Singularities of Noncommutative Surfaces

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Simon Philip Crawford*)

Publications

This thesis is based upon the work of the author contained in two preprints, [Cra16] and [Cra17]. The results of Chapter 3 are similar to those in [Cra16], although the overall argument has been shortened. The results in Chapters 5 and 6 are drawn from [Cra17], with some changes in presentation. Some of the preliminary material in Chapter 2 is also drawn from these preprints, but with significant changes to the presentation to ensure that this thesis has a cohesive narrative.

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Lay summary

In ring theory, a subfield of abstract algebra, we are interested in understanding rings, which are number systems that have similar properties to the integers: namely, there is a sensible notion of addition and multiplication. In particular, we can study number systems where the order of multiplication is important. While it is true that given any two integers we can multiply them in either order and get the same result, this fails to be the case in more exotic number systems. A *commutative ring* is a number system where the order in which we multiply two numbers does not affect the result, and a *noncommutative ring* is a number system where the order of multiplication matters.

Given a commutative ring, there is a mathematical procedure which produces a geometric shape associated to that ring, which for example might be (the outer edge of) a circle, or a pair of infinitely long straight lines which cross at some point. These two examples have quite different properties. With a circle, if we take any two points and zoom in sufficiently far then these points look roughly the same (after a rotation). However, the same is not true for a pair of crossed lines: no matter how far we zoom in at the crossing point, it still looks different to every other point on our pair of lines. We call this crossing point a *singular point*, and say that the pair of crossed lines is *singular*. On the other hand, a circle is said to be *nonsingular* because it has no singular points. These shapes are both examples of curves, and the definition can be generalised to higher dimensions, such as *surfaces*.

Suppose we have a commutative ring and we want to study the shape that the above procedure produces. A guiding principle is that we can study geometric properties of this shape by studying algebraic properties of the ring we started with, and vice versa. In particular, we can study the singularities (of singular points) of a shape by studying algebraic properties of the corresponding ring.

For the previous two paragraphs to make sense, it is vital that we begin with a ring which is commutative; if the ring is noncommutative, then the procedure which produces a shape no longer works. However, it is fruitful to pretend that this procedure does work even when the ring that we begin with is noncommutative. This allows us to, for example, study the singularities of a non-existent "noncommutative shape" by studying the corresponding algebraic properties of our noncommutative ring.

In this thesis, we apply this principle to understand the types of singular points that occur in *noncommutative surfaces*. These are noncommutative rings which have similar algebraic properties to commutative rings whose corresponding shape is a surface. By studying their singularities, we are able to better understand the structure and properties of these noncommutative rings.

Abstract

The primary objects of study in this thesis are noncommutative surfaces; that is, noncommutative noetherian domains of GK dimension 2. Frequently these rings will also be singular, in the sense that they have infinite global dimension. Very little is known about singularities of noncommutative rings, particularly those which are not finite over their centre. In this thesis, we are able to give a precise description of the singularities of a few families of examples. In many examples, we lay the foundations of noncommutative singularity theory by giving a precise description of the singularities of the fundamental examples of noncommutative surfaces. We draw comparisons with the fundamental examples of commutative surface singularities, called Kleinian singularities, which arise from the action of a finite subgroup of $\mathrm{SL}(2, \Bbbk)$ acting on a polynomial ring.

The main tool we use to study the singularities of noncommutative surfaces is the singularity category, first introduced by Buchweitz in [Buc86]. This takes a (possibly noncommutative) ring R and produces a triangulated category $\mathcal{D}_{\mathrm{sg}}(R)$ which provides a measure of "how singular" R is. Roughly speaking, the size of this category reflects how bad the singularity is; in particular, $\mathcal{D}_{\mathrm{sg}}(R)$ is trivial if and only if R has finite global dimension.

In [CBH98], Crawley-Boevey–Holland introduced a family of noncommutative rings which can be thought of as deformations of the coordinate ring of a Kleinian singularity. We give a precise description of the singularity categories of these deformations, and show that their singularities can be thought of as unions of (commutative) Kleinian singularities. In particular, our results show that deforming a singularity in this setting makes it no worse.

Another family of noncommutative surfaces were introduced by Rogalski–Sierra–Stafford in [RSS15b]. The authors showed that these rings share a number of ring-theoretic properties with deformations of type $\mathbb{A}$ Kleinian singularities. We apply our techniques to show that the "least singular" example has an $\mathbb{A}_1$ singularity, and conjecture that other examples exhibit similar behaviour.

In [CKWZ16a], Chan–Kirkman–Walton–Zhang gave a definition for a quantum version of Kleinian singularities. These require the data of a two-dimensional AS regular algebra A and a finite group G acting on A with trivial homological determinant. We extend a number of results in [CBH98] to the setting of quantum Kleinian singularities. More precisely, we show that one can construct deformations of the skew group rings $A \# G$ and the invariant rings A^G, and then determine some of their ring-theoretic properties. These results allow us to give a precise description of the singularity categories of quantum Kleinian singularities, which often have very different behaviour to their non-quantum analogues.

Contents

Chapter 1

Introduction

In this thesis we study *noncommutative surfaces*, which are noncommutative rings that can be viewed as noncommutative analogues of the coordinate ring of an affine (commutative) surface. In particular, we are interested in studying the singularities of these rings, where we say that a noncommutative ring is *singular* when it has infinite global dimension. One way of describing the types of singularities that occur in noncommutative surfaces is using *singularity categories*, as described by Buchweitz in [Buc86], and this is the approach we take. Given a (possibly noncommutative) ring R, the singularity category $\mathcal{D}_{\mathrm{sg}}(R)$ is a triangulated category which reflects "how singular" R is. In particular, $\mathcal{D}_{\mathrm{sg}}(R)$ is trivial if and only if R has finite global dimension. This category is often Krull-Schmidt, and so the number of indecomposables provides a measure of how bad the singularity is. In a number of examples we are able to describe the singularities of noncommutative rings by drawing comparisons with commutative rings. Throughout this chapter, let $\Bbbk$ be an algebraically closed field of characteristic 0.

1.1 Kleinian singularities and the McKay correspondence

One of the most important families of (commutative) singularities are *Kleinian singularities*, which are ubiquitous in algebraic geometry, representation theory, and singularity theory. Two of the main examples of noncommutative surfaces that we study in this thesis arise from these singularities: one by deforming Kleinian singularities, and one by generalising their construction to a noncommutative setting. We now recall the definition of the Kleinian singularities and discuss the *McKay correspondence*.

1.1.1 Kleinian singularities

Let G be a nontrivial finite subgroup of $\mathrm{SL}(2, \Bbbk)$. These are completely classified: up to conjugation, there are two infinite families and three exceptional examples [LW12, 6.11 Theorem]. (Presentations for these groups are not required for the purposes of this thesis so we do not provide them, but we provide some details in Table 1.1.) We can then define an action of G on $\Bbbk[u, v]$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u = au + cv, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = bu + dv.$$

The (coordinate rings of) Kleinian singularities are then defined to be the invariant rings

$$\mathbb{k}[u,v]^G \coloneqq \{p \in \mathbb{k}[u,v] \mid g \cdot p = p \text{ for all } g \in G\}. \tag{1.1.1}$$

One can show that the rings $\mathbb{k}[u,v]^G$ are isomorphic to $\mathbb{k}[x,y,z]/\langle f \rangle$ for some irreducible polynomial f, and so the varieties $\operatorname{Spec} \mathbb{k}[u,v]^G$ are surfaces. They are in fact singular, with a unique singular point at the origin. This geometric property corresponds to the rings $\mathbb{k}[u,v]^G$ having infinite global dimension, so when we speak of a (possibly noncommutative) ring being *singular*, we mean that it has infinite global dimension. Kleinian singularities are described as being type $\mathbb{A}, \mathbb{D}$, or $\mathbb{E}$, depending on the group G used to define them, and we explain the choice of names shortly. We summarise this information in Table 1.1.

Type	Group	$\lvert G \rvert$	f
$\mathbb{A}_n$	Cyclic, parametrised by $n \geqslant 1$	$n+1$	$x^2+y^2+z^{n+1}$
$\mathbb{D}_n$	Binary dihedral, parametrised by $n \geqslant 4$	$4(n-2)$	$x^2+y^2z+z^{n-1}$
$\mathbb{E}_6$	Binary tetrahedral	24	$x^2+y^3+z^4$
$\mathbb{E}_7$	Binary octahedral	48	$x^2+y^3+yz^3$
$\mathbb{E}_8$	Binary icosahedral	120	$x^2+y^3+z^5$

Table 1.1: The Kleinian singularities

1.1.2 Representation theory and the Auslander-McKay correspondence

Kleinian singularities have many interesting ring-theoretic, representation-theoretic, and geometric properties, and these are all intimately connected. The Auslander-McKay correspondence describes a large family of results associated to Kleinian singularities; we remark that many of the results we state below are true in greater generality. The *skew group ring* plays an important role in this correspondence, and we recall its definition below.

Definition 1.1.2. Let G be a group acting (on the left) on a ring R. The *skew group ring* $R\#G$ is the free left R-module with the elements of G as a basis, with multiplication extended linearly from the rule $(rg)(sh) = r(g \cdot s)gh$ for $r,s \in R$, $g,h, \in G$, where $g \cdot s$ is the image of s under the action of g.

Henceforth we write $R = \mathbb{k}[u,v]$ and assume that G is a finite subgroup of $\mathrm{SL}(2,\mathbb{k})$. We recall some basic facts connecting $R\#G$ and R^G, many of which are true in greater generality. We note that [Ben93, Theorem 1.3.1] implies that R is a finitely generated R^G-module, and hence the same is also true of $R\#G$. A famous result due to Auslander, often called Auslander's Theorem, is the following:

Theorem 1.1.3 ([Aus62]). *Suppose that G is a finite subgroup of $\mathrm{SL}(2,\mathbb{k})$. Then the ring homomorphism*

$$\phi : R\#G \to \operatorname{End}_{R^G}(R), \quad \phi(rg)(s) = r\,g \cdot s$$

is an isomorphism.

We also have the following representation-theoretic correspondences, which are established in part using Theorem 1.1.3. We draw attention to the fact that in the following result we work with *complete* Kleinian singularities, which are the rings $\Bbbk[\![u,v]\!]^G$ for a finite subgroup G of $\mathrm{SL}(2,\Bbbk)$.

Theorem 1.1.4 ([LW12, Corollary 6.4])**.** *Write $S = \Bbbk[\![u,v]\!]$ and let S^G be a complete Kleinian singularity. Then there are one-to-one correspondences between:*

- *irreducible G-modules;*
- *indecomposable direct summands of S as an S^G-module;*
- *indecomposable finitely generated projective $S\#G$-modules;*
- *indecomposable finitely generated projective $\mathrm{End}_{S^G}(S)$-modules; and*
- *indecomposable maximal Cohen-Macaulay S^G-modules.*

Another important representation-theoretic result involves the *McKay quiver*, which we now describe. Let $W_0, W_1, \ldots, W_n$ be a complete list of the isoclasses of irreducible representations of G, where W_0 is the trivial representation. Write $V = \Bbbk u \oplus \Bbbk v$ for the natural two-dimensional representation arising from the action of G on R. Then $V \otimes_\Bbbk W_i$ is a finite-dimensional representation of G, and so it decomposes as $V \otimes W_i \cong \bigoplus_{j=0}^n W_j^{m_{ij}}$ for some non-negative integers m_{ij}.

Definition 1.1.5. With the setup as above, the *McKay quiver* associated to the finite group G acting on R is defined to be the quiver (directed graph) with vertex set $\{0, 1, \ldots, n\}$ and m_{ij} arrows from vertex i to vertex j.

We remark that one can equivalently define

$$m_{ij} \coloneqq \dim_\Bbbk \mathrm{Hom}_{\Bbbk G}(W_j, V \otimes_\Bbbk W_i) = \dim_\Bbbk \mathrm{Hom}_{\Bbbk G}(V \otimes_\Bbbk W_i, W_j).$$

Writing $\chi_V, \chi_{W_0}, \ldots, \chi_{W_n}$ for the characters of $V, W_0, \ldots, W_n$ and using [FH13, (2.10)], we then find that

$$\begin{aligned} m_{ij} &= \dim_\Bbbk \mathrm{Hom}_{\Bbbk G}(W_j, V \otimes_\Bbbk W_i) = \frac{1}{|G|} \sum_{g \in G} \chi_{W_j}(g^{-1}) \chi_V(g) \chi_{W_i}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g^{-1}) \chi_{W_j}(g^{-1}) \chi_{W_i}(g) = \dim_\Bbbk \mathrm{Hom}_{\Bbbk G}(V \otimes_\Bbbk W_j, W_i) \\ &= m_{ji}, \end{aligned}$$

where the equality $\chi_V(g^{-1}) = \chi_V(g)$ follows from the fact that the trace of a matrix in $\mathrm{SL}(2,\Bbbk)$ is the same as that of its inverse. Therefore for Kleinian singularities we have $m_{ij} = m_{ji}$ for all i, j, and McKay showed [McK81] that if one replaces each opposed pair of arrows in the McKay quiver by a simple edge then we obtain an *extended Dynkin graph*, as shown in Figure 1.1.

If we remove vertex 0 from an extended Dynkin graph of type $\widetilde{\Delta}$ then we obtain a *Dynkin graph* of type Δ, and this explains the choice of names in Table 1.1. Given a Kleinian singularity R^G, we will often write $\widetilde{Q}$ for the quiver obtained from the corresponding graph $\widetilde{\Delta}$ by choosing an arbitrary orientation for each of the edges, and call this the quiver corresponding to R^G. We will also write Q for the full subquiver obtained by deleting the vertex labelled 0. We will always assume that Dynkin graphs and quivers have n vertices, and that extended Dynkin graphs and quivers have $n+1$ vertices.

Another component of the Auslander-McKay correspondence in which extended

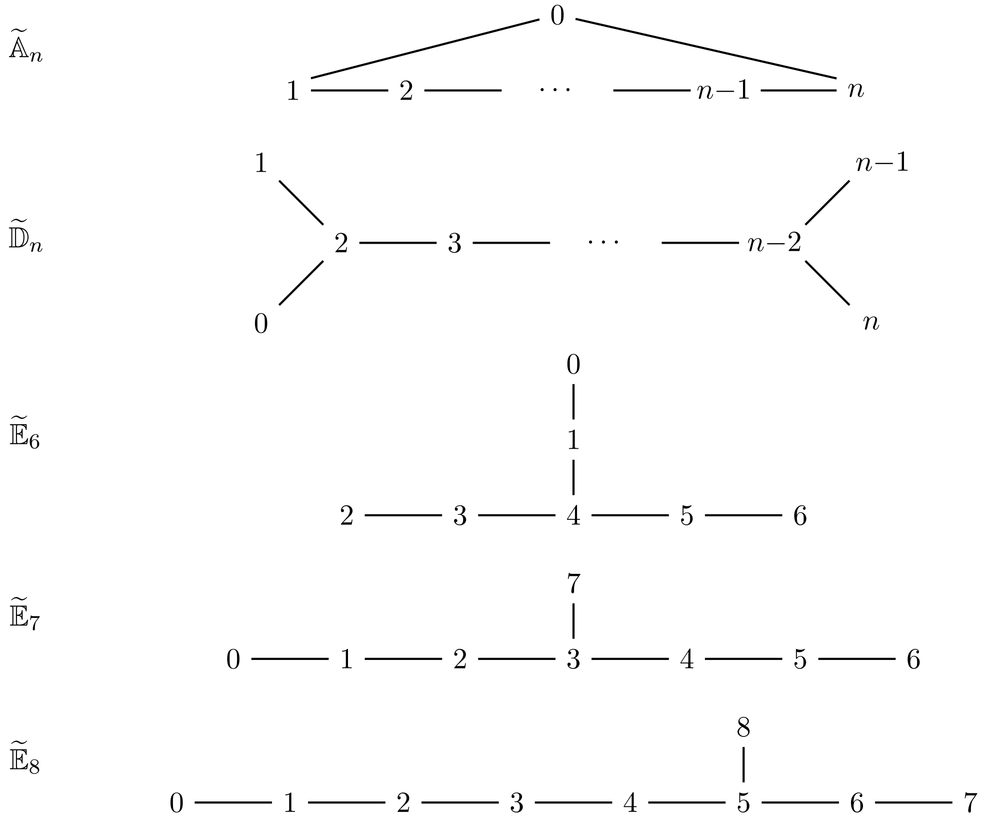


Figure 1.1: Extended Dynkin graphs.

Dynkin graphs make an appearance is the following result due to Reiten–Van den Bergh. This result provides an alternative perspective on the rings $R\#G$ and R^G from a representation-theoretic standpoint. We provide only an approximate statement of their result, and give a more precise statement in Theorem 2.5.4.

Theorem 1.1.6 ([RVdB89])**.** *Let R^G be a Kleinian singularity and let $\widetilde{Q}$ be the corresponding extended Dynkin quiver. Then there exists a path algebra with relations $\Pi(\widetilde{Q}) = \Bbbk\widetilde{Q}/I$, called the* preprojective algebra *of $\widetilde{Q}$, such that $R\#G$ is Morita equivalent to $\Pi(\widetilde{Q})$ and R^G is isomorphic to $e_0\Pi(\widetilde{Q})e_0$, where e_0 is the idempotent in $\Pi(\widetilde{Q})$ corresponding to the vertex 0 in $\widetilde{Q}$.*

This presentation of R^G and (an algebra Morita equivalent to) $R\#G$ was used by Crawley-Boevey–Holland in [CBH98] to define deformations of Kleinian singularities, which provide an important family of noncommutative surfaces and which we define in the next section.

1.1.3 The geometric Auslander-McKay correspondence

The Auslander-McKay correspondence also has a geometric aspect, which we now recall. Let G be a finite nontrivial subgroup of $\mathrm{SL}(2, \Bbbk)$ and $R = \Bbbk[u, v]$. Then the variety $\operatorname{Spec} R^G$ is an affine surface singularity with an isolated singular point at the origin. In algebraic geometry, every singular variety X defined over a field of characteristic 0 has a *resolution of singularities*: that is, there exists a nonsingular variety Y and a proper birational map $Y \to X$. A *minimal resolution* $\pi : \widetilde{X} \to X$ is a resolution through which any other resolution factors. Minimal resolutions do not exist in general, but for surfaces they exist and are unique, so in this case we can speak of *the* minimal resolution. In particular, this is the case for Kleinian singularities $X := \operatorname{Spec} R^G$. In the case of an $\mathbb{A}_1$ singularity, we are able to draw a picture of the resolution. In this case, $\operatorname{Spec} \Bbbk[u,v]^{C_2}$ is an infinite cone, and its resolution replaces the singular point at the origin by a circle, shown in Figure 1.2.

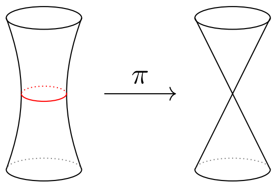


Figure 1.2: The minimal resolution of an $\mathbb{A}_1$ singularity.

To the minimal resolution $\pi : \widetilde{X} \to X = \operatorname{Spec} R^G$ we can associate the *exceptional divisor*, which is the preimage of the unique singular point at the origin (shown in red in Figure 1.2). It is known that, for Kleinian singularities, the exceptional divisor is a union of irreducible curves γ_i each of which is necessarily isomorphic to $\mathbb{P}^1$, and that these curves have self-intersection -2. Moreover, for $i \neq j$, $\gamma_i \cap \gamma_j$ is either empty or a point; see [LW12, Lemma 6.31] for details. In fact, if one forms the dual graph of the exceptional divisor by replacing each curve by a vertex, and connecting two vertices by an edge if they intersect, then the resulting graph is a Dynkin graph of type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, [LW12, Theorem 6.40]. McKay made the following observation, linking many of the above results:

Theorem 1.1.7 ([McK81])**.** *Let G be a finite subgroup of $\operatorname{SL}(2,\Bbbk)$, $R = \Bbbk[u,v]$ and let R^G be the corresponding Kleinian singularity. Then the dual graph and the McKay quiver are related as follows: the dual graph of the exceptional divisor of the minimal resolution $\pi : \widetilde{X} \to \operatorname{Spec} R^G$ is equal to the Dynkin graph obtained from the McKay quiver of G by removing the extending vertex 0 and replacing each pair of opposed arrows by an edge.*

In Chapter 3 we prove a result which may be viewed as a noncommutative analogue of the above theorem, which we will describe in the next section.

1.2 Deformations of Kleinian singularities

In [CBH98], Crawley-Boevey and Holland defined a family of (generically noncommutative) deformations of Kleinian singularities which we now define, and we also recall some of their basic properties. Their work can be viewed as an extension of the work of Hodges in [Hod93], in which deformations of type $\mathbb{A}$ Kleinian singularities were defined. Throughout, $R = \Bbbk[u,v]$ and G is a nontrivial finite subgroup of $\operatorname{SL}(2,\Bbbk)$.

Definition 1.2.1. Let R^G be a Kleinian singularity and let $\widetilde{Q}$ be the corresponding extended Dynkin quiver. Then G acts naturally on the free algebra $\Bbbk\langle u,v\rangle$, and so we can form the skew group ring $\Bbbk\langle u,v\rangle \# G$. Let $e = \frac{1}{|G|}\sum_{g \in G} g$ be the average of the group elements, viewed as an element of $\Bbbk\langle u,v\rangle \# G$. For each vertex i of $\widetilde{Q}$, choose $\lambda_i \in \Bbbk$ and write $\lambda = (\lambda_i)_{i \in \widetilde{Q}_0}$; we call λ a *weight* for $\widetilde{Q}$. Then, via the McKay correspondence, λ naturally gives rise to an element of $Z(\Bbbk G)$ which we also call λ (we make this correspondence more precise in Chapter 2). We then define $\Bbbk$-algebras

$$\mathcal{S}^\lambda(\widetilde{Q}) := \frac{\Bbbk\langle u, v\rangle \# G}{\langle vu - uv - \lambda\rangle} \quad \text{and} \quad \mathcal{O}^\lambda(\widetilde{Q}) := e\mathcal{S}^\lambda(\widetilde{Q})e.$$

We frequently write $\mathcal{S}^\lambda$ and $\mathcal{O}^\lambda$ when the precise choice of $\widetilde{Q}$ is unimportant.

After noting that $e(R\#G)e \cong R^G$, it is clear that if the central element λ is 0 (which happens precisely when the weight λ is $\mathbf{0}$) then $\mathcal{S}^\lambda = R\#G$ and $\mathcal{O}^\lambda \cong R^G$. Moreover, observe that we can filter $\mathcal{S}^\lambda$ by putting u and v in degree 1 and elements of G in degree 0, and that this also restricts to a filtration of $\mathcal{O}^\lambda$. We can then consider the associated graded rings $\operatorname{gr}\mathcal{S}^\lambda$ and $\operatorname{gr}\mathcal{O}^\lambda$, for which we have the following result:

Lemma 1.2.2 ([CBH98, Lemma 1.1])**.** *We have* $\operatorname{gr}\mathcal{S}^\lambda \cong R\#G$ *and* $\operatorname{gr}\mathcal{O}^\lambda \cong R^G$*.*

This means that the algebras $\mathcal{S}^\lambda$ and $\mathcal{O}^\lambda$ are PBW deformations of $R\#G$ and R^G. This result allows one to show that these deformations have nice ring-theoretic and homological properties, which we now recall. Any undefined terms will be defined in Chapter 2.

Proposition 1.2.3 ([CBH98, Lemma 1.2, Lemma 1.3])**.** *Both* $\mathcal{S}^\lambda$ *and* $\mathcal{O}^\lambda$ *are finitely generated noetherian* $\Bbbk$*-algebras of GK dimension 2 and are maximal orders. Moreover,* $\mathcal{S}^\lambda$ *is a prime ring while* $\mathcal{O}^\lambda$ *is a domain.*

While $\mathcal{S}^\lambda$ is always noncommutative, the deformations $\mathcal{O}^\lambda$ can be commutative. Recall that the McKay quiver $\widetilde{Q}$ corresponding to G is defined using the irreducible representations W_i of G. If we write $\delta_i = \dim_\Bbbk W_i$ and $\delta = (\delta_i)_{i\in\widetilde{Q}_0}$, then we can easily detect when $\mathcal{O}^\lambda$ is commutative:

Theorem 1.2.4 ([CBH98, Theorem 0.4])**.** *The algebra* $\mathcal{O}^\lambda$ *is commutative if and only if* $\lambda \cdot \delta := \sum_{i=0}^n \lambda_i\delta_i = 0$*.*

Crawley-Boevey–Holland also determined the global dimensions of the algebras $\mathcal{O}^\lambda$, and showed that the precise value depends on the weight λ, see [CBH98, Theorem 0.4]. In particular, for generic choices of λ, $\mathcal{O}^\lambda$ has finite global dimension. Of particular interest to us are the cases when $\mathcal{O}^\lambda$ has infinite global dimension, because we can then view $\mathcal{O}^\lambda$ as the coordinate ring of a (possibly noncommutative) surface which is singular. Unfortunately, it is difficult to give a concise statement of their result, but in Chapter 3 we will see that one can restrict attention to weights which have a particular form, and in this case it straightforward to detect when $\mathcal{O}^\lambda$ has infinite global dimension.

The main result of Chapter 3 gives a description of the singularities of the rings $\mathcal{O}^\lambda$ using singularity categories, which we now briefly define. The singularity category of a (possibly noncommutative) ring R is defined to be the Verdier quotient category

$$\mathcal{D}_{\mathrm{sg}}(R) := \frac{\mathcal{D}^{\mathrm{b}}(\text{mod-}R)}{\operatorname{Perf}(R)},$$

where $\operatorname{Perf}(R)$ is the full subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}R)$ consisting of perfect complexes. We are now able to state the main result of Chapter 3, where we write R_Q for the Kleinian singularity with corresponding Dynkin quiver Q:

Theorem 1.2.5 (Theorem 3.3.11)**.** *Let* $\widetilde{Q}$ *be an extended Dynkin quiver with vertex set* $\{0, 1, \ldots, n\}$*, where* 0 *is an extending vertex, and write* Q *for the full subquiver obtained* *by deleting vertex* 0. *Let* λ *be a weight for* $\widetilde{Q}$. *Then there exists a subset* $J = J(\lambda)$ *of* $\{1, \ldots, n\}$ *such that, if* $Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ *is the full subquiver of* Q *obtained by deleting the vertices in* J*, so that the* $Q^{(i)}$ *are connected and therefore necessarily Dynkin, there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^{\lambda}(\widetilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathcal{D}_{\text{sg}}(R_{Q^{(i)}}).$$

The above result coincides with the intuition coming from commutative singularity theory which says that deforming a singularity should make it no worse; in the present context, λ is some deformation parameter, and the theorem says that the corresponding deformation becomes less singular in a very precise sense. In particular, this theorem says that the singularities of $\mathcal{O}^{\lambda}$ are a union of commutative Kleinian singularities. The following is a corollary of Theorem 1.2.5 which identifies the "most singular" noncommutative deformation of R_Q.

Corollary 1.2.6. *Let* $\widetilde{Q}$ *be an extended Dynkin quiver with extending vertex* 0*, and write* Q *for the Dynkin subquiver obtained by deleting vertex* 0. *Consider the weight* $\lambda = (1, 0, 0, \ldots, 0) \in \Bbbk^{n+1}$. *Then the algebra* $\mathcal{O}^{\lambda}(\widetilde{Q})$ *is noncommutative, and there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^{\lambda}(\widetilde{Q})) \simeq \mathcal{D}_{\text{sg}}(R_Q).$$

It is therefore sensible to consider the algebra $\mathcal{O}^{\lambda}(\widetilde{Q})$ with $\lambda = (1, 0, \ldots, 0)$ as a noncommutative analogue of the algebra R_Q. This point of view is further supported by another one of our results which may be viewed as a noncommutative version of the geometric McKay correspondence of the previous section, and which we now explain.

Recall that in the geometric McKay correspondence we have the affine surface quotient singularity $X = \operatorname{Spec} R_Q = \operatorname{Spec} \Bbbk[u, v]^G$, the minimal resolution $\pi : \widetilde{X} \to X$, and the exceptional divisor $\pi^{-1}(0)$ consisting of irreducible curves γ_i. We call the matrix which records the intersections of the γ_i the *intersection matrix* of the resolution; Theorem 1.1.7 and the comments preceding it show that it is equal to $A - 2I$, where A is the adjacency matrix of Q. The matrix $A - 2I$ is well-known: it is -1 times the Cartan matrix C corresponding to Q.

To define a noncommutative version of the geometric McKay correspondence, we therefore need ring-theoretic analogues of a resolution of singularities, (curves in) an exceptional divisor, and intersection multiplicities. In general, if R is some singular noncommutative ring, we call a $\Bbbk$-algebra S a *noncommutative resolution* of R if $S = \operatorname{End}_R(M)$ for some reflexive generator M and $\operatorname{gl.dim} S < \infty$. The "exceptional objects" in this resolution, which are analogues of irreducible curves in the exceptional divisor, are finite-dimensional simple S-modules. If R is simple, this analogy is more precise because in this case R has no finite-dimensional representations, and so finite-dimensional representations of a resolution are "exceptional". Given two such modules M and N, following [MS01] we define their *intersection multiplicity* to be

$$M \bullet N \coloneqq \sum_{\ell \geqslant 0} (-1)^{\ell+1} \dim_{\Bbbk} \operatorname{Ext}_S^{\ell}(M, N),$$

provided that all terms in this sum are finite. We then have the following noncommutative version of the geometric McKay correspondence:

Theorem 1.2.7 (Theorem 3.5.11). *Let $\widetilde{Q}$ be an extended Dynkin quiver, Q the corresponding Dynkin quiver, and $\lambda = (1, 0, 0, \dots, 0)$. Then $\mathcal{O}^\lambda(\widetilde{Q})$ has a noncommutative resolution of the form $\mathcal{O}^\mu(\widetilde{Q})$ for some weight μ, and the exceptional objects in this resolution may be indexed so that the corresponding intersection matrix is $-C$, where C is the Cartan matrix corresponding to Q.*

1.3 Quantum Kleinian singularities

The majority of the work contained in Chapters 6, 7, and 8 relate to the so-called *quantum Kleinian singularities* of Chan–Kirkman–Walton–Zhang, [CKWZ16a]. Recall that Crawley-Boevey and Holland constructed noncommutative surfaces by deforming commutative Kleinian singularities $\Bbbk[u,v]^G$. On the other hand, in [CKWZ16a] the authors were able to construct new examples of noncommutative surfaces by replacing the commutative polynomial ring $\Bbbk[u,v]$ by a two-dimensional Artin-Schelter regular algebra A, and by replacing the finite subgroup of $\mathrm{SL}(2,\Bbbk)$ by a finite subgroup G of $\mathrm{GL}(2,\Bbbk)$ acting on A with *trivial homological determinant*. We will define all of these terms in Chapter 2. In [CKWZ14] the authors classified all such pairs (A, G), which we provide in Table 1.2. We remark that the classification achieved in [CKWZ14] also allowed for a Hopf algebra to act on an AS regular algebra A, but for the purposes of this thesis these cases are unimportant.

The complete classification is as follows, where for each case there is a corresponding quiver $\widetilde{Q}$, the relevance of which we will explain shortly:

Case	A	G	$\widetilde{Q}$
(0)	$\Bbbk[u,v]$	$G \underset{\text{fin}}{\leqslant} \mathrm{SL}(2,\Bbbk)$	$\widetilde{\mathbb{A}}\widetilde{\mathbb{D}}\widetilde{\mathbb{E}}$
(i)	$\Bbbk_q[u,v]$	C_n	$\widetilde{\mathbb{A}}_{n-1}$
(ii)	$\Bbbk_{-1}[u,v]$	S_2	$\widetilde{\mathbb{L}}_1$
(iii)	$\Bbbk_{-1}[u,v]$	D_n	$\begin{cases} \widetilde{\mathbb{D}}_{\frac{n+4}{2}} & \text{if } n \text{ is even} \\ \widetilde{\mathbb{DL}}_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$
(iv)	$\Bbbk_J[u,v]$	C_2	$\widetilde{\mathbb{A}}_1$

Table 1.2: The pairs (A, G) for quantum Kleinian singularities A^G.

We briefly explain the notation and how the groups act. The algebras $\Bbbk_q[u,v]$ (where $q \in \Bbbk^\times$) and $\Bbbk_J[u,v]$ are, respectively, the *quantum plane* and *Jordan plane*, and have respective presentations

$$\Bbbk_q[u,v] = \frac{\Bbbk\langle u,v\rangle}{\langle vu - quv\rangle} \quad \text{and} \quad \Bbbk_J[u,v] = \frac{\Bbbk\langle u,v\rangle}{\langle vu - uv - u^2\rangle}.$$

These algebras are both noetherian domains of global dimension 2, and may be thought of as noncommutative analogues of $\Bbbk[u,v]$.

The groups C_n, S_2, and D_n are, respectively, the cyclic group of order n, the symmetric group on two letters, and the dihedral group of order $2n$ (obviously $S_2 \cong C_2$, but our choice of notation will become clear soon). We will frequently make use of the abstract presentations

$$C_n = \langle g \mid g^n \rangle, \quad S_2 = \langle h \mid h^2 \rangle \quad \text{and} \quad D_n = \langle g, h \mid g^n, h^2, (hg)^2 \rangle.$$

Given a quantum Kleinian singularity A^G, g will denote an element of order n (where n should be determined from context) and h will denote an element of order 2. These elements will act on $u, v \in A$ via

$$g \cdot u = \omega u, \quad g \cdot v = \omega^{-1} v, \quad h \cdot u = v, \quad h \cdot v = u,$$

where ω is a primitive nth root of unity. One can verify that these give rise to well-defined actions of the groups from Table 1.2 on the corresponding algebras.

From Table 1.2, we notice that the classification includes the case of finite subgroups of $\mathrm{SL}(2, \Bbbk)$ acting on a polynomial ring in two variables. We will often refer to case (0) as the classical case, and the remaining cases as the quantum cases. For case (0), we refer to the rings A^G as classical Kleinian singularities, and following [CKWZ16a], for the remaining cases we call the rings A^G *quantum Kleinian singularities*. Henceforth, when we say that A^G is a quantum Kleinian singularity, we mean that the pair (A, G) is a pair from cases (i)-(iv) of Table 1.2.

Quantum Kleinian singularities have been shown to have similar ring-theoretic and representation-theoretic properties to classical Kleinian singularities, providing evidence that they are sensible generalisations to a noncommutative setting. For example, we have the following three results, which are generalisations of the observation after (1.1.1), and of Theorems 1.1.3 and 1.1.4.

Theorem 1.3.1 ([CKWZ16a, Theorem 5.2])**.** *Let A^G be a quantum Kleinian singularity. Then A^G is isomorphic to $B/\Omega B$, where B is an AS regular algebra of dimension 3 and $\Omega \in B$ is a normal nonzerodivisor.*

Theorem 1.3.2 ([CKWZ16a, Theorem 4.1])**.** *Let A^G be a quantum Kleinian singularity. Then the ring homomorphism*

$$\phi : A \# G \to \mathrm{End}_{A^G}(A), \quad \phi(ag)(b) = ag \cdot b$$

is an isomorphism.

In the following, an *initial* module is a graded module which is generated in degree 0 and satisfies $M_{<0} = 0$.

Theorem 1.3.3 ([CKWZ16b, Theorem A, Theorem C])**.** *Let A^G be a quantum Kleinian singularity. Then there are one-to-one correspondences between:*

- *irreducible G-modules;*
- *indecomposable direct summands of A as an A^G-module;*
- *indecomposable finitely generated initial projective $A \# G$-modules;*
- *indecomposable finitely generated initial projective $\mathrm{End}_{A^G}(A)$-modules; and*
- *indecomposable maximal Cohen-Macaulay graded A^G-modules, up to a degree shift.*

To each quantum Kleinian singularity we can associate a graph by constructing the McKay quiver and then replacing each pair of opposed arrows by an edge. The resulting graphs are examples of *Euclidean diagrams* and can be found in Table 1.2. In addition to extended Dynkin graphs this process also yields some new graphs, which are said to have types $\widetilde{\mathbb{L}}_1$ and $\widetilde{\mathbb{DL}}_n$; we provide them in Figure 1.3.

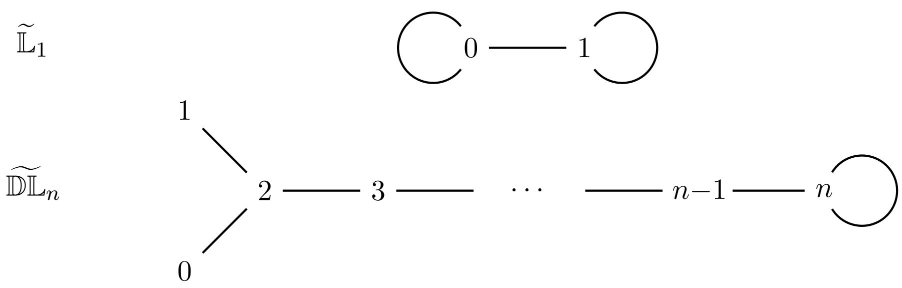


Figure 1.3: Euclidean diagrams occurring in the McKay correspondence for quantum Kleinian singularities.

It is then natural to ask whether an analogue of Theorem 1.1.6 holds in this setting; in Chapter 7, we establish a result that has the following as a special case.

Theorem 1.3.4 (Theorem 7.2.1)**.** *Let A^G be a quantum Kleinian singularity and let $\widetilde{Q}$ be the corresponding Euclidean diagram. Then there exists a path algebra with relations $\Pi_*(\widetilde{Q}) = \Bbbk\overline{\widetilde{Q}}/I$, called a* quantum preprojective algebra*, such that $A\#G$ is Morita equivalent to $\Pi_*(\widetilde{Q})$ and A^G is isomorphic to $e_0\Pi_*(\widetilde{Q})e_0$, where e_0 is an idempotent in $\Pi_*(\widetilde{Q})$ corresponding to the vertex 0 in $\widetilde{Q}$.*

It is also sensible to explore the extent to which the results of Crawley-Boevey–Holland extend to the quantum setting, and it is this that Chapters 6, 7, and 8 are devoted to. In complete anology with Definition 2.5.2, we make the following definition:

Definition 1.3.5. Let A^G be a quantum Kleinian singularity and let $\widetilde{Q}$ be the corresponding Euclidean diagram. Then G acts naturally on the free algebra $\Bbbk\langle u, v\rangle$, and so we can form the skew group ring $\Bbbk\langle u, v\rangle\#G$. Let $e = \frac{1}{|G|}\sum_{g\in G} g$ be the average of the group elements, viewed as an element of $\Bbbk\langle u, v\rangle\#G$. For $* \in \{q, -1, J\}$ define

$$\rho_*(u,v) = \begin{cases} vu - quv & \text{if } * = q \text{ (case (i))} \\ vu + uv & \text{if } * = -1 \text{ (cases (ii) and (iii))} \\ vu - uv - u^2 & \text{if } * = J \text{ (case (iv))} \end{cases} .$$

For each vertex i of $\widetilde{Q}$, choose $\lambda_i \in \Bbbk$ and write $\lambda = (\lambda_i)_{i\in\widetilde{Q}_0}$; we call λ a *weight* for $\widetilde{Q}$. As in Definition 2.5.2, the weight λ naturally gives rise to an element of $Z(\Bbbk G)$ which we also call λ. We then define $\Bbbk$-algebras

$$\mathcal{S}_*^\lambda(\widetilde{Q}) \coloneqq \frac{\Bbbk\langle u,v\rangle\#G}{\langle\rho_*(u,v)-\lambda\rangle} \quad \text{and} \quad \mathcal{O}_*^\lambda(\widetilde{Q}) \coloneqq e\mathcal{S}_*^\lambda(\widetilde{Q})e.$$

We frequently write $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ when the precise choice of $\widetilde{Q}$ is unimportant.

Lemma 1.2.2 and Proposition 1.2.3 then immediately generalise to these deformations:

Lemma 1.3.6 (Lemma 6.1.2)**.** *There exists a filtration of $\mathcal{S}_*^\lambda$ (and hence $\mathcal{O}_*^\lambda$) such that $\operatorname{gr}\mathcal{S}_*^\lambda \cong A\#G$ and $\operatorname{gr}\mathcal{O}_*^\lambda \cong A^G$.*

Proposition 1.3.7 (Lemma 6.1.3)**.** *Both $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ are finitely generated noetherian $\Bbbk$-algebras of GK dimension 2. Moreover, $\mathcal{S}_*^\lambda$ is a prime ring while $\mathcal{O}_*^\lambda$ is a domain.*

In Theorem 1.2.4, we saw that the analagous deformations of classical Kleinian singularities could be commutative, and also remarked that they generically have finite global dimension. Moreover, this behaviour was uniform across all Dynkin types. In contrast, whether or not similar properties hold for deformations of quantum Kleinian singularities depends on the particular case, although the most interesting behaviour occurs in cases (ii) and (iii):

Proposition 1.3.8 (Lemma 6.1.7, Theorem 8.5.1, Theorem 8.6.3)**.** *Suppose that* $\mathcal{O}^\lambda(\widetilde{Q})$ *is a deformation of a quantum Kleinian singularity in case* (ii) *or case* (iii) *(when n is odd). Then* $\mathcal{O}^\lambda(\widetilde{Q})$ *is always noncommutative and is always singular.*

The singularity theory of deformations of quantum Kleinian singularities also admits a less uniform description than in the classical case considered by Crawley-Boevey–Holland. Similar results to Theorem 1.2.5 can be found in Chapter 8, although they are less precise.

1.4 Other results

1.4.1 Azumaya skew group algebras

In Chapter 5, we digress to prove some general results which determine when a skew group ring (or more generally, a crossed product) is an Azumaya algebra (undefined terms will be defined in Chapter 2).

Theorem 1.4.1 (Theorem 5.1.3)**.** *Consider a crossed product* $T := A * G$*, where* A *is a prime noetherian* $\Bbbk$*-algebra and* G *is a finite group acting* X*-outer on* A *by* $\Bbbk$*-linear automorphisms. Then* T *is prime noetherian. Moreover,* T *is Azumaya if and only if*

(1) *A is Azumaya; and*
(2) *G acts freely on $Z(A)$; that is, the stabiliser of every maximal ideal of $Z(A)$ is trivial.*

If T is Azumaya, then the ranks of A and T satisfy $\operatorname{rank} T = |G|^2 \operatorname{rank} A$.

If A is commutative, the above theorem can be stated more concisely as follows:

Corollary 1.4.2 (Corollary 5.1.4)**.** *Consider a crossed product* $T := A * G$*, where* A *is a commutative noetherian* $\Bbbk$*-algebra which is a domain and where* G *is a finite group acting* $\Bbbk$*-linearly on* A*. Then* T *is prime noetherian. Moreover,* T *is Azumaya if and only if* G *acts freely on* A*, and in this case,* $\operatorname{rank} T = |G|^2$.

We now outline our main motivation for proving these results. An important property of the deformations $\mathcal{S}^\lambda$ and $\mathcal{O}^\lambda$ defined by Crawley-Boevey–Holland is that they are maximal orders; see Proposition 1.2.3. To establish this, they showed that the rings $\Bbbk[u,v] \# G$ and $\Bbbk[u,v]^G$ are maximal orders using [Mar95, Theorem 3.13], which implies the result for $\mathcal{S}^\lambda$ and $\mathcal{O}^\lambda$ by [VdBVO89, Theorem 5]. In particular, this property allows one to show that a version of Auslander's Theorem holds for their deformations; that is, there is an isomorphism

$$\operatorname{End}_{\mathcal{O}^\lambda}(\mathcal{S}^\lambda e) \cong \mathcal{S}^\lambda.$$

This result is plays an important role in the proof of Theorem 1.2.5, and so it was our hope to establish a similar result for quantum Kleinian singularities.

To show that quantum Kleinian singularities are maximal orders, one can successfully adopt the approach of [CBH98, Proposition 1.4] for case (i) (when q is not a root of unity) and case (iv). However, for the remaining cases this approach fails because the hypotheses of [Mar95, Theorem 3.13] are not satisfied or are difficult to verify. Our approach is to instead show that suitable localisations of the algebras $A\#G$ are Azumaya algebras. This allows us to better understand the prime spectrum of $A\#G$ which can be used to show that (deformations of) these algebras are maximal orders:

Theorem 1.4.3 (Theorem 6.3.2, Theorem 6.4.2)**.** *The algebras $\mathcal{S}_*^\lambda(\widetilde{Q})$ and $\mathcal{O}_*^\lambda(\widetilde{Q})$ are maximal orders. Moreover, they satisfy*

$$\operatorname{End}_{\mathcal{O}_*^\lambda(\widetilde{Q})}(\mathcal{S}_*^\lambda(\widetilde{Q})e) \cong \mathcal{S}_*^\lambda(\widetilde{Q}).$$

1.4.2 Singularities of blowups of Sklyanin algebras

In Chapter 4, we study the singularities of a family of noncommutative rings that were first studied in depth in [Rog11]. These algebras arise from Sklyanin algebras, which are three-dimensional AS regular algebras which depend on a parameter $[a:b:c] \in \mathbb{P}^2$ and which have the presentation

$$S(a,b,c) = \Bbbk\langle x,y,z\rangle/\langle axy+byx+cz^2, ayz+bzy+cx^2, azx+bxz+cy^2\rangle.$$

Sklyanin algebras are some of the most interesting algebras that arise in noncommutative algebraic geometry, and may be thought of as the coordinate rings of (non-existent) noncommutative $\mathbb{P}^2$. While the polynomial ring $\Bbbk[x,y,z]$ corresponds to projective space $\mathbb{P}^2$, we think of the Skylanin algebra $S(a,b,c)$ as corresponding to "noncommutative projective space" $\mathbb{P}^2_{\text{nc}}$. In [ATVdB91], it was shown that, for generic choices of $[a:b:c]$, S is AS regular, has centre generated by a single element g of degree 3, and that the ring $S/gS \cong B(E,\mathcal{L},\sigma)$ is a *twisted homogeneous coordinate ring.* We will define this last term later; in this case, it depends on the data of an elliptic curve E, an invertible sheaf $\mathcal{L}$, and an automorphism σ of E. Rogalski used this description of S/gS to construct subalgebras of S, denoted $T(\mathbf{d})$, by "blowing up" up a divisor $\mathbf{d}$ of E of degree at most 7, and showed that these subalgebras have nice ring-theoretic and homological properties.

Of particular interest are the algebras $A(\mathbf{d}) = (T(\mathbf{d})[g^{-1}])_0$ obtained by localising $T(\mathbf{d})$ at powers of g and considering only the degree 0 part. These are noetherian domains of GK dimension 2 and have been studied in [Rog11, RSS14, RSS15b, RSS17]. Interestingly, when $\mathbf{d}$ has degree 2 these algebras have similar properties to noncommutative deformations of an $\mathbb{A}_1$ singularity, and the algebra $A(\mathbf{d})$ can be thought of as a deformation of a (commutative) affine del Pezzo surface with an $\mathbb{A}_1$ singularity. In Chapter 4, we give the following precise description of the singularities of $A(2p)$:

Theorem 1.4.4 (Theorem 4.5.9)**.** *The algebra $A(2p)$ has an $\mathbb{A}_1$ singularity. That is, if we write $R_{\mathbb{A}_1}$ for the coordinate ring of an $\mathbb{A}_1$ singularity, then there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(A(2p)) \simeq \mathcal{D}_{\text{sg}}(R_{\mathbb{A}_1}).$$

This result, and the techniques used to prove it, has an application to the birational geometry of noncommutative projective surfaces; see [RSS17].

1.5 Organisation of this thesis

We briefly summarise the organisation of this thesis. Chapter 2 contains background material, and we recall a number of definitions and results that we will use throughout this thesis; additionally, we fix some notation. Chapter 3 is concerned with determining the singularity categories of the deformations $\mathcal{O}^\lambda$ of Crawley-Boevey–Holland. In Chapter 4 we consider a family of noncommutative surfaces which have previously been studied by Rogalski–Sierra–Stafford. We prove results relating to the singularities of these algebras, which includes a proof of their global dimensions (a result which was known to the aforementioned authors but does not appear in the literature) as well as the singularity category of an example which has infinite global dimension.

In Chapter 5 we make a slight detour to prove some results on when a skew group algebra is Azumaya, which are interesting results in their own right, although our main motivation is so that we can use them later in this thesis. In Chapter 6, we show how to adapt the definitions of Crawley-Boevey–Holland to define deformations of quantum Kleinian singularities. We also work out their first properties, with an emphasis on showing that they are maximal orders, which makes use of the results in Chapter 5.

The main result of Chapter 7 is Theorem 1.3.4, which shows that another result of Crawley-Boevey–Holland extends to deformations of quantum Kleinian singularities. We use this result in Chapter 8 to determine the global dimensions of our deformations, and to give a description of their singularity categories.

Chapter 2

Background

In this chapter we present the background material that is required in this thesis. We mainly recall important definitions and results that we will wish to refer to on a number of occasions later on. We will also fix our notation.

2.1 Notations and assumptions

Throughout this thesis, we write $\Bbbk$ for an algebraically closed field of characteristic 0. When we speak of a noetherian ring, we mean a ring which is both right and left noetherian. Given a ring R, we write Mod-R (respectively, mod-R) for the category of right R-modules (respectively, finitely generated right R-modules) and R-Mod (R-mod) for the category of (finitely generated) left R-modules. In general, a lower case leading letter means that we are considering a module category whose modules are finitely generated. If R is graded, we write gr-R (respectively, R-gr) for the category of finitely generated graded right (respectively, left) R-modules; we write $\operatorname{Hom}_{\text{gr-}R}(-,-)$ for the Hom spaces in this category. The rings we consider are almost always noetherian, so mod-R and gr-R are abelian categories. When there is a possibility for confusion, we write M_R (respectively, ${}_RN$) to emphasise that a module is a right (respectively, left) R-module. We will usually work with right modules. Given $M \in$ mod-R, we write $M^* := \operatorname{Hom}_R(M,R)$ for the *dual* of M, which is an $(R, \operatorname{End}_R(M))$-bimodule. We write $\operatorname{p.dim} M$ and $\operatorname{i.dim} M$ for the projective and injective dimensions of a module M, respectively. The centre of a ring R is denoted $Z(R)$.

2.2 Singularity categories and maximal Cohen-Macaulay modules

The principal objects of study in this thesis are certain rings which we refer to as *singular noncommutative surfaces*. Intuitively, a singular noncommutative surface is a noncommutative ring S which can be thought of as a noncommutative analogue of the coordinate ring of a singular affine (commutative) surface. To make this more precise, we recall some definitions.

Definition 2.2.1. The *Gelfand-Kirillov dimension* (or GK dimension) of a finitely generated $\Bbbk$-algebra A is

$$\operatorname{GKdim} A := \limsup_{n\to\infty} \log_n(\dim_{\Bbbk} V^n),$$

where V is any finite-dimensional $\Bbbk$-subspace of A which generates A as an algebra and where $1 \in V$. This definition does not depend on the choice of V.

We first note that if A is finite-dimensional then $\operatorname{GKdim} A = 0$. The GK dimension is meant to give a measure of the rate of growth of an algebra; in particular, the polynomial ring $\Bbbk[x_1, \dots, x_n]$ has GK dimension n, and so if $\operatorname{GKdim} A = n$ then we think of A as "growing like a polynomial ring". When A is commutative and finitely generated, the GK dimension and Krull dimension of A coincide [KL00, Theorem 4.5]. We can also define the notion of the GK dimension of an A-module, but as we only require this in the setting of graded $\Bbbk$-algebras, we will define it later. We remark that in the graded setting, the GK dimension has an alternative characterisation which makes it easier to determine.

We also need to define what it means for a ring to be singular:

Definition 2.2.2. Let R be a noetherian ring. The *global dimension* of R is

$$\operatorname{gl.dim} R \coloneqq \sup_{M \in \text{mod-}R} \operatorname{p.dim} M.$$

We say that R is *singular* if $\operatorname{gl.dim} R = \infty$, and say it is *nonsingular* otherwise.

The noetherian hypothesis in the above is used to simplify the definition. If R is not noetherian, then there are notions of left global dimension and right global dimension, and these values need not coincide.

If R is commutative, then the variety $\operatorname{Spec} R$ is singular if and only if R has infinite global dimension. It is therefore sensible to say that a (possibly noncommutative) ring R is *singular* if it has infinite global dimension. We are now able to give the definition of a (singular) noncommutative surface.

Definition 2.2.3. A *noncommutative surface* is a noncommutative noetherian domain R with $\operatorname{GKdim} R = 2$, and it is singular when it has infinite global dimension.

Having precisely defined what we mean by a singular noncommutative surface, we now work towards defining the main tool that we use to study the singularities of these rings. We first need to define what we mean by the *(bounded) derived category* of an abelian category, the definition of which is quite involved. We give an approximate definition below, and direct the reader to [Huy06, Chapter 2] for a more thorough treatment.

Definition 2.2.4. Let $\mathcal{A}$ be an abelian category. A *chain complex* $(A_\bullet, d_\bullet)$ is a sequence of objects of $\mathcal{A}$ and morphisms

$$\dots \xrightarrow{d_{-3}} A_{-2} \xrightarrow{d_{-2}} A_{-1} \xrightarrow{d_{-1}} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \dots$$

such that the composition of consecutive morphisms is the zero map. Given a chain complex $(A_\bullet, d_\bullet)$, its *nth homology group* is defined to be

$$H_n(A_\bullet) = \frac{\ker d_n}{\operatorname{im} d_{n-1}}.$$

Given two chain complexes $(A_\bullet, d_\bullet^A)$ and $(B_\bullet, d_\bullet^B)$, a *chain map* f between them is a sequence of homomorphisms $f_n : A_n \to B_n$ such that $d_n^B \circ f_n = f_{n+1} \circ d_n^A$ for all $n \in \mathbb{Z}$;

that is, the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d^A_{n-2}} & A_{n-1} & \xrightarrow{d^A_{n-1}} & A_n & \xrightarrow{d^A_n} & A_{n+1} & \xrightarrow{d^A_{n+1}} & \dots \\
 & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\
\dots & \xrightarrow{d^B_{n-2}} & B_{n-1} & \xrightarrow{d^B_{n-1}} & B_n & \xrightarrow{d^B_n} & B_{n+1} & \xrightarrow{d^B_{n+1}} & \dots
\end{array}$$

A chain map induces maps $H_n(f) : H_n(A_\bullet) \to H_n(B_\bullet)$ on homology. If all of these maps are isomorphisms, then we call f a *quasi-isomorphism.* The *category of chain complexes* is then defined to have chain complexes as its objects, and morphisms given by chain maps.

If $(A_\bullet, d^A_\bullet)$ is a chain complex then for $k \in \mathbb{Z}$ we define a new chain complex $(A[k]_\bullet, d^{A[k]}_\bullet)$ by setting $A[k]_n = A[n+k]$ and $d^{A[k]}_n = (-1)^k d^{n+k}_A$. This gives rise to a functor on the category of chain complexes over $\mathcal{A}$.

From the category of chain complexes we can form the *derived category* of $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, by "formally inverting" all quasi-isomorphisms. The *bounded derived category* of $\mathcal{A}$, denoted $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$, is the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of all chain complexes $A_\bullet$ with $H_n(A_\bullet) = 0$ for all $|n| \gg 0$.

The (bounded) derived category is an example of a *triangulated category.* By this, we mean an additive category $\mathcal{T}$ equipped with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, called the *translation functor*, and a set of *distinguished triangles.* A distinguished triangle is a sequence of objects and morphisms

$$X \to Y \to Z \to \Sigma X$$

satisfying a number of axioms. The reader is directed to [Huy06, Chapter 1] for a precise definition. The translation functor in $\mathcal{D}(\mathcal{A})$ is given by the shift of complexes functor [1]. A *triangulated functor* $F : \mathcal{T} \to \mathcal{T}'$ is an additive functor between triangulated categories which commutes with the translation functors in $\mathcal{T}$ and $\mathcal{T}'$, and which maps distinguished triangles to distinguished triangles.

We can now define the singularity category of a ring:

Definition 2.2.5. Let R be a noetherian ring. A complex of R-modules is said to be *perfect* if it quasi-isomorphic to a bounded complex of projective modules. Write $\mathrm{Perf}(R)$ for the full subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}R)$ consisting of perfect complexes. The *singularity category* of R is the Verdier quotient category (see [Kra08, Section 4] for details)

$$\mathcal{D}_{\mathrm{sg}}(R) := \frac{\mathcal{D}^{\mathrm{b}}(\text{mod-}R)}{\mathrm{Perf}(R)}.$$

By construction, this category possesses the structure of a triangulated category.

From this definition, it follows relatively quickly that R is nonsingular if and only if the singularity category of R is trivial (in the sense that it has only a zero object). This follows from the fact that a module M has finite projective dimension if and only if the complex with M concentrated in degree 0 is perfect, and perfect modules are isomorphic to the zero object in $\mathcal{D}_{\mathrm{sg}}(R)$. In general, when the singularity category is nontrivial it should be thought of as providing a measure of "how singular" R is. For example, when R has the mildest possible singularity, called an $\mathbb{A}_1$ singularity, then $\mathcal{D}_{\mathrm{sg}}(R)$ is equivalent to $\mathbf{FVect}_{\Bbbk}$, the category of finite-dimensional vector spaces over $\Bbbk$, and hence has only one indecomposable object. We will see a few examples of noncommutative rings which have an $\mathbb{A}_1$ singularity, in the sense that their singularity categories are also equivalent to $\mathbf{FVect}_{\Bbbk}$.

While the singularity category has a suggestive name (and it is for this reason we use it in the introduction), under relatively mild assumptions this category is triangle equivalent to another category which is much easier to work with. We now recall the definitions which are required to state this result.

Definition 2.2.6. Given R-modules M and N, write $\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/{\sim}$, where $f \sim f'$ if and only if $f - f'$ factors through a finitely generated projective module. The *stable module category* of R, denoted $\underline{\mathrm{mod}}$-R, is then the category whose objects are the same as those of mod-R, and for modules M, N, has morphisms $\underline{\mathrm{Hom}}_R(M,N)$. Given a full subcategory abc-R of mod-R, we write $\underline{\mathrm{abc}}$-R for the full subcategory of $\underline{\mathrm{mod}}$-R whose objects are the same as those of abc-R.

Noting that an element of $\sum_{i=1}^{k} n_i \otimes f_i$ of $N \otimes_R M^*$ gives rise to a homomorphism $M \to N$ via $m \mapsto \sum_{i=1}^{k} n_i f_i(m)$, it is not hard to show that a module homomorphism $f : M \to N$ factors through a projective module if and only if f is in the image of $N \otimes_R M^*$. This allows us to identify $\underline{\mathrm{Hom}}_R(M,N)$ with $\mathrm{Hom}_R(M,N)/(N \otimes_R M^*)$, which will be useful in later calculations. In this thesis, N and M^* will frequently sit inside the Goldie quotient ring $Q(R)$ (defined in the next subsection) of a prime noetherian ring R, in which case we can identify $N \otimes_R M^*$ with NM^*.

In the stable module category, we have a weaker notion of an isomorphism than in the usual module category. Indeed, [AB69, Proposition 1.44] shows that two R-modules M, N are isomorphic in $\underline{\mathrm{mod}}$-R if and only if there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$ in mod-R.

The *first syzygy* ΩM of $M \in$ mod-R is defined to be the kernel of any surjection $R^n \twoheadrightarrow M$. The observation in the previous paragraph combined with [Rot08, Proposition 8.5] implies that ΩM is uniquely determined in $\underline{\mathrm{mod}}$-R. Moreover, if $f : M \to N$ is any homomorphism, then we can form a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & R^m & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega N & \longrightarrow & R^n & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where g exists since R^m is projective, and h exists by diagram chasing. The map $h : \Omega M \to \Omega N$ depends on the choice of g, but one can show that if $g' : R^m \to R^n$ is any other such choice, which then uniquely determines a map $h' : \Omega M \to \Omega N$, then the map $h - h'$ factors through a projective module. Therefore this map h is uniquely determined in $\underline{\mathrm{mod}}$-R, and so we obtain a functor $\Omega : \underline{\mathrm{mod}}\text{-}R \to \underline{\mathrm{mod}}\text{-}R$.

We need a few more definitions:

Definition 2.2.7. A ring R is said to be *Gorenstein* if it is noetherian and both $\mathrm{i.dim}\, R_R$ and $\mathrm{i.dim}\, {}_R R$ are finite. By [Zak69, Lemma A], under these hypotheses the values $\mathrm{i.dim}\, R_R$ and $\mathrm{i.dim}\, {}_R R$ coincide, and we call this common value the *(injective) dimension* of R.

Definition 2.2.8. Suppose that R is Gorenstein. A finitely generated R-module M is said to be *maximal Cohen-Macaulay* (MCM) if it satisfies $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \geqslant 1$. We write MCM-R for the full subcategory of mod-R consisting of maximal Cohen-Macaulay R-modules.

For commutative local rings, the above definition coincides with the usual (commutative) definition of maximal Cohen-Macaulay modules in terms of depth [Buc86, Section 4.2]. Maximal Cohen-Macaulay modules have the following elementary properties, proofs of which can be found in [Buc86]:

Lemma 2.2.9.
(1) *Any finitely generated projective module is MCM.*
(2) *MCM modules are reflexive.*
(3) *Finite direct sums and direct summands of MCM modules are MCM.*
(4) *An MCM module is either projective or has infinite projective dimension.*

We are now able to state a theorem which identifies a category which is triangle equivalent to the singularity category in the case of a Gorenstein ring R:

Theorem 2.2.10 ([Buc86, Theorem 4.4.1])**.** *Suppose that R is Gorenstein. Then the full subcategory $\underline{\mathrm{MCM}}$-R of $\underline{\mathrm{mod}}$-R whose objects are MCM R-modules is a triangulated category, with translation functor Σ given by $\Sigma M = \Omega^{-1}M$. Moreover, there is a triangle equivalence $\mathcal{D}_{\mathrm{sg}}(R) \simeq \underline{\mathrm{MCM}}$-$R$.*

Every example that we consider in this thesis satisfies the hypotheses of this theorem, and so we instead focus our attention on determining $\underline{\mathrm{MCM}}$-R.

Theorem 2.2.10 is a specific example of a more general result due to Happel. An *exact category* $\mathcal{C}$ is an additive category possessing a class of *conflations* (sometimes called exact sequences) which are triples of objects connected by arrows $X \to Y \to Z$, and which satisfy a number of axioms; see [Che12, Section 2] for more details. An exact category $\mathcal{C}$ is said to be *Frobenius* provided that it has enough projectives and enough injectives, and the class of projective objects coincides with the class of injective objects. Given a Frobenius category $\mathcal{C}$, we may form its stable category $\underline{\mathcal{C}}$ in the same way we formed the stable category $\underline{\mathrm{MCM}}$-R. Then [Hap88] shows that this category is triangulated, and if $X \to Y \to Z$ is a conflation in $\mathcal{C}$ then there exists a triangle of the form $X \to Y \to Z \to \Sigma X$ in $\underline{\mathcal{C}}$. If $\mathcal{T}$ is a triangulated category which is triangle equivalent to the stable category of a Frobenius category, then we say that $\mathcal{T}$ is *algebraic.*

If R is a Gorenstein ring, then MCM-R is Frobenius and so Happel's result implies that $\underline{\mathrm{MCM}}$-R is triangulated; this triangulated structure is precisely the one given in Theorem 2.2.10. In MCM-R, every conflation $X \to Y \to Z$ arises from a short exact sequence $0 \to X \to Y \to Z \to 0$ of MCM R-modules.

Finally, we recall two useful results that will be helpful when identifying the maximal Cohen-Macaulay modules of a ring. Given an additive category $\mathcal{C}$ and an object $C \in \mathcal{C}$, we write $\mathrm{add}(C)$ for the full subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sums of C. This is the smallest additive subcategory of $\mathcal{C}$ which contains C and is closed under taking direct summands. The following result is due to Auslander, but we provide a proof.

Proposition 2.2.11 (Auslander)**.** *Suppose that R is Gorenstein and that $M \in$ MCM-R is a generator (for example, this occurs if M has R as a direct summand or if R is simple). If $\operatorname{gl.dim}\operatorname{End}_R(M) \leqslant 2$, then $\operatorname{add} M =$ MCM-R. Moreover, if R is of injective dimension at most 2, then the converse also holds.*

Proof. Write $\Lambda = \operatorname{End}_R(M)$. Since mod-$R$ has split idempotents (a fact which holds for any ring R), [Kra15, Proposition 2.3] implies that the functor $\operatorname{Hom}_R(M, -) : \text{mod-}R \to$ mod-Λ restricts to an equivalence

$$\operatorname{add} M \xrightarrow{\simeq} \text{proj-}\Lambda. \tag{2.2.12}$$

We also note that since M is a generator, $R^n \in \operatorname{add} M$ for any $n \geqslant 1$.

($\Rightarrow$) Assume that $\operatorname{gl.dim} \Lambda \leqslant 2$. That $\operatorname{add} M \subseteq \text{MCM-}R$ is clear, so suppose that $N \in \text{MCM-}R$. Since R is noetherian, N^* is finitely presented, so we have an exact sequence of left R-modules of the form

$$R^m \to R^n \to N^* \to 0.$$

Applying $\operatorname{Hom}_R(-, R)$ and noting that N is MCM and therefore reflexive, we obtain an exact sequence

$$0 \to N \to R^n \to R^m.$$

Applying $\operatorname{Hom}_R(M, -)$ then gives an exact sequence

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, R^n) \xrightarrow{\theta} \operatorname{Hom}_R(M, R^m) \to \operatorname{coker} \theta \to 0,$$

where, since M is a generator, $\operatorname{Hom}_R(M, R^n)$ and $\operatorname{Hom}_R(M, R^m)$ are both projective Λ-modules by (2.2.12). Since $\operatorname{gl.dim} \Lambda \leqslant 2$ we have $\operatorname{p.dim} \operatorname{coker} \theta \leqslant 2$, and therefore $\operatorname{Hom}_R(M, N)$ is also a projective Λ-module. By (2.2.12), it follows that $N \in \operatorname{add} M$.

($\Leftarrow$) Now assume that $\operatorname{add} M = \text{MCM-}R$ and that $\operatorname{i.dim} R \leqslant 2$. Let $N \in \text{mod-}\Lambda$, and consider the initial terms in a projective resolution of N,

$$P_1 \xrightarrow{f} P_0 \to N \to 0.$$

By (2.2.12), there exists a morphism $g : M_1 \to M_0$ in $\operatorname{add} M = \text{MCM-}R$ with

$$(f : P_1 \to P_0) = (g \circ - : \operatorname{Hom}_R(M, M_1) \to \operatorname{Hom}_R(M, M_0)).$$

Set $K = \ker g$. We have two short exact sequences

$$0 \to K \to M_1 \to \operatorname{im} g \to 0,$$
$$0 \to \operatorname{im} g \to M_0 \to \operatorname{coker} g \to 0.$$

Applying $\operatorname{Hom}_R(-, R)$ to each of these gives rise to exact sequences (where here $i \geqslant 1$),

$$\operatorname{Ext}^i_R(M_1, R) \to \operatorname{Ext}^i_R(K, R) \to \operatorname{Ext}^{i+1}_R(\operatorname{im} g, R) \to \operatorname{Ext}^{i+1}_R(M_1, R),$$
$$\operatorname{Ext}^{i+1}_R(M_0, R) \to \operatorname{Ext}^{i+1}_R(\operatorname{im} g, R) \to \operatorname{Ext}^{i+2}_R(\operatorname{coker} g, R).$$

Observe that the flanking terms are all 0: indeed, $\operatorname{Ext}^{i+2}_R(\operatorname{coker} g, R)$ vanishes since $\operatorname{i.dim} R \leqslant 2$, while the other three terms vanish because M_0 and M_1 are both MCM. Therefore $\operatorname{Ext}^i_R(K, R) \cong \operatorname{Ext}^{i+1}_R(\operatorname{im} g, R) = 0$ for all $i \geqslant 1$, and so K is MCM. Since $\operatorname{add} M = \text{MCM-}R$ by assumption, we have an exact sequence

$$0 \to K \to M_1 \to M_0,$$

where each term lies in $\operatorname{add} M$. Then applying $\operatorname{Hom}_R(M, -)$ gives an exact sequence

$$0 \to \operatorname{Hom}_R(M, K) \to P_1 \to P_0 \to N \to 0,$$

where $\operatorname{Hom}_R(M, K)$ is projective by (2.2.12). Therefore $\operatorname{p.dim} N \leqslant 2$, and so $\operatorname{gl.dim} \Lambda \leqslant 2$. $\square$

When R has (injective) dimension at most 2, we also have the following:

Lemma 2.2.13. *Let R be a Gorenstein ring of injective dimension at most 2. Then $M \in \text{mod-}R$ is reflexive if and only if it is maximal Cohen-Macaulay.*

Proof. ($\Leftarrow$) This is Lemma 2.2.9 (2), and doesn't require the hypothesis on injective dimension.

($\Rightarrow$) Suppose now that M is reflexive. Since R is noetherian, M^* is finitely presented, so we have an exact sequence of the form

$$R^m \to R^n \to M^* \to 0.$$

Applying $\text{Hom}_R(-, R)$ and noting that M is reflexive yields an exact sequence

$$0 \to M \to R^n \xrightarrow{\theta} R^m \to \operatorname{coker} \theta \to 0.$$

But then, by [Rot08, Corollary 6.55], $\text{Ext}^i_R(M, R) \cong \text{Ext}^{i+2}_R(\operatorname{coker} \theta, R) = 0$ for all $i \geqslant 1$, where the last equality follows since $\text{i.dim}\, R \leqslant 2$. That is, M is maximal Cohen-Macaulay. □

2.3 General ring-theoretic and homological definitions

This section is devoted to defining technical terminology from ring theory and homological algebra which will be used freely throughout this thesis. We first discuss graded rings and graded modules, giving special attention to the case of $\Bbbk$-algebras.

Definition 2.3.1. A ring A is $\mathbb{N}$-*graded* if there exists a decomposition $A = \bigoplus_{n \in \mathbb{N}} A_n$ as abelian groups which satisfies $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. An element $a \in A$ is said to be *homogeneous* if $a \in A_n$ for some n. If A is a $\Bbbk$-algebra, then we say that A is *connected* if $A_0 = \Bbbk$.

Definition 2.3.2. A (right) module M over a graded ring A is said to be $\mathbb{Z}$-*graded* if there exists a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_n$ as abelian groups which satisfies $M_i A_j \subseteq M_{i+j}$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. Given a graded A-module M, we define $M[i]$ to be the graded module which is isomorphic to M in Mod-A, but which satisfies $M[i]_n = M_{i+n}$.

It will occasionally be useful to work with $\mathbb{Z}$-graded rings (which have a similar definition), but unless otherwise stated we will always assume that we grade rings using $\mathbb{N}$ and modules using $\mathbb{Z}$, and simply refer to them as graded rings and graded modules.

In most situations of interest to us, our rings are finitely generated $\Bbbk$-algebras and all modules are finitely generated, and in this situation we can make the following defintions:

Definition 2.3.3. A connected graded $\Bbbk$-algebra A which is a finitely generated as a $\Bbbk$-algebra is called *finitely graded*. If this is the case, then $\dim_{\Bbbk} A_n < \infty$ for all n, and if $M \in \text{gr-}A$ then $\dim_{\Bbbk} M_n < \infty$ for all $n \in \mathbb{Z}$. We then define the *Hilbert series* of M (which of course allows $M = A_A$) to be the formal Laurent series

$$\text{hilb}\, M := \sum_{n \in \mathbb{Z}} (\dim_{\Bbbk} M_n) t^n.$$

In the previous subsection we defined the GK dimension of a ring, and this definition can be extended to modules in an obvious way. We neglect to give the precise definition, but note that if A is a finitely graded $\Bbbk$-algebra and $M \in \text{gr-}A$ (which forces $M_{\leqslant -n} = 0$ for $n \gg 0$) then [KL00, Proposition 6.6] implies that

$$\operatorname{GKdim} M = \limsup_{n\to\infty} \log_n(\dim_{\Bbbk} M_{\leqslant n}).$$

In particular, this formula holds when $M = A$.

We now discuss an important technique involving graded rings which can be used to determine the properties of related rings.

Definition 2.3.4. A *filtered ring* is a ring R with a family of additive subgroups $\{F_n \mid n \in \mathbb{N}\}$ such that:

(1) $F_iF_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{N}$;
(2) $F_i \subseteq F_j$ for all $i \leqslant j$; and
(3) $\bigcup_{n\in\mathbb{N}} F_n = R$.

If R is a filtered ring, then one can construct a graded ring $S := \operatorname{gr} R$, called the *associated graded ring*, by setting $S_n = F_n/F_{n-1}$ and $S = \bigoplus_{n\in\mathbb{N}} S_n$. We now define a multiplication in S, and it is enough to do so on homogeneous elements. If $a \in F_n \backslash F_{n-1}$ then we say that a has degree n and write $\overline{a} = a + F_{n-1} \in S_n$. Given another element b of degree m, we define the product $\overline{a}\overline{b}$ to be the element $ab + F_{m+n-1} \in S_{m+n}$. It is easy to check that this gives a well-defined multiplication which makes S into a graded ring.

The construction of associated graded rings is useful because many good properties of $\operatorname{gr} R$ pass to R. For example, if there exists a filtration of R such that $\operatorname{gr} R$ is noetherian, an integral domain, or prime, then the same is true of R. More complicated ring-theoretic and homological properties are also preserved under certain hypotheses; see [Lev92, Bjö87] for example.

We now define some algebras which appear in the Auslander-McKay correspondence, and in the process fix some conventions:

Definition 2.3.5. A *quiver* Q is a directed multigraph, and we write Q_0 for the set of vertices and Q_1 for the set of arrows. We equip Q with head and tail maps $h, t : Q_1 \to Q_0$ which take an arrow to the vertices that are its head and tail respectively. A *non-trivial path* in the quiver is a sequence of arrows $p = \alpha_1\alpha_2 \ldots \alpha_\ell$ with $h(\alpha_i) = t(\alpha_{i+1})$ for $1 \leqslant i \leqslant \ell - 1$ (that is, we compose arrows from left to right), and such a path is said to have *length* ℓ. Moreover, for each vertex $i \in Q_0$ there is a *trivial path* e_i of length 0, with head and tail vertex both equal to i.

Note that it is our convention that arrows are composed from left to right. However, in Chapter 7 only, this convention is broken and we will compose arrows from right to left (the reason for doing this will become clear at the time). All quivers in this thesis are assumed to be finite, in the sense that both Q_0 and Q_1 are finite.

Definition 2.3.6. Given a field $\Bbbk$ and a quiver Q, we define the path algebra $\Bbbk Q$ of Q as follows: as a $\Bbbk$-vector space, $\Bbbk Q$ has a basis given by paths in the quiver, and we define multiplication by concatenation of paths:

$$p \cdot q = \begin{cases} pq & \text{if } h(p) = t(q) \\ 0 & \text{otherwise} \end{cases}.$$

In this algebra, the lazy paths e_i are idempotents, called *vertex idempotents*. We will frequently make use of the fact that $\Bbbk Q$ can be graded by path length.

Since all quivers are assumed to be finite, every path algebra is unital, with $\sum_{i\in Q_0} e_i$ the multiplicative identity.

Skew group rings also play an essential role in the Auslander-McKay correspondence and in this thesis. We recall their definition, as well as a generalisation.

Definition 2.3.7. Let G be a group acting (on the left) on a ring R. The *skew group ring* $R\#G$ is the free left R-module with the elements of G as a basis, with multiplication extended linearly from the rule $(rg)(sh) = r(g\cdot s)gh$ for $r, s \in R$, $g, h, \in G$, where $g \cdot s$ is the image of s under the action of g.

Crossed products are a generalisation of skew group rings, and will be used in Chapter 5.

Definition 2.3.8. Suppose that R is a ring and G is a group. A *crossed product* $R*G$ of R by G is a ring containing a copy of R and a set of units $\overline{G} = \{\overline{g} \mid g \in G\}$ which is in bijection with G, such that:

(1) $R * G$ is a free left R-module with basis $\overline{G}$, and where $\overline{e} = 1$;
(2) if $g \in G$ then $\overline{g}R = R\overline{g}$; and
(3) if $g, h \in G$ then $R\overline{g}\,\overline{h} = R\overline{gh}$.

The second condition implies that each element $\overline{g}$ induces an automorphism α_g of R via $\overline{g}r = \alpha_g(r)\overline{g}$; however, in general, the set $\{\alpha_g \mid g \in G\}$ is not a group. Despite this, it will still be convenient to say that G acts on R, and to use standard group-theoretic terminology. The third condition implies that there exists a map $\tau : G \times G \to R^\times$, where $R^\times$ denotes the set of units of R, such that $\overline{g}\,\overline{h} = \tau(g, h)\overline{gh}$. A skew group ring is the special case where $\tau(g, h) = 1$ for all $g, h \in G$.

A useful observation is that, when restricted to the centre $Z(A)$ of A, we have $\alpha_g \circ \alpha_h = \alpha_{gh}$. Indeed, if $z \in Z(A)$ then

$$\alpha_g \circ \alpha_h(z) = \overline{g}\,\overline{h}z\overline{h}^{-1}\overline{g}^{-1} = \tau(g, h)\overline{gh}z\overline{gh}^{-1}\tau(g, h)^{-1} = \tau(g, h)\alpha_{gh}(z)\tau(g, h)^{-1} = \alpha_{gh}(z),$$

where the last equality follows since $\alpha_{gh}(z)$ is central. In particular, $\{\alpha_g\big|_{Z(A)} \mid g \in G\}$ is a group, a fact which we will make use of implicitly later on.

Crossed products and skew group rings often have nice properties when the automorphisms α_g are *X-outer*. To define what we mean by this, we first recall some preliminaries on quotient rings of noncommutative rings. For a more in depth treatment, the reader is directed to [GW04, Chapter 6]. We note that a *regular* element of a ring is an element which is a (left and right) nonzerodivisor.

Definition 2.3.9. A *classical right* (respectively, *left*) *quotient ring* Q for a ring R is an overring $S \supseteq R$ such that:

(1) every regular element of R is invertible in S; and
(2) every element of S can be expressed in the form rx^{-1} (respectively, $x^{-1}r$) for some $r \in R$ and regular element $x \in R$.

The conditions which determine when this construction is possible are quite subtle, and are provided by Goldie's Theorem. A more easily stated version of this theorem is as follows:

Theorem 2.3.10. *Suppose that R is a (semi)prime noetherian ring. Then R has a (semi)simple classical right quotient ring, and this is also a classical left quotient ring. Moreover, this ring is unique up to isomorphism; we call it the* Goldie quotient ring *of R, and denote it by $Q(R)$.*

We can now define what we mean by X-inner and X-outer automorphisms:

Definition 2.3.11. An automorphism of a prime noetherian ring R is said to be *X-inner* if it is equal to conjugation by an element of $Q(R)$, and it is called *X-outer* otherwise. If a group G acts on a prime noetherian ring R, we write $G_{X\text{-inn}}$ for the subgroup of G consisting of elements which act as X-inner automorphisms. The action is said to be *X-outer* if $G_{X\text{-inn}}$ is trivial.

We remark that this definition is different to the usual definition in the literature, which requires the notion of a *symmetric Martindale ring of quotients*. However, by [Mon78, Theorem 1.4], when R is prime noetherian the usual definition is equivalent to the one given above.

Much of Chapter 6 is concerned with showing that certain skew group rings are *maximal orders*, which we now define.

Definition 2.3.12. Let R be a semiprime noetherian ring with Goldie quotient ring $Q = Q(R)$.

(1) A subring S of Q is called a *right* (respectively, *left*) *order* in Q if each element of Q can be written as st^{-1} (respectively, $t^{-1}s$) for some $s, t \in S$.

(2) R is said to be a *maximal order* if there exists no order S with $R \subsetneq S \subseteq Q(R)$ and with $aSb \subseteq R$ for some nonzero $a, b \in R$.

One motivation for the definiton of a maximal order is the following: if R is a commutative noetherian domain, then it is a maximal order if and only if it is integrally closed, see [MR01, Proposition 5.1.3].

We have the following equivalent characterisations of the property of being a maximal order. One of these characterisations appears in [Mar95] but without proof, so we provide one. We recall that for a nonzero ideal I of R, we write

$$O_\ell(I) := \{q \in Q(R) \mid qI \subseteq I\}, \qquad O_r(I) := \{q \in Q(R) \mid Iq \subseteq I\}.$$

We remark that these are equal to $\operatorname{End}_R(I_R)$ and $\operatorname{End}_R({}_RI)$, respectively.

Lemma 2.3.13 ([Mar95, Lemma 2.1])**.** *Let R be a prime Noetherian ring. Then the following are equivalent:*

(1) *R is a maximal order;*

(2) $\operatorname{End}_R(I_R) = R = \operatorname{End}_R({}_RI)$ *for all nonzero ideals I of R; and*

(3) $\operatorname{End}_R(P_R) = R = \operatorname{End}_R({}_RP)$ *for all nonzero prime ideals P of R.*

Proof. Throughout write $Q = Q(R)$. The equivalence of (1) and (2) is well-known and is true under weaker hypotheses; see [MR01, Proposition 5.1.4]. It is clear that (2) implies (3), so it remains to show the reverse implication.

Suppose that (2) does not hold. By the noetherian hypothesis, choose an ideal I maximal among those ideals satisfying $R \subsetneq \operatorname{End}_R(I) \subseteq Q$. We claim that I is prime. Seeking a contradiction, suppose this is not the case, so there exist ideals $J, K \not\subseteq I$ with $JK \subseteq I$; without loss of generality, we may assume that $J, K \supsetneq I$. Set

$H = \{r \in R \mid rK \subseteq I\}$, so that H is an ideal of R with $H \supseteq J \supsetneq I$. Note that if $h \in H$ and $q \in \mathrm{End}_R(I)$ then

$$qhK \subseteq qI \subseteq I \subseteq K, \tag{2.3.14}$$

so that $qh \in \mathrm{End}_R(K)$. By the maximality hypothesis on I, we have $\mathrm{End}_R(K) = R$, so $qh \in R$. But (2.3.14) also shows that $qhK \subseteq I$, so that $qh \in H$ by the definition of H and hence $q \in \mathrm{End}_R(H)$. Therefore H is an ideal of R with $H \supsetneq I$ and $R \subsetneq \mathrm{End}_R(I) \subseteq \mathrm{End}_R(H) \subseteq Q$, contradicting our choice of I, and so I must be prime. Thus conditions (2) and (3) are equivalent. □

Finally, we define some homological notions that will be required in this thesis. For a right R-module M, the *grade* of M is defined to be

$$j(M) = \inf\{i \mid \mathrm{Ext}^i_R(M, R) \neq 0\} \in \mathbb{N} \cup \{\infty\},$$

and the grade of a left module is defined in the same way. A right R-module M satisfies the *Auslander conditon* if for all $i \geqslant 0$ and all left submodules $N \subseteq \mathrm{Ext}^i_R(M, R)$ one has $j(N) \geqslant i$; the definition for a left R-module is symmetric. A ring is called *Auslander-Gorenstein* if it is Gorenstein and all of its left and right modules satisfy the Auslander condition. If it also has finite global dimension then it is called *Auslander regular.* If R is a Gorenstein ring of finite GK dimension, then it is said to be *(GK-)Cohen-Macaulay* (CM) if $\mathrm{GKdim}\, M + j(M) = \mathrm{GKdim}\, R$ for all finitely generated left and right modules M.

2.4 PI theory and Azumaya algebras

Azumaya algebras will be of central importance in Chapter 5, and so we give a definition in this chapter. They are very closely related to PI (polynomial identity) rings, which we also define. The theory of PI rings is rich and well-developed, and so an interested reader is directed towards [BG12, MR01] for a more comprehensive treatment.

We begin by defining what a PI ring is:

Definition 2.4.1. A ring R is said to be a *polynomial identity (PI) ring* if there exists some $f(x_1, \dots, x_n) \in \mathbb{Z}\langle x_1, x_2, \dots \rangle$ with $f(r_1, \dots, r_n) = 0$ for all $r_i \in R$, where at least one word of highest degree in the support of f has coefficient 1. In this case, we call f a *polynomial identity* of R. The *minimal degree* of a PI ring R is the smallest degree of a polynomial identity for R.

We provide a few examples to demonstrate that this definition is not too abtruse.

Examples 2.4.2.
(1) Every commutative ring is PI and of minimal degree 2: indeed, they satisfy the identity $xy - yx = 0$.
(2) The Amitsur-Levitzki Theorem asserts that if R is a nonzero commutative ring, then $M_n(R)$ is a PI ring of minimal degree $2n$.
(3) Every quotient and subring of a PI ring R is PI, and is of minimal degree at most that of R.
(4) If R is a ring which is finitely generated over a commutative subring, then R is PI.

If we additionally assume that a PI ring R is prime, then Posner's Theorem (see [BG12, I.3.13]) implies some very strong properties of R. In particular, it tells us that the minimal degree of R is even, which allows us to make the following definition:

Definition 2.4.3. Let R be a prime PI ring. We define the *PI degree* of R, written $\mathrm{PIdeg}(R)$, to be

$$\mathrm{PIdeg}(R) = \tfrac{1}{2}(\text{minimal degree of } R).$$

Observe that if P is a prime ideal of a prime ring R, then we necessarily have $\mathrm{PIdeg}(R/P) \leqslant \mathrm{PIdeg}(R)$. This observation will be important shortly.

We now define what we mean by an Azumaya algebra.

Definition 2.4.4. Let A be a ring with centre Z. There is a natural map

$$\theta : A \otimes_Z A^{\mathrm{op}} \to \mathrm{End}(A_Z), \qquad a \otimes b \mapsto (r \mapsto arb).$$

We say that A is an *Azumaya algebra* (over its centre Z) if

(1) A is a finitely generated projective Z-module; and

(2) $\theta : A \otimes_Z A^{\mathrm{op}} \to \mathrm{End}(A_Z)$ is an isomorphism.

Again, we provide some examples, the last two of which are particularly important:

Examples 2.4.5.

(1) Every commutative ring is Azumaya.

(2) More generally, if R is a commutative ring then $M_n(R)$ is an Azumaya algebra.

(3) Every Azumaya algebra A is PI: indeed, since A_Z is finitely generated, this follows from Example 2.4.2 (4). The converse is not true, but the Artin-Procesi Theorem provides necessary and sufficient conditions for this to be the case.

(4) Suppose that q is a root of unity and let $\mathcal{O}_q(\Bbbk^n) = \Bbbk_q[x_1, \ldots, x_n]$ be the quantised coordinate ring of affine n-space, which is the algebra with generators x_i subject to the relation $x_j x_i = q x_i x_j$ for $j > i$. Also write $\mathcal{O}_q((\Bbbk^\times)^n) = \Bbbk_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ for the quantum torus of rank n, obtained from $\mathcal{O}_q(\Bbbk^n)$ by inverting the x_i. Then $\mathcal{O}_q(\Bbbk^n)$ is not Azumaya while $\mathcal{O}_q((\Bbbk^\times)^n)$ is.

(5) If A is Azumaya and X is an Ore set contained in Z, then AX^{-1} is Azumaya.

One of the most important properties of Azumaya algebras, for our purposes, is that there is a strong correspondence between the ideals of A and of its centre Z:

Proposition 2.4.6 ([MR01, Proposition 13.7.9])**.** *Let A be an Azumaya algebra with centre Z. Then there is a one-to-one order-preserving correspondence*

$$\begin{array}{ccc} \{\textit{ideals of } Z\} & \longleftrightarrow & \{\textit{ideals of } A\} \\ I & \longmapsto & IA \\ J \cap Z & \longleftarrow\!\shortmid & J \end{array}$$

which preserve primeness and maximality.

By [BG12, III.1.4], if A is a prime Azumaya algebra over its centre Z, then $n^2 := \dim_{Z/\mathfrak{m}}(A/\mathfrak{m}A)$ is constant as $\mathfrak{m}$ varies over maximal ideals of Z, and this value is necessarily square. In this case, we say that A has *rank* n^2.

As observed earlier, if R is a prime PI ring then every prime ideal P satisifies $\mathrm{PIdeg}(R/P) \leqslant \mathrm{PIdeg}(R)$. If we have equality here, then P is said to be *regular*. The following theorem shows that regularity of prime ideals is closely related to the Azumaya property.

Theorem 2.4.7 (Artin-Procesi)**.** *Let R be a prime ring with centre Z, and n a positive integer. Then the following are equivalent:*

(1) *R is an Azumaya algebra of rank n^2 over Z;*
(2) *R is a PI ring of PI degree n whose prime ideals are all regular; and*
(3) *R is a PI ring of PI degree n whose maximal ideals are all regular.*

One can actually characterise regular maximal ideals when R is sufficiently nice.

Theorem 2.4.8 ([BG12, Theorem III.1.6])**.** *Suppose that R is a prime $\Bbbk$-algebra which is finitely generated as a module over its centre Z. Let d be the PI degree of R. Let M be a maximal ideal of R and let $\mathfrak{m} := M \cap Z$, which is a maximal ideal of Z. Then the following are equivalent:*
(1) *M is a regular maximal ideal of R;*
(2) *$R/M \cong M_n(\Bbbk)$; and*
(3) *$M = \mathfrak{m}R$.*

By combining these two theorems, one obtains the following criterion which we will utilise later on:

Lemma 2.4.9. *Suppose that R is a prime $\Bbbk$-algebra which is finite over its centre Z (hence PI) and of PI degree n. Then R is an Azumaya algebra of rank n^2 if and only if $R/\mathfrak{m}R \cong M_n(\Bbbk)$ for all $\mathfrak{m} \in \operatorname{MaxSpec} Z$.*

Proof. ($\Rightarrow$) First suppose that R is Azumaya of rank n^2 and consider $\mathfrak{m} \in \operatorname{MaxSpec} Z$. By Proposition 2.4.6, $\mathfrak{m}R$ is a maximal ideal of R, and therefore regular by the Artin-Procesi Theorem. By Theorem 2.4.8, it follows that $R/\mathfrak{m}R \cong M_n(\Bbbk)$.

($\Leftarrow$) Now let M be a maximal ideal of R, and set $\mathfrak{m} = M \cap Z$, which is a maximal ideal of Z. Then $M/\mathfrak{m}R$ is a proper ideal of the ring $R/\mathfrak{m}R$, which is isomorphic to $M_n(\Bbbk)$ by hypothesis. Since this is a simple ring we must have $M/\mathfrak{m}R = 0$, and so $M = \mathfrak{m}R$. By Theorem 2.4.8, M is regular, and so by the Artin-Procesi Theorem, R is an Azumaya algebra of rank n^2. □

2.5 The deformations of Crawley-Boevey–Holland

As discussed in the introduction, in [CBH98], Crawley-Boevey–Holland introduced a family of deformations of Kleinian singularities. We now provide more details of the results in [CBH98], in particular those which will be used frequently in this thesis. For completeness, we repeat the definition of their deformations from the introduction.

Let $\Bbbk[u,v]^G$ be a Kleinian singularity with corresponding extended Dynkin quiver $\widetilde{Q}$; these quivers are provided in Figure 2.1 (along with two other types of quiver), where we also provide the labellings for the vertices and arrows as used throughout this thesis. We now establish a bijection between $\Bbbk^{n+1} = \Bbbk^{\widetilde{Q}_0}$ and $Z(\Bbbk G)$. The irreducible representations $W_0, W_1, \dots, W_n$ of G correspond to the vertices of the McKay quiver $\widetilde{Q}$, where W_0 is the trivial representation. Write $\delta_i = \dim_{\Bbbk} W_i$; explicitly (using the labelling of the vertices from Figure 2.1) we have

$$\widetilde{\mathbb{A}}_n: \quad \delta = (\underbrace{1, 1, \dots, 1, 1}_{n+1 \text{ times}})$$

$$\widetilde{\mathbb{D}}_n: \quad \delta = (1, 1, \underbrace{2, 2 \dots, 2, 2}_{n-3 \text{ times}}, 1, 1)$$

$$\widetilde{\mathbb{E}}_6: \quad \delta = (1, 2, 1, 2, 3, 2, 1)$$

$$\widetilde{\mathbb{E}}_7: \quad \delta = (1, 2, 3, 4, 3, 2, 1, 2)$$

$$\widetilde{\mathbb{E}}_8: \quad \delta = (1, 2, 3, 4, 5, 6, 4, 2, 3).$$

Writing χ_i for the character of the representation W_i, for $0 \leqslant i \leqslant n$ set

$$\eta_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) g,$$

which are central idempotents in $\Bbbk G$ and form a $\Bbbk$-basis for $Z(\Bbbk G)$. Then the map

$$\Bbbk^{n+1} \to Z(\Bbbk G), \qquad (\lambda_0, \lambda_1, \dots, \lambda_n) \mapsto \sum_{i=0}^{n} \frac{\lambda_i}{\delta_i} \eta_i \tag{2.5.1}$$

is a bijection, which we will henceforth use to identify $\Bbbk^{n+1}$ with $Z(\Bbbk G)$, often without mention.

Definition 2.5.2. Let $\Bbbk[u, v]^G$ be a Kleinian singularity and let $\widetilde{Q}$ be the corresponding extended Dynkin quiver. Then G acts naturally on the free algebra $\Bbbk\langle u, v\rangle$, and so we can form the skew group ring $\Bbbk\langle u, v\rangle \# G$. Let $e = \frac{1}{|G|} \sum_{g \in G} g$ be the average of the group elements, viewed as an element of $\Bbbk\langle u, v\rangle \# G$. Fix a weight $\lambda \in \Bbbk^{n+1}$ which, using (2.5.1), gives rise to an element of $Z(\Bbbk G)$ which we also call λ. Then define

$$\mathcal{S}^\lambda(\widetilde{Q}) \coloneqq \frac{\Bbbk\langle u, v\rangle \# G}{\langle vu - uv - \lambda\rangle} \quad \text{and} \quad \mathcal{O}^\lambda(\widetilde{Q}) \coloneqq e \mathcal{S}^\lambda(\widetilde{Q}) e.$$

We frequently write $\mathcal{S}^\lambda$ and $\mathcal{O}^\lambda$ when the precise choice of $\widetilde{Q}$ is unimportant.

We view $\mathcal{S}^\lambda$ as a deformation of $\Bbbk[u, v] \# G$ and $\mathcal{O}^\lambda$ as a deformation of $\Bbbk[u, v]^G$. It is easy to see that multiplying λ by a nonzero scalar does not change the isomorphism classes of $\mathcal{S}^\lambda$ or $\mathcal{O}^\lambda$, so we can assume that $\lambda \cdot \delta$ is either 0 or 1. Recalling that $\mathcal{O}^\lambda$ is commutative if and only if $\lambda \cdot \delta = 0$, the former case yields a deformation of $\Bbbk[u, v]^G$ that is commutative, while the latter gives a noncommutative deformation.

Crawley-Boevey–Holland showed that these deformations have many nice ring-theoretic and homological properties; see Proposition 1.2.3. In particular, the deformations $\mathcal{O}^\lambda$ are always noetherian domains of GK dimension 2, they are noncommutative if and only if $\lambda \cdot \delta \neq 0$, and they are frequently singular, which means that they provide examples of singular noncommutative surfaces.

To determine many of the properties of these rings, Crawley-Boevey–Holland showed that the deformations $\mathcal{S}^\lambda$ are Morita equivalent to *deformed preprojective algebras*, which we now define. These algebras provide a generalisation of the preprojective algebras which were mentioned in Chapter 1.

Definition 2.5.3. Let Q be a quiver. Define the double $\overline{Q}$ of Q to be the quiver obtained from Q by adding a reverse arrow $\overline{\alpha} : j \to i$ for each arrow $\alpha : i \to j$ in Q or, if $\alpha : i \to i$ is a loop, adding no arrows and declaring $\overline{\alpha} = \alpha$. We call the arrows in $\overline{Q}$ which are not reverse arrows *ordinary arrows*. Given a weight $\lambda \in \Bbbk^{Q_0}$ for Q, the corresponding *deformed preprojective algebra* is

$$\Pi^\lambda(Q) \coloneqq \Bbbk\overline{Q}/I,$$

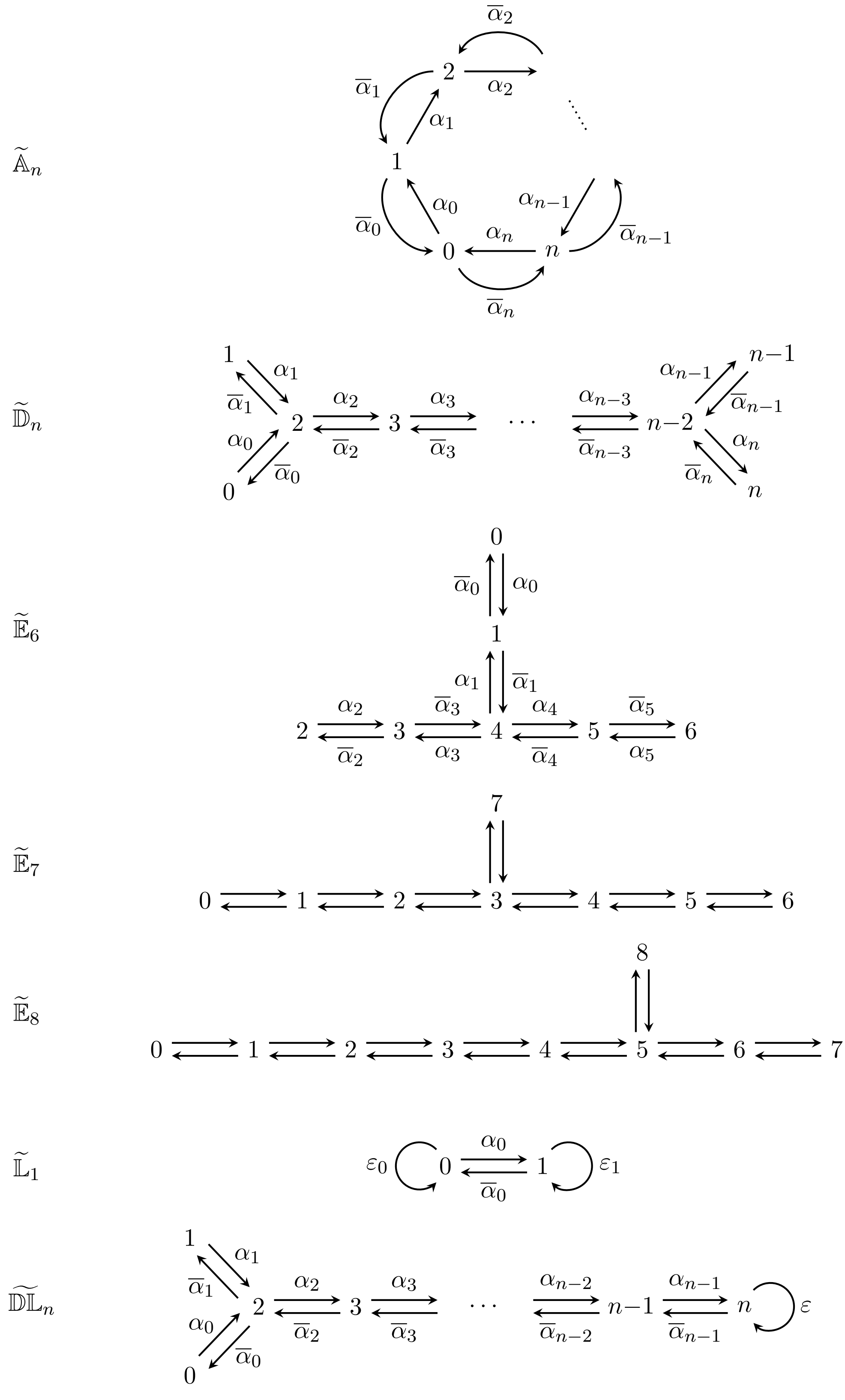


Figure 2.1: Doubles of extended Dynkin graphs and the Euclidean graphs $\widetilde{\mathbb{L}}_1$ and $\widetilde{\mathbb{DL}}_n$, with the labelling of vertices and arrows that will be used throughout this thesis. We have only provided arrow labelling for those quivers in which we will need to refer to specific paths.

where I is the two-sided ideal of $\Bbbk\overline{Q}$ with generators

$$\sum_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} \alpha\overline{\alpha} \;-\; \sum_{\substack{\alpha\in Q_1\\ h(\alpha)=i\\ t(\alpha)\neq i}} \overline{\alpha}\alpha \;-\; \lambda_i e_i$$

for each vertex $i \in Q_0$.

Theorem 2.5.4 ([CBH98, Theorem 0.1])**.** *Let $\Bbbk[u,v]^G$ be a Kleinian singularity with corresponding extended Dynkin quiver $\widetilde{Q}$, and let $\lambda \in \Bbbk^{n+1}$ be a weight for $\widetilde{Q}$. Then there is a Morita equivalence between $\mathcal{S}^\lambda(\widetilde{Q})$ and $\Pi^\lambda(\widetilde{Q})$, and moreover there is an isomorphism $\mathcal{O}^\lambda(\widetilde{Q}) \cong e_0\Pi^\lambda(\widetilde{Q})e_0$.*

To determine further properties of the deformed preprojective algebras Π^λ, Crawley-Boevey–Holland constructed *reflection functors* and showed that they provide Morita equivalences between different deformed preprojective algebras. In the following, we say that a vertex $i \in Q_0$ is *loop-free* if there are no arrows $\alpha : i \to i$.

Definition 2.5.5. Let Q be a quiver, and let $C = 2I - A$ be the generalised Cartan matrix of Q, where A is the adjacency matrix of the underlying graph of Q. For each loop-free vertex $i \in Q_0$, define the *dual reflection* $r_i : \Bbbk^{Q_0} \to \Bbbk^{Q_0}$ by

$$(r_i\lambda)_j = \lambda_j - C_{ij}\lambda_i.$$

It is easy to see that if $\widetilde{Q}$ is extended Dynkin then $\lambda \cdot \delta = (r_i\lambda) \cdot \delta$. We also have the following result:

Theorem 2.5.6 ([CBH98, Theorem 5.1])**.** *Let Q be a quiver and suppose that $i \in Q_0$ is loop-free. Then $\Pi^\lambda(Q)$ is Morita equivalent to $\Pi^{r_i\lambda}(Q)$.*

These Morita equivalences turn out to be very useful when our quiver is Dynkin or extended Dynkin. In these cases it is relatively easy to deduce some properties of Π^λ when λ has a particular form, and then these properties can be translated to the other Π^λ using the Morita equivalence of Theorem 2.5.6. For what follows, we need to fix a total ordering $\prec$ on $\Bbbk$ which also satisfies the following:

(1) If $a \prec b$, then $a + c \prec b + c$ for all $c \in \Bbbk$;
(2) On the integers, $\prec$ coincides with the usual order; and
(3) For any $a \in \Bbbk$, there exists $m \in \mathbb{Z}$ with $a \prec m$.

For example, when $\Bbbk = \mathbb{C}$ we may define $\prec$ by $z \prec z'$ if and only if $\operatorname{Re} z < \operatorname{Re} z'$, or $\operatorname{Re} z = \operatorname{Re} z'$ and $\operatorname{Im} z < \operatorname{Im} z'$.

Definition 2.5.7. We say that a weight $\lambda \in \Bbbk^{Q_0}$ is *dominant* if $\lambda_i \succeq 0$ for all $i \in Q_0$.

Lemma 2.5.8 ([CBH98, Lemma 7.2])**.**

(1) *Suppose that Q is Dynkin, and let λ be a weight for Q. Then there exists a unique dominant weight λ' such that λ' is the image of λ under a sequence of dual reflections.*
(2) *Suppose that $\widetilde{Q}$ is extended Dynkin, and let λ be a weight for $\widetilde{Q}$ with $\lambda \cdot \delta = 1$. Then there exists a unique dominant weight λ' such that λ' is the image of λ under a sequence of dual reflections.*

Therefore, to determine representation-theoretic properties of Π^λ when our quiver is (extended) Dynkin, it suffices to restrict our attention to dominant weights. For example, we have the following useful result which will be used frequently in Chapter 3:

Lemma 2.5.9 ([CBH98, Lemma 7.1]). *Suppose that Q is Dynkin, and let λ be a dominant weight for Q. Write Q_λ for the full subquiver supported on those vertices i with $\lambda_i = 0$. Then $\Pi^\lambda(Q) \cong \Pi(Q_\lambda)$. In particular, the projective modules of $\Pi^\lambda(Q)$ are the modules $e_i\Pi^\lambda(Q)$, where i is a vertex with $\lambda_i = 0$.*

In Chapter 3 we will also define what it means for a weight to be *quasi-dominant*, and show that one can restrict attention to quasi-dominant weights when determining the singularity categories $\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^\lambda(\widetilde{Q}))$.

2.6 Quantum Kleinian singularities

We now provide more details on quantum Kleinian singularities, as discussed in the introduction. Recall that these can be thought of as a generalisation of Kleinian singularities $\Bbbk[u,v]^G$ obtained by replacing $\Bbbk[u,v]$ by a two-dimensional AS regular algebra and replacing G with a suitable subgroup of $\mathrm{GL}(2,\Bbbk)$.

Definition 2.6.1. Let A be a finitely generated $\mathbb{N}$-graded $\Bbbk$-algebra with $A_0 = \Bbbk$, and let $\Bbbk = A/A_{\geqslant 1}$ be the trivial module. We say that A is *Artin-Schelter regular* (or AS regular) *of dimension d* if:

(1) $\mathrm{gl.dim}\, A = d < \infty$;
(2) $\mathrm{GKdim}\, A < \infty$; and
(3) $\mathrm{Ext}^i_{\text{gr-}A}(\Bbbk_A, A_A) \cong \begin{cases} 0 & \text{if } i \neq d \\ {}_A\Bbbk[\ell] & \text{if } i = d \end{cases}$ as left A-modules.

While there could be some confusion between which of global dimension and GK dimension we mean when we speak of a d-dimensional AS regular algebra, we remark that there are no known examples where these values differ.

In particular, the only commutative AS regular algebra of dimension d is the polynomial ring $\Bbbk[x_1, \dots, x_d]$. AS regular algebras of dimension d are meant to be viewed as noncommutative analogues of this polynomial ring and, in light of the above definition, they share good homological properties.

In dimension 2, it is relatively easy to classify all AS regular algebras that are generated in degree one; indeed, up to isomorphism, they are the quantum plane and Jordan plane from the introduction [BRS+16, Theorem I.2.2.1 (2)]:

$$\Bbbk_q[u,v] = \frac{\Bbbk\langle u,v\rangle}{\langle vu - quv\rangle} \quad \text{and} \quad \Bbbk_J[u,v] = \frac{\Bbbk\langle u,v\rangle}{\langle vu - uv - u^2\rangle}.$$

The classification problem in dimension 3 is much more difficult, and required the development of geometric techniques due to Artin–Tate–Van den Bergh [ATVdB90, ATVdB91]. The classification is more involved and not required for this thesis.

After replacing $\Bbbk[u,v]$ by a two-dimensional AS regular algebra, we also need to determine which finite subgroups of $\mathrm{GL}(2,\Bbbk)$ we should be allowed to act by. In the commutative setting, the condition we impose is that $G \leqslant \mathrm{SL}(2,\Bbbk)$, and a sensible noncommutative analogue of this is to require that every element of G has *trivial homological determinant*.

Definition 2.6.2. Suppose that a finite group $G \leqslant \mathrm{GL}(n,\Bbbk)$ acts on an AS regular algebra A of dimension d, and let $g \in G$. Then the *trace* of g on A is the power series

$$\mathrm{Tr}(g) \coloneqq \sum_{n\in\mathbb{N}} \mathrm{tr}(g|_{A_n})\, t^n,$$

where $\operatorname{tr}(g|_{A_n})$ is the usual trace of the linear map $g|_{A_n} : A_n \to A_n$.

Under our hypotheses, $\operatorname{Tr}(g)$ has a series expansion in $\Bbbk((t^{-1}))$ of the form

$$\operatorname{Tr}(g) = (-1)^d c^{-1} t^{-\ell} + \text{lower order terms},$$

for some $c \in \Bbbk$, where ℓ is as in Definition 2.6.1. We call this constant c the *homological determinant of* g, which we denote by $\operatorname{hdet} g$.

It is shown in [JZ00, Section 2] that hdet is multiplicative, and that if A is the commutative polynomial ring $\Bbbk[x_1, \dots, x_d]$, then $\operatorname{hdet} g = \det g$ for all $g \in \operatorname{GL}(d, \Bbbk)$. When we say that we require every element of G to have trivial homological determinant, we mean that $\operatorname{hdet} g = 1$ for all $g \in G$.

As mentioned in the introduction, Chan–Kirkman–Walton–Zhang classified all finite groups $G \leqslant \operatorname{GL}(2, \Bbbk)$ acting on a two-dimensional AS regular algebra A with trivial homological determinant in [CKWZ14]. Later work, see [CKWZ16a, CKWZ16b], shows that the resulting algebras $A\#G$ and A^G have many properties in common with the analogous algebras that arise for Kleinian singularities. The authors also showed how to associate an extended Euclidean diagram to these algebras, which is simply the McKay quiver of the group G. We repeat the classification from Chapter 1 below, this time excluding case (0) which covers the classical Kleinian singularities:

Case	A	G	$\widetilde{Q}$
(i)	$\Bbbk_q[u,v]$	C_n	$\widetilde{\mathbb{A}}_{n-1}$
(ii)	$\Bbbk_{-1}[u,v]$	S_2	$\widetilde{\mathbb{L}}_1$
(iii)	$\Bbbk_{-1}[u,v]$	D_n	$\begin{cases} \widetilde{\mathbb{D}}_{\frac{n+4}{2}} & \text{if } n \text{ is even} \\ \widetilde{\mathbb{DL}}_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$
(iv)	$\Bbbk_J[u,v]$	C_2	$\widetilde{\mathbb{A}}_1$

Table 2.1: The pairs (A, G) for quantum Kleinian singularities A^G, and their McKay quivers.

As discussed in the introduction, quantum Kleinian singularities share many ring-theoretic and representation-theoretic properties with classical Kleinian singularities, which justifies their suggestive name.

Chapter 3

Singularities of Deformations of Kleinian Singularities

In this chapter, we prove Theorem 1.2.5 from the introduction, which gives a precise description of the singularity category of a deformation $\mathcal{O}^\lambda(\widetilde{Q})$, and we also prove Theorem 1.2.7, which establishes a noncommutative version of the geometric McKay correspondence. In light of Theorem 2.5.4, throughout we identify $\mathcal{O}^\lambda(\widetilde{Q}) = e_0\Pi^\lambda(\widetilde{Q})e_0$.

3.1 Restriction to quasi-dominant weights

We first seek to simplify the problem by focussing our attention on certain choices of weights λ which occur in the deformations $\mathcal{O}^\lambda$. We make a definition:

Definition 3.1.1. If $\widetilde{Q}$ is extended Dynkin, we say that a weight λ is *quasi-dominant* if $\lambda_i \succeq 0$ for all $i \neq 0$, where $\prec$ is a total ordering on $\Bbbk$ as in Definition 2.5.7.

We now show that we can restrict attention to quasi-dominant weights for the remainder of this chapter. Recall the definition of dual reflections from Definition 2.5.5. We have the following lemma, which appears in unpublished work of Boddington and Levy [BL07].

Lemma 3.1.2. *Suppose that λ is a weight for an extended Dynkin quiver $\widetilde{Q}$, and let ρ be a sequence of dual reflections at vertices other than the extending vertex 0. Then $\mathcal{O}^\lambda \cong \mathcal{O}^{\rho(\lambda)}$.*

This is a strengthening of [CBH98, Lemma 7.9], in which the authors established only a Morita equivalence between these two rings, rather than an isomorphism. Combining Lemma 3.1.2 with [CBH98, Lemma 7.8], we have the following result:

Lemma 3.1.3. *Suppose that λ is a weight for an extended Dynkin quiver $\widetilde{Q}$. Then there exists a quasi-dominant weight λ' with $\mathcal{O}^\lambda \cong \mathcal{O}^{\lambda'}$.*

With this result in hand, we now state an assumption that will hold for the remainder of this chapter.

Assumption 3.1.4. *If λ is a weight for an extended Dynkin quiver $\widetilde{Q}$, then we always assume that the weight λ is quasi-dominant unless explicitly stated otherwise.*

We will see later that this assumption allows one to easily read off a number of useful facts about the module category of $\mathcal{O}^\lambda$, and ultimately its singularity category as well. As a first example, if we restrict our attention to quasi-dominant weights then it is easy to detect whether $\mathcal{O}^\lambda$ is singular:

Lemma 3.1.5. *If λ is a quasi-dominant weight for an extended Dynkin quiver $\widetilde{Q}$, then $\mathcal{O}^\lambda$ is singular if and only if $\lambda_i = 0$ for some $i \neq 0$.*

Proof. By [CBH98, Theorem 0.4 (4)], $\mathcal{O}^\lambda$ is singular if and only if $\lambda \cdot \alpha = 0$ for some *Dynkin root* α. The possible values of these Dynkin roots are not important to us; it suffices to know that they have the form $(0, \alpha') \in \mathbb{Z}^{n+1}$ where, in particular, α' has entirely non-negative or non-positive entries, and has at least one nonzero entry. In addition, $\varepsilon_i \in \mathbb{Z}^{n+1}$ for $1 \leqslant i \leqslant n$ is always a Dynkin root, where ε_i is the ith coordinate vector (here the entries are indexed from 0 to n). Therefore, if $\lambda_i = 0$ for some $i \neq 0$ then $\lambda \cdot \alpha = 0$ for the Dynkin root $\alpha = \varepsilon_i$, while if $\lambda_i \neq 0$ for all $i \neq 0$, then necessarily $\lambda \cdot \alpha \neq 0$ for all Dynkin roots α. The result then follows. □

3.2 The singularity category of $\mathcal{O}^\lambda(\widetilde{Q})$ as a $\Bbbk$-linear category

Our first step in determining $\underline{\text{MCM}}$-$\mathcal{O}^\lambda$ is to determine its structure as an additive category, or indeed as a $\Bbbk$-linear category. We first identify an important module.

Lemma 3.2.1. $\Pi^\lambda e_0$ *is a finitely generated* $\mathcal{O}^\lambda$*-module, and it satisfies* $\operatorname{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \Pi^\lambda$.

Proof. The first part of the statement follows from [MS81, Lemma 1]. To determine the endomorphism ring, first note that, by [CBH98, Lemma 1.4, Corollary 3.5], Π^λ is Morita equivalent to a ring which is a maximal order and hence is itself a maximal order. The claim then follows from the results in [CB, Section 5.4]. □

Write $V_i = e_i \Pi^\lambda e_0$; we shall refer to these $\mathcal{O}^\lambda$-modules as *vertex modules*, and they will play an important role in determining $\underline{\text{MCM}}$-$\mathcal{O}^\lambda$. Using Lemma 3.2.1, we are able to calculate the Hom spaces between the vertex modules.

Corollary 3.2.2. *We have* $\operatorname{Hom}_{\mathcal{O}^\lambda}(V_i, V_j) = e_j \Pi^\lambda e_i$*, and so* $\Pi^\lambda e_0$ *is a reflexive (and hence maximal Cohen-Macaulay)* $\mathcal{O}^\lambda$*-module.*

Proof. By Lemma 3.2.1, $\Pi^\lambda = \operatorname{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \bigoplus_{k,\ell} \operatorname{Hom}_{\mathcal{O}^\lambda}(e_k \Pi^\lambda e_0, e_\ell \Pi^\lambda e_0)$. Multiplying on the left by e_j kills each Hom space with $\ell \neq j$, while multiplying on the right by e_i kills each Hom space with $k \neq i$. It follows that

$$e_j \Pi^\lambda e_i = e_j \left(\bigoplus_{k,\ell} \operatorname{Hom}_{\mathcal{O}^\lambda}(V_k, V_\ell) \right) e_i = e_j \operatorname{Hom}_{\mathcal{O}^\lambda}(V_i, V_j)\, e_i = \operatorname{Hom}_{\mathcal{O}^\lambda}(V_i, V_j).$$

Since all of our results on Hom spaces are left-right symmetrical, we also find that

$$\begin{aligned}(\Pi^\lambda e_0)^{**} = \left(\operatorname{Hom}_{\mathcal{O}^\lambda} \left(\bigoplus_{i=0}^n V_i, V_0 \right) \right)^* &= \left(\bigoplus_{i=0}^n e_0 \Pi^\lambda e_i \right)^* \\ = \bigoplus_{i=0}^n \operatorname{Hom}_{\mathcal{O}^\lambda}(e_0 \Pi^\lambda e_i, e_0 \Pi^\lambda e_0) &= \bigoplus_{i=0}^n V_i = \Pi^\lambda e_0,\end{aligned}$$

and so $\Pi^\lambda e_0$ is a reflexive $\mathcal{O}^\lambda$-module. Therefore, since $\Pi^\lambda e_0$ is finitely generated, and since $\operatorname{i.dim} \mathcal{O}^\lambda \leqslant 2$ ([CBH98, Theorem 1.6]), Lemma 2.2.13 implies that $\Pi^\lambda e_0$ is maximal Cohen-Macaulay. □

This allows us to determine the stable endomorphism ring of $\Pi^\lambda e_0$. We fix some notation which will be used throughout the rest of this chapter: write Q_λ for the full subquiver of $\widetilde{Q}$ with vertex set $I_\lambda \coloneqq \{i \in \{1,\dots,n\} \mid \lambda_i = 0\}$.

Lemma 3.2.3. *We have* $\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi(Q_\lambda)$.

Proof. Write $\mu = (\lambda_1,\dots,\lambda_n)$. By Corollary 3.2.2, we have that

$$(\Pi^\lambda e_0)^* = \bigoplus_i \mathrm{Hom}_{\mathcal{O}^\lambda}(e_i\Pi^\lambda e_0, e_0\Pi^\lambda e_0) = \bigoplus_i e_0\Pi^\lambda e_i = e_0\Pi^\lambda.$$

Then, noting that $\Pi^\lambda e_0 \left(\Pi^\lambda e_0\right)^* = \Pi^\lambda e_0 \Pi^\lambda$, we have

$$\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \frac{\Pi^\lambda}{\Pi^\lambda e_0 \Pi^\lambda} \cong \Pi^\mu(Q).$$

Since the entries of μ are all $\succeq 0$ by Assumption 3.1.4 and Q is Dynkin, Lemma 2.5.9 tells us that $\Pi^\mu(Q)$ is isomorphic to the preprojective algebra supported on the vertices i of Q with $\mu_i = 0$; that is, $\Pi^\mu(Q) \cong \Pi(Q_\lambda)$. Therefore $\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi(Q_\lambda)$. □

We are also able to determine when a vertex module is projective. It turns out that this is the case precisely when the corresponding vertex is deleted when passing from $\widetilde{Q}$ to Q_λ.

Lemma 3.2.4. *If $i = 0$ or $\lambda_i \neq 0$, then V_i is a projective $\mathcal{O}^\lambda$-module.*

Proof. When $i = 0$ this is clear. So suppose that $i \neq 0$ and $\lambda_i \neq 0$. Then, as in the proof of Lemma 3.2.3, $e_i = 0$ in $\Pi^\lambda/\Pi^\lambda e_0 \Pi^\lambda$ and so $e_i \in \Pi^\lambda e_0 \Pi^\lambda$. But then, using Corollary 3.2.2,

$$V_i V_i^* = e_i \Pi^\lambda e_0 \Pi^\lambda e_i \ni e_i^3 = e_i,$$

where e_i is the identity element of $\mathrm{End}_{\Pi^\lambda}(V_i) = e_i\Pi^\lambda e_i$, and so V_i is projective by the dual basis lemma (see [Lam99, (2.9)]). □

It follows that the vertex modules V_i satisfying $\lambda_i \neq 0$ are equal to the zero object in the singularity category, so that $\Pi^\lambda e_0$ and $\bigoplus_{i\in I_\lambda} V_i$ define the same element in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$. When working in the stable module category, we will sometimes refer to those vertex modules whose corresponding weight is zero as non-projective vertex modules.

Proposition 3.2.5.
(1) $\mathrm{MCM}\text{-}\mathcal{O}^\lambda = \mathrm{add}\,\Pi^\lambda e_0$.
(2) $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \mathrm{add}\,\Pi^\lambda e_0 = \mathrm{add}\left(\bigoplus_{i\in I_\lambda} V_i\right)$.

Proof.
(1) First note that $\mathcal{O}^\lambda$ is Gorenstein and that, using [CBH98, Theorem 1.5],

$$\mathrm{gl.dim}\,\mathrm{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \mathrm{gl.dim}\,\Pi^\lambda \leqslant 2.$$

Since $\Pi^\lambda e_0$ has $\mathcal{O}^\lambda$ as a direct summand, the first claim then follows from Proposition 2.2.11.

(2) Part (1) immediately implies that $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda = \operatorname{add}\Pi^\lambda e_0 = \operatorname{add}\left(\bigoplus_i V_i\right)$. But projective modules become the zero object when passing to the stable module category, so the result follows by Lemma 3.2.4. □

We recall that an additive category is said to be *Krull-Schmidt* if every object decomposes into a finite direct sum of objects, each of which has a local endomorphism ring, and where this decomposition is unique up to reordering.

Theorem 3.2.6. *The functor* $\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$ *induces a* $\mathbb{k}$*-linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda).$$

Proof. By [Kra15, Proposition 2.3], the functor

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -) : \underline{\text{mod}}\text{-}\mathcal{O}^\lambda \to \text{mod-}\,\underline{\text{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \text{mod-}\Pi(Q_\lambda)$$

induces a fully faithful $\mathbb{k}$-linear functor $\operatorname{add}\Pi^\lambda e_0 \to \text{proj-}\Pi(Q_\lambda)$, where $\operatorname{add}\Pi^\lambda e_0 = \underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ by Proposition 3.2.5. Since $\Pi(Q_\lambda)$ is finite-dimensional [BES07, Proposition 2.1], mod-$\Pi(Q_\lambda)$ is Krull-Schmidt and hence so too is proj-$\Pi(Q_\lambda)$. Therefore, to establish essential surjectivity of the functor $\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$, it suffices to show that we can hit each indecomposable projective $e_i\Pi(Q_\lambda)$, where $i \in I_\lambda$. Indeed, we have

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, V_i) = \frac{e_i\Pi^\lambda}{e_i\Pi^\lambda e_0\Pi^\lambda} = \frac{e_i\Pi^\lambda}{e_i\Pi^\lambda \cap \Pi^\lambda e_0\Pi^\lambda} = e_i\,\frac{\Pi^\lambda}{\Pi^\lambda e_0\Pi^\lambda} = e_i\Pi(Q_\lambda),$$

and so the functor is also essentially surjective. We therefore have the claimed equivalence. □

We therefore see that $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ is nontrivial if and only if $\lambda_i = 0$ for some $i \neq 0$ which, by Lemma 3.1.5, happens precisely when $\mathcal{O}^\lambda$ is singular, which is consistent with the remark after Definition 2.2.5. Moreover, the vertex modules V_i with $i = 0$, or with $i \neq 0$ and $\lambda_i \neq 0$, are the vertex modules which are projective and hence vanish in $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$. This is reflected by the fact that these are the vertices which are deleted to obtain Q_λ.

As an immediate consequence of (the proof of) Theorem 3.2.6, we have the following result:

Corollary 3.2.7. $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ *is a Krull-Schmidt category.*

Remark 3.2.8. By Proposition 3.2.5, the objects of $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ are direct summmands of finite direct sums of the non-projective vertex modules. Since these vertex modules are indecomposable and $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ is Krull-Schmidt, in fact every object of $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ is isomorphic to a finite direct sum of vertex modules.

The following two corollaries are then immediate from Theorem 3.2.6:

Corollary 3.2.9. *Suppose that* $\widetilde{Q}$ *is an extended Dynkin quiver and* $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ *is a disjoint union of connected quivers* $Q^{(i)}$*, which are therefore necessarily Dynkin. Then there is a* $\mathbb{k}$*-linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}).$$

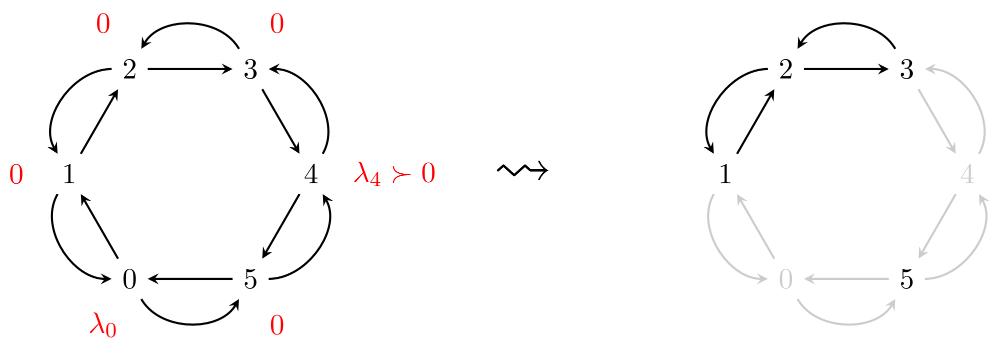
Proof. This follows from the isomorphism $\Pi(Q_\lambda) \cong \prod_{i=1}^r \Pi(Q^{(i)})$ and the fact that, for rings R_i, we have an equivalence $\text{proj-}\left(\prod_{i=1}^r R_i\right) \simeq \bigoplus_{i=1}^r \text{proj-}R_i$. □

Corollary 3.2.10. *Let $\widetilde{Q}$ and $\widetilde{Q}'$ be extended Dynkin quivers (not necessarily of the same type) and let λ and λ' be quasi-dominant weights for $\widetilde{Q}$ and $\widetilde{Q}'$, respectively. If $Q_\lambda \cong Q'_{\lambda'}$ then there is a $\Bbbk$-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{Q}) \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda'}(\widetilde{Q}'). \qquad \square$$

It is illustrative to apply Theorem 3.2.6 (and its corollaries) to an example:

Example 3.2.11. Suppose that $\widetilde{Q} = \widetilde{\mathbb{A}}_5$, and consider the deformation $\mathcal{O}^\lambda$ where the weight λ is indicated in red on the left hand quiver below:



By Theorem 3.2.9, there is a $\Bbbk$-linear equivalence $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(\mathbb{A}_3) \oplus \text{proj-}\Pi(\mathbb{A}_1)$. In more suggestive notation, we can write this equivalence as

$$\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^\lambda) \simeq \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_3}) \oplus \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1}),$$

and so it is sensible to consider $\mathcal{O}^\lambda$ as having an $\mathbb{A}_3$ and an $\mathbb{A}_1$ singularity.

If we more concretely set $\lambda = (-1,0,0,0,1,0)$ (respectively, $\lambda = (0,0,0,0,1,0)$) then $\mathcal{O}^\lambda$ is commutative (respectively, noncommutative), and it is easy to write down a presentation for $\mathcal{O}^\lambda$. In particular, we have a $\Bbbk$-linear equivalence

$$\mathcal{D}_{\mathrm{sg}} \frac{\Bbbk[x,y,z]}{\langle xy - z^4(z+1)^2 \rangle} \simeq \mathcal{D}_{\mathrm{sg}} \frac{\Bbbk\langle x,y,z \rangle}{\left\langle \begin{matrix} xz = (z+1)x, & xy = z^4(z+1)^2 \\ yz = (z-1)y, & yx = (z-1)^4 z^2 \end{matrix} \right\rangle}.$$

The ring appearing on the right hand side here is an example of a generalised Weyl algebra, as studied in [Bav92, Hod93].

If $\lambda = \mathbf{0}$, so that $\mathcal{O}^\lambda(\widetilde{Q})$ is isomorphic to the Kleinian singularity R_Q, then Theorem 3.2.6 gives a $\Bbbk$-linear equivalence $\underline{\mathrm{MCM}}\text{-}R_Q \simeq \text{proj-}\Pi(Q)$. Since $\underline{\mathrm{MCM}}\text{-}R_Q$ is triangulated, this equivalence induces a triangulated structure on proj-$\Pi(Q)$. Moreover, the $\Bbbk$-linear equivalences of Theorem 3.2.6 and Corollary 3.2.9 induce a triangulated structure on $\text{proj-}\Pi(Q_\lambda) \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$. To prove Theorem 1.2.5 from the introduction, it suffices to show that each of the proj-$\Pi(Q^{(i)})$ are triangulated subcategories of proj-$\Pi(Q_\lambda)$, and that we have triangle equivalences $\underline{\mathrm{MCM}}\text{-}R_{Q^{(i)}} \simeq \text{proj-}\Pi(Q^{(i)})$. We establish both of these in the next section.

3.3 The singularity category of $\mathcal{O}^\lambda(\widetilde{Q})$ as a triangulated category

We are now able to to prove our main theorem which gives a description of the singularity category of $\mathcal{O}^\lambda(\widetilde{Q})$. Recall that we have a $\mathbb{k}$-linear equivalence

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}) \tag{3.3.1}$$

where Q_λ, the subquiver of Q with vertex set $\{i \in \{1,\dots,n\} \mid \lambda_i = 0\}$, decomposes into connected components as $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$. Recall also that we have $\mathbb{k}$-linear equivalences proj-$\Pi(Q^{(i)}) \simeq \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$, where $R_{Q^{(i)}}$ is the Kleinian singularity corresponding to $Q^{(i)}$. We wish to show that each of the proj-$\Pi(Q^{(i)})$ is a triangulated subcategory of $\underline{\mathrm{MCM}}$-$\mathcal{O}^\lambda$ which is also triangle equivalent to $\mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$. To achieve this, we make use of the following result:

Theorem 3.3.2 ([Kel18, Corollary 2]). *Let $\mathcal{T}$ and $\mathcal{T}'$ be Krull-Schmidt $\mathbb{k}$-linear triangulated categories which are finite, connected, algebraic and standard. If $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as $\mathbb{k}$-linear categories, then they are in fact equivalent as triangulated categories.*

We note that, if Q is Dynkin, proj-$\Pi(Q)$ (and the $\mathbb{k}$-linearly equivalent category $\mathcal{D}_{\mathrm{sg}}(R_Q)$) are finite, connected, and standard since they are $\mathbb{k}$-linearly equivalent to certain orbit categories which are known to have these properties (see [AIR15, Remark 5.9]). Therefore, if we can show that each proj-$\Pi(Q^{(i)})$ is an algebraic triangulated subcategory under the $\mathbb{k}$-linear equivalence (3.3.1), then each $\mathbb{k}$-linear equivalence proj-$\Pi(Q^{(i)}) \simeq \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$ is in fact a triangle equivalence, which will prove Theorem 1.2.5 from the introduction.

We must first show that the translation functor induced on the right hand side by this equivalence *preserves connected components*, in the sense that it restricts to an autoequivalence of each of the subcategories proj-$\Pi(Q^{(i)})$. If the $Q^{(i)}$ are pairwise non-isomorphic, then the fact that the induced translation functor has to be a graph automorphism forces it to preserve connected components, completing the proof in these cases. This leaves only the cases where some of the $Q^{(i)}$ are isomorphic, and one might hope to abstractly prove that the translation functor preserves connected components. Unfortunately, the following example shows that one should not expect this to be the case.

Example 3.3.3. Let T be a Krull-Schmidt $\mathbb{k}$-linear category with only two indecomposable objects U and V, and suppose these objects satisfy $\mathrm{Hom}_T(U,V) = 0 = \mathrm{Hom}_T(V,U)$ and $\mathrm{End}_T(U) = \mathbb{k} = \mathrm{End}_T(V)$. For example, this is the case for $\underline{\mathrm{MCM}}$-$\mathcal{O}^\lambda(\widetilde{\mathbb{A}}_3)$ when $(\lambda_0,\lambda_1,\lambda_2,\lambda_3) = (0,0,1,0)$, since in this case it is $\mathbb{k}$-linearly equivalent to proj-$\Pi(\mathbb{A}_1) \oplus$ proj-$\Pi(\mathbb{A}_1)$. This category has two possible triangulated structures: the first has $\Sigma = \mathrm{id}$, and the distinguished triangles are isomorphic to direct sums and rotations of

$$U \xrightarrow{\mathrm{id}} U \to 0 \to U \quad \text{and} \quad V \xrightarrow{\mathrm{id}} V \to 0 \to V,$$

and the second option has $\Sigma U = V$ and $\Sigma V = U$, and the distinguished triangles are isomorphic to direct sums and rotations of

$$U \xrightarrow{\mathrm{id}} U \to 0 \to V.$$

The first example decomposes into a direct sum of two triangulated subcategories, while the second example does not.

While the above example shows that one should not expect to be able to abstractly prove that the translation functor preserves connected components, this is essentially the only counterexample. The following proof is due to Jeremy Rickard, and we thank him for allowing us to reproduce it:

Lemma 3.3.4. *Suppose that $\mathcal{T}$ is a Krull-Schmidt $\Bbbk$-linear triangulated category with finitely many indecomposables which decomposes as a $\Bbbk$-linear category as*

$$\mathcal{T} = \bigoplus_{i=1}^{n} \mathcal{T}_i.$$

Suppose that the translation functor Σ satisfies $\Sigma\mathcal{T}_i = \mathcal{T}_j$ for some $i \neq j$. Then $\mathcal{T}_i$ and $\mathcal{T}_j$ each have only one isoclass of indecomposable objects.

Proof. Let $\alpha : X \to Y$ be a nonzero morphism between two indecomposable objects of $\mathcal{T}_i$, and complete to a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X,$$

where $\Sigma X \in \mathcal{T}_j$ by assumption. We claim that every indecomposable summand of Z lies in $\mathcal{T}_j$. To this end, suppose that $Z = Z' \oplus Z''$ where $Z' \in \mathcal{T}_j$ and $Z'' \in \bigoplus_{k\neq j} \mathcal{T}_k$, and write $\gamma = (\gamma', 0)$. The map γ' gives rise to a triangle $Z' \xrightarrow{\gamma'} \Sigma X \to Y' \to \Sigma Z'$ and rotating yields the triangle $X \to \Sigma^{-1}Y' \to Z' \xrightarrow{\gamma'} \Sigma X$. The direct sum of this triangle with the triangle $0 \to Z'' \to Z'' \to 0$ is a triangle isomorphic to the triangle $X \to Y \to Z \to \Sigma X$, and so $Y \cong \Sigma^{-1}Y' \oplus Z''$. By indecomposability of Y, we therefore have $\Sigma^{-1}Y' = 0$ or $Z'' = 0$. If $\Sigma^{-1}Y' = 0$ then $Y \cong Z''$ and $\Sigma X \cong Z'$. Our original triangle becomes

$$X \xrightarrow{\alpha} Z'' \to \Sigma X \oplus Z'' \to \Sigma X$$

which is isomorphic to the direct sum of the triangles $X \to 0 \to \Sigma X \to \Sigma X$ and $0 \to Z'' \to Z'' \to 0$. This means that α is the zero map, contrary to our assumption, and so we must have $Z'' = 0$, establishing the claim. Now, since every indecomposable summand of Z lies in $\mathcal{T}_j$, β is the zero map. Applying $\mathrm{Hom}(Y, -)$, we get an exact sequence

$$\mathrm{Hom}(Y, X) \xrightarrow{\alpha\circ -} \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, Z)$$

where the last term is 0. By exactness, there exists $\alpha' : Y \to X$ with $\alpha\varepsilon = \mathrm{id}_Y$. Since $\mathcal{T}$ is Krull-Schmidt the endomorphism ring of X is local, which implies that the idempotent map $\alpha'\alpha$ is a unit and therefore equal to id_X. Therefore $\alpha : X \to Y$ is an isomorphism, and so $\mathcal{T}_i$ (and hence $\mathcal{T}_j$) has only one indecomposable object, up to isomorphism. □

Therefore, to show that the induced translation functor on $\bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)})$ from the $\Bbbk$-linear equivalence

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda} \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)})$$

preserves connected components, we only need to consider the case when there exist $Q^{(i)}$ and $Q^{(j)}$, $i \neq j$, with $Q^{(i)} = \mathbb{A}_1 = Q^{(j)}$. It suffices to show that, for the corresponding objects $V_i, V_j \in \underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{Q})$, we have $\Sigma V_i = V_i$ and $\Sigma V_j = V_j$. To this end, we first have the following result:

Proposition 3.3.5. *Let Q be a non-Dynkin quiver with no oriented cycles, and with vertices labelled $\{0, 1, \dots n\}$. Write $\Pi(Q)$ for the preprojective algebra of Q, and write $V_i = e_i \Pi(Q) e_0$, which is a right $e_0 \Pi(Q) e_0$-module. Then, for any $i \neq 0$, there exists a short exact sequence of $e_0 \Pi(Q) e_0$-modules*

$$0 \to V_i \to \bigoplus_{j \in \partial i} V_j \to V_i \to 0,$$

where ∂i for the set of vertices adjacent to i in Q.

Proof. By [BBK02, Proposition 4.2], there is an exact sequence of $\Pi(Q)$-modules

$$0 \to e_i \Pi(Q) \to \bigoplus_{j \in \partial i} e_j \Pi(Q) \to e_i \Pi(Q) \to S_i \to 0, \tag{3.3.6}$$

where S_i is the simple module at vertex i. Noting that $e_0 \Pi(Q)$ is a direct summand of $\Pi(Q)$ and hence projective, applying $\text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), -)$ yields an exact sequence

$$\begin{aligned} 0 \to \text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), e_i \Pi(Q)) &\to \bigoplus_{j \in \partial i} \text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), e_j \Pi(Q)) \\ &\to \text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), e_i \Pi(Q)) \to \text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), S_i) \to 0. \end{aligned}$$

Noting that $\text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), e_k \Pi(Q)) = V_k$ and, since $i \neq 0$, $\text{Hom}_{\Pi(Q)}(e_0 \Pi(Q), S_i) = 0$, we therefore have exactness of

$$0 \to V_i \to \bigoplus_{j \in \partial i} V_j \to V_i \to 0,$$

as claimed. □

In particular this result holds for extended Dynkin quivers, where we remark that if we wish to apply it to an $\widetilde{\mathbb{A}}_n$ quiver then we must orient the arrows so that there are no oriented cycles; this does not change the isomorphism class of $\Pi(\widetilde{\mathbb{A}}_n)$.

Remark 3.3.7. The above result may or may not fail for Dynkin quivers, depending on how the vertices are labelled. For example, when $Q = \mathbb{A}_3$ where the vertices are labelled as follows,

$$0 \longrightarrow 1 \longrightarrow 2$$

then the complexes of interest to us are

$$0 \to V_1 \to V_0 \oplus V_2 \to V_1 \to 0, \quad \text{and} \quad 0 \to V_2 \to V_1 \to V_2 \to 0.$$

Since $\dim_{\Bbbk} V_0 = 1$, $\dim_{\Bbbk} V_1 = 1$, and $\dim_{\Bbbk} V_2 = 1$, the first of these is exact while the second is not. If instead we label the vertices of Q as follows,

$$1 \longrightarrow 0 \longrightarrow 2$$

then the complexes of interest to us are

$$0 \to V_1 \to V_0 \to V_1 \to 0, \quad \text{and} \quad 0 \to V_2 \to V_0 \to V_2 \to 0,$$

and both of these are exact since $\dim_{\Bbbk} V_0 = 2$, $\dim_{\Bbbk} V_1 = 1$, and $\dim_{\Bbbk} V_2 = 1$.

Proposition 3.3.5 now allows us to show that the induced translation functor on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ maps connected components to themselves.

Proposition 3.3.8. *Let $\widetilde{Q}$ be an extended Dynkin quiver and λ be a quasi-dominant weight for $\widetilde{Q}$. Write $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ as a disjoint union of connected quivers $Q^{(i)}$, which are therefore necessarily Dynkin. Consider the triangulated structure on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ induced by the $\Bbbk$-linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$$

of Corollary 3.2.9, and let Σ be the translation functor. Then each $\text{proj-}\Pi(Q^{(i)})$ *is invariant under Σ.*

Proof. By Lemma 3.3.4 and the discussion following it, the only situation in which there exist $\text{proj-}\Pi(Q^{(i)})$ which are not necessarily invariant under Σ is when we have multiple $Q^{(i)}$ equal to $\mathbb{A}_1$. Working in $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{Q})$, this happens if and only if there is some vertex i with $\lambda_i = 0$, and if j is adjacent to i then either $j = 0$ or $\lambda_j \neq 0$; in particular, the modules V_j corresponding to these vertices are projective as $\mathcal{O}^\lambda(\widetilde{Q})$-modules by Lemma 3.2.4. By Proposition 3.3.5, we have an exact sequence of $e_0\Pi(\widetilde{Q})e_0$-modules

$$0 \to V_i \xrightarrow{\phi} \bigoplus_{j \in \partial i} V_j \xrightarrow{\psi} V_i \to 0. \tag{3.3.9}$$

Now consider (3.3.9) as a sequence of modules over $\mathcal{O}^\lambda(\widetilde{Q})$. It is a complex since the composition $\psi\phi$ is equal to the (undeformed) preprojective relation at vertex i, which is equal to $\lambda_i e_i = 0$. Filtering $\Pi(\widetilde{Q})$ and $\mathcal{O}^\lambda(\widetilde{Q})$ by path length we obtain a sequence of associated graded modules, which is in fact the exact sequence (3.3.9). It is standard (see [MR01, Proposition 1.6.7]) that this implies that (3.3.9) is exact as a sequence of modules over $\mathcal{O}^\lambda(\widetilde{Q})$. To summarise, we have an exact sequence of $\mathcal{O}^\lambda(\widetilde{Q})$-modules

$$0 \to V_i \to \bigoplus_{j \in \partial i} V_j \to V_i \to 0$$

whose middle term is projective. By definition of the translation functor, $\Sigma V_i = V_i$ in $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{Q})$. Thus each $\text{proj-}\Pi(Q^{(i)})$ is invariant under the induced translation functor on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$. □

We now seek to prove Theorem 1.2.5. Retaining all of the above notation, for each $1 \leqslant i \leqslant r$, define

$$\mathcal{W}_i := \{V_j \mid j \in Q_0^{(i)}\}, \quad \mathcal{C}_i := \text{add}\left(V_0 \oplus \bigoplus_{j \in Q_0^{(i)}} V_j\right) \quad \text{and} \quad \mathcal{T}_i := \text{add}\left(\bigoplus_{j \in Q_0^{(i)}} V_j\right),$$

where the latter two are viewed as subcategories of $\text{MCM}\text{-}\mathcal{O}^\lambda$ and $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$, respectively. It will also be convenient to write

$$M_i = \bigoplus_{j \in (Q_\lambda)_0 \setminus Q_0^{(i)}} V_j,$$

and to set

$$\mathcal{W}_i^{\mathrm{c}} := \{V_j \mid j \in (Q_\lambda)_0 \setminus Q_0^{(i)}\}, \quad \mathcal{C}_i^{\mathrm{c}} := \operatorname{add}(V_0 \oplus M_i) \quad \text{and} \quad \mathcal{T}_i^{\mathrm{c}} := \operatorname{add} M_i.$$

Observe that we can decompose $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ as

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \bigoplus_{i=1}^{r} \mathcal{T}_i$$

as $\Bbbk$-linear categories. We wish to show that this is also a decomposition into triangulated subcategories. To do this, we first prove a result which shows that the $\mathcal{C}_i$ are Frobenius subcategories of the Frobenius category $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$. Following [Che12], we call a subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ *extension-closed* if whenever we have a conflation $X \to Y \to Z$ with $X, Z \in \mathcal{B}$ then necessarily $Y \in \mathcal{B}$. Furthermore, an extension-closed subcategory $\mathcal{B}$ is called *admissible* provided that every $B \in \mathcal{B}$ fits into conflations $B \to P \to B'$ and $B'' \to Q \to B$ with $B', B'' \in \mathcal{B}$ and where P, Q are projective in $\mathcal{A}$. We remark that an admissible subcategory of a Frobenius category is itself Frobenius, see [Che12, §2].

Lemma 3.3.10. *For each i, the subcategory $\mathcal{C}_i$ satisfies the following property: if $X \to Y \to Z$ is a conflation in* $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$ *such that two of the three objects are in $\mathcal{C}_i$, then the third object is also in $\mathcal{C}_i$. Consequently, $\mathcal{C}_i$ is a Frobenius subcategory of* $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$.

Proof. We only show that if $X \to Y \to Z$ is a conflation with $X, Y \in \mathcal{C}_i$ then $Z \in \mathcal{C}_i$, with the other cases being similar. So suppose that we have such a conflation. Since $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda)$ and this category is Krull-Schmidt, we have $Z \oplus P \cong U \oplus U' \oplus Q$ where $U \in \mathcal{W}_i$, $U' \in \mathcal{W}_i^{\mathrm{c}}$, and P, Q are projective. This conflation gives rise to a triangle $X \to Y \to Z \to \Sigma X$ in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$, and applying the functor $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, -)$ yields an exact sequence

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, Y) \to \underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, \Sigma X).$$

Now $\Sigma X \in \mathcal{C}_i$ by Proposition 3.3.8 and $Y \in \mathcal{C}_i$ by definition, while $M_i \in \mathcal{C}_i^{\mathrm{c}}$, so both of the flanking terms are 0. This implies that the middle term, which is equal to $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, U')$, is also 0. But this means that $U' = 0$, and hence $Z \oplus P \in \mathcal{C}_i$. Since, by definition, $\mathcal{C}_i$ is closed under direct summands, it follows that $Z \in \mathcal{C}_i$ as required.

For the final claim, first notice that the above paragraph tells us that $\mathcal{C}_i$ is extension-closed. Moreover, given an object $C \in \mathcal{C}_i$, since $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$ is Frobenius we can always find conflations $C \to P \to Z$ and $X \to Q \to C$ with $X, Z \in \mathrm{MCM}\text{-}\mathcal{O}^\lambda$ and P, Q projective. Since projective $\mathcal{O}^\lambda$-modules are direct summands of sums of copies of $\mathcal{O}^\lambda$, we have $P, Q \in \mathcal{C}_i$ by definition, and then the previous paragraph tells us that $X, Z \in \mathcal{C}_i$. Therefore $\mathcal{C}_i$ is admissible and hence Frobenius. □

This allows us to prove our main theorem:

Theorem 3.3.11. *Let $\widetilde{Q}$, Q, and λ be as above, and suppose that $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$,* *where the $Q^{(i)}$ are connected and necessarily Dynkin. Then the $\mathbb{k}$-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}),$$

of Corollary 3.2.9 is a triangle equivalence, where the right hand side is a decomposition into triangulated subcategories satisfying proj-$\Pi(Q^{(i)}) \simeq \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$.

Proof. By Lemma 3.3.10, we know that $\mathcal{C}_i$ is a Frobenius subcategory of MCM-$\mathcal{O}^\lambda$. Using [AH17, Theorem 3.15 (2)], it follows that $\mathcal{T}_i$ is equal to the stable category of the Frobenius category $\mathcal{C}_i$ for $1 \leqslant i \leqslant r$, and so the decomposition

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \bigoplus_{i=1}^{r} \mathcal{T}_i$$

is in fact a decomposition into triangulated subcategories. If we set $e^{(i)} = \sum_{j \in Q_0^{(i)}} e_j$, [Kra15, Proposition 2.3] implies that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(e^{(i)}\Pi^\lambda e_0, -) : \underline{\mathrm{mod}}\text{-}\mathcal{O}^\lambda \to \mathrm{mod}\text{-}\,\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(e^{(i)}\Pi^\lambda e_0)$$

restricts to a $\mathbb{k}$-linear equivalence $\mathcal{T}_i \simeq$ proj-$\Pi(Q^{(i)})$. This equivalence also induces an algebraic triangulated structure on proj-$\Pi(Q^{(i)})$. Since this category is $\mathbb{k}$-linearly equivalent to $\mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$, Theorem 3.3.2 implies that they are triangle equivalent, completing the proof. □

3.4 Uniqueness of the translation functor on objects of proj-$\Pi(Q)$ when Q is Dynkin

In this section, we show that if Q is Dynkin and proj-$\Pi(Q)$ has the structure of a (not necessarily algebraic) triangulated category, then the translation functor Σ is uniquely determined on objects of proj-$\Pi(Q)$. In particular, this tells us how the translation functor acts on objects in Theorem 3.3.11, something which is already well-known.

For the remainder of this section, write $P_1, \dots, P_n$ for the n indecomposable projective right $\Pi(Q)$-modules corresponding to the vertices of Q. Write $W_0, \dots, W_n$ for the $n+1$ irreducible representations of the finite group G corresponding to Q. Since proj-$\Pi(Q)$ is Krull-Schmidt, it is easy to see that $\Sigma P_i = P_j$ for some j, so write σ for the permutation of the vertices of Q satisfying $\Sigma P_i = P_{\sigma(i)}$. The map $W_i \to W_i^*$ sending a representation to its dual is an involution of $\{W_1, \dots, W_n\}$ (where we intentionally omit W_0), and we can view this map as an automorphism ν of Q. Throughout this section, all Hom spaces are over $\Pi(Q)$, and we omit this subscript. The aim of this section is to prove the following result, which we achieve by analysing cases.

Theorem 3.4.1. *Consider the category* proj-$\Pi(Q)$ *with some triangulated structure with translation functor Σ. Then $\sigma = \nu$ as automorphisms of Q.*

Remark 3.4.2. Explicitly, ν is the identity automorphism of Q when Q is $\mathbb{A}_1, \mathbb{D}_n$ (n even), $\mathbb{E}_7$, or $\mathbb{E}_8$, and it is the unique graph automorphism of order 2 when Q is $\mathbb{A}_n$ ($n \geqslant 2$), $\mathbb{D}_n$ (n odd), or $\mathbb{E}_6$.

We first make the following observation:

Lemma 3.4.3. *With the above setup, σ is a graph automorphism of Q.*

Proof. First note that the spaces $\mathrm{Hom}(P_i, P_j) = e_j\Pi(Q)e_i$ can be graded by path length, and that vertex i and vertex j are adjacent in Q if and only if there is a degree 1 morphism in $\mathrm{Hom}(P_i, P_j)$. Applying Σ, this is equivalent to $\mathrm{Hom}(P_{\sigma(i)}, P_{\sigma(j)})$ containing a degree 1 morphism, which happens if and only if $\sigma(i)$ and $\sigma(i)$ are adjacent in Q. That is, σ is a graph automorphism of Q. □

Since the automorphism group of an $\mathbb{A}_1$, $\mathbb{E}_7$, or $\mathbb{E}_8$ graph is trivial, it immediately follows that σ is the identity in these cases. For the remaining cases, we argue that Σ is uniquely determined using Lemma 3.4.3 and by considering the dimensions of the Hom spaces between the P_i. We record the dimensions of these Hom spaces in the following lemma; the $\mathbb{E}_7$ and $\mathbb{E}_8$ cases are unnecessary and hence omitted, but they can be established in the same way.

Lemma 3.4.4. *Let Q be a Dynkin quiver with n vertices and let $P_1, \dots, P_n$ be the n indecomposable projective right $\Pi(Q)$-modules corresponding to the vertices of Q. Let $H(Q)$ be the matrix with*

$$H(Q)_{ij} = \dim_{\Bbbk} \mathrm{Hom}(P_j, P_i) = \dim_{\Bbbk} e_i\Pi(Q)e_j.$$

(1) *If $Q = \mathbb{A}_n$ then*

$$H(\mathbb{A}_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}.$$

(2) *If $Q = \mathbb{D}_n$ then*

$$H(\mathbb{D}_n) = \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & \left\lceil \frac{n-1}{2} \right\rceil & \left\lfloor \frac{n-1}{2} \right\rfloor \\ 1 & 2 & 3 & \cdots & n-2 & \left\lfloor \frac{n-1}{2} \right\rfloor & \left\lceil \frac{n-1}{2} \right\rceil \end{pmatrix}.$$

(3) *If $Q = \mathbb{E}_6$ then*

$$H(\mathbb{E}_6) = \begin{pmatrix} 4 & 2 & 4 & 6 & 4 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 6 & 4 & 8 & 12 & 8 & 4 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 2 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}.$$

Proof. These can be calculated using [ES98a, §4], [ES98b, 3.4], and [MOV06, Theorem 2.3.b]. □

We now begin our case-by-case argument. In each case, the technique is the same: seeking a contradiction, we show that if σ is a graph automorphism of Q different from the one given in Proposition 3.4.1 then we arrive at a contradiction. We begin with the type $\mathbb{A}$ case:

Proposition 3.4.5. *Let σ be the graph automorphism of $\mathbb{A}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{A}_n)$*. Then σ is the identity when $n = 1$, and it is the unique order 2 graph automorphism when $n \geqslant 2$.*

Proof. We have already established the $n = 1$ case, so suppose $n \geqslant 2$. By Lemma 3.4.3, σ is either the identity or has order 2 so, seeking a contradiction, suppose that σ is the identity; that is $\Sigma P_i = P_i$ for all i. Consider the nonzero morphism $P_1 \to P_n$ given by left multiplication by $\overline{\alpha}_{n-1}\overline{\alpha}_{n-2}\dots\overline{\alpha}_1$, which gives rise to a distinguished triangle

$$P_1 \to P_n \to M \to P_1$$

for some $M \in \text{proj-}\Pi(\mathbb{A}_n)$. Applying $\text{Hom}(-, P_n)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc}
\text{Hom}(P_n,P_n) & \xrightarrow{\cdot\overline{\alpha}_{n-1}\overline{\alpha}_{n-2}\dots\overline{\alpha}_1} & \text{Hom}(P_1,P_n) & \xrightarrow{\beta} & \text{Hom}(M,P_n) & \xrightarrow{\gamma} & \text{Hom}(P_n,P_n) & \xrightarrow{\cdot\overline{\alpha}_{n-1}\overline{\alpha}_{n-2}\dots\overline{\alpha}_1} & \text{Hom}(P_1,P_n) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_n & & \Bbbk\overline{\alpha}_{n-1}\overline{\alpha}_{n-2}\dots\overline{\alpha}_1 & & & & \Bbbk e_n & & \Bbbk\overline{\alpha}_{n-1}\overline{\alpha}_{n-2}\dots\overline{\alpha}_1
\end{array}$$

where we use Lemma 3.4.4 to write down bases for each of the Hom spaces. Now the left hand map is surjective, so exactness implies that β is the zero map, which forces γ to be injective. Moreover, the right hand map is injective, so that γ is the zero map. In particular, Lemma 3.4.4 implies that we have $\text{Hom}(M, P_n) = 0$ and so $M = 0$, but this tells us that $P_1 \cong P_n$ which is absurd. Therefore σ must be the unique order 2 graph automorphism of $\mathbb{A}_n$. □

We now turn our attention to the type $\mathbb{E}$ cases:

Proposition 3.4.6. *Let σ be the graph automorphism of $\mathbb{E}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{E}_n)$*, where $n \in \{6, 7, 8\}$. Then σ is the identity when $n \neq 6$, and it is the unique order 2 graph automorphism when $n = 6$.*

Proof. Again, the $\mathbb{E}_7$ and $\mathbb{E}_8$ cases are immediate from Lemma 3.4.3, so consider $\mathbb{E}_6$. By Lemma 3.4.3, σ is either the identity or has order 2 so, seeking a contradiction, suppose that σ is the identity. Consider the nonzero morphism $P_2 \to P_6$ given by left multiplication by $\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2$, which gives rise to a distinguished triangle

$$P_2 \to P_6 \to M \to P_2$$

for some $M \in \text{proj-}\Pi(\mathbb{E}_6)$. Applying $\text{Hom}(-, P_6)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc}
\text{Hom}(P_6,P_6) & \xrightarrow{\cdot\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2} & \text{Hom}(P_2,P_6) & \xrightarrow{\beta} & \text{Hom}(M,P_6) & \xrightarrow{\gamma} & \text{Hom}(P_6,P_6) & \xrightarrow{\cdot\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2} & \text{Hom}(P_2,P_6) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_6 & & \Bbbk\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2 & & & & \Bbbk e_6 & & \Bbbk\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2 \\
\oplus & & \oplus & & & & \oplus & & \oplus \\
\Bbbk\alpha_5\overline{\alpha}_4\alpha_1\overline{\alpha}_1\alpha_4\overline{\alpha}_5 & & \Bbbk\alpha_5\overline{\alpha}_4\alpha_1\overline{\alpha}_1\alpha_4\overline{\alpha}_5\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2 & & & & \Bbbk\alpha_5\overline{\alpha}_4\alpha_1\overline{\alpha}_1\alpha_4\overline{\alpha}_5 & & \Bbbk\alpha_5\overline{\alpha}_4\alpha_1\overline{\alpha}_1\alpha_4\overline{\alpha}_5\alpha_5\overline{\alpha}_4\alpha_3\overline{\alpha}_2
\end{array}$$

where again we use Lemma 3.4.4 to write down bases for each of the Hom spaces. We see that the left hand map is surjective and so β is the zero map, and exactness implies that γ is injective. Since the right hand map is injective it follows that γ is the zero map. Therefore by Lemma 3.4.4 $\mathrm{Hom}(M, P_6) = 0$ and so $M = 0$, but this tells us that $P_2 \cong P_6$ which is absurd. Therefore σ must be the unique order 2 graph automorphism of $\mathbb{E}_6$. ☐

Finally we consider the type $\mathbb{D}$ cases. Since we claim that Σ behaves differently depending on whether n is odd or even, we have to consider these two cases separately; additionally, we consider the $n = 4$ case separately since $\mathrm{Aut}(\mathbb{D}_4) \cong S_3$ instead of it having order 2.

Proposition 3.4.7. *Let σ be the graph automorphism of $\mathbb{D}_4$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{D}_4)$*. Then σ is the identity.*

Proof. By Lemma 3.4.3, σ is either the identity, a two-cycle which swaps a pair of vertices $i \neq 2 \neq j$, or it cycles the vertices $1, 3, 4$. We rule out the latter two possibilities.

First suppose that σ is a two-cycle: without loss of generality, $\sigma = (3\,4)$. Consider the nonzero morphism $P_3 \to P_4$ given by left multiplication by $\overline{\alpha}_4\alpha_3$, which gives rise to a distinguished triangle

$$P_3 \to P_4 \to M \to P_4$$

for some $M \in$ proj-$\Pi(\mathbb{D}_4)$. Applying $\mathrm{Hom}(-, P_3)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc} \mathrm{Hom}(P_3,P_3) & \xrightarrow{\cdot\overline{\alpha}_3\alpha_4} & \mathrm{Hom}(P_4,P_3) & \xrightarrow{\beta} & \mathrm{Hom}(M,P_3) & \xrightarrow{\gamma} & \mathrm{Hom}(P_4,P_3) & \xrightarrow{\cdot\overline{\alpha}_4\alpha_3} & \mathrm{Hom}(P_3,P_3) \\ \| & & \| & & & & \| & & \| \\ \Bbbk e_3 \oplus \Bbbk\overline{\alpha}_3\alpha_4\overline{\alpha}_4\alpha_3 & & \Bbbk\overline{\alpha}_3\alpha_4 & & & & \Bbbk\overline{\alpha}_3\alpha_4 & & \Bbbk e_3 \oplus \Bbbk\overline{\alpha}_3\alpha_4\overline{\alpha}_4\alpha_3 \end{array}$$

Clearly the left hand map surjects, so exactness forces β to be the zero map, which in turn implies that γ is injective. The right hand map is injective, and exactness forces γ to be the zero map. In particular we have $\mathrm{Hom}(M, P_3) = 0$ and so $M = 0$, but this tells us that $P_3 \cong P_4$ which is absurd. Therefore σ is not a two-cycle.

Now suppose that σ is a three-cycle: without loss of generality, $\sigma = (1\,3\,4)$. We now consider the triangle obtained from the morphism $\alpha_1\alpha_3\cdot : P_3 \to P_1$,

$$P_3 \to P_1 \to M \to P_4$$

and seek to obtain contradiction. Applying the functor $\mathrm{Hom}(-, P_3)$, we get exactness of the following sequence:

$$\begin{array}{ccccccccc} \mathrm{Hom}(P_3,P_3) & \xrightarrow{\cdot\overline{\alpha}_3\alpha_4} & \mathrm{Hom}(P_4,P_3) & \xrightarrow{\beta} & \mathrm{Hom}(M,P_3) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1,P_3) & \xrightarrow{\cdot\alpha_1\alpha_3} & \mathrm{Hom}(P_3,P_3) \\ \| & & \| & & & & \| & & \| \\ \Bbbk e_3 \oplus \Bbbk\overline{\alpha}_3\overline{\alpha}_1\alpha_1\alpha_3 & & \Bbbk\overline{\alpha}_3\alpha_4 & & & & \Bbbk\overline{\alpha}_3\overline{\alpha}_1 & & \Bbbk e_3 \oplus \Bbbk\overline{\alpha}_3\overline{\alpha}_1\alpha_1\alpha_3 \end{array}$$

Again the left hand map is surjective, forcing β to be the zero map and hence γ to be injective. Moreover, the right hand map is injective, and so γ must be the zero map. In particular we have $\mathrm{Hom}(M, P_3) = 0$ and so $M = 0$, but this tells us that $P_1 \cong P_3$ which is absurd. Therefore σ is not a three-cycle, and hence must be the identity. ☐

Proposition 3.4.8. *Let $n \geqslant 5$ be odd and let σ be the graph automorphism of $\mathbb{D}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{D}_n)$*. Then σ is the unique graph automorphism of order 2.*

Proof. By Lemma 3.4.3, σ is either the identity or $(n{-}1\ n)$ so, seeking a contradiction, assume it is the former; that is, $\Sigma P_i = P_i$ for all i. Consider the morphism $P_n \to P_1$ given by left multiplication by $\alpha_1\alpha_2\alpha_3 \dots \alpha_{n-3}\alpha_n$. This gives rise to a distinguished triangle

$$P_n \to P_1 \to M \to P_n$$

for some $M \in$ proj-$\Pi(\mathbb{D}_n)$. Applying $\mathrm{Hom}(-, P_1)$ gives rise to the following exact sequence,

$$\begin{array}{ccccccccc}
\mathrm{Hom}(P_1,P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n,P_1) & \xrightarrow{\beta} & \mathrm{Hom}(M,P_1) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1,P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n,P_1) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n & & & & \Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n
\end{array}$$

where here p is some path. The left hand map is surjective, so β is the zero map and therefore γ is injective. The kernel of the right hand map is one-dimensional, and exactness tells us that γ has rank 1. In particular we have $\dim \mathrm{Hom}(M, P_1) = 1$ and so M is either P_{n-1} or P_n by Lemma 3.4.4. If we instead apply $\mathrm{Hom}(-, P_n)$, the resulting exact sequence is

$$\begin{array}{ccccccccc}
\mathrm{Hom}(P_1,P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n,P_n) & \xrightarrow{\beta} & \mathrm{Hom}(M,P_n) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1,P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n,P_n) \\
\| & & \| & & & & \| & & \| \\
\Bbbk\overline{\alpha}_n\overline{\alpha}_{n-3}\dots\overline{\alpha}_2\overline{\alpha}_1 & & \Bbbk^{\frac{n-1}{2}} & & & & \Bbbk\overline{\alpha}_n\overline{\alpha}_{n-3}\dots\overline{\alpha}_2\overline{\alpha}_1 & & \Bbbk^{\frac{n-1}{2}}
\end{array}$$

Since n is odd, the shortest path from vertex n to vertex 1 and back to vertex n is zero in $\Pi(\mathbb{D}_n)$, so the first and the last maps both have rank zero. Therefore β has full rank, forcing the kernel of γ to have dimension $\frac{n-1}{2}$. Exactness also forces γ to have rank 1, and therefore $\dim \mathrm{Hom}(M, P_n) = \frac{n-1}{2} + 1$. Now we have already seen that M is either P_{n-1} or P_n, but $\dim \mathrm{Hom}(P_{n-1}, P_n) = \frac{n-1}{2} = \dim \mathrm{Hom}(P_n, P_n)$, so we have a contradiction. Therefore σ is the unique graph automorphism of order 2. □

Proposition 3.4.9. *Let $n \geqslant 6$ be even and let σ be the graph automorphism of $\mathbb{D}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{D}_n)$*. Then σ is the identity.*

Proof. By Lemma 3.4.3, σ is either the identity or $(n{-}1\ n)$ so, seeking a contradiction, assume it is the latter. Consider the morphism $P_n \to P_1$ given by left multiplication by $\alpha_1\alpha_2 \dots \alpha_{n-3}\alpha_n$, and extend this to a distinguished triangle

$$P_n \to P_1 \to M \to P_{n-1}$$

for some $M \in$ proj-$\Pi(\mathbb{D}_n)$. If we apply $\mathrm{Hom}(-, P_1)$ we get the following exact sequence

$$\begin{array}{ccccccccc}
\mathrm{Hom}(P_1,P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_{n-1}} & \mathrm{Hom}(P_{n-1},P_1) & \xrightarrow{\beta} & \mathrm{Hom}(M,P_1) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1,P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n,P_1) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_{n-1} & & & & \Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\dots\alpha_{n-3}\alpha_n
\end{array}$$

where p is some path. The left hand map surjects, so $\beta = 0$ and therefore γ injects. The right hand map has a one-dimensional kernel, so γ has rank 1. It follows that $\dim \operatorname{Hom}(M, P_1) = 1$, which implies that M is either P_{n-1} or P_n. If we instead apply $\operatorname{Hom}(-, P_n)$ we get

$$\begin{array}{ccccccccc}
\operatorname{Hom}(P_1,P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\ldots\alpha_{n-3}\alpha_{n-1}} & \operatorname{Hom}(P_{n-1},P_n) & \xrightarrow{\theta} & \operatorname{Hom}(M,P_n) & \xrightarrow{\eta} & \operatorname{Hom}(P_1,P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\ldots\alpha_{n-3}\alpha_n} & \operatorname{Hom}(P_n,P_n) \\
\| & & \| & & & & \| & & \| \\
\Bbbk\overline{\alpha}_n\overline{\alpha}_{n-3}\ldots\overline{\alpha}_2\overline{\alpha}_1 & & \Bbbk^{\frac{n}{2}-1} & & & & \Bbbk\overline{\alpha}_n\overline{\alpha}_{n-3}\ldots\overline{\alpha}_2\overline{\alpha}_1 & & \Bbbk^{\frac{n}{2}}
\end{array}$$

Since n is even, the shortest path from vertex n to vertex 1 and then to vertex $n-1$ is zero in $\Pi(\mathbb{D}_n)$, while the shortest path from vertex n to vertex 1 and back to vertex n is nonzero. It follows that the left hand map is the zero map, while the right hand map has rank 1. Therefore θ has full rank which implies that the kernel of η has dimension $\frac{n}{2} - 1$. Moreover, the right hand map is injective, so that η has rank 0 and so $\dim \operatorname{Hom}(M, P_n) = \frac{n}{2} - 1$. Combining this with our earlier restriction on M, this forces $M = P_{n-1}$, and our distinguished triangle is therefore

$$P_n \to P_1 \to P_{n-1} \to P_{n-1}.$$

Since $\operatorname{Hom}(P_1, P_{n-1})$ is spanned by $\overline{\alpha}_{n-1}\overline{\alpha}_{n-3}\ldots\overline{\alpha}_2\overline{\alpha}_1$ and γ is not the zero map, we can assume that the map $P_1 \to P_{n-1}$ in this triangle is given by left multiplication by (a scalar multiple of) $\overline{\alpha}_{n-1}\overline{\alpha}_{n-3}\ldots\overline{\alpha}_2\overline{\alpha}_1$. Moreover, $\operatorname{Hom}(P_{n-1}, P_{n-1}) = \operatorname{span}\{e_{n-1}, p_2, \ldots, p_{n/2}\}$ where the p_i are paths of length $\geqslant 4$. Since θ is not the zero map and the composition $P_1 \to P_{n-1} \to P_{n-1}$ must be zero, the map $P_{n-1} \to P_{n-1}$ in this triangle lies in $\operatorname{span}\{p_2, \ldots, p_{n/2}\}$. But then $\theta : \operatorname{Hom}(P_{n-1}, P_n) \to \operatorname{Hom}(P_{n-1}, P_n)$ maps the longest path in $\operatorname{Hom}(P_{n-1}, P_n)$ to zero, contradicting the fact that θ has trivial kernel. It follows that σ is not a two-cycle. □

3.5 A noncommutative geometric McKay correspondence

Let $\widetilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, let Q be the quiver obtained by removing the extending vertex, and let $\lambda = \varepsilon_0 = (1, 0, \ldots, 0)$; that is, the weight at the extending vertex is 1, and 0 for all of the other vertices. We may then consider $\mathcal{O}^\lambda(\widetilde{Q})$ to be a noncommutative analogue of $R_Q = \mathcal{O}(\widetilde{Q})$, the coordinate ring of the corresponding Kleinian singularity; indeed, we have just seen that these rings have the same singularity categories. We now provide another reason why $\mathcal{O}^\lambda$ may be considered to be a noncommutative analogue of R_Q.

Motivated by, for example, [VdB04], we say that a $\Bbbk$-algebra S is a *noncommutative resolution* of a $\Bbbk$-algebra R if $S = \operatorname{End}_R(M)$ for some reflexive generator M and $\operatorname{gl.dim} S < \infty$. Moreover, following [MS01], given $M, N \in \text{mod-}S$ which satisfy $\dim_\Bbbk \operatorname{Ext}^\ell_S(M, N) < \infty$ for all $\ell \geqslant 0$, we define the *intersection multiplicity* of M and N to be

$$M \bullet N := \sum_{\ell \geqslant 0} (-1)^{\ell+1} \dim_\Bbbk \operatorname{Ext}^\ell_S(M, N)$$

(note that this sum has finitely many terms since S is nonsingular). We first make the following observation:

Lemma 3.5.1. *If S is a noncommutative resolution of R and T is Morita equivalent to S, then T is also a noncommutative resolution of R.*

Proof. Since S is a noncommutative resolution of R, there exists a reflexive generator $M \in \text{mod-}R$ such that $S = \text{End}_R(M)$ and $\text{gl.dim}\, S < \infty$. Since S and T are Morita equivalent, there exists a progenerator $P \in \text{mod-}S$ with $T = \text{End}_S(P)$. We claim that the module $N := P \otimes_S M \in \text{mod-}R$ is a reflexive generator which satisfies $\text{End}_R(N) = T$. If we can show that this is the case, the fact that $\text{gl.dim}\, T = \text{gl.dim}\, S < \infty$ will mean that T is a noncommutative resolution of R.

It will be convenient to recall some general facts from Morita theory before proceeding with the proof; our main reference is [Lam99, Section 18C]. Since ${}_TP_S$ is a progenerator, $P^* \otimes_T P \cong S$ and there are natural isomorphisms of functors $\text{Hom}_S(P, -) \cong - \otimes_S P^*$ and $\text{Hom}_S(P^*, -) \cong P \otimes_S -$. Moreover, since M_R is a generator, we have $M^* \otimes_S M \cong R$.

We first show that N is reflexive. Indeed,

$$\begin{aligned}\text{Hom}_R(\text{Hom}_R(N, R), R) &= \text{Hom}_R(\text{Hom}_R(P \otimes_S M, R), R) \cong \text{Hom}_R(\text{Hom}_S(P, M^*), R)\\ &\cong \text{Hom}_R(M^* \otimes_S P^*, R) \cong \text{Hom}_S(P^*, \text{Hom}_R(M^*, R))\\ &\cong \text{Hom}_S(P^*, M) \cong P \otimes_S M = N,\end{aligned}$$

using the above facts, hom-tensor adjointness and reflexivity of M.

We now show that $\text{End}_R(N) \cong T$. Noting that projectivity of P implies that there is a natural isomorphism of bifunctors $\text{Hom}_R(-, P \otimes_S \square) \cong P \otimes_S \text{Hom}_R(-, \square)$ (as bifunctors $\text{mod-}R \times \text{bimod-}(S, R) \to \text{mod-}R \times \text{bimod-}(S, R)$), we have

$$\begin{aligned}\text{End}_R(N) &= \text{Hom}_R(P \otimes_S M, P \otimes_S M) \cong \text{Hom}_S(P, \text{Hom}_R(M, P \otimes_S M))\\ &\cong \text{Hom}_S(P, P \otimes_S \text{Hom}_R(M, M)) \cong \text{Hom}_S(P, P \otimes_S S) \cong \text{End}_S(P) \cong T,\end{aligned}$$

as required.

Finally, to see that N is a generator for mod-R we equivalently show that $N^* \otimes_T N \cong R$. Indeed,

$$\begin{aligned}N^* \otimes_T N &= \text{Hom}_R(P \otimes_S M, R) \otimes_T P \otimes_S M \cong \text{Hom}_S(P, M^*) \otimes_T P \otimes_S M\\ &\cong M^* \otimes_S P^* \otimes_T P \otimes_S M \cong M^* \otimes_S S \otimes_S M \cong M^* \otimes_S M \cong R,\end{aligned}$$

which completes the proof. □

3.5.1 Intersection theory for a family of noncommutative resolutions

We return now to the $\Bbbk$-algebra of interest, namely $\mathcal{O}^\lambda$ where $\lambda = \varepsilon_0$. Our first aim is to identify an appropriate noncommutative resolution, which we have in fact already done:

Lemma 3.5.2. Π^λ *is a noncommutative resolution of* $\mathcal{O}^\lambda$.

Proof. First note that the $\mathcal{O}^\lambda$-module $\Pi^\lambda e_0$ has $e_0 \Pi^\lambda e_0 = \mathcal{O}^\lambda$ as a direct summand, and so is a generator. It is also reflexive by Corollary 3.2.2. Moreover, $\Pi^\lambda = \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$ by Lemma 3.2.1, and this is nonsingular by [CBH98, Theorem 1.5]. Therefore Π^λ is a noncommutative resolution of $\mathcal{O}^\lambda$ by definition. □

We can actually obtain infinitely many noncommutative resolutions of $\mathcal{O}^\lambda$ using the dual reflections r_i of [CBH98], the definition of which was given in Definition 2.5.5. It is clear that the r_i preserve the $\mathbb{Z}^{n+1}$ lattice inside $\Bbbk^{n+1}$. Moreover, it is not difficult to show that $\lambda \cdot \delta = r_i\lambda \cdot \delta$ for all $\lambda \in \Bbbk^{\widetilde{Q}_0}$ and $i \in \widetilde{Q}_0$, so that the r_i preserve the affine hyperplanes $\{\lambda \in \Bbbk^{n+1} \mid \lambda \cdot \delta = c\}$ for each $c \in \Bbbk$; since $\varepsilon_0 \cdot \delta = 1$, we are primarily interested in the case $c = 1$. Then we have the following:

Lemma 3.5.3 ([CBH98, Corollary 5.2]). *Let ρ be a composition of dual reflections. Then Π^λ is Morita equivalent to $\Pi^{\rho(\lambda)}$.*

By combining Lemmas 3.5.1, 3.5.2 and 3.5.3, we obtain the following:

Corollary 3.5.4. *$\Pi^{\rho(\lambda)}$ is a noncommutative resolution of $\mathcal{O}^\lambda$ for any composition of dual reflections ρ.*

To establish a noncommutative version of the geometric McKay correspondence, we first identify an analogue of the exceptional curves appearing in the minimal resolution of a Kleinian singularity. When $\lambda = \varepsilon_0$, by [CBH98, Lemma 7.1 (2)], Π^λ has precisely n isoclasses of finite-dimensional simple modules, and hence by Morita equivalence so does $\Pi^{\rho(\lambda)}$ for any composition of dual reflections ρ. These will play the role of the exceptional objects in our noncommutative resolution.

This allows us to prove a preliminary version of Theorem 1.2.7 from the introduction:

Theorem 3.5.5. *Let $\widetilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, and let $\lambda = \varepsilon_0$. Let $\mu = \rho(\lambda)$, where ρ is any composition of dual reflections, so that Π^μ is a noncommutative resolution of $\mathcal{O}^\lambda$. Then Π^μ has precisely n finite-dimensional simple modules S_i up to isomorphism, and with a suitable indexing of them, the intersection matrix Γ with entries $\Gamma_{ij} = S_i \bullet S_j$ is $-C$, where C is the Cartan matrix corresponding to Q.*

Proof. The discussion after Corollary 3.5.4 shows that Π^μ has n finite-dimensional simple modules S_i up to isomorphism, so it remains to prove the result on the intersection multiplicities.

Since Morita equivalence preserves dimensions of Hom and Ext groups, we are able to calculate the intersection numbers of the finite-dimensional Π^μ-modules by doing the calculations over Π^λ instead. Identifying Π^λ-modules with representations of $\overline{\widetilde{Q}}$ which satisfy the relations coming from Π^λ, [CBH98, Lemma 7.2 (6), Theorem 7.4] tells us that the dimension vector of S_i is $\varepsilon_i \in \mathbb{N}^{n+1}$. It follows that

$$S_i \cong \frac{e_i\Pi^\lambda}{\bigoplus_{\substack{\alpha \in \overline{\widetilde{Q}}_1 \\ t(\alpha)=i}} \alpha\Pi^\lambda}. \tag{3.5.6}$$

Also observe that

$$\mathrm{Hom}_{\Pi^\lambda}(e_i\Pi^\lambda, S_j) = \begin{cases} \Bbbk e_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{3.5.7}$$

For each $i \neq 0$, it is easy to see that we have a complex

$$0 \to e_i\Pi^\lambda \xrightarrow{\phi} \bigoplus_{k \in \partial i} e_k\Pi^\lambda \xrightarrow{\psi} e_i\Pi^\lambda \to S_i \to 0,$$

where $\psi\phi$ is the preprojective relation at vertex i and hence equal to 0. Filtering by path length, the corresponding complex of $\Pi(Q)$-modules is precisely the exact sequence

(3.3.6), and hence (3.5.1) is also exact. Since the modules $e_k\Pi^\lambda$ are direct summands of Π^λ and hence projective, (3.5.1) is in fact a projective resolution of S_i. Now let $1 \leqslant j \leqslant n$. Seeking to calculate the extension groups between S_i and S_j, we apply $\mathrm{Hom}_{\Pi^\lambda}(-, S_j)$ to the corresponding deleted resolution to obtain the complex

$$0 \to \mathrm{Hom}_{\Pi^\lambda}(e_i\Pi^\lambda, S_j) \to \bigoplus_{k\in\partial i} \mathrm{Hom}_{\Pi^\lambda}(e_k\Pi^\lambda, S_j) \to \mathrm{Hom}_{\Pi^\lambda}(e_i\Pi^\lambda, S_j) \to 0. \tag{3.5.8}$$

We now consider three distinct cases when computing the homology of this complex. If $j = i$ then, using (3.5.7), as a complex of vector spaces (3.5.8) becomes

$$0 \to \Bbbk \to 0 \to \Bbbk \to 0$$

and so we can immediately read off that

$$\dim_{\Bbbk} \mathrm{Hom}_{\Pi^\lambda}(S_i, S_i) = 1 = \dim_{\Bbbk} \mathrm{Ext}^2_{\Pi^\lambda}(S_i, S_i),$$
$$\dim_{\Bbbk} \mathrm{Ext}^\ell_{\Pi^\lambda}(S_i, S_i) = 0 \quad \text{for } \ell = 1 \text{ or } \ell \geqslant 3,$$

and so $S_i \bullet S_i = -1 + 0 - 1 = -2$. If $j \in \partial i$, then (3.5.8) becomes

$$0 \to 0 \to \Bbbk \to 0 \to 0$$

and so

$$\dim_{\Bbbk} \mathrm{Ext}^1_{\Pi^\lambda}(S_i, S_i) = 1, \qquad \dim_{\Bbbk} \mathrm{Ext}^\ell_{\Pi^\lambda}(S_i, S_i) = 0 \quad \text{for } \ell = 0 \text{ or } \ell \geqslant 2.$$

That is, if i and j are adjacent in $\widetilde{Q}$, then $S_i \bullet S_j = 0 + 1 + 0 = 1$. Finally, if $j \neq i$ and $j \notin \partial i$ then (3.5.8) becomes

$$0 \to 0 \to 0 \to 0 \to 0$$

and clearly

$$\dim_{\Bbbk} \mathrm{Ext}^\ell_{\Pi^\lambda}(S_i, S_i) = 0 \quad \text{for } \ell \geqslant 0,$$

and so $S_i \bullet S_j = 0$ in this case. It follows that the intersection matrix Γ satisfies $\Gamma = -C$. □

The above result should be seen as a noncommutative analogue of the geometric McKay correspondence. However, we can strengthen this result by showing that $\mathcal{O}^\lambda$ possesses a noncommutative resolution which is actually a "deformation": that is, a noncommutative resolution of the form $\mathcal{O}^\mu$ for some weight μ. Since we are restricting our attention to quasi-dominant weights, the fact that $\mathcal{O}^\mu$ is nonsingular forces $\mu_i \succ 0$ for all $i \geqslant 1$ (see Lemma 3.1.5). It is not immediately clear that such a deformation exists; we prove its existence in the next subsection.

3.5.2 $\mathcal{O}^\lambda$ has a noncommutative resolution which is a deformation

The dual reflections defined in Definition 2.5.5 also appear in the so-called *numbers game* of [Moz90]. The relationship between this game and our setting is that the moves considered by Mozes can equivalently be described as an application of a dual reflection to a weight λ. This allows us to make use of some of the results from this paper; in particular we are able to prove that, for $\lambda = \varepsilon_0$, noncommutative resolutions of $\mathcal{O}^\lambda$ which are also deformations exist:

Lemma 3.5.9. *Let $\widetilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices. Then there exists a sequence of dual reflections ρ such that $\rho(\varepsilon_0)_i > 0$ for all $i \neq 0$; in particular, $\rho(\varepsilon_0)$ is quasi-dominant.*

Proof. It suffices to show that we can find such a sequence of dual reflections when we work over the field $\mathbb{R}$, since any such sequence will also have the desired effect on ε_0 when we work over our algebraically closed field $\Bbbk$ of characteristic 0. Write G for the group generated by the dual reflections. Lemma 5.5 of [Moz90], when translated into our notation, says that $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geqslant 0 \text{ for all } 0 \leqslant i \leqslant n\}$ is a fundamental domain for the action of G on $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta > 0\}$. Recalling that G preserves the affine hyperplane $V := \{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta = 1\}$, it follows that $V = \bigcup_{\rho \in G} \rho U$, where U is the n-simplex $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geqslant 0 \text{ for all } 0 \leqslant i \leqslant n \text{ and}, \lambda \cdot \delta = 1\}$. Let $H = \{\lambda \in V \mid \lambda_i > 0 \text{ for all } i \neq 0\}$, which is a convex subset of V containing open balls of arbitrarily large diameter. Since each ρU has the same finite diameter, there exists some $\rho \in G$ with $\rho U \subseteq H$. In particular, $\rho(\varepsilon_0) \in H$; that is, $\rho(\varepsilon_0)_i > 0$ for all $i \neq 0$. □

Remark 3.5.10. By playing Mozes' numbers game, one can often determine an explicit sequence of dual reflections ρ satisfying the hypotheses of Lemma 3.5.9. For example, if $\widetilde{Q} = \widetilde{\mathbb{A}}_4$, then the numbers game starting with the initial configuration $(-3, 1, 1, 1, 1)$ terminates at ε_0, and so by applying the corresponding dual reflections in reverse we obtain the desired ρ. More generally, [GSS12, Proposition 5.1] tells us that when $\widetilde{Q}$ is of type $\widetilde{\mathbb{A}}_{2m}$, $\widetilde{\mathbb{D}}_{4m}$, $\widetilde{\mathbb{D}}_{4m+1}$, $\widetilde{\mathbb{E}}_6$ or $\widetilde{\mathbb{E}}_8$, where m is a positive integer, then the numbers game starting with the initial configuration $(1 - \sum_{i=1}^{n} \delta_i, 1, 1, \dots, 1)$ terminates at ε_0, and so this determines a sequence of dual reflections ρ such that $\rho(\lambda)_i > 0$ for all $i \neq 0$.

We are now in a position to prove Theorem 1.2.7 from the introduction:

Theorem 3.5.11. *Let $\widetilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, and let $\lambda = \varepsilon_0$. Then $\mathcal{O}^\lambda$ has a noncommutative resolution of the form $\mathcal{O}^\mu$, where $\mathcal{O}^\mu$ has precisely n finite-dimensional simple modules S_i up to isomorphism. With a suitable indexing of the S_i, the intersection matrix Γ with entries $\Gamma_{ij} = S_i \bullet S_j$ is $-C$, where C is the Cartan matrix corresponding to Q.*

Proof. Lemma 3.5.9 tells us that there exists a sequence of dual reflections ρ such that $\mathcal{O}^\mu$ is nonsingular, where $\mu = \rho(\lambda)$. Since Π^λ is a resolution of $\mathcal{O}^\lambda$ and there are Morita equivalences between Π^λ, Π^μ, and $\mathcal{O}^\mu$ (by [CBH98, Corollary 5.2, Corollary 9.6]), it follows that $\mathcal{O}^\mu$ is a noncommutative resolution of $\mathcal{O}^\lambda$. Finally, these Morita equivalences combined with Theorem 3.5.5 tells us that $\mathcal{O}^\mu$ has precisely n finite-dimensional simple modules S_i up to isomorphism, and since Morita equivalences preserve dimensions of Hom and Ext groups, the claimed intersection multiplicities follow from Theorem 3.5.5 as well. □

Chapter 4

Singularities of Blowups of Sklyanin Algebras

In this chapter we determine the singularities of some noncommutative surfaces which have been studied by Rogalski–Sierra–Stafford in [Rog11, RSS15b, RSS17]. The authors have shown that these rings have similar properties to deformations of an $\mathbb{A}_1$ singularity and had conjectured that, in the cases where these rings are singular, their singularity categories were equivalent to $\mathcal{D}_{\text{sg}}(R_{\mathbb{A}_1})$. This is indeed the case, as we will see in Theorem 4.5.9. The intuition for this result is as follows: these rings depend, in particular, on an infinite order automorphism τ of an elliptic curve E. If we replace τ by the identity, then the ring we instead obtain is the anticanonical coordinate ring of a singular del Pezzo surface, and this is known to have an $\mathbb{A}_1$ singularity [CT88, Theorem C].

The preliminaries for this chapter are quite disjoint from those for the remainder of this thesis, so this chapter is relatively self-contained. We will first recall some definitions and results in Sections 4.1 and 4.2, before proving our main results in the remainder of this chapter. For this chapter, we assume that the reader has some understanding of algebraic geometry and sheaf theory.

4.1 Background material

Twisted homogeneous coordinate rings are one of the most important types of ring in noncommutative algebraic geometry. For example, these rings play a crucial role in the classification of three-dimensional AS regular algebras, a problem from which noncommutative algebraic geometry first developed. We now define these rings:

Definition 4.1.1. Let X be a projective $\Bbbk$-scheme, let $\mathcal{L}$ be an invertible sheaf on X, and let $\sigma : X \to X$ be an automorphism of X. Set $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ for $n \geqslant 1$, where for a sheaf $\mathcal{F}$ we use the notation $\mathcal{F}^\sigma := \sigma^*(\mathcal{F})$ for the pullback of $\mathcal{F}$ along an automorphism σ. Then there is also a natural pullback of global sections map $\sigma^* : \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(X, \mathcal{F}^\sigma)$. The *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$ is then the graded ring $B(X, \mathcal{L}, \sigma) = \bigoplus_{n\in\mathbb{N}} B_n$, where $B_n = \mathrm{H}^0(X, \mathcal{L}_n)$. Here, the multiplication on $B_m \otimes B_n$ is given by the composition

$$\mathrm{H}^0\left(X, \mathcal{L}_m\right) \otimes \mathrm{H}^0\left(X, \mathcal{L}_n\right) \xrightarrow{\mathrm{id}\otimes(\sigma^m)^*} \mathrm{H}^0\left(X, \mathcal{L}_m\right) \otimes \mathrm{H}^0\big(X, \mathcal{L}_n^{\sigma^m}\big) \xrightarrow{\mu} \mathrm{H}^0(X, \mathcal{L}_{m+n}),$$

where μ is the natural multiplication of global sections map, and where we note that $\mathcal{L}_m \otimes \mathcal{L}_n^{\sigma^m} = \mathcal{L}_{m+n}$.

We only consider this construction under the hypothesis that $\mathcal{L}$ is σ-*ample*; that is, for every coherent sheaf $\mathcal{F}$ on X, $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}_n) = 0$ for all $i \geqslant 1$ and all $n \gg 0$. This ensures that the ring $B(X, \mathcal{L}, \sigma)$ is noetherian, see [AVdB90, Theorem 1.4]. We remark then when X is a curve, $\mathcal{L}$ is σ-ample if and only if it is ample, [AVdB90, Corollary 1.6].

Recall from the introduction that for any $[a : b : c] \in \mathbb{P}^2_{\Bbbk}$, the *Sklyanin algebra* $S = S(a, b, c)$ is the $\Bbbk$-algebra with presentation

$$S(a, b, c) = \Bbbk\langle x, y, z\rangle/\langle axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2\rangle.$$

If we set the degrees of x, y and z to be 1, then the algebra S is finitely graded and is generated as a $\Bbbk$-algebra by S_1. For very general choices of the parameters a, b and c, it is known that S has the following properties [ATVdB90, ATVdB91]: it is noetherian, Artin-Schelter regular, and has Hilbert series equal to that of a commutative polynomial ring in three variables, namely $\operatorname{hilb} S = 1/(1-t)^3$; the centre of S is generated by a single central element $g \in S_3$; and $S/gS \cong B(E, \mathcal{L}, \sigma)$, where E is a nonsingular elliptic curve embedded in $\mathbb{P}^2_{\Bbbk}$, $\mathcal{L}$ is a degree three line bundle, and $\sigma : E \to E$ has infinite order. When S has all of the above properties, we say that S is a *generic* Sklyanin algebra, and henceforth every Sklyanin algebra in this chapter is assumed to be generic. Since σ has infinite order, it is known that if we fix some base point O for the group law on E, then σ is a translation $x \mapsto x + r$ for some point $r \in E$ which has infinite order in the group. We indicate the image of any subset V under the quotient map $S \to S/gS$ by $\overline{V}$.

We will frequently make use of the Riemann-Roch Theorem to calculate dimensions of spaces of global sections. We state it below in a form that will be of most use to us, and will frequently invoke it without mention.

Theorem 4.1.2 (Riemann-Roch Theorem for sheaves on an elliptic curve)**.** *Let E be an elliptic curve and let $\mathcal{L}$ be a sheaf with* $\deg \mathcal{L} \geqslant 1$*. Then* $\dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{L}) = \deg \mathcal{L}$*. In particular, if $\mathcal{L} = \mathcal{O}_E(\mathbf{d})$ for some divisor $\mathbf{d}$ of positive degree, then* $\dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{O}_E(\mathbf{d})) = \deg \mathbf{d}$.

We also have the following important lemma which we will frequently use, again often without mention.

Lemma 4.1.3 ([Rog11, Lemma 3.1])**.**
(1) *Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on an elliptic curve E with* $\deg \mathcal{L} \geqslant 2$ *and* $\deg \mathcal{M} \geqslant 2$*. Consider the natural map $\mu : \mathrm{H}^0(E, \mathcal{L}) \otimes \mathrm{H}^0(E, \mathcal{M}) \to \mathrm{H}^0(E, \mathcal{L} \otimes \mathcal{M})$. Then μ is surjective unless* $\deg \mathcal{L} = 2 = \deg \mathcal{M}$ *and $\mathcal{L} \cong \mathcal{M}$, in which case* $\dim_{\Bbbk} \operatorname{im} \mu = 3$.
(2) *Let $\mathcal{L}$ be an invertible sheaf on an elliptic curve E with* $\deg \mathcal{L} \geqslant 2$,*and let $\sigma : E \to E$ be an automorphism of infinite order. Then $B(E, \mathcal{L}, \sigma)$ is generated in degree* 1.

Finally, we recall some terminology and results for graded rings. If A is finitely graded and generated in degree 1 as a $\Bbbk$-algebra, then we say that $M \in \text{gr-}A$ is a *point module* if it is cyclic, generated in degree 1, and has Hilbert series $1/(1-t)$, while M is said to be a *line module* if it is cyclic, generated in degree 1, and has Hilbert series $1/(1-t)^2$.

If A is a finitely graded $\Bbbk$-algebra which is a noetherian domain, then we can form the *graded quotient ring* $Q_{\mathrm{gr}}(A)$ by localising at the set of all nonzero homogeneous elements of A, and this ring is $\mathbb{Z}$-graded, see [GS99]. If $M, N \subseteq Q_{\mathrm{gr}}(A)$ are graded right A-modules with $M \neq 0$, then we have an identification

$$\operatorname{Hom}_A(M, N) = \{q \in Q_{\mathrm{gr}}(A) \mid qM \subseteq N\}.$$

4.2 Blowup subalgebras of the Sklyanin algebra

In [Rog11], Rogalski introduced a method for constructing subalgebras of the Sklyanin algebra in terms of specific geometric data. This method was then extended to a family of algebras known as *elliptic algebras*, of which the Sklyanin algebra is an example, in [RSS15b]. The latter approach, while more general, was also more categorical and more difficult to perform explicit calculations with. The following exposition more closely follows that of [Rog11].

We now define the class of algebras that we are interested in:

Definition 4.2.1. Let S be a generic Sklyanin algebra with central element g, so that $S/gS \cong B(E, \mathcal{L}, \sigma)$. Now let $T = S^{(3)} = \bigoplus_{n \geqslant 0} S_{3n}$ be the 3-Veronese of S, so that $g \in T_1$ and $T/gT \cong B(E, \mathcal{M}, \tau)$, where $\mathcal{M} \coloneqq \mathcal{L}_3 = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \mathcal{L}^{\sigma^2}$ and $\tau = \sigma^3$. Let $\mathbf{d}$ be an effective divisor on E with $0 \leqslant \deg \mathbf{d} \leqslant 7$. Then $T(\mathbf{d}) \coloneqq \Bbbk\langle V \rangle$ is the $\Bbbk$-subalgebra of T generated by $V = \{x \in T_1 \mid \overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-\mathbf{d}) \otimes \mathcal{M})\}$, and we say that $T(\mathbf{d})$ is the *blowup* of T at $\mathbf{d}$.

These subalgebras have a number of nice properties, which we collect in the following theorem. We take this opportunity to introduce some useful terminology. A graded vector subspace V of T is said to be *g-divisible* if $V \cap gT = gV$. Given $M \in \text{gr-}T(\mathbf{d})$, we define the *g-torsion submodule* of M to be

$$\operatorname{tors}_g(M) = \{m \in M \mid mg^n = 0 \text{ for some } n \in \mathbb{N}\},$$

and we say that M is *g-torsionfree* if $\operatorname{tors}_g(M) = 0$.

Theorem 4.2.2 ([Rog11, Theorem 1.1])**.** *Let $\mathbf{d}$ be an effective divisor on E with $0 \leqslant d \coloneqq \deg \mathbf{d} \leqslant 7$. Then:*
(1) *$T(\mathbf{d})$ has Hilbert series $\operatorname{hilb} T(\mathbf{d}) = \frac{t^2+(7-d)t+1}{(1-t)^3}$;*
(2) *$T(\mathbf{d})$ is g-divisible, and satisfies $\overline{T(\mathbf{d})} = T(\mathbf{d})/gT(\mathbf{d}) = B(E, \mathcal{O}_E(-\mathbf{d}) \otimes \mathcal{M}, \tau)$;*
(3) *$T(\mathbf{d})$ is noetherian, Auslander-Gorenstein and Cohen-Macaulay; and*
(4) *$T(\mathbf{d})$ has infinite global dimension and has GK dimension 3.*

It turns out that we can iterate this blowing-up process. Indeed, suppose that $\mathbf{d}$ and $\mathbf{d}'$ are effective divisors on E with $0 \leqslant \deg \mathbf{d} + \deg \mathbf{d}' \leqslant 7$. Then $T(\mathbf{d})$ is a graded ring with $g \in T(\mathbf{d})_1$ and $T(\mathbf{d})/gT(\mathbf{d}) \cong B(E, \mathcal{O}_E(-\mathbf{d}) \otimes \mathcal{M}, \tau)$, and we may blow up $T(\mathbf{d})$ at $\mathbf{d}'$ in the same way as in Definition 4.2.1: namely, set $\big(T(\mathbf{d})\big)(\mathbf{d}')$ to be the $\Bbbk$-subalgebra of $T(\mathbf{d})$ generated by $V = \{x \in T(\mathbf{d})_1 \mid \overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-\mathbf{d} - \mathbf{d}') \otimes \mathcal{M})\}$. By [RSS15b, Proposition 5.4], we have $\big(T(\mathbf{d})\big)(\mathbf{d}') \cong T(\mathbf{d} + \mathbf{d}')$.

One of the main reasons for studying these subalgebras is to help solve the fundamental problem of the classification of noncommutative projective surfaces. A significant stepping stone towards a solution of this problem is to classify all algebras with the same graded quotient ring as a generic Sklyanin algebra. In particular, one wishes to classify all maximal orders inside a generic Skylanin algebra; it turns out that the subalgebras of $T = S^{(3)}$ generated in degree 1 which are maximal orders in $Q_{\mathrm{gr}}(T)$ are precisely the rings $T(\mathbf{d})$, [Rog11, Theorem 1.2].

To prove this result, Rogalski expended a lot of effort developing the theory behind so-called *sporadic ideals* (which were called *special ideals* in [Rog11]).

Definition 4.2.3. Let A be a finitely graded $\Bbbk$-algebra. If I is a graded ideal of A with $\operatorname{GKdim} A/I \leqslant 1$, then we say that I is a *sporadic ideal* of A, and if $\operatorname{GKdim} A/I = 1$ then we call I a *proper sporadic ideal* of A. A *minimal sporadic ideal* of A is a sporadic ideal I of A which, for any other sporadic ideal J of A, satisfies $I_{\geqslant n} \subseteq J_{\geqslant n}$ for some $n \geqslant 0$.

We have the following results on sporadic ideals:

Theorem 4.2.4 ([Rog11, Lemma 10.3, Theorem 10.4], [RSS15b, Example 9.5])**.**
(1) *Let* $\mathbf{d}$ *be an effective divisor on* E *with* $0 \leqslant \deg \mathbf{d} \leqslant 7$*. Then* $T(\mathbf{d})$ *has a minimal sporadic ideal.*
(2) T *has no proper sporadic ideals.*
(3) *Let* $p \in E$*. Then* $T(p)$ *has no proper sporadic ideals.*

To one-point blowups $T(\mathbf{d}+p) \subseteq T(\mathbf{d})$, where $0 \leqslant \deg \mathbf{d} \leqslant 6$, one can associate an *exceptional line ideal* and *exceptional line module.* The (right) exceptional line ideal is defined to be

$$J = \{y \in T(\mathbf{d}+p) \mid T(\mathbf{d})_1 y \subseteq T(\mathbf{d}+p)\},$$

while the (right) exceptional line module is the factor module $L = T(\mathbf{d}+p)/J$. The left-hand analogues are defined in the obvious way; unless otherwise noted, all subsequent instances of the exceptional line ideal or module will be the right hand version.

Proposition 4.2.5 ([Rog11, Lemma 9.1])**.** *Let* $\mathbf{d}$ *and* p *be as above, let* J *be the exceptional line ideal for* $T(\mathbf{d}+p) \subseteq T(\mathbf{d})$ *and let* $L = T(\mathbf{d}+p)/J$ *be the exceptional line module.*
(1) L *is a* g*-torsionfree line module, and* J *is* g*-divisible.*
(2) $T(\mathbf{d})/T(\mathbf{d}+p) \cong \bigoplus_{i\geqslant 1} L[-i]$ *as right* $T(\mathbf{d}+p)$*-modules.*

We now define particular subspaces of the Sklyanin algebra which are very useful for performing computations. Recall that $S/gS \cong B = B(E,\mathcal{L},\sigma)$. For any point $q \in E$, define $P(q) := \bigoplus_{n\geqslant 0} \mathrm{H}^0(E, \Bbbk(q) \otimes \mathcal{L}_n)$, where $\Bbbk(q)$ is the skyscraper sheaf at q. Since the tensor product of a skyscraper sheaf and an invertible sheaf is isomorphic to the original skyscraper sheaf [Har77, pp. 296], it follows that $P(q)$ is a point module for S/gS. Note also that $P(q) \cong B/J(q)$, where $J(q) = \bigoplus_{n\geqslant 0} \mathrm{H}^0(E, \mathcal{O}_E(-q) \otimes \mathcal{L}_n)$.

It is known, by [ATVdB91, Proposition 7.7 (ii)], that every point module for S is annihilated by g, so the point modules for S are the same as the point modules for B. In particular, the B-module $P(q)$ corresponds to the S-module $S/I(q)$, where $I(q) = \{x \in S \mid \overline{x} \in J(q)\}$. This allows us to make the following definition.

Definition 4.2.6. Let $q \in E$, and let $I(q)$ be as above. Then $W(q) = I(q)_1 \subseteq S_1$ is called a *point space.* Explicitly, $W(q) = \mathrm{H}^0(E, \mathcal{O}_E(-q) \otimes \mathcal{L})$.

Since $\dim_\Bbbk S_1 = 3$ and $S/I(q)$ is a point module, it follows that $\dim_\Bbbk W(q) = 2$. We now collect some useful properties of point spaces that will be used in later calculations.

Lemma 4.2.7 ([Rog11, Lemmas 4.1, 4.2, 4.6, Proposition 4.3])**.** *Let* $p, q \in E$*. Then:*
(1) $W(p)S_1 = S_1 W(\sigma(p))$*;*
(2) $\dim_\Bbbk W(p)W(q) = \begin{cases} 4 & \text{if } p \neq \sigma^2(q) \\ 3 & \text{if } p = \sigma^2(q) \end{cases}$.
In particular, $W(p)W(\sigma(q)) = W(q)W(\sigma(p))$ *if* $p \neq \sigma^3(q)$ *and* $q \neq \sigma^3(p)$*;*

(3) $\dim_{\Bbbk} W(p)W(q)S_1 = \begin{cases} 8 & \text{if } p \neq \sigma^2(q) \\ 7 & \text{if } p = \sigma^2(q) \end{cases}$.
In particular, $g \in W(p)W(q)S_1$ *if and only if* $p \neq \sigma^2(q)$.
(4) *Let* $X \subseteq S_2$. *Then* $\dim_{\Bbbk} W(p)X = 2\dim_{\Bbbk} X - \dim_{\Bbbk}\{s \in S_1 \mid W(\sigma^{-2}(p))s \subseteq X\}$.
(5) $W(p)$ *is generated in degree* 1*, so* $W(p)S = I(p)$. *Moreover, there exist* $w, x, y, z \in S_1$ *such that* $W(p) = \Bbbk w + \Bbbk x$ *and* $W(\sigma^{-2}(p)) = \Bbbk y + \Bbbk z$*, with* $wS \cap xS = wyS = xzS$*; and*
(6) $\dim_{\Bbbk}(W(p)S_2)^m S_k = \dim S_{3m+k} - \binom{m+1}{2}$ *for all* $m, k \in \mathbb{N}$.

The above properties of point spaces means that they can be used to calculate the dimension of subspaces of $T(\mathbf{d})$ when $\deg \mathbf{d}$ is small. We refer to property (1) as "moving point spaces".

4.3 $T(2p)$ has no proper sporadic ideals

In this section, we prove the following:

Theorem (*Theorem 4.3.13*)**.** *Let* $p \in E$. *Then* $T(2p)$ *has no proper sporadic ideals.*

In [RSS15b, Example 9.6] it was shown that if p and q lie on different orbits of $\tau = \sigma^3$ then $T(p+q)$ has no proper sporadic ideals, while the remark after [loc. cit.] shows that $T(p + \tau^i(p))$ always has a proper sporadic ideal when $i \neq 0$. The authors speculated that $T(2p)$ has no proper sporadic ideals, and the above result confirms that this is the case. Our main motivation for proving this theorem will become apparent in the next section, where we are able to use it to deduce that a related ring is simple.

The proof of this theorem is quite long and occupies the remainder of this section, so we outline our strategy. Rogalski's proof of Theorem 4.2.4 (1) shows that a minimal sporadic ideal of $T(2p)$ is given by $H'T(2p)H$, where H (respectively, H') is a sporadic ideal of $T(2p)$ which annihilates all g-torsionfree factors of the right (respectively, left) exceptional line module L (respectively, L') associated to the blowup $T(2p) \subseteq T(p)$. If we can show that L and L' have no proper g-torsionfree factors, then we may take $H = H' = T(2p)$ in the above. This tells us that any sporadic ideal I of $T(2p)$ is equal to $T(2p)$ in large degree, and therefore $\operatorname{GKdim} T(2p)/I = 0$, meaning that I is not a proper sporadic ideal.

Let $L = T(2p)/J$. To show that L has no g-torsionfree factors, we will see that it suffices to show that for any $x \in T(2p)_m T_n \setminus J_m T_n$ there exists $k \geqslant 0$ such that $J_m T_{n+k} + xT_k \supseteq g^m T_{n+k}$. To show this, one needs to know how spaces of the form $J_m T$, $g^m T$ and $g^{m-k} T(2p)_k T$ intersect. It is usually easy to write down a candidate space for such an intersection, but to establish an equality one needs to compare Hilbert series.

We have the following result which allows us to express certain blowups in terms of point spaces. We keep the notation from the previous section, and remind the reader that $\tau = \sigma^3$.

Lemma 4.3.1. *Let* $p \in E$. *Then, for* $n \geqslant 1$*:*
(1) $T(p)_n = \big(W(p)S_2\big)^n$*;*
(2) $T(2p)_n = \big(W(p)W(\sigma(p))S_1\big)^n$*; and*
(3) $T(p + \sigma^3(p))_n = \big(W(p)W(\sigma^4(p))S_1\big)^n$.

Proof. In each case it suffices to show that, for the appropriate divisor $\mathbf{d}$, we have equality in degree 1, since $T(\mathbf{d})_n = \big(T(\mathbf{d})_1\big)^n$.

(1) Recall first that $T(p)$ is defined to be the $\Bbbk$-subalgebra of T given by $\Bbbk\langle V\rangle$, where $V = \{x \in T_1 \mid \overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-p) \otimes \mathcal{M})\}$. If $x \in W(p)S_2$ then

$$\begin{aligned}\overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-p) \otimes \mathcal{L}) \cdot \mathrm{H}^0(E, \mathcal{L}_2) &= \mathrm{H}^0(E, \mathcal{O}_E(-p) \otimes \mathcal{L} \otimes \mathcal{L}_2^{\sigma}) \\ &= \mathrm{H}^0(E, \mathcal{O}_E(-p) \otimes \mathcal{M}),\end{aligned}$$

where the first equality follows from Lemma 4.1.3. Therefore $W(p)S_2 \subseteq V = T(p)_1$. But by Lemma 4.2.7 and Theorem 4.2.2, we know that $\dim_{\Bbbk} W(p)S_2 = 9 = \dim_{\Bbbk} T(p)_1$, and therefore we have $W(p)S_2 = T(p)_1$.
(2) We have that $T(2p) = \Bbbk\langle V\rangle$ where $V = \{x \in T_1 \mid \overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-2p) \otimes \mathcal{M})\}$. A similar calculation as in part (1) shows that $\overline{W(p)W(\sigma(p))S_1} = \mathrm{H}^0(E, \mathcal{O}_E(-2p) \otimes \mathcal{M})$ so that $W(p)W(\sigma(p))S_1 \subseteq T(2p)_1$. By Lemma 4.2.7, $\dim_{\Bbbk} W(p)W(\sigma(p))S_1 = 8$, while $\dim_{\Bbbk} T(2p)_1 = 8$ by Theorem 4.2.2, and so in fact $W(p)W(\sigma(p))S_1 = T(2p)_1$.
(3) We now have $T(p + \sigma^3(p)) = \Bbbk\langle V\rangle$ where

$$V = \{x \in T_1 \mid \overline{x} \in \mathrm{H}^0(E, \mathcal{O}_E(-p - \sigma^3(p)) \otimes \mathcal{M})\}$$

and $W(p)W(\sigma^4(p))S_1 \subseteq T(p + \sigma^3(p))$. That both spaces have the same dimension again follows from Lemma 4.2.7 and Theorem 4.2.2, and so we have equality. □

Remark 4.3.2. It might initially appear that $W(\sigma^3(p))W(\sigma(p))S_1$ is also a candidate for $T(p + \sigma^3(p))_1$, but we do not have equality here. Indeed, we certainly have $\overline{W(\sigma^3(p))W(\sigma(p))S_1} = \mathrm{H}^0(E, \mathcal{O}_E(-p - \sigma^3(p)) \otimes \mathcal{M})$ and so $W(\sigma^3(p))W(\sigma(p))S_1 \subseteq T(p + \sigma^3(p))_1$. However, the right hand side has dimension 8 while, by Lemma 4.2.7, the left hand side only has dimension 7, and so we have a strict inclusion.

The first step in proving the main theorem of this section is to identify the line ideal corresponding to the blowup $T(2p) \subseteq T(p)$. This has a simple description in terms of point spaces which makes computations involving it tractable.

Proposition 4.3.3. *Let J be the line ideal corresponding to the blowup $T(2p) \subseteq T(p)$. Then $J = VT(2p)$, where $V = W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))$.*

Proof. By the proof of [RSS17, Lemma 10.3 (2)], $J_1 = V$. Moreover, J is generated in degree 1 by [RSS17, Lemma 5.8 (2)], whence the result. □

Henceforth we take J to be the line ideal above. The next step in our proof makes use of the right T-modules $T(2p)_m T/J_m T$ and its submodules, which have some properties which simplify later calculations. However, to be able to exploit these properties we must first calculate the dimensions of various spaces involved. We make the reader aware that we will often use Theorem 4.1.2 and Lemma 4.1.3 without mention.

Proposition 4.3.4.
(1) $\dim_{\Bbbk} J_m = \dim_{\Bbbk} T_m - (m+1)^2$ *for all* $m \geqslant 0$.
(2) $\dim_{\Bbbk} T(2p)_m S_k = \dim_{\Bbbk} S_{3m+k} - m(m+1)$ *for all* $m \geqslant 0$ *and* $k \geqslant 1$.
(3) $\dim_{\Bbbk} J_m S_k = \dim_{\Bbbk} S_{3m+k} - (m+1)^2$ *for all* $m \geqslant 1$ *and* $k \geqslant 0$.

Proof.
(1) This follows from the fact that

$$\operatorname{hilb} J = \frac{t^2 + 6t}{(1-t)^3} = \operatorname{hilb} T - \frac{t+1}{(1-t)^3} = \frac{t^2 + 7t + 1}{(1-t)^3} - (1 + 4t + 9t^2 + 16t^3 + \dots).$$

(2) This follows from [RSS15b, Proposition 5.2].

(3) Write $h(m,k) = \dim_{\Bbbk} J_m S_k$ and $j(m,k) = \dim_{\Bbbk} S_{3m+k} - (m+1)^2$; we wish to prove that $h(m,k) = j(m,k)$ for all $m \geqslant 1$ and $k \geqslant 0$. By part (1), the result holds when $k = 0$, so we henceforth assume that $k \geqslant 1$.

Observe that we have the following recurrence relation, which follows from direct computation and the fact that $\dim_{\Bbbk} S_{3m+k} = \binom{3m+k+2}{2}$:

$$j(m,k) = j(m-1,k) + 7m + 3k - 1. \tag{4.3.5}$$

Seeking to prove the result by induction on m, we first establish the result for $m = 1$. Observe that $\dim_{\Bbbk} \overline{J_1 S_k} = 3k+6$ and that $J_1 S_k \cap gS$ contains $gW(\sigma^3(p))S_{k-1}$. Therefore,

$$\begin{aligned} h(1,k) &= \dim_{\Bbbk} J_1 S_k = \dim_{\Bbbk} J_1 S_k \cap gS + \dim_{\Bbbk} \overline{J_1 S_k} \\ &\geqslant \dim_{\Bbbk} W(\sigma^3(p))S_{k-1} + 3k + 6 \\ &= \dim_{\Bbbk} S_k - 1 + 3k + 6 = \frac{1}{2}(k+2)(k+1) + 3k + 5 \\ &= \frac{1}{2}(k+5)(k+4) - 4 = \dim_{\Bbbk} S_{3+k} - (1+1)^2 \\ &= j(1,k), \end{aligned}$$

where the first equality on the third line follows from the fact that $S/(W(\sigma^3(p))S)$ is a point module for S.

For the reverse inequality, notice that

$$J_1 S_k = W(\sigma^3(p))\Big(W(\sigma(p))W(\sigma^2(p))S_1 S_{k-1}\Big) = W(\sigma^3(p))T(2\sigma(p))_1 S_{k-1} = wU + xU$$

where $U = T(2\sigma(p))_1 S_{k-1}$ and $W(\sigma^3(p)) = \Bbbk w + \Bbbk x$. Here we have used Lemma 4.2.7 (5) to choose $w, x \in S_1$ such that there exist $y, z \in S_1$ with $W(\sigma(p)) = \Bbbk y + \Bbbk z$, $wy + xz = 0$, and $wS \cap xS = wyS = xzS$. We then have

$$\begin{aligned} wU \cap xU &= \{wys \mid s \in S, ys, zs \in U\} = \{wys \mid s \in S, W(\sigma(p))s \subseteq U\} \\ &\supseteq wyW(\sigma^2(p))S_k. \end{aligned}$$

This implies that

$$\begin{aligned} h(1,k) &= 2\dim_{\Bbbk} U - \dim_{\Bbbk} wU \cap xU \leqslant 2\dim_{\Bbbk} U - \dim_{\Bbbk} W(\sigma^2(p))S_k \\ &= 2(\dim_{\Bbbk} S_{k+2} - 2) - (\dim_{\Bbbk} S_{k+1} - 1) = (k+4)(k+3) - \frac{1}{2}(k+3)(k+2) - 3 \\ &= \frac{1}{2}(k+5)(k+4) - 4 = \dim_{\Bbbk} S_{k+3} - (1+1)^2 \\ &= j(1,k), \end{aligned}$$

where the first equality on the second line follows from the fact that $S/(W(\sigma^2(p))S)$ is a point module for S. Thus the claim holds for $m = 1$ and any $k \geqslant 1$, establishing the base case.

Now suppose that $m \geqslant 2$, $k \geqslant 1$ and that the result holds for smaller values of m. Observe that, by the Riemann-Roch theorem,

$$\dim_{\Bbbk} \overline{J_m S_k} = \dim_{\Bbbk} \overline{J_1 T(2p)_{m-1} S_k} = 6 + 7(m-1) + 3k = 7m + 3k - 1.$$

Moreover $J_2 = J_1 T(2p)_1 \supseteq gJ_1$, so $J_m S_k \cap gS$ contains $gJ_{m-1}S_k$ and therefore

$$\begin{aligned} h(m,k) &= \dim_{\Bbbk} J_m S_k = \dim_{\Bbbk} J_m S_k \cap gS + \dim_{\Bbbk} \overline{J_m S_k} \\ &\geqslant h(m-1,k) + 7m + 3k - 1 = j(m-1,k) + 7m + 3k - 1 \\ &= j(m,k), \end{aligned}$$

where the penultimate equality follows from the induction hypothesis, and the final equality follows from identity (4.3.5).

For the reverse inequality, first observe that we can write

$$\begin{aligned} J_m S_k &= W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))\Big(W(p)W(\sigma(p))S_1\Big)^{m-1} S_k \\ &= W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))S_1\Big(W(\sigma(p))W(\sigma^2(p))S_1\Big)^{m-1} S_{k-1} \\ &= W(\sigma^3(p))T(2\sigma(p))_m S_{k-1} \\ &= wU + xU, \end{aligned}$$

where $U = T(2\sigma(p))_m S_{k-1}$ and $W(\sigma^3(p)) = \Bbbk w + \Bbbk x$. Here we again use Lemma 4.2.7 (5) to choose $w, x \in S_1$ such that there exist $y, z \in S_1$ with $W(\sigma(p)) = \Bbbk y + \Bbbk z$, $wy + xz = 0$, and $wS \cap xS = wyS = xzS$. Letting $J' = W(\sigma^2(p))W(p)W(\sigma(p))T(2\sigma^{-1}(p))$ be the line ideal corresponding to the blowup $T(2\sigma^{-1}(p)) \subseteq T(\sigma^{-1}(p))$, we have

$$wU \cap xU = \{wys \mid s \in S, ys, zs \in U\} = \{wys \mid s \in S, W(\sigma(p))s \subseteq U\} \supseteq wyJ'_{m-1}S_{k+1},$$

where the last inclusion follows from the fact that

$$\begin{aligned} &W(\sigma(p))J'_{m-1}S_{k+1} \\ &\quad = W(\sigma(p))W(\sigma^2(p))W(p)W(\sigma(p))\Big(W(\sigma^{-1}(p))W(p)S_1\Big)^{m-2} S_{k+1} \\ &\quad = W(\sigma(p))W(\sigma^2(p))S_1W(\sigma(p))W(\sigma^2(p))S_1\Big(W(\sigma(p))W(\sigma^2(p))S_1\Big)^{m-2} S_{k-1} \\ &\quad = T(2\sigma(p))_m S_{k-1} \\ &\quad = U. \end{aligned}$$

Therefore,

$$\begin{aligned} h(m,k) &= \dim_{\Bbbk}(wU + xU) = 2\dim_{\Bbbk} U - \dim_{\Bbbk} wU \cap xU \\ &\leqslant 2\dim_{\Bbbk} T(2\sigma(p))_m S_{k-1} - \dim_{\Bbbk} J'_{m-1}S_{k+1} \\ &= 2\big(\dim_{\Bbbk} S_{3m+k-1} - m(m+1)\big) - \big(\dim_{\Bbbk} S_{3m+k-2} - m^2\big) \\ &= (3m+k+1)(3m+k) - \frac{1}{2}(3m+k)(3m+k-1) - (m^2+2m) \\ &= \frac{1}{2}(3m+k+2)(3m+k+1) - (m^2+2m+1) \\ &= \dim_{\Bbbk} S_{3m+k} - (m+1)^2 \\ &= j(m,k), \end{aligned}$$

where the equality on the third line uses part (2) and the induction hypothesis applied to $\dim_{\Bbbk} J'_{m-1}S_{k+1}$. We therefore have the desired equality for all $m \geqslant 1$ and $k \geqslant 1$. $\square$

Proposition 4.3.6. *We have $J_m T \cap g^m T = g^m W(\sigma^3(p))S_2 T$ for all $m \geqslant 1$.*

Remark 4.3.7. This tells us that $J_mT_n \cap g^mT$ is codimension 1 in g^mT_n for all $m, n \geqslant 1$, a fact which we will make use of later.

Proof of Proposition 4.3.6. First note that

$$\begin{aligned} J_mT_k &= W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))T(2p)_{m-1}T_k \\ &\supseteq g^{m-1}W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))S_1S_2T_{k-1} \\ &\supseteq g^{m-1}W(\sigma^3(p))gS_2T_{k-1} \\ &= g^mW(\sigma^3(p))S_2T_{k-1}, \end{aligned}$$

from which it follows that $g^mW(\sigma^3(p))S_2T \subseteq J_mT \cap g^mT$. We prove the reverse inclusion by induction on m, by showing that the nth piece of each side of the claimed equality has the same dimension. That is, we wish to show that $\dim_{\Bbbk} J_mT_{n-m} \cap gT = \dim_{\Bbbk} W(\sigma^3(p))S_2T_{n-m-1}$ for all $n > m$.

By Proposition 4.3.4, for $m \geqslant 1$ we have

$$\begin{aligned} \dim_{\Bbbk} J_mT_{n-m} \cap gT_{n-1} &= \dim_{\Bbbk} J_mT_{n-m} - \dim_{\Bbbk} \overline{J_mT_{n-m}} \\ &= \dim_{\Bbbk} T_n - (m+1)^2 - \big(9n - 3 - 2(m-1)\big) \\ &= \dim_{\Bbbk} T_n - 9n - m^2 \\ &= \dim_{\Bbbk} T_{n-1} - m^2. \end{aligned} \tag{4.3.8}$$

Since $S/(W(\sigma^3(p))S)$ is a point module, we also have

$$\dim_{\Bbbk} W(\sigma^3(p))S_2T_{n-2} = \dim_{\Bbbk} T_{n-1} - 1.$$

Substituting $m = 1$ into equation (4.3.8), we obtain the result for $m = 1$.

Now suppose that $m \geqslant 2$ and that the result holds for smaller values of m. We first prove an intermediate result; we claim that

$$J_mT \cap gT = gJ_{m-1}T \quad \text{for all } m \geqslant 2.$$

Since $gJ_{m-1} \subseteq J_m$, we have $gJ_{m-1}T \subseteq J_mT \cap gT$; we check dimensions to obtain the desired equality. For $n \geqslant m$ we have

$$\dim_{\Bbbk}(gJ_{m-1}T)_n = \dim_{\Bbbk} J_{m-1}T_{n-m} = \dim_{\Bbbk} T_{n-1} - m^2,$$

so that the equality $J_mT \cap gT = gJ_{m-1}T$ follows by comparison with equation (4.3.8). Finally,

$$\begin{aligned} J_mT \cap g^mT = (J_mT \cap gT) \cap g^mT = gJ_{m-1}T \cap g^mT = g(J_{m-1}T \cap g^{m-1}T) \\ = g^mW(\sigma^3(p))S_2T, \end{aligned}$$

where the last equality follows from the induction hypothesis. □

Lemma 4.3.9. *Suppose that $x \in T(2p)_mT_n \setminus J_mT_n$ for some $m \geqslant 0$, $n \geqslant 1$. Then there exists $k \geqslant 0$ such that $J_mT_{n+k} + xT_k \supseteq g^mT_{n+k}$.*

Proof. We prove this by induction on m. So let $m = 1$ and consider $x \in T(2p)_1T_n \setminus J_1T_n$. There are two cases to consider:

(1) $\overline{x} \notin \overline{J_1T_n}$, and so $\overline{J_1T_n} \oplus \Bbbk\overline{x} = \overline{T(2p)_1T_n}$ since $\overline{J_1T_n}$ is codimension 1 in $\overline{T(2p)_1T_n}$; or

(2) $\overline{x} \in \overline{J_1 T_n}$.

First suppose that (1) holds. If xg lies in $J_1 T_{n+1} \cap gT = gW(\sigma^3(p))S_2 T_n$ (where this equality follows from Proposition 4.3.6) then we find that $\overline{x}$ vanishes at $\sigma^3(p)$, but this can not happen since $\overline{x} \notin \overline{J_1 T_n}$. Therefore $xg \notin J_1 T_{n+1} \cap gT$, which is codimension 1 in gT_{n+1}, and so $gT_{n+1} = \big(J_1 T_{n+1} \cap gT\big) \oplus \Bbbk xg \subseteq J_1 T_{n+1} + xT_1$.

Now suppose that (2) holds, so that $x - y \in gT_n$ for some $y \in J_1 T_n$. Moreover, $x - y \notin J_1 T_n \cap gT = gW(\sigma^3(p))S_2 T_{n-1}$, which is codimension 1 in gT_n. It follows that $gT_n = \big(J_1 T_n \cap gT\big) \oplus \Bbbk(x - y)$, and so $gT_n \subseteq J_1 T_n + \Bbbk(x - y) \subseteq J_1 T_n + \Bbbk x$. This establishes the base case.

Now assume that $m \geqslant 2$ and that the result holds for smaller m, and let $x \in T(2p)_m T_n \setminus J_m T_n$. Again we consider two separate cases:

(a) $\overline{x} \notin \overline{J_m T_n}$, and so $\overline{J_m T_n} \oplus \Bbbk \overline{x} = \overline{T(2p)_m T_n}$ since $\overline{J_m T_n}$ is codimension 1 in $\overline{T(2p)_m T_n}$; or

(b) $\overline{x} \in \overline{J_m T_n}$.

Case (a) is proved in a similar way to case (1) of the base case, but instead we consider the element xg^m and arrive at the conclusion that $g^m T_{n+m} \subseteq J_m T_{n+m} + xT_m$. So suppose instead that case (b) holds, so that $x - y \in gT_{m+n-1}$ for some $y \in J_m T_n$. Then $x - y \in T(2p)_m T_n \cap gT = gT(2p)_{m-1} T_n$ by [RSS15b, Proposition 5.2 (2)], while $x - y \notin J_m T_n \cap gT = gJ_{m-1} T_n$. It follows that $x - y = gx'$ for some $x' \in T(2p)_{m-1} T_n \setminus J_{m-1} T_n$. By the inductive hypothesis, $g^{m-1} T_{n+k} \subseteq J_{m-1} T_{n+k} + x' T_k$ for some $k \in \mathbb{N}$, and hence $g^m T_{n+k} \subseteq gJ_{m-1} T_{n+k} + gx' T_k \subseteq J_m T_n + xT_k$, completing the proof. □

We now prove a short lemma which allows us to move all of our calculations in T back into $T(2p)$.

Lemma 4.3.10. *For any $p \in E$ and any $k \geqslant 1$, there exists $\ell \geqslant 1$ and $U \subseteq T_\ell$ such that $T_k U = T(2p)_{k+\ell}$.*

Proof. We prove this by induction on k. When $k = 1$, set

$$U = W(\sigma^3(p))W(\sigma^4(p))W(\sigma^2(p))W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p)) \subseteq T_2 \tag{4.3.11}$$

(so here $\ell = 2$), and then

$$\begin{aligned} T_1 U &= S_3 W(\sigma^3(p))W(\sigma^4(p))W(\sigma^2(p))W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p)) \\ &= (W(p)W(\sigma(p))S_1)^3 \\ &= T(2p)_3, \end{aligned}$$

establishing the base case.

Now suppose that $k \geqslant 2$ and that the result holds for smaller values of k. Applying the induction hypothesis to the point $q = \sigma^3(p)$, we deduce the existence of $\ell' \in \mathbb{N}$ and $U' \subseteq T_{\ell'}$ with $T_{k-1} U' = T(2\sigma^3(p))_{k+\ell'-1}$. With U as in (4.3.11), setting $\ell = \ell' + 2$ and $\widetilde{U} = U'U \subseteq T_\ell$ yields

$$\begin{aligned} T_k \widetilde{U} &= T_1 T_{k-1} U' U \\ &= S_3 (W(\sigma^3(p))W(\sigma^4(p))S_1)^{k+\ell'-1} U \\ &= (W(p)W(\sigma(p))S_1)^{k+\ell'-1} S_3 U \\ &= T(2p)_{k+\ell'-1} T(2p)_3 \\ &= T(2p)_{k+\ell}, \end{aligned}$$

finishing the proof. □

The proof outline for our main result given at the beginning of this section showed that we needed to understand the g-torsionfree factors of the exceptional line modules corresponding to the blowup $T(2p) \subseteq T(p)$. The key result is therefore the following:

Proposition 4.3.12. *Let $L = T(2p)/J$ be the exceptional line module corresponding to the blowup $T(2p) \subseteq T(p)$. Then L has no proper graded g-torsionfree factors.*

Proof. Any proper graded factor of L has the form $T(2p)/(M + J)$ for some graded right ideal M of $T(2p)$ not containing J, and moreover it suffices to consider right ideals generated by a single homogeneous element, $M = xT(2p)$, where $x \in T(2p)_m \setminus J_m$ for some $m \geqslant 1$. Therefore, we wish to show that each proper factor of the form $T(2p)/(xT(2p) + J)$ is not g-torsionfree.

Note that, by Lemma 4.3.9, we have $g^m T_k \subseteq J_m T_k + xT_k$ for some $k \geqslant 1$. By Lemma 4.3.10, there exists $\ell \in \mathbb{N}$ and $U \subseteq T_\ell$ such that $T_k U = T(2p)_{k+\ell}$. It then follows that $g^m T(2p)_{k+\ell} \subseteq xT(2p)_{k+\ell} + J_m T(2p)_{k+\ell} = xT(2p)_{k+\ell} + J_{m+k+\ell}$, and so $T(2p)/(xT(2p) + J)$ is not g-torsionfree. □

We are finally in a position to prove our main result:

Theorem 4.3.13. *$T(2p)$ has no proper sporadic ideals.*

Proof. As discussed at the beginning of this section, Rogalski's proof of Theorem 4.2.4 (1) tells us that a minimal sporadic ideal of $T(2p)$ has the form $H'(K \cap T(2p))H$. Here K is a minimal sporadic ideal of $T(p)$, and H (respectively, H') is a sporadic ideal of $T(2p)$ which, in particular, annihilates all g-torsionfree factors of the right (respectively, left) exceptional line module associated to the blowup $T(2p) \subseteq T(p)$. However, by Theorem 4.2.4 (2), $T(p)$ has no proper sporadic ideals, so $T(p)$ is a minimal sporadic ideal in itself, and we may therefore set $K = T(p)$. Moreover, Proposition 4.3.12 tells us that we may take $H = T(2p)$ in the above. Since all of our right hand results have left hand analogues, we may also take $H' = T(2p)$, and then the ideal $T(2p)\big(T(p) \cap T(2p)\big)T(2p) = T(2p)$ is a minimal sporadic ideal of $T(2p)$. It follows that any sporadic ideal I of $T(2p)$ is equal to $T(2p)$ in large degree. That is, $T(2p)$ has no proper sporadic ideals. □

4.4 The rings $A(\mathbf{d})$ and their properties

We now consider some algebras which have appeared previously in [ATVdB91, Rog11, RSS15b, RSS17] and work out some of their properties.

Definition 4.4.1. Let T be the 3-Veronese of a generic Sklyanin algebra, and let $\mathbf{d}$ be an effective divisor on E with $0 \leqslant \deg \mathbf{d} \leqslant 7$. We set $A(\mathbf{d}) = (T(\mathbf{d})[g^{-1}])_0$.

When $\mathbf{d}$ consists of no points then we write $A := A(\mathbf{d}) = (T[g^{-1}])_0$ and think of this as "coordinate ring of noncommutative affine space $\mathbb{P}^2_{\mathrm{nc}} \setminus E$", and otherwise we think of $A(\mathbf{d})$ as "coordinate ring of the blowup of noncommutative affine space $\mathrm{Bl}_{\mathbf{d}}(\mathbb{P}^2_{\mathrm{nc}}) \setminus E$". When $\mathbf{d}$ consists of multiple points then the resulting geometric picture depends on whether the points lie on distinct τ-orbits, and this is also reflected in the ring-theoretic properties of the $A(\mathbf{d})$.

Observing that $A(\mathbf{d}) \cong (T(\mathbf{d}) \otimes_{\Bbbk[g]} \Bbbk[g^{\pm 1}])_0$ one can also define a functor $(-)^\circ = (- \otimes_{\Bbbk[g]} \Bbbk[g^{\pm 1}])_0 : \text{Gr-}T(\mathbf{d}) \to \text{Mod-}A(\mathbf{d})$. Since $\Bbbk[g^{\pm 1}]$ is a flat $\Bbbk[g]$-module, this functor is exact. Moreover, standard localisation theory implies that $\mathrm{Hom}_{T(\mathbf{d})}(M, N)^\circ \cong \mathrm{Hom}_{A(\mathbf{d})}(M^\circ, N^\circ)$ for any $M, N \in \text{Gr-}T(\mathbf{d})$ and that $(-)^\circ$ eliminates degree shifts, in the sense that $(M[n])^\circ \cong M^\circ$ for all $n \in \mathbb{Z}$. Moreover, $A(\mathbf{d}) \cong T(\mathbf{d})/(g-1)T(\mathbf{d})$ by [RSS15b, Lemma 2.1], which allows one to deduce a number of properties of these rings. We first recall a result which is implicit in the proof of [CBH98, Theorem 1.5], but which is stated more precisely in [CB, Lemma 5.6].

Lemma 4.4.2 ([CB, Lemma 5.6])**.** *Let A be a finitely generated noetherian $\Bbbk$-algebra which is Auslander-regular and Cohen-Macaulay. Then*

$$\operatorname{gl.dim} A = \max\{\operatorname{GKdim} A - \operatorname{GKdim} M \mid 0 \neq M \in \text{mod-}A\}.$$

Proposition 4.4.3. *The rings $A(\mathbf{d})$ are finitely generated $\Bbbk$-algebras which are noetherian domains. They are Cohen-Macaulay and Auslander-Gorenstein with* $\operatorname{i.dim} A(\mathbf{d}) \leqslant 2$ *and* $\operatorname{GKdim} A(\mathbf{d}) = 2$. *If $A(\mathbf{d})$ has finite global dimension, then it has global dimension 1 if and only if it has no nonzero finite-dimensional modules, and global dimension 2 otherwise.*

Proof. Since $A(\mathbf{d})$ is the degree 0 part of the localisation of a domain, it is itself a domain. By [RSS14, Theorem 1.1], it is a finitely generated noetherian $\Bbbk$-algebra. By [RSS15b, Corollary 2.3], $A(\mathbf{d})$ has a filtration whose associated graded ring is isomorphic to $B = B(E, \mathcal{O}_E(-\mathbf{d}) \otimes \mathcal{M}, \tau)$, and it is known by [Lev92, Theorem 6.6], that $B \cong T(\mathbf{d})/gT(\mathbf{d})$ is Cohen-Macaulay and Auslander-Gorenstein with $\operatorname{i.dim} B = 2$. Then, by [Lev92, Proposition 3.2 (ii), Theorem 3.6 (2)], it follows that $A(\mathbf{d})$ is Cohen-Macaulay and Auslander-Gorenstein with $\operatorname{i.dim} A(\mathbf{d}) \leqslant 2$. The value of the GK dimension of $A(\mathbf{d})$ follows from [Lev92, (5.2.1)], and the value of the global dimension of $A(\mathbf{d})$ follows from Lemma 4.4.2. □

The ideal theory of $A(\mathbf{d})$ is closely related to the sporadic ideal theory of $T(\mathbf{d})$, as follows:

Proposition 4.4.4. *Suppose that $T(\mathbf{d})$ has no proper sporadic ideals. Then $A(\mathbf{d})$ is simple.*

Proof. By the discussion preceding [RSS15a, Lemma 7.8], $(-)^\circ$ gives a bijection between the set of graded g-divisible ideals of $T(\mathbf{d})$ and the set of ideals of $A(\mathbf{d})$. So suppose that J is a nonzero ideal of $A(\mathbf{d})$, so that $J = I^\circ$ for some nonzero graded g-divisible ideal I of $T(\mathbf{d})$. Then $\overline{I} = (I + gT(\mathbf{d}))/gT(\mathbf{d})$ is a graded ideal of $\overline{T(\mathbf{d})} = T(\mathbf{d})/gT(\mathbf{d}) = B(E, \mathcal{O}_E(-\mathbf{d}) \otimes \mathcal{M}, \tau)$. It then follows by [RRZ06, Example 1.4] that $\operatorname{GKdim} \overline{T}/\overline{I} = 0$. Thus, for $n \gg 0$, we have

$$\begin{aligned}
0 = \dim_\Bbbk \left(\overline{T}/\overline{I}\right)_n &= \dim_\Bbbk \left(T(\mathbf{d})/(I + gT(\mathbf{d}))\right)_n \\
&= \dim_\Bbbk T(\mathbf{d})_n - \dim_\Bbbk I_n - \dim_\Bbbk (gT(\mathbf{d}))_n + \dim_\Bbbk (I \cap gT(\mathbf{d}))_n \\
&= \dim_\Bbbk (T(\mathbf{d})/I)_n - \dim_\Bbbk T(\mathbf{d})_{n-1} + \dim_\Bbbk (Ig)_n \\
&= \dim_\Bbbk (T(\mathbf{d})/I)_n - \dim_\Bbbk T(\mathbf{d})_{n-1} + \dim_\Bbbk (I)_{n-1} \\
&= \dim_\Bbbk (T(\mathbf{d})/I)_n - \dim_\Bbbk (T(\mathbf{d})/I)_{n-1}.
\end{aligned}$$

That is, for $n \gg 0$ we have $\dim_\Bbbk (T(\mathbf{d})/I)_n = \dim_\Bbbk (T(\mathbf{d})/I)_{n-1}$, so $\operatorname{GKdim} T(\mathbf{d})/I \leqslant 1$. But $T(\mathbf{d})$ has no proper sporadic ideals, and so I is equal to $T(\mathbf{d})$ in all large degrees, which implies that $I^\circ = T(\mathbf{d})^\circ$. Therefore $J = A(\mathbf{d})$. □

As a corollary of this, we obtain the following:

Corollary 4.4.5. *Let $p, q \in E$, where p and q lie on distinct τ-orbits. Then A, $A(p)$, $A(p+q)$ and $A(2p)$ are all simple rings.*

Proof. By Theorem 4.2.4, [RSS15b, Example 9.6] and Theorem 4.3.13, in each case the corresponding $T(\mathbf{d})$ has no proper sporadic ideals, and so $A(\mathbf{d})$ is simple by Proposition 4.4.4. □

4.5 Singularities of two-point blowups

In this section we determine the global dimension of algebras of the form $A(p+q)$, and show that the only such algebras which are singular are those of the form $A(2p)$. In this case, we also calculate the singularity category. The fact that $A(2p)$ is simple is crucial in determining $\mathcal{D}_{\mathrm{sg}}(A(2p))$, justifying the effort expended in Section 4.3.

We will in fact see that the global dimension of $A(p+q)$ exhibits trichotomous behaviour:

$$\operatorname{gl.dim} A(p+q) = \begin{cases} 1 & \text{if } p \text{ and } q \text{ lie on distinct } \tau\text{-orbits} \\ 2 & \text{if } p = \tau^n(q) \text{ for some } n \in \mathbb{Z} \setminus \{0\} \\ \infty & \text{if } p = q \end{cases}.$$

At this point, it is worth drawing a comparison with some results in the literature. In [Sta82], Stafford studied primitive factors R_α of $\mathcal{U}(\mathfrak{sl}_2)$, which depend on a parameter $\alpha \in \Bbbk$ and have the presentation

$$R_\alpha = \frac{\Bbbk\langle e, f, h\rangle}{\left\langle \begin{matrix} he - eh = 2e, & hf - fh = -2f \\ ef - fe = h, & h^2 + 2h + 4fe = \alpha^2 + 2\alpha \end{matrix} \right\rangle}.$$

Stafford showed that the global dimension of R_α also exhibits trichotomous behaviour, as follows:

$$\operatorname{gl.dim} R_\alpha = \begin{cases} 1 & \text{if } \alpha \in \Bbbk \setminus \mathbb{Z} \\ 2 & \text{if } \alpha \in \mathbb{Z} \setminus \{-1\} \\ \infty & \text{if } \alpha = -1 \end{cases}.$$

Stafford also showed that there exist Morita contexts between many of the R_α, which he used to establish a number of properties of these algebras in addition to the above. The values for the global dimensions of $A(p+q)$ when $p \neq q$ and the existence of similar Morita contexts for these algebras have been known to the authors of [RSS15b] for some time.

Additionally, the results of the previous chapter show that for the singular primitive factor R_{-1} we have a triangle equivalence $\mathcal{D}_{\mathrm{sg}}(R_{-1}) \simeq \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$. This follows from the fact that the map

$$\frac{\Bbbk\langle x, y, z\rangle}{\left\langle \begin{matrix} xz = (z+1)x, & xy = z(z+\alpha+1) \\ yz = (z-1)y, & yx = (z-1)(z+\alpha) \end{matrix} \right\rangle} \to R_\alpha,$$

$$x \mapsto f, \quad y \mapsto -e, \quad z \mapsto \frac{1}{2}(h - \alpha)$$

is an isomorphism, where the domain is isomorphic to $\mathcal{O}^{(-\alpha,\alpha+1)}(\widetilde{\mathbb{A}}_1)$, and when $\alpha = -1$ we have seen that this ring has an $\mathbb{A}_1$ singularity.

We now work towards proving the claimed global dimensions of the algebras $A(p+q)$

which, while known, are not present in the literature. We first recall an important result which we will use frequently in this section:

Theorem 4.5.1 ([RSS17, Theorem 9.1, Lemma 6.9])**.** *Let L be the exceptional line module for a blowup $T(\mathbf{d}+p) \subseteq T(\mathbf{d})$. Then the following are equivalent:*
(1) $\operatorname{gl.dim} A(\mathbf{d}+p) < \infty$*;*
(2) $\operatorname{p.dim}_{A(\mathbf{d}+p)} L^\circ = 1$ *and* $\operatorname{gl.dim} A(\mathbf{d}) < \infty$*; and*
(3) $\operatorname{p.dim}_{A(\mathbf{d}+p)} L^\circ < \infty$ *and* $\operatorname{gl.dim} A(\mathbf{d}) < \infty$.

Using this, we are able to show that the algebras $A(p)$ are nonsingular, and are in fact hereditary. This will be used in conjuction with Theorem 4.5.1 to deduce the global dimensions of two-point blowups.

Proposition 4.5.2. *Let $p \in E$. Then $A(p)$ is hereditary.*

Proof. We seek to apply Theorem 4.5.1, and so wish to show that $\operatorname{p.dim} L^\circ < \infty$, where L° is the localised line module corresponding to the blowup $T(p) \subseteq T$. To do this, we show that the localisation of the line ideal J corresponding to the blowup $T(p) \subseteq T$ is projective.

We first determine an expression for J. Noting that

$$T_1 W(\sigma^3(p))W(\sigma(p))S_1 = (W(p)S_2)^2 = T(p)_2,$$

it follows that $W(\sigma^3(p))W(\sigma(p))S_1 \subseteq J_1$ by definition. Since $T(p)/J$ is a line module, it follows that $\dim_{\Bbbk} J_1 = 7$, while $\dim_{\Bbbk} W(\sigma^3(p))W(\sigma(p))S_1 = 7$ by Lemma 4.2.7, and so $J_1 = W(\sigma^3(p))W(\sigma(p))S_1$. Since J is generated in degree 1, [RSS17, Lemma 5.8 (2)], it follows that $J = W(\sigma^3(p))W(\sigma(p))S_1T(p)$.

We now calculate $J^* \coloneqq \operatorname{Hom}_{T(p)}(J, T(p))$. Clearly $\Bbbk \subseteq J^*$ and

$$T(p)T_1 \cdot J = T(p)S_3W(\sigma^3(p))W(\sigma(p))S_1T(p) = T(p)(W(p)S_2)^2T(p) \subseteq T(p),$$

so that $\Bbbk + T(p)T_1 \subseteq J^*$. Now [RSS15b, Proposition 5.2 (1)] shows that $\dim_{\Bbbk}(T(p)T_1)_n = \dim_{\Bbbk} T_n - \binom{n}{2}$, so that

$$\operatorname{hilb}\big(\Bbbk + T(p)T_1\big) = \frac{t^2+7t+1}{(1-t)^3} - \frac{t^2}{(1-t)^3} = \frac{7t+1}{(1-t)^3}.$$

But we also have

$$\operatorname{hilb} J^* = \operatorname{hilb} T(p) + \frac{t}{(1-t)^2} = \frac{7t+1}{(1-t)^3}$$

by [RSS17, Lemma 5.8 (3)], and since J^* and $\Bbbk + T(p)T_1$ have the same Hilbert series they must be equal.

Using these expressions for J and J^*, we find that

$$(JJ^*)_2 \supseteq W(\sigma^3(p))W(\sigma(p))S_1 \cdot S_3 = (W(\sigma^3(p)S_2)^2 = T(\sigma^3(p))_2 \ni g^2.$$

It follows that $1 \in (JJ^*)^\circ = J^\circ(J^\circ)^*$, and so J° is projective by the dual basis lemma.

Consider the short exact sequence coming from the definition of J:

$$0 \to J \to T(p) \to L \to 0.$$

Applying the exact functor $(-)^\circ$, we obtain a projective resolution of L°,

$$0 \to J^\circ \to A(p) \to L^\circ \to 0,$$

and since $\operatorname{gl.dim} A = 1 < \infty$ by [Aji99, Proposition 2.18], Theorem 4.5.1 implies that $\operatorname{gl.dim} A(p) < \infty$. Now, Corollary 4.4.5 tells us that $A(p)$ is simple, which implies that it has no finite-dimensional modules, and so $\operatorname{gl.dim} A(p) = 1$ by Proposition 4.4.3; that is, $A(p)$ is hereditary. □

We now turn our attention to two-point blowups, beginning with the cases where $p \neq q$. The following proof is similar to that of Proposition 4.5.2, and so some of the details are suppressed:

Proposition 4.5.3. *Let $p, q \in E$, where $p \neq q$. Then*

$$\operatorname{gl.dim} A(p+q) = \begin{cases} 1 & \text{if } p \text{ and } q \text{ lie on distinct } \tau\text{-orbits} \\ 2 & \text{if } p = \tau^i(q) \text{ for some } i \in \mathbb{Z} \setminus \{0\} \end{cases}.$$

Proof. If p and q lie on the same τ-orbit then, relabelling if necessary, we can assume that $q = \tau^i(p)$ for some $i \geqslant 1$. We initially treat both cases simultaneously and show that $A(p+q)$ has finite global dimension provided that $p \neq q$.

Let J (respectively, L) be the line ideal (respectively, module) for the blowup $T(p+q) \subseteq T(p)$. By mimicking the proof of Lemma 4.3.1, one can show that $T(p+q)_n = \big(W(p)W(\sigma(q))S_1\big)^n$. (We remark that if p and q lie on the same τ-orbit then, since $i \neq -1$, we are not in the situation where the right hand side is $\big(W(p)W(\sigma^{-2}(p))S_1\big)^n$, which is strictly contained in $T(p + \tau^{-1}(p))$.) As in Proposition 4.5.2 one can show that $J = W(\sigma^3(q))W(\sigma(p))W(\sigma^2(q))T(p+q)$ and $J^* = \Bbbk + T(p+q)T(p)_1$. We then find that

$$\begin{aligned}(JJ^*)_2 &\supseteq W(\sigma^3(q))W(\sigma(p))W(\sigma^2(q)) \cdot W(p)S_2 \\ &= \big(W(\sigma^3(q))W(\sigma(p))S_1\big)^2 \\ &= T(p+\sigma^3(q))_2 \\ &\ni g^2,\end{aligned}$$

where the hypothesis $i \geqslant 1$ ensures that we have an equality on the third line in the cases where p and q lie on the same τ-orbit. As in the proof of Proposition 4.5.2, we therefore find that J° is projective and that $\operatorname{p.dim} L^\circ < \infty$, and hence $\operatorname{gl.dim} A(p+q) < \infty$ for both cases.

Now suppose that p and q lie on distinct τ-orbits. Since $A(p+q)$ is simple by Corollary 4.4.5, it has no finite-dimensional modules, and therefore it is hereditary by Proposition 4.4.3. If instead p and q lie on the same τ-orbit then, by the remark preceding [RSS15b, Example 9.7], $T(p+q)$ has a proper sporadic ideal I, and by [Rog11, Lemma 6.4 (2)], we may assume that $T(p+q)/I$ is g-torsionfree. It follows that $\big(T(p+q)/I\big)^\circ$ is a finite-dimensional $A(p+q)$-module, and so $\operatorname{gl.dim} A(p+q) = 2$ by Proposition 4.4.3. □

It remains to prove the claim that $A(2p)$ is singular, and to determine its singularity category. This is more involved, and takes up majority of this section. For the remainder of this section, we will fix notation for the following three $T(2p)$-modules,

$$J = W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))T(2p)$$

$$M = \Bbbk + T(p)_1 T(2p)$$
$$N = \Bbbk + T_1 T(2p),$$

and we write J°, M° and N° for the corresponding $A(2p)$-modules. We note that J is the line ideal for the blowup $T(2p) \subseteq T(p)$. We first determine some Hom spaces.

Proposition 4.5.4.
(1) $\mathrm{End}_{T(2p)}(J) = T(p + \tau(p))$.
(2) $J^* = \Bbbk + T(2p)T(p)_1$.
(3) $M^* = T(2p)W(p)W(\sigma(p))W(\sigma^{-1}(p))$.
(4) $\mathrm{End}_{T(2p)}(N) = T(2\tau^{-1}(p))$.

Proof.
(1) This follows from [RSS17, Theorem 8.9 (1)].
(2) One inclusion is clear: noting that left multiplication by $\Bbbk$ gives an element of J^*, and that

$$\begin{aligned} T(2p)T(p)_1 \cdot J &= T(2p)W(p)S_2W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))T(2p) \\ &= T(2p)W(p)W(\sigma(p))S_1W(p)W(\sigma(p))S_1T(2p) \\ &\subseteq T(2p), \end{aligned}$$

it follows that we have $\Bbbk + T(2p)T(p)_1 \subseteq J^*$. We complete the proof by showing that both spaces have the same Hilbert series. By [RSS15b, Proposition 5.2 (1)], $\dim_\Bbbk(\Bbbk + T(2p)T(p)_1)_n = \dim_\Bbbk T(p)_n - \frac{1}{2}n(n-1)$ for $n \geqslant 0$. Therefore, we have

$$\mathrm{hilb}\,(\Bbbk + T(2p)T(p)_1) = \frac{t^2+6t+1}{(1-t)^3} - \frac{t^2}{(1-t)^3} = \frac{6t+1}{(1-t)^3}.$$

Then, by [RSS17, Lemma 5.8 (3)], $\mathrm{hilb}\, J^* = \mathrm{hilb}\, T(2p) + t/(1-t)^2 = (6t+1)/(1-t)^3$ and so $\mathrm{hilb}\, J^* = \mathrm{hilb}\,(\Bbbk + T(2p)T(p)_1)$, as required.
(3) By [RSS17, Lemma 5.8 (2)], J is reflexive, and the same is true of the left line module for the blowup $T(2p) \subseteq T(p)$, which we call $J^\vee$. Using the same methods as for J, it is easy to check that $J^\vee = T(2p)W(p)W(\sigma(p))W(\sigma^{-1}(p))$ and that $(J^\vee)^* = \Bbbk + T(p)_1T(2p) = M$. Reflexivity of $J^\vee$ then implies

$$M^* = (J^\vee)^{**} = J^\vee = T(2p)W(p)W(\sigma(p))W(\sigma^{-1}(p)).$$

(4) This follows from the calculations in [RSS17, Lemma 10.6]. □

Proposition 4.5.5.
(1) *N° is projective, and therefore reflexive.*
(2) *J° is reflexive, but not projective.*
(3) *M° is reflexive, but not projective.*

Proof.
(1) Since N and N^* may be viewed as left and right $\mathrm{End}_{T(2p)}(N)$-modules respectively, we may view NN^* as a two-sided ideal of $\mathrm{End}_{T(2p)}(N) \cong T(2\tau^{-1}(p))$. Since the functors $(-)^*$ and $(-)^\circ$ commute, so we find that $(NN^*)^\circ = (N^\circ)(N^\circ)^*$ is a nonzero two-sided ideal of $A(2\tau^{-1}(p))$. But $A(2\tau^{-1}(p))$ is simple by Theorem 4.3.13 and Theorem 4.4.4, so $1 \in (N^\circ)(N^\circ)^*$, and hence N° is projective by the dual basis lemma. The second claim follows from the fact that projective modules are always reflexive; see [Lam99, pp. 55, Example 7].

(2) As mentioned previously, J is reflexive by [RSS17, Lemma 5.8 (2)]. Applying the functor $(-)^\circ$, we see that J° is reflexive.

For the second claim, first note that we may view JJ^* as a two-sided ideal of $\mathrm{End}_{T(2p)}(J) \cong T(p+\tau(p))$, and that $J^* \cong \Bbbk + T(2p)T(p)_1$. We then have $(JJ^*)_1 = J_1$, while for $n \geqslant 2$,

$$\begin{aligned}(JJ^*)_n &= \big((J_1T(2p))(\Bbbk + T(2p)T(p)_1)\big)_n \\ &= J_1 T(2p)_{n-2} T(p)_1 \\ &= W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))\big(W(p)W(\sigma(p))S_1\big)^{n-2} W(\sigma(p))S_2 \\ &= W(\sigma^3(p))W(\sigma(p))S_1 W(\sigma^3(p))\big(W(\sigma(p))S_1W(\sigma^3(p))\big)^{n-2} W(\sigma(p))S_1 \\ &= \big(W(\sigma^3(p))W(\sigma(p))S_1\big)^n.\end{aligned}$$

By [Rog11, Proposition 11.2], it follows that $JJ^* = J_1 \oplus K_{\geqslant 2}$, where

$$K \coloneqq W(\sigma^3(p))W(\sigma(p))S_1T(p+\sigma^3(p))$$

is a sporadic ideal of $T(p+\sigma^3(p))$ with $g^n \notin K$ for all $n \geqslant 1$. It follows that $1 \notin K^\circ$, and so since $(J^\circ)(J^\circ)^* \subseteq K^\circ$, we have $1 \notin (J^\circ)(J^\circ)^*$, and therefore J° is not projective by the dual basis lemma.

(3) As we saw in Proposition 4.5.4 (3), M is the dual of the left line module $J^\vee$, and since $J^\vee$ is reflexive, the same is true of M.

To show that M° is not projective, we consider the two exact sequences

$$0 \to J \to T(2p) \to L \to 0,$$
$$0 \to T(2p) \to M \to L[-1] \to 0,$$

the first of which comes from the definition of the exceptional line module $L = T(2p)/J$, and the second of which comes from the proof of [Ro, Lemma 9.1 (1)]. Applying $(-)^\circ$ yields exact sequences

$$0 \to J^\circ \to A(2p) \to L^\circ \to 0,$$
$$0 \to A(2p) \to M^\circ \to L^\circ \to 0.$$

If M° were projective, then Schanuel's Lemma would imply that $M^\circ \oplus J^\circ \cong A(2p)^2$. In particular, J° would be a direct summand of a free module and would therefore be projective, contradicting part (2). Therefore M° is not projective. □

As an immediate corollary, we deduce that $A(2p)$ is singular:

Corollary 4.5.6. $\mathrm{p.dim}\, M^\circ = \infty$, *and therefore* $\mathrm{gl.dim}\, A(2p) = \infty$.

Proof. By Lemma 2.2.13 and Proposition 4.5.5, M° is a maximal Cohen-Macaulay module which is not projective. It therefore has infinite projective dimension by Lemma 2.2.9 (4), whence the result. □

We now wish to determine the singularity category of $A(2p)$, which we claim is equivalent to $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$, where $R_{\mathbb{A}_1}$ is the coordinate ring of an $\mathbb{A}_1$ singularity. To achieve this, we first prove a fairly general result which gives sufficient conditions for a (possibly noncommutative) ring to have an $\mathbb{A}_1$ singularity, and then show that $A(2p)$ satisfies all of its hypotheses.

Theorem 4.5.7. *Let R be a Gorenstein $\Bbbk$-algebra of injective dimension at most 2. Suppose that R has a maximal Cohen-Macaulay module M which is a generator such that $\operatorname{gl.dim} \operatorname{End}_R(M) \leqslant 2$ and $\underline{\operatorname{End}}_R(M) \cong \Bbbk$. Then R has infinite global dimension and has an $\mathbb{A}_1$ singularity.*

Proof. Since $\underline{\operatorname{End}}_R(M) \neq 0$, M is not projective. It therefore has infinite projective dimension by Lemma 2.2.9 (4), and therefore $\operatorname{gl.dim} R = \infty$. By [Kra15, Proposition 2.3], the functor

$$\underline{\operatorname{Hom}}_R(M, -) : \underline{\operatorname{mod}}\text{-}R \to \operatorname{Mod-}\underline{\operatorname{End}}_R(M) = \operatorname{Mod-}\Bbbk$$

induces a fully faithful functor $\operatorname{add}_{\underline{\operatorname{mod}}\text{-}R}(M) \to \operatorname{proj-}\Bbbk$. From Chapter 3, we know that proj-$\Bbbk$ is equivalent to $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$ as a $\Bbbk$-linear category, where $R_{\mathbb{A}_1}$ is the coordinate ring of an $\mathbb{A}_1$ singularity. Since $\operatorname{i.dim} R \leqslant 2$ and $\operatorname{gl.dim} \operatorname{End}_R(M) \leqslant 2$, Proposition 2.2.11 implies that $\operatorname{add}_{\operatorname{mod}\text{-}R}(M) = \operatorname{MCM}\text{-}R$. Since projective modules are killed when passing to the stable category, we find that $\operatorname{add}_{\underline{\operatorname{mod}}\text{-}R}(M) = \underline{\operatorname{MCM}}\text{-}R$. We therefore have a fully faithful functor

$$\underline{\operatorname{Hom}}_R(M, -) : \underline{\operatorname{MCM}}\text{-}R \to \operatorname{proj-}\Bbbk = \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1}),$$

which is also essentially surjective since $\Bbbk^n$ is the image of M^n under the functor. Therefore we have an $\Bbbk$-linear equivalence between these two categories, and this induces a triangulated structure on $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$. It is also a triangle equivalence because, by [Che11, Lemma 3.4], $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$ has a unique triangulated structure since the only $\Bbbk$-linear autoequivalence of $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$ is the identity. □

We now show that $A(2p)$ and the module M° satisfy the hypotheses of Theorem 4.5.7. We have already seen that $A(2p)$ is Gorenstein of injective dimension at most 2, and that M° is reflexive and hence MCM.

Proposition 4.5.8. *The module M° is a generator for* mod-$A(2p)$ *which satisfies* $\operatorname{gl.dim} \operatorname{End}_{A(2p)}(M^\circ) = 2$. *Moreover,* $\underline{\operatorname{End}}_{A(2p)}(M^\circ) \cong \Bbbk$.

Proof. Write $E := \operatorname{End}_{T(2p)}(M)$, so that M becomes an $(E, T(2p))$-bimodule. By similar arguments to those found in the proof of Proposition 4.5.4, $E \cong T(p + \tau(p))$. It follows that $\operatorname{End}_{A(2p)}(M^\circ) \cong A(p + \tau(p))$, which we know to have global dimension 2 by Proposition 4.5.3.

To show that M° is a generator for mod-$A(2p)$, we can equivalently show that M° is a projective left E°-module. Since $(M^\circ)^* M^\circ$ is a nonzero two-sided ideal of the simple ring $A(2p)$, we necessarily have $1 \in (M^\circ)^* M^\circ$. It follows from the dual basis lemma that M° is a projective left E°-module, as required.

For the final claim, we first show that M° and J° are isomorphic in $\underline{\operatorname{mod}}$-$A(2p)$. By the definition of L, the proof of [RSS17, Lemma 10.8], and exactness of $(-)^\circ$, we have exact sequences

$$0 \to J^\circ \to A(2p) \to L^\circ \to 0,$$
$$0 \to M^\circ \to N^\circ \to L^\circ \to 0,$$

where $A(2p)$ and N° are projective. Schanuel's Lemma implies that $J^\circ \oplus N^\circ \cong M^\circ \oplus A(2p)$, so that J° and M° are projectively equivalent and therefore isomorphic in $\underline{\operatorname{mod}}$-$A(2p)$.

Hence, to determine $\underline{\operatorname{End}}_{A(2p)}(M^\circ)$ it suffices to calculate $\underline{\operatorname{End}}_{A(2p)}(J^\circ)$. In the proof of Proposition 4.5.5, we saw that $(J^\circ)(J^\circ)^* \subseteq K^\circ$, where K is a sporadic ideal of $T(p+\sigma^3(p))$; we claim that this inclusion is in fact an equality. To see this, recall that $JJ^* = J_1 \oplus K_{\geqslant 2}$ and consider the short exact sequence

$$0 \to JJ^* \to K \to K/JJ^* \to 0,$$

where K/JJ^* is finite-dimensional. Applying the exact functor $(-)^\circ$ and noting that $(K/JJ^*)^\circ = 0$, we obtain an isomorphism $(J^\circ)(J^\circ)^* \cong K^\circ$, as claimed. Finally,

$$\underline{\mathrm{End}}_{A(2p)}(M^\circ) \cong \underline{\mathrm{End}}_{A(2p)}(J^\circ) = \frac{\mathrm{End}_{A(2p)}(J^\circ)}{(J^\circ)(J^\circ)^*} \cong \frac{\mathrm{End}_{T(2p)}(J)^\circ}{K^\circ} \cong \left(\frac{T(p+\sigma^3(p))}{K}\right)^\circ.$$

Then, since $T(p+\sigma^3(p))/K$ has $\Bbbk$-basis given by $\{1, g, g^2, g^3, \dots\}$, see [Ro, Proposition 11.2], it follows that $\underline{\mathrm{End}}_{A(2p)}(M^\circ) \cong \Bbbk$. □

We have therefore shown that $A(2p)$ and the module M° satisfy all the hypotheses of Theorem 4.5.7, and hence $A(2p)$ has an $\mathbb{A}_1$ singularity:

Theorem 4.5.9. *There is a triangle equivalence* $\underline{\mathrm{MCM}}\text{-}(A(2p)) \simeq \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$. □

Chapter 5

Azumaya Skew Group Algebras

In this chapter, we provide necessary and sufficient conditions for a skew group algebra, or more generally, a crossed product ring, to be an Azumaya algebra. The two particular cases of interest to us are that of a finite group acting X-outer on a prime noetherian $\Bbbk$-algebra, and a finite cyclic group acting inner on a prime $\Bbbk$-algebra. We recall from Chapter 2 that a group acts X-outer on a prime noetherian ring R if the only element acting as conjugation by an element of the Goldie quotient ring $Q(R)$ is the identity. The main results of this chapter will be used in subsequent chapters to establish that (deformations of) quantum Kleinian singularities, as defined in Chapters 1 and 2, are maximal orders.

5.1 Azumaya crossed products with an X-outer action

Suppose that $A * G$ is a crossed product where G is a finite group acting X-outer on A. Unsurprisingly, a necessary condition to ensure that $A * G$ is Azumaya is that A is itself Azumaya. Additionally, since properties of an Azumaya algebra A are closely related to the geometry of $\operatorname{Spec} Z(A)$, one might expect the action of G on the points of $\operatorname{Spec} Z(A)$ to influence whether $A * G$ is Azumaya. This is the case, as we will see in Theorem 5.1.3.

We first determine the centre $Z(A*G)$ of $A*G$, which is relatively straightforward.

Lemma 5.1.1. *Consider a crossed product $T := A * G$, where A is prime and G is a finite group. Suppose that the action of G on A is X-outer. Then $Z(A * G) = Z(A)^G$.*

Proof. The inclusion $Z(A)^G \subseteq Z(A * G)$ is easy to establish and holds without the X-outer hypothesis, so we now show the reverse inclusion. So suppose that $\sum_{g \in G} c_g \overline{g} \in Z(A * G)$. Then for any $b \in A$ we have

$$\sum_{g \in G} b c_g \overline{g} = \sum_{g \in G} c_g \overline{g} b = \sum_{g \in G} c_g \, \alpha_g(b) \overline{g}.$$

This forces $c_g \alpha_g(b) = b c_g$ for each $g \in G$ and each $b \in A$. In particular, c_g is a normal element of A, and so if it were nonzero then it would be an element of $Q(A)$ by [Pas87, Lemma 2.1 (ii)]. Since α_g acts X-outer when $g \neq e$ by hypothesis, this forces $c_g = 0$ for all $g \neq e$. Therefore we must have $Z(A * G) \subseteq Z(A)$, so now consider some $a \in Z(A * G) \subseteq Z(A)$. Then, for any $\overline{g}$ we have

$$a \overline{g} = \overline{g} a = \alpha_g(a) \overline{g},$$

which means that a lies in $Z(A)^G$. This gives the claimed equality. $\square$

We briefly recall some terminology. A *G-stable* ideal I of R is an ideal which satisfies $g \cdot I = I$ for all $g \in G$. In this case, I is said to be *G-prime* if it is proper and whenever $JK \subseteq I$ for G-stable ideals J and K, then $J \subseteq I$ or $K \subseteq I$. Moreover, I is said to be *G-maximal* if whenever $I \subsetneq J$ for some G-stable ideal J, then $J = R$. It is straightforward to check that the one-to-one correspondence between prime (respectively, maximal) ideals of an Azumaya algebra A and maximal ideals of its centre Z restricts to a one-to-one correspondence between G-prime (respectively, G-maximal) ideals.

To prove our main result, we will also need the following lemma:

Lemma 5.1.2. *Let R be a commutative finitely generated $\Bbbk$-algebra which is a domain. Suppose that G is a finite group acting $\Bbbk$-linearly and freely on R; that is, the stabiliser of every maximal ideal of R is trivial. If $\mathfrak{m} \in \operatorname{MaxSpec} R^G$, then $\mathfrak{m}R$ is a G-maximal ideal of R.*

Proof. Let $\mathfrak{m} \in \operatorname{MaxSpec} R^G$. It is straightforward to check that $\mathfrak{m}R$ is a G-stable ideal of R, so it remains to verify that it is maximal among all such ideals. Since $\Bbbk$ is algebraically closed, there is a one-to-one correspondence between orbits of points in $\operatorname{MaxSpec} R$ and G-maximal ideals of R, and by our hypothesis on maximal ideals, these orbits have size $|G|$. Since $\operatorname{Spec} R/\mathfrak{m}R$ is a G-stable subvariety of $\operatorname{Spec} R$, it is a union of orbits, and we wish to show that it consists of a single orbit. Therefore if we show that $\operatorname{Spec} R/\mathfrak{m}R$ has size $|G|$ then it follows that $\mathfrak{m}R$ is a G-maximal ideal.

To this end, consider the natural morphism $\pi : \operatorname{Spec} R \to \operatorname{Spec} R^G$ coming from the inclusion $i : R^G \hookrightarrow R$. Since G is finite, the action of G on $\operatorname{Spec} R$ is closed, and so [Mum65, Amplification 1.3] tells us that $(\operatorname{Spec} R^G, \pi)$ is a geometric quotient of R. By [Mum65, Proposition 0.9], it follows that π is flat and finite. Therefore π is a flat, finite, dominant morphism between integral schemes, and so writing $p = \operatorname{Spec} R^G/\mathfrak{m}$, we have $|\pi^{-1}(p)| = [\operatorname{Frac}(R) : \operatorname{Frac}(R^G)]$ by [Liu02, Exercise 5.1.25], and this value is equal to $|G|$ by [Ben93, Proposition 1.1.1]. But

$$\pi^{-1}(p) \cong p \times_{\operatorname{Spec} R^G} \operatorname{Spec} R \cong \operatorname{Spec}(R^G/\mathfrak{m} \otimes_{R^G} R) \cong \operatorname{Spec} R/\mathfrak{m}R,$$

and so $|\operatorname{Spec} R/\mathfrak{m}R| = |G|$. By the preceding paragraph, it now follows that $\mathfrak{m}R$ is a G-maximal ideal of R. $\square$

We remind the reader that, for a crossed product $A * G$, the automorphisms α_g act as a group on $Z(A)$. This fact will be used in the proof of the following, which is the main result of this section.

Theorem 5.1.3. *Consider a crossed product $T := A*G$, where A is a prime noetherian $\Bbbk$-algebra and G is a finite group acting X-outer on A by $\Bbbk$-linear automorphisms. Then T is prime noetherian. Moreover, T is Azumaya if and only if*

1. *A is Azumaya; and*
2. *G acts freely on $Z(A)$; that is, the stabiliser of every maximal ideal of $Z(A)$ is trivial.*

If T is Azumaya, then the ranks of A and T satisfy $\operatorname{rank} T = |G|^2 \operatorname{rank} A$.

Proof. First note that T is prime by [Pas89, Corollary 12.6] and noetherian by [MR01, Lemma 1.5.1]. Also, $Z(A)$ is a domain since A is prime.

Throughout this proof, we will be concerned with maximal ideals lying in a number of different rings: it will be our convention to write $\mathfrak{m}, \mathfrak{n}$, and M for maximal ideals in

$Z(T), Z(A)$, and T, respectively.

($\Rightarrow$) Assume that T is Azumaya. Since G acts X-outer, $Z(T) = Z(A)^G \subseteq A$ by Lemma 5.1.1. Therefore, [Car05, Proposition 2.4] implies that A is a separable extension of $Z(A)^G$. We then have a chain of inclusions $Z(A)^G \subseteq Z(A) \subseteq A$, and so A is a separable extension of $Z(A)$ by [HS66, Proposition 2.5]; that is, A is Azumaya. In particular, since A is Azumaya and noetherian this implies that $Z(A)$ is noetherian by [MR01, 13.7.10].

It remains to show that (2) holds, and that if $\operatorname{rank} T = r^2$ then $\operatorname{rank} A = (r/|G|)^2$. We establish these simultaneously. We claim that a maximal ideal $\mathfrak{n}$ of $Z(A)$ has trivial stabiliser if and only if $\dim_{\Bbbk} A/\mathfrak{n}A = (r/|G|)^2$. So suppose that $\mathfrak{n} \in \operatorname{MaxSpec} Z(A)$. Since $Z(A)$ is a module-finite extension of $Z(A)^G$, [Ben93, Lemma 1.4.2] implies that $\mathfrak{n} \cap Z(A)^G$ is a maximal ideal of $Z(A)^G = Z(T)$. Therefore, as T is Azumaya, $(\mathfrak{n} \cap Z(A)^G)T$ is a maximal ideal of T. Writing

$$\mathfrak{n}' = \bigcap_{g \in G} g \cdot \mathfrak{n},$$

which is the intersection of $|G|/|\operatorname{Stab}_G(\mathfrak{n})|$ maximal ideals of $Z(A)$, we have $\mathfrak{n} \cap Z(A)^G \subseteq \mathfrak{n}'$. Maximality of $(\mathfrak{n} \cap Z(A)^G)T$ forces $(\mathfrak{n} \cap Z(A)^G)T = \mathfrak{n}'T$, so $\mathfrak{n}'T$ is a maximal ideal of T. Since $\mathfrak{n}'A$ is G-stable and T is Azumaya of rank r^2, we have

$$\dim_{\Bbbk} \left(\frac{A}{\mathfrak{n}'A} * G \right) = \dim_{\Bbbk} \frac{T}{\mathfrak{n}'T} = r^2,$$

and so $\dim_{\Bbbk} A/\mathfrak{n}'A = r^2/|G|$. We now turn our attention to determining $A/\mathfrak{n}'A$. We clearly have an inclusion $\mathfrak{n}'A \subseteq \bigcap_{g \in G} \big((g \cdot \mathfrak{n})A \big)$ of G-stable ideals. Since $\mathfrak{n}'$ is a G-maximal ideal of $Z(A)$ and A is Azumaya, it follows that $\mathfrak{n}'A$ is a G-maximal ideal of A, and hence $\mathfrak{n}'A = \bigcap_{g \in G} \big((g \cdot \mathfrak{n})A \big)$. Using the fact that A is Azumaya, there is some $\ell \in \mathbb{N}$ such that $A/(g \cdot \mathfrak{n})A \cong M_\ell(\Bbbk)$ for each $g \in G$. Then, since $\mathfrak{n}'A$ is the intersection of $|G|/|\operatorname{Stab}_G(\mathfrak{n})|$ maximal ideals of A, the Chinese Remainder Theorem implies that

$$\frac{A}{\mathfrak{n}'A} \cong \underbrace{M_\ell(\Bbbk) \times \cdots \times M_\ell(\Bbbk)}_{|G|/|\operatorname{Stab}_G(\mathfrak{n})| \text{ copies}},$$

so that, taking dimensions on both sides,

$$\frac{r^2}{|G|} = \frac{|G|}{|\operatorname{Stab}_G(\mathfrak{n})|} \ell^2.$$

Therefore $\mathfrak{n}$ has trivial stabiliser if and only if $\ell^2 = r^2/|G|^2$, which happens if and only if $\dim_{\Bbbk} A/\mathfrak{n}A = (r/|G|)^2$, as claimed.

It follows that if we can find a single maximal ideal of $Z(A)$ with trivial stabiliser then A has rank $(r/|G|)^2$, and since A is Azumaya, this will imply that every maximal ideal of $Z(A)$ has trivial stabiliser.

To this end, write $X = \operatorname{Spec} Z(A)$, which is an irreducible affine variety. Since G acts X-outer it acts faithfully on $Z(A)$, so if $g \in G$ is not the identity, then $\{x \in X \mid g \cdot x = x\}$ is a proper subvariety of X and so has strictly smaller dimension than X. But X is irreducible and so cannot be a finite union of subvarieties of strictly smaller dimension, and so some point of X lies outside of $\bigcup_{g \in G} \{x \in X \mid g \cdot x = x\}$. That is, there exists some maximal ideal of $Z(A)$ having trivial stabiliser, completing the proof of necessity.

($\Leftarrow$) Now suppose that (1) and (2) both hold. As in the proof of necessity, noetherianity of A implies that $Z(A)$ is noetherian. Seeking to prove that T is Azumaya, first note that by Lemma 5.1.1, we have $Z(T) = Z(A)^G$. Let $\mathfrak{m}$ be a maximal ideal of $Z(T)$. Then, since $Z(A)$ is a noetherian domain and the stabiliser of every maximal ideal of $Z(A)$ is trivial, Lemma 5.1.2 implies that $\mathfrak{m}Z(A)$ is a G-maximal ideal of $Z(A)$. Since A is Azumaya, we find that $\mathfrak{m}A$ is a G-maximal ideal of A. Let Q be a prime of A minimal over $\mathfrak{m}A$; necessarily Q is a maximal ideal of A, and so by hypothesis $\operatorname{Stab}_G Q = \{e\}$. Since

$$\frac{A/\mathfrak{m}A}{Q/\mathfrak{m}A} * \operatorname{Stab}_G Q \cong A/Q$$

is prime, [Pas89, Corollary 14.8] implies that $(A/\mathfrak{m}A) * G$ is prime, and hence $\mathfrak{m}T$ is a prime ideal of T. We claim that $\mathfrak{m}T$ is in fact maximal. So suppose that I is a prime ideal of T with $\mathfrak{m}T \subseteq I$. Intersecting down with A we find that $\mathfrak{m}A \subseteq I \cap A$, where $I \cap A$ is a G-stable ideal of A, and so G-maximality of $\mathfrak{m}A$ forces $\mathfrak{m}A = I \cap A$. By [Pas89, Theorem 14.7], such prime ideals I are in one-to-one correspondence with primes J of $A * \operatorname{Stab}_G Q = A$ with $J \cap A = Q$. This forces $J = Q$, and so there is only one prime I with $\mathfrak{m}T \subseteq I$. Since $\mathfrak{m}T$ is prime, this forces $\mathfrak{m}T = I$, and so $\mathfrak{m}T$ is a maximal ideal of T.

Now let M be a maximal ideal of T, and write $\mathfrak{m} = M \cap Z(T)$. By [BG12, Lemma III.1.5], $\mathfrak{m}$ is a maximal ideal of $Z(T)$, and moreover it satisfies $\mathfrak{m}T \subseteq M$. By the previous paragraph, $\mathfrak{m}T$ is a maximal ideal of T, and so $\mathfrak{m}T = M$.

Since T is prime, we use the Artin-Procesi Theorem to show that T is Azumaya, and it suffices to show that T is PI and that every maximal ideal M of T satisfies $(M \cap Z(T))T = M$, see [BG12, Theorem III.1.6]. To see the first of these, note that we have a chain of inclusions

$$Z(T) = Z(A)^G \subseteq Z(A) \subseteq A \subseteq T$$

where each term is module-finite over the preceding term: indeed, since $Z(A)$ is a noetherian domain we find that $Z(A)$ is finite over $Z(A)^G$ by [LW12, Proposition 5.4], while A is finite over $Z(A)$ since A is Azumaya, and T is finite over A by definition. Therefore T is PI by [MR01, Corollary 13.1.13 (iii)]. Finally, the preceding paragraph shows that every maximal ideal M of T satisfies $(M \cap Z(T))T = M$, and so T is Azumaya. □

Noting that the action of a group on a commutative ring A is automatically X-outer, we obtain the following.

Corollary 5.1.4. *Consider a crossed product $T := A * G$, where A is a commutative noetherian $\Bbbk$-algebra which is a domain and where G is a finite group acting $\Bbbk$-linearly on A. Then T is prime noetherian. Moreover, T is Azumaya if and only if G acts freely on A, and in this case,* $\operatorname{rank} T = |G|^2$.

We now give an application of Theorem 5.1.3. This demonstrates an approach which we will use in Chapter 6 to analyse skew group rings arising from quantum Kleinian singularities.

Example 5.1.5. Let $A = \Bbbk[u, v]$, which is clearly a prime noetherian Azumaya algebra. Let $G = S_2 = \langle h \rangle$ act (X-outer) on A via $h \cdot u = v$, $h \cdot v = u$. Observe that a maximal ideal $\langle u-a, v-b \rangle$ of $Z(A) = A$ has nontrivial stabiliser if and only if $a = b$; in particular, the action is not free, and so Theorem 5.1.3 tells us that $T = A \# G$ is not Azumaya. Indeed, the maximal ideal $\mathfrak{m} = \langle u + v, uv \rangle$ of $Z(T) = \Bbbk[u + v, uv]$ does not extend to a maximal ideal of T since $\mathfrak{m}T \subsetneq \langle u+v, uv, h-1\rangle \subsetneq T$, and so T is not Azumaya. More generally, given $\mathfrak{m} = \langle u+v-\alpha, uv-\beta\rangle \in \operatorname{MaxSpec} Z(T)$, one can show that

$$T/\mathfrak{m}T \cong \begin{cases} M_2(\Bbbk) & \text{if } \alpha^2 \neq 4\beta \\ \Pi(\mathbb{A}_2) & \text{if } \alpha^2 = 4\beta \end{cases},$$

where $\Pi(\mathbb{A}_2)$ is the preprojective algebra of an $\mathbb{A}_2$ Dynkin quiver, so that $\mathfrak{m}T$ is not even prime when $\alpha^2 = 4\beta$. However, if we replace A by $A' = A[(u-v)^{-1}]$ then the maximal ideals of $Z(A') = A'$ have the form $\langle u-a, v-b\rangle$ with $a \neq b$, and Theorem 5.1.3 guarantees that $T' = A'\#G$ is Azumaya. Indeed, $Z(A'\#G) = \Bbbk[u+v, uv][(u-v)^{-2}]$ and

$$(u-v)^{-2}\left((u+v-\alpha)(u+v+\alpha) - 4\left(uv - \frac{\alpha^2}{4}\right)\right) = (u-v)^{-2}\left((u+v)^2 - 4uv\right) = 1$$

so that $\operatorname{MaxSpec} Z(T') = \{\langle u+v-\alpha, uv-\beta\rangle \mid \alpha^2 \neq 4\beta\}$. As before, one can then show that $T'/\mathfrak{m}T' \cong M_2(\Bbbk)$ for all $\mathfrak{m} \in \operatorname{MaxSpec} Z(T')$, so that $T' = A'\#G$ is Azumaya by Lemma 2.4.9.

As mentioned above, this example demonstrates a strategy that we will utilise frequently in Chapter 6. In that setting, we are frequently concerned with rings A which are finite over their centres, and where G acts X-outer. It will often be the case that a generic maximal ideal of the centre of A has trivial stabiliser, and we can remove those that do not by localising A at a suitable subset of $Z(A)$. The new algebra A' and the group G will then meet the hypothesis of Theorem 5.1.3, and so $A'\#G$ will be Azumaya.

5.2 Azumaya skew group algebras where a cyclic group acts inner

Now let G be a cyclic group acting inner on an algebra A. As in the previous section, one would expect that A must necessarily be Azumaya to ensure that $A\#G$ is Azumaya. Under our hypotheses, this turns out to be both necessary and sufficient.

We begin with a lemma that determines the centre of such a skew group ring.

Lemma 5.2.1. *Suppose that $G = \langle g \mid g^n\rangle$ acts inner on A, so there exists a unit $\eta \in A$ such that $g \cdot a = \eta a \eta^{-1}$ for all $a \in A$. Then*

$$Z(A\#G) = Z(A)[(\eta^{-1}g)^{\pm 1}].$$

Proof. First observe that if $z \in Z(A)$, $a \in A$ and $g^i \in C_n$ then

$$z\eta^{-1}g \cdot ag^i = z\eta^{-1}\eta a\eta^{-1}gg^i = za\eta^{-1}gg^i = az\eta^{-1}g^ig = ag^i\eta^{-i}z\eta^{-1}\eta^ig = ag^i \cdot z\eta^{-1}g,$$

which establishes the inclusion $Z(A)[(\eta^{-1}g)^{\pm 1})] \subseteq Z(A\#G)$. For the reverse inclusion, consider any element $x = \sum_{i=0}^{n-1} a_ig^i \in Z(A\#G)$. Then for any $b \in A$ we have

$$\sum_{i=0}^{n-1} ba_ig^i = bx = xb = \sum_{i=0}^{n-1} a_ig^ib = \sum_{i=0}^{n-1} a_i\eta^ib\eta^{-i}g^i,$$

and so $ba_i = a_i\eta^ib\eta^{-i}$ for each i. Equivalently, $ba_i\eta^i = a_i\eta^ib$ for each i and for each

$b \in A$, and so each $a_i\eta^i$ lies in $Z(A)$. Therefore

$$x = \sum_{i=0}^{n-1} (a_i\eta^i)\eta^{-i}g^i = \sum_{i=0}^{n-1} (a_i\eta^i)(\eta^{-1}g)^i \in Z(A)[(\eta^{-1}g)^{\pm 1})],$$

as required. □

We can now prove the main result of this section:

Theorem 5.2.2. *Let A be a prime $\Bbbk$-algebra and let $G = \langle g \mid g^n \rangle$ be a cyclic group acting inner on A, so there exists a unit $\eta \in A$ such that $g \cdot a = \eta a \eta^{-1}$ for all $a \in A$. Suppose also that $T = A\#G$ is prime. Then A is Azumaya if and only if T is Azumaya, and in this case A and T have the same rank.*

Proof. ($\Rightarrow$) Assume that A is Azumaya of rank d^2. First observe that T is PI: indeed, A is necessarily module-finite over its centre, and since $Z(T) = Z(A)[(\eta^{-1}g)^{\pm 1}]$ where $(\eta^{-1}g)^n = \eta^{-n}$, it follows that T is also module-finite over its centre, which implies that T is PI by [MR01, Corollary 13.1.13 (iii)]. Therefore, since T is also prime, by Lemma 2.4.9 it suffices to show that $T/\mathfrak{m}T \cong M_d(\Bbbk)$ for each $\mathfrak{m} \in \operatorname{MaxSpec} Z(T)$.

To this end, let $\mathfrak{m}$ be a maximal ideal of $Z(T)$. Then $\mathfrak{m} \cap Z(A)$ is a maximal ideal of $Z(A)$ by [Ben93, Lemma 1.4.2], and since A is Azumaya, $(\mathfrak{m} \cap Z(A))A$ is a maximal ideal of A. Therefore, $B \coloneqq A/(\mathfrak{m} \cap Z(A))A \cong M_d(\Bbbk)$ which, in particular, is a simple ring. Let $\pi : A \to B$ be the natural projection and define a linear map

$$\theta : B \to T/\mathfrak{m}T, \quad \theta(b) = a + \mathfrak{m}T, \text{ where } \pi(a) = b.$$

This map is well-defined in the sense that it does not depend on the choice of the preimage of b: if $\pi(a) = b = \pi(a')$, then $a - a' \in (\mathfrak{m} \cap Z(A))A \subseteq \mathfrak{m}T$, so that $a + \mathfrak{m}T = a' + \mathfrak{m}T$. The map θ is also clearly a ring homomorphism, and we claim that it is surjective. To this end, let $t = \sum_{i=0}^{n-1} a_i g^i \in T$. Since $\Bbbk$ is algebraically closed and A is a finitely generated $\Bbbk$-algebra, $\mathfrak{m}$ contains an element of the form $\eta^{-1}g - \lambda$, where $\lambda \in \Bbbk$. By replacing each instance of g in t by $(g - \lambda\eta) + \lambda\eta$ and noting that g and $\lambda\eta$ commute, we can write $t = \sum_{i=0}^{n-1} a'_i(g - \lambda\eta)^i$ for some $a'_i \in A$. Therefore $t + \mathfrak{m}T = a'_0 + \mathfrak{m}T$ so that $\theta(\pi(a'_0)) = a'_0 + \mathfrak{m}T = t + \mathfrak{m}T$, and hence θ is surjective. Since B is simple and θ is not the zero map, $\ker\theta$ is trivial, and hence θ is an isomorphism. Therefore $T/\mathfrak{m}T \cong B \cong M_d(\Bbbk)$, as required.

($\Leftarrow$) For the converse, suppose that T is Azumaya of rank d^2. Since T is PI of PI degree d, A is PI and $\operatorname{PIdeg} A \leqslant d$. Using Lemma 2.4.9, it suffices to show that if $\mathfrak{m}$ is a maximal ideal of $Z(A)$ then $A/\mathfrak{m}A \cong M_d(\Bbbk)$, since this will force $\operatorname{PIdeg} A = d$. So let $\mathfrak{m} \in \operatorname{MaxSpec} Z(A)$. Now, $Z(A)$ and $Z(T) = Z(A)[(\eta^{-1}g)^{\pm 1}] \cong Z(A)[t]/\langle t^n - \eta^{-n}\rangle$ are finitely generated $\Bbbk$-algebras, so $Z(T)$ has a maximal ideal of the form $\mathfrak{m}' = \mathfrak{m} + \langle \eta^{-1}g - \beta \rangle$ for some $\beta \in \Bbbk^\times$. Since T is Azumaya of rank d^2, $T/\mathfrak{m}'T \cong M_d(\Bbbk)$. Define a linear map

$$\widetilde{\phi} : T \to A/\mathfrak{m}A, \quad \widetilde{\phi}(ag^i) = a(\eta\beta)^i + \mathfrak{m}A,$$

which is easily checked to be a surjective ring homomorphism. It is clear that $\mathfrak{m}'T \subseteq \ker\widetilde{\phi}$, so we get a well-defined surjection $\phi : T/\mathfrak{m}'T \to A/\mathfrak{m}A$. But $T/\mathfrak{m}'T \cong M_d(\Bbbk)$ is a simple ring and ϕ is not the zero map, so $\ker\phi$ is trivial and ϕ is an isomorphism. Therefore $A/\mathfrak{m}A \cong T/\mathfrak{m}'T \cong M_d(\Bbbk)$, as required. □

Remark 5.2.3. It is not automatically the case that T is prime under the above hypotheses, even if the action is faithful. For example, set $A = M_2(\mathbb{C})$ and $G = C_2 = \langle g \rangle$, where g acts as conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} g, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g$$

are central elements of $T \coloneqq A\#G$ which satisfy $xy = 0$. Therefore $xTy = xyT = 0$ and so T is not prime.

Example 5.2.4. We provide an example of an application of Theorem 5.2.2 which will be generalised in Chapter 6. Let $A = \mathbb{k}_{-1}[u^{\pm 1}, v^{\pm 1}]$ and $G = C_2$, where the generator g acts via $g \cdot u = -u$, $g \cdot v = -v$. This is an inner action, since g acts as conjugation by $(uv)^{-1}$. We claim that $A\#G$ is Azumaya; since A is Azumaya, by Theorem 5.2.2 we only need to show that $A\#G$ is prime.

Since $A\#G$ is semiprime, see [Pas89, Theorem 4.4], and noetherian, $A\#G$ has a classical quotient ring $Q(A\#G)$, [GW04, Corollary 6.16]. By [GW04, Lemma 6.17], it suffices to show that $Q(A\#G) \cong \mathbb{k}_{-1}(u, v)\#G$ is simple. Since $\mathbb{k}_{-1}(u, v)$ is simple, by [Öin14, Theorem 1.2] $Q(A\#G)$ is simple if and only if $Z(\mathbb{k}_{-1}(u, v)\#G)$ is a field. Observing that g acts as conjugation by $(uv)^{-1}$, Lemma 5.2.1 implies that

$$Z(\mathbb{k}_{-1}(u, v)\#G) = \mathbb{k}(u^2, v^2)[(uvg)^{\pm 1}] \cong \frac{\mathbb{k}(u^2, v^2)[t]}{\langle t^2 - u^2v^2 \rangle}.$$

Since $t^2 - u^2v^2$ is irreducible over $\mathbb{k}[u^2, v^2][t]$, it is also irreducible over $\mathbb{k}(u^2, v^2)[t]$ by Gauss's Lemma. Therefore $Z(\mathbb{k}_{-1}(u, v)\#G)$ is a field, which implies that $A\#G$ is prime.

To close this example, note that $\mathbb{k}_{-1}[u^{\pm 1}, v]$ is not Azumaya, but one can show that the skew group ring $\mathbb{k}_{-1}[u^{\pm 1}, v]\#C_2$ is Azumaya. This does not contradict Theorem 5.2.2, since in this case the action is not inner, as v is not invertible.

Chapter 6

Deformations of Quantum Kleinian Singularities

In this chapter we show that one can deform the quantum Kleinian singularities A^G from Table 2.1 and the corresponding rings $A\#G$ in the same way as in [CBH98], and show that these deformations have nice ring-theoretic and homological properties. In later chapters we also determine properties related to the singularities and representation theory of these deformations. We remark that the existence of these deformations is guaranteed by [WW14, Theorem 3.1], so the main new results in this chapter relate to their properties.

6.1 The deformations $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ and their first properties

We deform the algebras $A\#G$ and A^G in the same manner as in [CBH98], and this section follows the same outline as that of [CBH98, Section 1]. We first repeat the classification of quantum Kleinian singularities below, drawing attention to the changes in notation for cases (i) and (iii); in particular, for the latter we now write

$$n = \begin{cases} \frac{m+4}{2} & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases} .$$

This allows us to always assume that our McKay quivers have $n+1$ vertices.

Case	A	G	$\widetilde{Q}$
(i)	$\Bbbk_q[u,v]$	C_{n+1}	$\widetilde{\mathbb{A}}_n$
(ii)	$\Bbbk_{-1}[u,v]$	S_2	$\widetilde{\mathbb{L}}_1$
(iii)	$\Bbbk_{-1}[u,v]$	D_m	$\begin{cases} \widetilde{\mathbb{D}}_n & \text{if } m \text{ is even} \\ \widetilde{\mathbb{DL}}_n & \text{if } m \text{ is odd} \end{cases}$
(iv)	$\Bbbk_J[u,v]$	C_2	$\widetilde{\mathbb{A}}_1$

Table 6.1: The pairs (A, G) for quantum Kleinian singularities A^G, and their McKay quivers.

Given a pair (A, G) from Table 6.1, we first establish a bijection between $\Bbbk^{n+1} = \Bbbk^{\widetilde{Q}_0}$ and $Z(\Bbbk G)$; this is essentially the same as in Section 2.5, but we repeat it for the reader's convenience. The irreducible representations $W_0, W_1, \ldots, W_n$ of G correspond to the vertices of the McKay quiver $\widetilde{Q}$, where W_0 is the trivial representation. Write $\delta_i = \dim_{\Bbbk} W_i$. The only cases where we have not already explicitly listed δ are when $\widetilde{Q}$ is $\widetilde{\mathbb{L}}_1$ or $\widetilde{\mathbb{DL}}_n$ $(n \geqslant 2)$. With the numbering of the vertices of $\widetilde{Q}$ as in Figure 2.1 we have

$$\begin{aligned} \widetilde{\mathbb{L}}_1 : \quad & \delta = (1,1) \\ \widetilde{\mathbb{DL}}_n : \quad & \delta = (1,1,\underbrace{2,2\ldots,2,2}_{n-1 \text{ times}}). \end{aligned}$$

Writing χ_i for the character of the representation W_i, for $0 \leqslant i \leqslant n$ set

$$\eta_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) g,$$

which are central idempotents in $\Bbbk G$ and form a $\Bbbk$-basis for $Z(\Bbbk G)$. Then the map

$$\Bbbk^{n+1} \to Z(\Bbbk G), \qquad (\lambda_0, \lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=0}^{n} \frac{\lambda_i}{\delta_i} \eta_i \tag{6.1.1}$$

is a bijection, which we will henceforth use to identify $\Bbbk^{n+1}$ with $Z(\Bbbk G)$, often without mention.

Now fix a pair (A, G) from Table 6.1 and view the action of G on A as an action on $\Bbbk\langle u, v\rangle$. For $* \in \{q, -1, J\}$ define

$$\rho_*(u,v) = \begin{cases} vu - quv & \text{if } * = q \text{ (case (i))} \\ vu + uv & \text{if } * = -1 \text{ (cases (ii) and (iii))} \\ vu - uv - u^2 & \text{if } * = J \text{ (case (iv))} \end{cases}.$$

Fix $\lambda \in \Bbbk^{n+1}$, which gives rise to an element of $Z(\Bbbk G)$ as in (6.1.1) which we also call λ, and let $e = \frac{1}{|G|} \sum_{g \in G} g \in \Bbbk G$ be the average of the group elements. Noting that $A^G \cong e(A\#G)e$ [BHZ16, Lemma 3.1 (3)], we then define our deformations of $A\#G$ and A^G to be, respectively,

$$\mathcal{S}_*^{\lambda}(\widetilde{Q}) \coloneqq \frac{\Bbbk\langle u, v\rangle \# G}{\langle \rho_*(u,v) - \lambda \rangle} \quad \text{and} \quad \mathcal{O}_*^{\lambda}(\widetilde{Q}) \coloneqq e\mathcal{S}_*^{\lambda}(\widetilde{Q})e.$$

Observe that the algebra we are deforming is uniquely determined by $*$ and $\widetilde{Q}$. When the precise algebra is unimportant, we will frequently omit the quiver $\widetilde{Q}$, but will retain the asterisk to make it clear that these are deformations in the quantum setting. Setting $\lambda = 0$, we recover the algebras $A\#G$ and A^G.

Note that $\mathcal{S}_*^{\lambda}$ and $\mathcal{O}_*^{\lambda}$ can be filtered by putting u and v in degree 1 and elements of G in degree 0. This also induces a filtration of $\mathcal{O}_*^{\lambda}$. We then have:

Lemma 6.1.2. *With respect to the above filtration,* $\operatorname{gr} \mathcal{S}_*^{\lambda} \cong A\#G$ *and* $\operatorname{gr} \mathcal{O}_*^{\lambda} \cong A^G$.

Proof. In each case, the relation $\rho_*(u,v) - \lambda$ enables any monomial in u and v to be rewritten as a linear combination of monomials of the form $u^i v^j$, and hence

$$\{u^i v^j g \mid g \in G,\, i,j \geqslant 0\}$$

is a $\Bbbk$-basis of $\operatorname{gr} \mathcal{S}_*^{\lambda}$. It follows that $\operatorname{gr} \mathcal{S}_*^{\lambda} \cong A\#G$. Since $e(A\#G)e \cong A^G$, the result for $\mathcal{O}_*^\lambda$ also follows. □

This lemma allows us to prove a number of ring-theoretic properties of these deformations by first proving them for the undeformed algebras $A\#G$ and A^G.

Lemma 6.1.3.
(1) $\mathcal{S}_*^\lambda$ *and* $\mathcal{O}_*^\lambda$ *are both noetherian, and are finitely generated as* $\Bbbk$*-algebras.*
(2) $\mathcal{S}_*^\lambda$ *is a prime ring and* $\mathcal{O}_*^\lambda$ *is a domain.*
(3) $\mathcal{S}_*^\lambda$ *and* $\mathcal{O}_*^\lambda$ *both have GK dimension* 2.

Proof. (1) Since A is either $\Bbbk_q[u,v]$ or $\Bbbk_J[u,v]$, both of which are noetherian, $A\#G$ is noetherian by [MR01, Lemma 1.5.11]. Noetherianity of A^G follows from [Mon93, Corollary 4.3.5]. It then follows from Lemma 6.1.2 and [MR01, Theorem 1.6.9] that $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ are both noetherian. Finally, $\mathcal{S}_*^\lambda$ is clearly finitely generated as a $\Bbbk$-algebra, while the corresponding result for $\mathcal{O}_*^\lambda$ follows from [MS81, Corollary 1].
(2) The claims follow by [MR01, Proposition 1.6.6] provided that we can show that $A\#G$ is prime and A^G is a domain in each case. The latter of these follows from the fact that both $\Bbbk_q[u,v]$ and $\Bbbk_J[u,v]$ are domains, so we turn our attention to showing that $A\#G$ is prime. We use a case-by-case argument noting that, by [Pas89, Corollary 12.6], it suffices to show that $A\#G_{X\text{-inn}}$ is prime.

We first consider case (i). By [BW99, Proposition 2.2], every X-inner automorphism of A is given by conjugation by a monomial in u and v. In particular, if q is not a root of unity, then $G_{X\text{-inn}}$ is trivial, whence the result. So assume that q has order k and write $\ell = \operatorname{lcm}(n+1,k)$ and $d = \gcd(n+1,k)$. Let g be a generator of C_{n+1}, acting via $g\cdot u = \omega u$, $g \cdot v = \omega^{-1} v$, where ω is a primitive $(n+1)$th root of unity. If g^i acts X-inner, say as conjugation by some monomial $f \in \Bbbk_q(u,v)$, then since u^k is central we have

$$\omega^{ik}u^k = g^i \cdot u^k = f u^k f^{-1} = u^k.$$

Therefore $ik \in (n+1)\mathbb{Z}$, so $i\frac{k/\gcd(n+1,k)}{(n+1)/\gcd(n+1,k)} \in \mathbb{Z}$, and so necessarily $i \in \frac{n+1}{\gcd(n+1,k)}\mathbb{Z} = \frac{\ell}{k}\mathbb{Z}$. But also $g^{\ell/k}\cdot a = (uv)^{\ell/(n+1)} a (uv)^{-\ell/(n+1)}$ for all $a \in \Bbbk_q(u,v)$, so that $G_{X\text{-inn}} = \langle g^{\ell/k}\rangle$. We therefore need to show that $\Bbbk_q[u,v]\#\langle g^{\ell/k}\rangle$ is prime, and by [GW04, Lemma 6.17] it suffices to show that $Q(A\#G_{X\text{-inn}}) \cong \Bbbk_q(u,v)\#\langle g^{\ell/k}\rangle$ is simple. Since $\Bbbk_q(u,v)$ is simple, by [Öin14, Theorem 1.2] we only need to show that $Z(\Bbbk_q(u,v)\#\langle g^{\ell/k}\rangle)$ is a field. Since $g^{\ell/k}$ acts inner on $\Bbbk_q(u,v)$ as conjugation by $(uv)^{-\ell/(n+1)}$, by Lemma 5.2.1 we have

$$\begin{aligned} Z(\Bbbk_q(u,v)\#\langle g^{\ell/k}\rangle) &= Z(\Bbbk_q(u,v))[(uv)^{-\ell/(n+1)}g^{\ell/k}] \\ &= \Bbbk(u^k,v^k)[(uv)^{-\ell/(n+1)}g^{\ell/k}] \\ &\cong \frac{\Bbbk(u^k,v^k)[t]}{\langle t^d - u^{-k}v^{-k}\rangle}. \end{aligned}$$

Now, $t^d - u^{-k}v^{-k}$ is irreducible over $\Bbbk[u^{-k},v^{-k}][t]$, and so Gauss's Lemma implies that it is also irreducible over $\Bbbk(u^k,v^k)[t]$. Therefore $\Bbbk(u^k,v^k)[t]/\langle t^d - u^{-k}v^{-k}\rangle$ is a field, and hence so too is $Z(\Bbbk_q(u,v)\#\langle g^{\ell/k}\rangle)$, from which it follows that $A\#G$ is prime.

Now consider case (ii). Since the action of the generator of S_2 interchanges u and v, it quickly follows from degree considerations that $G_{X\text{-inn}}$ is trivial. Therefore, since $\Bbbk_{-1}[u,v]$ is prime, so too is $A\#G$.

For case (iii), the analysis in the previous two paragraphs shows that

$$\Bbbk_{-1}[u,v]\#G_{X\text{-inn}} = \begin{cases} \Bbbk_{-1}[u,v]\#\langle g^{m/2}\rangle & \text{if } m \text{ is even} \\ \Bbbk_{-1}[u,v] & \text{if } m \text{ is odd} \end{cases},$$

where g is a generator for the group of rotations inside D_m. In either case, we already know that $A\#G_{X\text{-inn}}$ is prime, and so $A\#G$ is prime.

Finally, consider case (iv). We claim that every X-inner automorphism of $\Bbbk_J[u,v]$ is given by conjugation by some power of u. First observe that $\Bbbk_J[u,v]$ is the Ore extension $\Bbbk[u][v; u^2\frac{\mathrm{d}}{\mathrm{d}u}]$. Therefore by [Mon81, Theorem 2], if σ is X-inner then it is necessarily conjugation by some $u^k a/b \in \Bbbk(u)$ where $a, b \in \Bbbk[u]$ are coprime, $u \nmid a$, $u \nmid b$, and $k \in \mathbb{Z}$. Since u^k is normal, conjugation by it is clearly X-inner, and since X-inner automorphisms form a subgroup of the automorphism group of $\Bbbk_J[u,v]$, $\sigma u^{-k} = a/b$ must also act X-inner. By [Mon81, Theorem 2] again, the element

$$\frac{b}{a} \cdot u^2 \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{a}{b}\right) = \frac{u^2(ba' - ab')}{ab}$$

must lie in $\Bbbk[u]$. Since $u \nmid a$ and $u \nmid b$, it follows that $ab \mid (ba' - ab')$ but this is impossible by degree considerations unless $a' = b' = 0$, in which case $a/b \in \Bbbk$. Therefore σ is given by conjugation by u^k. But

$$u^k u u^{-k} = u \neq -u = g \cdot u$$

and so $G_{X\text{-inn}}$ is trivial, from which it follows that T is prime.
(3) The filtration of $\mathcal{S}_*^\lambda$ given above is a finite-dimensional filtration and so by [MR01, Proposition 8.6.5] and Lemma 6.1.2, $\operatorname{GKdim} \mathcal{S}_*^\lambda = \operatorname{GKdim} A\#G$ and $\operatorname{GKdim} \mathcal{O}_*^\lambda = \operatorname{GKdim} A^G$. Since $\operatorname{GKdim} A\#G = \operatorname{GKdim} A = 2$ by [MR01, Proposition 8.2.9], we find that $\operatorname{GKdim} \mathcal{S}_*^\lambda = 2$. Moreover, by [Mon93, Theorem 4.4.2], A is finitely generated as a module over A^G on either side, and so [MR01, Proposition 8.2.9 (ii)] tells us that $\operatorname{GKdim} A^G = 2$, and hence $\operatorname{GKdim} \mathcal{O}_*^\lambda = 2$ as well. □

Remark 6.1.4. One can alternatively show that $A\#G$ is prime by combining [BHZ16, Lemma 3.10 (2)] and [CKWZ16a, Theorem 4.1]. However, we later give an alternative proof of [CKWZ16a, Theorem 4.1] which relies on the fact that that $A\#G$ is prime, and so we can not invoke their results at this point.

If we were to follow the exposition of [CBH98], we would next like to show that these algebras are maximal orders. However, this result requires significantly more work in our new setting, and so we defer it until the next section. Instead, we now show that these deformations have nice homological properties. We also provide a value for the global dimensions of these algebras which depends on the existence of finite-dimensional modules. In Chapter 8, we are able to show when these exist and hence give a more precise value for the global dimension and, in particular, when they are singular.

Proposition 6.1.5. *The algebra $\mathcal{S}_*^\lambda$ is Auslander-regular and Cohen-Macaulay. It has global dimension* 1 *if and only if it has no nonzero finite-dimensional modules, and global dimension* 2 *otherwise.*

Proof. We first focus our attention on $A\#G$, which has global dimension 2 by [MR01, Theorem 7.5.6]. First note that in either case A is an Ore extension:

$$\Bbbk_q[u,v] = \Bbbk[u][v; \alpha : u \mapsto qu] \quad \text{and} \quad \Bbbk_J[u,v] = \Bbbk[u][v; u^2\tfrac{\mathrm{d}}{\mathrm{d}u}].$$

It follows from [Eks89, Theorem 4.2] that in either case A is Auslander-Gorenstein, and hence so too is $A\#G$ by [Yi95, Proposition 3.9 (1)]. Similarly, in either case A is Cohen-Macaulay by [Lev92, Theorem 5.10], and so $A\#G$ is also Cohen-Macaulay by [BHZ16, Proposition 3.3].

We now translate these results to $\mathcal{S}_*^\lambda$ using Lemma 6.1.2. By [MR01, Corollary 7.6.18] we find that $\operatorname{gl.dim} \mathcal{S}_*^\lambda \leqslant 2$, and by [Bjö87, Theorem 4.1], $\mathcal{S}_*^\lambda$ is Auslander-Gorenstein; combining these, we find that it is Auslander-regular. Finally, [Bjö87, Theorem 4.3] tells us that, given a finitely generated $\mathcal{S}_*^\lambda$-module M, we have $j_{\mathcal{S}_*^\lambda}(M) = j_{A\#G}(\operatorname{gr} M)$, while [MR01, Proposition 6.5] tells us that $\operatorname{GKdim}(M) = \operatorname{GKdim}(\operatorname{gr} M)$. Therefore, using the Cohen-Macaulay property of $A\#G$,

$$\begin{aligned}\operatorname{GKdim}(M) + j_{\mathcal{S}_*^\lambda}(M) &= \operatorname{GKdim}(\operatorname{gr} M) + j_{A\#G}(\operatorname{gr} M) \\ &= \operatorname{GKdim}(A\#G) = \operatorname{GKdim}(\mathcal{S}_*^\lambda),\end{aligned}$$

and so $\mathcal{S}_*^\lambda$ is Cohen-Macaulay.

For the final claim, we apply Lemma 4.4.2. Observe that $\operatorname{gl.dim} \mathcal{S}_*^\lambda \neq 0$, since otherwise this would imply that $\mathcal{S}_*^\lambda$ is artinian, which is not the case since it is infinite-dimensional. Since $\operatorname{GKdim} M = 0$ if and only if M is finite-dimensional, it now follows that $\mathcal{S}_*^\lambda$ has global dimension 1 (respectively, 2) if and only if it has no nonzero finite-dimensional modules (respectively, has a nonzero finite-dimensional module). □

Proposition 6.1.6. *The algebra $\mathcal{O}_*^\lambda$ is Auslander-Gorenstein of injective dimension at most 2 and is Cohen-Macaulay. It has global dimension* 1, 2, *or* ∞. *When it has finite global dimension, it has global dimension* 1 *if and only if it has no nonzero finite-dimensional modules.*

Proof. As above, we first consider $\operatorname{gr} \mathcal{O}_*^\lambda = A^G$. By [CKWZ16a, Theorem 5.2], $A^G \cong B/\Omega B$, where B is an AS regular algebra with $\operatorname{i.dim} B = 3$ and Ω is homogeneous, regular, and normal. By [Lev92, Corollary 6.2], B is Auslander-Gorenstein (in fact, Auslander-regular) and Cohen-Macaulay. Then [Lev92, Theorem 5.10] implies that A^G is Auslander-Gorenstein and Cohen-Macaulay with $\operatorname{i.dim} A^G = 2$.

The same arguments as in the second paragraph of Proposition 6.1.5 now show that $\mathcal{O}_*^\lambda$ is Auslander-Gorenstein and Cohen-Macaulay. The bound on the injective dimension follows from [Bjö87, Theorem 4.1]. Finally, A^G has infinite global dimension by [CKWZ14, Theorem 2.3], so we cannot a priori bound the global dimension of $\mathcal{O}_*^\lambda$. However, when it is finite, the same argument (using Lemma 4.4.2) as in the third paragraph of Proposition 6.1.5 establishes the last claim. □

In [CBH98], the authors had to work quite hard to show that the corresponding deformations of Kleinian singularities R^G are commutative precisely when $\lambda \cdot \delta = 0$. On the other hand, deformations of quantum Kleinian singularities are always noncommutative in many cases:

Lemma 6.1.7. *The algebra $\mathcal{O}_*^\lambda$ is noncommutative, except possibly in case* (i) *when $q^{n+1} = 1$ and case* (iii) *when m is even.*

Proof. Since $\operatorname{gr} \mathcal{O}_*^\lambda = A^G$ is noncommutative for all cases except for case (i) when $q^{n+1} = 1$ and case (iii) when m is even (see [CKWZ16a, Table 3]), it follows that $\mathcal{O}_*^\lambda$ is noncommutative. □

6.2 Azumaya localisations of quantum Kleinian singularities

To completely generalise the results of [CBH98, Section 1], it remains to show that deformations of quantum Kleinian singularities are maximal orders. In the classical setting this follows immediately from results in the literature, but the same is not true for quantum Kleinian singularities. The main result that Crawley-Boevey–Holland used was [Mar95, Theorem 3.13], which gives necessary and sufficient conditions for a skew group ring $A\#G$ to be a maximal order, provided that other hypotheses on A and G are met. In many cases, however, quantum Kleinian singularities do not satisfy these hypotheses, so we will employ an alternative strategy.

Our approach is to use the results of Chapter 5 to show that suitable localisations of the rings $A\#G$ are Azumaya, which will be used later to show that $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ are maximal orders. More specifically, we use Theorems 5.1.3 and 5.2.2 to show that suitable localisations of the rings $T := A\#G$ are Azumaya in cases (i) (when q is a root of unity), (ii), and (iii). The choice of Ore set at which we localise is influenced by both conditions (1) and (2) of Theorem 5.1.3, since firstly none of the algebras A are Azumaya, and since secondly we wish to remove maximal ideals of $Z(A)$ which have nontrivial stabiliser under the action of G.

For ease of notation, for the remainder of this chapter we return to our earlier notation for quantum Kleinian singularities from Table 2.1. In particular, this means that g will always denote an element of order n (where n is to be determined from context) and h will always denote an element of order 2. These elements will act on $u, v \in A$ via

$$g \cdot u = \omega u, \quad g \cdot v = \omega^{-1} v, \quad h \cdot u = v, \quad h \cdot v = u,$$

where ω is a primitive nth root of unity.

6.2.1 An Azumaya localisation for case (i) when q is a root of unity

We first show that, for case (i) when q is a root of unity, a suitable localisation of $T = A\#G = \Bbbk_q[u,v]\#C_n$ is Azumaya. Since A is not Azumaya, we instead consider the Azuamya algebra $A' = \Bbbk_q[u^{\pm 1}, v^{\pm 1}]$ and write $T' := A'\#G$. Note that $T' = T[u^{-1}, v^{-1}]$, which is prime since T is prime. By combining Theorems 5.1.3 and 5.2.2, we now show that T' is Azumaya.

Proposition 6.2.1. *Suppose that q is a kth root of unity. Then the algebra $T' := \Bbbk_q[u^{\pm 1}, v^{\pm 1}]\#C_n$ is Azumaya.*

Proof. Throughout the proof, it will be convenient to write $\ell = \operatorname{lcm}(n,k)$. We also let ε be a primitive ℓth root of unity, so that we may as well assume that $\omega = \varepsilon^{\ell/n}$ and $q = \varepsilon^{\ell/k}$.

Note that $g^{\ell/k}$ acts inner on A', since

$$\begin{aligned} g^{\ell/k} \cdot u &= \omega^{\ell/k} u = \varepsilon^{\ell^2/nk} u = q^{\ell/n} u = (uv)^{\ell/n} u (uv)^{-\ell/n} \\ g^{\ell/k} \cdot v &= \omega^{-\ell/k} v = \varepsilon^{-\ell^2/nk} v = q^{-\ell/n} v = (uv)^{\ell/n} v (uv)^{-\ell/n} \end{aligned}$$

and so the subgroup $\langle g^{\ell/k} \rangle$ acts inner on A'. By Lemma 6.1.3, $A\#\langle g^{\ell/k}\rangle$ is prime, and hence so too is the localisation $A'\#\langle g^{\ell/k}\rangle$. Therefore Theorem 5.2.2 ensures that $A'\#\langle g^{\ell/k}\rangle$ is Azumaya.

Since $C_n/\langle g^{\ell/k}\rangle \cong C_{\ell/k}$, as in [MR01, Lemma 1.5.9] we may define an isomorphism

$$T' \cong (A' \# \langle g^{\ell/k}\rangle) * C_{\ell/k}.$$

When defining this isomorphism, we think of $C_{\ell/k}$ as the group $\langle \sigma \mid \sigma^{\ell/k}\rangle$ and write $\overline{C_{\ell/k}} = \{\overline{\sigma^i} \mid 0 \leqslant i < \ell/k\}$, where $\overline{\sigma^i} = g^i$. The induced automorphisms of $A' \# \langle g^{\ell/k}\rangle$ are given by

$$\alpha_{\sigma^i}(u) = \omega^i u, \quad \alpha_{\sigma^i}(v) = \omega^{-i} v, \quad \alpha_{\sigma^i}(g^{\ell/k}) = g^{\ell/k}.$$

We wish to apply Theorem 5.1.3 to show that $(A' \# \langle g^{\ell/k}\rangle) * C_{\ell/k}$ is Azumaya. We first show that the action of $C_{\ell/k}$ on $R \coloneqq A' \# \langle g^{\ell/k}\rangle$ is X-outer, so suppose that α_{σ^r} acts as conjugation by an element of $Q(R)$. Since R is prime and PI, $Q(R)$ is obtained from R by inverting all nonzero central elements. Moreover, since central elements do not affect the result when conjugating, we can assume that α_{σ^r} is given by conjugation by an element of the form $\sum_t \lambda_t u^{i_t} v^{j_t} g^{m_t \ell/k}$, where $i_t, j_t, m_t \in \mathbb{Z}$ and $\lambda_t \in \Bbbk$. We therefore have

$$\alpha_{\sigma^r}(u) \cdot \sum_t \lambda_t u^{i_t} v^{j_t} g^{m_t \ell/k} = \sum_t \lambda_t u^{i_t} v^{j_t} g^{m_t \ell/k} \cdot u,$$

or equivalently,

$$\sum_t \omega^r \lambda_t u^{i_t+1} v^{j_t} g^{m_t \ell/k} = \sum_t q^{j_t} \omega^{m_t \ell/k} \lambda_t u^{i_t+1} v^{j_t} g^{m_t \ell/k}.$$

Therefore, for each t we require

$$\varepsilon^{r\ell/n} = \varepsilon^{j_t \ell/k + m_t \ell^2/nk}, \quad \text{or equivalently,} \quad j_t \tfrac{\ell}{k} + m_t \tfrac{\ell^2}{nk} - r\tfrac{\ell}{n} \in \ell\mathbb{Z}.$$

This forces $j_t + m_t \frac{\ell}{n} - r\frac{k}{n} \in k\mathbb{Z}$, and so necessarily $r\frac{k}{n} = r\frac{k/\gcd(n,k)}{n/\gcd(n,k)}$ is an integer. Since $k/\gcd(n,k)$ and $n/\gcd(n,k)$ are coprime, we must then have $r \in \frac{n}{\gcd(n,k)}\mathbb{Z} = \frac{\ell}{k}\mathbb{Z}$. But $0 \leqslant r < \ell/k$ so $r = 0$, which means that the induced action is X-outer.

We also need to show that $\operatorname{Stab}_{C_{\ell/k}}(\mathfrak{m})$ is trivial for each $\mathfrak{m} \in \operatorname{MaxSpec} Z(R)$. Note that $\mathfrak{m}$ has form

$$\langle u^k - \alpha, (uv)^{-\ell/n} g^{\ell/k} - \beta\rangle,$$

where $\alpha \neq 0 \neq \beta$. Therefore, to show that any such ideal has trivial stabiliser, it suffices to show that if $0 \leqslant r < \ell/k$ and $\alpha_{\sigma^r}(u^k) = u^k$, then $r = 0$. If $\alpha_{\sigma^r}(u^k) = u^k$, then $\varepsilon^{r\ell k/n} = 1$, and so we require $\frac{rk}{n} \in \mathbb{Z}$. But, arguing in the same way as before, this forces $r \in \frac{\ell}{k}\mathbb{Z}$ and hence $r = 0$, and so every maximal ideal has trivial stabiliser. Applying Theorem 5.1.3, we find that $(\Bbbk_q[u^{\pm 1}, v^{\pm 1}] \# \langle g^{\ell/k}\rangle) * C_{\ell/k}$ is Azumaya, and hence so too is T'. □

6.2.2 An Azumaya localisation for case (ii)

We now consider case (ii). It is easy to show that the action in this case is X-outer, and so we wish to apply Theorem 5.1.3. As with case (i), we replace $A = \Bbbk_{-1}[u, v]$ by the Azumaya algebra $A' = \Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}]$. We will also need to localise a second time to ensure that every maximal ideal of the centre has trivial stabiliser, as in Example

5.1.5.

Proposition 6.2.2. *Write $A' = \Bbbk_{-1}[u^{\pm1}, v^{\pm1}]$ and $A'' = A'[(u^2 - v^2)^{-1}]$. Then the algebra $T'' := A'' \# S_2$ is Azumaya.*

Proof. Seeking to apply Theorem 5.1.3, first note that A' is a prime noetherian $\Bbbk$-algebra which is Azumaya. Localisation preserves these properties, so the same is true of A''. Since the generator of S_2 interchanges u and v, degree considerations imply that the action is X-outer, so it remains to show that S_2 acts freely on $\operatorname{MaxSpec} Z(A'')$. Now, using the fact that $Z(RX^{-1}) = Z(R)X^{-1}$ for a noetherian ring R and a multiplicative set X of regular elements contained in $Z(R)$, we find that

$$Z(A'') = \Bbbk[u^{\pm2}, v^{\pm2}][(u^2 - v^2)^{-1}] = Z(A)[(u^2 - v^2)^{-1}].$$

The maximal ideals of $Z(A'')$ are in one-to-one correspondence with maximal ideals of $Z(A')$ which do not contain $u^2 - v^2$ [GW04, Theorem 10.20]. But $\operatorname{MaxSpec} Z(A') = \{\langle u^2 - \alpha, v^2 - \beta\rangle \mid \alpha \neq 0 \neq \beta\}$, and such a maximal ideal $\mathfrak{m}$ has nontrivial stabiliser if and only if $\alpha = \beta$ and this happens if and only if $u^2 - v^2$ lies in $\mathfrak{m}$. Therefore such $\mathfrak{m}$ do not give rise to maximal ideals of $Z(A'')$, so the stabiliser of every maximal ideal of $Z(A'')$ is trivial, and so Theorem 5.1.3 tells us that T'' is Azumaya. ☐

6.2.3 An Azumaya localisation for case (iii)

The final case we consider is case (iii). When showing that a suitable localisation of $A \# G$ is Azumaya in these cases, the set at which we localise depends on the parity of n, so we consider two separate cases. We first consider the case when n is odd, which is easier due to the action being X-outer.

Proposition 6.2.3. *Suppose that n is odd, and let $A' = \Bbbk_{-1}[u^{\pm1}, v^{\pm1}]$ and $A'' = A'[(u^{2n} - v^{2n})^{-1}]$. Then $T'' := A'' \# D_n$ is Azumaya.*

Proof. We check that the hypotheses of Theorem 5.1.3 are met. Firstly, as in the proof of Proposition 6.2.1, the nontrivial rotations g^i act X-outer since the order of $q = -1$ is coprime to n, and each of the reflections $g^i h$ acts X-outer by degree considerations. Moreover, A'' is a prime noetherian Azumaya $\Bbbk$-algebra, so it remains to check that D_n acts freely on maximal ideals of $Z(A'')$. These are in one-to-one correspondence with the maximal ideals of $Z(A')$ which do not contain $u^{2n} - v^{2n}$ [GW04, Theorem 10.20]. But $\operatorname{MaxSpec} Z(A') = \{\langle u^2 - \alpha, v^2 - \beta\rangle \mid \alpha \neq 0 \neq \beta\}$, and such a maximal ideal $\mathfrak{m}$ has nontrivial stabiliser if and only if $\alpha = \omega^i \beta$ for some i, and this happens if and only if $u^2 - \omega^i v^2$ lies in $\mathfrak{m}$. However, since $\prod_{0 \leqslant i < n}(u^2 - \omega^i v^2) = u^{2n} - v^{2n}$, such $\mathfrak{m}$ do not give rise to maximal ideals of $Z(A'')$. Therefore the stabiliser of every maximal ideal of $Z(A'')$ is trivial, and so Theorem 5.1.3 tells us that T'' is Azumaya. ☐

We now assume that n is even, in which case the action is not X-outer, and so we have to combine Theorems 5.1.3 and 5.2.2 as in the proof of Proposition 6.2.1.

Proposition 6.2.4. *Suppose that n is even, and let $A' = \Bbbk_{-1}[u^{\pm1}, v^{\pm1}]$. Then $T' := A' \# D_n$ is Azumaya.*

Proof. Write $m = n/2$. First note that we have an isomorphism

$$T' \cong (A' \# \langle g^m \rangle) * D_m,$$

where, as shown in the proof of Proposition 6.2.1, $A' \# \langle g^m \rangle$ is a prime noetherian Azumaya algebra with centre $Z = \Bbbk[u^{\pm 2}, v^{\pm 2}, uvg] \cong \Bbbk[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]/\langle xy + z^2 \rangle$. Here, we think of D_m as the group $\langle \sigma, \tau \mid \sigma^m, \tau^2, \tau\sigma = \sigma^{-1}\tau \rangle = \{\sigma^i \tau^j \mid 0 \leqslant i < m, j = 0, 1\}$, and our copy of the set D_m inside T' is given by $\overline{\sigma^i \tau^j} = g^i h^j$. Tracing through this isomorphism, one finds that the induced automorphisms of $A' \# \langle g^m \rangle$ are given by

$$\alpha_{\sigma^i}(u) = \omega^i u, \quad \alpha_{\sigma^i}(v) = \omega^{-i} v, \quad \alpha_{\sigma^i}(g^m) = g^m,$$
$$\alpha_{\sigma^i \tau}(u) = -\omega^i v, \quad \alpha_{\sigma^i \tau}(v) = \omega^i u, \quad \alpha_{\sigma^i \tau}(g^m) = g^m.$$

The same argument as in the proof of Proposition 6.2.1 shows that each of these automorphisms is X-outer. It is easy to see that the restrictions of the above automorphisms to Z are given by

$$\alpha_{\sigma^i}(x) = \omega^{2i} x, \quad \alpha_{\sigma^i}(y) = \omega^{-2i} y, \quad \alpha_{\sigma^i}(z) = z,$$
$$\alpha_{\sigma^i \tau}(x) = \omega^{-2i} y, \quad \alpha_{\sigma^i \tau}(y) = \omega^{2i} x, \quad \alpha_{\sigma^i \tau}(z) = -z.$$

Therefore if $\mathfrak{m} = \langle u^2 - \alpha, v^2 - \beta, uvg - \gamma \rangle$, where $\alpha \neq 0 \neq \beta$ and $\alpha\beta + \gamma^2 = 0$, is a maximal ideal of Z, then

$$\alpha_{\sigma^i \tau^j}(\mathfrak{m}) = \begin{cases} \langle u^2 - \omega^{-2i}\alpha, v^2 - \omega^{2i}\beta, uvg - \gamma \rangle & \text{if } j = 0 \\ \langle u^2 - \omega^{-2i}\beta, v^2 - \omega^{2i}\alpha, uvg + \gamma \rangle & \text{if } j = 1 \end{cases}$$

which in either case is not equal to $\mathfrak{m}$ unless $i = 0 = j$. Therefore D_m acts freely on $Z(A' \# \langle g^m \rangle)$ so, by Theorem 5.1.3, $(\Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}] \# \langle g^m \rangle) * D_m$ is Azumaya, and hence so too is T'. □

6.3 Deformations of quantum Kleinian singularities are maximal orders

We now show that an analogue of [CBH98, Lemma 1.4] holds for deformations of quantum Kleinian singularities. The majority of this subsection is devoted to proving the following result:

Theorem 6.3.1. *Suppose that A^G is a quantum Kleinian singularity. Then $A \# G$ is a maximal order.*

Our proof of this result considers each case in turn. Assuming that Theorem 6.3.1 holds, it is not difficult to prove the appropriate analogue of [CBH98, Lemma 1.4]:

Theorem 6.3.2. *The deformations $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ are maximal orders.*

Proof. By [VdBVO89, Theorem 5], we only need to prove that $\operatorname{gr} \mathcal{S}_*^\lambda = A \# G$ and $\operatorname{gr} \mathcal{O}_*^\lambda = A^G$ are maximal orders. The first of these is the content of Theorem 6.3.1, so we now show that A^G is a maximal order. Letting $e = \frac{1}{|G|} \sum_{g \in G} g$, we have $e(A \# G)e \cong A^G$ by [BHZ16, Lemma 3.1]. Then A^G is a maximal order by [MZY98, Corollary 1.7]. □

It remains to prove Theorem 6.3.1. We first recall and prove some preliminary results which will be used in the proof of Theorem 6.3.1. We have the following result, which we state in a form most suited to our use:

Theorem 6.3.3 ([Mar95, Theorem 3.13]). *Let A be a prime noetherian ring and G a finite group acting on A such that the action of G is X-outer. Write $T = A\#G$ and write*

$$\Psi = \left\{ \bigcap_{g \in G} g \cdot p \;\middle|\; p \text{ is a reflexive height } 1 \text{ prime ideal of } A \right\}.$$

Suppose also that the following two conditions hold:

(1) *A is a maximal order in its quotient ring; and*

(2) *$PT \in \operatorname{Spec} T$ for all $P \in \Psi$.*

Then T is a prime maximal order.

It will turn out that the above result is useful in only some of the cases of interest to us, since our actions are frequently not X-outer. Instead, we will make use of Lemma 2.3.13 from Chapter 2. The following result will allow us to reduce the amount of work we need to do when applying it. We remind the reader that for a nonzero ideal I of R, we write

$$O_\ell(I) := \{q \in Q(R) \mid qI \subseteq I\}, \qquad O_r(I) := \{q \in Q(R) \mid Iq \subseteq I\}.$$

Lemma 6.3.4. *Let T be a finitely generated prime noetherian PI $\Bbbk$-algebra which is Cohen-Macaulay and of GK dimension d. Then the (classical) Krull dimension of T is d and $O_\ell(P) = T = O_r(P)$ for every prime ideal P of T with $\operatorname{ht} P \geqslant 2$.*

Proof. That $\text{cl.Kdim} = d$ under these hypotheses is well-known; see [KL00, Theorem 10.10]. If P is a height r prime of T with $r \geqslant 2$, then T/P is a finitely generated prime PI algebra and so by [KL00, Theorem 10.10], we find that $\operatorname{GKdim} T/P = \text{cl.Kdim}\, T/P = d - r$. Now, applying the functor $\operatorname{Hom}_T(-, T)$ to the short exact sequence of right T-modules

$$0 \to P \to T \to T/P \to 0,$$

we obtain the following exact sequence:

$$\operatorname{Hom}_T(T/P, T) \to T \to \operatorname{Hom}_T(P, T) \to \operatorname{Ext}^1_T(T/P, T).$$

By the Cohen-Macaulay property of T, the grade of T/P satisfies

$$j(T/P) = \operatorname{GKdim}(T) - \operatorname{GKdim}(T/P) = d - (d - r) = r \geqslant 2.$$

In particular, $\operatorname{Hom}_T(T/P, T) = 0 = \operatorname{Ext}^1_T(T/P, T)$, and so $\operatorname{Hom}_T(P, T) = T$. Therefore,

$$T \subseteq \operatorname{End}_T(P) \subseteq \operatorname{Hom}_T(P, T) = T,$$

and so $O_\ell(P) = \operatorname{End}_T(P) = T$. The proof that $O_r(P) = T$ is symmetrical. □

We now explain why the above lemma is useful. If A^G is a quantum Kleinian singularity, then Lemma 6.1.3 tells us that $T = A\#G$ is a prime noetherian $\Bbbk$-algebra which is Cohen-Macaulay. Moreover, other than for case (i) when q is not a root of unity and for case (iv), T is PI since it is finite over its centre. Finally, each such skew group ring is a noetherian finitely generated $\Bbbk$-algebra and so, by [KL00, Theorem 10.10], its (classical) Krull dimension is equal to its GK dimension, namely 2. Therefore, since Lemma 2.3.13 tells us that T is a maximal order provided that $\operatorname{End}_T(P_T) = T =$

$\mathrm{End}_T({}_TP)$ for all prime ideals P, Lemma 6.3.4 implies that we only need to check that $\mathrm{End}_T(P) = T$ for all height 1 primes P of T. We formalise this observation into the following lemma, to which we will refer back multiple times.

Lemma 6.3.5. *Let $T := A\#G$, where the pair (A, G) is as in cases* (i) *(when q is a root of unity),* (ii)*, or* (iii)*. Then T is a maximal order if and only if* $\mathrm{End}(P_T) = T = \mathrm{End}({}_TP)$ *for all height 1 prime ideals P of T.* □

We now seek to prove Theorem 6.3.1 for each case, beginning with the cases where the conditions of Theorem 6.3.3 are easily verified.

6.3.1 Proof of Theorem 6.3.1 for case (iv)

That $A\#G$ is a maximal order in this case follows relatively quickly from Theorem 6.3.3 since, as we saw in the proof of Lemma 6.1.3, the action is X-outer.

Theorem 6.3.6. *The algebra $T = \Bbbk_J[u, v]\#C_2$ in case* (iv) *is a maximal order.*

Proof. First recall that $A = \Bbbk_J[u, v]$ is a noetherian domain. By [Sta94, Theorem 2.10], $\Bbbk_J[u, v]$ is a maximal order, and so condition (1) of Theorem 6.3.3 is satisfied. Moreover, by [Irv79b, Theorem 5.2], the only height one prime of $\Bbbk_J[u, v]$ is $P = \langle u\rangle$, which is also reflexive and G-stable. We claim that PT is a prime ideal of T, or equivalently that T/PT is prime. Since P is G-stable, this latter ring is isomorphic to $(\Bbbk_J[u, v]/P)\#C_2 \cong \Bbbk[v]\#C_2$, and so by [GW04, Lemma 6.17] it suffices to show that $Q(\Bbbk[v]\#C_2)$ is a simple ring. But $Q(\Bbbk[v]\#C_2) \cong \Bbbk(v)\#C_2$, and this ring is simple by [MR01, Proposition 7.8.12]. Thus PT is a prime ideal of T. Therefore condition (2) of Theorem 6.3.3 holds, so T is a maximal order. □

6.3.2 Proof of Theorem 6.3.1 for case (i)

By the proof of Lemma 6.1.3, in this case the action is X-outer if and only if q is not a root of unity or when the orders of g and q are coprime. We first consider the former case:

Theorem 6.3.7. *Suppose q is not a root of unity. Then the algebra $T = \Bbbk_q[u, v]\#C_n$ in case* (i) *is a maximal order.*

Proof. It is well-known that $A = \Bbbk_q[u, v]$ is a noetherian domain. Moreover, since A is AS regular, it follows from [Sta94, Theorem 2.10] that it is a maximal order, and so condition (1) of Theorem 6.3.3 is satisfied.

We now show that condition (2) holds. Since q is not a root of unity, by [GW04, Exercise 10P] the height 1 prime ideals of A are $\langle u\rangle$ and $\langle v\rangle$. Moreover, these ideals are reflexive and G-stable, so $\Psi = \{\langle u\rangle, \langle v\rangle\}$. The same argument as in the proof of Theorem 6.3.6 shows that both $\langle u\rangle T$ and $\langle v\rangle T$ are prime ideals of T, so condition (2) of Theorem 6.3.3 holds. Therefore T is a maximal order. □

We now turn our attention to the case when q is a root of unity. Despite the fact that the action is X-outer when the order of q is coprime to n, it is difficult to check condition (2) of Theorem 6.3.3 in this case. This is due to the fact that $\Bbbk_q[u, v]$ has many more G-prime ideals than when q is not a root of unity, see [Irv79a, Section 8]. We therefore take advantage of the fact that $\Bbbk_q[u^{\pm 1}, v^{\pm 1}]\#C_n$ is Azumaya.

Theorem 6.3.8. *Suppose q is a kth root of unity. Then the algebra $T = \Bbbk_q[u, v]\#C_n$ in case* (i) *is a maximal order.*

Proof. By Lemma 6.3.5, it suffices to show that $O_\ell(P) = T = O_r(P)$ for all height 1 primes. So suppose that P has height 1. If P contains u, then since $\langle u \rangle$ is a prime ideal of T we have $P = \langle u \rangle$ (that $\langle u \rangle$ is prime follows from the same argument as in the proof of Theorem 6.3.7, which does not make use of q having infinite order). Since u is a normal nonzerodivisor, $O_\ell(P) = T$. Similarly, if P contains v then $P = \langle v \rangle$ is a prime ideal of T and $O_\ell(P) = T$. Write $A' := \Bbbk_q[u^{\pm 1}, v^{\pm 1}]$ and $T' := A' \# G$, both of which are Azumaya by Proposition 6.2.1. Now let P be a height 1 prime of T which does not contain u or v, so that P corresponds to a height 1 prime PT' of T'. Since T' is Azumaya, there exists a height 1 prime $\mathfrak{p}$ of $Z(T')$ such that $PT' = \mathfrak{p}T'$. Writing $\ell = \operatorname{lcm}(n, k)$, it is not difficult to show that

$$Z(T') = \Bbbk[u^{\pm \ell}, ((uv)^{-\ell/n} g^{\ell/k})^{\pm 1}],$$

which is a Laurent polynomial ring in two variables and is hence a UFD. Therefore its height 1 primes are principal, and so $\mathfrak{p} = zZ(T')$ for some $z \in Z(T')$. Since z is a central nonzerodivisor, we have $\operatorname{End}_{T'}(zT') = T'$, and therefore we have a chain of inclusions

$$T \subseteq \operatorname{End}_T(P) \subseteq \operatorname{End}_{T'}(PT') = \operatorname{End}_{T'}(zT') = T'.$$

It remains to show that this forces $\operatorname{End}_T(P) = T$. To this end, let $t' \in \operatorname{End}_T(P) \subseteq T'$ and choose $i \geqslant 0$ minimal such that $t := (uv)^i t' \in T$. We claim that $i = 0$, forcing $t' \in T$. Seeking a contradiction, suppose that $i \geqslant 1$; then

$$tP = (uv)^i t' P \subseteq (uv)^i P \subseteq \langle u \rangle.$$

Since $P \nsubseteq \langle u \rangle$ and, as noted previously, $\langle u \rangle$ is a prime ideal of T, we find that $t \in \langle u \rangle$; similarly, $t \in \langle v \rangle$. Therefore $t \in \langle u \rangle \cap \langle v \rangle = \langle uv \rangle$, contradicting minimality of i. Hence $t' \in T$, and so $\operatorname{End}_T(P) = T$.

It follows that every nonzero prime ideal of P of T satisfies $O_\ell(P) = T$, and similarly also satisfies $O_r(P) = T$. Thus T is a maximal order. □

We will use the same approach as in the above proof to show that $T = A \# G$ is a maximal order in cases (ii) and (iii). As before, it suffices to show that $\operatorname{End}_T(P) = T$ for all height 1 primes, which we show to be true for a few carefully chosen primes. These primes are chosen so that when we invert powers of their generators, the resulting algebra T' is Azumaya. We then show that $Z(T')$ is a UFD, which will allow us to deduce that $T \subseteq \operatorname{End}_T(P) \subseteq T'$ for all remaining primes P. Finally, we argue that necessarily $\operatorname{End}_T(P) = T$.

It turns out that we must work harder to prove Theorem 6.3.1 for the remaining two cases, mainly because it is more difficult to show that our chosen ideals are in fact prime. Moreover, showing that the centres of our Azumaya skew group algebras are UFDs is more involved.

6.3.3 Proof of Theorem 6.3.1 for case (ii)

We have already seen that the action is X-outer in this case, but again Theorem 6.3.3 is difficult to apply for the same reasons as for case (i). Recall that, by Proposition 6.2.2, $T'' := A'' \# S_2$ is Azumaya, where $A' = \Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}]$ and $A'' = A'[(u^2 - v^2)^{-1}]$.

We write $A = \Bbbk_{-1}[u, v]$ and $G = S_2 = \langle h \rangle$ throughout this subsection, where $h \cdot u = v$, $h \cdot v = u$. We will need the following result:

Lemma 6.3.9. *Let $T = \Bbbk_{-1}[u,v] \# S_2$, as in case* (ii)*. Then $\langle uv \rangle$ and $\langle u^2 - v^2 \rangle$ are prime ideals of T.*

Proof. We first consider $\langle uv \rangle$. Since this ideal is G-stable, it suffices to show that the quotient $T/\langle uv \rangle \cong (\Bbbk_{-1}[u,v]/\langle uv \rangle) \# S_2$ is prime. Noting that $\Bbbk_{-1}[u,v]/\langle uv \rangle$ is semiprime, [Pas89, Theorem 4.4] implies that the same is true of $(\Bbbk_{-1}[u,v]/\langle uv \rangle) \# S_2$; since this latter ring is also noetherian, it necessarily has a classical quotient ring. Therefore to show that $(\Bbbk_{-1}[u,v]/\langle uv \rangle) \# S_2$ is prime, we can equivalently show that its classical quotient ring

$$Q\big((\Bbbk_{-1}[u,v]/\langle uv \rangle) \# S_2\big) \cong Q\big(\Bbbk_{-1}[u,v]/\langle uv \rangle\big) \# S_2$$

is simple. First note that the natural map

$$\phi : \Bbbk_{-1}[u,v] \to \Bbbk[u] \times \Bbbk[v], \quad f(u,v) \mapsto (f(u,0), f(0,v))$$

gives rise to an isomorphism $Q\big(\Bbbk_{-1}[u,v]/\langle uv \rangle\big) \cong \Bbbk(u) \times \Bbbk(v)$. Tracing through this isomorphism, we find that the corresponding S_2-action is given by

$$h \cdot (f_1(u), f_2(v)) = (f_2(u), f_1(v)),$$

and so we wish to show that $\big(\Bbbk(u) \times \Bbbk(v)\big) \# S_2$ is simple. By [Öin14, Theorem 1.2 (c)], it suffices to show that $\Bbbk(u) \times \Bbbk(v)$ is G-simple, and that the centre of $\big(\Bbbk(u) \times \Bbbk(v)\big) \# S_2$ is a field. For the first of these, let I be a nonzero G-stable ideal of $\Bbbk(u) \times \Bbbk(v)$, and let $0 \neq (f_1(u), f_2(v)) \in I$, where, acting by h if necessary and using G-stability, we may assume that $f_1(u) \neq 0$. Multiplying by $(f_1(u)^{-1}, 0)$, we find that $(1,0) \in I$, and then acting by h shows that $(0,1) \in I$, so that $(1,1) \in I$ and hence $I = \Bbbk(u) \times \Bbbk(v)$. Therefore, $\Bbbk(u) \times \Bbbk(v)$ is G-simple. By Lemma 5.1.1, the centre of $\big(\Bbbk(u) \times \Bbbk(v)\big) \# S_2$ equals $\big(\Bbbk(u) \times \Bbbk(v)\big)^{S_2}$, which is easily seen to be $\{(f(u), f(v)) \mid f(t) \in \Bbbk(t)\}$, and this is clearly a field. Therefore $\big(\Bbbk(u) \times \Bbbk(v)\big) \# S_2$ is simple, and hence $(\Bbbk_{-1}[u,v]/\langle uv \rangle) \# S_2$ is prime.

We now show that $\langle u^2 - v^2 \rangle$ is a prime ideal of T. As before, it suffices to show that $Q(\Bbbk_{-1}[u,v]/\langle u^2 - v^2 \rangle) \# S_2$ is simple. We first claim that we have an isomorphism

$$Q\left(\frac{\Bbbk_{-1}[u,v]}{\langle u^2 - v^2 \rangle}\right) \# S_2 \cong M_2(\Bbbk(t)) \# S_2$$

with an appropriate action of S_2 on $M_2(\Bbbk(t))$. To this end, define an algebra homomorphism

$$\phi : \frac{\Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}]}{\langle u^2 - v^2 \rangle} \to M_2(\Bbbk[t^{\pm 1}]),$$
$$\phi(u) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}, \quad \phi(v) = i \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix},$$

where $i^2 = -1$, which is easily checked to be well-defined. Since $\Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}]$ and $M_2(\Bbbk[t^{\pm 1}])$ are both free modules of rank 4 over $R := \Bbbk[u^{\pm 2}]$ and $\phi(R) = \Bbbk[t^{\pm 1}]$ respectively, we see that ϕ is an isomorphism. Chasing through the definition of ϕ, one can verify that it is an isomorphism of S_2-modules provided that we define

$$h \cdot e_{11} = e_{22}, \quad h \cdot e_{12} = -ite_{21}, \quad h \cdot e_{21} = it^{-1}e_{12}, \quad h \cdot e_{22} = e_{11}, \quad h \cdot t = t.$$

With this action, we therefore have a chain of isomorphisms

$$Q\left(\frac{\Bbbk_{-1}[u,v]}{\langle u^2-v^2\rangle}\right)\# S_2 \cong Q\left(\frac{\Bbbk_{-1}[u^{\pm 1},v^{\pm 1}]}{\langle u^2-v^2\rangle}\right)\# S_2 \cong Q(M_2(\Bbbk[t^{\pm 1}])\# S_2 \cong M_2(\Bbbk(t))\# S_2.$$

By direct calculation, one can verify that the centre of $M_2(\Bbbk(t))\# S_2$ is

$$Z \coloneqq \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -itb & 0 \end{pmatrix} h \;\middle|\; a,b \in \Bbbk(t) \right\}.$$

This is a field because the map

$$\psi : \Bbbk(t)[x] \to Z, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ -it & 0 \end{pmatrix} h$$

gives rise to an isomorphism $Z \cong \Bbbk(t)[x]/\langle x^2+it\rangle$, where the right hand side is a field. Additionally, $M_2(\Bbbk(t))$ is simple and therefore, by [Öin14, Theorem 1.2 (c)], it follows that $M_2(\Bbbk(t))\# S_2$ is simple. Hence $\langle u^2-v^2\rangle$ is a prime ideal of T, as claimed. □

Theorem 6.3.10. *The algebra $T = \Bbbk_{-1}[u,v]\# S_2$ in case* (ii) *is a maximal order.*

Proof. Again, by Lemma 6.3.5 it suffices show that $O_\ell(P) = T = O_r(P)$ for all height 1 primes. So suppose that P has height 1. We first remark that $\langle u\rangle = \langle v\rangle$ is not a prime ideal of T, since the quotient of T by this ideal is isomorphic to $\Bbbk S_2$, which is not prime. If P contains u^2-v^2, then by Lemma 6.3.9 $P = \langle u^2-v^2\rangle$, and since u^2-v^2 is a normal nonzerodivisor, we have $O_\ell(P) = T$ in this case. Similarly, using Lemma 6.3.9 again, if P contains uv then $P = \langle uv\rangle$ and $O_\ell(P) = T$. Now let P be a height 1 prime of T not containing u^2-v^2 or uv, so that P corresponds to a height 1 prime PT'' of $T'' \coloneqq \Bbbk_{-1}[u^{\pm 1},v^{\pm 1}][(u^2-v^2)^{-1}]\# S_2$. As established in Proposition 6.2.2, T'' is Azumaya, so there exists a height 1 prime $\mathfrak{p}$ of $Z(T'')$ such that $PT'' = \mathfrak{p}T''$. But using the fact that $Z(RX^{-1}) = Z(R)X^{-1}$ if $X \subseteq Z(R)$ and that $(RX^{-1})^G = R^G X^{-1}$ if $X \subseteq R^G$, and since clearly $A'[(u^2-v^2)^{-1}] = A'[(u^2-v^2)^{-2}]$,

$$\begin{aligned}
Z(T'') &= Z\Big(\Bbbk_{-1}[u^{\pm 1},v^{\pm 1}][(u^2-v^2)^{-2}]\# S_2\Big) \\
&= Z\big(\Bbbk_{-1}[u^{\pm 1},v^{\pm 1}][(u^2-v^2)^{-2}]\big)^{S_2} \\
&= \big(\Bbbk[u^{\pm 2},v^{\pm 2}][(u^2-v^2)^{-2}]\big)^{S_2} \\
&= \big(\Bbbk[u^2,v^2][(u^2v^2)^{-1}]\big)^{S_2}[(u^2-v^2)^{-2}] \\
&= \Bbbk[(u^2v^2)^{\pm 1}, u^2+v^2][(u^2-v^2)^{-2}] \\
&\cong \Bbbk[x^{\pm 1},y][(y^2-4x)^{-1}].
\end{aligned}$$

The last ring is the localisation of a UFD which implies that $Z(T'')$ is a UFD, and so height 1 primes are principal, which means that $\mathfrak{p} = zZ(T'')$ for some $z \in Z(T'')$. Since z is a central nonzerodivisor, we have $\operatorname{End}_{T''}(zT'') = T''$, and therefore we have a chain of inclusions

$$T \subseteq \operatorname{End}_T(P) \subseteq \operatorname{End}_{T''}(PT'') = \operatorname{End}_{T''}(zT'') = T''.$$

It remains to show that this forces $\operatorname{End}_T(P) = T$. To this end, let $t'' \in \operatorname{End}_T(P) \subseteq T''$ and choose $i \geqslant 0$ minimal such that $t \coloneqq (uv(u^2-v^2))^i t'' \in T$. We claim that $i = 0$, forcing $t'' \in T$. Seeking a contradiction, suppose that $i \geqslant 1$; then

$$tP = (uv(u^2 - v^2))^i t'' P \subseteq (uv(u^2 - v^2))^i P \subseteq \langle uv \rangle.$$

Since $P \nsubseteq \langle uv \rangle$ and, by Lemma 6.3.9, $\langle uv \rangle$ is a prime ideal of T, we find that $t \in \langle uv \rangle$; similarly, $t \in \langle u^2 - v^2 \rangle$. Hence $t \in \langle uv \rangle \cap \langle u^2 - v^2 \rangle = \langle uv(u^2 - v^2) \rangle$, contradicting minimality of i. Therefore $t'' \in T$, and so $\mathrm{End}_T(P) = T$.

It follows that every nonzero prime ideal of P of T satisfies $O_\ell(P) = T$, and similarly also satisfies $O_r(P) = T$, and so T is a maximal order. $\square$

6.3.4 Proof of Theorem 6.3.1 for case (iii)

We finally come to what turns out to be the most involved case. Again, we use the same approach as in the proof of Theorem 6.3.8, but we must first make some preliminary calculations. Most notably, computing the centres of the algebras of interest is quite involved, and so we state these as independent lemmas.

Throughout, we write $A = \Bbbk_{-1}[u, v]$, $A' = A[u^{-1}, v^{-1}]$, $A'' = A'[(u^{2n} - v^{2n})^{-1}]$, and T, T', T'' for the corresponding skew group rings coming from the action of $G = D_n$.

Proposition 6.3.11. *Suppose that n is odd. Then there is an isomorphism*

$$Z(T'') = \Bbbk[u^2v^2, u^{2n} + v^{2n}][(u^2v^2)^{-1}, ((u^{2n} - v^{2n})^2)^{-1}] \cong \Bbbk[x^{\pm 1}, y][(y^2 - 4x^n)^{-1}],$$

and this ring is a UFD.

Proof. By the proof of Proposition 6.2.3, T satisfies the hypotheses of Lemma 5.1.1. Writing $a = u^2$, $b = v^2$, we therefore have

$$Z(T) = Z(A)^{D_n} = \Bbbk[a, b]^{D_n}.$$

where the D_n-action on $\Bbbk[a, b]$ is given by

$$g \cdot a = \omega^{2i} a, \quad g \cdot b = \omega^{-2i} b, \quad h \cdot a = b, \quad h \cdot b = a.$$

By [Ben93, Appendix A],

$$\Bbbk[a, b]^{D_n} = \Bbbk[ab, a^n + b^n]$$

is a polynomial ring in two variables, and so

$$Z(T) = \Bbbk[u^2v^2, u^{2n} + v^{2n}] \cong \Bbbk[x, y],$$

where $x := u^2v^2, y := u^{2n} + v^{2n}$.

From here we can quickly determine $Z(T'')$. To do this, first note that $T'' = T[(u^2v^2)^{-1}, ((u^{2n} - v^{2n})^2)^{-1}]$, where the multiplicative set generated by u^2v^2 and $(u^{2n} - v^{2n})^2$ is contained in $Z(T)$. With the notation for x and y as above, we have $(u^{2n} - v^{2n})^2 = y^2 - 4x^n$, and so

$$Z(T'') = Z(T)[(u^2v^2)^{-1}, ((u^{2n} - v^{2n})^2)^{-1}] \cong \Bbbk[x^{\pm 1}, y][(y^2 - 4x^n)^{-1}].$$

This, being the localisation of a UFD, is itself a UFD. $\square$

Determining the centre of T'' is more involved when n is even, essentially because the action is not X-outer.

Proposition 6.3.12. *Suppose that n is even and write $m = n/2$. Then*

$$Z(T'') \cong \frac{\Bbbk[x,y,z]}{\langle x^2y + y^{m+1} + z^2\rangle}[y^{-1}, (x^4 + x^2y^m)^{-1}].$$

Moreover, $Z(T'')$ is a UFD.

Proof. We first determine the centre of T. As in the proof of Proposition 6.2.4, we have an isomorphism

$$T \cong (\Bbbk_{-1}[u,v] \# \langle g^m\rangle) * D_m.$$

Now,

$$Z(\Bbbk_{-1}[u,v] \# \langle g^m\rangle) \cong \Bbbk[a,b,c]/\langle ab - c^2\rangle$$

where $a := u^2, b := v^2, c := iuvg^m$. As in the proof of Proposition 6.2.4, the set of automorphisms $\{\alpha_{\sigma^i\tau^j} \mid 0 \leqslant i < m, j = 0, 1\}$ acts as a group of X-outer automorphisms on $\Bbbk[a,b,c]/\langle ab - c^2\rangle$ via

$$\alpha_{\sigma^i}(a) = \varepsilon^i a, \quad \alpha_{\sigma^i}(b) = \varepsilon^{-i} b, \quad \alpha_{\sigma^i}(c) = c,$$
$$\alpha_{\sigma^i\tau}(a) = \varepsilon^{-i} b, \quad \alpha_{\sigma^i\tau}(b) = \varepsilon^i a, \quad \alpha_{\sigma^i\tau}(c) = -c,$$

where $\varepsilon = \omega^2$. Therefore, using Lemma 5.1.1, we have

$$\begin{aligned} Z(\Bbbk_{-1}[u,v] \# D_n) &= Z((\Bbbk_{-1}[u,v] \# \langle g^m\rangle) * D_m) \\ &= Z(\Bbbk_{-1}[u,v] \# \langle g^m\rangle)^{D_m} \\ &\cong \left(\frac{\Bbbk[a,b,c]}{\langle ab - c^2\rangle}\right)^{D_m}. \end{aligned}$$

We first determine $R := \Bbbk[a,b,c]^{D_m}$, where the action is as above. We use Molien's formula [Ben93, Theorem 2.5.2] to work out the Hilbert series of R:

$$\operatorname{hilb} R = \frac{1}{|D_m|} \sum_{\alpha \in G} \frac{1}{\det(I - \alpha t)}.$$

In matrix form, the elements of D_m and the relevant determinants are as follows:

$\alpha \in G$	Matrix	Number	$\det(I - \alpha t)$
$\sigma^i, \quad 0 \leqslant i \leqslant m-1$	$\begin{pmatrix} \varepsilon^i & 0 & 0 \\ 0 & \varepsilon^{-i} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	m	$(1-t)(1-\varepsilon^i t)(1-\varepsilon^{-i} t)$
$\sigma^i\tau, \quad 0 \leqslant i \leqslant m-1$	$\begin{pmatrix} 0 & \varepsilon^i & 0 \\ \varepsilon^{-i} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	m	$(1-t)(1+t)^2$

We therefore have

$$\operatorname{hilb} R = \frac{1}{2m}\left(\frac{m}{(1-t)(1+t)^2} + \sum_{i=0}^{m-1} \frac{1}{(1-t)(1-\varepsilon^i t)(1-\varepsilon^{-i} t)}\right)$$

$$= \frac{1}{2(1-t)(1+t)^2} + \frac{1}{2(1-t)} \cdot \underbrace{\frac{1}{m}\sum_{i=0}^{m-1} \frac{1}{(1-\varepsilon^i t)(1-\varepsilon^{-i} t)}}_{(*)} .$$

Now $(*)$ is, by Molien's formula, the Hilbert series of the coordinate ring of an $\mathbb{A}_{m-1}$ singularity, and so is known to equal $(1-t^{2m})/(1-t^2)(1-t^m)^2$. A routine calculation then shows that

$$\operatorname{hilb} R = \frac{1-t^{2(m+1)}}{(1-t^2)^2(1-t^m)(1-t^{m+1})}.$$

This implies that R has four generators, of degrees $2, 2, m, m+1$, and that there is a single relation of degree $2(m+1)$ between these generators. It is easy to check that

$$x \coloneqq \frac{i}{2}(a^m + b^m), \quad y \coloneqq ab, \quad y' \coloneqq c^2, \quad z \coloneqq \frac{1}{2}(a^m c - b^m c),$$

are D_m-invariants and that $x^2y' + y^m y' + z^2 = 0$, so that

$$R = \Bbbk[a,b,c]^{D_m} \cong \frac{\Bbbk[x,y,y',z]}{\langle x^2y' + y^m y' + z^2 \rangle}.$$

Since $y = y'$ in $\Bbbk[a,b,c]/\langle ab - c^2 \rangle$, it immediately follows that

$$\left(\frac{\Bbbk[a,b,c]}{\langle ab - c^2 \rangle} \right)^{D_m} \cong \frac{\Bbbk[x,y,z]}{\langle x^2y + y^{m+1} + z^2 \rangle},$$

where x, y, and z are as above. Finally, since $Z(\Bbbk_{-1}[u,v] \# D_n) \cong (\Bbbk[a,b,c]/\langle ab - c^2 \rangle)^{D_m}$ where $a \coloneqq u^2, b \coloneqq v^2, c \coloneqq iuvg^m$, we find that

$$Z(\Bbbk_{-1}[u,v] \# D_n) \cong \frac{\Bbbk[x,y,z]}{\langle x^2y + y^{m+1} + z^2 \rangle},$$

where we have set

$$x \coloneqq \frac{i}{2}(u^n + v^n), \quad y \coloneqq u^2v^2, \quad z \coloneqq \frac{i}{2}(u^{n+1}vg^m - uv^{n+1}g^m).$$

To determine the centre of T'', observe that we have

$$T'' = T[(u^2v^2)^{-1}, (u^{2n} - v^{2n})^{-2}],$$

where the multiplicative set generated by u^2v^2 and $(u^{2n} - v^{2n})^2$ is contained in $Z(T)$. With the notation for x, y, z as above, we find

$$\frac{1}{16}(u^{2n} - v^{2n})^2 = x^4 + x^2y^m,$$

and so

$$Z(T'') = Z(T)[y^{-1}, (x^4 + x^2y^m)^{-1}] \cong \frac{\Bbbk[x,y,z]}{\langle x^2y + y^{m+1} + z^2 \rangle}[y^{-1}, (x^4 + x^2y^m)^{-1}].$$

For the final claim, observe that $S \coloneqq Z(T'')$ is a localisation of the coordinate ring R of a $\mathbb{D}_{m+2}$ singularity. Write $\operatorname{Cl}(R)$ for the divisor class group of R. By Nagata's theorem, the canonical map

$$\phi : \mathrm{Cl}(R) \to \mathrm{Cl}(S), \quad \langle \alpha_1, \ldots, \alpha_r \rangle_R \mapsto \langle \alpha_1, \ldots, \alpha_r \rangle_S$$

is a surjection. Therefore, if ϕ maps every element of $\mathrm{Cl}(R)$ to $[S]$ in $\mathrm{Cl}(S)$ then $\mathrm{Cl}(S)$ is trivial. Since R is a noetherian integrally closed domain [Ben93, Proposition 1.1.1], we can define $\mathrm{Cl}(R)$ to be the set of isomorphism class of rank one reflexive modules, with multiplication given by $[I][J] = [(I \otimes_R J)^{**}]$. Since $\mathrm{i.dim}\, R = 2$, these are precisely the rank one maximal Cohen-Macaulay modules. These modules are known: by [LW12, 9.21], up to isomorphism they are

$$\begin{cases} R,\ \langle y, z\rangle,\ \langle z, xy - iy^{m/2+1}\rangle,\ \langle z, xy + iy^{m/2+1}\rangle & \text{if } m \text{ is even} \\ R,\ \langle y, z\rangle,\ \langle x, z + iy^{(m+1)/2}\rangle,\ \langle xy, z + iy^{(m+1)/2}\rangle & \text{if } m \text{ is odd} \end{cases}.$$

But x, y, and z are all invertible in S since

$$\begin{aligned} x \cdot (x^3 + xy^m)(x^4 + x^2y^m)^{-1} &= 1, \\ y \cdot y^{-1} &= 1, \\ z \cdot -x^2zy^{-1}(x^4 + x^2y^m)^{-1} = (x^4y + x^2y^{m+1})y^{-1}(x^4 + x^2y^m)^{-1} &= 1. \end{aligned}$$

Therefore each of the above ideals gets sent to $[S]$ under ϕ, and so $\mathrm{Cl}(S)$ is trivial, which implies that S is a UFD. □

The remainder of the proof of Theorem 6.3.1 for case (iii) does not depend on the parity of n. We now show that certain ideals of T are prime:

Lemma 6.3.13. *Let $T = \Bbbk_{-1}[u, v] \# D_n$, as in case* (iii). *Then $\langle uv\rangle$ and $\langle u^{2n} - v^{2n}\rangle$ are prime ideals of T.*

Proof. Since $\langle uv\rangle$ is G-stable, we need to show that $T/\langle uv\rangle \cong (\Bbbk_{-1}[u, v]/\langle uv\rangle) \# D_n$ is prime. Equivalently, we show that $Q(\Bbbk_{-1}[u, v]/\langle uv\rangle) \# D_n$ is simple, where we note that the classical quotient ring of $(\Bbbk_{-1}[u, v]/\langle uv\rangle) \# D_n$ exists by the same argument as in Lemma 6.3.9. As in the proof of Lemma 6.3.9, we have

$$Q(\Bbbk_{-1}[u, v]/\langle uv\rangle) \# D_n \cong (\Bbbk(u) \times \Bbbk(v)) \# D_n,$$

where D_n acts via

$$g \cdot (u, 0) = (\omega u, 0), \quad g \cdot (0, v) = (0, \omega^{-1}v), \quad h \cdot (u, 0) = (0, v), \quad h \cdot (0, v) = (u, 0).$$

To show that this latter ring is simple, we show that $\Bbbk(u) \times \Bbbk(v)$ is G-simple and that the centraliser C of $\Bbbk(u) \times \Bbbk(v)$ in $((\Bbbk(u) \times \Bbbk(v)) \# D_n$ is $\Bbbk(u) \times \Bbbk(v)$, see [Öin09, Theorem 6.13]. G-simplicity of $\Bbbk(u) \times \Bbbk(v)$ follows from the same argument as in the proof of Lemma 6.3.9, so now let $c = \sum_{0 \leqslant i < n, 0 \leqslant j \leqslant 1} f_{ij} g^i h^j \in C$, where $f_{ij} = (f_{ij}^{(1)}(u), f_{ij}^{(2)}(v)) \in \Bbbk(u) \times \Bbbk(v)$. Then

$$\sum_{\substack{0 \leqslant i < n \\ 0 \leqslant j \leqslant 1}} (uf_{ij}^{(1)}, 0) g^i h^j = (u, 0)c = c(u, 0) = \sum_{0 \leqslant i < n} (\omega^i u f_{i0}^{(1)}, 0) g^i + \sum_{0 \leqslant i < n} (0, \omega^{-i} v f_{i1}^{(2)}) g^i h,$$

which implies that $f_{i0}^{(1)} = 0$ for all $1 \leqslant i < n$ and that $f_{i1}^{(1)} = 0 = f_{i1}^{(2)}$ for all $0 \leqslant i < n$. A similar calculation with $(0, v)$ in place of $(u, 0)$ shows that $f_{i0}^{(2)} = 0$ for all $1 \leqslant i < n$, and so $c = (f_{00}^{(1)}, f_{00}^{(2)})$. It follows that $C = \Bbbk(u) \times \Bbbk(v)$, and so $(\Bbbk(u) \times \Bbbk(v)) \# D_n$ is simple. Therefore $\langle uv \rangle$ is a prime ideal of T.

We now consider $\langle u^{2n} - v^{2n} \rangle$. Since $\langle u^2 - v^2 \rangle$ is a prime ideal of $\Bbbk_{-1}[u, v]$ [Irv79a, Section 8], and $\langle u^{2n} - v^{2n} \rangle = \bigcap_{f \in D_n} f \cdot \langle u^2 - v^2 \rangle$, it follows that $\langle u^{2n} - v^{2n} \rangle$ is a G-prime ideal of A, and so we wish to show that $T/\langle u^{2n} - v^{2n} \rangle \cong (A/\langle u^{2n} - v^{2n} \rangle) \# D_n$ is prime. Now, $T/\langle u^{2n} - v^{2n} \rangle$ is a G-prime ring and $\langle u^2 - v^2 \rangle / \langle u^{2n} - v^{2n} \rangle$ is a minimal prime of $T/\langle u^{2n} - v^{2n} \rangle$ with stabiliser $H = \langle h \rangle$, and so by [Pas89, Corollary 14.8], it suffices to show that $(\Bbbk_{-1}[u, v]/\langle u^2 - v^2 \rangle) \# H$ is prime (where here we recall that H acts via $h \cdot u = v$, $h \cdot v = u$). But this is established in Lemma 6.3.9, and so $\langle u^{2n} - v^{2n} \rangle$ is a prime ideal of T. □

This allows us to prove that the remaining case is a maximal order, which completes the proof of Theorem 6.3.1.

Theorem 6.3.14. *The algebra $T = \Bbbk_{-1}[u, v] \# D_n$ in case* (iii) *is a maximal order.*

Proof. Again, we only show that $O_\ell(P) = T$ for all height 1 primes. So suppose that P is a height 1 prime. We first remark that $\langle u \rangle = \langle v \rangle$ is not a prime ideal of T, since the quotient of T by this ideal is isomorphic to $\Bbbk D_n$, which is not prime. If P contains $u^{2n} - v^{2n}$, then by Lemma 6.3.13 $P = \langle u^{2n} - v^{2n} \rangle$, and since $u^{2n} - v^{2n}$ is a normal nonzerodivisor, we have $O_\ell(P) = T$ in this case. Similarly, using Lemma 6.3.13 again, if P contains uv then $P = \langle uv \rangle$ and $O_\ell(P) = T$. Now let P be a height 1 prime of T not containing $u^{2n} - v^{2n}$ or uv, so that P corresponds to a height 1 prime PT'' of $T'' := \Bbbk_{-1}[u^{\pm 1}, v^{\pm 1}][(u^{2n} - v^{2n})^{-1}] \# D_n$. As established in Propositions 6.2.3 and 6.2.4, T'' is Azumaya, and so there exists a height 1 prime $\mathfrak{p}$ of $Z(T'')$ such that $PT'' = \mathfrak{p}T''$. But $Z(T'')$ is a UFD by Propositions 6.3.11 and 6.3.12, so height 1 primes are principal, and so $\mathfrak{p} = zZ(T'')$ for some $z \in Z(T'')$. Since z is a central nonzerodivisor, we have $\operatorname{End}_{T''}(zT'') = T''$, and therefore we have a chain of inclusions

$$T \subseteq \operatorname{End}_T(P) \subseteq \operatorname{End}_{T''}(PT'') = \operatorname{End}_{T''}(zT'') = T''.$$

It remains to show that this forces $\operatorname{End}_T(P) = T$. To this end, let $t'' \in \operatorname{End}_T(P) \subseteq T''$ and choose $i \geqslant 0$ minimal such that $t := (uv(u^{2n} - v^{2n}))^i t'' \in T$. We claim that $i = 0$, forcing $t'' \in T$. Seeking a contradiction, suppose that $i \geqslant 1$; then

$$tP = (uv(u^{2n} - v^{2n}))^i t'' P \subseteq (uv(u^{2n} - v^{2n}))^i P \subseteq \langle uv \rangle.$$

Since $P \nsubseteq \langle uv \rangle$ and, by Lemma 6.3.9, $\langle uv \rangle$ is a prime ideal of T, we find that $t \in \langle uv \rangle$; similarly, $t \in \langle u^{2n} - v^{2n} \rangle$. Therefore $t \in \langle uv \rangle \cap \langle u^{2n} - v^{2n} \rangle = \langle uv(u^{2n} - v^{2n}) \rangle$, contradicting minimality of i. Hence $t'' \in T$, and so $\operatorname{End}_T(P) = T$.

It follows that every nonzero prime ideal of P of T satisfies $O_\ell(P) = T$, and similarly also satisfies $O_r(P) = T$, and so T is a maximal order. □

6.4 A proof of Auslander's Theorem for deformations of quantum Kleinian singularities

Using the fact that the deformations $\mathcal{S}_*^\lambda$ are maximal orders, we now show that Auslander's Theorem holds for these algebras. We use the following quite general result, which is implicit in [CBH98, Lemma 1.4], but is proven in full generality in [CB, Section 5.4]:

Lemma 6.4.1. *Let R be a prime Goldie maximal order and let $e \in R$ an idempotent. Then* $\operatorname{End}_{eRe}(Re) \cong R$.

We can now prove a version of Auslander's Theorem:

Theorem 6.4.2. *There is an isomorphism*

$$\operatorname{End}_{\mathcal{O}_*^\lambda}(\mathcal{S}_*^\lambda e) \cong \mathcal{S}_*^\lambda.$$

In particular, $\operatorname{End}_{A^G}(A) \cong A\#G$ *for a quantum Kleinian singularity* A^G.

Proof. The ring $R = \mathcal{S}_*^\lambda$ is a prime noetherian maximal order by Lemma 6.1.3 and Theorem 6.3.2. Setting $e = \frac{1}{|G|}\sum_{g\in G} g$, we have $e\mathcal{S}_*^\lambda e = \mathcal{O}_*^\lambda$ by definition, and so the first isomorphism is immediate from Lemma 6.4.1. The second isomorphism then follows from the first after setting $\lambda = 0$ and noting that $(A\#G)e \cong A$ as an $(A\#G, A^G)$-bimodule by [BHZ16, Lemma 3.1]. □

We remark that this result was established for the undeformed algebras $A\#G$ and A^G in [CKWZ16a, Theorem 4.1], but their proof uses very different techniques. It does not appear to be possible to use their result to prove Auslander's Theorem for the deformations $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$ using, for example, associated graded techniques.

Chapter 7

Deformations of Quantum Preprojective Algebras

The aim of this chapter is to show that analogues of the results of [CBH98, §3] hold for quantum Kleinian singularities. Using the notation of the previous chapter, we show that for a deformation $\mathcal{S}_*^\lambda(\widetilde{Q})$ of $A\#G$, where A^G is a quantum Kleinian singularity, there exists a path algebra with relations $\Pi_*^\lambda(\widetilde{Q})$ which is Morita equivalent to $\mathcal{S}_*^\lambda(\widetilde{Q})$. Moreover, we show that the corresponding deformation $\mathcal{O}_*^\lambda(\widetilde{Q})$ of A^G is isomorphic to $e_0\Pi_*^\lambda(\widetilde{Q})e_0$, where e_0 is the idempotent corresponding to vertex 0 in $\widetilde{Q}$.

7.1 Deformed quantum preprojective algebras

We first define the algebras of interest in this chapter.

Definition 7.1.1. Let Q be a quiver and λ a weight for Q. Define the double $\overline{Q}$ of Q to be the quiver obtained from Q by adding a *reverse arrow* $\overline{\alpha} : j \to i$ for each arrow $\alpha : i \to j$ in Q or, if $\alpha : i \to i$ is a loop, adding no arrows and declaring $\overline{\alpha} = \alpha$. We call the arrows in $\overline{Q}$ which are not reverse arrows *ordinary arrows*. Let $q \in \Bbbk^\times$. We then define the *deformed quantum preprojective algebra* of Q to be

$$\Pi_q^\lambda(Q) := \Bbbk\overline{Q}/I,$$

where I is the two-sided ideal of $\Bbbk\overline{Q}$ with generators

$$\sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - q \sum_{\substack{\alpha \in Q_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \overline{\alpha}\alpha - \lambda_i e_i$$

for each vertex $i \in Q_0$. If $Q = \widetilde{\mathbb{A}}_1$, then we additionally allow a subscript J in place of q, in which case we define

$$\Pi_J^\lambda(\widetilde{\mathbb{A}}_1) = \frac{\Bbbk\overline{\widetilde{\mathbb{A}}}_1}{\left\langle \begin{matrix} \alpha_0\overline{\alpha}_0 - \overline{\alpha}_1\alpha_1 - \alpha_0\alpha_1 - \lambda_0 e_0 \\ \alpha_1\overline{\alpha}_1 - \overline{\alpha}_0\alpha_0 - \alpha_1\alpha_0 - \lambda_1 e_1 \end{matrix} \right\rangle}.$$

We write $\Pi_*^\lambda(Q)$ to mean an arbitrary deformed quantum preprojective algebra.

When $q = 1$, this is just the deformed preprojective algebra of [CBH98]. We often omit Q when it is unimportant or understood from context. If we write $\Pi^\lambda(Q)$ then we have implicitly set $q = 1$, and if we write $\Pi_*(Q)$ then we have implicitly set $\lambda = \mathbf{0}$. When $\lambda = \mathbf{0}$, more general versions of these algebras have been studied in [Kle04] and [Kal09].

We note that the ideal I can equivalently be defined as being generated by a single element, namely the sum of the given generators. In general, for our purposes it will be more convenient to think of Π_*^λ as the path algebra of a quiver with a relation at each vertex, and our definition has been chosen to emphasise this.

When the underlying graph of Q is a tree, the following lemma (which is similar to [Kal09, Fakt 4.1.9]) shows that the deformed quantum preprojective algebra $\Pi_q^\lambda(Q)$ of Q is simply a deformed preprojective algebra:

Lemma 7.1.2. *Suppose that Q is a quiver without cycles (either oriented or unoriented). Then there exists a weight λ' such that $\Pi_q^\lambda(Q) \cong \Pi^{\lambda'}(Q)$, where $\lambda_i' = 0$ if and only if $\lambda_i = 0$.*

Proof. For this proof, it will be convenient to generalise Definition 7.1.1 as follows. Let $\boldsymbol{q} : Q_0 \to \Bbbk^\times$ be some function and write $\Pi_{\boldsymbol{q}}^\lambda(Q) = \Bbbk\overline{Q}/I$, where I is the two-sided ideal of $\Bbbk\overline{Q}$ with generators

$$\sum_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} \alpha\overline{\alpha} - \boldsymbol{q}(i) \sum_{\substack{\alpha\in Q_1\\ h(\alpha)=i}} \overline{\alpha}\alpha - \lambda_i e_i$$

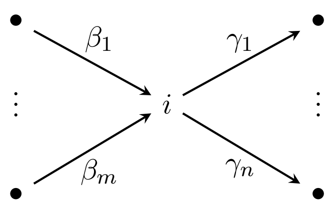
for each vertex $i \in Q_0$. (Observe that setting $\boldsymbol{q}(i) = q$ for all $i \in Q_0$ recovers our previous definition.)

Now assume that Q is as in the statement of the lemma, λ is a weight for Q and $\boldsymbol{q} : Q_0 \to \Bbbk^\times$. Fix $i \in Q_0$. We claim that $\Pi_{\boldsymbol{q}}^\lambda(Q) \cong \Pi_{\boldsymbol{q}'}^{\lambda'}(Q)$ where

$$\boldsymbol{q}'(j) = \begin{cases} 1 & \text{if } j = i \\ \boldsymbol{q}(j) & \text{if } j \neq i \end{cases}$$

and where λ' will be defined in the course of the proof; in particular, we will see that it satisfies $\lambda' = 0$ if and only if $\lambda = 0$. This is sufficient to prove the result.

At vertex i, Q has the form



Since Q has no cycles, removing vertex i splits Q into $m + n$ connected components. For each j, let Γ_j^L be the connected component containing the vertex $t(\beta_j)$ and let Γ_j^R be the connected component containing the vertex $h(\gamma_j)$. Also write Γ^L and Γ^R for the full subquivers of Q with respective vertex sets $\{i\} \sqcup \bigsqcup_{j=1}^m \left(\Gamma_j^L\right)_0$ and $\{i\} \sqcup \bigsqcup_{j=1}^n \left(\Gamma_j^R\right)_0$. Finally, define a weight λ' by

$$\lambda_j' = \begin{cases} \boldsymbol{q}(i)\lambda_j & \text{if } j \in (\Gamma^L)_0 \setminus \{i\} \\ \lambda_j & \text{if } j \in (\Gamma^R)_0 \end{cases}.$$

We now define an algebra homomorphism $\phi : \Pi_{\boldsymbol{q}}^{\lambda}(Q) \to \Pi_{\boldsymbol{q}'}^{\lambda'}(Q)$ on the generators as follows:

$$\phi(e_j) = e_j \text{ for all } j \in Q_0, \quad \phi(\alpha) = \alpha \text{ for all } \alpha \in Q_1, \quad \phi(\overline{\alpha}) = \begin{cases} \frac{1}{\boldsymbol{q}(i)}\overline{\alpha} & \text{if } \alpha \in (\Gamma^L)_1 \\ \overline{\alpha} & \text{if } \alpha \in (\Gamma^R)_1 \end{cases}.$$

We verify that this map is well-defined: at vertex i, we have

$$\phi\left(\sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - \boldsymbol{q}(i) \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=i}} \overline{\alpha}\alpha - \lambda_i e_i \right) = \sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=i}} \overline{\alpha}\alpha - \lambda_i e_i = 0,$$

while for any vertex $j \in \Gamma_0^L \setminus \{i\}$ we have

$$\phi\left(\sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=j}} \alpha\overline{\alpha} - \boldsymbol{q}(j) \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=j}} \overline{\alpha}\alpha - \lambda_j e_j \right) = \frac{1}{\boldsymbol{q}(i)} \left(\sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=j}} \alpha\overline{\alpha} - \boldsymbol{q}(j) \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=j}} \overline{\alpha}\alpha - \boldsymbol{q}(i)\lambda_j e_j \right) = 0,$$

and for $j \in \Gamma_0^R \setminus \{i\}$,

$$\phi\left(\sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=j}} \alpha\overline{\alpha} - \boldsymbol{q}(j) \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=j}} \overline{\alpha}\alpha - \lambda_j e_j \right) = \sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=j}} \alpha\overline{\alpha} - \boldsymbol{q}(j) \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=j}} \overline{\alpha}\alpha - \lambda_j e_j = 0,$$

as required. In the same way, it is easy to write down a map which is inverse to ϕ and so ϕ is an isomorphism. □

Lemma 2.2 of [CBH98] shows that the orientation of the arrows of Q does not change the isomorphism class of $\Pi^{\lambda}(Q)$, but in general this is not the case for $\Pi_q^{\lambda}(Q)$. However, it is still the case that multiplying λ by a unit does not affect $\Pi_*^{\lambda}(Q)$:

Lemma 7.1.3. *$\Pi_*^{\lambda}(Q)$ is unchanged up to isomorphism if λ is multiplied by $c \in \Bbbk^{\times}$.*

Proof. It is easy to see that the map

$$\phi : \Pi_*^{\lambda}(Q) \to \Pi_*^{c\lambda}(Q), \qquad e_i \mapsto e_i, \quad \alpha \mapsto \frac{1}{\sqrt{c}}\alpha, \quad \overline{\alpha} \mapsto \frac{1}{\sqrt{c}}\overline{\alpha}$$

is an isomorphism. □

Finally, one can grade $\Bbbk\overline{Q}$ by putting the e_i in degree 0 and the arrows α and $\overline{\alpha}$ in degree 1. This induces a grading of Π_* and a filtration on Π_*^{λ}. We then have the following result, the statement and proof of which are identical to [CBH98, Lemmas 2.3 and 2.4].

Lemma 7.1.4. *There is a natural surjective homomorphism $\Pi_* \to \operatorname{gr}\Pi_*^{\lambda}$. Consequently, if $* = q$ and Q is a (disjoint union of) Dynkin or type $\mathbb{L}$ quiver(s), then $\Pi_q^{\lambda}(Q)$ is finite-dimensional.*

7.2 Morita equivalence between $\Pi_*^{\lambda}(\widetilde{Q})$ and $\mathcal{S}_*^{\lambda}(\widetilde{Q})$

At this point, the definition of deformed quantum preprojective algebras is somewhat unmotivated, especially in the case of $\Pi_J^{\lambda}(\widetilde{\mathbb{A}}_1)$. We will show that the deformations

$\mathcal{S}^{\lambda}_{*}(\widetilde{Q})$ (respectively, $\mathcal{O}^{\lambda}_{*}(\widetilde{Q})$) from the previous chapter are Morita equivalent to (respectively, isomorphic to corner rings of) deformed quantum preprojective algebras. Throughout, we assume that all quivers are labelled as in Figure 2.1; this is especially important in case (i), where the ordinary arrows of $\widetilde{\mathbb{A}}_n$ must be cyclically ordered.

Theorem 7.2.1. *Let A^G be a quantum Kleinian singularity with corresponding Euclidean diagram $\widetilde{Q}$, and let $\lambda \in \Bbbk^{n+1}$ be a weight for $\widetilde{Q}$. Then $\mathcal{S}^{\lambda}_{*}(\widetilde{Q})$ is Morita equivalent to $\Pi^{\lambda}_{*}(\widetilde{Q})$ and there is an isomorphism $\mathcal{O}^{\lambda}_{*}(\widetilde{Q}) \cong e_0 \Pi^{\lambda}_{*}(\widetilde{Q}) e_0$.*

When $\lambda = \mathbf{0}$ this theorem says that $A\#G$ is Morita equivalent to some $\Pi_{*}(\widetilde{Q})$ and $A^G \cong e_0 \Pi_{*}(\widetilde{Q}) e_0$, which is itself a new result. We also remark that in any case where $\widetilde{Q} = \widetilde{\mathbb{A}}_n$, the Morita equivalence of Theorem 7.2.1 is in fact an isomorphism, a fact which comes out during the course of the proof.

For this section only, in all (quotients of) path algebras we will compose arrows from right to left, rather than using our usual convention of composing from left to right. This will make our calculations easier, and is justified by the following lemma:

Lemma 7.2.2.
(1) *Let Q be a quiver, λ a weight for Q, and $q \in \Bbbk^{\times}$. Then $\Pi^{\lambda}_{q}(Q) \cong \Pi^{\lambda}_{q}(Q)^{\mathrm{op}}$.*
(2) $\Pi^{\lambda}_{J}(\widetilde{\mathbb{A}}_1) \cong \Pi^{\lambda}_{J}(\widetilde{\mathbb{A}}_1)^{\mathrm{op}}$.

Proof. (1) Define an antihomomorphism $\phi : \Pi^{\lambda}_{q}(Q) \to \Pi^{\lambda}_{q}(Q)$ on arrows via $\phi(\alpha) = \overline{\alpha}$ and $\phi(\overline{\alpha}) = \alpha$. This is well-defined since for each vertex i we have

$$\begin{aligned}
\phi\Bigg(\sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - q \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \overline{\alpha}\alpha - \lambda_i e_i \Bigg) &= \sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=i}} \phi(\overline{\alpha})\phi(\alpha) - q \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \phi(\alpha)\phi(\overline{\alpha}) - \lambda_i e_i \\
&= \sum_{\substack{\alpha\in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - q \sum_{\substack{\alpha\in Q_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \overline{\alpha}\alpha - \lambda_i e_i \\
&= 0.
\end{aligned}$$

Since ϕ is an involution, the claim follows.
(2) Define an antihomomorphism $\phi : \Pi^{\lambda}_{J}(\widetilde{\mathbb{A}}_1) \to \Pi^{\lambda}_{J}(\widetilde{\mathbb{A}}_1)$ on arrows via $\phi(\alpha_i) = \alpha_{1-i}$ and $\phi(\overline{\alpha}_i) = -\overline{\alpha}_{1-i}$ for $i = 0, 1$. Since ϕ is an involution, we only need to check that this is well-defined. Indeed, for $i = 0, 1$ we have

$$\begin{aligned}
&\phi(\alpha_i\overline{\alpha}_i - \overline{\alpha}_{1-i}\alpha_{1-i} - \alpha_i\alpha_{1-i} - \lambda_i e_i) \\
&\qquad = \phi(\overline{\alpha}_i)\phi(\alpha_i) - \phi(\alpha_{1-i})\phi(\overline{\alpha}_{1-i}) - \phi(\alpha_{1-i})\phi(\alpha_i) - \lambda_i e_i \\
&\qquad = -\overline{\alpha}_{1-i}\alpha_{1-i} + \alpha_i\overline{\alpha}_i - \alpha_i\alpha_{1-i} - \lambda_i e_i \\
&\qquad = 0,
\end{aligned}$$

as required. □

The following paragraph is essentially a repeat of the material required for (6.1.1) but with some additional details that will be important for this chapter. Let A^G be a quantum Kleinian singularity as in Table 6.1 and let $V = \Bbbk u \oplus \Bbbk v$. The irreducible representations $W_0, W_1, \ldots, W_n$ of G correspond to the vertices of the McKay quiver $\widetilde{Q}$ of G, where W_0 is the trivial representation. Writing χ_i for the character of the representation W_i, for $0 \leqslant i \leqslant n$ set

$$\eta_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) g,$$

which are central idempotents in $\Bbbk G$ and form a $\Bbbk$-basis for $Z(\Bbbk G)$. Alternatively, if we write $\delta_i = \dim_{\Bbbk} W_i$, fix an isomorphism

$$\Bbbk G \cong \prod_{i=0}^{n} M_{\delta_i}(\Bbbk),$$

and write $E_{jk}^{(i)}$, where $0 \leqslant i \leqslant n$ and $1 \leqslant j,k \leqslant \delta_i$, for the element of $\Bbbk G$ which is mapped to the corresponding matrix unit under this isomorphism, then we can equivalently define

$$\eta_i = \sum_{j=1}^{\delta_i} E_{jj}^{(i)}.$$

We then identify $\Bbbk^{n+1}$ with $Z(\Bbbk G)$ as in (6.1.1):

$$\Bbbk^{n+1} \to Z(\Bbbk G), \qquad (\lambda_0, \lambda_1, \dots, \lambda_n) \mapsto \sum_{i=0}^{n} \frac{\lambda_i}{\delta_i} \eta_i,$$

Let $e = \frac{1}{|G|} \sum_{g \in G} g \in \Bbbk G$ be the average of the group elements, and for $0 \leqslant i \leqslant n$ set $f_i = E_{11}^{(i)}$. Then we have the following facts:

(1) $\Bbbk G f_i \cong W_i$;

(2) The f_i are pairwise orthogonal, so $f = f_0 + \cdots + f_n$ is an idempotent;

(3) $f_0 = e$, so $e = fef$;

(4) $\Bbbk G f \Bbbk G = \Bbbk G$, since $1 = \sum_{i,j} E_{j1}^{(i)} E_{1j}^{(i)} = \sum_{i,j} E_{j1}^{(i)} f E_{1j}^{(i)} \in \Bbbk G f \Bbbk G$. Therefore f is a full idempotent of $\Bbbk G$, and so $\Bbbk G$ is Morita equivalent to $f \Bbbk G f$; and

(5) $f_j \sum_{i=0}^{n} \lambda_i \eta_i = \lambda_j f_j$.

To prove Theorem 7.2.1, we first establish some intermediate lemmas for each of the cases. The exposition from now on closely follows that of [CBH98, §3], although we also provide some of the proofs that were left to the reader there.

Lemma 7.2.3 (Case (i))**.** *Let $q \in \Bbbk^{\times}$. Suppose $(A, G) = (\Bbbk_q[u,v], C_{n+1})$. For each arrow $\alpha_i : i \to i+1$ in $\widetilde{Q} = \widetilde{\mathbb{A}}_n$, define linear maps*

$$\theta_{\alpha_i} : W_{i+1} \to V \otimes W_i, \qquad \theta_{\alpha_i}(f_{i+1}) = u \otimes f_i,$$
$$\overline{\theta}_{\alpha_i} : W_i \to V \otimes W_{i+1}, \qquad \overline{\theta}_{\alpha_i}(f_i) = v \otimes f_{i+1}.$$

Then these maps are $\Bbbk G$-module homomorphisms which satisfy

$$(\mathrm{id}_V \otimes \theta_{\alpha_i})\overline{\theta}_{\alpha_i} - q(\mathrm{id}_V \otimes \overline{\theta}_{\alpha_{i-1}})\theta_{\alpha_{i-1}} = (v \otimes u - q u \otimes v) \otimes \mathrm{id}_{W_i}$$

as maps $W_i \to V \otimes V \otimes W_i$, and moreover these maps combine to give a basis for $\bigoplus_{i,j} \mathrm{Hom}_{\Bbbk G}(W_i, V \otimes W_j)$.

Proof. We assume that the f_i are ordered so that $g \cdot f_i = \omega^i f_i$, where $C_{n+1} = \langle g \rangle$ and

ω is a primitive $(n+1)$th root of unity. The θ_{α_i} are $\Bbbk G$-module homomorphisms since

$$\begin{aligned}\theta_{\alpha_i}(g\cdot f_{i+1}) &= \omega^{i+1}\theta_{\alpha_i}(f_{i+1}) = \omega^{i+1}u\otimes f_i = (\omega u)\otimes(\omega^i f_i) = g\cdot(u\otimes f_i)\\ &= g\cdot\theta_{\alpha_i}(f_{i+1}),\end{aligned}$$

and similarly one can show that the $\overline{\theta}_{\alpha_i}$ are $\Bbbk G$-module homomorphisms. The following calculation shows that these maps satisfy the claimed relation:

$$\begin{aligned}&\Big((\mathrm{id}_V\otimes\theta_{\alpha_i})\overline{\theta}_{\alpha_i} - q(\mathrm{id}_V\otimes\overline{\theta}_{\alpha_{i-1}})\theta_{\alpha_{i-1}}\Big)(f_i)\\ &\qquad= (\mathrm{id}_V\otimes\theta_{\alpha_i})(v\otimes f_{i+1}) - q(\mathrm{id}_V\otimes\overline{\theta}_{\alpha_{i-1}})(u\otimes f_{i-1})\\ &\qquad= v\otimes u\otimes f_i - q(u\otimes v\otimes f_i)\\ &\qquad= (v\otimes u - q\,u\otimes v)\otimes\mathrm{id}_{W_i}(f_i).\end{aligned}$$

Finally, these maps form a basis for $\bigoplus_{i,j}\mathrm{Hom}_{\Bbbk G}(W_i, V\otimes W_j)$ since the number of arrows between vertex i and j in the double of $\widetilde{Q}$ is $\dim_{\Bbbk}\mathrm{Hom}_{\Bbbk G}(W_i, V\otimes W_j)$, and this is the number of linearly independent maps we have written down for these spaces. □

The next lemma is an analogue of Lemma 7.2.3 for case (iv). The proof is similar to that of Lemma 7.2.3, so we omit it.

Lemma 7.2.4 (Case (iv))**.** *Suppose* $(A,G) = (\Bbbk_J[u,v], C_2)$*. For* $i = 0,1$*, define linear maps*

$$\begin{aligned}\theta_{\alpha_i} &: W_{1-i}\to V\otimes W_i, &\qquad \theta_{\alpha_i}(f_{1-i}) &= u\otimes f_i,\\ \overline{\theta}_{\alpha_i} &: W_i\to V\otimes W_{1-i}, &\qquad \overline{\theta}_{\alpha_i}(f_i) &= v\otimes f_{1-i}.\end{aligned}$$

Then these maps are $\Bbbk G$*-module homomorphisms which satisfy*

$$(\mathrm{id}_V\otimes\theta_{\alpha_i})\overline{\theta}_{\alpha_i} - (\mathrm{id}_V\otimes\overline{\theta}_{\alpha_{1-i}})\theta_{\alpha_{1-i}} - (\mathrm{id}_V\otimes\theta_{\alpha_i})\theta_{\alpha_{1-i}} = (v\otimes u - u\otimes v - u\otimes u)\otimes\mathrm{id}_{W_i}$$

as maps $W_i\to V\otimes V\otimes W_i$*, and moreover these maps combine to give a basis for* $\bigoplus_{i,j}\mathrm{Hom}_{\Bbbk G}(W_i, V\otimes W_j)$*.*

We would also like to prove an analogue of Lemmas 7.2.3 and 7.2.4 for cases (ii) and (iii). This is more involved, so we adapt the approach in [CBH98]. In these cases, since $G\leqslant O(2,\Bbbk)$, there is a G-invariant symmetrical bilinear form B on V given by

$$B(u,v) = 1 = B(v,u), \qquad B(u,u) = 0 = B(v,v),$$

and hence this induces a homomorphism $B: V\otimes V\to\Bbbk$. There is also a homomorphism

$$\nu:\Bbbk\to V\otimes V, \qquad \nu(1) = u\otimes v + v\otimes u.$$

We then have the following technical lemma:

Lemma 7.2.5 ([CBH98, Lemma 3.1])**.** *Suppose that* M *and* N *are* $\Bbbk G$*-modules. Then there are mutually inverse bijections*

$$\begin{aligned}\sharp &: \mathrm{Hom}_{\Bbbk G}(M, V\otimes N)\to\mathrm{Hom}_{\Bbbk G}(V\otimes M, N), &\quad \theta &\mapsto \theta^\sharp = (B\otimes\mathrm{id}_N)(\mathrm{id}_V\otimes\theta)\\ \flat &: \mathrm{Hom}_{\Bbbk G}(V\otimes M, N)\to\mathrm{Hom}_{\Bbbk G}(M, V\otimes N), &\quad \phi &\mapsto \phi^\flat = (\mathrm{id}_V\otimes\phi)(\nu\otimes\mathrm{id}_M).\end{aligned}$$

Proof. Let $\theta \in \mathrm{Hom}_{\Bbbk G}(M, V \otimes N)$ and $m \in M$, and write $\theta(m) = u \otimes n + v \otimes n'$ for some $n, n' \in N$. Then

$$\begin{aligned}
(\theta^\sharp)^\flat(m) &= (\mathrm{id}_V \otimes \theta^\sharp)(\nu \otimes \mathrm{id}_M)(m) \\
&= u \otimes \theta^\sharp(v \otimes m) + v \otimes \theta^\sharp(u \otimes m) \\
&= u \otimes (B \otimes \mathrm{id}_N)(v \otimes \theta(m)) + v \otimes (B \otimes \mathrm{id}_N)(u \otimes \theta(m)) \\
&= u \otimes (B \otimes \mathrm{id}_N)(v \otimes u \otimes n + v \otimes v \otimes n') + v \otimes (B \otimes \mathrm{id}_N)(u \otimes u \otimes n + u \otimes v \otimes n') \\
&= u \otimes n + v \otimes n' \\
&= \theta(m).
\end{aligned}$$

Now suppose that $\phi \in \mathrm{Hom}_{\Bbbk G}(V \otimes M, N)$ and $m \in M$. Then

$$\begin{aligned}
(\phi^\flat)^\sharp(u \otimes m) &= (B \otimes \mathrm{id}_N)(\mathrm{id}_V \otimes \phi^\flat)(u \otimes m) \\
&= (B \otimes \mathrm{id}_N)(u \otimes \phi^\flat(m)) \\
&= (B \otimes \mathrm{id}_N)(u \otimes u \otimes \phi(v \otimes m) + u \otimes v \otimes \phi(u \otimes m)) \\
&= \phi(u \otimes m),
\end{aligned}$$

and similarly $(\phi^\flat)^\sharp(v \otimes m) = \phi(v \otimes m)$. Therefore the maps $\sharp$ and $\flat$ are mutually inverse bijections, as claimed. □

We are now in a position to prove an analogue of Lemmas 7.2.3 and 7.2.4 for cases (ii) and (iii), the proof of which is essentially the same as [CBH98, Lemma 3.2]. We recall that $\delta_i = \dim_\Bbbk W_i$.

Lemma 7.2.6 (Cases (ii) and (iii))**.** *Suppose (A, G) is as in cases* (ii) *or* (iii) *with associated McKay quiver $\widetilde{Q}$. To each arrow $\alpha : i \to j$ in $\widetilde{Q}$ with $i \neq j$, one can associate homomorphisms*

$$\theta_\alpha : W_j \to V \otimes W_i, \qquad \overline{\theta}_\alpha : W_i \to V \otimes W_j,$$

and to each loop $\alpha : i \to i$ in $\widetilde{Q}$ one can associate a homomorphism

$$\theta_\alpha = \overline{\theta}_\alpha : W_i \to V \otimes W_i,$$

such that, for each vertex i,

$$\sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha)\overline{\theta}_\alpha + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha)\theta_\alpha = \delta_i(\nu \otimes \mathrm{id}_{W_i})$$

as maps $W_i \to V \otimes V \otimes W_i$, and moreover these maps combine to give a basis for $\bigoplus_{i,j} \mathrm{Hom}_{\Bbbk G}(W_i, V \otimes W_j)$.

Proof. Since the double of $\widetilde{Q}$ is the McKay quiver of G corresponding to the representation V, for each arrow α in $\widetilde{Q}$ one can choose nonzero homomorphisms $\theta_\alpha : W_{h(\alpha)} \to V \otimes W_{t(\alpha)}$ and $\overline{\theta}_\alpha : W_{t(\alpha)} \to V \otimes W_{h(\alpha)}$ (if α is a loop, take $\theta_\alpha = \overline{\theta}_\alpha$). Then, since the irreducible representation $W_{h(\alpha)}$ (respectively, $W_{t(\alpha)}$) appears as a summand of $W_{t(\alpha)} \otimes V$ (respectively, $W_{h(\alpha)} \otimes V$) precisely once, $\overline{\theta}^\sharp_\alpha \theta_\alpha$ and $\theta^\sharp_\alpha \overline{\theta}_\alpha$ are nonzero endomorphisms of $W_{h(\alpha)}$ and $W_{t(\alpha)}$, respectively; in particular, they are scalar multiples of the identity, and by rescaling the θ_α, one can rescale these endomorphisms. We claim that it can be arranged so that there is a nonzero scalar m_i for each vertex i such that, for each arrow α which is not a loop,

$$\overline{\theta}^\sharp_\alpha \theta_\alpha = m_{t(\alpha)} \mathrm{id}_{W_{h(\alpha)}}, \quad \theta^\sharp_\alpha \overline{\theta}_\alpha = m_{h(\alpha)} \mathrm{id}_{W_{t(\alpha)}}. \tag{7.2.7}$$

(Note that if α is a loop then these two equations are the same.) This is possible essentially because $\widetilde{Q}$ is a tree which possibly has loops. Indeed, fix the value of m_i for a vertex i, and consider some arrow $\alpha : i \to j$. If α is not a loop then one of the two equations (7.2.7) fixes θ_α up to multiplication by a scalar, and the other equation determines m_j. If α is a loop then we only need to satisfy a single equation, and this only requires $\Bbbk$ to be quadratically closed. One then repeats this process recursively for the other vertices in the quiver.

It follows that $\theta_\alpha \overline{\theta}^\sharp_\alpha$ is given by $m_{t(\alpha)}$ times the projection $V \otimes W_{t(\alpha)} \to W_{h(\alpha)}$ followed by the inclusion $W_{h(\alpha)} \to V \otimes W_{t(\alpha)}$, and similarly $\overline{\theta}_\alpha \theta^\sharp_\alpha$ is $m_{h(\alpha)}$ times the composition $V \otimes W_{h(\alpha)} \twoheadrightarrow W_{t(\alpha)} \hookrightarrow V \otimes W_{h(\alpha)}$. Therefore, for any vertex i we have

$$\sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} \theta_\alpha \overline{\theta}^\sharp_\alpha + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \overline{\theta}_\alpha \theta^\sharp_\alpha = m_i \mathrm{id}_{V\otimes W_i}. \tag{7.2.8}$$

Call this map ψ, and consider the map $(\mathrm{id}_V \otimes \psi)(\nu \otimes \mathrm{id}_{W_i})$. Calculating using the right hand side of (7.2.8) we obtain $(m_i \mathrm{id}_V \otimes \mathrm{id}_{V \otimes W_i})(\nu \otimes \mathrm{id}_{W_i}) = m_i(\nu \otimes \mathrm{id}_{W_i})$ while, using instead the left hand side and applying Lemma 7.2.5,

$$\begin{aligned}
&\sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha \overline{\theta}^\sharp_\alpha)(\nu \otimes \mathrm{id}_{W_i}) + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha \theta^\sharp_\alpha)(\nu \otimes \mathrm{id}_{W_i}) \\
&\quad = \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha)(\mathrm{id}_V \otimes \overline{\theta}^\sharp_\alpha)(\nu \otimes \mathrm{id}_{W_i}) + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha)(\mathrm{id}_V \otimes \theta^\sharp_\alpha)(\nu \otimes \mathrm{id}_{W_i}) \\
&\quad = \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha)(\overline{\theta}^\sharp_\alpha)^\flat + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha)(\theta^\sharp_\alpha)^\flat \\
&\quad = \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha)\overline{\theta}_\alpha + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha)\theta_\alpha.
\end{aligned}$$

Therefore we have an equality

$$\sum_{\substack{\alpha \in \widetilde{Q}_1 \\ t(\alpha)=i}} (\mathrm{id}_V \otimes \theta_\alpha)\overline{\theta}_\alpha + \sum_{\substack{\alpha \in \widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} (\mathrm{id}_V \otimes \overline{\theta}_\alpha)\theta_\alpha = m_i(\nu \otimes \mathrm{id}_{W_i}). \tag{7.2.9}$$

We now claim that that vector $(m_i)_{i\in\widetilde{Q}_0}$ is necessarily a nonzero scalar multiple of δ. Composing both sides of (7.2.9) with the map $B \otimes \mathrm{id}_{W_i} : V \otimes V \otimes W_i \to W_i$ and using Lemma 7.2.5, the resulting endomorphism of W_i is multiplication by

$$\sum_{\substack{\alpha\in\widetilde{Q}_1\\ t(\alpha)=i}} m_{h(\alpha)} + \sum_{\substack{\alpha\in\widetilde{Q}_1\\ h(\alpha)=i\\ t(\alpha)\neq i}} m_{t(\alpha)} = 2m_i.$$

That is, writing ∂i for the set of vertices adjacent to vertex i (where $i \in \partial i$ if there is a loop at vertex i), we have an identity

$$\sum_{j\in\partial i} m_j = 2m_i.$$

In the terminology of [HPR80], ths means that $(m_i)_{i\in\widetilde{Q}_0}$ gives an additive function of $\widetilde{Q}$. But, by [HPR80, Theorem 2], any additive function of a graph of type $\widetilde{\mathbb{L}}_1$, $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{DL}}_n$ is necessarily a scalar multiple of δ, forcing $(m_i)_{i\in\widetilde{Q}_0}$ to be a nonzero scalar multiple of δ. The result now follows after rescaling all of the θ_α suitably.

Finally, we have the claimed basis for the same reason as in the proof of Lemma 7.2.3. □

Write $S = \Bbbk\langle u, v\rangle \# G$, graded with u and v in degree 1 and G in degree 0. Consider $V \otimes \Bbbk G$ as a $\Bbbk G$-bimodule with action $h_1(w \otimes g)h_2 = h_1 w \otimes h_1 g h_2$. Then, as algebras, S is isomorphic to $T_{\Bbbk G}(V \otimes \Bbbk G)$, the tensor algebra of the bimodule $V \otimes \Bbbk G$ over $\Bbbk G$, via the map

$$\phi : S \to T_{\Bbbk G}(V \otimes \Bbbk G), \qquad w_1 \dots w_k g \mapsto (w_1 \otimes 1) \otimes \cdots \otimes (w_{k-1} \otimes 1) \otimes (w_k \otimes g).$$

Since f is a full idempotent of $\Bbbk G$ it is also a full idempotent of fSf and so S is Morita equivalent to fSf. Therefore fSf is isomorphic to the tensor algebra of the bimodule fS_1f over $f\Bbbk Gf$. We also have an isomorphism $fS_0f \cong \Bbbk^{Q_0}$ since $S_0 \cong \Bbbk G$, and we also have an identification

$$\mathrm{Hom}_{\Bbbk G}(W_i, V \otimes W_j) \cong \mathrm{Hom}_{\Bbbk G}(\Bbbk G f_i, V \otimes \Bbbk G f_j) \cong \mathrm{Hom}_{\Bbbk G}(\Bbbk G f_i, S_1 f_j) \cong f_i S_1 f_j.$$

Noting that $\delta_i = 1$ for each vertex i in cases (i), (ii) and (iv), and recalling the notation

$$\rho_*(u,v) = \begin{cases} vu - quv & \text{if } * = q \text{ (case (i))} \\ vu + uv & \text{if } * = -1 \text{ (cases (ii) and (iii))} \\ vu - uv - u^2 & \text{if } * = J \text{ (case (iv))} \end{cases}$$

from Chapter 6, we can restate Lemmas 7.2.3, 7.2.4, and 7.2.6 as follows:

Lemma 7.2.10. *Let A^G be a quantum Kleinian singularity. To each arrow $\alpha : i \to j$ in $\widetilde{Q}$ with $i \neq j$, one can associate elements*

$$\theta_\alpha \in f_j S_1 f_i, \qquad \overline{\theta}_\alpha \in f_i S_1 f_j,$$

and to each loop $\alpha : i \to i$ in $\widetilde{Q}$ one can associate an element

$$\theta_\alpha = \overline{\theta}_\alpha \in f_i S_1 f_i,$$

such that, for each vertex i,

$$\delta_i f_i \rho_*(u,v) = \begin{cases} \overline{\theta}_{\alpha_i}\theta_{\alpha_i} - q\theta_{\alpha_{i-1}}\overline{\theta}_{\alpha_{i-1}} & \textit{for case (i)} \\ \sum_{\substack{\alpha\in\widetilde{Q}_1 \\ t(\alpha)=i}} \overline{\theta}_\alpha\theta_\alpha + \sum_{\substack{\alpha\in\widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \theta_\alpha\overline{\theta}_\alpha & \textit{for cases (ii) and (iii)} \\ \overline{\theta}_{\alpha_i}\theta_{\alpha_i} - \theta_{\alpha_{1-i}}\overline{\theta}_{\alpha_{1-i}} - \theta_{\alpha_{1-i}}\theta_{\alpha_i} & \textit{for case (iv)} \end{cases}$$

and moreover these elements combine to give a basis for $\bigoplus_{i,j} f_i S_1 f_j$. □

This allows us to prove the following theorem, from which Theorem 7.2.1 follows:

Theorem 7.2.11. *Let A^G be a quantum Kleinian singularity with corresponding Euclidean diagram $\widetilde{Q}$, and let $\lambda \in \Bbbk^{n+1}$. Then there is an isomorphism $f\mathcal{S}^\lambda_* f \cong \Pi^\lambda_*(\widetilde{Q})$. Under this isomorphism, e is sent to the trivial path e_0.*

Proof. By the discussion above, there is an isomorphism $\phi : fSf \to \Bbbk\overline{\widetilde{Q}}$ sending

$$f_i \mapsto e_i, \qquad \theta_{\alpha_i} \mapsto \alpha_i, \qquad \overline{\theta}_{\alpha_i} \mapsto \overline{\alpha}_i,$$

where if α is a loop then $\alpha = \overline{\alpha}$ and $\theta_\alpha = \overline{\theta}_\alpha$. Identifying λ with a central element of $\Bbbk G$ via (6.1.1), we see that every element of $\Bbbk G$ commutes with $\rho_*(u,v)-\lambda$. In particular, f commutes with $\rho_*(u,v)-\lambda$, and so $f(\rho_*(u,v)-\lambda)f = f(\rho_*(u,v)-\lambda) = (\rho_*(u,v)-\lambda)f$.

Now let $I = \langle \rho_*(u,v)-\lambda\rangle_S$, where the subscript S indicates that we are considering this as the two-sided ideal of S generated by $\rho_*(u,v)-\lambda$. We claim that $I = \langle f(\rho_*(u,v)-\lambda)\rangle_S$. Indeed, since f is a full idempotent of $\Bbbk G$, there exist $g_i, h_i \in \Bbbk G$ with $\sum_i g_i f h_i = 1$, and so $\langle f(\rho_*(u,v)-\lambda)\rangle_S$ contains

$$\sum_i g_i f(\rho_*(u,v)-\lambda)h_i = \sum_i g_i f h_i(\rho_*(u,v)-\lambda) = \rho_*(u,v)-\lambda.$$

Therefore $\langle \rho_*(u,v)-\lambda\rangle_S \subseteq \langle f(\rho_*(u,v)-\lambda)\rangle_S$, and the other inclusion is clear, so we have the claimed equality. It follows that, as an ideal of fSf, we have $I \cap fSf = \langle f(\rho_*(u,v)-\lambda)\rangle_{fSf}$. One also observes that this ideal is generated by the elements

$$\begin{aligned} f_i(\rho_*(u,v)-\lambda) &= f_i\left(\rho_*(u,v) - \sum_{j=0}^{n}\frac{\lambda_j}{\delta_j}\eta_j\right) = f_i\left(\rho_*(u,v) - \frac{\lambda_i}{\delta_i}\right) \\ &= \frac{1}{\delta_i}(\delta_i f_i \rho_*(u,v) - \lambda_i f_i). \end{aligned}$$

Therefore, using Lemma 7.2.10,

$$I \cap fSf = \begin{cases} \left\langle \overline{\theta}_{\alpha_i}\theta_{\alpha_i} - q\theta_{\alpha_{i-1}}\overline{\theta}_{\alpha_{i-1}} - \lambda_i f_i \;\middle|\; i \in \widetilde{Q}_0 \right\rangle & \text{for case (i)} \\ \left\langle \sum_{\substack{\alpha\in\widetilde{Q}_1 \\ t(\alpha)=i}} \overline{\theta}_\alpha\theta_\alpha + \sum_{\substack{\alpha\in\widetilde{Q}_1 \\ h(\alpha)=i \\ t(\alpha)\neq i}} \theta_\alpha\overline{\theta}_\alpha - \lambda_i f_i \;\middle|\; i \in \widetilde{Q}_0 \right\rangle & \text{for cases (ii) and (iii)} \\ \left\langle \overline{\theta}_{\alpha_i}\theta_{\alpha_i} - \theta_{\alpha_{1-i}}\overline{\theta}_{\alpha_{1-i}} - \theta_{\alpha_{1-i}}\theta_{\alpha_i} - \lambda_i f_i \;\middle|\; i \in \widetilde{Q}_0 \right\rangle & \text{for case (iv)} \end{cases}$$

The image under ϕ of this ideal is the ideal of relations defining $\Pi^\lambda_*(\widetilde{Q})$, and so we have the claimed isomorphism. □

Proof of Theorem 7.2.1. Since f is a full idempotent and $f\mathcal{S}_*^\lambda(\widetilde{Q})f \cong \Pi_*^\lambda(\widetilde{Q})$, this immediately implies that $\mathcal{S}_*^\lambda(\widetilde{Q})$ and $\Pi_*^\lambda(\widetilde{Q})$ are Morita equivalent. Moreover, since e is mapped to e_0 under this isomorphism, pre- and post-multiplying the left hand side (respectively, right hand side) of the isomorphism by e (respectively, e_0) shows that $\mathcal{O}_*^\lambda(\widetilde{Q}) \cong e_0\Pi_*^\lambda(\widetilde{Q})e_0$. □

Remark 7.2.12. We remarked previously that the Morita equivalence between $\mathcal{S}_*^\lambda(\widetilde{Q})$ and $\Pi_*^\lambda(\widetilde{Q})$ is in fact an isomorphism in cases (i), (ii) and (iv). This is because our proof shows that $f\mathcal{S}_*^\lambda(\widetilde{Q})f \cong \Pi_*^\lambda(\widetilde{Q})$, and when $\widetilde{Q} = \widetilde{\mathbb{A}}_n$, f is actually equal to 1 since G is abelian.

Theorem 7.2.1 has the following corollary, the proof of which is identical to that of [CBH98, Corollary 3.6].

Corollary 7.2.13. *Let $\widetilde{Q}$ be the quiver corresponding to a quantum Kleinian singularity A^G and let λ be a weight for $\widetilde{Q}$. Then the natural map $\Pi_*(\widetilde{Q}) \to \operatorname{gr}\Pi_*^\lambda(\widetilde{Q})$ is an isomorphism and $\Pi_*^\lambda(\widetilde{Q})$ is a prime noetherian ring of GK dimension 2.*

Chapter 8

Singularities of Deformations of Quantum Kleinian Singularities

In this chapter, we seek to determine the global dimensions of the deformations of quantum Kleinian singularities $\mathcal{O}_*^\lambda$ from Chapter 6. In the cases where $\mathcal{O}_*^\lambda$ is singular, we also provide a description of its singularity category.

We will see that, in many cases, the singularity theory of deformations of quantum Kleinian singularities is quite different from that of deformations of classical Kleinian singularities, as considered in [CBH98]. For example, Crawley-Boevey–Holland showed that their deformations are generically nonsingular, and that this behaviour was uniform across all Dynkin types. Moreover, they showed that generically $\mathcal{O}^\lambda(\widetilde{Q})$ and $\mathcal{S}^\lambda(\widetilde{Q})$ are Morita equivalent for deformations of Kleinian singularities. On the other hand, deformations of quantum Kleinian singularities in case (ii) and in case (iii) (when m is odd) are *always* singular. The corresponding deformations $\mathcal{S}_*^\lambda$ of the skew group rings $A\#G$ also exhibit new behaviour in these cases, as well as in case (iv), since they always have global dimension 2; the analagous deformations in [CBH98] generically have global dimension 1.

8.1 Preliminary results

We refer the reader to Table 6.1 for the classification of quantum Kleinian singularities in the form that we will need for this chapter. The main aim of this section is to prove weaker a version of Theorem 3.2.6 for deformations of quantum Kleinian singularities. In subsequent sections we are then able to make more precise statements by considering each case separately.

In light of Theorem 7.2.1, in this chapter it will be convenient to view deformations of quantum Kleinian singularities A^G as having the form $\mathcal{O}_*^\lambda(\widetilde{Q}) = e_0\Pi_*^\lambda(\widetilde{Q})e_0$, where $\widetilde{Q}$ is the Euclidean diagram corresponding to A^G and λ is a weight for $\widetilde{Q}$. We also write Q for the quiver obtained by removing the extending vertex 0 and we set $\mu = (\lambda_1, \dots, \lambda_n)$, which may be viewed as a weight for Q. Many of the results in this section are similar to those found in Chapter 3 and so some of the details are left to the reader.

Lemma 8.1.1. *$\Pi_*^\lambda e_0$ is a finitely generated $\mathcal{O}_*^\lambda$-module, and it satisfies $\operatorname{End}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0) = \Pi_*^\lambda$.*

Proof. This is proved in the same way as Lemma 3.2.1 after noting that Π_*^λ is Morita equivalent to the maximal order $\mathcal{S}_*^\lambda$. □

As in Chapter 3, write $V_i = e_i\Pi_*^\lambda e_0$. We again call the V_i vertex modules.

Corollary 8.1.2. *We have* $\mathrm{Hom}_{\mathcal{O}_*^\lambda}(V_i, V_j) = e_j\Pi_*^\lambda e_i$*, and so* $\Pi_*^\lambda e_0$ *is a reflexive (and hence maximal Cohen-Macaulay)* $\mathcal{O}_*^\lambda$*-module.*

Proof. This is similar to Corollary 3.2.2. □

This allows us to determine the stable endomorphism ring of $\Pi_*^\lambda e_0$.

Lemma 8.1.3. *We have*

$$\underline{\mathrm{End}}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0) \cong \begin{cases} \Pi_*^\mu(Q) & \textit{for cases } \text{(i), (ii) } \textit{and } \text{(iii)} \\ \Pi^\mu(\mathbb{A}_1) & \textit{for case } \text{(iv)} \end{cases},$$

where $\mu = (\lambda_1, \ldots, \lambda_n)$.

Proof. By Corollary 3.2.2, we have that

$$(\Pi_*^\lambda e_0)^* = \bigoplus_i \mathrm{Hom}_{\mathcal{O}_*^\lambda}(e_i\Pi_*^\lambda e_0, e_0\Pi_*^\lambda e_0) = \bigoplus_i e_0\Pi_*^\lambda e_i = e_0\Pi_*^\lambda.$$

Then, noting that $\Pi_*^\lambda e_0 \left(\Pi_*^\lambda e_0\right)^* = \Pi_*^\lambda e_0 \Pi_*^\lambda$, we have

$$\underline{\mathrm{End}}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0) \cong \frac{\Pi_*^\lambda}{\Pi_*^\lambda e_0 \Pi_*^\lambda} \cong \begin{cases} \Pi_*^\mu(Q) & \text{for cases (i), (ii) and (iii)} \\ \Pi^\mu(\mathbb{A}_1) & \text{for case (iv)} \end{cases},$$

as claimed. □

We now identify all maximal Cohen-Macaulay $\mathcal{O}_*^\lambda$-modules. We remark that in [CKWZ16b, Theorem C], the authors were able to give a more precise description of the maximal Cohen-Macaulay modules in the case where $\lambda = 0$, i.e., when $\mathcal{O}_*^\lambda = A^G$. In particular, they showed that there is a bijective correspondence between indecomposable MCM modules and vertices of $\widetilde{Q}$. We do not know if similar results hold for arbitrary λ. (In [CKWZ16b], the authors use an alternative definition of maximal Cohen-Macaulay, but this agrees with our definition by the remark after Definition 2.4 in [Uey17].)

Proposition 8.1.4. *We have* $\mathrm{MCM}\text{-}\mathcal{O}_*^\lambda = \mathrm{add}\,\Pi_*^\lambda e_0$.

Proof. First note that $\mathcal{O}_*^\lambda$ is Gorenstein and that, using Proposition 6.1.5

$$\mathrm{gl.dim}\,\mathrm{End}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0) = \mathrm{gl.dim}\,\Pi_*^\lambda \leqslant 2.$$

Since $\Pi_*^\lambda e_0$ has $\mathcal{O}_*^\lambda$ as a direct summand, the claim then follows from Proposition 2.2.11. □

Finally, we prove a weaker version of Theorem 3.2.6.

Theorem 8.1.5. *The functor* $\underline{\mathrm{Hom}}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0, -)$ *induces a fully faithful functor*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}_*^\lambda \to \begin{cases} \text{proj-}\Pi_*^\mu(Q) & \textit{for cases } \text{(i), (ii) } \textit{and } \text{(iii)} \\ \text{proj-}\Pi^\mu(\mathbb{A}_1) & \textit{for case } \text{(iv)} \end{cases},$$

where $\mu = (\lambda_1, \ldots, \lambda_n)$.

Proof. By [Kra15, Proposition 2.3], the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0, -) : \underline{\mathrm{mod}}\text{-}\mathcal{O}_*^\lambda \to \mathrm{mod}\text{-}\,\underline{\mathrm{End}}_{\mathcal{O}_*^\lambda}(\Pi_*^\lambda e_0),$$

induces a fully faithful $\Bbbk$-linear functor

$$\mathrm{add}\,\Pi_*^\lambda e_0 \to \begin{cases} \text{proj-}\Pi_*^\mu(Q) & \text{for cases (i), (ii) and (iii)} \\ \text{proj-}\Pi^\mu(\widetilde{\mathbb{A}}_1) & \text{for case (iv)} \end{cases},$$

where $\mathrm{add}\,\Pi_*^\lambda e_0 = \underline{\mathrm{MCM}}\text{-}\mathcal{O}_*^\lambda$ by Proposition 8.1.4. □

As stated earlier, we are not able to make this result more precise without considering each case separately. We do this now, and in the process determine the global dimensions of $\mathcal{S}_*^\lambda$ and $\mathcal{O}_*^\lambda$, as well as their singularity categories where appropriate.

8.2 Singularities in case (i)

We begin be considering case (i), so we are interested in the singularities of deformations of $\Bbbk_q[u,v]^{C_{n+1}}$. By Theorem 7.2.1, we can write our deformation as $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) = e_0\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)e_0$ for some weight λ, where the arrows of $\widetilde{\mathbb{A}}_n$ are cyclically oriented. We first show that one can restrict attention to quasi-dominant weights, in the same sense as Definition 3.1.1. We achieve this using an analogue of the dual reflections of [CBH98].

We make the following definition, which is valid for any loop-free quiver Q but where we are particularly interested in the case of an $\widetilde{\mathbb{A}}_n$ quiver. Define a matrix C with rows and columns indexed by vertices of Q as follows:

$$C_{ij} = \begin{cases} 2 & \text{if } i = j \\ -kq - \frac{\ell}{q} & \text{if there are } k \text{ arrows } i \to j \text{ and } \ell \text{ arrows } j \to i \end{cases}. \tag{8.2.1}$$

Then for each vertex i of Q we define a *quantum dual reflection* $r_i^q : \Bbbk^{Q_0} \to \Bbbk^{Q_0}$ by

$$(r_i^q\lambda)_j = \lambda_j - C_{ij}\lambda_i.$$

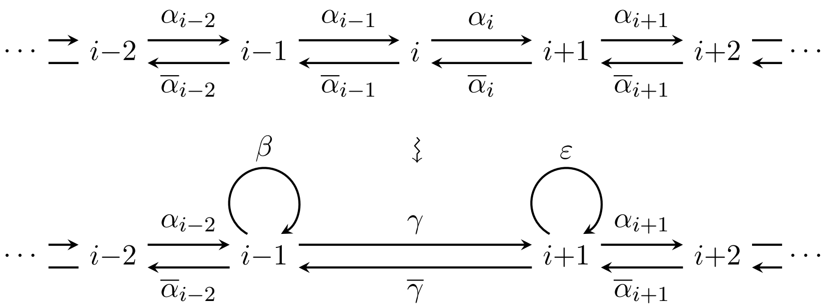
We note that applying a reflection to a weight at vertex i is the same as a move in a generalised version of Mozes' numbers game (as discussed in Chapter 3) which has been studied in [DE08], for example.

We now wish to show that these quantum dual reflections give rise to isomorphisms of the deformations in case (i) as follows: we claim that if $i \neq 0$, then $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \cong \mathcal{O}_q^{r_i^q\lambda}(\widetilde{\mathbb{A}}_n)$. We first require some technical results:

Lemma 8.2.2. *Let $n \geqslant 1$ and $i \in \{0, 1, \ldots, n\}$. Let Γ be the quiver obtained from the double of an $\widetilde{\mathbb{A}}_n$ quiver by deleting vertex i and the arrows $\alpha_{i-1}, \overline{\alpha}_{i-1}, \alpha_i, \overline{\alpha}_i$, and adding arrows*

$$\beta : i{-}1 \to i{-}1, \quad \gamma : i{-}1 \to i{+}1, \quad \overline{\gamma} : i{+}1 \to i{-}1, \quad \varepsilon : i{+}1 \to i{+}1$$

as follows:



Then there is an isomorphism

$$(1-e_i)\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)(1-e_i) \cong \frac{\Bbbk\Gamma}{\left\langle \begin{array}{c} \alpha_j\overline{\alpha}_j - q\overline{\alpha}_{j-1}\alpha_{j-1} - \lambda_j e_j \quad \text{for } j \neq i{-}1, i, i{+}1 \\ \beta - q\overline{\alpha}_{i-2}\alpha_{i-2} - \lambda_{i-1}e_{i-1}, \quad \alpha_{i+1}\overline{\alpha}_{i+1} - q\varepsilon - \lambda_{i+1}e_{i+1} \\ \gamma\overline{\gamma} - q\beta^2 - \lambda_i\beta, \quad \varepsilon^2 - q\overline{\gamma}\gamma - \lambda_i\varepsilon \\ \gamma\varepsilon - q\beta\gamma - \lambda_i\gamma, \quad \varepsilon\overline{\gamma} - q\overline{\gamma}\beta - \lambda_i\overline{\gamma} \end{array} \right\rangle}.$$

Proof. It is clear that $(1-e_i)\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)(1-e_i)$ is generated by the idempotents e_j for $j \neq i$ and the arrows α of the double of $\widetilde{\mathbb{A}}_n$ which satisfy $t(\alpha) \neq i \neq h(\alpha)$, as well as the paths $\alpha_{i-1}\overline{\alpha}_{i-1}$, $\overline{\alpha}_i\alpha_i$, $\alpha_{i-1}\alpha_i$, $\overline{\alpha}_i\overline{\alpha}_{i-1}$ and so, identifying these latter four paths with the elements $\beta, \varepsilon, \gamma, \overline{\gamma}$ respectively, we see that we can write $(1-e_i)\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)(1-e_i)$ as a quotient $\Bbbk\Gamma/J$ for some ideal J.

Let I be the ideal of relations for $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$, so that $I = \langle r \rangle$ where

$$r = \sum_{i=0}^{n} \alpha_i\overline{\alpha}_i - q\overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i.$$

Clearly there is a bijection θ between $(1-e_i)\Bbbk\overline{\widetilde{\mathbb{A}}}_n(1-e_i) \cap I$ and J which sends a linear combination of paths in $\overline{\widetilde{\mathbb{A}}}$ to the corresponding linear combination of paths in $\Bbbk\Gamma$ (replacing $\alpha_{i-1}\overline{\alpha}_{i-1}$ with β and so on). Since $e_j r e_k \neq 0$ if and only if $j = k$, elements of $(1-e_i)\Bbbk\overline{\widetilde{\mathbb{A}}}_n(1-e_i) \cap I$ are linear combinations of elements of the form prq where p and q are paths with $h(p) = t(q)$ and $t(p) \neq i \neq h(q)$. If $h(p) = t(q) = j \neq i$, then the element $\theta(prq) \in J$ will lie in the ideal of $\Bbbk\Gamma$ generated by $e_j\theta(r)e_j$. On the other hand, if $h(p) = t(q) = i$, then the pair of arrows (ℓ, f), where ℓ is the last arrow of p and f is the first arrow of q, is one of

$$(\alpha_{i-1}, \overline{\alpha}_{i-1}), \quad (\overline{\alpha}_i, \alpha_i), \quad (\alpha_{i-1}, \alpha_i), \quad (\overline{\alpha}_i, \overline{\alpha}_{i-1})$$

and the element $\theta(prq) \in J$ will lie in the ideal of $\Bbbk\Gamma$ generated by $\theta(\ell r f)$. It follows that $(1-e_i)\Bbbk\overline{\widetilde{\mathbb{A}}}_n(1-e_i) \cap I$ is generated by

$$\begin{gathered} \alpha_j\overline{\alpha}_j - q\overline{\alpha}_{j-1}\alpha_{j-1} - \lambda_j e_j \quad \text{for } j \neq i-1, i, i+1 \\ \alpha_{i-1}\overline{\alpha}_{i-1} - q\overline{\alpha}_{i-2}\alpha_{i-2} - \lambda_{i-1}e_{i-1} \\ \alpha_{i+1}\overline{\alpha}_{i+1} - q\overline{\alpha}_i\alpha_i - \lambda_{i+1}e_{i+1} \\ \alpha_{i-1}(\alpha_i\overline{\alpha}_i - q\overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i)\overline{\alpha}_{i-1} \\ \overline{\alpha}_i(\alpha_i\overline{\alpha}_i - q\overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i)\alpha_i \\ \alpha_{i-1}(\alpha_i\overline{\alpha}_i - q\overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i)\alpha_i \\ \overline{\alpha}_i(\alpha_i\overline{\alpha}_i - q\overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i)\overline{\alpha}_{i-1} \end{gathered}$$

and the images under θ of these elements are precisely the generators of the ideal in the statement of the lemma. $\square$

Proposition 8.2.3. *Let $n \geqslant 1$ and $i \in \{0, 1, \ldots, n\}$. Then there is an isomorphism*

$$(1 - e_i)\Pi_q^{\lambda}(\widetilde{\mathbb{A}}_n)(1 - e_i) \cong (1 - e_i)\Pi_q^{r_i^q \lambda}(\widetilde{\mathbb{A}}_n)(1 - e_i)$$

satisfying $e_j \mapsto e_j$ for $j \neq i$.

Proof. Retaining the notation of Lemma 8.2.2, write J and J' for the ideals of $\Bbbk\Gamma$ satisfying $(1 - e_i)\Pi_q^{\lambda}(\widetilde{\mathbb{A}}_n)(1 - e_i) \cong \Bbbk\Gamma/J$ and $(1 - e_i)\Pi_q^{r_i^q \lambda}(\widetilde{\mathbb{A}}_n)(1 - e_i) \cong \Bbbk\Gamma/J'$. It suffices to show that we have an isomorphism $\Bbbk\Gamma/J \cong \Bbbk\Gamma/J'$. To this end, define a ring homomorphism

$$\begin{gathered}\phi : \Bbbk\Gamma \to \Bbbk\Gamma, \\ e_j \mapsto e_j, \;\; \alpha_j \mapsto \alpha_j, \;\; \overline{\alpha}_j \mapsto \overline{\alpha}_j \quad \text{for all } j, \\ \gamma \mapsto \gamma, \quad \overline{\gamma} \mapsto \overline{\gamma}, \quad \beta \mapsto \beta - \tfrac{1}{q}\lambda_i e_{i-1}, \quad \varepsilon \mapsto \varepsilon + \lambda_i e_{i+1}.\end{gathered}$$

By direct calculation, we show that ϕ maps the generators of the J of to those of J'. Indeed, for $j \neq i{-}1, i, i{+}1$, we see that ϕ is the identity on $\alpha_j\overline{\alpha}_j - q\overline{\alpha}_{j-1}\alpha_{j-1} - \lambda_j e_j$, while

$$\begin{aligned}
\phi(\beta - q\overline{\alpha}_{i-2}\alpha_{i-2} - \lambda_{i-1}e_{i-1}) &= \beta - \tfrac{1}{q}\lambda_i e_{i-1} - q\overline{\alpha}_{i-2}\alpha_{i-2} - \lambda_{i-1}e_{i-1} \\
&= \beta - q\overline{\alpha}_{i-2}\alpha_{i-2} - (\lambda_{i-1} + \tfrac{1}{q}\lambda_i)e_{i-1} \\
\phi(\alpha_{i+1}\overline{\alpha}_{i+1} - q\varepsilon - \lambda_{i+1}e_{i+1}) &= \alpha_{i+1}\overline{\alpha}_{i+1} - q(\varepsilon + \lambda_i e_{i+1}) - \lambda_{i+1}e_{i+1} \\
&= \alpha_{i+1}\overline{\alpha}_{i+1} - q\varepsilon - (\lambda_{i+1} + q\lambda_i)e_{i+1} \\
\phi(\gamma\overline{\gamma} - q\beta^2 - \lambda_i\beta) &= \gamma\overline{\gamma} - q(\beta - \tfrac{1}{q}\lambda_i e_{i-1})^2 - \lambda_i(\beta - \tfrac{1}{q}\lambda_i e_{i-1}) \\
&= \gamma\overline{\gamma} - q\beta^2 + 2\tfrac{q}{q}\lambda_i\beta - \tfrac{q}{q^2}\lambda_i^2 e_{i-1} - \lambda_i\beta + \tfrac{1}{q}\lambda_i e_{i-1} \\
&= \gamma\overline{\gamma} - q\beta^2 - (-\lambda_i)\beta \\
\phi(\varepsilon^2 - q\overline{\gamma}\gamma - \lambda_i\varepsilon) &= (\varepsilon + \lambda_i e_{i+1})^2 - q\overline{\gamma}\gamma - \lambda_i(\varepsilon + \lambda_i e_{i+1}) \\
&= \varepsilon^2 + 2\lambda_i\varepsilon + \lambda_i^2 e_{i+1} - q\overline{\gamma}\gamma - \lambda_i\varepsilon - \lambda_i^2 e_{i+1} \\
&= \varepsilon^2 - q\overline{\gamma}\gamma - (-\lambda_i)\varepsilon \\
\phi(\gamma\varepsilon - q\beta\gamma - \lambda_i\gamma) &= \gamma(\varepsilon + \lambda_i e_{i+1}) - q(\beta - \tfrac{1}{q}\lambda_i e_{i-1})\gamma - \lambda_i\gamma \\
&= \gamma\varepsilon - q\beta\gamma - (-\lambda_i)\gamma \\
\phi(\varepsilon\overline{\gamma} - q\overline{\gamma}\beta - \lambda_i\overline{\gamma}) &= (\varepsilon + \lambda_i e_{i+1})\overline{\gamma} - q\overline{\gamma}(\beta - \tfrac{1}{q}\lambda_i e_{i-1}) - \lambda_i\overline{\gamma} \\
&= \varepsilon\overline{\gamma} - q\overline{\gamma}\beta - (-\lambda_i)\overline{\gamma},
\end{aligned}$$

and these are precisely the generators of J'. $\square$

The desired result now follows quickly:

Corollary 8.2.4. *Let $n \geqslant 1$ and $i \in \{1, \ldots, n\}$. Then there is an isomorphism*

$$\mathcal{O}_q^{\lambda}(\widetilde{\mathbb{A}}_n) \cong \mathcal{O}_q^{r_i^q \lambda}(\widetilde{\mathbb{A}}_n).$$

Proof. Pre- and post-multiply both sides of the isomorphism of Proposition 8.2.3 by e_0 to obtain $e_0\Pi_q^{\lambda}(\widetilde{\mathbb{A}}_n)e_0 \cong e_0\Pi_q^{r_i^q \lambda}(\widetilde{\mathbb{A}}_n)e_0$, which is precisely the claimed isomorphism. $\square$

Using this, we can show that it is sufficient to restrict our attention to quasi-dominant weights for the remainder of this section:

Lemma 8.2.5. *Suppose that λ is a weight for $\widetilde{\mathbb{A}}_n$. Then there exists a quasi-dominant weight λ' with $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \cong \mathcal{O}_q^{\lambda'}(\widetilde{\mathbb{A}}_n)$.*

Proof. Consider $\mu = (\lambda_1, \ldots, \lambda_n)$ as a weight for the $\mathbb{A}_n$ obtained by removing vertex 0. In the notation of [DE08, Section 4], the underlying graph of $\mathbb{A}_n$ with matrix C as in (8.2.1) is an E-GCM graph. By [DE08, Theorem 4.1], it follows that there exists a sequence of quantum dual reflections at the vertices of $\mathbb{A}_n$ which, when applied to μ, yields a dominant weight μ'. (We remark that [DE08] considers the problem of reflecting to a weight where every entry is $\preceq 0$, but this is an equivalent problem if we instead consider $-\mu$.) This same sequence of reflections, now viewed as a sequence of reflections at vertices $i \neq 0$ of $\widetilde{\mathbb{A}}_n$, yields a quasi-dominant weight λ' when applied to λ. It then follows from Corollary 8.2.4 that we have an isomorphism $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \cong \mathcal{O}_q^{\lambda'}(\widetilde{\mathbb{A}}_n)$, as required. □

Remark 8.2.6.
(1) The results on dual reflections given above hold in greater generality than stated here. If Q is any quiver with vertex set $\{0, 1, \ldots n\}$ and i is a loop free vertex, then we have an isomorphism $e_0 \Pi_q^\lambda(Q) e_0 \cong e_0 \Pi_q^{r_i^q \lambda}(Q) e_0$. The proof is similar to the one given above, but it is easier to write down a proof in the specific case of an $\widetilde{\mathbb{A}}_n$ quiver, and this is sufficient for our purposes.
(2) We believe that a similar result to [CBH98, Theorem 5.1] holds for quantum dual reflections; that is, if Q is any quiver and i is a loop-free vertex then there is a Morita equivalence between $\Pi_q^\lambda(Q)$ and $\Pi_q^{r_i^q \lambda}(Q)$.

For the remainder of this section, we will assume that all weights are quasi-dominant. This allows us to make Theorem 8.1.5 more precise:

Theorem 8.2.7. *Suppose that λ is a quasi-dominant weight for $\widetilde{\mathbb{A}}_n$. Let Q_λ be the full subquiver of $\widetilde{Q}$ obtained by deleting vertex 0 and deleting each vertex i with $\lambda_i \succ 0$. Then $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ is a disjoint union of connected Dynkin quivers and, writing R_Q for the Kleinian singularity with corresponding Dynkin quiver Q, there is a triangle equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \simeq \bigoplus_{i=1}^r \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}}).$$

Proof. By Theorem 8.1.5 there is a fully faithful functor

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \to \mathrm{proj}\text{-}\Pi_q^\mu(\mathbb{A}_n).$$

By Lemma 7.1.2, $\Pi_q^\mu(\mathbb{A}_n) \cong \Pi^{\mu'}(\mathbb{A}_n)$ for some weight μ' with $\mu_i' = 0$ precisely when $\mu_i = 0$. By [CBH98, Lemma 7.1 (1)], $\Pi^{\mu'}(\mathbb{A}_n) \cong \Pi(Q_\lambda)$, where Q_λ is as claimed in the statement of the theorem. Therefore we in fact have a fully faithful functor

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \to \mathrm{proj}\text{-}\Pi(Q_\lambda),$$

and one can show that it is essentially surjective in the same way as in the proof of Theorem 3.2.6. Noting that the right hand side is equivalent to $\bigoplus_{i=1}^r \mathrm{proj}\text{-}\Pi(Q^{(i)})$, we therefore have a $\Bbbk$-linear equivalence

$$\underline{\text{MCM}}\text{-}\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}).$$

Since the left hand side is triangulated, this equivalence induces a triangulated structure on the right hand side, and arguments similar to those found Section 3.3 show that each proj-$\Pi(Q^{(i)})$ is an algebraic triangulated subcategory of $\bigoplus_{i=1}^{r}$ proj-$\Pi(Q^{(i)})$. Since each proj-$\Pi(Q^{(i)})$ is $\Bbbk$-linearly equivalent to $\mathcal{D}_{\text{sg}}(R_{Q^{(i)}})$, Theorem 3.3.2 implies that they are triangle equivalent, whence the result. □

Using this we are able to give crude estimates of the global dimensions of the deformations $\mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n)$ and $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$:

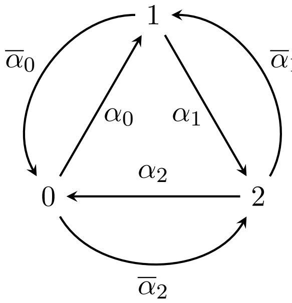
Corollary 8.2.8. *Suppose that λ is a quasi-dominant weight for $\widetilde{\mathbb{A}}_n$. Then $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ has finite global dimension if and only if $\lambda_i \neq 0$ for all $i \neq 0$. If $\operatorname{gl.dim} \mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) = \infty$, then $\operatorname{gl.dim} \mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n) = 2$. If $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ has finite global dimension, then it is Morita equivalent to $\mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n)$, in which case*

$$\begin{aligned}\operatorname{gl.dim} \mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n) &= \operatorname{gl.dim} \mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n) \\ &= \begin{cases} 1 & \text{if } \Pi_q^\lambda(\widetilde{\mathbb{A}}_n) \text{ has no finite-dimensional modules} \\ 2 & \text{otherwise} \end{cases}.\end{aligned}$$

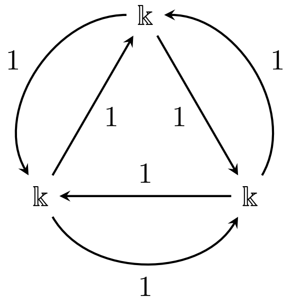
Proof. The first claim follows from Theorem 8.2.7, which implies that the singularity category of $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ is trival if and only if $\lambda_i \neq 0$ for all $i \neq 0$. When this is the case, if we write Q_λ for the full subquiver of $\mathbb{A}_n$ supported on those vertices $i \in \{1, \dots, n\}$ with $\lambda_i = 0$, then $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ has the nonzero finite-dimensional module $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)/\langle e_0 \rangle \cong \Pi_q(Q_\lambda)$, and so by the Morita equivalence of Theorem 7.2.1, $\mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n)$ has at least one finite-dimensional module. By Proposition 6.1.5, in this case $\operatorname{gl.dim} \mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n) = 2$.

Now suppose that $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ has finite global dimension. Consider the right $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$-module $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)e_0$, which has $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ as a direct summand and is hence a generator for mod-$\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$. Since this module is maximal Cohen-Maculay and has finite projective dimension, it is necessarily projective. Therefore $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)e_0$ is a progenerator, and so the isomorphism $\operatorname{End}_{\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)}(\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)e_0) \cong \Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ from Lemma 8.1.1 implies that $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ and $\mathcal{O}_q^\lambda(\widetilde{\mathbb{A}}_n)$ are Morita equivalent. As $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ is also Morita equivalent to $\mathcal{S}_q^\lambda(\widetilde{\mathbb{A}}_n)$, all three of these algebras have the same global dimension, and its precise value can be calculated using Proposition 6.1.5. □

We have been unable to give precise conditions as to when $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ has finite-dimensional modules. If $\lambda_i = 0$ for some i then it is easy to see that $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ has a one-dimensional representation. However, it is possible to have finite-dimensional representations when $\lambda_i \succ 0$ for all i if $q \neq 1$, which is in contrast with the $q = 1$ case considered by Crawley-Boevey–Holland. For example, if $q < 1$ and $\lambda = (1-q, 1-q, 1-q)$ (so each λ_i is positive) then $\Pi_q^\lambda(\widetilde{\mathbb{A}}_2)$ (shown below on the left with its relations) has the finite-dimensional representation given below on the right:



$$\begin{aligned} \alpha_0\overline{\alpha}_0 - q\overline{\alpha}_2\alpha_2 &= (1-q)e_0 \\ \alpha_1\overline{\alpha}_1 - q\overline{\alpha}_0\alpha_0 &= (1-q)e_1 \\ \alpha_2\overline{\alpha}_2 - q\overline{\alpha}_1\alpha_1 &= (1-q)e_2 \end{aligned}$$



8.3 Singularities in case (iv)

Now consider case (iv), in which case we are deforming $\Bbbk_J[u,v]^{C_2}$. In this case, we can write our deformation as $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) = e_0\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)e_0$ for some weight λ. Theorem 8.1.5 can then be refined as follows:

Theorem 8.3.1. *Suppose that λ is a weight for $\widetilde{\mathbb{A}}_1$. The functor*

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)}(\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)e_0, -) : \underline{\mathrm{mod}}\text{-}\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) \to \mathrm{Mod}\text{-}\,\underline{\mathrm{End}}_{\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)}(\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)e_0)$$

induces a triangle equivalence

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) \xrightarrow{\simeq} \begin{cases} \text{proj-}\Bbbk & \text{if } \lambda_1 = 0 \\ 0 & \text{if } \lambda_1 \neq 0 \end{cases}.$$

Proof. By Theorem 8.1.5 there is a fully faithful functor

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) \to \text{proj-}\Pi^{\lambda_1}(\mathbb{A}_1).$$

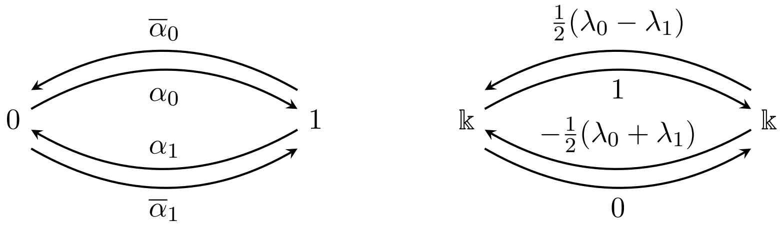
By [CBH98, Lemma 7.1 (1)], $\Pi^{\lambda_1}(\mathbb{A}_1) = 0$ if $\lambda_1 \neq 0$, whence the result in this case. If $\lambda_1 = 0$ then to establish a $\Bbbk$-linear equivalence it remains to show essential surjectivity, but this is immediate since the only indecomposable of $\text{proj-}\Pi^{\lambda_1}(\mathbb{A}_1) \simeq \mathbf{FVect}_\Bbbk$ is $\Bbbk$, and $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)}(\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)e_0, \Pi^\lambda_J(\widetilde{\mathbb{A}}_1)e_0) \cong \Bbbk$. Finally, this is a triangle equivalence because, by [Che11, Lemma 3.4], $\mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1})$ has a unique triangulated structure since the only $\Bbbk$-linear autoequivalence of $\mathbf{FVect}_\Bbbk$ is the identity. ☐

In particular, Theorem 8.3.1 shows that $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$ has an $\mathbb{A}_1$ singularity when $\lambda_1 = 0$. Using this theorem, we are able to determine the global dimensions of $\mathcal{S}^\lambda_J(\widetilde{\mathbb{A}}_1)$ and $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$:

Theorem 8.3.2. *Suppose that λ is a weight for $\widetilde{\mathbb{A}}_1$. Then*

$$\mathrm{gl.dim}\,\mathcal{S}^\lambda_J(\widetilde{\mathbb{A}}_1) = 2 \quad \textit{and} \quad \mathrm{gl.dim}\,\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) = \begin{cases} \infty & \text{if } \lambda_1 = 0 \\ 2 & \text{if } \lambda_1 \neq 0 \end{cases}.$$

Proof. We first show that $\mathcal{S}^\lambda_J(\widetilde{\mathbb{A}}_1)$ always has global dimension 2. By Proposition 6.1.5, it suffices to show that it always has finite-dimensional modules, and by Theorem 7.2.1, we only need to show that this is the case for $\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)$. But it is easy to check that $\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)$ (shown below on the left with its relations) has the finite-dimensional representation given below on the right,



$$\alpha_0\overline{\alpha}_0 - \overline{\alpha}_1\alpha_1 - \alpha_0\alpha_1 = \lambda_0 e_0$$
$$\alpha_1\overline{\alpha}_1 - \overline{\alpha}_0\alpha_0 - \alpha_1\alpha_0 = \lambda_1 e_1$$

and so $\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)$ has finite-dimensional modules.

We now seek to determine the global dimension of $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$. If $\lambda_1 = 0$, then Theorem 8.3.1 implies that $\operatorname{gl.dim} \mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1) = \infty$. So instead suppose that $\lambda_1 \neq 0$. By Theorem 8.3.1, $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$ has finite global dimension (in particular, this value is either 1 or 2 by Proposition 6.1.6). As in the proof of Corollary 8.2.8, $\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)$ and $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$ are Morita equivalent. But the previous paragraph shows that $\Pi^\lambda_J(\widetilde{\mathbb{A}}_1)$ always has global dimension 2, and hence the same is true of $\mathcal{O}^\lambda_J(\widetilde{\mathbb{A}}_1)$ when $\lambda_1 \neq 0$. □

8.4 Singularities in case (iii), m even

For case (iii) when m is even, we are interested in deformations of the singularity arising from D_m acting on $\Bbbk_{-1}[u, v]$. Writing $n = \frac{m+4}{2}$, Theorem 7.2.1 allows us to view our deformations as having the form $\mathcal{O}^\lambda_{-1}(\widetilde{\mathbb{D}}_n) = e_0\Pi^\lambda_{-1}(\widetilde{\mathbb{D}}_n)e_0$ for some weight λ. In fact, since $\widetilde{\mathbb{D}}_n$ is a tree, Lemma 7.1.2 implies that $\Pi^\lambda_{-1}(\widetilde{\mathbb{D}}_n)$ is isomorphic to $\Pi^{\lambda'}(\widetilde{\mathbb{D}}_n)$ for some weight λ', and hence $\mathcal{O}^\lambda_{-1}(\widetilde{\mathbb{D}}_n) \cong \mathcal{O}^{\lambda'}(\widetilde{\mathbb{D}}_n)$. The global dimension of $\Pi^{\lambda'}(\widetilde{\mathbb{D}}_n)$ and $\mathcal{O}^{\lambda'}(\widetilde{\mathbb{D}}_n)$ can then be determined from the results of [CBH98].

We now consider the singularity category of $\mathcal{O}^{\lambda'}(\widetilde{\mathbb{D}}_n)$. By the results of Section 3.1, there exists a quasi-dominant weight λ'' such that $\mathcal{O}^{\lambda'}(\widetilde{\mathbb{D}}_n) \cong \mathcal{O}^{\lambda''}(\widetilde{\mathbb{D}}_n)$. A description of the singularity category of this deformation then follows from Theorem 3.3.11.

8.5 Singularities in case (ii)

We now turn our attention to deformations of $\Bbbk_{-1}[u, v]^{S_2}$. We will see that this is the first example which exhibits new behaviour, in the sense that we obtain singularity categories which are distinct from those we have seen so far. We will also see that deforming these singularities never makes them nonsingular, as was the case for deformations of Kleinian singularities. In particular, this means that $\mathcal{O}^\lambda_{-1}(\widetilde{\mathbb{L}}_1)$ and $\mathcal{S}^\lambda_{-1}(\widetilde{\mathbb{L}}_1)$ are never Morita equivalent, since the latter always has global dimension at most 2.

To this end, suppose $\lambda = (\lambda_0, \lambda_1)$ is a weight for $\widetilde{\mathbb{L}}_1$ and write $\mu = (\lambda_1)$. We can then view our deformation as having the form $\mathcal{O}^\lambda_{-1}(\widetilde{\mathbb{L}}_1) = e_0\Pi^\lambda_{-1}(\widetilde{\mathbb{L}}_1)e_0$. Theorem 8.1.5 tells us that there is a fully faithful functor

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda_{-1}(\widetilde{\mathbb{L}}_1) \to \text{proj-}\Pi^\mu_{-1}(\mathbb{L}_1).$$

Now $\Pi^\mu_{-1}(\mathbb{L}_1)$ has a single vertex with a loop ε_1, and the only relation is $\varepsilon_1^2 = \lambda_1 e_1$. It follows that

$$\Pi^\mu_{-1}(\mathbb{L}_1) \cong \begin{cases} \Bbbk[x]/\langle x^2 \rangle & \text{if } \lambda_1 = 0 \\ \Bbbk \times \Bbbk & \text{if } \lambda_1 \neq 0 \end{cases}.$$

If we combine these observations then we are able to determine the global dimension of $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$, and then use this information to determine the global dimension of $\mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$:

Theorem 8.5.1. *Suppose that λ is a weight for $\widetilde{\mathbb{L}}_1$. Then*

$$\operatorname{gl.dim} \mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1) = 2 \quad \textit{and} \quad \operatorname{gl.dim} \mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1) = \infty.$$

Proof. Under the fully faithful functor $\underline{\operatorname{Hom}}_{\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)}(\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)e_0, -)$, the object $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)e_0$ is sent to $\Pi^{\mu}_{-1}(\mathbb{L}_1) \neq 0$, so the category $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ is nontrivial. This implies that $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ always has infinite global dimension.

We now show that $\mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ always has global dimension 2, and as in the proof of Theorem 8.3.2, it suffices to show that $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ always has finite-dimensional modules. But this is clear since $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)/\langle e_0 \rangle$ is always two-dimensional. □

This result contrasts with those [CBH98], which shows that deformations of Kleinian singularities are generically nonsingular and that the corresponding deformations of their skew group rings are generically hereditary. While $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ is never nonsingular, the fact that proj-$\mathbb{k} \times \mathbb{k}$ (obtained when $\lambda_1 \neq 0$) is homologically nicer than proj-$\mathbb{k}[x]/\langle x^2 \rangle$ provides some evidence towards λ acting as a "smoothing parameter".

8.5.1 Singularities when $\lambda_1 = 0$

In the cases where $\lambda_1 = 0$, we are able to give a more precise description of the singularity category.

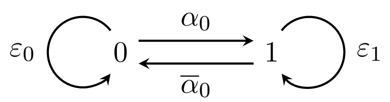
Theorem 8.5.2. *Let λ be a weight for $\widetilde{\mathbb{L}}_1$ with $\lambda_1 = 0$. Then there is a triangle equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1) \simeq \text{proj-}\Pi_{-1}(\mathbb{L}_1).$$

The induced translation functor on proj-$\Pi_{-1}(\mathbb{L}_1)$ *is the identity on objects, and satisfies $\Sigma\varepsilon_1 = -\varepsilon_1$. In particular, if λ and λ' are both weights with $\lambda_1 = 0 = \lambda'_1$, then there is a triangle equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1) \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda'}_{-1}(\widetilde{\mathbb{L}}_1).$$

Proof. We recall that in this setting, $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ is the following path algebra with relations:



$$\varepsilon_0^2 + \alpha_0\overline{\alpha}_0 = \lambda_0 e_0, \quad \varepsilon_1^2 + \overline{\alpha}_0\alpha_0 = 0.$$

By Theorem 8.1.5, the functor

$$\underline{\operatorname{Hom}}_{\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)}(\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)e_0, -) : \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1) \to \text{proj-}\Pi_{-1}(\mathbb{L}_1),$$

is fully faithful. Now proj-$\Pi_{-1}(\mathbb{L}_1)$ is Krull-Schmidt and has only a single indecomposable, namely $P_1 := \Pi_{-1}(\mathbb{L}_1)$ itself, which is the image of $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)e_0$ under the above functor, and so we in fact have a $\mathbb{k}$-linear equivalence. In particular, this induces an algebraic triangulated structure on proj-$\Pi_{-1}(\mathbb{L}_1)$, and Theorem 3.3.2 implies that such a triangulated structure is unique. This immediately implies the final claim.

Write Σ for the translation functor on proj-$\Pi_{-1}(\mathbb{L}_1)$, which is necessarily the identity on objects. To determine the translation of $\varepsilon_1 : P_1 \to P_1$, it suffices to determine this for $\varepsilon_1 : V_1 \to V_1$ viewed as a morphism in $\underline{\mathrm{MCM}}$-$\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$. To do this, we must first identify a short exact sequence of $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$-modules ending with V_1 whose middle term is projective. To do this, observe that we have a complex of $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$-modules

$$0 \to V_1 \xrightarrow{\begin{pmatrix} \alpha_0\varepsilon_1 \\ -\alpha_0 \end{pmatrix}} V_0 \oplus V_0 \xrightarrow{\begin{pmatrix} \overline{\alpha}_0 & \varepsilon_1\overline{\alpha}_0 \end{pmatrix}} V_1 \to 0. \tag{8.5.3}$$

Filtering by path length, we can consider the associated graded complex of $e_0\Pi_{-1}(\widetilde{\mathbb{L}}_1)e_0$-modules

$$0 \to V_1[-3] \xrightarrow{\begin{pmatrix} \alpha_0\varepsilon_1 \\ -\alpha_0 \end{pmatrix}} V_0[-1] \oplus V_0[-2] \xrightarrow{\begin{pmatrix} \overline{\alpha}_0 & \varepsilon_1\overline{\alpha}_0 \end{pmatrix}} V_1 \to 0, \tag{8.5.4}$$

and we claim that this sequence is in fact exact. Since any path from vertex 1 to vertex 0 in $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ begins with either $\overline{\alpha}_0$ or $\varepsilon_1\overline{\alpha}_0$, the right hand map in (8.5.4) is surjective. Similarly, the map

$$\begin{pmatrix} \alpha_0\varepsilon_1 & -\alpha_0 \end{pmatrix} : e_0\Pi_{-1}(\widetilde{\mathbb{L}}_1)e_0 \oplus e_0\Pi_{-1}(\widetilde{\mathbb{L}}_1)e_0 \to e_0\Pi_{-1}(\widetilde{\mathbb{L}}_1)e_1$$

given by multiplication on the right is surjective. Dualising and applying Corollary 8.1.2 shows that the left hand map in (8.5.4) is injective. To establish exactness, it therefore suffices to show that the alternating sum of Hilbert series in (8.5.4) is equal to 0. Using [MOV06, Theorem 2.3.b], one calculates that V_0 and V_1 have Hilbert series

$$\operatorname{hilb} V_0 = \frac{t^2 - t + 1}{(1+t^2)(1-t)^2} \qquad \text{and} \qquad \operatorname{hilb} V_1 = \frac{t}{(1+t^2)(1-t)^2}.$$

It follows that the alternating sum of the Hilbert series in the complex (8.5.4) is

$$\frac{t(1+t^3)}{(1+t^2)(1-t)^2} - \frac{(t+t^2)(t^2-t+1)}{(1+t^2)(1-t)^2} = 0,$$

as required. Finally, since (8.5.4) is exact, [MR01, Proposition 1.6.7] implies that the sequence of $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$-modules in (8.5.3) is also exact. This also tells us that $\Sigma V_1 = V_1$, which is in line with our earlier observation.

We are now able to determine $\Sigma\varepsilon_1$. It is easy to check that the following diagram of $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$-modules, the rows of which are exact, commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \xrightarrow{\left(\begin{smallmatrix} \alpha_0\varepsilon_1 \\ -\alpha_0 \end{smallmatrix}\right)} & V_0 \oplus V_0 & \xrightarrow{\left(\begin{smallmatrix} \overline{\alpha}_0 & \varepsilon_1\overline{\alpha}_0 \end{smallmatrix}\right)} & V_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\varepsilon_1} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & -\alpha_0\overline{\alpha}_0 \\ e_0 & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \varepsilon_1} & & \\
0 & \longrightarrow & V_1 & \xrightarrow[\left(\begin{smallmatrix} \alpha_0\varepsilon_1 \\ -\alpha_0 \end{smallmatrix}\right)]{} & V_0 \oplus V_0 & \xrightarrow[\left(\begin{smallmatrix} \overline{\alpha}_0 & \varepsilon_1\overline{\alpha}_0 \end{smallmatrix}\right)]{} & V_1 & \longrightarrow & 0
\end{array}$$

By the definition of Σ on morphisms, it follows that $\Sigma\varepsilon_1 = -\varepsilon_1$. $\square$

In [ES08], the authors show that the singularity category of a simple curve singularity of type $\mathbb{A}_2$, namely $\Bbbk[\![x,y]\!]/\langle x^2+y^3\rangle$, is $\Bbbk$-linearly equivalent to proj-$\Pi(\mathbb{L}_1)$. By Theorem 3.3.2 again, we therefore have a triangle equivalence

$$\underline{\mathrm{MCM}}\text{-}\Bbbk[\![x,y]\!]/\langle x^2+y^3\rangle \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$$

for any weight λ with $\lambda_1 = 0$.

We have been unable to give a more precise description of the singularity category when λ_1 is nonzero. In this case, the category proj-$\Pi^{\mu}_{-1}(\mathbb{L}_1)$ has two indecomposables. We have been unable to identify indecomposables of $\underline{\mathrm{MCM}}$-$\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{L}}_1)$ which are mapped to these indecomposables under the functor from Theorem 8.1.5.

8.6 Singularities in case (iii), m odd

Finally, we consider deformations of $\Bbbk_{-1}[u,v]^{D_m}$ and $\Bbbk_{-1}[u,v]\#D_m$ when m is odd. We will see that these behave in a similar way to the singularities in case (ii). This similar behaviour is related to the common feature linking these deformations: their dependence on the data of an affine version of an $\mathbb{L}_n$ quiver.

As has been the case throughout this chapter, we view our deformation of $\Bbbk_{-1}[u,v]^{D_m}$ as having the form $\mathcal{O}^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n) = e_0\Pi^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n)e_0$. For notational convenience, we have the following lemma:

Lemma 8.6.1. *If λ is a weight for $\widetilde{\mathbb{DL}}_n$, then there exists a weight λ' such that $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n) \cong \Pi^{\lambda'}(\widetilde{\mathbb{DL}}_n)$, where $\lambda'_i = 0$ if and only if $\lambda_i = 0$.*

Proof. By applying the procedure outlined in the proof of Lemma 7.1.2 to loop-free vertices and borrowing the notation, we can construct an isomorphism $\Pi^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n) \cong \Pi^{\lambda''}_{\boldsymbol{q}}(\widetilde{\mathbb{DL}}_n)$ where

$$\boldsymbol{q}(i) = \begin{cases} 1 & \text{if } i \neq n \\ -1 & \text{if } i = n \end{cases} .$$

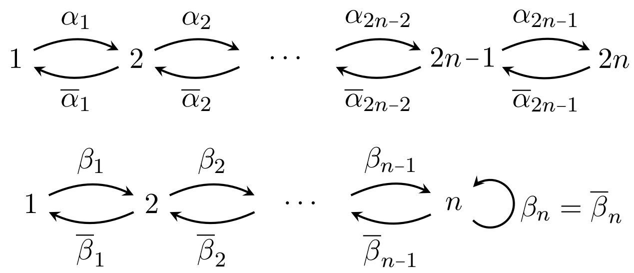
Setting $\lambda'_n = -\lambda''_n$ and $\lambda'_i = \lambda'_i$ for $i \neq n$, it is then easy to check that the map $\Pi^{\lambda''}_{\boldsymbol{q}}(\widetilde{\mathbb{DL}}_n) \to \Pi^{\lambda'}(\widetilde{\mathbb{DL}}_n)$ sending $\varepsilon \mapsto \sqrt{-1}\varepsilon$ and fixing every other arrow is an isomorphism. □

Without loss of generality, we therefore instead work with the algebras $\Pi^{\lambda}(\widetilde{\mathbb{DL}}_n)$ and $\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n) = e_0\Pi^{\lambda}(\widetilde{\mathbb{DL}}_n)e_0$ for some weight λ for the rest of this section.

We begin by showing that deformed preprojective algebras of $\mathbb{L}_n$ quivers are always nonzero, which is in contrast with the Dynkin case, cf. [CBH98, Lemma 7.1 (1)].

Proposition 8.6.2. *Let $\Pi^{\lambda}(\mathbb{L}_n)$ be a deformed preprojective algebra of type $\mathbb{L}$. Then $\dim \Pi^{\lambda}(\mathbb{L}_n) = \dim \Pi(\mathbb{L}_n) = \frac{1}{3}n(n+1)(2n+1)$; in particular, $\Pi^{\lambda}(\mathbb{L}_n)$ is nonzero.*

Proof. For this proof, label the arrows of the quivers $\overline{\mathbb{A}}_{2n}$ and $\overline{\mathbb{L}}_n$ as follows,



and write e_i and f_i for the vertex idempotents in $\Bbbk\overline{\mathbb{A}}_{2n}$ and $\Bbbk\overline{\mathbb{L}}_n$ respectively. Define a linear map $\phi : \Bbbk\overline{\mathbb{A}}_{2n} \to \Bbbk\overline{\mathbb{L}}_n$ (which we emphasise is not an algebra homomorphism) on a basis of paths as follows: on vertices and arrows it takes values

$$\begin{aligned} e_i &\mapsto f_i, & e_{n+i} &\mapsto f_{n+1-i}, & 1 \leqslant i \leqslant n, \\ \alpha_i &\mapsto \beta_i, & \overline{\alpha}_i &\mapsto \overline{\beta}_i, & 1 \leqslant i \leqslant n \\ \alpha_{n+i} &\mapsto \overline{\beta}_{n-i}, & \overline{\alpha}_{n+i} &\mapsto \beta_{n-i}, & 1 \leqslant i \leqslant n-1, \end{aligned}$$

and on longer paths we extend the above rule multiplicatively. (Graphically, we can think of ϕ as "folding" $\overline{\mathbb{A}}_{2n}$ to obtain $\overline{\mathbb{L}}_n$.) In particular, ϕ is a surjective map such that every basis element of $\Bbbk\overline{\mathbb{L}}_n$ has exactly two preimages under ϕ. Let U_1 be the space of paths in $\Bbbk\overline{\mathbb{A}}_{2n}$ beginning at vertices $1, \dots, n$ and let U_2 be the space of paths starting at vertices $n+1, \dots, 2n$. We can then think of ϕ as a map $U_1 \oplus U_2 \to \Bbbk\overline{\mathbb{L}}_n$ such that, for $i = 1, 2$, $\phi|_{U_i}$ is a vector space isomorphism. Let W be the subspace of $\Bbbk\overline{\mathbb{A}}_{2n}$ spanned by elements of the form

$$\begin{aligned} p(\alpha_i\overline{\alpha}_i - \overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i)q, \qquad 1 \leqslant i \leqslant n \\ p(\alpha_{n+i}\overline{\alpha}_{n+i} - \overline{\alpha}_{n+i-1}\alpha_{n+i-1} + \lambda_{n+1-i}e_{n+i})q, \qquad 1 \leqslant i \leqslant n \end{aligned}$$

where p, q are arbitrary paths and where $\alpha_0 = \overline{\alpha}_0 = \alpha_{2n} = \overline{\alpha}_{2n} := 0$. As vector spaces, we have an isomorphism $\Bbbk\overline{\mathbb{A}}_{2n}/W \cong \Pi^\mu(\mathbb{A}_{2n})$, where $\mu = (\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1)$, and also $\Bbbk\overline{\mathbb{A}}_{2n}/W \cong U_1/(W \cap U_1) \oplus U_2/(W \cap U_2)$. Now, ϕ gives rise to a linear map

$$\widetilde{\phi} : \frac{U_1}{W \cap U_1} \oplus \frac{U_2}{W \cap U_2} \to \frac{\Bbbk\overline{\mathbb{L}}_n}{\phi(W)}$$

which is an isomorphism when restricted to either summand. Writing $\beta_0 = \overline{\beta}_0 := 0$, observe that for $1 \leqslant i \leqslant n$ we have

$$\begin{aligned} \phi(\alpha_i\overline{\alpha}_i - \overline{\alpha}_{i-1}\alpha_{i-1} - \lambda_i e_i) &= \beta_i\overline{\beta}_i - \overline{\beta}_{i-1}\beta_{i-1} - \lambda_i f_i \\ \phi(\alpha_{n+i}\overline{\alpha}_{n+i} - \overline{\alpha}_{n+i-1}\alpha_{n+i-1} + \lambda_{n+1-i}e_{n+i}) &= \overline{\beta}_{n-i}\beta_{n-i} - \beta_{n+1-i}\overline{\beta}_{n+1-i} \\ &\qquad + \lambda_{n+1-i}f_{n+1-i} \\ &= -(\beta_j\overline{\beta}_j - \overline{\beta}_{j-1}\beta_{j-1} - \lambda_j f_j) \end{aligned}$$

where $j = n+1-i$, and so $\Bbbk\overline{\mathbb{L}}_n/\phi(W) \cong \Pi^\lambda(\mathbb{L}_n)$. It follows that

$$2\dim_\Bbbk \Pi^\lambda(\mathbb{L}_n) = \dim_\Bbbk \Pi^\mu(\mathbb{A}_{2n}).$$

Finally, by [EE07, Proposition 5.0.2],

$$\dim_\Bbbk \Pi^\mu(\mathbb{A}_{2n}) = \dim_\Bbbk \Pi(\mathbb{A}_{2n}) = \frac{1}{6}(2n)(2n+1)(2n+2) = \frac{2}{3}n(n+1)(2n+1),$$

and therefore $\dim_{\Bbbk} \Pi^{\lambda}(\mathbb{L}_n) = \frac{1}{3}n(n+1)(2n+1)$. $\square$

We can now deduce that the deformations $\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)$ are always singular, as well as determine the global dimensions of the algebras $\mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n)$:

Theorem 8.6.3. *Suppose that λ is a weight for $\widetilde{\mathbb{DL}}_n$. Then*

$$\operatorname{gl.dim} \mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n) = 2 \quad \textit{and} \quad \operatorname{gl.dim} \mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n) = \infty.$$

Proof. We begin with $\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)$. Writing $\mu = (\lambda_1, \ldots, \lambda_n)$, Theorem 8.1.5 tells us that there is a fully faithful functor

$$\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n) \to \text{proj-}\Pi^{\mu}(\mathbb{L}_n).$$

Notice that $\Pi^{\mu}(\mathbb{L}_n)$ nonzero by Proposition 8.6.2. Therefore under the fully faithful functor $\underline{\text{Hom}}_{\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)}(\Pi^{\lambda}(\widetilde{\mathbb{DL}}_n)e_0, -)$, the object $\Pi(\widetilde{\mathbb{DL}}_n)e_0$ is sent to $\Pi^{\mu}(\mathbb{L}_n) \neq 0$, so the category $\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)$ is nontrivial. This implies that $\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)$ has infinite global dimension.

To show that $\mathcal{S}^{\lambda}_{-1}(\widetilde{\mathbb{DL}}_n)$ always has global dimension 2, it suffices to show that $\Pi^{\lambda}(\widetilde{\mathbb{DL}}_n)$ has finite-dimensional modules. As in Theorem 8.5.1, this follows from the fact that $\Pi^{\lambda}(\widetilde{\mathbb{DL}}_n)/\langle e_0 \rangle \cong \Pi^{\mu}(\mathbb{L}_n)$ is nonzero and finite-dimensional. $\square$

As with case (ii), this behaviour contrasts with that of deformations of classical Kleinian singularities, since deformations of the invariant rings and skew group rings in the classical setting are generically Morita equivalent and hereditary.

8.6.1 Singularities when $\lambda_i = 0$ for all $i \neq 0$

As with case (ii), when $\lambda_i = 0$ for $i \neq 0$, we can give a more precise description of the singularity category of $\mathcal{O}^{\lambda}(\widetilde{\mathbb{DL}}_n)$. We first need some preparatory results.

Lemma 8.6.4. *Grading $\Pi(\widetilde{\mathbb{DL}}_n)$ by path length, the Hilbert series of the vertex modules $V_i = e_i\Pi(\widetilde{\mathbb{DL}}_n)e_0$ are given by*

$$\begin{aligned}
\operatorname{hilb} V_0 &= \frac{1 + t^{2n+1}}{(1 - t^4)(1 - t^{2n+1})} \\
\operatorname{hilb} V_1 &= \frac{t^2(1 + t^{2n-3})}{(1 - t^4)(1 - t^{2n-1})} \\
\operatorname{hilb} V_i &= \frac{t^{i-1}(1 + t^{2(n-i)+1})}{(1 - t^2)(1 - t^{2n-1})} \qquad \textit{for } 2 \leqslant i \leqslant n
\end{aligned}$$

Proof. By [MOV06, Theorem 2.3.b], if we write A for the adjacency matrix of $\widetilde{\mathbb{DL}}_n$, the matrix Hilbert series of $\Pi(\widetilde{\mathbb{DL}}_n)$ is given by $H(t) = \big((1+t^2)I - tA\big)^{-1}$. In particular, $\operatorname{hilb} V_i$ is the $(i,0)$th entry of this matrix. To verify that the Hilbert series are as claimed, if we write h_i for the claimed Hilbert series of V_i then it suffices to check that

$$\begin{pmatrix} h_0 & h_1 & \cdots & h_n \end{pmatrix} C_0 = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}^{\mathrm{T}},$$

where C_0 is the 0th column of $\big((1+t^2)I - tA\big)$. This amounts to verifying the equalities

$$\begin{aligned}
(1+t^2)h_0 - th_2 &= 1,\\
(1+t^2)h_1 - th_2 &= 0,\\
(1+t^2)h_2 - t(h_0+h_1+h_3) &= 0,\\
(1+t^2)h_i - t(h_{i-1}+h_{i+1}) &= 0 \qquad \text{for } 3 \leqslant i \leqslant n-1,\\
(1+t^2)h_n - th_{n-1} &= 0.
\end{aligned}$$

This is entirely routine, so we verify only the third equality:

$$\begin{aligned}
&(1+t^2)h_2 - t(h_0+h_1+h_3)\\
&= \frac{t(1+t^2)(1+t^{2n-3})}{(1-t^2)(1-t^{2n-1})} - t\left(\frac{1+t^{2n+1}}{(1-t^4)(1-t^{2n-1})} + \frac{t^2(1+t^{2n-3})}{(1-t^4)(1-t^{2n-1})} + \frac{t^2(1+t^{2n-5})}{(1-t^2)(1-t^{2n-1})}\right)\\
&= \frac{t(1+t^2)^2(1+t^{2n-3}) - t(1+t^{2n+1}) - t^3(1+t^{2n-3}) - t^3(1+t^2)(1+t^{2n-5})}{(1-t^4)(1-t^{2n-1})}\\
&= \frac{(t+2t^3+t^5)(1+t^{2n-3}) - t(1+t^{2n+1}) - t^3(1+t^{2n-3}) - t^3(1+t^2+t^{2n-5}+t^{2n-3})}{(1-t^4)(1-t^{2n-1})}\\
&= \frac{t+t^{2n-2}+2t^3+2t^{2n}+t^5+t^{2n+2}-t-t^{2n+2}-t^3-t^{2n}-t^3-t^5-t^{2n-2}-t^{2n}}{(1-t^4)(1-t^{2n-1})}\\
&= 0.
\end{aligned}$$

The other calculations are similar, and so the Hilbert series are as claimed. □

Lemma 8.6.5. *Write p for the path $\alpha_2\alpha_3\ldots\alpha_{n-1}$ in $\widetilde{\mathbb{DL}}_n$, and $\overline{p}$ for its reverse. Then*

$$V_n = \overline{p}\overline{\alpha}_0V_0 \oplus \varepsilon\overline{p}\overline{\alpha}_0V_0 \oplus \varepsilon^2\overline{p}\overline{\alpha}_0V_0 \oplus \varepsilon^3\overline{p}\overline{\alpha}_0V_0. \tag{8.6.6}$$

Proof. We need to show that the map $V_0^4 \to V_n$ given by left multiplication by

$$\begin{pmatrix}\overline{p}\overline{\alpha}_0 & \varepsilon\overline{p}\overline{\alpha}_0 & \varepsilon^2\overline{p}\overline{\alpha}_0 & \varepsilon^3\overline{p}\overline{\alpha}_0\end{pmatrix}$$

is surjective and, by [MR01, Corollary 6.14], it suffices to show this for $\operatorname{gr}\Pi^\lambda(\widetilde{\mathbb{DL}}_n) = \Pi(\widetilde{\mathbb{DL}}_n)$. Consider any path γ from vertex n to vertex 0 in $\widetilde{\mathbb{DL}}_n$, where it suffices to assume that γ visits vertex 0 only at its end. We need to show that γ can be written as an element of the right hand side of (8.6.6). By using the relation at vertex 2, we can assume that γ does not visit vertex 1. Furthermore, if γ has a subpath from vertex n to itself which does not pass through vertex 2 (respectively, a subpath from vertex 2 to itself which does not pass through vertex n), by using the relations $\alpha_i\overline{\alpha}_i = \overline{\alpha}_{i-1}\alpha_{i-1}$ $(3 \leqslant i \leqslant n-1)$ and $\varepsilon^2 = \overline{\alpha}_{n-1}\alpha_{n-1}$ we can assume that this subpath is given by ε^i for some i (respectively, $(\alpha_2\overline{\alpha}_2)^j$ for some j). We can therefore assume that γ has the form

$$\gamma = \varepsilon^{i_1}\overline{p}(\alpha_2\overline{\alpha}_2)^{j_1}p\varepsilon^{i_2}\overline{p}(\alpha_2\overline{\alpha}_2)^{j_2}p\ldots\varepsilon^{i_k}\overline{p}(\alpha_2\overline{\alpha}_2)^{j_k}\overline{\alpha}_0.$$

Using the easily-established identities $\overline{p}\alpha_2\overline{\alpha}_2 = \overline{\alpha}_{n-1}\alpha_{n-1}\overline{p}$ and $\overline{p}p = \varepsilon^{n-2}$ repeatedly, $\gamma = \varepsilon^i\overline{p}\overline{\alpha}_0$ in $\Pi(\widetilde{\mathbb{DL}}_n)$. If $i \leqslant 3$ then we're done, so suppose that $i \geqslant 4$. Then, noting that $\alpha_0\overline{\alpha}_0 = 0 = \alpha_1\overline{\alpha}_1$ in $\Pi(\widetilde{\mathbb{DL}}_n)$,

$$\gamma = \varepsilon^{i-4}\varepsilon^4\overline{p}\overline{\alpha}_0 = \varepsilon^{i-4}\overline{p}\alpha_2\overline{\alpha}_2\alpha_2\overline{\alpha}_2\overline{\alpha}_0 = \varepsilon^{i-4}\overline{p}\alpha_2\overline{\alpha}_2\alpha_1\overline{\alpha}_1\overline{\alpha}_0 = \varepsilon^{i-4}\overline{p}\alpha_0\overline{\alpha}_0\alpha_1\overline{\alpha}_1\overline{\alpha}_0,$$

which lies in the right hand side of (8.6.6). The result then follows by induction on $i \geqslant 4$. □

Theorem 8.6.7. *Let λ be a weight for $\widetilde{\mathbb{L}}_n$ with $\lambda_i = 0$ for $i \neq 0$. Then there is a triangle equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n}) \simeq \text{proj-}\Pi(\mathbb{L}_n).$$

The induced translation functor on proj-$\Pi(\mathbb{L}_n)$ *is the identity on objects, and satisfies $\Sigma\varepsilon = -\varepsilon$. In particular, if λ and λ' are both weights with $\lambda_i = 0 = \lambda'_i$ for all $i \neq 0$, then there is a triangle equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n}) \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda'}(\widetilde{\mathbb{DL}_n}).$$

Proof. By Theorem 8.1.5, the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})}(\Pi^\lambda(\widetilde{\mathbb{L}}_n)e_0, -) : \underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{\mathbb{L}}_n) \to \text{proj-}\Pi(\mathbb{L}_n),$$

is fully faithful. Now proj-$\Pi(\mathbb{L}_n)$ is Krull-Schmidt and has n indecomposables $P_i = e_i\Pi(\mathbb{L}_n)$. As in Theorem 3.2.6, we have

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})}(\Pi^\lambda(\widetilde{\mathbb{L}}_n)e_0, V_i) \cong P_i,$$

and so this functor is essentially surjective and hence a $\Bbbk$-linear equivalence. As in Theorem 8.5.2, this induces an algebraic triangulated structure on proj-$\Pi(\mathbb{L}_n)$, which is unique by Theorem 3.3.2. The final claim then follows.

Write Σ for the translation functor on proj-$\Pi(\mathbb{L}_n)$. By Lemma 3.4.3, Σ must induce a graph automorphism of the vertices of $\mathbb{L}_n$, and hence it is the identity on the objects P_i. To determine $\Sigma\varepsilon : P_n \to P_n$, it suffices to determine the translation of the corresponding morphism $\varepsilon : V_n \to V_n$ in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})$.

We first determine a short exact sequence of $\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})$-modules ending with V_n and whose middle term is projective. (This will also determine ΣV_n, which we already know to be V_n.) Adopting the notation from Lemma 8.6.5, we claim that the sequence

$$0 \to V_n \xrightarrow{\begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix}} V_0^4 \xrightarrow{\begin{pmatrix} \overline{p}\overline{\alpha}_0 & \varepsilon\overline{p}\overline{\alpha}_0 & \varepsilon^2\overline{p}\overline{\alpha}_0 & \varepsilon^3\overline{p}\overline{\alpha}_0 \end{pmatrix}} V_n \to 0 \tag{8.6.8}$$

is exact. The following calculation confirms that it is a complex, where we note that $\overline{\alpha}_{n-1}\alpha_{n-1}\overline{p} = \overline{p}\alpha_2\overline{\alpha}_2$ and use the fact that $\alpha_1\overline{\alpha}_1 = 0$:

$$\begin{aligned}
\overline{p}\overline{\alpha}_0 \cdot \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon + \varepsilon^2\overline{p}\overline{\alpha}_0 \cdot \alpha_0 p\varepsilon &= \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_1\alpha_1 p\varepsilon + \overline{\alpha}_{n-1}\alpha_{n-1}\overline{p}\overline{\alpha}_0\alpha_0 p\varepsilon \\
&= \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_1\alpha_1 p\varepsilon + \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_0\alpha_0 p\varepsilon \\
&= \overline{p}\alpha_2\overline{\alpha}_2\alpha_2\overline{\alpha}_2 p\varepsilon \\
&= \varepsilon^{2n+1} \\
&= \varepsilon\overline{p}\alpha_2\overline{\alpha}_2\alpha_2\overline{\alpha}_2 p \\
&= \varepsilon\overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_1\alpha_1 p + \varepsilon\overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_0\alpha_0 p \\
&= \varepsilon\overline{p}\overline{\alpha}_0 \cdot \alpha_0\overline{\alpha}_1\alpha_1 p + \varepsilon\overline{\alpha}_{n-1}\alpha_{n-1}\overline{p}\overline{\alpha}_0\alpha_0 p \\
&= \varepsilon\overline{p}\overline{\alpha}_0 \cdot \alpha_0\overline{\alpha}_1\alpha_1 p + \varepsilon^3\overline{p}\overline{\alpha}_0 \cdot \alpha_0 p
\end{aligned}$$

Filtering by path length, we can consider the associated graded complex of $\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})$-

modules

$$0 \to V_n[-(2n+1)] \xrightarrow{\begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix}} V_0[-(n-1)] \oplus V_0[-n] \oplus V_0[-(n+1)] \oplus V_0[-(n+2)]$$

$$\xrightarrow{\begin{pmatrix} \overline{p}\overline{\alpha}_0 & \varepsilon\overline{p}\overline{\alpha}_0 & \varepsilon^2\overline{p}\overline{\alpha}_0 & \varepsilon^3\overline{p}\overline{\alpha}_0 \end{pmatrix}} V_n \to 0 \qquad (8.6.9)$$

and we claim that this sequence is in fact exact. By Lemma 8.6.5, the right hand map is surjective. Similarly, one can show that the map

$$\begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon & -\alpha_0\overline{\alpha}_1\alpha_1 p & \alpha_0 p\varepsilon & -\alpha_0 p \end{pmatrix} : (e_0\Pi(\widetilde{\mathbb{DL}}_n)e_0)^4 \to e_0\Pi(\widetilde{\mathbb{DL}}_n)e_n$$

given by right multiplication is surjective. Dualising and applying Corollary 8.1.2 shows that the left hand map in (8.6.9) is injective. To establish exactness, it therefore suffices to show that the alternating sum of the Hilbert series of the modules in this complex is equal to 0. By direct calculation:

$$\begin{aligned}
(1+t^{2n+1})\,\mathrm{hilb}V_n - (t^{n-1}+t^n+t^{n+1}+t^{n+2})\,\mathrm{hilb}\,V_0 = \\
\frac{t^{n-1}(1+t)(1+t^{2n+1})}{(1-t^2)(1-t^{2n-1})} - \frac{(1+t^{2n+1})(t^{n-1}+t^n+t^{n+1}+t^{n+2})}{(1-t^4)(1-t^{2n+1})} \\
= \frac{(1+t^{2n+1})\big(t^{n-1}(1+t)(1+t^2) - (t^{n-1}+t^n+t^{n+1}+t^{n+2})\big)}{(1-t^4)(1-t^{2n+1})} \\
= \frac{(1+t^{2n+1})\big((t^{n-1}+t^n+t^{n+1}+t^{n+2}) - (t^{n-1}+t^n+t^{n+1}+t^{n+2})\big)}{(1-t^4)(1-t^{2n+1})} \\
= 0,
\end{aligned}$$

and so (8.6.9) is exact. Then [MR01, Proposition 1.6.7] implies that the sequence of $\mathcal{O}^\lambda(\widetilde{\mathbb{DL}}_n)$-modules in (8.6.8) is also exact. We also see that $\Sigma V_n = V_n$, as expected.

We are finally able to determine $\Sigma\varepsilon$. We claim that the following diagram of $\mathcal{O}^\lambda(\widetilde{\mathbb{DL}}_n)$-modules, the rows of which are exact, commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_n & \xrightarrow{\begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix}} & V_0^4 & \xrightarrow{\begin{pmatrix} \overline{p}\overline{\alpha}_0 \\ \varepsilon\overline{p}\overline{\alpha}_0 \\ \varepsilon^2\overline{p}\overline{\alpha}_0 \\ \varepsilon^3\overline{p}\overline{\alpha}_0 \end{pmatrix}^{\mathrm{T}}} & V_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle -\varepsilon} & & \Big\downarrow{\scriptstyle \begin{pmatrix} 0 & 0 & 0 & \alpha_0\overline{\alpha}_1\alpha_1\overline{\alpha}_0 \\ e_0 & 0 & 0 & 0 \\ 0 & e_0 & 0 & \alpha_0\overline{\alpha}_0 \\ 0 & 0 & e_0 & 0 \end{pmatrix}} & & \Big\downarrow{\scriptstyle \varepsilon} & & \\
0 & \longrightarrow & V_n & \xrightarrow[\begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix}]{} & V_0^4 & \xrightarrow[\begin{pmatrix} \overline{p}\overline{\alpha}_0 \\ \varepsilon\overline{p}\overline{\alpha}_0 \\ \varepsilon^2\overline{p}\overline{\alpha}_0 \\ \varepsilon^3\overline{p}\overline{\alpha}_0 \end{pmatrix}^{\mathrm{T}}]{} & V_n & \longrightarrow & 0
\end{array}$$

To show that the right hand square commutes, the only calculation that requires some effort is the following:

$$\overline{p}\overline{\alpha}_0 \cdot \alpha_0\overline{\alpha}_1\alpha_1\overline{\alpha}_0 + \varepsilon^2\overline{p}\overline{\alpha}_0 \cdot \alpha_0\overline{\alpha}_0 = \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_1\alpha_1\overline{\alpha}_0 + \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_0\alpha_0\overline{\alpha}_0$$

$$= \overline{p}\alpha_2\overline{\alpha}_2\overline{\alpha}_2\alpha_2\overline{\alpha}_0$$
$$= \varepsilon \cdot \varepsilon^3 \overline{p}\overline{\alpha}_0.$$

The following calculation verifies that the left hand square commutes:

$$\begin{pmatrix} 0 & 0 & 0 & \alpha_0\overline{\alpha}_1\alpha_1\overline{\alpha}_0 \\ e_0 & 0 & 0 & 0 \\ 0 & e_0 & 0 & \alpha_0\overline{\alpha}_0 \\ 0 & 0 & e_0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix} = \begin{pmatrix} -\alpha_0\overline{\alpha}_1\alpha_1\overline{\alpha}_0\alpha_0 p \\ \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p - \alpha_0\overline{\alpha}_0\alpha_0 p \\ \alpha_0 p\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha_0\overline{\alpha}_1\alpha_1\alpha_2\overline{\alpha}_2 p \\ \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_2\alpha_2 p \\ \alpha_0 p\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon^2 \\ \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0 p\varepsilon^2 \\ \alpha_0 p\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_0\overline{\alpha}_1\alpha_1 p\varepsilon \\ -\alpha_0\overline{\alpha}_1\alpha_1 p \\ \alpha_0 p\varepsilon \\ -\alpha_0 p \end{pmatrix} (-\varepsilon)$$

By the definition of Σ on morphisms, it follows that $\Sigma\varepsilon = -\varepsilon$. □

By [ES08] again, the singularity category of $\mathbb{k}[\![x, y]\!]/\langle x^2 + y^{2n+1}\rangle$, the coordinate ring of a simple curve singularity of type $\mathbb{A}_{2n}$, is $\mathbb{k}$-linearly equivalent to proj-$\Pi(\mathbb{L}_n)$. By Theorem 3.3.2 again, we therefore have a triangle equivalence

$$\underline{\mathrm{MCM}}\text{-}\mathbb{k}[\![x, y]\!]/\langle x^2 + y^{2n+1}\rangle \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\widetilde{\mathbb{DL}_n})$$

for any weight λ with $\lambda_i = 0$ for $i \neq 0$.

As with case (ii), we have been unable to give a more precise description of the singularity category when some of the λ_i are nonzero for $i \neq 0$. In particular, the number of indecomposables of $\Pi^\mu(\mathbb{L}_n)$ appears to depend on λ in a highly nontrivial way. Investigating this will be the subject of future work.

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