SPONTANEOUS SYMMETRY-BREAKING IN QUANTUM FIELD THEORY

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GEORGE F. REID, B.Sc.

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C O N T E N T S

INTRODUCTION

The idea that the observed masses of elementary particles may be regarded as being generated totally by the interactions to which the particles are subject, is, in terms of practical description, a relatively recent one. Coupled with the observation that the world is, loosely speaking, approximately symmetric, as indicated by the mass differences of particles, such an idea would suggest the possibility that dynamics may, itself, be the mechanism whereby the symmetry is broken. Such is the underlying thought behind the recent interest in what has come to be termed "the spontaneous breakdown of symmetry", characterised by non-symmetrical solutions to symmetric field equations.

The central ideas, as presented by Goldstone[1], and Nambu and Jona-Lasinio[2], stem almost completely from the solution of the superconductor problem given by Bardeen, Cooper and Schrieffer[3], and its subsequent refinements by Bogoliubov, Tolmacev and Shirkov[4], Valatin[5], Nambu[6], and others, the main inspiration of these preliminary investigations having been the close formal analogy existing between the 'superconducting' electron, which possesses an energy gap in its spectrum by virtue of the breakdown of simple gauge symmetry,[×] and the relativistic Dirac electron, having a mass gap derivable from the breakdown of chiral symmetry. These, and subsequent attempts[7] to account for mass splittings by assuming the spontaneous breakdown of internal symmetries, have been handicapped by the apparently general prediction, via a theorem due to Goldstone[1],

× More correctly, both the energy gap and symmetry breakdown arise as a consequence of the dynamics.

that any theory admitting solutions of the type associated with the spontaneous breakdown of symmetry must contain massless bosons in the spectrum. The local conservation law of current, derived from the original symmetry, plays the essential role in establishing the theorem, for which various formal proofs have been given[8-11]. Since, by observation no such boson excitations could be accounted for, recent interest tends to have been centred on ways and means of removing them from such theories, with reference to simple models. In this respect, the theory of superconductivity has played a major part, in view of the proven absence of such corresponding excitations in that context[12]; the long-range Coulomb force has proved to be the agent responsible for their non-appearance[13,14]. The immediate generalisation of this effect in relativistic theory suggests[15,16,17] the conclusion that when gauge invariance of the second kind on local symmetry, necessitating the introduction of a long range force in the form of a gauge field, is combined with the spontaneous breakdown of global symmetry, then neither the Goldstone particles nor the anticipated massless vector particles appear; they are replaced by massive vector particles. This conforms with the contention of Schwinger[18] that gauge invariance need not imply massless vector particles.

The main object of this thesis is to consider the question of the Goldstone Theorem, and its domain of applicability, both in non-relativistic and relativistic quantum field theory.

To this end, the first chapter is of an introductory nature, the emphasis being on outlining the main ideas relating to spontaneous symmetry breakdown, while at the same time surveying the recent literature on the subject. In Sections (1.1) and (1.2), using the scalar field doublet model of Goldstone as our basic example, we indicate how the notion of vacuum degeneracy appears as a more general possibility in field theory, and demonstrate its connection with inequivalent representations of the field operator algebra, Analogous arguments for fermion systems are mentioned. Still arguing from Goldstone's model, in (1.3), we indicate the likelihood of such solutions from a classical basis, and then proceed to show the apparent equivalence between the presence of massless particles and the assumption of vacuum degeneracy by the formally exact arguments of Goldstone, Salam, Weinberg, and others. Objections to these arguments are mentioned. In (1.4), the fact that superconductivity, as a non-relativistic field theory, runs counter to these arguments is discussed, and the reasons, suggested by Lange, and others, are given. Finally, in (1.5) and (1.6), the work of Guralnik, Hagen, and Kibble, Higgs, and others, indicating how the theorem can be invalidated by extending Goldstone's model, is described.

The remaining chapters deal with topics relating to the models described in Chapter I.

In Chapter II, it is shown how the quasi-particle spectrum of the Goldstone Model may be derived by a formal argument based on the Lehmann-Kallen representation, the results so obtained being equivalent to those obtained by other well-known methods. The difficulty of drawing any conclusion about the validity, or otherwise, of Goldstone's Theorem, from this viewpoint, is indicated.

The content of Chapter III concerns the distinction to be drawn between manifest Lorentz covariance and non-covariance as they are related to the validity of Goldstone's Theorem. It is demonstrated how the extension of global to local invariance in Goldstone-type models leads to the inapplicability of the theorem, on account of the necessarily non-covariant description then demanded for consistency.

In Chapter IV, the connection between the Hartree-Fock method in the theory of superconductivity, and the appearance or non-appearance of Goldstone excitations is discussed.

Finally, in Chapter V, the spontaneous breakdown of chiral symmetry in the context of electrodynamics is considered.

CHAPTER 1

PRELIMINARY SURVEY

In this chapter we undertake to review some of the current literature relating to the ideas of spontaneously broken symmetry, gauge invariance, and mass, and the connections which have been deduced between them. Before proceeding, however, it may be relevant to dwell a little on the motivation behind the recent interest in these topics.

The most satisfactory application of quantum field theory to elementary particle processes, has been that to ordinary electrodynamics, the interaction between electrons and photons. The relative success of this theory seems to be built upon two fortunate occurrences, viz. the all but exact specification of the form of the interaction Lagrangian, and also the convenient apparent applicability of perturbation theory. The former is based on a principle of gauge invariance, which has its origin in two verifiable facts: there exists a conservation law of electric charge, and the photon is massless. The latter, although the existence of the perturbation series has not been proved, may be viewed as a consequence of the smallness of the coupling constant, and the mathematical procedures (e.g. renormalization) involved are accepted, if only because the correct results (in particular, the Lamb Shift) can be extracted. We note that the field theory employed in quantum electrodynamics is phenomenological only to the extent that the 'bare' masses of the interacting fields are inserted.

Accepting that the use of quantum field theory must have some validity, even if obscure, in the domain of electrodynamics, we compare the situation in the region of strong interactions. The difficulty with any perturbation theoretic framework presents itself immediately, for the coupling constants are known to be large, notwithstanding the question of the existence of the perturbation series. Further, the strongly interacting particles are regarded as deviating from unitary symmetry, this being exhibited through the observed mass differences. To account for this deviation would in the usual theory mean having to define specifically the form of interaction believed to be responsible. The difficulties associated with perturbation theory would follow any satisfactory statement of how the symmetry is broken.

However, in the strong (and weak) interactions there do exist conservation laws, and many authors[19] have been led to consider how it might be possible to view the interaction of these particles through a gauge principle, similar to that appearing in electrodynamics[20]. But the required massiveness of the vector particles produces the main barrier to such an approach. The demand for gauge invariance of the second kind seems to necessitate the introduction of massless vector fields; the presence of any bare mass term in the Lagrangian function will destroy the invariance, and with it the required conservation law. Thus, if we take the comparison with electrodynamics seriously, we would have to reconcile the massiveless of the vector particles with the existence of a conservation law, as well as accounting for the mass splitting in the multiplets. This is the basic motivation in asking about the connections between gauge invariance, mass, and broken symmetry.

A conservation law in a field theory arises if the Lagrangian function, or equivalently the field equations, is invariant under an internal symmetry group of transformations, the conserved quantity being a generator of the transformations. The question which has been asked recently is whether one can have such a conserved current, when it is known that the symmetry of the physical particles must be broken. The emphasis on the term 'physical' is important, for the field equations do not in themselves constitute physics, but rather the solutions of the field equations characterized by the spectrum of states built on the vacuum or ground state determine the physical situation. Thus, in an alternative form, the question is: can one have solutions of symmetrical field equations, which do not possess the same symmetry, and if so, have they any physical significance? Because of the general lack of mathematical rigour inherent in present field theory arguments, the answers to these questions must be conjectural to some degree, but there do exist physical many-particle systems, in particular the superconductor, described by non-relativistic field theory, which appear to be necessarily described by such solutions. Furthermore, the fact that they happen to be non-perturbative in the coupling parameters is, perhaps, a reason, in itself, warranting investigation.

1.1 Vacuum Degeneracy

The symmetries of elementary particle physics fall roughly into two categories, those connected with space-time related to the dynamics, and the internal symmetries, described by continuous symmetry groups. It is in the latter category that interest has centred so far as spontaneous symmetry breakdown is concerned, although attempts have been made to include the Lorentz group in such considerations[21]. We shall restrict ourselves here to a discussion of the implications of spontaneous breakdown as they appear in the simplest of models.

The models which serve as the source of the current investigations are due to the work of Nambu and Jona-Lasinio who consider the self-interacting fermion system, and Goldstone who, in addition, considers the boson case. Both models, commonly referred to in the literature as 'superconductor' models, are invariant under continuous symmetry groups, respectively the γ_5-group and the U(1) gauge group. Because the latter has a direct classical analogy, exploited by Goldstone himself, and others, we shall discuss this first with a view to defining in the quantum theory the nature of the symmetry breakdown.

The simplest non-trivial continuous symmetry group describing the interaction of spinless, Hermitian scalar fields is U(1). If $\phi(x)$ is a complex scalar field, where

$$\phi = \frac{1}{2^{1/2}}(\phi_1 + i\,\phi_2)\ , \qquad \phi^* = \frac{1}{2^{1/2}}(\phi_1 - i\,\phi_2)$$

(ϕ_1, ϕ_2) being a Hermitian scalar doublet. Then we may describe the interaction of the real fields under U(1) invariance by the general Lagrangian

$$\begin{aligned}\mathcal{L}(x) &= -\partial_\mu\phi^*(x)\,\partial^\mu\phi(x) - V(\phi^*\phi)\\ &= -\tfrac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - V\{\tfrac{1}{2}(\phi_1^2+\phi_2^2)\}\end{aligned} \tag{1}$$

where $V(\tfrac{1}{2}(\phi_1^{\,2} + \phi_2^{\,2}))$ represents the interaction and possibly contains a bare mass term. The related field equations are

$$\Box\phi_i(x) - V'\{\tfrac{1}{2}(\phi_1^2+\phi_2^2)\}\,\phi_i(x) = 0 \tag{2}$$

In the actual model considered by Goldstone,

$$V_G\,(\phi^*\phi) = -\lambda^2\phi^*\phi + \mu^2(\phi^*\phi)^2$$

λ^2, μ^2 being understood as coupling constants.

The theory is invariant under the (global) transformations:

$$\phi(x) \to \phi'(x) = e^{-i\alpha}\phi(x) \quad ; \quad \phi_i(x) \to \phi_i'(x) = R_{ij}(\alpha)\,\phi_j(x)$$

$$R_{ij}(\alpha) = \delta_{ij} + \alpha\,T_{ij} + \ldots \quad , \quad R_{ij}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad (3)$$

and so by the Noether Theorem, we have a microscopically conserved current

$$j^\mu_{(x)} = g^{\mu\rho} j_\rho(x) = \alpha\,\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}}\,T_{ij}\,\phi_j(x) = \alpha\,\pi^{i\mu}\,T_{ij}\,\phi_j$$

$$j^\mu(x) = -\alpha\left[\phi_2(x)\,\partial^\mu\phi_1(x) - \phi_1(x)\,\partial^\mu\phi_2(x)\right] \qquad (4)$$

Applying the divergence theorem (assuming for the moment its application to be valid) yields the globally conserved quantity (charge)

$$\alpha F_\sigma = \int_\sigma d\sigma_\mu\, j^\mu(x) = \int d^3\underline{x}\; j^0(x) \quad (\text{independent of } \sigma) \qquad (5)$$

which generates the original field transformation

$$\phi_i(x) \to \phi_i{}'(x) = U\,\phi_i(x)\,U^{-1} = R_{ij}(\alpha)\,\phi_j(x) \qquad (6)$$

where U is a unitary operator, given by

$$U = e^{-i\alpha F} .$$

This is the usual argument leading to Noether's Theorem, and is normally taken to be independent of V, the interaction. If we compare terms of first order in α in equation (6), then we find the commutation rule

$$i\left[F, \phi_i(x)\right] = T_{ij}\,\phi_j(x) \tag{7}$$

and, using equations (4) and (5), we find consistency with the usual boson equal-time commutation relations

$$\left[\pi^i(x), \phi^j(x')\right]_{x_0=x_0'} = -i\delta^{ij}\delta^{(3)}(\underline{x}-\underline{x}') \; ; \; \left[\phi^i(x), \phi^j(x')\right]_{x_0=x_0'} = 0 \tag{8}$$

We point out that the equations of motion, the Noether Theorem, and the canonical commutation relations, all may be viewed as a consequence of the Generalised Action Principle. In any normal theory, we would expect that all may be consistent. However, it may be untrue, in certain circumstances, that the generator F is independent of x_0. This is a dynamical assumption which depends ultimately on V. In fact it turns out that the occurrence of vacuum degeneracy must be accompanied by a failure of the global law.

We now invoke the usual axioms of field theory. The solutions of the equations of motion (2) may be characterised by the complete set of states $|p\rangle$, labelled by the 4-momentum, with a normalisable vacuum state (p = 0). In any normal theory, we would regard the vacuum to be unique, so that

$$U|0\rangle = |0\rangle$$

up to a phase factor, in which case F is a good quantum number, $[F,H] = i\dot{F} = 0$, with the vacuum and the other excited states as eigenstates.

Now, we suppose that the solutions do not obey the symmetry of the field equations. This implies a restriction on V, of course. We may express the existence of such anomalous solutions by

$$U(\alpha)\,|\,0,\,0\rangle \;=\; |\,0,\,\alpha\rangle \tag{9}$$

where $|\,0,\,\alpha\rangle$ differs from $|\,0\rangle$ by something other than a trivial phase factor. We say that the vacuum is degenerate with respect to the symmetry group, in this case U(1), taking up a chosen direction in 'charge-space'. Mathematically, in general, it transforms according to a non-trivial representation of the group.

We shall now show how this degeneracy can account for a mass difference between the particles described by the fields $(\phi_1,\,\phi_2)$. The spectral function $\langle 0\,|\,\phi(x)\,\phi(x')\,|\,0\rangle$, for the complex field, can, by virtue of (9) be written†

$$\langle 0\,|\,\phi(x)\,\phi(x')\,|\,0\rangle = \langle 0,\,\alpha\,|\,\phi(x)\,\phi(x')\,|\,0,\,\alpha\rangle e^{-2i\alpha} \quad .$$

If the vacuum were invariant, then we would have to conclude that this spectral function were zero, which would, in turn, imply the equality of $\langle 0|\phi_1(x)\,\phi_1(x')|\,0\rangle$ and $\langle 0|\phi_2(x)\,\phi_2(x')|0\rangle$. Thus, the existence of a degenerate vacuum implies the inequality of these functions in a particular gauge ($\alpha = 0$), and hence the inequality of the associated spectral weight functions, if one accepts that $(\phi_1,\,\phi_2)$ are the physical fields. That they need not be the physical fields can be inferred by applying the same argument to the one-point function $\langle\phi(x)\rangle_0$, which could be a non-vanishing constant in the event that the vacuum was degenerate, but translationally invariant.× (It should be noted that the occurrence of vacuum degeneracy would be indicated by the non-vanishing of the expectation value of any product of field operators[1] of the form $\phi_1^m\,\phi_2^n$ (m or n odd))

† **Note 1:** if $\langle 0,\alpha|\phi(x)\phi(x')|0,\alpha\rangle$ is independent of α then $(e^{-2i\alpha} - 1)\,\langle 0|\phi(x)\phi(x')|0\rangle = 0$.

× [...]t order symmetry breaking: $\langle\phi\rangle \neq 0$, $\langle\phi\phi\rangle \neq 0$ etc. 2nd order symmetry breaking: $\langle\phi\rangle = 0$, $\langle\phi\phi\rangle \neq 0$ e[...]

Thus, in terms of the usual formulation of field theories through the n-point functions, the anomalous solution may be defined generally by the non-vanishing of the one-point function. We could thus redefine the complex field by one whose vacuum expectation value vanished.

$$\phi(x) = \langle \phi(x) \rangle_0 + \chi(x) \qquad (10)$$

and so also for the real and imaginary components. What we call the 'physical gauge' (say, $\alpha = 0$) may be fixed by having either $\langle \phi_1 \rangle_0$ or $\langle \phi_2 \rangle_0$ non-zero, as we see from the relation

$$\langle \phi(x)\phi(x') \rangle_0 = \langle \phi_1 \rangle_0^2 - \langle \phi_2 \rangle_0^2 + \langle \chi_1(x)\chi_1(x') \rangle_0 - \langle \chi_2(x)\chi_2(x') \rangle_0$$
$$+ i \left(2\langle \phi_1 \rangle_0 \langle \phi_2 \rangle_0 + 2\langle \chi_1(x)\chi_2(x') \rangle_0 \right)$$

the imaginary part being required to vanish.* We take $\langle \phi_2 \rangle_0 \neq 0$. Thus, the condition we obtain on the 2-point function is

$$\langle \chi_1(x)\chi_1(x') \rangle_0 - \langle \chi_2(x)\chi_2(x') \rangle_0 - \langle \phi_2 \rangle_0^2 \neq 0$$

which, by itself, does not say that the propagators cannot be equal, but certainly envisages the possibility.

If such a solution exists, it necessarily follows by taking the vacuum expectation value of equation (7), that the vacuum cannot be an eigenstate of the generator, F. We recall equation (9), in which the unitary transformation generated by F had the effect of 'rotating' the vacuum. In fact it can be shown that F is undefined, the vacuum being, as it were, able to mop up 'charge'. Further, we might expect a formal time-dependence in F, because of the linear (physical) field dependence possessed by the current.

Note: $[F(t), H] = \lim_{\Omega, \Omega' \to \infty} \int d^3\underline{x}\, d^3\underline{x}' \, [j^0(x), \mathcal{H}(x')] \neq 0$.

* We assume there is no further spontaneous breakdown of the discrete symmetry $\chi \to -\chi$.

The scalar model of Goldstone provides the simplest non-trivial example of the spontaneous symmetry breakdown we have just discussed. Without yet discussing the why and the wherefore of the vacuum degeneracy, we have indicated the likely consequences of it, in particular, the strong likelihood of 'mass-splitting'. These simple considerations can be generalised to O_N and other continuous groups.

In the case of self-interacting fermion systems, the situation is not quite so simple. Nambu and Jona-Lasinio considered a theory invariant under both the ordinary and γ_5-gauge transformations. Being a single field model, there is not the same interpretation as in the scalar model (essentially a 2-field problem). However, if one thinks of the model in terms of primary (or quasi-particle) and secondary (or collective) excitations, as in the many-body problem, then one effectively has a mass-difference between the associated particles, if one assumes the interaction is capable of maintaining vacuum degeneracy. The requirement of broken symmetry should be placed on bilinear products of fermion operators, rather than on that of the single operator. This, alternatively, means a condition on the field propagator. It has been pointed out that the degeneracy condition should be imposed on objects which can be said to be observables, automatically valid in the boson case.

The argument of Nambu and Jona-Lasinio and others is based on analogy with the B.C.S. model of supercondutivity - hence the terminology 'superconductor' solution. In that case, the quasi-particle excitations are regarded as a coherent mixture of particle-hole pairs around the Fermi-level, where there exists a strong virtual phonon coupling. In momentum space, the wave functions satisfy the

the Bogoliubov-Valatin Relations:

$$E_P \psi_P = \varepsilon_P \psi_{P\uparrow} + \Delta \psi^+_{-P\downarrow}$$

$$E_P \psi^+_{-P} = -\varepsilon_P \psi^+_{-P\downarrow} + \Delta \psi_{P\uparrow}$$

where E_P is the quasi-particle energy, ε_P effectively the free-particle energy, and Δ is the symmetry-breaking parameter, related to the ground state expectation value $\langle \psi^+_1(x) \psi^+_2(x) \rangle_o$. The eigenvalues give

$$E_P = \pm \left[\varepsilon_P^2 + \Delta^2 \right]^{1/2}$$

which shows that there is an energy gap 2Δ, determined self-consistently, separating the states below the Fermi-Surface from those above. The resemblance to the Dirac Theory of the electron is self-evident. There, there is a 'mass gap', 2m, separating the infinite sea of negative energy states from the physical positive energy states. Further the Dirac equation can be decomposed in terms of its eigenstates of chirality, analagous to the Bogoliubov equations

$$E\psi_1 = \sigma.p\, \psi_1 + m\psi_2$$

$$E\psi_2 = -(\sigma.p)\psi_2 + m\psi_1$$

whence $$E_P = \pm \left[P^2 + m^2 \right]^{1/2} .$$

Hence, in analogy with B.C.S., if one considers an interaction between, say, massless fermions, which is capable of possessing non-γ_5-symmetric solutions, then one is led to a mechanism for mass generation in the quasi-particle spectrum.

It should be emphasised that the existence of this type of degeneracy of the vacuum implies degeneracy of all states built upon the vacuum, in which case the breakdown of symmetry may be described by the more general condition on the field operator (or product of field operators) that its expectation value with respect to the state $|P\rangle$ is non-vanishing.

Finally, we note that inherently within relativistic quantum field theory there already exists the possibility of vacuum degeneracy in an exact sense[22], although it seems to manifest itself in only the most trivial cases, namely that of the massless field, satisfying the field equation (corresponding to (2) above)

$$\Box \phi = 0 .$$

Such a system is invariant under the trivial continuous symmetry, the field translation $\phi(x) \rightarrow \phi(x) + C$ so that, in an obvious notation, the connection between vacuum degeneracy and the existence of a non-vanishing field expectation value can be established thus:

$$\langle 0, C | \phi(x) | 0, C \rangle = C .$$

The field equation simultaneously embodies the microscopic conservation law, the current being $C\partial^{\mu}\phi$. The associated generator of the translation is therefore $C \int d^3\underline{x}\ \partial^0\phi$, and, using the canonical equal-time commutation relations in the form (8), we achieve consistency provided the quantity $\int d^3\underline{x}\ \partial^0\phi$ is independent of time (as would normally be implied by the field equation). A similar argument can be applied to the more realistic example of the free electromagnetic field, once again pertaining to the massless case.

It in fact turns out that it is no mere coincidence that massless particles should be singled out when vacuum degeneracy is involved.

These trivially exact examples do nothing more than obey what is known as the Goldstone Theorem, which essentially predicts the presence of massless particles within the excitation spectra of any system which undergoes spontaneous breakdown of symmetry, in the manner described above.

1.2 The Bogoliubov-Valatin Transformation

Having defined what we mean in this context by symmetry-breaking, it is useful to relate the definition to the representations of the operator algebra of the fields. Such considerations apply to both fermion[23] and boson systems[24], but our main interest here will be in the model of Goldstone.

The decomposition (corresponding to the particular gauge choice $\langle \phi_1 \rangle_0 = 0$, $\langle \phi_2 \rangle_0 \neq 0$)

$$\phi_1(x) = \chi_1(x) \quad , \quad \phi_2(x) = \langle \phi_2(x) \rangle_0 + \chi_2(x)$$

may be viewed as a general representation of the canonical commutation relations, $\langle \phi_2(x) \rangle_0$ requiring to be determined in order to fix the physical representation. If we consider the system to be confined to a volume Ω , then the most general plane-wave representation of the fields is

$$\phi_r(\underline{x}) = \langle \phi_r \rangle_0 + \Omega^{-1/2} \sum_{\underline{k}} \left(u_k^r \alpha_k^r e^{i\underline{k}.\underline{x}} + u_k^{r*} \alpha_k^{r\dagger} e^{-i\underline{k}.\underline{x}} \right)$$

where $\langle \phi_r \rangle_0$, U_k^r are parameters to be found.

(The particular case

$$\langle \phi_r \rangle_0 = 0 \quad , \quad u_k^r = (2W_k^r)^{-1/2} \quad , \quad W_k^{r\,2} = \underline{k}^2 + m^2 \quad \text{(all } r\text{)}$$

corresponds to the usual representation for the Klein-Gordon field with $\alpha_k^r \equiv a_k^r$, say, and associated ground state $(\bar{\Upsilon})$, such that $a_k^r |\bar{\Upsilon}\rangle = 0$.)

This form may be understood to be built up as follows. The first term, being C-no., obviously cannot alter the commutation rules. The second term arises through the unitary Bogoliubov-Transformation, performed on the usual creation and annihilation operators

$$\alpha_k^r = \cosh\theta_k \, a_k^r + e^{i\varphi_k^r} \sinh\theta_k^r \, a_{-k}^{r\,\dagger}$$

$$\alpha_{-k}^{r\,\dagger} = e^{-i\varphi_k^r} \sinh\theta_k^r \, a_k^r + \cosh\theta_k^r \, a_{-k}^{r\,\dagger}$$

and also preserves the commutation rules, i.e.

$$\text{if } [a_k^r, a_{k'}^{s\,\dagger}] = \delta_{kk'}\delta^{rs} \quad \text{then} \quad [\alpha_k^r, \alpha_{k'}^{s\,\dagger}] = \delta_{kk'}\delta^{rs}$$

The associated unitary transformations are:

$$\alpha_k^r = G_1 a_k^r G_1^{-1} = a_k^r - \Omega^{1/2}\left(\frac{m}{2}\right)^{1/2} \bar{\eta}_r \, \delta_{k0}$$

$$\alpha_k^r = G_2 a_k^r G_2^{-1} = \cosh\theta_k^r \, a_k^r + e^{i\varphi_k^r} \sinh\theta_k^r \, a_{-k}^{r\,\dagger}$$

where

$$G_1 = e^{-\left(\frac{\Omega m}{2}\right)^{1/2}\left(\bar{\eta}\, a_0^r - \bar{\eta}^* a_0^{r\,\dagger}\right)}$$

$$G_2 = e^{\frac{1}{2}\sum_k \theta_k^r \left(e^{-i\varphi_k^r} a_k^r a_{-k}^r - e^{i\varphi_k^r} a_k^{r\,\dagger} a_{-k}^{r\,\dagger}\right)}$$

The combination of these transformations (generalised Bogoliubov-Valatin transformation) with associated operator G_1, G_2, yields the desired form, with

$$u_k = (2w_k)^{-\frac{1}{2}}\left(\cosh\theta_k - e^{-i\varphi_k}\sinh\theta_k\right) \; ; \; Re.(\bar{\eta}_r) = \langle\phi_r\rangle_0$$

The 'new' creation and annihilation operators now are associated with the ground states

$$|\Phi_0(\bar{\eta})\rangle = G_1|\Psi\rangle \ ; \quad |\Phi_0(\theta,\varphi)\rangle = G_2|\Psi\rangle \ ; \quad |\Phi_0(\theta,\varphi;\eta)\rangle = G_1G_2|\Psi\rangle$$

and, on forming the matrix elements

$$\langle\Psi|\Phi_0(\eta)\rangle = e^{-\frac{1}{4}\Omega m|\bar{\eta}|^2}$$

$$\langle\Psi|\Phi_0(\theta,\varphi)\rangle = e^{-\Omega(2\pi)^{-3}\int d^3k \log\cosh\theta_k}$$

we observe they vanish in the limit of infinite volume. Furthermore, we can show that

$$\langle\Phi_0(\theta,\varphi;\eta)|\Phi(\theta',\varphi';\eta')\rangle = 0 \quad ; \quad \theta'\neq\theta,\ \varphi'\neq\varphi,\ \eta'\neq\eta$$

as $\Omega \to \infty$. The effect of the transformations G_1 and G_2, separately or combined, is to render the vacuum states orthogonal in the limit of infinite volume. What we originally regarded as unitary operators are in fact improperly unitary, since their expectation values, in any representation, are zero. The representations are said to be inequivalent.

The physical representation meaning a specification of ϕ and θ is still to be derived. This may be done by demanding, in the infinite volume limit, that the Hamiltonian be of the diagonal form $\sum_k E_k a^+_{rk} a_{rk}$ [24]. However, even after this has been done, there remains an inequivalence, due to the vacuum degeneracy, and as such is of mathematical significance only.

We had, from the last section, that

$$U(\alpha)\,|0, 0\rangle = |0, \alpha\rangle .$$

In the above, because for $\phi = \phi'$, the orthogonality still persists, we may write

$$\begin{aligned} \langle 0, \alpha \,|\, 0, \beta \rangle &= 1, \quad \text{if} \quad \alpha = \beta \\ &= 0, \quad \text{if} \quad \alpha \neq \beta \end{aligned} \tag{11}$$

and so $U(\alpha)$ is improperly unitary, with the implication that the 'generator' F cannot be well-defined.

Exactly the same arguments hold in the theory of Nambu and Jona-Lisinio, and any other fermion model, relativistic or non-relativistic, in which the symmetry is broken by assuming non-vanishing expectation values for bilinear products of fermion fields. In these cases, the existence of the inequivalent representation is intimately connected with the Hartree-Fock variational calculation, in which one starts from a trial ground state vector, whose form manifestly lacks the symmetry of the Hamiltonian, and is always a variate of the B.C.S. state vector. It was shown, by Bogoliubov[25] that in the infinite volume limit, the Hartree-Fock treatment is exact. Thus, the inequivalent representations appear to be a relevant facet of the problem of describing broken symmetry theories.

Without as yet introducing any detailed discussion of dynamics, we have indicated how the assumption of broken symmetry, defined through vacuum degeneracy, points to the existence of inequivalent representations in the theory. The usual symmetric situation ($\langle\phi\rangle_0 = 0$) is but a special case within this framework. Whether or not we actually have such representations in a particular model must be a question of detailed dynamics.

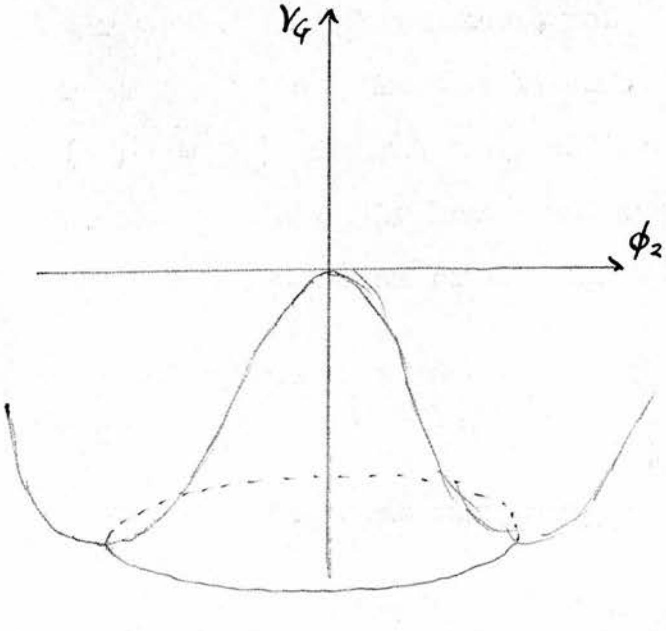
1.3 The Theorem of Goldstone

Originally a conjecture based on formal arguments, but now proved to general satisfaction, the theorem makes an assertion of an exact nature about the spectrum of states in any theory suffering spontaneous breakdown of symmetry (defined by a Lie group). That it should provide an exact result of field theory, has been the main reason for the interest aroused.

The statement of the theorem is as follows.

Given a Lorentz Invariant interacting field system, symmetric under a continuous Lie Group of Transformations, such that spontaneous breakdown of the internal symmetry occurs, then there must exist excitations, corresponding to zero mass, within the associated spectrum.

We shall see later that it is important to restrict the statement to continuous Lie Groups, as well as to manifestly Lorentz covariant theories. The Goldstone Model, governed by the interaction Lagrangian V_G , falls into this category, and was first discussed by Goldstone from a classical viewpoint. We may regard $V(\frac{1}{2}(\phi_1^2 + \phi_2^2))$, the interaction Lagrangian as a 'quasi-potential', so that the 'equilibrium' solution is given by the vanishing of V' , say (ϕ_1^0, ϕ_2^0) or by rotation, $(0, \phi_2^0)$. The quartic model obviously satisfies this result, with $(\phi_2^0)^2 = \frac{\lambda^2}{2\mu^2}$. Also, the 'stability' requirement, $V'' > 0$, is, in that case, satisfied, so that this solution is the physically acceptable one. The corresponding



field equations for small amplitude first order displacements from 'equilibrium' take the form

$$\Box \chi_1 = 0, \qquad (\Box - (\phi_2^0)^2 V'')\chi_2 = 0 ,$$

so that the degree of freedom associated with 'rotation' is massless, while that associated with the radial oscillations acquires mass. The absolute necessity of having a massless mode, in this approximation, is seen to be dictated by the original symmetry requirement imposed. For, by Noether's Theorem, the microscopic current is, to first order, $j_\mu(x) = -\phi_2^0\, \partial_\mu \chi_1$ and conservation yields just the above massless equation for χ_1.

This classical argument, it is argued, should serve as a reliable guide to the quantum field situation, since we are here dealing with a boson system which possesses a classical limit. The results, therefore, should, at least, correspond to the quasi-particle approximation of the quantum theory.

In attempting to set the argument on a quantum footing, Goldstone, Salam and Weinberg study the 2-point spectral function $\langle [J_\mu(x), \phi_1(x')] \rangle_0$, and render what can be termed a formally exact proof of the theorem which, as we shall see, depends ultimately on the manifest Lorentz covariance of the theory. By translational invariance, we may define the Fourier Transform as follows

$$f^\mu(k) = -i \int d^4x\, e^{ikx} \langle [j^\mu(x), \phi_1(0)] \rangle_0$$

where, because of current conservation,

$$k_\mu f^\mu(k) = 0 . \tag{12}$$

Also, from equation (8), the sum rule

$$\int \frac{dk_o}{(2\pi)} \quad f^o(k_o, \underline{k}) \quad = \quad \langle \phi_2 \rangle_o \tag{13}$$

must be satisfied for all $\underline{k}$.

Now, if we demand manifest Lorentz covariance, as we might reasonably expect to do in a Lorentz Invariant theory, then the most general form for $f^\mu(k)$ is

$$f^\mu(k) \; = \; k^\mu \left[\varepsilon(k_o)\, P_1(k^2) + P_2(k^2) \right],$$

where the functions $P_1(k^2)$, $P_2(k^2)$ are defined in terms of the intermediate states $|k_s\rangle$,

$$k^\mu \left[\varepsilon(k_o)\, P_1(k^2) + P_2(k^2) \right] = \sum_{\underline{k}k_o} \langle 0 | j^\mu(0)\, k_s \rangle\langle k_s | \phi_1(0) | 0 \rangle - \text{h.c.} \quad .$$

The imposition of microscopic conservation immediately gives, for all k^μ,

$$k^2 \left[\varepsilon(k_o)\, P_1(k^2) + P_2(k^2) \right] \; = \; 0$$

where $P_1(k^2)$ and $P_2(k^2)$ must either contain singularities at $k^2 = 0$, or vanish there identically. That is, they must contain contributions of the form

$$P_1(k^2) \; = \; C_1\, \delta(k^2)\,, \quad P_2 = C_2\, \delta(k^2)$$

where C_1, C_2 are constants which may or may not vanish. However, substitution in the sum rule shows that while no restriction need be applied to C_2, C_1 has to be given by

$$C_1 \; = \; \frac{\langle \phi_2 \rangle_o}{(2\pi)^{-1}}$$

and so if the asymmetrical solution exists, then there must definitely be a δ-function singularity at $k^2 = 0$, which we can only conclude comes from the presence of massless particles in the intermediate states which go to make up the Fourier representation.

Other formal proofs of the theorem have been offered by Bludman and Klein, Jona-Lasinio, and Domokos and Suranyl. These authors employ the functional formulation of field theory, and their conclusions concern the actual propagator of the ϕ_1-mode, rather than the above spectral function. All agree that the inverse momentum-space Green's Function must be zero, in the limit of vanishing 4-momentum, or when combined with Lorentz invariance, that there is a pole singularity at $p^2 = 0$, which in turn implies the presence of massless particles, as before.

Arguments against the result have been voiced, in the main by advocates of the use of the Bogoliubov-Valatin Transformation in the context of Goldstone's Model[23, 24]. These, however, have suffered from the problem of divergences, and since they appear in an approximation procedure, it would seem better to question the procedure, rather than the above formally exact arguments.

Of course, it should be stated that these 'proofs' are essentially non-rigorous, in that they envisage a decomposition of the particular spectral functions which may, in fact, not be meaningful. For example, in the proof of Goldstone, Salam and Weinberg, the decomposition takes place in terms of matrix elements of the type $\langle 0 | F | P_s \rangle$, which, on account of what has been said earlier, might be expected to lack definition.[x] If, however, the distinct parts do not have a meaning, then it is believed that the

x Note: $\langle\lambda_1|F|\lambda_2\rangle = \lim_{\Omega\to\infty} \int d^3\underline{x} \, \langle\lambda_1|j^0|\lambda_2\rangle$ may exist for many states $|\lambda_1\rangle$, $|\lambda_2\rangle$.

actual spectral function does. Furthermore, the conclusion that if the Green's Function inverse has a zero for zero 4-momentum, then it describes the propagation of massless particles, is not really proved. For, as pointed out by Schwinger[26], a possible representation of the propagator having the correct properties is

$$\int d^4x\, e^{-ikx} \langle T(\phi(x)\phi(0))\rangle_0 = G(k^2) = \left[k^2+\mu^2+k^2\int_0^\infty \frac{dm^2 A(m^2)}{k^2+m^2}\right]^{-1}$$

where $\mu^2 \geqslant 0$ and $A(m^2) \geqslant 0$. Thus, for a zero mass particle, it is not only necessary to require $G^{-1}(0) = 0$, but also to show that $\int_0^\infty \frac{A(m^2)}{m^2}\, dm^2$ is finite.

Provided, then, we are willing to ignore this lack of rigour, we may conclude the validity of the theorem.

One further important point should be noted, namely, that the formal arguments are not explicitly dependent on the interaction, as was the case in the classical argument. Thus, while we say that $\langle \phi_2 \rangle_0 \neq 0$, we have no positive indication of how this is brought about. We must rely on analogy with the classical treatment.

The only massless particles known to exist are the photon, (and the graviton?), and possibly the neutrino. The validity of this theorem would therefore seem to cast extreme doubt on the feasibility of accounting for observed mass-differences (in SU(3) for example) between particles by such a mechanism as spontaneous symmetry breakdown, unless there exists a way of removing the massless quanta without simultaneously destroying the ability of the theorem to predict the presence of particles.

1.4 The Non-Relativistic Case

It was this apparent lack of dependence on the interaction which has prompted various authors to question the general validity of Goldstone's Conjecture. By relaxing the requirement of Lorentz Invariance, the resulting non-relativistic system so described could well be regarded as a model for a superconducting fermion system, or superfluid boson system. Thus, the theorem would have us believe that the excitation spectra in such systems contain Goldstone modes, i.e. modes whose dispersion law $\omega(\underline{k})$, is such that the frequency vanishes in the long wavelength limit. But in the real superconductor, for example, by which we mean one in which the long-range Coulomb force is taken into account, no such excitations exist, although, in the event that the Coulomb force is neglected, they do indeed appear.

Let us re-write the spectral function in the more appropriate form for a discussion of aspects connected with the non-relativistic case:

$$\langle 0|[J^\mu(x),\phi_I(x')]|0\rangle = \sum_{k_s} \langle 0|J^\mu(0)|k_s\rangle\langle k_s|\phi_I(0)|0\rangle\, e^{ik_s(x-x')} - c.c.$$

$$= \int d^4k \sum_{k_s} \langle 0|J^\mu(0)|k_s\rangle\langle k_s|\phi_I(0)|0\rangle\, \delta^{(4)}(k_s-k)\, e^{ik_s(x-x')} - c.c.$$

$$\langle 0|[J^\mu(x),\phi_2(x')]|0\rangle = \int d^4k \sum_s \langle 0|J^\mu(0)|\underline{k}, w_s(\underline{k})\rangle\langle \underline{k}, w_s(\underline{k})|\phi_2(0)|0\rangle\, \delta(w-w_s(\underline{k}))\, e^{ik_s(x-x')}$$

$$- c.c.$$

where $|0\rangle$ now is interpreted as the ground state, rather than as the physical vacuum state, and S refers to the various branches of the spectrum. The Fourier representation is then expressed

$$-i\langle[J^{\mu}(x),\phi_1(0)]\rangle_0 = \int \frac{d^4k}{(2\pi)^4} f^{\mu}(k)\, e^{ik\cdot x}$$

where

$$\frac{f^{\mu}(k)}{(2\pi)^4} = \sum_s \Big\{ \langle 0|J^{\mu}(0)|\underline{k},\omega_s(\underline{k})\rangle\langle\underline{k},\omega_s(\underline{k})|\phi_1(0)|0\rangle\ \delta(\omega-\omega_s(\underline{k}))$$

$$-\langle 0|\phi_1(0)|\underline{k},\omega_s(\underline{k})\rangle\langle\underline{k},\omega_s(\underline{k})|J^{\mu}(0)|0\rangle\ \delta(\omega+\omega_s(\underline{k}))\Big\}$$

The continuity equation (12), and sum rule (13), hold as before with $\int d^3\underline{x}\ J^0(x)$ being interpreted as the particle number operator. The former relation can be written in the more convenient non-relativistic form

$$k_0 f_0(k_0,\underline{k}) - \underline{k}\cdot\underline{f}(k_0,\underline{k}) = 0$$

so that, provided $\underline{f}(k_0, \underline{k})$ is non-singular as $\underline{k} \to 0$, we have

$$\lim_{\underline{k}\to 0} f_0(k_0,\underline{k}) = 2\pi\langle\phi_2\rangle\,\delta(k_0)$$

on using the sum-rule. Comparing with the decomposition above, we conclude that there must exist a branch of the spectrum for which $\omega(\underline{0}) = 0$, and the theorem apparently comes through in fact, quite independently of the interaction potential.

Klein and Lee[(27)], in an effort to avoid this blatant contradiction with the explicit calculations of Anderson, amongst others, who shows that the plasmon mode which would correspond to that of Goldstone were it not for the existence of the Coulomb potential, postulate the existence of a 'spurious' ground state, an essentially non-relativistic construct, by requiring the presence of an additional term, $C\ \delta^{(3)}(\underline{k})\ \delta(k_0)$, in the quantity $f_0(k)$. This term, it is argued, is then sufficient to guarantee the consistency of the argument of Goldstone, Salam and Weinberg, without necessarily having Goldstone Bosons, provided C is non-zero. But, as is pointed out by Kibble[28], such a term is, in fact, excluded by the sum-rule relation[13], in which obviously no explicit $\underline{k}$ dependence can appear.

The answer to the question of how the superconductor manages to escape the Goldstone phenomenon must clearly be an important consideration in determining the general validity, or otherwise, of the theorem. The solution has been put forward by Guralnik, Hagen and Kibble in the relativistic context, and by Lange for non-relativistic systems. Both cases turn out to be very closely connected, as we shall see. For the moment, though, we shall proceed with the non-relativistic case, as it is instrumental in leading to the proper relativistic result.

If we take the results of the explicit calculation of the collective modes in superconductivity, viz., that they oscillate at the plasma frequency $\omega_p(\underline{k})$ where $\omega_p(\underline{0}) \neq 0$, then we would expect $\lim_{k\to 0} f_o(k_o, \underline{k})$ to be modified thus:

$$\lim_{\underline{k}\to 0} f_0(k_0,\underline{k}) = C_1\,\delta(k_0) + C_2\,\delta(k_0+\omega_p) + C_3\,\delta(k_0-\omega_p)$$

whereupon it would no longer be essential that C_1 be non-vanishing, since the sum-rule may now be maintained by requiring C_2 , C_3 to be non-zero. Assuming, then, that C_1 vanishes, we find $C_2, C_3 = 2\pi \langle\phi_2\rangle_0$.
The Fourier Transform of the sum-rule now gives

$$-i\int d^3\underline{x}\,\langle[J_0(x), \phi_1(x')]\rangle_0 = \int\frac{dk_0}{2\pi}\lim_{\underline{k}\to 0} f_0(k_0,\underline{k})\,e^{-ik_0(x_0-x_0')}$$

$$= \frac{1}{2}\langle\phi_2\rangle_0\left\{e^{-i\omega_p(x_0-x_0')} + e^{i\omega_p(x_0-x_0')}\right\}$$

showing an explicit x_0-dependence. We note that for equal-times, $x_0 = x'_0$, we retrieve the expected relation, so that the appearance of this time-dependence does not at all contradict the equal time canonical commutation rules.

Thus, while the microscopic conservation law, a consequence of the symmetry, would continue to be valid, the global law would have to break down if the theorem were not to hold. This is equivalent to saying that $\underline{f}(k_0, \underline{k})$ becomes singular in the limit that $\underline{k}$ vanishes, for only then will the surface term make a contribution in the integrated form of the continuity equation, as it will have to if the above time-dependence is to occur*.

The connection between the breakdown of the global conservation law and the presence of long-range forces, has been neatly illustrated in an exactly soluble bilinear model discussed by Kibble[28].

Consider the non-relativistic model of a self-interacting field $\phi(\underline{x}, x_0)$ described by the Hamiltonian

$$H = \frac{1}{2}\int d^3\underline{x}\,\left[\pi^2(x) + (\nabla\phi(x))^2\right] + \frac{1}{2}\int d^3\underline{x}\, d^3\underline{y}\;\pi(x)V(\underline{x}-\underline{y})\pi(y)$$

with $V(\underline{x} - \underline{y})$ effectively playing the role of an instantaneous potential. The associated equations of motion are

$$[H, \pi(x)] = -i\dot{\pi}(x) = -i\nabla^2\phi(x)$$

$$[H, \phi(x)] = -i\dot{\phi}(x) = -\left[i\pi(x) + \int d^3\underline{y}\, V(\underline{x}-\underline{y})\pi(y)\right]$$

the first of which is identified as a microscopic conservation law, a consequence of the invariance of the system under the simple field translation

$$\phi(x) \to \phi(x) + C, \qquad \pi(x) \to \pi(x).$$

---

* Since, wven without the explicit inclusion of Coulomb forces, the global law for the operator $\int d^3\underline{x}\, J_0(x)$ may be expected to break down in addition to its lack of definition, the terms 'global conservation', as used here, applies to the spectral function $\langle[J_\mu(x), \phi_1(x')]\rangle_0$ rather than $J_\mu(x)$.

The conserved current is thus

$$j_\mu(x) = (-\pi(x), \nabla\phi(x)),$$

and the equal-time commutation rules

$$[\pi(x), \phi(x')]_{x_0 = x'_0} = -i\delta^{(3)}(\underline{x}-\underline{x}') \quad ; \quad [\phi(x), \phi(x')]_{x_0 = x'_0} = 0$$

allow us to deduce that

$$-i\int d^3\underline{x}\ [j_0(x), \phi(x')]_{x_0 = x'_0} = 1$$

From the equations of motion, we deduce the wave equation for ϕ:

$$\Box\phi(x) + \int d^3\underline{x}'\, V(\underline{x}-\underline{x}')\nabla^2\phi(x') = 0$$

If $\tilde{V}(\underline{k})$ is the Fourier Transform of the potential, then the spectrum for the model is given by

$$\omega^2(\underline{k}) = \underline{k}^2(1 + \tilde{V}(\underline{k})).$$

and we can therefore write the spectral functions

$$\langle[\phi(x), \phi(0)]\rangle_0 = \int d^4k\, e^{ikx}\,\delta(k_0^2 - w_{\underline{k}}^2)\, A(\underline{k}, k_0)$$

$$\langle[\dot\phi(x), \phi(0)]\rangle_0 = \int d^4k\, e^{ikx}(-ik_0)\,\delta(k_0^2 - w_{\underline{k}}^2)\, A(\underline{k}, k_0)$$

$$\langle[\nabla\phi(x), \phi(0)]\rangle_0 = \int d^4k\, e^{ikx}(i\underline{k})\,\delta(k_0^2 - w_{\underline{k}}^2)\, A(\underline{k}, k_0)$$

Using the equal-time C-rules, and the equations of motion, we can now deduce the results

$$\underline{f}(k_0, \underline{k}) = -i\int d^4x\, e^{-ikx}\langle[\underline{j}(x), \phi(0)]\rangle_0 = \underline{k}\,\frac{k_0^2}{|\underline{k}|^2}\,\epsilon(k_0)\,\delta(k_0^2 - w_{\underline{k}}^2)$$

$$f_0(k_0, \underline{k}) = -i\int d^4x\, e^{-ikx}\langle[j_0(x), \phi(0)]\rangle_0 = |k_0|\,\delta(k_0^2 - w_{\underline{k}}^2)$$

We see that $\underline{f}(k_0, \underline{k})$ is singular in $\underline{k}$, when $\omega(0) \neq 0$, and that this renders $\int d^3\underline{x} \langle [j_o(x), \phi(x')] \rangle_o$ time-dependent, oscillating at the frequency $\omega(\underline{0})$ as expected. More importantly the limiting case in which $\omega(\underline{0})$ is non-vanishing, is when $V(\underline{k}) \propto (\underline{k}^2)^{-1}$, i.e. the long-range Coulomb force.

So, we have a simple example of a system possessing a degenerate vacuum, $|0, C\rangle$, which nevertheless, because of the dynamics, does not describe Goldstone bosons. (As in the case of the free massless particles already discussed, it is not necessary actually to have the expectation value of ϕ non-zero, so that there is no new parameter actually present in this model).

The crucial result of this section is that when long-range forces are present in a system which undergoes spontaneous breakdown of symmetry, the assumption of global conservation is not permissible, in which case the Goldstone Theorem fails to be applicable. This is just the situation in superconductivity, as was shown by Lange. Interpreted physically, it means that if we distort the phase parameter specifying the ground state so that it acquires a space dependence, then, provided the forces are of short (meaning finite) range, no energy will be required, and so the existence of oscillations of vanishing frequency is assured; under the action of long-range forces, however, even a small disturbance must produce an effect at a great distance, so that a finite amount of energy is necessary; hence the energy of the resulting oscillations is pushed by a wavelength independent amount. In other words, no matter how large a surface in the system we take, there will always be a flux oscillating through it.

1.5 The Extension to Local Invariance

In the non-relativistic case, we have concluded that the theorem will not be applicable when, as a result of the dynamics, global conservation of $\langle [F, \phi] \rangle_0$ cannot be inferred. The same will certainly be true in the relativistic context, only the relevant dynamics remains to be specified.

According to Gilbert[29], unhappy with the original suggestion by Klein and Lee that the existence of a spurious vacuum state would serve as an 'escape hatch' from the theorem, the most general form of $f^\mu(k)$ incorporating a spurion contribution is necessarily non-relativistic. Retaining a pseudo-covariant formulation by introducing the special time-like vector $n_\mu = (\underline{0}, 1)$, we write it as

$$f^\mu(k) = k^\mu \rho_1(k^2, n\cdot k) + n^\mu \rho_2(k^2, n\cdot k) + n^\mu C_3 \delta^{(4)}(k)$$

Since the continuity equation must be satisfied, a more appropriate form, allowing for the possibility of Goldstone bosons, is

$$f^\mu(k) = k^\mu \rho_3(k^2, n\cdot k)\,\delta(k^2) + [k^\mu (n\cdot k) - n^\mu k^2]\,\rho_4(k^2, n\cdot k) + n^\mu C_3 \delta^{(4)}(k)$$

As already pointed out, because it violates the sum rule, the last term in fact must be excluded, so that the point of writing in this form seems lost. Paradoxically, it is the introduction of this non-manifestly covariant description which provides the link-up between the relativistic and non-relativistic cases. For, as pointed out by Higgs[30], the one situation in which loss of manifest covariance is palatable in a Lorentz invariant theory occurs whenever the system possesses gauge invariance (of the second kind). But, of course, there is an intimate connection linking gauge invariance with the presence of long range forces; the carrier of the long-range force in a relativistic field theory is the vector gauge field.

The relevant generalisation of the model Lagrangian, is obtained by demanding invariance under the extended transformation

$$\phi(x) \rightarrow e^{-ie\alpha(x)}\,\phi(x)$$

necessitating the introduction of the vector gauge field $A_\mu(x)$, transforming simultaneously as

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x)$$

through the prescription $\partial_\mu \rightarrow \partial_\mu - ieA_\mu(x)$. This is known as the condition of minimal electromagnetic coupling of the conserved current to the electromagnetic field, and e defines the coupling constant. Thus, the Lagrangian describes a form of scalar electrodynamics:

$$\mathcal{L}(x) = \mathcal{L}_\phi(x) + e\,\partial_\mu\phi_i\,T_{ij}\,\phi_j\,A^\mu - \tfrac{1}{2}e^2 A_\mu A^\mu \phi_i\phi_i - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with associated equations of motion

$$\Box\phi_i - V'\{\tfrac{1}{2}(\phi_1^2+\phi_2^2)\}\phi_i + e\,T_{ij}\,\partial_\mu(\phi_j A^\mu) + e^2 A_\mu A^\mu = 0$$

$$\partial_\nu F^{\mu\nu} = J^\mu = e\phi_i T_{ij}\,\partial^\mu\phi_j + e^2\phi_i\phi_i A^\mu \;;\quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

The canonical quantisation rules are, as before,

$$[\pi_i(x), \phi_j(x')]_{x_0=x_0'} = -i\,\delta^{(3)}(\underline{x}-\underline{x}') \quad ; \quad [\phi_i(x), \phi_j(x')]_{x_0=x_0'} = 0$$

$$[\pi_i(x),\ \phi_j(x')]_{x_0=x_0'} = -i\delta^{(3)}(\underline{x}-\underline{x}') \quad ; \quad [\phi_i(x),\ \phi_j(x')]_{x_0=x_0'} = 0$$

with the momentum density given by

$$\pi_i(x) = \partial_0\phi_i(x) - eA_0(x)\ \phi_i(x)$$

Using them, we can deduce the relation

$$-i\int d\sigma_\mu\ [J^\mu(x),\ \phi_i(x')] = T_{ij}\ \phi_j(x')$$

or

$$-i\int d^3\underline{x}\ [J^0(x),\ \phi_i(x')]_{x_0=x_0'} = T_{ij}\ \phi_j(x')$$

We now postulate that the quartic interaction is capable of maintaining vacuum degeneracy, as before. Since the system is still invariant under the usual global transformation, $\phi \rightarrow \phi e^{-i\alpha}$, we choose, as before

$$\langle\phi_1\rangle_0 = 0, \qquad \langle\phi_2\rangle_0 \neq 0 \ ,$$

so that the spectral function $\langle[J^\mu(x),\ \phi_1(x')]\rangle_0$ has to be consistent with the condition

$$i\int d^3\underline{x}\, \langle[J^0(x),\ \phi_1(x')]\rangle_0 \neq 0 \ ,$$

and, in particular

$$i\int d^3\underline{x}\, \langle[J^0(x),\ \phi_1(x')]\rangle_{0\ x_0=x_0'} = \langle\phi_2\rangle_0 \ .$$

Now, however, the required spectral function may be derived through the second equation of motion from the more fundamental function $\langle[A_\mu(x),\ \phi_1(x')]\rangle_0$, whose Fourier Transform will possess the general form above, and it turns out that there can be no contributions from the term associated with Goldstone bosons[30], although the consistency condition can still be met.

The situation is best summed up by the generalised classical argument due to Higgs[15]. Assuming that the self-interaction of the scalar field, $V(\phi^*\phi)$, is still dominant in defining the stability of the system, then the equilibrium solution may be assumed to be

$$\phi_1{}^{\circ} = 0, \qquad \phi_2{}^{\circ} \neq 0, \qquad A_\mu{}^{\circ} = 0$$

and, for small amplitude deviations, the equations of motion assume the form

$$\Box \chi_1 - e\,\phi_2^{\circ}\,\partial_\mu A^\mu = 0$$

$$\Box \chi_2 - V''(\phi_2^{\circ})^2\,\chi_2 = 0$$

$$\partial_\nu F^{\mu\nu} = e\,\phi_2^{\circ}\,\partial^\mu \chi_1 - e^2(\phi_2^{\circ})^2 A^\mu \quad .$$

Defining the gauge transformed field

$$B_\mu = A_\mu - \frac{\partial^\mu \chi_1}{e\phi_2{}^{\circ}}$$

yields

$$\partial_\nu G^{\mu\nu} = -e^2(\phi_2^{\circ})^2 B^\mu, \qquad G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \quad .$$

$$\Box \chi_2 - V''(\phi_2^{\circ})^2\,\chi_2 = 0 \; .$$

The former is the Proca field equation for a massive spin 1 vector field, while the latter describes a massive scalar meson. The ϕ_1-

mode has thus ceased to exist as a physical entity, being now nothing more than an 'invisible' gauge function in the theory. What were the Goldstone bosons possessing 1 degree of freedom, appear to have combined with the photon field, possessing two degrees of freedom, to form the massive Proca field, possessing three degrees of freedom.

Arguments have been advanced by Guralnik, Hagen and Kibble[16], and by Englert and Brout[17] to substantiate the claim, that the classical result holds good in the quantum theory. The first named collaborators consider an approximation to the field equations, which essentially boils down to the zero-order approximation; they establish the violation of global conservation as the essential cause of the inapplicability of the theorem. On the other hand, Englert and Brout, work in terms of Feynman diagrams and in an approximation corresponding to zero order, obtain the result of Higgs.

1.6 Gauge Invariance and Zero Mass

There would appear to be something of a contradiction between the results mentioned in the last section, and the conventionally held view that for a non-trivial model gauge invariance implies the presence of zero-mass vector particles. As pointed out by Schwinger[18], the relation between gauge invariance and mass is one which cannot be entirely divorced from the dynamics of the system, and it is argued that all that can be safely deduced from the imposition of gauge invariance is the absence of any term donating a bare mass to the associated vector field. The above extension of the Goldstone model can be regarded as exemplifying the contention that the existence of massless vector fields (in a necessarily interacting world) cannot be guaranteed simply by demanding gauge invariance of the second kind.

Consider the unordered Green's Function associated with the vector field

$$\langle A_\mu(x)\, A_\nu(x')\rangle_0 = \frac{1}{(2\pi)^3}\int_0^\infty dm^2\, B(m^2)\int d^4k\ \theta(k_0)\,\delta(k^2+m^2)\, G_{\mu\nu}(k)\, e^{ik(x-x')}$$

where $G_{\mu\nu}(k)$ is some gauge dependent factor present in any theory admitting gauge invariance. Normally, we would expect the exact result, $B(m^2) = Z_3\,\delta(m^2) + \sigma(m^2)$ and this is certainly formally provable in the case where $G_{\mu\nu}(k)$ takes a manifestly covariant form[31]. In that case there always exist redundant modes (associated with longitudinal photons) associated with the electromagnetic field, which have no physical content. To ensure the absence of such modes, requires a manifestly non-covariant formulation, while the overall Lorentz Invariance of the theory is really maintained.

The radiation gauge is defined by the transversality condition

$$\nabla\cdot\underline{A} = 0 .$$

The canonical variables being the transverse components of $\underline{A}$,

$$[A_i(x),\ \partial_0' A_j(x')]_{x_0=x_0'} = +i\int\frac{d^3p}{(2\pi)^3}\,\frac{(p^2\delta_{ij}-p_ip_j)}{p^2}\, e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}$$

with the field equations

$$P^2 A_0 = J_0 , \qquad P^2 A_i = J_i^T .$$

We note the relations

$$\langle 0 | [A^0(x), A^i(x')] | 0 \rangle = 0$$

$$\langle 0 | [A^0(x), A^0(x')] | 0 \rangle = \int_0^\infty dm^2\, B(m^2) \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2+m^2)\,\theta(p_0)\,\frac{m^2}{p^2}$$

In order that the commutation rules* and field equations may be satisfied, we require the sum rule

$$\int_0^\infty dm^2\, B(m^2) = 1 \quad ,$$

$B(m^2)$ being positive definite.

Through the field equations, the same spectral function, $B(m^2)$, which determines the vacuum expectation values of the fields, also determines those of the currents. In particular,

$$\langle 0 | J_\mu(x) J_\nu(x') | 0 \rangle = \int_0^\infty dm^2\, m^2 B(m^2) \int \frac{d^4p}{(2\pi)^3}\, e^{ip(x-x')} (p_\mu p_\nu - g_{\mu\nu} p^2)\,\theta(p_0)\,\delta(p^2+m^2)$$

so that the vacuum fluctuations determine the behaviour of $B(m^2)$ everywhere except at zero-mass. If we write

$$B(m^2) = B_0\, \delta(m^2) + B_1(m^2)$$

then, through the sum rule

$$1 = B_0 + \int_0^\infty dm^2\, B_1(m^2)$$

the case of no interactions corresponds to $B_1 = 0$, $B_0 = 1$, while as the coupling builds up, B_0 reduces in value from unity. It is asked whether the coupling can be such that B_0, in fact, vanishes, while B_1 possesses a massive contribution.

An exactly soluble one-dimensional model of electrodynamics exhibiting this very result has been put forward by Schwinger, and

* $[F_{0i}(x), A_j(0)]_{x_0=0} = -g_{ij}(\partial)\,\delta^{(3)}(x) = [\partial_0 A_i(x) - \partial_i A_0(x), A_j(0)]_{x_0=0}$

subsequently discussed by Brown[31]. One important fact to emerge from their calculations was the failure of the global conservation of charge. This is, perhaps, a general result of requiring the dynamics of a gauge invariant theory to allow for a massive mode. For, according to Anderson[13], superconductivity is a non-relativistic example agreeing with Schwinger's conjecture, and as has been indicated, the violation of the global law there is absolutely necessary.

Intended as a confirmation of Schwinger's idea, Bouleware and Gilbert[32] investigate the general theory of a system involving vector fields, possessing a non-vanishing bare mass. The gauge invariant case can be realised as the limit in which the bare mass vanishes; they find that a ncessary condition for this limit to be well-defined is that we must decompose so that the radiation modes are selected out. Thus, a non-manifestly covariant description is necessary, if we choose to view the problem in this way. They further conclude that the appearance or non-appearance of photons cannot be stated in advance, and proceed to illustrate by way of an exactly soluble bilinear model, which, in fact, is the approximation of the extension of Goldstone's model discussed by Guralnik, Hagen and Kibble. One of their conclusions was that a solution could only be found in the radiation gauge if consistency of the canonical quantisation rules was to be maintained, and this clearly agrees with the result of Boulware and Gilbert's argument.

In conclusion, then, the hope of Anderson, namely that there can occur an effective 'cancellation' of the massless vector modes usually originating in gauge invariance, and those expected by Goldstone's theorem, seems justified, at least in the zero-order approximation considered so far.

CHAPTER II

ON THE QUASI-PARTICLE SOLUTION OF GOLDSTONE'S MODEL

The quartic coupling between charged boson fields, which is the essential element in the model of Goldstone, has been known for some time in quantum field theory in a rather different context. In scalar electrodynamics, such a term enters in order to cancel all divergences from the Möller Scattering, the coupling constant being necessarily chosen infinite. This is in addition to the usual terms required for mass and charge renormalisation. In the present consideration, however, such a term is supposed to represent a definite interaction, and as such is quite distinct from any considerations of renormalisation, although, of course, the question of the well-definedness of any computation one might make within perturbation theory is inevitably bound up with the problem of renormalisation.

From the classical argument, which one might believe contains all the essential ingredients, one should expect to find in the quantum analogue, one deduces the important fact that the solution sought after is not obtainable by conventional perturbation theory, renormalisation difficulties apart. It therefore seems reasonable, in the first instance, to look at the problem in a quite general way outside perturbation theory, with a view to extracting any relevant information.

Kamafuchi and Umezawa, and Marx[24] amongst others, in this respect have utilised the non-perturbative Hartree-Fock method drawn from the many-body problem, which essentially consists of applying the Bogoliubov-Valatin Transformation, and minimising the ground state energy to determine the parameters (θ and $\langle \phi_2 \rangle_0$). (See Chapter I, Section 2.). It is the purpose of this chapter to show that the quasi-particle spectrum, so obtained, can be derived by a more direct, albeit formal, method.

In any usual field theory, if we add to Lorentz Invariance the further assumptions of the spectral conditions, i.e. the existence of a unique, normalisable vacuum, and positive energy states with time-like momenta, and that the physical states span a Hilbert space possessing a Hermitian scalar product so that they have positive norm, then we may assume the existence of the Lehmann-Kallen representation for the two-point functions of the theory. The argument is quite general.

In this example, it might seem that the condition of a unique vacuum is not appropriate, in view of the assumed degeneracy. Nevertheless it could be argued that once we have chosen to work in a particular representation, inequivalent to all others, then this condition will be satisfied, and we need not have any qualms about using the usual procedure. This will be the attitude adopted here.

The Two-Point Functions

The intermediate states in any gauge will satisfy the completeness condition:

$$|0,\alpha\rangle\langle 0,\alpha| + \sum_{ks}{}' |ks,\alpha\rangle\langle ks,\alpha| = 1$$

and it follows that the functions can be expressed

$$\langle 0,\alpha|\phi_i(x)\,\phi_j(x')|0,\alpha\rangle = \langle 0,\alpha|\phi_i(x)|0,\alpha\rangle\langle 0,\alpha|\phi_j(x')|0,\alpha\rangle + \int_0^\infty dm^2\,\rho_{ij}^\alpha(m^2)\,\Delta^{(+)}(x-x'|m^2)$$

where

$$\rho_{ij}^\alpha(k^2) = \sum_{ks}\langle\alpha;0|\phi_i(0)|ks,\alpha\rangle\langle ks,\alpha|\phi_j(0)|0,\alpha\rangle$$
$$\Delta^{(+)}(x|m^2) = \frac{1}{(2\pi)^3}\int d^4k\,\theta(k_0)\,\delta(k^2+m^2)\,e^{ik\cdot x}$$

Choosing the physical gauge

$$\langle\phi_1\rangle_0 = 0 \quad , \qquad \langle\phi_2\rangle_0 \neq 0$$

corresponding to $\alpha = 0$, gives

$$\langle 0|\phi_i(x)\,\phi_j(x')|0\rangle = \langle\phi_2\rangle_0^2\,\delta_{i2}\,\delta_{j2} + \int dm^2\,\rho_{ij}(m^2)\,\Delta^{(+)}(x-x'|m^2)$$

In terms of what we call the physical fields, defined by

$$\phi_i = \langle\phi_i\rangle_0 + \chi_i$$

we have

$$\langle\chi_i(x)\,\chi_j(x')\rangle_0 = \int_0^\infty dm^2\,\rho_{ij}(m^2)\,\Delta^{(+)}(x-x'|m^2)$$

where it is presupposed that in the gauge $\alpha = 0$, the off-diagonal terms vanish, i.e.

$$\rho_{ij}(m^2) = \rho_{(i)}(m^2)\,\delta_{ij}$$

(where the summation convention does not operate. We note that this last assumption is dependent on the fact that in this gauge the theory remains invariant under the discrete transformation

$\chi_1 \to -\chi_1$ and no spontaneous breakdown of this 'residual' symmetry is contemplated. Thus the transformation which takes us from $\langle 0,\alpha|\phi_i(x)\,\phi_j(x')|0,\alpha\rangle$ to $\langle 0,0|\phi_i(x)\,\phi_j(x')|0,0\rangle$ also takes ρ^{α}_{ij} to the diagonal matrix ϕ_{ij}. (If the 'residual' symmetry were also broken, then ρ_{ij} would possess off-diagonal elements.) The $\rho_{(i)}(m^2)$ are the weight functions associated with the physical particles, and will therefore generally assume the form

$$\rho_{(i)}(m^2) = Z^{(3)}_{(i)}\,\delta(m^2-\mu^2_{(i)}) + \sigma_{(i)}(m^2)$$

where $\mu^2_{(i)}$ is the exact renormalised mass associated with the ϕ_i-field, and $\sigma_{(i)}(m^2)$ takes account of the many-particle states. In any other gauge, $(\alpha \neq 0)$, the weight functions would be appropriate linear combinations of these (produced simply by rotation).

Further, the commutation and time-ordered functions are defined as usual

$$\langle[\phi_i(x),\phi_j(x')]\rangle_0 = \langle[\chi_i(x),\chi_j(x')]\rangle_0 = \int_0^\infty dm^2\,\rho_{ij}(m^2)\,\Delta(x-x'|m^2)$$

$$\langle T\,(\chi_i(x)\,\chi_j(x'))\rangle_0 = \int_0^\infty dm^2\,\rho_{ij}(m^2)\,\Delta_F(x-x'|m^2)$$

It is easily seen that the weight functions, as usual, must satisfy the normalisation conditions

$$\int_0^\infty dm^2\,\rho_{(i)}(m^2) = 1$$

in accord with canonical quantisation.

$$\langle[\phi_i(x),\,\dot{\phi}_j(x')]\rangle_{x_0=x_0'} = i\,\delta_{ij}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

Some Formal Relations

With the aid of the above representation, and the field equations of Goldstone's model, one can deduce certain formally exact relations in the same way by which one establishes the connection between the bare mass and the renormalised mass in in the more usual type of theory.

We take the field equations in the form

$$\Box \phi_i(x) + 2\lambda^2 \phi_i(x) - 4\mu^2 \phi_k(x) \phi_k(x) \phi_i(x) = 0 \tag{1}$$

In any theory of interacting fields it is necessary to ensure that a relation such as this between field operators is defined. This can be carried out by introducing some particular ordering prescription, and for the usual type of theory this suffices. Here, however, things are complicated by the appearance of the parameter $\langle \phi_2 \rangle_0$, and so for the time being we impose no definition, with the implication that any relations we might obtain are sure to involve divergences.

We have the commutation relation

$$\langle [\phi_i(x), \phi_j(x')] \rangle_0 = \int_0^\infty dm^2 \rho_{ij}(m^2) \Delta(x-x'|m^2)$$

Applying the d'Alembertian operator,

$$\langle [-2\lambda^2 \phi_i(x) + 4\mu^2 \phi_k(x) \phi_k(x) \phi_i(x), \phi_j(x')] \rangle_0 = \int_0^\infty dm^2 \rho_{ij}(m^2) m^2 \Delta(x-x'|m^2)$$

We note that for equal times the relation is identically satisfied. Now, taking the time derivative, and then imposing equal times, we find, on using the result

$$\partial_0 \Delta_{ij} (x-x'|m^2)\Big|_{x_0 = x'_0} = i\, \delta^{(3)}(\underline{x}-\underline{x}')\, \delta_{ij}$$

that

$$\int_0^\infty dm^2 \rho_{ij}(m^2) m^2 = \left(-2\lambda^2 + 4\mu^2 \langle \phi_k(x)\phi_k(x)\rangle_0\right)\delta_{ij} + 8\mu^2 \langle \phi_j(x)\phi_i(x)\rangle_0$$

involving the quantities

$$\langle \chi_i(x)\chi_j(x')\rangle_0 = \int_0^\infty dm^2 \rho_{ij}(m^2) \int d^4k\, \theta(k_0)\, \delta(k^2+m^2)\, e^{ik(x-x')}$$

which diverge, as expected, for $x = x'$.

So, we have the two relations connecting ρ_1, ρ_2 and $\langle \phi_2\rangle_0$

$$\int_0^\infty dm^2 \rho_1(m^2) m^2 = -2\lambda^2 + 4\mu^2 \langle\phi_2\rangle_0^2 + 4\mu^2 \int_0^\infty dm^2 \rho_2(m^2)\Delta^{(+)}(0|m^2) + 12\mu^2 \int_0^\infty dm^2 \rho_1(m^2)\Delta^{(+)}(0|m^2)$$

$$\int_0^\infty dm^2 \rho_2(m^2) m^2 = -2\lambda^2 + 12\mu^2 \langle\phi_2\rangle_0^2 + 12\mu^2 \int_0^\infty dm^2 \rho_2(m^2)\Delta^{(+)}(0|m^2) + 4\mu^2 \int_0^\infty dm^2 \rho_1(m^2)\Delta^{(+)}(0|m^2)$$

Further, taking the expectation value of the equation of motion

$$2\lambda^2 \langle\phi_i\rangle_0 = 4\mu^2 \langle \phi_k\phi_k\phi_i\rangle_0 \tag{3}$$

ie. $$2\lambda^2 \langle\phi_i\rangle_0 = 4\mu^2 \langle\{(\langle\phi_k\rangle_0 + \chi_k)(\langle\phi_k\rangle_0 + \chi_k)(\langle\phi_i\rangle_0 + \chi_i)\}\rangle_0$$

$$= 4\mu^2 \langle\{(\langle\phi_k\rangle_0^2 + \chi_k\chi_k)(\langle\phi_i\rangle_0 + \chi_i)\}\rangle_0 + 8\mu^2 \langle\phi_k\rangle_0 \langle\chi_k\chi_i\rangle_0$$

$$2\lambda^2\langle\phi_i\rangle_0 = 4\mu^2 \{\langle\phi_k\rangle_0^2 \langle\phi_i\rangle_0 + \langle\phi_i\rangle_0\langle\chi_k\chi_k\rangle_0 + 2\langle\phi_k\rangle_0\langle\chi_k\chi_i\rangle_0 + \langle\chi_k\chi_k\chi_i\rangle_0\}$$

With the condition $\langle\phi_1\rangle_0 = 0$, $\langle\phi_2\rangle_0 \neq 0$. We find

$$2\lambda^2\langle\phi_2\rangle_0 = 4\mu^2\{\langle\phi_2\rangle_0^3 + \langle\phi_2\rangle_0\langle\chi_k\chi_k\rangle_0 + 2\langle\phi_k\rangle_0\langle\chi_k\chi_2\rangle_0 + \langle\chi_k\chi_k\chi_2\rangle_0\}$$

and $$0 = \langle\chi_k\chi_k\chi_1\rangle_0$$

Thus, by virtue of solely the equations of motion, and the assumption of canonical commutation relations, we arrive at relations which involve the 'unknowns', the $\rho_{(i)}(m^2)$ and $\langle\phi_2\rangle_0$.

In particular, we may define any satisfactory approximation as one which satisfies all the appropriate relations, viz. (2) and (3) and the normalisation conditions

$$\int_0^\infty dm^2 \rho_I(m^2)m^2 = -2\lambda^2 + 4\mu^2\langle\phi_2\rangle_0^2 + 4\mu^2\int_0^\infty dm^2\rho_2(m^2)\Delta^{(+)}(0/m^2) + 12\mu^2\int_0^\infty dm^2\rho_I(m^2)\Delta^{(+)}(0/m^2)$$

$$\int_0^\infty dm^2\rho_2(m^2)m^2 = -2\lambda^2 + 12\mu^2\langle\phi_2\rangle_0^2 + 12\mu^2\int_0^\infty dm^2\rho_2(m^2)\Delta^{(+)}(0/m^2) + 4\mu^2\int_0^\infty dm^2\rho_I(m^2)\Delta^{(+)}(0/m^2)$$

$$2\lambda^2\langle\phi_2\rangle_0^2 = 4\mu^2\langle\phi_2\rangle_0\left\{\langle\phi_2\rangle_0^2 + \int_0^\infty dm^2\left[\rho_I(m^2) + 3\rho_2(m^2)\right]\Delta^{(+)}(0/m^2)\right\} + 4\mu^2\langle\chi_k\chi_k\chi_i\rangle_0$$

$$0 = \langle\chi_k\chi_k\chi_I\rangle$$

$$\int_0^\infty dm^2\rho_I(m^2) = \int_0^\infty dm^2\rho_2(m^2) = 1 \tag{4}$$

In the quasi-particle approximation, we obtain equations for the unknowns, which are theoretically enough to provide a solution. We take

$$\rho_2(m^2) = \delta(m^2 - m_2^2) \quad , \quad \rho_I(m^2) = \delta(m^2 - m_I^2)$$

which must automatically satisfy the normalisation relations. In that case the ordinary product is that appropriate to the free case.

$$\langle\chi_i(x)\chi_j(x')\rangle_0 = \int_0^\infty dm^2\rho_{(i)}(m^2)\delta_{ij}\Delta^{(+)}(x-x'/m^2) = \Delta^{(+)}(x-x'/m_i^2)\delta_{ij}$$

This particular choice for the weight functions implies that the χ- fields satisfy free field equations, and may therefore be decomposed in terms of creation and annihilation operators, thus

$$\chi_i(\underline{x}) = \Omega^{-1/2} \sum_{\underline{k}}{}' (2w^i_{\underline{k}})^{-1/2} (\alpha^i_{\underline{k}} e^{i\underline{k}.\underline{x}} + \alpha^{i\dagger}_{\underline{k}} e^{-i\underline{k}.\underline{x}})$$

where

$$(w^i_{\underline{k}})^2 = \underline{k}^2 + m_i^2$$

and

$$\alpha^i_{\underline{k}} |0\rangle = 0$$

The original field operators $\phi_2(x)$ are expressible likewise in terms only of m_i, $\langle\phi_i\rangle_0$ where these have still to be determined.

As a result of this decomposition, it must follow that all vacuum expectation values of an odd number of the physical field operators must vanish, in which case the m_i, ϕ_i are determined completely by the relations

$$m_1^2 = -2\lambda^2 + 4\mu^2\langle\phi_2\rangle_0^2 + 4\mu^2\Delta^{(+)}(0/m_2^2) + 12\mu^2\Delta^{(+)}(0/m_1^2)$$

$$m_2^2 = -2\lambda^2 + 12\mu^2\langle\phi_2\rangle_0^2 + 12\mu^2\Delta^{(+)}(0/m_2^2) + 4\mu^2\Delta^{(+)}(0/m_1^2) \tag{5}$$

$$\langle\phi_2\rangle_0 \left\{2\lambda^2 - 4\mu^2\langle\phi_2\rangle_0^2 - 4\mu^2\left[\Delta^{(+)}(0/m_1^2) + 3\Delta^{(+)}(0/m_2^2)\right]\right\} = 0$$

Since

$$\Delta^{(+)}(0/m^2) = \frac{1}{(2\pi)^3}\int d^3\underline{k}\,(2w_{\underline{k}})^{-1}$$

is without cut-off divergent, these relations are relatively meaningless as they stand, but they do serve to illustrate the possibility of the existence of anomalous solutions. The possibility that $\langle\phi_2\rangle_0 = 0$ has the effect of rendering the equations for m_1, m_2

symmetrical so that the result $m_1 = m_2 = m$ is implied; n will satisfy, in that case

$$m^2 = -2\lambda^2 + 16\mu^2 \Delta^{(+)}(0/m^2) \tag{6}$$

On the other hand, if $\langle\phi_2\rangle_0 \neq 0$, then after re-arrangement of (5) we obtain the relations

$$\begin{aligned} m_1^2 &= 8\mu^2 \left\{ \Delta^{(+)}(0/m_2^2) - \Delta^{(+)}(0/m_1^2) \right\} \\ m_2^2 &= 8\mu^2 \langle\phi_2\rangle_0^2 \\ 2\lambda^2 &= 4\mu^2 \langle\phi_2\rangle_0^2 + 4\mu^2 \Delta^{(+)}(0/m_1^2) + 12\mu^2 \Delta^{(+)}(0/m_2^2) \end{aligned} \tag{7}$$

We have now effectively selected out the 'non-perturbative part' of the solution. We observe from (7) that the classical expression for m_2 is obtained as anticipated, but that for m_1 (zero, by the classical argument) is not. No confirmation of the Goldstone result is evident in the argument, because no ordering procedure in the equation of motion (1), from which we started, has been applied. Had we applied normal ordering in the usual way, i.e. by demanding that all bilinear products of free fields of the type $\chi_i \chi_j$ are written explicitly with all annihilation operators on the right and all creation operators on the left so that $\Delta^{(+)}(0/m^2)$ is effectively neglected, then the classical result would appear. The difficulty with this proposition is that this ordering is applied <u>after</u>. the quasi-particle approximation has been made. What we would more reasonably require would be to rid ourselves of the divergences at the root, viz. by giving an appropriate definition to the field equation (1), that is, in effect, the product of operators at coincident points.

However, the Goldstone Theorem claims the existence of the zero-mass χ_1 - boson as an exact result for the interacting model, independently of renormalisation and ordering prescriptions,

$$\text{i.e.} \quad \mu^2_{(1)} = 0 \qquad ; \quad \rho_1(m^2) = Z^{(3)}_{(1)}\,\delta(m^2) + \sigma_1(m^2)$$

It seems therefore that this approximation scheme, on its own, while showing some degree of conformity with the results of the classical argument, cannot have much relevance now with regard to drawing a conclusion about the theorem. To discuss the theorem would now require a treatment of the residual interactions between the quasi-particles just derived. The use of perturbation theory, based on the modified equivalent Lagrangian(24), would then be the only technique at our disposal, and this would necessarily involve us in the problem of renormalisation.

We can, perhaps, draw some qualitative conclusions from the quasi-particle solution. In agreement with the classical case, the question of whether $\langle\phi_2\rangle_0 \neq 0$ or not, depends on the signs of the coupling constants λ^2, μ^2. From our final equations (7), it is clear that if m_1^2, m_2^2 are to be positive, then λ^2, μ^2 had better be positive also, and more precisely $\lambda^2/\mu^2 > 8\Delta^{(+)}(0|0)$ Should it be that $\lambda^2/\mu^2 < 8\Delta^{(+)}(0|0)$, then the symmetrical situation described by (6) will prevail. Without cut-off, of course, these expressions are not defined, but they would tend to indicate, qualitatively at least, the important role played by the couplings in deciding which solutions are appropriate. We note that with normal ordering (implying $\Delta^{(+)} = 0$), the classical requirements on the couplings (viz. that they are both positive) are reproduced.

CHAPTER III

ON THE EXTENSION TO GOLDSTONE'S MODEL

In this chapter we shall discuss the extension of Goldstone's model which has the effect of rendering the theorem inapplicable. Such extensions must always possess the simple invariance property of the original model, and the particular significance of gauge invariance of the second kind, in this context, we have noted in Chapter I (Section 1.5).

We consider first the use of perturbation theory in the model, having first decomposed in terms of the physical fields, i.e. those whose vacuum expectation values vanish. Thus, there enters the Lagrangian the unknown parameter $\langle\phi_2\rangle_0$ which should be determined self-consistently. In order to be quite general, we allow the vector field to possess a bare mass m_o, which we shall ultimately allow to vanish. Under the assumption that the theory is perturbative in the coupling of the vector to the scalar fields, as opposed to the self-coupling of the scalar fields, we consider the lowest order effects. In particular, the spectral function $\langle[J_\mu(x), \phi_1(x')]\rangle_0$ is 'calculated' in the limit of zero coupling and zero mass m_o with a view to demonstrating the connection between the existence of the Goldstone Boson and the Lorentz covariance of the model. We then proceed to consider the interaction effects to lowest order, in order to observe the mechanism by which the Goldstone boson 'disappears'.

3.1. The Model

The system to be investigated is described by the following Lagrangian:

$$\mathcal{L}(x) = \mathcal{L}_G(x) + \mathcal{L}_I(x) + \mathcal{L}_V(x)$$

where

$$\mathcal{L}_G(x) = -\tfrac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - V\{\tfrac{1}{2}(\phi_1^2+\phi_2^2)\}$$

$$\mathcal{L}_V(x) = -\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} - \tfrac{1}{2}m_0^2 B_\mu B^\mu \qquad (1)$$

$$\mathcal{L}_I(x) = e\,\partial_\mu\phi_i T_{ij}\phi_j B^\mu - \tfrac{1}{2}e^2\phi_i\phi_i B_\mu B^\mu$$

admitting only the global symmetry: $\phi_i \rightarrow R_{ij}(\alpha)\,\phi_j$.

It describes the interaction of scalar fields, already interacting symmetrically with themselves, with a vector field of bare mass m_o, through couplings of the electromagnetic type. Indeed, if we permit m_o to vanish, the system effectively describes a variation on scalar electrodynamics, with

$$m_o \rightarrow 0, \qquad G_{\mu\nu} \rightarrow F_{\mu\nu}, \qquad B_\mu \rightarrow A_\mu$$

and the Lagrangian possesses local invariance

$$\phi_i(x) \rightarrow R_{ij}\{\alpha(x)\}\,\phi_j(x) \qquad ; \quad A_\mu(x) \rightarrow A_\mu(x) - e\,\partial_\mu\alpha(x)$$

The related field equations are

$$\Box\phi_i(x) - V'\{\tfrac{1}{2}(\phi_1^2+\phi_2^2)\}\,\phi_i(x) - e\,\partial_\mu(T_{ij}\phi_j B^\mu) - e^2 B_\mu B^\mu\phi_i = 0$$

$$\partial_\nu G^{\mu\nu}(x) + m_0^2 B^\mu(x) = J^\mu(x) = e\,\partial^\mu\phi_i T_{ij}\phi_j - e^2 B^\mu\phi_i\phi_i \qquad (2)$$

$$J^\mu(x) = e\,j^\mu(x) - e^2 B^\mu(x)\phi_i(x)\phi_i(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$$

In the limit of vanishing coupling constant, e, the system more or less reduces to that of self-interacting scalar fields, which we shall demand is of the type envisaged by Goldstone, so that, associated with the Lagrangian $\mathcal{L}_G(x)$, there will be solutions such that

$$\langle \phi_1 \rangle_o = 0 \;, \qquad \langle \phi_2 \rangle_o \neq 0 \;.$$

We shall further assume that the effect of the interaction Lagrangian $\mathcal{L}_1(x)$, whether m_o is zero or not, does not affect the existence of such a possibility; $\langle \phi_2 \rangle_o$ will be changed, only to the extent that it carries a dependence on the coupling, e, and on the mass, m_o, if it exists.

Use of Perturbation Theory:

Let us decompose the fields $\phi_i(x)$ in terms of the physical components $\chi_i(x)$, through the relations

$$\phi_1(x) = \chi_1(x), \qquad \phi_2(x) = \langle \phi_2 \rangle_o + \chi_2(x)$$

so that

$$\langle \chi_1(x) \rangle_o = 0, \qquad \langle \chi_2(x) \rangle_o = 0 \;.$$

On inserting these relations into $\mathcal{L}(x)$ (or $\mathcal{L}_G(x)$, alone, for that matter), we would obtain a rather complicated expression involving, as well as the inserted coupling constants (and masses), the additional unknown quantity $\langle \phi_2 \rangle_o$. At least, theoretically, the requirement that $\langle \chi_2(x) \rangle_o = 0$ could then be looked upon as supplying us with the self-consistent equation for $\langle \phi_2 \rangle_o$, within the approach of perturbation theory. The trouble lies in the fact that the equation is not only a polynomial of infinite degree, but also, in any case, must be undefined unless some appropriate renormalisation technique can be formulated to remove the divergent contributions which will inevitably appear in the coefficients of the series. Thus, while the use of perturbation theory seems no less justifiable than usual once the decomposition has been made, its usefulness in the context, vis. that of evaluating $\langle\phi_2\rangle_0$, would seem to be doubtful in view of the general intractability entailed. On the other hand, if we are prepared to forego an explicit evaluation of the quantity $\langle\phi_2\rangle_0$, assuming that it, in fact, exists, then we can perhaps hope, through perturbation theory, to discuss the modification to the basic self-interacting system, $\mathcal{L}_G(x)$, which the coupling to the vector field might be expected to bring about. The appropriate interaction Lagrangian

$$\mathcal{L}_I(x) = -e\left\{(\langle\phi_2\rangle_0 + \chi_2)\,\partial^\mu\chi_1 - \chi_1\,\partial^\mu\chi_2\right\}B_\mu - \tfrac{1}{2}e^2 B_\mu B^\mu\left\{\chi_1^2 + (\langle\phi_2\rangle_0 + \chi_2)^2\right\}$$

we now view as a perturbation to the unperturbed system, described by $\mathcal{L}_G(x)$. In assuming that perturbation theory is here applicable, we must presume that all relevant quantities, understood to be derived from $\mathcal{L}_G(x)$, achieve an analytic dependence on the coupling e, so that the limit $e \to 0$ is defined. We have no reason to believe otherwise, although we should note that our ability to give rigorous meaning to the statement ultimately depends on whether the theory can be made well-defined or not. The conventional view of derivative coupling theories, involving massive vector mesons, of which the above is an example, has been that they must lack definition; they are not renormalised in the usual sense. Even if m_0 vanishes in this case, it turns out that the assumed non-vanishing of $\langle\phi_2\rangle_0$ is instrumental in leaving the situation unchanged, as might be inferred from the presence of an effective mass term $\frac{1}{2}e^2 \langle\phi_2\rangle_0^2 A_\mu A^\mu$, in $\mathcal{L}_1(x)$.

Setting aside the problem of renormalisation, we now briefly indicate the formulation of the perturbation theory, which we expect to be basically little different from that encountered in ordinary scalar electrodynamics[33].

The interaction Hamiltonian density, derivable by the usual operation from $\mathcal{L}_1(x)$, is not appropriate in establishing a covariant equation of motion, from which the interaction series can be deduced. This is so because the derivative terms make it non-invariant. In order that the equation of motion, the Tomonaga-Schwinger Equation,

$$i \frac{\delta}{\delta\sigma(x)} |\Psi(\sigma)\rangle = \mathcal{H}_I(x) |\Psi(\sigma)\rangle$$

where $|\Psi(\sigma)\rangle$ represents the state of the system, and σ any space-like surface, be soluble, the integrability condition

$$\left[\frac{\delta}{\delta\sigma(x)} , \frac{\delta}{\delta\sigma(x')} \right] |\Psi(\sigma)\rangle = 0$$

or alternatively

$$\left[H_1(x), \quad H_1(x') \right] = 0$$

must be satisfied, the Hamiltonian density, $H_1(x)$, to be used in this case, has to depend explicitly on the surface $\sigma(x)$: otherwise the integrability condition will not be satisfied. We must choose

$$\mathcal{H}_1'(x) = \mathcal{H}_1(x) - e^2 \phi_i(x) \phi_i(x) n_\mu(x) n_\nu(x) A^\mu(x) A^\nu(x)$$

where $\eta_\mu(x)$ is the normal to σ at x. The specific case of the flat space-like surface for σ gives us the equation of motion as usual in terms of derivatives with respect to time; it then follows that the interaction picture can be defined as usual, with the S-operator defined by

$$S = T.\left\{e^{i \; d^4x H_1'(x)}\right\} .$$

The evaluation of any matrix element now automatically involves the disappearance of n_μ from the scene, as can usually be proved to all orders(34). This happens because of the effect of the derivative terms on the T-ordering operator; thus, the covariance is apparent, and permits the rule whereby the effect of the time derivatives on the time-ordering symbol may be neglected, along with the surface dependent term of H_1, so that η_μ need never appear explicitly. However, to be quite sure that this is so, here, we shall retain such terms in whatever approximation we might employ.

Written out in full, the interaction Hamiltonian density we require is

$$\mathcal{H}_I'(x) = e\left\{-(\langle\phi_2\rangle_0 + \chi_2)\,\partial^\mu\chi_1 + \chi_1\,\partial^\mu\chi_2\right\}B_\mu$$
$$-\tfrac{1}{2}e^2\left\{\chi_1^2 + (\langle\phi_2\rangle_0 + \chi_2)^2\right\}B_\mu B^\mu - \tfrac{1}{2}e^2\left\{\chi_1^2 + (\langle\phi_2\rangle_0 + \chi_2)^2\right\}B_\mu B_\nu n^\mu n^\nu \qquad (3)$$

The Gauge Factors (for $m_0 = 0$)

In the case for which m_0 vanishes, the unordered two-point function associated with the gauge field is generally

$$\langle A_\mu(x)\, A\ (0)\rangle_0^{(e)} = \int_0^\infty dm^2 B(m^2)\ d^4k\ \mathcal{G}_{\mu\nu}(k)\ \theta(k_0)\ \partial(k^2+m^2)e^{ikx}$$

where the symmetric factor $\mathcal{G}_{\mu\nu}(k)$ must be of a form consistent with the invariance of the theory under gauge transformations of the second kind.

The general properties of $\mathcal{G}$ may be deduced from the requirement of canonical quantisation

$$\langle [F_{0i}(x),\ A_j(0)]\rangle_{0\ x_0=0}^{(e)} = -i\mathcal{G}_{ij}\delta^{(3)}(\underline{x})$$

leading to unit normalisation of $B(m^2)$, and the requirement that the current spectral function $\langle J_\mu(x), J_\nu(0)\rangle_0^{(e)}$ must be Lorentz covariant, and of such a form that it satisfies the local conservation law. Defining the idempotent quantity

$$\mathcal{G}^L_{\mu\nu}(k) = g_{\mu\nu} - \frac{1}{k^2}k_\mu k_\nu \qquad ; \qquad \mathcal{G}^L = (\mathcal{G}^L)^2$$

and using the current field relation in (2), we take the restriction on $\mathcal{G}$ in the form

$$\mathcal{G}^L\ \mathcal{G}\ \mathcal{G}^L = \mathcal{G}^L$$

implying that $\mathcal{G}^L\mathcal{G}$ and $\mathcal{G}\mathcal{G}^L$ like $\mathcal{G}^L$, must be idempotent, i.e.

$$(\mathcal{G}^L\mathcal{G})^2 = \mathcal{G}^L\mathcal{G}\ , \qquad (\mathcal{G}\mathcal{G}^L)^2 = \mathcal{G}\mathcal{G}^L\ .$$

It follows, without inconsistency, that $\mathcal{G}_{\mu\nu}(k)$ itself may be chosen to be idempotent.

For a covariant gauge

$$\mathcal{G}_{\mu\nu}(k) = \lambda_1(k^2)\, g_{\mu\nu} + \lambda_2(k^2)\, k_\mu k_\nu$$

the imposition of idempotency leads to the well-known gauges of Feynman and Landau.

$$\mathcal{G}^{F}_{\mu\nu}(k) = g_{\mu\nu}$$

$$\mathcal{G}^{L}_{\mu\nu}(k) = g_{\mu\nu} - \frac{1}{k^2} k_\mu k_\nu$$

However, an equally possible description may be given in terms of a non-covariant gauge, typified by the radiation gauge operator

$$\mathcal{G}^{R}_{\mu\nu}(k,n) = g_{\mu\nu} - \frac{(k_\mu n_\nu + k_\nu n_\mu)(n\cdot k) + k_\mu k_\nu}{k^2 + (n\cdot k)^2}$$

where n_μ is the special time-like unit vector. $(\underline{0}, 1)$, which we note is not idempotent, although $\mathcal{G}^R\mathcal{G}^L$ is so. This pseudo-covariant form for $\mathcal{G}^R$ is equivalent to the relations given in (1.6) of Chapter I.

Usually, if the radiation gauge is used, then although manifest covariance is lost, the calculation of physical quantities should show no difference in the final analysis. It may be said that the freedom offered by gauge invariance allows for the description of a Lorentz invariant theory, in a manifestly non-covariant way.

We note the other properties of $\mathcal{G}$:

$$\mathcal{G}_{\mu\nu}(k) = \mathcal{G}_{\nu\mu}(k) = \mathcal{G}_{\nu\mu}(-k) = \mathcal{G}^{*}_{\mu\nu}(k) \quad ; \quad \mathcal{G}\mathcal{G}^{L}\mathcal{G} = \mathcal{G}$$

<u>The Unperturbed Propagators:</u>

The unperturbed propagators which will occur in the Dyson-Wick perturbation expansion of any matrix element, will be those appropriate to the Lagrangian density, $\mathcal{L}_G + \mathcal{L}_V$, defined by (1). As those associated with $\mathcal{L}_G$ are unknown, we describe them generally in Lehmann form

$$\langle T.\left(\chi_i(x)\,\chi_j(x')\right)\rangle_0 = \int_0^\infty dm^2\,\rho_{ij}(m^2)\,\Delta_F(x-x'|m^2)$$

with

$$\Delta_F(x-x'|m^2) = \int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m^2-i\varepsilon}\,e^{ik(x-x')}$$

and ρ^G_{ij} diagonal, and normalised to unity, while that associated with the free vector bosons is

$$\langle T.\left(B_\mu(x)\,B_\nu(x')\right)\rangle_0 = \int\frac{d^4k}{(2\pi)^4}\,\mathcal{G}^{m_0}_{\mu\nu}(k)\,\frac{1}{k^2+m^2-i\varepsilon}\,e^{ik(x-x')}$$

with

$$\mathcal{G}^{m_0}_{\mu\nu}(k) = g_{\mu\nu} + \frac{1}{m_0^2}\,k_\mu k_\nu.$$

In the case for which m_0 vanishes, the vector field $B_\mu(x)$ becomes the gauge field $A_\mu(x)$, and $\mathcal{G}^{m_0}_{\mu\nu}(k, 0)$ is undefined, as a result of the onset of gauge invariance. Then, the vector field propagator becomes gauge-dependent, taking the form

$$\langle T.\left(A_\mu(x)\,A_\nu(x')\right)\rangle_0 = \int\frac{d^4k}{(2\pi)^4}\,\mathcal{G}_{\mu\nu}(k)\,\frac{1}{k^2-i\varepsilon}\,e^{ik(x-x')}$$

with $\mathcal{G}_{\mu\nu}(k)$ possibly being given by $\mathcal{G}^L$, $\mathcal{G}^F$, or $\mathcal{G}^R$, as given above.

We note that these will be the only non-vanishing two-point functions formed from the fields and corresponding to $\mathcal{L}_G + \mathcal{L}_V$. The other possibility $\langle T.(B_\mu(x)\,\chi_i(x'))\rangle_0$ will vanish in this case, as for any free field theory.

Also we shall have cause to consider the two-point spectral functions involving the current, j^μ, associated with the Goldstone-type model with the normalisations

$$\int_0^\infty dm^2 \rho_1(m^2) = \langle \phi_2 \rangle_0 , \qquad \int_0^\infty dm^2 \rho_2(m^2) = 0$$

The Goldstone Theorem tells us that $\rho_1(m^2) = \langle \phi_2 \rangle_0 \, \delta(m^2)$ and this we expect to be deducible from the problem described by (1), in the limit $e \to 0$.

<u>The Spectral Function $\langle [B_\mu(x), \phi_1(x')] \rangle_0$</u>

In any normal theory with a derivative coupling of this type i.e. bilinear in scalar fields, the function $\langle T\,(B_\mu(x)\,\phi_1(x'))\rangle_0$ would be taken to be zero, as for the non-interacting case. Here, however, because of the presence of terms in the Hamiltonian (3) effectively linear in the scalar fields χ, this function will not vanish.

Before resorting to perturbation theory, let us consider the commutator $[B_\mu(x), \phi_1(x')]$. We recall that in the GSW 'proof' of Goldstone's Theorem, our interest centred on the function $\langle [J_\mu(x), \phi_1(x')] \rangle_0$ which in this case may be related to $\langle [B_\mu(x), \phi_1(x')] \rangle_0$ through the equations of motion. We have the operator relation from (2)

$$(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)[B^\nu(x), \phi_1(x')] + m_0^2 [B_\mu(x), \phi_1(x')]$$

$$= [J_\mu(x), \phi_1(x')] \qquad (4)$$

$$= e[j_\mu(x), \phi_1(x')] - e^2 [B_\mu(x)\phi_i^2(x), \phi_1(x')]$$

Thus, a knowledge of the function $\langle [B_\mu(x),\ \phi_1(x')]\rangle_0^{(e)}$ gives us $\langle [J_\mu(x),\ \phi_1(x')]\rangle_0^{(e)}$, through

$$\{\partial_\mu \partial_\nu - g_{\mu\nu}(\partial^2 - m_0^2)\} \langle [B^\nu(x),\ \phi_1(x')]\rangle_0^{(e)} = \langle [J_\mu(x),\ \phi_1(x')]\rangle_0^{(e)}$$

and the former could be obtained within some perturbation theoretic approximation scheme to determine $\langle T.(B_\mu(x)\ \phi_1(x'))\rangle_0^{(e)}$.

Further, we may look upon the Goldstone Model as the 'situation' surviving in the limit as the coupling e goes to zero. Thus, the spectral function for the Goldstone Model may be expressed

$$\langle [j_\mu(x),\ \phi_1(x')]\rangle_0 = \lim_{e\to 0} \frac{1}{e} \{\partial_\mu \partial_\nu - g_{\mu\nu}(\partial^2 - m_0^2)\} \langle [B^\nu(x),\ \phi_1(x')]\rangle_0^{(e)} \qquad (5)$$

The only stipulation we have made in writing the above expressions is the assumption

$$\lim_{e\to 0} |0\rangle^{(e)} = |0\rangle^{G}$$

that the effect of the interaction on the vacuum appropriate to Goldstone's Model is perturbative in the coupling (e). Thus, any enquiry concerning the vector function $\langle [J_\mu(x),\ \phi_1(x')]\rangle_0$ or that appropriate to Goldstone's Model can be translated into the more pra ctical investigation of the two-point function $\langle [B_\mu(x),\ \phi_1(x')]\rangle_0$, which, in turn, may be obtained via perturbation theory from $\langle T.(B_\mu(x)\ \phi_1(x'))\rangle_0$.

We now proceed to discuss the limit as $e \to 0$.

The limit $e \to 0$: The Goldstone Limit.

This limit is interesting, because it serves not only to establish the Goldstone Boson, but to exhibit the connection with manifest covariance, which, of course, the Goldstone Model must possess.

We make use of the expression[5]

$$\langle [j_\mu(x), \phi_1(x')] \rangle_0 = \lim_{e \to 0} \frac{1}{e} \{ \partial_\mu \partial_\nu - (\partial^2 - m_0^2) g_{\mu\nu} \} \langle [B^\nu(x), \phi_1(x')] \rangle_0^{(e)}$$

Because of the presence of the factor $\frac{1}{e}$, and because of our basic assumption that the presence of the vector field has a perturbative effect, which at this point we may argue is manifested in $\langle \phi_2 \rangle_0^{(e)}$ as a series expansion in e, we need only consider those contributions explicitly of first order in e.

We see, from the interaction Hamiltonian, that we must have

$$\langle T.(B_\mu(x)\phi_1(x')) \rangle_0^{(e)} = ie \int d^4y \, \langle T.(B_\mu(x) B_\nu(y)) \rangle_0 \langle T.(j^\nu(y)\phi_1(x')) \rangle_0$$

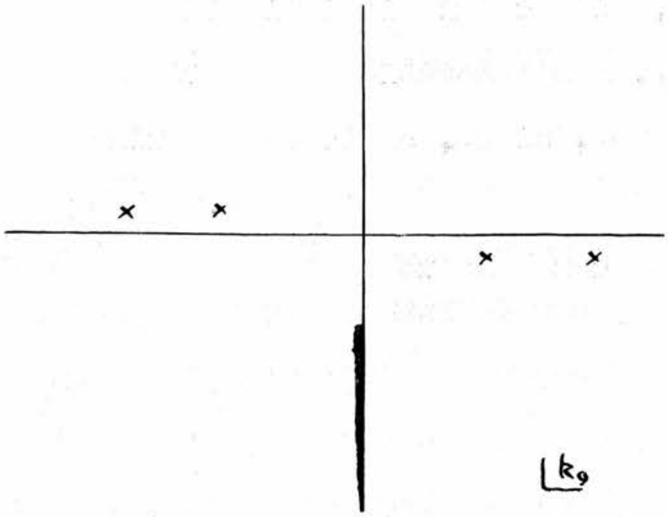
to first order in e, in terms of momentum-space variables we find

$$\langle T.(B_\mu(x)\phi_1(x')) \rangle_0 = e \int_0^\infty dm^2 \rho_1(m^2) \int \frac{d^4k}{(2\pi)^4} \, g_{\mu\lambda}^{m_0}(k) \frac{k^\lambda}{(k^2+m_0^2-i\varepsilon')} \frac{e^{ik(x-x')}}{(k^2+m^2-i\varepsilon)} \tag{6}$$

In the case in which $m_0 = 0$, the corresponding expression for the above contribution must be gauge dependent (through $G_{\mu\lambda}$),

$$\langle T.(A_\mu(x)\phi_1(x')) \rangle_0 = e \int_0^\infty dm^2 \rho_1(m^2) \int \frac{d^4k}{(2\pi)^4} \, g_{\mu\lambda}(k) \frac{k^\lambda}{(k^2-i\varepsilon')} \frac{e^{ik(x-x')}}{(k^2+m^2-i\varepsilon)} \tag{7}$$

We may obtain the commutation function from the time-ordered counterpart by the well-known prescription. We have two poles in



the upper and lower half-planes (for the variable k_0), and depending on the sign of the time interval $x_0 - x_0'$, the integral is defined by choosing a contour enveloping one or other of the half-planes. Thus, for $x_0 - x_0' > 0$, the function is $\Gamma^{(+)}(xx')$, obtained by closing the contour in the lower half plane:

$$\Gamma_\nu^{(+)}(x,x') = \frac{-ie}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2)\int d^4k\; \mathcal{G}^{m_0}_{\nu\lambda}(k)k^\lambda \theta(k_0)\left\{\frac{\delta(k^2+m_0^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2+m_0^2}\right\} e^{ik(x-x')}$$

while, for $x_0 - x_0' < 0$, it is $\Gamma^{(-)}(xx')$, closing it in the upper half-plane

$$\Gamma_\nu^{(-)}(x,x') = \frac{ie}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2)\int d^4k\; \mathcal{G}^{m_0}_{\nu\lambda}(k)\,k^\lambda \theta(-k_0)\left\{\frac{\delta(k^2+m_0^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2+m_0^2}\right\} e^{ik(x-x')}$$

It then follows that the commutator can be expressed

$$\begin{aligned}\langle[B_\mu(x), \phi_1(x')]\rangle_0 &= \Gamma^{(+)}(xx') - \Gamma^{(+)*}(xx') \\ &= \Gamma^{(-)*}(xx') - \Gamma^{(-)}(xx')\end{aligned}$$

Thus, our final expression explicitly to first order in e is

$$\langle[B_\mu(x), \phi_1(x')]\rangle_0^{(e)} = \frac{-ie}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2)\int d^4k\; \mathcal{G}^{m_0}_{\nu\lambda}(k)\,k^\lambda \varepsilon(k_0)\left\{\frac{\delta(k^2+m_0^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2+m_0^2}\right\} e^{ik(x-x')} \tag{8}$$

When m_0 is zero, we have similarly

$$\langle[A_\mu(x), \phi_1(x')]\rangle_0 = \frac{ie}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2)\int d^4k\, \mathcal{G}_{\nu\lambda}(k) k^\lambda \varepsilon(k_0)\left\{\frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2}\right\} e^{ik(x-x')} \tag{9}$$

In either case, the operator $\mathcal{G}_{\mu\lambda}(k)$ satisfies the property

$$\mathcal{G}_{\mu\lambda}(k) = \mathcal{G}^{*}_{\mu\lambda}(-k)$$

which we have used above.

Application of the differential operator $\lim_{e\to 0}\frac{1}{e}(\partial_\mu\partial_\nu - \partial^2 g_{\mu\nu} + m_0^2 g_{\mu\nu})$ must yield, in both cases, the spectral function $\langle[j_\mu(x), \phi_1(x')]\rangle_0$ associated with the Goldstone model, by (5).

Considering the case $m_0 \neq 0$ first, we find

$$\langle[j_\mu(x), \phi_1(x')]\rangle_0$$

$$= \frac{i}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2)\int d^4k\left\{k_\mu k_\nu + [k^2+m_0^2]g_{\mu\nu}\right\}\mathcal{G}^{m_0}_{\nu\lambda}(k) k^\lambda \varepsilon(k_0)\left\{\frac{\delta(k^2+m_0^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2+m_0^2}\right\} e^{ik(x-x')}$$

which must reduce to

$$\begin{aligned}\langle[j_\mu(x), \phi_1(0)]\rangle_0 &= \frac{i}{(2\pi)^3}\int dm^2 \rho_1(m^2)\int d^4k\, k_\mu\, \epsilon(k_0)\delta(k^2+m^2) e^{ikx} \\ &= \int dm^2 \rho_1(m^2)\, \partial_\mu \Delta(x/m^2)\end{aligned} \tag{10}$$

We now require to impose broken symmetry in the form

$$\int d^3\underline{x}\ \langle[j_0(x), \phi_1(x')]\rangle_0 \neq 0$$

Substitution of (10) then yields the condition on the weight function $\rho_1(m^2)$

$$\int_0^\infty dm^2 \rho_1(m^2) \cos m(x_0 - x_0') \neq 0 \quad .$$

The whole argument for $m_0 \neq 0$ is covariant as we observe; provided we assume $\int d^3\underline{x} \langle [j_0(x), \phi_1(x')] \rangle_0$ is time-independent. However, we note that in taking the limit $e \to 0$ in this case, we have fome from a situation in which there is a conserved current $J_\mu - m_0 B_\mu$ to one where it is j_μ. In the expression for $\langle [j_\mu(x), \phi_1(x')] \rangle_0$ in terms of $\langle [B_\mu(x), \phi_1(x')] \rangle_0$, we do not see the immediate microconservation of $j_\mu(x)$.

On the other hand, if $m_0 = 0$ to start with, then by gauge invariance $J_\mu(x)$ is conserved microscopically, and thus $j_\mu(x)$ is also. Applying $\lim_{e \to 0} \frac{1}{e}(\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2)$ to $\langle [A_\mu(x), \phi_1(x')] \rangle_0$, we find

$$\langle [j^\mu(x), \phi_1(x')] \rangle_0$$

$$= \frac{i}{(2\pi)^3} \int_0^\infty dm^2 \rho_1(m^2) \int d^4k \, (k^\mu k^\nu - g^{\mu\nu} k^2) \mathcal{G}_{\nu\lambda}(k) k^\lambda \varepsilon(k_0) \left\{ \frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2} \right\} e^{ik(x-x')}$$

Now, in any covariant gauge (such as Landau or Feynman)

$$\mathcal{G}_{\nu\lambda}(k) = \lambda_1(k^2) g_{\nu\lambda} + \lambda_2(k^2) k_\nu k_\lambda$$

We observe that $\langle [j_\mu(x), \phi_1(x')] \rangle_0$ appears to vanish automatically in general, signifying inconsistency with the broken symmetry condition. It thus appears, if we are to achieve consistency, that the gauge must be chosen to be non-covariant. Choosing radiation gauge, we find

$$\langle [j_\mu(x), \phi_1(x')] \rangle_0$$

$$= \frac{-i}{(2\pi)^3} \int_0^\infty dm^2 \rho_1(m^2) \int d^4k \, \frac{(n\cdot k) k^2}{k^2 + (n\cdot k)^2} \left[k_\mu (n\cdot k) - n_\mu k^2 \right] \varepsilon(k_0) \left\{ \frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2} \right\} e^{ik(x-x')}$$

$$\langle [j_\mu(x), \phi_1(x')] \rangle_0$$

$$= \frac{-i}{(2\pi)^3}\int_0^\infty dm^2 \rho_1(m^2) \int d^4k \; \frac{(n.k)k^2}{k^2+(n.k)^2}\left[k_\mu(n.k) - n_\mu k^2\right]\varepsilon(k_0)\left\{\frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2}\right\} e^{ik(x-x')} \tag{11}$$

i.e. the Fourier Transform has the manifestly non-covariant form

$$(k^\mu(n.k) - k^2 n^\mu)\, f_4'(n, k) \quad \text{where} \quad f_4'(n,k) = \int_0^\infty \frac{(n.k)k^2\varepsilon(k_0)}{k^2+(n.k)^2}\left\{\frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2}\right\}\rho_1(m^2)\,dm^2$$

This particular form, we recall from Section 1.5 was one of the possible terms allowed for simply on the basis of conservation of current in the case where a loss of covariance was contemplated(29,30).

It has been shown, in fact, that this would be the only term to survive in the case where a gauge field was present.30) But the deduction that the non-appearance of the term proportional to $k_\mu \delta(k^2)$ has the effect of ridding us of the Goldstone Boson, does not stand in the present case $(m_0, e \to 0)$. In fact, forming the space integral of the time component, we find

$$i\int d^3\underline{x}\, \langle [j_0(x), \phi_1(x')] \rangle_0 = \int_0^\infty dm^2 \rho_1(m^2) \cos m(x_0 - x_0') \tag{12}$$

as for the case $m_0 \neq 0$.

We note that this result is consistent with the equal time commutation relations for any $\rho_1(m^2)$ normalised to $\int_0^\infty dm^2 \rho_1(m^2) = \langle \phi_2 \rangle_0$ This is apparently as far as we can go without more detailed calculation.

Closer inspection of the case $m_0 = 0$, however, shows that only when $\rho_1(m^2) \propto \delta(m^2)$ can the theory be manifestly Lorentz covariant, as it, in fact, has to be (otherwise the model would depend on the limit $e \to 0$). This can be deduced quite simply if we demand the following condition

$$[k^\mu(n\cdot k) - n^\mu k^2]\, \rho_4'(k, n\cdot k) \propto k^\mu \xi(k^2)$$

that is

$$k_o \underline{k} \propto \underline{k}\ \xi(k^2)$$

and $$\underline{k}^2 \propto k_o\ \xi(k^2)$$

from which we see that it is consistent only for $k_o = |\underline{k}|$ and so $\xi(k^2) \propto \delta(k^2)$; thus only for massless particles will such a non-covariant form be in fact covariant. This could not have been ascertained from the case $m_o \neq 0$, since there was no question of Lorentz covariance being violated there. Further, when massless particles are present, it automatically follows that $\int d^3\underline{x} \langle [j^o(x), \phi_1(x')] \rangle_o$ is independent of time. This, however, is not to say that the 'operator' $\int d^3\underline{x}\ j^o(x)$ is time-independent, although the fact that it is undefined really forbids any discussion of time-dependence.

By virtue of the microscopic conservation law, we would normally assume the global conservation law of 'charge'. But we point out that this is not a foregone conclusion, and should really be derived as a consequence of the theory. In the models under discussion, when we take $e \to 0$, we would believe that the global law for the 'operator' is violated, although the presence of massless particles prevents the same statement being applied to commutators involving the 'charge'.

To conclude then, in regarding the Goldstone model as the limiting case of zero electromagnetic interaction, we are forced into an apparantly non-covariant description, the Fourier Transform of the spectral function $\langle 0 | [j^{\mu}(x), \phi_1(x')] | 0 \rangle$ being of the form

$$\left[k^{\mu}(n.k) - n^{\mu} k^2 \right] f_4'(n.k, k^2)$$

a possible form allowed by microscopic current conservation. Other possible terms $Ck_{\mu} \; \delta(k^2)$, $C_3 n_{\mu} \; \delta^{(4)}(k)$ which would allow us to infer directly the presence of massless particles or a spurious vacuum state, are absent. Nevertheless, we have shown that the GSW result is contained in this expression, since $f_4'(n.k, k^2)$ depends on the boson propagator, and in such a way that it is equivalent to the usual manifestly covariant form. Thus one can view the Goldstone Boson as a necessity in preserving the manifest Lorentz Covariance of the theory in the first instance.

Manifest Covariance as an Essential Assumption in Goldstone's Theorem:

For the sake of clarity we now state the theorem as it concerns relativistic theories in general, and indicate the main points of the argument in the order which we think most logical.

Any Lorentz invariant field theory whose solutions are associated with a spontaneous breakdown of some continuous internal symmetry, and for which there can be no other but a manifestly covariant description, must describe massless particles. This must imply the space integral of the time-component of any two point spectral function involving the micro-conserved current in such a theory, is time-independent.

Let $f_{\mu}(x)$ be a vector two-point function which is microscopically conserved, $\partial_\mu f^\mu(x) = 0$ and such that $\int d^3\underline{x}\, f_o(x)$ is non-zero. A suitable example of $f_\mu(x)$ is the function $\langle [J_\mu(x), \phi_1(0)] \rangle_o$ in the usual notation.

Manifest Lorentz Covariance would imply the general form

$$f_\mu(x) = \int_0^\infty dm^2 \rho(m^2)\ \partial_\mu \Delta(x/m^2)$$

and applying the micro-conservation law:

$$\begin{aligned}
\partial_\mu f^\mu(x) &= \int_0^\infty dm^2 \rho(m^2)\ \Box\Delta(x/m^2) = 0 \\
&= \int_0^\infty dm^2 \rho(m^2)\ m^2 \Delta(x/m^2) = 0 \\
&= \int_0^\infty dm^2 \rho(m^2)\, m^2 \int d^3\underline{k}\ \frac{e^{i\underline{k}\cdot\underline{x}}}{w_{\underline{k}}} \sin(w_{\underline{k}} x_0) = 0
\end{aligned}$$

where $\omega_{\underline{k}}^2 = \underline{k}^2 + m^2$.

Integrating over all space, we find

$$\int_0^\infty dm^2 \rho(m^2)\, m \sin(m x_0) = 0$$

and differentiating with respect to x_o, and imposing the equal time condition $x_o = 0$,

$$\int_o^\infty dm^2 \rho(m^2)\ (m^2)^N = 0, \qquad \text{for } N > 0$$

or

$$\int_o^\infty dm^2 \rho(m^2)\ (m^2)^N = \left(\int_o^\infty dm^2 \rho(m^2)\right) \delta_{NO} \quad (N = 1,2,\ldots)$$

implying $$\rho(m^2) = \left(\int_o^\infty dm'^2 \rho(m'^2)\right) \delta(m^2)\ .$$

This is enough to guarantee the time-independence of

$$\int d^3\underline{x}\ f_o(\underline{x},\ x_o) = \int d^3\underline{x} \int_0^\infty dm^2\ \rho(m^2)\ \partial_o \Delta(x\ m^2) = \int_0^\infty dm^2\ \rho(m^2) \cos m\ x_o ,$$

Working from the spectral function $\langle [J_\mu(x), \phi_1(0)] \rangle_0 = f_\mu(x)$

the appropriate order of events is that Lorentz invariance implies that $\int d^3\underline{x}\ f_o(\underline{x},\ x_o)$ is time-independent, which in turn implies a Goldstone Boson; this fact was first pointed out by Guralnik, Hagen, Kibble(16) and Lange(14).

The Quasi-Particle Approximation of the Extended Model:

Through the use of perturbation theory, we would hope, for example to determine the appropriate two-point functions for the full interacting system. The Dyson Equations for the fields, in momentum space, are

$$\Delta_F^{ij}(p) = \Delta_{F0}^{ij}(p) + \Delta_{F0}^{ik}(p) \sum{}^{k\ell}(p)\ \Delta_F^{\ell j}(p)$$

$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + D^0_{\mu k}(p)\ \Pi^{k\lambda}(p)\ D_{\lambda\nu}(p)$$

$$\Delta_F^{ij}(p) = \int d^4x\ \langle T.\ (\phi_i(x)\phi_j(0))\rangle_0\ e^{-ipx}$$

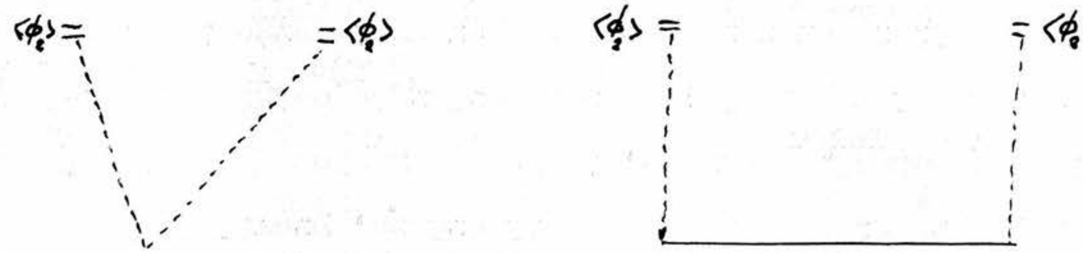
$$D_{\mu\nu}(p) = \int d^4x\ \langle T.\ (A_\mu(x) A_\nu(0))\rangle_0\ e^{-ipx}$$

where $\Sigma(p)$, $\Pi(p)$ are the respective proper self-energy and polarisation operators and Δ_0 and D_0 are the propagators for the Goldstone Model and the free vector field. Any particular approximate choice of Σ or Π will enable the corrected propagators to be determined up to that approximation. Thus, although Σ or Π may be taken to be first order in the coupling, the propagators will always have a more complicated dependence on the coupling through these relations.

Having established the existence of the Goldstone Boson as a requirement of Lorentz Invariance in the limit $e \to 0$, we are in a position to make use of this result to calculate at least some of the properties of the fully interacting model.

It has been understood in the past that in a gauge invariant theory, such as scalar electrodynamics, the self-energy contributions arising in a perturbation treatment of the electromagnetic field cannot alter the mass of the photon. In other words, the photon remains a photon to all orders, provided gauge invariance is maintained. However, it has been shown that this cannot be guaranteed generally; for in order to obtain a well-defined theory in the limit of zero bare mass of an interacting vector meson theory, one should reduce to a non-covariant description, in which case the usual arguments would no longer be valid[32].

In the present model, because of the assumption of spontaneous breakdown of symmetry, there will occur additional contributions to those expected in ordinary scalar electrodynamics. To the second order in the explicit electromagnetic coupling these may be represented by the Feynman diagrams associated with the polarisation tensor, as suggested by Englert and Brout[17]



where the unperturbed system is effectively the Goldstone Model, for which we know one of the propagators, and the electromagnetic field.

Thus, the polarisation tensor, in this approximation is

$$\Pi^{\lambda\sigma}(k) = -ie^2 \langle\phi_2\rangle_0^2 \left[g^{\lambda\sigma} - \frac{1}{k^2} k^\lambda k^\sigma\right]$$

the surface dependent terms of the same order cancelling out.

In terms of the Landau Gauge photon propagator

$$\Pi^{\lambda\sigma}(k) = -ie^2 \langle\phi_2\rangle_0^2 \; G_L^{\lambda\sigma}(k)$$

In the second contribution, it is accepted that the Goldstone boson, now taken as uncoupled, has zero mass, $\rho_1^G = \delta(m^2)$. We note that the polarisation tensor in this approximation has the form required by gauge invariance, and is covariant.

We may now construct the corrected photon propagator via the Dyson Equation

$$D_{\mu\nu}(k) = D^0_{\mu\nu}(k) + D^0_{\mu\lambda}(k)\, \Pi^{\lambda\sigma}(k)\, D_{\sigma\nu}(k)$$

$D^0_{\mu\nu}(k)$ being that propagator associated with the Lagrangian for the free electromagnetic field. We may assume $D^0_{\mu\nu}$ is given by

$$D^0_{\mu\nu}(k) = -i\, \mathcal{G}_{\mu\nu}(k)\, \Delta_c(k) \qquad ; \qquad \Delta_c(k) = \frac{1}{k^2 - i\varepsilon}$$

and, because of the property,

$$\mathcal{G}_{\mu\lambda}(k)\, \mathcal{G}_L^{\lambda\sigma}(k)\, \mathcal{G}_{\sigma\nu}(k) = \mathcal{G}_{\mu\nu}(k)$$

where $G_{\mu\lambda}(k)$ may be radiation, Landau or Feynman Gauge operators, we have a solution

$$D_{\mu\nu}(k) = -i\, \mathcal{G}_{\mu\nu}(k)\, \Delta(k)$$

where

$$\Delta(k) = \frac{1}{k^2 - i\varepsilon} - \frac{1}{k^2 - i\varepsilon}\, e^2 \langle \phi_2 \rangle_0^2\, \Delta(k)$$

or

$$\Delta(k) = \frac{1}{k^2 + e^2 \langle \phi_2 \rangle_0^2 - i\varepsilon}$$

Thus the photon would appear to have acquired a mass in this approximation, consistent with that expected from classical arguments. We observe that the calculation is gauge independent, as we would expect. In a higher approximation, of course, Π would generally contain the bare photon propagators, and would thus generally become gauge dependent. The usefulness of the present approximation lies in its connection with the quasi-particle approximation.

The connected propagator for the 'photon' is thus

$$D_{\mu\nu}(k) = -i\, \mathcal{G}_{\mu\nu}(k)\, \frac{1}{k^2 + e^2 \langle \phi_2 \rangle_0^2 - i\varepsilon}$$

This, of course, is not the form associated with a free vector meson, being non-covariant as it is. The point of interest, at present, is the appearance of a pole at a non-zero value.

To compare with the argument of Schwinger[18], we note that in radiation gauge, the unordered products $\langle A_\mu(x)\, A_\nu(x')\rangle_0$ are given by

$$\langle A_\mu(x) A_\nu(x')\rangle_0 = \frac{1}{(2\pi)^3}\int_0^\infty dm^2\, B(m^2)\int d^4k\; g_{\mu\nu}(k)\,\delta(k^2+m^2)\,\theta(k_0)\, e^{ik(x-x')}$$

and by virtue of the field equations, we find the gauge independent quantity involving the current

$$\langle J_\mu(x) J_\nu(x')\rangle_0 = \frac{1}{(2\pi)^3}\int d^4k\int_0^\infty dm^2\, B(m^2)(k_\mu k_\nu - k^2 g_{\mu\nu})\,\delta(k^2+m^2)\,\theta(k_0)\,e^{ik(x-x')}$$

Thus, the vacuum fluctuations of the currents determine the weight function $B(m^2)$, except at $m^2 = 0$.

In our case, it is obvious that the present approximation corresponds to the quasi-particle weight function

$$B(m^2) = \delta(m^2 - e^2\langle\phi_2\rangle_0^2)$$

so that

$$\langle J_\mu(x) J_\nu(x')\rangle_0 = e^2\langle\phi_2\rangle_0^2\int \frac{d^4k}{(2\pi)^3}\,(k_\mu k_\nu - k^2 g_{\mu\nu})\,\delta(k^2+e^2\langle\phi_2\rangle_0^2)\,\theta(k_0)\,e^{ik(x-x')}$$

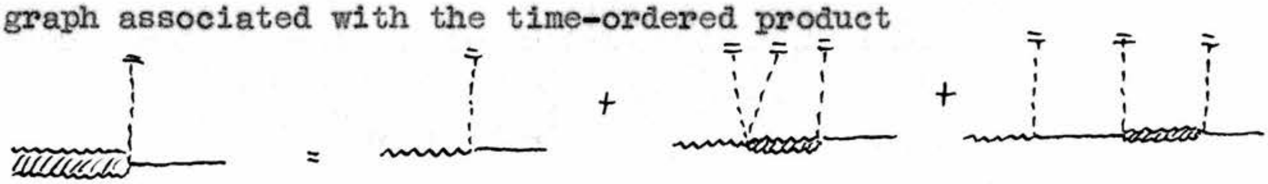
In this approximation, it appears that Schwinger's contention that it is primarily a dynamical question whether $B(m^2)$ contains a zero mass pole or not, is borne out. The extended Goldstone Model appears to have the ingredients leading to the required dynamics such that no zero mass pole appears.

We now calculate, in the same approximation, the other relevant spectral functions of the model.

We have already had cause to consider the function $\langle[J^\mu(x), \phi_1(x')]\rangle_0$

$\langle [J^\mu(x), \phi_1(x')] \rangle_0$ in order to argue the existence of the Goldstone Boson for the limit $e \to 0$. There, for the sake of consistency with canonical commutation relations, we were forced to introduce a non-covariant gauge description, although the subsequent conclusion that the massless boson in fact was necessary to preserve manifest covariance saved the situation.

In the present case, the same argument can be applied, but this time the quasi-particle approximation above suggests that in general we must use a non-covariant gauge in order to avoid inconsistency. Associating the weight function $B(m^2)$ with the vector field, we may write down the expression corresponding to the graph associated with the time-ordered product



$$\langle [A_\mu(x), \phi_1(x')] \rangle_0 = \frac{-ie\langle\phi_2\rangle_0}{(2\pi)^3} \int_0^\infty dm^2 B(m^2) \int d^4k\, g_{\nu\lambda}(k) k^\lambda [\theta(k_0) - \theta(-k_0)] \left\{ \frac{\delta(k^2)}{k^2+m^2} + \frac{\delta(k^2+m^2)}{k^2} \right\} e^{ik(x-x')}$$

which is virtually identical to the expression (9) obtained previously prior to taking the limit $e \to 0$. Effectively $\langle\phi_2\rangle_0^2\, B(m^2)$ replaces $\rho_1(m^2)$, and there is a massless contribution from the Goldstone Boson, which before came from the unperturbed photon field. Here, however, the indications are that $B(m^2) = \delta(m^2 - e^2\langle\phi_2\rangle_0^2)$. Application of $(\partial_\mu\partial_\nu - g_{\mu\nu}\,\partial^2)$ gives $\langle [J_\mu(x), \phi_1(x')] \rangle_0$, and taking the space integral of the time component, as before, we must get

$$i \int d^3\underline{x}\, \langle [J_0(x), \phi_1(x')] \rangle_0 = \langle\phi_2\rangle_0 \int_0^\infty dm^2 B(m^2) \cos\{m(x_0 - x_0')\}$$

which we can only do if we use radiation gauge. We note the equal time commutation relations are obeyed as before.

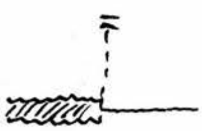
In general, outside any approximation scheme, a non-covariant gauge is essential. For, from equation (4) (for $m_o = 0$), the broken symmetry requirement

$$\int d^3\underline{x}\ \langle [J_o(x),\ \phi_1(x')]\rangle_o \neq 0$$

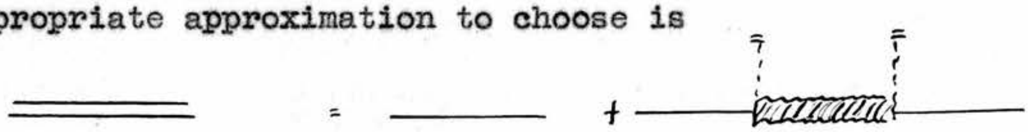
can never be maintained, in perturbation theory at least if a covariant gauge is employed. This fact alone might lead us to suspect the absence bf both a massless vector particle, and scalar particle(32).

The approximate calculation of the time-ordered product $\langle T.(A_\mu(x)\ \phi_1(x')\rangle_o$ consistent with that carried out for the photon propagator must be carried out in radiation gauge.

Diagrammatically, it is



Similarly, for the boson propagator $\langle T.(\chi_1(x)\ \chi_1(x'))\rangle_o$ the appropriate approximation to choose is



with the value

$$\left(1 + \frac{e^2\langle\phi_2\rangle_o^2}{k^2+(1\cdot k)^2}\right)\Delta(k) \quad , \quad \Delta(k) = \frac{1}{k^2+e^2\langle\phi_2\rangle_o^2 - i\varepsilon}$$

after some calculation. Thus, the boson field described by the field function χ_1 has acquired a mass in the same approximation as the 'photon' does.

$$\langle T.\left(\chi_1(x)\chi_1(x')\right)\rangle_0 = \int \frac{d^4k}{(2\pi)^4}\left(1+\frac{e^2\langle\phi_2\rangle_0^2}{k^2+(n.k)^2}\right)\frac{1}{k^2+e^2\langle\phi_2\rangle_0^2-i\varepsilon}\, e^{ik(x-x')}$$

Thus, we have, in this quasi-particle approximation, expressed all the relevant propagation functions, and we may derive from them the commutation functions

$$\langle[A_\mu(x), A_\nu(x')]\rangle_0 = \frac{i}{(2\pi)^3}\int d^4k\; \mathcal{G}^R_{\mu\nu}(k,n)\,\varepsilon(k_0)\,\delta(k^2+e^2\langle\phi_2\rangle_0^2)\, e^{ik(x-x')}$$

$$\langle[A_\mu(x), \chi_1(x')]\rangle_0 = e\langle\phi_2\rangle_0\int\frac{d^4k}{(2\pi)^3}\frac{n_\mu(n.k)}{k^2+(n.k)^2}\,\varepsilon(k_0)\,\delta(k^2+e^2\langle\phi_2\rangle_0^2)\, e^{ik(x-x')}$$

$$\langle[\chi_1(x), \chi_1(x')]\rangle_0 = \int\frac{d^4k}{(2\pi)^3}\frac{(n.k)^2}{k^2+(n.k)^2}\,\varepsilon(k_0)\,\delta(k^2+e^2\langle\phi_2\rangle_0^2)\, e^{ik(x-x')}$$

as obtained by Higgs(35).
These three relations may be combined so as to eliminate the special time-like vector n_μ, in the following way.

Defining the vector field $B_\mu(x)$ through the relation

$$B_\mu(x) = A_\mu(x) - \frac{1}{e^2\langle\phi_2\rangle_0^2}\,\partial_\mu\chi_1(x)$$

which appeared in the first order classical context in (1.5) of Chapter I, we find the covariant commutator function

$$\langle[B_\mu(x), B_\nu(x')]\rangle_0 = \int\frac{d^4k}{(2\pi)^3}\,\mathcal{G}_{\mu\nu}^{e\langle\phi_2\rangle_0}(k)\,\varepsilon(k_0)\,\delta(k^2+e^2\langle\phi_2\rangle_0^2)\, e^{ik(x-x')} \qquad (13)$$

independent of n_μ, and immediately recognisable as the propagator associated with the free Proca spin 1 massive field.

We conclude that, under the effect of the scalar field self-coupling, what was the massless gauge field, comprising two states of polarisation, has combined with what was the massless Goldstone boson, to form a massive Proca field, with three polarisation states. This effect was anticipated in the small amplitude classical model(15). In order to obtain it here we were forced into radiation gauge, a requirement which has no real analogue in that classical argument.

Further, from above we have in this approximation

$$i\int d^3\underline{x}\ \langle[J_0(x), \phi_1(x')]\rangle_0 = \langle\phi_2\rangle_0 \cos\{e\langle\phi_2\rangle_0(x_0 - x_0')\}$$

which is now time-dependent, as opposed to the Goldstone case, in which the masslessness of the particle ensures the time-independence of this quantity. (In this case, it is apparent that the 'operator' $\int d^3\underline{x}\ \ J_0(x)$ must be time-dependent.)

Some Classical Considerations of Gauge Invariance

In order to try to understand the mechanism which renders the GSW argument inapplicable whenever a gauge field is present, let us consider the old question of extending the simple gauge invariance to that of the second kind in the context of the classical Goldstone Model, in the form

$$\mathcal{L}_G(x) = -(\partial_\mu\phi^*)(\partial^\mu\phi) - V(\phi^*\phi)$$

The demand for invariance under the extended transformation

$$\phi(x) \to e^{+i\alpha(x)}\phi(x) \quad ; \quad \phi^*(x) \to \phi^*(x)\, e^{-i\alpha(x)}$$

implies the minimal prescription

$$\partial_\mu \rightarrow \partial_\mu - ie A_\mu$$

where A_μ must simultaneously transform as

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha .$$

At this stage $A_\mu(x)$ need not be an independent field, and in fact there already exists a candidate for A_μ in terms of the scalar fields ϕ, in the form of the longitudinal term

$$-\frac{i}{e} \partial_\mu \log \left(\frac{\phi}{\phi^*}\right)^{1/2} .$$

Thus, a possible formal extension of the model, without requiring the presence of an independent vector field, would be described by the Lagrangian

$$\mathcal{L}(x) = -\left[\left\{\partial_\mu + e\partial_\mu \tfrac{1}{e}\log\left(\frac{\phi}{\phi^*}\right)\right\}\phi^*\right]\left[\left\{\partial^\mu - e\partial^\mu \tfrac{1}{e}\log\left(\frac{\phi}{\phi^*}\right)^{\frac{1}{2}}\right\}\phi\right] - V(\phi^*\phi)$$

However, a closer investigation reveals that the consequences of the apparent gauge invariance here are trivial. The current associated must vanish identically. In fact, through the transformation $\Phi^2 = \phi^*\phi$, one can see that the model really describes the single component field Φ.

$$\mathcal{L}(x) = -(\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi^2)$$

Thus, the demand for gauge invariance of the second kind without the introduction of an independent vector field is equivalent to eliminating one of the field components, in fact, that which would be associated with the Goldstone particle. In so doing, we have also lost the continuous symmetry of the model. If $\langle \Phi \rangle_0 \neq 0$ the symmetry $\Phi \rightarrow -\Phi$ is spontaneously broken. Therefore, this possible extension is trivial, although it does indicate a method of eliminating the Goldstone Boson.

The point made by Nambu(36) that the addition of the coupling $g J_\mu J^\mu$ to $L_G(x)$ eliminates the Goldstone mode to first order in the fields, provided $g = \frac{1}{2\langle\phi_2\rangle_0}$, is therefore seen to be but a special case of the above more general prescription. There, in first order, as here exactly, the current, conserved on account of the global symmetry, vanishes.

The non-trivial generalisation of this method occurs whenever we introduce a vector gauge field, $A_\mu(x)$(15). We write A_μ in such a way that it may be expressed in terms of a part which possesses all the gauge variation, and a part which is invariant

$$A_\mu(x) = B_\mu(x) - \frac{i}{e} \partial_\mu \log \left(\frac{\phi}{\phi^*}\right) .$$

This equation may alternatively be viewed as the definition of a gauge invariant field, $B_\mu(x)$. Expressed in terms of B_μ, the Lagrangian becomes

$$\mathcal{L} = -\left\{\partial_\mu - ie\left[-B_\mu + \frac{i}{e}\partial_\mu \log\left(\frac{\phi}{\phi^*}\right)\right]\right\}\phi^*\left\{\partial^\mu + ie\left[-B^\mu + \frac{i}{e}\partial^\mu \log\left(\frac{\phi}{\phi^*}\right)\right]\right\}\phi$$

$$- \left\{ V(\phi^*\phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}$$

which is gauge invariant as required.

In this case, making the transformation $\Phi^2 = \phi^*\phi$

$$\mathcal{L} = -\left\{(\partial_\mu\Phi)(\partial^\mu\Phi) + V(\Phi^2) + \frac{1}{2}e^2 B_\mu B^\mu \Phi^2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}$$

and we now effectively are describing the interaction of singlet particles with the vector field B_μ. What would have been the Goldstone particles no longer exist as physical fields, being absorbed in the vector field of the problem.

Thus, it is apparent that the mechanism which rids us of the Goldstone particles (at least in this classical context) can be incorporated at the very start, and in an exact sense, through the field transformation

$$B_\mu(x) = A_\mu(x) - \frac{i}{e}\partial_\mu \log \left(\frac{\phi}{\phi^*}\right)^{1/2} .$$

To first order in the field amplitudes, $B_\mu(x)$ is the form used previously in the quasi-particle approximation, or in the classical small amplitude theory.

To what extent we actually have a theory with gauge invariance of the second kind is now not quite apparent. What we have succeeded in doing is to construct a model, possessing a conserved current (here $B_\mu \Phi^2$) in which the gauge invariance is concealed so that we may not talk of invariance at all. The question is to whether $\langle \phi_2 \rangle_0$ or $\langle \Phi \rangle_0 \neq 0$ still remains a dynamical problem, but not now connected with broken continuous symmetry as previously defined, since the question of 'infinite' vacuum degeneracy does not enter, if we start from the new Lagrangian. The broken symmetry is the discrete transformation $\Phi \to -\Phi$.

A final remark may be in order concerning the connection between gauge invariance and the necessity of having vector fields possessing bare mass zero. It would appear that, provided B_μ is still defined in terms of A_μ and ϕ as above, we may add on a term $-\frac{1}{2} m_o'^2 B_\mu B^\mu$ to the above Lagrangian, without in any way destroying the 'hidden' gauge invariance of the theory. The conserved current will then be enhanced by an amount $m_o'^2 B^\mu$. Written in terms of the fields $A_\mu \phi$ the Lagrangian has the form classically

$$\mathcal{L}(x) = -(\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^*\phi) - e^2 A_\mu A^\mu \phi^*\phi$$

$$-\frac{ie}{2} A_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_o'^2 A_\mu A^\mu$$

$$-\frac{i m_o'^2}{2e} A_\mu \frac{(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)}{\phi^*\phi} - \frac{m_o'^2}{8e^2} \frac{(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)}{(\phi^*\phi)^2}$$

which is formally gauge invariant, although there appears the bare mass term $m_o^2 A_\mu A^\mu$. (Of course, m_o also appears as a coupling parameter). We conclude that there is a loophole in the usual argument which implies zero bare mass in a gauge invariant theory. Certain additional types of interactions can be introduced so that the requirement of gauge invariance can accommodate vector fields possessing finite bare mass.

<u>Conclusion</u>:

The classical considerations of the final section indicate the mechanism behind the disappearance of the Goldstone boson. By introducing the field dependent gauge transformation, the global continuous symmetry of the ϕ_i is observed to be spurious, reducing in fact to the discrete reflection invariance of $(\phi_i \phi_i)^{1/2}$, which along with the vector field B_μ, becomes a physical mode of the problem. The Goldstone boson has been transformed away, as has been the original continuous symmetry. Quantisation of the resulting Lagrangian would now not at all be expected to involve

Goldstone particles in the associated spectrum.

The sign that something of the sort happens in a direct quantisation of the theory involving the variables A_μ, ϕ_i is given by the apparent necessity to use the physical non-covariant gauge (e.g. radiation gauge), since, for this model, (as described by (1) for $m_o = 0$), any covariant gauge is inconsistent. It has recently been argued by Kibble[36] that a consistent covariant gauge formalism is possible using the extended operator formalism of Schwinger, but this would demand the introduction of a subsidiary field, and alter the problem sufficiently for the arguments given to be inapplicable. In the radiation gauge, we have shown that the classical approximation may be obtained by a procedure of selective summation of diagrams, as suggested by Englert and Brout, the result being summed up in (13), the commutator for a free Proca field.

Finally, we note that should the vector field possess any bare mass whatever $(m_o \neq 0)$, the accompanying necessity for manifest Lorentz Covariance dictates the presence of the Goldstone boson.

CHAPTER IV

ON THE NON-RELATIVISTIC CASE

We have discussed briefly, in Chapter I, how the theory of Superconductivity, as proposed by B.C.S., and refined by B.T.S., Valatin, and others, has in the main been responsible for much of the interest for broken symmetry solutions in quantum field theory in general. To the extent that the Goldstone Theorem does not apply in this case, the superconductor model offers the clue to the domain of applicability of the theorem.

The main physical properties of any many-particle system, of which a superconductor may be considered an example, are governed by the spectra of quasi-particle and collective excitations, the derivation of which, one may consider to be the essential problem. The techniques of quantum field theory have proved invaluable in this respect, the two central methods employed being perturbation theoretic, and the Hartree-Fock Variational Principle, in which one postulates the form of the ground state vector. In the case of superconductivity the direct use of the former is forbidden, and the theory largely relies on what has been termed a generalisation of the Hartree-Fock procedure, yielding effectively a non-perturbative result. Explicit calculations based on this method give the results which have formed the argument against the general applicability of the theorem. The collective mode which one would expect to be 'massless' by the theorem, turns out to be massive (the plasmon).

Turning now to the general arguments associated with the theorem, we have so far been careful to emphasise that these are independent of the forces involved. It would therefore seem desirable, in the light of the explicit calculations, to arrive at an argument, relating to the theorem in the same spirit as that of G.S.W., but which incorporated the forces explicitly. Then, perhaps, one might be able to see more clearly the connection between the forces and the collective mode spectrum.

In this chapter, we outline the generalised Hartree-Fock Method in the theory of Superconductivity. Drawing on general arguments advanced by Goldstone and Baker, Johnson and Lee in the relativistic context, (and employing the more appropriate (for the present purpose) formulation of Nambu and Gorkov), we demonstrate the equivalence between the usual Hartree-Fock procedure, incorporating the Bogoliubov Transformation explicitly, and another version which is more easily related to the Goldstone question, culminating in an interpretation of the well known energy gap equation as a homogeneous Bethe-Salpeter equation associated with zero momentum-energy. The forces thus appear explicitly in the argument. With this interpretation it is more immediately apparent that no statement of the sort conjectured by Goldstone for relativistic theories can be forthcoming.

Outline of the Hartree-Fock Procedure

The many-fermion problem is generally described by the Lagrangian function

$$\mathcal{L}(x) = \frac{\nabla\psi_\alpha^\dagger \cdot \nabla\psi_\alpha}{2m} - \frac{i}{2}\left(\psi_\alpha^\dagger \dot{\psi}_\alpha - \dot{\psi}_\alpha^\dagger \psi_\alpha\right) - \frac{1}{2}\int d^3\underline{x}'\, \psi_\alpha^\dagger(x)\,\psi_\beta^\dagger(x')\,V(\underline{x}-\underline{x}')\,\psi_\beta(x')\,\psi_\alpha(x) \tag{1}$$

where the fermion creation and annihilation fields satisfy the usual anti-commutation relations

$$\{\psi_\alpha(x), \psi^\dagger_\beta(x')\}_{x_0=x_0'} = \delta_{\alpha\beta}\delta^{(3)}(\underline{x}-\underline{x}') \quad ; \quad \{\psi_\alpha(x), \psi_\beta(x')\}_{x_0=x_0'} = 0$$

The forces acting between every pair of fermions of the system are represented by the two-body potential function $V(\underline{x} - \underline{x}')$; for simplicity they are here assumed to be instantaneous. The actual agency producing the interaction in a superconductor, in most cases, is the lattice of ions in the metal, the problem being then ultimately viewed in terms of the interaction of a phonon field with the electron field.

As a quantum field problem, without the assumption of spontaneous breakdown, we may regard the task of solving it as reducing to a determination of the N-particle propagator

$$i \langle \Theta | T \cdot \{ \psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2) \ldots \psi_{\alpha_N}(x_N)\psi^\dagger_{\alpha_1'}(x_1') \ldots \psi^\dagger_{\alpha_N'}(x_N') \} | \Theta \rangle$$

$|\Theta\rangle$ being the translationally invariant ground state vector. In practice, we are usually only interested in the cases $N = 1,2$, the single, and two-particle propagators. A knowledge of these allows us to infer the excitation spectra for the system.

The requirement that the system be superfluid, however, forces upon us the necessity to invoke the Cooper pairing hypothesis, now more or less an experimental fact. In the ground state of a superfluid fermion system, particles are correlated in pairs of equal and opposite spin and equal and opposite momenta. The mathematical representation of this statement is taken to be

$$\langle \Theta | \psi_1(x)\, \psi_2(x') | \Theta \rangle \neq 0$$

implying the degeneracy of the ground state with respect to simple gauge transformations.

To take account of this condition, B.C.S. chose a trial ground state vector in the form:

$$|\Theta\rangle = \prod_p \left(\cos\theta_p + \sin\theta_p\, a^+_{p1} a^+_{p2} \right) |0\rangle$$

which may be written more suggestively as

$$|\Theta\rangle = U|0\rangle = e^{-\sum_p \theta_p (a^+_{p1} a^+_{p2} - a_{p2} a_{p1})} |0\rangle$$

a_{ps} , a^+_{ps} being respectively annihilation, creation operators for electrons of spin s, momentum p.

This leads automatically to the linear Bogoliubov-Valatin relations, defined through the unitary transformations

$$\xi_{p1} = U a_{p1} U^{-1} \qquad , \qquad \xi_{p2} = U a_{p2} U^{-1}$$

such than

$$\xi_{ps} |\Theta\rangle = 0$$

Minimisation of the ground state expectation value of the Hamiltonian with respect to the parameter θ_k after linearisation leads to the familiar energy gap equation, and the quasi-particle approximation. (See, for example, reference 5).

Further, allowing the pairs to possess a net non-zero momentum leads to the collective spectrum.

The direct use of this approach, however, conceals the relation of the approximation to the Goldstone problem.

The Nambu-Gorkov Formalism and Spontaneous Symmetry Breakdown

In order to facilitate the application of the more conventional mathematical apparatus of quantum field theory, Nambu and Gorkov have reformulated the problem in a way which brings out the underlying matrix structure of the theory.

Introducing the two-component field operators defined by

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2^+(x) \end{pmatrix} \quad , \quad \Psi^+(x) = (\psi_1^+(x), \psi_2(x))$$

we observe that the Lagrangian function may be expressed (up to an infinite quantity) as

$$\mathcal{L}(x) = \frac{\nabla \Psi^+ \cdot \sigma_3 \nabla \Psi}{2m} - i(\Psi \dot{\Psi}^+ - \dot{\Psi}\Psi^+) - \int d^3\underline{x}'\, \Psi^+(x)\sigma_3 \Psi(x) V(\underline{x}-\underline{x}')\Psi^+(x')\sigma_3 \Psi(x')$$

We note that the anti-commutation relations are preserved

$$\{\Psi_\alpha(x), \Psi_\beta^+(x')\}_{x_0 = x_0'} = \delta_{\alpha\beta}\,\delta^{(3)}(\underline{x}-\underline{x}') \quad , \quad \{\Psi_\alpha(x), \Psi_\beta(x')\}_{x_0 = x_0'} = 0$$

The problem is still one of interacting Fermi fields.

It is interesting to note that the hermitian bilinear constructs defined in terms of the Nambu-Gorkov fields

$$\phi_i(x) = \Psi^+(x)\,\sigma_i\,\Psi(x)$$

and satisfying the 'angular momentum' commutation rules

$$[\phi_i(x), \phi_j(x')]_{x_0 = x_0'} = 2i\,\varepsilon_{ijk}\,\phi_k(x')\,\delta^{(3)}(\underline{x}-\underline{x}')$$

in fact, for the choice ijh = 312, give us, on taking the ground state expectation value, the spectral function of central importance in the argument og G.S.W., namely $\langle 0|\,[J_0(x), \phi_1(x)]\,|0\rangle$. Under certain conditions their argument would imply the existence of intermediate states whose energy vanishes in the long wavelength limit. We recall from Section (4) of Chapter I, and Chapter III, that these conditions have not been explicitly formulated in terms of the forces, (although we have good evidence that the long-range force must be excluded). For convenience, we now restate the relevant points.

The General Non-Relativistic Statement

Given any function $f_0(\underline{x}, x_0)$, such that
(a) $f_0(\underline{x}, 0) \propto \delta^{(3)}(\underline{x})$, and (b) $\int d^3\underline{x}(f_0(\underline{x}, x_0)$ is independent of x_0, then the Fourier Transform $f_0(\underline{k}, k_0)$ is such that
(a') $\int dk_0\, f_0(\underline{k}, k_0)$ is independent of $\underline{k}$, and (b') $f_0(\underline{0}, k_0) \propto \delta(k_0)$
(a') follows directly from (a), as (b') does from (b), both independently. The proofs are straight forward and quite independent of any local conservation law which in the relativistic case, we have seen implies (b).

It is apparent that (b) must be relaxed if $f_0(\underline{0}, k_0)$ is to have a singularity at a non-zero value of k_0.

The application of this general statement to the present problem is obvious, with $f_0(\underline{x}, x_0) = i\langle 0|\,[J_0(x), \phi_1(0)]\,|0\rangle$.
(However, it might be worth noting that the statements
(a) $\rightarrow$ (a'), (b) $\rightarrow$ (b') are of interest in field theory

generally (outside consideration of spontaneous symmetry breakdown in quantum field theory) where one has a global conservation law.

For example in the classical field problem of the quantum mechanics of a single particle, we may take $f_o(\underline{x}, x_o)$ to be the probability density required to satisfy (b), in which case (b') must follow. Whether in fact (b) is satisfied must depend on the potential to which the particle is subject.)

A form satisfying (a), (a'), (b), (b') is

$$f_o(\underline{x}, x_o) \propto \int d^3\underline{k}\, e^{i(\underline{k}.\underline{x} - W_{\underline{k}} x_o)}$$

where $W_{\underline{o}} = 0$, indicating the presence of a Goldstone excitation.

Thus the forces responsible for a Goldstone excitation, also must be such that $\langle \Theta | [J_o(x), \phi_1(x)] | \Theta \rangle$ is time-independent.

<u>The Green Functions</u>:

An immediate advantage of the present formalism, is that the assumption of the breakdown of the gauge symmetry possessed by the Lagrangian, $\Psi \rightarrow e^{i\alpha\sigma_3}\Psi$, may be expressed completely as a condition on the associated single-particle propagator, defined as the matrix

$$\mathcal{G}_{\alpha\beta}(x, x') = i \langle \Theta | T\left(\Psi_\alpha(x)\, \Psi^\dagger_\beta(x')\right) | \Theta \rangle$$

If the interaction is 'switched off', i.e. if $V = 0$, then we effectively have the 'free' propagator, whose Fourier Transform is

$$\mathcal{G}^0(p) = \left(p_0 1 - \sigma_3 \, p^2/2m\right)^{-1} = \frac{p_0 1 + \sigma_3 \, p^2/2m}{p_0^2 - (p^2/2m)^2}$$

where energies are measured from the Fermi level.

If the ground state is degenerate, then the off-diagonal elements, which would otherwise vanish, now exist. Since is generally expressible in the form

$$\mathcal{G}_{\alpha\beta}(x,x') = \sum_{i=0}^{3} G_i(x,x')\sigma_i \qquad (\sigma_0 = 1)$$

The broken symmetry condition may be conveniently stated as

$$[\sigma_3, \mathcal{G}(x,x')] \neq 0 \tag{2}$$

It is now our concern to attempt to show how this assumption is related if at all to the collective spectrum. We might expect to be able to make a statement referring to the two particle propagator

$$\mathcal{G}_{\alpha\beta,\gamma\delta}(x_1,x_2;x_1',x_2') = i^2 \langle \Theta | T\{\Psi_\alpha(x_1)\Psi_\beta(x_2)\Psi_\gamma^\dagger(x_1')\Psi_\delta^\dagger(x_2')\} | \Theta \rangle$$

The Consistency Condition:

As in any field theory problem, the exact single-particle propagator may be related to the 'free' propagator through the Dyson-Schwinger Equation:

$$\mathcal{G}(x,x') = \mathcal{G}_0(x,x') + \int dy\,dy'\,\mathcal{G}_0(x,y)\,\Sigma(y,y')\,\mathcal{G}(y',y)$$

or in momentum-space

$$\mathcal{G}(p) = \mathcal{G}_0(p) + \mathcal{G}_0(p)\,\Sigma(p)\,\mathcal{G}(p)$$

While in a normal theory, with no spontaneous breakdown, this equation may be interpreted directly in terms of diagrammatic perturbation theory, even when spontaneous breakdown does exist there is no logical objection to its use. $\Sigma(P)$ is the self-energy operator associated with the exact propagator, and the relation may be regarded as either a definition for G in terms of Σ, or vice versa. Its form is suggested by that of the field equations derived from the Lagrangian.

For brevity, we use the notation, in coordinate space

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0\,\Sigma\,\mathcal{G}$$

where it is understood that factors multiplied together incorporate an integration over the continuous space-time variables as well as summation over the discrete 'pseudo-spin' labels.

Regarded in an exact sense, the self-energy operator is a functional of the single-particle propagator, so that given $\Sigma(\mathcal{G})$ the Dyson Equation would then serve to determine G. Of course, we cannot know $\Sigma(G)$ in any exact sense, and even if we could, our ability to solve the resulting equation for G would be in question.

However, from this general viewpoint, it is possible to extract some useful points which are connected with the consistency of the theory, and must be taken into account in any approximation procedure which might be employed.

Whereas in a theory in which spontaneous breakdown does not occur, under a gauge transformation the propagator transforms as

$$\mathcal{G} \to \mathcal{G}' = e^{i\alpha\sigma_3}\mathcal{G}e^{-i\alpha\sigma_3} = \mathcal{G}$$

in the present case, because of the condition (2), this cannot be so

i.e. $G' \neq G$.

However, by virtue of the symmetry in the theory, we may express the transformed Dyson Equation in the two forms

$$\mathcal{G}' = \mathcal{G}_0 + \mathcal{G}_0\Sigma'(\mathcal{G})\mathcal{G}'$$

$$\mathcal{G}' = \mathcal{G}_0 + \mathcal{G}_0\Sigma(\mathcal{G}')\mathcal{G}'$$

and so deduce that

$$\Sigma'(\mathcal{G}) = \Sigma(\mathcal{G}') \quad , \quad e^{i\alpha\sigma_3}\Sigma(\mathcal{G})e^{-i\alpha\sigma_3} = \Sigma(e^{i\alpha\sigma_3}\mathcal{G}e^{-i\alpha\sigma_3}) \tag{3}$$

This may be regarded as nothing more than a statement that the theory is symmetrical under the gauge transformation $\Psi \to e^{i\alpha\sigma_3}\Psi$

We may formally expand both sides of (3), in terms of powers of the parameter α. The left hand side is

$$e^{i\alpha\sigma_3}\Sigma(\mathcal{G})e^{-i\alpha\sigma_3} = \sum_{r=0}^{\infty}\frac{(i\alpha)^r}{r!}[\sigma_3, \Sigma(\mathcal{G})]^{(r)} \quad , \quad [\sigma_3, \Sigma(\mathcal{G})]^{(0)} = \Sigma(\mathcal{G})$$

while the right hand side takes the form

$$\Sigma'(e^{i\alpha\sigma_3}\mathcal{G}e^{-i\alpha\sigma_3}) = \Sigma'\left(\mathcal{G} + \sum_{r=1}^{\infty} \frac{(i\alpha)^r}{r!}[\sigma_3, \mathcal{G}]^{(r)}\right)$$

$$\Sigma(e^{i\alpha\sigma_3}\mathcal{G}e^{-i\alpha\sigma_3}) = \sum_{s=0}^{\infty}\left(\sum_{r=1}^{\infty} \frac{(i\alpha)^r}{r!}[\sigma_3, \mathcal{G}]^{(r)}\right)\frac{\alpha^s}{s!}\left(\frac{\delta}{\delta\mathcal{G}}\right)^s \Sigma'(\mathcal{G})$$

involving the functional derivatives of $\Sigma'(G)$.

We need, in fact, only concern ourselves with the first order terms, which when equated give

$$[\sigma_3, \Sigma'(\mathcal{G})] = [\sigma_3, \mathcal{G}]\,\frac{\delta\Sigma'(\mathcal{G})}{\delta\mathcal{G}}$$

Written out in full, this relation means

$$[\sigma_3, \Sigma'(x,y)]_{\alpha\beta} = \int d^4\xi\, d^4\eta\, [\sigma_3, \mathcal{G}(\xi,\eta)]_{\lambda\mu}\, \frac{\delta\Sigma_{\alpha\beta}(x,y)}{\delta\mathcal{G}_{\lambda\mu}(\xi,\eta)}$$

the summation convention on the discrete labels being operative.

Now, from the Dyson Equation it is possible to relate $[\sigma_3, \Sigma'(\mathcal{G})]$ to $[\sigma_3, \mathcal{G}]$ by taking it in the form

$$\mathcal{G}_0^{-1} = \mathcal{G}^{-1} + \Sigma$$

and taking the commutator with σ_3, we get

$$[\sigma_3, \Sigma] = -[\sigma_3, \mathcal{G}^{-1}]$$

or $$[\sigma_3, \mathcal{G}] = \mathcal{G}[\sigma_3, \mathcal{G}]\mathcal{G}$$

We thus find, on substitution, the relation

$$[\sigma_3, \mathcal{G}] = \mathcal{G}[\sigma_3, \mathcal{G}]\,\frac{\delta\Sigma'(\mathcal{G})}{\delta\mathcal{G}}\,\mathcal{G} \qquad (4)$$

or written out in non-abbreviated form

$$[\sigma_3, \mathcal{G}(x,y)] = \int d^4x' d^4y' d^4\xi d^4\eta \; \mathcal{G}(x,x') \, [\sigma_3, \mathcal{G}(\xi,\eta)] \, \frac{\delta \Sigma(x',y')}{\delta \mathcal{G}(\xi,\eta)} \, \mathcal{G}(y',y)$$

This relation, formally exact, must be satisfied in general. The specific (in fact, very specific) case when the ground state is non-degenerate, i.e. when $[\sigma_3, G] = 0$, is automatically a possibility, for then the relation is identically satisfied. However, we might argue that the existence of this case would depend eventually on the dynamics, and from a purely mathematical point of view should not be conjectured, but derived. The assumption that $[\sigma_3, G]$ does not vanish is seen, in fact to be the relaxation of what is usually a severe restriction.

Finally, it must be stated that there is nothing in this relation which is not already present in the Dyson Equation, which alone has been used to obtain it. Any approximation used to determine G from the Dyson Equation (as carried out by Nambu) will automatically ensure that (4) is satisfied.

The Energy Gap Equation:

We now show that within a certain well-known formulation of the Hartree-Fock approximation, the relation (4) may be interpreted as the energy gap equation.

Defining the Fourier Transform of the quantity $\frac{\delta \Sigma(x',y')}{\delta \mathcal{G}(\xi,\eta)}$ through the relation

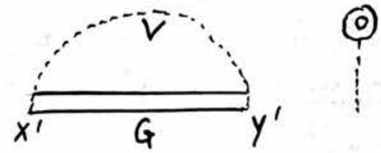
$$\frac{\delta \Sigma_{\alpha\beta}(x',y')}{\delta \mathcal{G}_{\lambda\mu}(\xi,\eta)} = \int d^4k \, d^4q \, d^4p \; I_{\alpha\beta,\lambda\mu}(p,q;k) \, e^{i\left\{\frac{k}{2}(x'+y'-\xi-\eta) + p(x'-y') + q(\xi-\eta)\right\}}$$

the momentum-space form of the relation is

$$\left[\sigma_3, \mathcal{G}(p)\right]_{\alpha\beta} = \mathcal{G}_{\alpha\rho}(p)\int d^4q \left[\sigma_3, \mathcal{G}(q)\right]_{\lambda\mu} I_{\rho\sigma,\lambda\mu}(p,q;0)\, \mathcal{G}_{\sigma\beta}(p)$$

Following Nambu, we approximate Σ (x'y') by the second-order perturbation theoretic expression, linear in the propagator, already containing the self-consistent self-energy given diagrammatically by

(We neglect the tadpole contribution)



Thus

$$\Sigma_{\alpha\beta}(x',y') = i\, V(\underline{x}'-\underline{y}')\, \delta(x_0'-y_0')\, \sigma_3^{\alpha a}\, i\mathcal{G}_{ab}(x',y')\, \sigma_3^{b\beta}$$

from which we derive the functional derivative

$$\frac{\delta}{\delta \mathcal{G}_{\lambda\mu}(\xi,\eta)} \Sigma_{\alpha\beta}(x',y') = i^2 V(\underline{x}'-\underline{y}')\, \delta(x_0'-y_0')\, \sigma_3^{\alpha a}\, \delta_{\lambda a}\, \delta_{\mu b}\, \delta^{(4)}(x'-\xi)\, \delta^{(4)}(y'-\eta)$$

$$= i^2 V(\underline{x}'-\underline{y}')\, \delta(x_0'-y_0')\, \sigma_3^{\alpha\lambda}\, \sigma_3^{\mu\beta}\, \delta^{(4)}(x'-\xi)\, \delta^{(4)}(y'-\eta)\, \sigma_3^{b\beta}$$

We conclude that in this approximation

$$I_{\alpha\beta,\lambda\mu}(p,q;k) = -\sigma_3^{\alpha\beta}\, \sigma_3^{\lambda\mu}\, V(\underline{p}-\underline{q})$$

and is independent of the variable k. In other words, we have chosen I such that

$$I_{\alpha\beta,\lambda\mu}(p,q;k) = I_{\alpha\beta,\lambda\mu}(p,q;0)$$

The consistency relation, in this approximation, now is of the form

$$[\sigma_3, \mathcal{G}(P)]_{\alpha\beta} = - \mathcal{G}_{\alpha\rho}(P) \int d^4q \; \sigma_3^{\rho\lambda} \sigma_3^{\sigma\mu} \, \gamma(P-q) \, [\sigma_3, \mathcal{G}(q)]_{\lambda\mu} \, \mathcal{G}_{\sigma\beta}(P)$$

Because of the symmetry, we may assume that G(P) can be expressed

$$\mathcal{G}(P) = G_0' 1 + G_3' \sigma_3 + G_2' \sigma_2$$

since by rotation in the 'pseudo-spin' space, it is always possible to get to the more general form. Alternatively, the inverse has the form

$$\mathcal{G}(P)^{-1} = G_0 + G_3 \sigma_3 + G_2 \sigma_2$$

giving

$$\mathcal{G}(P) = \frac{G_0 1 - G_3 \sigma_3 - G_2 \sigma_2}{G_0^2 - (G_3^2 + G_2^2)}$$

With this choice, we observe that

$$[\sigma_3, \mathcal{G}(P)] = - \frac{G_2}{G_0^2 - (G_3^2 + G_2^2)} \sigma_2$$

If we now make the following identification

$$G_0(P) = p_0 \quad , \qquad G_3(P) = \bar{\epsilon}(P), \quad G_2(P) = \Delta(P)$$

suggested by the transition to the 'free' fermion case

$$\Delta(P) \to 0 \quad ; \qquad \bar{\epsilon}(P) \to \epsilon(P) = P^2/2m$$

we find that $\Delta(P)$ must be given by

$$\Delta(\underline{p}) = \int d^4q \; V(\underline{p}-\underline{q}) \; \frac{\Delta(q)}{q_0^2 - (\bar{\epsilon}^2(\underline{q}) + \Delta^2(\underline{q}))}$$

and carrying out the q_o-integration, we have

$$\Delta(\underline{p}) = -\int d^3\underline{q} \; V(\underline{p}-\underline{q}) \; \frac{\Delta(\underline{q})}{2\sqrt{\bar{\epsilon}^2(\underline{p}) + \Delta^2(\underline{q})}}$$

which is the well-known energy gap equation. We might note that $\bar{\epsilon}(\underline{p})$ is given by the Dyson Equation, which also yields the gap equation. From the expression for the propagator, we see that the poles give the quasi-particle energies at

$$E(\underline{p}) = \pm\sqrt{\bar{\epsilon}^2(\underline{p}) + \Delta^2(\underline{p})}$$

measured from the Fermi surface.

Thus, the quasi-particle approximation of B.C.S. is seen to arise directly from equation (4). In other words, inherent in the original variational Hartree-Fock procedure, and its more field theoretic refinements, there is the assumption that a non-trivial solution of the equation

$$\Phi_{\alpha\beta}(p) = \mathcal{G}_{\alpha\rho}(p) \int d^4q \; \Phi_{\lambda\mu}(q) \; I_{\rho\sigma,\lambda\mu}(p,q;0) \; \mathcal{G}_{\sigma\beta}(p)$$

exists.

Meaning of $\dfrac{\delta\Sigma(x'y')}{\delta\mathcal{G}(\xi\eta)}$

The quantity $\quad I(x'y';\xi'\eta') = \dfrac{\delta\Sigma(x'y')}{\delta\mathcal{G}(\xi\eta)}$

$$I(x', y'; \xi', \eta') = \frac{\delta}{\delta \mathcal{G}(\xi', \eta')} \Sigma(x'; y')$$

through which we have obtained the Hartree-Fock approximation, has been used in the relativistic context, and in many-body theory by Baym.[38] We shall endeavour in this section to describe its significance, although it is essentially no different than in the relativistic domain.

Here, we follow the argument of Baker, Johnson and Lee, outlining the main points as applied to the present case. Through the medium of an additional interaction of the Fermi field Ψ (x), with a classical (spinor) external source $\lambda(x)$

$$\mathcal{L}'(x) = \Psi^{+}(x)\,\lambda(x) + \lambda^{+}(x)\,\Psi(x)$$

it is possible to obtain some quite general relations, which are obtained by allowing $\lambda(x)$ ultimately to vanish. For consistency, the sources are restricted so that the following anticommutation rules are satisfied

$$\{\lambda(x), \lambda(y)\} = \{\lambda(x), \lambda^{+}(y)\} = \{\lambda(x), \Psi(y)\} = 0$$

etc.

If, for the Lagrangian density $\mathcal{L}(x) + \mathcal{L}'(x)$, the associated propagators are $G(x\ x';\ \lambda)$, then by the procedure of functional differentiation we may obtain $G(x\ x';\ \lambda)$ from the S-matrix as follows.

Since the ground state expectation value of the S-matrix is given by

$$\mathcal{S}(\lambda) = \langle\Theta|T.\{e^{i\int d^4x\,\mathcal{L}'(x)}\}|\Theta\rangle = \langle\Theta|S|\Theta\rangle = \langle S\rangle$$

We have

$$\frac{\delta\langle S\rangle}{\delta\lambda^{\dagger}(x)} = i\langle\Theta|T.(\Psi(x)S)|\Theta\rangle$$

and $$\frac{-i}{\langle S\rangle}\frac{\delta^2\langle S\rangle}{\delta\lambda^{\dagger}(x)\delta\lambda(y)} = \frac{i}{\langle S\rangle}\langle\Theta|T.(\Psi(x)\Psi^{\dagger}(y)S)|\Theta\rangle = \mathcal{G}(x,y;\lambda) \tag{5}$$

Further

$$\frac{\delta^2\mathcal{G}(xy;\lambda)}{\delta\lambda^{\dagger}(y')\delta\lambda(x')} = -i\,\mathcal{G}(x,y)\,\mathcal{G}(y,x') - i\,\mathcal{G}^{(2)}(xy;y'x')$$

where $$\mathcal{G}^{(2)}(x,y;x',y') = (i)^2\langle T.(\Psi(x)\Psi(y')\Psi^{\dagger}(y)\Psi^{\dagger}(x'))\rangle_0$$

ie. $$i\,\frac{\delta^2\mathcal{G}(xy;\lambda)}{\delta\lambda^{\dagger}(y)\delta\lambda(x')} = \mathcal{G}^{(2)}(x,y;x',y') + \mathcal{G}^{(1)}(x,y)\,\mathcal{G}^{(1)}(y',x') = \mathcal{H}(x,y;x',y'). \tag{5a}$$

In the limit $\lambda(x) \to 0$, the above expressions reduce to the appropriate Green's Functions for the given problem.

Now regarding $\mathcal{L}(x)$ as describing the unperturbed system the single particle Green's Function given by (5) may be decomposed in terms of disconnected diagrams, not cancelled by the ground to ground contribution to the S-matrix.

We may write

$$\mathcal{G}(\lambda) = \mathcal{G}'(\lambda) + \mathcal{G}'(\lambda)\, \lambda^{+}\lambda\, \mathcal{G}(\lambda) \tag{6}$$

where $G'(\lambda)$ may be defined by a Dyson type equation

$$\mathcal{G}'(\lambda) = \mathcal{G}_0 + \mathcal{G}_0\, \Sigma'(\lambda)\, \mathcal{G}'(\lambda)$$

$\Sigma'(\lambda)$ is the self-energy associated with the fully interacting system, but excluding the contributions mentioned above, and already taken account of in (6).

Using (5), (5a) and (6), we can now derive the following relation for F

$$\mathcal{F} = \mathcal{G}\mathcal{G} - \mathcal{G}\, \frac{\delta\Sigma}{\delta\mathcal{G}}\, \mathcal{F}\, \mathcal{G} \tag{7}$$

- the inhomogeneous Bethe-Salpeter Equation for the two-particle amplitude.

Thus $\delta\Sigma/\delta\mathcal{G}$ represents the interaction kernel for the two-particle amplitude ($G^{(2)}$ or F).

The coordinate and momentum space versions of the Bethe-Salpeter Equation are, respectively

$$\mathcal{F}(xy,x'y') = \mathcal{G}(xx')\,\mathcal{G}(yy') + \int d\xi\, d\eta\, d\xi'\, d\eta'\, \mathcal{G}(x\xi)\, I(\xi\eta,\xi'\eta')\, \mathcal{F}(\xi'\eta' x'y')\, \mathcal{G}(\eta y)$$

$$\mathcal{F}(pq;k) = \mathcal{G}(p-\tfrac{1}{2}k)\,\mathcal{G}(p+\tfrac{1}{2}k)\,\delta(p-q) - \mathcal{G}(p-\tfrac{1}{2}k)\int d^4s\, I(p,s;k)\, \mathcal{F}(qs;k)\, \mathcal{G}(p+\tfrac{1}{2}k)$$

F(pqk) being defined similarly to I(psk)

Statement of 'Goldstone's Theorem' involving the Forces:

The collective modes of the system are described by the two particle amplitude F(p,q; k), the excitation frequencies being given by that value of $k_o = W(p,q; \underline{k})$ at which F is singular. Near the point of singularity we may therefore assume the form:

$$\mathcal{F}(p,q;k) = \frac{\xi(p,k)\,\xi'(q,k)}{k_0 - W(\underline{k})} + \text{------}$$

Using translational invariance of the intermediate collective states of momentum $\underline{k}$.

It follows from (7) that when $k_o = W(\underline{k})$, there exists a solution to the homogeneous Bethe-Salpeter Equation:

$$\xi(p,k) = \mathcal{G}(p-\tfrac{1}{2}k)\int d^4q\, I(p,q;k)\,\xi(q,k)\,\mathcal{G}(p-\tfrac{1}{2}k)$$

The assumption of spontaneous symmetry breakdown is equivalent to the statement that there exists a solution, viz. $[\sigma_3, G(P)]$ to the homogeneous Bethe-Salpeter equation, for k = 0. The question of the Goldstone Theorem is thus equivalent to asking for the conditions under which this fact implies the statement $W(\underline{0}) = 0$.

In the Hartree-Fock approximation, if, for a given potential, solutions of the energy gap equation exist, corresponding to the existence of the homogeneous Bethe-Salpeter equation for zero momentum-energy, then what further conditions (over and above that which allows a solution of the gap equation) should be placed on the potential so that the collective mode energy vanishes for zero wave number?

While it is true that if the two-particle amplitude has a pole at $\omega_{\underline{k}}$, where $\omega_o = 0$, then there exists a solution of the homogeneous Bethe-Salpeter equation for $k = 0$, the converse is not necessarily true; the conditions necessary for the converse to be true would supply the important corollary to the theorem.

From the known results, we suspect that a long-range contribution in the potential is sufficient to invalidate the converse. This, we believe, should be provable by a detailed investigation of both the Bethe-Salpeter equations in the Hartree-Fock approximation. Recently, it has been shown by Lange, for the example of the ferromagnet,(of which the super conductor may be regarded as the continuous limit) that short-range forces, at least, lead to Goldstone-type spin waves.

CHAPTER V

'ELECTRODYNAMICS' AND CHIRAL SYMMETRY BREAKDOWN

Introduction

The possibility that field theoretic models supposedly describing elementary particle processes, may possess solutions of the type associated with spontaneously broken internal symmetries, has led many authors to conjecture that such solutions might account for the approximate symmetries actually observed. Thus, for example, while the interactions of the hadrons may be viewed, in the first place, as possessing the full SU(3) symmetry, the dynamics is such as to render physical particles of equal mass[30, 11]. Such a view, however, must be subject to the Goldstone Theorem, and so, unless one can account for the consequent massless particles, must be unacceptable. Even if one were prepared to incorporate gauge field effects through the extension of the internal symmetry, in which case, from Chapter III, we might expect at least some of the Goldstone particles to be 'accommodated', the difficulties of making sense out of the resulting 'multiplet' structure would remain.

Another context in which the possibility has been contemplated is in that of the high energy interactions of the leptons, or the electrodynamics of electrons and muons[40, 41]. It is argued that the electron and muon are almost identical in all respects (meaning essentially their scattering properties), except their masses, such that the problem might plausibly be considered as an ideal candidate for description in terms of spontaneously broken symmetry, the symmetries involved being the chiral gauge and the 'isospin' SU(2) groups. The large electron-muon mass difference is to be viewed as a dynamical consequence of the electromagnetic interaction of initially massless fermions ('neutrinos'). Authors advocating this viewpoint and the related view that ordinary electrodynamics may be described as the interaction of neutrinos with the electromagnetic field (implying chiral symmetry breakdown), have allowed themselves the relative luxury of ignoring the accompanying Goldstone bosons, in the hope that an argument which will eradicate them, may emerge.

We might do well to point out, at this stage, the error in the conclusion that since a long-range force (viz. the electromagnetic field) is in evidence the massless particles 'disappear'. For, while the essential broken symmetry in the theory related to the chiral gauge group (and emphatically not to the ordinary gauge group, as was the case in the argument of Chapter III), the electromagnetic field is coupled to that current which is conserved by virtue of ordinary gauge invariance of the second kind.

A recent report has claimed that[43], in the event that the chiral symmetry is broken, there is an accompanying invalidation of the local conservation law of the chiral current to the extent that, in an approximation scheme established for the ordinary electrodynamics case[42], the matrix elements of the divergence of the relevant currents are non-vanishing. The result is absolutely dependent on the use of the Schwinger[44] limiting procedure applied to bilinear products of fermion operators at coincident points. Were this result correct in an exact sense, then the Goldstone Theorem, dependent as it is on the microconservation of the current associated with the broken symmetry, could not be applied.

In this chapter, we argue that, apart from the obvious difficulty in understanding what one means by manifest symmetry without an accompanying local conservation law, we are forced by general considerations of gauge invariance and consistency to adopt a more general definition of the current as the limit of a product of fermion operators at spatially separated points, but incorporating a dependence on the electromagnetic field. Through such a definition, the current operator should be conserved identically, in which case there could be no escape from the consequences of Goldstone's Theorem.

Models involving Chiral Symmetry Breakdown

As we have already mentioned in Chapter I, Section 1, the massive fermion may be viewed in terms of the chirality invariant interactions of 'neutrinos'. In the Nambu-Heisenberg model[2], for example, a quartic interaction of the form $\left[\bar{\psi}(x)\Gamma_{\mu}\,\psi(x)\right]\left[\bar{\psi}(x)\Gamma^{\mu}\,\psi(x)\right]$ where $\Gamma^{\mu} = \gamma^{\mu}\gamma_5$ or γ^{μ} is chosen, and spontaneous breakdown assumed. However, another more physically relevant mechanism which could possibly support such solutions has been conjectured recently[41,42,37], namely the electromagnetic field.

The problem, so far as we will be concerned, is the interaction of a one-component fermion field of zero bare mass with the electromagnetic field, minimal coupling being the essential requirement.

The field equations take the form

$$\gamma_\mu (\partial^\mu - ie\, A^\mu(x))\ \Psi(x) = 0$$

$$\partial_\nu F^{\mu\nu}(x) = \partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = e\, J^\mu_{e-m}(x) \qquad (1)$$

where $J^\mu_{e-m}(x)$ is the e-m current (coupled to $A^\mu(x)$), formally $i\,\bar{\Psi}(x)\,\gamma_\mu\Psi(x)$, with $\Psi(x)$ satisfying the usual anti-commutation relations

$$\{\Psi_\alpha(x),\ \Psi^\dagger_\beta(x')\}_{x_0=x_0'} = \delta_{\alpha\beta}\,\delta^{(3)}(\underline{x}-\underline{x}') \ ; \ \{\Psi_\alpha(x), \Psi_\beta(x')\}_{x_0=x_0'} = 0$$

$$\{\Psi(x),\ \bar{\Psi}(x')\}_{x_0=x_0'} = -i\gamma^0\,\delta^{(3)}(\underline{x}-\underline{x}') = i\gamma_0\,\delta^{(3)}(\underline{x}-\underline{x}') \qquad (2)$$

These field equations (1), in addition to the usual space-time invariances (such as proper Lorentz Transformations), possess invariance under the following gauge transformations

(i) gauge transformation of the second kind

$$\Psi(x) \to \Psi(x)\, e^{ie\alpha(x)} \ ; \qquad A_\mu(x) \to A_\mu(x) + e\,\partial_\mu\alpha(x)$$

which includes the simple gauge group $U(1)$ (α constant). The associated microconserved current is $J^\mu_{e-m}(x)$, the conservation law being incorporated in the field equation.

(ii) The chiral gauge group

$$\Psi(x) \to e^{i\beta\gamma_5}\,\Psi(x)$$

with an expected conserved current, formally $i\,\bar{\Psi}(x)\,\gamma_\mu\gamma_5\Psi(x)$

In the last case, we use the term 'expected' to take account of any dubiety there may be about the existence of a conserved current, and the term 'formally' because of the inherent lack of definition involved in expressing a product of field operators at coincident points. Certainly, from formal consideration of the Action Principle and Noether's Theorem, we would anticipate such a locally conserved current, although the precise definition we have still to give.

For later reference, we might note that the electron-muon problem may also be described by (1) and (2), provided $\psi(x)$ is interpreted as a two-component operator $(\psi_e(x), \psi_\mu(x))$. In that case, in addition to the symmetries (i) and (ii), we have invariance under

(iii) internal SU(2) symmetries

$$\psi(x) \rightarrow e^{i\underline{\alpha}\cdot\underline{\tau}}\,\psi(x)$$

with an expected conserved current, formally $i\bar{\psi}(x)\gamma_\mu\underline{\tau}\psi(x)$;

(iv) chiral-isospin symmetry

$$\psi(x) \rightarrow e^{i\underline{\beta}\cdot\underline{\tau}\gamma_5}\,\psi(x)$$

with an expected conserved current, formally $i\bar{\psi}(x)\gamma_\mu\gamma_5\underline{\tau}\psi(x)$.

Whether $\psi(x)$ is one or two component, we assume that there is a breakdown of chiral symmetry, indicating that the fermion field $\psi(x)$ acquires mass due to the interaction. If $\psi(x)$ is two component, then a further breakdown of SU(2) symmetry must be assumed if the masses associated with the individual components (electron-muon) are to be different.

Unless otherwise stated, we take $\psi(x)$ to be a one-component field, so that the broken symmetry assumed is chirality alone (case ii) and the statement of this can best be made as a condition on the Fourier transform, $S(\rho)$, of the fermion propagator $\langle T(\psi(x)\,\bar{\psi}(y)\rangle_0$, or the unordered function.

By relativistic invariance, we have the general form

$$S^{-1}(p) = 1\, S_1(p^2) + \gamma\cdot p\, S_2(p^2) \tag{3A}$$

and here we demand $S_1(p^2) \neq 0$ emphatically, indicating that the vacuum is degenerate with respect to the chiral group parameter β. More compactly

$$\left\{\gamma_5,\ S^{-1}(p)\right\} \neq 0 \quad . \tag{3B}$$

Similarly for the unordered functions.

<u>Goldstone's Theorem</u>

In the preceding chapter, we have discussed the relation of spontaneous breakdown of symmetry to the existence of Goldstone particles in contexts not immediately related to the present case. However, the arguments applied there may be easily adopted for this case, the statement of broken symmetry now involving the vacuum expectation value of bilinear products of fermion operators (as in superconductivity theory).

If $J_{\mu 5}(x)$ is the chiral current, then, on using the equal-time commutator

$$\left[\psi^{\dagger}_{\alpha}(x)\psi_{\beta}(x),\ \psi_{\rho}(y)\psi^{\dagger}_{\sigma}(z)\right]_{x_0=y_0=z_0} = \psi_{\rho}(y)\psi^{\dagger}_{\alpha}(x)\,\delta_{\beta\sigma}\,\delta^{(3)}(\underline{x}-\underline{z}) - \psi_{\beta}(x)\psi^{\dagger}_{\sigma}(z)\,\delta_{\rho\alpha}\,\delta^{(3)}(\underline{x}-\underline{y})$$

$$\left[\psi^{\dagger}_{\alpha}(x)\psi_{\beta}(x),\ \psi_{\rho}(y)\psi^{\dagger}_{\sigma}(z)\right]_{-\,x_0=y_0=z_0} = -\psi^{\dagger}_{\alpha}(x)\psi_{\rho}(y)\,\delta_{\beta\sigma}\,\delta^{(3)}(\underline{x}-\underline{z}) + \psi^{\dagger}_{\sigma}(z)\psi_{\beta}(x)\,\delta_{\alpha\rho}\,\delta^{(3)}(\underline{x}-\underline{y})$$

(4)

we find, for equal times (suppressing the Dirac indices)

$$i \int d^3\underline{x} \, \langle [J_{05}(x), \, \psi(y)\bar{\psi}(z)] \rangle_0 \quad = \quad \{ \gamma_5, \langle \psi(y)\bar{\psi}(z) \rangle_0 \}$$

which is non-zero by the condition (3B).

It thus follows, applying the argument of GSW (See Chapter I, Section 3) to the spectral function $\langle [J_{\mu 5}(x), \, \psi(y) \, \bar{\psi}(z)] \rangle_0$, that, provided that $J_{\mu 5}(x)$ is conserved, and a Lorentz covariant description possible, then $\int d^3\underline{x} \, \langle [J_{05}(x), \, \psi(y) \, \bar{\psi}(z)] \rangle_0$ is x_0-independent and so Goldstone particles must be present.

Another argument which makes obvious the absolute dependence of the theorem on local current conservation involves the time-ordered counterpart of $\langle [J_{\mu 5}(x), \, \psi(y) \, \bar{\psi}(z)] \rangle_0$, for which a Ward Identity may be written down[45].

We have

$$\partial^\mu \langle T. \{ J_{\mu 5}(x) \, \psi_\alpha(y) \, \bar{\psi}_\beta(z) \} \rangle_0$$

$$= \langle T. \{ \partial^\mu J_{\mu 5}(x) \, \psi_\alpha(y) \bar{\psi}_\beta(z) \} \rangle_0 \; + \langle T. \{ [J_{05}(x), \psi_\alpha(y)] \bar{\psi}_\beta(z) \} \rangle_0 \, \delta(x_0 - y_0)$$

$$+ \langle T. \{ \psi_\alpha(y) \, [J_{05}(x), \bar{\psi}_\beta(z)] \} \rangle_0 \, \delta(x_0 - z_0)$$

and on using (2), we find the identity

$$\partial^\mu \langle T. \{ J_{\mu 5}(x) \, \psi_\alpha(y) \, \bar{\psi}_\gamma(z) \} \rangle_0$$

$$= \langle T. \{ \partial^\mu J_{\mu 5}(x) \, \psi_\alpha(y) \bar{\psi}_\gamma(z) \} \rangle_0 + i \langle T. (\psi_\alpha(y) \bar{\psi}_\sigma(x)) \rangle_0 \, \gamma^0_{\sigma\rho} \, \gamma^5_{\rho\beta} \, \gamma^0_{\beta\gamma} \, \delta^{(4)}(x-z)$$

$$- i \gamma^5_{\alpha\sigma} \langle T. (\psi_\sigma(x) \, \bar{\psi}_\gamma(z)) \rangle_0 \quad \delta^{(4)}(x-y)$$

(5A)

Defining the Fourier Transform

$$\langle T.\left(J_{\mu 5}(x)\,\psi_\alpha(y)\,\bar{\psi}_\beta(z)\right)\rangle_0 = \int \frac{d^4p\,d^4q}{(2\pi)^8}\,\Lambda_{\mu 5}(p+q,p)\,e^{ip(y-z)+iq(y-x)}$$

we obtain, in momentum space

$$q^\mu \Lambda_{\mu 5}(p+q,p) = -S(p)\gamma_5 - \gamma_5 S(p+q) + \int d^4y\,d^4z\,e^{-i(p+q)x+ipz}\langle T.\left(\partial^\mu J_{\mu 5}(0)\,\psi(y)\,\bar{\psi}(z)\right)\rangle_0 \qquad (5B)$$

Thus, as $q_\mu \to 0$, provided $\partial^\mu J_{\mu 5}(x) = 0$, equation (5B) tells us on using (3) that $\Lambda_{\mu 5}(p+q,p)$ develops a singularity at $q = 0$, which we take to indicate that Goldstone particles are present.

(We note, in passing, that a similar argument may be applied in the Nambu formalism in superconductivity theory, although the existence there of such a singularity does not imply a 'massless' state, on account of the presence of the long-range force. While, here, it is true that the e-m field is involved, we are not dealing with the e-m current.

In relativistic theories in general, we may conclude that the existence of a zero momentum-energy singularity in any spectral function involving the current associated with the symmetry which is broken, indicates the presence of zero mass particles, unless that current derives from local symmetry, in which case the accompanying gauge fields invalidate one of the assumptions of Goldstone's Theorem, namely that the theory may be formulated directly in a manifestly covariant way. The assumption of spontaneous breakdown of chiral symmetry would not appear to forbid a covariant description, so that our interpretation would be valid.)

From equation (4), it is seen that, in order not to have zero mass particles in a covariant theory, not only must local current conservation be relaxed, but also, the following identity must be satisfied

$$\{\gamma_5, S(p)\} = \int d^4y\, d^4z\, e^{-ip(y-z)} \langle T.\left(\partial^\mu J_{\mu 5}(0)\, \psi(y)\, \bar{\psi}(z)\right)\rangle_0$$

or in coordinate space form, from (5A),

$$\langle T.\left(\partial^\mu J_{\mu 5}(x)\, \psi_\alpha(y)\, \bar{\psi}_\gamma(z)\right)\rangle_0 = -S(y-x)\,\gamma_5\,\delta^{(4)}(x-z) - \gamma_5\, S(x-z)\,\delta^{(4)}(x-y)$$

which would appear to be a fairly strong restriction, even if we could accept that $\partial^\mu J_{\mu 5}(x)$ was non-vanishing.

Current Definitions

Generally, in any problem involving the interaction of Fermi fields in the form of a coupling of a bilinear Fermi current $J_\mu(x)$ (which need not be conserved) to a boson field, care must be taken in the way these currents are defined[46]. In particular, according to Schwinger,[44] the following current-current commutation relation must be satisfied

$$\left[J_0(x), J_i(0)\right]_{x_0=0} = -\kappa\, \partial_i\, \delta^{(3)}(\underline{x}) \tag{6}$$

where κ is some non-vanishing constant operator.

We may see this in the present case by considering the photon spectral function

$$\langle [A_\mu(x), A_\nu(0)] \rangle_0 = \frac{1}{(2\pi)^3}\int dm^2\, B(m^2) \int d^4k\, \mathcal{G}_{\mu\nu}(k)\, \epsilon(k_0)\, \delta(k^2+m^2)\, e^{ikx} \tag{7}$$

where $\mathcal{G}_{\mu\nu}(k)$ specifies the gauge, as in Chapter III and $\int_0^\infty dm^2 B(m^2) = 1$ by canonical quantisation. Applying the equations of motion in (1) to (7), and noting the property of $\mathcal{G}_{\mu\nu}(k)$

$$\mathcal{G}^L\ \mathcal{G}\ \mathcal{G}^L = \mathcal{G}$$

as stated in Chapter III, we arrive at the spectral function

$$\langle [J^\mu_{e\text{-}m}(x), J^\nu_{e\text{-}m}(0)] \rangle_0 = \int_0^\infty dm^2\, B(m^2) \left[g_{\mu\nu} - \frac{1}{m^2}\partial^\mu \partial^\nu \right] (m^2)^2\, \Delta(x/m^2) \tag{8}$$

from which we may deduce that

$$\langle [J^0_{e\text{-}m}(x), J^i_{e\text{-}m}(0)]_{x_0=0} \rangle_0 = -\int_0^\infty dm^2\, B(m^2)\, \partial^0 \partial^i\, \Delta(x/m^2)\Big|_{x_0=0} m^2$$

$$\langle [J^0_{e\text{-}m}(x), J^i_{e\text{-}m}(0)]_{x_0=0} \rangle_0 = -i\int_0^\infty dm^2\, B(m^2)\, m^2\ \partial^i \delta^{(3)}(\underline{x})$$

We thus see that (6) is reproduced provided we identify

$$\langle K \rangle_0 = i\int_0^\infty dm^2\, B(m^2)\, m^2$$

and vanishes only if $B(m^2)$ vanishes, or is exactly proportional to $\delta(m^2)$, which would in turn imply the vanishing of the current operator (i.e. effectively there could be no coupling to the e-m field).

Thus, in the present case, it is obvious that the formal definition of the e-m current $i\bar{\psi}(x)\,\gamma_\mu \psi(x)$ would, on using (2), give $k = 0$, which would be inconsistent with a non-vanishing interaction. This has been discussed generally by Boulware and Deser[46], who further argue that the current need not be conserved in order to draw such a conclusion. Also Okubo[47] has shown that we must require (6) for any current.

These results lead us to define such currents generally as the symmetric limit of the bilinear Fermi product at space separated points, such that the commutation rule (6) is realised in terms of matrix elements. For, taking the current as

$$J_\mu(x) = \underset{\underline{\varepsilon} \to 0}{\mathrm{Lt}}\; i\,\bar{\psi}(\underline{x} + \underline{\varepsilon})\, \Gamma_\mu\, \psi(x) \tag{9A}$$

(where Γ_μ may be γ_μ or $\gamma_\mu \gamma_5$, for example, i.e. J_μ may be e-m or chiral) we find for vacuum expectation values, from (4)

$$\langle [J_0^{e\text{-}m}(x), \bar{\psi}(z)\Gamma^i \psi(y)] \rangle_0 = \mathrm{Trace}\left(\langle \psi(y)\bar{\psi}(x)\rangle_0 \Gamma^i\right) \delta^{(3)}(\underline{x}-\underline{z}) - \mathrm{Trace}\left(\langle \psi(x)\bar{\psi}(z)\rangle_0 \Gamma^i\right) \delta^{(3)}(\underline{x}-\underline{y})$$

and with $\underline{z} = \underline{y} + \underline{\varepsilon}$,

$$\langle [J_0^{e\text{-}m}(x), \bar{\psi}(y+\underline{\epsilon})\Gamma^i \psi(y)] \rangle_0 = \mathrm{Trace}\left(\langle \psi(y)\bar{\psi}(y+\underline{\epsilon})\rangle_0 \Gamma^i\right) \left\{\delta^{(3)}(\underline{x}-\underline{y}-\underline{\epsilon}) - \delta^{(3)}(\underline{x}-\underline{y})\right\}$$

$$\langle [J_0^{e\text{-}m}(x), \bar{\psi}(y+\underline{\epsilon})\Gamma^i \psi(y)] \rangle_0 = -\mathrm{Trace}\left(\langle \psi(y)\bar{\psi}(y+\epsilon)\rangle_0 \Gamma^i\right)\; \underline{\epsilon}\cdot\nabla\delta^{(3)}(\underline{x}-y) \tag{9B}$$

and if the limit $\underline{\varepsilon} \to 0$ is taken symmetrically (i.e. same in all directions), this is equal to the form (6), with, on average

$$\langle \kappa \rangle_0 = -\tfrac{1}{3}\,\mathrm{Trace}\left(\langle \psi(y)\bar{\psi}(y+\underline{\epsilon})\,\Gamma^k\right) \epsilon_k \qquad (\to \infty, \text{ as } \underline{\varepsilon} \to 0)$$

as required. This necessity for symmetric averaging was first suggested by Schwinger, for the electric current in particular. We see however that it may be invoked to justify the use of (6) generally. Using the general spectral form

$$\langle \psi(x) \bar{\psi}(x+\epsilon) \rangle_0 = S^{(+)}(-\epsilon) = \int_0^\infty dm^2 \left[\rho_1(m^2) + \rho_2(m^2) \gamma \cdot \partial \right] \Delta^{(+)}(-\epsilon, m^2)$$

and the properties of the Dirac matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \qquad \{\gamma_\mu, \gamma_5\} = 0 \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

we may obtain $\langle K \rangle_0$ explicitly in terms of the weight functions ρ_1, ρ_2.

A useful formal device by means of which this may be achieved is the 'smearing function' $f(x)$, in terms of which the current may be expressed

$$J_\mu(x) = \lim_{\epsilon \to 0} i \bar{\psi}(x,\epsilon) \gamma_\mu \psi(x,\epsilon) \tag{9C}$$

with

$$\psi(x,\epsilon) = \int d^3\underline{x}' f(\underline{x}-\underline{x}', \underline{\epsilon}) \psi(\underline{x}', x_0)$$

Since the anticommutation relations (2) must be satisfied in the limit that $\underline{\epsilon} \to 0$, we observe that a convenient candidate for $f(x, \underline{\epsilon})$ would be any symmetric function which approaches the Dirac δ-function in that limit. The Gaussian form $(\pi^{1/2}\epsilon)^{-3} e^{-x^2/\epsilon^2}$ would be an obvious choice[43].

While such a device is sufficient to ensure the presence of the 'Schwinger term', the question of overall consistency must still be considered. It has been shown, on general grounds that a dependence on the field to which the given current couples must be built in to the definition of the current. We may see the necessity for this in the present case by deriving from (7) the commutation functions $\langle [J_\mu^{e-m}(x), A_\nu(0)] \rangle_0$ and $\langle [J_\mu^{e-m}(x), F_{\nu\lambda}(0)] \rangle_0$ We find, using the non-covariant radiation gauge

$$\langle [J^\mu_{e-m}(x), A^\nu(0)] \rangle_0 = \frac{1}{(2\pi)^3} \int_0^\infty dm^2 B(m^2) m^2 \int d^4k \, \mathcal{G}_L^{\mu\lambda}(k) \left[\delta^\nu_\lambda - \frac{k^\nu n_\lambda (n\cdot k)}{k^2 + (n\cdot k)^2} \right] \epsilon(k_0) \delta(k^2+m^2) e^{ikx} \tag{10}$$

$$\langle [J^\mu_{e-m}(x), F^{\sigma\nu}(0)] \rangle_0 = \frac{1}{(2\pi)^3} \int_0^\infty dm^2 B(m^2) m^2 \int d^4k \, (g^{\mu\sigma} k^\nu - g^{\mu\nu} k^\sigma) \epsilon(k_0) \delta(k^2+m^2) e^{ikx} \tag{11}$$

the latter being gauge independent (i.e. identical to that obtained using a covariant gauge.

While from (10) we see that, in radiation gauge

$$\langle [J^0_{e-m}(x), A^i(0)]_{x_0=0} \rangle_0 = 0 \; ; \qquad \langle [J^i_{e-m}(x), A^j(0)]_{x_0=0} \rangle_0 = 0 \tag{11A}$$

(but not so in a covariant gauge)

on the other hand, from (11), we deduce that in a general gauge

$$\langle [J^0_{e-m}(x), F^{0i}(0)]_{x_0=0} \rangle_0 = 0$$

$$\langle [J^i_{e-m}(x), F^{0j}(0)]_{x_0=0} \rangle_0 = \delta_{ij} \int_0^\infty dm^2 B(m^2) m^2 \, \delta^{(3)}(\underline{x}) \tag{11B}$$

We conclude that if, as we require, $\langle [J^0_{e-m}(x), J^i_{e-m}(0)]_{x_0=0} \rangle_0$ is to be non-zero, then an explicit e-m field dependence must be contained in the definition of $J^\mu_{e-m}(x)$. In particular, the dependence is essential in the space components, although the time component does not require it (i.e. it may possess it without any noticeable effect).

An appropriate condition, as implied by the field equation $\partial_\nu F^{\mu\nu} = J^\mu_{e-m}$, which should be imposed in determining the form of the dependence, is that the expression prior to taking the limit should be gauge invariant. The form given by Johnson[46] for the case in which $A_\mu(x)$ is an external field, satisfies our requirements

$$J_\mu^{e-m}(x) = \underset{\underline{\epsilon}\to 0}{Lt}\, J_\mu^{e-m}(x,\underline{\epsilon}) = i \underset{\underline{\epsilon}\to 0}{Lt}\, \bar{\Psi}(x+\underline{\epsilon})\gamma_\mu e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)} \Psi(x) \tag{12}$$

in that the result (11B) may be obtained, $\int_0^\infty dm^2 B(m^2)m^2$ being the divergent. This form (12) should be automatically conserved, as is demanded by the field equations (1).

Forming $\partial_\mu J^\mu_{e-m}(x)$, we have

$$\begin{aligned}\partial_\mu J^\mu_{e-m}(x,\underline{\epsilon}) = {} & i\,\partial_\mu \bar{\Psi}(x+\underline{\epsilon})\gamma^\mu e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\Psi(x)\\ & + i\,\bar{\Psi}(x+\underline{\epsilon})\gamma^\mu e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\partial_\mu\Psi(x)\\ & + i\,\bar{\Psi}(x+\epsilon)\gamma^\mu (ie)\int_0^{\epsilon} d\underline{\xi}\cdot\partial_\mu\underline{A}(\underline{x}+\underline{\xi},x_0)\, e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\Psi(x)\end{aligned}$$

and using (1)

$$\begin{aligned}\partial_\mu J^\mu_{e-m}(x,\underline{\epsilon}) = {} & i\,\bar{\Psi}(x+\underline{\epsilon})\gamma_\mu (ie)\left[-A^\mu(x+\epsilon)+A^\mu(x)\right] e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\Psi(x)\\ & + i\,\bar{\Psi}(x+\epsilon)\gamma_\mu (ie)\int_0^{\underline{\epsilon}} d\underline{\xi}\cdot\partial^\mu\underline{A}(x+\underline{\xi},x_0)\, e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\Psi(x)\end{aligned}$$

$$\partial_\mu J^\mu_{e-m}(x,\epsilon) = i(ie)\,\bar{\Psi}(x+\epsilon)\gamma^\mu\int_{\underline{x}}^{x+\underline{\epsilon}} d\xi_i\, F_{\mu i}(\underline{\xi},x_0)\, e^{ie\int_{\underline{x}}^{\underline{x}+\epsilon} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}\Psi(x)$$

which is manifestly gauge invariant, as required.

More compactly

$$\partial_\mu J^\mu_{e-m}(x,\epsilon) = (ie)\, J^\mu_{e-m}(x,\epsilon)\int_{\underline{x}}^{x+\epsilon} d\xi_i\, F_{\mu i}(\underline{\xi},x_0) \tag{13}$$

We note that in carrying out the above differentiation no attention need be paid to order, since for equal times we assume the fields involved commute.

We now turn our attention to the chiral current, $J^{\mu 5}(x)$. In order to establish the necessity of a field dependence, in similar fashion to the above, we clearly cannot use the same argument directly, since there is no coupling of this current to the field $A_\mu(x)$ (and so no field equation directly connecting with A_μ).

However, we would anticipate such a field dependence on the following grounds of consistency. Through (4), we may deduce from (9B), analogously to the e-m current case

$$\left[J_0^{e\text{-}m}(x),\ \bar{\psi}(z)\gamma_k\gamma_5\psi(y)\right]_{x_0=y_0=z_0} = -\mathrm{Trace}\left(\psi(y)\bar{\psi}(z)\gamma_k\gamma_5\right)\underline{\epsilon}\cdot\nabla\delta^{(3)}(\underline{x}-\underline{y})$$

$$\left[J_0^{e\text{-}m}(x),\ \bar{\psi}(z)\gamma_k\gamma_5\psi(y)\right]_{x_0=y_0=z_0} = -\tfrac{1}{3}\mathrm{Trace}\left(\psi(y)\bar{\psi}(y+\underline{\epsilon})\gamma^i\gamma_5\right)\epsilon_i\,\partial_k\delta^{(3)}(\underline{x}-\underline{y})$$

For vacuum expectation values the right-hand side vanishes when the trace is taken (using the spectral form for $\langle\psi\bar{\psi}\rangle_0$, and the properties of the γ-matrices), but will not do so for general matrix elements. Using (1), we must therefore conclude

$$\left[\partial^i F_{0i}(x),\ J_{k5}(x')\right]_{x_0=x_0'} \neq 0$$

or

$$\left[F_{0i}(x),\ J_{k5}(x')\right]_{x_0=x_0'} \neq 0 .$$

If we assume a 'smeared' current in the first instance, then it must be such as to include a dependence on the e-m field.

Thus we conclude that what we do with one current (by way of definition) we do to the other also, i.e. to demand gauge invariance (and chiral invariance) prior to taking the limit to coincident points, in which case we would define $J_{\mu 5}(x)$ as

$$J_{\mu 5}(x) = \lim_{\underline{\epsilon}\to 0} i\bar{\psi}(x+\epsilon)\gamma_\mu\gamma_5 e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\xi_i A_i(\underline{\xi},x_0)}\psi(x) \tag{14}$$

In a general notation, where Γ_μ may be γ_μ or $\gamma_\mu\gamma_5$, we may combine both currents (13) and (14) in the single expression

$$J_\mu(x,\underline{\epsilon}) = i\bar{\psi}(x,\underline{\epsilon})\Gamma_\mu e^{ie\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\xi_i A_i(\underline{\xi},x_0)}\psi(x) \tag{15}$$

(with limit $\underline{\epsilon} \to 0$ understood)

satisfying

$$\partial_\mu J^\mu(x,\underline{\epsilon}) = (ie)\, J^\mu(x,\underline{\epsilon})\int_{\underline{x}}^{\underline{x}+\underline{\epsilon}} d\xi_i F_{\mu i}(\underline{\xi},x_0) \tag{16}$$

We now compare (9A) and (15), and note the general procedure for dealing with bilinear products of fermion operators at coincident points, when there is a coupling to the e-m field; namely, perform the gauge transformation

$$\psi(x) \rightarrow \psi(x)\, e^{ie\int^{\underline{x}} d\underline{\xi}\cdot\underline{A}(\underline{\xi},x_0)}$$

This will always ensure a manifestly gauge invariant current, bilinear in the fields, and eliminate inconsistencies associated with equal time commutation rules, provided at least we work in radiation gauge.

For the purpose of calculational ease, it is useful to recast (15) and (16) in terms of a smearing function f in a similar way to that mentioned in the field independent definition (9A).

We define (limit $\epsilon \to 0$ understood)

$$J_\mu(x,\epsilon) = i\ \bar{\psi}(x,\epsilon)\ \Gamma_\mu\ \psi(x,\epsilon)$$

where

$$\psi(x,\epsilon) = \int d^3\underline{x}'\, f(\underline{x}-\underline{x}')\, e^{-ie\int_{\underline{x}}^{\underline{x}'} d\xi_i\, A_i(\xi,x_0)}\, \psi(x')$$

so that

$$J_\mu(x,\epsilon) = i\int d^3\underline{x}''\, d^3\underline{x}'\, f(\underline{x}-\underline{x}'')\, f(\underline{x}-\underline{x}')\, \bar{\psi}(x'')\, \Gamma_\mu\, \psi(x')\, e^{ie\int_{\underline{x}'}^{\underline{x}''} d\xi_i\, A_i(\xi,x_0)} \quad (17)$$

and

$$\partial_\mu J^\mu(x,\epsilon) = i\,(ie)\int d^3\underline{x}''\, d^3\underline{x}'\, f(\underline{x}-\underline{x}'')\, f(\underline{x}-\underline{x}')\int_{\underline{x}'}^{\underline{x}''} d\xi_i\, F_{\mu i}(\xi,x_0)\, \bar{\psi}(x'')\, \Gamma^\mu \psi(x')\, e^{ie\int_{\underline{x}'}^{\underline{x}''} d\xi_i\, A_i(\xi,x_0)} \quad (18)$$

$$(x_0'' = x_0' = x_0)$$

With the above general definition, since the e-m current is to be conserved in the limit $\epsilon \to 0$, then we must also expect the chiral current to be likewise conserved. We would anticipate that the result of any calculation, in some consistent approximation, of the matrix elements of the current J_μ given by (17) would be independent of F_μ. However, this has to be explicitly verified, in view of the fact, established by Maris and Jacob[43], in a consistent approximation scheme (which we shall in fact utilise) that while the 'non field-dependent' electromagnetic current vanishes as required, the 'non field-dependent' chiral current does not. It is not obvious, at this stage, that the same may not be true for the field dependent definition.

The Single Particle Approximation

We now consider a lowest order approximation (in electromagnetic coupling) of the matrix elements $\langle p | \partial^{\mu} J_{\mu}(x) | q \rangle$, where $| p \rangle$, $| q \rangle$ are single particle fermion states (satisfying $p^2 = -m^2$, $q^2 = -m^2$), and $J_{\mu}(x)$ is given by (17) (in limit ε 0).

The assumption of chiral symmetry breakdown gives rise to the propagator, whose inverse is of the form (3A), which is satisfied by that for a free particle of mass m .

$$S^{-1}(p) = i\gamma.p + m \tag{19}$$

As has been suggested by Baker, Johnson and Willey[42], this may be considered as a quasi-particle solution to (1), corresponding to a certain class of diagrams in perturbation theory. They argue that their procedure is divergence free.

We shall assume that (19) is an acceptable approximation to the true propagator, and now proceed to take account of the residual interaction of these quasi particles with $A_{\mu}(x)$, which we shall take to be explicitly perturbative; the approximation we shall consider will be that which is explicitly of lowest order in the coupling (corresponding implicitly to a summation over infinitely many fermion self-energy diagrams.)

We have, using (18)

$$\langle p | \partial^{\mu} J_{\mu}(x) | q \rangle = e^{-i(p-q)x} \langle p | \partial^{\mu} J_{\mu}(0) | q \rangle$$

$$\langle p | \partial^\mu J_\mu(x) | q \rangle$$
$$= e \int d^3\underline{x}'' d^3\underline{x}' f(\underline{x}'') f(\underline{x}') e^{-i(p-q)x} \langle p | \int_{\underline{x}'}^{\underline{x}''} d\xi_i F_{\mu i}(\underline{\xi}, x_0) \bar{\psi}(x'') \Gamma^\mu \psi(x') e^{ie\int_{\underline{x}'}^{\underline{x}''} d\xi_i A_i(\underline{\xi}, x_0)} \Big| q \rangle \Big|_{x_0''=x_0'=x_0} \tag{20}$$

Introducing the S-matrix in the form

$$S = T\left(e^{i\int d^4y \, \mathcal{H}_I(y)} \right)$$

where the interaction Hamiltonian density is

$$i\mathcal{H}_I(y) = (ie) \, J_\mu^{e\text{-}m}(y) \, A^\mu(y)$$

and, making the usual transition to the interaction picture, we find from (20), to lowest order (in e):

$$\langle p | \partial^\nu J_\nu(x) | q \rangle \big|_{x=0}$$
$$= (ie)^2 i \int d^3\underline{x}'' d^3\underline{x}' f(\underline{x}'') f(\underline{x}') \int d^4y \langle p | \int_{\underline{x}'}^{\underline{x}''} d\xi_i \, T\left\{ F_{\mu i}(\underline{\xi}, x_0) \bar{\psi}(x'') \Gamma^\mu \psi(x') \bar{\psi}(y) \gamma^\nu \psi(y) A_\nu(y) \right\} | q \rangle \Big|_{x_0''=x_0'=x_0=0} \tag{21}$$

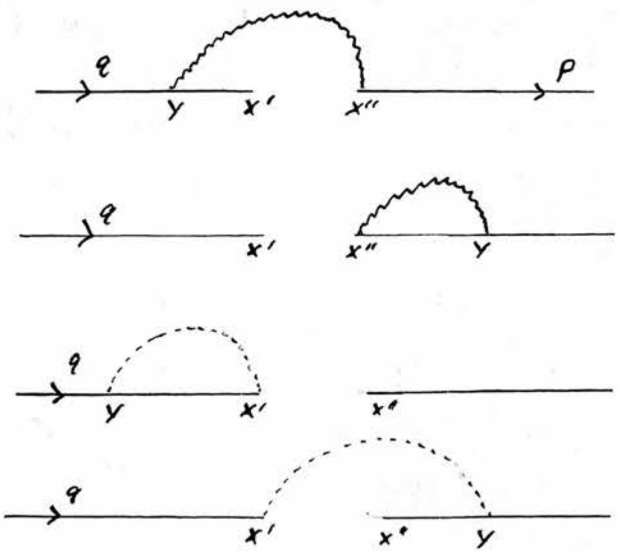
The introduction of the time-ordering symbol, acting overall in (21), does not lead to any ambiguity, since only space-like separations are involved in the original expression (20).

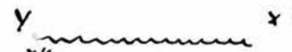
Further, it should be noted that we have not 'smeared' the electromagnetic current appearing in H_1, assuming it to be unnecessary for the present purpose.

We find, on carrying out the appropriate contractions, that

$$\begin{aligned}
&\int_{x'}^{x''} d\xi_i \, \langle p | T\left\{ F_{\mu i}(\xi, x_0)\, \bar{\psi}(x'')\, \Gamma^\mu \psi(x') \right\} | q \rangle \\
&= \int_{x'}^{x''} d\xi_i \int d^4y \, e^{-ipx'' + iqy}\, \bar{u}(p)\, \Gamma^\mu S(x'-y)\, \gamma^\nu u(q) \, \left\langle T\left(F_{\mu i}(\xi, x_0)\, A_\nu(y)\right)\right\rangle_0 \\
&- \int_{x'}^{x''} d\xi_i \int d^4y \, e^{iqx' - ipy}\, \bar{u}(p)\, \gamma^\nu S(y - x'')\, \Gamma^\mu u(q) \, \left\langle T\left(A_\nu(y)\, F_{\mu i}(\xi, x_0)\right)\right\rangle_0
\end{aligned} \tag{22}$$

An appropriate graphical representation which we might associate with this lowest order approximation would be





$$\int_{x'}^{x''} d\xi_i \left\langle T\left(F_{\mu i}(\xi, x_0)\, A_\nu(y)\right)\right\rangle_0$$

$$- \int_{y}^{x'} d\xi_i \left\langle T\left(F_{\mu i}(\xi, x_0)\, A_\nu(y)\right)\right\rangle_0$$

Going into momentum space through the Fourier Transformation defined by

$$S(x) = \int \frac{d^4k}{(2\pi)^4}\, S(k)\, e^{ikx}$$

$$F_{\mu i \nu}(x) = \left\langle T\left(F_{\mu i}(x)\, A_\nu(0)\right)\right\rangle_0 = \int \frac{d^4k}{(2\pi)^4}\, F_{\mu i \nu}(k)\, e^{ikx}$$

where $S(k)$ is the fermion propagator given by (19), and $F_{\mu i \nu}(k)$ may be derived from the photon propagator associated with (7), viz.

$$D_{\mu\nu}(k) = \int d^4x \langle T\cdot(A_\mu(x)A_\nu(0))\rangle_0 e^{-ikx} = -ig_{\mu\nu}(k)\frac{1}{k^2-i\lambda}$$

In any gauge (and we are required to use radiation gauge) we find

$$\mathcal{F}_{\mu\lambda\nu}(k) = (k_\lambda g_{\mu\nu} - k_\mu g_{\lambda\nu})\frac{1}{k^2-i\lambda}$$

(which is gauge independent).

Thus, in lowest order, (22) may be rewritten

$$\int_{x'}^{x''} d\xi_i \langle p|T\cdot\{F_{\mu i}(\xi,x_0)\bar\psi(x'')\Gamma^\mu\psi(x')\}|q\rangle (2\pi)^4$$

$$= \bar u(p)\int_{x'}^{x''} d\xi_i\, e^{-ipx''}\int d^4k\, \Gamma^\mu S(q-k)\gamma^\nu e^{ik\xi}\mathcal{F}_{\mu i\nu}(k)\, e^{i(q-k)x'}u(q)$$

$$+ \bar u(p)\int_{x'}^{x''} d\xi_i\, e^{iqx'}\int d^4k\, \gamma^\nu S(p-k)\Gamma^\mu e^{-ik\xi}\mathcal{F}_{\mu i\nu}(k)\, e^{-i(p-k)x''}u(q)$$

$$= \bar u(p)\int_0^1 dt\, e^{-ipx''+iqx'}\int d^4k\, x''_i\,\mathcal{F}_{\mu i\nu}(k)\Gamma^\mu S(q-k)\gamma^\nu e^{ik(x''t-x'')}u(q)$$

$$+ \bar u(p)\int_0^1 dt\, e^{-ipx''+iqx'}\int d^4k\, x''_i\,\mathcal{F}_{\mu i\nu}(k)\gamma^\nu S(p-k)\Gamma^\mu e^{-ikx''(t-1)}u(q)$$

$$- \bar u(p)\int_0^1 dt\, e^{-ipx''+iqx'}\int d^4k\, x'_i\,\mathcal{F}_{\mu i\nu}(k)\gamma^\nu S(p-k)\Gamma^\mu e^{-ik(x't-x'')}u(q)$$

$$- \bar u(p)\int_0^1 dt\, e^{-ipx''+iqx'}\int d^4k\, x'_i\,\mathcal{F}_{\mu i\nu}(k)\Gamma^\mu S(q-k)\gamma^\nu e^{ikx'(t-1)}u(q)$$

on choosing the linear paths $\xi_i = x''_i t,\ x'_i t$.

Hence the total contribution to (22) from all terms explicitly of lowest order in the coupling is

$$\int_{x'}^{x''} d\xi_i \langle p|T\cdot\{F_{\mu i}(\xi,x_0)\bar\psi(x'')\Gamma^\mu\psi(x')\}|q\rangle$$

$$\int_{\underline{x}'}^{\underline{x}''} d\xi_i \, \langle p| T.\{ F_{\mu i}(\underline{\xi}, x_0) \bar{\psi}(x'') \Gamma^\mu \psi(x') \} |q\rangle \; (2\pi)^4$$

$$= \bar{u}(p) \int_0^1 dt \int d^4k \, x_i'' \, \mathcal{F}_{\mu i \nu}(k) \, e^{i(q-k)x'} \Gamma^\mu S(q-k) \gamma^\nu e^{i(kt-p)x''} u(q)$$

$$+ \bar{u}(p) \int_0^1 dt \int d^4k \, x_i'' \, \mathcal{F}_{\mu i \nu}(k) \, e^{iqx'} \gamma^\nu S(p-k) \Gamma^\mu e^{i(k-kt-p)x''} u(q)$$

$$- \bar{u}(p) \int_0^1 dt \int d^4k \, x_i' \, \mathcal{F}_{\mu i \nu}(k) \, e^{i(k-p)x''} \gamma^\nu S(p-k) \Gamma^\mu e^{i(q-kt)x'} u(q)$$

$$- \bar{u}(p) \int_0^1 dt \int d^4k \, x_i' \, \mathcal{F}_{\mu i \nu}(k) \, e^{-ipx''} \Gamma^\mu S(q-k) \gamma^\nu e^{i(q+kt-k)x'} u(q) \qquad (23)$$

Using the result

$$\frac{1}{i} \tilde{f}_i(\underline{p}) = \frac{1}{i} \frac{\partial}{\partial p_i} \tilde{f}(\underline{p}) = \int d^3\underline{x} \; x_i \, f(\underline{x}) \, e^{i\underline{p}\cdot\underline{x}}$$

we thus obtain, on substituting (23) into (20),

$$\langle p| \partial_\nu J^\nu(0) |q\rangle$$

$$= \int_0^1 dt \; \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \left[\tilde{f}(\underline{q}-\underline{k}) \tilde{f}_i(\underline{k}t-\underline{p}) - \tilde{f}(\underline{p}) \tilde{f}_i(\underline{q}+\underline{k}t-\underline{k}) \right] \Gamma^\mu S(q-k) \gamma^\nu \mathcal{F}_{\mu i \nu}(k) u(q)$$

$$- \int_0^1 dt \; \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \left[\tilde{f}(\underline{k}-\underline{p}) \tilde{f}_i(\underline{q}-\underline{k}t) - \tilde{f}(\underline{q}) \tilde{f}_i(-\underline{p}-\underline{k}t+\underline{k}) \right] \gamma^\nu S(p-k) \Gamma^\mu \mathcal{F}_{\mu i \nu}(k) u(q)$$

(24)

The required matrix elements depend on <u>derivatives</u> of the smearing functions, in this approximation. With the smearing function of Gaussian form

$$\tilde{f}(\underline{p}) = e^{-\underline{p}^2 \epsilon^2/4} \qquad (25)$$

so that

$$\tilde{f}_i(\underline{p}) = (-\tfrac{1}{2} p_i) \, \epsilon^2 \tilde{f}(\underline{p})$$

we now require to investigate the $\underline{\varepsilon}$ - dependence of (24), for small $\underline{\varepsilon}$.

Rewriting (24) in the form

$$\langle p|\,\partial_\nu J^\nu(0)\,|q\rangle = \frac{(-ie)^2}{(2\pi)^4}\,\bar{u}(p)\ \mathcal{H}(p,q)\,u(q) \tag{26A}$$

where

$$\mathcal{H}(p,q) = \int d^4k\,\left[F_i(p,q;k)\,\Gamma^\mu S(q-k)\gamma^\nu - F_i(q,p;k)\,\gamma^\nu S(p-k)\Gamma^\mu\right]\mathcal{F}_{\mu i\nu}(k) \tag{26B}$$

with, on using (25)

$$F_i(p,q;k) = \int_0^1 dt\,\left[\tilde{f}(\underset{\sim}{q}-\underset{\sim}{k})\,\tilde{f}_i(\underset{\sim}{k}t-\underset{\sim}{p}) - \tilde{f}(\underset{\sim}{p})\,\tilde{f}_i(\underset{\sim}{q}+\underset{\sim}{k}t-\underset{\sim}{k})\right]$$

$$F_i(p,q;k) = (-\tfrac{1}{2}\epsilon^2)\int_0^1 dt\,\left[\tilde{f}(\underset{\sim}{q}-\underset{\sim}{k})\tilde{f}(\underset{\sim}{k}t-\underset{\sim}{p})(k_i t-p_i) - \tilde{f}(\underset{\sim}{p})\tilde{f}(\underset{\sim}{q}+\underset{\sim}{k}t-\underset{\sim}{k})(q_i+k_i t-k_i)\right]$$

we note the following identities

$$\gamma_5\,\mathcal{H}_{e\text{-}m}(p,q) = -\mathcal{H}_5(p,q) + \int d^4k\,F_i(p,q;k)\,\gamma^\nu\{\gamma_5, S(p-k)\}\gamma^\mu\,\mathcal{F}_{\mu i\nu}(k)$$

$$\mathcal{H}_{e\text{-}m}(p,q)\,\gamma_5 = \mathcal{H}_5(p,q) - \int d^4k\,F_i(p,q;k)\,\gamma^\mu\{\gamma_5, S(q-k)\}\gamma^\nu\,\mathcal{F}_{\mu i\nu}(k) \tag{27}$$

Here, H_5 and H_{e-m} are the matrix quantities appropriate to the respective choices $\Gamma^\mu = \gamma^\mu\gamma_5$, and $\Gamma^\mu = \gamma^\mu$.

We further obtain from (27)

$$2\mathcal{H}_5(p,q) = [\mathcal{H}_{e\text{-}m}(p,q), \gamma_5] + \int d^4k\,\left[F_i(p,q;k)\,\gamma^\mu\{\gamma_5, S(q-k)\}\gamma^\nu + F_i(q,p;k)\{\gamma_5, S(p-k)\}\gamma^\mu\right]\mathcal{F}_{\mu i\nu} \tag{28A}$$

$$0 = \{\gamma_5, \mathcal{H}_{e\text{-}m}(p,q)\} + \int d^4k\,\left[F_i(p,q;k)\,\gamma^\mu\{\gamma_5, S(q-k)\}\gamma^\nu - F_i(q,p;k)\,\gamma^\nu\{\gamma_5, S(p-k)\}\gamma^\mu\right]\mathcal{F}_{\mu i\nu} \tag{28B}$$

We would now require to determine $\mathcal{H}_{e-m}(p,q)$ as $\underline{\varepsilon} \to 0$, and hence $\mathcal{H}_5(p,q)$ likewise; the procedure for doing so we now indicate.

From (26B) and (26C), we may deduce the expression

$$\int d^4k\, F_i(p,q;k)\, \Gamma^\mu S(q-k)\gamma^\nu \mathcal{H}_{\mu i\nu}(k)$$
$$= (-\tfrac{1}{2}\epsilon^2)\int_0^1 dt\; e^{-\left\{\underline{p}^2+\underline{q}^2-\frac{(\underline{p}t+\underline{q})^2}{1+t^2}\right\}\epsilon^2/4}\int d^3\underline{k}\; \xi_1(\underline{k},t;p,q)\, e^{-\left\{\underline{k}\sqrt{1+t^2}-\frac{(\underline{p}t+\underline{q})}{\sqrt{1+t^2}}\right\}^2\epsilon^2/4}$$
$$(+\tfrac{1}{2}\epsilon^2)\int_0^1 dt\; e^{-\underline{p}^2\epsilon^2/4}\int d^3\underline{k}\; \xi_2(\underline{k},t;p,q)\; e^{-\{\underline{k}(t-1)+\underline{q}\}^2\epsilon^2/4}$$

where

$$\xi_1(\underline{k},t;p,q) = (k_i t - p_i)\int dk_0\, \mathcal{H}_{\mu i\nu}(k)\,\Gamma^\mu S(q-k)\gamma^\nu$$

$$\xi_2(\underline{k},t;p,q) = (q_i + k_i t - k_i)\int dk_0\, \mathcal{H}_{\mu i\nu}(k)\,\Gamma^\mu S(q-k)\gamma^\nu$$

(29)

Making the respective changes of variable

$$\underline{k}_1' = \left[\underline{k}\sqrt{1+t^2} - \frac{(\underline{p}t+\underline{q})}{\sqrt{1+t^2}}\right]\epsilon/2 \quad ; \quad \underline{k}_2' = \left[\underline{k}(t-1)+\underline{q}\right]\epsilon/2$$

in the two contributions

$$\int d^4k\; F_i(p,q;k)\,\Gamma^\mu S(q-k)\gamma^\nu \mathcal{H}_{\mu i\nu}(k)$$
$$= (-\tfrac{1}{2}\epsilon^2)\int_0^1 dt\; e^{-\left\{\underline{p}^2+\underline{q}^2-\frac{(\underline{p}t+\underline{q})^2}{1+t^2}\right\}\epsilon^2/4}\int d^3\underline{k}_1'\; \xi_1(\underline{k},t;p,q)\Big|_{\underline{k}(\underline{k}_1')} e^{-k_1'^2}\left[\frac{8}{\epsilon^3(1+t^2)^{3/2}}\right]$$
$$(+\tfrac{1}{2}\epsilon^2)\int_0^1 dt\; e^{-\underline{p}^2\epsilon^2/4}\int d^3\underline{k}_2'\; \xi_2(\underline{k},t;p,q)\Big|_{\underline{k}(\underline{k}_2')} e^{-k_2'^2}\left[\frac{8}{\epsilon^3(t-1)^3}\right] \qquad (30)$$

from which we observe that the dependence we require to find is now concentrated in the functions ξ_1, ξ_2.

Carrying out the k_0-integration in (29), to determine $\xi_\alpha(\underline{k}, t; p, q)$ ($\alpha = 1,2$) (taking a contour enveloping the lower half-plane, and containing within it poles at $k_0 = |\underline{k}| - i\varepsilon$, $k_0 = q_0 + \sqrt{(\underline{q} - \underline{k})^2 + m^2} - i\varepsilon$), we could then determine the limit as $\underline{\varepsilon} \to 0$ of (30) by expanding the integrands in each term. The use of a Taylor expansion of ξ_α will be justified on the grounds that each coefficient of ε is defined (by virtue of the exponential factor $e^{-k'^2}$ in the integrand associated with the $\underline{k}'$-integration.)

Rather than go through the rather heavy algebra involved in carrying this through directly for both the e-m and chiral currents, we shall anticipate that, by Lorentz invariance and e-m current conservation, $\mathcal{H}_{e-m}(p,q)$ takes the form where α and β are constants

$$\mathcal{H}_{e-m}(p,q) = \alpha\gamma\cdot(p-q) + \beta(p^2-q^2) \tag{31}$$

in the limit $\underline{\varepsilon} \to 0$. (From the form (26B), note that we can deduce immediately that $\beta = \{\gamma_5, \mathcal{H}_{e-m}\} = 0$). Were it not to turn out so, then we would have to conclude the inconsistency of the approximation scheme considered. The limiting procedure and the approximation scheme chosen are seen to be inextricably linked through the requirement of e-m current conservation.

It then follows from (28B) that

$$\int d^4k \left[\frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\mu\gamma^\nu - \frac{F_i(q,p;k)}{(p-k)^2+m^2}\gamma^\nu\gamma^\mu \right]\gamma_5 \mathcal{H}_{\mu i\nu}(k) = 0 \tag{32A}$$

while (28A) gives

$$2\mathcal{H}_5(p,q) = 2\alpha\gamma\cdot(p-q)\gamma_5 - 2m\int d^4k\left[\frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\mu\gamma^\nu + \frac{F_i(q,p;k)}{(p-k)^2+m^2}\gamma^\nu\gamma^\mu\right]\gamma_5\mathcal{F}_{\mu i\nu}(k)$$

(32B)

On using the anti-commutation relations $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ and the relation

$$\mathcal{F}_{\mu i\nu}(k) = (k_i g_{\mu\nu} - k_\mu g_{i\nu})\frac{1}{k^2}$$

we may deduce from (32A) and (32B) the expressions (33)

$$\mathcal{H}_5(p,q) = \alpha\gamma\cdot(p-q)\gamma_5 - 2m\int d^4k\,\frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\mu\gamma^\nu\gamma_5\,\mathcal{F}_{\mu i\nu}(k) \tag{A}$$

$$\mathcal{H}_5(p,q) = \alpha\gamma\cdot(p-q)\gamma_5 - 2m\int d^4k\,\frac{F_i(q,p;k)}{(p-k)^2+m^2}\gamma^\nu\gamma^\mu\gamma_5\,\mathcal{F}_{\mu i\nu}(k) \tag{B}$$

$$\mathcal{H}_5(p,q) = \alpha\gamma\cdot(p-q)\gamma_5 - 2m\int d^4k\,\frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma_5\,\mathcal{F}_{\mu i}{}^{\mu}(k) \tag{C}$$

$$\mathcal{H}_5(p,q) = \alpha\gamma\cdot(p-q)\gamma_5 - 2m\int d^4k\,\frac{F_i(p,q;k)}{(q-k)^2+m^2}\,\frac{3k_i}{k^2}\gamma_5 \tag{D}$$

However, from (26B), we require

$$\mathcal{H}_{e-m}(p,q) = i\int d^4k\left[\frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\mu\gamma\cdot(q-k)\gamma^\nu - \frac{F_i(q,p;k)}{(p-k)^2+m^2}\gamma^\nu\gamma\cdot(p-k)\gamma^\mu\right]\mathcal{F}_{\mu i\nu}(k)$$

(34A)

which may be re-expressed

$$\mathcal{H}_{e\text{-}m}(p,q) = \int d^4k \left[\frac{F_i(p,q;k)}{(q-k)^2+m^2} \gamma^\nu \gamma\cdot(q-k)\gamma^\mu - \frac{F_i(q,p;k)}{(p-k)^2+m^2} \gamma^\mu \gamma\cdot(p-k)\gamma^\nu \right] \mathcal{J}_{\mu i \nu}(k) \tag{34B}$$

since from (31),

$$\mathcal{H}_{e\text{-}m}(p,q) = - \mathcal{H}_{e\text{-}m}(q,p)$$

Now, since

$$\gamma^\mu \gamma\cdot(q-k)\gamma^\nu = 2\gamma^\mu (q-k)^\nu - \gamma^\mu\gamma^\nu\gamma\cdot(q-k) = 2(q-k)^\mu\gamma^\nu - \gamma\cdot(q-k)\gamma^\mu\gamma^\nu$$

we may obtain the two alternative expressions

$$\int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2} \gamma^\mu\gamma\cdot(q-k)\gamma^\nu \mathcal{J}_{\mu i\nu}(k) = \int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2} \left\{ -\frac{k_i}{k^2}\gamma\cdot q + \frac{2k_i}{k^2}\gamma\cdot k - \frac{2q_i}{k^2}\gamma\cdot k + \gamma_i \right\}$$

$$= \int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2} \left\{ -\frac{k_i}{k^2}\gamma\cdot q + \frac{2k_i}{k^2}\gamma\cdot k - \frac{2k\cdot q}{k^2}\gamma_i + \gamma_i \right\} \tag{35}$$

where we have used the results

$$k^\nu \mathcal{J}_{\mu i\nu}(k) = 0 \qquad \text{and} \qquad \int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\mu\gamma^\nu \mathcal{J}_{\mu i\nu}(k) = \int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\frac{3k_i}{k^2}$$

The other quantity $\int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\gamma^\nu\gamma\cdot(p-k)\gamma^\mu \mathcal{J}_{\mu i\nu}(k)$ appearing in (34) is obtained from (35) simply by interchanging p and q. Thus, we have from (34)

$$-i\,\mathcal{H}_{e\text{-}m}(p,q) = \int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\frac{k_i}{k^2}\gamma\cdot(p-q)$$

$$+ \int d^4k \left[\frac{F_i(p,q;k)}{(q-k)^2+m^2} - \frac{F_i(q,p;k)}{(p-k)^2+m^2} \right] \left(\frac{2k_i}{k^2}\gamma\cdot k + \gamma_i \right) \tag{36}$$

$$-2\int d^4k \left[\frac{\gamma_i F_i(p,q;k)}{(q-k)^2+m^2}\frac{k\cdot q}{k^2} - \frac{\gamma_i F_i(q,p;k)}{(p-k)^2+m^2}\frac{k\cdot p}{k^2} \right]$$

We note that this form allows us to infer the value of α, by equating the coefficients of the γ-matrices: we obtain

$$\alpha(p_0-q_0) = i\int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\frac{k_i}{k^2}(p_0-q_0) - i\int d^4k\left[\frac{F_i(p,q;k)}{(q-k)^2+m^2} - \frac{F_i(q,p;k)}{(p-k)^2+m^2}\right]\frac{2k_i k_0}{k^2} \tag{37A}$$

and

$$\alpha(p_j-q_j) = i\int d^4k \frac{F_i(p,q;k)}{(q-k)^2+m^2}\frac{k_i}{k^2}(p_j-q_j) + i\int d^4k\left[\frac{F_i(p,q;k)}{(q-k)^2+m^2} - \frac{F_i(q,p;k)}{(p-k)^2+m^2}\right]\left(\frac{2k_i k_j}{k^2}+\delta_{ij}\right)$$

$$-2i\int d^4k\left[\frac{F_j(p,q;k)}{(q-k)^2+m^2}\frac{k\cdot q}{k^2} - \frac{F_j(q,p;k)}{(p-k)^2+m^2}\frac{k\cdot p}{k^2}\right] \tag{37B}$$

Some Integrals and Determination of α

We observe that there is quite a bit of information in (37) about the various integrals involved. However, our immediate aim is to determine α, and we may best do this using (A), which demands that we evaluate integrals of the type

$$I = \int d^4k\, \frac{F_i(p,q;k)}{(q-k)^2+m^2}\,\frac{k_i f(k)}{k^2}$$

Using (26C), we have immediately on carrying out the parametric t-integration

$$I = \lim_{\epsilon\to 0}\left\{e^{-(\underline{p}^2+\underline{q}^2)\epsilon^2/4}\int d^4k\left[e^{-\{\underline{k}-\frac{1}{2}(\underline{p}+\underline{q})\}^2\epsilon^2/2+\{\frac{1}{2}(\underline{p}+\underline{q})\}^2\epsilon^2/4} - 1\right]\frac{f(k)}{(q-k)^2+m^2}\cdot\frac{1}{k^2}\right\}$$

$$= \lim_{\epsilon\to 0}\int d^4k\, e^{-\{\underline{k}-\frac{1}{2}(\underline{p}+\underline{q})\}^2\epsilon^2/2}\frac{f(k)}{[(q-k)^2+m^2]k^2} - \int d^4k\frac{f(k)}{[(q-k)^2+m^2]k^2}$$

$$I = I_1 + I_2$$

$$I_1 = -\int d^3\underline{k}\left[\frac{f(\underline{k},q_0+w_{qk})}{w_{qk}(q_0^2+q_0w_{qk}-\underline{q}\cdot\underline{k})} + \frac{f(\underline{k},k)}{k(\underline{q}\cdot\underline{k}-q_0k)}\right]; \quad I_2 = \lim_{\epsilon\to 0}\int d^3\underline{k}\, e^{-\{\underline{k}-\frac{1}{2}(\underline{p}+\underline{q})\}^2\epsilon^2/2}\left[\frac{f(\underline{k},q_0+w_{qk})}{w_{qk}(q_0^2+q_0w_{qk}-\underline{q}\cdot\underline{k})} + \frac{f(\underline{k},k)}{k(\underline{q}\cdot\underline{k}-q_0k)}\right]$$

$$w_{qk}^2 = (\underline{q}-\underline{k})^2+m^2$$

on carrying out the k_o-integration. In both the cases of interest, viz. $f(k) = 1$ and $f(k) = k_o$, the contribution I_1 is divergent, but will not appear in (37A); we assume cancellation of these divergences. We are then left with the calculation of the limit

$$I_2 = \lim_{\epsilon \to 0} \int d^3\underline{k}\; e^{-\{\underline{k}-\frac{1}{2}(\underline{p}+\underline{q})\}^2\epsilon^2/2} \left[\frac{f(\underline{k}, q_0+\omega_{qk})}{\omega_{qk}(q_0^2+q_0\omega_{qk}-\underline{q}.\underline{k})} + \frac{f(\underline{k},k)}{k(\underline{q}.\underline{k}-q_0k)}\right] \pi i^2$$

$$= \lim_{\epsilon \to 0} \left(\frac{\sqrt{2}}{\epsilon}\right)^3 \int d^3\underline{k}'\; e^{-k'^2} \left[\frac{f(\underline{k}, q_0+\omega_{qk})}{\omega_{qk}(q_0^2+q_0\omega_{qk}-\underline{q}.\underline{k})} + \frac{f(\underline{k},k)}{k(\underline{q}.\underline{k}-q_0k)}\right]_{\underline{k}=\frac{\sqrt{2}}{\epsilon}\underline{k}'+\frac{1}{2}(\underline{p}+\underline{q})} \pi i^2$$

For $f = 1$, we find

$$\left[\frac{1}{\omega_{qk}(q_0^2+q_0\omega_{qk}-\underline{q}.\underline{k})} + \frac{1}{k(\underline{q}.\underline{k}-q_0k)}\right]_{\underline{k}=\frac{\sqrt{2}}{\epsilon}\underline{k}'+\frac{1}{2}(\underline{p}+\underline{q})} = -\left(\frac{\epsilon}{2\sqrt{2}}\right)^3 \frac{1}{k'^3} + O(\epsilon^4)$$

while for $f = k_o$,

$$\left[\frac{q_0+\omega_{qk}}{\omega_{qk}(q_0^2+q_0\omega_{qk}-\underline{q}.\underline{k})} + \frac{k}{k(\underline{q}.\underline{k}-q_0k)}\right]_{\underline{k}=\frac{\sqrt{2}}{\epsilon}\underline{k}'+\frac{1}{2}(\underline{p}+\underline{q})} = -\left(\frac{\epsilon}{2\sqrt{2}}\right)^3 \frac{q_0}{2k'^3} + O(\epsilon^4)$$

It follows that

$$\alpha = +\frac{\pi}{4} i \int d^3\underline{k}'\; \frac{e^{-k'^2}}{k'^3}$$

$$\alpha = i \int d^4k\; \frac{F_i(p,q;k)}{(q-k)^2+m^2} \cdot \frac{2k_i}{k^2}$$

which is divergent.

Thus, we appear to have verified the assumed Lorentz covariance (31); at least our result indicates no inconsistency so far. To be completely sure, we would have to obtain the same value for α from equation (37B), but the integrals involved there are a bit more difficult to handle than the above. The method summed up in equation

(30) might conceivably offer a way of obtaining (31) directly, although one would have to go to higher orders of ε in order to deduce the limit.

Concluding Remarks

Forming the matrix elements of 33D,

$$\langle p|\partial^{\mu} J_{\mu 5}(0)|q\rangle = \frac{2me^2 i}{(2\pi)^4}\int d^4k \, \frac{F_i(p,q;k)}{(q-k)^2+m^2}\,\frac{k_i}{k^2}\quad \bar{u}(p)\gamma_5 u(q)$$

which does not vanish, contrary to what might have been expected. While this result is in agreement with that of Maris et al., for a non-field dependent current definition, we might have thought that the definition we have taken would have led to conservation. If our result is accepted we would be required to put $\partial^{\mu} J_{\mu 5}(x) \neq 0$, so that from (5B) the Goldstone Theorem would not come through, i.e. Λ_{μ} (p+q, p) would not possess a singularity at q = 0, indicating the absence of massless bosons. This agrees with a recent report by Johnson, in which he points out that in the exactly soluble one-dimensional case of QED with zero bare mass (Thirring's Model), there also with the gauge invariant definition of the current, the chiral current is not locally conserved. In that case, however, as has been pointed out by Hagen, there is ambiguity in the model, to the extent that the solution discussed by Johnson is one only of a family of possible solutions, and it turns out to be possible to have a solution with chiral current conservation.

That it turns out not to be so here is pleasing in that the claim that it might be possible to have symmetry without local conservation and therefore symmetry breakdown without Goldstone Boson still stands.

However, it is also true that the following basic objections to this possibility still stand.

(1) Accepting that the current definitions (17) and (18) are the appropriate ones to choose, and the single particle approximation scheme employed is valid, even then it may be necessary to include several factors which we have ignored. For example, the ℓ-m current appearing in the interaction $J_\mu A^\mu$ was not 'smeared', when perhaps it should have been.

(2) The approximation scheme itself and the current definition may not be consistent with each other. For example, it is conceivable that a more 'consistent' gauge invariant definition could be found, (i.e. one which conserved the chiral current in this approximation).

(3) Forgetting about (1) and (2) and accepting the result, the need to revise our view of the Noether Theorem would arise. Since it is connected closely to the Action Principle, it would be difficult to contemplate a generalisation consistent with the usual requirements.

Finally, it should be possible to generalise the above arguments to the electron-muon problem, with an additional spontaneous breakdown of 'isospin' symmetry ($m_e \neq m_m$), and we might expect to see the same lack of local conservation in that case also in the single particle approximation.

General Conclusion

We have been concerned in this thesis with the attempt to account for mass differences in elementary particle physics by the mechanism of spontaneous symmetry breakdown. The main stumbling block to such an approach has been the Goldstone Theorem, which asserts that in such theories zero mass bosons must always be present.

From our considerations, there would appear to be only two possible means of escape from the theorem; namely, (a) the coupling of long-range gauge fields to the locally conserved current associated with the spontaneously broken symmetry, or (b) the breakdown of the local conservation law.

It would seem from Chapters III and V that while (a) definitely results in the inapplicability of the theorem, there is dubiety about the occurrence of (b). In the context of electrodynamics at least, the question would appear to be open.

Thus (a) seems, at present, to be the only one definite way of removing the unwanted particles. Only then do we have the relative freedom of a non-manifestly covariant description which appears to be the single condition allowing escape from the theorem.

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