

THESIS

ON THE

INTERSECTION OF TWO CONIC SECTIONS.

A descriptive treatment of the general problem, with modifications of previous results; accompanied by some special metrical illustrations and verifications of the foregoing.

Submitted to the Faculty of Science in the University of Edinburgh for the Degree of Doctor of Science.

BY

J. A. H. JOHNSTON, M.A.,

HEADMASTER OF HIGHGATE SCHOOL.





THE SCHOOL HOUSE,
HIGHGATE, N.

Statement of eligibility
for the Degree of D.Sc. and
declaration that the
accompanying Thesis was
composed of its subject
matter original work
undertaken by the
Candidate.

I, the undersigned,
beg to state that I took
the Degree of M.A. with
First Class Honours in
Mathematics & Natural
Philosophy at the

University of Edinburgh
in the year 1894.

I beg further to state
that the enclosed Thesis
has been compared by
myself & that it
treats of matters which
have been entirely the
results of my own
original investigations,
the conclusions arrived at
& the method adopted
having been finally
completed before any
comparison was made
with the work of
previous writers on the
Subject.

4th April - 1900.

THE INTERSECTION OF TWO CONIC SECTIONS.

By J. A. H. JOHNSTON.

As the general problem discussed in this paper is a purely projective one, the treatment of the subject has been purely descriptive in method, and the results have been expressed in forms which are independent of the plane of projection.

The metrical appendices, however, not only illustrate the class of question to which the general results may be appropriately applied, but also confirm the validity of the results themselves by an appeal to special cases which admit of independent verification.

It is stated in Salmon's *Conic Sections*, sixth edition, p. 337, that the cases of four real and four imaginary intersections of two conics have not been distinguished by any simple criterion.

In the notes (p. 391) of the same volume it is further stated that this discrimination has been made by Kemmer (Giessen, 1878), and his results are quoted.

The problem has been discussed subsequently by Storey (*American Journal*, Vol. VI.) and by Gundelfinger (*Vorlesungen*, Teubner, Leipzig, 1895), whose results differ from each other and are distinct from those of Kemmer.

It is the object of this paper, whose treatment is independent of theirs in method, to establish simple criteria for the cases of intersection of two conics, with corresponding results for real and imaginary tangents.

The forms in which both Kemmer and Storey expressed their results will be derived at once from the treatment of this paper; of Kemmer's four conditions one will be shown to be unnecessary, and the results of Storey will likewise call for modification. By a singular coincidence the results obtained by both these writers contain the same redundant condition, though expressed in one case in point and in the other in line coordinates. This condition asserts the necessity for the positiveness of an expression which the subsequent pages will show to be the square of a real quantity.

The following notation will be adopted:—The two conics $(abcfgh) (xyz)^2$ and $(a'b'c'f'g'h') (xyz)^2$ will be called S and S' .

The minors of the determinants Δ and Δ' will be styled, as usual, A, B, C, F, G, H and A', B', C', \dots

$$\begin{aligned} (bc' + b'c - 2ff') & \text{ will be called } K, \\ (ac' + ca' - 2gg') & \text{ ,, ,, } L, \\ (ab' + a'b - 2hh') & \text{ ,, ,, } M, \\ (gh' + g'h - af' - a'f) & \text{ ,, ,, } K', \\ (fh' + f'h - bg' - b'g) & \text{ ,, ,, } L', \\ (fg' + f'g - ch' - c'h) & \text{ ,, ,, } M'. \end{aligned}$$

The tangential equations of S and S' , viz., $(ABCFGH) (uvw)^2$ and $(A'B'C'F'G'H') (uvw)^2$, will be called Σ and Σ' , and the contravariant $(KLMK'L'M') (uvw)^2$ called, as usual, ϕ .

The invariants of the conics will be Δ , θ , θ' , and Δ' , the cubic determining the line pairs $\lambda S + S'$, $\Delta\lambda^3 + \theta\lambda^2 + \theta'\lambda + \Delta' = 0$, and its discriminant $\theta^2\theta'^2 + 18\Delta\Delta'\theta\theta' - 27\Delta^2\Delta'^2 - 4\Delta\theta'^3 - 4\Delta'\theta^3 = D$.

The question of the intersection of the two conics S and S' may be placed upon the simple basis of the nature of the several line pairs $\lambda S + S'$.

If $\lambda S + S'$ break up into straight lines $(lx + my + nz)$ and $(l'x + m'y + n'z)$, then l, m, n and l', m', n' can be so determined that

$$\begin{aligned} a\lambda + a' &= ll', \\ b\lambda + b' &= mm', \\ c\lambda + c' &= nn', \\ 2(f\lambda + f') &= mn' + m'n, \\ 2(g\lambda + g') &= nl' + n'l, \\ 2(h\lambda + h') &= lm' + l'm. \end{aligned}$$

It readily follows that

$$\left. \begin{aligned} (1) \quad -(A\lambda^2 + K\lambda + A') &= \{(f\lambda + f')^2 - (b\lambda + b')(c\lambda + c')\} = \frac{1}{4}(mn' - m'n)^2 \\ (2) \quad -(B\lambda^2 + L\lambda + B') &= \{(g\lambda + g')^2 - (c\lambda + c')(a\lambda + a')\} = \frac{1}{4}(nl' - n'l)^2 \\ (3) \quad -(C\lambda^2 + M\lambda + C') &= \{(h\lambda + h')^2 - (a\lambda + a')(b\lambda + b')\} = \frac{1}{4}(lm' - l'm)^2 \\ (4) \quad -(F\lambda^2 + K'\lambda + F') &= \{(a\lambda + a')(f\lambda + f') - (g\lambda + g')(h\lambda + h')\} \\ &= \frac{1}{4}(lm' - l'm)(nl' - n'l) \\ (5) \quad -(G\lambda^2 + L'\lambda + G') &= \{(b\lambda + b')(g\lambda + g') - (f\lambda + f')(h\lambda + h')\} \\ &= \frac{1}{4}(mn' - m'n)(lm' - l'm) \\ (6) \quad -(H\lambda^2 + M'\lambda + H') &= \{(c\lambda + c')(h\lambda + h') - (f\lambda + f')(g\lambda + g')\} \\ &= \frac{1}{4}(nl' - n'l)(mn' - m'n) \end{aligned} \right\} \quad (\text{I.})$$

From (1), (2), and (3) of (I.) it is readily seen that

$$\begin{aligned} u\sqrt{\{-(A\lambda^2 + K\lambda + A')\}} + v\sqrt{\{-(B\lambda^2 + L\lambda + B')\}} \\ + w\sqrt{\{-(C\lambda^2 + M\lambda + C')\}} = \frac{1}{2} \begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}. \end{aligned} \quad (\text{II.})$$

Now, if both sides of (1), (2), (3), (4), (5), and (6) be multiplied by u^2 , v^2 , w^2 , $2vw$, $2uw$, and $2uv$ respectively, and the results be added,

$$\begin{aligned} -(\Sigma\lambda^2 + \phi\lambda + \Sigma') &= \frac{1}{4} \begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}^2 \quad \text{by (II.)} \\ &= [u\sqrt{\{-(A\lambda^2 + \dots)\}} + v\sqrt{\{-(B\lambda^2 + \dots)\}} \\ &\quad + w\sqrt{\{-(C\lambda^2 + \dots)\}}]^2. \end{aligned} \quad (\text{III.})$$

If the line pair $\lambda S + S'$ be real, equations (III.), where u, v, w are any real variable line coordinates, show that, since the squared expressions are positive, $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ is essentially negative in sign.

If the line pair be coincident, then, since $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ vanishes, it follows

that $(\Sigma\lambda^2 + \phi\lambda + \Sigma') = 0$.

Again, if the line pair be imaginary, and the values of λ be real, then, since $a\lambda + a' = ll'$, $b\lambda + b' = mm'$, $c\lambda + c' = nn'$, it follows that l and l' , m and m' , n and n' are all pairs of conjugate complex numbers, and therefore $(mn' - m'n)$, $(lm' - l'm)$, ... are all entirely imaginary and of the form ti , where t is real, and therefore $\frac{1}{2} \begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = t'i$, where t' is real. Its square is therefore negative,

$$\frac{1}{2} \begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = t'i$$

and consequently, by (III.), the value of $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ is essentially positive.

In equation (III.) $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$ is plainly the equation to a vertex of

the common self-polar triangle, and the three values of $\sqrt{\{-(A\lambda^2 + \dots)\}}$, $\sqrt{\{-(B\lambda^2 + \dots)\}}$, $\sqrt{\{-(C\lambda^2 + \dots)\}}$ are proportional to the coordinates of its three vertices, and so, if we contemplate the case of $\lambda S + S'$ a parallel pair, and these vertices at infinity, the vanishing of $\begin{vmatrix} \sin A & \sin B & \sin C \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ indicates also

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

the vanishing of $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ for these special coordinates, and includes this case in the above.

The nature of the line pair $\lambda S + S'$, therefore, where λ is real, depends solely upon the sign of $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$. The cases of four real and four imaginary intersections of two conics, it is well known, have thus much in common, that D , the discriminant of the cubic $\Delta\lambda^3 + \theta\lambda^2 + \theta'\lambda + \Delta' = 0$, is positive, and that λ has three real values; if there be two real and two imaginary intersections, D is negative and λ has only one real value.

To distinguish the first two cases, we notice that

(a) For four real intersections, identified geometrically by the existence of three real line pairs $\lambda S + S'$, $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ must by the foregoing have *three real negative values*. (IV.)

(\beta) For four imaginary intersections, given by

$$(1) \quad s + it, \quad s' + it';$$

$$(2) \quad s - it, \quad s' - it';$$

$$(3) \quad \sigma + i\tau, \quad \sigma' + i\tau';$$

$$(4) \quad \sigma - i\tau, \quad \sigma' - i\tau';$$

there is clearly still one real pair of common chords, *i.e.*, the lines joining (1) to (2) and (3) to (4), but the other two pairs are imaginary. The values of λ are all real, and, as $\lambda S + S'$ imaginary implies that $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ is positive, it follows that of the three values of $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ *two are positive and one is negative*. (V.)

The three values of $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ may now be regarded as the roots of a cubic equation in z , where

$$z = \Sigma\lambda^2 + \phi\lambda + \Sigma',$$

subject to

$$\Delta\lambda^3 + \theta\lambda^2 + \theta'\lambda + \Delta' = 0.$$

The elimination of λ yields

$$\begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma' - z & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' - z & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma' - z \end{vmatrix} = 0, \quad (\text{VI.})$$

in which we note that

$$z_1 z_2 z_3 = \Pi(\Sigma\lambda^2 + \phi\lambda + \Sigma') = 1/\Delta^2 \begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma' & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma' \end{vmatrix} = \delta/\Delta^2,$$

and we propose to show that, in all cases of possible intersection, the determinant δ is essentially negative in sign.

By (IV.) the case of four real intersections was distinguished by the fact that $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ had three real negative values. The product of its values is therefore negative.

By (V.), four imaginary intersections required that $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ should have one negative and two positive values. The product of the three values is therefore again negative.

In the case of two real and two imaginary intersections the product is also negative; for the cubic in λ has now two imaginary roots, but one real pair of common chords remains, *i.e.*, the line through the two real intersections and the line through the two conjugate imaginary intersections. $(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')$ is therefore negative for the real pair $\lambda_1 S + S'$. The product of $(\Sigma\lambda_2^2 + \phi\lambda_2 + \Sigma')$ and $(\Sigma\lambda_3^2 + \phi\lambda_3 + \Sigma')$, being the sum of two squares, is positive. This is clearly the case, since λ_2 and λ_3 are conjugate complex numbers. It follows that the product $\Pi(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ is again negative.

The foregoing having shown that $\Pi - (\Sigma\lambda^2 + \phi\lambda + \Sigma')$ is proportional to $\Pi(ux + vy + wz)^2$, *i.e.*, the square of the tangential equation to the vertices (x_1, y_1, z_1, \dots) of the common self-polar triangle, it is now possible to identify this product $-\delta$ with Γ^2 , where Γ is the well-known contravariant of the two conics.

The cubic equation (VI.) when written in full, is

$$\begin{aligned} z^3 + \left\{ \frac{\Sigma(-\theta^2 + 2\theta'\Delta) - 3\Delta^2\Sigma' + \theta\Delta\phi}{\Delta^2} \right\} z^2 \\ + \left\{ \frac{\Sigma^2(\theta^2 - 2\theta\Delta') + \Sigma'^2 3\Delta^2 + \Sigma\Sigma'(2\theta^2 - 4\theta'\Delta) + \Sigma\phi(3\Delta\Delta' - \theta\theta') - \Sigma'\phi 2\theta\Delta + \phi^2\theta'\Delta}{\Delta^2} \right\} z \\ - \left\{ \frac{-\Delta\Delta'\phi^3 + (\theta\Sigma\Delta' + \theta'\Sigma'\Delta)\phi^2 - \{\Sigma^2\Delta'\theta' + \Sigma'^2\Delta\theta + \Sigma\Sigma'(\theta\theta' - 3\Delta\Delta')\}\phi + \Sigma^3\Delta'^2 + \Sigma'^3\Delta^2 - \Sigma^2\Sigma'(2\Delta'\theta - \theta'^2) - \Sigma'^2\Sigma(2\Delta\theta' - \theta^2)}{\Delta^2} \right\} = 0 \quad (\text{VI.})' \end{aligned}$$

$$\text{which we shall call} \quad z^3 + pz^2 + qz + r = 0, \quad (\text{VI.})''$$

where $r = -\delta/\Delta^2$ has been shown to be *positive* in all cases of intersection.

The distinction between the cases of four real and four imaginary intersections is now apparent.

In the former case the cubic has three real negative roots; in the latter one real negative and two real positive roots.

Therefore, for four real points, p and q must both be positive.

For four imaginary points, p or q at least must be negative. (VII.)

This twofold condition may be embodied formally in one, if we note that, at the minimum point of the z cubic, the value is negative in the former case and positive in the latter, inasmuch as r is positive.

This value of z is the common root of the equations

$$(1) \quad z^3 + pz^2 + qz + (r - \zeta) = 0,$$

$$(2) \quad 3z^2 + 2pz + q = 0,$$

for the minimum case. The elimination of z between (1) and (2) gives a quadratic for $(r - \zeta)$ whose greater root must be chosen, and $(p^2 - 3q)$ is positive by the conditions. The common value of z in (1) and (2) is easily found to be

$$z = \frac{pq - 9(r - \zeta)}{-2(p^2 - 3q)}, \quad (3)$$

and therefore $27(r - \zeta)^2 + (4p^3 - 18pq)(r - \zeta) + q^2(4q - p^2) = 0$;

$$\text{so} \quad 9(r - \zeta) = -\frac{(2p^3 - 9pq) + 2(p^2 - 3q)^{\frac{3}{2}}}{3},$$

and therefore, by (3) $z = \frac{1}{3}[-p + \sqrt{(p^2 - 3q)}]$.

For four real points D is positive, $+\sqrt{(p^2 - 3q)} - p$ is negative. For four imaginary points D is positive, $+\sqrt{(p^2 - 3q)} - p$ is positive, which embodies the previous twofold conditions.

It can be easily shown that the four points of intersection are given tangentially by

$$\begin{aligned} (\lambda_2 - \lambda_3)\sqrt{(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')} \pm (\lambda_3 - \lambda_1)\sqrt{(\Sigma\lambda_2^2 + \phi\lambda_2 + \Sigma')} \\ \pm (\lambda_1 - \lambda_2)\sqrt{(\Sigma\lambda_3^2 + \phi\lambda_3 + \Sigma')} = 0, \end{aligned}$$

and this suggests the derivation of another cubic in z' whose coefficients will display collateral symmetry in Σ , ϕ , Σ' and the invariants.

If, therefore, the roots of the cubic in z be multiplied respectively by $\Delta^2(\lambda_2 - \lambda_3)^2$, $\Delta^2(\lambda_3 - \lambda_1)^2$, and $\Delta^2(\lambda_1 - \lambda_2)^2$, we shall arrive at a cubic in z' whose coefficients are the expressions used in Kemmer's conditions.

In this case, to form

$$z'^3 + p'z'^2 + q'z' + r' = 0,$$

we have

$$\begin{aligned} \{\Delta^2(\lambda_2 - \lambda_3)^2(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')\} + 2 \text{ similar expressions,} \\ = 2\Sigma(\theta'^2 - 3\theta\Delta') + \phi(\theta\theta' - 9\Delta\Delta') + 2\Sigma'(\theta^2 - 3\theta'\Delta); \end{aligned}$$

and therefore

$$p' = \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix},$$

and similarly

$$\begin{aligned} q' &= \{\Delta^4(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')(\Sigma\lambda_2^2 + \phi\lambda_2 + \Sigma')\} + 2 \text{ similar expressions.} \\ &= \frac{1}{4}\{p'^2 - (\phi^2 - 4\Sigma\Sigma')D\}, \end{aligned}$$

and
$$r' = rD\Delta^2 = -D\delta = -D \begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta & \Delta' \\ \Sigma & \phi & \Sigma' & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma' \end{vmatrix},$$

and the z' cubic appears as

$$z'^3 + \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix} z'^2 + \frac{1}{4} \left\{ \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix}^2 - (\phi^2 - 4\Sigma\Sigma')D \right\} z' + rD\Delta^2 = 0 \quad (\text{VIII.})$$

or
$$z'^3 + p'z'^2 + q'z' + r' = 0,$$

and all the former conclusions still hold from the positive nature of the multipliers of the former roots in z .

Criticism of Kemmer's Results.

Kemmer's results are that for four real intersections the following four conditions must be satisfied, viz., D , p' , q' , and r' must all be positive. Now $r' = rD\Delta^2$, and, if D be positive, this asserts that r must also be positive.

But we have shown clearly that in all possible cases of intersection r is positive and equal to Γ^2/Δ^2 ; and therefore this fourth condition of Kemmer is superfluous.

The case of four real intersections may now be distinguished with advantage by (1) D positive, (2) $+\sqrt{(p'^2 - 3q')} - p'$ must be negative.

The Results of Storey.

From the original identities (I.) we can deduce very simply the results which Storey obtained by a different method.

$$\left. \begin{aligned} (1) \quad & -(A\lambda^2 + K\lambda + A') = \frac{1}{4}(mn' - m'n)^2 \\ (2) \quad & -(B\lambda^2 + L\lambda + B') = \frac{1}{4}(nl' - n'l)^2 \\ (3) \quad & -(C\lambda^2 + M\lambda + C') = \frac{1}{4}(lm' - l'm)^2 \\ (4) \quad & -(F\lambda^2 + K'\lambda + F') = \frac{1}{4}(lm' - l'm)(nl' - n'l) \\ (5) \quad & -(G\lambda^2 + L'\lambda + G') = \frac{1}{4}(mn' - m'n)(lm' - l'm) \\ (6) \quad & -(H\lambda^2 + M'\lambda + H') = \frac{1}{4}(nl' - n'l)(mn' - m'n) \end{aligned} \right\} \dots \dots (I)'$$

If we choose for u , v , w the quantities $(ax + hy + gz)$, $(hx + by + fz)$, and $(gx + fy + cz)$ as multipliers, and multiply (1) to (6) respectively by $(ax + hy + gz)^2$, \dots , $2(hx + by + fz)(gx + fy + cz)$, \dots , and add once more, we shall find the sums to give, as may be readily verified for the two simplest conics,

$$-\{\Delta S\lambda^2 + (\theta S - \Delta S')\lambda + (\theta' S - F)\} = \frac{1}{4}\{\Sigma(mn' - m'n)(ax + hy + gz)\}^2 \dots \dots (IX.)$$

where F is the covariant conic of S and S' , and the right-hand side now represents the square of the equation of a side of the common self-polar triangle when equated to zero.

The apparent anomaly, common to the right-hand sides of (IX.) and (III.) that, despite the fact of *real* vertices and sides of the self-polar triangle in the case of four imaginary intersections, the square of $\Sigma(mn' - m'n)$ *u* should be negative is explained by the statement that $(mn' - m'n), \dots$ are only *proportional* to the values of the coordinates of the vertices, e.g., $(mn' - m'n) = kx_i$, where x_i is real, &c.

The function $-\{\Delta S\lambda^2 + (\theta S - \Delta S')\lambda + (\theta' S - F)\}$ now simply replaces the former $-(\Sigma\lambda^2 + \phi\lambda + \Sigma')$, and all the conclusions with respect to the sign of the latter apply equally to that of the former.

The most symmetrical form of these new criteria will now be obtained by the simple summations, &c., used for the *z* cubic, and, just as the sum of three values of $-(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ appeared as *p*, so does the sum of the three values of $-\{\Delta S\lambda^2 + (\theta S - \Delta S')\lambda + (\theta' S - F)\}$ appear as Storey's

$$S_1 = (-\theta' S - \theta S' + 3F).$$

Similarly the equivalent of *q* is Storey's

$$S_2 = (\theta\Delta' S^2 + \theta'\Delta S'^2 + 3F^2 + (\theta\theta' - 3\Delta\Delta')SS' - 2\theta'SF - 2\theta S'F),$$

and, lastly, the equivalent of *r* appears as Storey's

$$S_3 = F^3 - F^2(\theta S' + \theta' S) + F\{\Delta'\theta S^2 + \Delta\theta' S'^2 + (\theta\theta' - 3\Delta\Delta')SS'\} - \Delta\Delta'^2 S^3 \\ - \Delta'\Delta^2 S'^3 + \Delta'(2\Delta\theta' - \theta^2)S^2 S' + \Delta(2\Delta'\theta - \theta'^2)S S'^2,$$

Storey states that for four real intersections *D* must be positive, $S_2 > 0$, $S_1 S_3 \geq 0$. Now S_3 , in harmony with the interpretation of (IX.), and on independent grounds, is clearly equal to J^2 , where *J* is the Jacobian of *S* and *S'*, and is, moreover, the exact equivalent of *r*, which we showed to be positive in all cases.

The same reasoning applied to S_3 as to *r* shows that it is also necessarily *positive*. The appearance of J^2 is what we should be led to expect from the reciprocal method implied in the use of the above multipliers $(ax + hy + gz), \dots$, and, just as we showed $r = \Gamma^2/\Delta^2$ to be always positive and Γ always real, so does S_3 remain always positive and *J* always real. The results of Storey should consequently be modified to read *D* positive, $S_1 \geq 0$, $S_2 > 0$. N.B.—In cases of 4-pointed osculation $S_1 \equiv 0$.

Gundelfinger's Results.

By different methods Gundelfinger arrives at the expression $(\Sigma\lambda^2 + \phi\lambda + \Sigma')$, and thence deduces Storey's

$$S_1 = (-\theta' S - \theta S' + 3F),$$

the equivalent of *p* in point coordinates. This function he treats as a "combinant" conic ψ , gives it a geometrical meaning, and deduces three conditions for four real intersections from the fact that it must represent an imaginary conic in this case, so as to preserve constantly a positive sign for all values of the variables.

These three conditions, however, contain, in addition to the invariants, the specific constants of the two conics *S* and *S'*, and do not present the results in invariant contravariant or invariant covariant forms appropriate to the general projective problem.

The following alternative criteria have now been established :—

For four real points

(1) D must be positive ;

(2) $p' = \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix}$ must be positive ;

(3) $q' = \frac{1}{4} \{p'^2 - D(\phi^2 - 4\Sigma\Sigma')\}$ must be positive ;

for all values of the variables.

For four imaginary points (1) D must be positive, (2) p' or q' at least negative. (X.)

Or for four real points

(1) D must be positive,

(2) $S_1 = (-\theta'S - \theta S' + 3F)$ must be positive or zero,

(3) $S_2 = \{\theta\Delta'S^2 + \theta'\Delta S'^2 + 3F^2 + (\theta\theta' - 3\Delta\Delta') SS' - 2\theta'SF - 2\theta S'F\}$ must be positive,

for all values of the variables.

For four imaginary points (1) D must be positive, but not at once $S_1 > 0$, $S_2 > 0$. (XI.)

Both sets of criteria require alternatives for special cases of intersection, as follows :—

In (X.), if $D=0$, or if there be contact, the other intersections are real or imaginary, according as $p' > 0$ or < 0 . If $D=0$ and $p'=0$, q' is also $=0$, and double contact is easily inferred. The distinction between real and imaginary double contact is given by the sign of p in the z cubic, or more symmetrically thus. The λ cubic has equal roots, one common to $\Sigma\lambda^2 + \phi\lambda + \Sigma' = 0$. Two values of $\Sigma\lambda^2 + \phi\lambda + \Sigma'$ vanish, and the third value ($= -p$) is easily found to be, since the unequal root of λ is $\Delta'(2\theta^2 - 6\Delta\theta')/\Delta(2\theta'^2 - 6\Delta'\theta)$,

$$-p = \frac{\Sigma\Delta'^2(2\theta^2 - 6\Delta\theta')^2 + \phi\Delta\Delta'(\theta\theta' - 9\Delta\Delta')^2 + \Sigma'\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}{\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}; \quad \text{(XII.)}$$

and therefore the double contact is real or imaginary, according as $p > 0$ or < 0 .

The cases of osculation require no criteria.

Similarly in (XI.). For four real intersections S_1 must generally be positive for all values of the variables; but in the case of four-pointed osculation it is easily shown to vanish identically, requiring $S_1 > 0$.

In conclusion, the criteria for real and imaginary common tangents may now be developed.

If we reciprocate the original conics S and S' with respect to

$$x^2 + y^2 + z^2 = 0,$$

and apply the foregoing criteria for real intersections to

$$R = Ax^2 + By^2 + Cz^2 + 2Fyz, \dots,$$

$$R' = A'x^2 + B'y^2 + \dots,$$

we shall evidently get the criteria for four real common tangents.

The identities analogous to (I.) now assume the types

$$\left. \begin{aligned} -\{a\Delta\lambda^2 + (BC' + B'C - 2FF')\lambda + a'\Delta'\} &= \frac{1}{4}(mn' - m'n)^2, \dots, \text{etc.} \\ -\{f\Delta\lambda^2 + (GH' + G'H - AF' - A'F)\lambda + f'\Delta'\} &= \frac{1}{4}(lm' - l'm)(n'l' - n'l), \text{etc.} \end{aligned} \right\} \text{(XIII.)}$$

and, if we multiply these six identities on both sides by

$$(Au + Hv + Gw)^2, \dots, \quad 2(Hu + Bv + Fw)(Gu - Fv + Cw), \dots,$$

and add once more, we shall obtain the new equivalent of

$$-\{\Sigma\lambda^2 + \phi\lambda + \Sigma'\}$$

as

$$-\{\Delta^2\Sigma\lambda^2 + \Delta(\theta'\Sigma - \Delta\Sigma')\lambda + \Delta'(\Sigma\theta - \Delta\phi)\},$$

where the new cubic determining $\lambda R + R'$ is

$$\Delta^2\lambda^3 + \Delta\theta'\lambda^2 + \Delta'\theta\lambda + \Delta'^2 = 0.$$

As before, the product of the three values of this new quantity is positive in all cases and is equal to $r\Delta^4\Delta'^2$, and we must simply express that the sum of its three values is positive and the sum of its product pairs positive for four real common tangents.

$$\text{The sum} \quad = P = (-\theta\Delta'\Sigma - \theta'\Delta\Sigma' + 3\Delta\Delta'\phi).$$

The sum of the product pairs

$$\begin{aligned} = Q = \Delta\Delta' \{ & \theta'\Delta'\Sigma^2 + \theta\Delta\Sigma'^2 + 3\Delta\Delta'\phi^2 + (\theta\theta' - 3\Delta\Delta')\Sigma\Sigma' \\ & - 2\theta\Delta'\Sigma\phi - 2\theta'\Delta\Sigma'\phi \}, \end{aligned}$$

and the product

$$= r\Delta^4\Delta'^2 = \Gamma^2\Delta^2\Delta'^2,$$

and the new cubic whose roots are the values of

$$-\{\Delta^2\Sigma\lambda^2 + \Delta(\theta'\Sigma - \Delta\Sigma')\lambda + \Delta'(\Sigma\theta - \Delta\phi)\},$$

is

$$\zeta^3 + P\zeta^2 + Q\zeta + \Gamma^2\Delta^2\Delta'^2 = 0.$$

The discriminant of the new λ cubic is now $\Delta^2\Delta'^2D$, where D has its old meaning.

And therefore for four real common tangents (1) D must be positive, (2) P must be positive, (3) Q must be positive.

For four imaginary common tangents (1) D must be positive, and (2) P or Q at least must be negative. (XIV.)

If D be negative, there are two real and two imaginary common tangents. These tangent conditions can be thrown into the alternative forms involving S , S' , and the covariant F , by multiplying the identities of (XIII.) by x^2 , y^2 , z^2 , $2yz$, $2zx$, and $2xy$, and then adding.

The result gives $-(\Delta S\lambda^2 + F\lambda + \Delta'S')$ as the analogue of the original

$$-(\Sigma\lambda^2 + \phi\lambda + \Sigma').$$

Since this new quantity stands in relation to the cubic

$$\Delta^2\lambda^3 + \Delta\theta'\lambda^2 + \Delta'\theta\lambda + \Delta'^2 = 0,$$

in precisely the same way as $-(\Sigma\lambda^2 + \phi\lambda + \Sigma')$ stood with regard to the original λ cubic, the new results may be at once inferred from (X.) by writing ΔS , F , and $\Delta'S'$ for Σ , ϕ , and Σ' , and making corresponding changes in the invariants.

The criteria appear as follows:—

For four real common tangents

(1) D must be positive,

$$(2) \quad S_1 = \Delta\Delta' \begin{vmatrix} S & F & S' \\ 3\Delta & 2\Delta\theta' & \theta \\ \theta' & 2\Delta'\theta & 3\Delta' \end{vmatrix} \text{ must } \gtrsim 0,$$

$$(3) \quad S_2 = \frac{1}{4} \{S_1^2 - \Delta^2\Delta'^2 D (F^2 - 4\Delta\Delta' S S')\} > 0, \quad (\text{XV.})$$

for all real values of the variables.

For four imaginary common tangents (1) D must > 0 ,

(2) not at once $S_1 \gtrsim 0$, $S_2 > 0$.

The results of (X.) and (XI.), of (XIV.) and (XV.) constitute a complete solution of the problem proposed. While it is obvious that the results may be expressed in an unlimited number of modes by varying the positive symmetrical multipliers of the roots of the cubics discussed, an effort has been made to present them in their simplest and most symmetrical forms.

APPENDIX I.

The foregoing criteria embody many interesting geometrical results, if line infinity co-ordinates be employed for u, v, w in Σ, Σ' and ϕ . In this case the coefficients p', q' , etc., in the z' cubic (VIII.) are invariants for projections in which the line at infinity is unaltered, and the metrical significance of this cubic in z' can now be developed. In all that follows $\sin A, \sin B$, and $\sin C$ must be understood to replace u, v , and w .

THE GEOMETRICAL SIGNIFICANCE OF THE ROOTS OF THE EQUATION

$$z'^3 + p'z'^2 + q'z' + r' = 0.$$

The formula for the area of the quadrilateral given by

$$x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4;$$

is known to be $(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3) \dots \dots \dots$ (i.)

(1) *Real points.*—If we consider the 6 values which the formula (i.) receives, when one point is fixed and the other three suffixes are permuted in every possible way, it is clear that the six reduce to three different pairs, the members of each pair being equal and of opposite sign.

If this area be called A'' , there are, therefore, three values for $-A''^2$ implied in the formula, dependent upon the order in which the corners are taken.

The geometrical equivalents of these three values are easily seen to be—
(a) the sum of the triangles 1, 2, 3 and 1, 3, 4 (the conventional *area*); (b) the difference between the triangles 1, 2, 3 and 1, 2, 4; and (c) the difference between the triangles 1, 2, 4 and 1, 3, 4; where 1, 2, 3 and 4 are used to designate the corners for brevity.

This is only a special case of a more general theorem for any polygon, and we must expect that the analytical formula for A''^2 (if the 4 points are given as the intersection of two conic sections), will, *a priori*, appear in the shape of a cubic equation, whose roots, when the points are all real, will necessarily all be negative, since the equal members of each pair possess opposite signs.

(2) *Imaginary points.*—If the points be given by two pairs of conjugate complex numbers, let them be

$$(1)' \quad m + in, \mu + i\nu$$

$$(2)' \quad m - in, \mu - i\nu$$

$$(3)' \quad m' + in', \mu' + i\nu'$$

$$(4)' \quad m' - in', \mu' - i\nu'$$

Of the 3 possible values for the area *one* is immediately found to be real, viz.—

$$(x_1 - x_2)(y_3 - y_4) - (x_3 - x_4)(y_1 - y_2)$$

which is

$$2in \times 2i\nu' - 2in' \times 2i\nu = 4(\nu n' - \nu' n)$$

Each of the other two is *wholly* imaginary

$$\begin{aligned} \text{e.g.} \quad & \{(m - m') + i(n - n')\} \{(\mu - \mu') - i(\nu - \nu')\} \\ & - \{(m - m') - i(n - n')\} \{(\mu - \mu') + i(\nu - \nu')\} \\ & = 2i\{(\mu - \mu')(n - n') - (m - m')(\nu - \nu')\} \end{aligned}$$

It follows that, of the three values of $-A''^2$, still all real, one is negative and the other two are positive.

The parallelism between these two cases and the discrimination given by the z' cubic renders it highly probable that the roots of this equation are proportional of A''^2 and such in effect turns out to be the case.

Let us investigate an expression for the value of A'' by referring the two conics to their common self-polar triangle.

$$\text{Let } S = ax^2 + by^2 + cz^2 \text{ and } S' = a'x^2 + b'y^2 + c'z^2$$

Then the 4 points are given by the intersection of the line pairs

$$\left. \begin{aligned} (ab' - a'b) x^2 &= (bc' - b'c)z^2 \\ (ab' - a'b) y^2 &= (ca' - c'a)z^2 \\ (ca' - c'a) x^2 &= (bc' - b'c)y^2 \end{aligned} \right\} \dots \dots \dots \text{(ii.)}$$

The λ cubic is $\Pi(a\lambda + a') = 0$, $\Delta = abc$, $\Delta' = a'b'c'$, etc.

Moreover

$$\left. \begin{aligned} bc \sin^2 A + ac \sin^2 B + ab \sin^2 C &= \Sigma \\ b'c' \sin^2 A + a'c' \sin^2 B + a'b' \sin^2 C &= \Sigma' \\ (bc' + b'c) \sin^2 A + (ac' + a'c) \sin^2 B + (ab' + a'b) \sin^2 C &= \phi \end{aligned} \right\} \dots \text{(iii.)}$$

Let the quadrilateral formed by the 4 points of intersection be called $P_1P_2P_3P_4$; let P_1P_4 and P_2P_3 meet in A_0 , P_1P_2 and P_3P_4 in B_0 , and P_1P_3 and P_2P_4 in C_0 where $A_0B_0C_0$ is the triangle of reference whose area will be called T' .

$$\begin{aligned} \text{The equation to } B_0P_3P_4 \text{ is } \sqrt{ab' - a'b}x &= \sqrt{bc' - b'c} z \\ \text{,, ,, } B_0P_2P_1 \text{ is } \sqrt{ab' - a'b}x &= -\sqrt{bc' - b'c} z \\ \text{etc., etc.} \end{aligned}$$

Let the first value of A'' , viz. (Tri. $P_1P_2P_3$ + Tri. $P_1P_3P_4$), be calculated by the use of Walker's formula for the area of a triangle, which may be applied successively to the triangles $A_0P_1P_2$ and $A_0P_3P_4$, whose difference will give A'' .

This formula for the area of a triangle determined by the lines $lx + my + nz$, $l'x + m'y + n'z$, $l''x + m''y + n''z$ is :

$T' \sin A \sin B \sin C$	l	m	n	2
	l'	m'	n'	
	l''	m''	n''	
Π	$\sin A$	$\sin B$	$\sin C$	
	l	m	n	
	l'	m'	n'	

This value of A'' appears as

$$\frac{-8T' \sin A \sin B \sin^2 C (ab' - a'b) (bc' - b'c)^{\frac{1}{2}} (ca' - c'a)^{\frac{1}{2}}}{\{(bc' - b'c)^2 \sin^4 A + \text{two similar terms} - 2 (bc' - b'c) (ca' - c'a) \sin^2 A \sin^2 B - \text{two similar terms}\}}$$

whose denominator by (iii) is easily recognised as $(\phi^2 - 4\Sigma\Sigma')$

$$\text{(iv.) and } \therefore A''^2 = \frac{16 \{4T'^2 \sin^2 A \cdot \sin^2 B \cdot \sin^2 C\} (ab' - a'b)^2 (bc' - b'c) (ca' - c'a) \sin^2 C}{(\phi^2 - 4\Sigma\Sigma')^2}$$

with two symmetrical expressions of a similar kind for the two other values of A''^2 .

Now $4T'^2 \sin^2 A \sin^2 B \sin^2 C$ is clearly equal to μ^4 where μ^* is the modulus of transformation between the present system of coordinates and a Cartesian system, and so one may conveniently write the last results in the form :

$$(v.) \quad \frac{A''^2 (\phi^2 - 4\Sigma\Sigma')^2}{16\mu^4} = (ab' - a'b)^2 (ca' - c'a) (bc' - b'c) \sin^2 C$$

with two similar expressions.

If equations (iii.) be solved for $\sin^2 A$, $\sin^2 B$, $\sin^2 C$, it is easily found that

$$(ab' - a'b)^2 (ca' - c'a) (bc' - b'c) \sin^2 C = -(ab' - a'b)^2 \{c'^2 \Sigma - c^2 \Sigma' - cc' \phi\}$$

$$(vi.) \quad \text{And } \therefore \frac{A''^2 (\phi^2 - 4\Sigma\Sigma')^2}{16\mu^4} = -(ab' - a'b)^2 \{c'^2 \Sigma - c^2 \Sigma' - cc' \phi\}$$

$$\text{but } -\frac{a'}{a} = \lambda_1, -\frac{b'}{b} = \lambda_2, -\frac{c'}{c} = \lambda_3$$

and $abc = \Delta$, etc.

It follows at once by utilising these values is the right-hand side of (vi.) that

$$(vii.) \quad -\frac{A''^2 (\phi^2 - 4\Sigma\Sigma')^2}{16\mu^4} = \Delta^2 (\lambda_1 - \lambda_2)^2 (\Sigma \lambda_3^2 + \phi \lambda_3 + \Sigma')$$

= a root of the z' cubic (VIII) See page 6.

with two similar expressions.

This interesting result shows that if

$$z' = -\frac{A''^2 (\phi^2 - 4\Sigma\Sigma')^2}{16\mu^4}$$

the three values for the square of the area of the quadrilateral formed by the four points of intersection of two conic sections can be obtained from the cubic equation

$$(viii.) \quad z'^3 + \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix} z'^2 + \frac{1}{4} \left\{ \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix}^2 - D(\phi^2 - 4\Sigma\Sigma') \right\} z' + D\Gamma'^2 = 0$$

and the metrical significance of the criteria for four real and four imaginary intersections is at once apparent.

We found that for four real intersections the above cubic had three real negative roots, and this corresponds perfectly to three real positive values for A''^2 as we should geometrically anticipate.

If the cubic had one negative and two positive roots there were four imaginary intersections. This is again in harmony with the values for A''^2 , still all real, one of them positive and two negative, which we should be led to expect.

The present appendix not only illustrates and verifies the general theory, but has also solved a special metrical problem, whose results can be expressed very simply in certain particular cases.

$$* \text{ If } \begin{cases} x = X \cos \alpha + Y \sin \alpha - p_1 \\ y = X \cos \beta + Y \sin \beta - p_2 \\ z = X \cos \gamma + Y \sin \gamma - p_3 \end{cases} \quad \mu = \begin{vmatrix} \cos \alpha & \sin \alpha & -p_1 \\ \cos \beta & \sin \beta & -p_2 \\ \cos \gamma & \sin \gamma & -p_3 \end{vmatrix} = \frac{T'}{R}$$

$$\therefore \mu^3 = \frac{2T'^3}{R} \cdot \sin A \sin B \sin C$$

$$\text{and } \mu^2 = 2T' \sin A \sin B \sin C.$$

For example, if the conics touch, $D=0$, and the area of the triangle to which the quadrilateral reduces, is given by

$$z' \left\{ z' + \frac{1}{2} \left| \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{array} \right| \right\}^2 = 0$$

If two corners of the quadrilateral coincide, one value of A'' clearly vanishes, and the other two coincide,

$$\text{and } A''^2 = 8\mu^4 \frac{\left| \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{array} \right|^2}{(\phi^2 - 4\Sigma\Sigma')^2}$$

Again, a simple case arises when the intersections form a parallelogram, *i.e.*, when the conics are concentric, for in this case $\Gamma'=0$, and by virtue of the fact that $\Sigma\lambda^2 + \phi\lambda + \Sigma'$ becomes a factor of the λ cubic, it can be readily shown that

$$\left| \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{array} \right|^2 = D(\phi^2 - 4\Sigma\Sigma')$$

Two of the values of A'' vanish and the third appears at once from the reduced cubic,

$$z'^2 \left\{ z' + \left| \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta' & 2\theta' & 3\Delta' \end{array} \right| \right\} = 0,$$

$$\text{as } A''^2 = 16\mu^4 \frac{\left| \begin{array}{ccc} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{array} \right|^2}{(\phi^2 - 4\Sigma\Sigma')^2}$$

It is possible to deduce many other similar results.

APPENDIX II.

The distinctions in the signs of $\Sigma\lambda^2 + \phi\lambda + \Sigma'$ for the different values of λ and their bearing on the nature of the intersections of the two conics may be further exemplified by a reference to the value T of the area of the common self-polar triangle which may be found either synthetically or more instructively as follows :—

If x, y, z , etc., be vertices of this triangle we have

$$(i.)' \quad 2\mu T = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Multiply this by Δ and we get

$$(ii.)' \quad 2\mu\Delta T = \begin{vmatrix} ax_1 + by_1 + gz_1 & ax_2 + by_2 + gz_2 & ax_3 + by_3 + gz_3 \\ hx_1 + by_1 + fz_1 & hx_2 + by_2 + fz_2 & hx_3 + by_3 + fz_3 \\ gx_1 + fy_1 + cz_1 & gx_2 + fy_2 + cz_2 & gx_3 + fy_3 + cz_3 \end{vmatrix}$$

The multiplication of (i.)' and (ii.)' yields

$$(iii.)' \quad 4\mu^2\Delta T^2 = S_1 S_2 S_3, \text{ where } S_1 = (abcfgh) (x_1 y_1 z_1)^2, \text{ etc.} \\ \text{(the other constituents vanishing, etc.)}$$

Equations (I) of this paper show that

$$(iv.)' \quad \begin{aligned} x_1^2 & \text{ may be written } -m^2 (A\lambda_1^2 + K\lambda_1 + A'), \text{ etc., etc.} \\ x_1 y_1 & \text{ ,, ,, } -m^2 (H\lambda_1^2 + M'\lambda_1 + H'), \text{ etc., etc.} \end{aligned} \\ \text{where } m^2 \text{ must be expressed appropriately.}$$

$$\text{Now } T' \text{ (Tri. of reference)} = R (\sin A x_1 + \sin B y_1 + \sin C z_1)$$

$$\therefore \text{ by (iv.)}' \quad \mu^2 = (\sin A \sqrt{-(A\lambda_1^2 + K\lambda_1 + A')} + \text{similar terms})^2 \cdot m^2$$

$$\text{or } m^2 = \frac{\mu^2}{(\sin A \sqrt{-(A\lambda_1^2 + K\lambda_1 + A')} + \text{similar terms})^2}$$

The values given by (iv.)' may now be used to calculate S_1 , etc., and it appears that

$$S_1 = \frac{-\mu^2 (3\Delta\lambda_1^2 + 2\theta\lambda_1 + \theta')}{(\sin A \sqrt{-(A\lambda_1^2 + K\lambda_1 + A')} + \text{similar terms})^2}$$

Wherefore

$$(v.)' \quad 4\mu^2\Delta T^2 = S_1 S_2 S_3 = + \frac{\mu^6 \Pi(3\Delta\lambda_1^2 + 2\theta\lambda_1 + \theta')}{\Pi(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')}$$

by Equations III

$$\text{or } T^2 = \frac{\mu^4}{4\Delta} \cdot \frac{\Pi(3\Delta\lambda_1^2 + 2\theta\lambda_1 + \theta')}{\Pi(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma')}$$

Now
$$3\Delta\lambda^2 + 2\theta\lambda + \theta = \frac{d}{d\lambda} (\Delta\lambda^3 + \theta\lambda^2 + \theta\lambda + \Delta)$$

$$= \Delta \{ (\lambda - \lambda_2)(\lambda - \lambda_3) + \&c. \}$$

\therefore
$$\Pi(3\Delta\lambda_1^2 + 2\theta\lambda_1 + \theta) = -\Delta^3\Pi(\lambda_1 - \lambda_2)^2$$

$$= -\frac{D}{\Delta}$$

(vi)' or
$$T^2 = -\frac{\mu^4}{4} \cdot \frac{D}{\Delta^2\Pi(\Sigma\lambda_1^2 + \phi\lambda_1 + \Sigma)}$$

and finally $T^2 = \frac{\mu^4}{4} \cdot \frac{D}{I^2}$, a result which could be obtained synthetically without any of the illumination revealed by a dissection of its denominator.

The result of (vi)', when D is positive, agrees with the fact that whether there be four real or four imaginary intersections, T^2 is real and positive, and T real, in virtue of what the paper has proved concerning the denominator.

When D is negative for two real and two imaginary intersections, T appears again as imaginary.