_2}}{\partial \bar{x}^{a_2}} \dots \frac{\partial \bar{x}'^{\nu_r}}{\partial \bar{x}^{a_r}} \frac{\partial \bar{x}^{b_1}}{\partial \bar{x}'^{\mu_1}} \frac{\partial \bar{x}^{b_2}}{\partial \bar{x}'^{\mu_2}} \dots \frac{\partial \bar{x}^{b_s}}{\partial \bar{x}'^{\mu_s}} T^{\lambda_1\dots\lambda_r(p_1\dots p_r)}_{\mu_1\dots\mu_\sigma(b_1b_2\dots b_s)}$$

for a transformation of the (\bar{x}^i) to the (\bar{x}'^i) with the (x^μ) unaltered;

$$\text{III. } T^{\lambda_1\dots\lambda_r(p_1\dots p_r)}_{\mu_1\dots\mu_\sigma(a_1\dots a_s)} = \frac{\partial x'^{\lambda_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\lambda_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}} \dots \frac{\partial x^{\beta_\sigma}}{\partial x'^{\mu_\sigma}} \frac{\partial \bar{x}'^{\nu_1}}{\partial \bar{x}^{a_1}} \dots \frac{\partial \bar{x}'^{\nu_r}}{\partial \bar{x}^{a_r}} \frac{\partial \bar{x}^{b_1}}{\partial \bar{x}'^{\mu_1}} \dots \frac{\partial \bar{x}^{b_s}}{\partial \bar{x}'^{\mu_s}} T^{\alpha_1\dots\alpha_r(p_1\dots p_r)}_{\beta_1\dots\beta_\sigma(b_1\dots b_s)}$$

for a simultaneous transformation of the (x^μ) and of the (\bar{x}^i) . When greater explicitness is required, $T^{\lambda_1\dots\lambda_r(p_1\dots p_r)}_{\mu_1\dots\mu_\sigma(a_1\dots a_s)}$ will be described as a mixed x -tensor of rank $\rho + \sigma$ and a mixed \bar{x} -tensor of rank $r + s$.

In particular, if a set of functions of the (x^μ) and of the (\bar{x}^i) behave like scalars for transformations of one of the sets of variables, and like the components of a vector or tensor for transformations of the other set, they will be described as forming a scalar-vector or scalar-

* More briefly, a double-vector is a set of functions of the x 's and \bar{x} 's which transforms according to (3) for transformations of the x 's with the \bar{x} 's unaltered, according to (4) when the \bar{x} 's are transformed but not the x 's, and according to (5) for a simultaneous transformation of both sets of variables.

tensor as the case may be. Thus, for example, a set of n functions T_μ transforming according to the laws

$$\begin{aligned}
T'_\mu &= \frac{\partial x^\alpha}{\partial x'^\mu} T_\alpha \quad [(\bar{x}^i) \text{ unaltered}], \\
{}'T_\mu &= T_\mu \quad [(x^\mu) \text{ unaltered}], \\
{}'T'_\mu &= \frac{\partial x^\alpha}{\partial x'^\mu} T_\alpha \quad [\text{simultaneous transformation}],
\end{aligned}$$

are the components of a scalar-vector, or, more explicitly, of a *covariant x -vector and an \bar{x} -scalar*.*

Similarly, a contravariant x -vector and a covariant \bar{x} -tensor such as $T^\mu_{(pq)}$ will sometimes be called a *vector-tensor*; and so on.

An accent placed after a functional symbol will always, as above, denote the system obtained by transforming from the (x^μ) to the (x'^μ) with the (\bar{x}^i) unaltered; one placed *before* the symbol will indicate a transformation from the (\bar{x}^i) to the (\bar{x}'^i) with the (x^μ) unaltered; and one before and one after will represent the system obtained by a simultaneous transformation of both sets of variables.

It should not be necessary to show that the definitions given above are self-consistent, or that the transformation-laws possess the reciprocal and transitive properties, for the proofs would be very similar to the corresponding theory in the ordinary tensor calculus. Moreover, it is evident that sums of like double-tensors are double-tensors of the same kind, that products of double-tensors are double-tensors, and so on. Obvious properties of this kind will be assumed without comment in the subsequent paragraphs.

3. Scalar-connexions

A *scalar-connexion* is defined to be a set of functions of the x 's and \bar{x} 's which behave like scalars for transformations of one set of variables and like the components of an affine connexion for transformations of the other set.

For example, let $\Gamma^\lambda_{\mu\nu}$ be a set of n^3 functions of both sets of variables which transform according to the laws:

$$\left. \begin{aligned}
\Gamma'^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \\
{}'\Gamma^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} \\
{}'\Gamma'^\lambda_{\mu\nu} &= \Gamma'^\lambda_{\mu\nu}
\end{aligned} \right\} \tag{6}$$

* From the point of view of the ordinary tensor theory there are of course n of these scalars, namely, T_1, T_2, \dots, T_n .

for transformations of the variables of the types indicated by the accents. Then the $\Gamma_{\mu\nu}^{\lambda}$ are the components of a scalar-connexion, or, as we shall sometimes say, of an *x-connexion and \bar{x} -scalar*.

Similarly, a set of m^3 functions L_{qr}^p such that

$$\left. \begin{aligned} L'_{qr} &= L_{qr}^p \\ 'L_{qr}^p &= \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} \frac{\partial \bar{x}^b}{\partial \bar{x}'^q} \frac{\partial \bar{x}^c}{\partial \bar{x}'^r} L_{bc}^a + \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} \frac{\partial^2 \bar{x}^a}{\partial \bar{x}'^q \partial \bar{x}'^r} \end{aligned} \right\} \quad (7)$$

constitute, by definition, an *x-scalar and an \bar{x} -connexion*.

4. Covariant derivatives

Let $T_{\nu(q)}^{\mu(p)}$ be a double-tensor of the type indicated by the suffixes. For the sake of simplicity a double-tensor possessing only one contravariant and one covariant suffix of each kind is considered, the formulae being easily generalized to double-tensors of any rank.

Let $\Gamma_{\mu\nu}^{\lambda}$ be a given *x-connexion and \bar{x} -scalar*, and L_{qr}^p a given *x-scalar and \bar{x} -connexion*. Then the double-tensors defined by

$$T_{\nu(q);\lambda}^{\mu(p)} = \frac{\partial T_{\nu(q)}^{\mu(p)}}{\partial x^{\lambda}} + \Gamma_{\alpha\lambda}^{\mu} T_{\nu(q)}^{\alpha(p)} - \Gamma_{\nu\lambda}^{\alpha} T_{\alpha(q)}^{\mu(p)} \quad (8)$$

and
$$T_{\nu(q);(r)}^{\mu(p)} = \frac{\partial T_{\nu(q)}^{\mu(p)}}{\partial \bar{x}^r} + L_{ar}^p T_{\nu(q)}^{\mu(a)} - L_{qr}^a T_{\nu(a)}^{\mu(p)} \quad (9)$$

will be called respectively *the covariant x-derivative of $T_{\nu(q)}^{\mu(p)}$ with respect to the scalar-connexion $\Gamma_{\mu\nu}^{\lambda}$* , and *the covariant \bar{x} -derivative of $T_{\nu(q)}^{\mu(p)}$ with respect to the scalar-connexion L_{qr}^p* . That $T_{\nu(q);\lambda}^{\mu(p)}$ and $T_{\nu(q);(r)}^{\mu(p)}$ are in fact double-tensors of the types indicated by their suffixes follows easily from the definitions of the preceding paragraphs.

It will be observed that the definitions (8) and (9) are exactly similar to those of covariant derivatives in the ordinary tensor theory, so that there will be no difficulty in writing down the formulae for double-tensors of any rank. It may, however, be remarked that one of the covariant derivatives of a scalar-vector or scalar-tensor is the set of ordinary partial derivatives. Thus, for example, the covariant derivatives of the scalar-vector T^{μ} are given by the formulae

$$\begin{aligned} T^{\mu}_{;\lambda} &= \frac{\partial T^{\mu}}{\partial x^{\lambda}} + \Gamma_{\alpha\lambda}^{\mu} T^{\alpha}, \\ T^{\mu}_{;(r)} &= \frac{\partial T^{\mu}}{\partial \bar{x}^r}. \end{aligned}$$

Second-order covariant derivatives of double-tensors.

In order to avoid the introduction of unnecessarily complicated formulae we now consider simply a contravariant double-vector $T^{\mu(\nu)}$. The results obtained can easily be generalized to more complicated cases.

By definition,

$$\begin{aligned} T^{\mu(\nu)}_{,\lambda} &= \frac{\partial T^{\mu(\nu)}}{\partial x^\lambda} + \Gamma_{\alpha\lambda}^\mu T^{\alpha(\nu)}, \\ T^{\mu(\nu)}_{,\lambda,\sigma} &= \frac{\partial T^{\mu(\nu)}_{,\lambda}}{\partial x^\sigma} + \Gamma_{\beta\sigma}^\mu T^{\beta(\nu)}_{,\lambda} - \Gamma_{\lambda\sigma}^\beta T^{\mu(\nu)}_{,\beta} \\ &= \frac{\partial^2 T^{\mu(\nu)}}{\partial x^\sigma \partial x^\lambda} + \Gamma_{\alpha\lambda}^\mu \frac{\partial T^{\alpha(\nu)}}{\partial x^\sigma} + \frac{\partial \Gamma_{\alpha\lambda}^\mu}{\partial x^\sigma} T^{\alpha(\nu)} + \Gamma_{\beta\sigma}^\mu \frac{\partial T^{\beta(\nu)}}{\partial x^\lambda} + \\ &\quad + \Gamma_{\beta\sigma}^\mu \Gamma_{\alpha\lambda}^\beta T^{\alpha(\nu)} - \Gamma_{\lambda\sigma}^\beta \frac{\partial T^{\mu(\nu)}}{\partial x^\beta} - \Gamma_{\lambda\sigma}^\beta \Gamma_{\alpha\beta}^\mu T^{\alpha(\nu)}. \end{aligned}$$

Interchanging λ, σ and subtracting,* we have

$$T^{\mu(\nu)}_{,\lambda,\sigma} - T^{\mu(\nu)}_{,\sigma,\lambda} = P_{\alpha\lambda\sigma}^\mu T^{\alpha(\nu)} - 2 \Sigma_{\lambda\sigma}^\beta T^{\mu(\nu)}_{,\beta} \quad (10)$$

$$\text{where} \quad P_{\alpha\lambda\sigma}^\mu = \frac{\partial \Gamma_{\alpha\lambda}^\mu}{\partial x^\sigma} - \frac{\partial \Gamma_{\alpha\sigma}^\mu}{\partial x^\lambda} + \Gamma_{\beta\sigma}^\mu \Gamma_{\alpha\lambda}^\beta - \Gamma_{\beta\lambda}^\mu \Gamma_{\alpha\sigma}^\beta \quad (11)$$

$$\text{and} \quad \Sigma_{\lambda\sigma}^\beta = \frac{1}{2}(\Gamma_{\lambda\sigma}^\beta - \Gamma_{\sigma\lambda}^\beta). \quad (12)$$

$P_{\alpha\lambda\sigma}^\mu$ and $\Sigma_{\lambda\sigma}^\beta$ are x -tensors as in the ordinary tensor-theory.† Since $\Gamma_{\mu\nu}^\lambda$ is an \bar{x} -scalar, they are also \bar{x} -scalars. $P_{\alpha\lambda\sigma}^\mu$ may be called the scalar-tensor of curvature formed from $\Gamma_{\mu\nu}^\lambda$, and $\Sigma_{\lambda\sigma}^\beta$ the scalar-tensor of torsion formed from $\Gamma_{\mu\nu}^\lambda$.

Similarly

$$T^{\mu(\nu)}_{;(q);(r)} - T^{\mu(\nu)}_{;(r);(q)} = R_{aqr}^\mu T^{\mu(a)} - 2S_{qr}^a T^{\mu(\nu)}_{;(a)} \quad (13)$$

$$\text{where} \quad R_{aqr}^\mu = \frac{\partial L_{aq}^\mu}{\partial \bar{x}^r} - \frac{\partial L_{ar}^\mu}{\partial \bar{x}^q} + L_{br}^\mu L_{aq}^b - L_{bq}^\mu L_{ar}^b \quad (14)$$

$$\text{and} \quad S_{qr}^a = \frac{1}{2}(L_{qr}^a - L_{rq}^a) \quad (15)$$

* It is assumed that the components of $T^{\mu(\nu)}$ are continuous functions of both sets of variables for the ranges of values considered, so that

$$\partial^2 T^{\mu(\nu)} / \partial x^\sigma \partial x^\lambda = \partial^2 T^{\mu(\nu)} / \partial x^\lambda \partial x^\sigma.$$

† See, for example, Eisenhart, *Non-Riemannian Geometry* (American Math. Soc. Colloquium Publications, vol. viii, 1927), p. 7, (4.2).

are scalar-tensors which may be called respectively *the scalar-tensors of curvature and torsion formed from L_{pq}^p* .

These formulae, which of course are exactly similar to those of the ordinary tensor theory, give the permutation rules for successive covariant x -derivatives and \bar{x} -derivatives.

Consider now the covariant \bar{x} -derivative of $T^{\mu(p)},_{\lambda}$. We have

$$\begin{aligned} T^{\mu(p)},_{\lambda};(q) &= \frac{\partial T^{\mu(p)},_{\lambda}}{\partial \bar{x}^q} + L_{aq}^p T^{\mu(a)},_{\lambda} \\ &= \frac{\partial^2 T^{\mu(p)}}{\partial \bar{x}^q \partial x^\lambda} + \Gamma_{\alpha\lambda}^\mu \frac{\partial T^{\alpha(p)}}{\partial \bar{x}^q} + L_{aq}^p \frac{\partial T^{\mu(a)}}{\partial x^\lambda} + \\ &\quad + \Gamma_{\alpha\lambda}^\mu L_{aq}^p T^{\alpha(a)} + T^{\alpha(p)} \frac{\partial \Gamma_{\alpha\lambda}^\mu}{\partial \bar{x}^q}, \end{aligned} \quad (16)$$

by definition.

Similarly,

$$\begin{aligned} T^{\mu(p)},_{\lambda};(q),\lambda &= \frac{\partial^2 T^{\mu(p)}}{\partial x^\lambda \partial \bar{x}^q} + L_{aq}^p \frac{\partial T^{\mu(a)}}{\partial x^\lambda} + \Gamma_{\alpha\lambda}^\mu \frac{\partial T^{\alpha(p)}}{\partial \bar{x}^q} + \\ &\quad + L_{aq}^p \Gamma_{\alpha\lambda}^\mu T^{\alpha(a)} + T^{\mu(a)} \frac{\partial L_{aq}^p}{\partial x^\lambda}. \end{aligned} \quad (17)$$

Subtracting,

$$T^{\mu(p)},_{\lambda};(q) - T^{\mu(p)},_{\lambda};(q),\lambda = \frac{\partial \Gamma_{\alpha\lambda}^\mu}{\partial \bar{x}^q} T^{\alpha(p)} - \frac{\partial L_{aq}^p}{\partial x^\lambda} T^{\mu(a)}. \quad (18)$$

Hence, *covariant x -derivatives are not in general permutable with covariant \bar{x} -derivatives*. In the particular case when $\Gamma_{\alpha\lambda}^\mu$ is independent* of the (\bar{x}^i) and L_{aq}^p of the (x^μ) , the right-hand side is zero and the covariant derivatives are permutable.

5. Geometrical considerations

The present paragraph contains remarks of a general character concerning the geometrical aspect of the analysis outlined above, and should give some indication of the kind of ideas to which the present theory leads.

Let $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ be two distinct symmetrical† scalar-tensors of the types indicated by the suffixes. Let $(g^{\mu\nu})$, $(\bar{g}^{(pq)})$ be the matrices respectively reciprocal to $(g_{\mu\nu})$, $(\bar{g}_{(pq)})$. Then it is easily proved (as in the ordinary tensor theory) that the elements $g^{\mu\nu}$, $\bar{g}^{(pq)}$ are the components of symmetrical contravariant scalar-tensors.

* In a case such as this we shall say that the Γ 's are x -connexions and \bar{x} -constants, and that the L 's are \bar{x} -connexions and x -constants.

† i.e. such that $g_{\mu\nu} = g_{\nu\mu}$ and $\bar{g}_{(pq)} = \bar{g}_{(qp)}$.

Suppose now that (x^μ) and (\bar{x}^i) are systems of coordinates in the Riemannian spaces whose metrics are respectively specified by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\text{A})$$

$$d\bar{s}^2 = \bar{g}_{(pq)} d\bar{x}^p d\bar{x}^q, \quad (\text{B})$$

the former therefore being of n dimensions and the latter of m .

Since $g_{\mu\nu}$ is an x -tensor and an \bar{x} -scalar, it follows in general that (A) will not determine a definite Riemannian space unless the (\bar{x}^i) are kept constant; it is only when the $g_{\mu\nu}$ are actually independent of the (\bar{x}^i) (i.e. are \bar{x} -constants) that $g_{\mu\nu}$ does not depend on the particular set of values possessed by the (\bar{x}^i) . Similarly, (B) will in general determine a definite Riemannian space only when the (x^μ) are kept constant. In other words, to each point of a Riemannian space (B) corresponds a Riemannian space (A), and vice versa. The formulae (A) and (B) thus really specify two multiply-infinite sets of Riemannian spaces.

Any double-tensor, $T^{\mu(\nu)}$ for example, will belong to both sets. In a space (A) it will be a set of m contravariant vectors $T^{\mu(1)}, T^{\mu(2)}, \dots, T^{\mu(m)}$, and in a space (B) a set of n contravariant vectors $T^{1(\nu)}, T^{2(\nu)}, \dots, T^{n(\nu)}$.

A special case may be of interest. Suppose that $n = m$ and that the $g_{\mu\nu}$ are \bar{x} -constants, so that (A) determines a single Riemannian space. Further, let the $\bar{g}_{(pq)}$ be the functions obtained from the $g_{\mu\nu}$ by replacing x^1, x^2, \dots, x^n respectively by $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$. Then (A) and (B) define the same space and $(x^\mu), (\bar{x}^\nu)$ are the coordinates of two points of it, any double-tensor being a set of functions of the two sets of coordinates. For example, the length s of the arc of the geodesic joining the points (x^μ) and (\bar{x}^ν) is a double-scalar; the derivatives $\partial s / \partial x^\mu, \partial s / \partial \bar{x}^\nu$ are scalar-vectors; the derivative $\partial^2 s / \partial x^\mu \partial \bar{x}^\nu$ a double-vector, and so on.*

Return now to the general case when m and n are not necessarily equal and $g_{\mu\nu}, \bar{g}_{(pq)}$ are distinct scalar-tensors. The Christoffel symbols

$$\{\lambda\mu, \nu\} = \frac{1}{2} g^{\nu\alpha} \left(\frac{\partial g_{\lambda\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\alpha} \right), \quad (19)$$

$$\{\overline{pq}, r\} = \frac{1}{2} \bar{g}^{(ra)} \left(\frac{\partial \bar{g}_{(pa)}}{\partial \bar{x}^q} + \frac{\partial \bar{g}_{(qa)}}{\partial \bar{x}^p} - \frac{\partial \bar{g}_{(pq)}}{\partial \bar{x}^a} \right) \quad (20)$$

* Some properties of the function $\Omega = \frac{1}{2} s^2$ have been investigated in earlier papers. See, in particular, *Proc. London Math. Soc.*, 32 (1931), 87, in which derivatives of Ω with respect to both sets of variables are employed. It was the consideration of these derivatives which led to the ideas of the present paper.

are then scalar-connexions, and could be made the basis of a theory of covariant differentiation. Examples of another type of scalar-connexion are given below.

Double-ennuples.

As above, let

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (A')$$

$$d\bar{s}^2 = \bar{g}_{(pq)} d\bar{x}^p d\bar{x}^q \quad (B')$$

characterize two infinite sets of Riemannian spaces, $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ being scalar-tensors.* For the purpose of this section it is necessary to suppose that $m = n$ (so that repeated Latin suffixes sum from 1 to n as well as Greek), but it is not now being assumed that $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ are derivable from one another by an interchange of the x 's and \bar{x} 's, though that may happen in particular cases.

Let $\lambda^{\mu(p)}$ be a contravariant double-vector belonging to these spaces, and write

$$\left. \begin{array}{l} \lambda_{\mu}^{(p)} \text{ for } g_{\mu\alpha} \lambda^{\alpha(p)} \\ \lambda_{\mu(p)} \text{ for } g_{\mu\alpha} \bar{g}_{(pq)} \lambda^{\alpha(q)} \\ \lambda_{(p)}^{\mu} \text{ for } \bar{g}_{(pq)} \lambda^{\mu(q)} \end{array} \right\}. \quad (21)$$

Let the double-vector be such that

$$\bar{g}_{(pq)} \lambda^{\mu(p)} \lambda^{v(q)} = g^{\mu\nu}, \quad (22)$$

which, in view of the assumed symmetry of $g_{\mu\nu}$, is equivalent to imposing $\frac{1}{2}n(n+1)$ conditions on the n^2 components $\lambda^{\mu(p)}$, a legitimate proceeding, since $n^2 > \frac{1}{2}n(n+1)$ if $n > 1$.

Multiplying (22) by $g_{\sigma\nu}$ and summing for ν , we get, in the notation of (21),

$$\lambda_{(q)}^{\mu} \lambda_{\sigma}^{(q)} = \delta_{\sigma}^{\mu}, \quad (23)$$

where δ_{σ}^{μ} is Kronecker's symbol. The matrices $(\lambda_{(q)}^{\mu})$, $(\lambda_{\sigma}^{(q)})$ are therefore reciprocal to one another, so it follows that

$$\lambda_{(q)}^{\mu} \lambda_{\mu}^{(p)} = \delta_{(q)}^{(p)}. \quad (24)$$

If both sides of this equation are multiplied by $\bar{g}^{(rq)}$ (and a summation performed for q), the resulting equation may be written

$$g_{\mu\nu} \lambda^{\mu(r)} \lambda^{v(p)} = \lambda^{(rp)}, \quad (25)$$

a formula analogous to (22).

If for the moment the (\bar{x}^i) are kept fixed, it follows from (24) that each of the n x -vectors $\lambda_{(1)}^{\mu}$, $\lambda_{(2)}^{\mu}$, ..., $\lambda_{(n)}^{\mu}$ is orthogonal (in the sense of the ordinary tensor theory) to all but one of the n x -vectors

* $g_{\mu\nu}$ and $\bar{g}^{(pq)}$ may, in particular cases, be respectively \bar{x} -constants and x -constants, so that (A') and (B') may sometimes define unique spaces.

$\lambda^{\mu(1)}, \lambda^{\mu(2)}, \dots, \lambda^{\mu(n)}$. Similarly, (23) shows that, the x 's being constant, each of the \bar{x} -vectors $\lambda_{(q)}^{\mu}$ is orthogonal to all but one of the \bar{x} -vectors $\lambda_{\mu(q)}$.

A double-vector $\lambda^{\mu(p)}$ which satisfies equations (22), and hence also (25), will be said to constitute a *reciprocally orthogonal double-ennuple*.*

Such a double-ennuple can be used to define what may be called the *scalar-tensors of rotation*, namely,

$$\bar{\gamma}_{(pqr)} = \lambda_{\mu(p),v} \lambda_{(q)}^{\mu} \lambda_{(r)}^v \quad (26)$$

and

$$\gamma_{\mu\nu\sigma} = \lambda_{\mu(p);(q)} \lambda_{\nu}^{(p)} \lambda_{\sigma}^{(q)}, \quad (27)$$

where

$$\lambda_{\mu(p),v} = \frac{\partial \lambda_{\mu(p)}}{\partial x^v} - \{\mu\nu, \alpha\} \lambda_{\alpha(p)}$$

and

$$\lambda_{\mu(p);(q)} = \frac{\partial \lambda_{\mu(p)}}{\partial \bar{x}^q} - \{\overline{pq}, a\} \lambda_{\mu(a)}$$

the Christoffel symbols being defined by (19) and (20). The name given to $\bar{\gamma}_{(pqr)}$ and $\gamma_{\mu\nu\sigma}$ is of course chosen on account of their similarity in Ricci's coefficients of rotation in the ordinary tensor theory,† to which indeed they reduce in certain particular cases. That they are actually scalar-tensors follows easily from the definitions.

It is worth noticing that, unlike Ricci's coefficients of rotation, $\bar{\gamma}_{(pqr)}$ and $\gamma_{\mu\nu\sigma}$ are *not* in general skew-symmetric in the first two suffixes. That is to say, the statements

$$\bar{\gamma}_{(pqr)} + \bar{\gamma}_{(qpr)} \neq 0 \quad (28)$$

and

$$\gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} \neq 0 \quad (29)$$

are true except in special cases. For it follows easily from (24) that

$$\lambda_{\mu(p)} \lambda_{(q)}^{\mu} = \bar{g}_{(pq)},$$

and hence, by differentiating covariantly with respect to the x 's and the Christoffel symbols $\{\lambda_{\mu}, \nu\}$,

$$\lambda_{\mu(p),v} \lambda_{(q)}^{\mu} + \lambda_{\mu(p)} \lambda_{(q),v}^{\mu} = \bar{g}_{(pq),v},$$

where, of course, $\bar{g}_{(pq),v}$ is the first covariant x -derivative $\partial \bar{g}_{(pq)} / \partial x^v$ of

* It should be noticed that, the \bar{x} 's being fixed, the n x -vectors $\lambda^{\mu(1)}, \dots, \lambda^{\mu(n)}$ do not in general constitute an orthogonal ennuple in the strict sense, for that would require $\lambda^{\mu(p)} \lambda_{\mu}^{(q)} = \delta_q^p$, which is not true. A similar remark is applicable, if the x 's are kept fixed and $\lambda^{\mu(p)}$ is regarded as an \bar{x} -ennuple. It is for this reason that the word *reciprocal* is introduced into the definition.

† Levi-Civita, *Absolute Differential Calculus* (Blackie, 1927), p. 271.

the scalar-tensor $\bar{g}_{(pq)}$. Multiplying both sides of the last equation by $\lambda_{(r)}^v$ and summing for v , we get

$$\bar{\gamma}_{(pqr)} + \bar{\gamma}_{(qpr)} = \bar{g}_{(pq),v} \lambda_{(r)}^v \quad (30)$$

Similarly,

$$\gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} = g_{\mu\nu;(q)} \lambda_{\sigma}^{(q)}, \quad (31)$$

so that the truth of (28) and (29) is established.

Lastly, it may be noted that the sets of functions defined by

$$\Delta_{\mu\nu}^{\sigma} = \lambda_{(a)}^{\sigma} \frac{\partial \lambda_{\mu}^{(a)}}{\partial x^{\nu}} \quad (32)$$

and

$$L_{qr}^p = \lambda_{\mu}^{(p)} \frac{\partial \lambda_{(q)}^{\mu}}{\partial \bar{x}^r} \quad (33)$$

constitute scalar-connexions of the types indicated by the suffixes. This is easily shown to be true by direct transformation of the x 's and \bar{x} 's. For example,

$$' \Delta_{\mu\nu}^{\sigma} = \lambda_{(a)}^{\sigma} \frac{\partial \lambda_{\mu}^{(a)}}{\partial x^{\nu}} = \lambda_{(b)}^{\sigma} \frac{\partial \bar{x}^b}{\partial \bar{x}'^a} \frac{\partial}{\partial x^{\nu}} \left(\lambda_{\mu}^{(c)} \frac{\partial \bar{x}'^a}{\partial \bar{x}^c} \right) = \lambda_{(b)}^{\sigma} \frac{\partial \bar{x}^b}{\partial \bar{x}'^a} \frac{\partial \bar{x}'^a}{\partial \bar{x}^c} \frac{\partial \lambda_{\mu}^{(c)}}{\partial x^{\nu}},$$

since the derivatives $\partial \bar{x}'^a / \partial \bar{x}^c$ are independent of the (x^{μ}) . But

$$\frac{\partial \bar{x}^b}{\partial \bar{x}'^a} \frac{\partial \bar{x}'^a}{\partial \bar{x}^c} = \delta_c^b,$$

whence it follows at once that

$$' \Delta_{\mu\nu}^{\sigma} = \Delta_{\mu\nu}^{\sigma},$$

and $\Delta_{\mu\nu}^{\sigma}$ is therefore an \bar{x} -scalar. By a similar method it can be shown that it is also an x -connexion,* and that L_{qr}^p is a scalar-connexion.

Other scalar-connexions of a similar character are those defined by

$$\Gamma_{\mu\nu}^{\sigma} = \lambda^{\sigma(a)} \frac{\partial \lambda_{\mu}^{(a)}}{\partial x^{\nu}}, \quad (34)$$

and

$$H_{qr}^p = \lambda^{\mu(p)} \frac{\partial \lambda_{\mu}^{(q)}}{\partial \bar{x}^r}. \quad (35)$$

Since $\frac{\partial \lambda_{\mu}^{(a)}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}} (\bar{g}_{(ab)} \lambda_{\mu}^{(b)}) = \bar{g}_{(ab)} \frac{\partial \lambda_{\mu}^{(b)}}{\partial x^{\nu}} + \bar{g}_{(ab),\nu} \lambda_{\mu}^{(b)}$,

we get $\Gamma_{\mu\nu}^{\sigma} = \Delta_{\mu\nu}^{\sigma} + \bar{g}_{(ab),\nu} \lambda_{\mu}^{(b)} \lambda^{\sigma(a)}$. (36)

Similarly, $H_{qr}^p = L_{qr}^p + g_{\alpha\beta;(r)} \lambda_{(q)}^{\beta} \lambda^{\alpha(p)}$. (37)

6. Conclusion

The remark of § 1 concerning the extension of the foregoing theory to tensor-functions of more than two sets of variables may now con-

* Cf. Eisenhart, op. cit., p. 48, (18.6).

veniently be referred to. A double-tensor requires two kinds of suffixes; a triple-tensor (a tensor-function of three sets of variables) would require three kinds, and so on. It is therefore obvious that a radical alteration in notation, if not in terminology, would be required if such a generalized theory were not to be submerged by the weight of its own symbolism.

In § 3 scalar-connexions are defined. It might be asked whether there would be any advantage in defining such concepts as those of 'double-connexions' (sets of functions behaving like affine-connexions for transformations of both sets of variables), of 'tensor-connexions', or of other hybrids of the same sort. The answer appears to be in the negative; for the covariant derivative of a double-tensor, formed with respect to a double-connexion (or a tensor-connexion) would not in general be a double-tensor.

The work contained in this paper has been presented as a generalization of the Absolute Differential Calculus of Ricci and Levi-Civita. From another point of view* it can be regarded as a *particular case* of the same Calculus. Suppose, for example, that $T_{\lambda(p)}$ is a covariant double-vector, the components being functions of the n variables (x^μ) and of the m variables (\bar{x}^i). Write $\xi^1, \xi^2, \dots, \xi^n$ respectively for x^1, x^2, \dots, x^n , and $\xi^{n+1}, \xi^{n+2}, \dots, \xi^{n+m}$ for $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m$; also $\xi'^1, \xi'^2, \dots, \xi'^n$ for x'^1, x'^2, \dots, x'^n and $\xi'^{n+1}, \xi'^{n+2}, \dots, \xi'^{n+m}$ for $\bar{x}'^1, \bar{x}'^2, \dots, \bar{x}'^m$. Then the laws of transformation of $T_{\lambda(p)}$ give

$${}'T'_{\lambda(p)} = T_{\alpha(a)} \frac{\partial \xi^\alpha}{\partial \xi'^\lambda} \frac{\partial \xi^a}{\partial \xi'^p}, \quad (38)$$

where it is understood that Greek indices sum from 1 to n and Latin from $n+1$ to $n+m$, and where $T_{\alpha(n+r)}$ is taken to mean what has hitherto been denoted by $T_{\alpha(r)}$.

Now if this restriction on the summations did not exist, and if $\xi^1, \xi^2, \dots, \xi^{n+m}$ could be functions of *all* the ξ' 's (instead of the first n of the ξ 's being functions of the first n only of the ξ' 's, and the last m of the ξ 's functions of the last m only of the ξ' 's), then (38) would give the ordinary law of transformation of the tensor $T_{\lambda(p)}$ of the second rank whose components are functions of the $n+m$ variables $\xi^1, \xi^2, \dots, \xi^{n+m}$. So a double-vector $T_{\lambda(p)}$ can be regarded as an ordinary tensor of the second rank, specialized by the fact that the variables

* The remarks in the remainder of this paragraph are due to Professor T. Levi-Civita, to whom I owe my best thanks for his kind interest in this paper.

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are divided into two separate groups of n and m respectively. A similar remark applies to any double-tensor.

This dividing of the variables into two groups is not unfamiliar, for it is commonly done when the four-dimensional space-time of Relativity is partitioned into 'space' and 'time'. Suspending now the summation convention, let

$$ds^2 = \sum_{\mu, \nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \quad (39)$$

specify the metric of space-time, the $g_{\mu\nu}$ being functions of the four coordinates x^0, x^1, x^2, x^3 . Suppose that x^0 is known, intuitively or otherwise, to be identifiable with the time t , and that (x^1, x^2, x^3) are a given set of spatial coordinates. Then (39) may be written

$$ds^2 = V^2 dt^2 + 2dt \sum_{\mu=1}^3 w_\mu dx^\mu - \sum_{\mu, \nu=1}^3 a_{\mu\nu} dx^\mu dx^\nu, \quad (40)$$

where* $V^2 = g_{00}$, $w_\mu = g_{0\mu}$, $a_{\mu\nu} = -g_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$).

For transformations of the spatial coordinates (x^1, x^2, x^3) only, with t unaltered, V^2 behaves like a scalar, the w_μ like the three components of a covariant vector, and the $a_{\mu\nu}$ like the nine components of a symmetric covariant tensor of the second rank. And, if it were possible to attach a physical meaning to transformations of t only (x^1, x^2, x^3 remaining unaltered), V^2 would behave for such transformations like a covariant tensor of the second rank, having of course only one component since there is only one variable t ; while w_1, w_2, w_3 would each behave like covariant vectors, and the $a_{\mu\nu}$ like scalars. So, having thus partitioned space-time into space and time, V^2 and $a_{\mu\nu}$ are scalar-tensors, and w_μ a double-vector according to the definitions of this paper, one of the two sets of variables being, in this special case, the single variable t .

Note: As stated in the introduction, I believe that it is the point of view rather than the mathematics itself which is new in the present chapter. In writing the paper of which the chapter is composed, I had intended to develop the work further, possibly applying it to the geometry of the sub-spaces of a riemannian space. But while the paper was in the press, there appeared one by A.W. Tucker "On generalised covariant differentiation" (*Annals of Math.*, 32 (1931) 451) containing a theory which from the

* The notation is borrowed from Levi-Civita, op. cit., p. 338.

mathematical standpoint is similar to but more general than mine. Dr Tucker has, for example, devised a notation which renders practicable the consideration of tensor-functions of more than two sets of variables, which, unlike mine, are not necessarily independent of one another. The treatment of sub-spaces by Duschek and Mayer in their "Lehrbuch der Differentialgeometrie" Bd. II, (1930) Ch. VII (of which no copy was available until some months after my paper was published) also involves analysis similar to that in this chapter.

CHAPTER IV

The integration of certain tensor differential equations.

Paper 1: "On the 'elementary' solution of Laplace's Equation."
(Page 48).

Paper 2: "Generalised solutions of Laplace's Equation."
(Page 56).

Paper 3: "The potential of an electron in a space-time of constant curvature". (Page 64).

Paper 4: "Generalised solutions of some of the partial differential equations of mathematical physics." (Page 74).

Foreword.

§1 of Paper 4 contains a few general remarks (referred to in the Introduction) concerning the integration of tensor differential equations. Had it been possible I should therefore have made this paper the first instead of the last section of the present chapter, but I was unable to do so since it contains references to theorems in Paper III. I have felt unwilling to separate the paragraph in question from its context and place it (as could have been done if slight alterations were made in it) at the beginning of the chapter.

CHAPTER IV, Paper I.

On the "elementary" solution of Laplace's Equation

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§1. Introduction.

Hadamard defines¹ the "elementary solution" of the general linear partial differential equation of the second order, namely

$$F(u) \equiv \sum_{i,k=1}^n A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^n B_i \frac{\partial u}{\partial x_i} + Cu = 0, \quad (1)$$

(A_{ik} , B_i , C being functions of the n variables x_1, x_2, \dots, x_n , which may be regarded as coordinates in a space of n dimensions), to be one of those solutions which are infinite to as low an order as possible at a given point and on every bicharacteristic through that point.² He then proceeds to find formulae for the elementary solution of equation (1), his result being as follows:

Let (H_{ik}) be the matrix reciprocal to (A_{ik}), and consider the Riemannian space whose metric is specified by

$$ds^2 = \sum_{i,k} H_{ik} dx_i dx_k. \quad (2)$$

Let Γ denote the square of the geodesic distance between the point (x_i) and a fixed point (q_i) of this space. Then if n be odd, the elementary solution of (1) is given by

$$u = \Gamma^{-\frac{1}{2}(n-2)} [U_0 + U_1 \Gamma + U_2 \Gamma^2 + \dots + U_r \Gamma^r + \dots],$$

where

$$U_0 = k \exp \left[- \int_0^s \frac{1}{4s} \{F(\Gamma) - C\Gamma - 2n\} ds \right],$$

k being a certain constant, $s = \sqrt{\Gamma}$, the integral being taken along the arc of the geodesic from (q_i) to (x_i). The functions U_r ($r > 0$) are then determined by the recurrence-formula

$$U_r = - \frac{U_0}{(4r - 2n + 4) s^r} \int_0^s \frac{s^{r-1}}{U_0} F(U_{r-1}) ds.$$

¹ *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale, 1923), p. 70, *et seq.*

² The solutions satisfying this condition differ only in the values of arbitrary constants, the elementary solution being obtained by choosing these according to a certain rule. For the purpose of this paper it suffices to say that the elementary solution is the one which reduces to $u = 1/r$, where $r = \sqrt{\{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}}$, when the differential equation (1) is of the particular form $\nabla^2 V = 0$.

A similar formula holds when n is even, but involves a term in $\log \Gamma$.

The object of the present paper is to establish a formula for the elementary solution for the particular case in which equation (1) is the tensor generalisation, with respect to the metric (2), of the ordinary Laplace's equation. The result obtained lacks the generality of Hadamard's, but may be of interest on account of its comparative simplicity and because a single formula holds whether n is odd or even.

§ 2. Laplace's Equation in Tensor Form.

Let

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 1, 2, \dots, n), \tag{3}$$

define the metric¹ of a general Riemannian space of n dimensions.

If V be any scalar and $V_{\mu\nu}$ its second covariant derivative, viz.

$$V_{\mu\nu} = \frac{\partial^2 V}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\} \frac{\partial V}{\partial x^\alpha},$$

then the partial differential equation of which we seek the elementary solution is

$$g^{\mu\nu} V_{\mu\nu} = 0. \tag{4}$$

(When $n = 3$ and $ds^2 = dx^2 + dy^2 + dz^2$, this reduces to the ordinary Laplace's Equation $\nabla^2 V = 0$).

Let Ω be one half the square of the geodesic distance² between the point $(x^i) \equiv (x^1, x^2, \dots, x^n)$ and the fixed point $(\bar{x}^i) \equiv (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$. Thus

$$\Omega = \frac{1}{2} s^2 \tag{5}$$

where s is the length of the arc of the geodesic joining (x^i) and (\bar{x}^i) .

Ω is a function of the x 's and of the \bar{x} 's. We shall write Ω_μ for $\partial \Omega / \partial x^\mu$, $\Omega_{(\mu)}$ for $\partial \Omega / \partial \bar{x}^\mu$, $\Omega_{\mu(\nu)}$ for $\partial^2 \Omega / \partial x^\mu \partial \bar{x}^\nu$. Further, $\bar{g}_{\mu\nu}$, $\bar{g}^{\mu\nu}$ will be used respectively to denote the values at (\bar{x}^i) of $g_{\mu\nu}$, $g^{\mu\nu}$, while \bar{g} will represent the value at this point of the determinant $g = \|g_{\mu\nu}\|$.

¹ The summation convention is adopted throughout. The notation of the succeeding paragraphs will differ to some extent from that of Hadamard, in order that it should be brought into conformity with the notation now usual in the Tensor Calculus.

² Thus if in § 1 (q_i) is taken at (\bar{x}) , we have $\Omega = \frac{1}{2} \Gamma$.

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Let J denote the determinant $\|\Omega_{\mu(\nu)}\|$.^{*} Then we shall prove the following theorem:

The elementary solution of the partial differential equation $g^{\mu\nu} V_{\mu\nu} = 0$ is given by

$$V = A \int^s \frac{J}{g^{\frac{1}{2}} \bar{g}^{\frac{1}{2}}} \frac{ds}{s^{n-1}} + B, \tag{6}$$

where A, B are suitable constants, and the integral is taken along an arc of the geodesic joining (\bar{x}^i) to (x^i) .

It should be remarked that (\bar{x}^i) must not itself be a point of the arc of integration, for if it were the integral would in general be divergent. Further, the constant B must be so chosen that the solution is unaltered by an interchange of the the x^i with the \bar{x}^i . The actual value given to A is not of fundamental importance.

§ 3. Proof of the theorem of § 2.

It is a known fact that by transferring to a normal coordinate-system (y^i) , the equations of any geodesic through (\bar{x}^i) can be put in the form¹

$$y^i = a^i s, \tag{7}$$

where the constants a^i are the values at (\bar{x}^i) (which is the origin of the normal coordinates) of dx^i/ds for the geodesic in question. Thus

$$a^i = \frac{d\bar{x}^i}{ds}. \tag{8}$$

Components of tensors corresponding to the y -coordinate-system will be denoted by the affixing of an asterisk. For example, ${}^*\Omega_{\mu}$ will denote the vector $\partial\Omega/\partial y^{\mu}$.

By (3) and (8), $\bar{g}_{\mu\nu} a^{\mu} a^{\nu} = 1$. Multiplying by $\frac{1}{2} s^2$ and using (5), we get

$$\Omega = \frac{1}{2} \bar{g}_{\mu\nu} y^{\mu} y^{\nu}. \tag{9}$$

Hence

$$\frac{\partial\Omega}{\partial y^{\mu}} = \bar{g}_{\mu\nu} y^{\nu},$$

that is,

$${}^*\Omega_{\mu} = {}^*g_{\mu\nu} y^{\nu}, \tag{10}$$

since²

$$\bar{g}_{\mu\nu} y^{\nu} = {}^*g_{\mu\nu} y^{\nu}.$$

¹ The y^i are the normal variables of Lipschitz: Hadamard, *loc. cit.*, p. 87. A full account of them is given by Veblen, *Invariants of Quadratic Differential Forms* (Cambridge Tract no. 24, 1927), ch. VI.

² Veblen, *loc. cit.*, ch. VI (14.8).

* It may be remarked that this determinant J is \bar{g} times the determinant $J = \|\Omega_{\mu}^{(\nu)}\|$ introduced on page 27.

Raising the suffix μ in (10),

$$*\Omega^\mu = y^\mu.$$

Hence

$$*\Omega^\mu_\nu = \frac{\partial *\Omega^\mu}{\partial y^\nu} + *\{a\nu, \mu\} *\Omega^a \\ = \delta^\mu_\nu + *\{a\nu, \mu\} y^a.$$

Contracting,

$$*\Omega^\mu_\mu = n + \frac{1}{2} y^a \frac{\partial}{\partial y^a} (\log *g)$$

or¹, by (7),

$$*\Omega^\mu_\mu = n + \frac{1}{2} s \frac{d}{ds} (\log *g). \tag{11}$$

Again, it has been shown² that

$$y^\mu = -\bar{g}^{\mu a} \Omega_{(a)}, \tag{12}$$

the repetition of a implying a summation.

By (5), $*\Omega^\mu = s *s^\mu$, and it quickly follows from (10), (9) and (5), that

$$*s_\mu *s^\mu = 1. \tag{13}$$

And it further follows from (11) and (13) that

$$*s^\mu_\mu = \frac{n-1}{s} + \frac{1}{2} \frac{d}{ds} (\log *g). \tag{14}$$

We are now in a position to solve the partial differential equation (4). Transferring to normal coordinates, the equation becomes

$$*g^{\mu\nu} \left[\frac{\partial^2 V}{\partial y^\mu \partial y^\nu} - *\{\mu\nu, a\} \frac{\partial V}{\partial y^a} \right] = 0. \tag{15}$$

But

$$\frac{\partial V}{\partial y^\mu} = \frac{dV}{ds} \frac{\partial s}{\partial y^\mu} = *s_\mu \frac{dV}{ds} \\ \frac{\partial^2 V}{\partial y^\mu \partial y^\nu} = *s_\mu *s_\nu \frac{d^2 V}{ds^2} + \frac{\partial *s_\mu}{\partial y^\nu} \frac{dV}{ds}.$$

Substituting in (15), we get

$$*s_\mu *s^\mu \frac{d^2 V}{ds^2} + *s^\mu_\mu \frac{dV}{ds} = 0,$$

and hence, by (13) and (14),

$$\frac{d^2 V}{ds^2} + \left[\frac{n-1}{s} + \frac{1}{2} \frac{d}{ds} (\log *g) \right] \frac{dV}{ds} = 0.$$

¹ This equation is essentially the same as Hadamard's, *loc. cit.*, p. 91 (37).

² Page 18, Equation 13.



Multiplying by the integrating factor $*g^{\frac{1}{2}} s^{n-1}$, we quickly get the solution

$$V = K \int^s \frac{1}{*g^{\frac{1}{2}}} \frac{ds}{s^{n-1}} + B, \quad (16)$$

K, B being arbitrary constants.

Now $*g^{\frac{1}{2}}$ is a scalar density, hence

$$\begin{aligned} *g^{\frac{1}{2}} &= g^{\frac{1}{2}} \frac{\partial(x^i)}{\partial(y^j)} \\ &= g^{\frac{1}{2}} / \frac{\partial(y^i)}{\partial(x^j)}, \end{aligned} \quad (17)$$

by a well-known property of Jacobians.

But

$$\begin{aligned} \frac{\partial(y^i)}{\partial(x^j)} &= \left\| \frac{\partial y^\mu}{\partial x^\nu} \right\| \\ &= \left\| -\bar{g}^{\mu\alpha} \Omega_{\nu(\alpha)} \right\| \quad \text{by (12),} \\ &= (-1)^n \bar{g} \left\| \Omega_{\nu(\alpha)} \right\| \\ &= (-1)^n \bar{g} J. \end{aligned} \quad (18)$$

Substituting from (17) and (18) in (16), and putting A for the arbitrary constant $(-1)^n K \bar{g}^{\frac{1}{2}}$, we get

$$V = A \int^s \frac{J}{g^{\frac{1}{2}} \bar{g}^{\frac{1}{2}}} \frac{ds}{s^{n-1}} + B,$$

the result stated; $\bar{g}^{\frac{1}{2}}$ is of course a constant, being a function of the \bar{x}^i only. Since we have made no supposition regarding n , the solution holds whether n be odd or even.

§4. Application.

An example of the above theorem may be of interest. Consider the two-dimensional spherical space defined by

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (19)$$

$$\left. \begin{aligned} \text{for which } g_{11} &= 1, \quad g_{22} = \sin^2 \theta, \quad g_{12} = g_{21} = 0, \\ g^{11} &= 1, \quad g^{22} = \frac{1}{\sin^2 \theta}, \quad g^{12} = g^{21} = 0. \end{aligned} \right\} \quad (20)$$

The equation to be solved is now

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} + \cot \theta \frac{\partial V}{\partial \theta} = 0. \quad (21)$$

The square of the geodesic distance between the points (θ, ϕ) and $(\bar{\theta}, \bar{\phi})$ is of course the

square of the length of the arc of the great-circle joining them, and therefore given by

$$2\Omega = (\text{arc } \cos Q)^2 \dots \dots \dots (22)$$

where $Q = \cos \theta \cos \bar{\theta} + \sin \theta \sin \bar{\theta} \cos(\phi - \bar{\phi}) \dots (23)$

So $\Omega_{\lambda} = - \frac{\text{arc } \cos Q}{\sqrt{1-Q^2}} Q_{\lambda}$

and $\Omega_{\lambda(\mu)} = - \frac{\text{arc } \cos Q}{\sqrt{1-Q^2}} Q_{\lambda(\mu)} + \left[\frac{1}{1-Q^2} - \frac{Q \text{ arc } \cos Q}{(1-Q^2)^{3/2}} \right] Q_{\lambda} Q_{(\mu)}$
 $= P \cdot Q_{\lambda(\mu)} + R \cdot Q_{\lambda} Q_{(\mu)}, \text{ say.}$

Hence

$$J = \|\Omega_{\lambda(\mu)}\|$$

$$= P^2 \|Q_{\lambda(\mu)}\| + P \cdot R \cdot Q_{(2)} \begin{vmatrix} Q_{1(1)} & Q_1 \\ Q_{2(1)} & Q_2 \end{vmatrix}$$

$$+ P \cdot R \cdot Q_{(1)} \begin{vmatrix} Q_1 & Q_{1(2)} \\ Q_2 & Q_{2(2)} \end{vmatrix} \dots (24)$$

Differentiating (23), it is easily proved that

$$\|Q_{\lambda(\mu)}\| = \sin \theta \sin \bar{\theta} \cdot Q, \dots \dots \dots (25)$$

$$\begin{vmatrix} Q_{1(1)} & Q_1 \\ Q_{2(1)} & Q_2 \end{vmatrix} = -\sin^2 \theta \sin(\phi - \bar{\phi}),$$

$$\begin{vmatrix} Q_1 & Q_{1(2)} \\ Q_2 & Q_{2(2)} \end{vmatrix} = \sin \theta \sin \bar{\theta} [\cos \theta \sin \bar{\theta} - \sin \theta \cos \bar{\theta} \cos(\phi - \bar{\phi})].$$

So since

$$Q_{(2)} = \frac{\partial Q}{\partial \bar{\phi}} = \sin \theta \sin \bar{\theta} \sin(\phi - \bar{\phi})$$

and $Q_{(1)} = \frac{\partial Q}{\partial \theta} = -\cos \theta \sin \bar{\theta} + \sin \theta \cos \bar{\theta} \cos(\phi - \bar{\phi}),$

it follows from the last four equations that the coefficient of P.R in (24)

$$= -\sin^3 \theta \sin \bar{\theta} \sin^2(\phi - \bar{\phi}) - \sin \theta \sin \bar{\theta} [\cos \theta \sin \bar{\theta} - \sin \theta \cos \bar{\theta} \cos(\phi - \bar{\phi})]^2$$

$$= -\sin \theta \sin \bar{\theta} (1 - Q^2) \dots \dots \dots (26)$$

Substituting from (25) and (26) in (24), and putting in

the values of P and R ,

$$\begin{aligned} J &= -\sin\theta \sin\bar{\theta} \frac{\arccos Q}{\sqrt{1-Q^2}} \left[-\frac{Q \arccos Q}{\sqrt{1-Q^2}} - 1 + \frac{Q \arccos Q}{\sqrt{1-Q^2}} \right] \\ &= \sin\theta \sin\bar{\theta} \frac{\arccos Q}{\sqrt{1-Q^2}}. \end{aligned}$$

Now by (20),

$$g = g_{11} g_{22} - g_{12}^2 = \sin^2\theta$$

and $\bar{g} = \bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2 = \sin^2\bar{\theta},$

so $g^{\frac{1}{2}} \bar{g}^{\frac{1}{2}} = \sin\theta \sin\bar{\theta}.$

Hence

$$\begin{aligned} \frac{J}{g^{\frac{1}{2}} \bar{g}^{\frac{1}{2}}} &= \frac{\arccos Q}{\sqrt{1-Q^2}} \\ &= \frac{s}{\sin s}, \end{aligned}$$

where $s = \sqrt{2\Omega} = \arccos Q$ is the geodesic distance between the points (θ, ϕ) and $(\bar{\theta}, \bar{\phi})$. So putting $n=2$ in (6), the elementary solution of (21) is obtained, namely

$$\begin{aligned} V &= A \int^s \frac{s}{\sin s} \cdot \frac{ds}{s} + B \\ &= A \log\left(\frac{\sin s}{1+\cos s}\right) + B \\ &= A \log\left(\frac{\sqrt{1-Q^2}}{1+Q}\right) + B \\ &= \frac{A}{2} \log\left(\frac{1-Q}{1+Q}\right) + B \dots \dots \dots (27) \end{aligned}$$

To choose the most suitable values for the constants A, B , we observe that when $\bar{\theta} = 0$, $Q = \cos\theta$ by (23). Also if θ is small, the space becomes that defined by

$$ds^2 = d\theta^2 + \theta^2 d\phi^2,$$

so that θ, ϕ are ordinary plane polar coordinates

and the elementary solution is of course $V = \log \theta$, since the partial differential equation (21) is now the ordinary Laplace's equation $\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\theta^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{\theta} \frac{\partial V}{\partial \theta} = 0$ in two dimensions. But with θ small (27) reduces (with $\bar{\theta} = 0$) to

$$\begin{aligned} V &= \frac{A}{2} \log(\tan^2 \frac{1}{2} \theta) + B \\ &= A \log(\frac{1}{2} \theta) + B \end{aligned}$$

to the first order. So choosing $A = 1$ and $B = \log 2$, we obtain $V = \log \theta$ approximately. So, finally, the elementary solution of the original partial differential equation (21) is

$$\begin{aligned} V &= \frac{1}{2} \log \frac{1-Q}{1+Q} + \log 2 \\ &= \frac{1}{2} \log \left\{ 4 \left(\frac{1-Q}{1+Q} \right) \right\}, \end{aligned}$$

where Q is given by (23). It is a simple matter to verify that this is indeed a solution of (21).

Generalised Solutions of Laplace's Equation

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The present paper contains solutions of the tensor generalisation of Laplace's Equation. The results obtained are summarised in the two theorems enunciated in §1. They apply only to the case when the Riemannian space forming the background of the theory is flat. In the concluding paragraph a special case is considered, and it is shown that the present theory is closely connected with Whittaker's well known general solution of the ordinary Laplace's Equation.¹

§1. INTRODUCTION.

Let

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.1)$$

define the metric of an n -dimensional Euclidean space; that is, one for which the Riemann-Christoffel tensor is everywhere zero. Let Ω be one half of the square of the geodesic distance between the two points (x^i) and (\bar{x}^i) of this space,² so that Ω is a scalar function of the two sets of variables (x^i) , (\bar{x}^i) . Let further the coordinates (\bar{x}^i) be each functions of a variable τ . Then Ω is a function of x^1, x^2, \dots, x^n , and τ , and we shall later define τ as a function of the x^i by means of the equation

$$\Omega = 0. \quad (1.2)$$

Greek suffixes will be used to denote covariant differentiations with respect to the x^i , with τ kept constant, the only exception to this rule being that the suffix τ will denote ordinary partial differentiations with respect to τ . Thus, for example,

$$\left. \begin{aligned} \Omega_{,\mu} &= \partial \Omega / \partial x^\mu, & \Omega_{,\tau} &= \partial \Omega / \partial \tau, \\ \Omega_{,\mu\nu} &= \frac{\partial^2 \Omega}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\} \frac{\partial \Omega}{\partial x^\alpha}, & \Omega_{,\tau\tau} &= \frac{\partial^2 \Omega}{\partial \tau^2}, \\ \Omega_{,\tau\mu\nu} &= \frac{\partial^2 \Omega_{,\tau}}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\} \frac{\partial \Omega_{,\tau}}{\partial x^\alpha}, & \Omega_{,\tau\mu} &= \frac{\partial^2 \Omega}{\partial \tau \partial x^\mu}, \end{aligned} \right\} \quad (1.3)$$

¹ Whittaker and Watson, "Modern Analysis" (1920), §18.3.

² Some properties of this function have been investigated in earlier papers, particularly (i) *Proc. London Math. Soc.*, 31 (1930), 225; (ii) *ibid.*, 32 (1931), 87. These will be referred to as papers 1 and 2 respectively of Chapter II.

and so on. It must be emphasised that in these definitions the partial differentiations with respect to the x^i are strict, that is, they treat τ as a constant as well as the other x 's. The summation convention does not of course hold for the suffix τ .

For convenience the Christoffel symbol $\{\lambda\mu, \nu\}$ will be denoted by $\Gamma_{\lambda\mu}^\nu$. Further, the evaluation at (\bar{x}^i) of any function of the x^i will be indicated by the superposing of a bar on the functional symbol.

The partial differential equation of which solutions are sought is

$$V_\lambda^\lambda \equiv g^{\lambda\mu} \left\{ \frac{\partial^2 V}{\partial x^\lambda \partial x^\mu} - \Gamma_{\lambda\mu}^\alpha \frac{\partial V}{\partial x^\alpha} \right\} = 0. \tag{1.4}$$

The following are the theorems proved.

THEOREM I. *If the functions $\bar{x}^i(\tau)$ are chosen to satisfy the differential equations*

$$\bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} = 0, \tag{1.5}$$

$$\frac{d^2 \bar{x}^\mu}{d\tau^2} + \bar{\Gamma}_{\alpha\beta}^\mu \frac{d\bar{x}^\alpha}{d\tau} \frac{d\bar{x}^\beta}{d\tau} = 0, \tag{1.6}$$

then, for all values of n , a solution of the partial differential equation $V_\lambda^\lambda = 0$ is given by

$$V = f(\Omega_\tau), \tag{1.7}$$

where, after differentiating,¹ τ is expressed as a function of x^1, x^2, \dots, x^n by means of the equation $\Omega = 0$, and where $f(\Omega_\tau)$ is an arbitrary function of Ω_τ .

That the equations (1.5) and (1.6) are compatible is well known.²

THEOREM II. *A solution of the equation $V_\lambda^\lambda = 0$ is given by*

$$V = \phi(\tau) / \Omega_\tau^{\frac{1}{2}(n-2)}, \tag{1.8}$$

where $\phi(\tau)$ is an arbitrary function of τ , and τ is expressed as a function of the x 's by means of the equation $\Omega = 0$. If the number n of the

¹ $\partial\Omega/\partial\tau$ is in general a function of τ as well as of the x 's, so τ must be eliminated in order that the solution should be expressed as a function of the x 's only.

² See, for example, Veblen, "Invariants of Quadratic Differential Forms" *Camb. Math. Tract. No. 24* (1927), 95.

variables is equal to 2 or 4 there is no limitation on the choice of the functions $\bar{x}^i(\tau)$; but if n has any other value, these functions must satisfy the conditions (1.5) and (1.6).

§ 2. PRELIMINARY FORMULAE.

It is well known that Ω satisfies¹ the partial differential equation

$$\Omega^\lambda \Omega_\lambda = 2 \Omega, \quad (2.1)$$

where $\Omega^\lambda = g^{\lambda\alpha} \Omega_\alpha$.

Moreover, it is shown elsewhere² that, the space being flat,

$$\Omega_{\mu\nu} = g_{\mu\nu}. \quad (2.2)$$

Furthermore, since we have put

$$\bar{\Omega} = 0 \quad (2.3)$$

it follows, by differentiating partially with respect to x^λ , that

$$\Omega_\lambda + \Omega_\tau \tau_\lambda = 0,$$

where $\tau_\lambda = \partial\tau/\partial x^\lambda$. Hence

$$\tau_\lambda = -\Omega_\lambda/\Omega_\tau. \quad (2.4)$$

Differentiating (2.1) twice in succession with respect to τ , we get

$$\Omega_{\tau\lambda} \Omega^\lambda = \Omega_\tau, \quad (2.5)$$

$$\Omega_{\tau\tau\lambda} \Omega^\lambda + \Omega_{\tau\lambda} \Omega_\tau^\lambda = \Omega_{\tau\tau}. \quad (2.6)$$

From (2.2), raising the suffix ν and contracting,

$$\Omega_\mu^\mu = n, \quad (2.7)$$

and the differentiation of this equation with respect to τ gives

$$\Omega_{\tau\mu}^\mu = 0. \quad (2.8)$$

By (2.1), (2.3), (2.4), it follows that

$$\tau^\lambda \tau_\lambda = 0. \quad (2.9)$$

By (2.4),

$$\begin{aligned} \tau_\lambda^\lambda &= g_a^\lambda \tau_{\lambda a} = -g^{\lambda a} \left[\left(\frac{\Omega_\lambda}{\Omega_\tau} \right)_a + \left(\frac{\Omega_\lambda}{\Omega_\tau} \right)_\tau \tau_a \right] \\ &= \Omega_\tau^{-2} (\Omega_\tau^\lambda \Omega_\lambda - \Omega_\tau \Omega_\lambda^\lambda - \Omega_\tau \Omega_{\tau\lambda} \tau^\lambda + \Omega_{\tau\tau} \Omega_\lambda \tau^\lambda), \end{aligned}$$

whence, by (2.4), (2.5), (2.7) and (2.9),

$$\tau_\lambda^\lambda = -(n-2) \Omega_\tau^{-1}. \quad (2.10)$$

¹ This in fact follows at once from equations (1) and (10) of paper 2, Chapter II, pp. 15 and 17.

² Paper 1, § 2, where Ω denotes twice the function here represented by Ω .

Lastly, it follows from (2.2), by interchanging the x 's and the \bar{x} 's, that

$$\frac{\partial^2 \Omega}{\partial \bar{x}^\mu \partial \bar{x}^\nu} - \bar{\Gamma}_{\mu\nu}^{\alpha} \frac{\partial \Omega}{\partial \bar{x}^\alpha} = \bar{g}_{\mu\nu}. \quad (2.11)$$

But
$$\frac{\partial \Omega}{\partial \tau} = \frac{\partial \Omega}{\partial \bar{x}^a} \frac{\partial \bar{x}^a}{\partial \tau}$$

and
$$\frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\partial \Omega}{\partial \bar{x}^a} \frac{d^2 \bar{x}^a}{d\tau^2} + \frac{\partial^2 \Omega}{\partial \bar{x}^a \partial \bar{x}^b} \frac{d\bar{x}^a}{d\tau} \frac{d\bar{x}^b}{d\tau}$$

and therefore, by (2.11),

$$\Omega_{\tau\tau} = \bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} + \frac{\partial \Omega}{\partial \bar{x}^a} \left(\frac{d^2 \bar{x}^a}{d\tau^2} + \bar{\Gamma}_{\mu\nu}^a \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} \right). \quad (2.12)$$

§3. PROOF OF THEOREM I.

We now show, by direct substitution, that any function of Ω_τ , say

$$U = f(\Omega_\tau), \quad (3.1)$$

is a solution of the equation

$$V_\lambda^\lambda = 0 \quad (3.2)$$

provided that the conditions (1.5) and (1.6) are satisfied.

For

$$U_\lambda = f'(\Omega_\tau) (\Omega_{\tau\lambda} + \Omega_{\tau\tau} \tau_\lambda), \quad (3.3)$$

and

$$\begin{aligned} U_\lambda^\lambda &= f''(\Omega_\tau) (\Omega_{\tau\lambda} + \Omega_{\tau\tau} \tau_\lambda) (\Omega_\tau^\lambda + \Omega_{\tau\tau} \tau^\lambda) \\ &\quad + f'(\Omega_\tau) (\Omega_{\tau\lambda}^\lambda + 2 \Omega_{\tau\tau\lambda} \tau^\lambda + \Omega_{\tau\tau\tau} \tau_\lambda \tau^\lambda + \Omega_{\tau\tau} \tau_\lambda^\lambda) \\ &= f''(\Omega_\tau) (\Omega_{\tau\lambda} \Omega_\tau^\lambda + 2 \Omega_{\tau\tau} \Omega_\tau^\lambda \tau_\lambda) + f'(\Omega_\tau) (\Omega_{\tau\lambda}^\lambda + 2 \Omega_{\tau\tau\lambda} \tau^\lambda + \Omega_{\tau\tau} \tau_\lambda^\lambda), \end{aligned}$$

using equation (2.9).

By (2.4), (2.5), (2.6), (2.8) and (2.10), it quickly follows that

$$U_\lambda^\lambda = -f''(\Omega_\tau) (\Omega_{\tau\tau\lambda} \Omega^\lambda + \Omega_{\tau\tau}) - \Omega_\tau^{-1} f'(\Omega_\tau) \{2 \Omega_{\tau\tau\lambda} \Omega^\lambda + (n-2) \Omega_{\tau\tau}\}. \quad (3.4)$$

By (2.12), if the functions $\bar{x}^i(\tau)$ satisfy the relations

$$\bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} = 0, \quad (3.5)$$

$$\frac{d^2 \bar{x}^\mu}{d\tau^2} + \bar{\Gamma}_{\alpha\beta}^{\mu} \frac{d\bar{x}^\alpha}{d\tau} \frac{d\bar{x}^\beta}{d\tau} = 0, \quad (3.6)$$

then

$$\Omega_{\tau\tau} = 0. \quad (3.7)$$

Differentiating with respect to x^λ ,

$$\Omega_{\tau\tau\lambda} + \Omega_{\tau\tau\tau} \tau_\lambda = 0,$$

and therefore

$$\Omega_{\tau\tau\lambda} \Omega^\lambda = 0, \quad (3.8)$$

since $\Omega^\lambda \tau_\lambda = 0$ by (2.4) and (2.9).

By (3.4), (3.7), (3.8) it follows that

$$U_\lambda^\lambda = 0$$

provided that the conditions (3.5) and (3.6) are satisfied. This concludes the proof of Theorem I.

§ 4. PROOF OF THEOREM II.

The next problem is to show that, if

$$V = \phi(\tau) / \Omega_\tau^{\frac{1}{2}(n-2)}, \quad (4.1)$$

where $\phi(\tau)$ is an arbitrary function of τ , then V satisfies the equation $V_\lambda^\lambda = 0$. It will be shown that the restrictions (3.5) and (3.6) must still be imposed except when $n = 2$ or $n = 4$.

Consider first the function W defined by

$$W = \Omega_\tau^{-\frac{1}{2}(n-2)}. \quad (4.2)$$

Putting $f(\Omega_\tau) = \Omega_\tau^{-\frac{1}{2}(n-2)}$ in (3.3) and (3.4),

$$W_\lambda = -\frac{1}{2}(n-2) \Omega_\tau^{-\frac{1}{2}n} (\Omega_{\tau\lambda} + \Omega_{\tau\tau} \tau_\lambda), \quad (4.3)$$

$$W_\lambda^\lambda = \frac{1}{4}(n-2)(n-4) \Omega_\tau^{-\frac{1}{2}(n+2)} (\Omega_{\tau\tau} - \Omega_{\tau\tau\lambda} \Omega^\lambda). \quad (4.4)$$

Hence, if $n = 2$ or $n = 4$, we have

$$W_\lambda^\lambda = 0 \quad (4.5)$$

without any restrictions being placed on the choice of the functions $\bar{x}^i(\tau)$. But, if n has neither of these values, it follows as in the previous paragraph that we shall still have

$$W_\lambda^\lambda = 0 \quad (4.5)^*$$

provided that the $\bar{x}^i(\tau)$ satisfy the relations

$$\bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} = 0, \quad (4.6)$$

$$\frac{d^2\bar{x}^\mu}{d\tau^2} + \bar{\Gamma}_{\alpha\beta}^\mu \frac{d\bar{x}^\alpha}{d\tau} \frac{d\bar{x}^\beta}{d\tau} = 0. \quad (4.7)$$

Now consider the function V defined by

$$V = W\phi(\tau), \quad (4.8)$$

where $\phi(\tau)$ is any function of τ . Covariant differentiation gives

$$\begin{aligned} V_\lambda &= W_\lambda \phi(\tau) + W\tau_\lambda \phi'(\tau), \\ V_\lambda^\lambda &= W_\lambda^\lambda \phi(\tau) + 2W_\lambda \tau^\lambda \phi'(\tau) + W\tau_\lambda \tau^\lambda \phi''(\tau) + W\tau_\lambda^\lambda \phi'(\tau) \\ &= \phi'(\tau) (2W_\lambda \tau^\lambda + W\tau_\lambda^\lambda), \end{aligned}$$

by (4.5), (4.5)* and (2.9).

Equation (4.8) therefore gives a solution of $V_\lambda^\lambda = 0$ provided that

$$2W_\lambda \tau^\lambda + W\tau_\lambda^\lambda = 0.$$

But by (4.3), (4.2) and (2.10), the left-hand side of this equation is equal to

$$-(n-2)\Omega_\tau^{-\frac{1}{2}n}(\Omega_{\tau\lambda}\tau^\lambda + \Omega_{\tau\tau}\tau_\lambda\tau^\lambda) - (n-2)\Omega_\tau^{-\frac{1}{2}n},$$

which is zero in virtue of equations (2.4), (2.5) and (2.9), for all values of n . We deduce finally, therefore, that

$$V = \phi(\tau) / \Omega_\tau^{\frac{1}{2}(n-2)}$$

is a solution of $V_\lambda^\lambda = 0$ provided that, when n has a value other than 2 or 4, the choice of the functions $x^i(\tau)$ is restricted by the equations (4.6) and (4.7).

When $n=2$ this theorem is the tensor generalisation of the well-known fact that any function of $x \pm iy$ is a solution of the equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$. When $n=4$ it gives a generalisation of a solution, due to Conway, of the classical wave-equation of mathematical physics.¹

§ 5. CONNECTION WITH WHITTAKER'S SOLUTION OF LAPLACE'S EQUATION.

Apply Theorem I to the case when $n=3$ and the metric is given by

$$ds^2 = dx^2 + dy^2 + dz^2; \quad (x^1 = x, x^2 = y, x^3 = z).$$

¹ See Bateman, "Electrical and Optical Wave Motion" (1915), 115.

The equation to be solved is then the ordinary Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{5.1}$$

Also, of course,

$$2 \Omega = (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2. \tag{5.2}$$

The restrictions (4.6) and (4.7) placed on the choice of \bar{x} , \bar{y} , \bar{z} as functions of τ reduce in this case to

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 + \left(\frac{d\bar{y}}{d\tau}\right)^2 + \left(\frac{d\bar{z}}{d\tau}\right)^2 = 0, \tag{5.3}$$

and
$$\frac{d^2 \bar{x}}{d\tau^2} = 0 = \frac{d^2 \bar{y}}{d\tau^2} = \frac{d^2 \bar{z}}{d\tau^2}. \tag{5.4}$$

The most general solutions of (5.3) and (5.4) are

$$\left. \begin{aligned} \bar{x} &= a + \lambda i \tau \cos u \\ \bar{y} &= b + \lambda i \tau \sin u \\ \bar{z} &= c + \lambda \tau \end{aligned} \right\} \tag{5.5}$$

where $i = \sqrt{-1}$ and a, b, c, λ, u are arbitrary constants. Take $a = b = c = 0$ and $\lambda = -1$. Substituting from (5.5) in (5.2), we have

$$2 \Omega = r^2 + 2\tau (ix \cos u + iy \sin u + z),$$

where
$$r^2 = x^2 + y^2 + z^2,$$

and hence
$$\Omega_\tau = ix \cos u + iy \sin u + z.$$

A solution of equation (5.1) is therefore, by Theorem I,

$$V = f(ix \cos u + iy \sin u + z, u), \tag{5.6}$$

where f is an arbitrary function and u an arbitrary constant.¹

It follows therefore that

$$\int f(ix \cos u + iy \sin u + z, u) du \tag{5.7}$$

is also a solution of (5.1), provided that the limits of integration are such that differentiation under the integral sign is permissible.

¹Since u is an arbitrary constant, the function f of the two arguments $ix \cos u + iy \sin u + z$ and u , is (regarded as a function of x, y, z), an arbitrary function of the former argument only; that is, of $\partial\Omega/\partial\tau$ only.

Whittaker has shown¹ that the most general solution of Laplace's Equation is of this form.

An application of Theorem II to the same special case leads ultimately to the conclusion that the integral

$$\int \frac{1}{r} \psi \left(\frac{ix \cos u + iy \sin u + z}{r^2}, u \right) du \quad (5.8)$$

gives a solution of (5.1), ψ being an arbitrary function of its arguments. It is however a well known fact that if a function $\chi(x, y, z)$ satisfies Laplace's Equation, so also does $\frac{1}{r} \chi \left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right)$. The solution (5.8) is therefore deducible from (5.7).

¹ Whittaker and Watson, *loc. cit.*

CHAPTER $\overline{\text{IV}}$, Paper 3.

THE POTENTIAL OF AN ELECTRON IN A SPACE-TIME OF CONSTANT CURVATURE*

By H. S. RUSE

[Received 14 May 1930]

§ 1. Introduction

THE present paper contains a solution of the equations of electromagnetism as they existed before the introduction of Einstein's Unified Field-Theory.†

Let $X_{\mu\nu}$ denote the electromagnetic six-vector in a space-time specified by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3).$$

Then it is well known that Maxwell's equations may be reduced to the simple form

$$X_{\mu\nu} = \frac{\partial \phi_\mu}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\mu}, \quad (1.1)$$

and

$$X_{\nu}^{\mu\nu} = J^\mu, \quad (1.2)$$

where ϕ_μ is the electromagnetic potential-vector, J^μ is the charge-and-current vector, and $X_{\nu}^{\mu\nu}$ is the divergence of $X^{\mu\nu}$.

Substituting from (1.1) in (1.2), we are quickly led to the familiar equations

$$g^{\alpha\beta} \phi_{\mu\alpha\beta} - (\phi_\nu)_{,\mu} + G_\mu^\epsilon \phi_\epsilon = J_\mu, \quad (1.3)$$

where G_μ^ϵ denotes the gravitational tensor, and all suffixes not otherwise defined denote covariant or contravariant differentiations.

It is usual to impose the conventional divergence-condition $\phi_\nu^{\cdot\nu} = 0$; but this we shall replace by the condition

$$\phi_\nu^{\cdot\nu} = \text{constant}. \quad (1.4)$$

At a point of space-time where there are no electrons, $J_\mu = 0$, and equations (1.3) reduce, in virtue of (1.4), to

$$g^{\alpha\beta} \phi_{\mu\alpha\beta} + G_\mu^\epsilon \phi_\epsilon = 0. \quad (1.5)$$

We propose to solve the partial differential equations (1.4) and (1.5), expressing the solution in tensor form, for the case of an electron in a space-time of constant curvature K . For this purpose we ignore the distortion of the metric due to the gravitational effect

* The term 'electron' is used in this paper in a purely conventional sense, and is defined to mean an ideal point charge.

† This was written at a time when the Unified Theory seemed likely to gain a wider acceptance than has actually proved to be the case.

of the electron itself, this effect being in general small.* By so doing we are enabled to regard the components of the fundamental tensor $g_{\mu\nu}$ as given functions of the co-ordinates, so that we have a determinate set of partial differential equations to solve.

The solution of these equations for an electron in a *flat* space-time (i.e. a space-time in which the Riemann-Christoffel Tensor is everywhere zero) has already been found,† the result being enunciated as follows:

Let an electron e be moving in any manner, and let τ be its proper-time at the point where its world-line intersects the null-cone of any point P . Then the electromagnetic potential-vector at P , due to the electron e , is

$$\left. \begin{aligned} \phi_p &= \frac{e}{c} g^{qr} \sigma_{pqr}, \\ \sigma_p &= \tau_p |g^{lm} \tau_{lm}, \end{aligned} \right\} \quad (1.6)$$

where

and where the subscripts p, q, r, l, m on the right-hand side represent covariant differentiations.

It is shown that, when space-time is Galilean, the formulae (1.6) give the classical formulae for the scalar and vector-potential of a moving electron. By an application to special cases, it can, however, be shown that they do *not* give a solution of the equations (1.4) and (1.5) when the Riemann-Christoffel Tensor is not everywhere zero.

The formula which we seek to establish for a space-time of constant curvature may be regarded as a generalization of (1.6); but, as we shall show, it is in fact much simpler, and gives as a special case a new formula for the potential of an electron in a flat space-time, which is, however, directly transformable into (1.6).

Before proceeding to establish the theorem (§ 3) which forms the main subject-matter of this paper, it is necessary to prove a theorem in the Tensor Calculus.

§ 2. Theorem

Let $(x^0, x^1, x^2, \dots, x^{n-1})$ be generalized co-ordinates in an n -dimensional space of constant curvature K whose metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, \dots, n-1).$$

Then if $B_{\mu\rho, \sigma\nu}$ be the Riemann-Christoffel Tensor, we shall have

$$B_{\mu\rho, \sigma\nu} = K(g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}),$$

* E. T. Whittaker, *Proc. Roy. Soc., A*, 116 (1927), 720, § 1.

† *Ibid.* 120 (1928), 1

and hence also

$$G_{\mu\nu} = g^{\rho\sigma} B_{\mu\rho, \sigma\nu} = -(n-1)K g_{\mu\nu}. \quad (2.1)$$

Let $(\bar{x}^0, \bar{x}^1, \dots, \bar{x}^{n-1})$ be a fixed point of the space, and $(x^0, x^1, \dots, x^{n-1})$ a variable point. Let s denote the length of the arc of the geodesic (assumed unique) joining these points, so that s is a function of the x^i .

Let Q denote the scalar

$$Q = \cos(K^{\frac{1}{2}}s). \quad (2.2)$$

Then we shall prove that

Q satisfies the partial differential equations

$$Q_{\mu} Q^{\mu} = K(1-Q^2), \quad (2.3)$$

and

$$Q_{\mu\nu} = -K g_{\mu\nu} Q, \quad (\mu, \nu = 0, 1, 2, \dots, n-1), \quad (2.4)$$

where the suffixes on the left-hand side denote covariant or contravariant differentiations, and where (as always) repeated suffixes imply summations.

The proof of (2.3) is simple if we quote the well-known theorem that the arc s of a geodesic satisfies the partial differential equation

$$s_{\mu} s^{\mu} = 1. \quad (2.5)$$

For since

$$s = K^{-\frac{1}{2}} \arccos Q,$$

we have

$$s_{\mu} = -K^{-\frac{1}{2}} \cdot (1-Q^2)^{-\frac{1}{2}} \cdot Q_{\mu},$$

with a similar formula for s^{μ} .

Substitution in (2.5) immediately gives (2.3).

For the proof of (2.4) we take as the canonical form of the metric of a space of constant curvature that defined by

$$ds^2 = -K^{-1}(x^0)^{-2}[(dx^0)^2 + (dx^1)^2 + \dots + (dx^{n-1})^2]. \quad (2.6)$$

By a real or imaginary transformation, the metrics of all such spaces can be reduced to this form.

Suspending temporarily the summation-convention, we have

$$\begin{aligned} g_{\mu\mu} &= -1/K(x^0)^2, & g_{\mu\nu} &= 0, \\ g^{\mu\mu} &= -K(x^0)^2, & g^{\mu\nu} &= 0, \quad (\mu \neq \nu; \mu, \nu = 0, 1, \dots, n-1). \end{aligned}$$

The only non-vanishing Christoffel symbols are

$$\begin{aligned} \{00, 0\} &= -1/x^0; \quad \{\mu\mu, 0\} = 1/x^0; \quad \{0\mu, \mu\} = \{\mu 0, \mu\} = -1/x^0, \\ &(\mu = 1, 2, \dots, n-1, \text{ but } \mu \neq 0). \end{aligned}$$

Substituting in the equations of the geodesics, namely,

$$\frac{d^2 x^{\mu}}{ds^2} + \{\alpha\beta, \mu\} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = 0,$$

(the summation-convention being now restored), and integrating, we get ultimately the formula

$$s = K^{-1/2} \arccos \left[\frac{\left\{ (x^0)^2 + (\bar{x}^0)^2 + \sum_{\mu=1}^{n-1} (x^\mu - \bar{x}^\mu)^2 \right\}}{2x^0\bar{x}^0} \right]$$

for the geodesic distance between the points (x^i) and (\bar{x}^i) .

Hence for this metric,

$$Q = \frac{1}{2x^0\bar{x}^0} \left\{ (x^0)^2 + (\bar{x}^0)^2 + \sum (x^\mu - \bar{x}^\mu)^2 \right\}. \tag{2.7}$$

It is now a matter of straightforward differentiation and substitution to prove that

$$Q_{\mu\nu} = \frac{\partial^2 Q}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\} \frac{\partial Q}{\partial x^\alpha}, = -K g_{\mu\nu} Q$$

for

$$\mu, \nu = 0, 1, 2, \dots, n-1.$$

And since this is a relation between tensors, it must be true for all spaces derivable from (2.6) by point-transformations, i.e. for all spaces of constant curvature K .

§ 3. Solution of the partial differential equations of electromagnetism

Consider now a space-time of constant curvature K whose metric is specified by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3), \tag{3.1}$$

x^0 being the 'time' co-ordinate.

Let the null-cone of the point (x^i) cut the world-line of an electron e in the point (\bar{x}^i) ; so that, in the language of the older physics, the electron was at the point $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ at the time \bar{x}^0 when radiation left it to arrive at the point (x^1, x^2, x^3) at time x^0 .

Suppose the motion of the electron to be given. Then we shall know $\bar{x}^1, \bar{x}^2, \bar{x}^3$ as functions of the 'time' \bar{x}^0 , say

$$\bar{x}^r = f_r(\bar{x}^0), \quad (r = 1, 2, 3). \tag{3.2}$$

Let τ be the proper-time of the electron at the point (\bar{x}^i) ; then since

$$d\tau^2 = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu,$$

$\bar{g}_{\mu\nu}$ being the value of $g_{\mu\nu}$ at (\bar{x}^i) , we have, on substituting from (3.2),

$$d\tau^2 = (d\bar{x}^0)^2 \times \text{a function of } \bar{x}^0.$$

Integration of this equation gives \bar{x}^0 as a function of τ , and hence, by (3.2), $\bar{x}^1, \bar{x}^2, \bar{x}^3$ are also given as functions of τ ; thus

$$\bar{x}^r = \bar{x}^r(\tau), \quad (r = 0, 1, 2, 3). \tag{3.3}$$

Let $s(x^i; \xi^i)$ be the geodesic distance between the points (x^i) and (ξ^i) .

Then, regarding the ξ^i as current co-ordinates, the equation of the null-cone of (x^i) is

$$s(x^i; \xi^i) = 0.$$

Since this passes through (\bar{x}^i) , we have

$$s(x^i; \bar{x}^i) = 0.$$

Hence

$$\cos\{K^{\frac{1}{2}}s(x^i; \bar{x}^i)\} = 1,$$

that is,

$$Q = 1, \quad (3.4)$$

Q being the function defined in § 2. This equation, combined with (3.3), expresses τ as a function of x^0, x^1, x^2, x^3 .

It should be noticed (i) that Q is a function of the x^i and of the \bar{x}^i ; (ii) that in virtue of (3.3) it is therefore a function of the x^i and of τ .

We shall now show how to prove the following theorem:

The potential of the electron e (whose motion is specified) is given by

$$\phi_\mu = -\frac{e}{c} \frac{\partial}{\partial x^\mu} \left[\log \left(\frac{\partial Q}{\partial \tau} \right) \right], \quad (3.5)$$

where (i) c is a constant; (ii) the partial differentiation with respect to x^μ treats τ as a constant, i.e. ignores the fact that τ is given as a function of the x^i by the equation (3.4); (iii) τ is eliminated after the performance of the differentiations by means of (3.4).

Our justification for describing this as the potential of an electron is that in the particular case when $K = 0$ and the co-ordinates are Galilean, (3.5) reduces to the classical formula for the potential of a moving electron.

Before indicating the general proof of the theorem, it will perhaps be as well to apply the formula to a particular case.

Consider an electron at rest* in the space-time specified by

$$ds^2 = (1 + \mu x)^{-2} \left[dt^2 - \frac{1}{c^2} (dx^2 + dy^2 + dz^2) \right],$$

$$(x^0 = t, x^1 = x, x^2 = y, x^3 = z),$$

μ being a constant. This is a space-time of constant curvature.

Since the electron is at rest,

$$\bar{x} = 0 = \bar{y} = \bar{z}; \quad (3.6)$$

hence

$$d\tau^2 = (1 + \mu \bar{x})^{-2} \left[d\bar{t}^2 - \frac{1}{c^2} (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) \right]$$

$$= d\bar{t}^2,$$

* A certain confusion of language arises on account of the admixture of the ideas of the older physics with those of the newer.

and we may therefore take $\bar{t} = \tau$, (3.7)

For this space [cf. (2.7)],

$$Q = \frac{1}{2}\{(1+\mu x)(1+\mu \bar{x})\}^{-1}[(1+\mu x)^2 + (1+\mu \bar{x})^2 - \mu^2\{c^2(t-\bar{t})^2 - (y-\bar{y})^2 - (z-\bar{z})^2\}]$$

$$= \frac{1}{2(1+\mu x)}[(1+\mu x)^2 + 1 - \mu^2\{c^2(t-\tau)^2 - y^2 - z^2\}],$$

by (3.6) and (3.7).

Hence
$$\frac{\partial Q}{\partial \tau} = \mu^2 c^2 (t-\tau) / (1+\mu x),$$

and therefore

$$\phi_0 = -\frac{e}{c} \frac{\partial}{\partial t} \left[\log \left(\frac{\partial Q}{\partial \tau} \right) \right] = -e/c(t-\tau), \quad (3.8)$$

$$\phi_1 = -\frac{e}{c} \frac{\partial}{\partial x} \left[\log \left(\frac{\partial Q}{\partial \tau} \right) \right] = e\mu/c(1+\mu x), \quad (3.9)$$

$$\phi_2 = \phi_3 = 0. \quad (3.10)$$

Now τ is obtained as a function of (x, y, z) by the equation $Q = 1$; this gives

$$x^2 + y^2 + z^2 = c^2(t-\tau)^2,$$

and thus $\tau = t - r/c$, where $r^2 = x^2 + y^2 + z^2$.

Substituting in (3.8), we get

$$\phi_0 = -e/r, \quad \phi_1 = e\mu/c(1+\mu x), \quad \phi_2 = 0, \quad \phi_3 = 0.$$

It is a matter of straightforward algebra to show that this is indeed a solution of equations (1.4) and (1.5) for this particular metric.

It will be seen that the constant c has the dimensions of a velocity, and can (in the case considered) be interpreted as that of light.

§ 4. Proof of the theorem

A complete proof of the theorem of the last paragraph cannot be given on account of its length; but the following should serve to indicate the method adopted.

Ignoring the multiplicative constant $-e/c$, we have to show that the vector defined by

$$\phi_\mu = \frac{\partial}{\partial x^\mu} \left[\log \left(\frac{\partial Q}{\partial \tau} \right) \right] \quad (4.1)$$

satisfies the partial differential equations

$$g^{\alpha\beta} \phi_{\mu\alpha\beta} + G_\mu^\epsilon \phi_\epsilon = 0, \quad (4.2)$$

$$\phi_\mu^\mu = \text{constant}. \quad (4.3)$$

This may be done by direct substitution of (4.1) in the last two equations, making use of the results of § 2, namely,

$$Q_{\mu}Q^{\mu} = K(1-Q^2), \quad (4.4)$$

and

$$Q_{\mu\nu} = -Kg_{\mu\nu}Q. \quad (4.5)$$

A certain confusion may arise on account of the fact that Q is a function of both the x^i and of τ , where τ itself is a function of the x^i defined by the equation

$$Q = 1. \quad (4.6)$$

Covariant differentiations of Q with respect to the x^i which treat τ as a constant will be denoted by the simple addition of suffixes. Thus

$$Q_{\mu} = \frac{\partial Q}{\partial x^{\mu}}, \quad (\tau \text{ constant}).$$

Covariant differentiations which treat τ as a function of the x^i defined by (4.6) will be indicated by the bracketing* of the corresponding suffix; thus

$$Q_{(\mu)} = \frac{\partial Q}{\partial x^{\mu}}, \quad (\tau \text{ not constant}).$$

Hence

$$\begin{aligned} Q_{(\mu)} &= Q_{\mu} + \frac{\partial Q}{\partial \tau} \frac{\partial \tau}{\partial x^{\mu}} \\ &= Q_{\mu} + Q_{\tau} \tau_{\mu}, \end{aligned} \quad (4.7)$$

where the suffix τ denotes partial differentiation with respect to τ .

With this notation we have to show that the vector ϕ_{μ} defined by

$$\phi_{\mu} = Q_{\tau\mu} / Q_{\tau}, \quad (4.8)$$

where $Q_{\tau\mu} = \frac{\partial^2 Q}{\partial x^{\mu} \partial \tau} = \frac{\partial^2 Q}{\partial \tau \partial x^{\mu}}$, satisfies the equations

$$g^{\alpha\beta} \phi_{\mu(\alpha)(\beta)} + G_{\mu}^{\epsilon} \phi_{\epsilon} = 0 \quad (4.9)$$

and

$$g^{\mu\alpha} \phi_{\mu(\alpha)} = \text{constant}. \quad (4.10)$$

Equations (4.4), (4.5), (4.6) remain unaltered.

Differentiating (4.4) twice with respect to τ , we get

$$Q_{\tau\mu} Q^{\mu} = -K Q Q_{\tau}, \quad (4.11)$$

$$Q_{\tau\tau\mu} Q^{\mu} + Q_{\tau\mu} Q_{\tau}^{\mu} = -K Q Q_{\tau\tau} - K Q_{\tau}^2. \quad (4.12)$$

In (4.11) use is made of the fact that $Q_{\mu} Q_{\tau}^{\mu} = Q^{\mu} Q_{\tau\mu}$. Differentiating (4.5) with respect to τ , we have

$$Q_{\tau\mu\nu} = -K g_{\mu\nu} Q_{\tau}. \quad (4.13)$$

* It may be noticed that bracketed suffixes in this paper have a different meaning from that which they have elsewhere.

Also, differentiating (4.6) with respect to x^μ , we have

$$Q_{(\mu)} = 0,$$

and hence, by (4.7), $\tau_\mu = -Q_\mu/Q_\tau$. (4.14)

Finally, on raising the suffix ν in (4.13) and contracting,

$$Q_{\tau\mu}^\mu = -4KQ_\tau. \tag{4.15}$$

Now $\phi_{\mu(\alpha)} = \phi_{\mu\alpha} + \frac{\partial\phi_\mu}{\partial\tau}\tau_\alpha$

$$= (Q_{\tau\mu}/Q_\tau)_\alpha + (Q_{\tau\mu}/Q_\tau)_{\tau\alpha}, \text{ by (4.8),}$$

$$= Q_\tau^{-1}(Q_{\tau\mu\alpha} + Q_{\tau\tau\mu}\tau_\alpha) - Q_\tau^{-2}(Q_{\tau\mu}Q_{\tau\alpha} + Q_{\tau\mu}Q_{\tau\tau}\tau_\alpha).$$

On multiplying by $g^{\mu\alpha}$, summing for μ and α , and using (4.14),

$$g^{\mu\alpha}\phi_{\mu(\alpha)} = Q_\tau^{-1}Q_{\tau\mu}^\mu - Q_\tau^{-2}(Q_{\tau\tau\mu}Q^\mu + Q_{\tau\mu}Q_\tau^\mu) + Q_\tau^{-3}Q_{\tau\mu}Q^\mu Q_{\tau\tau}.$$

Using (4.15), (4.12), and (4.11) we at once get

$$g^{\mu\alpha}\phi_{\mu(\alpha)} = -3K = \text{constant},$$

so that ϕ_μ , thus defined, satisfies (4.10).

By a similar process, using (2.1) (with $n = 4$), and remembering that we may put $Q = 1$ after all differentiations have been performed, it can be shown that the expression given for ϕ_μ also satisfies (4.9). The work is lengthy, but straightforward if the distinction between bracketed and unbracketed suffixes is borne in mind.

§ 5. The case $K = 0$

We now deduce a formula which is valid for a flat space-time, i.e. a space-time of zero curvature.

Let P be the scalar defined by the equation

$$P = (1 - Q)/K \\ = 2\{\sin^2(\frac{1}{2}K^{\frac{1}{2}}s)\}/K. \tag{5.1}$$

Then

$$\lim_{K \rightarrow 0} P = \frac{1}{2}s^2 \\ = \Omega, \tag{5.2}$$

where Ω denotes one-half the square of the geodesic distance (in a flat space) between the points (x^i) and (\bar{x}^i) .

By (5.1), $Q = 1 - KP$, (5.3)

and substitution in (4.4) and (4.5) shows that P satisfies the equations

$$P_\mu P^\mu = 2P - KP^2, \tag{5.4}$$

$$P_{\mu\nu} = g_{\mu\nu}(1 - KP). \tag{5.5}$$

Making $K \rightarrow 0$, and assuming that

$$\lim_{K \rightarrow 0} \frac{\partial P}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \left(\lim_{K \rightarrow 0} P \right),$$

we find that the scalar Ω as above defined satisfies the partial differential equations*

$$\Omega_\mu \Omega^\mu = 2\Omega, \tag{5.6}$$

$$\Omega_{\mu\nu} = g_{\mu\nu}. \tag{5.7}$$

Furthermore, (4.6) becomes

$$\Omega = 0. \tag{5.8}$$

Also, substituting from (5.3) in (3.5), and proceeding to the limit,

$$\phi_\mu = \lim_{K \rightarrow 0} [-eP_{\tau\mu}/cP_\tau]$$

in the notation of § 4.

Hence, for a flat space,

$$\phi_\mu = -e\Omega_{\tau\mu}/c\Omega_\tau,$$

which may be written

$$\phi_\mu = -\frac{e}{c} \frac{\partial}{\partial x^\mu} \left[\log \frac{\partial \Omega}{\partial \tau} \right]. \tag{5.9}$$

With the help of (5.6), (5.7), (5.8), this formula can be transformed directly into (1.6). It has the advantage of being considerably simpler than (1.6), since its application involves only simple partial differentiations instead of an elaborate series of covariant differentiations.

Although we have deduced (5.9) by a limiting process from (3.5), it can nevertheless be established independently by a method similar to that used in proving (3.5).

Application.

Apply (5.9) to the case in which the co-ordinates are Galilean, that is, when the metric is given by

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2).$$

Then, of course,

$$\Omega = \frac{1}{2} \left[(t-\bar{t})^2 - \frac{1}{c^2} \{ (x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2 \} \right].$$

* These equations are established independently in Chapter II, Paper I.

Hence

$$\begin{aligned}\frac{\partial \Omega}{\partial \tau} &= \frac{\partial \Omega}{\partial \bar{t}} \frac{d\bar{t}}{d\tau} + \frac{\partial \Omega}{\partial \bar{x}} \frac{d\bar{x}}{d\tau} + \frac{\partial \Omega}{\partial \bar{y}} \frac{d\bar{y}}{d\tau} + \frac{\partial \Omega}{\partial \bar{z}} \frac{d\bar{z}}{d\tau} \\ &= -(t-\bar{t})v_0 + \frac{1}{c^2} \{(x-\bar{x})v_1 + (y-\bar{y})v_2 + (z-\bar{z})v_3\} \\ &= \psi, \text{ say,}\end{aligned}$$

where $v_0 = \frac{d\bar{t}}{d\tau}, \quad v_1 = \frac{d\bar{x}}{d\tau}, \quad v_2 = \frac{d\bar{y}}{d\tau}, \quad v_3 = \frac{d\bar{z}}{d\tau}.$

Hence, by (5.9),

$$\phi_0 = -\frac{e}{c} \frac{\partial}{\partial t} (\log \psi) = \frac{ev_0}{c\psi}, \quad (5.10)$$

with similar expressions for $\phi_1, \phi_2, \phi_3.$

Now τ is given as a function of x, y, z by the equation $\Omega = 0,$ which gives the familiar relation

$$\bar{t} = t - \bar{r}/c,$$

where $\bar{r}^2 = (x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2.$

Hence $\psi = -\left[\frac{\bar{r}}{c} v_0 - \frac{1}{c^2} \{(x-\bar{x})v_1 + (y-\bar{y})v_2 + (z-\bar{z})v_3\} \right].$

Substituting in (5.10), and writing $v_x = \frac{v_1}{v_0} = \frac{dx}{d\bar{t}},$ etc., we get

$$\phi_0 = -e / \left[\bar{r} - \frac{1}{c} \{(x-\bar{x})v_x + (y-\bar{y})v_y + (z-\bar{z})v_z\} \right],$$

with similar expressions for $\phi_1, \phi_2, \phi_3.$ This is the classical formula for the potential of an electron which was moving with the velocity (v_x, v_y, v_z) at the point $(\bar{x}, \bar{y}, \bar{z})$ at time $\bar{t},$ the radiation then emitted arriving at (x, y, z) at time $t.$

CHAPTER IV , Paper 4.

GENERALIZED SOLUTIONS OF SOME OF THE
PARTIAL DIFFERENTIAL EQUATIONS OF
MATHEMATICAL PHYSICS

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THE present paper contains an expression in tensor form of some theorems due to A. R. Forsyth* concerning solutions of the partial differential equation

$$\nabla^2 V + \kappa^2 V = 0, \quad (1)$$

where ∇^2 is the Laplacian operator $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ in n variables,

and κ is a constant. The theorems are given in § 2. Equation (1) includes a number of equations important in mathematical physics; for example, if $n = 4$, $\kappa = 0$ and $ct\sqrt{-1}$ is written for x_4 , the classical wave-equation is obtained.

§ 1 contains some general remarks about the ideas underlying the subsequent work; §§ 3 and 4 contain the generalizations of Forsyth's theorems, §§ 5 and 6 being devoted to proofs.

1. Tensor differential equations

The absolute differential calculus of Ricci and Levi-Civita has proved a powerful instrument in the development of a number of branches of pure mathematics, a fact which suggests the desirability of developing an 'absolute integral calculus'. This idea leads one to inquire whether it is possible to define a unique operation inverse to that of covariant differentiation. The answer is fairly obviously in the negative, since an attempt to define such an operation leads almost immediately to the necessity of solving partial differential equations.

The most fruitful way of beginning the development of an absolute integral calculus would therefore seem to be to study tensor differential equations and their solutions. Perhaps the simplest of such differential equations are those involving the tensor generalization of the Laplacian operator ∇^2 , the easiest method of finding solutions

* *Messenger of Math.*, 27 (1897), 99.

being to express in tensor form solutions already known for the non-tensorized equation. This is in effect the method adopted below.

The method has, however, certain serious limitations. The expression of formulae in tensor form implies the introduction of an associated riemannian geometry, and therefore the selection of an appropriate metric. As a rule it is necessary in the first place to choose the metric of a flat space, further investigation being required to show whether the solutions obtained are valid in a non-flat space. This investigation is often a matter of no little difficulty. Indeed, in the case of the theorems of the present paper, it has been possible to obtain results only for flat spaces and for spaces of constant curvature.

The difficulty arises in this way. In the differential equation $\nabla^2 V + \kappa^2 V = 0$ the variables are assumed to be rectangular cartesian, the associated metric being therefore defined by

$$ds^2 = dx_1^2 + dx_2^2 + \dots + dx_n^2.$$

Now rectangular cartesian coordinates are a special case of the so-called normal coordinates of general riemannian geometry. But if $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ defines the metric of a general riemannian space, then the Riemann-Weblen normal coordinates (y^μ) (which in a flat space are cartesian, though in general non-rectangular) are defined as functions of the coordinates (x^μ) by the equation*

$$y^\mu = -\bar{g}^{\mu\nu} \frac{\partial \Omega}{\partial \bar{x}^\nu},$$

where Ω is one-half of the square of the geodesic distance between a fixed point (\bar{x}^μ) and the variable point (x^μ), and $\bar{g}^{\mu\nu}$ is the value of $g^{\mu\nu}$ at (\bar{x}^μ). Consequently any generalization to tensors of a solution of the given partial differential equation will almost inevitably involve the derivatives of the function Ω thus defined. Now in the case of a flat space this function possesses certain very simple properties, similar ones being enjoyed† in a space of constant curvature K by the function $Q = \cos(2K\Omega)^{\frac{1}{2}}$. But no such simple properties appear to exist in the case of a general riemannian space, and hence arises the difficulty of obtaining completely generalized solutions of the differential equations.

* Page 18, Equation (13).

† Chapter IV, Paper 3 (Page 64 et seq.)

2. Forsyth's theorems

The principal results of the paper quoted above may be summarized as follows:

Let p_1, p_2, \dots, p_n denote n arbitrary functions of a variable u , subject to the single condition

$$p_1^2 + p_2^2 + \dots + p_n^2 = 0, \quad (2.1)$$

and let u be determined as a function of n independent variables x_1, x_2, \dots, x_n by the equation

$$au = x_1 p_1 + x_2 p_2 + \dots + x_n p_n, \quad (2.2)$$

where a is any constant. Let also

$$\eta = x_1 p'_1 + x_2 p'_2 + \dots + x_n p'_n, \quad (2.3)$$

$$\theta = p_1'^2 + p_2'^2 + \dots + p_n'^2, \quad (2.4)$$

the accents denoting differentiation with respect to u . Then

I. If $f(u)$ is an arbitrary function of u ,

$$V = f(u) \quad (2.5)$$

is a solution of the partial differential equation

$$\nabla^2 V = 0, \quad (2.6)$$

where, as above, ∇^2 is the Laplacian operator $\sum_1^n \frac{\partial^2}{\partial x_r^2}$.

II. If $\phi(u)$ and $\psi(u)$ are arbitrary functions of u , then

$$V = \frac{\phi(u)}{a-\eta} \exp\left(\frac{i\kappa\eta}{\theta}\right) + \frac{\psi(u)}{a-\eta} \exp\left(-\frac{i\kappa\eta}{\theta}\right) \quad (2.7)$$

is a solution of the equation

$$\nabla^2 V + \kappa^2 V = 0. \quad (2.8)$$

When $\kappa = 0$ it follows that, if $\chi(u)$ is an arbitrary function of u , then $V = \chi(u)(a-\eta)^{-1}$ is a solution of $\nabla^2 V = 0$, and since this equation is linear it follows by I that $V = f(u) + \chi(u)(a-\eta)^{-1}$ is also a solution. A remark similar to this applies to the formulae obtained below, which are a generalization of I and II.

The truth of the above theorems, the proofs of which are remarkably simple in view of their great generality, will be assumed in the following paragraphs.

3. Tensor formulae for a flat space

$$\text{Let} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (3.1)$$

define the metric of an n -dimensional flat (euclidean) space.* Then the partial differential equation of which solutions are sought is

$$g^{\mu\nu} V_{\mu\nu} + \kappa^2 V = 0, \quad (3.2)$$

$$\text{where} \quad V_{\mu\nu} = \frac{\partial^2 V}{\partial x^\mu \partial x^\nu} - \{\mu\nu, \alpha\} \frac{\partial V}{\partial x^\alpha}.$$

Equation (3.2) is a scalar equation, since $V_{\mu\nu}$ is the second covariant derivative of the scalar V .

Let $(\bar{x}^\mu) \equiv (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ be any given fixed point of the space and let Ω , as above, represent one-half of the square of the geodesic distance between (\bar{x}^μ) and a variable point (x^μ) , so that Ω is a function of the x 's and of the \bar{x} 's. Let $\bar{g}^{\mu\nu}$ be the value at (\bar{x}^μ) of the contravariant tensor $g^{\mu\nu}$, and write $\Omega^{(\mu)}$ for $\bar{g}^{\mu\alpha} \partial \Omega / \partial \bar{x}^\alpha$.

Now suppose that $p_{(\alpha)} \equiv [p_{(1)}, p_{(2)}, \dots, p_{(n)}]$ denote \dagger n arbitrary functions of a variable u , subject to the single condition

$$\bar{g}^{\mu\nu} p_{(\mu)} p_{(\nu)} = 0, \quad (3.3)$$

(there being a summation with respect to μ and ν), and let u be determined as a scalar function of the n independent variables (x^μ) by the equation

$$au = -\Omega^{(\alpha)} p_{(\alpha)} \quad (3.4)$$

where a is any constant. Let also

$$\eta = -\Omega^{(\alpha)} p'_{(\alpha)} \quad (3.5)$$

and

$$\theta^2 = \bar{g}^{\mu\nu} p'_{(\mu)} p'_{(\nu)}, \quad (3.6)$$

the accents denoting, as always hereafter, differentiation with respect to u .

Then the following theorems are true:

THEOREM A. *If $f(u)$ is an arbitrary function of u , then*

$$V = f(u) \quad (3.7)$$

is a solution of the partial differential equation

$$g^{\mu\nu} V_{\mu\nu} = 0. \quad (3.8)$$

* No assumption is being made as to the definiteness or indefiniteness of the quadratic form in (3.1).

† The suffixes are written in brackets to indicate that they are not ordinary tensor-suffixes.

THEOREM B. If $\phi(u)$ and $\psi(u)$ are arbitrary functions of u , then

$$V = \frac{\phi(u)}{a-\eta} \exp\left(\frac{i\kappa\eta}{\theta}\right) + \frac{\psi(u)}{a-\eta} \exp\left(-\frac{i\kappa\eta}{\theta}\right) \tag{3.9}$$

is a solution of the partial differential equation

$$g^{\mu\nu}V_{,\mu\nu} + \kappa^2 V = 0. \tag{3.10}$$

The proof is simple if it is observed that all the equations involved, namely (3.3) to (3.10), are *scalar* relations. Hence, if the theorems are true for any one system of coordinates, they will be true for all sets derivable from them by point-transformations. Hence, since the space is flat, it is necessary to demonstrate the truth of the theorems only for the case when the fundamental quadratic form is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2. \tag{3.11}$$

For this
$$g_{\mu\nu} = g^{\mu\nu} = \bar{g}_{\mu\nu} = \bar{g}^{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}, \tag{3.12}$$

and
$$\Omega = \frac{1}{2}[(x^1 - \bar{x}^1)^2 + (x^2 - \bar{x}^2)^2 + \dots + (x^n - \bar{x}^n)^2], \tag{3.13}$$

whence
$$\begin{aligned} \Omega^{(\mu)} &= \bar{g}^{\mu\alpha} \partial\Omega / \partial\bar{x}^\alpha && (\mu = 1, 2, \dots, n) \\ &= \partial\Omega / \partial\bar{x}^\mu && \text{by (3.12)} \\ &= -(x^\mu - \bar{x}^\mu) && \text{by (3.13)}. \end{aligned}$$

So (3.3), (3.4), (3.5), and (3.6) become respectively

$$p_{(1)}^2 + p_{(2)}^2 + \dots + p_{(n)}^2 = 0 \tag{3.14}$$

$$au = (x^1 - \bar{x}^1)p_{(1)} + (x^2 - \bar{x}^2)p_{(2)} + \dots + (x^n - \bar{x}^n)p_{(n)} \tag{3.15}$$

$$\eta = (x^1 - \bar{x}^1)p'_{(1)} + (x^2 - \bar{x}^2)p'_{(2)} + \dots + (x^n - \bar{x}^n)p'_{(n)} \tag{3.16}$$

$$\theta^2 = p'_{(1)}{}^2 + p'_{(2)}{}^2 + \dots + p'_{(n)}{}^2, \tag{3.17}$$

and the partial differential equations (3.8) and (3.10) reduce to $\nabla^2 V = 0$ and $\nabla^2 V + \kappa^2 V = 0$ respectively.

By I and II of § 2 the truth of theorems A and B is now obvious. The fact that the differences $x^1 - \bar{x}^1, x^2 - \bar{x}^2$, etc., appear in (3.15) and (3.16), instead of simply the x^1, x^2, \dots in (2.2) and (2.3), is of no consequence since the \bar{x} 's are constants and the replacement of the x 's by the differences $x - \bar{x}$ amounts therefore to a mere change of origin.

4. Corresponding theorems for spaces of constant curvature

Let now
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.1}$$

define the metric of a space of constant positive or negative curvature K . The equation to be solved is still formally the same, namely,

$$g^{\mu\nu}V_{,\mu\nu} + \kappa^2 V = 0, \tag{4.2}$$

though no point-transformation will now reduce this to the form $\nabla^2 V + \kappa^2 V = 0$.

Let s be the geodesic distance between a fixed point (\bar{x}^μ) and the variable point (x^μ) , and let

$$Q = \cos(K^{\frac{1}{2}}s). \quad (4.3)$$

It will be convenient to employ this function Q rather than the function $\Omega = \frac{1}{2}s^2$ used in the case of a flat space.

As before, let $p_{(\alpha)}$ ($\alpha = 1, 2, \dots, n$) be n functions of a parameter u , subject to the condition

$$\bar{g}^{\mu\nu} p_{(\mu)} p_{(\nu)} = 0. \quad (4.4)$$

Define u as a scalar function of the variables (x^μ) by the equation

$$Q^{(\alpha)} p_{(\alpha)} = 0, \quad (4.5)$$

where

$$Q^{(\alpha)} = \bar{g}^{\alpha\beta} \partial Q / \partial \bar{x}^\beta.$$

Equation (4.5) replaces equation (3.4) of the previous section. Let also

$$\eta = Q^{(\alpha)} p'_{(\alpha)} \quad (4.6)$$

and

$$\theta^2 = \bar{g}^{\mu\nu} p'_{(\mu)} p'_{(\nu)}. \quad (4.7)$$

Then the theorems to be proved are as follows:

THEOREM C. *A solution of the partial differential equation*

$$g^{\mu\nu} V_{\mu\nu} = 0 \quad (4.8)$$

is

$$V = f(u), \quad (4.9)$$

where $f(u)$ is an arbitrary function of u .

THEOREM D. *A solution of the partial differential equation*

$$g^{\mu\nu} V_{\mu\nu} + \kappa^2 V = 0 \quad (4.10)$$

is

$$V = \phi(u) H(a, b; c; \xi), \quad (4.11)$$

where $\phi(u)$ is an arbitrary function of u , and $H(a, b; c; \xi)$ is any convergent solution of the hypergeometric equation

$$\xi(1-\xi) \frac{d^2 H}{d\xi^2} + \{c - (a+b+1)\xi\} \frac{dH}{d\xi} - abH = 0 \quad (4.12)$$

in which

$$\left. \begin{aligned} a &= \frac{1}{4}[n-1 + \{(n-1)^2 + 4\kappa^2 K^{-1}\}^{\frac{1}{2}}] \\ b &= \frac{1}{4}[n-1 - \{(n-1)^2 + 4\kappa^2 K^{-1}\}^{\frac{1}{2}}] \\ c &= \frac{3}{2}, \quad \xi = \eta^2 \theta^{-2} K^{-1} \end{aligned} \right\}. \quad (4.13)$$

5. Proof of Theorem C

In the following work ordinary subscripts will denote covariant differentiations with respect to the variables (x^μ) , while bracketed

suffixes not otherwise defined will represent covariant differentiations with respect to the \bar{x} 's, which, though the coordinates of a fixed point, may be regarded as parameters which enter into the discussion. Moreover, the operations of raising and lowering suffixes will be resorted to freely, but it must be remembered that the raising of an ordinary suffix implies contracted multiplication by the coefficients $g^{\lambda\mu}$, while the raising of a bracketed suffix denotes similar multiplication by the constants $\bar{g}^{\lambda\mu}$. Dummy suffixes will sometimes be changed without comment. Since the x 's and \bar{x} 's are entirely independent of one another, bracketed suffixes representing covariant differentiations may be permuted with unbracketed. As an illustration of these remarks the following should suffice:

$$Q_\lambda = \partial Q / \partial x^\lambda, \quad Q^\lambda = g^{\lambda\mu} Q_\mu, \quad Q_{(\lambda)} = \partial Q / \partial \bar{x}^\lambda, \quad Q^{(\lambda)} = \bar{g}^{\lambda\mu} Q_{(\mu)},$$

$Q_{\lambda\mu}$ denotes the second covariant derivative of Q with respect to the x 's,

$$Q_{(\lambda)(\mu)} = \partial Q_{(\lambda)} / \partial \bar{x}^\mu - \{\bar{\lambda}\mu, \alpha\} Q_{(\alpha)}, \quad \text{where the Christoffel symbol is evaluated at } (\bar{x}^\mu), \text{ and}$$

$$Q_{\lambda(\mu)} = Q_{(\mu)\lambda} = \partial^2 Q / \partial x^\lambda \partial \bar{x}^\mu.$$

We first observe, then, that Q satisfies the identical relations*

$$Q^\lambda Q_\lambda = K(1 - Q^2), \quad (5.1)$$

$$Q_{\mu\nu} = -K g_{\mu\nu} Q, \quad (5.2)$$

whence, raising the suffix ν and contracting,

$$Q^\mu_\mu = -n K Q. \quad (5.3)$$

Differentiating (5.1) with respect to \bar{x}^α and raising the suffix (α) ,

$$Q_\lambda^{(\alpha)} Q^\lambda = -K Q Q^{(\alpha)}, \quad (5.4)$$

since $Q_\lambda Q^{(\alpha)\lambda} = Q_\lambda^{(\alpha)} Q^\lambda$. Differentiating again (covariantly) with respect to \bar{x}^β and raising the suffix,

$$Q_\lambda^{(\alpha)} Q^{(\beta)\lambda} + Q_\lambda^{(\alpha)(\beta)} Q^\lambda = -K Q^{(\alpha)} Q^{(\beta)} - K Q Q^{(\alpha)(\beta)}. \quad (5.5)$$

Since Q by its definition is unchanged when the x 's and \bar{x} 's are interchanged, it follows from (5.2) that

$$Q_{(\alpha)(\beta)} = -K \bar{g}_{\alpha\beta} Q,$$

so, raising the suffixes,

$$Q^{(\alpha)(\beta)} = -K \bar{g}^{\alpha\beta} Q, \quad (5.6)$$

whence

$$Q_\lambda^{(\alpha)(\beta)} = -K \bar{g}^{\alpha\beta} Q_\lambda. \quad (5.7)$$

* *Quart. J. of Math.* (Oxford), 1 (1930), 148. (Page 66 above).

Substituting from (5.6) and (5.7) in (5.5), we deduce that

$$Q_{\lambda}^{(\alpha)}Q^{(\beta)\lambda} = K\bar{g}^{\alpha\beta}Q_{\lambda}Q^{\lambda} - KQ^{(\alpha)}Q^{(\beta)} + K^2\bar{g}^{\alpha\beta}Q^2,$$

whence, by (5.1),

$$Q_{\lambda}^{(\alpha)}Q^{(\beta)\lambda} = K^2\bar{g}^{\alpha\beta} - KQ^{(\alpha)}Q^{(\beta)}. \quad (5.8)$$

Again, differentiating (5.3) with respect to \bar{x}^{α} and raising the suffix,

$$Q_{\mu}^{(\alpha)\mu} = -nKQ^{(\alpha)}. \quad (5.9)$$

We are now in a position to begin the proof of Theorem C. Differentiating the given relation (4.4) with respect to u ,

$$\bar{g}^{\mu\nu}p'_{(\mu)}p_{(\nu)} = 0. \quad (5.10)$$

We now show that the scalar u defined by (4.5) satisfies the relation

$$u_{\lambda}u^{\lambda} = 0. \quad (5.11)$$

For, differentiating (4.5) with respect to x^{λ} ,

$$Q^{(\alpha)}p'_{(\alpha)}u_{\lambda} + Q_{\lambda}^{(\alpha)}p_{(\alpha)} = 0,$$

so

$$u_{\lambda}\eta = -Q_{\lambda}^{(\alpha)}p_{(\alpha)}, \quad (5.12)$$

where η is defined by (4.6). Hence

$$u^{\lambda}\eta = -Q^{(\beta)\lambda}p_{(\beta)}, \quad (5.13)$$

whence

$$\begin{aligned} u_{\lambda}u^{\lambda}\eta^2 &= Q_{\lambda}^{(\alpha)}Q^{(\beta)\lambda}p_{(\alpha)}p_{(\beta)} \\ &= K^2\bar{g}^{\alpha\beta}p_{(\alpha)}p_{(\beta)} - KQ^{(\alpha)}Q^{(\beta)}p_{(\alpha)}p_{(\beta)}, \end{aligned}$$

by (5.8). The first term on the right vanishes on account of the condition (4.4) imposed on the arbitrary functions $p_{(\alpha)}$, and the second in virtue of (4.5). Hence the truth of (5.11) is established if we assume, as we must, that the functions $p_{(\alpha)}$ are chosen so that $\eta \neq 0$.

The next problem is to show that

$$u_{\lambda}^{\lambda} = 0, \quad (5.14)$$

where $u_{\lambda}^{\lambda} = g^{\lambda\mu}u_{\lambda\mu}$, the suffixes as usual denoting covariant differentiations.

Differentiating (5.12) covariantly with respect to x^{μ} , raising the suffix and contracting,

$$\begin{aligned} u_{\lambda}^{\lambda}\eta + u_{\lambda}\eta^{\lambda} &= -Q^{(\alpha)\lambda}p_{(\alpha)} - Q_{\lambda}^{(\alpha)}p'_{(\alpha)}u^{\lambda} \\ &= nKQ^{(\alpha)}p_{(\alpha)} + Q_{\lambda}^{(\alpha)}Q^{(\beta)\lambda}p'_{(\alpha)}p_{(\beta)}\eta^{-1} \end{aligned}$$

by (5.9) and (5.13). By (4.5) the first term on the right is zero, and by (5.8),

$$Q_{\lambda}^{(\alpha)}Q^{(\beta)\lambda}p'_{(\alpha)}p_{(\beta)} = K^2\bar{g}^{\alpha\beta}p'_{(\alpha)}p_{(\beta)} - KQ^{(\alpha)}Q^{(\beta)}p'_{(\alpha)}p_{(\beta)} = 0 \quad (5.15)$$

by (5.10) and (4.5).

So
$$u_\lambda^\lambda \eta + u_\lambda \eta^\lambda = 0. \quad (5.16)$$

But, by (4.6),
$$\eta^\lambda = Q^{(\alpha)\lambda} p'_{(\alpha)} + Q^{(\alpha)} p''_{(\alpha)} u^\lambda. \quad (5.17)$$

Hence, using (5.11),

$$u_\lambda \eta^\lambda = Q^{(\alpha)\lambda} u_\lambda p'_{(\alpha)} = -Q^{(\alpha)\lambda} Q^{(\beta)} p'_{(\alpha)} p_{(\beta)} \eta^{-1}, \text{ by (5.12),}$$

and this vanishes by (5.15). Hence

$$u_\lambda \eta^\lambda = 0, \quad (5.18)$$

whence $u_\lambda^\lambda = 0$ by (5.16).

The proof of Theorem C is now immediate. For if $V = f(u)$ is an arbitrary function of u , then

$$V_\lambda = f'(u) u_\lambda, \quad V_\lambda^\lambda = f''(u) u_\lambda^\lambda + f'(u) u^\lambda u_\lambda = 0,$$

by (5.11) and (5.14). That is, V satisfies the partial differential equation

$$g^{\lambda\mu} V_{\lambda\mu} = 0.$$

6. Proof of Theorem D

The equation to be solved is

$$V_\lambda^\lambda + \kappa^2 V = 0. \quad (6.1)$$

The scalar u is defined as a function of the x 's by

$$Q^{(\alpha)} p_{(\alpha)} = 0, \quad (6.2)$$

where
$$\bar{g}^{\alpha\beta} p_{(\alpha)} p_{(\beta)} = 0; \quad (6.3)$$

whence, as above,
$$\bar{g}^{\alpha\beta} p'_{(\alpha)} p_{(\beta)} = 0. \quad (6.4)$$

Differentiating the last equation with respect to u , we obtain

$$\bar{g}^{\alpha\beta} p''_{(\alpha)} p_{(\beta)} = -\bar{g}^{\alpha\beta} p'_{(\alpha)} p'_{(\beta)} = -\theta^2, \quad (6.5)$$

where θ is the function introduced above in (4.7). Also, we have already defined

$$\eta = Q^{(\alpha)} p'_{(\alpha)}. \quad (6.6)$$

For the purpose of the discussion it is assumed that the arbitrary functions $p_{(\alpha)}$ are so chosen that neither of the quantities θ , η is zero.

Now assume
$$V = f(u, \eta)$$

as a solution of (6.1), where f is a function of u and η to be determined. Differentiating with respect to x^λ ,

$$V_\lambda = f_u u_\lambda + f_\eta \eta_\lambda,$$

where the suffixes u and η denote partial differentiations of f with respect to u and η respectively. Differentiating the last equation covariantly with respect to x^μ , raising the suffix and contracting,

$$V_\lambda^\lambda = f_{uu} u_\lambda u^\lambda + f_u u_\lambda^\lambda + 2f_{u\eta} u_\lambda \eta^\lambda + f_{\eta\eta} \eta_\lambda \eta^\lambda + f_\eta \eta_\lambda^\lambda.$$

so by (5.11), (5.14), and (5.18),

$$V_\lambda^\lambda = f_{\eta\eta} \eta^\lambda \eta_\lambda + f_\eta \eta_\lambda^\lambda. \quad (6.7)$$

Now multiplying (5.17) by η_λ and using (5.18),

$$\eta^\lambda \eta_\lambda = Q^{(\alpha\lambda)} \eta_\lambda p'_{(\alpha)}.$$

But by (5.17),

$$\eta_\lambda = Q_\lambda^{(\beta)} p'_{(\beta)} + Q^{(\beta)} p''_{(\beta)} u_\lambda. \quad (6.8)$$

$$\begin{aligned} \text{so } \eta^\lambda \eta_\lambda &= Q^{(\alpha\lambda)} Q_\lambda^{(\beta)} p'_{(\alpha)} p'_{(\beta)} + Q^{(\alpha\lambda)} Q^{(\beta)} p'_{(\alpha)} p''_{(\beta)} u_\lambda \\ &= Q^{(\alpha\lambda)} Q_\lambda^{(\beta)} p'_{(\alpha)} p'_{(\beta)} - Q^{(\alpha\lambda)} Q^{(\beta)} Q_\lambda^{(\gamma)} p'_{(\alpha)} p'_{(\beta)} p_{(\gamma)} \eta^{-1}, \text{ by (5.12),} \\ &= Q^{(\alpha\lambda)} Q_\lambda^{(\beta)} \{ p'_{(\alpha)} p'_{(\beta)} - Q^{(\gamma)} p'_{(\alpha)} p_{(\beta)} p''_{(\gamma)} \eta^{-1} \}, \end{aligned}$$

by a change of dummy suffixes in the second term. Putting in the value of $Q^{(\alpha\lambda)} Q_\lambda^{(\beta)}$ obtained from (5.8), multiplying out and using (6.2), (6.3), (6.4), (6.5), and (6.6), we get

$$\eta^\lambda \eta_\lambda = K^2 \theta^2 - K \eta^2. \quad (6.9)$$

It is now necessary to evaluate η_λ^λ . Differentiating (6.8) covariantly with respect to x^μ , raising the suffix and contracting,

$$\begin{aligned} \eta_\lambda^\lambda &= Q^{(\beta\lambda)} p'_{(\beta)} + 2Q_\lambda^{(\beta)} p''_{(\beta)} u^\lambda + Q^{(\beta)} p'''_{(\beta)} u_\lambda u^\lambda + Q^{(\beta)} p''_{(\beta)} u_\lambda^\lambda \\ &= -nK Q^{(\beta)} p'_{(\beta)} + 2Q_\lambda^{(\beta)} p''_{(\beta)} u^\lambda, \end{aligned}$$

by (5.9), (5.11), and (5.14). Inserting the value of u^λ obtained from (5.13) and using (5.8), we get

$$\begin{aligned} \eta_\lambda^\lambda &= -nK Q^{(\beta)} p'_{(\beta)} - 2K^2 \bar{g}^{\alpha\beta} p_{(\alpha)} p''_{(\beta)} \eta^{-1} + 2K Q^{(\alpha)} Q^{(\beta)} p_{(\alpha)} p''_{(\beta)} \eta^{-1} \\ &= -nK \eta + 2K^2 \theta^2 \eta^{-1} \end{aligned} \quad (6.10)$$

by (6.6), (6.5), and (6.2). Hence, substituting from (6.9) and (6.10) in (6.7) we deduce that

$$V_\lambda^\lambda = (K^2 \theta^2 - K \eta^2) f_{\eta\eta} + (2K^2 \theta^2 \eta^{-1} - nK \eta) f_\eta. \quad (6.11)$$

Hence, if $V = f(u, \eta)$ satisfies the partial differential equation (6.1), f must satisfy the partial differential equation

$$(K^2 \theta^2 - K \eta^2) \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{2K^2 \theta^2}{\eta} - nK \eta \right) \frac{\partial f}{\partial \eta} + \kappa^2 f = 0. \quad (6.12)$$

Putting for convenience

$$K \theta^2 = h^2, \quad \kappa^2 K^{-1} = \lambda^2, \quad (6.13)$$

this becomes

$$(h^2 - \eta^2) \frac{\partial^2 f}{\partial \eta^2} + \frac{1}{\eta} (2h^2 - n\eta^2) \frac{\partial f}{\partial \eta} + \lambda^2 f = 0.$$

Since only derivatives with respect to η appear in this equation, u may be treated as a constant in integrating it. Noticing that h is independent of η , introduce the new variable ξ defined by

$$\eta^2 = h^2 \xi, \quad (6.14)$$

and (6.12) reduces to

$$\xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{3}{2} - \frac{n+1}{2}\xi\right)\frac{\partial f}{\partial \xi} + \frac{\lambda^2}{4}f = 0. \quad (6.15)$$

Writing this in the form

$$\xi(1-\xi)\frac{\partial^2 f}{\partial \xi^2} + \{c - (a+b+1)\xi\}\frac{\partial f}{\partial \xi} - abf = 0, \quad (6.16)$$

$$\text{so that } c = \frac{3}{2}, \quad a+b+1 = \frac{1}{2}(n+1), \quad ab = -\frac{1}{4}\lambda^2, \quad (6.17)$$

$$\text{it is evident that } f = \phi(u)H(a, b; c; \xi), \quad (6.18)$$

where $\phi(u)$ is an arbitrary function of u , and H is any convergent solution of the hypergeometric equation. Putting the values of λ , h given by (6.13) in (6.14) and (6.17), and solving the latter for a and b , the truth of Theorem D becomes obvious.

7. Conclusion

In Theorem D consider in particular the solution

$$V = \phi(u)F(a, b; c; \xi) \quad (7.1)$$

where F is the hypergeometric function, convergent when $|\xi| < 1$. Make $K \rightarrow 0$, so that the space tends to a flat space. By (4.3),

$$Q = \cos(K^{\frac{1}{2}}s),$$

$$\text{hence } Q^{(\alpha)}K^{-1} = -\sin(K^{\frac{1}{2}}s)s^{(\alpha)}K^{-\frac{1}{2}} \rightarrow -ss^{(\alpha)} = -\Omega^{(\alpha)}, \quad (7.2)$$

since by definition $\Omega = \frac{1}{2}s^2$. So dividing (6.2) by K , the relation defining u as a function of the x 's becomes

$$\Omega^{(\alpha)}p_{(\alpha)} = 0, \quad (7.3)$$

which is the same as (3.4) with $a = 0$. Moreover, it is a matter of mere algebra to show that

$$F(a, b; c; \xi) \rightarrow (\theta/\kappa\eta)\sin(\kappa\eta/\theta), \quad (7.4)$$

where η now has the meaning it had in § 3, namely, $\eta = -\Omega^{(\alpha)}p'_{(\alpha)}$. So, since θ/κ is a function of u only, it may be absorbed into the arbitrary function $\phi(u)$, and the solution (7.1) thus tends to

$$V = \phi(u)\eta^{-1}\sin(\kappa\eta\theta^{-1}), \quad (7.5)$$

which is obviously in agreement with Theorem B.

Similarly, it can be shown that the complementary solution

$$V = \phi(u)(\xi)^{1-c}F(a-c+1, b-c+1; 2-c; \xi) \quad (7.6)$$

$$\text{tends to the form } V = \chi(u)\eta^{-1}\cos(\kappa\eta\theta^{-1}), \quad (7.7)$$

where $\chi(u)$ is arbitrary.

CHAPTER V.

An application to Relativity.

("On the definition of spatial distance
in General Relativity").

VI.—On the Definition of Spatial Distance in General Relativity.

By H. S. Ruse, B.A.(Oxon.).

(M.S. received October 15, 1931. Read November 2, 1931.)

IN a recent paper,* Professor E. T. Whittaker discussed the problem of defining, in a general riemannian space-time, the concept of spatial distance between material particles. It is the object of this paper to give an alternative definition, and to compare the new formula with that of Whittaker.

§ 1. WHITTAKER'S DEFINITION OF SPATIAL DISTANCE.

It is pointed out in the above-mentioned paper that if an observer O in a general riemannian space-time makes an assertion regarding the distance from himself of a particle S, he is really stating a relation between the world-point of S at the instant when light left it, and his own world-point at the instant when the light arrives. In other words, he is giving a relation between two world-points which lie on the same null geodesic. It is indeed obvious that, if O is to measure the "distance" between himself and a distant point S, he must be able to *see* S; that is, O and S must lie on the world-line of a light-pulse, which is a null geodesic.

Now astronomers sometimes determine the distance of a star from the earth by finding the square root of the ratio of its absolute brightness to its apparent brightness. The geometrical statement of this fact constitutes Whittaker's definition of spatial distance, which may be expressed as follows:—

The spatial distance between a star S and an observer O (on the same null geodesic) is proportional to the square root of the two-dimensional cross-section made by a thin pencil of null geodesics, with vertex S and passing near O, on the instantaneous three-dimensional space of the observer.†

The definition is further particularised by requiring that when O and S are near one another the "spatial distance" shall reduce to the element of length at O of the instantaneous three-dimensional space.

* *Proc. Roy. Soc., A*, 133 (1931), 93.

† The instantaneous three-dimensional space of the observer consists of those world-points in his immediate neighbourhood which he regards as simultaneous. Geometrically, it is a small portion near O of the hypersurface formed by the geodesics through O which are perpendicular at this point to the observer's world-line.

It is shown that an application of this general definition to the space-time of constant curvature $-1/R^2$ specified by

$$(1.1) \quad \frac{ds^2}{R^2} = \frac{du^2 - dx^2 - dy^2 - dz^2}{1 + x^2 + y^2 + z^2 - u^2} + \frac{(udu - xdx - ydy - zdz)^2}{(1 + x^2 + y^2 + z^2 - u^2)^2}$$

leads to the following conclusion:

The spatial distance between a star whose world-coordinates are (u, x, y, z) and an observer whose world-line is the geodesic $x = \bar{x}$, $y = \bar{y}$, $z = \bar{z}$, where \bar{x} , \bar{y} , \bar{z} are constants, is

$$(1.2) \quad \Delta = \frac{R \sin \rho}{\cos(\sigma + \rho)},$$

where

$$\rho = \arccos \left\{ \frac{1 + x\bar{x} + y\bar{y} + z\bar{z}}{(1 + x^2 + y^2 + z^2)^{\frac{1}{2}}(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}} \right\}, \quad (0 \leq \rho \leq \pi)$$

and

$$\sigma = \arcsin \left\{ \frac{u}{(1 + x^2 + y^2 + z^2)^{\frac{1}{2}}} \right\}, \quad \left(-\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2} \right)$$

the radicals being taken positively. Substitution of these values of ρ and σ in (1.2) gives

$$(1.3) \quad \Delta = \frac{R(1 + \Sigma x^2)^{\frac{1}{2}} \{ \Sigma(x - \bar{x})^2 + \Sigma(y\bar{z} - \bar{y}z)^2 \}^{\frac{1}{2}}}{(1 + \Sigma x\bar{x})(1 - u^2 + \Sigma x^2)^{\frac{1}{2}} - u \{ \Sigma(x - \bar{x})^2 + \Sigma(y\bar{z} - \bar{y}z)^2 \}^{\frac{1}{2}}}$$

the summations in each case being for x, y, z ; so, for example, Σx^2 denotes $x^2 + y^2 + z^2$.

§ 2. SPATIAL DISTANCE IN GALILEAN SPACE-TIME.

In restricted relativity, if an observer O is at the point $(\bar{x}, \bar{y}, \bar{z})$ with respect to a galilean frame of reference relative to which he is at rest, then his distance from a star at the point (x, y, z) is

$$(2.1) \quad \Delta = \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}^{\frac{1}{2}}.$$

Suppose that his measurement of the distance is made at time \bar{t} , and that the light from the star which reaches him at this instant left it at time t . Then, as is well known,

$$(2.2) \quad \bar{t} = t + \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}^{\frac{1}{2}},$$

the velocity of light being unity.

In geometrical language the above statements may be expressed as follows. If

$$(2.3) \quad ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

defines the metric of space-time, and if the world-line of the observer O is the geodesic $x = \bar{x}$, $y = \bar{y}$, $z = \bar{z}$, where \bar{x} , \bar{y} , \bar{z} are constants, then the spatial distance between a star S at the world-point (t, x, y, z) and the observer is

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given by (2.1). If at the instant of making the observation the world-coordinates of the observer are $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$, then the relation between them and the world-coordinates of the star is given by equating to zero the geodesic distance (interval) s between $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ and (t, x, y, z) , thus expressing the fact that the two points lie on the same null geodesic. But

$$(2.4) \quad s^2 = (t - \bar{t})^2 - (x - \bar{x})^2 - (y - \bar{y})^2 - (z - \bar{z})^2,$$

so $s = 0$ gives

$$(2.5) \quad t - \bar{t} = \pm \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}^{\frac{1}{2}}.$$

The negative sign must be chosen since we are dealing with rays of light from the star to the observer (so that $\bar{t} > t$), and not vice versa.

Any definition of spatial distance in a general riemannian space-time must reduce to (2.1) when applied to the above particular case, which will be referred to as the "fundamental galilean case."

It may be remarked that the corresponding formulæ for an observer who is moving relative to the given system of reference may be obtained from the above by applying an appropriate Lorentz transformation.

§ 3. A DEFINITION OF SPATIAL DISTANCE.

Let

$$(3.1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

define the metric of space-time in terms of a coordinate system (x^0, x^1, x^2, x^3) selected by an observer O.

It will be assumed that the quadratic form on the right-hand side of (3.1) is indefinite and of signature -2 ; that is, that it can be reduced by a real point-transformation $x^\mu = x^\mu(\xi^\nu)$ to the form

$$(\lambda_0 d\xi^0)^2 - (\lambda_1 d\xi^1)^2 - (\lambda_2 d\xi^2)^2 - (\lambda_3 d\xi^3)^2,$$

where the λ 's are real functions of the ξ 's. It will be seen that this assumption involves no real loss of generality. It will be convenient to use the symbol e_a defined by

$$(3.2) \quad \begin{cases} e_a = 1 & (a = 0) \\ e_a = -1 & (a = 1, 2, 3). \end{cases}$$

Let the world-coordinates of the star S be $(x^\mu) \equiv (x^0, x^1, x^2, x^3)$, and those of the observer be $(\bar{x}^\mu) \equiv (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$. Then, if Ω denote one-half of the square of the geodesic distance between (x^μ) and (\bar{x}^μ) , the fact that these two points lie on a null geodesic may be expressed by the equation

$$(3.3) \quad \Omega = 0.$$

Let τ be the proper-time of the observer at the point (\bar{x}^μ) . That is, let τ be the length of the arc of his world-line (a geodesic) measured from

some given fixed point on it to the point (\bar{x}^μ) . Then each of the \bar{x} 's is a function of τ , say

$$(3.4) \quad \bar{x}^\mu = x^\mu(\tau), \quad (\mu = 0, 1, 2, 3)$$

while by (3.1),

$$(3.5) \quad d\tau^2 = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu,$$

where $\bar{g}_{\mu\nu}$ is the value of $g_{\mu\nu}$ at (\bar{x}^μ) . Moreover, since in a space of four dimensions ∞^3 geodesics pass through any point, equations (3.4) will involve three given constants which may be eliminated, so that τ is expressible as a function of $\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3$.

Let

$$(3.6) \quad \bar{k}^\mu = \frac{d\bar{x}^\mu}{d\tau} \quad (\mu = 0, 1, 2, 3)$$

be the contravariant components of the unit vector in the direction of the tangent at (\bar{x}^μ) to the world-line of O. That \bar{k}^μ is a unit vector follows from (3.5), which gives

$$(3.7) \quad \bar{g}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu = 1.$$

It must be observed that the \bar{k}^μ are each functions of τ , and therefore of $\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3$.

It will now be shown that the observer can set up a system of reference with his world-point (\bar{x}^μ) as origin such that

- (i) the "coordinate-axes" are geodesics which are mutually perpendicular at (\bar{x}^μ) , one of these axes being his own world-line;
- (ii) the corresponding coordinate-system * $(\eta^a) \equiv (\eta^0, \eta^1, \eta^2, \eta^3)$ is such that the four-dimensional distance (interval) from the origin to the point (η^a) is $[(\eta^0)^2 - (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2]^{\frac{1}{2}}$;
- (iii) in terms of the new coordinates the equations of the geodesics (and in particular of the null geodesics) through the observer's world-point are linear.

This system of coordinates is the nearest approximation to a galilean system that can be obtained in his curved space; in fact, in the case when space-time is flat, $(\eta^0, \eta^1, \eta^2, \eta^3)$ are nothing other than the galilean coordinate-system (t, x, y, z) , with respect to which the observer's world-line is $x=0, y=0, z=0$. Now in the galilean case the four-dimensional distance of (t, x, y, z) from the origin $(0, 0, 0, 0)$ is $(t^2 - x^2 - y^2 - z^2)^{\frac{1}{2}}$, and the spatial distance is $(x^2 + y^2 + z^2)^{\frac{1}{2}}$. It will therefore be argued that a reasonable definition of spatial distance in the more general case is

* The η 's are in fact a particular set of Riemann normal coordinates. Systems of this type have recently been employed by T. Y. Thomas, *Proc. Nat. Acad. Sci.*, 16 (1930), 761.

ates* having (\bar{x}^μ) as origin. These are defined as functions of the original coordinates (x^μ) by the equation †

$$(3.12) \quad y^\mu = -\bar{g}^{\mu\alpha} \frac{\partial \Omega}{\partial \bar{x}^\alpha}, \quad (\mu = 0, 1, 2, 3)$$

where Ω , as above, represents one-half of the square of the geodesic distance between (\bar{x}^μ) and (x^μ) . For the moment Ω is not zero, since the x^μ are being employed as current coordinates instead of as the world-coordinates of the star.

In the fundamental galilean case,

$$\begin{aligned} \Omega &= \frac{1}{2} \{ (t - \bar{t})^2 - (x - \bar{x})^2 - (y - \bar{y})^2 - (z - \bar{z})^2 \}, \\ \bar{g}^{00} &= 1, \bar{g}^{11} = \bar{g}^{22} = \bar{g}^{33} = -1, \bar{g}^{\mu\nu} = 0 \text{ if } \mu \neq \nu, \text{ so} \\ y^0 &= -\partial \Omega / \partial \bar{t} = t - \bar{t} \end{aligned}$$

and similarly $y^1 = x - \bar{x}$, $y^2 = y - \bar{y}$, $y^3 = z - \bar{z}$. Hence in this case the transference to normal coordinates reduces to a mere change of origin without rotation.

Expressed in terms of the new variables, the equations of any geodesic through (\bar{x}^μ) are of the form

$$(3.13) \quad y^\mu = a^\mu s, \quad (\mu = 0, 1, 2, 3)$$

where s is the length of its arc from the origin $(y^\mu) = 0$ to the point (y^μ) , and the constants a^μ determine the direction of the geodesic. In fact

$$(3.14) \quad a^\mu = \frac{dy^\mu}{ds} = \left(\frac{dx^\mu}{ds} \right)_0$$

where $\left(\frac{dx^\mu}{ds} \right)_0$ is the value at (\bar{x}^μ) of $\frac{dx^\mu}{ds}$ for the geodesic in question. In particular, the observer's world-line has the equations

$$(3.15) \quad y^\mu = k^\mu s, \quad (\mu = 0, 1, 2, 3)$$

It should be noticed that, in terms of the new coordinates, the equations (3.13) of the geodesics through (\bar{x}^μ) are linear.

By (3.14),

$$\begin{aligned} \bar{g}_{\mu\nu} a^\mu a^\nu &= \bar{g}_{\mu\nu} \left(\frac{dx^\mu}{ds} \right)_0 \left(\frac{dx^\nu}{ds} \right)_0 \\ &= 1 \end{aligned} \quad \text{by (3.1).}$$

So, multiplying by s^2 and using (3.13), we deduce that the square of the four-dimensional distance from the observer of a star whose normal world-coordinates are (y^μ) is given by

$$(3.16) \quad s^2 = \bar{g}_{\mu\nu} y^\mu y^\nu.$$

* Veblen, *Invariants of Quadratic Differential Forms* (Camb. Math. Tract No. 24, 1927), ch. vi.

† Ruse, *Proc. London Math. Soc.*, 32 (1931), 90. (Page 18, Equation 13, above).

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Now transform to the coordinates $(\eta^a) \equiv (\eta^0, \eta^1, \eta^2, \eta^3)$ defined by

$$(3.17) \quad \eta^a = \bar{a}^{\bar{\mu}} y^{\bar{\mu}}. \quad (a=0, 1, 2, 3)$$

In terms of these the equations of the geodesics remain linear, and the observer's world-line (3.15) has the equations

$$\eta^a = \bar{a}^{\bar{\mu}} \bar{k}^{\bar{\mu}} s, \quad (a=0, 1, 2, 3)$$

or, using (3.8),

$$\eta^a = \bar{a}^{\bar{\mu}} \bar{l}_{\bar{\mu}} s,$$

whence, by (3.10),

$$\eta^a = e_a \delta_{a0} s.$$

So, putting $a=0, 1, 2, 3$, the four equations of the observer's world-line are

$$(3.18) \quad \eta^0 = s, \quad \eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = 0.$$

His world-line is therefore the " η^0 -axis of coordinates," just as in the fundamental galilean case, with the observer's world-point as origin, the t -axis is the world-line of the observer. Since the observer recognises the direction of his world-line as the time-direction,* it follows that he will regard η^1, η^2, η^3 as spatial coordinates. Now

$$\begin{aligned} (\eta^0)^2 - (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2 &= \sum_{a=0}^3 e_a (\eta^a)^2 \\ &= \sum_{a=0}^3 e_a \bar{a}^{\bar{\mu}} \bar{a}^{\bar{\nu}} y^{\bar{\mu}} y^{\bar{\nu}} \quad \text{by (3.17)} \\ &= \bar{g}_{\bar{\mu}\bar{\nu}} y^{\bar{\mu}} y^{\bar{\nu}} \quad \text{by (3.11)} \\ &= s^2 \quad \text{by (3.16)} \end{aligned}$$

So, in terms of the coordinates (η^a) , the square of the four-dimensional interval between the star and the observer is

$$(3.19) \quad s^2 = (\eta^0)^2 - (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2.$$

As stated above, in the galilean case in which the observer is at the world-point $(0, 0, 0, 0)$ and the star at (t, x, y, z) , the square of the interval is

$$t^2 - x^2 - y^2 - z^2,$$

and the "spatial distance" δ from the star to the observer is defined by

$$\delta^2 = x^2 + y^2 + z^2.$$

So, in the more general case we define the spatial distance δ by the equation

$$(3.20) \quad \delta^2 = (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2.$$

* It is necessary to assume that he measures time by a clock in his possession, so that the physical time is identical with his proper-time τ .

Hence

$$\begin{aligned} \delta^2 &= - \sum_{a=0}^3 e_a (\eta^a)^2 + (\eta^0)^2 \\ &= -s^2 + (\eta^0)^2 \quad \text{by (3.19)}. \end{aligned}$$

We can now use the fact that the star and the observer lie on the same null geodesic, so that $s=0$. Hence $\delta^2 = (\eta^0)^2$ and therefore

$$(3.21) \quad \delta = -\eta^0,$$

the negative sign being chosen since in the fundamental galilean case $\eta^0 = t - \bar{t}$ and $t < \bar{t}$. So

$$\begin{aligned} \delta &= -\bar{g}_{\mu\nu} y^\mu && \text{by (3.17)} \\ &= -\bar{g}_{\mu\nu} \bar{k}^\nu y^\mu && \text{by (3.9) and (3.8)} \\ &= \bar{g}_{\mu\nu} \bar{g}^{\mu\alpha} \bar{k}^\nu \frac{\partial \Omega}{\partial \bar{x}^\alpha} && \text{by (3.12)} \\ &= \bar{k}^\nu \frac{\partial \Omega}{\partial \bar{x}^\nu} \end{aligned}$$

since $\bar{g}_{\mu\nu} \bar{g}^{\mu\alpha}$ is equal to the Kronecker symbol δ_ν^α .

Hence, by (3.6),

$$(3.22) \quad \delta = \frac{\partial \Omega}{\partial \bar{x}^\nu} \frac{d\bar{x}^\nu}{d\tau}.$$

Since Ω is a function of the world-coordinates (x^μ) of the star and of the world-coordinates (\bar{x}^μ) of the observer, it is, by (3.4), a function of the x^μ and of the observer's proper-time τ . So, by (3.22),

$$\delta = \frac{\partial \Omega}{\partial \tau}.$$

We can therefore finally frame the following definition of spatial distance in a general riemannian space-time:

The spatial distance between a star at the world-point (x^μ) and an observer at the world-point (\bar{x}^μ) , these points being on the same null geodesic, is

$$(3.23) \quad \delta = \frac{\partial \Omega}{\partial \tau},$$

where (i) Ω is one-half the square of the interval between (x^μ) and (\bar{x}^μ) ;

(ii) τ is the proper-time of the observer at (\bar{x}^μ) ;

(iii) one of the variables may be eliminated, after performing the differentiation required by (3.23), by means of the equation $\Omega = 0$.

It must be remembered that in solving the equation $\Omega = 0$ for one of the variables, the sign of the radical which in general appears must be determined from the consideration that the light travels *from* and not *to* the star.

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In the fundamental galilean case,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2,$$

the star is at (t, x, y, z) and the observer at $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ where $\bar{x}, \bar{y}, \bar{z}$ are constants ; so

$$\begin{aligned} d\tau^2 &= d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 \\ &= d\bar{t}^2 \end{aligned}$$

and we may therefore take $\tau = \bar{t}$. Then

$$\begin{aligned} 2\Omega &= (t - \bar{t})^2 - (x - \bar{x})^2 - (y - \bar{y})^2 - (z - \bar{z})^2 \\ &= (t - \tau)^2 - (x - \bar{x})^2 - (y - \bar{y})^2 - (z - \bar{z})^2. \end{aligned}$$

Hence

$$\begin{aligned} \delta &= \partial\Omega / \partial\tau \\ &= -(t - \tau). \end{aligned}$$

The equation $\Omega = 0$ gives

$$t - \tau = \pm \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}^{\frac{1}{2}},$$

and we choose the negative sign since $\tau = \bar{t} > t$.

Hence

$$\delta = \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}^{\frac{1}{2}}.$$

§ 4. COMPARISON WITH WHITTAKER'S FORMULA.

We now apply the formula of the last section to the de Sitter world, whose metric in Beltrami's coordinates is given by (I.1), namely

$$(4.1) \quad \frac{ds^2}{R^2} = \frac{du^2 - dx^2 - dy^2 - dz^2}{1 + x^2 + y^2 + z^2 - u^2} + \frac{(udu - xdx - ydy - zdz)^2}{(1 + x^2 + y^2 + z^2 - u^2)^2}.$$

Suppose the star to be at (u, x, y, z) and the observer at $(\bar{u}, \bar{x}, \bar{y}, \bar{z})$, where $\bar{x}, \bar{y}, \bar{z}$ are constants* and \bar{u} is a function of the observer's proper-time τ . Then †

$$(4.2) \quad 2\Omega = R^2(\arg \cosh Q)^2$$

where

$$Q = \frac{1 + x\bar{x} + y\bar{y} + z\bar{z} - u\bar{u}}{(1 + x^2 + y^2 + z^2 - u^2)^{\frac{1}{2}}(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)^{\frac{1}{2}}}.$$

So

$$\begin{aligned} \delta &= \partial\Omega / \partial\tau \\ &= \frac{R^2 \arg \cosh Q}{(Q^2 - 1)^{\frac{1}{2}}} \frac{\partial Q}{\partial \bar{u}} \frac{d\bar{u}}{d\tau}. \end{aligned}$$

The equation $\Omega = 0$ gives $Q = 1$. Since $\lim_{Q \rightarrow 1} \frac{\arg \cosh Q}{(Q^2 - 1)^{\frac{1}{2}}} = 1$, we get

$$(4.3) \quad \delta = R^2 \frac{\partial Q}{\partial \bar{u}} \frac{d\bar{u}}{d\tau}.$$

* $x = \bar{x}, y = \bar{y}, z = \bar{z}$ are then the equations of a geodesic, since the equations of geodesics in this space are all of the form $x = a'u + b, y = a''u + b', z = a'''u + b''$, where the a 's and b 's are constants.

† Whittaker, *loc. cit.*, equation (4).

Now by (4.1),

$$\frac{d\tau^2}{R^2} = \frac{d\bar{u}^2}{1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2} + \frac{\bar{u}^2 d\bar{u}^2}{(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)^2},$$

whence

$$\frac{d\bar{u}}{d\tau} = \frac{1}{R} \frac{1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2}{(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}}.$$

Also

$$\frac{\partial Q}{\partial \bar{u}} = \frac{\bar{u}(1 + x\bar{x} + y\bar{y} + z\bar{z}) - u(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2)}{(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)^{\frac{1}{2}}(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)^{\frac{3}{2}}}.$$

Substituting in (4.3) we get

$$(4.4) \quad \frac{\delta}{R} = \frac{\bar{u}(1 + \Sigma x\bar{x}) - u(1 + \Sigma \bar{x}^2)}{(1 + \Sigma \bar{x}^2)^{\frac{1}{2}}(1 - u^2 + \Sigma \bar{x}^2)^{\frac{1}{2}}(1 - \bar{u}^2 + \Sigma \bar{x}^2)^{\frac{1}{2}}}$$

the summations being for x, y, z . From this \bar{u} (say) may be eliminated by means of the equation $Q=1$, which gives

$$(4.5) \quad \bar{u} = \frac{u(1 + \Sigma x\bar{x}) + (1 - u^2 + \Sigma \bar{x}^2)^{\frac{1}{2}}\{\Sigma(x - \bar{x})^2 + \Sigma(y\bar{z} - \bar{y}\bar{z})^2\}^{\frac{1}{2}}}{1 + \Sigma \bar{x}^2}.$$

Now compare this formula for δ with the formula for Whittaker's spatial distance Δ . Referring to (1.3), it is easily seen by the use of (4.5) that

$$(4.6) \quad \frac{\Delta}{R} = -\frac{u(1 + \Sigma x\bar{x}) - \bar{u}(1 + \Sigma \bar{x}^2)}{(1 + \Sigma \bar{x}^2)^{\frac{1}{2}}(1 - \bar{u}^2 + \Sigma \bar{x}^2)^{\frac{1}{2}}(1 - u^2 + \Sigma \bar{x}^2)^{\frac{1}{2}}}.$$

A comparison of (4.4) and (4.6) shows that in this special case the formula for Δ may be obtained from that of δ by interchanging the observer's and the star's world-coordinates, and altering the sign.

In terms of the quantities ρ and σ appearing in (1.2), equation (4.5) may be written

$$(4.7) \quad \bar{u} = (1 + \Sigma \bar{x}^2)^{\frac{1}{2}} \sin(\sigma + \rho),$$

and it is easily shown that

$$(4.8) \quad \frac{\delta}{R} = \frac{\sin \rho}{\cos \sigma}.$$

This may be compared with (1.2), namely

$$\frac{\Delta}{R} = \frac{\sin \rho}{\cos(\sigma + \rho)}.$$

§ 5. SPATIAL DISTANCE IN THE DE SITTER WORLD WHEN THE METRIC IS GIVEN IN THE STANDARD FORM.

The more usual form of the metric of the de Sitter world is

$$(5.1) \quad ds^2 = \left(1 - \frac{r^2}{R^2}\right) dt^2 - \frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.$$

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This is derivable from the form (4.1) by putting *

$$(5.2) \quad \begin{cases} x = r(R^2 - r^2)^{-\frac{1}{2}} \sin \theta \cos \phi \operatorname{sech} (t/R) \\ y = r(R^2 - r^2)^{-\frac{1}{2}} \sin \theta \sin \phi \operatorname{sech} (t/R) \\ z = r(R^2 - r^2)^{-\frac{1}{2}} \cos \theta \operatorname{sech} (t/R) \\ u = \tanh (t/R). \end{cases}$$

If $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$ are the world-coordinates of the star, then of course they are connected with the original coordinates $(\bar{u}, \bar{x}, \bar{y}, \bar{z})$ by relations corresponding to (5.2); in fact

$$(5.3) \quad \begin{cases} \bar{x} = \bar{r}(R^2 - \bar{r}^2)^{-\frac{1}{2}} \sin \bar{\theta} \cos \bar{\phi} \operatorname{sech} (\bar{t}/R) \\ \dots \dots \dots \end{cases}$$

Using this, it is easily shown by substitution in (4.2) that

$$(5.4) \quad \dots \dots \dots 2\Omega = R^2(\arg \cosh Q)^2$$

where

$$Q = \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} \left(1 - \frac{\bar{r}^2}{R^2}\right)^{\frac{1}{2}} \cosh \frac{t - \bar{t}}{R} + \frac{r\bar{r}}{R^2} \{ \cos \theta \cos \bar{\theta} + \sin \theta \sin \bar{\theta} \cos (\phi - \bar{\phi}) \},$$

the star being at the world-point (t, r, θ, ϕ) .

Suppose that the observer is "at rest" at the spatial origin; that is, suppose that $\bar{r} = 0$, an assumption consistent with the requirement that the observer's world-line shall be a geodesic. By (5.1) his proper-time τ is then given by

$$d\tau^2 = d\bar{t}^2$$

so we may take

$$(5.5) \quad \dots \dots \dots \tau = t.$$

Hence

$$(5.6) \quad \dots \dots \dots Q = \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} \cosh \frac{t - \tau}{R}.$$

Now

$$\begin{aligned} \delta &= \partial\Omega / \partial\tau \\ &\rightarrow R^2 \partial Q / \partial\tau \end{aligned}$$

as above; so by (5.6),

$$(5.7) \quad \dots \dots \dots \delta = R \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} \sinh \frac{\tau - t}{R}.$$

Making use of the fact that $Q = 1$, which gives

$$\cosh \frac{\tau - t}{R} = \left(1 - \frac{r^2}{R^2}\right)^{-\frac{1}{2}}$$

and therefore

$$\sinh \frac{\tau - t}{R} = \frac{r}{R} \left(1 - \frac{r^2}{R^2}\right)^{-\frac{1}{2}},$$

we at once get

$$(5.8) \quad \dots \dots \dots \delta = r.$$

* Whittaker, *loc. cit.*, 96, and Eddington, *Mathematical Theory of Relativity* (1924), 161.

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Hence:

The coordinate r measures spatial distance from the (spatial) origin according to the definition of this paper.

It is interesting to compare this with Eddington's remark that "in so far as the distances of celestial objects are determined by parallaxes or parallactic motions, the coordinate r will agree with their accepted distances."*

To find the formula for Δ corresponding to (5.8), it is necessary to substitute from (5.2) and (5.3) in (1.3). Since we are assuming $\bar{r}=0$, we get by (5.3)

$$(5.9) \quad \begin{cases} \bar{x} = \bar{y} = \bar{z} = 0 \\ \bar{u} = \tanh(t/R). \end{cases}$$

So, by (1.3),

$$\Delta = \frac{R(1 + \sum x^2)^{\frac{1}{2}} (\sum x^2)^{\frac{1}{2}}}{(1 - u^2 + \sum x^2)^{\frac{1}{2}} - u(\sum x^2)^{\frac{1}{2}}}.$$

Substituting from (5.2), we get

$$\Delta = \frac{rR}{(R^2 - r^2)^{\frac{1}{2}}} \frac{(R + r \tanh(t/R))^{\frac{1}{2}}}{(R - r \tanh(t/R))^{\frac{1}{2}}}.$$

This therefore is Whittaker's formula for the spatial distance of a star from the origin in the de Sitter world.

§ 6 The Einstein World.

The metric of Einstein's space-time is given by[†]

$$(6.1) \quad ds^2 = dt^2 - R^2 [dX^2 + \sin^2 X (d\theta^2 + \sin^2 \theta d\phi^2)]$$

where R is a constant. Putting $r = R \sin X$, this reduces to the more convenient form

$$(6.2) \quad ds^2 = dt^2 - \left(\frac{dr^2}{1 - \mu r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

where $\mu = \frac{1}{R^2}$. Writing

$$L = t'^2 - \frac{r'^2}{1 - \mu r^2} - r^2 \theta'^2 + r^2 \sin^2 \theta \phi'^2,$$

dashes denoting derivatives with respect to s , the differential equations of the geodesics are

$$\frac{d}{ds} \left(\frac{\partial L}{\partial t'} \right) - \frac{\partial L}{\partial t} = 0,$$

$$\frac{d}{ds} \left(\frac{\partial L}{\partial r'} \right) - \frac{\partial L}{\partial r} = 0,$$

and two similar equations. So the geodesics are

* *Op. cit.*, 163. † *Ibid.*, 156, (67.12).

given by

$$(6.3) \quad \frac{d^2 t}{ds^2} = 0$$

$$(6.4) \quad \frac{d}{ds} \left(\frac{1}{1-\mu r^2} \frac{dr}{ds} \right) - \frac{\mu r}{(1-\mu r^2)^2} \left(\frac{dr}{ds} \right)^2 - r \left(\frac{d\theta}{ds} \right)^2 - r \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 = 0$$

$$(6.5) \quad \frac{d}{ds} \left(r^2 \frac{d\theta}{ds} \right) - r^2 \sin \theta \cos \theta \left(\frac{d\phi}{ds} \right)^2 = 0$$

$$(6.6) \quad \frac{d}{ds} \left(r^2 \sin^2 \theta \frac{d\phi}{ds} \right) = 0.$$

The last three equations are satisfied by $r=0$, and the first by $t=s$. So it will be supposed that the observer, at the instant of making his observation, is at rest at the spatial origin, so that $\bar{t} = \tau$, $\bar{r} = 0$.

To find the geodesic distance \underline{s} between the observer and a star at the world-point (t, r, θ, ϕ) , we assume that \underline{s} is independent of θ and ϕ , an assumption which is justified below.* So making θ and ϕ both constant, equations (6.5) and (6.6) are immediately satisfied. Equation (6.3) gives

$$(6.7) \quad t = A + Bs,$$

where A, B are arbitrary constants. To avoid the necessity of integrating (6.4), we observe that $L=1$ is an integral of the equations of the geodesics, a relation which, combined with the assumptions that $\theta = \text{const.}$, $\phi = \text{const.}$, gives

$$(6.8) \quad \left(\frac{dr}{ds} \right)^2 = (B^2 - 1)(1 - \mu r^2).$$

Since it is required that $s=0$ when $t=\tau$ and $r=0$, it follows that $A = \tau$; and hence, rewriting (6.7)

* That the assumption is a reasonable one is evident from the form of the metric, the "spatial" part of which has spherical symmetry about $r=0$.

and integrating (6.8), we get

$$(t - \tau) = B s$$

$$\begin{aligned} \text{and } (B^2 - 1)^{\frac{1}{2}} s &= \int_0^r \frac{dr}{1 - \mu r^2} \\ &= \frac{1}{\mu^{\frac{1}{2}}} \arcsin(\mu^{\frac{1}{2}} r). \end{aligned}$$

Eliminating B between the last two equations and solving for s , we get

$$\begin{aligned} 2\Omega &= s^2 \\ &= (t - \tau)^2 - \frac{1}{\mu} (\arcsin \mu^{\frac{1}{2}} r)^2 \end{aligned}$$

or

$$(6.9) \quad 2\Omega = (t - \tau)^2 - R^2 (\arcsin \frac{r}{R})^2,$$

restoring the original constant $R = \frac{1}{\mu^{\frac{1}{2}}}$. This is therefore the four-dimensional interval between the star at the world-point (t, r, θ, ϕ) and the observer for whom $\bar{t} = \tau$, $\bar{r} = 0$. It is easily verified that Ω , defined thus, satisfies the partial differential equation $\Omega^\lambda \Omega_\lambda = 2\Omega$ (which in this case is $(\frac{\partial \Omega}{\partial t})^2 - (1 - \mu r^2) (\frac{\partial \Omega}{\partial r})^2 - \frac{1}{r^2} (\frac{\partial \Omega}{\partial \theta})^2 - \frac{1}{r^2 \sin^2 \theta} (\frac{\partial \Omega}{\partial \phi})^2 = 2\Omega$).

Moreover, as $\mu \rightarrow 0$ the space defined by (6.2) becomes galilean, and (6.9) gives $2\Omega = (t - \tau)^2 - r^2$, as it should.

The assumption that Ω is independent of θ, ϕ is therefore justified.

By (6.9),

$$\frac{\partial \Omega}{\partial r} = -(t - \tau)$$

Putting $\Omega = 0$, we obtain

$$(t - \tau)^2 = R^2 (\arcsin \frac{r}{R})^2,$$

whence the spatial distance of the observer from the star is (according to my own definition)

$$\delta = \left(\frac{\partial \Omega}{\partial r} \right)_{\Omega=0}$$

that is,

$$(6.10) \quad \delta = R \arcsin \frac{r}{R}.$$

The sign is so chosen that as $\mu \rightarrow 0$ (i.e. $R \rightarrow \infty$) and the space to galilean form, $\delta \rightarrow r$.

So (6.10) is the formula for spatial distance from the origin in the Einstein World. Taking it as radius-vector by writing $\rho = \delta = R \arcsin \frac{r}{R}$, the equation (6.2) defining the metric becomes

$$(6.11) \quad ds^2 = dt^2 - [d\rho^2 + R^2 \sin^2\left(\frac{\rho}{R}\right) d\theta^2 + R^2 \sin^2\left(\frac{\rho}{R}\right) \sin^2\theta d\phi^2].$$

Concluding Note.

I have not had the leisure to pursue further the problem of defining spatial distance in a curved world. The definition of this chapter, based as it is on an argument which is essentially mathematical (in that it is deduced from a geometrical partitioning of space-time into space and time) must be correlated with a practical means of measuring spatial distances if it is to be of any real use. My suggestion that the definition may correspond to the measurement of spatial distance by parallaxes is insufficient in itself. My friend Mr. J. M. H. Etherington is however at present engaged in an investigation of the connection between the two definitions of spatial distance and the actual methods adopted by astronomers, and I hope that when his investigation is completed some real light may be thrown on the nature of spatial distance as defined by myself.

22 March 1932.