Periodic homogenization of Dirichlet problem for divergence type elliptic operators

Hayk Aleksanyan

Doctor of Philosophy
University of Edinburgh
2015

Declaration

The candidate declares that this thesis was composed by himself and that the work contained herein is his own, except where explicitly stated otherwise in the text. Where the work included in the thesis is from jointly-authored publications, the contribution of the candidate is substantial, so that to be eligible for using in the thesis. At places where the work of others has been used, an appropriate reference has been given.

The candidate also declares that the work has not been submitted for any other degree or professional qualification.

Hayk Aleksanyan

Contents

Acknowledgements

Forunately one is given an opportunity to acknowledge various inputs and lucky arrangement of events that have lead to this day. I would like to start by thanking my family back in Armenia, for their constant love and support, and for being a great source of motivation at critical times for me.

I express my sincere gratitude to my scientific advisor Dr Aram Karakhanyan for a large amount of time and energy he has invested in my mathematical development. Numerous discussions on various topics and his kind openness for discussions are gratefully appreciated. At several places Aram's insistence and encouragement to pursue a certain line of thought were instrumental. I also thank Aram for providing me with the necessary freedom I needed to work, while guiding me with his advice, and encouraging in me the search for beauty in mathematics.

My thanks are also due to Dr Tadahiro Oh for acting as my second advisor, and to Dr Pieter Blue for mentoring my scholarship-related activities.

It is my great pleasure to thank Professors Henrik Shahgholian, and Per Sjölin for their collaboration, and the amount of knowledge and expertise they have so generously shared with me. While still a graduate student, they treated me as a colleague, showing me the joy of a mathematical collaboration. I am deeply indebted to Professor Henrik Shahgholian for introducing me to the topic of homogenization, for countless discussions we had on different pieces of mathematics, and for his constant support, as well as advice. His encouragement, and flux of positive energy have certainly kept me afloat several times in recent years.

I also thank all my teachers from Yerevan State University and especially Professors Arthur Sahakian, Grigori Karagulyan, and Michael Poghosyan, for teaching me mathematics over the years, and inspiring by their example.

Special thanks are due to Mrs Gillian Law, our Graduate School Administrator, for her very kind and professional help with all sorts of paperwork starting from the admission process up to now.

During my PhD studies I was financially supported by The University of Edinburgh through Principal's Career Development Scholarship, and Edinburgh's Global Research Scholarship. This generous support is gratefully acknowledged. I have also had a privilege to enjoy the hospitality of several institutions during my long-term visits to KTH Royal Institute of Technology at Stockholm, Issac Newton Institute for Mathematical Sciences at Cambridge, and Institut Mittag-Leffler at Djursholm. I have benefited from these visits enormously, and wish to thank the organizers of the corresponding mathematical events for inviting me, as well as these institutions, for their support and for providing excellent working conditions.

The unconditional support I had from all my friends made a lot of things an order of magnitude easier for me. In particular, since the very first day of my arrival in Edinburgh my dear friend Ani Karapetyan has been irreplaceable. I thank her for the support and encouragement she gave to me, and for all the happiness our friendship brought into my years in Edinburgh. I also thank all my friends that are a large distance away from me now, but we still cherish our firm friendship over the seas.

I am grateful to my cousin Tigran Aleksanyan, for his invaluable help with settling in UK, and his kind support.

Expressions, and ideas literally in any wording can all be improved, some important points may have been forgotten, thus it is really very hard to have a text that one can feel fully satisfied with. In conclusion, to the people who gave so much to me, I hope I can be as generous in return as they were with me, should any need arise.

Thank you all!

To Ani, for making Edinburgh home to me,
and to my family, for so many reasons.

Lay summary

It is well-known that a vast amount of natural, scientific, and technological phenomena in mathematical terms are modeled by problems involving partial differential operators. The systems which are being studied, in many cases have an intrinsic structure, the simplest of which being a certain pattern repeating itself periodically. One then naturally thinks about the given pattern as a building block of the entire system, and having the way they are arranged (say periodically) one can expect to get an information about the behavior of the entire system at larger scales. In other words, we are interested in a passage from *micro* to *macro* scales. These type of problems are being addressed by the *theory of homogenization.* To have a simple example in mind for such a situation, think about composites, which have different individual materials as their building blocks. At larger (macro) scales the composite looks homogeneous, and characteristics of the individual components are somewhat dissolved into each other by that determining the behavior of the composite, while at microscopic level one may spot properties of each component separately.

The thesis is focused on periodic homogenization of Dirichlet problem for elliptic operators in divergence form. These type of boundary value problems are involved in material physics, in particular in linear elasticity and thermics. In the first part of the thesis we develop a Fourier-analytic framework that enables us to handle a class of homogenization problems admitting integral representations of solutions. As one particular outcome of our methods, we were able to establish for the first time homogenization of Dirichlet boundary value problem in its optimal form with respect to the speed of convergence. In the second part of the thesis we study boundary layer phenomenon associated with the Dirichlet problem. When microscopic scale of the system is too small, solutions to boundary value problems show strong concentration near the boundaries. This is called *boundary layer phenomenon* and is perhaps the most serious obstacle in the study of homogenization problems considered here. In this regard we study two main issues. First, we analyze how regularly the effect of the boundary layers is changing along the boundary, and show that there is a certain regular pattern for media that have layered structure. Second, we examine how fast the effect of boundary layers is evolving when we approach the boundaries. Here by a novel construction we establish that it may take arbitrarily large time for the effect of boundary layers to regularize.

Abstract

The thesis studies homogenization of Dirichlet boundary value problems for divergence type elliptic operators, and the associated boundary layer issues. This type of problems for operators with periodically oscillating coefficients, and fixed boundary data are by now a classical topic largely due to the celebrated work by Avellaneda and Lin from late 80's. The case when the operator and the Dirichlet boundary data exhibit periodic oscillations simultaneously was a longstanding open problem, and a progress in this direction has been achieved only very recently, in 2012, by Gérard-Varet and Masmoudi who proved a homogenization result for the simultaneously oscillating case with an algebraic rate of convergence in L^2.

Aimed at understanding the homogenization process of oscillating boundary data, in the first part of the thesis we introduce and develop Fourier-analytic ideas into the study of homogenization of Dirichlet boundary value problems for elliptic operators in divergence form. In smooth and bounded domains, for fixed operator and periodically oscillating boundary data we prove pointwise, as well as L^p convergence results the homogenization problem. We then investigate the optimality (sharpness) of our L^p upper bounds. Next, for the above mentioned simultaneously oscillating problem studied by Gérard-Varet and Masmoudi, we establish optimal L^p bounds for homogenization in some class of operators.

For domains with non smooth boundary, we study similar boundary value homogenization problems for scalar equations set in convex polygonal domains. In the vein of smooth boundaries, here as well for problems with fixed operator and oscillating Dirichlet data we prove pointwise, and L^p convergence results, and study the optimality of our L^p bounds. Although the statements are somewhat similar with the smooth setting, challenges for this case are completely different due to a radical change in the geometry of the domain.

The second part of the work is concerned with the analysis of boundary layers arising in periodic homogenization. A key difficulty toward the homogenization of Dirichlet problem for elliptic systems in divergence form with periodically oscillating coefficients and boundary condition lies in identification of the limiting Dirichlet data corresponding to the effective problem. This question has been addressed in the aforementioned work by Gérard-Varet and Masmoudi on the way of proving their main homogenization result. Despite the progress in this direction, some very basic questions remain unanswered, for instance the regularity of this effective data on the boundary. This issue is directly linked with the up to the boundary regularity of homogenized solutions, but perhaps more importantly has a potential to cast light on the homogenization process. We initiate the study of this regularity problem, and prove certain Lipschitz continuity result. The work also comprises a study on asymptotic behavior of solutions to boundary layer systems set in halfspaces. By a new construction we show that depending on the normal direction of the hyperplane, convergence of the solutions toward their tails far away from the boundaries can be arbitrarily slow. This last result, combined with the previous studies gives an almost complete picture of the situation.

Notation

Here we list some notation and conventions that are commonly used throughout the text. Notation, specific for a particular chapter or section, will be introduced as needed.

- **(Constants)** We denote by c, C, C_1, C_2,... generic positive constants that may vary from formula to formula. These constants are allowed to depend on problem-related ingredients, however they are independent of varying parameters of the problem (if any) under consideration. We write $x \asymp y$ if there are absolute constants $a, b > 0$ such that $ax \leq y \leq by$. Likewise, we let $x \lesssim_\delta y$ if there is a constant $a_\delta > 0$, depending on parameter δ and otherwise considered absolute, for which $x \leq a_\delta y$. We may also drop the parameter δ in the last notation, meaning that we have a similar inequality with an absolute constant.

- **(Matrices)** For a positive integer n we denote by $M_n(\mathbb{R})$ the set of $n \times n$ matrices with real entries. Similarly $O_n(\mathbb{R})$ denotes the set of $n \times n$ real orthogonal matrices. For a matrix M having dimensions $n \times p$, $n, p \in \mathbb{N}$ by M^t (or M^T, if t is in use, and vice versa) we will denote the transposed matrix of M.

- **(Some special subsets of $\mathbb{R}^d$)** Throughout the text, $d \in \mathbb{N}$ stands for the dimension of the Euclidean space $\mathbb{R}^d$. By a *domain* in $\mathbb{R}^d$ we mean open and connected subset of $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$ we denote by ∂D its boundary. A *hypersurface* in $\mathbb{R}^d$ is a $(d-1)$-dimensional embedded submanifold of $\mathbb{R}^d$.

  $B_r(x)$ and $B(x, r)$ both denote an open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$.

  $\mathbb{RQ}^d$ is the set of all vectors of $\mathbb{R}^d$ that are scalar multiples of vectors with all components being rational numbers. We refer to elements of the complement of $\mathbb{RQ}^d$ as *irrational* vectors (or irrational directions if they have length one).

  $\mathbb{S}^d$ denotes the unit sphere of $\mathbb{R}^{d+1}$.

  $\mathbb{T}^d$ is the unit torus of $\mathbb{R}^d$, i.e. the quotient space $\mathbb{R}^d/\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the integer lattice.

- **(Measures and norms on $\mathbb{R}^d$)** For a vector $x \in \mathbb{R}^d$ by $|x|$ we denote its standard Euclidean norm. If a confusion may arise, we use $||\cdot||$ to denote the norm of x. In $\mathbb{R}^d$ we will be working with the standard Lebesgue measure, and on smooth hypersurfaces $S \subset \mathbb{R}^d$ by default the surface measure would be the usual one, i.e. induced by the Lebesgue measure of $\mathbb{R}^d$. For a subset $E \subset \mathbb{R}^d$, we may write $vol_d(E)$ or $vol_{d-1}(E)$ if we wish to emphasize the dimensionality of the measure under the consideration.

- **(Functional spaces)** Let $D \subset \mathbb{R}^d$ be a domain. For $1 \leq p \leq \infty$, and $N \in \mathbb{N}$ we let $L^p(D; \mathbb{R}^N)$ be the standard Lebesgue space of $\mathbb{R}^N$-valued functions. We may drop $\mathbb{R}^N$ from the notation, if the dimension of the set of values is clear from the context. For $k \in \mathbb{N}$ by $C^k(D)$ we denote the space of k times continuously differentiable functions defined on D. If $0 < \alpha \leq 1$ then $C^{k,\alpha}(D)$ is the subset of $C^k(D)$ where the all k-th order derivatives are α-Hölder continuous on D. By $C^\infty(D)$ we denote the space of infinitely differentiable functions, and by $C^\infty(\mathbb{T}^d)$-the space of infinitely differentiable $\mathbb{Z}^d$-periodic functions. In the text for $k \in \mathbb{N}$, and $1 \leq p \leq \infty$, $W^{k,p}(D)$ denotes the standard Sobolev space with smoothness index k and integrability index p. When $p = 2$ we set $H^k := W^{k,2}$, and we let H^k_0 be the subspace of H^k consisting of trace 0 elements.

  The same convention of adding or dropping $\mathbb{R}^N$ as we have mentioned for L^p applies to all the functional spaces defined above.

- **(Miscellaneous)** By ∇ we denote the gradient operator, and by ∂x^k, ∂_{x_k}, or ∂_k we denote the differentiation in x_k coordinate.

Throughout the text, if not stated otherwise, we adopt the convention that summation is taken over repeated indices (*Einstein summation convention*). For example, writing $a_{ij}b_{pj}$ will mean $\sum_j a_{ij}b_{pj}$.

Chapter 1

Introduction

The objective of the present chapter is twofold. First, it gives the reader a general idea of homogenization, and then glimpses into the mathematical theory behind that. Next, it discusses the class of problems addressed by the thesis along with some motivation and background. The chapter ends by giving an outline of our main results.

1.1 The idea of homogenization: a prelude

A plethora of phenomena across science and technology are modeled by boundary value problems involving partial differential operators. In many cases systems under the consideration carry some structure, such as periodicity, almost periodicity, or stationary ergodicity. If, for example, in a presence of the periodic structure the size of the period is very small compared to the size of a region where the study is being conducted, then the problem can be seen as an attempt to determine macroscopic characteristics of the system from the information on microscopic scales. If the size of a period is too small, then applying numerical analysis directly is practically out of reach, since then a typical mesh must have size even smaller than the period, which would result in enormous computational costs. This raises a need for mathematical tools that would address the passage from *microscopic* to *macroscopic*, bringing us to the realm of the mathematical theory of *homogenization.* In brief this theory develops tools for asymptotic analysis of problems that naturally have widely varying descriptive scales. As we will see below problems of these type are ubiquitous in nature and technology.

Let us consider few examples to put the discussion on a more palpable grounds. Perhaps one of the most illuminating and easily described scenarios where the concept of homogenization appears in a natural way, is the construction of *composite materials.* Assume we have some given number of components of which we wish to create a composite. Let Ω be the fixed domain which is to be occupied by the final material, and for simplicity let us assume that Ω is the unit square in $\mathbb{R}^2$. Suppose our goal is to create a uniform mixture out of the given ingredients. This will mean that if we take a small piece of Ω it should have the same proportion of ingredients as the entire Ω itself. To achieve this one may take some $N \in \mathbb{N}$ and partition Ω into N^2 equal squares. Next we put the original ingredients in a certain configuration into one of the smaller squares and then copy the same structure into the rest of the squares of our partition. If N is large enough this construction will to some extent resemble the uniformity criteria we were aiming at. Also if we let $\varepsilon := 1/N$ then the material will have a *periodic* structure with periodicity ε by our construction. We thus have $\varepsilon > 0$ as some quantifier of the mixing process, and taking $\varepsilon \to 0$ will result in ideal mix of components. We have that our intermediate composites, determined by ε, have heterogeneity at scale ε, but if ε is small enough the material will be seen as somewhat *homogeneous* at unit scales if we zoom out. The problem here is to understand the properties of the mixture, i.e. the final homogeneous material, given the information on individual components and the structure of the material (e.g. periodicity in this case). Now, if $\varepsilon > 0$ is small but stays above a certain threshold, say we do not reach the molecular level, then the intermediate composites can be characterized by equations of continuum mechanics. This will bring us to the study of certain PDE problems posed in domain Ω, and with an operator depending on ε. As our ultimate goal we will be concerned with the asymptotics of solutions as $\varepsilon \to 0$.

Of course the range of applications of the theory of homogenization goes far beyond the studies of composites. To illustrate the wide range of spectrum covered by the theory we briefly list without further details some small selection of areas where the theory is used.

The next example is borrowed from the recent book by Chechkin, Piatnitski, and Shamaev [20]. Consider the movement of a small satellite of Earth; it orbits Earth, and by the same time is moving rapidly around its center of mass. Apart from the gravity of Earth the satellite is also subject to some number of weak forces, such as the magnetic field of the planet, or the pressure of light, etc. For any small time interval the cumulative effect of these forces is negligible, however these effects can slowly evolve to something considerable given a sufficiently large time span. We thus have two very different descriptive scales of the dynamics of the satellite. On one hand it is the rapid movement of the object around itself, on the other hand we have a very slowly evolving process that can affect the first movement. Due to a disparity in the scales we again need an asymptotic analysis, which is being covered by the theory of homogenization.

Another nice example, considered in [51], comes from the problem of sound propagation through a liquid populated sparsely by bubbles. Here the sparsity assumption is incorporated in a fact that the bubble spacing is much greater than the radius of bubbles, and the aim is to find an effective equation for the propagation of sound which has wavelength much larger than the bubble spacing.

In his book [8] G. Allaire discusses applications of homogenization theory to shape optimization problems. These include, for example, problems of finding the optimal shape of a domain that would be of maximal conductivity, given some loading conditions.

Continuing on the theme of applications, let us mention a recent thesis [19] which develops a macroscopic model of low complexity for the ventilation process of the human lung, where the ventilation is seen as air transport through porous medium, and the mathematical apparatus is based on the concept of two-scale convergence in homogenization. Finally we refer to [24] for applications of homogenization theory in ocean-atmospheric science, and to [58] for plasma physics.

Due to the broad range of applications the theory has developed a variety of ideas and techniques, and currently has a very rich and diverse mathematical theory. To outline a typical setup of homogenization problems, let us get back to the search of the procedure addressing the passage from *micro* to *macro* scale. As we have discussed, given some structure, heterogeneous materials can be regarded as homogeneous at large scales. In other words making the size of heterogeneity smaller and smaller while preserving the structure in which they are arranged (say periodicity), amounts to smoothing effects at larger scales, and the problem is then to describe these final states. To fix the ideas, assume the macroscopic scale has unit length, and microscopic structure has length scale $\varepsilon > 0$, where ε is much smaller than 1. Then properties of the microstructure will determine the effective properties of the macro scale through some differential equation or system of equations. Suppose the model is given by

$$\mathcal{A}_\varepsilon u(x) = f(x), \qquad x \in D,$$

where $\mathcal{A}_\varepsilon$ models the microstructure at scale ε in domain D, and u is subject to some boundary conditions. Now, for each ε we let u_ε be the solution to the aforementioned problem from some functional space $\mathcal{H}$ (which is very much problem dependent). The following list represents some typical issues addressed by the theory of homogenization.

- Does the limit of u_ε exist in some given topology?
- Assume the answer to the first question is positive, and let u_0 be the limit in question. Then, does it solve some differential operator?
- Here as well, suppose that the answer to the first question is positive, and let u_0 be the limit. Then is it possible to derive some effective estimates on the rate of convergence of u_ε toward u_0 ?

Let us finally mention that arguably the first appearance of ideas involving homogenization dates back to the work [50] from 1881 by J.C. Maxwell, where he studied effective conductivity of media with small concentrations of randomly arranged inclusions. Though the theory has been in development since then, the term "homogenization" was introduced much later, in 1974 by I. Babuška [14]. A long list of prominent mathematicians largely contributed to the theory, including E. De Giorgi, J.L. Lions, P.L. Lions, L. Caffarelli, P. Souganidis, and there is now a vast literature on the subject. Due to a limited space we have no chance to list even the portion of it, so our selection is slightly arbitrary. We refer to monographs [15], [16], [20], [23], and [43], survey articles [17], [26], and papers [18], [25], [37], and [49] for some nice and diverse mathematics.

1.1.1 An important example

To give the reader a flavor of the mathematical idea of homogenization let us consider the following simple, yet nontrivial example. The analysis to be presented is very well known, and can be found in practically any monograph on the subject, see for example [16].

Assume we are given a bounded interval $\Omega = (x_0, x_1) \subset \mathbb{R}$, and $f \in L^2(\Omega)$. Let also $a(x)$ be 1-periodic, bounded, measurable function such that $a(x) \geq \alpha_0 > 0$ almost everywhere in $\mathbb{R}$. For $\varepsilon > 0$ set $a^\varepsilon(x) := a(x/\varepsilon)$ and consider the following problem

$$-\left(a^\varepsilon(x) u_\varepsilon'(x)\right)' = f(x) \quad \text{in } \Omega, \quad \text{where } u_\varepsilon \in H_0^1(\Omega). \tag{1.1}$$

This problem models stationary heat conduction in composite material whose microscopic properties vary rapidly. Here a^ε represents the thermal conductivity of the material, and Ω is the domain occupied by the material. The size of Ω defines the macroscopic length-scale of the problem, whereas ε shows the size of heterogeneity of the material. The goal is to freed the problem from ε.

For $u, v \in H^1(\Omega)$ define

$$a^\varepsilon(u, v) := \int\limits_\Omega a^\varepsilon(x) u'(x) v'(x) dx.$$

Clearly $a^\varepsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is a bounded and coercive bilinear form by means of which one may write the variational formulation of (1.1) in the following way:

$$\text{find } u_\varepsilon \in H_0^1(\Omega) \text{ satisfying } a^\varepsilon(u_\varepsilon, v) = (f, v), \ \forall v \in H_0^1(\Omega), \tag{1.2}$$

where $(f, v) := \int_\Omega f v dx$. Observe that in view of Lax-Milgram's lemma the problem (1.2) admits a unique solution u_ε, for any $\varepsilon > 0$. Here we will be interested in the limit behavior of u_ε as $\varepsilon \to 0$, and determination of the homogenized (effective) operator.

By coercivity and Hölder's inequality we have

$$\alpha_0 ||u_\varepsilon'||_{L^2}^2 \leq a^\varepsilon(u_\varepsilon, u_\varepsilon) \leq ||f||_{L^2} ||u_\varepsilon||_{L^2}.$$

From this, and Poincaré inequality we obtain

$$||u_\varepsilon||_{H^1(\Omega)} \leq C ||f||_{L^2(\Omega)}, \tag{1.3}$$

where the constant C is independent of ε and f. Since we have a bounded sequence in reflexive Banach space we may extract a subsequence, still denoted by u_ε, weakly converging to some $u_0 \in H_0^1(\Omega)$. On the other hand, $a(x)$ being 1-periodic, implies that a^ε converges weak* to $\int_0^1 a(x) dx := \mathcal{M}(a)$ in $L^\infty(\Omega)$. Given these, one may guess that the homogenized equation should read $\mathcal{M}(a) \int_\Omega u_0' v' dx = (f, v)$, for all $v \in H_0^1(\Omega)$, however, in general this is not the case!

To find the correct homogenized equation we proceed as follows. Set $\xi^\varepsilon := a^\varepsilon u_\varepsilon'$, and observe that due to (1.3) we have $||\xi^\varepsilon||_{L^2} \leq C ||f||_{L^2}$. On the other hand $-(\xi^\varepsilon)' = f$, therefore $||\xi^\varepsilon||_{H^1} \leq C ||f||_{L^2}$. It follows that there exists a subsequence of ξ^ε, still relabeled as ξ^ε, weakly converging to some element $\xi^0 \in H^1(\Omega)$. By Rellich's theorem the identity mapping compactly embeds $H^1(\Omega)$ into $L^2(\Omega)$, from which we get $\xi^\varepsilon \to \xi^0$ strongly in $L^2(\Omega)$. It follows that

$$\frac{1}{a^\varepsilon(x)}\xi^\varepsilon \to \mathcal{M}\left(\frac{1}{a}\right)\xi^0, \text{ weakly in } L^2(\Omega).$$

On the other hand we have weak convergence of derivatives of u_ε to derivative of u_0, hence by the uniqueness of weak limit, from the last expression we obtain

$$\xi^0 = \frac{1}{\mathcal{M}\left(\frac{1}{a}\right)}u_0'.$$

This equality, and the definitions of ξ^ε, and ξ^0 imply

$$-\frac{d}{dx}\left(\frac{1}{\mathcal{M}\left(\frac{1}{a}\right)}\frac{du_0}{dx}\right) = f,$$

which is the *homogenized* equation for the limit $u_0 \in H_0^1(\Omega)$.

Finally, observe that since the homogenized operator is independent of the weakly converging subsequence of u_ε, we get weak convergence in H^1 of u_ε to u_0 without extracting a subsequence.

1.2 The class of problems we study

The thesis is mainly concerned with homogenization of boundary value problems for divergence type elliptic operators. To motivate the problem, and fix some ideas let us start by introducing the method of *two-scale expansion*, a classical tool in mathematical theory of homogenization.

Assume we have a bounded domain $D \subset \mathbb{R}^d$, $(d \geq 2)$, $N \in \mathbb{N}$ is fixed, along with a family of measurable functions $A = A^{\alpha\beta}(x) \in M_N(\mathbb{R})$, $x \in \mathbb{R}^d$, indexed by $1 \leq \alpha, \beta \leq d$ and with values in the set of matrices $M_N(\mathbb{R})$. For each $\varepsilon > 0$ we let $\mathcal{L}_\varepsilon$ be a differential operator, where the i-th component of its action on a vector function $u = (u_1, ..., u_N)$ is defined as follows

$$(\mathcal{L}_\varepsilon u)_i(x) = -\left(\nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right)\nabla u\right)_i(x) = -\partial_{x_\alpha}\left[A_{ij}^{\alpha\beta}\left(\frac{\cdot}{\varepsilon}\right)\partial_{x_\beta}u_j\right], \qquad (1.4)$$

$1 \leq i \leq N$. We impose the following conditions:

- **(Ellipticity)** there exists a constant $\lambda > 0$ such that for any $x \in \mathbb{R}^d$, and any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{dN}$ one has
$$\lambda\xi_i^\alpha\xi_i^\alpha \leq A_{ij}^{\alpha\beta}(x)\xi_i^\alpha\xi_j^\beta \leq \frac{1}{\lambda}\xi_i^\alpha\xi_i^\alpha. \qquad (1.5)$$
- **(Periodicity)** A is $\mathbb{Z}^d$-periodic, i.e. $A(x+h) = A(x)$, for all $x \in \mathbb{R}^d$, and any $h \in \mathbb{Z}^d$.

For a fixed $f \in L^2(D;\mathbb{R}^N)$ consider the following problem

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x) = f(x),\ x \in D \qquad \text{and} \qquad u^\varepsilon = 0,\ x \in \partial D. \tag{1.6}$$

By Lax-Milgram's lemma for each $\varepsilon > 0$ we have a unique weak solution $u^\varepsilon \in H_0^1(D;\mathbb{R}^N)$ to (1.6). Then, standard energy estimates give $||u^\varepsilon||_{H^1(D)} \leq C||f||_{L^2(D)}$, with C independent of $\varepsilon > 0$. This shows that we may extract a subsequence from u^ε weakly converging in $H^1(D)$ and hence strongly in $L^2(D)$. The aim is to describe the possible limits (we do not know *à priori* if there is only one weak limit as $\varepsilon \to 0$), along with some error estimates for convergence; in short we are interested in *homogenization* of problem (1.6). For that, we will invoke one of the classical methods, namely *two-scale expansion* of solutions, the main idea of which is to attribute the effects that come from rapid oscillations to a new, independent variable. To illustrate the method, we will mostly follow the exposition of [16], [7], and [54].

1.2.1 The homogenized operator and some error estimates

Assume the solution u^ε to (1.6) adimts the following expansion

$$u^\varepsilon(x) = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}\right) + \dots, \tag{1.7}$$

where for each $k = 0, 1, \dots$ the profile $u^k(x, y)$ is $\mathbb{Z}^d$-periodic in y-variable. We do not specify any modes of convergence for the expansion, since for now it is formal, and is only used to guess a reasonable approximation for u^ε. Looking ahead let us fix here that the initial terms of (1.7) will be justified in due course.

We will treat $x \in D$ and $y \in \mathbb{R}^d$ as two independent variables, keeping in mind that y represents the oscillatory variable x/ε, and thus for a function $\phi(x, y)$ the following differentiation rule will take place $\nabla\phi(x, x/\varepsilon) = \nabla_x\phi(x, x/\varepsilon) + \varepsilon^{-1}\nabla_y\phi(x, x/\varepsilon)$. With this separation of scales we get a decomposition of the original operator into

$$\mathfrak{L}_\varepsilon = \varepsilon^{-2}\mathfrak{L}_0 + \varepsilon^{-1}\mathfrak{L}_1 + \mathfrak{L}_2, \tag{1.8}$$

where

$$\begin{aligned}
\mathfrak{L}_0 &= -\nabla_y \cdot A(y)\nabla_y, \\
\mathfrak{L}_1 &= -\nabla_y \cdot A(y)\nabla_x - \nabla_x \cdot A(y)\nabla_y, \\
\mathfrak{L}_2 &= -\nabla_x \cdot A(y)\nabla_x.
\end{aligned}$$

Next, plugging expansion (1.7) into (1.6) and identifying powers of ε on both sides of the equation, leads to the following cascade of equalities

$$\begin{aligned}
&\mathcal{L}_0 u^0 = 0,\\
&\mathcal{L}_0 u^1 + \mathcal{L}_1 u^0 = 0,\\
&\mathcal{L}_0 u^2 + \mathcal{L}_1 u^1 + \mathcal{L}_2 u^0 = f,\\
&\mathcal{L}_0 u^3 + \mathcal{L}_1 u^2 + \mathcal{L}_2 u^1 = 0,\\
&\dots
\end{aligned} \tag{1.9}$$

where one can easily compute the solutions to systems involved in (1.9). Recall that the periodic y-variable lives on $\mathbb{T}^d$, and we solve each of systems in (1.9) in y, treating x as a parameter. With this in mind, we see from the first line of (1.9) that $u^0(x, y)$ is independent of y, and hence[1] $u^0(x, y) = u^0(x)$. Consequently, from the second system in (1.9) we obtain that $u^1(x, y) = \chi^\alpha(y)\partial_{x_\alpha} u^0(x) + \widetilde{u}^1(x)$, where $\chi = \chi^\beta(y) \in M_N(\mathbb{R})$, $1 \leq \beta \leq d$ is the family of solutions to the following *cell* problem

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y \chi^\beta(y) = \partial_{y_\alpha} A^{\alpha\beta}, & y \in \mathbb{T}^d,\\ \int_{\mathbb{T}^d} \chi^\beta(y)dy = 0. \end{cases} \tag{1.10}$$

Using the information we have for u^0 and u^1, and writing down the compatibility condition for the third equation in (1.9), which in this case will be that the average of the source term over the unit cell of periodicity of $u_2(x, \cdot)$ equals 0, we obtain that u^0 must solve the following problem

$$-\nabla \cdot A^0 \nabla u^0(x) = f, \ x \in D \qquad \text{and} \qquad u^0 = 0, \ x \in \partial D, \tag{1.11}$$

where $A^0 = A^{0,\alpha,\beta} \in M_N(\mathbb{R})$ is a constant coefficient tensor determined by

$$A^{0,\alpha\beta} = \int_{\mathbb{T}^d} A^{\alpha\beta}(y)dy + \int_{\mathbb{T}^d} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y)dy. \tag{1.12}$$

In the form of (1.11) we have obtained the **homogenized** system corresponding to (1.6). We will call the operator in (1.11) *homogenized operator* corresponding to the family of operators $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$, and will refer to A^0 as *homogenized coefficients*. It can be shown that A^0 is elliptic in the sense formulated above.

Recall that the expansion in (1.7) was *formal*, and now we proceed to justification of its initial terms, showing that u^ε actually converges to u^0. Getting back to (1.9), from the third equality we have

$$u^2(x, y) = \Theta^{\alpha\beta}(y)\partial_{x_\alpha}\partial_{x_\beta} u^0(x) - \chi^\alpha(y)\partial_\alpha \widetilde{u}^1(x) + \widetilde{u}^2(x),$$

where $\Theta = \Theta^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is the family of solutions to another *cell* problem

---

[1] Let us emphasize that at this stage u^0 can be any function in x, since it is nothing but the additive constant that emerges when solving the first line of (1.9) in y. We will get more precise information on u^0, when dealing with the next order approximations in (1.7).

given by

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y \Theta^{\alpha\beta}(y) = B^{\alpha\beta} - \int_{\mathbb{T}^d} B^{\alpha\beta} dy, & y \in \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \Theta^{\alpha\beta} dy = 0, \end{cases} \tag{1.13}$$

where

$$B^{\alpha\beta} := A^{\alpha\beta} - A^{\alpha\gamma}\partial_{y_\gamma}\chi^\beta - \partial_{y_\gamma}(A^{\gamma\alpha}\chi^\beta).$$

Set

$$r^\varepsilon(x) := u^\varepsilon(x) - u^0(x) - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \varepsilon^2\Theta\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2 u^0(x),$$

and let us choose $\widetilde{u}^1(x) \equiv \widetilde{u}^2(x) \equiv 0$. As we have remarked for u^0, these constants in y play no role, as long as we are concerned with initial terms of the expansion in (1.7), however in order to deal with higher order terms, they must be taken into consideration. To proceed we will assume that u^0, solving (1.11) is from the class $H^4(D)$.

From the construction above, we easily see that r^ε satisfies

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla r^\varepsilon(x) = f^\varepsilon(x), & x \in D \\ r^\varepsilon(x) = g^\varepsilon(x), & x \in \partial D, \end{cases} \tag{1.14}$$

where $||f^\varepsilon||_{H^1(D)} \leq C\varepsilon||u_0||_{H^4(D)}$, and $||g^\varepsilon||_{H^{1/2}(\partial D)} \leq C\varepsilon^{1/2}||u_0||_{H^4(D)}$.[2] With these bounds at hand, one can see by energy considerations that $||r^\varepsilon||_{H^1(D)} \leq C\varepsilon^{1/2}||u^0||_{H^4(D)}$. We are thus lead to

$$\left|\left|u^\varepsilon - u^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^0\right|\right|_{H^1(D)} \leq ||r^\varepsilon||_{H^1(D)} + \varepsilon^2\left|\left|\Theta\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2 u^0\right|\right|_{H^1(D)} \leq C\varepsilon^{1/2}||u^0||_{H^4(D)}, \tag{1.15}$$

which in its turn implies

$$||u^\varepsilon - u^0||_{L^2(D)} \leq C\varepsilon^{1/2}||u^0||_{H^4(D)}. \tag{1.16}$$

Estimates obtained in (1.15) and (1.16) illustrate that the first two terms in formal expansion (1.7) are indeed valid!

In conclusion, let us remark that for problems with oscillating operator, and fixed source term, and boundary data, there is a well established theory largely

---

[2] The trace space $H^s(\partial D)$, where $0 < s < 1$, is equipped with the norm (see [28])

$$||u||_{H^s(\partial D)} := \left(||u||^2_{L^2(\partial D)} + \iint\limits_{\partial D\times\partial D} \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}} d\sigma(x)d\sigma(y)\right)^{1/2}.$$

From here, one can easily derive that $||u(\cdot/\varepsilon)||_{H^{1/2}(\partial D)} = O(\varepsilon^{-1/2})$, for $\varepsilon > 0$, which we do for g^ε defined above. Let us also note, that trace spaces are defined under some regularity assumptions on the boundary, however, for our purposes we may assume that everything is sufficiently smooth.

stemming from the series of works by Avellaneda and Lin from late 80's. In the seminal paper [12] they introduced *compactness methods* from calculus of variations into the theory of homogenization, which in particular allowed them to prove the following result. Assume for $\varepsilon > 0$, u^ε solves

$$\mathcal{L}_\varepsilon u^\varepsilon = 0 \text{ in } D \qquad \text{and} \qquad u^\varepsilon = g \text{ on } \partial D,$$

then for the solution u^0 of the homogenized operator and with the same prescribed Dirichlet data g, one has

$$||u^\varepsilon - u^0||_{L^\infty(D)} \leq C_g \varepsilon, \tag{1.17}$$

where the constant C_g depends on some smoothness norm of g. This result, which is the concern of Theorem 5 of [12], is established under some regularity conditions on the problem-related ingredients.

Some important advances in this direction have been achieved very recently. In particular Kenig, Lin, and Shen [44] proved homogenization results for Green's and Poisson's kernels of the oscillating operator. On other direction, uniform Lipschitz estimates of Avellaneda-Lin on solutions to ε-problems have been extended to include the case of almost-periodic coefficients by Armstrong and Shen in [10].

1.2.2 Emergence of boundary layers and related issues

The reader may have noticed that the error estimate obtained in (1.15) is not really the one that is anticipated from expansion (1.7) at first glance, since if the first two terms of (1.7) provide a correct approximation to u^ε, it may seem natural to expect an error estimate of order ε in $H^1(D)$. Surprisingly, the $\varepsilon^{1/2}$-loss in (1.7) is generally inevitable, for the reason that the Dirichlet data of u^ε does not agree with that of the approximation. Recall that $u^1(x, x/\varepsilon) = \chi(x/\varepsilon) \cdot \nabla u^0(x)$, where χ is $\mathbb{Z}^d$-periodic. This means that u^1 has rapid oscillations along the boundary of D, forcing u^ε to concentrate in the neighborhood of ∂D. Moreover, since ∂D in general has no alignment with the lattice $\mathbb{Z}^d$, one can not expect these oscillations to have any periodic structure. This effect of strong concentration of solutions in the vicinity of the boundaries is referred to as *boundary layer phenomenon*, and is a cause of serious mathematical difficulties. The following quote is from the preface of the classical book [16] by Bensoussan-Lions-Papanicolaou.

> *"Of particular importance is the analysis of the behavior of solutions near the boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem."*

As another way to emphasize the effect of boundary layers, we bring into attention a result due to Allaire and Amar [7], Theorem 2.3, which states that if D_0 is compactly inside D, then under some regularity conditions on coefficients A and boundary of D, one has

$$\left|\left| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) \right|\right|_{H^1(D_0)} \leq C\varepsilon,$$

where C depends on D_0, but is independent of ε. This estimate, which should be compared with (1.15), shows that indeed oscillations near the boundaries are filtered out inside the domain.

To handle the boundary layers in general, the obvious thing to try is to introduce some new profiles (correctors) into expansion (1.7) that will adjust the Dirichlet data of the approximation. To this end we consider a modified expansion into two-scales as follows

$$u^\varepsilon(x) = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\left[u^1\left(x, \frac{x}{\varepsilon}\right) + u_{bl}^{1,\varepsilon}(x)\right] + \varepsilon^2\left[u^2\left(x, \frac{x}{\varepsilon}\right) + u_{bl}^{2,\varepsilon}(x)\right] + \dots, \quad (1.18)$$

where we have the same notation as in (1.7) and the newly introduced terms satisfy

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u_{bl}^{k,\varepsilon}(x) = 0, & x \in D \\ u_{bl}^{k,\varepsilon}(x) = -u^k\left(x, \frac{x}{\varepsilon}\right), & x \in \partial D, \end{cases} \quad (1.19)$$

for $k = 1, 2, \dots$. Here *'bl'* in the index stands for boundary layers. It can be observed from (1.18) and (1.19) that each $u_{bl}^{k,\varepsilon}$ is designed to kill the trace of the corresponding u^k on the boundary ensuring by this a homogenous Dirichlet condition for the approximations. To see that this idea can actually improve the approximation, we as before consider the error term $r^\varepsilon(x) := u^\varepsilon(x) - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon u_{bl}^{1,\varepsilon}(x)$, which now has homogeneous Dirichlet condition on ∂D. Then, using the equations in (1.9), and noticing that $\mathfrak{L}_\varepsilon r^\varepsilon = -\varepsilon \mathfrak{L} u^1 + \mathfrak{L}_0 u^2$, one can easily derive that under the the assumption $u^0 \in H^2(D)$ we have

$$||u^\varepsilon(x) - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon u_{bl}^{1,\varepsilon}(x)||_{H^1(D)} \leq C\varepsilon. \quad (1.20)$$

Let us also note that the last estimate combined with

$$||u_{bl}^{1,\varepsilon}||_{H^1(D)} \leq C||u_{bl}^{1,\varepsilon}||_{H^{1/2}(\partial D)} \leq C\varepsilon^{-1/2}$$

implies (1.15) under milder regularity on u^0. We also refer to [55] for refined error estimates with more effective usage of the regularity assumption on u^0.

Summarizing the observations made above, we highlight two points here.

- For better understanding of problems with oscillating operator and fixed data, one is naturally lead to the study of problems with simultaneously oscillating operator and boundary data.
- Any type of improvement in error estimates on account of boundary layer correctors remain useless, as long as one can not homogenize systems of type (1.19).

At first sight one may have an impression that problems of type (1.19) should not be too far away from those with fixed data. However, the main technique that was available to address problems with oscillating operator and fixed data, namely the compactness methods of Avellaneda and Lin, had as its starting point *à priori* bounds in H^1 of solutions to ε-problem. These bounds are clearly not available for problems having oscillating boundary conditions, and thus compactness methods break down at the very first step. The case with simultaneously oscillating operator and boundary data was a longstanding open problem, and a breakthrough in this direction came in 2012, when D. Gérard-Varet and N. Masmoudi [34] proved homogenization result for problems like (1.19) in smooth and strictly convex domains. More precisely, if we consider

$$\mathcal{L}_\varepsilon u^\varepsilon(x) = 0 \text{ in } D \qquad \text{and} \qquad u^\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) \text{ in } \partial D, \tag{1.21}$$

where $g(x, \cdot)$ is $\mathbb{Z}^d$-periodic, then there exists a fixed $g^* \in L^\infty(\partial D)$ such that if u^0 satisfies $\mathcal{L}_0 u^0 = 0$ in D and has Dirichlet data g^*, then

$$||u^\varepsilon - u^0||_{L^2(D)} \leq C_\alpha \varepsilon^\alpha, \qquad \forall \alpha \in \left(0, \frac{d-1}{3d+5}\right).$$

The result is proved under strict convexity of D, the standard periodicity conditions on A, and smoothness of A, ∂D, and g. The following quote from the same work [34] regards the exponent of convergence.

> *"The value $(d-1)/(3d+5)$ in the theorem comes from the optimization of several small parameters and hence is not sharp. Finding the sharp rate seems a very interesting open problem."*

We will address this question later in the thesis. An idea that was introduced in [33], and developed in [34] by the same authors, was to approximate the solution to (1.21) near the boundary of D by functions admitting separation of scales in the spirit of two-scale expansion discussed above. More precisely, we fix $x_0 \in \partial D$, and assume that D lies locally on one side of its tangent plane, i.e. we let $D \subset \{x \in \mathbb{R}^d : (x - x_0) \cdot n > 0\}$ in a neighborhood of x_0. Then, in this neighborhood one tries to approximate the solution u^ε to problem (1.21) by a function of the form $v\left(x, \frac{x}{\varepsilon}\right)$. Heuristically, plugging such a v into (1.21) and using expansion of $\mathcal{L}_\varepsilon$ as in (1.8) we obtain that v should solve

$$\begin{cases} -\nabla_y \cdot A(y) \nabla_y v(x_0, y) = 0, & y \cdot n > \frac{x_0 \cdot n}{\varepsilon}, \\ v(x_0, y) = g(x_0, y), & y \cdot n = \frac{x_0 \cdot n}{\varepsilon}, \end{cases} \tag{1.22}$$

Note that here x_0 is only a parameter, and essentially we have a problem of the form

$$\begin{cases} -\nabla_y \cdot A(y) \nabla_y v(y) = 0, & y \cdot n > a, \\ v(y) = v_0(y), & y \cdot n = a, \end{cases} \tag{1.23}$$

where $a \in \mathbb{R}$, and v_0 is smooth and $\mathbb{Z}^d$-periodic. We will refer to systems of the form (1.23) as **boundary layer systems**. Regarding this we first need to understand the well-posedness of the these systems, and next, recalling that y was representing the oscillatory variable x/ε, we should study the asymptotics of the solutions far away from the boundary of the corresponding halfspace. Let us also remark that in a passage from (1.22) to (1.23) by fixing a we slightly abused the notation, dropping the dependence on the position of the halfspace that comes through x_0/ε. As we will see in a moment, for a certain class of directions n, the asymptotics of solutions to (1.23) depends only on n, and is independent of the position of the halfspace, hence fixing a is indeed a valid step.

It was proved in [33] that boundary layer systems are well-posed in the class of quasi-periodic functions. The next step, namely the asymptotics of solutions as $y \cdot n \to \infty$, was studied initially by Gérard-Varet and Masmoudi in [33], and with more detailed analysis by the same authors in [34], and by Prange in [54]. It is proved that under certain Diophantine condition on the normal n (see Section 1.2.3 below, or Chapter 4 for this condition) the solution to (1.23) converges, as $y \cdot n \to \infty$, to a constant vector field which is called a **boundary layer tail**, and this constant field depends upon the normal n only and is independent of a. As one can observe from the aforementioned work of Prange, the components of this constant field should be seen as certain *ergodic* constants that capture the averaging properties of the problem. Having this information at hand, at the end one is trying to glue the all approximations by boundary layer systems to obtain an approximation in the vicinity of the boundary for the original problem (1.21). For the analysis strict convexity of the domain plays an important role. It first assures that almost all boundary points of the domain possess a normal satisfying the Diophantine criteria mentioned above, and secondly, convexity puts the domain on one side of its tangent planes, and hence makes the approximation argument by boundary layer systems viable. As a rough summary of the idea discussed above, one may see the argument from [34] as an approximation of the original domain by polygonal domains from outside that have some suitable normal directions for their bounding faces. Then, one tries to transfer the problem from the original domain to the approximating polygon, where only finitely many correctors (approximants) should be considered, namely one for each face. Let us emphasize that this is an ultra-simplified sketch and the detailed analysis is in fact extremely involved.

From what we have discussed so far, a picture of the verge of our understanding regarding *periodic* homogenization of Dirichlet problem for elliptic operators in divergence form by 2012 was as follows.

(1) **(Fixed data, oscillating coefficients)** We have a well establish theory around the compactness methods of Avellaneda and Lin, and a problem is reasonably well understood.

(2) **(Simultaneously oscillating coefficients and boundary data)** The only general result here, due to Gérard-Varet and Masmoudi, proves a homogenization result with an algebraic rate of convergence.

It is not known what is the optimal rate of convergence. Also, there is no regularity theory for the homogenized boundary data, and hence no results on up to the boundary regularity for the homogenized solutions.

(3) **(Fixed operator, and oscillating boundary data)** A result due to Lee and Shahgholian [46] shows homogenization in this case, but without any error estimates. It should be emphasized that the simultaneously oscillating case does not cover this one, since there we do not have dependence of the coefficients on slow variable, and the only intersection of these two cases is that of the constant coefficient operators.

Here we do not have any method that proves homogenization with a rate of convergence.

(4) **(Boundary layer systems)** Understanding these systems, as was discussed above, is one of the key steps toward homogenization of simultaneously oscillating case, and the analysis has been initiated by Gérard-Varet and Masmoudi first in [33], and later in greater details in [34]. Both papers study these systems under the Diophantine condition on the normal direction. In particular, the papers establish convergence toward boundary layer tails faster than any polynomial rate, where the rate is with respect to the distance of the point from the boundary of the corresponding hyperplane. Let us stress that having effective statements on the speed of convergence is vital for establishing error estimates for the underlying homogenization problem. A refinement in this aspect came with a work of Prange [54], which freed the analysis from the Diophantine condition, and proved convergence of solutions toward their boundary layer tails for all irrational directions, however without any speed of convergence in this generality. The work [54] also demonstrates that for irrational directions, which are non Diophantine, convergence toward boundary layer tail can be slower than any power rate.

Here we do not know if the boundary layer tails vary regularly with respect to the normal directions. This is another key step for homogenization procedure of the simultaneously oscillating case, as well as for the regularity problem of the homogenized solutions. Also, it is not known if the normal is irrational and non Diophantine, is it possible to go beyond power rates in slow-convergence counterexamples, for instance can we have convergence slower than a logarithmic speed? Nothing is known concerning the slow-convergence phenomenon for variable coefficient operators.

The objective of the thesis is to present the developments, a part of which has been achieved in [3]-[6], which were meant to address the problems highlighted in (2), (3), and (4). In some of the cases we will give an almost complete answer, while for some, in spite of a non trivial progress, our understanding is still partial. In the next section we are going discuss our main results in details.

1.2.3 Outline of the main results

The results of the current work are clustered into two groups, *homogenization of boundary value problems*, and *analysis of boundary layer phenomenon.* We will start with the first group of results.

Boundary value homogenization

As we have discussed above there was virtually neither a steady method nor a general tool to address the problems of boundary value homogenization. Aiming at understanding the circle of problems around boundary layers' analysis and boundary value homogenization, in a series of papers [4]-[6], in collaboration with *Henrik Shahgholian*, and *Per Sjölin* we introduced and developed Fourier-analytic framework for homogenization of boundary value problems in periodic setting. This in particular enabled us to handle the case of fixed operator and oscillating boundary data, as well as to apply our method to some particular class of operators exhibiting simultaneous oscillations in coefficients and boundary data.

We now proceed to formulations. Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain, $A(x) = (A_{ij}^{\alpha\beta}(x))$ be $\mathbb{R}^{N^2 \times d^2}$-valued function defined on $\mathbb{R}^d$, where $1 \leq \alpha, \beta \leq d$, $1 \leq i, j \leq N$ with $N \in \mathbb{N}$, and fix some $g : \partial D \times \mathbb{T}^d \to \mathbb{R}^N$, that is we assume $g(x, \cdot)$ is $\mathbb{Z}^d$-periodic for any $x \in \partial D$. Now for small $\varepsilon > 0$ we consider the following Dirichlet problem

$$-\nabla \cdot A(x)\nabla u(x) = 0 \text{ in } D \qquad \text{and} \qquad u(x) = g\left(x, \frac{x}{\varepsilon}\right) \text{ on } \partial D. \tag{1.24}$$

The operator in (1.24) defines an elliptic system containing N equations; more precisely the i-th component of the operator is defined as

$$(\mathcal{L}u)_i := (-\nabla \cdot (A\nabla u)(x))_i = -\frac{\partial}{\partial x^\alpha}\left[A_{ij}^{\alpha\beta}\frac{\partial u_j}{\partial x^\beta}\right](x),$$

where $u = (u_1, ..., u_N)$, $1 \leq i \leq N$ and N represents the number of equations in the system. We assume that A and g are *sufficiently smooth*, and that A is *elliptic* in the sense of (1.5) discussed in the beginning of the Chapter.

For each $\varepsilon > 0$ we let $u_\varepsilon \in W^{1,2}(D; \mathbb{R}^N)$ be the unique solution to (1.24). The existence and uniqueness in the class $W^{1,2}(D; \mathbb{R}^N)$ are standard and follow from Lax-Milgram lemma. The goal is to investigate the behavior of u_ε as $\varepsilon \to 0$. To this end we let $u_0 \in W^{1,2}(D; \mathbb{R}^N)$ be the solution to

$$\mathcal{L}u_0(x) = 0 \text{ in } D \qquad \text{and} \qquad u_0(x) = \overline{g}(x) \text{ on } \partial D, \tag{1.25}$$

where for $x \in \partial D$ we have set $\overline{g}(x) = \int_{\mathbb{T}^d} g(x, y)dy$.

Smooth boundaries. In the following theorem we collect our main results for boundary value homogenization in smooth domains.

Theorem A (Boundary value homogenization in smooth domains). *Let u_ε and u_0 be as above. In addition to ellipticity and smoothness assumptions made on A and g, assume as well that D is strictly convex and has smooth boundary. Then*

(a) **(Pointwise estimates; Theorem 2.2.1)** *For each $\kappa > d-1$ there exists a constant C_κ such that*

$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{(d-1)/2}}{d(x)^\kappa}\right\} \qquad \forall x \in D,$$

where $d(x)$ is the distance of x from the boundary of D.

(b) **(L^p-estimates; Theorem 2.3.1)** *For any $1 \leq p < \infty$ there is a constant*

C_p such that

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_p \begin{cases} \varepsilon^{1/2p}, & d=2, \\ (\varepsilon|\ln\varepsilon|)^{1/p}, & d=3\ , \\ \varepsilon^{1/p}, & d \geq 4. \end{cases}$$

(c) **(Optimality; Theorem 2.3.2)** *Take $N=1$ and let the boundary data g depend only on its oscillating variable. Then for each $1 \leq p < \infty$ there exists a constant C_p such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \geq C_p \varepsilon^{1/p} ||g - \overline{g}||_{L^\infty(\mathbb{T}^d)}.$$

We next discuss the problem of the simultaneously oscillating case. For $\varepsilon > 0$ let $\mathfrak{L}_\varepsilon$ be the periodically oscillating operator defined by (1.4) and let $\mathfrak{L}_0$ be the corresponding homogenized operator. Let us emphasize that now we assume that the coefficient tensor A is $\mathbb{Z}^d$-periodic. By $\mathfrak{L}_\varepsilon^*$ we denote the formal adjoint to $\mathfrak{L}_\varepsilon$, that is the coefficient tensor of $\mathfrak{L}_\varepsilon^*$ is $A_{ji}^{\beta\alpha}$. Following Kenig-Lin-Shen [44] set $P_\gamma^k(x) = x_\gamma(0,...,1,0,...) \in \mathbb{R}^N$, with 1 in the k-th position, where $1 \leq \gamma \leq d$ and $1 \leq k \leq N$. Our next result, which is the subject of Theorem 2.3.3 of Chapter 2, proves homogenization of elliptic systems with optimal error estimate, however for some class of operators only.

Theorem B (Homogenization of elliptic systems in smooth domains). *Assume $d \geq 3$, and let the smoothness, ellipticity, periodicity assumptions be in force, and assume also that D is strictly convex. For each $\varepsilon > 0$ let u_ε be the solution to the following problem*

$$\mathfrak{L}_\varepsilon u_\varepsilon(x) = 0,\ x \in D \qquad \textit{and} \qquad u_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right),\ x \in \partial D.$$

If $\mathfrak{L}_\varepsilon^*(P_\gamma^k) = 0$ in D for all $1 \leq k \leq N$, $1 \leq \gamma \leq d$, and $\varepsilon > 0$ then there exists a fixed boundary data g^* depending on operator, domain and boundary data g so that if u_0 is the solution to the homogenized problem

$$\mathfrak{L}_0 u_0(x) = 0,\ x \in D \qquad \textit{and} \qquad u_0(x) = g^*(x),\ x \in \partial D,$$

then

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_p[\varepsilon(\ln(1/\varepsilon))^2]^{1/p}.$$

We note that the condition on the operator $\mathfrak{L}_\varepsilon$ involving projections P_γ^k can be simplified and formulated in terms of coefficients A only, without any ε-dependence. To avoid repetition we will skip this reduction here, and will refer to Chapter 2 for the details.

Polygonal domains. For non smooth boundaries we study the problem for bounded convex *polygonal domains*. We call a domain $D \subset \mathbb{R}^d$ ($d \geq 2$) *polygonal* if it is bounded by some finite number of halfspaces. This is precisely in analogy with the usual polygon on $\mathbb{R}^2$, but now in any dimensions. The operator $\mathfrak{L}$ is defined as in (1.24), however we only consider the case of *scalar equations*, i.e. we set $N = 1$. The usual ellipticity assumption on A, and periodicity requirement of g remain in force. We will impose some conditions regarding the boundary of D. For that we introduce the notion of a *Diophantine* vector which is a nonzero element $\nu \in \mathbb{R}^d$ such that for some parameters $\kappa > 0$ and $c > 0$ we have $|\nu \cdot m| > c||m||^{-\kappa}$ for any lattice point $m \in \mathbb{Z}^d \setminus \{0\}$. As will be shown in Chapter 3 for a suitable choice of parameters this condition is satisfied for almost all vectors.

Concerning the polygon D we will assume that it is convex, and the normal vector for each flat piece of its boundary is Diophantine. We also let $\alpha_* > 0$ be such that $\pi/(1+\alpha_*)$ is the largest angle between any two adjacent faces (flat pieces) of the boundary of D. For $\varepsilon > 0$ we let u_ε be the solution to (1.24) (recall that now $N = 1$), and let u_0 solve (1.25). In the next statement we collect our main results concerning homogenization in polygonal domains.

Theorem C (Boundary value homogenization in polygonal domains). *Keeping the notation and assumptions concerning the polygonal setting, we have:*

(a) **(Pointwise estimates; Theorem 3.1.1)** *If $\alpha_* > 1$ set $\beta = 1$, otherwise, let $0 < \beta < \alpha_*$ be any number. Then for each $\delta > 0$ small there exists a constant C_δ such that*

$$|u_\varepsilon(x) - u_0(x)| \leq C_\delta \left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}} \right)^{\frac{d-1}{d-1+\beta}} \qquad \forall x \in D,$$

where $d(x)$ denotes the distance of x to the boundary of D.

(b) **(L^p-estimates; Theorem 3.1.2)** *Define $\gamma := \frac{(d-1)\min\{1,\alpha_*\}}{d-1+\min\{1,\alpha_*\}}$. Then for each $1 \leq p < \infty$, and $\delta > 0$ there exists a constant $C_{p,\delta}$ such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_{p,\delta} \varepsilon^{\min\{\gamma, \frac{1}{p}\} - \delta}.$$

(c) **(Optimality; Theorem 3.1.3)** *Assume in addition that the boundary data g depends on its oscillating variable only. Then for each $1 \leq p < \infty$ there exists a constant C_p such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \geq C_p \varepsilon^{1/p} ||g - \overline{g}||_{L^\infty(\mathbb{T}^d)}.$$

The results of Theorems **A**, **B**, and **C** are obtained in collaboration with Henrik Shahgholian, and Per Sjölin. In particular *Pointwise estimates* of Theorem **A** have appeared in our paper

[4] Aleksanyan, H., Shahgholian, H., Sjölin, P.: *Applications of Fourier analysis in homogenization of Dirichlet problem I. Pointwise estimates.* Journal of Differential Equations, **254**(6), 2626-2637 (2013).

The results on L^p*-convergence* and *Optimality* of Theorem **A**, Theorem **B**, as well as an example showing optimality in dimension 2 and for $p = 1$ included in Chapter 2, are all part of our paper

[5] Aleksanyan, H., Shahgholian, H., Sjölin, P.: *Applications of Fourier analysis in homogenization of Dirichlet problem.* L^p *estimates.* Archive for Rational Mechanics and Analysis (ARMA), **215**(1), 65-87 (2015)

Statements of Theorem **C** are contained in our paper

[6] Aleksanyan, H., Shahgholian, H., Sjölin, P.: *Applications of Fourier analysis in homogenization of Dirichlet problem III. Polygonal domains.* Journal of Fourier Analysis and Applications **20**(3), 524-546 (2014)

The detailed arguments are given in Chapter 2 for smooth domains, and in Chapter 3 for polygonal domains. We just remark that at some places in the text the arguments may appear slightly different from those contained originally in our papers [4]-[6]. The reason for this is twofold. First, whenever it was possible we tried to give a unified approach for pointwise and L^p estimates. In particular, for pointwise bounds, and for the setting of polygonal domains we allowed the boundary data to have dependence on slow variable as well. This addition does not require any new ideas, and is handled exactly as in the case of only oscillating variable. However we chose to do so, in order to have uniformity in the formulations of our statements concerning pointwise and L^p estimates. The second reason is that at few places the arguments and discussions are a bit more elaborated in order to make it more transparent for the interested reader.

The contribution. The results presented above provide full or partial answers to problems formulated in items (2) and (3) at the end of the previous section. In particular, *Optimality* statement of Theorem **A** shows that in general one can not have homogenization with an algebraic rate of convergence larger than $1/p$ in L^p. Also, Theorem **C** illustrates that indeed the rate $1/p$ can be achieved for some class of operators. Moreover, Theorem **C** provides an alternative and more direct way of proving the homogenization result by Gérard-Varet and Masmoudi and establishes the statement in its best possible form with respect to the speed of convergence.

On the other direction Theorems **A** and **C** give a complete answer for homogenization of boundary value problems with optimal error estimates. Furthermore, methods developed here are of independent interest, and have a potential to be used for homogenization problems admitting integral representations. We will not attempt to explore the limits of these methods in the text, but to point out one particular application we refer to our paper [5] for a treatment of Neumann problem with the methods developed here.

A word on strategies of proofs. The main strategy of proofs of convergence results in Theorems **A** and **C** is to start with the integral representation of solutions via Poisson's kernel, and then expand the boundary data into Fourier series with respect to its periodic (oscillating) variable. We are then lead to the study of oscillatory integrals with singular weights (singularity here is due to the Poisson kernel), and it is here that the geometry of the domain comes into play. First, it determines the regularity properties of the kernel. While for smooth boundaries this regularity is not an issue, it is a challenge in the case of polygonal domains in view of the corner points on the boundary and needs a delicate care. Once we have an understanding of the regularity of the representation kernel, we next use the decay properties of Fourier transform of the surface carried measure of the boundary. This appears naturally from the expansion of the boundary data, and its decay is again determined by the geometry of the domain, such as convexity, or Diophantine property of the normal vectors in polygonal case. At the end we get two competing quantities in the averaging process, namely the singularity of the fundamental kernel *versus* cancellations in the integral due to oscillations of the boundary data which are quantified in terms of Fourier transform of the surface measure. One then makes a careful trade-off between these two quantities assuring homogenization at the end.

For the *Optimality* statements we first show that solutions to ε-problem concentrate in a strip of width comparable to ε near the boundary of our domain D. Then we show that the boundary of D scaled by a factor of $1/\varepsilon$ if considered modulo the lattice $\mathbb{Z}^d$ foliates the unit cell of periodicity of the boundary data in somewhat uniform fashion as $\varepsilon \to 0$. This gives an information on distribution of boundary values of solutions to ε-problems, using which we show that on a fixed portion of the aforementioned strip solutions stay uniformly away from the homogenized limit. Integrating on this subset only, leads to the desired lower bound.

Regarding Theorem **B** it should be noted that the strategy of proof of Theorem **A** is not applicable since now the Poisson kernel depends on the parameter $\varepsilon > 0$ as well, and hence we do not have uniform bounds on its order of singularity which was necessary to run the procedure from the setting of a fixed operator. To overcome this, we use a recent result due to Kenig, Lin, and Shen [44] where they prove a homogenization result for the family of Poisson kernels corresponding to the oscillating operator. By [44] one may compare the Poisson kernel of the ε-problem with that of the homogenized operator, by this effectively reducing the problem with oscillating coefficients to fixed one, however changing the original boundary data in a rather drastic way. This reduction to fixed operator, combined with the structural assumption on the operator made in the formulation of our Theorem **B** makes it possible to apply our methods for fixed operator here as well.

A localization principle. Let us conclude this part with some further remarks about the methods developed here. The reader may wonder why we study smooth convex domains and polygonal domains only. In fact, as will be seen in a moment, these two are important prototype geometries for developing our techniques. Regarding this in Chapter 2 we discuss possible ways to relax the convexity assumption. By our methods we can still prove homogenization for domains with smooth boundaries where at each boundary point at least one of the *principal curvatures is nonzero*. Nonetheless, the L^p-estimates will be worse than in strictly convex case. Next, as will be seen in Chapters 2 and 3 our analysis has local nature on the boundary. This as a consequence enables one to combine different types of geometric conditions where the boundary value homogenization takes place. In particular we may allow a completely flat piece on the smooth boundary of a domain (not necessarily convex), provided its normal vector is from the Diophantine class, and on the non-flat part we have at least one non-vanishing principal curvature.

However, it should be noted that the smoothness of the boundary is essential for our arguments, and at this stage we do not have a way to reduce the regularity requirements from the boundary.

Analysis of boundary layer phenomenon

Here we discuss the second set of results contained in the thesis. We will study two problems: first, concerning the regularity of the boundary data corresponding to homogenized elliptic system, and second, the convergence speed of solutions to boundary layer systems toward their tails.

Recall that from the analysis by Gérard-Varet and Masmoudi we know that for a strictly convex domain D the simultaneously oscillating problem (1.21) discussed above can be homogenized with some fixed boundary data $g^* \in L^\infty(\partial D)$. Assuming that all the problem-related ingredients are smooth, we can not even conclude whether this data g^* has to be regular, say continuous at least at one point on the boundary! The last remark shows how scarce is our understanding of the issue. In our paper [3] we adopted this regularity problem, but before presenting the setup, let us briefly review some prior knowledge we had on the matter.

For a unit vector $n \in \mathbb{S}^{d-1}$ denote by $P_{n^\perp}$ the operator of orthogonal projection on the hyperplane orthogonal to n. Fix some $l > 0$ satisfying $(d-1)l > 1$, and for $\kappa > 0$ define

$$\mathcal{A}_\kappa := \{ n \in \mathbb{S}^{d-1} : |P_{n^\perp}(\xi)| \geq \kappa |\xi|^{-l} \text{ for all } \xi \in \mathbb{Z}^d \setminus \{0\} \}.$$

Elements of $\mathcal{A}_\kappa$ are called *Diophantine* vectors[3]. It follows from the analysis of [34] that g^* is Lipschitz continuous on the subset of ∂D where the normal lies in $\mathcal{A}_\kappa$, and Lipschitz norm can be bounded by a constant multiple of κ^{-2}. From the strict convexity and smoothness of D one can conclude that as $\kappa \to 0$ almost any boundary point falls into some $\mathcal{A}_\kappa$, as well as one may show that the complement of each $\mathcal{A}_\kappa$, while a set of small surface measure on ∂D, is everywhere dense and is an open subset of ∂D. Since the Lipschitz constant blows-up as we cover

---

[3] Note that it is now the second time we are introducing the notion of a *Diophantine* vector, where the first appearance was for the polygonal domains. The reader can easily see that in fact these two formulations are essentially the same, where the first one does not let the normal n to get closer to orthogonal directions of lattice points, while this new one keeps the normal direction away from directions of lattice points. But a nice fact about the lattice $\mathbb{Z}^d$ is that with each non zero $\xi \in \mathbb{Z}^d$ it contains an element $\xi^* \in \mathbb{Z}^d$ with equivalent norm (constants depending on dimension d only) and orthogonal to ξ. Hence looking for lattice directions or orthogonal directions to lattice points is the same, and is a matter of a suitable choice of parameters in two definitions of *Diophantine* condition. However, we do not merge these two conditions into one, since we will use both of the formulations in their own way later in the text.

the boundary we can not conclude continuity of g^* even at a single point on the boundary.

Why one would need to study this problem? Of course this is a very interesting and challenging mathematical problem on its own right, but there are some outside motivating factors as well, a straightforward one of them being that the regularity of g^* governs up to the boundary regularity of the homogenized solutions. But perhaps more interestingly, the regularity of g^* carries an important information concerning the whole homogenization procedure pursued in [34]. The essential difficulty in homogenization of the simultaneously oscillating case, as we have seen in the previous section, are the boundary layers where solutions have very strong concentration of practically no traceable structure. To analyze the boundary layers one fixes a boundary point and in the neighborhood of that point tries to approximate the boundary layer corrector by a function admitting scale separation. After scaling, the averaging problem is transformed from the small neighborhood near the boundary to a halfspace having normal direction as the original point on the boundary. Then the homogenization of the corrector essentially corresponds to the asymptotics of the corresponding halfspace problem away from the boundary. As it is shown in [34] the solution to halfspace problem for Diophantine normals converges to some constant, the boundary layer tail, by means of which one then constructs the homogenized data g^*. From [34] and a later work of Prange [54] we see that the averaging process is somehow encoded in these ergodic constants, and that the regularity of g^* is closely tied with the regularity of these constants with respect to normal field of the boundary.

Here we will study this problem for *layered media*, i.e. for operators that are independent of one of the coordinates. Namely, in addition to the smoothness and periodicity of the coefficient tensor A we also assume that A is independent of the direction $e_d = (0, ..., 0, 1) \in \mathbb{R}^d$. Regarding the domain D we assume it is bounded, strictly convex and has smooth boundary. For $x \in \partial D$ by $n(x)$ we will denote the unit normal vector to ∂D. The following is our main result concerning the regularity of the homogenized boundary data.

Theorem D (Regularity of the homogenization boundary data). *Under the notation and assumptions made above we have:*

(a) **(Theorem 4.1.1)** *for any $\kappa > 0$ the boundary data g^* is Lipschitz continuous on*

$$\{x \in \partial D : \ n(x) \notin \mathbb{R}\mathbb{Q}^d \text{ and } |n(x) \cdot e_d| > \kappa\}.$$

(b) **(Corollary 4.1.2)** *g^* can be extended continuously on*

$$\{x \in \partial D : \ n(x) \cdot e_d \neq 0\}.$$

The details are given in Chapter 4. Theorem **D** have appeared in our preprint

[3] Aleksanyan, H.: *Regularity of boundary data in periodic homogenization of elliptic systems in layered media.* arXiv:1409.7344 [math.AP] (2014)

Let us also note that although homogenization problems involving layered media have been studied in the literature and have independent interest (see e.g. [53]), in our case the structural restriction on the operator is rather technical and is due to our proof. However, we will outline a program, which should lead to a proper understanding of the regularity issues around g^*. The results in Theorem **D** should be seen as the initial steps toward completion of the program.

We finally describe the last problem we study in this thesis. For scalar $a \in \mathbb{R}$, unit vector $n \in \mathbb{S}^{d-1}$, and smooth and $\mathbb{Z}^d$-periodic function v_0 consider the boundary layer problem for *scalar* equations. Namely we let v be the solution to

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y v(y) = 0, & y \cdot n > a, \\ v(y) = v_0(y), & y \cdot n = a. \end{cases} \tag{1.26}$$

By a solution here we mean in a sense of [34], a variational solution possessing some energy estimates. The details will be made more precise in Chapter 4. As we have discussed above, boundary layer systems play a key role in homogenization theory for simultaneously oscillating case and it is of primary importance to understand the speed of convergence of solutions toward their tails in order to prove error estimates for homogenization. Let us fix that here when referring to the speed of convergence we mean with respect to $y \cdot n$. Interestingly, this convergence is linked to some number-theoretic properties of the normal n, which in their turn determine the position of the halfplane $y \cdot n = a$ with respect to the microstructure which is the lattice $\mathbb{Z}^d$.

When $n \in \mathbb{R}\mathbb{Q}^d$ the analysis is classical. For example, one may consult a well-known paper by Moskow and Vogelius [52] where they show that convergence toward boundary layer tail holds with exponential speed, however this tail may depend on a, i.e. the position of the halfspace. When n is Diophantine, and hence irrational, Gérard-Varet and Masmoudi [33] proved convergence faster than any polynomial rate. Later, Prange [54] proved convergence for any irrational direction, however without any effective bounds on the speed of convergence. He also proved that if the normal is irrational and non Diophantine then for Laplace operator, and in dimension 2, one may have convergence slower than any *power rate*. In case when n is irrational, the boundary layer tail is independent of a.

Our main concern here is to close this spectrum of dependence of the speed of convergence on the properties of the normal. Clearly one only needs to consider the case of non Diophantine irrational normals. We will study (1.26) under some condition on the coefficients, namely we assume that

$$\text{there is} \quad 1 \leq \gamma \leq d \quad \text{satisfying} \quad \partial_\alpha A^{\gamma\alpha} = 0. \tag{1.27}$$

The main result in this direction is the following.

Theorem E (Arbitrarily slow convergence). *Let $R > 0$ be fixed, and take any continuous, one-to-one function $g : [0, \infty) \to (0, \infty)$ decreasing to 0 at infinity. Then there exists a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$, a smooth function $v_0 : \mathbb{T}^d \to \mathbb{R}$, and a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers growing to infinity, such that if v*

solves (1.26) under the condition (1.27), and with n *and* v_0 *as specified here, then for any* $k = 1, 2, \ldots$*, and all* $y' \in \partial\Omega_n \cap B(0, R)$ *one has*

$$|v(y' + \lambda_k n) - v^\infty| \geq g(\lambda_k),$$

where the constant v^∞ *is the corresponding boundary layer tail.*

Theorem **E** is proved in Section 4.4 of Chapter 4. Ibid, for the Laplace operator we investigate to which extent our examples of slow convergence are typical. Regarding this we construct a probability measure on $C^\infty(\mathbb{T}^d)$ with respect to which almost any initial boundary data leads to a slow convergence example. This probability measure, while somewhat tailored for our purposes, suggests anyway that in a simple case of the Laplace operator, the slow convergence phenomenon is a genuine property of the normal direction rather than the initial boundary data.

Our Theorem **E** shows in particular that one can not hope to obtain boundary value homogenization results with uniform error estimates relying merely on the smoothness of the boundary of the domain. The results of Chapter 4 answer in full or partially the problems outlined in (2) and (4) at the end of the previous section.

Chapter 2

Homogenization in smooth domains

In this chapter we study homogenization of Dirichlet boundary value problem for elliptic systems in divergence form. For the problems with fixed operator and oscillating boundary data we prove pointwise as well as L^p convergence results for the homogenization process. Next, we study optimality (sharpness) of our L^p bounds. For the problem with simultaneously oscillating coefficients and boundary data we prove homogenization results for a class of operators. In our analysis the Gaussian curvature of the boundary of the domain plays a crucial role.

The main results of this chapter are obtained in collaboration with Henrik Shahgholian, and Per Sjölin, and have appeared in our papers [4] and [5].

2.1 Introduction

The chapter is mainly concerned with the study of asymptotic behavior of solutions to elliptic systems in divergence form

$$-\nabla \cdot (A(x)\nabla u(x)) = 0,\ x \in D \tag{2.1}$$

set in a bounded domain $D \subset \mathbb{R}^d$ ($d \geq 2$) and with oscillating Dirichlet data

$$u(x) = g\left(x, \frac{x}{\varepsilon}\right),\ x \in \partial D. \tag{2.2}$$

As is usual $\varepsilon > 0$ here is a small parameter, $A(x) = (A_{ij}^{\alpha\beta}(x))$ is $\mathbb{R}^{N^2 \times d^2}$-valued function defined on $\mathbb{R}^d$, where $1 \leq \alpha, \beta \leq d$, $1 \leq i, j \leq N$, and $g(x, y)$ is $\mathbb{R}^N$-valued function defined on $\partial D \times \mathbb{T}^d$. Using the summation convention for repeated indices the i-th component of the operator in (2.1) is defined as

$$(\mathcal{L}u)_i := (-\nabla \cdot (A\nabla u)(x))_i = -\frac{\partial}{\partial x^\alpha}\left[A_{ij}^{\alpha\beta}\frac{\partial u_j}{\partial x^\beta}\right](x), \tag{2.3}$$

where $u = (u_1, ..., u_N)$ and $1 \leq i \leq N$. Along with (2.1)-(2.2), let us also introduce the corresponding homogenized problem, which reads

$$-\nabla \cdot (A(x)\nabla u(x)) = 0, \ x \in D \qquad \text{and} \qquad u(x) = \overline{g}(x), \ x \in \partial D, \tag{2.4}$$

where $\overline{g}(x) = \int_{\mathbb{T}^d} g(x,y)dy$. For each $\varepsilon > 0$ let u_ε be the solution to Dirichlet problem (2.1)-(2.2), and let u_0 be the solution to (2.4). Under certain assumptions on domain, operator, and boundary data involved in the problem we will prove convergence results for u_ε to u_0 as $\varepsilon \to 0$. In particular in Section 2.2 we will prove pointwise convergence of u_ε, while Section 2.3 will be concerned with the study of L^p bounds and their optimality.

Along with the case of fixed operator, we will also consider the situation when both the operator and the boundary data exhibit oscillations at the same scale. For this case let us introduce the family of operators $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$, where the i-th component of $\mathcal{L}_\varepsilon$ is defined as follows

$$(\mathcal{L}_\varepsilon u)_i := -(\nabla \cdot A(x/\varepsilon)\nabla u)_i = -\frac{\partial}{\partial x^\alpha}\left[A_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_j}{\partial x^\beta}(x)\right], \tag{2.5}$$

where $u = (u_1, ..., u_N)$ and $1 \leq i \leq N$. In Section 2.3 we will show on a particular class of operators, how to obtain optimal L^p convergence for Dirichlet problem corresponding to $\mathcal{L}_\varepsilon$ and oscillating boundary data as in (2.2).

2.1.1 Assumptions and basic preliminaries

We will study the Dirichlet problem (2.1)-(2.2), as well as that corresponding to oscillating operator (2.5) under the following hypotheses.

(i) (Periodicity) The boundary vector-valued function g is $\mathbb{Z}^d$-periodic with respect to its second variable, i.e.

$$g(x, y+h) = g(x,y), \qquad \forall x \in \partial D, \ \forall y \in \mathbb{R}^d, \ \forall h \in \mathbb{Z}^d.$$

When working with operator (2.5), we assume that the coefficient tensor A is $\mathbb{Z}^d$-periodic.

(ii) (Ellipticity) We assume that the coefficient tensor A is strongly elliptic, that is there exists a constant $c > 0$ such that

$$c\xi_\alpha^i\xi_\alpha^i \leq A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \leq c^{-1}\xi_\alpha^i\xi_\alpha^i, \qquad \forall x \in \mathbb{R}^d, \ \forall \xi \in \mathbb{R}^{d\times N}.$$

(iii) (Convexity) Domain D is a strictly convex.

(iv) (Smoothness) We suppose that the all components of the coefficient tensor A, the boundary of D, and the boundary data g in both variables are sufficiently smooth.

For the family $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ we let $\mathcal{L}_0$ be the homogenized (effective) operator in the standard sense of the theory (see Section 1.2.1 for the construction). Let us now recall some of the very few known results concerning homogenization problems that are directly related to those studied in this chapter. First, if we consider homogenization of Dirichlet problem for operator $\mathcal{L}_\varepsilon$ defined by (2.5) and with fixed (non-oscillating) Dirichlet data on the boundary, then as we have mentioned in the introduction, due to the classical work by M. Avellaneda and F. Lin [12] from late 80's this case is well understood. In particular, if we let u_ε be the solution to ε-problem and u_0 be the solution of the homogenized problem, then under some mild conditions on the operator, domain, and boundary data, it is proved in [12], Theorem 5, that $||u_\varepsilon - u_0||_{L^\infty(D)} \leq C\varepsilon$, where the constant depend on some smoothness norm of g that blows-up if we let g to be ε-periodic.

Handling the homogenization with simultaneously oscillating operator and Dirichlet data was a longstanding open problem, and a step forward has been done only very recently in 2012 by D. Gérard-Varet and N. Masmoudi in [34]. The following is their main result from [34].

Theorem 2.1.1. (Gérard-Varet and Masmoudi [34], Theorem 1.1) *Under the notation and assumptions (i)-(iv) formulated above, let u_ε be the solution to*

$$\mathcal{L}_\varepsilon u_\varepsilon(x) = 0 \text{ in } D \qquad \text{and} \qquad u_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) \text{ on } \partial D.$$

Then there exists $g^* \in L^\infty(\partial D)$ such that if u_0 solves

$$\mathcal{L}_0 u_0(x) = 0 \text{ in } D \qquad \text{and} \qquad u_0(x) = g^*(x) \text{ on } \partial D,$$

then

$$||u_\varepsilon - u_0||_{L^2(D)} \leq C_\alpha \varepsilon^\alpha \qquad \forall \alpha \in \left(0, \frac{d-1}{3d+5}\right).$$

It should be stressed that the setting of Theorem 2.1.1 is very different form that of [12] considered by Avellaneda and Lin. The main point of the difference is that having fixed boundary data one gets *à priori* boundedness of the family $\{u_\varepsilon\}_{\varepsilon>0}$ in $H^1(D)$, where u_ε is the solution for the operator $\mathcal{L}_\varepsilon$ and attains fixed boundary values on ∂D. Then one can use compactness ideas to study the limit behavior of u_ε. In contrast to the latter case, in the setting of Theorem 2.1.1 the family $\{u_\varepsilon\}_{\varepsilon>0}$ is not necessarily bounded in $H^1(D)$ due to rapid oscillations on the boundary, which makes the problem with oscillating boundary data much harder. Recalling some highlights of our discussion from the introductory chapter, we remark that a heuristic reason for mathematical difficulties here is that oscillations of the boundary data force solutions u_ε to concentrate near the boundaries. However, since the boundary of the domain intersects the microstructure, which in this case is $\mathbb{Z}^d$, in a non-periodic manner, it is very hard to trace back the nature of concentration of solutions in vicinity of the boundaries.

We also note that being an interesting mathematical problem on its own right, a proper understanding of the simultaneously oscillating case will lead to a refined estimates for the case with oscillating operator and fixed boundary data, which can be achieved through the standard two-scale expansions of solutions (see [34], Section 5). This fact can be considered as one particular motivation for studying problems with oscillating Dirichlet data.

Finally, let us mention that when the operator is fixed and only the boundary data is oscillating, a convergence result was proved in [46] for some general class of domains, however, without any speed of convergence. In contrast to [46], here we will aim at quantitative results, putting the geometry of the boundary in a central place in our analysis.

More on assumptions. Before going into details, let us comment on some of the assumptions above. The starting point for our analysis will be integral representation of solutions, where the integrals are taken over the boundary of the domain. As will be seen from the analysis below we will work locally on the boundary, and what we will really need for our localized analysis is a piece of boundary with non-vanishing Gaussian curvature. To define the Gaussian curvature at $x_0 \in \partial D$, we may assume after translation and rotation of the coordinate system of $\mathbb{R}^d$, that x_0 is transformed into the origin and the tangent plane to the hypersurface ∂D at the origin is given by $x_d = 0$. In a neighborhood of zero we can represent our surface ∂D by a graph $x_d = \varphi(x_1, ..., x_{d-1})$, where φ is smooth and $\varphi(0) = \nabla\varphi(0) = 0$, and our domain D lies locally above this graph, i.e. inside the domain we have $x_d > \varphi(x_1, ..., x_{d-1})$. Then, the eigenvalues of the following matrix (which is called the Hessian)

$$\left[\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0)\right]_{i,j=1}^{d-1}$$

are called the *principal curvatures* of ∂D at x_0 and their product is called the *Gaussian curvature*[1]. So, for our analysis it will be enough to assume that Gaussian curvature is nonzero everywhere. However, since we need this condition everywhere on the boundary, and the boundary is assumed to be smooth, these force D to be strictly convex. Let us formulate and prove this fact here[2].

Claim 2.1.2. *Assume $D \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded domain with smooth boundary, and suppose that the Gaussian curvature of ∂D is nowhere vanishing. Then all the principal curvatures of ∂D are everywhere positive, and D is strictly convex.*

Proof. We call D *weakly locally convex* at $x_0 \in \partial D$ if there exists a halfspace $\mathbb{H}$ with $x_0 \in \partial\mathbb{H}$, and $r > 0$ such that $B(x_0, r) \cap D \subset \mathbb{H}$. Since in our case D is smooth, the formulated local convexity condition means that D stays locally on one side of its tangent plane at x_0. Moreover, still relying on smoothness of the domain it is easy to see that local convexity of D at x_0 is equivalent to the fact that the all principal curvatures of ∂D at x_0 are positive. Let us first show that D is locally convex at least at one point. Note, that geometrically the

---

[1] It should be remarked that implicitly we have fixed an orientation on ∂D, when saying that the domain lies above the graph. First of all, the fact that the boundary of D is orientable is not given *à priori*, however it is well known that codimension 1 smooth hypersurface in $\mathbb{R}^d$ is orientable, see e.g. [56] for a short proof when the surface is smooth. Also observe, that our choice of orientation on ∂D implies that the normal to the boundary is pointing inwards the domain.

[2] We believe that this fact is classical and should be very well known. However, since we could not find a concise reference for it, we decided to present the proof here.

obvious candidate for that point is the "farthermost" point on the boundary. To find it, we let B be the ball of the smallest radius containing D, which exists in view of the boundedness of D. Due to the choice of B its boundary must touch ∂D at some point, assume at $x_0 \in \partial D$. Since ∂D and ∂B are smooth, it follows that the tangent plane to the boundary of B at x_0 is the tangent plane for ∂D as well, but as $D \subset B$, we get that D must stay on one side of this plane. We get that D is locally convex at x_0 and hence all its principal curvatures are positive at x_0. Due to the assumption of non-vanishing curvature, we get that the principal curvatures must stay positive everywhere, and hence D is locally convex everywhere. Now the passage from local to global follows by Tietze-Nakajima's theorem (see [63], Theorem 4.4), which states that if domain D is weakly locally convex, then it is convex. The claim is proved. $\square$

The statement of Claim 2.1.2 is the reason as why we have chosen to state Assumption (iii) in its current form, rather than in terms of the curvature. However it is useful to keep in mind that what we really need is the curvature condition rather than convexity, since as will be seen from the analysis below, one may allow some of the principal curvatures to vanish and still get a quantitative homogenization result but with slower speed. This may enable one to include domains such as torus, or a "smooth" cylinder (see e.g. Corollary 2.2.8 and Theorem 2.2.9), which are now ruled out by the strict convexity assumption.

As regards the last assumption (iv), we remark that there we need g to be smooth with respect its first variable, which lives on ∂D only. By smoothness here we understand differentiability with respect to local coordinate charts. Namely, assume we have some $f : \partial D \to \mathbb{R}$, and let $x_0 \in \partial D$ be fixed. The surface ∂D is locally a graph, so after translating and permuting the coordinate system we may assume that in some neighborhood $\mathcal{M}$ of x_0, ∂D is given by $\{(x', \psi(x')) : |x'| \leq a\}$, where $x' = (x_1, ..., x_{d-1})$, $a > 0$ is small and ψ is a smooth function in a neighborhood of $0 \in \mathbb{R}^{d-1}$. Thus $x_0 = (0, \psi(0))$, and for $x \in \mathcal{M}$ we have $f(x) = f(x', \psi(x')) := h(x')$. Then when referring to differentiability (derivatives) of f at x_0 we have in mind those of h at 0.

Let us also note that under the ellipticity and smoothness assumptions, the questions of existence, uniqueness and regularity of solutions to (2.1)-(2.2) are classical (see e.g. [35], Theorem 3.39 and Theorem 5.21).

2.1.2 Green and Poisson

Throughout the text we will need the notions of Green and Poisson kernels which will be used for integral representations of solutions. The Green's kernel in domain D for the operator $\mathcal{L}$ defined by (2.3) is an $\mathbb{R}^{N\times N}$-valued function defined on $D \times D \setminus \{(x,y) \in D \times D : \ x = y\}$ such that for any $f \in L^2(D; \mathbb{R}^N)$ the function

$$u(x) = \int_D G(x,y) f(y) dy$$

is the unique solution of

$$\mathfrak{L}u = f \text{ in } D \qquad \text{and} \qquad u = 0 \text{ on } \partial D.$$

Under the ellipticity assumption (ii) and smoothness assumption (iv) for the domain and the operator, existence and uniqueness of the matrix G is proved in [29], along with the following basic properties.

(a) For any $y \in D$, $G(\cdot, y)$ solves the following system

$$-\nabla_x \cdot A(x)\nabla_x G(x,y) = \delta(x-y)I_N \text{ in } D \quad \text{and} \quad G(x,y) = 0 \text{ for } x \in \partial D,$$

in a variational sense, where δ is Dirac delta function and I_N is the $N \times N$ identity matrix.

(b) Set $G^*(x,y) := G^T(y,x)$, then G^* is the Green's kernel for the formal adjoint to $\mathfrak{L}$, i.e. a divergence type operator with coefficient tensor equal to $A_{ji}^{\beta\alpha}$.

(c) For any multi-index $m \in \mathbb{Z}_+^d$ we have

$$|\partial_x^m G(x,y)| \leq C_m |x-y|^{2-d-|m|} \text{ for } d + |m| > 2, \tag{2.6}$$

and

$$|G(x,y)| \leq C \log \frac{C}{|x-y|} \text{ for } d = 2.$$

Observe that due to condition (b), we have that G vanishes on the boundary with respect to both variables. We also get that (2.6) holds true for derivatives with respect to y as well.

Now using the Green's matrix we introduce the Poisson's kernel. For $y \in \partial D$ we denote by $n(y) = (n^\alpha(y))_{\alpha=1}^d$ the unit exterior normal to ∂D at y. Then the Poisson kernel $P(x,y) = (P^{ij}(x,y))_{i,j=1}^N$ is defined through

$$P^{ij}(x,y) = n^\alpha(y) A_{ij}^{\alpha\beta}(y) \partial_{y_\beta} G^{kj}(x,y).$$

If u solves $\mathfrak{L}u = 0$ in D and $u = g$ on ∂D, then it can be represented in terms of the Poisson integral as follows $u(x) = \int_{\partial D} P(x,y)g(y)d\sigma(y)$.

The next result provides the main estimates on P that we are going to use. Its proof relies on the Green's matrix estimate and the scheme from [12].

Lemma 2.1.3. *Assume the ellipticity and smoothness conditions of Section 2.1.1, and let P be the Poisson kernel for the operator $\mathfrak{L}$ defined in D. Then*

$$|P(x,y)| \leq C \frac{d(x)}{|x-y|^d}, \qquad \forall x \in D, \ \forall y \in \partial D, \tag{2.7}$$

where $d(x)$ is the distance of x from the boundary of D, and

$$|\partial_y^m P(x,y)| \leq C_m |x-y|^{1-d-|m|}, \qquad \forall x \in D, \ \forall y \in \partial D. \tag{2.8}$$

Proof. Let us first observe that the estimates on derivatives of P follow trivially from (2.6), the symmetry property of Green's matrix (c) formulated above, combined with smoothness of the coefficients A, and domain D. Let us also stress that derivatives are understood with respect to convention made at the end of Section 2.1.1.

We now proceed to the proof of the estimate with distance. First consider the case when $d \geq 3$, and let $G(x, y)$ be the corresponding Green's kernel. We now show that for $x, y \in D$ with $x \neq y$ one has

$$|G(x,y)| \leq C \frac{d(x)}{|x-y|^{d-1}}. \tag{2.9}$$

By (2.6) we have $|G(x,y)| \leq C|x-y|^{2-d}$, hence (2.9) is trivial if $d(x) > \frac{1}{3}|x-y|$, thus we will assume that $d(x) \leq \frac{1}{3}|x-y|$. Now fix $\overline{x} \in \partial D$ such that $d(x) = |x-\overline{x}|$. Since G vanishes on the boundary of D with respect to both variables, using the mean value theorem and estimate (2.6) for G^* we get

$$|G(x,y)| = |G(x,y) - G(\overline{x},y)| \leq |\nabla_x G(\widetilde{x},y)||x-\overline{x}| \leq C \frac{d(x)}{|\widetilde{x}-y|^{d-1}}. \tag{2.10}$$

Here, for estimating the derivative of G we used symmetry relation (b) above. Since $d(x) \leq \frac{1}{3}|x-y|$, and $\widetilde{x}$ lies on the segment connecting x and $\overline{x}$, by the triangle inequality we have

$$|\widetilde{x}-y| \geq |x-y| - |x-\widetilde{x}| \geq |x-y| - |x-\overline{x}| \geq \frac{2}{3}|x-y|,$$

which combined with (2.10) implies (2.9).

Now fix x_0, y_0 and let $r := |x_0 - y_0| > 0$. Consider $\widetilde{G}(z) := G(x_0, rz + x_0)$ in a scaled and shifted domain $D_r := r^{-1}(D - x_0)$. We have that $\widetilde{G}$ is a solution to the adjoint operator in $D_r \cap (B(0,3) \setminus B(0,1/3))$, hence in view of the smoothness of the coefficients, from the elliptic regularity estimates (see [1]-[2]) we obtain

$$|\nabla_z \widetilde{G}(z)| = |r\nabla_y G(x_0, rz + x_0)| \leq C||\widetilde{G}||_{L^\infty(D_r \cap (B(0,3)\setminus B(0,1/3)))},$$

for all $z \in D_r \cap (B(0,2) \setminus B(0,1/2))$. The last inequality combined with (2.9) implies

$$|\nabla_y G(x_0, y_0)| \leq C \frac{d(x_0)}{|x_0-y_0|^d}. \tag{2.11}$$

Now the estimate (2.7) follows from the last inequality and the definition of the Poisson kernel.

It is left to prove the Lemma for dimension two, which can be done as in Lemma 21 of [12]. For the reader's convenience we sketch the proof. The point is to reduce the analysis to 3-dimensional case by adding a "dummy" variable to the plane. Namely, we consider $\widetilde{D} = D \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$ and is identified with the interval $[0,1)$. We have that $\widetilde{D}$ is 3-dimensional manifold with smooth boundary[3]. In a similar vein consider the operator $\widetilde{\mathfrak{L}} := \mathfrak{L} + \frac{\partial^2}{\partial t^2}$, and let $\widetilde{G}(x,t;y,s)$, and $\widetilde{P}(x,t;y,s)$ be correspondingly Green's and Poisson's kernels for $\widetilde{\mathfrak{L}}$ in $\widetilde{D}$. Due to locality of elliptic estimates we still get the estimates for Green's kernel for $\widetilde{D}$ and using similar arguments as for $d \geq 3$ we get

$$|\widetilde{P}(x,t;y,s)| \leq C \frac{d(x)}{[|x-y|^2 + (t-s)^2]^{3/2}},$$

where we have used that $d(x) = d((x,t))$. It is clear from the definition of $\widetilde{\mathfrak{L}}$ and $\widetilde{D}$ that

$$P(x,y) = \int_0^1 \int_0^1 \widetilde{P}(x,t;y,s) ds dt,$$

and hence using the estimate for $\widetilde{P}$ we obtain

$$|P(x,y)| \leq \int_0^1 \int_0^1 \frac{Cd(x)}{(|x-y|^2 + (t-s)^2)^{3/2}} ds dt \leq \int_0^\infty \frac{Cd(x)}{[|x-y|^2 + s^2]^{3/2}} ds \leq$$
$$\frac{Cd(x)}{|x-y|^2} \int_0^\infty \frac{1}{[1 + (s/|x-y|)^2]^{3/2}} \frac{ds}{|x-y|} \leq C \frac{d(x)}{|x-y|^2}.$$

The Lemma is proved. □

The next result, which is a corollary to Lemma 2.1.3 will be used later on to establish uniform bounds on solutions to Dirichlet problem (2.1)-(2.2).

Lemma 2.1.4. *Retain the ellipticity and smoothness assumptions of Section 2.1.1, and let P be the Poisson kernel for $\mathfrak{L}$ in domain D. Then*

$$\int_{\partial D} |P(x,y)| d\sigma(y) \leq C, \qquad \forall x \in D. \tag{2.12}$$

Proof. Fix $x \in D$. Without loss of generality we will assume that $d(x) = |x|$, and that the tangent plane to ∂D at 0 is $\{x \in \mathbb{R}^d : \ x_d = 0\}$, since otherwise we may bring x and ∂D to these positions by translation and rotation of the coordinate system. Since $d(x) = |x|$ it is not hard to see that x is orthogonal to the tangent plane of ∂D at 0. Next, in view of the smoothness of the domain there exists a smooth function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ so that for some $0 < \delta < 1$ small, which can be chosen independently of x due to compactness, we have

$$\partial D \cap B(0,\delta) = \{(y', \varphi(y')) : \ |y'| \leq 10\delta\} \cap B(0,\delta),$$

where $y' = (y_1, ..., y_{d-1})$. Also, it is clear that $\varphi(0) = \nabla\varphi(0) = 0$, from which we get that $|\varphi(y')| \leq C|y'|^2$, where $|y'| \leq \delta$.

---

[3] In fact $\widetilde{D}$ is a solid torus with base D.

It follows from the estimate of Lemma 2.1.3 that in order to prove the current Lemma it is enough to show that

$$\int\limits_{\partial D\cap B(0,\delta)} \frac{d\sigma(y)}{|x-y|^d} \leq C\frac{1}{|x|},$$

where the constant C is independent of x. Now, making a change of variables in the last integral we get

$$\int\limits_{\partial D\cap B(0,\delta)} \frac{d\sigma(y)}{|x-y|^d} \leq C \int\limits_{|y'|\leq\delta} \frac{dy'}{|x-(y',\varphi(y'))|^d}. \tag{2.13}$$

From orthogonality of x to $\{x\in\mathbb{R}^d:\ x_d=0\}$ and the mentioned properties of φ we have

$$|x-(y',\varphi(y'))|^2 = |x'|^2+|y'|^2+x_d^2-2x_d\varphi(y')+\varphi^2(y') \geq |x|^2+\frac{1}{2}|y'|,$$

if $|x|$ and $\delta>0$ are sufficiently small. Using the last inequality from (2.13), and integrating in the spherical coordinates we get

$$\int\limits_{|y'|\leq\delta} \frac{dy'}{|x-(y',\varphi(y'))|^d} \lesssim \int\limits_{|y'|\leq\delta} \frac{dy'}{(|x|^2+|y'|^2)^{d/2}} \lesssim \int\limits_0^\delta \frac{t^{d-2}}{(|x|^2+t^2)^{d/2}}dt \lesssim$$

$$\int\limits_0^\delta \frac{dt}{|x|^2+t^2} = C\frac{1}{|x|}\arctan\frac{\delta}{|x|}.$$

Since $d(x)=|x|$ the last expression completes the proof. □

An important consequence which follows trivially from the last lemma and the Poisson representation of solutions to Dirichlet problem is the following.

Corollary 2.1.5. *Let u be a solution to Dirichlet problem (2.1)-(2.2). Then*

$$||u||_{L^\infty(D)} \leq C||g||_{L^\infty(\partial D\times\mathbb{R}^d)}, \tag{2.14}$$

where C is independent of the boundary data g.

It should be noted that since we are working with systems of equations the maximum principle is not available anymore. This is the reason as why the bound in (2.14) is non trivial and needs some care.

2.2 Pointwise estimates

For each $\varepsilon>0$ we let u_ε be the solution to (2.1)-(2.2), and let u_0 be the solution to homogenized problem (2.4). The main task of the current section is to prove the following results.

Theorem 2.2.1. (Pointwise convergence) *Let assumptions 2.1.1 be in force. Then, for each $\kappa > d-1$ there exists a constant C_κ such that*

$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{(d-1)/2}}{d(x)^\kappa}\right\}, \qquad \forall x \in D,$$

where $d(x)$ denotes the distance of x from the boundary of D. The constant C_κ depends on domain, operator and dimension of the space.

Corollary 2.2.2. *For each $1 \leq p < \infty$ and each $\kappa < 1/(2p)$ there exists a constant C_κ such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_\kappa \varepsilon^\kappa.$$

The main idea of the proof is to use integral representation of solutions, and expand the boundary data into Fourier series with respect to its oscillating variable. This reduces the analysis to a study of oscillatory integrals with singular weights over the surface of the domain.

2.2.1 Preliminary tools

Let us recall some standard notation for multi-indices that we are going to use. If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+^d$ and $x = (x_1, ..., x_d) \in \mathbb{R}^d$ then we set $x^\alpha := x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$. Also, if we treat $\alpha \in \mathbb{Z}^d$ as a multi-index, then we denote $|\alpha| := |\alpha_1| + ... + |\alpha_d|$.

Lemma 2.2.3. *There exists a constant $c > 0$ depending on dimension d only such that*

$$\sum_{\substack{\alpha \in \mathbb{Z}_+^d \\ \alpha_1 + ... + \alpha_d = k}} |m^{2\alpha}| \geq c||m||^{2k},$$

where $k \in \mathbb{N}$ and $m \in \mathbb{Z}^d$.

Proof. For $x \neq 0$ consider the function

$$f(x) = \frac{1}{||x||^{2k}} \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha| = k} |x_1|^{2\alpha_1} \cdot ... \cdot |x_d|^{2\alpha_d},$$

which is continuous and positive function on the unit sphere of $\mathbb{R}^d$, hence we have $c := \min_{||x||=1} f(x) > 0$. The statement now follows from the fact that $f(tx) = f(x)$ for any $t > 0$. □

Lemma 2.2.4. *Let $f \in C^k(\mathbb{T}^d)$ and $\beta \in \mathbb{R}$. If $k + \beta > d/2$, then*

$$\sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \frac{1}{||m||^\beta} |c_m(f)| \leq C_{k+\beta} \left(\sum_{\alpha \in \mathbb{Z}_+^d, |\alpha| = k} ||D^\alpha f||^2_{L^2(\mathbb{T}^d)} \right)^{1/2},$$

where $c_m(f) := \int_{\mathbb{T}^d} f(x) e^{-2\pi i m \cdot x} dx$ is the m-th Fourier coefficient of f.

Proof. Let $m \in \mathbb{Z}^d \setminus \{0\}$ be fixed, and pick a multi-index $\alpha \in \mathbb{Z}_+^d$ satisfying $|\alpha| \leq k$. Then integration by parts leads to $(2\pi i m)^\alpha c_m(f) = c_m(D^\alpha(f))$, where we are interpreting 0^0 as 1. Since $D^\alpha(f) \in L^2(\mathbb{T}^d)$ by Parseval we have

$$\sum_{m \neq 0} \sum_{\alpha_1 + \ldots + \alpha_d = k} |m^\alpha c_m(f)|^2 \leq C \sum_{\alpha_1 + \ldots + \alpha_d = k} ||D^\alpha f||_2^2. \tag{2.15}$$

Now using Lemma 2.2.3 and Hölder's inequality we get

$$\sum_{m \neq 0} \frac{|c_m(f)|}{||m||^\beta} \leq C \sum_{m \neq 0} \frac{|c_m(f)|}{||m||^\beta} \left(||m||^{-2k} \sum_{\alpha_1 + \ldots + \alpha_d = k} |m_1|^{2\alpha_1} \cdot \ldots \cdot |m_d|^{2\alpha_d} \right)^{1/2} =$$

$$C \sum_{m \neq 0} \left[|c_m(f)| \left(\sum_{\alpha_1 + \ldots + \alpha_d = k} |m_1|^{2\alpha_1} \cdot \ldots \cdot |m_d|^{2\alpha_d} \right)^{1/2} \right] ||m||^{-(k+\beta)} \leq$$

$$C \left(\sum_{m \neq 0} \sum_{\alpha_1 + \ldots + \alpha_d = k} |m^\alpha|^2 |c_m(f)|^2 \right)^{1/2} \left(\sum_{m \neq 0} ||m||^{-2(k+\beta)} \right)^{1/2}.$$

The second factor is finite due to condition $k + \beta > d/2$ of the lemma. The conclusion now follows from (2.15). ☐

The following result is a direct consequence of the last lemma and will be used later on.

Corollary 2.2.5. *Let $\tau \in \mathbb{R}$, Ω be a compact subset of $\mathbb{R}^d$, and a function $f(x, y) : \Omega \times \mathbb{T}^d \to \mathbb{C}$ be periodic in its second variable. Suppose that for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \leq k$, $D_y^\alpha f(x, y)$ exists and is continuous on $\Omega \times \mathbb{T}^d$. Then*

$$\sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \frac{|c_m(f; x)|}{||m||^\tau} \leq C_{k+\tau, f}, \qquad \forall x \in \Omega,$$

provided $k + \tau > d/2$, where $c_m(f; x)$ is the m-th Fourier coefficient of $f(x, \cdot)$.

Lemma 2.2.6. *Assume $\psi : \mathbb{R}^d \to \mathbb{R}$ have all its derivatives up to second order which are continuous in a ball $B(0, r)$, for some $r > 0$, and assume the matrix*

$$\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)_{i,j=1}^d (0) := D^2 \psi(0)$$

has full rank. Then there exist constants $a > 0$, c_1, and c_2 such that

$$c_1 |x - y| \leq |\nabla \psi(x) - \nabla \psi(y)| \leq c_2 |x - y|, \qquad \forall x, y \in B(0, a).$$

The constants depend on dimension, eigenvalues of $D^2\psi(0)$, modulus of continuity, and upper bound of the second order derivatives of ψ.

Proof. The right-hand side inequality is a direct consequence of the boundedness of second derivatives, together with mean value theorem, we thus proceed to the proof of the first inequality. For an orthogonal matrix $M \in M_d(\mathbb{R})$ define $g(x) = \psi(Mx)$. Then it is easy to see that $\nabla g(x) = M^T \nabla\psi(Mx)$, and that $D^2 g(x) = M^T D^2\psi(Mx)M$, where D^2 denotes the Hessian matrix. We now fix M so that $D^2 g(0)$ is diagonal, this choice is nonempty since a symmetric matrix can be diagonalized by orthogonal transformation. Also, we choose $a > 0$ small enough so that $D^2 g$ is uniformly diagonally dominant in $B(0, a)$, i.e. for some constant $c > 0$ one has

$$|\partial^2_{ii} g(x)| - \sum_{i=1,\ i\neq j}^{d} |\partial^2_{ij} g(x)| \geq c, \quad \forall x \in B(0, a),\ \forall 1 \leq i \leq d. \tag{2.16}$$

Clearly a can be chosen depending on the modulus of continuity of second order derivatives of ψ. We now prove the lower bound for g. Using the equivalence of norms in $\mathbb{R}^d$ and mean value theorem we have

$$|\nabla g(x) - \nabla g(y)| \geq C \sum_{i=1}^{d} |\partial_i g(x) - \partial_i g(y)| = C \sum_{i=1}^{d} |\nabla(\partial_i g(\tau_i)) \cdot (x - y)|,$$

where τ_i is a point on the segment connecting x and y. Fix $1 \leq i \leq d$ so that $|x_i - y_i| \geq c|x - y|$, then by the last inequality and (2.16) we have

$$\begin{aligned} |\nabla g(x) - \nabla g(y)| \geq C|\nabla(\partial_i g(\tau_i)) \cdot (x - y)| \geq \\ C\left(|\partial^2_{ii} g(\tau_i)| - \sum_{i\neq j} |\partial^2_{ji} g(\tau_i)|\right)|x_i - y_i| \geq C|x - y|, \end{aligned}$$

for all $x, y \in B(0, a)$. From here, using the orthogonality of M and definition of g we have

$$\begin{aligned} |\nabla\psi(x) - \nabla\psi(y)| = |M^T(\nabla\psi(x) - \nabla\psi(y))| = \\ |\nabla g(M^T x) - \nabla g(M^T y)| \geq c|M^T x - M^T y| = c|x - y|. \end{aligned}$$

The proof is now complete. □

We now carry on to the formulation and proof of the main lemma of this section. Fix nonzero $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$ $(d \geq 2)$, set $\xi' = (\xi_1, ..., \xi_{d-1})$, and let a_0 and ρ be small positive numbers. Assume that for a function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ we have $\psi \in C^\infty(B(0, b))$, $|D^\alpha \psi(z')| \leq C_\alpha$, for each $\alpha \in \mathbb{Z}^{d-1}_+$ and $z' \in B(0, b)$, and that the matrix $\left(\frac{\partial^2\psi}{\partial z'_i \partial z'_j}\right)_{i,j=1}^{d-1}(0') := D^2\psi(0')$ is invertible.

Assume further $u : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies $u \in C^\infty(B(0, a_0))$, $\operatorname{supp}(u) \subset B(0, \frac{3}{4}a_0)$, and $|D^\alpha u(z')| \leq C_\alpha$ for each $\alpha \in \mathbb{Z}^{d-1}_+$ and any $z' \in B(0, a_0)$.

For $x \in \mathbb{R}$ we will set $\exp(x) := e^{2\pi i x}$.

Lemma 2.2.7. *Under the assumptions and notation above for $(d-1)$-dimensional* *integral*

$$\mathcal{J} = \int\limits_{|z'|<a_0} \exp[\xi' \cdot z' + \xi_d \frac{1}{\rho} \psi(\rho z')] u(z') dz'$$

one has

$$|\mathcal{J}| \leq C(\rho|\xi|)^{-(d-1)/2}.$$

Dependence of the constant C on u is bounded by supremum norms of some finite number of derivatives of u, and the highest order of derivatives is independent of the function u.

Proof. Set $\lambda(\eta', \eta_d) = (\xi', \rho\xi_d)$ with $\lambda = (|\xi'|^2 + \rho^2\xi_d^2)^{1/2}$ and $|\eta'|^2 + \eta_d^2 = 1$. Clearly $\lambda\eta' = \xi'$ and $\lambda\eta_d = \rho\xi_d$. Note that since $\lambda \geq (\rho^2|\xi'|^2 + \rho^2\xi_d^2)^{1/2} = \rho|\xi|$, it is enough to prove that

$$|\mathcal{J}| \leq C\lambda^{-(d-1)/2}. \tag{2.17}$$

Denote $F(z') := \eta' \cdot z' + \eta_d \frac{1}{\rho^2} \psi(\rho z')$. Clearly $\nabla F(z') = \eta' + \eta_d \frac{1}{\rho} \nabla\psi(\rho z')$, and with this notation we have

$$\mathcal{J} = \int\limits_{|z'|<a_0} \exp[\lambda F(z')] u(z') dz'.$$

Case 1. $\nabla\psi(0) = 0$ and $d = 2$.

For $|\eta_d| \leq c_1$, where c_1 is a small constant, we have $|F'(z')| > c_2 > 0$. Here we can invoke integration by parts in $\mathcal{J}$, and using the fact that the derivatives of u are bounded get an estimate $|\mathcal{J}| \leq C\lambda^{-M}$, where $M > 0$ is large, and hence also (2.17) for $d = 2$.

If $|\eta_d| > c_1$ we get $|F''(z')| \geq c_2 > 0$ since $F''(z') = \eta_d \psi''(\rho z')$ and $\psi''(0) \neq 0$. We therefore can apply van der Corput's Lemma (see [59], p. 334) and obtain the estimate $|\mathcal{J}| \leq C\lambda^{-1/2}$.

Case 2. $\nabla\psi(0) = 0$ and $d > 2$.

As in two-dimensional case let us first assume that $|\eta_d| \leq c_1$ for some small constant $c_1 > 0$. Then using the relation $|\eta'|^2 + \eta_d^2 = 1$ we integrate by parts in $\mathcal{J}$ and get $|\mathcal{J}| \leq C\lambda^{-M}$ where $M > 0$ is large, and (2.17) follows.

We now study the case when $|\eta_d| \geq c_1$. Setting $Q_\rho(z') = -\frac{1}{\rho}\nabla\psi(\rho z')$, we obtain

$$\nabla F(z') = \eta_d \left(\frac{\eta'}{\eta_d} + \frac{1}{\rho} \nabla\psi(\rho z') \right) = \eta_d \left(\frac{\eta'}{\eta_d} - Q_\rho(z') \right),$$

hence z' is a critical point of F if and only if $Q_\rho(z') = \frac{\eta'}{\eta_d}$. Observe that since the Hessian matrix of ψ at $z' = 0$ has full rank, it follows by Lemma 2.2.6 that the mapping $z' \mapsto Q_\rho(z')$ is one-to-one close to the origin for every small $\rho > 0$. Denoting

$$f(z') := \frac{\eta'}{\eta_d} \cdot z' + \frac{1}{\rho^2} \psi(\rho z'),$$

we have

$$\nabla f(z') = \frac{\eta'}{\eta_d} + \frac{1}{\rho} \nabla\psi(\rho z') = \frac{\eta'}{\eta_d} - Q_\rho(z'),$$

and $D^2 f(z') = D^2\psi(\rho z')$, where D^2 denotes the Hessian matrix. As $F(z') =$

$\eta_d f(z')$ we get

$$\mathcal{J} = \int\limits_{|z'|<a_0} \exp[\lambda \eta_d f(z')] u(z') dz',$$

where $\operatorname{supp}(u) \subset B(0, \frac{3}{4}a_0)$. Since the determinant of the Hessian matrix of ψ is nonzero at $z' = 0$, by Lemma 2.2.6 there exist constants $R > 0$ and $C_1 > 0$ such that for any $x', z' \in B(0, R)$ one has

$$\frac{1}{C_1}|x' - z'| \leq |Q_\rho(x') - Q_\rho(z')| \leq C_1|x' - z'|. \tag{2.18}$$

Clearly we may also assume that $|D^2 f(x')| \geq c > 0$ for $x' \in B(0, R)$. It follows from (2.18) that

$$\frac{1}{C_1}|x' - z'| \leq |\nabla f(x') - \nabla f(z')| \leq C_1|x' - z'|,$$

and using the fact that $\nabla\psi(0) = 0$ we obtain

$$|Q_\rho(z')| \leq C_1|z'|, \tag{2.19}$$

for each $x', z' \in B(0, R)$. One can see that there exists a neighborhood $\mathcal{M}$ of 0 such that if $x' \in \mathcal{M}$ then there exists $z' \in \overline{B(0, \frac{1}{4}R)}$ with[4] $Q_\rho(z') = x'$. Here the constants R, C_1, and the neighborhood $\mathcal{M}$ are independent of η and ρ. Then choose a_0 so that $B(0, a_0) \subset B(0, \frac{1}{4}R)$ and $B(0, 2C_1 a_0) \subset \mathcal{M}$.

First assume that $\left|\frac{\eta'}{\eta_d}\right| \geq 2C_1 a_0$. Using (2.19) we have

$$\begin{aligned} |\nabla f(z')| = \left|\frac{\eta'}{\eta_d} - Q_\rho(z')\right| \geq \left|\frac{\eta'}{\eta_d}\right| - |Q_\rho(z')| \geq 2C_1 a_0 - C_1|z'| \geq \\ 2C_1 a_0 - C_1 a_0 = C_1 a_0, \qquad |z'| \leq a_0. \end{aligned}$$

Hence we can integrate by parts in $\mathcal{J}$ and obtain the inequality (2.17). We then assume that $\left|\frac{\eta'}{\eta_d}\right| < 2C_1 a_0$. In this case $\frac{\eta'}{\eta_d} \in \mathcal{M}$ and there exists $z'_0 \in \overline{B(0, \frac{1}{4}R)}$ such that $Q_\rho(z'_0) = \frac{\eta'}{\eta_d}$, that is z'_0 is a critical point for f. We have $\nabla f(z'_0) = 0$ and therefore

$$\frac{1}{C_1}|z' - z'_0| \leq |\nabla f(z')| \leq C_2|z' - z'_0| \text{ for } z' \in B(0, R).$$

---

[4] A direct way to see the existence of the neighborhood $\mathcal{M}$ is to attribute it to the principle of *invariance of domain*, which states that an image of an open set in $\mathbb{R}^d$ under injective and continuous map into $\mathbb{R}^d$ is open. However this is a deep result, and a typical proof would touch upon algebraic topology by using Brouwer's fixed point theorem. In our particular case the claim (the existence of $\mathcal{M}$) can be proven directly along the lines of Lemma 2.2.6. We have that Jacobian of Q_ρ at 0, which is the Hessian of f at 0, is invertible. We then diagonalize it by a rigid motion of the ambient space, and see that in a small neighborhood of zero, the i-th component of Q_ρ behaves like scaling in the i-direction of $\mathbb{R}^{d-1}$, for all $i = 1, 2, ..., d-1$. The size of the scaling will be determined by eigenvalues of the Hessian of f at 0. This observation implies the existence of $\mathcal{M}$ independent of parameters η and ρ.

Now we can use Theorem 7.7.5 from [41] to obtain the estimate (2.17), and thus completing the proof when $\psi(0) = \nabla\psi(0) = 0$.

Case 3. $\nabla\psi(0) \neq 0$.

In this case we set $\psi_1(z') := \psi(z') - \nabla\psi(0)\cdot z'$, so that $\psi_1(0) = \nabla\psi_1(0) = 0$. Further, for

$$H(z') := \xi'\cdot z' + \xi_d\frac{1}{\rho}\psi(\rho z').$$

we have

$$H(z') = \xi'\cdot z' + \xi_d\frac{1}{\rho}(\psi_1(\rho z') + \rho\nabla\psi(0)\cdot z') =$$

$$\xi'\cdot z' + \xi_d\nabla\psi(0)\cdot z' + \xi_d\frac{1}{\rho}\psi_1(\rho z') = (\xi' + \xi_d\nabla\psi(0))\cdot z' + \xi_d\frac{1}{\rho}\psi_1(\rho z').$$

Next setting

$$\begin{cases} v' = \xi' + \xi_d\nabla\psi(0), \\ v_d = \xi_d. \end{cases}$$

or

$$\begin{cases} \xi' = v' - v_d\nabla\psi(0), \\ \xi_d = v_d, \end{cases}$$

with $c|\xi| \leq |v| \leq C|\xi|$, we shall obtain

$$H(z') = v'\cdot z' + v_d\frac{1}{\rho}\psi_1(\rho z').$$

Since $\nabla\psi_1(0) = 0$ we arrive at

$$|\mathfrak{I}| \leq C(\rho|v|)^{-(d-1)/2} \leq C(\rho|\xi|)^{-(d-1)/2},$$

which completes the proof of the main estimate of the Lemma.

Finally, the remark on dependence of C on u follows from the proof presented above, and is due to integration by parts. The proof of the Lemma is complete. $\square$

The astute reader could have observed from the proof of the last lemma, that one may relax the invertibility condition of the Hessian of ψ at the origin, requiring it to have nonzero rank only, while still getting a decay estimate for the integral $\mathfrak{I}$ in consideration. However, if the Hessian does not have full rank, one will get a worse upper bound for the integral. To keep the proof of Lemma 2.2.7 relatively less technical and in order not to diverge a lot from our paper [4], we will formulate the mentioned observation below in a separate statement.

Corollary 2.2.8. *Retain the notation of Lemma 2.2.7 and let all its assumptions be in force, except only instead of invertibility of the matrix $D^2\psi(0')$ we assume that it has rank k, where $1 \leq k \leq d-1$. Then, for the integral $\mathfrak{I}$ one has*

$$|\mathfrak{I}| \leq C(\rho|\xi|)^{-k/2},$$

with the same remark about the dependence of the constant C on the function u

as in Lemma 2.2.7.

Proof. We will use the proof of Lemma 2.2.7, and clearly it is enough to check the validity of the Case 2 of the proof above. Since $D^2\psi(0')$ is a symmetric matrix, we may diagonalize it by orthogonal transformation. Thus we fix a matrix $M \in O_{d-1}(\mathbb{R}^{d-1})$ such that $M^T D^2\psi(0')M$ is diagonal and has its first k entries on diagonal being nonzero. Then, instead of working in $\mathbb{R}^{d-1}$ we work in $\mathbb{R}^k \times \mathbb{R}^{d-1-k}$ treating the last $d-1-k$ coordinates as parameters. We set $\mathbb{R}^d \ni z' := (z''; z''') \in \mathbb{R}^k \times \mathbb{R}^{d-1-k}$. For z'' we have the all conditions of Lemma 2.2.7 to run the proof of Case 2 up to inequality (2.19), since there we do not have $\nabla''\psi(0''; z''') = 0''$ in general. But we still have $|Q_\rho(0''; z''')| \leq C|z'''|$ in view of the fact that $\psi(0) = 0$. The bound on Q_ρ is enough to proceed the proof. We note that it is easy to see that the all estimates can be done independently from the last $d-1-k$ coordinates. The proof of the corollary is complete. $\square$

Finally, let us remark, that the situation discussed in Corollary 2.2.8 will appear if we allow the domain D to have some of its principal curvatures vanishing. This will allow us to enlarge the class of domains where one has quantitative homogenization results. In fact, it will be enough for us to have a domain where at each point of its boundary there is at least one non-vanishing principal curvature. For example torus, or cylindrical-type domains will be included in that class.

2.2.2 Proofs of the main results

Proof of Theorem 2.2.1 (Pointwise convergence). Fix $r > 0$ and cover ∂D by the following family of balls $\mathcal{B} = \{B(z, \frac{1}{5}r) : z \in \partial D\}$. By the covering lemma of Vitali we extract a finite collection of disjoint balls

$$\mathcal{B}_0 := \{B(z_j, \frac{1}{5}r) : j = 1, 2, ..., M\} \subset \mathcal{B}$$

such that $\partial D \subset \bigcup_{j=1}^{M} B(z_j, r)$. We now fix a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ everywhere, $\varphi(x) = 0$ for $|x| \geq 2$ and $\varphi(x) = 1$ for $|x| \leq 1$. Next, we set $\varphi_r(x) := \varphi(x/r)$ and for $j = 1, 2, ..., M$ we let $\varphi_{r,j}(x) := \varphi_r(x - z_j)$. We thus get $\mathrm{supp}(\varphi_{r,j}) \subset B_j := B(z_j, 2r)$, for $1 \leq j \leq M$ and set $\varphi_j := \left(\sum_{n=1}^{M} \varphi_{r,n}\right)^{-1} \varphi_{r,j}$. By the definition of φ_j we have that it is defined on a neighborhood of ∂D, is supported in B_j and also $\sum_{j=1}^{M} \varphi_j \equiv 1$ on ∂D. Moreover, it follows form the definition of φ_j that for any $\alpha \in \mathbb{Z}_+^d$ there exists a constant C_α depending on α and φ, but independent of r such that

$$|D^\alpha \varphi_j(x)| \leq Cr^{-|\alpha|}, \qquad x \in B_j. \tag{2.20}$$

Let us also observe that for each point $x \in \partial D$ if M_x is the number of balls having center at some z_j, radius r and containing x, then $M_x \leq 10^d$. Indeed, for $x \in \partial D$ fixed we have that $B(x, 2r)$ contains all these balls, from which using the disjointness property of the family $\mathcal{B}_0$ we obtain $M_x(r/5)^d \le (2r)^d$, and the estimate on M_x follows.

We start by fixing a small number $a > 0$. Take $x \in D$ and for some constant $c_0 > 0$ let $0 < \rho \le c_0 d(x)$, where $d(x)$ is the distance of x from ∂D. Then take $r = a\rho$, and thus the support of φ_j which is the ball B_j has radius $2a\rho$. By (2.20) we have

$$|D^\alpha \varphi_j(x)| \le C_\alpha \frac{1}{(a\rho)^{|\alpha|}} = C_\alpha \frac{1}{\rho^{|\alpha|}}, \qquad x \in B_j.$$

For $\varepsilon > 0$ and $y \in \partial D$ set $g_\varepsilon(y) := g(y, y/\varepsilon)$ and recall that $\overline{g}(x) = \int\limits_{\mathbb{T}^d} g(x,y)dy$. By Poisson's representation we have

$$u_\varepsilon(x) - u_0(x) = \int\limits_{\partial D} P(x,y)[g_\varepsilon(y) - \overline{g}(y)] \left(\sum_{j=1}^{M} \varphi_j(y) \right) d\sigma(y) =$$

$$\sum_{j=1}^{M} \int\limits_{\Gamma_j} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_j(y)d\sigma(y),$$

where $\Gamma_j = \partial D \cap B_j$. We now analyze each piece of Γ_j separately. After a permutation and shift of the coordinate system, we may assume that for some constant $b > 0$ and smooth real-valued function ψ defined on $|z'| < b$, where $z' = (z_1, ..., z_{d-1})$ and satisfying $\psi(0) = 0$, Γ_j is represented by the following graph

$$\Gamma_j = \{(z', \psi(z')) : \ |z'| < 10a\rho\} \cap B_j,$$

where B_j is a ball in $\mathbb{R}^d$ with radius $2a\rho$ and centered at some point of Γ_j. We may also assume that $\operatorname{supp} \varphi_j \subset B_j$, $|D^\alpha \psi(z')| \le C_\alpha$, and $|\det(\frac{\partial^2 \psi}{\partial z_i \partial z_j})_{i,j=1}^{d-1}(z')| \ge c_0 > 0$ for $|z'| < b$, where constant c_0 is independent of z' and Γ_j. The last condition is due to strict convexity of the domain.

For $y^{(j)} \in \Gamma_j$ fixed we have

$$\int\limits_{\Gamma_j} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_j(y)d\sigma(y) = \frac{1}{|x - y^{(j)}|^{d-1}} I_j(x), \tag{2.21}$$

where

$$I_j(x) := I_j := \int\limits_{\Gamma_j} |x - y^{(j)}|^{d-1} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_j(y)d\sigma(y).$$

Due to the representation of Γ_j we have

$$I_j = \int\limits_{|z'|<10a\rho} |x - y^{(j)}|^{d-1} P(x, (z', \psi(z')))[g_\varepsilon - \overline{g}](z', \psi(z')) \times$$

$$\varphi_j(z', \psi(z'))(1 + |\nabla\psi(z')|^2)^{1/2} dz'.$$

Setting $z' = \rho y'$ we get

$$I_j = \rho^{d-1} \int\limits_{|y'|<10a} |x - y^{(j)}|^{d-1} P[x, (\rho y', \psi(\rho y'))][g_\varepsilon - \overline{g}](\rho y', \psi(\rho y')) \times$$
$$\varphi_j(\rho y', \psi(\rho y'))(1 + |\nabla \psi(\rho y')|^2)^{1/2} dy'.$$

Set $a_0 := 10a$. Since the boundary data g is $\mathbb{Z}^d$-periodic and smooth in its second variable, we have $g(x, y) = \sum\limits_{m \in \mathbb{Z}^d} c_m(x) \exp(m \cdot y)$, for $x \in \partial D$ and $y \in \mathbb{R}^d$, where $c_m(x)$ is the m-th Fourier coefficient of $g(x, \cdot)$. From here we get

$$[g_\varepsilon - \overline{g}](\rho y', \psi(\rho y')) = \sum_{m \neq 0} c_m(\rho y', \psi(\rho y')) \exp\left[(m' \cdot y') \frac{\rho}{\varepsilon} + \frac{m_d}{\varepsilon} \psi(\rho y')\right].$$

Plugging this expansion into I_j we obtain

$$I_j = \rho^{d-1} \sum_{m \neq 0} \int\limits_{|y'|<a_0} |x - y^{(j)}|^{d-1} P[x, (\rho y', \psi(\rho y'))] c_m(\rho y', \psi(\rho y')) \times$$
$$\varphi_j(\rho y', \psi(\rho z'))(1 + |\nabla \psi(\rho y')|^2)^{1/2} \exp\left[(m' \cdot y') \frac{\rho}{\varepsilon} + \frac{m_d}{\varepsilon} \psi(\rho y')\right] =$$
$$\rho^{d-1} \sum_{m \neq 0} \int\limits_{|z'|<a_0} \exp\left[(m' \cdot y') \frac{\rho}{\varepsilon} + \frac{m_d}{\varepsilon} \psi(\rho y')\right] u_{x,m}(y') dy',$$

where we have set

$$u_{x,m}(y') := |x - y^{(j)}|^{d-1} P[x, (\rho y', \psi(\rho y'))] c_m(\rho y', \psi(\rho y')) \times$$
$$\varphi_j(\rho y', \psi(\rho y'))(1 + |\nabla \psi(\rho y')|^2)^{1/2}.$$

Now observe that due to the choice of ρ, estimate (2.8) for the Poisson kernel and smoothness of ψ, φ_j, and the boundary data g, for any $\alpha \in \mathbb{Z}^d_+$ we have $|D^\alpha u_{x,m}(y')| \leq C_{\alpha,m}$ for all $|z'| < a_0$, where $C_{\alpha,m}$ is independent of x. Moreover we have

$$C_{\alpha,m} \lesssim_\alpha ||c_m||_{L^\infty(\partial D)} + \sum_{|\beta| \leq |\alpha|} ||D^\beta c_m||_{L^\infty(\partial D)}, \tag{2.22}$$

where $c_m(x) = \int_{\mathbb{T}^d} g(x, y) e^{2\pi i m \cdot y} dy$, and $m \in \mathbb{Z}^d \setminus \{0\}$. Denoting $\xi = (\rho m)/\varepsilon = ((\rho m')/\varepsilon, (\rho m_d)/\varepsilon) := (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ we obtain

$$I_j = \rho^{d-1} \sum_{m \neq 0} \int\limits_{|y'|<a_0} \exp[\xi' \cdot y' + \frac{\xi_d}{\rho} \psi(\rho y')] u_{x,m}(y') dy' := \rho^{d-1} \sum_{m \neq 0} \mathcal{I}_j(x; m),$$

where $\mathcal{I}_j(x; m)$ denotes the m-th integral in the sum. By Lemma 2.2.7 we have

$$|\mathcal{I}_j(x; m)| \leq C_m (\rho |\xi|)^{-(d-1)/2} = C_m \rho^{-(d-1)} \varepsilon^{(d-1)/2} \frac{1}{||m||^{(d-1)/2}}. \tag{2.23}$$

Moreover, by the same Lemma 2.2.7 and (2.22) we have that each C_m can be bounded by some fixed (independent of m) set of derivatives of $c_m(x)$. From this observation, and Corollary 2.2.5 we get that

$$\sum_{m\neq 0} \frac{|C_m|}{||m||^{(d-1)/2}} < \infty,$$

and thus by (2.23) we get $|I_j| \leq C\varepsilon^{(d-1)/2}$. From here, getting back to (2.21) we obtain

$$\left| \int\limits_{\Gamma_j} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_j(y)d\sigma(y) \right| \leq C\frac{1}{|x-y^{(j)}|^{d-1}}\varepsilon^{(d-1)/2}.$$

In view of definition of Γ_j we have $vol_{d-1}(\Gamma_j) \asymp \rho^{d-1}$, from which and the last inequality we obtain

$$|u_\varepsilon(x) - u_0(x)| \leq C\varepsilon^{(d-1)/2} \sum_{j=1}^{M} \frac{1}{|x-y^{(j)}|^{d-1}} \leq$$

$$C\varepsilon^{(d-1)/2}\rho^{-(d-1)} \sum_{j=1}^{M} \frac{vol_{d-1}(\Gamma_j)}{|x-y^{(j)}|^{d-1}} \leq C\varepsilon^{(d-1)/2}\rho^{-(d-1)} \int\limits_{\partial D} \frac{d\sigma(y)}{|x-y|^{d-1}} \leq$$

$$C\varepsilon^{(d-1)/2}\rho^{-(d-1)} \int\limits_{\partial D} \frac{1}{|x-y|^{d-1}} \frac{|x-y|^{\delta}}{d(x)^{\delta}} d\sigma(y) \leq C\varepsilon^{(d-1)/2}\rho^{-(d-1)} \frac{1}{d(x)^{\delta}},$$

where $\delta > 0$ is any small number. Also note that Γ_j-s are not necessarily pairwise disjoint, nevertheless the sum above is controlled by the corresponding surface integral, due to the fact noted in the beginning of the proof, that for any $y \in \partial D$ the number of j-s for which $y \in \Gamma_j$ is bounded by 10^d. Now we take $\rho = c_0 d(x)$ where $c_0 > 0$ is a small constant, and get

$$|u_\varepsilon(x) - u_0(x)| \leq C\varepsilon^{(d-1)/2} \frac{1}{d(x)^{d-1+\delta}}, \qquad x \in D,$$

where $\delta > 0$ is arbitrarily small. This estimate, in combination with uniform bound for u_ε and u_0 due to Corollary 2.1.5 completes the proof of the Theorem. □

Proof of Corollary 2.2.2. Let us first consider the case when $p = 1$. Fix some $\delta > 0$ small, then by Theorem 2.2.1 we have

$$||u_\varepsilon - u_0||_{L^1(D)} \lesssim \int\limits_0^{\varepsilon^{1/2}} 1dt + \int\limits_{\varepsilon^{1/2}}^{1} \varepsilon^{(d-1)/2} t^{1-d-\delta} dt \lesssim$$

$$\varepsilon^{1/2} + \varepsilon^{(d-1)/2}\varepsilon^{1/2(2-d-\delta)} \lesssim \varepsilon^{1/2-\delta/2}.$$

The case for $1 < p < \infty$ follows by interpolating between $p = 1$ and $p = \infty$. We get

$$||u_\varepsilon - u_0||_{L^p(D)} \leq ||u_\varepsilon - u_0||_{L^1(D)}^{1/p} ||u_\varepsilon - u_0||_{L^\infty(D)}^{1-1/p} \leq C\varepsilon^{(1/2-\delta)/p},$$

completing the proof. □

Let us now close this section with a few remarks. First, in the light of Corollary 2.2.8 we get the next result as a direct implication of the proof of Theorem 2.2.1.

Theorem 2.2.9. *Retain the assumptions (i), (ii) and (iv) of Section 2.1.1, and assume that there exists an integer* $1 \leq m \leq d-1$ *such that for any* $x \in \partial D$ *at least* m *of the principal curvatures of* ∂D *are non-vanishing at* x*.*

Under these assumptions, for $\varepsilon > 0$ *let* u_ε *be the solution to (2.1)-(2.2) and let* u_0 *be the solution of (2.4). Then we have the following results.*

(a) *For each* $\kappa > m$ *there exists a constant* C_κ *such that*

$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{m/2}}{d(x)^\kappa}\right\}, \qquad \forall x \in D,$$

where $d(x)$ *denotes the distance of* x *from the boundary of* D*.*

(b) *For each* $1 \leq p < \infty$ *and each* $\kappa < 1/(2p)$ *there exists a constant* C_κ *such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_\kappa \varepsilon^\kappa.$$

Proof. We only need to prove (a), since part (b) follows from (a) as in Corollary 2.2.2. To obtain (a) observe that having at least $1 \leq m \leq d-1$ non-vanishing curvatures at $x \in \partial D$ amounts to requiring the Hessian of the graph representing ∂D near x, to have rank at least m. Now we run the proof of Theorem 2.2.1, and for (2.23) we apply Corollary 2.2.8 instead of Lemma 2.2.7. The rest follows by obvious modifications. □

The next observation concerns the question of optimality of convergence rate obtained in Theorem 2.1.1. This natural problem was raised in [34] (see Remark 3, after Theorem 1.1 in [34]). Let us now consider the case of constant coefficient operators, then the settings of Theorem 2.1.1 and our Theorem 2.2.1 become identical. When $p = 2$ Theorem 2.1.1 gives $||u_\varepsilon - u_0|| \leq C_\alpha \varepsilon^\alpha$ for any $\alpha < (d-1)/(3d+5)$, while by Corollary 2.2.2 the upper bound for the exponent α can be replaced by $1/4$. We have $1/4 > (d-1)/(3d+5)$ if and only if $d \leq 8$, getting an improvement in the rate of convergence in dimensions up to 8, while in dimensions larger than 9, the rate provided by Theorem 2.1.1 is better. This shows that the convergence exponent of Theorem 2.1.1 is not optimal in general! However, neither is $1/4$ given by Corollary 2.2.2 for L^2 convergence.

At this stage, we are left out with the question of finding the optimal rate of convergence for homogenization, which we will address in the next section.

Let us also stress that convexity of the domain appears in Theorem 2.1.1 and in our Theorem 2.2.1, however the usage of this condition is completely different in these two cases. As we discussed in the introductory chapter, strict convexity for Theorem 2.1.1 is needed in order to guarantee that almost all points of the boundary have normal directions with a certain Diophantine property. Also, convexity was used to put the domain on one side of its supporting planes in order to make the approximation argument with boundary layer correctors workable.

In our case we only use strict convexity to get everywhere non vanishing Gaussian curvature, and in view of Theorem 2.2.9 convexity can be essentially relaxed while still ensuring homogenization with error estimates. This last remark enables one to incorporate large class of domains with smooth boundaries where boundary value homogenization holds. We also refer the reader to discussions on *localization principle* made in Section 1.2.3 of the first chapter.

2.3 L^p estimates

This section is devoted to the study of L^p convergence results for solutions to problem (2.1)-(2.2). We keep the same notation as before, and for $\varepsilon > 0$ we let u_ε be the solution (2.1)-(2.2) and let u_0 be the solution of the corresponding homogenized problem (2.4). We now state the main results of this section, where as before we will assume that assumptions 2.1.1 hold true.

Theorem 2.3.1. (L^p-convergence) *Let u_ε and u_0 be as above. Then, for all $1 \leq p < \infty$ one has*

$$\|u_\varepsilon - u_0\|_{L^p(D)} \leq C_p \begin{cases} \varepsilon^{1/2p}, & d = 2, \\ (\varepsilon |\ln \varepsilon|)^{1/p}, & d = 3 \\ \varepsilon^{1/p}, & d \geq 4. \end{cases},$$

The heuristic reason for getting better convergence rates in Theorem 2.3.1 compared to Corollary 2.2.2 lies in the following. If we let $x \in D$ to stay compactly inside D, we have a very fast convergence of u_ε toward u_0 by Theorem 2.2.1. But as we let x approach the boundary, we have to deal with the singularity that arises from the Poisson's kernel. To handle this, we trade-off some speed that comes from the averaging process of the boundary data to balance the singularity. But since we are working on the surface which is $(d-1)$-dimensional, the integrability threshold for singularities of the form $|z|^{-\alpha}$ (which is that of the Poisson kernel), with $\alpha > 0$ and z near 0, is $d-1$. However, if we could instead work in the domain D, we could have a larger allowance for integrable singularity, that can potentially lead to a lesser loss from the averaging process of the oscillating boundary data. We will implement this idea in the proof of Theorem 2.3.1.

Next, we study the question of optimality of the L^p convergence rate provided by Theorem 2.3.1. For simplicity we will consider the case of scalar equations rather than systems, and will assume that the boundary data g has no slow dependence, that is $g : \mathbb{T}^d \to \mathbb{C}$, and $\overline{g} = \int_{\mathbb{T}^d} g(y)dy$.

Theorem 2.3.2. (Optimality) *Let* $N = 1$, *and* g *be as above. Then for each* $1 \leq p < \infty$ *there exists a constant* C_p *independent of* ε, *such that*

$$\|u_\varepsilon - u_0\|_{L^p(D)} \geq C_p \varepsilon^{1/p} \|g - \overline{g}\|_{L^\infty(\mathbb{T}^d)}.$$

Theorems 2.3.1-2.3.2 imply that the convergence rate of homogenization of the Dirichlet problem with fixed operator and oscillating boundary data is optimal when $d \geq 4$.

The next result concerns the setting of Theorem 2.1.1, i.e. when both the coefficients and the boundary data exhibit oscillations simultaneously. Thus we are working with the operator $\mathcal{L}_\varepsilon$, and $\mathcal{L}_0$ is the corresponding homogenized operator both defined in the beginning of the current Chapter. Let us also recall that when working with $\mathcal{L}_\varepsilon$ we assume that coefficient tensor A is $\mathbb{Z}^d$-periodic.

Following [44] we set $P_\gamma^k(x) = x_\gamma(0, ..., 1, 0, ...)$, with 1 in the k-th position, where $1 \leq \gamma \leq d$ and $1 \leq k \leq N$. We also let $\mathcal{L}_\varepsilon^*$ to be the formal adjoint to $\mathcal{L}_\varepsilon$, that is the matrix of coefficients of $\mathcal{L}_\varepsilon^*$ is $A_{ji}^{\beta\alpha}$.

Theorem 2.3.3. (Homogenization of elliptic systems) *Assume* $d \geq 3$, *and for each* $\varepsilon > 0$ *let* u_ε *be the solution to the following problem*

$$\mathcal{L}_\varepsilon u_\varepsilon(x) = 0, \ x \in D \qquad \text{and} \qquad u_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right), \ x \in \partial D. \tag{2.24}$$

If $\mathcal{L}_\varepsilon^*(P_\gamma^k) = 0$ *in* D *for all* $1 \leq k \leq N$, $1 \leq \gamma \leq d$, *and* $\varepsilon > 0$ *then there exists a fixed boundary data* g^* *depending on operator, domain and boundary data* g *so that if* u_0 *is the solution to the homogenized problem*

$$\mathcal{L}_0 u_0(x) = 0, \ x \in D \qquad \text{and} \qquad u_0(x) = g^*(x), \ x \in \partial D, \tag{2.25}$$

then

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_p[\varepsilon(\ln(1/\varepsilon))^2]^{1/p}.$$

The restriction on the operator in the last theorem may be interpreted in different ways. First, let us reformulate it without ε, for which we observe from the definition of $\mathcal{L}_\varepsilon^*$ and P_γ^k that

$$0 = \mathcal{L}_\varepsilon^*(P_\gamma^k) = \frac{\partial}{\partial x^\alpha}\left[A_{ji}^{\beta\alpha}\left(\frac{\cdot}{\varepsilon}\right)\frac{\partial (P_\gamma^k)_j}{\partial x^\beta}\right](x) = \frac{1}{\varepsilon}\frac{\partial A_{ki}^{\gamma\alpha}}{\partial x^\alpha}\left(\frac{x}{\varepsilon}\right), \quad \forall x \in D, \ \forall \varepsilon > 0. \tag{2.26}$$

If $\varepsilon > 0$ is small enough, the domain $(1/\varepsilon)D$ will contain a lattice cube, hence in view of periodicity of A the condition (2.26) is equivalent to

$$\frac{\partial A_{ki}^{\gamma\alpha}}{\partial x^\alpha}(x) = 0, \qquad \forall x \in \mathbb{R}^d.$$

Now set $v_{k,i}^\gamma(x) = (A_{ki}^{\gamma 1}, ..., A_{ki}^{\gamma d})(x)$, where $x \in \mathbb{R}^d$, and $1 \leq k, i \leq N$, $1 \leq \gamma \leq d$. We obtain that condition (2.26) of Theorem 2.3.3 is equivalent to

$$\nabla \cdot v_{k,i}^\gamma(x) = 0, \qquad \forall x \in \mathbb{R}^d, \ 1 \leq k, i \leq N, \ 1 \leq \gamma \leq d.$$

Note that for scalar equations ($N = 1$) the last condition simply means that the rows of the coefficient matrix A considered as vector fields in $\mathbb{R}^d$ must be divergence free. On the other hand, this reformulation demonstrates that the cell problem for adjoint operator has only trivial solution.

Another interpretation of the mentioned condition of Theorem 2.3.3, that might be interesting, is through *harmonic coordinates* (see [27] for more details on harmonic coordinates, and in particular their usage in inverse homogenization problems). For each P_γ^k let $F_{\gamma,\varepsilon}^k$ be the solution to

$$\mathcal{L}_\varepsilon^*(F_{\gamma,\varepsilon}^k) = 0 \text{ in } D \qquad \text{and} \qquad F_\gamma^k = P_\gamma^k \text{ on } \partial D.$$

Then $\{F_\gamma^k\}$ are called harmonic coordinates associated to operator $\mathcal{L}_\varepsilon^*$. In this terms the structural condition requires all harmonic coordinates to be linear.

2.3.1 Proofs of convergence results

Proof of Theorem 2.3.1. We divide the proof into some steps.
Step 1. Reduction to local graphs. Let $z \in \partial D$, and $r > 0$ be small. Then there exists an orthogonal transformation $\mathcal{R}$ such that

$$(\mathcal{R}(\partial D - z)) \cap B(0,r) = \{(y', \psi(y')) : \ |y'| \leq 10r\} \cap B(0,r), \tag{2.27}$$

where $y' = (y_1, ..., y_{d-1})$, $\psi(0) = \nabla\psi(0) = 0$ and

$$\text{Hess}\psi(0) = \text{diag}(a_1, ..., a_{d-1}) \in M_{d-1}(\mathbb{R})$$

with $0 < c \leq a_1 \leq a_2 \leq ... \leq a_{d-1} \leq C$, where the lower bound c is due to strict convexity of D. We also have $|D^\alpha \psi| \leq C_\alpha$, and

$$K_1|y'| \leq |\nabla\psi(y')| \leq K_2|y'|, \tag{2.28}$$

where K_1 and K_2 do not depend on z. Now choose $\delta > 0$ so small that

(a) $\delta < \frac{r}{1000}$ and $K_1\delta < 1$,

(b) (2.28) holds for $|y'| \leq \frac{K_1}{4K_2}\delta$,

(c) $\left|\frac{\partial^2\psi}{\partial y_i \partial y_j}\right| \leq \frac{a_1}{1000d}$ for $i \neq j$ and $\left|\frac{\partial^2\psi}{\partial y_j^2} - a_j\right| \leq \frac{a_j}{100}$ for $|y'| \leq 100\delta$ when $j = 1, 2, ..., d-1$,

(d) $|n'| \leq K_1\delta$ implies that there exists a unique $|y'| \leq \delta$ so that $\nabla\psi(y') = n'$.

We remark that δ is a constant that does not depend on z. We have $\partial D \subset \bigcup\limits_{z\in\partial D} B(z, \frac{1}{2}L\delta)$ where $L = \frac{K_1}{4K_2 d}$, hence $\partial D \subset \bigcup\limits_{k=1}^{M} B(z^k, \frac{1}{2}L\delta)$ for some $z^1, ..., z^M \in \partial D$. Set $B_k := B(z^k, L\delta)$, and take a smooth partition of unity $\sum\limits_{k=1}^{M} \varphi_k = 1$ on ∂D, where $\text{supp}(\varphi_k) \subset B_k$. Recall that $\overline{g}(x)$ is the average of g on the unit torus with respect to its periodic variable, also denote $g_\varepsilon(x) := g(x, x/\varepsilon)$.

We have

$$u_\varepsilon(x) - u_0(x) = \sum_{k=1}^{M} \int_{\partial D} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_k(y)d\sigma(y) := \sum_{k=1}^{M} I_k(x),$$

where

$$I_k(x) := I_k := \int_{\partial D} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi_k(y)d\sigma(y).$$

Step 2. Reduction to volume integrals. Set $z = y - z^k$, then

$$I_k = \int_{(\partial D \cap B_k) - z^k} P(x, z^k + z)[g_\varepsilon - \overline{g}](z^k + z)\varphi_k(z^k + z)d\sigma(z).$$

We have $(\partial D \cap B_k) - z^k = (\partial D - z^k) \cap B(0, L\delta)$. By setting $y = \mathcal{R}z$ we obtain

$$I_k = \int_{\mathcal{R}(\partial D - z^k) \cap B(0,L\delta)} P(x, z^k + \mathcal{R}^{-1}y)[g_\varepsilon - \overline{g}](z^k + \mathcal{R}^{-1}y)\varphi_k(z^k + \mathcal{R}^{-1}y)d\sigma(y).$$

By (2.27) and condition (a) of Step 1 we may assume that

$$\mathcal{R}(\partial D - z^k) \cap B(0, L\delta) = \{(y', \psi(y')) : \ |y'| < 100\delta\} \cap B(0, L\delta),$$

and hence

$$I_k = \int_{|y'| < L\delta} P[x, z^k + \mathcal{R}^{-1}(y', \psi(y'))][g_\varepsilon - \overline{g}](z^k + \mathcal{R}^{-1}(y', \psi(y')))\times$$
$$\varphi_k(z^k + \mathcal{R}^{-1}(y', \psi(y')))(1 + |\nabla\psi(y')|^2)^{1/2}dy'.$$

Step 3. Reduction to oscillatory integrals. Since g is $\mathbb{Z}^d$-periodic in its second variable and is sufficiently smooth, we have

$$g(x,y) = \sum_{m \in \mathbb{Z}^d} c_m(x)\exp(m \cdot y),$$

and hence

$$g_\varepsilon(x) = \sum_{m \in \mathbb{Z}^d} c_m(x)\exp\left(\frac{m}{\varepsilon} \cdot x\right),$$

where for $m \in \mathbb{Z}^d$ we let $c_m : \partial D \to \mathbb{C}^N$ be the m-th Fourier coefficient of $g(x, \cdot)$. Using this and orthogonality of $\mathcal{R}$ we have

$$[g_\varepsilon - \overline{g}](z^k + \mathcal{R}^{-1}(y', \psi(y'))) =$$
$$\sum_{m \neq 0} c_m(z^k + \mathcal{R}^{-1}(y', \psi(y')))\exp\left(\frac{m}{\varepsilon} \cdot z^k\right)\exp\left[\frac{1}{\varepsilon}\langle \mathcal{R}m, (y', \psi(y'))\rangle\right], \quad (2.29)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. By setting $n := \mathfrak{R}m$ and $n = |n|(n', n_d)$ with $|(n', n_d)| = 1$ from (2.29) we obtain

$$[g_\varepsilon - \overline{g}](z^k + \mathfrak{R}^{-1}(y', \psi(y'))) = \\ \sum_{m \neq 0} c_m(z^k + \mathfrak{R}^{-1}(y', \psi(y')))\mathrm{exp}\left(\frac{m}{\varepsilon} \cdot z^k\right) \mathrm{exp}[\lambda F(y')],$$

where

$$F(y') = n' \cdot y' + n_d \psi(y'),$$

and $\lambda := |n|/\varepsilon = |m|/\varepsilon$ due to orthogonality of $\mathfrak{R}$. Next, by setting

$$\Phi_k(y') = \varphi_k(z^k + \mathfrak{R}^{-1}(y', \psi(y')))(1 + |\nabla \psi(y')|^2)^{1/2},$$

and

$$I_{k,m} = \int\limits_{|y'| < L\delta} c_m(z^k + \mathfrak{R}^{-1}(y', \psi(y')))P(x, z^k + \mathfrak{R}^{-1}(y', \psi(y')))\Phi_k(y')\mathrm{exp}[\lambda F(y')]dy',$$

we obtain

$$|I_k| \leq \sum_{m \neq 0} |I_{k,m}|.$$

Step 4. Decay of I_k. We split the study of decay of the integrals $I_{k,m}$ into two cases.

Case 1. $|n'| \geq K_1\delta/2$.

We have $\nabla F(y') = n' + n_d \nabla \psi(y')$. Then $|n'_j| \geq K_1\delta/2d$ for some $1 \leq j \leq d-1$, hence by (2.28) on $\mathrm{supp}(\Phi_k)$ we have

$$\left|\frac{\partial F}{\partial y_j}\right| \geq \frac{K_1\delta}{2d} - K_2|y'| \geq \frac{K_1\delta}{2d} - K_2 L\delta = \frac{K_1\delta}{2d} - \frac{K_1\delta}{4d} \geq \frac{K_1\delta}{4d}. \tag{2.30}$$

Now integrating by parts in $I_{k,m}$ in the j-th coordinate twice, by virtue of (2.30) and Lemma 2.1.3, for all $x \in D$ we conclude

$$|I_{k,m}(x)| \leq C\lambda^{-2} \int\limits_{|y'| \leq L\delta} \frac{\left[|c_m| + \left|\frac{\partial c_m}{\partial y_j}\right| + \left|\frac{\partial^2 c_m}{\partial y_j^2}\right|\right](z^k + \mathfrak{R}^{-1}(y', \psi(y')))}{|x - z^k - \mathfrak{R}^{-1}(y', \psi(y'))|^{d-1+2}} dy'. \tag{2.31}$$

Recall that $d(x)$ is the distance of $x \in D$ from the boundary of D, and set

$$D_\varepsilon = \{x \in D : \ d(x) \geq \varepsilon\}. \tag{2.32}$$

Now observe that

$$\int\limits_{D_\varepsilon} \frac{1}{|x - z^k - \mathfrak{R}^{-1}(y', \psi(y'))|^{d+1}} dx \leq C \int\limits_{|w| \geq \varepsilon} \frac{dw}{|w|^{d+1}} \leq C \int\limits_{\varepsilon}^{C} \frac{r^{d-1}}{r^{d+1}} dr = \frac{C}{\varepsilon}. \tag{2.33}$$

Combining this and (2.31) we obtain

$$\int\limits_{D_\varepsilon} |I_{k,m}(x)|dx \le C\lambda^{-2}\varepsilon^{-1} \int\limits_{|y'|\le L\delta} \left[|c_m| + \left|\frac{\partial c_m}{\partial y_j}\right| + \left|\frac{\partial^2 c_m}{\partial y_j^2}\right| \right] (z^k + \mathcal{R}^{-1}(y', \psi(y')))dy'.$$

Now taking into account the smoothness properties of g and applying Corollary 2.2.5 to g and to its derivatives to sum up c_m and its derivatives, from the last estimate we obtain

$$\sum_{m\neq 0} \|I_{k,m}\|_{L^1(D_\varepsilon)} \le C\varepsilon. \tag{2.34}$$

Case 2. $|n'| < K_1\delta/2$.

Since $\delta > 0$ is small and $|(n', n_d)| = 1$ we have $|n_d| > 1/2$ and hence

$$\left|\frac{n'}{n_d}\right| < \frac{K_1\delta}{2\frac{1}{2}} = K_1\delta.$$

By (d) there exists a unique $\widetilde{y'}$ with $|\widetilde{y'}| \le \delta$ and $\nabla\psi(\widetilde{y'}) = -\frac{n'}{n_d}$. Clearly $\nabla F(\widetilde{y'}) = 0$, and using (c) we arrive at

$$\left|\frac{\partial}{\partial y_j}\left(\frac{\partial F}{\partial y_j}\right)\right| = \left|n_d \frac{\partial^2 \psi}{\partial y_j^2}\right| \ge \frac{1}{2}\frac{99}{100}a_1.$$

From the latter by mean value theorem we get

$$\left|\frac{\partial F}{\partial y_j}(\widetilde{y'} + se'_j)\right| \ge \frac{99}{200}a_1|s|, \qquad |s| \le 4\delta,$$

where s is scalar and e'_j is the j-th unit vector of $\mathbb{R}^{d-1}$. By (c) for $i \neq j$ we have

$$\left|\frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial y_j}\right)\right| = \left|n_d \frac{\partial^2 \psi}{\partial y_i \partial y_j}\right| \le \frac{a_1}{1000d},$$

from which we obtain

$$\left|\frac{\partial F}{\partial z_j}(\widetilde{y'} + z')\right| \ge c|z_j|, \tag{2.35}$$

for $z' \in \mathcal{C}_j$ and $\widetilde{y'} + z' \in \mathrm{supp}(\Phi_k)$ where

$$\mathcal{C}_j = \{z' \in \mathbb{R}^{d-1} : \ |z'_j| \ge \frac{1}{2\sqrt{d-1}}|z'|\}, \qquad 1, 2, ..., d-1.$$

Clearly the cones $\mathcal{C}_j$ cover $\mathbb{R}^{d-1}$. For $j = 1, 2, ..., d-1$ there exists ω_j supported in $\mathcal{C}_j$, smooth away from the origin and homogeneous of degree 0 such that[5]

$$\sum_{j=1}^{d-1} \omega_j(z') = 1, \qquad \forall z' \neq 0.$$

Now fix a nonnegative function $h \in C^\infty(\mathbb{R}^{d-1})$ such that $h(y') = 0$ for $|y'| \geq 2$ and $h(y') = 1$ for $|y'| \leq 1$. Setting $y' = \widetilde{y'} + z'$ and $z^* := z^k + \mathcal{R}^{-1}(\widetilde{y'} + z', \psi(\widetilde{y'} + z'))$ we obtain

$$I_{k,m} = \int\limits_{|\widetilde{y'}+z'|<L\delta} c_m(z^*)P(x,z^*)\Phi_k(\widetilde{y'} + z')\exp[\lambda F(\widetilde{y'} + z')]dz'.$$

Set

$$I^1_{k,m} = \int\limits_{|\widetilde{y'}+z'|<L\delta} h(\varepsilon^{-1/2}z')c_m(z^*)P(x,z^*)\Phi_k(\widetilde{y'} + z')\exp[\lambda F(\widetilde{y'} + z')]dz',$$

and

$$I^2_{k,m} = \int\limits_{|\widetilde{y'}+z'|<L\delta} (1-h(\varepsilon^{-1/2}z'))c_m(z^*)P(x,z^*)\Phi_k(\widetilde{y'} + z')\exp[\lambda F(\widetilde{y'} + z')]dz',$$

so that $I_{k,m} = I^1_{k,m} + I^2_{k,m}$. It follows from Lemma 2.1.3 that

$$\int\limits_{D_\varepsilon} |P(x,y)|dx \leq C \int\limits_{D_\varepsilon} \frac{dx}{|x-y|^{d-1}} \leq C,$$

uniformly with respect to $y \in \partial D$ and $\varepsilon > 0$, which together with the smoothness condition on g and Corollary 2.2.5 gives

$$\sum_{m\neq 0} \int\limits_{D_\varepsilon} |I^1_{k,m}(x)|dx \leq C \sum_{m\neq 0} \int\limits_{|z'|\leq 2\varepsilon^{1/2}} |c_m(z^*)|dz' \leq C\varepsilon^{(d-1)/2}.$$

For the second part we have $I^2_{k,m} = \sum\limits_{j=1}^{d-1} I^{2,j}_{k,m}$ where

$$I^{2,j}_{k,m} = \int \omega_j(z')(1-h(\varepsilon^{-1/2}z'))c_m(z^*)P(x,z^*)\Phi_k(\widetilde{y'} + z')\exp[\lambda F(\widetilde{y'} + z')]dz'.$$

---

[5] To see the existence of ω_j-s, let $\mathcal{S}_j := \mathcal{C}_j \cap \mathbb{S}^{d-2}$, where $j = 1, 2, ..., d-1$. Then fix some smooth partition of unity, say $\{\tau_j\}_{j=1}^{d-1}$, subordinate to these intersections. Restrict τ_j onto $\mathcal{S}_j$ and extend it to $\mathbb{R}^{d-1} \setminus \{0\}$ as homogeneous of degree zero. Call this extension ω_j.

Now integrating by parts with respect to z_j in $I^{2,j}_{k,m}$ twice we obtain

$$|I^{2,j}_{k,m}(x)| \leq C\lambda^{-2}\cdot \\ \int \left| \frac{\partial}{\partial z_j} \left\{ \frac{1}{\frac{\partial F}{\partial z_j}} \frac{\partial}{\partial z_j} [\frac{1}{\frac{\partial F}{\partial z_j}} \omega_j(z')(1-h(\varepsilon^{-1/2}z'))c_m(z^*)P(x,z^*)\Phi_k] \right\} \right| dz' \quad (2.36)$$

Observe that since ω_j is homogeneous of degree 0, for each $j = 1, 2, ..., d-1$ and small $|z'|$ we have

$$\left| \frac{\partial^k \omega_j}{\partial z_j^k}(z') \right| \leq C\frac{1}{|z'|^k}, \qquad k = 1, 2, ...$$

Using this, (2.35), (2.36), Lemma 2.1.3, and applying Corollary 2.2.5 we obtain

$$\sum_{m\neq 0} |I^{2,j}_{k,m}(x)| \leq C\varepsilon^2 \sum_{k=1}^{6} A_k(x),$$

where

$$\begin{aligned}
A_1(x) &= \int\limits_{\varepsilon^{1/2}\leq|z'|\leq C} \frac{1}{|z'|^4}\frac{1}{|x-z^*|^{d-1}}dz',\\
A_2(x) &= \varepsilon^{-1/2}\int\limits_{\varepsilon^{1/2}\leq|z'|\leq 2\varepsilon^{1/2}} \frac{1}{|z'|^3}\frac{1}{|x-z^*|^{d-1}}dz',\\
A_3(x) &= \int\limits_{\varepsilon^{1/2}\leq|z'|\leq C} \frac{1}{|z'|^3}\frac{1}{|x-z^*|^{d}}dz',\\
A_4(x) &= \varepsilon^{-1}\int\limits_{\varepsilon^{1/2}\leq|z'|\leq 2\varepsilon^{1/2}} \frac{1}{|z'|^2}\frac{1}{|x-z^*|^{d-1}}dz',\\
A_5(x) &= \varepsilon^{-1/2}\int\limits_{\varepsilon^{1/2}\leq|z'|\leq 2\varepsilon^{1/2}} \frac{1}{|z'|^2}\frac{1}{|x-z^*|^{d}}dz',\\
A_6(x) &= \int\limits_{\varepsilon^{1/2}\leq|z'|\leq C} \frac{1}{|z'|^2}\frac{1}{|x-z^*|^{d+1}}dz'.
\end{aligned}$$

Let us note that z^* depends on z', however we have $z^* \in \partial D$. Having this in mind, an easy calculation shows that

$$\int\limits_{D_\varepsilon} \sum_{k=1}^{6} A_k(x)dx \leq C \begin{cases} \varepsilon^{-3/2}, & d=2,\\ \varepsilon^{-1}|\ln\varepsilon|, & d=3,\\ \varepsilon^{-1}, & d\geq 4,\end{cases}$$

where we used Fubini's theorem to change the volume and surface integrations.

Using this we obtain

$$\sum_{m\neq 0}\int_{D_\varepsilon} |I^{2,j}_{k,m}(x)|dx \leq C\begin{cases}\varepsilon^{1/2}, & d=2,\\ \varepsilon|\ln\varepsilon|, & d=3,\\ \varepsilon, & d\geq 4.\end{cases}$$

Combining together the estimates for $I^{1,j}_{k,m}$ and $I^{2,j}_{k,m}$, and using (2.34) we arrive at

$$\int_{D_\varepsilon} |I_k(x)|dx \leq C\begin{cases}\varepsilon^{1/2}, & d=2,\\ \varepsilon|\ln\varepsilon|, & d=3,\\ \varepsilon, & d\geq 4.\end{cases} \tag{2.37}$$

Step 5. L^p estimates. By virtue of Corollary 2.1.5 we have $\sup\limits_{\varepsilon>0}\|u_\varepsilon - u_0\|_{L^\infty(D)} < \infty$ and hence

$$\int_{D\setminus D_\varepsilon} |u_\varepsilon(x)-u_0(x)|dx \leq C\varepsilon, \tag{2.38}$$

which combined with (2.37) gives the claim when $p=1$.

Now for $1<p<\infty$ using the boundedness of $u_\varepsilon - u_0$ we obtain

$$\int_D |u_\varepsilon - u_0|^p dx \leq C\int_D |u_\varepsilon - u_0|dx = C\|u_\varepsilon - u_0\|_{L^1(D)},$$

hence

$$\|u_\varepsilon - u_0\|_{L^p(D)} \leq C\|u_\varepsilon - u_0\|^{1/p}_{L^1(D)}. \tag{2.39}$$

Theorem is proved. □

Simultaneously oscillating case

We now proceed to the proof of L^p estimates when both the operator and the boundary data are oscillating. For the proof we will use a result due to Kenig-Lin-Shen [44] to reduce the setting of rapidly oscillating operators to the fixed operator with oscillating Dirichlet condition, where our method can be applied. We start with some preliminaries.

For $y\in\partial D$ set $n(y)=(n_1(y),...,n_d(y))$ to be the unit outward normal to ∂D at the point y. Let $\widehat{A}^{\alpha\beta}_{ij}$, $1\leq\alpha,\beta\leq d$, $1\leq i,j\leq N$ be the (constant) coefficient matrix of the homogenized operator $\mathcal{L}_0$, and set $h(y)=(h_{ij}(y))_{N\times N}$ to be the inverse matrix of $(\widehat{A}^{\alpha\beta}n_\alpha(y)n_\beta(y))_{N\times N}$, where $y\in\partial D$. Recall that the operator $\mathcal{L}_0$ is elliptic in a sense of Section 2.1.1 (see [16]) hence $h(y)$ is correctly defined. Recall also that $P^k_\gamma(x)=x_\gamma(0,...,1,0,...)\in\mathbb{R}^N$, with 1 in the k-th position, where $1\leq\gamma\leq d$, $1\leq k\leq N$, and $\mathcal{L}^*_\varepsilon$ is the formal adjoint to $\mathcal{L}_\varepsilon$, that is the matrix of coefficients of $\mathcal{L}^*_\varepsilon$ is $A^{\beta\alpha}_{ji}$. We now let $\Phi^{*k}_{\varepsilon,\gamma}:=(\Phi^{*1k}_{\varepsilon,\gamma},...,\Phi^{*Nk}_{\varepsilon,\gamma})$ be the solution of

$$\mathcal{L}^*_\varepsilon\Phi^{*k}_{\varepsilon,\gamma}(x)=0,\ x\in D \qquad \text{and} \qquad \Phi^{*k}_{\varepsilon,\gamma}(x)=P^k_\gamma(x),\ x\in\partial D. \tag{2.40}$$

For $\varepsilon > 0$ and $y \in \partial D$ set

$$\omega_\varepsilon^{ij}(y) = h_{ik}(y) \cdot \frac{\partial}{\partial n(y)} \{\Phi_{\varepsilon,\gamma}^{*lk}(y)\} \cdot n_\gamma(y) \cdot n_\alpha(y) n_\beta(y) A_{lj}^{\alpha\beta}(y/\varepsilon). \tag{2.41}$$

Also set $g_\varepsilon(x) = g(x, x/\varepsilon)$, where $x \in \partial D$. We are now ready to formulate the result we will use from [44].

Theorem 2.3.4. (Kenig, Lin, and Shen [44], Theorem 3.9) *Let $d \geq 3$ and assumptions (i)-(iv) of Section 2.1.1 hold. Let also $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D and $u_\varepsilon = g_\varepsilon$ on ∂D. Then for any $1 \leq p < \infty$ one has*

$$||u_\varepsilon - v_\varepsilon||_{L^p(D)} \leq C\{\varepsilon(\ln[\varepsilon^{-1}M + 2])^2\}^{1/p} ||g_\varepsilon||_{L^p(\partial D)},$$

where $\mathcal{L}_0(v_\varepsilon) = 0$ in D and $v_\varepsilon = \omega_\varepsilon g_\varepsilon$ on ∂D, with ω_ε defined by (2.41), and M is the diameter of D.

We remark that this theorem is proved under some mild regularity conditions on the operator, domain and boundary data.

Proof of Theorem 2.3.3. Under the condition of the theorem on has $\mathcal{L}_\varepsilon^*(P_\gamma^k) = 0$ in D from which we get that $\Phi_{\varepsilon,\gamma}^{*k} \equiv P_\gamma^k$ where $1 \leq \gamma \leq d$ and $1 \leq k \leq N$. Using this and (2.41) we get

$$\omega_\varepsilon^{ij}(y) = h_{ik}(y) n_\gamma(y) n_\gamma(y) n_\alpha(y) n_\beta(y) A_{kj}^{\alpha\beta}(y/\varepsilon) = \\ h_{ik}(y) n_\alpha(y) n_\beta(y) A_{kj}^{\alpha\beta}(y/\varepsilon), \tag{2.42}$$

where the last equality is due to the fact that $n(y)n(y) = |n(y)|^2 = 1$ for all $y \in \partial D$. We now proceed to identification of the homogenized boundary data $g^*(x)$. Recall that since we are working with the family $\mathcal{L}_\varepsilon$, the coefficient matrix A is now assumed to be $\mathbb{Z}^d$-periodic. Set $c_m(A_{kj}^{\alpha\beta})$ to be the m-th Fourier coefficient of $A_{kj}^{\alpha\beta}$. For the boundary vector-valued function $g(x,y)$ let g_j be its j-th component, $1 \leq j \leq N$, and set $c_m(g_j; x)$ to be the m-th Fourier coefficient of the function $g_j(x, \cdot)$, where $x \in \partial D$.

Now observe that by virtue of Theorem 2.3.4 to get the homogenization of problem (2.24) it is enough to homogenize v_ε. Using (2.42) and Fourier expansion of A and $g(x, \cdot)$ for the boundary data of v_ε we get

$$v_\varepsilon(y) = \omega_\varepsilon(y) g_\varepsilon(y) = h_{ik}(y) n_\alpha(y) n_\beta(y) A_{kj}^{\alpha\beta}(y/\varepsilon) g_j(y, y/\varepsilon) = \\ h_{ik}(y) n_\alpha(y) n_\beta(y) \sum_{m \in \mathbb{Z}^d} c_m(A_{kj}^{\alpha\beta}) c_{-m}(g_j; y) + \\ h_{ik}(y) n_\alpha(y) n_\beta(y) \sum_{\substack{m,n \in \mathbb{Z}^d \\ m+n \neq 0}} c_m(A_{kj}^{\alpha\beta}) c_n(g_j; y) \exp\left[\frac{y}{\varepsilon} \cdot (m+n)\right]. \tag{2.43}$$

Due to the smoothness conditions on A and g their Fourier series converge absolutely, hence rearrangements in (2.43) are correct. Set $g_i^*(y)$ to be the first term in the right hand side of (2.43), we claim that the homogenized boundary data is

$g^*(x) = (g_i^*(x))_{i=1}^N$. To see this define u_0 as the solution to the following problem

$$\mathcal{L}_0 u_0(x) = 0, \ x \in D \qquad \text{and} \qquad u_0(x) = g^*(x), \ x \in \partial D.$$

By the smoothness of the domain, operator and boundary data, the definition of v_ε and u_0, it follows from the proof of Theorem 2.3.1 that

$$||v_\varepsilon - u_0||_{L^p(D)} \leq C_p \begin{cases} (\varepsilon |\ln \varepsilon|)^{1/p}, & d = 3 \ , \\ \varepsilon^{1/p}, & d \geq 4. \end{cases}$$

This in combination with Theorem 2.3.4 finishes the proof of our Theorem 2.3.3 with homogenized boundary data g^* defined explicitly in terms of operator, domain and boundary data g. □

Remark 2.3.5. *Observe that the homogenized boundary data g^* computed in the proof of Theorem 2.3.3 is smooth on the boundary, and depends non-trivially on the all problem-related ingredients. This is the first example of homogenization problem with non constant operator such that the boundary data corresponding to the limiting problem is regular. For general operators the regularity issue of g^* will be addressed in Chapter 4.*

2.3.2 Optimality of upper bounds

Throughout this section instead of systems we will consider equations, so the operator $\mathcal{L}$ is considered only in the case $N = 1$. The main idea of the proof of Theorem 2.3.2 is to show that solutions to ε-problem concentrate in a strip of width ε near the boundary of D, and to prove that the portion of this strip where we have concentration is bounded from below independently of ε.

We start with a simple lemma.

Lemma 2.3.6. (Concentration near the boundary) *Let u be the solution to the Dirichlet problem for the operator $\mathcal{L}$ in the domain D with boundary data $g : \mathbb{R}^d \to \mathbb{C}$ which is Lipschitz with constant $Lip(g)$.*

Then there exist constants C_1, C_2 depending on dimension, domain, operator, but independent of g, so that for any $x \in D$, $\xi \in \partial D$, and small enough $\delta > 0$ one has

$$|u(x) - g(\xi)| \leq C_1 \delta Lip(g) + \frac{1}{8} \|g\|_\infty,$$

provided $|x - \xi| \leq C_2 \delta$.

Proof. By the Poisson representation we have

$$u(x) = \int\limits_{\partial D} P(x, y) g(y) d\sigma(y), \qquad x \in D.$$

Fix $\xi \in \partial D$ and $x \in D$. If $|x - \xi| \leq \delta/2$, and $|\xi - y| > \delta$ where $y \in \partial D$, then clearly $|x - y| > \delta/2$. Using this, Lemma 2.1.4, the distance estimate of

Lemma 2.1.3, and the fact that the Poisson kernel has integral equal to one over the boundary of D, we obtain

$$|u(x) - g(\xi)| = \left| \int\limits_{\partial D} P(x,y)[g(y) - g(\xi)]d\sigma(y) \right| \leq$$
$$\int\limits_{|y-\xi|<\delta} |P(x,y)||g(y) - g(\xi)|d\sigma(y) + \int\limits_{|y-\xi|\geq\delta} |P(x,y)||g(y) - g(\xi)|d\sigma(y) \leq$$
$$C\delta Lip(g) + C\|g\|_{\infty} d(x) \int\limits_{|y-\xi|\geq\delta} \frac{d\sigma(y)}{|x-y|^d}, \tag{2.44}$$

where the constant C is determined by the Poisson kernel. The last integral is estimated in a similar way as we proved Lemma 2.1.4, and uniformly with respect to x we obtain

$$\int\limits_{|y-\xi|\geq\delta} \frac{d\sigma(y)}{|x-y|^d} \leq C\frac{1}{\delta}.$$

It is left to take x so that $|x - \xi| < C_2\delta$, where C_2 is a sufficiently small constant independent of $x \in D$, $\xi \in \partial D$ and g, hence the claim. $\square$

The next Lemma is essentially the Weyl's equidistribution theorem, in our case concerning equidistribution of scaled surfaces modulo $\mathbb{Z}^d$.

Definition 2.3.1. *For $x, y \in \mathbb{R}^d$ we say that they are equal modulo $\mathbb{Z}^d$, and write $x \equiv y \pmod{\mathbb{Z}^d}$ if $x - y \in \mathbb{Z}^d$.*

If $x \in \mathbb{R}^d$, by $x \bmod \mathbb{Z}^d$ we denote the unique point y in the unit torus of $\mathbb{R}^d$ which is equal to x modulo one.

Lemma 2.3.7. (Equidistribution of scaled surfaces) *Suppose $D \subset \mathbb{R}^d$ $(d \geq 2)$ is strictly convex, and $\partial D := \Gamma$ is smooth. Then for any Riemann integrable function $g : \mathbb{T}^d \to \mathbb{R}$ one has*

$$\int\limits_{\mathbb{T}^d} g(x)dx = \lim_{\lambda\to\infty} \frac{1}{vol_{d-1}(\Gamma)} \int\limits_{\Gamma} g(\lambda y)d\sigma(y). \tag{2.45}$$

Proof. We first prove the Lemma for smooth functions. Suppose $g \in C^{\infty}(\mathbb{R}^d)$ and is $\mathbb{Z}^d$-periodic. Then

$$g(x) = \sum_{m\in\mathbb{Z}^d} c_m \exp(m \cdot x), \qquad x \in \mathbb{T}^d,$$

where the series converges absolutely due to smoothness of g. Plugging this expansion into (2.45) we see that it is enough to prove that

$$a_\lambda := \sum_{m\neq 0} c_m \int\limits_{\Gamma} \exp(\lambda m \cdot y)d\sigma(y)$$

converges to 0, as $\lambda \to \infty$. Denote by $\widehat{\sigma}(\xi)$ the Fourier transform of the surface measure σ. Since Γ is smooth and has non-vanishing Gaussian curvature, we have the following well-known estimate (see [59], chapter VIII, Theorem 1)

$$|\widehat{\sigma}(\xi)| \leq C|\xi|^{-(d-1)/2}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Using this estimate we obtain

$$|a_\lambda| \leq \sum_{m \neq 0} |c_m||\widehat{\sigma}(\lambda m)| \leq C\lambda^{-(d-1)/2} \sum_{m \neq 0} \frac{|c_m|}{\|m\|^{(d-1)/2}}.$$

The last sum converges due to smoothness of g, and thus we get the claim for smooth functions.

Now if g is a characteristic function of some rectangle in the unit torus, then it is easy to see that there exist a sequence of smooth functions f_n and F_n, $n = 1, 2, ...$ so that

1. $f_n(x) \leq g(x) \leq F_n(x), \qquad x \in \mathbb{R}^d$,
2. $\lim\limits_{n\to\infty} \int_{\mathbb{T}^d} [F_n(x) - f_n(x)]dx = 0$,

from which it follows that (2.45) holds true for characteristic functions of rectangles. Clearly it will hold true also for their linear combinations, i.e. step-functions. Now observe that when g is Riemann integrable function, then the same pointwise bounds from above and below hold true by means of step-functions, hence the statement $\square$

Applying Lemma 2.3.7 to characteristic functions we obtain the following result.

Corollary 2.3.8. *Let Γ be as above, and $A \subset \mathbb{T}^d$ be a ball. Then*

$$\mu(A) = \lim_{\lambda\to\infty} \frac{vol_{d-1}\{y \in \Gamma : \ \lambda y \bmod \mathbb{Z}^d \in A\}}{vol_{d-1}(\Gamma)},$$

where μ denotes the Lebesgue measure in $\mathbb{R}^d$.

Now we are ready to complete the proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. Without loss of generality we may assume that the boundary data g has mean value 0, and hence $u_0 = 0$.

By the Poisson representation we have

$$u_\varepsilon(x) = \int_{\partial D} P(x,y) g(y/\varepsilon) d\sigma(y), \qquad x \in D. \tag{2.46}$$

If $g \equiv 0$, then we are done, otherwise set $E := \{x \in \mathbb{T}^d : \ |g(x)| > \frac{1}{2}\|g\|_\infty\}$. Clearly E is an open set, and by passing to a subset of positive measure, we may assume that E is a ball.

Due to Corollary 2.3.8 there exists a constant $c_0 > 0$ so that for all $\varepsilon > 0$ small enough one has

$$\frac{1}{vol_{d-1}(\partial D)} vol_{d-1}\{y \in \partial D : \ |g_\varepsilon(y)| > \frac{1}{2}\|g\|_\infty\} > \frac{1}{2}\mu(E).$$

Now fix $y \in \partial D$, so that $|g_\varepsilon(y)| > 1/2\|g\|_\infty$, and apply Lemma 2.3.6 with

$$\delta = \frac{1}{8C_1} \frac{\|g\|_\infty}{Lip(g_\varepsilon)} = \frac{1}{8C_1} \frac{\varepsilon\|g\|_\infty}{Lip(g)}.$$

We obtain

$$|u_\varepsilon(x) - g_\varepsilon(y)| < \|g\|_\infty/4, \text{ if } x \in D, \text{ and } |x - y| \leq C\varepsilon, \tag{2.47}$$

where the constant C is independent of ε. Since $|g_\varepsilon(y)| > \|g\|_\infty/2$ on a fixed portion of the boundary for all small enough $\varepsilon > 0$, inequality (2.47) implies that on a fixed portion of the strip $B_\varepsilon := \{x \in D : \ d(x) < C\varepsilon\}$ one has $|u_\varepsilon(x)| > \|g\|_\infty/4$, where $x \in B_\varepsilon$.

Now for $1 \leq p < \infty$ taking the L^p norm of u_ε only on that strip we obtain $\|u_\varepsilon\|_{L^p(B)} \geq C\varepsilon^{1/p}\|g\|_\infty$, which proves the theorem. □

We conclude here that Theorem 2.3.2 gives sharp bounds on convergence rate of the homogenization process in dimensions 4 and higher, and sharp up to logarithmic correction in dimension 3.

Two dimensional case

The aim of this section is to study optimality of the L^p convergence rate for boundary homogenization established in Section 2.3. In contrast to higher dimensions, the statement of generic optimality fails in dimension two as we will illustrate on some examples below.

The case of $p = 1$. Here we show on a particular example that L^1 convergence rate in dimension two provided by Theorem 2.3.1 can not be improved in general. Let B be the unit disc in $\mathbb{R}^2$, and set $g(x_1, x_2) = e^{2\pi i x_2}$. We have that g is periodic with respect to $\mathbb{Z}^2$ and has mean value 0 over its unit cell of periodicity. For $\lambda > 1$ large consider the following problem

$$\Delta u_\lambda(x) = 0 \text{ in } B \qquad \text{and} \qquad u_\lambda(x) = g(\lambda x) \text{ on } \partial B. \tag{2.48}$$

In order to estimate u_λ on B we are going to use the method of stationary phase (see e.g. [59], Chapter VIII). For $|x| < 1$ and $|y| = 1$ let $P(x, y)$ be the Poisson kernel for the Laplace operator in B. We then let $n_\pm := (0, \pm 1)$ be north and south poles of the disc, and fix some small open neighborhoods of $n_\pm$ on the unit circle denoting correspondingly by $\Delta_\pm$. Now let functions ψ_+, ψ_- and $1 - (\psi_+ + \psi_-)$ be a smooth partition of unity of the unit circle subordinate to Δ_+ and Δ_-. For the sake of our example it is enough to consider u_ε only at the points $|x| < 1/2$ where we have that for any y, $P(\cdot, y)$ is a smooth function with bounded derivatives. Representing u_λ by the Poisson integral, we see from the definition of g that the only critical points of the phase function of the exponential are $n_\pm$, hence by the principle of the non-stationary phase (see [59], p. 341, Prop. 4) we get

$$u_\lambda(x) = \int\limits_{|y|=1} P(x,y)\psi_+(y)g(\lambda y)d\sigma(y) + \int\limits_{|y|=1} P(x,y)\psi_-(y)g(\lambda y)d\sigma(y) + O(\lambda^{-2}),$$

where the constants are uniform for $|x| < 1/2$. In each of the integrals above, the phase function of the exponential in g has non-degenerate critical point at the corresponding pole, thus using the principle of the stationary phase (see [59], Chapter VIII, Prop. 3) uniformly for $|x| < 1/2$ we obtain[6]

$$u_\lambda(x) = C\lambda^{-1/2}[P(x,n_+)e^{2\pi i\lambda} - P(x,n_-)e^{-2\pi i\lambda}] + O(\lambda^{-3/2}).$$

To see that the two terms in the brackets do not cancel, it is enough to restrict x to $\{x \in B : \ |x| < 1/2, \ 1/4 < x_2 < 1/2\}$, since $P(x,y) = C(1-|x|^2)/|x-y|^2$. Considering u_λ on the mentioned subset only, we obtain $||u_\lambda||_{L^1(B)} \geq C\lambda^{-1/2}$ showing that the convergence rate given by Theorem 2.3.1 is sharp in general for $p = 1$ and $d = 2$.

The case of $p = 2$. Here we study the case of optimality of L^2-convergence rate in dimension two. We will consider the same problem as for $p = 1$, however for notational convention we will change the periodicity of the boundary data from $\mathbb{Z}^2$ to $2\pi\mathbb{Z}^2$, namely for $\lambda > 1$ we let u_λ be the solution to (2.48), where $g(x_1, x_2) = e^{ix_2}$.

Claim 2.3.9. *Let u_λ be as above. Then $||u_\lambda||_{L^2(B)} \leq C\lambda^{-1/3}(\ln\lambda)^{1/2}$.*

Let us remark that by Theorem 2.3.1 we have $||u_\lambda||_{L^2(B)} \leq C\lambda^{-1/4}$, while Theorem 2.3.2 gives asymptotic lower bound of $\lambda^{-1/2}$. Thus, this example shows that convergence rate of $\lambda^{-1/4}$ is not generically optimal.

Proof. As before let $P(x,y)$ be the Poisson kernel for Laplacian in the unit ball, where $|x| < 1$ and $|y| = 1$. We have the following expansion (see [13], p. 99)

$$P(re^{it}, e^{i\theta}) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in(t-\theta)} \quad \text{where } 0 \leq r < 1 \text{ and } t,\theta \in [0,2\pi].$$

---

[6] Here we use the following fact for the constants appearing in the stationary phase analysis. For oscillatory integral of the form $\int_a^b e^{i\lambda\varphi(x)}\psi(x)dx$, let x_0 be the unique point in (a,b) where $\varphi'(x_0) = 0$, and assume that $\varphi''(x_0) \neq 0$. Also, let ψ be a smooth cut-off function. Then the constant of the principal term in the asymptotic expansion of the integral with respect to λ equals

$$\left(\frac{2\pi}{-i\varphi''(x_0)}\right)^{1/2} e^{i\lambda\varphi(x_0)}\psi(x_0),$$

where the principal branch of a complex root function is fixed along negative half-axis. This fact can be seen from the analysis of [59], Chapter VIII, Prop. 3.

Then representing u_λ by Poisson integral and making a change of variable we obtain

$$u_\lambda(re^{it}) = \sum_{n\in\mathbb{Z}} r^{|n|} \int_0^{2\pi} e^{in(t-\theta)} e^{i\lambda\sin\theta} d\theta = \sum_{n\in\mathbb{Z}} r^{|n|} e^{int} J_n(\lambda),$$

where for $n \in \mathbb{Z}$, $J_n(\lambda) := \int_0^{2\pi} e^{i(\lambda\sin\theta - n\theta)} d\theta$ is the Bessel function of order n. Due to orthogonality of exponentials, from the last expression we obtain

$$||u_\lambda(x)||^2_{L^2(B)} = \int_0^1 \int_0^{2\pi} r u_\lambda(re^{it}) \overline{u_\lambda(re^{it})} dt dr =$$

$$\sum_{n,m\in\mathbb{Z}} J_n(\lambda)\overline{J_m(\lambda)} \int_0^1 \int_0^{2\pi} r^{|n|+|m|+1} e^{i(n-m)t} dt dr = \sum_{n\in\mathbb{Z}} \frac{|J_n(\lambda)|^2}{2(1+|n|)}. \quad (2.49)$$

We have that $J_n(\lambda) = \int_0^{2\pi} e^{i\lambda\varphi(\theta)} d\theta$ where $\varphi(\theta) = \sin\theta - \frac{n}{\lambda}\theta$. Since $\varphi''(\theta) = -\sin\theta$ and $\varphi'''(\theta) = -\cos\theta$ we can partition $[0, 2\pi]$ into some finite number of subintervals so that in each of them either $|\varphi''(\theta)| > c_0$ or $|\varphi'''(\theta)| > c_0$ where $c_0 > 0$ is some absolute constant. Hence by van der Corput estimate (see [59], Chapter VIII, Prop. 2) we obtain

$$|J_n(\lambda)| \le C\lambda^{-1/3} \quad (2.50)$$

uniformly[7] in $n \in \mathbb{Z}$ and $\lambda > 1$. On the other hand for $n \neq 0$ integrating by parts we have

$$J_n(\lambda) = \frac{1}{-in} \int_0^{2\pi} e^{i\lambda\sin\theta} de^{-in\theta} = \frac{\lambda}{n} \int_0^{2\pi} e^{-in\theta} \cos\theta e^{i\lambda\sin\theta} d\theta. \quad (2.51)$$

Using (2.50) and (2.51) in (2.49) we obtain

$$||u_\lambda||^2_{L^2(B)} = \sum_{|n|\le\lambda^2} + \sum_{|n|>\lambda^2} \lesssim \lambda^{-2/3} \sum_{|n|\le\lambda^2} \frac{1}{1+|n|} + \sum_{|n|>\lambda^2} \frac{\lambda^2}{n^2(1+|n|)} \lesssim$$

$$\lambda^{-2/3}\ln\lambda + \lambda^2 \frac{1}{(\lambda^2)^2} \lesssim \lambda^{-2/3}\ln\lambda,$$

completing the proof. □

As these examples show the two dimensional case is somewhat different, and possibly needs a specified approach.

---

[7] We note that for n fixed, we can replace $1/3$ by $1/2$, however for uniform estimates in n and λ the exponent $1/3$ here is the best possible as can be shown for $\lambda = n$ (see [59], p. 357).

2.4 Finale

We end this chapter with a few remarks and problems.

L^p convergence of the simultaneously oscillating case. The structural condition imposed on the coefficients in Theorem 2.3.3 is due to our proof and it is a very interesting problem to see if one can remove it at the same time preserving the optimal speed of convergence for the homogenization process. To achieve it one would most probably need to develop some new ideas and tools. Let us finally stress that the exponent for L^p convergence can not go beyond $1/p$ as Theorem 2.3.2 shows, and it seems natural that in dimensions greater than two indeed $1/p$ should serve as the optimal exponent governing the algebraic rate of convergence for homogenization.

Pointwise bounds for the oscillating problem. It is also a very interesting problem to study pointwise convergence in the spirit of Theorem 2.2.1 of u_ε toward u_0 where u_ε solves the oscillating problem (2.24) and u_0 is the solution to the corresponding homogenized problem (2.25). Note that the methods of Theorem 2.2.1 will not apply in this situation, since one does not have appropriate bounds on the Poisson kernel corresponding to oscillating problem.

Change of the microstructure. Observe that when dealing with boundary value homogenization we have assumed that the boundary data is $\mathbb{Z}^d$-periodic. While this rigidity assumption is necessary for our arguments to proceed, we may as well change the microstructure from $\mathbb{Z}^d$ to any other co-compact lattice of $\mathbb{R}^d$ and consider periodicity with respect to that structure. More precisely take any collection of linearly independent vectors $v_1, ..., v_d \in \mathbb{R}^d$ and set $G := v_1\mathbb{Z} + ... + v_d\mathbb{Z}$. Clearly $(G, +)$ as a discrete subgroup of $(\mathbb{R}^d, +)$ is isomorphic to $\mathbb{Z}^d$, and by co-compactness here we mean that the fundamental domain of the quotient $\mathbb{R}^d/G$ has finite volume. Then one can in the same way as for $\mathbb{Z}^d$ consider periodicity with respect to G. Again, in a similar fashion we will get expansion into series but now with respect to the dual lattice of G, which is defined as the set of all vectors $v^* \in \mathbb{R}^d$ such that $v \cdot v^* \in \mathbb{Z}$ for any $v \in G$ (see [47] for the introduced notions concerning lattices). The entire analysis of boundary value homogenization considered in this chapter will work in like manner as for $\mathbb{Z}^d$.

An interesting situation comes into view if we allow the microstructure to vary independently. For this we may restrict ourselves to the set of unimodular lattices, i.e. full rank lattices having co-volume equal to one. The space of unimodular lattices of $\mathbb{R}^d$ can be identified with the homogeneous space $\mathrm{X} := \mathrm{SL}(\mathbb{Z}, d)/\mathrm{SL}(\mathbb{R}, d)$, where SL stands for the *special linear group*. The space X comes equipped with a natural Haar measure (see e.g. [60] for the definition of this measure and its Iwasawa decomposition in particular) and it is very interesting to understand what would be the probabilistic analogues of the homogenization results contained in this chapter if the microstructure is sampled at random from X according to its Haar measure.

When Fourier analysis meets compactness methods. In the problem with both oscillating coefficients and boundary data we consider oscillations on the same scale ε. Assume now the coefficients oscillate at scale ε^α, and the boundary data at scale ε^β, for some $\alpha, \beta > 0$. It seems interesting to understanding what is the true complexity of the problem with respect to scales. As some preliminary analysis shows if we let $\beta > \alpha$, meaning that the boundary data oscillates *faster* than the coefficients, then the process of boundary data homogenization dominates, and essentially by Fourier analysis methods, as developed in this chapter, one may show that the homogenized problem has the usual constant coefficient operator, and boundary data equal to the average of the original data with respect to its periodic variable. For the range $\beta < \alpha$, i.e. the operator oscillates *faster* than the boundary data, we may show similar results basically by compactness methods if $\beta < \alpha + \kappa$ for some properly chosen $0 \leq \kappa < 1$, however we do not know if we may let $\kappa = 0$. These remarks manifest that there is a range of oscillations covered by *Fourier analysis* and there is one covered by *compactness methods*. It is interesting to see when they both break down. Certainly they do at the same scale, i.e. $\alpha = \beta$, but is it the only case?

A decomposition argument. Keeping the standard notation and assumptions introduced in Section 2.1.1, for the simultaneously oscillating problem let us consider the following class of operators. For fixed $\alpha > 0$ we say that the coefficient tensor A is from the class $\mathcal{A}_\alpha$ if the problem with the oscillating operator $-\nabla \cdot A(\cdot/\varepsilon)\nabla$, and any oscillating boundary data $g(\cdot, \cdot/\varepsilon)$ admits homogenization in $L^2(D)$ with the rate of convergence at least ε^α. By Theorem 2.1.1 of Gérard-Varet and Masmoudi we know that for any $\alpha < (d-1)/(3d+5)$ the class $\mathcal{A}_\alpha$ contains all smooth and periodic coefficients A. By our Theorem 2.3.3 we have that in dimensions greater than two, for any $\alpha < 1/2$ the class $\mathcal{A}_\alpha$ is non empty, and by our Theorem 2.3.2 we have that $\mathcal{A}_\alpha$ is empty for any $\alpha > 1/2$. What is interesting to understanding here is whether the classes $\mathcal{A}_\alpha$ have additive structure, i.e. if $A, B \in \mathcal{A}_\alpha$, can we claim that $A + B \in \mathcal{A}_\alpha$? Understanding this will pave a way to use decomposition arguments for handling the simultaneously oscillating case.

Chapter 3

Flat geometries

The chapter is devoted to the study of homogenization of Dirichlet boundary value problems for divergence type elliptic equations set in bounded convex polygonal domains. In the spirit of Chapter 2, for problems with fixed operator and oscillating Dirichlet data we prove pointwise, as well as L^p convergence results for the homogenization process. Along with upper L^p bounds, we also study lower bounds for L^p convergence. It should be stressed that due to a radical change in the geometry of the domain, the current setting is very different from that considered in the previous chapter for smooth domains. While the starting point is the same, namely, the representation of solutions via Poisson's kernel, the main difficulties in this case lie in a proper understanding of the regularity issues of the representation kernels. Let us finally note that our methods here work for scalar equations only, since at several places the maximum principle plays an essential role.

The main results presented here are from our paper [6] which is joint with Henrik Shahgholian, and Per Sjölin.

3.1 Introduction

The objective of this chapter is the study of the asymptotic behavior of solutions to Dirichlet problem set in polygonal domains for divergence type elliptic equation, and oscillating Dirichlet data. To fix the ideas we let D be a convex polygonal domain in $\mathbb{R}^d$ $(d \geq 2)$, that is D is a bounded convex domain which is an intersection of some finite number of halfspaces

$$D = \bigcap_{j=1}^{N} \{x \in \mathbb{R}^d : \ x \cdot \nu_j > c_j\}, \tag{3.1}$$

where $c_j \in \mathbb{R}$ and $\nu_j \in \mathbb{S}^{d-1}$. Denote by Γ the boundary of D. Let also $A(y) = (A^{ij}(y))$, $1 \leq i, j \leq d$, be an $\mathbb{R}^{d^2}$-valued function defined on $\mathbb{R}^d$, and g be a complex-valued function defined on $\Gamma \times \mathbb{T}^d$. For a small parameter $\varepsilon > 0$ consider the following problem

$$\begin{cases} \mathcal{L}u_\varepsilon(x) = 0, & \text{in } D, \\ u_\varepsilon(x) = g(x, x/\varepsilon), & \text{on } \Gamma, \end{cases} \tag{3.2}$$

where using the summation convention of repeated indices the operator $\mathcal{L}$ is defined as

$$\mathcal{L}u := -\frac{\partial}{\partial x_i}\left[A^{ij}(x)\frac{\partial u}{\partial x_j}(x)\right] = -\nabla \cdot [A(x)\nabla u(x)]\,. \tag{3.3}$$

For (3.2) we consider the corresponding homogenized problem

$$\begin{cases} \mathcal{L}u_0(x) = 0, & \text{in } D, \\ u_0(x) = \overline{g}(x), & \text{on } \Gamma, \end{cases} \tag{3.4}$$

where $\overline{g}(x) = \int\limits_{\mathbb{T}^d} g(x,y)dy$. In this chapter we will prove convergence results for solutions u_ε of the problem (3.2) toward u_0 as $\varepsilon \to 0$. Recall that in the previous chapter we had a similar problem, but with the assumption that boundary of the domain was somewhat "curved" in all directions. This chapter studies the other extreme, where the boundaries are completely flat, which as we will see below, sets the problem into a qualitatively different framework. We also note that the homogenization of the Dirichlet problem for elliptic systems in divergence form set in convex polygonal domains with periodically oscillating operator, zero Dirichlet data, and fixed source term is studied by Gérard-Varet and Masmoudi in [33]. The main goal of [33] is to analyze higher order two-scale approximations to solutions, which is carried out under certain Diophantine condition on the normal vectors of the bounding hyperplanes of the domain, and restrictive regularity assumptions on initial terms in the two-scale expansion. We refer the interested reader to [33] for the details.

3.1.1 Standing Assumptions

We make the following assumptions:

(i) (Periodicity) The boundary data g is $\mathbb{Z}^d$-periodic with respect to its second variable, i.e.

$$g(x, y+h) = g(x,y), \ \forall x \in \partial D, \ \forall x \in \mathbb{R}^d, \ \forall h \in \mathbb{Z}^d.$$

(ii) (Ellipticity) There exists a constant $c > 0$ such that

$$c\xi_i\xi_i \leq A^{ij}(x)\xi_i\xi_j \leq c^{-1}\xi_i\xi_i, \ \forall x \in \mathbb{R}^d, \ \forall \xi \in \mathbb{R}^d.$$

(iii) (Convexity and normal directions) D is convex and for any bounding hyperplane of D its normal vector is Diophantine in a sense of Definition 3.1.1 below.

(iv) For the convex polygonal domain D choose $\alpha_* > 0$ so that $\pi/(1+\alpha_*)$ be the maximal angle between any two adjacent faces of D.

(v) (Smoothness) The boundary value g and all elements of A are sufficiently smooth.

3.1.2 The Main results

The following are the main results of this chapter.

Theorem 3.1.1. (Pointwise convergence) *Retain the standing assumptions in Section 3.1.1, and if $\alpha_* > 1$ set $\beta = 1$, otherwise, let $0 < \beta < \alpha_*$ be any number. Then for each $\delta > 0$ small there exists a constant C depending on δ, β, D, $\mathfrak{L}$, but independent of $\varepsilon > 0$, such that for all $x \in D$ one has*

$$|u_\varepsilon(x) - u_0(x)| \leq C \left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}} \right)^{\frac{d-1}{d-1+\beta}},$$

where $d(x)$ denotes the distance of x to the boundary of D.

Keeping the notation of Theorem 3.1.1 set

$$\gamma := \frac{(d-1)\min\{1,\alpha_*\}}{d-1+\min\{1,\alpha_*\}}. \tag{3.5}$$

Using the pointwise estimate we will get the following result.

Theorem 3.1.2. (L^p convergence) *Retain the standing assumptions in Section 3.1.1. Then for each $1 \leq p < \infty$, and $\delta > 0$ there exists a constant C depending on p, D, $\mathfrak{L}$, δ but independent of $\varepsilon > 0$ such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\gamma, \frac{1}{p}\}-\delta},$$

where γ is defined by (3.5).

Theorem 3.1.3. (Optimality) *Keeping the same conditions and notation of Theorem 3.1.1, assume in addition that the boundary data g depends only on its oscillating variable. Then for each $1 \leq p < \infty$ there exists a constant C depending on p, D, $\mathfrak{L}$, but independent of ε, such that*

$$||u_\varepsilon - u_0||_{L^p(D)} \geq C\varepsilon^{\frac{1}{p}} ||g - \overline{g}||_{L^\infty(\mathbb{T}^d)}.$$

Remark 3.1.4. *Observe that Theorem 3.1.3 shows that for larger exponents p the L^p convergence rate is close to optimal independently of the structure of the domain. This fact for all $1 \leq p < \infty$ is due to concentration of solutions near the boundary of the domain. However for smaller values of p there are some limitations in the speed of convergence in Theorem 3.1.2 due to the largest angle of the polygon. We do not know if this convergence rate is optimal as well.*

The reader may notice that the formulation of the problem and the main results presented here are visually very similar to those of Chapter 2. However, there are some crucial differences from the smooth setting. First of all it is the geometry of the boundary. Here we do not have smoothness, moreover, at the points where the boundary is smooth, it is flat, and hence has vanishing principal curvatures. This means that the methods of the previous chapter, that were heavily using the fact that the boundary was non-flat, break down immediately. However, here it is still the geometry of the boundary that will determine the asymptotic properties of the solutions to ε-problems. Namely, in this flat setting the averaging is due to the Diophantine condition on the normal directions of bounding hyperplanes, which puts the domain into a special position with respect to the microstructure, which is the lattice $\mathbb{Z}^d$. The second main difference that arises here in comparison to the smooth case, concerns the regularity issues of the fundamental solutions of the operators. The problem is of course due to the lack of regularity of the boundary of the domain.

Let us now formulate the mentioned condition on the normal directions of bounding hyperplanes, and prove that almost all directions satisfy our condition.

Definition 3.1.1. (Diophantine vector) *We call a vector* $\nu = (\nu_1, ..., \nu_d) \in \mathbb{R}^d$ ***Diophantine*** *with parameters* $0 < \tau(\nu) < \infty$ *and* $c > 0$, *and write* $\nu \in DC(\tau, c)$, *if*

$$|m \cdot \nu| > \frac{c}{||m||^{\tau(\nu)}},$$

for all $m = (m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\}$, *where* $m \cdot \nu$ *is the usual scalar product.*

The next result shows that almost any direction is Diophantine for suitable parameters.

Lemma 3.1.5. *For* $\tau > d - 1$ *the set* $\bigcup\limits_{c>0} DC(\tau, c)$ *has full measure in each ball of* $\mathbb{R}^d$.

Proof. The claim is clearly true for $d = 1$, so we will assume that $d \geq 2$. We fix $\tau > d - 1$, $R > 0$ and show that the complement of the set of Diophantine vectors in $B(0, R)$ form a set of measure 0. For $N \in \mathbb{N}$, and $m \in \mathbb{Z}^d \setminus \{0\}$ set

$$E_{m,N} := \left\{ x \in \mathbb{R}^d : |m_1 x_1 + ... + m_d x_d| \leq \frac{1}{N} \frac{1}{||m||^{\tau}} \right\},$$

and let $E_N := \bigcup\limits_{m \in \mathbb{Z}^d \setminus \{0\}} E_{m,N}$. Clearly the vector $x \in \mathbb{R}^d$ is not Diophantine with parameter τ if and only if $x \in \bigcap\limits_{N=1}^{\infty} E_N$. Since $E_N \supset E_{N+1}$, for $N = 1, 2, ...$, it is enough to show that $vol_d(E_N \cap B(0, R)) \to 0$, as $N \to \infty$.

Observe that $E_{m,N}$ is a strip bounded by two parallel hyperplanes with normals m, and thus the size of the strip, which is the distance between these hyperplanes, is equal to $2 \frac{1}{||m||} \frac{1}{N} \frac{1}{||m||^{\tau}}$. From this we get that the volume of $E_{m,N}$ inside the ball satisfies

$$vol_d(E_{m,N} \cap B(0,R)) \leq CR^{d-1}\frac{1}{||m||}\frac{1}{N}\frac{1}{||m||^{\tau}}. \tag{3.6}$$

Since $\tau > d-1$ we have that $\sum_{m\in\mathbb{Z}^d\setminus\{0\}} ||m||^{-(1+\tau)} < C_\tau$. Using this and summing (3.6) over all nonzero $m \in \mathbb{Z}^d$ we obtain that

$$vol_d(E_N \cap B(0,R)) \leq C_\tau \frac{R^{d-1}}{N},$$

and the claim follows. □

Before delving into the proofs let us point out some remarks regarding the condition on the normal directions. The reader may wonder what happens when the normals are not Diophantine. Here we should distinguish between two cases, namely *rational* directions, and *irrational non Diophantine* directions.

With rational directions one can not guarantee a unique limit for solutions u_ε as $\varepsilon \to 0$. Examples where the limit will depend on a subsequence are not hard to construct using for instance Lemma 3.2.1 and the Poisson representation of solutions. We also refer the reader to Lee-Shahgholian [46] for more detailed discussions on the possible scenarios of the limits of subsequences, and also to Moskow-Vogelius [52], and Allaire-Amar [7] for some interesting consequences of the rationality assumption in other situations.

The case of irrational non Diophantine normals will lead to homogenization practically by the same approach as presented here, with the same homogenized limit as considered above, however, without any effective estimates on the speed of convergence. Moreover, one may construct directions, which if set as normals of the bounding hyperplanes, will lead to convergence slower than any given speed in advance. For this type of constructions we refer the reader to Section 4.4 of the next chapter.

Let us finally give some rough intuition around the situation *rational* versus *irrationl*. The setup described below is not quite identical to our case, but is very similar, and can hopefully give more insights into the problem. Assume we have a face of D with normal $\nu \in \mathbb{S}^{d-1}$. Then we are essentially dealing with a distribution of the boundary data g on a hyperplane with normal ν. Note that the scaling by $1/\varepsilon$ will produce parallel hyperplanes all orthogonal to ν, but for now let us keep the position of the hyperplane fixed. Due to periodicity we should consider the hyperplane modulo $\mathbb{Z}^d$, which will produce some subset $E \subset \mathbb{T}^d$. Clearly we will "see" g on E only, and if we require averaging to the same value independently of the subsequence of ε, this means that translating hyperplanes in the direction of ν should produce the same set E up to measure 0. This is indeed the case for irrational ν since then E equals $\mathbb{T}^d$ up to measure 0, or equivalently slices of the hyperplane foliate almost entirely the cell of periodicity of g.

On the other hand for $\nu \in \mathbb{R}\mathbb{Q}^d$ our set E will capture a subtorus of $\mathbb{T}^d$, and thus translations in the direction of ν will each produce different subtori. So each translation will capture a different piece of information on the distribution of g and hence different subsequences will lead to different averaging properties resulting in a non uniqueness of the limit. In other words for rational direction ν the translation flow on the unit torus in the direction of ν is not ergodic, hence the system loses averaging properties. In a similar vein the Diophantine property of the normal should be seen as a way to quantify the ergodicity.

3.2 Proofs

The aim of this section is to present the proofs of the main results formulated above. The proofs are based on a series of preliminary tools which we will collect and classify into subsections.

3.2.1 Exploiting the geometry of the boundary

The following statements are primarily based on the geometry of D.

Lemma 3.2.1. *Let $m \in \mathbb{Z}^d$ be nonzero, and assume that $m_k \neq 0$, for some $1 \leq k \leq d$. For a vector $\nu = (\nu_1, \nu_2, ..., \nu_d) \in DC(\tau, c_0)$ consider $\Pi = \{x \in \mathbb{R}^d : \nu \cdot x = c,\ x_j \in [a_j, b_j], j = 1, 2, ..., d, j \neq k\}$, and for $\lambda > 1$ set*

$$\mathfrak{I}_\lambda := \int\limits_{\Pi} e^{2\pi i \lambda m \cdot y} d\sigma(y).$$

Then for all $\lambda > 1$ one has

$$|\mathfrak{I}_\lambda| \leq C \lambda^{-(d-1)} ||m||^{(d-1)\tau},$$

where the constant C depends on ν and dimension d only.

Proof. Without loss of generality we will assume that $k = d$, that is $m_d \neq 0$. Since ν is Diophantine, all its components are nonzero. In the domain of integration we have $\nu_1 y_1 + ... + \nu_{d-1} y_{d-1} + \nu_d y_d = c$, hence

$$y_d = \frac{c}{\nu_d} - \frac{1}{\nu_d}(\nu_1 y_1 + \nu_2 y_2 + ... + \nu_{d-1} y_{d-1}),$$

and substituting this in the integral we obtain

$$\mathfrak{I}_\lambda = C \prod_{j=1}^{d-1} \int\limits_{a_j}^{b_j} \exp\left[2\pi i \lambda \left(m_j - m_d \frac{\nu_j}{\nu_d}\right) y_j\right] dy_j. \tag{3.7}$$

From the Diophantine condition and the fact that $m_d \neq 0$ we have

$$\left| m_j - m_d \frac{\nu_j}{\nu_d} \right| = \frac{1}{|\nu_d|} |m_j \nu_d - m_d \nu_j| \geq \frac{C_\nu}{|\nu_d|} \frac{1}{(|m_j| + |m_d|)^\tau}, \tag{3.8}$$

for all $j = 1, 2, ..., d-1$. We now compute each of the integrals in (3.7), and applying (3.8) we get the desired estimate, finishing the proof. □

We now introduce some notation that will be used in the sequel. Let D be given as in (3.1). We say that $\Pi \subset \partial D$ is a ($(d-1)$-dimensional) *face* of the polygon D if for some $1 \leq k \leq N$ one has

$$\Pi = \{x \in \mathbb{R}^d : \nu_k \cdot x = c_k\} \cap \bigcap_{j=1, j\neq k}^{N} \{x \in \mathbb{R}^d : \nu_j \cdot x > c_j\}, \tag{3.9}$$

i.e., Π is just one of the flat portions of ∂D. For a given face Π, and a number $\rho > 0$ consider a strip of width ρ near the $(d-2)$-dimensional boundary of Π, and denote it by

$$\Pi_\rho = \{y \in \Pi : \mathrm{dist}(y, \partial\Pi) \leq \rho\}. \tag{3.10}$$

For $1 \leq k \leq d$ set π_k to be the projection operator in the k-th direction, namely

$$\pi_k(x) = (x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_d), \text{ where } x \in \mathbb{R}^d.$$

Lemma 3.2.2. *Let D be a polygon as defined in (3.1), and $\Pi \subset \{x \in \mathbb{R}^d : \nu \cdot x = c\}$ be a face of D. Fix $1 \leq k \leq d$, then for any small number $\rho > 0$ there exist a finite number of measurable sets $\Gamma_j \subset \Pi$, $j = 1, 2, ..., M$ with disjoint $(d-1)$-dimensional interiors, and a measurable set $E \subset \Pi$ such that*

(i) *$E \subset \Pi_{c_0\rho}$, for some constant c_0 depending on Π and dimension d, but independent of ρ,*

(ii) *$\Pi \setminus E = \bigcup_{j=1}^{M} \Gamma_j$, and $\pi_k(\Gamma_j)$ is a $(d-1)$-dimensional cube of side length ρ with vertices in the lattice $\pi_k(\rho\mathbb{Z}^d)$, for $j = 1, 2, ..., M$.*

(iii) *for $j = 1, 2, ..., M$ one has $vol_{d-1}(\Gamma_j) \asymp \rho^{d-1}$, and $\mathrm{diam}(\Gamma_j) \asymp \rho$, where constants in the equivalence depend on Π and dimension d, but are independent of ρ.*

Proof. We first construct the projections of the required sets in the projection of Π, and then lift it up to Π. To have a control on the lifted sets we need some control on the projection π_k, for which we will use the following two inequalities, namely for any $x, y \in \Pi$ one has

$$\frac{|\nu_k|}{||\nu||} ||x - y|| \leq ||\pi_k(x) - \pi_k(y)|| \leq ||x - y||, \tag{3.11}$$

where $\nu = (\nu_1, ..., \nu_d)$ is the unit outward normal vector of Π. The second inequality is obvious, for the first one observe that if $x \in \Pi$ then $x_k = \frac{c}{\nu_k} - \frac{1}{\nu_k} \sum_{i \neq k} \nu_i x_i$, from which we get

$$||x-y||^2=\sum_{i\neq k}(x_i-y_i)^2+\frac{1}{\nu_k^2}\left(\sum_{i\neq k}\nu_i(x_i-y_i)\right)^2=||\pi_k(x)-\pi_k(y)||^2+$$
$$\frac{1}{\nu_k^2}\left(\sum_{i\neq k}\nu_i(x_i-y_i)\right)^2\leq||\pi_k(x)-\pi_k(y)||^2+\frac{1}{\nu_k^2}\sum_{i\neq k}\nu_i^2\sum_{i\neq k}(x_i-y_i)^2=$$
$$||\pi_k(x)-\pi_k(y)||^2+||\pi_k(x)-\pi_k(y)||^2\frac{1}{\nu_k^2}\sum_{i\neq k}\nu_i^2,$$

and the first inequality in (3.11) follows. Note that the first inequality shows that $\pi_k:\Pi\longrightarrow\pi_k(\Pi)$ is a bijection.

Now consider the projection $\pi_k(\Pi)$, and let $\mathcal{C}=\{\mathcal{C}_j\}_{j=1}^M$ be a maximal family of lattice cubes of size ρ and vertices from $\pi_k(\rho\mathbb{Z}^d)$, such that $\mathcal{C}_j\subset\pi_k(\Pi)$. Set $\mathcal{S}=\{x\in\pi_k(\Pi):\operatorname{dist}(x,\partial\pi_k(\Pi))\leq 2\sqrt{d-1}\rho\}$-a strip near the $(d-2)$-dimensional boundary of $\pi_k(\Pi)$. Since the diameter of each $(d-1)$-dimensional cube of size ρ is $\sqrt{d-1}\rho$, it is clear that the set $\pi_k(\Pi)\setminus\mathcal{S}$ is entirely covered by the family of cubes $\mathcal{C}$. Now set $E_0:=\pi_k(\Pi)\setminus\bigcup\limits_{\mathcal{C}_j\in\mathcal{C}}\mathcal{C}_j$, which is the part not covered by the cubes, it follows that $E_0\subset\mathcal{S}$.

We define $E=\pi_k^{-1}(E_0)$, and $\Gamma_j=\pi_k^{-1}(\mathcal{C}_j)$, for $j=1,2,...,M$. Using the fact that π_k is a bijection from Π to $\pi_k(\Pi)$, and the mentioned properties of E_0, and the family of cubes $\mathcal{C}$, the assertions $(i)-(iii)$ follow immediately from double inequality (3.11). The proof is now complete. □

3.2.2 Properties of the Poisson kernel

Here we shall prove some basic estimates for Green's function and Poisson's kernel for the operator $\mathcal{L}$, defined by (3.3), in bounded convex polygonal domains. Our starting point will be to fix the domain D and the operator $\mathcal{L}$, as defined in Section 3.1, along with the corresponding Green's function $G(x,y)$ defined on $D\times D\setminus\{(x,x):x\in D\}$. Also, recall that by Γ we denote the boundary of D. Next, for $x\in D$ and $y\in\Gamma$ we denote by $P(x,y)$ the Poisson kernel corresponding to operator $\mathcal{L}$ in D. It is proved in Lemma 2 of [11], in a more general setting (namely for domains, satisfying uniform exterior sphere condition), that for all $x\in D$, one has

$$|P(x,y)|\leq C\frac{d(x)}{|x-y|^d},\ y \text{ a.e. in } \Gamma, \tag{3.12}$$

where the y null set is independent of x. We remark that the estimate (3.12) from [11] is proved in the case when the matrix A is periodic. However, a careful inspection of the proof from [11] shows that (3.12) continues to hold for non-periodic operators defined on the entire space $\mathbb{R}^d$ with some mild smoothness condition on the coefficients of the operator. To be more detailed, in [11], the authors consider the operator with ε-periodically oscillating coefficients, and aim at obtaining uniform estimates on the Poisson kernel(s) independent of the oscillations. For that a certain variant of boundary gradient estimates is used (Lemma 5 of [11]), and this is where one needs periodicity assumption, in order to gain uniform control over vast oscillations of the operator. The idea is that high oscillations of the coefficients basically amount to a lack of smoothness, but on the other hand enable one to compare the oscillating operator with the homogenized one, which has constant coefficients. This is why one needs periodicity in [11] to be able to control the entire family of operators. Since in our case we have a bounded domain with fixed, however not necessarily periodic operator, the proof of Lemma 2 of [11] goes through without basically any changes.

The next Lemma will be used to control the contribution of a small piece of the boundary in the integral representation of solutions to (3.2).

Lemma 3.2.3. *Let $\rho > 0$ be a small number, $x \in D$ be fixed with $d(x) \geq 2\rho$, and let Π be one of the faces of D. Then, there exists a constant C, independent of x and ρ such that*

$$\int\limits_{\Pi_\rho} |P(x,y)| d\sigma(y) \leq C \frac{\rho}{d(x)}.$$

Proof. If $d = 2$ then Π is a segment, and Π_ρ is a union of two segments of size ρ. It follows from (3.12) that

$$\int\limits_{\Pi_\rho} |P(x,y)| d\sigma(y) \leq C \frac{1}{d(x)} \int\limits_{\Pi_\rho} d\sigma(y) \leq C \frac{\rho}{d(x)}.$$

We now consider the case $d \geq 3$. After a rotation of the coordinates we may assume that Π is contained in the plane $\{x_d = 0\}$. Note that the boundary of Π is a subset of $(d-2)$-dimensional boundary of D, that is its edges. We let $L_1, ..., L_M$ to be the flat portions of $\partial \Pi$, i.e. the edges of the polygon D that form the $(d-2)$-dimensional boundary of Π. Next, we consider strips of size ρ near each L_i, namely for $i = 1, 2, ..., M$ we set

$$\mathcal{S}_i = \{ y = (y_1, ..., y_{d-1}, 0) \in \Pi : \operatorname{dist}(y, L_i) \leq \rho \}.$$

It is clear that $\Pi_\rho = \bigcap\limits_{i=1}^{M} \mathcal{S}_i$, from which we have

$$\int\limits_{\Pi_\rho} |P(x,y)| d\sigma(y) \leq \sum_{i=1}^{M} \int\limits_{\mathcal{S}_i} |P(x,y)| d\sigma(y).$$

We now set $\mathcal{S}$ to be one of the $\mathcal{S}_i$-s, and L to be the corresponding edge. It is enough to prove the estimate of the Lemma for $\mathcal{S}$. After a rotation we may assume that L lies in $(d-2)$-dimensional subspace $\{y_{d-1} = y_d = 0\}$, from which we get that

$$\mathcal{S} \subset \{ y = (y_1, y_2, ..., y_{d-1}, 0) \in \mathbb{R}^{d-1} : \ |y_{d-1}| \leq \rho \}. \tag{3.13}$$

For $x \in D$ we denote by x_Π the orthogonal projection of x onto the hyperplane containing Π. It is clear that $d(x) \le |x - x_\Pi| = |x_d|$, where x_d is the last coordinate of x. Using this, the estimate (3.12) for the Poisson kernel and (3.13) we get

$$\int\limits_{\mathcal{S}} |P(x,y)| d\sigma(y) \le C|x_d| \int\limits_{\mathcal{S}} \frac{d\sigma(y)}{[(x_1-y_1)^2 + ... + (x_{d-1}-y_{d-1})^2 + x_d^2]^{d/2}} \le$$

$$\frac{C}{|x_d|^{d-1}} \int\limits_{\substack{\overline{y}=(y_1,...,y_{d-1})\in\mathbb{R}^{d-1} \\ |y_{d-1}|\le\rho}} \frac{d\overline{y}}{\left[1 + \left(\frac{x_1-y_1}{x_d}\right)^2 + ... + \left(\frac{x_{d-1}-y_{d-1}}{x_d}\right)^2\right]^{d/2}} \le$$

$$\left(\text{setting } z_i := \frac{x_i - y_i}{x_d},\ 1 \le i \le d-1\right) \quad C \int\limits_{\substack{\overline{z}=(z_1,...,z_{d-1})\in\mathbb{R}^{d-1} \\ \left|z_{d-1}-\frac{x_{d-1}}{x_d}\right|\le\frac{\rho}{x_d}}} \frac{d\overline{z}}{(1+|\overline{z}|^2)^{d/2}} \le$$

$$C\frac{\rho}{|x_d|} \le C\frac{\rho}{d(x)}.$$

The proof of the lemma is complete. $\square$

We now turn on to the proof of a certain Hölder regularity for the Poisson kernel, for which we will first prove some general estimates for nonnegative solutions to the operator $\mathcal{L}$ in polygonal domains.

By Γ^* we denote the "singular" boundary of D, i.e. the set of all points of Γ that belong to more than one face of D. Let us fix a boundary point $z \in \Gamma^*$, and let Π_1 and Π_2 be any two supporting hyperplanes of D at z. Choose $\alpha > 0$ so that the angle between these two planes, i.e. $\arccos(\nu_1 \cdot \nu_2)$ equals $\pi/(1+\alpha)$, where ν_i denotes the outward unit normal to Π_i. Then obviously a rotated and translated version of the function $\mathbf{Im}(x_1 + \mathbf{i}x_2)^{1+\alpha}$ will be harmonic in the convex cylindrical cone generated by the two bounding halfspaces of D whose boundaries are Π_1 and Π_2 correspondingly. Let us note that here a *cone with vertex at 0 (say)* is a domain $\mathcal{C} \subset \mathbb{R}^d$ such that for any $x \in \mathcal{C}$ the ray ("open" half-line) starting from the origin and passing through x lies in $\mathcal{C}$, and $\mathbb{R}^d \setminus \overline{\mathcal{C}}$ is required to be nonempty. It is well known that positive harmonic functions in cone-like domains (with zero boundary values) behave as r^λ where λ is the first eigenvalue to the Laplace-Beltrami operator of surface which is the intersection of the cone with the unit sphere (see e.g. [9]). This fact can be used along with freezing coefficient techniques to show similar behavior for the solutions to variable coefficients elliptic equations. We now formalize this discussion in the next lemma[1], variations of which can be found in [45], and [9].

For the given convex polygonal domain D let $x_0 \in \Gamma^*$ be fixed. Choose $\alpha > 0$ so that $\pi/(1+\alpha)$ be the maximal angle between any two supporting planes of D at the point x_0.

---

[1] For the interested reader, it is instructive to consider the function $f(z) = \mathbf{Im} z^{1+\alpha}$ in the complex plane, where $\alpha > 0$.

Lemma 3.2.4. *With the above notation, consider any nonnegative solution h to $\mathfrak{L}h = 0$ in $D \cap B_1(x_0)$ with zero boundary data on $B_1(x_0) \cap \partial D$, and nonnegative values on $D \cap \partial B_1(x_0)$. Then for any $\beta < \alpha$ there exists a constant c_0 depending on β, and independent of solution h and point x_0 such that*

$$0 \leq h(x) \leq c_0 M |x - x_0|^{1+\beta}, \qquad \forall\, x \in D \cap B_1(x_0),$$

where $M = \sup\limits_{B_1(x_0) \cap D} h$.

Proof. The proof is based on scaling and Phragmén-Lindelöf type argument. After a translation of the coordinate system we may assume that $x_0 = 0$. Next, if A is the matrix of the operator $\mathfrak{L}$, then the matrix $\frac{1}{2}(A(0) + A(0)^T)$ is positive definite and is symmetric, hence by a composition of orthogonal transformation and scaling we may bring it to a scalar multiple of an identity matrix. Thus, doing a linear change of variables in the operator, we will still get a divergence form operator, but with matrix of coefficients having its symmetric component as a scalar multiple of Laplacian at the origin. Since the orthogonal transformation and scaling will transform D to a new polygonal domain with the same angles between its faces as the original one, without loss of generality we will assume that $\frac{1}{2}(A(0) + A(0)^T)$ is the identity matrix.

Let $\Pi_i = \{x \in \mathbb{R}^d : x \cdot \nu_i = 0\}$, $i = 1, 2$ be two supporting planes to D at the origin, so that the angle between Π_1 and Π_2 is $\pi/(1+\alpha)$. Set $D_\alpha = \{x \in \mathbb{R}^d : x \cdot \nu_i > 0,\ i = 1, 2\}$, then clearly $D \subset D_\alpha$. Now, for any $\gamma \in (\beta, \alpha)$ we denote by D_γ a convex region containing D_α, bounded by two hyperplanes passing through the origin and forming an angle equal to $\pi/(1+\gamma)$. Let us finally set H_γ to be the positive barrier function supported in D_γ, which is a rotation of $\mathbf{Im}(x_1 + \mathbf{i}x_2)^{1+\gamma}$. Clearly for some constant $C > 0$ we have

$$\sup_{B_R \cap D_\gamma} H_\gamma(x) = C R^{1+\gamma}. \tag{3.14}$$

To simplify the notation, we will set h to be 0 outside D, where h is any solution considered in the formulation of the lemma. With this preliminary setup the claim of the lemma is equivalent to existence of constant c_0, independent of h, such that

$$\sup_{B_r} h \leq c_0 M r^{1+\beta},\ \forall r \in (0, 1] \qquad \text{where } M = \sup_{B_1} h. \tag{3.15}$$

Assume (3.15) fails, then there are sequences $c_j \nearrow \infty$, $r_j \searrow 0$, and solutions h_j to our equation such that

$$\sup_{B_{r_j}} h_j = c_j M_j r_j^{1+\beta}, \tag{3.16}$$

and

$$\sup_{B_r} h_j < c_j M_j r^{1+\beta},\ \forall r \in (r_j, 1], \tag{3.17}$$

where $M_j = \sup_{B_1} h_j$, and $j = 1, 2, \ldots$. To see this, we proceed by induction. For $j = 1$ take $c_1 = 2$, then since (3.15) is false, there exists a solution h_1 such that $\sup_{B_r} h_1 \geq c_1 M_1 r^{1+\beta}$ for some $0 < r < 1$. We now choose r_1 to be the largest of these numbers r and get that (3.16)-(3.17) is satisfied when $j = 1$. To pass from j to $j+1$ we choose $c_{j+1} > c_j + 1$ satisfying $c_{j+1} r_j^{1+\beta} > 2$, then the argument goes as in the case $j = 1$. It easily follows from our construction that $r_j \to 0$.

We next consider the scaled version of our problem, namely for each $j = 1, 2, \ldots$ we set

$$\widetilde{h}_j(x) := \frac{h_j(r_j x)}{c_j M_j r_j^{1+\beta}}, \qquad |x| \leq 1/r_j.$$

In view of (3.16)-(3.17) we get

$$1 \leq \sup_{B_R} \tilde{h}_j \leq R^{1+\beta} \qquad \forall 1 \leq R \leq \frac{1}{r_j}. \tag{3.18}$$

Furthermore, $\tilde{h}_j$ satisfies the scaled equation $\mathfrak{L}_j \tilde{h}_j = 0$ in the scaled domain $\frac{1}{r_j}(B_1 \cap D)$, and with zero boundary data on $\frac{1}{r_j}(\partial D \cap B_1)$. Here the matrix of coefficients of $\mathfrak{L}_j$ is given by $A_j(x) := A(r_j x)$, where A is the matrix of the original operator. By Arzelà-Ascoli theorem we may extract a locally uniformly converging subsequence from $\widetilde{h}_j$, labeled again as $\widetilde{h}_j$, such that[2]

$$\widetilde{h}_j \to \widetilde{h}_0 \qquad \text{and} \qquad \mathfrak{L}_j \to \mathfrak{L}_0,$$

where $\mathfrak{L}_0$ is divergence type operator with constant coefficient matrix equal to $A(0)$. Also, by convergence of operators above, we mean locally uniform convergence of the corresponding coefficients. Clearly we have $\mathfrak{L}_0 \widetilde{h}_0 = 0$ in the cone $D_0 := \bigcup_{j=1}^{\infty} \frac{1}{r_j}(D \cap B_1)$. Since $\frac{1}{2}(A(0) + A(0)^T)$ is the identity matrix, we get that $\tilde{h}_0$ is harmonic in D_0. Moreover by (3.18) we also have

$$1 \leq \sup_{B_R \cap D_0} \tilde{h}_0 \leq R^{1+\beta}, \quad \forall R \geq 1. \tag{3.19}$$

The blow-up cone D_0 has vertex at the origin, and its boundary consists of k flat pieces, for some positive integer k, where each piece lies on some bounding hyperplane of the original domain D. Note that D_0 may be cylindrical (i.e. translation invariant) in some directions. In this case we want to reduce the dimension by showing that the function $\tilde{h}_0$ is independent of the cylindrical directions. It should be remarked that such a reduction is needed only because of our barrier argument to follow; the argument does not work with cylindrical domains, and needs the cone to have only one vertex. One may see this as asking for the intersection of the cone and the unit sphere to be compactly inside the upper hemisphere after rotation if necessary.

To this end we claim that positive harmonic functions in cones (with vertex at the origin) with zero Dirichlet data on the boundary of the cone must be homogeneous of some fixed positive degree if the cone is NTA-domain (non-tangentially accessible). This is proved in Theorem 1 of [45], and since Lipschitz domains are NTA, we get the claim for D_0 (for the definition of NTA-domain

---

[2] By elliptic regularity we may bound the gradient of $\widetilde{h}_j$ by its supremum norm, which, by (3.18) is locally bounded, uniformly for the whole sequence. This is the reason one can use Arzelà-Ascoli here.

and the fact that Lipschitz domains are NTA see [42], Section 3). Next, we show that solution $\tilde{h}_0$ is independent of the cylindrical directions. For simplicity, and without loss of generality assume that D_0 is cylindrical with respect to the last coordinate. Set $e_d = (0, ..., 0, 1) \in \mathbb{R}^d$, then for any $a > 0$ we have that $\tilde{h}_1(x) := \tilde{h}_0(x + ae_d)$ is also a positive harmonic function in D_0 with zero Dirichlet data on the boundary, and hence is homogeneous of the same degree as $\tilde{h}_0$, say $p > 0$. Now for any $\lambda > 0$ we get

$$\lambda^p \tilde{h}_1(x) = \tilde{h}_1(\lambda x) = \tilde{h}_0(\lambda x + ae_d) = \lambda^p \tilde{h}_0\left(x + \frac{a}{\lambda} e_d\right),$$

hence $\tilde{h}_0(x + ae_d) = \tilde{h}_0(x + \frac{a}{\lambda} e_d) \to \tilde{h}_0(x)$, as $\lambda \to \infty$. Thus $\tilde{h}_0$ is independent of the cylindrical directions. In particular, there will be no loss of generality, if we assume that our cone D_0 has the origin as the only vertex. On the other hand, this means that up to rotation, we may assume that the intersection of the unit sphere with D_0 lies compactly inside the convex cone D_γ introduced above. Note that this is only possible if there is no line passing through the origin (which is a vertex of D_0) and lies entirely on the boundary of D_0. This was the reason we ruled out translation invariant directions (i.e. lines on the boundaries). We thus have

$$\partial B_1(0) \cap \overline{D}_0 \subset \partial B_1(0) \cap \overline{D}_\alpha \subset \partial B_1(0) \cap D_\gamma. \tag{3.20}$$

Let us now take the two-dimensional barrier H_γ in the convex (cylindrical) cone D_γ introduced in (3.14). Now choose $\delta > 0$ such that $\beta + \delta < \gamma$. Define a new function $H_\gamma^\delta := R^{-\delta} H_\gamma$, and observe that by (3.20) there is a $c_0 > 0$ such that $H_\gamma(x) \geq c_0$ over the set $\partial B_1(0) \cap \overline{D}_0$ (e.g., by Harnack's inequality). From this we infer that for R sufficiently large

$$\inf_{D_0 \cap \partial B_R} H_\gamma^\delta(x) = R^{1+\gamma-\delta} \inf_{D_0 \cap \partial B_1} H_\gamma^\delta(x) \geq c_0 R^{1+\gamma-\delta} > R^{1+\beta} \geq \sup_{D_0 \cap \partial B_R} \tilde{h}_0.$$

Hence by the maximum principle (both functions are harmonic) we conclude that $H_\gamma^\delta \geq \tilde{h}_0$ in the truncated cone $D_0 \cap B_R$. In particular as R becomes large we arrive at $1 \leq \sup_{B_1} \tilde{h}_0 \leq \sup_{B_1} H_\gamma^\varepsilon \leq R^{-\delta} \sup_{B_1} H_\gamma < 1/2$ (say). This is a contradiction and we conclude that our claim (3.15) must be true[3]. The proof of the lemma is complete.

□

Using Lemma 3.2.4 we establish gradient bounds in the next result.

Lemma 3.2.5. *Let D, and h be as in Lemma 3.2.4. Then, for any $\beta < \alpha$ there*

---

[3] To emphasize once more the usage of the fact, that our cone has a unique vertex, we point out an alternative, rather a brute force way of constructing barriers that control the growth of $\tilde{h}_0$. Namely, if one does not rotate D_0 to a position that (3.20) is satisfied, we could just consider all possible pairs of bounding hyperplanes of D_0 at the origin, and as a barrier function take the sum of all two-dimensional barriers corresponding to the pairs of bounding hyperplanes. Then one could finalize in a similar fashion. Note that in this case as well, we need to cover all possible directions of the growth of $\tilde{h}_0$ which is only possible if we do not have translation invariant directions in our cone.

exists a constant c_0 *depending on* β*, so that*

$$|\nabla h(x)| \leq c_0 M d_*(x)^\beta, \qquad \forall x \in D \cap B_{1/2}(x_0),$$

where $M = \sup\limits_{D\cap B_1(x_0)} h(x)$*, and* $d_*(x)$ *is the distance from* x *to* Γ^**-the singular boundary of* D*.*

Proof. We translate the origin of the coordinate system to x_0, and we will normalize solutions by their supremum norm, hence assuming that all solutions are bounded by 1. The proof of the lemma proceeds by contradiction. Assume the claim is false, then there exists a sequence of solutions h_j, and a sequence of points $x_j \in D \cap B_{1/2}$ such that we have

$$|\nabla h_j(x_j)| \geq j d_*(x_j)^\beta. \tag{3.21}$$

For each j fix $y_j \in \Gamma^*$ so that $|x_j - y_j| = d_*(x_j) := d_j$. We have that $B(y_j, 1/4) \subset D \cap B_1$ thus applying Lemma 3.2.4 for h_j and y_j we obtain

$$|h_j(x)| \leq c_0 |x - y_j|^{1+\beta}, \qquad \forall x \in B(y_j, 1/4). \tag{3.22}$$

For the scaled solutions

$$v_j(x) := \frac{h_j(d_j x + x_j)}{d_j |\nabla h_j(x_j)|} \quad \text{in} \quad D_j := \frac{1}{d_j}(D - x_j),$$

we have

$$\mathcal{L}_j v_j = 0 \text{ in } D_j \qquad \text{and} \qquad |\nabla v_j(0)| = 1,$$

where $\mathcal{L}_j$ is the scaled operator. According to (3.22) and (3.21) we get

$$0 \leq v_j(x) \leq C \frac{|d_j x + x_j - y_j|^{1+\beta}}{j d_j^{1+\beta}} \leq \frac{C}{j} \qquad \text{if } |x| \leq 2,$$

where we have used the fact that $d_j \to 0$. Now observe that $B_{1/2} \cap \partial D_j$ has no points on singular boundaries, and hence is either empty or consists of a piece of hyperplane. From here by elliptic regularity we have

$$1 = |\nabla v_j(0)| \leq C \sup_{B_2 \cap D_j} v_j(x) \leq \frac{C}{j} \to 0 \qquad \text{as } j \to \infty.$$

The latter leads to a contradiction, and completes the proof of the lemma. $\square$

We are now ready to formulate and prove our main regularity estimate for the Poisson kernel.

Lemma 3.2.6. *Retain the hypothesis of the Standing Assumptions in Section 3.1.1, and if* $\alpha_* > 1$ *set* $\beta = 1$*, otherwise, let* $0 < \beta < \alpha_*$ *be any number. Fix any* $\delta \geq 0$*,* $x \in D$*, and* $y_1, y_2 \in \Pi \setminus \Gamma^*$*, where* Π *is a face of* D*, and* $|y_1 - y_2| \leq c_0 d(x)$*, where* c_0 *is a small, universal constant. Then, there exists a constant* C *depending* *on β, and δ, and independent of x, y_1, y_2 such that*

$$|P(x,y_1) - P(x,y_2)| \leq C \frac{|y_1 - y_2|^{\beta}}{|x - y_1|^{d-1+\beta+\delta}},$$

where δ can be taken arbitrarily small positive nonzero number in dimension two, and zero in dimensions greater than two.

Proof. Let $G(x,y)$ be the Green's kernel for the operator $\mathfrak{L}$ defined by (3.3) in domain D. Then the corresponding Poisson kernel has the form $P(x,y) = n(y)^T A(y) \nabla_y G(x,y)$ where $x \in D$ and $y \in \Gamma$ where $n(y)$ is the outward unit normal of Γ at y. We will study the regularity properties of the Green's function, which together with smoothness of A will imply the result. We will need the following estimates on the Green's function of $\mathfrak{L}$,

$$|G(x,y)| \leq C \begin{cases} \log \frac{C}{|x-y|}, & d = 2, \\ |x-y|^{2-d} & d \geq 3\ , \end{cases} \tag{3.23}$$

for all $(x,y) \in D \times D$, with $x \neq y$, where for $d = 2$ the estimate is proved in [29], and for $d \geq 3$ in [38]. Now fix any $x_0 \in D$, such that $|y_1 - y_2| \leq c_0 d(x_0)$, and set $R = |x_0 - y_1|$, $D_R = \frac{1}{R}(D - x_0)$, and let $G_R(\cdot, \cdot)$ be the Green's function for the scaled domain and the scaled operator. It is clear that $G_R(w,z) = R^{d-2} G(Rw + x_0, Rz + x_0)$, for $w, z \in D_R$, where in the exponent of R, d comes from the volume scaling, and 2 is due to the scaling of the operator. Consider $h_R(z) := G_R(0,z)$ in the set $\widetilde{D}_R := D_R \cap (B_4(0) \setminus B_{1/4}(0))$. Then h_R is a solution to our PDE in this set and zero on $\partial \widetilde{D}_R \setminus (\partial B_4(0) \cup \partial B_{1/4}(0))$. We claim that

$$h_R \in C^{1,\beta}(D_R \cap (B_3(0) \setminus B_{1/2}(0))) \tag{3.24}$$

with uniform norm bounded by a constant multiple of the supremum norm of h_R on the set $\widetilde{D}_R$. In the sequel, when proving (3.24) we will keep in mind the mentioned relation of constants with the supremum norm of h_R.

We first show that (3.24) implies the desired estimate of the lemma. Let $y_1, y_2 \in \Pi \setminus \Gamma^*$, and $x_0 \in D$ be as in the formulation, then we have $|y_1 - y_2| \leq c_0 d(x_0)$, and $n(y_1) = n(y_2)$. On the other hand if we let $z_i := R^{-1}(y_i - x_0)$, $i = 1, 2$ then by the choice of y_i and x_0, we get $z_i \in D_R \cap (B_3(0) \setminus B_{1/2}(0))$. Also observe that from the definition of h_R we have

$$\nabla_z h_R(z) = R^{d-1} \nabla_y G(x_0, y), \text{ where } y = Rz + x_0. \tag{3.25}$$

Using (3.25), and smoothness of A, from the Poisson representation we obtain

$$\begin{aligned} |P(x_0,y_1) - P(x_0,y_2)| \leq |n(y_1)^T [A(y_1) - A(y_2)] \nabla_y G(x_0,y_1)| + \\ |n(y_1)^T A(y_2) [\nabla_y G(x_0,y_1) - \nabla_y G(x_0,y_2)]| \leq \\ ||\nabla A||_{L^{\infty}(\overline{D})} |y_1 - y_2| |\nabla_y G(x_0,y_1)| + ||A||_{L^{\infty}(\overline{D})} |\nabla_y G(x_0,y_1) - \nabla_y G(x_0,y_1)| \lesssim_A \\ R^{2-d} |z_1 - z_2| |\nabla_z h_R(z_1)| + R^{1-d} |\nabla_z h_R(z_1) - \nabla_z h_R(z_2)|. \end{aligned}$$

Now we invoke (3.24), which combined with (3.23), and the choice of z_i leads to the estimate of the Lemma. We just remark that in dimension two we may trade-off the logarithmic singularity in the supremum norm of h_R by slightly increasing the power in the denominator of the estimate in the Lemma by means of the small parameter δ introduced in the formulation, while in dimensions greater than two, the supremum norm of h_R is uniformly bounded away from the origin.

In what follows we prove (3.24). Observe that due to Schauder estimates we locally have

$$h_R \in C^{1,\beta}(D_R \cap (B_3(0) \setminus B_{1/2}(0))). \tag{3.26}$$

It remains to show that when approaching the boundary of $\widetilde{D}_R$ the norm does not blow-up. From boundary regularity for elliptic equations, we also know that solutions are smooth at regular boundaries (see e.g. Theorem 6.19 in [36]). In particular in our case we have (at least) C^2 regularity for h_R on the flat boundaries, $\partial D_R \setminus \partial^* D_R$, where $\partial^* D_R$ denotes the set of all points of the boundary of D_R that belong to more than one face of D_R, i.e. the corner points. Again the norm may blow up when approaching the corners $\partial^* D_R$. Since we can approach the corner points both tangentially and non-tangentially, we may consider two cases for $z_j \to \partial^* D_R$:

(i) non-tangential to the boundary, (ii) tangential to the boundary.

Case (i). Consider two points z_i $(i = 1, 2)$, with the property that

$$\mathrm{dist}(z_i, \partial D_R) \geq c_0 \mathrm{dist}(z_i, \partial^* D_R),$$

for some $c_0 > 0$, i.e. they approach $\partial^* D_R$ non-tangentially. Clearly we may assume that $\mathrm{dist}(z_i, \partial D_R) \leq c_0/4$, $i = 1, 2$ since away from the boundaries we have at least C^2 regularity. Now let $z_i^* \in \partial^* D_R$ be so that $|z_i - z_i^*| = \mathrm{dist}(z_i, \partial^* D_R)$ for $i = 1, 2$. Observe that due to the properties of z_i we have $z_i \in B(z_1^*, 1/4)$ for $i = 1, 2$.

Assume first that $|z_1 - z_2| \geq (1/4)\mathrm{dist}(z_1, \partial^* D_R)$. Then, we apply 3.2.5 for $B(z_i^*, 1)$, $i = 1, 2$ and obtain

$$|\nabla h_R(z_1) - \nabla h_R(z_2)| \leq |\nabla h_R(z_1)| + |\nabla h_R(z_2)| \leq \\ C \max_{i=1,2} \mathrm{dist}^\beta(z_i, \partial^* D_R) \leq C|z_1 - z_2|^\beta \ .$$

Next, if $|z_1 - z_2| \leq (1/3)\mathrm{dist}(z_1, \partial^* D_R)$ then we scale h_R at z_1 with the distance to the corner $\widetilde{h}_R(y) = h_R(z_1 + d_1 y)/d_1^{1+\beta}$, where $d_1 = |z_1 - z_1^*|$. Again, due to the properties of z_i we have $z_i \in B(z_1^*, 1/2)$, $i = 1, 2$ and thus Lemma 3.2.4 applied in $B(z_1^*, 1)$ leads to uniform bound for $\widetilde{h}_R$ in $B_1(0)$. Since in $B_{1/2}(0)$ we have no corner points but only smooth boundary, the elliptic regularity implies that $\widetilde{h}_R$ is uniformly C^2, say, independently of z_i. But then the $C^{1,\beta}$ norm of $\widetilde{h}_R$ is uniformly bounded, hence with $y_2 := (z_2 - z_1)/d_1 \in B_{1/3}(0)$ we get

$$|\nabla h_R(z_1) - \nabla h_R(z_2)| = d_1^\beta |\nabla \widetilde{h}_R(0) - \nabla \widetilde{h}_R(y_2)| \leq C d_1^\beta |y_2|^\beta = C|z_2 - z_1|^\beta.$$

Case (ii). It is left to cover the tangential regions near the flat boundaries.

Here we start by taking any point z_0 on the flat boundary and consider the half ball $B_s^+(z_0)$ which is inside the domain $\widetilde{D}_R$. For simplicity assume that the flat portion of the boundary, with z_0 on it, is part of the hyperplane $\{x_d = 0\}$, such that $B_s^+ = \{x_d \geq 0\} \cap B_s(z_0)$. Now we let s denote the largest real number such that $B_{2s}^+(z_0) \subset \overline{\widetilde{D}_R}$. Obviously $\partial^* D_R \cap \overline{B_s^+(z_0)} = \emptyset$, and

$$c_0 s \geq \operatorname{dist}(z_0, \partial^* D_R) \tag{3.27}$$

for some $c_0 > 0$, due to Lipschitz character of the domain. We may also assume that s is sufficiently small, i.e. z_0 is in a vicinity of corner points, since otherwise we have regularity for smooth boundaries. Invoking Lemma 3.2.4 and using (3.27) we have that for $z \in B_1^+(0)$ the function $v_s(z) := h_R(sz + z_0)/s^{1+\beta}$ satisfies the bound

$$0 \leq v_s(z) \leq C \frac{(\operatorname{dist}(sz + z_0, \partial^* D_R))^{1+\beta}}{s^{1+\beta}} \leq \\ C \frac{(\operatorname{dist}(z_0, \partial^* D_R) + s)^{1+\beta}}{s^{1+\beta}} \leq C(c_0 + 1)^{1+\beta}$$

which is uniformly bounded in $B_1^+(0)$. Hence classical Schauder estimates can be applied to conclude uniform $C^{1,\beta}$-estimates for v_s in $B_{1/2}^+(0)$, i.e.

$$||h_R||_{C^{1,\beta}(B_{s/2}^+(z_0))} = ||v_s||_{C^{1,\beta}(B_{1/2}^+(z_0))} \leq C_0.$$

This in particular means that the $C^{1,\beta}$ norm of h_R is uniformly bounded up to any flat boundary point, which is the desired result.

The proof of the lemma is now complete. □

3.2.3 Proving the Theorems

Pointwise estimates

Proof of Theorem 3.1.1. By the Poisson representation we have

$$u_\varepsilon(x) - u_0(x) = \int_\Gamma P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y) = \\ \sum_{j=1}^N \int_{\Pi_j} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y),$$

hence it is enough to study the integrals over one particular face. Let Π be one of the faces of Π with Diophantine normal vector $\nu \in DC(\tau, c)$. Since g is smooth and $\mathbb{Z}^d$-periodic we have

$$g(x,y) = \sum_{m \in \mathbb{Z}^d} c_m(x) e^{2\pi i m \cdot y},$$

where $c_m(x)$ is the m-th Fourier coefficient of $g(x,\cdot)$, and the order of smoothness of g assures that the series converges absolutely. Define $\mathcal{J}_1 = \{m \in \mathbb{Z}^d : m_1 \neq 0\}$ and for $k = 2, 3, ..., d$ set $\mathcal{J}_k = \{m \in \mathbb{Z}^d : m_k \neq 0\} \setminus (\mathcal{J}_1 \cup ... \cup \mathcal{J}_{k-1})$. We get

$$\int\limits_{\Pi} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]d\sigma(y) = \sum_{k=1}^{d} \sum_{m \in \mathcal{J}_k} \int\limits_{\Pi} P(x,y) c_m(y) e^{\frac{2\pi i}{\varepsilon} m\cdot y} d\sigma(y).$$

We fix $x \in D$, $1 \leq k \leq d$, and a small parameter $0 < \rho \leq c d(x)$, where the constant c will be chosen from (3.28) below. Applying Lemma 3.2.2 we get a set $E \subset \Pi$, and a family $\{\Gamma_j^\rho\}_{j=1}^M$ with properties $(i) - (iii)$ of the Lemma, and let c_0 be the constant from part (i). Since $E \subset \Pi_{c_0 \rho}$ from Lemma 3.2.3 we get

$$\int\limits_{E} |P(x,y)| d\sigma(y) \leq C \frac{\rho}{d(x)}, \quad \text{for } x \in D \text{ with } d(x) \geq 2c_0\rho. \tag{3.28}$$

Now for $j = 1, 2, ..., M$ fix some $y_j \in \Gamma_j^\rho$, then outside E we have

$$\begin{aligned} &\int\limits_{\Pi \setminus E} P(x,y) c_m(y) e^{\frac{2\pi i}{\varepsilon} m\cdot y} d\sigma(y) = \\ &\sum_{j=1}^{M} \int\limits_{\Gamma_j^\rho} [P(x,y)c_m(y) - P(x,y_j)c_m(y_j)] e^{\frac{2\pi i}{\varepsilon} m\cdot y} d\sigma(y) + \\ &\sum_{j=1}^{M} P(x,y_j) c_m(y_j) \int\limits_{\Gamma_j^\rho} e^{\frac{2\pi i}{\varepsilon} m\cdot y} d\sigma(y) := A_m^{(1)}(x) + A_m^{(2)}(x). \end{aligned}$$

Estimate of $A_m^{(1)}$. Since $\operatorname{diam}(\Gamma_j^\rho) \leq C d(x)$, for any $y \in \Gamma_j^\rho$ from Lemma 3.2.6, estimate (3.12), and smoothness of g we obtain

$$\begin{aligned} &|P(x,y)c_m(y) - P(x,y_j)c_m(y_j)| \leq \\ &\qquad ||c_m||_{L^\infty(\Pi)} |P(x,y) - P(x,y_j)| + |P(x,y_j)||c_m(y) - c_m(y_j)| \lesssim \\ &\qquad ||c_m||_{L^\infty(\Pi)} \frac{|y-y_j|^\beta}{|x-y_j|^{d-1+\beta+\delta/2}} + ||\nabla c_m||_{L^\infty(\Pi)} \frac{d(x)}{|x-y_j|^d} |y - y_j| \lesssim \\ &\qquad\qquad \mathcal{C}_m \frac{|y-y_j|^\beta}{|x-y_j|^{d-1+\beta+\delta/2}}, \end{aligned}$$

where we have set $\mathcal{C}_m := ||c_m||_{L^\infty(\Pi)} + ||\nabla c_m||_{L^\infty(\Pi)}$. In view of $|y - y_j| \leq$

$\operatorname{diam}(\Gamma_j^\rho) \leq C\rho$, the last estimate implies

$$|A_m^{(1)}(x)| \lesssim \mathcal{C}_m \sum_j \int\limits_{\Gamma_j^\rho} \frac{|y-y_j|^\beta}{|x-y_j|^{d-1+\beta+\delta/2}} d\sigma(y) \lesssim \mathcal{C}_m \frac{\rho^\beta}{d(x)^{\beta+\delta}} \sum_j \frac{vol_{d-1}(\Gamma_j^\rho)}{|x-y_j|^{d-1-\delta/2}}, \quad (3.29)$$

where $\delta > 0$ is any small number. The sum in (3.29) is bounded, up to multiplication by some constant depending on $\delta > 0$, by the integral $\int\limits_\Gamma \frac{d\sigma(y)}{|x-y|^{d-1-\delta/2}}$, and hence is uniformly bounded with respect to x. We conclude that

$$|A_m^{(1)}(x)| \lesssim_\delta \mathcal{C}_m \frac{\rho^\beta}{d(x)^{\beta+\delta}}. \quad (3.30)$$

Estimate of $A_m^{(2)}$. Observe that $m_k \neq 0$, and $\pi_k(\Gamma_j^\rho)$ is a $(d-1)$-dimensional rectangle with sides parallel to the coordinate axes, hence we may apply Lemma 3.2.1, and using the fact that $vol_{d-1}(\Gamma_j^\rho) \asymp \rho^{d-1}$ we get

$$\left| \int\limits_{\Gamma_j^\rho} e^{\frac{2\pi i}{\varepsilon} m \cdot y} d\sigma(y) \right| \leq C \varepsilon^{d-1} ||m||^{(d-1)\tau(\nu)} \leq C \left(\frac{\varepsilon}{\rho}\right)^{d-1} vol_{d-1}(\Gamma_j^\rho) ||m||^{(d-1)\tau(\nu)}.$$

Using this for $A_m^{(2)}$ we obtain

$$|A_m^{(2)}(x)| \leq C \left(\frac{\varepsilon}{\rho}\right)^{d-1} ||m||^{(d-1)\tau} \sum_j |P(x,y_j)||c_m(y_j)| vol_{d-1}(\Gamma_j^\rho) \lesssim \left(\frac{\varepsilon}{\rho}\right)^{d-1} ||m||^{(d-1)\tau} \mathcal{C}_m, \quad (3.31)$$

where due to estimate (3.12) we have a uniform in x bound over the sum with the Poisson kernel through its surface integral. Combining the estimates for $A_m^{(1)}$ and $A_m^{(2)}$, for the integral on $\Pi \setminus E$ we get

$$\left| \int\limits_{\Pi \setminus E} P(x,y)[g_\varepsilon(y) - \overline{g}] d\sigma(y) \right| \lesssim_\delta \sum_{k=1}^d \sum_{m \in \mathfrak{I}_k} \mathcal{C}_m \left(\frac{\rho^\beta}{d(x)^{\beta+\delta}} + \left(\frac{\varepsilon}{\rho}\right)^{d-1} ||m||^{(d-1)\tau} \right) \lesssim_\delta \frac{\rho^\beta}{d(x)^{\beta+\delta}} + \left(\frac{\varepsilon}{\rho}\right)^{d-1}, \quad (3.32)$$

where the convergence of series involving Fourier coefficients is due to the smoothness of g and Corollary 2.2.5. Since $\beta \leq 1$ clearly the estimate (3.28) is better than (3.32), thus we have

$$|u_\varepsilon(x) - u_0(x)| \lesssim_\delta \frac{\rho^\beta}{d(x)^{\beta+\delta}} + \left(\frac{\varepsilon}{\rho}\right)^{d-1}, \tag{3.33}$$

for all $x \in D$ satisfying $d(x) \geq 2c_0\rho$. Equalizing the estimates we obtain

$$\frac{\rho^\beta}{d(x)^{\beta+\delta}} = \left(\frac{\varepsilon}{\rho}\right)^{d-1} \Longleftrightarrow \rho = \varepsilon^{\frac{d-1}{d-1+\beta}} d(x)^{\frac{\beta+\delta}{d-1+\beta}}.$$

Comparing this with $d(x) \geq 2c_0\rho$, we get that (3.33) holds true if $d(x) \geq C\varepsilon^{\frac{d-1}{d-1-\delta}}$, where C is some absolute constant, thus we conclude that

$$|u_\varepsilon(x) - u_0(x)| \leq C_\delta \left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}}\right)^{\frac{d-1}{d-1+\beta}}.$$

When $d(x) < C\varepsilon^{\frac{d-1}{d-1-\delta}}$ the estimate of the theorem follows by the uniform bound $|u_\varepsilon(x) - u_0(x)| \leq C||g||_{L^\infty(\mathbb{T}^d)}$ which is due to the maximum principle. Theorem is proved. □

L^p estimates

Proof of Theorem 3.1.2. For $\beta > 0$ set $\kappa = \frac{d-1}{d-1+\beta}$. By Theorem 3.1.1 we have

$$|u_\varepsilon(x) - u_0(x)| \leq C\frac{\varepsilon^{\beta\kappa}}{d(x)^{(\beta+\delta)\kappa}}, \qquad x \in D. \tag{3.34}$$

Set $p_0 = \frac{1}{\beta\kappa}$, and fix $1 \leq p < p_0$. Then for $\delta > 0$ small enough we have $p(\beta+\delta)\kappa = p\beta\kappa + \delta p\kappa < 1$. This, together with (3.34) implies that

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\beta\kappa}, \ 1 \leq p < p_0. \tag{3.35}$$

Now fix $p_0 \leq r < \infty$, and let $1 \leq p < p_0$. Using the uniform boundedness of $|u_\varepsilon - u_0|$, and estimate (3.35) we obtain

$$||u_\varepsilon - u_0||_{L^r(D)} = \left(\int_D |u_\varepsilon - u_0|^{r-p}|u_\varepsilon - u_0|^p\right)^{1/r} \leq C||u_\varepsilon - u_0||_{L^p(D)}^{\frac{p}{r}} \leq C\varepsilon^{\frac{\beta\kappa p}{r}}.$$

Take $p = p_0 - \delta$, where $\delta > 0$ is small enough. Since $p_0\beta\kappa = 1$, from the last estimate we get

$$||u_\varepsilon - u_0||_{L^r(D)} \leq C\varepsilon^{\beta\kappa\frac{p_0-\delta}{r}} = C\varepsilon^{\frac{1-\beta\kappa\delta}{r}} = C\varepsilon^{\frac{1}{r}-\delta_1},$$

where $\delta_1 = \frac{\beta\kappa\delta}{r}$. Combining this with (3.35), for $1 \leq p < \infty$ we get

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\beta\kappa, \frac{1}{p}\}-\delta}. \tag{3.36}$$

Thus, if $\beta = 1$, then we are done, otherwise, we have $\alpha_* \leq 1$, and (3.36) holds true for each $0 < \beta < \alpha_*$, and $\delta > 0$. Observe that for all $d \geq 2$ we have

$$0 < \alpha_* \kappa - \beta \kappa < \alpha_* - \beta, \text{ where } 0 < \beta < \alpha_* \leq 1.$$

Using this, for each $\delta > 0$ we choose $0 < \beta < \alpha_*$ such that $\alpha_* - \beta < \delta/2$, and from (3.36) we get

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C \varepsilon^{\min\{\gamma, \frac{1}{p}\} - \frac{3}{2}\delta},$$

completing the proof of the theorem. □

Optimality

Proof of Theorem 3.1.3. First recall that here we only consider boundary data that depend on the periodic variable only. For the proof we will follow the same strategy as in the proof of Theorem 2.3.2 for the smooth case. The only part that needs to be modified in this setting is Lemma 2.3.7, which proves certain type of equidistribution result for the family $\lambda\Gamma \mod \mathbb{Z}^d$, as $\lambda \to \infty$, where for $x \in \mathbb{R}^d$, and $(x \mod \mathbb{Z}^d)$ denotes the unique point $y \in \mathbb{T}^d$ satisfying $x - y \in \mathbb{Z}^d$. On the other hand, the proof of Lemma 2.3.7 is based on the following fact: for any smooth function $g : \mathbb{T}^d \to \mathbb{C}$ one has

$$\int\limits_{\mathbb{T}^d} g(x) dx = \lim_{\lambda \to \infty} \frac{1}{vol_{d-1}(\Gamma)} \int\limits_{\Gamma} g(\lambda y) d\sigma(y). \tag{3.37}$$

So, to complete the proof of the current theorem, we need to prove (3.37), which is now due to the Diophantine property imposed on the normals of the faces of D. Observe that since the linear combinations of exponentials $e_m(y) := e^{2\pi i m \cdot y}$, $m \in \mathbb{Z}^d$, $y \in \mathbb{T}^d$ are dense in the uniform metric in the space of smooth functions on $\mathbb{T}^d$, it is enough to prove (3.37) for each e_m, $m \in \mathbb{Z}^d$. When $m = 0$ then (3.37) is trivial, now fix some nonzero $m \in \mathbb{Z}^d$. We need to show that the limit in (3.37) is 0, which is enough to establish on each face of D. Let Π be a face of D with a normal vector $\nu \in DC(\tau, c)$. The proof will be complete once we show that

$$\mathfrak{I}_\lambda := \int\limits_{\Pi} e_m(\lambda y) d\sigma(y) \to 0, \text{ as } \lambda \to \infty. \tag{3.38}$$

Since $m \in \mathbb{Z}^d$ is nonzero, then $m_k \neq 0$ for some $1 \leq k \leq d$. Take any $\delta > 0$ small and apply Lemma 3.2.2 for k and δ. We will get a partition of Π into a set E, and a finite family of sets $\{\Gamma_j\}_{j=1}^M$ with properties $(i) - (iii)$ of Lemma 3.2.2. It is easy to see from the definition of sets Π_δ that $vol_{d-1}(\Pi_\delta) \leq C\delta$, and since $E \subset \Pi_{c_0\delta}$, for some absolute constant c_0, we have $vol_{d-1}(E) \leq C\delta$. We then use the properties of the partition and applying Lemma 3.2.1 on each of the Γ_j's we get

$$|\mathfrak{I}_\lambda| = \left| \int\limits_E + \sum_{j=1}^M \int\limits_{\Gamma_j} \right| \leq C\delta + C\lambda^{-(d-1)} ||m||^{(d-1)\tau} M \leq C\delta,$$

provided λ is large enough. Since $\delta > 0$ is arbitrarily small, the last expression shows (3.38), and hence completes the proof of the theorem. $\square$

Chapter 4

Analysis of boundary layers

The main purpose of the present chapter is to analyze the *boundary layers* arising in periodic homogenization. One of the primary difficulties toward the homogenization of Dirichlet problem for elliptic systems in divergence form with periodically oscillating coefficients and boundary condition, lies in determination of the limiting Dirichlet data corresponding to the effective problem. This question has been addressed only very recently in [34], where it was shown in particular, that this limiting data depends on all ingredients involved in the problem, nevertheless without constructive characterization of the dependence. A more refined description in this direction has been obtained later in [54]. However, some very basic issues remain unhandled. For example, it is not known whether the boundary data corresponding to the effective problem may develop singularities at the end of the homogenization process, if one starts with a problem with all the ingredients being smooth. Understanding this issue can shed a new light into the homogenization process, and one direct consequence would be up to the boundary regularity of homogenized solutions. In the first part of this chapter we initiate the study of this regularity problem, and prove certain Lipschitz continuity result. The second part of the chapter deals with *boundary layer systems* set in halfspaces, and studies asymptotic behavior of solutions far away from the boundaries. These are systems, that arise in the analysis of boundary layer correctors. By a new construction, here we show that depending on the normal direction of the hyperplane, convergence of the solutions toward constant vector fields away from the boundaries can be arbitrarily slow. This last result, combined with the previous studies from [33], [34], and [54] gives almost complete understanding of dependence of convergence speed of boundary layer correctors toward their tails, from the position of the corresponding hyperplane with respect to the microstructure.

The first part of this chapter concerning the regularity problem, is from our preprint [3]. The problem of slow convergence of boundary layer tails, which is the subject of study of Section 4.4, has originated from discussions with Christophe Prange, to whom we express our gratitude.

4.1 Introduction to the regularity problem

For a bounded domain $D \subset \mathbb{R}^d$ $(d \geq 2)$ consider the following problem

$$-\nabla \cdot \left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u \right)(x) = 0, \qquad x \in D, \tag{4.1}$$

with oscillating Dirichlet data

$$u(x) = g\left(x, \frac{x}{\varepsilon}\right), \qquad x \in \partial D. \tag{4.2}$$

Here, as usual $\varepsilon > 0$ is a small parameter, $A(x) = (A_{ij}^{\alpha\beta}(x))$ is $\mathbb{R}^{N^2 \times d^2}$-valued function defined on $\mathbb{R}^d$, where $1 \leq \alpha, \beta \leq d$, $1 \leq i, j \leq N$, and $g(x, y)$ is $\mathbb{R}^N$-valued function defined on $\partial D \times \mathbb{R}^d$. The operator in (4.1) is defined as

$$-(\mathcal{L}_\varepsilon u)_i(x) := \left[\nabla \cdot \left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u \right) \right]_i (x) = \frac{\partial}{\partial x^\alpha} \left[A_{ij}^{\alpha\beta}\left(\frac{\cdot}{\varepsilon}\right) \frac{\partial u_j}{\partial x^\beta} \right](x),$$

where $u = (u_1, ..., u_N)$ and $1 \leq i \leq N$. For the family of operators $\{\mathcal{L}_\varepsilon\}_{\varepsilon > 0}$ we let $\mathcal{L}_0$ be the homogenized (effective) operator in the standard sense of the theory of homogenization (see Section 1.2.1 in the Introduction to the thesis).

4.1.1 Assumptions

We start by formulating the hypotheses under which the problem (4.1)-(4.2) will be studied. Note that the homogenization problem considered here has already appeared in Chapter 2, and the assumptions (H1)-(H4) below are exactly the same under which we undertook the study of the problem before. For the purpose of this chapter we are going to add another assumption into the list, namely (H5) below, by that getting the following complete account of hypotheses.

(H1) (Periodicity) The coefficient tensor A and the boundary data g in its second (oscillating) variable are $\mathbb{Z}^d$-periodic, that is $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$ and $\forall x \in \partial D$ one has

$$A(y + h) = A(y), \;\; g(x, y + h) = g(x, y).$$

(H2) (Ellipticity) Coefficients are uniformly elliptic and bounded, that is there exist constants $\Lambda, \lambda > 0$ such that

$$\lambda \xi_\alpha^i \xi_\alpha^i \leq A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \leq \Lambda \xi_\alpha^i \xi_\alpha^i, \qquad \forall x \in \mathbb{R}^d, \; \forall \xi \in \mathbb{R}^{d \times N}.$$

(H3) (Convexity) We assume that D is strictly convex.

(H4) (Smoothness) We suppose that the boundary data g in both variables, the all elements of A, and ∂D are sufficiently smooth.

(H5) (Layered medium structure) We will assume that the coefficient tensor A is independent of the direction $e_d = (0, ..., 0, 1) \in \mathbb{R}^d$.

The last hypothesis (H5), which will also be discussed in Section 4.3.3, models the so-called laminates, i.e. when the media has layered structure. The reader may consult to [53] for some results on homogenization in layered media. We also note that the direction e_d is of no particular importance, but is rather fixed to simplify the notation. The whole point is that we require A to be independent of some rational direction, i.e. instead of (H5) one may assume that there exists a nonzero vector $\nu \in \mathbb{Q}^d$ such that $A(x+t\nu) = A(x)$ for any $x \in \mathbb{R}^d$ and any $t \in \mathbb{R}$. Then there exists a rotation and scaling of the coordinate system by an invertible matrix with rational entries, such that the new operator in still uniformly elliptic, is independent of e_d and is periodic with respect to lattice $k\mathbb{Z}^d$ with $k \in \mathbb{N}$ possibly large. The entire analysis presented here remains valid in this case too. To avoid repetition, we refer the reader to Section 4.3.3 for a similar treatment concerning change of variables.

4.1.2 Preliminary discussions and the main result

We now proceed to description of the regularity problem considered in this chapter. Denote by $g^* : \partial D \to \mathbb{R}^N$ the Dirichlet data of the homogenized problem corresponding to (4.1)-(4.2). Existence of g^* from the class $L^\infty(\partial D)$ under hypotheses (H1)-(H4) is due to Theorem 2.1.1. We also refer the reader to Theorem 2.3.3 for a particular class of problems (4.1)-(4.2), when this data can be computed explicitly (see formula (2.43) in the proof of Theorem 2.3.3), and proven to be smooth on the boundary of D. However, even in the case of the restrictive structural assumption considered in Theorem 2.3.3, one can observe that g^* is somehow subject to strong correlations between different ingredients of the problem, including the operator, boundary normal field, as well as the original boundary data. The main purpose we have here is the study of regularity of g^* in general.

Let us start by recalling some known facts from [34] concerning g^*. For a unit vector $n \in \mathbb{S}^{d-1}$ set $P_{n^\perp}$ to be the operator of orthogonal projection on the hyperplane orthogonal to n. Fix $l > 0$ so that $(d-1)l > 1$ and for $\kappa > 0$ set

$$\mathcal{A}_\kappa = \{n \in \mathbb{S}^{d-1} : |P_{n^\perp}(\xi)| \geq \kappa |\xi|^{-l} \text{ for all } \xi \in \mathbb{Z}^d \setminus \{0\}\}. \tag{4.3}$$

It is shown in [34] that $\sigma(\mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa) \leq C\kappa^{d-1}$, where σ denotes the Lebesgue measure on the unit sphere of $\mathbb{R}^d$. Now, for $x \in \partial D$ let $n(x)$ be the unit inward normal at x and for $\kappa > 0$ set $\partial D_\kappa = \{x \in \partial D : n(x) \in \mathcal{A}_\kappa\}$. With some little extra work one can show from the analysis of [34] that g^* is Lipschitz continuous on ∂D_κ with Lipschitz constant bounded by $C\kappa^{-2}$. However one can see that the complement $\mathcal{A}_\kappa^c = \mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa$, while a set of small measure, is everywhere dense and is an open subset of the unit sphere[1]. Let us illustrate this simple fact here, essentially following [34], and then discuss some consequences of this fact.

To start with, fix $n \in \mathbb{S}^{d-1}$, $\kappa > 0$ and assume that $n \notin \mathcal{A}_\kappa$. This means that for some nonzero $\xi \in \mathbb{Z}^d$ we have $|P_{n^\perp}(\xi)| < \kappa |\xi|^{-l}$, and in view of the linearity

---

[1] Here the sphere has a topology inherited from $\mathbb{R}^d$.

of the projection we get

$$|P_{n^\perp}(\xi/|\xi|)| < \kappa|\xi|^{-(l+1)}. \tag{4.4}$$

Geometrically this condition means that n stays inside a spherical neighborhood of $\pm\xi/|\xi|$ of radius some constant times $\kappa|\xi|^{-(l+1)}$. More precisely, set $\xi_1 = \xi/|\xi|$, and complete the unit vector ξ_1 by ξ_2,..., ξ_d to form an orthonormal basis in $\mathbb{R}^d$. Then for some scalars n_1,...,n_d we have $n = n_1\xi_1 + ... + n_d\xi_d$. Since $P_{\xi_1^\perp}(n)$ is the orthogonal projection of n on a subspace orthogonal to ξ_1 we obtain $P_{\xi_1^\perp}(n) = n_2\xi_2 + ... + n_d\xi_d$ and hence $|P_{\xi_1^\perp}(n)| = (1-n_1^2)^{1/2}$. By symmetry considerations we get

$$|P_{n^\perp}(\xi_1)| = |P_{\xi_1^\perp}(n)| = (1-n_1^2)^{1/2} = (n_2^2 + ... + n_d^2)^{1/2}.$$

From the last expression and (4.4) we conclude that

$$\mathcal{A}_\kappa^c = \bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}} \{(n_2, ..., n_d) \in \mathbb{R}^{d-1} : (n_2^2 + ... + n_d^2)^{1/2} < \kappa|\xi|^{-(l+1)}\}, \tag{4.5}$$

which implies

$$\sigma(\mathcal{A}_\kappa^c) \leq C\kappa^{d-1} \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} |\xi|^{-(d-1)(l+1)} \leq C\kappa^{d-1}, \tag{4.6}$$

where convergence of the series is due to the fact that $l(d-1) > 1$. Now observe that (4.5) shows that $\mathcal{A}_\kappa^c$ is an open set on the unit sphere, containing all rational directions, implying in particular that $\mathcal{A}_\kappa^c$ is dense on $\mathbb{S}^{d-1}$. On the other hand the strict convexity and smoothness of D implies that ∂D is diffeomorphic to $\mathbb{S}^{d-1}$ through its normal field[2], from which we conclude that ∂D_κ has essentially the same structure as $\mathcal{A}_\kappa$ had on the unit sphere, i.e. $\partial D \setminus \partial D_\kappa$ is open and is dense in ∂D. This fact prevents one to extend g^* continuously on any given open subset of the boundary of D relying on the Lipschitz continuity of g^*, since the Lipschitz constant blows-up as $\kappa \to 0$.

For a given domain D with smooth boundary, let $\partial D_{\mathrm{irr}}$ be the subset of ∂D where the unit normal field has irrational direction, that is for $x \in \partial D$ we let $x \in \partial D_{\mathrm{irr}}$ if and only if $n(x) \notin \mathbb{R}\mathbb{Q}^d$. For $\kappa > 0$ set

$$\partial D_{\kappa,+} = \{x \in \partial D_{\mathrm{irr}} : \ (n(x))_d > \kappa\} \text{ and } \partial D_{\kappa,-} = \{x \in \partial D_{\mathrm{irr}} : \ (n(x))_d < -\kappa\},$$

---

[2] This is a standard fact, but since we could not find a reference, for the readers' convenience we sketch an argument here. To see the claim, let $\mathcal{N} : \partial D \to \mathbb{S}^{d-1}$ be the Gauss map, i.e. at each point $x \in \partial D$ it assigns the normal vector as its value. As the domain is strictly convex, the Hessian of the graph, representing the surface, has full rank for any $x \in \partial D$ (see the discussion preceding Claim 2.1.2). From here one can see that the Jacobian of $\mathcal{N}$ has nonzero determinant, hence by the inverse function theorem the Gauss map is locally a diffeomorphism. The latter shows that the image of ∂D under $\mathcal{N}$ is open in $\mathbb{S}^{d-1}$. On the other hand since ∂D is compact we have that $\mathcal{N}(\partial D)$ is compact in $\mathbb{S}^{d-1}$ and thus is closed. The sphere is connected, and we conclude that $\mathcal{N}(\partial D) = \mathbb{S}^{d-1}$. Finally, the strict convexity of the domain trivially implies that $\mathcal{N}$ is one-to-one, since otherwise one would get two different tangent planes of ∂D having the same normal, violating the condition that D must lie strictly on one side of its support planes. It now follows that the Gauss map is a diffeomorphism between ∂D and $\mathbb{S}^{d-1}$.

also denote

$$\partial D_0 = \{x \in \partial D : \ (n(x))_d = 0\}.$$

Here the subscript d means the last coordinate of the vector. Since as we saw ∂D is diffeomorphic to $\mathbb{S}^{d-1}$ we have that ∂D_0 has surface measure 0. The following is our main theorem concerning the regularity problem introduced above.

Theorem 4.1.1. **(The Main result)** *Under the conditions (H1)-(H5) above, for any $0 < \kappa < 1$ the boundary data g^* is Lipschitz continuous on $\partial D_{\kappa,+}$ and $\partial D_{\kappa,-}$.*

An immediate consequence of this result is the following.

Corollary 4.1.2. *Under the conditions and notation of Theorem 4.1.1, g^* can be extended continuously on $\partial D \setminus \partial D_0$.*

Proof. In view of Theorem 4.1.1, g^* is uniformly continuous on $\partial D_{\kappa,+} \cup \partial D_{\kappa,-}$ for any $\kappa > 0$. Taking κ to 0 implies the result. □

We end this section with a few remarks and a brief review. First, it should be noted that while problems with operators of the form (H5) have been considered in the literature, in our case this new restriction is rather technical and allows at some points for our argument to go through. However, on the positive side, most of the arguments of the current chapter do not need hypotheses (H5). Moreover, building upon the previous works [34] and [54], the chapter sets up a program, which if completed, will lead to a sufficiently well understanding of the regularity issues considered above. The results here can be seen as initial steps toward the completion of the aforementioned program.

Around the theme of regularity. Let us also mention a very few existing results in the literature that to a certain extent relate to our regularity problem. Keeping the operator as in (4.1), instead of Dirichlet data, one may consider Neumann boundary condition, prescribing the co-normal derivative of solutions; that is to say instead of (4.2) we set

$$n(x) \cdot \left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u \right)(x) = g\left(x, \frac{x}{\varepsilon}\right), \qquad x \in \partial D,$$

where $n(x)$ is the unit inward normal. In this case if ∂D does not have flat pieces, or has finitely many with each flat portion of ∂D having normal direction not included in $\mathbb{R}\mathbb{Q}^d$, then the homogenized boundary condition is as follows

$$n(x) \cdot (A^0 \nabla u^0)(x) = \overline{g}(x), \qquad x \in \partial D,$$

where A^0 denotes the constant coefficient tensor of the homogenized operator, u^0 is the solution to the homogenized problem, and $\overline{g}(x) := \int_{\mathbb{T}^d} g(x,y)dy$. This result is classical, and the details can be found in [16]. We see that here there is no problem of regularity of the homogenized boundary data.

The nonlinear version of this problem has been studied by Choi, and Kim [21]. Specifically, they consider homogenization problem for fully nonlinear uniformly elliptic PDE in a bounded domain with periodically oscillating normal derivative on the boundary. In particular, they show that in case the homogenized operator is rotation-invariant, the homogenized boundary data is continuous with respect to normal directions. If the operator is homogeneous a similar result was proved by Choi, Kim, and Lee [22].

The case of Dirichlet data, for fully nonlinear non divergence form operators has been studied by Feldman in [31], and very recently by Feldman, and Kim in [32]. The continuity of the homogenized boundary data on the set of irrational directions is established in [31]. Continuing in this direction the recent work [32] shows that if the homogenized operator is rotation-invariant then the homogenized boundary data is Hölder continuous for strictly convex domains, while if the condition on rotation-invariance fails, the homogenized data can be discontinuous at every point of the boundary where the normal direction is rational.

4.2 Boundary layer systems and construction of homogenized data g^*

We first fix some notation, that will be used in the sequel. For a unit normal $n \in \mathbb{S}^{d-1}$ and scalar $a \in \mathbb{R}$ set $\Omega_{n,a} = \{x \in \mathbb{R}^d : \ x \cdot n > a\}$, where $\cdot$ is the usual dot product in $\mathbb{R}^d$. For $a = 0$ denote $\Omega_n := \Omega_{n,0}$.

For a smooth and $\mathbb{Z}^d$-periodic function v_0, unit vector $n \in \mathbb{R}^d$, and scalar $a \in \mathbb{R}$ consider the following problem

$$\begin{cases} -\nabla \cdot A(y) \nabla v(y) = 0, & y \in \Omega_{n,a}, \\ v(y) = v_0(y), & y \in \partial \Omega_{n,a}. \end{cases} \tag{4.7}$$

We will refer to problems of the form (4.7) as *boundary layer systems.* As we have discussed in Section 1.2.2, these systems play a central role in homogenization of problem (4.1)-(4.2). Concerning (4.7) we will need the following result.

Theorem 4.2.1. (Prange [54], Theorem 1.2) *Assume $n \notin \mathbb{R}\mathbb{Q}^d$. Then*

1. *there exists a unique solution $v \in C^\infty(\overline{\Omega_{n,a}}) \cap L^\infty(\Omega_{n,a})$ of (4.7) such that*

$$||\nabla v||_{L^\infty(\{y \cdot n > t\})} \to 0, \ \text{as } t \to \infty,$$

$$\int_a^\infty ||\partial_n v||^2_{L^\infty(\{y\cdot n - t = 0\})} dt < \infty,$$

2. *and a boundary layer tail $v^\infty \in \mathbb{R}^N$ independent of a so that for $y \in \Omega_{n,a}$*

$$v(y) \to v^\infty, \ \text{as } y \cdot n \to \infty,$$

and convergence is locally uniformly in tangential variable.

Now, following [34] and [54] we describe the construction of the homogenized boundary data. First, consider the case when boundary data g in (4.2) can be factored into independent components depending on x and y. Namely, assume that there exist a smooth v_0 defined on $\mathbb{T}^d$ with values in $M_N(\mathbb{R})$ and some smooth g_0 defined on ∂D and with values in $\mathbb{R}^N$ so that $g(x,y) = v_0(y)g_0(x)$. Next, take any $x \in \partial D_{\mathrm{irr}}$ and for $n(x)$ consider the boundary layer system (4.7) with boundary data v_0. Then let $v^\infty(x)$ be the constant field[3] provided by Theorem 4.2.1. Finally, set

$$g^*(x) := v^\infty(n(x))g_0(x), \qquad x \in \partial D_{\mathrm{irr}}.$$

The general case proceeds by approximation. Using periodicity of g in y and its smoothness we have the following expansion

$$g(x,y) = \sum_{m\in\mathbb{Z}^d} c_m(x)e^{2\pi i y\cdot m} := \sum_{m\in\mathbb{Z}^d} g_m(x,y),$$

where the series converge uniformly and absolutely. Here $g_m(x,y)$ is factored, since $c_m \in \mathbb{R}^N$ and we may identify the exponential $e^{2\pi i m\cdot y}$ with $e^{2\pi i m\cdot y}I_N$, where $I_N \in M_N(\mathbb{R})$ is the identity matrix. We let v_m^∞ be the constant field corresponding to the m-th exponential. Then, it is shown in [34] that the homogenized boundary data for g is given by

$$g^*(x) = \sum_{m\in\mathbb{Z}^d} c_m(x)v_m^\infty(n(x)) := \sum_{m\in\mathbb{Z}^d} g_m^*(x), \qquad x \in \partial D_{\mathrm{irr}}. \tag{4.8}$$

We refer the reader to [34], Section 4.2 for the details. What we see from here is the fact that the regularity of g^* depends on the regularity of v^∞ with respect to the normal directions. To analyze this dependence we will use a formula for v^∞ computed in [54]. For its introduction we need some preliminaries.

Recall that A^* is the coefficient tensor for the transposed operator, i.e. $(A^*)_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Next, for all $1 \leq \gamma \leq d$ we let $v^{*,\gamma} \in M_N(\mathbb{R})$ be the solution (in the sense of Theorem 4.2.1) to the following system

$$\begin{cases} -\nabla_{\widetilde{y}} \cdot A^*(\widetilde{y})\nabla_{\widetilde{y}} v^{*,\gamma}(\widetilde{y}) = 0, & \widetilde{y} \in \Omega_n, \\ v^{*,\gamma}(\widetilde{y}) = -\chi^{*,\gamma}(\widetilde{y}), & \widetilde{y} \in \partial\Omega_n, \end{cases} \tag{4.9}$$

where $\chi^{*,\gamma} \in M_N(\mathbb{R})$ is the solution of the following *cell* problem

$$\begin{cases} -\nabla_y \cdot A^*(y)\nabla_y \chi^{*,\gamma}(y) = \partial_{y_\alpha} A^{*,\alpha\gamma}, & y \in \mathbb{T}^d, \\ \int\limits_{\mathbb{T}^d} \chi^{*,\gamma}(y)dy = 0. \end{cases} \tag{4.10}$$

To proceed, we need the notion of quasi-periodicity. By $C_b(\mathbb{R}^d)$ denote the Banach algebra of complex-valued continuous and bounded functions on $\mathbb{R}^d$, with

---

[3] Note that technically Theorem 4.2.1 is formulated for the case when the boundary data is an N-dimensional vector, while here we need an $N \times N$ matrix. Clearly this is not an issue, since one may treat each column of the matrix separately, as is mentioned e.g. in [34].

the norm $||f||_{C_b(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)|$.

Definition 4.2.1. *We say that a function $f \in C_b(\mathbb{R}^d)$ is quasi-periodic, if the set of shifts $\{f(\cdot + a)\}_{a \in \mathbb{R}^d}$ is precompact in $C_b(\mathbb{R}^d)$.*

The set of all quasi-periodic functions forms a closed subalgebra of $C_b(\mathbb{R}^d)$. The next lemma provides a certain analogue of a mean-value for quasi-periodic functions[4].

Lemma 4.2.2. (Šubin [61], Theorem S.3) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be quasi-periodic. Then there exists a scalar $\mathcal{M}(f)$ such that for any $\varphi \in L^1(\mathbb{R}^d)$ one has*

$$\int_{\mathbb{R}^d} \varphi(y) f(\lambda y) dy \to \mathcal{M}(f) \int_{\mathbb{R}^d} \varphi(y) dy, \qquad \text{as } \lambda \to \infty.$$

The following formula for $v^\infty(n)$ defined by Theorem 4.2.1 is due to Prange (see [54], formula (6.4)). Keeping the notation of Theorem 4.2.1 and Lemma 4.2.2 we have

$$\begin{aligned} v^\infty(n) = \int_{\partial\Omega_n} \partial_{y_\alpha} G^0(n,y) d\sigma(y) \times \Big[&\mathcal{M}\{A^{\beta\alpha}(y) v_0(y) n_\beta\} + \\ &\mathcal{M}\left\{\partial_{y_\beta} (\chi^{*,\alpha})^t(y) A^{\beta\gamma}(y) v_0(y) n_\gamma\right\} + \\ &\mathcal{M}\left\{\partial_{y_\beta} (v^{*,\alpha})^t(y) A^{\beta\gamma}(y) v_0(y) n_\gamma\right\} \Big]. \end{aligned} \tag{4.11}$$

Here G^0 is the Green's kernel corresponding to the homogenized constant coefficient operator $-\nabla \cdot A^0 \nabla$ in domain Ω_n. Also, the averages $\mathcal{M}(\cdot)$ are understood for restrictions of functions on the hyperplane Ω_n, that is one may apply Lemma 4.2.2 after rotating the hyperplane $\partial\Omega_n$ to $\mathbb{R}^{d-1} \times \{0\}$. We finish this section by establishing a uniform bound on the constant field of Theorem 4.2.1 in terms of the corresponding boundary data.

Lemma 4.2.3. *Keeping the assumptions and notation of Theorem 4.2.1, for a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$ and boundary data v_0 let v^∞ be the corresponding constant field. Then there exists a constant C such that $|v^\infty| \leq C ||v_0||_{L^\infty(\mathbb{T}^d)}$.*

Proof. The proof relies on an integral representation of solutions to (4.7). Since v^∞ is independent of the scalar a in (4.7) we will assume that $a = 0$. By [54] Section 3.2, for the solution of (4.7) we have

$$v(y) = \int_{\partial\Omega_n} P(y, \widetilde{y}) v_0(\widetilde{y}) d\sigma(\widetilde{y}), \qquad y \in \Omega_n,$$

---

[4] To see the analogy with the ordinary mean-value of a function and the one given by Lemma 4.2.2, we note that quasi-periodic functions can be identified with continuous functions defined on Bohr compactification of $\mathbb{R}^d$. Then the mean-value in a sense of Lemma 4.2.2 agrees with the ordinary mean-value but with respect to the Haar measure of the compactification of $\mathbb{R}^d$.

where P is the Poisson kernel corresponding to (4.7) and satisfies the following estimate (see [34], Lemma 2.5)

$$|P(y,\widetilde{y})| \leq C\frac{y\cdot n}{|y-\widetilde{y}|^d}, \tag{4.12}$$

for all $d \geq 2$, $y \in \Omega_n$ and $\widetilde{y} \in \partial\Omega_n$. Using (4.12) we obtain

$$|v(y)| \leq C||v_0||_{L^\infty(\mathbb{T}^d)} \int\limits_{\widetilde{y}\cdot n=0} \frac{y\cdot n}{|y-\widetilde{y}|^d} d\sigma(\widetilde{y}).$$

Now take any orthogonal matrix $M \in O_d(\mathbb{R})$ such that $n = Me_d$, and make a change of variables in the last integral by $y = Mz$ and $\widetilde{y} = M\widetilde{z}$. Due to orthogonality of M, for $y \in \Omega_n$ we have $y \cdot n = z \cdot M^t n = z \cdot e_d = z_d > 0$ from which it follows that

$$|v(Mz)| \leq C||v_0||_{L^\infty(\mathbb{T}^d)} z_d \int\limits_{\widetilde{z}_d=0} \frac{d\widetilde{z}}{|z-\widetilde{z}|^d} =$$

$$C\frac{||v_0||_{L^\infty(\mathbb{T}^d)}}{z_d^{d-1}} \int\limits_{\widetilde{z}_d=0} \frac{d\widetilde{z}}{\left[\left(\frac{z_1-\widetilde{z}_1}{z_d}\right)^2 + ... + \left(\frac{z_{d-1}-\widetilde{z}_{d-1}}{z_d}\right)^2 + 1\right]^{d/2}}.$$

Setting $\tau_i := (z_i - \widetilde{z}_i)/z_d,\ i = 1, 2, ..., d-1$ in the last integral, we obtain

$$|v(Mz)| \leq C||v_0||_{L^\infty(\mathbb{T}^d)} \int\limits_{\mathbb{R}^{d-1}} \frac{d\tau}{(1+|\tau|^2)^{d/2}} \leq C||v_0||_{L^\infty(\mathbb{T}^d)},$$

finishing the proof. □

4.3 Stability of averages along hyperplanes

In this section we prove some stability results with respect to normal directions, for certain averages involved in formula (4.11).

4.3.1 Averages of periodic functions

We start with a particular case of Lemma 4.2.2, which is of special importance to us.

Lemma 4.3.1. *Assume $f : \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic, $n \notin \mathbb{R}\mathbb{Q}^d$ is a unit vector and $M \in O_d(\mathbb{R})$ is an orthogonal matrix satisfying $Me_d = n$. Then for $g(z') = f(M(z', 0))$, where $z' \in \mathbb{R}^{d-1}$, one has*

$$\mathcal{M}(g) = \int\limits_{\mathbb{T}^d} f(x)dx.$$

Proof. Observe that g is quasi-periodic, hence by Lemma 4.2.2 we have the existence of $\mathcal{M}(g)$. Due to periodicity and smoothness of f we have

$$g(z') = f(M(z',0)) = \sum_{k\in\mathbb{Z}^d} c_k(f) e^{-2\pi i k\cdot M(z',0)},$$

where $c_k(f)$ is the k-th Fourier coefficient of f. To compute $\mathcal{M}(g)$, we fix some $\varphi \in C_0^\infty(\mathbb{R}^{d-1})$, set $\phi_k(z') = k \cdot M(z',0)$ for $k \in \mathbb{Z}^d \setminus \{0\}$ and let

$$\mathcal{I}_k(\lambda) = \int\limits_{\mathbb{R}^{d-1}} \varphi(z') e^{-2\pi i \lambda \phi_k(z')} dz', \qquad \lambda > 1.$$

It follows from the definition of the matrix M that $M = [N|n]$, where N is $d \times (d-1)$ matrix. It is easy to see that $\nabla' \phi_k(z') = N^t k$, for all $z' \in \mathbb{R}^{d-1}$ where ∇' is the gradient in $\mathbb{R}^{d-1}$, but since M is orthogonal we have

$$|k| = |M^t k| = |(N^t k, n\cdot k)| = |(\nabla'\phi_k(z'), n\cdot k)|,$$

hence if $\nabla'\phi_k(z') = 0' \in \mathbb{R}^{d-1}$ it follows that $|n \cdot k| = |k|$, which implies that $n \in \mathbb{R}\mathbb{Q}^d$, thus leading to a contradiction. We conclude that $\nabla'\phi_k(z') \neq 0'$, hence by the principle of the non-stationary phase (see e.g. [59], p. 341, Prop. 4) we get that $\lim\limits_{\lambda\to\infty} \mathcal{I}_k(\lambda) = 0$, for any nonzero $k \in \mathbb{Z}^d$. This shows that

$$\int\limits_{\mathbb{R}^{d-1}} \varphi(z') g(\lambda z') dz' \to c_0(f) \int\limits_{\mathbb{R}^{d-1}} \varphi(z') dz', \qquad \text{as } \lambda \to \infty,$$

hence the claim. □

An important consequence of the previous Lemma is the independence of the first two averages involved in the formula (4.11) from the normal n. Namely, since A, v_0, and $\chi^{*,\gamma}$ are all $\mathbb{Z}^d$-periodic, from Lemma 4.3.1 we get

$$\mathcal{M}\{A^{\beta\alpha}(y)v_0(y)n_\beta\} = \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)\}n_\beta = c_0(A^{\beta\alpha}v_0)n_\beta, \tag{4.13}$$

and

$$\mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)n_\gamma\right\} = \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)\right\}n_\gamma = c_0[(\chi^{*,\alpha})^t A^{\beta\gamma}v_0]n_\gamma, \tag{4.14}$$

where we take $n \notin \mathbb{R}\mathbb{Q}^d$, and $c_0(f)$ denotes the 0-th Fourier coefficient of $\mathbb{Z}^d$-periodic function f, i.e. the average over the cell of periodicity. It is very important to note that at this stage we are not able to apply Lemma 4.3.1 to the last average in (4.11) since $v^{*,\gamma}$ is generally not $\mathbb{Z}^d$-periodic. This fact gives rise to serious mathematical difficulties.

4.3.2 Stability of Green's averages

Here we study the regularity with respect to normals of the integrals involving Green's kernel in formula (4.11). For a coefficient tensor A and a halfspace $\Omega \subset \mathbb{R}^d$, the Green's kernel $G = G(y, \widetilde{y}) \in M_N(\mathbb{R})$ corresponding to the operator $-\nabla \cdot A(y)\nabla$ in domain Ω is a matrix function satisfying the following elliptic system

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y G(y,\widetilde{y}) = \delta(y-\widetilde{y})I_N, & y \in \Omega, \\ G(y,\widetilde{y}) = 0, & y \in \partial\Omega, \end{cases} \tag{4.15}$$

for any $\widetilde{y} \in \Omega$, where δ is the Dirac distribution and $I_N \in M_N(\mathbb{R})$ is the identity matrix. To have a quick reference to this situation, we will say that G corresponds to *the pair* (A, Ω). The existence and uniqueness of Green's kernels for elliptic systems in halfspaces is proved in [40], Theorem 5.4 for $d \geq 3$, and in dimension two in [30], Theorem 2.21. Moreover, if A^* is the coefficient tensor for the transposed operator, and G^* is the corresponding Green's kernel, then one has the following symmetry relation

$$G^t(y,\widetilde{y}) = G^*(\widetilde{y}, y), \qquad y, \widetilde{y} \in \Omega_n. \tag{4.16}$$

Let B^0 be a constant coefficient elliptic matrix and $G^0(z, \widetilde{z})$ be the Green's kernel corresponding to the pair $(B^0, \mathbb{R}^d_+)$. Fix a unit vector $n \in \mathbb{S}^{d-1}$, along with a matrix $M \in O_d(\mathbb{R})$ satisfying $Me_d = n$. For $y, \widetilde{y} \in \Omega_n$ set $G^n(y,\widetilde{y}) := G^0(M^t y, M^t \widetilde{y})$, we now find a system of equations that is satisfied by the matrix G^n.

Clearly, for any $y \in \partial\Omega_n$ we have $M^t y \in \partial\mathbb{R}^d_+$ and hence $G^n(y,\widetilde{y}) = 0$, so we get a homogeneous boundary conditions for G^n on Ω_n for any $\widetilde{y} \in \Omega_n$. To get the system for G^n, let us rewrite the system in the definition of the Green's kernel in (4.15). Let $G^0 = G^0_{kj} \in M_N(\mathbb{R})$, then according to (4.15) for all $1 \leq i, j \leq N$, we have

$$-\partial_{z_\alpha}(B^{0,\alpha\beta}_{ij}\partial_{z_\beta}G^0_{kj}(z,\widetilde{z})) = \delta(z-\widetilde{z})\delta_{ik}, \qquad z \in \mathbb{R}^d_+, \tag{4.17}$$

where δ_{ik} is the Kronecker delta. For fixed $1 \leq i, j \leq N$ denote $B^0_{ij} := B^{0,\alpha\beta}_{ij} \in M_d(\mathbb{R})$, then by this notation (4.17) becomes

$$-\nabla_z \cdot B^0_{ij}\nabla G^0_{kj}(z,\widetilde{z}) = \delta(z-\widetilde{z})\delta_{ik}.$$

Now, fix $\widetilde{y} \in \Omega_n$, then for any $1 \leq \alpha \leq d$ we have

$$\partial_{y_\alpha}G^n_{kj}(y,\widetilde{y}) = \partial_{z_1}G^0_{kj}(M^t y, M^t\widetilde{y})m_{\alpha 1} + \dots + \partial_{z_d}G^0_{kj}(M^t y, M^t\widetilde{y})m_{\alpha d},$$

and hence $\nabla_y G^n_{kj}(y,\widetilde{y}) = M\nabla_z G^0_{kj}(M^t y, M^t\widetilde{y})$, from which we obtain

$$\nabla_y \cdot B^0_{ij}\nabla_y G^n_{kj}(y,\widetilde{y}) = \nabla_z \cdot M^t B^0_{ij} M \nabla_z G^0_{kj}(z,\widetilde{z}), \tag{4.18}$$

where $z = M^t y$ and $\widetilde{z} = M^t\widetilde{y}$. Observe that by non-degeneracy of M we have $\delta(z-\widetilde{z}) = \delta(M^t(y-\widetilde{y})) = \delta(y-\widetilde{y})$, which in combination with (4.18) implies the following.

Claim 4.3.2. *Let $n \in \mathbb{R}^d$ be a unit vector and $M \in O_d(\mathbb{R})$ be such that $Me_d = n$. If $G^{0,n}(z, \widetilde{z})$ is the Green's kernel for the pair $(M^t B^0 M, \mathbb{R}^d_+)$, then $G^n(y, \widetilde{y}) := G^{0,n}(M^t y, M^t \widetilde{y})$ is the Green's kernel for the pair (B^0, Ω_n), where $M^t B^0 M$ is understood in accordance with (4.18).*

We now let $G^n(y, \widetilde{y})$ be the Green's kernel for the pair (A^0, Ω_n), where A^0 is the homogenized tensor corresponding to $A(y)$. For $1 \leq \alpha \leq d$, set

$$\mathcal{I}^\alpha(n) = \int\limits_{\partial\Omega_n} \partial_{\widetilde{y}_\alpha} G^n(n, \widetilde{y}) d\sigma(\widetilde{y}), \tag{4.19}$$

which is precisely the term involved in the formula (4.11). Our goal is to study the regularity of $\mathcal{I}^\alpha$ as a function from the unit sphere $\mathbb{S}^{d-1}$ to $M_N(\mathbb{R})$. Let $G^{0,n}(z, \widetilde{z})$ be the Green's kernel for the pair $(M^t A^0 M, \mathbb{R}^d_+)$, then by Claim 4.3.2 and the computations preceding that we have

$$\partial_{\widetilde{y}_\alpha} G^n_{jk}(n, \widetilde{y}) = \partial_{\widetilde{z}_1} G^{0,n}_{jk}(e_d, M^t \widetilde{y}) m_{\alpha 1} + ... + \partial_{\widetilde{z}_d} G^{0,n}_{jk}(e_d, M^t \widetilde{y}) m_{\alpha d}.$$

Using this we make a change of variables in (4.19) by the formula $\widetilde{y} = M\widetilde{z}$, where $\widetilde{z} \in \mathbb{R}^d_+$. Since $G^{0,n}$ has homogeneous boundary conditions with respect to both variables, we get that all tangential derivatives in the last expression vanish. Also, since $Me_d = n$ it follows that $m_{\alpha d} = n_\alpha$ for any $1 \leq \alpha \leq d$. We thus get

$$\mathcal{I}^\alpha(n) = n_\alpha \int\limits_{\partial\mathbb{R}^d_+} \partial_{\widetilde{z}_d} G^{0,n}(e_d, \widetilde{z}) d\sigma(\widetilde{z}). \tag{4.20}$$

The following bound is proved in [34], Lemma 2.5

$$|G^{0,n}(z, \widetilde{z})| \leq C \frac{z_d \widetilde{z}_d}{|z - \widetilde{z}|^d}, \qquad z \neq \widetilde{z} \text{ in } \mathbb{R}^d_+, \tag{4.21}$$

where C is independent of n. From the last estimate we have $|\nabla_{\widetilde{z}} G^{0,n}(e_d, \widetilde{z})| \leq C|e_d - \widetilde{z}|^{-d}$, for all $\widetilde{z} \in \partial\mathbb{R}^d_+$, and hence the integral in (4.20) is absolutely convergent, and is uniformly bounded with respect to n. We are now ready to formulate and prove the main result of this section.

Lemma 4.3.3. *For any $1 \leq \alpha \leq d$ the function $\mathcal{I}^\alpha(n)$ is Lipschitz continuous function from the unit sphere $\mathbb{S}^{d-1}$ to the set of matrices $M_N(\mathbb{R})$.*

Proof. Since we have a uniform bound with respect to n on the integral in the definition of $\mathcal{I}^\alpha$, it is enough to show that the function

$$\mathcal{J}(n) = \int\limits_{\partial\mathbb{R}^d_+} \partial_{\widetilde{z}_d} G^{0,n}(e_d, \widetilde{z}) d\sigma(\widetilde{z})$$

is Lipschitz. For $k = 1, 2$ fix $n^k \in \mathbb{S}^{d-1}$, along with orthogonal matrices $M_k \in O_d(\mathbb{R})$ such that $M_k e_d = n^k$. Next, denote $A^k = M_k^t A^0 M_k$, and let $G^k(z, \widetilde{z})$ be the Green's kernel corresponding to the pair $(A^k, \mathbb{R}^d_+)$. It is clear that we may choose matrices M_k varying smoothly with n^k, hence we will assume that they are chosen so that $|A^1 - A^2| \leq C|n^1 - n^2|$.

For any $0 \leq \mu < 1$ there exists a constant C depending on μ, such that

$$|G^1(e_d, \widetilde{z}) - G^2(e_d, \widetilde{z})| \leq C|n^1 - n^2| \frac{(\widetilde{z}_d)^\mu}{|e_d - \widetilde{z}|^{d-2+2\mu}}, \qquad e_d \neq \widetilde{z} \text{ in } \mathbb{R}^d_+. \tag{4.22}$$

This estimate follows from the proof of Lemma 6.3 of [54]. By symmetry property (4.16) we have

$$\nabla_{\widetilde{z}}(G^k)^t(e_d, \widetilde{z}) = \nabla_{\widetilde{z}}(G^{k,*})(\widetilde{z}, e_d), \;\; k = 1, 2. \tag{4.23}$$

Fix $z_0 \in \partial\mathbb{R}^d_+$, and for $r > 0$ denote $D(z_0, r) := \mathbb{R}^d_+ \cap B(z_0, r)$, $\Gamma(z_0, r) := \{z_d = 0\} \cap B(z_0, r)$, and set $G(z) := G^{1,*}(z, e_d) - G^{2,*}(z, e_d)$. From the definition of the Green's kernel we obtain

$$\begin{cases} -\nabla \cdot A^{1,*}\nabla G(z) = \nabla \cdot (A^{1,*} - A^{2,*})\nabla G^{2,*}(z, e_d), & z \in D(z_0, 1/4), \\ G(z) = 0, & z \in \Gamma(z_0, 1/4), \end{cases} \tag{4.24}$$

Since $G^{2,*}$ is a Green's kernel corresponding to constant coefficient operator we have (see [57], V.4.2, Satz 3)

$$|\partial^2_z G^{2,*}(z, e_d)| \leq C \frac{1}{|z - e_d|^d}, \qquad e_d \neq z \in \mathbb{R}^d_+. \tag{4.25}$$

By boundary gradient estimates of Avellaneda and Lin (see [12], Lemma 20) for any $\delta > 0$ we get

$$||\nabla G||_{L^\infty(D(z_0,1/8))} \lesssim_\delta ||G||_{L^\infty(D(z_0,1/4))} + \\ ||\nabla \cdot (A^{1,*} - A^{2,*})\nabla G^{2,*}(\cdot, e_d)||_{L^{d+\delta}(D(z_0,1/4))}.$$

Using (4.22) with $\mu = 3/4$, the fact that the coefficients are constant and (4.25), from the last inequality we have

$$||\nabla(G^{1,*}(z, e_d) - G^{2,*}(z, e_d))||_{L^\infty(D(z_0,1/8))} \lesssim \frac{|n^1 - n^2|}{|e_d - z|^{d-1/2}} + \frac{|n^1 - n^2|}{|e_d - z|^d}. \tag{4.26}$$

By continuity the last estimate holds up to the boundary. Combining (4.26) with (4.16) we obtain

$$|\nabla_{\widetilde{z}} G^1(e_d, \widetilde{z}) - \nabla_{\widetilde{z}} G^2(e_d, \widetilde{z})| \leq C \frac{|n^1 - n^2|}{|e_d - \widetilde{z}|^{d-1/2}}, \qquad \widetilde{z} \in \partial\mathbb{R}^d_+, \tag{4.27}$$

which completes the proof of the Lemma, since the function $|e_d - \widetilde{z}|^{-(d-1/2)}$ is integrable on the boundary of $\mathbb{R}^d_+$. □

Remark 4.3.4. *Let us note that the Lemma is immediate in the case of scalar equations. To see this, fix $n \in \mathbb{S}^{d-1}$, and $M \in O_d(\mathbb{R})$ such that $Me_d = n$. Now, if $G^{0,n}$ is the Green's kernel for to the pair $(M^t A^0 M, \mathbb{R}^d_+)$, then the corresponding*

Poisson kernel is the following

$$P^{0,n}(z,\widetilde{z}) = -e_d^t(M^tA^0M)\nabla_{\widetilde{z}}G^{0,n}(z,\widetilde{z}) = -(Me_d)^tA^0(Me_d)\partial_{\widetilde{z}_d}G^{0,n}(z,\widetilde{z}) = \\ -n^tA^0n\partial_{\widetilde{z}_d}G^{0,n}(z,\widetilde{z}), \quad (4.28)$$

where $z \in \mathbb{R}^d_+$ and $\widetilde{z} \in \partial\mathbb{R}^d_+$. The claim of the Lemma can be easily deduced from (4.28), ellipticity of A^0 and the fact that $P^{0,n}(e_d,\widetilde{z})$ has integral equal to one over the boundary of $\mathbb{R}^d_+$. However, this simple idea is not working for systems, since the coupling between the coefficients and the Green's matrix does not allow one to control individual entries of the Green's matrix from the information on the Poisson's kernel.

4.3.3 Boundary layers with exponential decay and the main result

The objective of this section is to study the dependence of solutions to (4.9) on the normal field under the additional assumption (H5) of Section 4.1.1. The extra condition (H5) concerning layered structure allows us to transform the problem (4.9) to the standard halfspace $\mathbb{R}^d_+$ while keeping the periodicity of the operator and that of the boundary data. With this extra information we construct a solution with a certain exponential decay and periodicity properties and then show that it actually coincides with the solution to (4.9) given by Theorem 4.2.1. This idea of exploiting layered structure of the problem is motivated by [53], from where we also take some ideas for the proofs of our Lemmas 4.3.5 and 4.3.7 below.

Combining the preliminary results of this section, with stability statements proved in Section 4.3 we will give the proof our main result, Theorem 4.1.1.

Change of variables

Let $y \in \mathbb{R}^d$ and $\mathcal{L} = -\nabla_y \cdot A^*(y)\nabla_y$ be the operator involved in the boundary layer system (4.9). For $x \in \mathbb{R}^d$ set $y = Tx$, where $T \in M_d(\mathbb{R})$ and has nonzero determinant. For $1 \leq i,j \leq N$ let A^*_{ij} be the $d \times d$ matrix formed from the (i,j)-th entries of the matrices $A^{*,\alpha\beta}$. Then by a direct computation we see that the operator $\mathcal{L}$ in the new variable x can be written as $\mathcal{L} = -\nabla_x \cdot B(Tx)\nabla_x$, where correspondingly

$$B_{ij}(Tx) = T^{-1}A^*_{ij}(Tx)(T^{-1})^t \quad (4.29)$$

for all $1 \leq i,j \leq N$. To keep track of the ellipticity constant of the new operator we take a family of vectors $\xi = \xi^\alpha \in \mathbb{R}^N$, set $\omega_i = (\xi_i^1, ..., \xi_i^d)^t$ where $1 \leq i \leq N$ and compute

$$B_{ij}^{\alpha\beta}\xi_j^\beta\xi_i^\alpha = \omega_i^tB_{ij}\omega_j = \omega_i^tT^{-1}A^*_{ij}(T^{-1})^t\omega_j = [(T^{-1})^t\omega_i]^tA^*_{ij}(T^{-1})^t\omega_j \geq \\ \lambda_A\sum_{i=1}^N ||(T^{-1})^t\omega_i||^2 \geq \lambda_A\sigma^2_{\min}(T^{-1})\omega_i\cdot\omega_i = \lambda_A\sigma^2_{\min}(T^{-1})\xi^\alpha\cdot\xi^\alpha, \quad (4.30)$$

where λ_A is the ellipticity constant of the original operator and $\sigma_{\min}(T^{-1})$ is the least singular value of the matrix T^{-1}, that is the square root of the smallest eigenvalue of $T^{-1}(T^{-1})^t$.

From a general halfspace to upper halfspace

In this section we make an extensive use of the spaces V_τ which are defined in the Appendix. Consider the problem (4.9) and let $T \in M_d(\mathbb{R})$ be so that T maps bijectively the standard upper halfspace to Ω_n. Transformations with the mentioned properties exist in abundance, later on we will make a specific choice for such T. We then set $y = Tx$, where $x \in \mathbb{R}^d_+$ and make a change of variables in (4.9) transforming it to the following problem

$$\begin{cases} -\nabla \cdot B(Tx)\nabla w(x) = 0, & x \in \mathbb{R}^d_+, \\ w(x) = w_0(x), & x \in \partial\mathbb{R}^d_+, \end{cases} \tag{4.31}$$

where B is given as in Section 4.3.3, $w(x) = v^{*,\gamma}(Tx)$ and $w_0(x) = -\chi^{*,\gamma}(Tx)$. The ellipticity of B follows by (4.30). Due to condition (H5) on A, from (4.10) we see that $\chi^{*,\gamma}$, i.e. solutions to cell problem, enjoy the same property as A, that is $\chi^*(y)$ is independent of y_d. Given this, it is easy to see that if the matrix T has the form $T = [Z|\nu]^t$ where Z is $(d-1)\times d$ matrix with integer entries and $\nu \in \mathbb{R}^d$ is arbitrary, then both B and w_0 in (4.31) are $\mathbb{Z}^d$-periodic.

To a given unit vector $n = (n_1, ..., n_d)$ we attach such a matrix T, which will be denoted by T_n. As we will see later, for the purpose of the main result there is no loss of generality if we restrict ourselves to the case when $n_d > 0$, thus this will be assumed in the sequel unless specified otherwise. We set

$$T_n = \begin{pmatrix} 1 & 0 & ... & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 1 & 0 \\ -\frac{n_1}{n_d} & ... & -\frac{n_{d-1}}{n_d} & 1 \end{pmatrix} \in M_d(\mathbb{R}), \tag{4.32}$$

then it is clear that a linear transformation associated with T_n is a bijection from $\overline{\mathbb{R}^d_+}$ to $\overline{\Omega_n}$. It is also clear that

$$T_n^{-1} = \begin{pmatrix} 1 & 0 & ... & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 1 & 0 \\ \frac{n_1}{n_d} & ... & \frac{n_{d-1}}{n_d} & 1 \end{pmatrix} \tag{4.33}$$

is the inverse of T_n. We now need to estimate the smallest singular value of T_n^{-1} so that to keep track of the ellipticity constant in (4.30). For this we will use the notation and result of Section A.0.4. We have $\det(T_n^{-1}) = 1$ and using the fact that $|n| = 1$ we obtain $r_{\min(T_n^{-1})}/\Pi_{i=1}^d r_i(T_n^{-1}) = n_d$. Now by virtue of (A.1) it follows that

$$\sigma_{\min}(T_n^{-1}) \geq \left(\frac{d-1}{d}\right)^{(d-1)/2} n_d. \tag{4.34}$$

With this choice of T_n both the operator and the boundary data in (4.31) are $\mathbb{Z}^d$-periodic. Hence, due to Corollary A.0.17 we get that the following problem

$$\begin{cases} -\nabla \cdot B(Tx)\nabla w(x) = 0, & x \in G := Y' \times \mathbb{R}_+, \\ w(x',0) = w_0(x',0), & x' \in Y' := (0,1)^{d-1}. \end{cases} \tag{4.35}$$

admits a unique solution in the space $V_\tau(G)$, for each $0 < \tau < \frac{\lambda_B}{2||B||_\infty}$, where λ_B is the ellipticity constant of B. Since all the data are smooth in (4.35) it follows by the standard elliptic regularity for weak solutions that $w \in C^\infty(G)$ (see e.g. [35], Corollary 4.12). Also, the solution is smooth at smooth boundaries (see e.g. [35], Theorem 5.21), so we get that w is smooth up to flat boundaries of G. Using the periodicity of the operator and that of the boundary data, we extend w to the entire halfspace $\mathbb{R}^d_+$, by this getting a smooth solution w to (4.31) such that $w(\cdot, x_d)$ is $\mathbb{Z}^{d-1}$-periodic for each $x_d \geq 0$ and with the mentioned exponential decay condition in the cells $(Y' + m') \times \mathbb{R}_+$, for all $m' \in \mathbb{Z}^{d-1}$. We now need to show that the solution w obtained in this way coincides with the one given by Theorem 4.2.1.

Lemma 4.3.5. *Let w be as above. Then it satisfies the following two properties*

(a) $||\nabla w||_{L^\infty(\mathbb{R}^{d-1}\times[t,\infty))} \to 0, \qquad$ *as $t \to \infty$,*

(b) $\int\limits_0^\infty ||\nabla w(\cdot,t)||^2_{L^\infty(\mathbb{R}^{d-1})} dt < \infty.$

Proof. We will fix some $\tau > 0$ so that $w \in V_\tau$. For $y \in G$ and $r > 0$ let $K(y,r)$ be the closed cube centered at y and side length r. Assume $K(y,2r) \subset \mathbb{R}^d_+$, then in view of the standard Schauder estimates for $W^{1,2}$ solutions to elliptic systems (see [35], Theorem 5.19) we have

$$||\nabla w||_{L^\infty(K(y,r))} \leq C||\nabla w||_{L^2(K(y,2r))}. \tag{4.36}$$

Now, let us fix some $r > 0$ and modify (4.36) slightly. Set $K'(y,r)$ to be the $(d-1)$-dimensional cube which is a projection of $K(y,r)$ onto $\mathbb{R}^{d-1} \times \{0\}$. We have

$$||\nabla w||^2_{L^2(K(y,2r))} = \int\limits_{K'(y,2r)} \int\limits_{y_d-2r}^{y_d+2r} |\nabla w(z',z_d)|^2 dz' dz_d \leq$$

$$e^{4\tau r} e^{-2\tau y_d} \int\limits_{K'(y,2r)} \int\limits_{y_d-2r}^{y_d+2r} |e^{\tau z_d} \nabla w(z',z_d)|^2 dz' dz_d \leq C(r) e^{-2\tau y_d} ||w||^2_{V_\tau(G)}, \tag{4.37}$$

where we have used the periodicity of w to get a bound in $V_\tau(G)$ norm. From

(4.37) and (4.36) we obtain

$$||\nabla w||_{L^\infty(K(y,r))} \leq C(r)e^{-\tau y_d}||w||_{V_\tau(G)}, \tag{4.38}$$

Both assertions of the lemma follow directly from (4.38). ☐

Corollary 4.3.6. *Let v be the solution to (4.35) given by the Theorem 4.2.1, and let w be the solution of the same problem (4.35) such that $w \in V_\tau(G)$ for some $\tau > 0$. Then $v = w$.*

Proof. The proof proceeds by comparing variational solution with the one obtained by Poisson's representation, using duality arguments. The details follow the lines of the proof of Theorem 3.5 of [54], using the estimates (a) and (b) of Lemma 4.3.5. ☐

What we gain from Lemma 4.3.5 is that solutions to problem (4.9) enjoy the exponential decay condition, and the periodicity property that comes from the definition of V_τ spaces. Note that by definition the solution to (4.9) is a matrix function, but clearly we can treat each column separately.

Let $n \in \mathbb{S}^{d-1}$ be fixed, such that $n_d \geq \delta > 0$, and let T_n be the matrix defined by (4.32). For $1 \leq \gamma \leq d$ fixed let $v_n^{*,\gamma}$ be the solution to (4.9) in the space Ω_n. We write n in the definition of solutions to (4.9) to emphasize their dependence on the normals. From the argument above we have that $v_n^{*,\gamma}(y) = v_n^{*,\gamma}(T_n x) = w_n(x)$, for $x \in \mathbb{R}^d_+$, where w_n solves the corresponding problem (4.35). By construction we have that w_n is smooth in $\mathbb{R}^d_+$ up to the boundary, and $w_n(\cdot, x_d)$ is $\mathbb{Z}^{d-1}$-periodic for any $x_d \geq 0$. We thus have the following expansion

$$w_n(x) = w_n(x', x_d) = \sum_{m' \in \mathbb{Z}^{d-1}} c_{m'}(n; x_d)e^{2\pi i m' \cdot x'} = \sum_{m \in \mathbb{Z}^{d-1}\times\{0\}} c_m(n; x_d)e^{2\pi i m \cdot x}, \tag{4.39}$$

where $x_d \geq 0$, $x' \in \mathbb{R}^{d-1}$ and $c_m(n; x_d) = \int_{\mathbb{T}^{d-1}} w_n(x', x_d)e^{-2\pi i m \cdot x} dx'$. Since $x = T_n^{-1} y$ for $y \in \Omega_n$, from (4.39) we have

$$v_n^{*,\gamma}(y) = w_n(T_n^{-1} y) = \sum_{m \in \mathbb{Z}^{d-1}\times\{0\}} c_m\left(n; \frac{y \cdot n}{n_d}\right) e^{2\pi i m \cdot y}, \tag{4.40}$$

where we used that $m \cdot T_n^{-1} y = m \cdot y$ which is due to construction of T_n and the fact that $m_d = 0$. In view of (4.40), for $1 \leq \alpha \leq d$ we obtain

$$\partial_{y_\alpha} v_n^{*,\gamma}(y) = \sum_{m \in \mathbb{Z}^{d-1}\times\{0\}} \left[\frac{n_\alpha}{n_d} c_m'\left(n; \frac{y \cdot n}{n_d}\right) + c_m\left(n; \frac{y \cdot n}{n_d}\right) 2\pi i m_\alpha\right] e^{2\pi i m \cdot y} := g_n^\alpha(y).$$

Now, we choose $M \in O_d(\mathbb{R})$ such that $Me_d = n$, and make a change of variables in the last expression by the formula $y = Mz$, with $z \in \mathbb{R}^d_+$. On the boundary of Ω_n we have

$$g_n^\alpha(M(z', 0)) = \sum_{m \in \mathbb{Z}^{d-1}\times\{0\}} \left[\frac{n_\alpha}{n_d} c_m'(n; 0) + 2\pi i m_\alpha c_m(n; 0)\right] e^{2\pi i m \cdot M(z', 0)}. \tag{4.41}$$

It follows from (4.41) that we can apply Lemma 4.3.1 for the last average in the formula (4.11). Moreover, given the smoothness of A, v_0 and v^* in (4.11), the regularity of the average with respect to n would follow from regularity of coefficients in (4.41) with respect to normal field. More precisely we need to show that for irrational directions ν and μ one has the following bound

$$\sup_{m\in\mathbb{Z}^{d-1}\times\{0\}}\Big[|c_m(\nu;0)-c_m(\mu;0)|+|c_m'(\nu;0)-c_m'(\mu;0)|\Big]\leq C|\nu-\mu|.$$

The latter, when restricted to a certain set of normals, follows immediately from the definition of the coefficients c_m and the next lemma.

Lemma 4.3.7. *Let $\delta>0$ be small and $\nu,\mu\in\mathbb{S}^{d-1}$ be such that $\nu_d,\mu_d\geq\delta$. Let v_ν^* and v_μ^* be the solutions to (4.9) for unit vectors ν and μ correspondingly, and define w_ν and w_μ to be the corresponding solutions for the system (4.35). Then there exists a constant C_δ such that*

$$||w_\nu-w_\mu||_{L^\infty(\mathbb{R}^d_+)}+||\nabla(w_\nu-w_\mu)||_{L^\infty(\mathbb{R}^d_+)}\leq C_\delta|\nu-\mu|.$$

Proof. We first transform the boundary layer problems (4.9) to $\mathbb{R}^d_+$. Let T_ν and T_μ be the transformation matrices as defined by (4.32), we have $w_\nu(x)=v_\nu^*(T_\nu x)$ and $w_\mu(x)=v_\mu^*(T_\mu x)$ where $x\in\mathbb{R}^d_+$. Throughout this lemma we will set $G=\mathbb{T}^{d-1}\times\mathbb{R}_+$. For $x\in\mathbb{R}^d_+$ denote $u(x):=w_\nu(x)-w_\mu(x)$ and

$$F(x):=(B_\nu(T_\nu x)-B_\mu(T_\mu x))\nabla w_\mu(x)=(B_\nu(x)-B_\mu(x))\nabla w_\mu(x),$$

where the last equality comes from (4.32) and hypothesis (H5) of Section 4.1.1. Observe that $u(\cdot,x_d)$, as well as $F(\cdot,x_d)$ are $\mathbb{Z}^{d-1}$-periodic for any $x_d\geq0$ and u solves the following problem

$$\begin{cases}-\nabla\cdot B_\nu(x)\nabla u(x)=\nabla\cdot F(x), & x\in\mathbb{R}^d_+,\\ u(x',0)=0, & x'\in\mathbb{R}^{d-1}.\end{cases}\tag{4.42}$$

Also note that due to the construction we have $u\in V_\tau(G)$ for some $\tau>0$ which will be specified below, hence we may apply estimate (A.8) of Theorem A.0.16 to get

$$||u||_{V_\tau(G;\mathbb{R}^{d\times N})}\leq\frac{C}{\tau}\frac{||e^{\tau x_d}F(x)||_{L^2(G;\mathbb{R}^{d\times N})}}{\lambda_{B_\nu}-2\tau||B_\nu||_\infty},\tag{4.43}$$

where in (4.43) we are following notation of Theorem A.0.16. By (4.29) we have

$$||B_\nu(x)-B_\mu(x)||_\infty\leq\\ C||A||_\infty(||T_\nu^{-1}||_\infty+||T_\mu^{-1}||_\infty)||T_\nu^{-1}-T_\mu^{-1}||_\infty\leq C_\delta|\nu-\mu|.\tag{4.44}$$

It follows by (4.30) and (4.29) that

$$||B_\nu||_\infty\leq C||A||_\infty\delta^{-2}\qquad\text{and}\qquad\lambda_{B_\nu}\geq C\lambda_A\delta^2,\tag{4.45}$$

which implies that $\tau=\frac{\lambda_A}{4||A||_\infty}\delta^4$ is a valid choice in (4.43), thus we will keep in mind that we have a uniform control over τ in terms of the threshold δ. With this choice of τ combining (4.45) with (4.43) and (4.44) we get that

$$||u||_{V_\tau(G)} \leq C_\delta |\nu - \mu| \times ||w_\mu||_{V_\tau(G)}. \tag{4.46}$$

Clearly (4.45) holds with μ replaced by ν. By (A.10) and definition of w_μ we have

$$||w_\mu||_{V_\tau(G)} \leq C_\delta ||\chi^*||_{H^1(Y')} \leq C_\delta. \tag{4.47}$$

By Schauder estimates we have (see [35], Theorem 5.19)

$$||\nabla u||_{L^\infty(K(y,r))} \lesssim_\delta ||\nabla u||_{L^2(K(y,2r))} + ||F||_{C^{0,\sigma}(K(y,2r))}, \tag{4.48}$$

where $0 < \sigma < 1$, and $K(y, 2r) \subset G$. As in (4.37) we obtain

$$||\nabla u||_{L^2(K(y,2r))} \leq C_{\delta,r} e^{-\tau y_d} ||u||_{V_\tau(G)}. \tag{4.49}$$

Next, using the definition of F we have

$$||F||_{C^{0,\sigma}(K(y,2r))} \leq ||B_\nu(x) - B_\mu(x)||_{C^{0,\sigma}(K(y,2r))} ||\nabla w_\mu||_{C^{0,\sigma}(K(y,2r))}. \tag{4.50}$$

By a similar argument as above, the first product is bounded by $C_\delta|\nu - \mu|$, while the second one can be controlled by its L^2 norm on a slightly larger cube, and hence is handled as in (4.38). We end up with

$$||F||_{C^{0,\sigma}(K(y,2r))} \leq C_\delta |\nu - \mu| e^{-\tau y_d} ||w_\mu||_{V_\tau(G)}.$$

From this, together with (4.48), (4.49) and (4.47) we obtain

$$||\nabla u||_{L^\infty(K(y,r))} \leq C_\delta e^{-\tau y_d} |\nu - \mu|. \tag{4.51}$$

The Schauder estimates discussed above are valid at smooth boundaries too (see [35], Theorem 5.21), thus estimate (4.51) holds when the cube $K(y, r)$ is replaced by $K(y, r) \cap G$ near the base of G. Since u has homogeneous boundary conditions on the base of G, by (4.51) we have

$$|u(y', y_d)| \leq \int_0^{y_d} |\nabla u(y', t)| dt \leq C_\delta |\nu - \mu|. \tag{4.52}$$

By (4.51) and (4.52) the proof of the lemma is complete. $\square$

Getting back to the constant field $v^\infty(n) : \mathbb{S}^{d-1} \to \mathbb{R}^N$, we know by (4.13), (4.14) and Lemma 4.3.3 that all the terms in representation (4.11) are Lipschitz continuous on subset of $\mathbb{S}^{d-1}$ of irrational directions, except possibly the last average involving boundary layers v^*. However, the hypothesis (H5) with Lemma 4.3.7 guarantees Lipschitz continuity of the average involving v^* for normals n satisfying $|n_d| \geq \delta > 0$ for some δ. Keeping the notation of the last Lemma and

Theorem 4.2.1, we conclude the following

$$|v^\infty(\nu) - v^\infty(\mu)| \leq C_\delta |\nu - \mu|, \tag{4.53}$$

for any $\nu, \mu \in \mathbb{S}^{d-1}$ satisfying $|\nu_d|, |\mu_d| \geq \delta > 0$. We are now ready to prove the main theorem.

Proof of Theorem 4.1.1 (The main result). We will use the notation of Section 4.2. Fix $\kappa > 0$, then for any $x, y \in \partial D_{\kappa,+}$ by (4.8) we have

$$|g^*(x) - g^*(y)| \leq \sum_{m\in\mathbb{Z}^d} |g^*_m(x) - g^*_m(y)| \leq$$
$$\sum_{m\in\mathbb{Z}^d} |c_m(x)| \times |v^\infty_m(n(x)) - v^\infty_m(n(y))| + \sum_{m\in\mathbb{Z}^d} |v^\infty_m(n(x))| \times |c_m(x) - c_m(y)| =:$$
$$\Sigma_1 + \Sigma_2. \tag{4.54}$$

Since $g(x, y)$ is a smooth function in both variables, then for any nonzero $m \in \mathbb{Z}^d$ and $k \in \mathbb{N}$ for the Fourier coefficient we have

$$|c_m(x)| \leq C_{g,k}|m|^{-k}, \text{ for all } x \in \partial D, \tag{4.55}$$

where the constant can be bounded by supremum norm of derivatives of g of order k (see Corollary 2.2.5). By (4.53), combined with the construction of g^* and smoothness of the domain we have

$$|v^\infty_m(n(x)) - v^\infty_m(n(y))| \leq C_\kappa |x - y|,$$

Using (4.55) we get that $\Sigma_1 \leq C_\kappa |x - y|$. For Σ_2, by the smoothness of g and Lemma 4.2.3, we have

$$|v^\infty_m(n(x))| \times |c_m(x) - c_m(y)| \leq C|x - y| \times ||\nabla c_m||_{L^\infty(\partial D)}.$$

We remark that since c_m and $g(\cdot, y)$ are only defined on the boundary of D, their derivatives should be understood by means of local coordinate charts of the boundary. Given the boundedness and smoothness of the domain, and smoothness of g, it is clear that the estimate (4.55) holds for the derivatives of g with respect to x, hence from the last inequality we get $\Sigma_2 \leq C|x - y|$, which in combination with the bound for Σ_1 implies that $|g^*(x) - g^*(y)| \leq C_\kappa |x - y|$.

Obviously, the same argument works for $\partial D_{\kappa,-}$. Theorem is proved.

□

Remark 4.3.8. *An explicit upper bound for the Lipschitz constant of g^* on sets $\partial D_{\kappa,\pm}$ can be determined from the proof of Lemma 4.3.7 and Theorem 4.1.1. However, this constant blows up at a rather high rate, as κ tends to 0, and hence is of no practical use as compared with the Lipschitz constant for g^* obtained in [34] on the Diophantine sets $\mathcal{A}_\kappa$, where the blow-up rate is κ^{-2}. What we gain here, as compared with [34], is that the set where we have Lipschitz continuity is* *very regular on the boundary of the domain, allowing us to extend g^* continuously on the boundary away from some lower dimensional set.*

Epilogue

Let us finally turn to some discussion about the hypotheses (H5) and regularity of g^* in general. To get more intuition as why (H5) comes as a handy tool here, we need to recall from [34] the construction of solutions to boundary layer systems of the form (4.7).

Keeping the notation of problem (4.7), fix some matrix $M \in O_d(\mathbb{R})$ such that $Me_d = n$. This matrix is of course not unique, and is in fact defined modulo the group $O_{d-1}(\mathbb{R})$, however for our purposes this non-uniqueness is not an issue. Then in system (4.7) we make a change of variables by setting $y = Mz$ and transforming it to

$$\begin{cases} -\nabla_z \cdot B(Mz)\nabla_z \widetilde{v}(z) = 0, & z_d > a, \\ \widetilde{v}(z) = v_0(Mz), & z_d = a, \end{cases} \tag{4.56}$$

where $\widetilde{v}(z) = v(Mz)$ and the (i,j)-th component of B is defined as

$$M_d(\mathbb{R}) \ni (B_{ij}^{\alpha\beta})_{\alpha,\beta=1}^d := B_{ij} = M^t A_{ij} M.$$

By the definition of M we have $M = [N|n]$, where $N \in M_{d\times(d-1)}(\mathbb{R})$. Then the solution to (4.56) is being sought of the form

$$\widetilde{v}(z) = V(Nz', z_d), \quad \text{where} \quad V(\cdot, t) \text{ is } \mathbb{Z}^d\text{-periodic for any } t \geq a,$$

and $z = (z', z_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. This leads to the following system

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B(\theta + tn) \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V(\theta, t) = 0, & t > a, \\ V(\theta, t) = v_0(\theta + an), & t = a\ . \end{cases} \tag{4.57}$$

The authors of [34] then show that the system in (4.57) has a smooth solution V in the infinite cylinder $\mathbb{T}^d \times [a, \infty)$ satisfying certain energy estimates. Moreover, if V solves (4.57), then $\widetilde{v}(z) = V(Nz', z_d)$ gives a solution to (4.56).

Now observe that one of the terms which is involved in representation formula (4.11), namely $v^{*,\alpha}$, solves the system (4.9) which is exactly a system of type considered above, and its solution is understood in terms of the reduced problem on the infinite cylinder $\mathbb{T}^d \times [a, \infty)$ described in (4.57). Our assumption (H5) on layered structures comes into help when analyzing the contribution of $v^{*,\alpha}$ in the limiting boundary data g^*. In particular as our analysis shows, this extra condition allows us to work on cylinder with the base $\mathbb{T}^{d-1}$, one dimension less than in general. This fact lets us to some extent fell into the setting when the normal to the bounding hyperplane in boundary layer systems has rational direction. This in turn opens up a possibility of deploying a different method for constructing solutions to systems of the (4.7), which lets us to gain some extra control. However, we believe that the methods developed in this chapter, in combination with previous works [12], [33], [34], and [54] should lead to a complete understanding of the regularity problem for g^* initiated here. Moreover, as some preliminary analysis shows, a sufficient regularity of g^* may also have some positive effects on the speed of convergence of the homogenization problem for (4.1)-(4.2).

We will now leave off the discussion on this positive note, and will implement several ideas that stem out from here in the future work.

4.4 Convergence speed of boundary layers toward constant fields

We have seen in the previous chapter, as well as in the introductory part of this work, that boundary layer systems of the form (4.7) play a key role in the analysis of Dirichlet problem (4.1)-(4.2). Moreover, as the analysis of [34] shows the convergence speed of these solutions to their corresponding constant fields far away from the boundaries is intimately tied up with the speed of convergence of homogenization process of the Dirichlet problem just mentioned. Let us briefly review the problem and our current knowledge about it. Keeping the notation of Section 4.2 we consider the following system

$$\begin{cases} -\nabla \cdot A(y)\nabla v(y) = 0, & y \in \Omega_{n,a}, \\ v(y) = v_0(y), & y \in \partial\Omega_{n,a}, \end{cases} \tag{4.58}$$

where $v_0 : \mathbb{R}^d \to \mathbb{R}^N$ is $\mathbb{Z}^d$-periodic and smooth, $n \in \mathbb{S}^{d-1}$ and for scalar $a \in \mathbb{R}$ we have set $\Omega_{n,a} = \{x \in \mathbb{R}^d : \ x \cdot n > a\}$. The existence and uniqueness of smooth solutions to (4.58) satisfying some energy estimates is established in [33], Proposition 2. The main question here is the analysis of asymptotic behavior of these solutions far away from the boundary of $\Omega_{n,a}$. Interestingly this problem is linked with certain number-theoretic properties of the normal n.

Rational directions. If $n \in \mathbb{R}\mathbb{Q}^d$ then it is well-known (see e.g. [7], Lemma 4.4) that there exists a smooth (variational) solution v to (4.58), which is unique given some decay conditions on the gradient, and such that there is a constant vector $v^{a,\infty}$ for which $v(y) \to v^{a,\infty}$ exponentially fast as $y \cdot n \to \infty$, and convergence is uniform with respect to tangential directions. Although having these nice convergence properties, the drawback of the rational directions is that the constant field may depend on a, i.e. translating the hyperplane in the direction of n can lead to different limits at infinity.

Diophantine directions. These are directions n that for some $\kappa > 0$ fall into the class $\mathcal{A}_\kappa$ introduced in (4.3). Clearly elements of $\mathcal{A}_\kappa$ are non rational directions. This case has been studied only recently in [33], where it was proved (ibid, Propositions 4 and 5) that there exists a smooth (variational) solution v to (4.58) which is unique, given some growth conditions, and such that for some constant vector v^∞ one has $v(y) \to v^\infty$ as $y \cdot n \to \infty$. Moreover, the convergence is locally uniform with respect tangential directions, and is faster than any polynomial rate in $y \cdot n$. We point out that the effective constant v^∞ (the boundary layer tail) depends on the direction n only, and is independent of a in contrast to the rational case.

Non rational directions in general. Note that not all irrational directions are Diophantine, thus the previous two cases do not cover $\mathbb{S}^{d-1}$. In a recent work [54], it was proved that (4.58) has a smooth variational solution satisfying certain growth conditions, for which one has convergence toward its boundary layer tail far away from the boundaries (see Theorem 4.2.1 for the precise statement). However, the result of [54] does not provide any estimates on the rate of convergence given this generality on the normals. It does however show that for irrational directions which are non Diophantine (meaning they fail to satisfy (4.3) for any parameters κ and l involved in the definition), one may have convergence slower than any power rate in $y \cdot n$. More precisely, for a smooth and $\mathbb{Z}^d$-periodic function $v_0 : \mathbb{T}^d \to \mathbb{R}$ consider the following boundary value problem

$$\Delta v = 0 \text{ in } \Omega_n \qquad v = v_0 \text{ on } \partial\Omega_n, \tag{4.59}$$

where $\Omega_n = \{x \in \mathbb{R}^d : \ x \cdot n > 0\}$. In (4.59) we consider scalar equations, and not systems, i.e. $N = 1$. Clearly, this problem is of type (4.58) with matrix of coefficients equal to $d \times d$-identity matrix. Then, [54] gives the following partial answer.

Theorem 4.4.1. (Prange [54], Theorem 1.3) *Let $d = 2$, and normal $n \notin \mathbb{R}\mathbb{Q}^2$ be any non-Diophantine direction. Then for any $l > 0$, any $R > 0$ there exists a smooth function $v_0 : \mathbb{T}^d \to \mathbb{R}$ and a sequence $t_k \nearrow \infty$ such that if v solves (4.59) with boundary data v_0, then for any $k = 1, 2, \ldots$ and all $y' \in \partial\Omega_n \cap B(0, R)$ we have*

$$|v(y' + t_k n) - v^\infty| \geq t_k^{-l},$$

where the constant v^∞ is the corresponding boundary layer tail.

The main advantage of considering Laplace operator in Theorem 4.4.1 is that one has more or less an explicit form of the solution v. The proof proceeds by constructing v_0 with Fourier spectrum supported in the subset of $\mathbb{Z}^2$ on which the normal n fails to satisfy the Diophantine condition. Then choosing coefficients with appropriate decay condition combined with the special structure of the support of v_0, and the explicit form of the solution, immediately leads to the conclusion. We stress that Theorem 4.4.1 works for any irrational, non-Diophantine direction, however, it leaves out the question, whether one can go beyond algebraic rates of convergence, and perhaps more intriguing, it does not give any clue for the case of variable coefficient operators.

Motivated by these results, and the importance of boundary layer systems in homogenization problems considered here, the main purpose of this section is to analyze in details the case of irrational, non Diophantine directions. Most notably, we will provide a new method of constructing solutions with slowly converging tails, which will enable us to show that the convergence speed can be slower than any given speed in advance.

Organization. The rest of the section is organized as follows. In Section 4.4.1 we study the slow convergence phenomenon for Laplace operator, and give a relatively self-contained treatment. Next, in Section 4.4.2, still for the Laplace operator, we discuss to which extent our examples for slow convergence are generic (typical) based on a probabilistic standpoint. In the last part, Section 4.4.3 we consider the problem for variable coefficient operators. The main result of Section 4.4.3 covers the case of Laplacian as a particular case, however, the proof there is based on a different idea, and is technically more involved.

4.4.1 Convergence can be arbitrarily slow

The main aim of this section is to prove the following result.

Theorem 4.4.2. *Let $R > 0$ be fixed, and take any continuous, one-to-one function $g : [0, \infty) \to (0, \infty)$ decreasing to 0 at infinity. Then there exists a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$, a smooth function $v_0 : \mathbb{T}^d \to \mathbb{R}$, and a sequence $\{t_k\}_{k=1}^{\infty}$ of positive numbers growing to infinity, such that if v solves (4.59) with n and v_0 as specified here, then for any $k = 1, 2, ...,$ and all $y' \in \partial\Omega_n \cap B(0, R)$ one has*

$$|v(y' + t_k n) - v^{\infty}| \geq g(t_k),$$

where the constant v^{∞} is the corresponding boundary layer tail.

The result shows that there is *no* lower bound for the speed of convergence on the set of irrational directions, in other words convergence can be in fact arbitrarily slow! This case is in strong contrast with the case of Diophantine normals, where one has convergence faster than any power rate. Also note that in Theorem 4.4.2 we have fixed the position of the halfspace Ω_n, but since we are working with irrational directions, the boundary layer tail depends on the normal only, i.e. it does not change if we translate Ω_n keeping it orthogonal to n. Thus there is no loss of generality in fixing the halfspace through 0.

The proof of Theorem 4.4.2 is based on a series of observations and preliminary statements. We will use the connection of problem (4.59) with the corresponding problem (4.57) set on a cylinder $\mathbb{T}^d \times [0, \infty)$. For that fix an orthogonal matrix $M \in M_d(\mathbb{R})$ such that $Me_d = n$, clearly M is of the form $M = [N|n]$, where N is a matrix of size $d \times (d-1)$. The corresponding problem (4.57) for (4.59) reads

$$\begin{cases} \left|\begin{matrix} N^T\nabla_\theta \\ \partial_t \end{matrix}\right|^2 V(\theta, t) = 0, & t > 0, \ \theta \in \mathbb{T}^d, \\ V(\theta, 0) = v_0(\theta), & \theta \in \mathbb{T}^d, \end{cases} \tag{4.60}$$

where as before $V(\cdot, t)$ is $\mathbb{Z}^d$-periodic for all $t \geq 0$, and the action of the operator on V should be understood as $(N^T\nabla_\theta, \partial_t) \cdot (N^T\nabla_\theta, \partial_t)V$. As we have seen in Section 4.3.3, the unique solution v of (4.59) (in a sense of Theorem 4.2.1) is given by

$$v(y) = v(Mz) = V(Nz', z_d), \text{ where } y = Mz \text{ with } y \in \Omega_n \text{ and } z \in \mathbb{R}^d_+. \tag{4.61}$$

In the case of Laplace operator, the solution V of (4.60) can be computed explicitly. It is easy to see that this solution is given by

$$V(\theta,t)=\sum_{\xi\in\mathbb{Z}^d}c_\xi(v_0)e^{-2\pi|N^T\xi|t}e^{2\pi i\xi\cdot\theta}, \tag{4.62}$$

where $c_\xi(v_0)$ is the ξ-th Fourier coefficient of v_0. In view of (4.62) it is clear the boundary layer tail equals $c_0(v_0)$. We will first establish a slow convergence result for V, and then using that, will prove the main result, Theorem 4.4.2. Note, that by (4.62) and Parseval's identity we have

$$||V(\theta,t)-c_0(v_0)||^2_{L^\infty(\mathbb{T}^d)}\geq||V(\theta,t)-c_0(v_0)||^2_{L^2(\mathbb{T}^d)}=\\ \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}|c_\xi(v_0)|^2e^{-4\pi|N^T\xi|t}:=\mathcal{S}(t;v_0),\ t\geq 0. \tag{4.63}$$

Theorem 4.4.3. *For any continuous, one-to-one function $g:[0,\infty)\to(0,\infty)$ decreasing to 0 at infinity, there exists a unit vector $n\notin\mathbb{RQ}^d$, a smooth function $v_0:\mathbb{T}^d\to\mathbb{R}$, and a sequence of positive numbers $t_k\nearrow\infty$, $k=1,2,...$ such that*

$$\mathcal{S}(t_k;v_0)\geq g(t_k),\qquad k=1,2,...,$$

where $\mathcal{S}$ is defined by (4.63).

As we can observe from (4.63) convergence properties depend on the quantity $|N^T\xi|$, thus we will start with the following result.

Lemma 4.4.4. *Given any continuous, one-to-one function $\omega:[0,\infty)\to(0,\infty)$ decreasing to 0 at infinity, there exists a unit vector $n\notin\mathbb{RQ}^d$, and an infinite set $\Lambda\subset\mathbb{Z}^d\setminus\{0\}$ such that*

$$|N^T\xi|\leq\omega(|\xi|),\qquad\forall\xi\in\Lambda,$$

where N is $d\times(d-1)$ matrix such that the matrix $M=[N|n]$ is orthogonal and satisfies $Me_d=n$.

Proof. Set $\xi^{(1)}:=e_1=(1,0,...,0)\in\mathbb{R}^d$ and let $\Gamma_1\subset\mathbb{S}^{d-1}$ be a neighborhood of $\xi^{(1)}$ centered at $\xi^{(1)}$ and with diameter less than $\omega^2(|\xi^{(1)}|)/(10|\xi^{(1)}|^2)$. Due to the density of rational directions there exists a nonzero $\xi^{(2)}\in\mathbb{Z}^d$ such that $|\xi^{(2)}|\geq 2$ and

$$0<\left|\frac{\xi^{(2)}}{|\xi^{(2)}|}-\frac{\xi^{(1)}}{|\xi^{(1)}|}\right|\leq\operatorname{diam}(\Gamma_1)\leq\frac{\omega^2(|\xi^{(1)}|)}{10|\xi^{(1)}|^2}.$$

Using the same idea, we then inductively construct a sequence $\xi^{(k)}\in\mathbb{Z}^d\setminus\{0\}$ such that we have $k\leq|\xi^{(k)}|<|\xi^{(k+1)}|$, and for unit vectors $r_k=\frac{\xi^{(k)}}{|\xi^{(k)}|}$ we have

$$0<|r_{k+1}-r_k|\leq\frac{\omega^2(|\xi^{(k)}|)}{10^k|\xi^{(k)}|^2} \tag{4.64}$$

for $k=1,2,...$. It is clear by (4.64) that the sequence r_k is convergent, and we let n to be its limit, which is obviously a unit vector. We claim that $n\notin\mathbb{RQ}^d$, which is due to fast convergence of the sequence[5] $\{r_k\}$. Indeed, assume it is rational, but since n has length one, it is easy to see that there exists some $\xi_0 \in \mathbb{Z}^d$ such that $n = \xi_0/|\xi_0|$. By monotonicity of $|\xi^{(k)}|$ and ω, along with (4.64) we get

$$|(n\cdot e_1)^2 - (r_k\cdot e_1)^2| \leq 2|n\cdot e_1 - r_k\cdot e_1| \leq 2|n - r_k| \leq 2\sum_{j=k}^{\infty}|r_{j+1} - r_j| \leq$$

$$2\sum_{j=k}^{\infty}\frac{\omega^2(|\xi^{(j)}|)}{10^j|\xi^{(j)}|^2} \leq \frac{20}{9}\frac{1}{10^k}\frac{\omega^2(|\xi^{(k)}|)}{|\xi^{(k)}|^2}.$$

By rationality of n we have $n\cdot e_1 = \frac{p_0}{|\xi_0|}$, with $p_0 \in \mathbb{Z}$, and for r_k we have $r_k\cdot e_1 = \frac{p_k}{|\xi^{(k)}|}$ for some $p_k \in \mathbb{Z}$. Hence, from the last inequality we get

$$\left|p_0^2|\xi^{(k)}|^2 - p_k^2|\xi_0|^2\right| \leq \frac{20}{9}\frac{1}{10^k}\omega^2(|\xi^{(k)}|)|\xi_0|^2.$$

Since the left hand side is an integer less than one by absolute value, it must be 0. From here we conclude that the sequence $|r_k\cdot e_1|$ is constant. By our notation this implies

$$\frac{p_k}{|\xi^{(k)}|} = \pm\frac{p_{k+1}}{|\xi^{(k+1)}|}, \quad k = 1, 2, ..., \tag{4.65}$$

where p_k is an integer, representing the first coordinate of $\xi^{(k)}$. Now, if we have equality in the last expression with minus sign, we get

$$\frac{1}{10^k|\xi^{(k)}|^2} \geq |r_{k+1} - r_k| \geq |r_{k+1}\cdot e_1 - r_k\cdot e_1| = \frac{2|p_k|}{|\xi^{(k)}|},$$

which implies that $p_k = 0$, and hence p_{k+1} is 0 as well by (4.65). We thus see that (4.65) in either case of the signs, forces equality within the first components of r_k and r_{k+1}. But in this case the same argument with e_1 replaced by the remaining vectors of the standard basis of $\mathbb{R}^d$, would lead to equality for all corresponding components of r_k and r_{k+1} contradicting the fact that $r_k \neq r_{k+1}$. This completes the proof that n is not a rational direction.

We now set $\Lambda = \{\xi^{(k)} : \ k = 1, 2, ...\}$, and proceed to the proof of the claimed estimate of the lemma. By orthogonality of M for any $\xi \in \Lambda$ we have

$$|\xi|^2 = |M^T\xi|^2 = |N^T\xi|^2 + |n\cdot\xi|^2,$$

from which we obtain

$$|N^T\xi|^2 = |\xi|^2 - |n\cdot\xi|^2 \leq 2|\xi|^2\left(1 - \frac{|n\cdot\xi|}{|\xi|}\right). \tag{4.66}$$

---

[5] This is in analogy with a standard fact in Diophantine approximation theory, that the sum of a very fast converging series of rationals is irrational.

Now choose $k \in \mathbb{N}$ such that $\xi = \xi^{(k)}$. We get

$$\xi \cdot n = \xi^{(k)} \cdot n = \xi^{(k)} \cdot \left[\frac{\xi^{(k)}}{|\xi^{(k)}|} + \sum_{j=k}^{\infty} (r_{j+1} - r_j) \right] = |\xi^{(k)}| + \sum_{j=k}^{\infty} \xi^{(k)} \cdot (r_{j+1} - r_j).$$

Hence by (4.64) we have

$$|\xi^{(k)} \cdot n - |\xi^{(k)}|| \leq \sum_{j=k}^{\infty} |\xi^{(k)}||r_{j+1} - r_j| \leq \sum_{j=k}^{\infty} |\xi^{(k)}| \frac{\omega^2(|\xi^{(j)}|)}{10^j |\xi^{(j)}|^2} \leq \frac{1}{2} \frac{\omega^2(|\xi^{(k)}|)}{|\xi^k|}.$$

We thus get

$$\left| 1 - \frac{|n \cdot \xi^{(k)}|}{|\xi^{(k)}|} \right| \leq \frac{\omega^2(|\xi^{(k)}|)}{2|\xi^k|^2}.$$

From here, getting back to (4.66) we obtain

$$|N^T \xi|^2 \leq \omega^2(|\xi|).$$

The proof of the lemma is now complete. □

The following remark will be used later on when proving Lemma 4.4.13.

Remark 4.4.5. *One may easily observe from the proof that given any $\tau > 1$ it is possible to construct Λ such that for any $\xi, \eta \in \Lambda$ if $|\xi| < |\eta|$ then $\tau|\xi| < |\eta|$, meaning that Λ is τ-lacunar.*

We now give a proof of Theorem 4.4.3 based on the previous lemma.

Proof of Theorem 4.4.3. Define $\omega(t) : [1, \infty) \to \mathbb{R}_+$ as follows

$$\omega(t) = \frac{1}{4\pi g^{-1} \left(\frac{1}{e} t^{-2t} \right)}, \qquad t \geq 1.$$

Here g^{-1} stands for the inverse function of g, which exists since g is assumed to be one-to-one. Moreover, without loss of generality we will assume that ω is well-defined for $t \geq 1$, since otherwise we will just replace the lower bound of t by a sufficiently large number. It is easy to see from the definition of $\omega(t)$ that it is decreasing to 0, as $t \to \infty$, and that it is one-to-one. We now apply Lemma 4.4.4 for this choice of ω, and let n be the normal, and Λ be the index set given by Lemma 4.4.4. We define $v_0 \in C^\infty(\mathbb{T}^d)$ as follows. First arrange elements of Λ in increasing order of their norms, i.e. we let $\Lambda = \{\xi^{(k)} : \ k = 1, 2, ...\}$, where $|\xi^{(k)}| < |\xi^{(k+1)}|$, for $k = 1, 2, ...$. Observe that $|\xi_k| \geq k$ for all $k \in \mathbb{N}$, due to the construction of Lemma 4.4.4. Next, if $\xi \in \Lambda$ is the k-th element of Λ according to the mentioned arrangement, set $c_{\pm\xi}(v_0) = |\xi|^{-k}$, otherwise, if $\pm\xi \notin \Lambda$ let $c_\xi(v_0) = 0$. The sequence of coefficients defined in this way gives a smooth function v_0, since the coefficients decay faster than any polynomial rate. Furthermore, since $c_{-\xi} = c_\xi \in \mathbb{R}$, v_0 is a real-valued function. By (4.63) we get

$$\mathcal{S}(t; v_0) = \sum_{k=1}^{\infty} \frac{2}{|\xi^{(k)}|^{2k}} e^{-4\pi |N^T \xi^{(k)}| t}.$$

For each $k \in \mathbb{N}$, choose t_k from the condition that

$$g(t_k) = \frac{1}{e}|\xi^{(k)}|^{-2k}, \qquad k = 1, 2, \dots . \tag{4.67}$$

By construction we have $t_k \nearrow \infty$. To prove that $\mathcal{S}(t_k; v_0) \geq g(t_k)$ it is enough to show that $e^{-4\pi|N^T\xi^{(k)}|t_k}|\xi^{(k)}|^{-2k} \geq g(t_k)$, while for this one it is enough to prove that

$$e^{-4\pi\omega(|\xi^{(k)}|)t_k}|\xi^{(k)}|^{-2k} \geq g(t_k), \tag{4.68}$$

since $|N^t\xi^{(k)}| \leq \omega(|\xi^{(k)}|)$. We now use (4.67) and definition of ω, by which (4.68) is equivalent to

$$1 \geq 4\pi\omega(|\xi^{(k)}|)t_k = 4\pi\omega(|\xi^{(k)}|)g^{-1}\left(\frac{1}{e}|\xi^{(k)}|^{-2k}\right) = \\ 4\pi\frac{1}{4\pi g^{-1}\left(\frac{1}{e}|\xi^{(k)}|^{-2|\xi^{(k)}|}\right)}g^{-1}\left(\frac{1}{e}|\xi^{(k)}|^{-2k}\right).$$

But the last expression is equivalent to

$$g^{-1}\left(\frac{1}{e}|\xi^{(k)}|^{-2|\xi^{(k)}|}\right) \geq g^{-1}\left(\frac{1}{e}|\xi^{(k)}|^{-2k}\right),$$

which holds true, since $|\xi^{(k)}| \geq k$ and g^{-1} is decreasing.

Theorem is proved.

$\square$

Remark 4.4.6. *It is clear from the proof of Theorem 4.4.3 that given any $\delta > 0$ in advance, we may drop some finite number of initial terms from $\Lambda \subset \mathbb{Z}^d$, the Fourier spectrum of v_0, ensuring that $|N^T\xi| \leq \delta$ for all $\xi \in \Lambda$.*

Remark 4.4.7. *Normals, with the property as discussed in Theorem 4.4.3, are dense on the unit sphere, however with measure zero. The latter is due to the fact that on the set of Diophantine vectors, which is of full measure, the convergence is faster than any polynomial rate. It is also clear that any prescribed uniform lower bound on the sequence $|N^T\xi|$ for nonzero $\xi \in \mathbb{Z}^d$, would transform to a certain upper bound on the speed of convergence of V to its tail.*

The following question that is not covered by our analysis, seems to be interesting, namely, is it possible to get $\mathcal{S}(t; v_0) \gtrsim g(t)$ for all $t \geq 1$ rather than on a discrete sequence ? This will most probably require deeper understanding of approximation properties of a normal n by rational directions.

Remark 4.4.8. *Although in the formulation of Theorem 4.4.3 along with the normal n we also choose the boundary data v_0, we wish to stress that the slow convergence phenomena is essentially a property of direction n and is almost independent of the choice of v_0. To be more precise, once we have chosen the index set Λ, we could have let v_0 to have spectrum outside Λ as well, since that would only increase the L^2 norm in consideration, increasing by that the lower bound of $\mathcal{S}$. Effectively, all we need here is to have v_0 with coefficients not extremely small* *on Λ, which is in some sense a typical event. A peculiar manifestation of this typicality phenomena will be discussed in Section 4.4.2.*

We now proceed to the proof of our main result of this section.

Proof of Theorem 4.4.2. Let a unit vector $n \notin \mathbb{RQ}^d$, a $\mathbb{Z}^d$-periodic function v_0, and a sequence of positive numbers $\{t_k\}_{k=1}^{\infty}$ be obtained by applying Theorem 4.4.3 for the function g given in Theorem 4.4.2. Also let V be defined by (4.62) for this choice of v_0. As we have seen in (4.61) the unique solution v to problem (4.59) is given by $v(y) = V(Nz', z_d)$, where as usual $y \in \Omega_n$, and $z = Mz$ with $M = [N|n] \in O_d(\mathbb{R})$. By orthogonality of M we have

$$y \cdot n = Mz \cdot n = z \cdot M^T n = z \cdot e_d = z_d.$$

Thus if we let $y = y' + (y \cdot n)n$, with $y' \in \partial\Omega_n$, then $Nz' = y'$ for the tangential component. We now need to derive some bounds on the new tangential variable z'. Observe that N has rank $d-1$, hence some of its $d-1$ rows are linearly independent. Set N' to be a $(d-1) \times (d-1)$ matrix formed by these $d-1$ rows of N. From the overdetermined linear system $Nz' = y'$ we have $N'z' = y''$, where $y'' \in \mathbb{R}^{d-1}$ is the corresponding part of y'. Using the assumption that $|y'| \leq R$, we get the following bound

$$|z'| = |(N')^{-1} y''| \leq c_N |y| \leq c_N R. \tag{4.69}$$

Now if $\Lambda \subset \mathbb{Z}^d$ is the Fourier spectrum of v_0, by Remark 4.4.6 we may assume that $|N^T \xi| \leq 1/(8c_N R)$, for any $\xi \in \Lambda$, from which and (4.69) one has

$$|\xi \cdot Nz'| = |N^T \xi \cdot z'| \leq \frac{1}{8c_N R} c_N R = \frac{1}{8}.$$

The last expression shows that $\cos(2\pi\xi \cdot Nz') \geq \sqrt{2}/2$ for all $\xi \in \Lambda$ and any z' satisfying (4.69). By construction of Theorem 4.4.3, Λ is symmetric with respect to the origin, $c_\xi(v_0) = c_{-\xi}(v_0)$ for any $\xi \in \Lambda$, and all nonzero coefficients of V are positive and less than one. Hence for any $y' \in \partial\Omega_n \cap B(0, R)$ we get

$$\begin{aligned} v(y' + (y \cdot n)n) = V(Nz', z_d) = \sum_{\xi \in \Lambda} c_\xi(v_0) e^{-2\pi |N^T \xi| z_d} e^{2\pi i \xi \cdot Nz'} = \\ \sum_{\xi \in \Lambda} c_\xi(v_0) e^{-2\pi |N^T \xi| z_d} \cos(2\pi\xi \cdot Nz') \geq \frac{1}{\sqrt{2}} \mathcal{S}(z_d; v_0), \end{aligned} \tag{4.70}$$

where $\mathcal{S}$ is defined by (4.63). Finally, recall that $y \cdot n = z_d$, and hence in the last inequality restricting z_d to the sequence $\{t_k\}_{k=1}^{\infty}$, and using the estimate of Theorem 4.4.3 we complete the proof of the current theorem. $\square$

It follows from the proof of Theorem 4.4.2 that we may have local uniformity for slow convergence with respect to tangential directions. By this we mean that the sequence on which the convergence is slow, once chosen for the modulus of continuity g, can be used for any R. However, in this case one should start at a very large index (depending on R) in the sequence.

4.4.2 How typical are examples of slow convergence?

This section develops the discussion of Remark 4.4.8, and tries to address the question in the title. In particular, we wish to emphasize that in a setting of Theorem 4.4.3 once we have constructed the normal n corresponding to the given speed of decrease $g(t)$, then practically any choice of the boundary data v_0 would result to solutions with slowly converging tails. In this regard we will need some notion of "rare" and "typical" on the space $C^\infty(\mathbb{T}^d)$, where v_0 lives. Clearly, one may have different measurements here such as through Baire categories, in terms of probability, or through subspaces. In this section, we will follow a probabilistic approach, which will to some extent clarify the point of Remark 4.4.8. In particular, we will show that once the normal is constructed as in Theorem 4.4.3, then one can have a way of sampling boundary data from $C^\infty(\mathbb{T}^d)$ that will almost surely result in slow convergence.

We start by construction of a probability measure on the space of $C^\infty(\mathbb{T}^d)$, which is somewhat tailored to fit the construction of Theorem 4.4.3. To proceed first observe that a smooth periodic function is uniquely determined by the sequence of its Fourier coefficients, hence we may construct a probability measure on the space of sequences indexed by $\mathbb{Z}^d$ which are Fourier coefficients of smooth periodic functions. For the purpose of our problem, we may without loss of generality consider only the subspace of $C^\infty(\mathbb{T}^d)$ consisting of functions with mean value 0, since the 0-th coefficient will correspond to boundary layer tail, which is not involved in (4.63). Set

$$\mathcal{C}^\infty := \{(c_\xi)_{\xi\in\mathbb{Z}^d\setminus\{0\}} : \ c_\xi \in \mathbb{R}, \ c_{-\xi} = c_\xi, \ \text{and} \ \lim_{|\xi|\to\infty} |\xi|^k |c_\xi| = 0 \text{ for any } k>0\}.$$

It is clear that for any $(c_\xi) \in \mathcal{C}^\infty$, the function $v(\theta) = \sum_{\xi\neq 0} c_\xi e^{2\pi i\xi\cdot\theta}$ defines a smooth and $\mathbb{Z}^d$-periodic function with mean value 0. Also, due to condition $c_{-\xi} = c_\xi$ we get a real-valued function[6]. On the other hand the sequence of coefficients of any smooth and $\mathbb{Z}^d$-periodic function with mean value 0 is from $\mathcal{C}^\infty$, thus we get a natural correspondence between elements of $\mathcal{C}^\infty$ and functions we are interested in[7].

The idea now is to have a way to generate a random element of $\mathcal{C}^\infty$, and hence an element of $C^\infty(\mathbb{T}^d)$. The structure of $\mathcal{C}^\infty$ hints that we may choose each component of the element of $\mathcal{C}^\infty$ separately and then combine our choices to get an element of $\mathcal{C}^\infty$. Now, for all nonzero $\xi\in\mathbb{Z}^d$ we choose a family of probability density functions $\{\rho_\xi\}$ satisfying the following properties:

(1) each $\rho_\xi : \mathbb{R}\to\mathbb{R}_+$ is even and smooth,

---

[6] Observe that if v is smooth and has expansion $v(\theta) = \sum_{\xi\in\mathbb{Z}^d} c_\xi e^{2\pi i\xi\cdot\theta}$, where $\theta\in\mathbb{T}^d$, then v is real-valued if and only if $c_{-\xi} = \overline{c_\xi}$, for all $\xi\in\mathbb{Z}^d$. Since in $\mathcal{C}^\infty$ we choose coefficients to be real, we need to impose the symmetry condition $c_{-\xi} = c_\xi$ to get a real-valued function. Of course this will not cover entirely the space $C^\infty(\mathbb{T}^d)$, however it is not essential for our purposes, since one may easily modify the analysis, to allow complex coefficients. To keep the discussion more conceptual and less technical, we restrict ourselves to the case of real coefficients only.

[7] It should be noted that the set $\mathbb{Z}^d$ does not have a natural ordering, thus we might enumerate sequences in $\mathcal{C}^\infty$ in various ways. However, due to absolute convergence of Fourier series of smooth functions, these rearrangements does not affect the sum.

(2) for any ξ we have $\int\limits_{|t|\leq|\xi|^{-|\xi|}} \rho_\xi(t)dt = |\xi|^{-|\xi|}$,

(3) $\sum\limits_{\xi\neq 0}\int\limits_{t>10|\xi|^{-|\xi|}} \rho_\xi(t)dt < \infty$.

The second condition shows that it is unlikely that ξ-th coefficient is extremely small. This is in some sense natural, since for any fixed $\xi \in \mathbb{Z}^d$ the set of functions from $C^\infty(\mathbb{T}^d)$ that have its ξ-th coefficient equal to 0, forms a proper subspace. The third one is imposed to get an element of $\mathcal{C}^\infty$ at the end, since we need a decay of coefficients faster than any polynomial rate. The reason for the sum in (3) to be finite, is that it implies through Borel-Cantelli argument that chances of infinitely many of the coefficients being larger than $10|\xi|^{-|\xi|}$ are zero, ensuring the desired decay of coefficients. We remark that this condition can of course be relaxed, say one can have a very slowly varying (but growing to infinity) powers of $|\xi|^{-1}$, but still keep the condition involving the sum.

For $\xi \in \mathbb{Z}^d\setminus\{0\}$ we let $\mathbb{P}_\xi$ be a probability measure on $\mathbb{R}$ with density function ρ_ξ as above, and consider the probability space $X_\xi = (\mathbb{R}, \mathcal{B}, \mathbb{P}_\xi)$, where $\mathcal{B}$ is the family of Borel sets on $\mathbb{R}$. Here X_ξ will be the sampling space for the ξ-th Fourier coefficient. We also keep in mind that the spaces X_ξ and $X_{-\xi}$ are identified, due to the symmetry condition in the definition of $\mathcal{C}^\infty$, thus if c_ξ is chosen, we let $c_{-\xi}$ be the same. Next, we consider the product space

$$X := (\mathbb{R}^{\mathbb{Z}^d\setminus\{0\}}, \mathcal{F}, \mathbb{P}) = \prod_{\xi\in\mathbb{Z}^d\setminus\{0\}} X_\xi,$$

where $\mathcal{F}$ is the natural product σ-algebra, and $\mathbb{P}$ is the product probability measure generated from $\mathbb{P}_\xi$-s. We think of an element of X as a sequence of Fourier coefficients of a periodic function on $\mathbb{T}^d$. The first thing we need to check, is that almost surely we get a smooth function sampling by the measure $\mathbb{P}$.

Claim 4.4.9. *Consider the set of all smooth random functions*

$$E_\infty := \{\mathbf{c} = (c_\xi)_{\xi\in\mathbb{Z}^d\setminus\{0\}} \in X : \text{ s.t. if } v(\theta) = \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} c_\xi e^{2\pi i\xi\cdot\theta}, \text{ then } v \in C^\infty(\mathbb{T}^d)\},$$

then E_∞ *is* $\mathbb{P}$*-measurable and* $\mathbb{P}(E_\infty) = 1$.

Proof. We will show that the complement of E_∞ is a set of measure 0 in X. For nonzero $\xi \in \mathbb{Z}^d$ define

$$E_\xi = \{c_\xi \in \mathbb{R} : |c_\xi| \leq 10|\xi|^{-|\xi|}\},$$

it is clear that $E_\xi \in \mathcal{B}$. Observe that if $\mathbf{c} \in X \setminus E_\infty$, then necessarily infinitely many components of $\mathbf{c}$ belong to the complement of their corresponding E_ξ, since if each, except possible finite number of components of $\mathbf{c}$ are from the corresponding E_ξ, then summing them up as in E_∞ we will get a smooth function due to very fast decay of the coefficients. Let us enumerate $\mathbb{Z}^d \setminus \{0\} = \{\xi^{(k)} :$

$k \in \mathbb{N}\}$, then we have

$$X \setminus E_\infty \subset \limsup_{k\to\infty}(\mathbb{R}\setminus E_{\xi^{(k)}}) = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{c_{\xi^{(n)}} \in \mathbb{R} : \ |c_{\xi^{(n)}}| > 10|\xi^{(n)}|^{-|\xi^{(n)}|}\}. \quad (4.71)$$

We note that each individual set on the right hand side of (4.71) should be identified with its pullback by projections to the product space X, and hence is measurable in X. By this convention since we have a countable number of measurable sets in (4.71), we get that the right hand side of (4.71) is $\mathbb{P}$-measurable and hence so is $X \setminus E_\infty$ as a subset of a measurable set. It is left to show that $\mathbb{P}(X \setminus E_\infty) = 0$, but this follows by (4.71), the property (3) of densities ρ_ξ and Borel-Cantelli lemma.

The proof is now complete. □

At this stage we know that almost surely the elements of X generate smooth, real-valued functions on $\mathbb{T}^d$. What we need now is to estimate the probability that a randomly constructed smooth function will have a slow converging boundary layer tail. We let g be as in Lemma 4.4.4, and assume n is the normal, and $\Lambda \subset \mathbb{Z}^d$ is the index set constructed in Lemma 4.4.4 corresponding to g. It easily follows from the construction of Lemma 4.4.4 and proof of Theorem 4.4.3 that if $v \in C^\infty(\mathbb{T}^d)$ has the property that $|c_\xi(v)| \geq |\xi|^{-|\xi|}$ for infinitely many (not necessarily all) $\xi \in \Lambda$, then the solution to boundary layer system generated by v as boundary data, converges to its tail slower than g. We thus need to consider the following event

$$E_{slow} = \{\mathbf{c} = (c_\xi)_{\xi\in\mathbb{Z}^d\setminus\{0\}} \in X : \ |c_\xi| \geq |\xi|^{-|\xi|}, \ \forall \xi \in \Lambda_0, \ \text{for some infinte } \Lambda_0 \subset \Lambda\}.$$

It is clear that if $\mathbf{c} \in E_{slow}$ then the boundary data which has coefficient sequence $\mathbf{c}$, satisfies Theorem 4.4.3 for the normal n, in other words generates a slowly converging tail. We have the following result.

Lemma 4.4.10. *The set E_{slow} has full $\mathbb{P}$ measure.*

Proof. Observe that the complement of E_{slow} consists of all sequences having all elements, except possibly finite number of them, less than $|\xi|^{-|\xi|}$ for the corresponding ξ. Now let $\Lambda = \{\xi^{(k)} : \ k = 1, 2, ...\}$, and for $k = 1, 2, ...$ set

$$E^k_{slow} = \{\mathbf{c} = (c_\xi)_{\xi\in\mathbb{Z}^d\setminus\{0\}} \in X : \ |c_\xi| < |\xi|^{-|\xi|}, \ \forall \xi \in \Lambda \setminus \{\xi^{(1)}, ..., \xi^{(k)}\}\}.$$

That each E^k_{slow} is measurable can be argued along the lines of the proof of Claim 4.4.9. Now observe that

$$X \setminus E_{slow} = \bigcup_{k=1}^{\infty} E^k_{slow},$$

from which we have that E_{slow} is measurable. The sets E^k_{slow} are increasing, however each of them has measure 0 due to condition (1) of the probability densities ρ_ξ. The proof is now complete. □

What we obtain from the analysis above, is the existence of somewhat nice probability measures with respect to which almost all smooth functions, when chosen as boundary data in the context of Theorem 4.4.3, generate solutions which converge slowly far away from the boundaries. This probabilistic viewpoint, although artificial on its own, suggests anyway that the slow convergence phenomena is a genuine property of the direction, and is essentially independent of the choice of the boundary data.

4.4.3 The case of variable coefficients

In Section 4.4.1 we have discussed the slow convergence phenomenon for Laplace operator. Our argument used in a crucial way the existence of explicit forms for the solutions to boundary layer systems (see (4.62)). However, for variable coefficient case one does not possess explicit representation for the solutions, which necessitates a rather different approach. The main aim of this section is to show that the slow converge of boundary layer tail also holds with variable coefficient operators, albeit we will need some extra structural condition on the coefficients.

Keeping the notation of the current chapter, consider the following boundary layer problem for scalar equation

$$\begin{cases} -\nabla \cdot A(y)\nabla v(y) = 0, & y \in \Omega_n, \\ v(y) = v_0(y), & y \in \partial\Omega_n. \end{cases} \tag{4.72}$$

The structural restriction we impose on A is as follows:

$$\text{there exists } \ 1 \leq \gamma \leq d \ \text{ such that } \ \partial_{y_\alpha} A^{\gamma\alpha} = 0. \tag{4.73}$$

Observe, that this condition, which in particular covers the case of constant coefficient operators, have also appeared in a stronger form in Section 2.3 concerning L^p-estimates for homogenization of the simultaneously oscillating problem. The following is the main result of the current section.

Theorem 4.4.11. *Let $g : [0,\infty) \to (0,\infty)$ be continuous, one-to-one function, decreasing to 0 at infinity, and let $R > 0$ be fixed. Then, there exists a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$, a smooth function $v_0 : \mathbb{T}^d \to \mathbb{R}$, and a sequence of positive numbers $\{\lambda_k\}_{k=1}^{\infty}$ growing to infinity, such that if v solves (4.72) under condition (4.73) on A, and with n and v_0 as specified here, then for any $k = 1, 2, \dots$ and any $y' \in \partial\Omega_n \cap B(0,R)$ one has*

$$|v(y' + \lambda_k n) - v^\infty| \geq g(\lambda_k),$$

where v^∞ is the corresponding boundary layer tail.

As will be seen from the proof of this theorem, the result remains valid under slightly weaker requirement on A, however, that condition is more technical to state. Also, it is clear that Theorem 4.4.11 is much stronger than Theorem 4.4.2 proved above. Nonetheless the proofs of these two results use different ideas, and

Theorem 4.4.2 is relatively more self-contained and more transparent. This is the reason we have separated these two cases. The proof of Theorem 4.4.11 uses the representation formula (4.11) for boundary layer tails and constructs directions along which the averaging of integrable functions is arbitrarily slow. We start with some preliminary setup.

Recall that by A^* we denote the coefficient matrix of the transposed operator, i.e. $A^{*,\alpha\beta} = A^{\beta\alpha}$, for $1 \le \gamma \le d$, we let $\chi^{*,\gamma}$ be the solution to the cell-problem (4.10) for the transposed operator, and by $v^{*,\gamma}$ we denote the solution to the boundary layer system for the triple (coefficients, domain, boundary data) given by $(A^*, \Omega_n, -\chi^{*,\gamma})$ as was defined in (4.9).

For a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$, and a smooth function v_0 let v be the solution to (4.72). Set $\lambda := y \cdot n$ for $y \in \Omega_n$, and let $M \in O_d(\mathbb{R})$ be so that $Me_d = n$. Then for any $0 < \kappa < 1/(2d)$ we have

$$\begin{aligned} v(y) = \int_{\partial\mathbb{R}^d_+} \partial_{2,\alpha} G^n(n, Mz) \times \Big[& A^{\beta\alpha}(\lambda Mz) n_\beta + \\ & A^{\beta\gamma}(\lambda Mz) \left(\partial_{y_\beta} \chi^{*,\alpha}(\lambda Mz) + \partial_{y_\beta} v^{*,\alpha}(\lambda Mz) \right) n_\gamma \Big] v_0(\lambda Mz) d\sigma(z) + \\ & O(\lambda^{-\kappa}), \end{aligned} \tag{4.74}$$

where G^n is the Green's kernel for the pair (A^0, Ω_n), $\partial_{2,\beta}$ denotes differentiation with respect to the β-th coordinate of the second variable of G^n, and the error term $O(\lambda^{-\kappa})$ is locally uniform in tangential variable $y' := y - (y \cdot n)n$. The asymptotic formula (4.74) is proved in Prange [54] Section 6, for systems of equations. Here, since we are working with scalar equations, we have a slightly simpler form of it. Note that taking the limit as λ goes to infinity, implies the formula for the boundary layer tail discussed in (4.11).

We are going to switch from the differentiation in y to z-variable. Since $y = Mz$, it is easy to see that $\nabla_y = M \nabla_z$. Set $\boldsymbol{\chi}^{*,\alpha}(z) := \chi^{*,\alpha}(Mz)$, and $\mathbf{v}^{*,\alpha}(z) := v^{*,\alpha}(Mz)$. On the boundary of $\mathbb{R}^d_+$, for each $1 \le \alpha \le d$, we have $\boldsymbol{\chi}^{*,\alpha} + \mathbf{v}^{*,\alpha} = 0$ by the definition, hence taking into account the connection between y and z variables, and the fact that M has n as its last column, we obtain

$$\partial_{y_\beta}(\chi^{*,\beta} + v^{*,\beta})(y) = n_\beta \partial_{z_d}(\boldsymbol{\chi}^{*,\beta} + \mathbf{v}^{*,\beta})(z).$$

Next, we handle G^n as we did in (4.20) of Section 4.3.2, namely, if $\partial\Omega_n \ni y = Mz$, with $z \in \partial\mathbb{R}^d_+$ then

$$\partial_{y_\alpha} G^n(n, y) = \partial_{2,\alpha} G^n(n, Mz) = n_\alpha \partial_{z_d} G^{0,n}(e_d, z), \tag{4.75}$$

where $G^{0,n}$ is the Green's kernel for the pair $(M^T A^0 M, \mathbb{R}^d_+)$. Applying these observations in (4.74) we obtain

$$v(y) = \int\limits_{\mathbb{R}^{d-1}} \partial_{z_d} G^{0,n}(e_d,(z',0)) \times \Big[A^{\beta\alpha}(\lambda M(z',0)) n_\alpha n_\beta + A^{\beta\gamma}(\lambda M(z',0)) \times$$
$$n_\alpha n_\beta \left(\partial_{z_d} \boldsymbol{\chi}^{*,\alpha}(\lambda(z',0)) + \partial_{z_\beta} \mathbf{v}^{*,\alpha}(\lambda(z',0))\right) n_\gamma \Big] \times v_0(\lambda M(z',0)) dz' +$$
$$O(\lambda^{-\kappa}) = \int\limits_{\mathbb{R}^{d-1}} \partial_{z_d} G^{0,n}(e_d,(z',0)) \times n^T A(\lambda M(z',0)) n \times$$
$$\Big[1 + n_\alpha \big(\partial_{z_d} \boldsymbol{\chi}^{*,\alpha}(\lambda(z',0)) + \partial_{z_d} \mathbf{v}^{*,\alpha}(\lambda(z',0))\big) \Big] \times v_0(\lambda M(z',0)) dz' + O(\lambda^{-\kappa}). \tag{4.76}$$

In the next statement we collect the necessary information concerning the Green's kernel involved in (4.76).

Lemma 4.4.12. *For a vector $n \in \mathbb{S}^{d-1}$ let $G^{0,n}$ be as in (4.76). Then*

(i) *the value $z \mapsto \partial_{z_d} G^{0,n}(e_d, z)$, where $z \in \partial\mathbb{R}^d_+$, is independent of the matrix $M \in O_d(\mathbb{R})$.*

(ii) *$\partial_{z_d} G^{0,n}(e_d,(z',0)) \in L^1(\mathbb{R}^{d-1})$ and $\int_{\mathbb{R}^{d-1}} \partial_{z_d} G^{0,n}(e_d,(z',0)) dz' \neq 0$.*

(iii) *There exists an absolute constant C such that for any $\nu \in \mathbb{S}^{d-1}$ one has*

$$\int_{\mathbb{R}^{d-1}} \left| \partial_{z_d} G^{0,n}(e_d,(z',0)) - \partial_{z_d} G^{0,\nu}(e_d,(z',0)) \right| dz' \leq C|n-\nu|.$$

Proof. By our notation, $G^{0,n}(z,\widetilde{z})$ is the Green's kernel for the pair $(M^T A^0 M, \mathbb{R}^d_+)$, hence there is an implicit dependence on M. Fix some $1 \leq \alpha \leq d$ so that $n_\alpha \neq 0$. Then using (4.75) for this particular α implies (i).

The first assertion of (ii) is due to (4.21). For the second statement we will argue as in Remark 4.3.4. Set $P^{0,n}(z,\widetilde{z})$ to be the Poisson kernel for the pair $(M^T A^0 M, \mathbb{R}^d_+)$. Then for $z \in \mathbb{R}^d_+$ we have

$$\begin{aligned} P^{0,n}(z,\widetilde{z}) &= -e_d^T (M^T A^0 M) \nabla_{\widetilde{z}} G^{0,n}(z,\widetilde{z}) \\ &= -(M e_d)^T A^0 (M e_d) \partial_{\widetilde{z}_d} G^{0,n}(z,\widetilde{z}) \\ &= -n^T A^0 n \partial_{\widetilde{z}_d} G^{0,n}(z,\widetilde{z}). \end{aligned}$$

Since $\int_{\partial\mathbb{R}^d_+} P^{0,n}(e_d,\widetilde{z}) d\widetilde{z} = 1$, the last expression, combined with the ellipticity of A^0 completes the proof of the second claim of (i). Finally, (iii) follows by (4.27). The proof of the lemma is complete. $\square$

We now turn to the discussion of the core of averaging process of (4.76), which is of type discussed in Lemma 4.3.1. Our next result is the key step towards the proof of the main result of this section.

Lemma 4.4.13. *Let $\Xi \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a compact set, and assume we are given a family of functions $\mathfrak{F} = \{F_i\}_{i\in\Xi}$, with the following properties:*

(a) *for any $F \in \mathfrak{F}$ we have $F \in L^1(\mathbb{R}^{d-1})$ and $\int_{\mathbb{R}^{d-1}} F(x)dx \neq 0$,*

(b) *there is a universal constant C such that for any $i, j \in \Xi$ we have*

$$\int\limits_{\mathbb{R}^{d-1}} |F_i(x) - F_j(x)|dx \leq C|i-j|.$$

Let also $g(t) : [0,\infty) \to (0,\infty)$ be continuous one-to-one function decreasing to 0 as $t \to \infty$. Then, there exists a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$, an unbounded, and strictly increasing sequence of positive numbers $\{\lambda_k\}_{k=1}^{\infty}$ with the following properties:

for any $F \in \mathfrak{F}$ there is a function $v_0 \in C^\infty(\mathbb{T}^d)$ such that for any $M \in O_d(\mathbb{R})$ satisfying $Me_d = n$ one has

$$\left| \int\limits_{\mathbb{R}^{d-1}} F(x)v_0(\lambda_k M(x,0))dx - c_0(v_0) \int\limits_{\mathbb{R}^{d-1}} F(x)dx \right| \geq g(\lambda_k), \quad k = 1, 2, \dots .$$

Remark 4.4.14. *The lemma shows that the direction, and the sequence along which convergence is slow can be chosen uniformly for the entire family $\mathfrak{F}$. Moreover, as will be seen from the proof of the Lemma for any F and G from $\mathfrak{F}$ the corresponding functions $v_0(F)$ and $v_0(G)$ have identical Fourier spectra, and the corresponding Fourier coefficients of these two functions are equal up to the sign.*

Also, it will be clear from the proof and Remark 4.4.7 that normals satisfying Lemma 4.4.13 are everywhere dense on $\mathbb{S}^{d-1}$.

Let us also note that the index set Ξ can be any compact set in any topological space, and not necessarily from some $\mathbb{R}^k$; also dimensions k and d are independent of each other.

Proof of Lemma 4.4.13. For $F \in \mathfrak{F}$ let $\mathfrak{I}_F$ be the absolute value of the integral of F over $\mathbb{R}^{d-1}$, and set $\tau_0 := \inf\limits_{F\in\mathfrak{F}} \mathfrak{I}_F$. It follows from conditions (a) and (b) of the current lemma that the function $i \mapsto \mathfrak{I}_{F_i}$ is positive and continuous on Ξ, and since Ξ is compact, we get that $\tau_0 > 0$. We now show that

$$\sup_{F\in\mathfrak{F}} \int\limits_{|x|\geq A} |F(x)|dx \to 0 \qquad \text{as} \qquad A \to \infty. \tag{4.77}$$

Fix some $\varepsilon > 0$, by compactness of Ξ and condition (b) of the Lemma there is a finite $\frac{\varepsilon}{2}$-net in $\mathfrak{F}$, that is a set of functions F_1,...,F_{k_ε} from $\mathfrak{F}$ such that for any $F \in \mathfrak{F}$ there is $1 \leq j \leq k_\varepsilon$ satisfying

$$\int\limits_{\mathbb{R}^{d-1}} |F_j(x) - F(x)|dx \leq \frac{\varepsilon}{2}. \tag{4.78}$$

We now choose $A_0 > 0$ large enough so that for any $A \geq A_0$ and all $1 \leq j \leq k_\varepsilon$ one has $\int_{|x|\geq A} |F_j(x)|dx \leq \varepsilon/2$. This, combined with (4.78) and the triangle inequality implies (4.77). It is now easy to see that by (4.77) and the fact that $\tau_0 > 0$ we may choose $A_0 > 0$ large such that

$$\left| \int\limits_{|x|\leq A_0} F(x)dx \right| \geq 2 \int\limits_{|x|\geq A_0} |F(x)|dx + \frac{1}{2}\mathfrak{I}_F, \tag{4.79}$$

for any $F \in \mathfrak{F}$. For this choice of A_0, we denote $\varepsilon_F := \frac{1}{||F||_{L^1(\mathbb{R}^{d-1})}} \left| \int_{|x|\leq A_0} F(x)dx \right|$, where $F \in \mathfrak{F}$. In a similar vein as we did above, using that $\tau_0 > 0$, (4.77), and condition (b) we easily see that

$$0 < \varepsilon_0 := \inf_{F\in\mathfrak{F}} \varepsilon_F \leq 1. \tag{4.80}$$

We now fix some small constant $\delta_0 > 0$ such that

$$|\cos(t) - 1| \leq \varepsilon_0/4 \ \text{ for any } t \in \mathbb{R} \text{ with } |t| \leq \delta_0. \tag{4.81}$$

Assume that $n \in \mathbb{S}^{d-1}$ and let $M \in O_d(\mathbb{R})$ be so that $Me_d = n$. We then have $M = [N|n]$ where N is the matrix formed from the first $(d-1)$-columns of M. Observe that for any $\xi \in \mathbb{Z}^d$ and any $x \in \mathbb{R}^{d-1}$ we have $\xi \cdot M(x,0) = x^T N^T \xi$. Therefore if for some $\lambda > 0$ we have $2\pi\lambda A_0 |N^T\xi| \leq \delta_0$ then (4.79) and (4.81) imply

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^{d-1}} F(x)\cos[2\pi\lambda\xi \cdot M(x,0)]dx \right| \geq \left| \int\limits_{|x|\leq A_0} F(x)dx \right| - \\ \int\limits_{|x|\leq A_0} |F(x)| \times |\cos[2\pi\lambda x^T N^T \xi] - 1|dx - \int\limits_{|x|>A_0} |F(x)|dx \geq \frac{3}{8}\mathfrak{I}_F, \end{aligned} \tag{4.82}$$

for any $F \in \mathfrak{F}$. Now define

$$\omega(t) := \frac{\delta_0}{2\pi A_0} \frac{1}{g^{-1}\left(\frac{3}{8}\tau_0 \frac{1}{t^t}\right)}, \qquad t \geq 1,$$

where g^{-1} stands for the inverse function of g. Obviously ω is one-to-one, continuous, and decreasing to 0 as $t \to \infty$. It is also clear that ω is well-defined for large enough t, thus without loss of generality we will assume that ω is defined for all $t \geq 1$. Applying Lemma 4.4.4 for ω we obtain $\Lambda \subset \mathbb{Z}^d$, and a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$ such that if $M \in O_d(R)$ is any matrix satisfying $Me_d = n$, we get

$$|N^T\xi| \leq \omega(|\xi|), \qquad \forall \xi \in \Lambda,$$

where $d \times (d-1)$ matrix N is formed from the first $(d-1)$-columns of M. We arrange elements of Λ in increasing order of their norms, thus $\Lambda = \{\xi^{(k)} : k = 1, 2, ...\}$, where by construction we have $k \leq |\xi^{(k)}| < |\xi^{(k+1)}|$ for any $k \geq 1$. Moreover, according to Remark 4.4.5 we may also assume that for any $k \in \mathbb{N}$ we have

$$|\xi^{(k)}| < \varrho|\xi^{(k+1)}|, \tag{4.83}$$

where $0<\varrho<1$ satisfies

$$2\sup_{F\in\mathcal{F}}||F||_{L^1(\mathbb{R}^{d-1})}\frac{\varrho}{1-\varrho} < \frac{3}{8}\tau_0. \tag{4.84}$$

Set

$$\lambda_k := g^{-1}\left(\frac{3}{8}\tau_0\frac{1}{|\xi^{(k)}|^k}\right), \qquad k=1,2,\dots\,. \tag{4.85}$$

It is clear that λ_k is unbounded and is strictly increasing. Note that n, and the sequence $\{\lambda_k\}$ are uniform for the entire family $\mathcal{F}$. We proceed to construction of the function $v_0\in C^\infty(\mathbb{T}^d)$ for the given $F\in\mathcal{F}$, for which it is enough to construct the sequence of Fourier coefficients of v_0, which we will denote by $\{c_\xi(v_0)\}_{\xi\in\mathbb{Z}^d}$.

Let $F\in\mathcal{F}$ be fixed. For $\xi\in\mathbb{Z}^d$ if we have $\xi\in\Lambda$ then set $c_\xi(v_0)=c_{-\xi}(v_0)=\varepsilon_k(F)|\xi|^{-k}$, where $k\in\mathbb{N}$ is the index of ξ in Λ according to the increasing rearrangement made above, and $\varepsilon_k(F)\in\{-1,1\}$ which will be chosen below. It is important to note that this sign is the same for c_ξ and $c_{-\xi}$. Otherwise, if $\pm\xi\notin\Lambda$ we let $c_\xi(v_0)=0$. It is clear that the sequence $\{c_\xi\}$ decays faster than any polynomial rate in $|\xi|$, hence v_0 is smooth. Also, since $c_\xi(v_0)=c_{-\xi}(v_0)$ for any $\xi\in\mathbb{Z}^d$ we have that v_0 is real-valued. Observe that $c_0(v_0)=0$ by construction, and expanding v_0 into Fourier series we get

$$\int_{\mathbb{R}^{d-1}}F(x)v_0(\lambda M(x,0))dx = \sum_{m=1}^{\infty}\frac{2\varepsilon_m(F)}{|\xi^{(m)}|^m}\int_{\mathbb{R}^{d-1}}F(x)\cos\left(2\pi\lambda x^TN^T\xi^{(m)}\right)dx := \frac{2\varepsilon_k(F)}{|\xi^{(k)}|^k}\mathcal{I}_k(\lambda)+\Sigma_1(\lambda)+\Sigma_2(\lambda), \tag{4.86}$$

where $k\geq 1$, $\mathcal{I}_k(\lambda):=\int_{\mathbb{R}^{d-1}}F(x)\cos(2\pi\lambda x^TN^T\xi^{(k)})dx$, $\Sigma_1(\lambda)$ contains the part of sum where $m<k$ and $\Sigma_2(\lambda)$ respectively sums over the range $m>k$. Note that in view of our construction the sums $\Sigma_i(\lambda)$, $i=1,2$ are real-valued for any λ. By the definition of λ_k, the fact that $|\xi^{(k)}|\geq k$ and that g is decreasing we easily see that $2\pi\lambda_kA_0|N^T\xi^{(k)}|\leq\delta_0$ for any k, hence applying (4.82) we obtain

$$\frac{2}{|\xi^{(k)}|^k}|\mathcal{I}_k(\lambda_k)| \geq \frac{3}{4}\mathcal{I}_F\frac{1}{|\xi^{(k)}|^k}. \tag{4.87}$$

On the other hand, by (4.83) and (4.84) we easily get

$$|\Sigma_2(\lambda)| \leq 2||F||_{L^1(\mathbb{R}^{d-1})}\sum_{m=k+1}^{\infty}\frac{1}{|\xi^{(m)}|^m} \leq \frac{3}{8}\frac{1}{|\xi^{(k)}|^k}\tau_0, \tag{4.88}$$

for any $\lambda\geq 1$. We now estimate the contribution of the range $m<k$. The triangle inequality implies

$$\left|\Sigma_1(\lambda_k)+\frac{2}{|\xi^{(k)}|^k}\mathcal{I}_k(\lambda_k)\right| + \left|\Sigma_1(\lambda_k)-\frac{2}{|\xi^{(k)}|^k}\mathcal{I}_k(\lambda_k)\right| \geq \frac{4}{|\xi^{(k)}|^k}|\mathcal{I}_k(\lambda_k)|,$$

hence at least one of the terms in l.h.s. of the last inequality is not less than half of the r.h.s., and we choose the sign ε_k so that to get the largest term from the l.h.s. of the above inequality. This choice of ε_k, combined with estimates (4.87) and (4.88), and the definition of λ_k given by (4.85) imply

$$\left| \frac{2\varepsilon_k(F)}{|\xi^{(k)}|^k} \mathcal{I}_k(\lambda_k) + \Sigma_1(\lambda_k) + \Sigma_2(\lambda_k) \right| \geq \frac{3}{8} \mathcal{I}_F \frac{1}{|\xi^{(k)}|^k} \geq \frac{3}{8} \tau_0 \frac{1}{|\xi^{(k)}|^k} = g(\lambda_k), \tag{4.89}$$

for any $k = 1, 2, \ldots$. The statement of the Lemma obviously follows from the last inequality and (4.86). The proof is complete. □

We are now ready to prove the main result of this section, which will easily follow from the representation (4.76) and Lemmas 4.4.12 and 4.4.13.

Proof of Theorem 4.4.11. Without loss of generality we will assume that $tg(t) \to \infty$ as $t \to \infty$. The reason for this is to have slower speed of decay than the error term involved in (4.76).

Recall that for $n \in \mathbb{S}^{d-1}$ we let $G^{0,n}(\cdot, \cdot)$ be the Green's kernel for the pair $(M^T A^0 M, \mathbb{R}^d_+)$, where as is customary $M \in O_d(\mathbb{R})$ satisfying $Me_d = n$. Consider the family of functions $\mathcal{F} := \{F_n\}_{n \in \mathbb{S}^{d-1}}$, where we set $F_n(x) := \partial_d G^{0,n}(e_d, (x, 0))$ for $x \in \mathbb{R}^{d-1}$. Again, there is an implicit dependence of F_n on M, however, as Lemma 4.4.12 (i) shows, each function F_n is in fact independent of the choice of M. Next, by Lemma 4.4.12 we have that $\mathcal{F}$ satisfies conditions (a) and (b) of Lemma 4.4.13.

Now let $1 \leq \gamma \leq d$ be fixed from (4.73). We thus get $\chi^{*,\gamma} = 0$ from the definition of the cell-problem (4.10), and hence for the corresponding boundary layer corrector, by (4.9) we have[8] $v^{*,\gamma} = 0$ for any $n \in \mathbb{S}^{d-1}$. For any $1 \leq \alpha \leq d$, and any $n \in \mathbb{S}^{d-1}$, $v^{*,\alpha}$ solves a uniformly elliptic PDE in Ω_n with boundary data $\chi^{*,\alpha}$, hence standard elliptic regularity implies that there is a constant C_0 independent of n, such that $|\nabla_y v^{*,\alpha}(y)| \leq C_0$ for any $y \in \partial\Omega_n$, and all $1 \leq \alpha \leq d$. From this, and the fact that $v^{*,\gamma} = 0$ it follows that there exists an open subset of the sphere $\mathbb{S}_\gamma \subset \mathbb{S}^{d-1}$ such that for any $n \in \mathbb{S}_\gamma$ and any $M \in O_d(\mathbb{R})$ with $Me_d = n$, one has

$$\left| 1 + n_\alpha \left[\partial_{z_d} \boldsymbol{\chi}^{*,\alpha}(\lambda(z', 0)) + \partial_{z_d} \mathbf{v}^{*,\alpha}(\lambda(z', 0)) \right] \right| \geq \frac{1}{2}, \tag{4.90}$$

where $z' \in \mathbb{R}^{d-1}$ and $\lambda > 0$. Concretely, we choose $\mathbb{S}_\gamma$ so that any $n \in \mathbb{S}_\gamma$ has its γ-th component sufficiently close to 1.

Recall that solutions to boundary layer problems are constructed via the reduced boundary layer systems of form (4.57), hence we have

$$v^{*,\alpha}(y) = v^{*,\alpha}(Mz) = \mathbf{v}^{*,\alpha}(z) = V^\alpha(Nz', z_d) \tag{4.91}$$

---

[8] It should be noted that $v^{*,\alpha}$, the boundary layer corrector, depend on normal direction n through halfspace Ω_n, for all $1 \leq \alpha \leq d$. However, for the ease of notation we do not incorporate this dependence into the definition of $v^{*,\alpha}$. Whenever this dependence will play some role, we will make an appropriate comment on that.

where V^α solves the corresponding problem (4.57). In particular we have that each $V^\alpha(\cdot, t)$ is $\mathbb{Z}^d$-periodic for any $t \geq 0$, and is smooth with respect to all variables. Here as well, one should take into account the subtlety, that V^α, and hence also $\mathbf{v}^{*,\alpha}$, implicitly depend on the matrix M. To keep track of this dependence we take any two matrices $M_0 = [N_0|n]$, $M = [N|n] \in O_d(\mathbb{R})$, and in the next diagram we collect the corresponding solutions that these two matrices generate.

$$\begin{tikzcd}
y \arrow[d] & = & M_0\widetilde{z} \arrow[d] & = & Mz \arrow[d] & \\
v^{*,\alpha}(y) & = & v_0^{*,\alpha}(M_0\widetilde{z}) \arrow[d, equal] & = & v^{*,\alpha}(Mz) \arrow[d, equal] & \\
V_0^\alpha(N_0\widetilde{z}', \widetilde{z}_d) & = & \mathbf{v}_0^{*,\alpha}(\widetilde{z}) & = & \mathbf{v}^{*,\alpha}(z) & = V^\alpha(Nz', z_d).
\end{tikzcd} \tag{4.92}$$

It is clear from the first row that $z_d = \widetilde{z}_d$, and hence $N_0\widetilde{z}' = N_1 z'$. For $n \in \mathbb{S}^{d-1}$ consider the function

$$\Psi_n(y) := 1 + n_\alpha\big(n \cdot \nabla_y \chi^\alpha(y) + \partial_t V_0^\alpha(y, 0)\big), \qquad y \in \mathbb{R}^d,$$

where V_0^α is fixed from (4.92). Clearly, $\Psi_n \in C^\infty(\mathbb{T}^d)$. Recall that when switching from y to z we had $\nabla_y = M\nabla_z$, which implies $\partial_{z_d} = n \cdot \nabla_y$ giving the relation between normal derivatives. Now, if $y \in \partial\Omega_n$, we get

$$\partial_t V_0^\alpha(Nz', 0) = \partial_t V_0^\alpha(N_0\widetilde{z}', 0) = \partial_{\widetilde{z}_d}\mathbf{v}_0^{*,\alpha}(\widetilde{z}', 0) = n \cdot \nabla_y v^{*,\alpha}(y). \tag{4.93}$$

From here, the definition of Ψ_n and (4.90), let us show that for any *irrational* $n \in \mathbb{S}_\gamma$ one has

$$|\Psi_n(y)| \geq 1/2 \qquad \text{for all} \qquad y \in \mathbb{R}^d. \tag{4.94}$$

The small nuance, that (4.94) needs the normal to be irrational as compared with (4.90) lies in the fact that (4.90) holds on the boundary of Ω_n, while here we need the entire space $\mathbb{R}^d$. To see (4.94), observe that for $y = Nz'$ with $z' \in \mathbb{R}^{d-1}$ the lower bound we need is due to (4.90) and (4.93). Now, if the normal n is irrational, then $\{Nz' : \ z' \in \mathbb{R}^{d-1}\}$ is everywhere dense in $\mathbb{T}^d$, which is the unit cell of periodicity of Ψ_n, hence the continuity of Ψ_n completes the proof of (4.94).

We now apply Lemma 4.4.13 for the family $\mathcal{F}$ and decreasing function g, and let $\nu \notin \mathbb{R}\mathbb{Q}^d$ be the unit vector and $\{\lambda_k\}_{k=1}^\infty$ be the increasing sequence given by the Lemma. By Remark 4.4.14 we may assume that $\nu \in \mathbb{S}_\gamma$, hence for ν we have (4.94). Next, for a function $F_\nu(x)$ let $\widetilde{v}_0 \in C^\infty(\mathbb{T}^d)$ be the function given by Lemma 4.4.13 for which

$$\left| \int_{\mathbb{R}^{d-1}} F_\nu(x)\widetilde{v}_0(\lambda_k M(x, 0))dx - c_0(\widetilde{v}_0) \int_{\mathbb{R}^{d-1}} F_\nu(x)dx \right| \geq g(\lambda_k), \tag{4.95}$$

where $c_0(\widetilde{v}_0)$ is the 0-th Fourier coefficient, and $k = 1, 2, \ldots$. Ellipticity of A implies that $n^T A(y) n \geq c_0|n|^2 = c_0$ for any $y \in \mathbb{R}^d$, with absolute constant

$c_0 > 0$, hence taking into account (4.94) we define

$$v_0(y) := \frac{1}{\nu^T A(y)\nu} \frac{1}{\Psi_\nu(y)} \widetilde{v}_0(y), \qquad y \in \mathbb{R}^d, \tag{4.96}$$

and get $v_0 \in C^\infty(\mathbb{T}^d)$.

Finally, we claim that ν, $\{\lambda_k\}_{k=1}^\infty$, and v_0 defined by (4.96) satisfy the Theorem. Indeed by (4.76) we have that the solution to boundary layer problem with these parameters has the form

$$v(y) = \int\limits_{\mathbb{R}^{d-1}} \partial_{z_d} G^{0,\nu}(e_d, (z', 0)) \widetilde{v}_0(\lambda M(z', 0)) dz' + O(\lambda^{-\kappa}),$$

hence (4.95) completes the proof of the Theorem. ☐

Let us conclude with a comment on the condition (4.73) concerning the coefficients. As we saw, the proof of the main Theorem of this section proceeds by preparing a stage for applying Lemma 4.4.13. In this lemma we know how to construct the function $v_0 \in C^\infty(\mathbb{T}^d)$ that slows down the averaging for a fixed family of functions. However in the proof of Theorem 4.4.11 the averaging process comes with extra terms, namely we have an averaging of the form $\int_{\mathbb{R}^{d-1}} F(x) w_0(\lambda M(x,0)) v_0(\lambda M(x,0)) dx$ where $w_0 \in C^\infty(\mathbb{T}^d)$ is fixed, and we are allowed to choose v_0. The advantage of (4.73) is that it allows this extra term w_0 to preserve its sign, hence incorporating $1/w_0$ into v_0 we recover the case of $w_0 \equiv 1$. In more abstract terms, we have some subset of $C^\infty(\mathbb{T}^d)$ as a subset of suitable v_0-s, and we want to see if the principal ideal generated by w_0 has an intersection with this subset. When w_0 preserves the sign, this ideal coincides with the entire space $C^\infty(\mathbb{T}^d)$, and we trivially get the claim, while for sign-changing w_0 our method seems to be not working. It is very interesting to understanding this problem in general.

Appendix A

Some tools concerning boundary layers

A.0.4 The least singular value

For a matrix $A \in M_d(\mathbb{C})$ let $r_i(A)$ be the Euclidean norm of its i-th row, $c_i(A)$ be the Euclidean norm of its i-th column, and set $r_{\min}(A) = \min\limits_{1 \le i \le d} r_i(A)$ and $c_{\min}(A) = \min\limits_{1 \le i \le d} c_i(A)$. Also let $\sigma_{\min}(A)$ be the smallest singular value of A. Then the following estimate holds true (see [39], Theorem 1)

$$\sigma_{\min}(A) \geq \left(\frac{d-1}{d}\right)^{(d-1)/2} |\mathrm{det} A| \max \left\{ \frac{c_{\min}(A)}{\Pi_{i=1}^{d} c_i(A)}, \frac{r_{\min}(A)}{\Pi_{i=1}^{d} r_i(A)} \right\}. \tag{A.1}$$

A.0.5 Tartar's Lemma and existence of boundary layers

Here we discuss the problem of existence of boundary layers with exponential decay. The analysis is due to L. Tartar and is based on the following Lemma.

Lemma A.0.15. (see [62] Lemma 18.1, also [48] Lemma 10.1) *Assume V and W are Banach spaces, $a(\cdot,\cdot)$ is a continuous bilinear form on $V \times W$, $M : V \to W$ is a bounded linear operator which is onto and such that there exists a constant $\alpha > 0$ satisfying*

$$a(v, Mv) \geq \alpha ||v||_V, \qquad \forall v \in V. \tag{A.2}$$

Then for any $L \in W^*$ there exists a unique element $v \in V$ satisfying

$$a(v, w) = L(w), \qquad \forall w \in W. \tag{A.3}$$

Moreover one has the following estimate

$$||v||_V \leq \frac{||L||_{W^*}}{\alpha} ||M||_{op}. \tag{A.4}$$

We now apply Lemma A.0.15 to the study of boundary layers with exponential decay. Let Y' be an open parallelepiped in $\mathbb{R}^{d-1}$, $f = f_i$, and $F = F_i^{\alpha}$, where

$i = 1, 2, ..., N$ and $\alpha = 1, 2, ..., d$. For unknown vector function $u = u_i$ consider the following problem

$$\begin{cases} -\nabla \cdot A(y)\nabla u(y) = f - \nabla \cdot F(y), & \text{in } G = Y' \times \mathbb{R}_+, \\ u(y', 0) = 0, & y' \in Y', \\ u(\cdot, y_d), & \text{is } Y'\text{-periodic for any } y_d > 0, \end{cases} \tag{A.5}$$

where the system of equations is understood as follows

$$-\frac{\partial}{\partial y_\alpha}\left(A_{ij}^{\alpha\beta}(y)\frac{\partial u_j}{\partial y_\beta}(y)\right) = f_i - \frac{\partial F_i^\alpha}{\partial y_\alpha}, \qquad i = 1, 2, ..., N.$$

We assume that there exists $\gamma_0 > 0$ such that

$$e^{\gamma_0 y_d} f(y) \in L^2(G; \mathbb{R}^N) \qquad \text{and} \qquad e^{\gamma_0 y_d} F(y) \in L^2(G; \mathbb{R}^{d\times N}), \tag{A.6}$$

and

$$F(\cdot, y_d) \text{ is } Y'\text{-periodic for any } y_d > 0. \tag{A.7}$$

The existence of solutions to (A.5) with exponential decay is given in the following result.

Theorem A.0.16. (see [62] Chapter 18, and [48] Theorem 10.1) *Assume (A.6), (A.7) and that the coefficient matrix in (A.5) is bounded and is uniformly elliptic with ellipticity constant $\lambda_A > 0$. Then for any $0 < \gamma < \min\{\gamma_0, \frac{\lambda_A}{2||A||_\infty}\}$ there exists a unique solution u to system (A.5) satisfying $e^{\gamma y_d}\nabla u(y) \in L^2(G)$. More precisely, for any such $\gamma > 0$ one has the following estimate*

$$||e^{\gamma y_d}\nabla u||_{L^2(G;\mathbb{R}^{d\times N})} \le \frac{C}{\gamma}\frac{1}{\lambda_A - 2\gamma||A||_\infty}[||e^{\gamma y_d} f||_{L^2(G;\mathbb{R}^N)} + ||e^{\gamma y_d} F||_{L^2(G;\mathbb{R}^{d\times N})}], \tag{A.8}$$

where the constant C depends on dimension and Y'.

Let us first prove the following useful corollary we get from the Theorem.

Corollary A.0.17. *Let $g = g_i$ be smooth and periodic vector function defined on Y'. Then the following problem*

$$\begin{cases} -\nabla \cdot A(y)\nabla u(y) = 0, & \text{in } G = Y' \times [0, \infty), \\ u(y', 0) = g(y'), & y' \in Y', \\ u(\cdot, y_d), & \text{is } Y'\text{-periodic for any } y_d > 0, \end{cases} \tag{A.9}$$

has a unique solution $u = u_i$ such that $e^{\gamma y_d}\nabla u \in L^2(G; \mathbb{R}^{d\times N})$ for any $0 < \gamma < \frac{\lambda_A}{2||A||_\infty}$. Moreover, the solution u satisfies the following estimate

$$||e^{\gamma y_d}\nabla u||_{L^2(G;\mathbb{R}^{d\times N})} \le \frac{C}{\gamma}\frac{||A||_\infty}{\lambda_A - 2\gamma||A||_\infty}||g||_{H^1(Y';\mathbb{R}^N)}. \tag{A.10}$$

Proof. Take any nonnegative, compactly supported smooth function $\varphi(y_d)$, such that $\varphi = 1$ near $y_d = 0$. Set $\widetilde{g}(y) = \varphi(y_d)g(y')$, for $y = (y', y_d) \in G$. Since $\widetilde{g}$ has compact support in the direction of e_d we have that $e^{\gamma y_d}\nabla\widetilde{g} \in L^2(G;\mathbb{R}^{d\times N})$, for any $\gamma > 0$. Now let $\widetilde{u}$ be the unique solution to the problem (A.5) with the right hand side $\nabla\cdot A(y)\nabla\widetilde{g}$. Defining $u = \widetilde{u}+\widetilde{g}$ we get a solution to (A.9). The estimate (A.10) follows easily from the corresponding estimate of Theorem A.0.16. $\square$

Remark A.0.18. *The formulation of Theorem A.0.16 is slightly more general than the original one as given e.g. in [48] or [62]. Namely, here it is stated for elliptic systems rather than scalar equations, and involves detailed estimates of V_γ norms of solutions. The proof however, follows the lines of the original proof, making necessary changes to deal with systems of equations, and keeping track of the norms of the quantities involved in norm estimate of the Theorem. For the sake of exposition, here we present the complete proof of Theorem A.0.16.*

Proof of Theorem A.0.16. The proof is based on Tartar's Lemma and for that we start with some preliminary set up. Denote $V(Y') = H^1_{per}(Y')$, that is the set of all functions from $H^1(Y')$ with equal traces on opposite faces of Y'. Next, set

$$V_\gamma = \{v:\ v \in L^2_{loc}(\mathbb{R}_+;V(Y')),\ e^{\gamma y_d}\nabla v(y) \in L^2(G) \text{ and } v(y',0) \equiv 0\}.$$

It is easy to see that V_γ is a Hilbert space with scalar product defined as

$$[u,v]_\gamma = \int\limits_G e^{2\gamma y_d}\nabla u(y)\cdot\nabla v(y)dy.$$

We also define $V_{0,\gamma} = \{v \in V:\ e^{\gamma y_d}v \in L^2(G)\}$ and note that V_0 is also a Hilbert space with scalar product defined by

$$[u,v]_{0,\gamma} = [u,v]_\gamma + \int\limits_G e^{2\gamma y_d}u(y)v(y)dy.$$

To deal with systems of equations for $N \in \mathbb{N}$ we consider the direct sum of Hilbert spaces V_γ and $V_{0,\gamma}$. Namely, we set

$$V_\gamma^N = \underbrace{V_\gamma \oplus ... \oplus V_\gamma}_{N\ times} \qquad \text{and} \qquad V_{0,\gamma}^N = \underbrace{V_{0,\gamma} \oplus ... \oplus V_{0,\gamma}}_{N\ times}, \tag{A.11}$$

with the scalar product defined in a standard way, that is as a sum of scalar products of corresponding components. Clearly the spaces defined by (A.11) are Hilbert.

For $u \in V_\gamma^N$ and $v \in V_{0,\gamma}^N$ we then define the bilinear form

$$a_\gamma(u,v) = \int\limits_G a_{ij}^{\alpha\beta}\frac{\partial u_j}{\partial y_\beta}\frac{\partial(e^{2\gamma y_d}v_i)}{\partial y_\alpha}dy$$

and the functional

$$\mathcal{F}(v) = \int\limits_G e^{2\gamma y_d} f_i v_i dy + \int\limits_G F_i^\alpha \frac{\partial(e^{2\gamma y_d} v_i)}{\partial y_\alpha} dy.$$

It is clear that a_γ is bounded on $V_\gamma^N \times V_{0,\gamma}^N$. Next, an easy computation using Hölder's inequality implies that $\mathcal{F}$ is a bounded linear functional on $V_{0,\gamma}^N$ for any $0 < \gamma \leq \gamma_0$, with the following bound on its norm

$$||\mathcal{F}||_{(V_{0,\gamma}^N)^*} \leq C[||e^{\gamma y_d} f||_{L^2(G;\mathbb{R}^N)} + ||e^{\gamma y_d} F||_{L^2(G;\mathbb{R}^{d\times N})}], \tag{A.12}$$

with $C = 1 + 2\gamma_0$. Then the weak formulation of (A.5) reads: there exists a unique $u \in V_\gamma^N$ such that $a_\gamma(u,v) = \mathcal{F}(v)$, for all $v \in V_{0,\gamma}^N$. Hence the Theorem will be proven if we can apply Lemma A.0.15. For this we need to construct the operator M of the Lemma.

For a given $u \in V_\gamma^N$ set

$$\overline{u}(y_d) = \frac{1}{|Y'|} \int\limits_{Y'} u(y', y_d) dy',$$

and define $\widetilde{u} = \widetilde{u}_i$, where $\widetilde{u}_i : \mathbb{R}_+ \to \mathbb{R}$, by the following problem

$$\frac{d\widetilde{u}_i}{dt} + 2\gamma\widetilde{u}_i = 2\gamma\overline{u_i}, \quad \text{for } t > 0 \text{ and } \widetilde{u}_i(0) = 0, \quad i = 1, 2, ..., N. \tag{A.13}$$

We then define $Mu = u - \widetilde{u}$ and prove that M satisfies Lemma A.0.15. Set $E_\gamma(t) = 2\gamma e^{-2\gamma t}\mathbb{I}_{t>0}$ for $t \in \mathbb{R}$, and extend $\overline{u}(t)$ as zero function for $t \leq 0$. Then it is easy to see that $\widetilde{u}_i(t) = (E_\gamma \star \overline{u_i})(t)$, $t \geq 0$ solves (A.13), hence we have

$$Mu = u - E_\gamma \star \overline{u}.$$

Step 1. We prove that $M \in \mathcal{L}(V_\gamma^N; V_{0,\gamma}^N)$ and is *onto*. Due to the definition of M it is enough to verify the claim for each component of M. Fix $u = u_i \in V_\gamma^N$, then by Poincare's inequality[1] we have

$$||u_i(\cdot, t) - \overline{u_i}(t)||_{L^2(Y')} \leq c(d)||\nabla_{y'} u_i||_{L^2(Y')}.$$

By definition of V_γ and using the last inequality we get

$$||e^{\gamma t}(u_i - \overline{u_i})||_{L^2(G)} \leq c(d)||u_i||_{V_\gamma}. \tag{A.14}$$

Let δ denote Dirac delta operator at 0. By definition of M we have

$$(Mu)_i = Mu_i = u_i - \overline{u_i} + (\delta - E_\gamma) \star u_i.$$

---

[1] Since we apply Poincare's inequality for a cube of fixed length the constant can be chosen to be dimension dependent only.

On the other hand we have

$$\frac{d}{dt}E_\gamma + 2\gamma E_\gamma = 2\gamma\delta,$$

hence we get

$$Mu_i = u_i - \overline{u_i} + \frac{1}{2\gamma}\frac{dE_\gamma}{dt} \star \overline{u_i} = u_i - \overline{u_i} + \frac{1}{2\gamma}E_\gamma \star \frac{d\overline{u_i}}{dt}. \tag{A.15}$$

Since $u_i \in V_\gamma$ it is easy to see that $||e^{\gamma t}\frac{d\overline{u_i}}{dt}||_{L^2(0,\infty)} \leq |Y'|^{1/2}||u_i||_{V_\gamma}$. Fix $1 \leq i \leq N$ and set $\frac{d\overline{u_i}}{dt} = g(t)$, we now show that $e^{\gamma t}(E_\gamma \star g)(t) \in L^2(G)$. It is easy to see that $e^{\gamma t}(E_\gamma \star g)(t) = \widetilde{E}_\gamma \star \widetilde{g}(t)$ where

$$\widetilde{E}_\gamma(t) = e^{\gamma t}E_\gamma(t) \text{ and } \widetilde{g}(t) = e^{\gamma t}g(t).$$

Using the Young's inequality for convolutions and the estimate for L^2 norm of $\widetilde{g}$ obtained above we get

$$\begin{aligned} ||e^{\gamma t}(E_\gamma \star g)(t)||_{L^2(0,\infty)} \leq ||\widetilde{E}_\gamma \star \widetilde{g}(t)||_{L^2(0,\infty)} &\leq ||\widetilde{E}_\gamma||_{L^1}||\widetilde{g}||_{L^2} \\ &= 2||\widetilde{g}||_{L^2} \leq 2|Y'|^{1/2}||u_i||_{V_\gamma}. \end{aligned}$$

This implies that

$$||e^{\gamma t}(E_\gamma \star g)(t)||_{L^2(G)} \leq 2|Y'|||u_i||_{V_\gamma}. \tag{A.16}$$

Applying inequalities (A.14) and (A.16) on (A.15) we arrive at

$$||e^{\gamma t}Mu_i||_{L^2(G)} \leq \frac{1}{\gamma}|Y'|||u_i||_{V_\gamma} + c(d)||u_i||_{V_\gamma}. \tag{A.17}$$

By the definition of M we have

$$\frac{\partial Mu_i}{\partial y_\alpha} = \frac{\partial u_i}{\partial y_\alpha} - \delta_{\alpha,d}E_\gamma \star \frac{\partial \overline{u_i}}{\partial y_d}.$$

From here we get

$$||e^{\gamma t}\nabla Mu_i||_{L^2(G)} \leq 2(||u_i||_{V_\gamma} + 2|Y'|||u_i||_{V_\gamma}). \tag{A.18}$$

By (A.17) and (A.18) we have that

$$M \text{ is bounded from } V_\gamma^N \text{ to } V_{0,\gamma}^N \text{ and } ||M||_{op} \leq C\frac{1}{\gamma}, \tag{A.19}$$

where we assume that $\gamma < 1$ and C is a constant that depends on $|Y'|$ and dimension. This gives the *blow up rate in* γ for the operator M.

Step 2. Here we show that M is *onto*. Again, due to the definition of M we may argue component-wise. For this, fix $v \in V_{0,\gamma}$, we want to find $u \in V_\gamma$ satisfying $u - E_\gamma \star \overline{u} = v$. Let Y be the Heaviside function, that is $Y(t) = 1$ for $t > 0$ and 0 otherwise. Then set $u = v + 2\gamma Y \star \overline{v}$. It is easy to see that $u \in V_\gamma$. To see that it also satisfies $Mu = v$ we observe that

$$\overline{u} = (\delta + 2\gamma Y) \star \overline{v},$$

and hence

$$Mu = u - E_\gamma \star \overline{u} = v + [2\gamma Y - E_\gamma \star (\delta + 2\gamma Y)] \star \overline{v} = v,$$

since $2\gamma Y - E_\gamma \star (\delta + 2\gamma Y) = 0$.

Step 3. We now prove the coercivity assumption involving a and M. Since $\widetilde{u}$ depends on y_d only, from the definition of a, M and relation (A.13) we have

$$a(u, Mu) = \int\limits_G a_{ij}^{\alpha\beta} e^{2\gamma y_d} \frac{\partial u_j}{\partial y_\beta} \frac{\partial u_i}{\partial y_\alpha} dy + 2\gamma \int\limits_G a_{ij}^{d\beta} e^{2\gamma y_d} \frac{\partial u_j}{\partial y_\beta} (u_i - \overline{u_i}) dy := \mathcal{I}_1 + \mathcal{I}_2.$$

Using Hölder's inequality and then Poincare's we get

$$|\mathcal{I}_2| \leq 2\gamma ||A||_\infty ||u||^2_{V_\gamma^N}.$$

For $\mathcal{I}_1$ by ellipticity of A we have that $\mathcal{I}_1 \geq \lambda_A ||u||^2_{V_\gamma^N}$. Combining these two estimates we obtain

$$a(u, Mu) \geq (\lambda_A - 2\gamma ||A||_\infty) ||u||^2_{V_\gamma^N}, \tag{A.20}$$

hence for γ small enough we may apply Tartar's Lemma and prove the existence and uniqueness part. To get the bound (A.8) we apply the corresponding bound from Tartar's Lemma using (A.19), (A.20) and (A.12) to bound the quantities involved in the estimate (A.4).

Theorem is proved. □

Bibliography

[1] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.* **12**(4), 623-727 (1959)

[2] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.* **17**(1), 35-92 (1964)

[3] Aleksanyan, H.: Regularity of boundary data in periodic homogenization of elliptic systems in layered media. *arXiv:1409.7344* [math.AP] (2014)

[4] Aleksanyan, H., Shahgholian, H., Sjölin, P.: Applications of Fourier analysis in homogenization of Dirichlet problem I. Pointwise estimates. *J. Differential Equations* **254**(6), 2626-2637 (2013)

[5] Aleksanyan, H., Shahgholian, H., Sjölin, P.: Applications of Fourier analysis in homogenization of Dirichlet problem. L^p estimates. *Arch. Ration. Mech. Anal. (ARMA)* **215**(1), 65-87 (2015)

[6] Aleksanyan, H., Shahgholian, H., Sjölin, P.: Applications of Fourier analysis in homogenization of Dirichlet problem III. Polygonal domains. *J. Fourier Anal. Appl.* **20**(3), 524-546 (2014)

[7] Allaire, G., Amar, M.: Boundary layer tails in periodic homogenization. *ESAIM, Control, Optim. Calc. Var.* **4**, 209-243 (1999)

[8] Allaire, G.: *Shape optimization by the homogenization method*, Springer Verlag, New York (2002)

[9] Ancona, A.: On Positive Harmonic Functions in Cones and Cylinders. *Rev. Mat. Iberoam.* **28**(1), 201-230 (2012)

[10] Armstrong, S., Shen, Z.: Lipschitz estimates in almost-periodic homogenization. *arXiv:1409.2094* [math.AP] (2014)

[11] Avellaneda, M., Lin, F.: Homogenization of Elliptic Problems with L^p Boundary Data. *Appl. Math. Optim.* **15**, 93-107 (1987)

[12] Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.* **40**(6), 803-847 (1987)

[13] Axler, S., Bourdon, P., Wade, R.: *Harmonic function theory*, Vol. 137, Springer Science and Business Media (2001)

[14] Babuška, I.: Solution of problem with interfaces and singularities. *C. Boor (Ed.), Mathematical Aspects of Finite Elements in Partial Differential Equations* Academic Press, 213-277 (1974)

[15] Bakhvalov,N., Panasenko,G.: *Homogenization: averaging processes in periodic media*, Mathematics and its applications, Vol.36, Kluwer Academic Publishers, Dordrecht (1990)

[16] Bensoussan, A; Lions, J.-L.; Papanicolaou, G.: *Asymptotic Analysis For Periodic Structures.* Amsterdam; New York; North-Holland Pub. Co. (1978)

[17] Caffarelli, L. The homogenization of surfaces and boundaries. *Bull. Braz. Math. Soc. (N.S.)* **44**(4), 755-775 (2013)

[18] Caffarelli, L. A., Souganidis, P. E., Wang, L.: Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Comm. Pure Appl. Math.* **58**(3), 319-361 (2005)

[19] Cazeaux, P.: *Quelques modèles mathématiques homogénéisés appliqués à la modélisation du parenchyme pulmonaire.* Doctoral dissertation, Analysis of PDEs. Université Pierre et Marie Curie - Paris VI (2012)

[20] Chechkin, G., Piatnitski, A., Shamaev, A.: *Homogenization, Methods and Applications.* Translations of Mathematical Monographs, AMS 2007

[21] Choi, S.; Kim, I.: Homogenization for nonlinear pdes in general domains with oscillatory Neumann boundary data. *J. Math. Pures Appl. (9)* **102**(2), 419-448 (2014)

[22] Choi, S.; Kim, I.; Lee, K.-A.: Homogenization of Neumann boundary data with fully nonlinear operator. Anal. PDE **6**(4) 951-972 (2013)

[23] Cioranescu, D., Donato, P.: *An introduction to Homogenization.* Oxford Lecture Series in Mathematics and its applications 17, Oxford University Press (1999)

[24] Cushman-Roisin, B., McLaughlin, D. W., Papanicolaou, G.: Interactions between mean flow and finite-amplitude mesoscale eddies in a barotropic ocean. *Geophys. Astrophys. Fluid Dynamics* **29**, 333-353 (1984)

[25] Daubechies, I., Runborg, O., Zou, J.: A sparse spectral method for homogenization multiscale problems. Multiscale Modeling and Simulation **6**(3), 711-740 (2007)

[26] De Giorgi, E.: G-operators and Γ-convergence. *Proc. Int. Congr. Math. (Warszawa, August 1983)* 1175-1191. PWN Polish Scientific Publishing and North-Holland (1984)

[27] Desbrun, M. , Donaldson, R., Owhadi, H.: Discrete Geometric Structures in Homogenization and Inverse Homogenization with application to EIT. *arXiv:0904.2601* [math.AP] (2009)

[28] Ding, Z.: A proof of the trace theorem of sobolev spaces on Lipschitz domains, *Proc. Amer. Math. Soc.* **124**(2), 591-600 (1996)

[29] Dolzmann, G., Müller, S.: Estimates for Green's matrices of elliptic systems by L^p theory. *Manuscripta Math.* **88**(1), 261-273 (1995)

[30] Dong, H.; Kim, S.: Green's matrices for second order elliptic systems with measurable coefficients in two dimensional domains. *Trans. Amer. Math. Soc.* **361**, 3303-3323 (2009)

[31] Feldman, W.: Homogenization of the Oscillating Dirichlet Boundary Condition in General Domains. *J. Math. Pures Appl. (9)* **101**(5), 599-622 (2014)

[32] Feldman, W., Kim, I.: Continuity and discontinuity of the boundary layer tail. *arXiv:1502.00966* [math.AP] (2015)

[33] Gérard-Varet, D., Masmoudi, N.: Homogenization in polygonal domains. *J. Eur. Math. Soc.(JEMS)* **13**(5), 1477-1503 (2011)

[34] Gérard-Varet, D., Masmoudi, N.: Homogenization and boundary layers. *Acta Math.* **209**, 133-178 (2012)

[35] Giaquinta, M., Martinazzi, L.: *An introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs.* Scuola Normale Superiore Pisa (Lecture Notes) (2012)

[36] Gilbarg, D., Trudinger, N.: *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 224. Springer-Verlag, Berlin (1983)

[37] Gloria, A., Otto, F.: An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* **39**(3) 779-856 (2011)

[38] Grüter, M., Widman, K.-O.: The Green function for uniformly elliptic equations. *Manuscripta Math.* **37**, 303-342 (1982)

[39] Hong, Y.-P., Pan, C.-T.: A lower bound for the smallest singular value. *Linear Algebra Appl.* **172**, 27-32 (1992)

[40] Hofmann, S.; Kim, S.: The Green function estimates for strongly elliptic systems of second order. *Manuscripta Math.* **124**, 139-172 (2007)

[41] Hörmander, L.: *The analysis of linear partial differential operators I.* 2nd ed. Springer-Verlag (1980)

[42] Jerison, D., Kenig, C.: Boundary behavior of harmonic functions in non-tangentially accessible domains, *Adv. Math.* **46**(1), 80-147 (1982)

[43] Jikov, V., Kozlov, S., Oleinik,O.: *Homogenization of differential operators and integral functionals*, Springer, Berlin (1995)

[44] Kenig, C., Lin, F., Shen, Z.: Periodic Homogenization of Green and Neumann Functions. *Commun. Pure Appl. Math.* **67**(8), 1219-1262 (2014)

[45] Kuran, Ü.: On NTA-conical domains. *J. Lond. Math. Soc.* **40**(2), 467-475 (1989)

[46] Lee, K.-A., Shahgholian, H.: Homogenization of the boundary value for the Dirichlet problem. *arXiv:1201.6683* [math.AP] (2012)

[47] Lenstra, H. Jr.: Lattices. *Algorithmic Number Theory MSRI Publications* Vol. 44, 127-181 (2008)

[48] Lions, J.-L.: *Some methods in mathematical analysis of systems and their control.* Science Press, Beijing, Gordon and Breach, New York (1981)

[49] Lions, P.-L., Papanicolaou, G., Varadhan, S. R.: Homogenization of hamilton-Jacobi equations. Unpublished preprint (1986)

[50] Maxwell, J.C.: *A treatise on electricity and magnetism*, 3rd Ed., Clarendon Press, Oxford, 1881.

[51] Mei, C., Auriault, J.-L., Ng, C.-O.: Some Applications of the Homogenization Theory, Advances in Applied Mechanics, Vol. 32, 277-348 (1996)

[52] Moskow, S., Vogelius, M.: First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof. *Proc. Edinb. Math. Soc.* Section A Mathematics **127**(06), 1263-1299 (1997)

[53] Neuss-Radu, M.: The boundary behavior of a composite material. *ESAIM Math. Model. Numer. Anal.* **35**(3), 407-435 (2001)

[54] Prange, C.: Asymptotic analysis of boundary layer correctors in periodic homogenization. *SIAM J. Math. Anal.* **45**(1), 345-387 (2012)

[55] Prange, C.: First-order expansion for the Dirichlet eigenvalues of an elliptic system with oscillating coefficients. *Asymptot. Anal.* **83**(3), 207-235 (2013)

[56] Samelson, H.: Orientability of hypersurfaces in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* **22**, 301-302 (1969)

[57] Schulze, B.-W., Wildenhain G.; *Methoden der Potentialtheorie für elliptische Differentialgleichnungen beliebiger Ordnung*, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften: Mathematische Reihe 60, Birkäuser-Verlag, Basel (1977)

[58] Soward, A. M., Childress, S.: Large magnetic Reynolds number dynamo action in a spatially periodic flow with mean motion. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 331(1621), 649-733 (1990)

[59] Stein, E.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)

[60] Strömbergsson, A.: On the probability of the random lattice avoiding a large convex set. *Proc. Lond. Math. Soc. (3)* **103**, 950-1006 (2011)

[61] Šubin, M.: Differential and pseudodifferential operators in spaces of almost periodic functions, *Math. Sb. (N.S.)*, **95(137)** 4(12), 560-587 (1974)

[62] Tartar, L.: *The general theory of homogenization: a personalized introduction* Vol. 7. Springer (2009)

[63] Valentine, F.: *Convex sets*, McGraw-Hill (1964)