Mathematical programming for single- and multi-location non-stationary inventory control

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Declaration

This thesis has been composed by myself and contains no material that has been accepted for the award of any other degree at any university.

Parts of this thesis have been published in the following journals:

- Ma, X., Rossi, R., and Archibald, T. W. (2019). Stochastic inventory control: A literature review. *IFAC-PaperOnline*, 52(13):1490-1495.
- Ma, X., Rossi, R., and Archibald, T. W. (2022). Approximations for non-stationary stochastic lot-sizing under (s, Q)-type policy. *European Journal of Operational Research*, 298(2):573-584.

Permission to include text from these papers has been gained from the publisher and the authors.

To the best of my knowledge and belief this thesis contains no other material previously published by any other person except where due acknowledgement has been made.

(Xiyuan Ma)

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Abstract

Stochastic inventory control investigates strategies for managing and regulating inventories under various constraints and conditions to deal with uncertainty in demand. This is a significant field with rich academic literature which has broad practical applications in controlling and enhancing the performance of inventory systems. This thesis focuses on non-stationary stochastic inventory control and the computation of near-optimal inventory policies for single- and two-echelon inventory systems. We investigate the structure of optimal policies and develop effective mathematical programming heuristics for computing near-optimal policy parameters. This thesis makes three contributions to stochastic inventory control.

The first contribution concerns lot-sizing problems controlled under a static-dynamic uncertainty strategy. From a theoretical standpoint, I demonstrate the optimality of the non-stationary (s, Q) form for the single-item single-stocking location non-stationary stochastic lot-sizing problem in a static-dynamic setting; from a practical standpoint, I present a stochastic dynamic programming approach to determine optimal (s, Q)-type policy parameters, and I introduce mixed integer non-linear programming heuristics that leverage piecewise linear approximation of the cost function. The numerical study demonstrates that the proposed solution method efficiently computes near-optimal parameters for a broad class of problem instances.

The second contribution is to develop computationally efficient approaches for computing near-optimal policy parameters for the single-item single-stocking location non-stationary stochastic lot-sizing problem under the static-dynamic uncertainty strategy. I develop an efficient dynamic programming approach that, starting from a relaxed shortest-path formulation, leverages a state space aug-

mentation procedure to resolve infeasibility with respect to the original problem. Unlike other existing approaches, which address a service-level-oriented formulation, this method is developed under a penalty cost scheme. The approach can find a near-optimal solution to any instance of relevant size in negligible time by implementing simple numerical integrations.

This third contribution addresses the optimisation of the lateral transshipment amongst various locations in the same echelon from an inventory system. Under a proactive transshipment setting, I introduce a hybrid inventory policy for two-location settings to re-distribute the stock throughout the system. The policy parameters can be determined using a rolling-horizon technique based on a two-stage dynamic programming formulation and a mixed integer linear programme. The numerical analysis shows that the two-stage formulation can well approximate the optimal policy obtained via stochastic dynamic programming and that the rolling-horizon heuristic leads to tight optimality gaps.

Lay summary

The core question in the field of inventory control is to determine the optimal timing and size of replenishment orders. This thesis answers these questions under the non-stationary stochastic demand for inventory systems with single and multiple stocking locations.

Three strategies for stochastic systems control the uncertainty inherent in customer demand: “static”, “dynamic” and “static-dynamic”. As a combination, the static-dynamic strategy enjoys sufficient flexibility for most practical situations. It allows subsequent decisions to be determined based on information that may become available later and is further widely applied in practice.

The first contribution of this thesis is the formulation of the non-stationary stochastic lot-sizing problem in a type of static-dynamic inventory policies, where the model allows some decisions to be fixed at the start of the planning horizon and other decisions are based on information that may become available later in time in a wait-and-see fashion. These policies have received little attention in the literature, as well as the development of a model-based heuristic for approximating near-optimal policies.

The successful approximation of optimal policies naturally follows a desire for efficient computation. The second contribution is, therefore, the improvement of computational performance for inventory policies under the static-dynamic strategy, which is conducted through a classical mathematical programming problem and a graph-based heuristic.

Having considered single-location non-stationary stochastic inventory systems, the third contribution of this thesis is the extension of the problem to a multi-location setting. Warehouse replenishment is not the only supply source anymore;

lateral transshipment is allowed among retailers if applicable. This thesis introduces a rolling horizon heuristic with a static model and approximates a near-optimal inventory policy.

Extensive computational studies demonstrate that these newly proposed methods are competitive in terms of cost performance for single- and multi-location inventory systems. The core questions mentioned above can be answered by near-optimal policy parameters for a broad class of problems.

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Chapter 1

Introduction

1.1 Preliminaries

This section presents the motivation of the thesis and introduces the topics discussed, followed by the structure of the rest of this chapter.

1.1.1 Motivations

The rapid expansion of industrial consensus through cooperative manufacturing has brought increasing attention to improving the effectiveness of supply chains. As an essential component in the supply chain, inventory control optimises storage and holdings to guarantee that a company has the optimal inventory levels required to fill customer demands promptly.

Its essentiality is justified by prioritising efficient and effective inventory control for its potential impact on operating and financial performance (Capkun et al., 2009; Chuang et al., 2019; Mishra et al., 2013). A considerable amount of capital is tied up in inventory, with accounts receivable and accounts payable tied up $1.1 trillion in cash – equivalent to 7% of the U.S. GDP (REF, 2015).

Between 2013 to 2017, according to the Bureau of Labor Statistics of the U.S., the number of operating warehouses climbed 10.4% —– an increase of over 1,600 new warehouses in the U.S. alone. These domestic warehouses brought benefits by saving shipping costs and international taxes, tariffs, and fees; customers get their items faster simultaneously and hence have a high level of satisfaction.

In May 2020, when much of the world was still in the grip of the first wave of the pandemic and every part of the economy was disrupted, Mckinsey surveyed senior supply-chain executives from across industries and found that 93% of respondents wanted more flexible, agile, and robust supply chains. In a follow-up survey in May 2021, companies originally planned to increase near-shoring of suppliers to boost supply-chain resilience but wound up increasing inventory (Alicke et al., 2021).

And in the near future, the global supply chain market is expected to experience a compound annual growth rate (CAGR) of 11.2% from 2020 to 2027 (Zippa, 2022). More and more businesses are increasing the range of products they stock to meet consumer "long tail" demands. 54% of firms plan to expand the number of warehouses in the next five years (Motorola, 2021).

They all indicate the importance of smartly controlling the inventory in the whole process of production and manufacturing in the past and future. In a general picture, inventory control aims to help a company maintain only the appropriate quantities on hand without incurring excessive upfront costs or compromising customer satisfaction. The main challenges are two-fold as follows.

- **Demand variability.** Numerous variables, such as the season, location and market trends, can affect the unpredictability of product demand, which results in a shortage of goods when demand is high or an excess of goods when it is low.

- **Replenishment policy.** Effective inventory replenishment ensures that order fill rates can be achieved whilst keeping inventory carrying costs under control. In contrast, the decision rule of triggering a replenishment strongly depends on inventory system characteristics, which is complicated for large supply-chain networks.

In the view of academia, the earliest published academic study on inventory control, or more specifically, [1]lot-sizing problems, was carried out by Harris (1913).

[1]Inventory control encompasses the overall management of inventory levels to meet customer

This research came from the author's working experience determining the most economic quantity of replenishment orders. Six factors (unit cost, set-up cost, interest and depreciation on stock, movement and manufacturing interval) are considered to investigate how the total cost varied with the change of order size and derived the square-root formula for reordering quantity, which is now known as Economic Order Quantity (EOQ).

Over the last century, effective inventory control has received significant attention from the operations research community. A wealth of literature has been generated in this area to answer the critical questions at the heart of inventory control, namely (as summarised by Silver (1978)):

- replenishment timing: when should an order be placed?
- replenishment size: what should be the order quantity?

Over decades, academics and practitioners have worked on the problem of controlling inventory using a mathematically idealised context for analysis to make decisions in a complex and disorganised environment (Kumar et al., 2013). Lot-sizing problems, answering directly to the questions above, progressively embed more realistic assumptions; this boosts the development of inventory control (Graves, 1999), which has become one of the most advanced areas of Operational Research with strong theoretical underpinnings and real-world implementations.

Many gaps between the theory and the practice of inventory control stem from demand uncertainty. Most lot-sizing literature assumes demand to be deterministic rather than stochastic. Consequently, deterministic time series are employed to anticipate future demand, which makes it difficult to achieve customer satisfaction. On the other hand, stochastic lot-sizing, which is strongly related to the deterministic single-item dynamic uncapacitated lot-sizing problem (Wagner and Whitin, 1958), captures the randomness in demand and its optimal inventory policies have been investigated by Arrow et al. (1951) and Scarf (1960). Moreover, according to practical industry circumstances, demand is usually stochastic

demand, including the use of various policies and techniques to maintain appropriate stock levels, whereas lot-sizing specifically addresses determining the optimal order or production batch size for each replenishment cycle.

and non-stationary due to seasonal fluctuations and customers' preference shifts. Inventory policies, which capture three uncertainty strategies ("static", "static-dynamic" and "dynamic") developed by Bookbinder and Tan (1988), are investigated under non-stationary stochastic demand (Tarim and Kingsman, 2006; Rossi et al., 2015; Xiang et al., 2018).

Furthermore, inventory systems are designed to incorporate multiple stocking locations, where lateral transshipment is typically permitted to pool their inventories Gross (1963) among the locations in the same layer in a supply chain. This movement of items is categorised according to the timing of transshipment (Agrawal et al., 2004; Paterson et al., 2011) and is widely studied to fit various inventory assumptions on (non-)identical locations and cost parameters (Hoadley and Heyman, 1977; Tagaras and Vlachos, 2002; Paterson et al., 2012; Abouee-Mehrizi et al., 2015; Meissner and Senicheva, 2018). The resulting optimisation problems are particularly challenging and topical.

This thesis aims to develop near-optimal policies to answer questions regarding the timing and size of replenishment orders. By applying [2]mathematical programming, this thesis formulates inventory problems and develops approximation methods that are easy to implement and highly flexible in solving a broad class of non-stationary stochastic inventory problems. This statement is supported by the results in the later chapters. The newly developed approaches in this thesis are:

- **Innovative in modelling.** This thesis employs new mathematical pro-

[2]In this thesis, stochastic programming is utilised to address the lot-sizing problem under uncertainty, offering a more flexible and adaptive approach compared to both traditional and distributionally robust optimisation. While it is possible to incorporate time-dependent uncertainty sets into robust optimisation, making it more adaptable to non-stationary inventory control problems, the method may still be more conservative than stochastic programming. Distributionally robust optimisation strikes a balance between robustness and flexibility; however, the focus of this thesis remains on stochastic programming, which inherently captures demand randomness over time. This results in cost-effective solutions that better represent the uncertainty and variation in real-world situations, making it a more suitable choice for addressing non-stationary inventory control problems.

grammes to address non-stationary stochastic inventory control challenges.

- **Computationally efficient.** The approaches use modelling and approximation techniques to obtain policy parameters more efficiently without sacrificing quality.

- **Near-optimal.** The policies derived from the proposed approaches lead to reduced optimality gaps compared to existing policies in the literature.

- **Applicable for generalised inventory system.** The proposed approaches relax the demand assumptions in stochastic inventory control and are easily adaptable to other inventory contexts.

As was previously discussed, businesses invest a considerable amount of capital in controlling inventories. It is consequently of high necessity to develop more accurate, effective and efficient methods for complex inventory optimization. These goals are attempted by the research presented in this thesis.

Topic: *This thesis employs mathematical programming and heuristic techniques on non-stationary stochastic lot-sizing problems to investigate replenishment policies to optimally control single- and multi-location inventory systems. Approximation approaches are developed to find near-optimal policy parameters and further establish answers to the core questions of lot-sizing problems, the timing and order quantity for two classes of policies regarding stocking locations.*

To be more specific, this thesis explores the determination of near-optimal policies under (s, Q)-type policies and the improvement of computational efficiency under the (R_t, S_t) policy with penalty cost for single-location settings, and the formulation of a hybrid replenishment policy with lateral transshipment for multi-location settings. All these approaches can be easily implemented on large-scale instances. Comprehensive numerical experiments show that these approaches yield reasonable optimality gaps.

1.1.2 Structure

The rest of this chapter is structured as follows:

- Section 1.2 begins with an introduction to inventory control, followed by preliminaries in stochastic programming, piecewise linear approximation, mixed integer linear programming and the shortest path problem.
- Section 1.3 presents a comprehensive literature review on the stochastic lot-sizing problem (particularly on inventory policies) and lateral transshipment problems with research gaps and challenges indicated.
- Section 1.4 summarises the thesis's substance, emphasises the thesis's contributions, and gives the respective contributions for each of the subsequent chapters.
- Section 1.5 covers future prospects for study. Specifically, it examines which problems remain unanswered for each of the subsequent chapters, followed by potential study topics where the modelling techniques given in this thesis may be effectively used.
- Section 1.6 presents the conclusions of the thesis.

1.2 Formal background

This section provides a formal background for the inventory control-related topics and methodologies employed in this thesis. It begins with a discussion on the classification of inventory control (Section 1.2.1), where various inventory-related assumptions and how they are utilised in this thesis are introduced. Based on these assumptions, it follows classical models for deterministic (Section 1.2.2) and stochastic (Section 1.2.3) single-item single-location inventory problems. In Section 1.2.4, multiple-location transshipment problems are presented. In later chapters, single-location problems are discussed in chapters 2 and 3, and multi-location issues are addressed in chapter 4.

To tackle these problems, this thesis utilises a wide variety of programming methodologies. In Section 1.2.5, the fundamentals of stochastic programming are reviewed, followed by Section 1.2.6 with piecewise linear approximation techniques, which are consistently employed in later chapters to formulate non-stationary stochastic inventory systems with the mixed integer linear programming introduced in Section 1.2.7. Section 1.2.8 reviews a specific type of problem, the shortest-path problem, which is applied in Chapter 3.

1.2.1 Inventory control classifications

Inventory control is a prominent research area aiming to optimise inventory investments in supply chain management. To answer the questions summarised by Silver (1981) for the order timing and quantity, inventory systems are mathematically modelled and can be classified by the following dimensions.

- **Demand.** Based on the nature of demand, it is classified as deterministic when demand is known and stochastic when demand is unknown but follows a certain distribution. The demand can also be distribution-free, while this is not in the scope of this thesis. This thesis will focus on non-stationary stochastic demand in Chapters 2, 3 and 4.

- **Review frequency.** The inventory is reviewed continuously or periodically. In continuous-reviewed systems, as soon as the inventory position is sufficiently low, a replenishment order is triggered, whilst the periodic-reviewed systems only consider the inventory position at certain given points; the intervals between these reviews are constant. This thesis will refer to the replenishment cycle R as the number of periods a single order covers to fulfil the demand.

- **The number of items.** It can be single or multiple items considered in an inventory system. This thesis will discuss the inventory control problem involving only a single item.

- **The number of locations** This refers to the total count of distinct points in a supply chain where inventory is held, such as warehouses, distribution centers, retail stores, or manufacturing facilities. The number of locations influences the complexity of inventory management as more locations require increased coordination and information sharing. As the number of locations increases, decisions about replenishment, transshipments, and inventory allocation become more complex, and the system needs more sophisticated models and algorithms to optimize inventory policies. This thesis will consider the single-location problems in Chapters 2 and 3 and the two-location problem in Chapter 4.

- **The number of echelons** An echelon represents a level in a supply chain hierarchy, typically categorised as upper, middle, or lower echelons. The number of echelons affects the overall structure of the supply chain and the flow of goods, information, and funds within the system. As the number of echelons increases, the system becomes more intricate, and the lead times between various stages of the supply chain may also increase, potentially affecting responsiveness and overall efficiency. In Chapter 4, for the two-location problem, two retailers are in the same echelon.

- **Lead time.** The time between when a replenishment order is placed and when the order is received. In inventory literature, the lead time is classified into constant and stochastic types (Axsäter, 2015). This thesis will assume a zero lead time for any replenishment or transshipment in Chapters 2, 3 and 4.

- **Planning horizon.** The planning horizon can be finite (over periods $1, \ldots, T$) or infinite. This thesis discusses lot-sizing problems within finite planning horizons in the following context.

- **Stock-out types.** Any unmet demand in inventory control can be considered as back-orders or lost sales. It is a back-order when the customer waits for the stock to be replenished, whereby the demand is subsequently filled. It is a lost sale when the customer does not wait for the stock to be replenished and drops the demand. This thesis will consider unmet demand at the end of each period as back-orders in Chapters 2, 3 and 4.

- **Service levels.** It refers to the expected probability of being able to satisfy all possible inventory requirements within a particular period. [3]α and β service levels are widely discussed in the literature (Bashyam and Fu, 1998; Schneider and Ringuest, 1990; Sethi and Cheng, 1997; Rossi et al., 2015; Tunc et al., 2014, 2018) This thesis will address the difference between the inventory systems with service levels and penalty cost schemes in Chapter 3.

- Inventory Capacity. Inventory capacity refers to the maximum amount of items that can be stored in a facility or warehouse at a given time. It directly impacts the ability to maintain optimal inventory levels, meet customer demands, and minimise costs associated with overstocking or stockouts. The thesis is to explore and analyse the near-optimal inventory policies under non-stationary demand, which does not directly involve capacity considerations; in fact, capacity constraints can impact inventory policies, such as the economic order quantity and reorder point (discussed in Chapter 1.2), as they may need to be adjusted to account for the limited storage space available.

This thesis does not elaborate on other classifications of inventory systems such as perishable inventory, as they are not in the scope.

[3]α service level (Type 1 service level or Fill rate): This is the proportion of demand that is met directly from stock on hand without any delay or backorders. It is calculated as the total quantity of items delivered immediately upon request divided by the total demand over a specific time period. A high α service level indicates that a large percentage of customer orders are fulfilled without delays. β service level (Type 2 service level or Cycle service level): This is the probability that there will be no stockouts during a replenishment cycle, or equivalently, the proportion of replenishment cycles without stockouts. β service level is a measure of how effectively the inventory system can prevent stockouts from occurring, considering the entire order cycle. A high β service level indicates that the inventory system is effective at maintaining stock availability across replenishment cycles.

Inventory-related costs. The primary goal of inventory control is to optimise the investment spending on inventory, which consists of the following costs in most cases.

- Holding cost h: the cost of carrying a unit of the item in inventory and is usually assumed linear; is dependent on the cost of storage space, taxes, insurance, breakage, opportunity cost, and so on.

- Fixed ordering cost K: the fixed cost of placing an order; is independent of the order quantity and usually accounts for the mobilisation and administrative fees in logistics, and so on.

- Unit ordering cost z: the cost of ordering each unit of the item; usually is linear to the order quantity.

- Penalty cost b: when the unmet demand is allowed, each unit of unmet demand incurs a penalty cost.

Inventory-related measures. The performance of an inventory can be evaluated according to various parameters. This thesis focuses on the following measures, which will be discussed throughout this section and later chapters.

- on-hand inventory/stock (denoted positive): the amount of stock remaining in the system.

- inventory level: equals to the sum of on-hand inventory and back-ordered inventory. A positive inventory level represents that the system carries the remaining inventory; a negative inventory level indicates a back-order and vice versa.

- inventory position: the sum of inventory level and outstanding order (the order that has been placed but not received).

Overall, in the following contexts of this section, the discussion will be based on the classification presented in Figure 1.1 applying costs and measures as above.

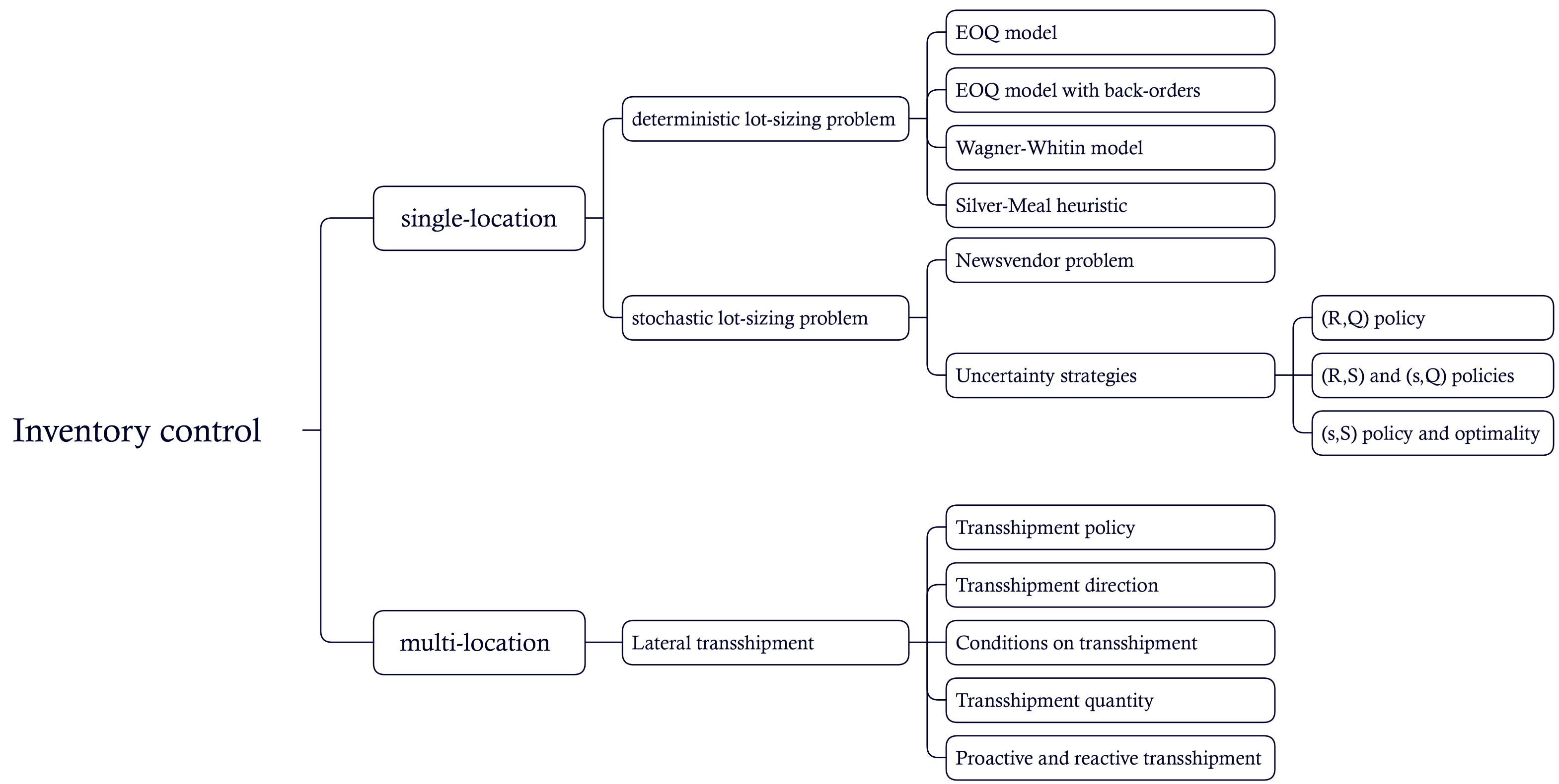


Figure 1.1: Topics in inventory control discussed in the thesis.

1.2.2 Single-item single-location inventory control with deterministic demand

This subsection starts the background from one of the classical deterministic problems, the Economic Order Quantity(EOQ) model, to illustrate the inventory systems and cost relationships with and without backorders, which is extensively applied for unmet demand in later chapters. The discussion is then generalised to time-varying demand through the Wagner-Whitin model and an approximate dynamic model through Silver-Meal heuristic, where the Wagner-Whitin model can be formulated as a shortest path problem (introduced in Section 1.2.8). In this thesis, Chapter 3 employs this idea with shortest-path formulation improve computational efficiency.

1.2.2.1 EOQ model

EOQ is the first and most well-known single-item continuous-review deterministic inventory management model, introduced by Harris (1913). It assumes that demand is predictable and constant at a rate of d and that back-orders are prohibited. The demand is known during the fixed lead time; hence no lead time is applied to replenishment. EOQ model calculates the best order quantity each time a replenishment is made to minimise the relevant average cost comprising of holding cost, fixed and unit ordering costs.

In deterministic inventory management, since the entire order quantity across the planning horizon is constant under the assumption of fixed unit cost, the total unit ordering cost is also constant[4]. Hence, it does not affect the best replenishment plan. For simplicity, the unit ordering cost is often neglected while optimising deterministic inventory management issues.

Figure 1.2 illustrates the inventory level of EOQ model as a function of time and the replenishment cycle length $R = \frac{Q}{d}$. Assume that the on-hand inventory is 0 at the beginning of the planning horizon, when a replenishment order is placed and arrives instantaneously. The inventory level then decreases at a constant rate

[4]Note that here we neglect the unit ordering cost as it is a constant.

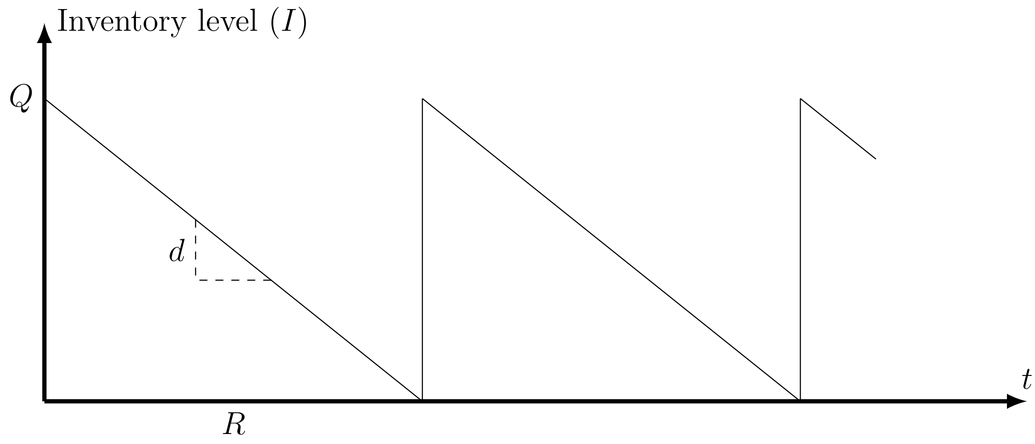


Figure 1.2: EOQ inventory model

d until the next order is placed, and the process repeats. Any optimal solution for the EOQ model has two important properties:

Theorem 1.2.1. *Zero inventory property.* The optimal replenishment plan of the EOQ model is to place orders when the inventory level is exactly at zero.

Theorem 1.2.2. *Constant order sizes.* If Q is the optimal order size at time 0, then it will also be the optimal order size every other time an order is placed.

Let $C(Q)$ represent the average cost per unit time unit. The cost functions are illustrated in Figure 1.3, from which the optimal order quantity Q^* to the EOQ model is solved as

$$Q^* = \sqrt{\frac{2Kd}{h}}, \tag{1.1}$$

and the minimised cost per time unit is $\sqrt{2Khd}$.

1.2.2.2 EOQ with back-orders.

Back-orders refer to unmet demand in the system that will be satisfied when the replenishment order arrives. In the deterministic EOQ model, the value of back-orders in each replenishment cycle is the same since the demand does not change. An EOQ inventory system with back-orders is illustrated in Figure 1.4, where γ represents the fraction of demand that is back-ordered.

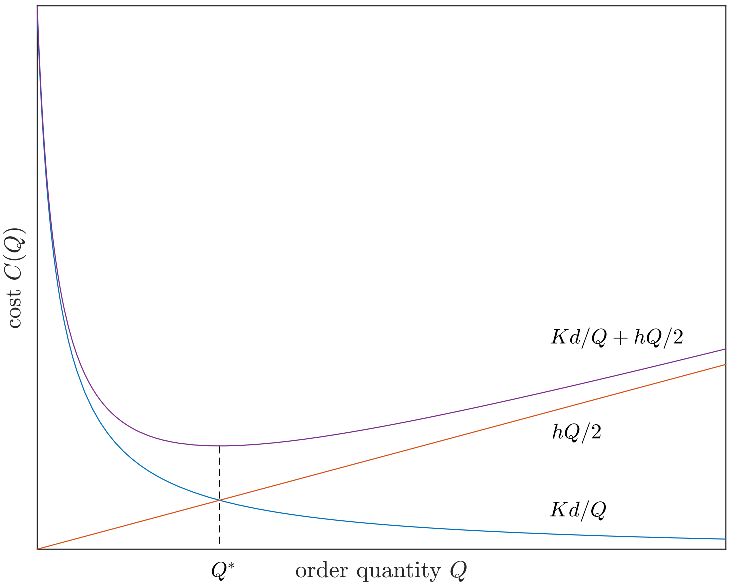


Figure 1.3: Cost functions of EOQ model.

The goal of the EOQ model with back-orders is to find the optimal replenishment plan to minimise the total average cost consisting of fixed ordering costs, holding and penalty costs. The optimal solution to this problem is given in Theorem 1.2.3.

Theorem 1.2.3. The optimal order quantity Q^* for EOQ with back-orders is

$$Q^* = \sqrt{\frac{2Kd(h+b)}{hb}}, \tag{1.2}$$

and the optimal fraction of demand that is back-ordered is

$$\gamma^* = \frac{h}{h+b}. \tag{1.3}$$

The classical EOQ model is a special case with back-backorders in which $\gamma = 0$. Managers can use the backorder quantity to assess the trade-offs between inventory holding costs and backorder costs. By analysing the impact of different holding and penalty costs on the back-ordered quantity, managers can make informed decisions about adjusting their inventory policies to achieve a better balance between

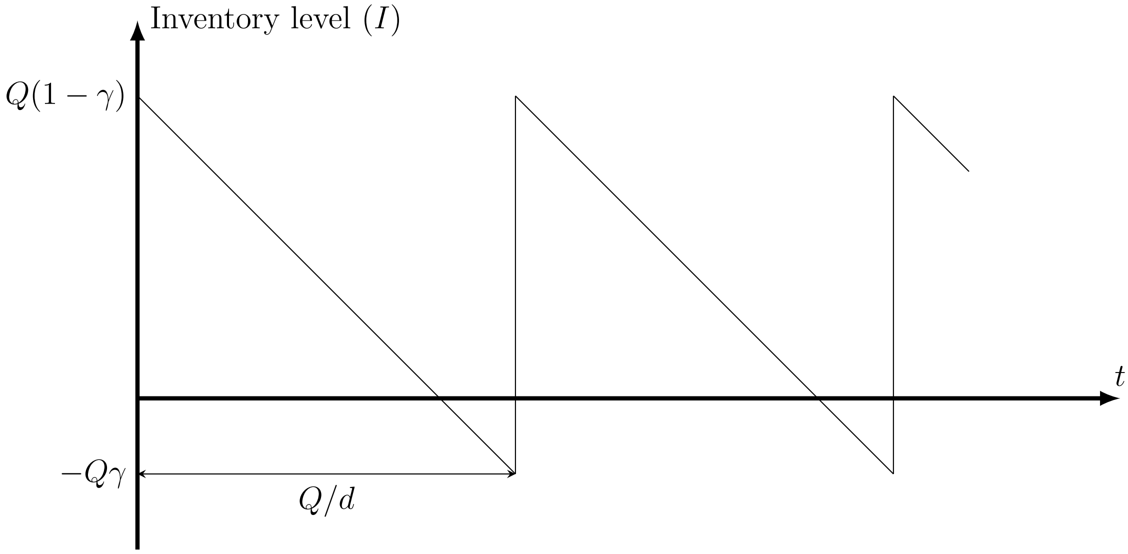


Figure 1.4: EOQ inventory model

inventory levels and customer service Q^* and γ^* contributes to the optimal average cost per time unit of $\sqrt{\frac{2Kdhb}{h+b}}$.

1.2.2.3 Wagner-Whitin model.

So far, this discussion only considered constant demand, while time-varying demand is widely applied in practice. The variations can be due to different reasons, such as seasonal variations in demand. Note that the thesis here still assumes deterministic demand, which means that all variations are known beforehand and have a zero lead time.

The Wagner-Whitin model (Wagner and Whitin, 1958) is one of the well-known inventory models for time-varying demand. Consider a $T-$period planning horizon and demand rate d_t for each period. For any replenishment order placed at the beginning of a period, a fixed ordering cost of K, and a unit ordering cost charged on each unit are incurred. At the end of each period, a holding cost of h is charged on each unit of items carried from one period to the next. *No back-orders are allowed.*

The Wagner-Whitin algorithm has a time complexity of $\mathcal{O}(T^2)$, where T is the number of periods in the planning horizon. This complexity arises from the fact that the algorithm needs to evaluate all possible combinations of order quantities for each period, creating a considerable number of potential solutions.

Similar to the EOQ model, the optimal solution holds the *zero inventory property* as discussed in Theorem 1.2.1. Therefore, the determination of timing and order quantity is equivalent to determining at which period to reorder, where the order quantity is the demand between two consecutive periods that reorder.

Let C_t denote the total cost over the horizon (t, T) given an order is placed at the beginning of period t, where

$$C_t \quad = \min_{t<j\leq T+1} \{C_j + K + h \sum_{i=t}^{j-1} d_i(i-t)\}, \tag{1.4}$$

which is recursively defined by C_j for subsequent periods $j, \ldots, T$ and the total cost incurred over periods $t, \ldots, j-1$ given an order placed in period t. Define $C_{T+1} = 0$ is set as the boundary condition.

Example 1.2.1. Consider a 4-period instance with demand rate $d_t = \{20, 40, 60, 40\}$ and $K = 100$, $h = 1$. The Wagner-Whitin model finds the minimum total cost of 280 with orders placed in period 1 and 3 with quantities 60 and 100, respectively.

The dynamic Wagner-Whitin model is equivalent to searching for the shortest path through a network with $T+1$ nodes from node 1 to node $T+1$. Each arc (i, j) in the graph represents ordering at the beginning of period i to satisfy demand over periods $i, \ldots, j-1$. This thesis will apply the shortest path formulation to non-stationary stochastic inventory problems, and details of shortest path problems will be presented in Section 1.2.8.

1.2.2.4 Silver-Meal heuristic.

The Wagner-Whitin model provides the ideal replenishment plan; however, it has limitations from a practitioner's perspective, including the extensive computing effort and the programming complexity. In this context, Silver and Meal (1973) introduces a simple variant of the basic EOQ model and a heuristic, which is widely used in practice.

The Silver-Mean heuristic is, like most other lot-sizing heuristics, a sequential method. The idea of the Silver-Meal heuristic is to choose to have a new delivery when the average per period costs increase for the first time. The procedure of the Silver-Meal heuristic is presented as follows.

Let $\overline{C}_{t,j}$ indicate the average cost of periods $t, \ldots, t+j$ given an order placed at the beginning of period t to cover demand over $j-t+1$ periods. The optimal number of periods that a replenishment order in period t covers is said to be the smallest value of j satisfying $\overline{C}_{t,t+j-1} < \overline{C}_{t,t+j}$; the optimal order quantity is obtained as the convolution of demand over j periods as $\sum_{k=t}^{t+j-1} d_k$.

Silver-Meal heuristic procedures

1. Starting from period t, compute the average cost of period t, where

$$\overline{C}_{t,t} = K. \tag{1.5}$$

2. Compute the average cost of periods $t, \ldots, j$, where

$$\overline{C}_{t,j} = \frac{K + \sum_{k=t+1}^{j} h \cdot d_k}{j-t+1}. \tag{1.6}$$

3. Stop the procedure until find a j such that $\overline{C}_{t,t+j-1} < \overline{C}_{t,t+j}$.

Note that this strategy only ensures a local minimum for the current replenishment's average cost per unit of time. Since the search operation is terminated at the first rise in costs per unit of time, it is possible to discover higher values of j that result in lower costs per unit of time.

The time complexity of the Silver-Meal heuristic method is $\mathcal{O}(T^2)$, where T is the number of periods in the planning horizon. This complexity arises from the iterative nature of the algorithm, as it evaluates the average cost per unit time for each possible order quantity combination over the entire planning horizon. Although the Silver-Meal heuristic has a time complexity of $\mathcal{O}(T^2)$, it is typically faster than the Wagner-Whitin algorithm in practice, as it does not require an exhaustive search of all possible combinations of order quantities for each period.

Example 1.2.2. Consider the same 4-period instance with demand rate $d_t = \{20, 40, 60, 40\}$ and $K = 100$, $h = 1$. By applying Silver-Meal's heuristic algorithm, the minimised total cost is 280 with order placed in periods 1 and 3 with quantities 60 and 100, respectively.

Additional overviews on deterministic inventory control literature can be found in Silver (1981); Pentico et al. (2009) and Drake and Marley (2014).

1.2.3 Single-item single-location inventory control with stochastic demand

As introduced above, deterministic demand is well-developed in the early studies, while stochastic demand subject to uncertainty prevails in real life. This thesis addresses stochastic inventory problems and heavily relies on the uncertainty strategies proposed by Bookbinder and Tan (1988) in later chapters. Therefore, this subsection reviews the newsvendor problem as a classical model for stochastic inventory control, followed by a brief introduction to uncertainty strategy and inventory policies.

1.2.3.1 Newsvendor problem.

The newsvendor problem is a classical problem in inventory control initially addressed by Edgeworth (1888), who uses the Central Limit Theorem to determine the amount of cash to keep at a bank to satisfy random cash withdrawals from depositors with high probability.

The newsvendor problem is formulated with the background of a newsvender who orders newspapers in the early morning and sells them until the end of the day. During the day,

- a unit ordering cost is charged to purchase a copy of the newspaper;
- a revenue is received from selling a copy of the newspaper;

- and a salvage profit is received from selling unsold newspapers back to the supplier.

The vendor should properly control the quantity brought in the early morning: if too large, then unsold copies are scrapped; if too small, then not all customers are satisfied, and potential profits are lost. The goal, therefore, is to find the optimal order quantity to maximise the expected profit given the uncertain demand.

Following the notations above, let h and b represent the holding and penalty costs, respectively. The holding cost h in the newsvendor problem consists of the purchase cost minus the salvage profit. Note that the inventory cannot be carried over to the next period; technically, this cost is not a "holding" cost, but the thesis will refer to it as such. Similarly, the penalty cost equals the selling price minus the unit ordering cost. Therefore, the objective of the original newsvendor problem is equivalent to identifying the best order quantity that minimises total cost.

Given the demand d is a random variable with probability density function $f(d)$ and the cumulative density function $F(d)$, let $\widetilde{C}(Q)$ denote the expected total cost at the end of the day with order quantity Q purchased in the early morning, where

$$\widetilde{C}(Q) = h\mathbb{E}[\max(Q-d,0)] + b\mathbb{E}[\max(d-Q,0)]; \tag{1.7}$$

for a continuous demand,

$$\widetilde{C}(Q) = h\int_0^Q (Q-\zeta)f(\zeta)\mathrm{d}\zeta + b\int_Q^\infty (\zeta-Q)f(\zeta)\mathrm{d}\zeta \tag{1.8}$$

and a discrete demand,

$$\widetilde{C}(Q) = h\sum_{\zeta=0}^{Q}(Q-\zeta)f(\zeta) + b\sum_{\zeta=Q}^{\infty}(\zeta-Q)f(\zeta). \tag{1.9}$$

Theorem 1.2.4. The optimal order quantity Q^* of the newsvendor problem is given by

$$Q^* = F^{-1}\left(\frac{b}{h+b}\right). \tag{1.10}$$

This ratio is an α *service level*, indicating the probability of satisfying all demand. This optimal order quantity Q^* represents the minimum copies of the newspaper to purchase in the morning to satisfy all customer demand in $100 \times \alpha\%$ of days.

Further, the single-period Newsvendor problem is extended to multi-period, where the remaining inventory is carried over to the next time period. The uncertainty strategies introduced by Bookbinder and Tan (1988) can effectively tackle this class of stochastic inventory problems.

1.2.3.2 Bookbinder and Tan's uncertainty strategies.

Bookbinder and Tan (1988) introduce three uncertainty models for stochastic inventory control as follows. In later chapters, this thesis discusses inventory control problems relying on these strategies.

The static uncertainty strategy. The static uncertainty decision rule stipulates that the values of all decision variables must be fixed at the start of the planning horizon. Timing and order quantities must be determined definitively before any demand is known. This model has organisational benefits if extensive preparation is required before production (Peters 1977). The (R, Q) inventory policy captures this strategy.

The dynamic uncertainty strategy. The dynamic uncertainty rule is sufficiently flexible that the timing and order quantity can be revised in response to outcomes of random variables observed during the time horizon. This strategy is captured by the (s, S) policy.

The static-dynamic uncertainty strategy. The static-dynamic uncertainty model allows some decisions to be fixed at the start of the planning horizon, and other decisions to be based on information that may become available later in time in a wait-and-see fashion. Once the realised value of demand in the last period is known (d_{t-1}, and thus I_{t-1}), the resulting deterministic problem has already been solved. This strategy is captured by (R, S) and (s, Q) policies depending on which parameter is first determined.

1.2.3.3 Inventory policies

These uncertainty strategies are reflected in the following inventory policies. In what follows, decision rules and inventory systems with corresponding policies are illustrated.

(R, Q) policy. The (R, Q) policy generates a replenishment order with quantity Q every R periods, where R is interpreted as the time interval between two consecutive replenishment orders. Figure 1.5 illustrates an inventory system using an (R, Q) policy.

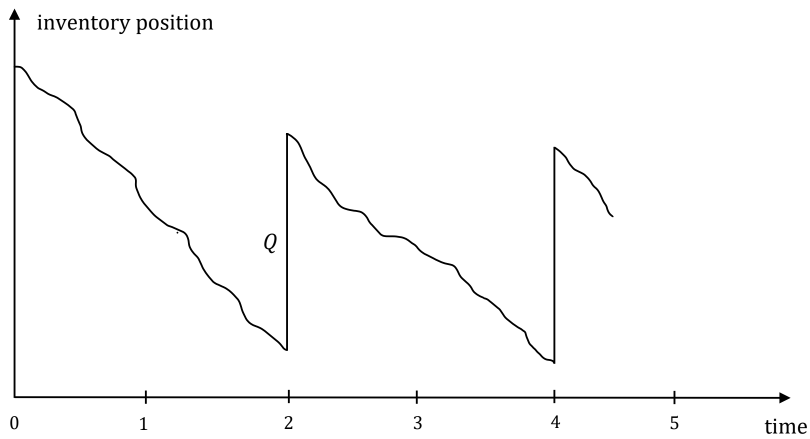


Figure 1.5: Inventory position curve of (R, Q) policy.

(R, S) policy. The (R, S) policy, illustrated in Figure 1.6, places a replenishment order every R periods to raise inventory up to the order-up-to level S. These two policy parameters are determined at the beginning of the planning horizon.

(s, Q) policy. The (s, Q) policy (also known as (r, Q) policy) places a replenishment order of quantity Q when the inventory position falls at or below the reorder point s. Note that it is the inventory position (rather than inventory level) that

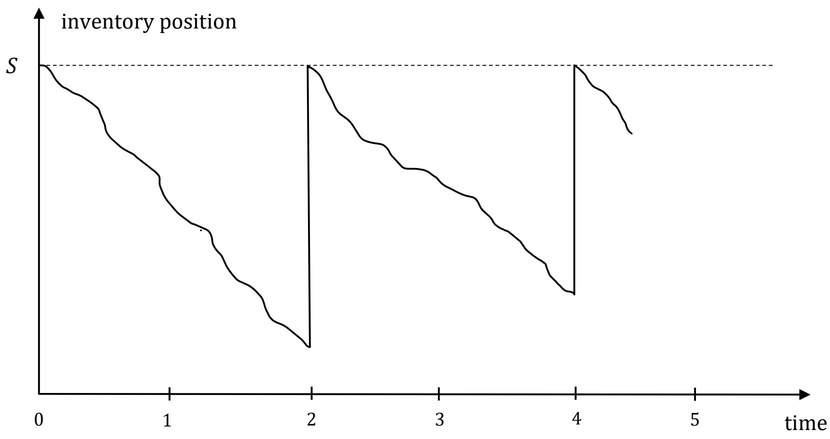


Figure 1.6: Inventory position curve of (R, S) inventory system.

triggers the replenishment[5] since the system takes into account the outstanding replenishment order. Figure 1.7 illustrates an inventory system using an (s, Q) policy.

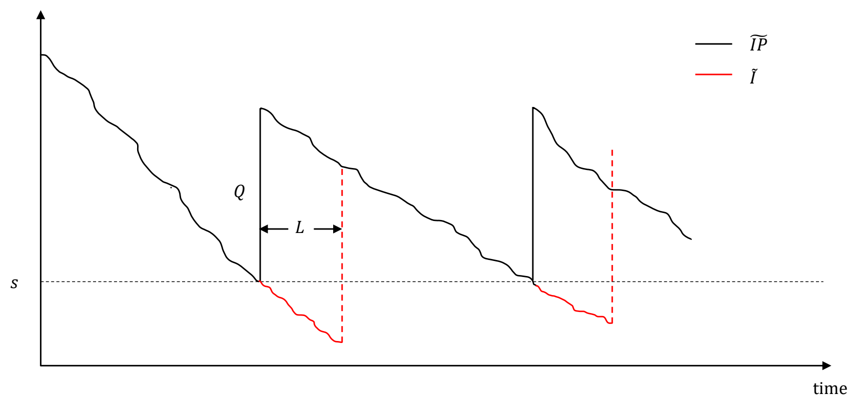


Figure 1.7: Curves of inventory position $\widetilde{IP}$ and inventory level I under an (s, Q) inventory system, where L indicates a lead time.

(s, S) policy. The (s, S) policy comprises two policy parameters: a reorder point s and an order-up-to level S. The inventory system places a replenishment when

[5] When the lead time is neglected, the inventory position is equal to inventory level

the inventory level falls below the reorder point, illustrated by Figure 1.8.

Figure 1.8: Inventory position curve of an (s, S) inventory system.

(s, S) policy's optimality. Consider a dynamic lot-sizing problem of single-item single-location with an ordering cost composed of a unit and a fixed cost. Scarf (1960) show that if the holding and penalty costs are linear, the optimal policy in each period is always of the (s, S) type.

Scarf's model assumes the stock level immediately after purchases are delivered is y. Therefore, the expected holding and penalty costs $L(y)$ are

$$L(y) = \begin{cases} \int_0^y h(y-\xi)f(\xi)\mathrm{d}\xi + \int_y^\infty b(\xi - y)f(\xi)\mathrm{d}\xi, & y \geq 0, \\ \int_0^\infty b(\xi - y)f(\xi)\mathrm{d}\xi, & y < 0. \end{cases} \tag{1.11}$$

where f is the density of the demand distribution. Let $c(Q)$ represent the ordering cost, where

$$c(Q) \triangleq \begin{cases} K + z \cdot Q, & Q > 0; \\ 0, & Q = 0. \end{cases} \tag{1.12}$$

The expected total cost $C_t(x)$ over periods t to T with the opening inventory level x is determined as

$$C_t(x) = \min_{y \geq x}\{c(y-x) + L(y) + \int_0^\infty C_{t-1}(y-\xi)f(\xi)\mathrm{d}\xi\}. \tag{1.13}$$

Let $G_t(y)$ represent the expected total cost over period t to T starting from inventory level y and no replenishment is placed at the beginning of period t, where

$$G_t(y) = cy + L(y) + \int_0^{\infty} C_{t-1}(y-\xi)f(\xi)\mathrm{d}\xi. \tag{1.14}$$

The optimality of (s, S) policy is shown based on $G_t(y)$ and is described as the following lemma.

Lemma 1.2.1. Scarf (1960) Let $K \leq 0$ and $g(x)$ be a differentiable function. $g(x)$ is K-*convex* if

$$K + g(a+x) - g(x) - ag'(x) \geq 0 \tag{1.15}$$

for all positive a and all x. If differentiability is not assumed, then the K-convexity is

$$K + g(a+x) - g(x) - a\left[\frac{g(x) - g(x-b)}{b}\right] \geq 0. \tag{1.16}$$

Lemma 1.2.2. The following properties are introduced in Scarf (1960):

1. 0-convexity is equivalent to ordinary convexity;

2. If $g(x)$ is K-convex, then $g(x+\epsilon)$ is K-convex for all constants ϵ;

3. If g_1 and g_2 are K_1-convex and K_2-convex, then $\alpha g_1 + \beta g_2$ is $(\alpha K_1 + \beta K_2)$-convex when α and β are positive.

Lemma 1.2.3. Let f be a continuous, K-convex function. There exists a unique s^* such that $f(x) = f(S^*) + K$, where S^* is the global optima of f.

By applying Lemma 1.2.3, there exists an S^* that minimises $G_t(I_{t-1})$, and there is a unique $s \leq S^*$ such that $G_t(s) = G_t(S^*) + K$. Therefore,

$$C_t(I_{t-1}) = \begin{cases} -cI_{t-1} + G_t(I_{t-1}), & s \leq I_{t-1} \leq S^*, \\ -cI_{t-1} + G_t(S^*) + K, & 0 \leq I_{t-1} \leq s, \end{cases} \tag{1.17}$$

and this is utilised to determine the reorder point of the (s, S) policy (Xiang et al., 2018) as illustrated in Figure 1.9.

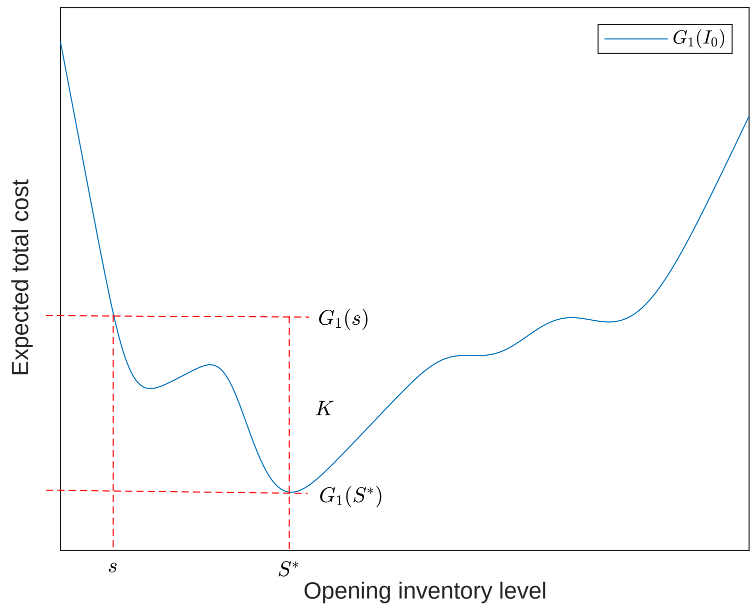


Figure 1.9: K-convexity of an (s, S) inventory system.

As discussed above, first, (s, S) policy is proved to be optimal, although it suffers from "nervousness" of control actions (Kilic and Tarim, 2011; Tunc et al., 2013). This thesis applies the expected total costs obtained from (s, S) policy as benchmarks in computation analysis in Chapters 2 and 3.

Secondly, (s, Q) and (R, S) policies under static-dynamic strategy enjoy characteristics of both static and dynamic strategies, where one parameter of order timing and quantity is pre-determined, and the other is determined in a wait-and-see fashion. This thesis, for the first time in the literature, investigates the (s, Q)-type policies under the non-stationary stochastic demand in Chapter 2 and improves the computation efficiency for static-dynamic policies under a penalty cost scheme in Chapter 3.

Meanwhile, (R, Q) policy under the static strategy is not sufficiently flexible but is appealing in material requirement planning systems (Kilic and Tarim, 2011). For two-location inventory systems with lateral transshipment, the complexity of the problem is increased with two more dimensions (the replenishment for the second location and the transshipment source and quantities). In Chapter 4, this thesis applies the (R, Q) policy to reduce the complexity and develops a rolling horizon heuristic to determine a near-optimal inventory policy.

1.2.4 Multi-location lot-sizing problem with lateral transshipment

So far, a single stocking location has been considered. In practice, however, it is common to face multi-location inventory systems in which additional supply chain management characteristics such as multiple echelons, lateral transshipment, and pooling are incorporated. This thesis in Chapter 4 addresses inventory systems with lateral transshipment, and this subsection presents characteristics of lateral transshipment.

In the inventory systems with lateral transshipment, the local stocks of retailers are normally replenished from an outside supplier. But in an emergency or at some predefined time point, it is also possible to use lateral transshipment between adjacent retailers. This transshipment is faster but incurs additional costs. However, models with lateral transshipment are usually more difficult to handle, and the available results are less general. Key questions to consider in the lateral transshipment problem are discussed as follows.

Transshipment policy. Different lateral transshipment policies exist regarding transshipment direction, conditions, and quantity. In addition, a transshipment decision is made either before or after demand is fully realised or in response to stock-outs.

Transshipment direction. There are two types of lateral transshipment within the inventory control system in terms of direction: unidirectional and bidirectional transshipment. Specifically, for a multi-location system with k locations, $k = 1, \ldots, n$, unidirectional transshipment is only permitted from location $k + 1$ to location k, $k = 1, \ldots, n - 1$. On the other hand, bidirectional transhipment permits transshipment between any two adjacent depots.

Conditions on the transshipment. The policies of complete pooling and no pooling are commonly used. A complete pooling policy means that when a stock-out occurs at one location, a transshipment request will be fulfilled whenever the requested location has enough stock to satisfy the request; no pooling policy means that the requested location will never fulfil a stock-out-related transshipment request. In addition to these two pooling policies, we may define a partial pooling policy that allows a transshipment to be delivered if a certain condition is met, e.g. the holdout pooling policy, in which a transshipment request is fulfilled when the inventory level at the requested location is greater than or equal to a specific holdout threshold value. Otherwise, the request for transshipment is denied.

Transshipment quantity. In certain instances, the function of the transshipment quantity may depend on inventory conditions between the transshipment requesting and requested locations.

Overall, two problems associated with lateral transshipment are essential to effectively control a multi-location inventory system (Axsäter, 2015):

- What is a suitable decision rule for deciding on the source and the size of a lateral transshipment?
- Given a certain decision rule for lateral transshipment, how can this policy be evaluated and how does it affect the policy for normal replenishment?

The first question relates to optimising the transshipment decisions. Early research to answer it includes Das (1975), Robinson (1990) and Archibald et al. (1997) with stochastic dynamic programming. Further details in literature are provided in Section 1.3.2. Many earlier papers dealing with lateral transshipment consider the second problem and assume that some simple decision rule is given. A frequent assumption is that demand at a local stock point that can not be met by stock on hand is, if possible, supplied by a lateral transshipment from some other stock point with inventory on hand. Examples of such approaches are Lee (1987) and Grahovac and Chakravarty (2001).

In this thesis, the classification of lateral transshipment is based on its timing, resulting in two types of transshipment between locations in the same echelon: *proactive* and *reactive* transshipment.

Proactive lateral transshipment. In proactive transshipment models, lateral transshipment is utilised to redistribute inventory between all stocking sites in an echelon at specified times. This can be arranged in advance and organised in a way that minimises the handling costs. This type of lateral transshipment is most advantageous in the retail industry, where handling costs are typically the most significant.

Reactive lateral transshipment. Reactive transshipment occurs when one of the stocking points faces a stock out (or the risk of a stock out) while another stocking point has sufficient inventory. This type of lateral transshipment is appropriate when the transshipment expenses are relatively low compared to the costs associated with keeping large quantities of inventory and failing to meet urgent demand. This is frequently the case in the spare parts industry.

The literature refers to the policy that applies both proactive and reactive transshipment as *joint transshipment* and that applies transshipment and replenishment as a *hybrid policy.* This thesis investigates the optimal replenishment and transshipment policy that applies the proactive transshipment in Chapter 4.

The context above introduced the inventory control background. In what follows, methodologies applied in this thesis are discussed through fundamental formulations. To have a general picture of the following subsections, Section 1.2.5 presents stochastic programming to deal with stochastic inventory problems that are focused on in Chapters 2, 3 and 4; a piecewise linear approximation technique is introduced in Section 1.2.6, which is incorporated in later chapters with the mixed integer linear programming (Section 1.2.7) and a specific model on shortest path problem (Section 1.2.8).

1.2.5 Stochastic programming

As discussed above, stochastic inventory control deals with uncertainty in demand. This section presents stochastic programming and its solution methods, which are extensively adopted in this thesis.

Stochastic programmes are far more complex than their deterministic counterparts. Deterministic models may yield excellent answers for certain data sets in particular models, but it is typically impossible to infer that they are good without comparing them to the results of stochastic programmes. The general formulation of a stochastic programme is presented as follows (Kall et al., 1994).

$$\min \quad g_0(x, \xi), \tag{1.18}$$

$$\text{s.t.} \quad g_i(x, \xi) \leq 0, \ i = 1, \ldots, m, \tag{1.19}$$

$$x \in X \subset \mathbb{R}^n, \tag{1.20}$$

where ξ is a random vector varying over a set $\Xi \subset \mathbb{R}^k$. More precisely, we assume throughout that a family $\mathcal{F}$ of events, i.e., subsets of Ξ and the probability distribution on $\mathcal{F}$ are given. Then, for every subset $A \subset \Xi$ that is an event ($A \in \mathcal{F}$), the probability $\Pr(A)$ is known. Furthermore, we assume that the functions $g_i(x, \cdot) : \Xi \to \mathbb{R}, \forall x, i$ are random variables themselves and that the probability distribution P is independent of x.

1.2.5.1 Stochastic dynamic programming

Stochastic programmes can be solved dynamically for an exact solution, and this leads to stochastic dynamic programming (SDP). This thesis applies results obtained from SDP as the benchmark for stochastic inventory problems in Chapter 2, 3 and 4.

In general, the objective of stochastic dynamic programming is to find a strategy for a stochastic optimisation problem that minimises the expected total cost incurred over a given planning horizon. For a discrete system with T stages, the following concepts define a stochastic dynamic program (Bellman, 1957).

- State, $x_t \in X_t$, where X_t is the set of feasible states of stage t.

- Action, $a_t(x_t) \in \mathcal{A}_t(x_t)$, where $A_t(x_t)$ is the set of feasible actions given a state x_t in stage t.

- Expected immediate cost, $\varphi_t(x_t, a_t(x_t))$ describes the expected cost during stage t when the system is in state x_t and action $a_t(x_t)$ is chosen.

- Transition probability, $\Pr(x_{t+1}|(x_t, a_t(x_t)))$ denotes the probability that the state evolves from x_t to x_{t+1}, if action a_t is chosen in state x_t at stage t, where $x_{t+1} \in X_t$.

- Objective function, $V_t(x_t)$ indicates the optimal expected total cost, where

$$V_t(x_t) = \min_{a_t(x_t) \in \mathcal{A}_t(x_t)} \{\varphi_t(x_t, a_t(x_t)) + \sum_{x_{t+1} \in X_{t+1}} \Pr(x_{t+1}|(x_t, a_t(x_t))) \cdot V_{t+1}(x_{t+1})\}, \tag{1.21}$$

  and

$$V_T(x_T) = \min_{a_T(x_T) \in \mathcal{A}_T(x_T)} \{\varphi_T(x_T, a_T(x_T))\}. \tag{1.22}$$

SDP is a generic technique for solving stochastic optimisation problems. It is often used in fields of operational research to derive the ideal solution, particularly decision-making problems under uncertainty. However, it is computationally inefficient due to the so-called "curse of dimensionality" (Bellman and Dreyfus, 2015). This thesis applies the SDP model to provide benchmark expected total cost as an optimal policy in Chapters 2, 3 and 4 to yield the comparison in terms of the optimality gaps.

1.2.5.2 Stochastic programming bounding techniques

A widely-applied method for modelling optimisation with uncertainty is stochastic programming. This subsection introduces stochastic programming and its bounding techniques, Jensen's lower bound and Edmundson-Madansky upper bound, which are mainly based on Kall et al. (1994) and Birge and Louveaux (2011).

Recall the expected immediate cost $\varphi_t(x_t, a_t(x_t))$ at stage t given state x_t and action $a_t(x_t)$. A T-stage stochastic programming can be presented as

$$\min \sum_{t=1}^{T} \varphi_t(x_t, a_t). \tag{1.23}$$

The conventional approach to calculating lower and upper bounds for a stochastic program objective is to establish a deterministic problem by substituting all of the random variables with their expected values and then utilising bounding techniques, Jensen's lower bound and Edmundson-Madansky upper bound.

Jensen's lower bound Consider a random variable ζ and a convex function $\phi(\zeta)$ defined on the support $\Omega = [a, b]$. $\phi(\zeta)$ is bounded by a linear function of ζ illustrated in Figure 1.10. The best lower bound is tangent to $\phi(\zeta)$ at point $\mathbb{E}[\zeta]$, where

$$\mathbb{E}[\phi(\zeta)] \geq \phi(\mathbb{E}[\zeta]). \tag{1.24}$$

Edmundson-Madansky's upper bound Consider a random variable ζ and a convex function $\phi(\zeta)$ defined on the support $\Omega = [a, b]$. $\phi(\zeta)$ is bounded by a linear function $U(\zeta)$ between points $(a, \phi(a))$ and $(b, \phi(b))$ illustrated in Figure 1.10, where

$$U(\zeta) = \frac{1}{b-a} \times [\phi(b) - \phi(a) + b\phi(a) - a\phi(b)]. \tag{1.25}$$

These bounding techniques have been extended to higher dimensions by Frauendorfer (1996), Kuhn (2006) and Natarajan and Teo (2017). In this thesis, these two bounds are applied in Chapters 2, 3, and 4 to approximate the expected holding and penalty costs through piecewise linear approximation techniques, as introduced in the next subsection.

1.2.6 Piecewise linear approximation technique

This subsection introduces a piecewise linear approximation based on the first order loss function and its complementary that applies Jensen's lower bound and Edmundson-Madanski's upper bound, respectively, as discussed above. This technique is developed by Rossi et al. (2014) and is applied in stochastic inventory

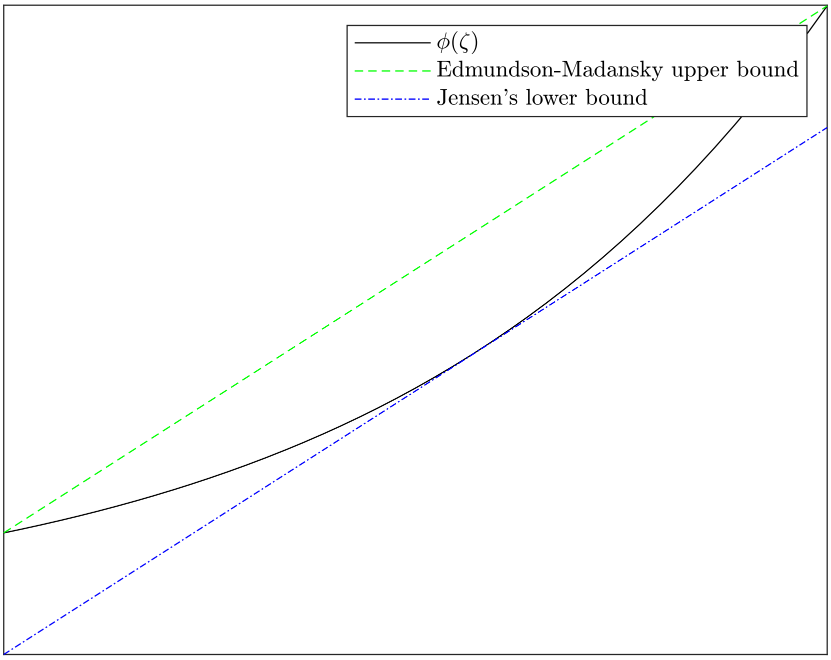


Figure 1.10: Jensen's lower bound and Edmundson-Madansky's upper bound.

control by Rossi et al. (2015) and Xiang et al. (2018). This thesis extensively uses these techniques in later chapters.

1.2.6.1 First order loss function and its complementary

Consider a random variable ω and a scalar variable x. The first order loss function is defined as

$$\mathcal{L}(x, \omega) = \mathbb{E}[\max(\omega - x, 0)] \tag{1.26}$$

and its complementary as

$$\hat{\mathcal{L}}(x, \omega) = \mathbb{E}[\max(x - \omega, 0)], \tag{1.27}$$

where $\mathbb{E}$ denotes taking expectation with respect to ω.

Lemma 1.2.4. (Rossi et al., 2014) The first order loss function $\mathcal{L}(x, \omega)$ can be expressed as

$$\mathcal{L}(x,\omega) = \hat{\mathcal{L}}(x,\omega) - (x - \mathbb{E}[\omega]) \tag{1.28}$$

Consider a support Ω and W-set partitions of $\Omega_1,\ldots,\Omega_W$, $\Omega_i \subset \Omega$ for $i = 1,\ldots,W$. Let $g_\omega(\cdot)$ denote the probability density function of ω, we define

$$p_i = \Pr(\omega \in \Omega_i) = \int_{\Omega_i} g_{\omega_i}(t)\mathrm{d}t, \tag{1.29}$$

and

$$\mathbb{E}[\omega|\Omega_i] = \frac{1}{p_i}\int_{\Omega_i} t g_\omega(t)\mathrm{d}t. \tag{1.30}$$

Let $\hat{\mathcal{L}}_{lb}(x,\omega)$ be the lower bound of the complementary of the first order loss function $\hat{\mathcal{L}}(x,\omega)$. Applying Jensen's lower bound and the partition above,

$$\hat{\mathcal{L}}_{lb}(x,\omega) = \sum_{i=1}^{W} p_i \max(x - \mathbb{E}[\omega|\Omega_i], 0), \tag{1.31}$$

and it is equivalent to a piecewise linear function of x with breakpoints at $\mathbb{E}[\omega|\Omega_1]$, $\mathbb{E}[\omega|\Omega_2]$, $\cdots$, $\mathbb{E}[\omega|\Omega_W]$ as

$$\hat{\mathcal{L}}_{lb}(x,\omega) = \begin{cases} 0, & x \le \mathbb{E}[\omega|\Omega_1]; \\ p_1 x - p_1\mathbb{E}[\omega|\Omega_1], & \mathbb{E}[\omega|\Omega_1] \le x \le \mathbb{E}[\omega|\Omega_2]; \\ (p_1+p_2)x - (p_1\mathbb{E}[\omega|\Omega_1] + p2\mathbb{E}[\omega|\Omega_1]), & \mathbb{E}[\omega|\Omega_2] \le x \le \mathbb{E}[\omega|\Omega_3]; \\ \cdots, & \cdots; \\ x\sum_{i=1}^{W} p_i - \sum_{i=1}^{W} p_i\mathbb{E}[\omega|\Omega_i], & \mathbb{E}[\omega|\Omega_W] \le x. \end{cases}$$

Lemma 1.2.5. (Rossi et al., 2014) The i-th segment of $\hat{\mathcal{L}}_{lb}(x,\omega)$ is

$$\hat{\mathcal{L}}^i_{lb}(x,\omega) = x\sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k\mathbb{E}[\omega|\Omega_k], \qquad \mathbb{E}[\omega|\Omega_i] \le x \le \mathbb{E}[\omega|\Omega_{i+1}], \tag{1.32}$$

where $i = 1,\ldots,W$; $x = 0$ for $x \le \mathbb{E}[\omega|\Omega_1]$.

Lemma 1.2.6. (Rossi et al., 2014) According Lemma 1.2.4, The i-th segment of $\mathcal{L}_{lb}(x, \omega)$ is

$$\mathcal{L}^i_{lb}(x, \omega) = x \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k \mathbb{E}[\omega|\Omega_k] - (x - \mathbb{E}[\omega]) \qquad \mathbb{E}[\omega|\Omega_i] \le x \le \mathbb{E}[\omega|\Omega_{i+1}], \tag{1.33}$$

where $i = 1, \dots, W$; $x = 0$ for $x \le \mathbb{E}[\omega|\Omega_1]$.

$\hat{\mathcal{L}}^i_{lb}(x, \omega)$ and $\mathcal{L}^i_{lb}(x, \omega)$ are directly obtained by applying Jensen's inequality. Upper bounds of $\hat{\mathcal{L}}(x, \omega)$ and $\mathcal{L}(x, \omega)$, can be obtained according to Rossi et al. (2014) by applying Edmundson-Madanski's upper bound with Lemma 1.2.5 and 1.2.6.

Lemma 1.2.7. Consider the upper bound of $\hat{\mathcal{L}}(x, \omega)$

$$\hat{\mathcal{L}}_{ub}(x, \omega) = \sum_{i=1}^{W} p_i \max(x - \mathbb{E}[\omega|\Omega_i], 0) + e_W \tag{1.34}$$

as a piecewise linear function with $W + 1$ segments. The i-th segment of $\hat{\mathcal{L}}^i_{ub}(x, \omega)$ is

$$\hat{\mathcal{L}}^i_{ub}(x, \omega) = \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k \mathbb{E}[\omega|\Omega_i] + e_W \tag{1.35}$$

for $\mathbb{E}[\omega|\Omega_i] \le x \le \mathbb{E}[\omega|\Omega_{i+1}]$; and $x = e_W$ for $x \le \mathbb{E}[\omega|\Omega_1]$. The i-th segment of $\mathcal{L}^i_{ub}(x, \omega)$ is

$$\mathcal{L}^i_{ub}(x, \omega) = \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k \mathbb{E}[\omega|\Omega_i] - (x - \mathbb{E}[\omega]) + e_W \tag{1.36}$$

for $\mathbb{E}[\omega|\Omega_i] \le x \le \mathbb{E}[\omega|\Omega_{i+1}]$; and $x = e_W$ for $x \le \mathbb{E}[\omega|\Omega_1]$.

Lemma 1.2.8. (Rossi et al., 2014) Given the number of W, $\Omega_1, \dots, \Omega_W$, $\Omega_i \subset \Omega$ for $i = 1, \dots, W$. Let $g_\omega(\cdot)$ is an optimal partition of the support Ω under a minimax strategy, if and only if approximation errors at breakpoints are all equal.

Lemma 1.2.9. (Rossi et al., 2014) Assume that the probability density function is symmetric about a mean value. Then under a minimax strategy, if $\Omega_1, \dots, \Omega_W$, $\Omega_i \subset \Omega$ is an optimal partition of the support Ω, breakpoints will be symmetric about the mean value.

Rossi et al. (2014) propose an efficient approach to split the support Ω into W disjoint regions with uniform probability mass, $p_i = \Pr\{\omega \in \Omega_i\} = \frac{1}{W}$ for $i = 1, 2, \ldots, W$, which uniquely determines $\mathbb{E}[\omega|\Omega_i]$. The maximum approximation error e_W is observed at breakpoints of this piecewise linear approximation.

This piecewise linear approximation is applied in Chapters 2, 3 and 4 of this thesis to formulate the problem based on which near-optimal policies are derived efficiently. In what follows, this thesis discusses two types of demands (Normal and Poisson distributions) that are considered in the following chapters.

1.2.6.2 Piecewise linear approximation in normal distribution

Consider a special case of the standard Normal random variable ξ. The piecewise linear approximation of ξ can be extended to the general case of a general Normal random variable ζ with mean μ and standard deviation σ as below.

Lemma 1.2.10. $\hat{\mathcal{L}}(x, \zeta)$ can be expressed in terms of the standard Normal cumulative distribution function

$$\hat{\mathcal{L}}(x, \zeta) = \sigma \int_{-\infty}^{\frac{x-\mu}{\sigma}} \Phi(x)\mathrm{d}t = \sigma \hat{\mathcal{L}}(\frac{x - \mu}{\sigma}, \xi). \tag{1.37}$$

Example 1.2.3. Consider a standard Normal random variable ξ on $\mathbb{R}$ partitioned into two segments. We can easily obtain the probability $p_1 = 0.5$, the breakpoint 0 with the maximum approximation error $e_W = \frac{1}{\sqrt{2\pi}}$. Figure 1.11 and 1.12 illustrates lower and upper bounds of loss function $\mathcal{L}(x, \xi)$ and its complementary $\hat{\mathcal{L}}(x, \xi)$.

1.2.6.3 Piecewise linear approximation in Poisson distribution

This thesis focuses on applying a lower bound for Poisson distribution to avoid massive computation, as will be discussed later. According to the technique in (Rossi et al., 2014), $\Omega_1 = [0, a_1]$, $\Omega_i = [a_{i-1}, a_i]$ for $i = 2, \cdots, N-1$ and $\Omega_N = [a_{N-1}, \infty]$. Let the probability density function of d_t be $g_{\lambda_{\lambda_t}}(k) = e^k/\lambda_t!$ with its cumulative function $G_{\lambda_t}(k)$, $g^{-1}_{\lambda_t}$ and $G^{-1}_{\lambda_t}$ be their inverse functions, which returns the value of k satisfying $g_{\lambda_t}(k) = p$, then

$$a_i = \lceil G^{-1}_{\lambda_t}(\frac{i}{N}) \rceil, \tag{1.38}$$

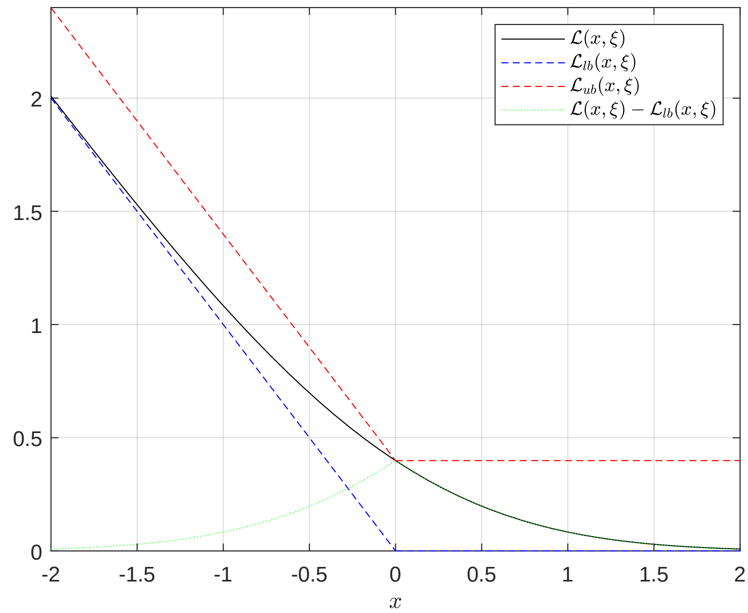


Figure 1.11: Loss function $\mathcal{L}(x, \xi)$ with its two-segment approximation in lower bound $\mathcal{L}_{lb}(x, \xi)$ and upper bound $\mathcal{L}_{ub}(x, \xi)$.

and the probability p_i that a realisation of the Poisson random variable d_t (i.e. a value of demand d_t) locates within the subregion i is

$$p_i = \Pr\{d_t \in \Omega_i\} = \sum_{\Omega_i} g_{\lambda_t}(u) \; \mathrm{d}u, \tag{1.39}$$

and

$$\mathbb{E}[d_t|\Omega_i] = \frac{N}{i} \sum_{\Omega_i} u g_{\lambda_t}(u) \; \mathrm{d}u, \tag{1.40}$$

where $i = 1, 2, \cdots, N$.

Example 1.2.4. Consider a Poisson random variable ζ with $\lambda = 10$ partitioned into two segments. According to Equations (1.38) the breakpoint is obtained at $x = 11$. Figure 1.13 and 1.14 illustrates the lower bound of loss function $\mathcal{L}(x, \zeta)$ and its complementary $\hat{\mathcal{L}}(x, \zeta)$. Note that these curves are connected to better present the trends, while the loss function and its complementary are formulated as discrete functions.

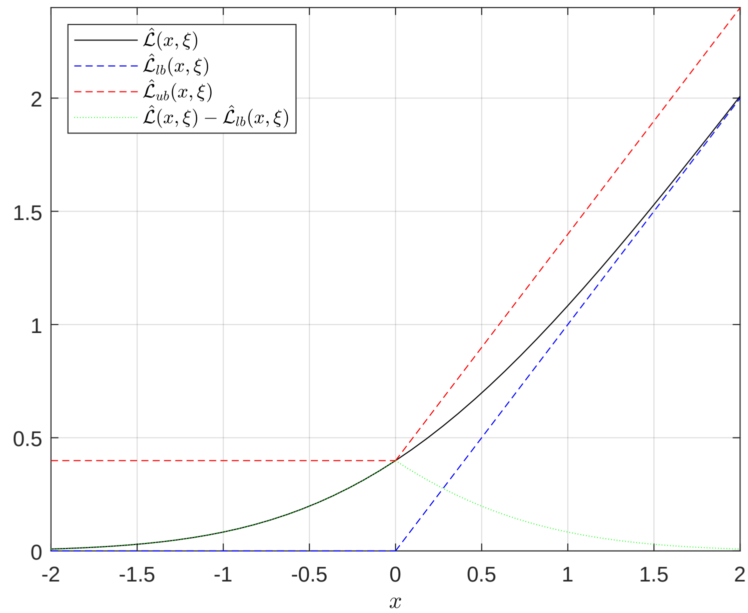


Figure 1.12: Complementary of the loss function $\hat{\mathcal{L}}(x, \xi)$ with its two-segment approximation in lower bound $\hat{\mathcal{L}}_{lb}(x, \xi)$ and upper bound $\hat{\mathcal{L}}_{ub}(x, \xi)$.

The existing application of piecewise linear approximation is built on Normal distributions. However, since Poisson demand is also common in practice and captures real-world discreteness, in Chapters 2 and 4, non-stationary Poisson demand will also be considered in computational analysis. As will be explained later, Poisson-piecewise will require a large amount of pre-computation, so the long-horizon problem instances are conducted on Normal demand.

Meanwhile, the piecewise linear approximation models the expected holding and penalty cost in stochastic inventory problems. In later chapters, they are incorporated into mathematical programming as objective functions to solve near-optimal policies. The next subsection will present this programming approach and its solution methods.

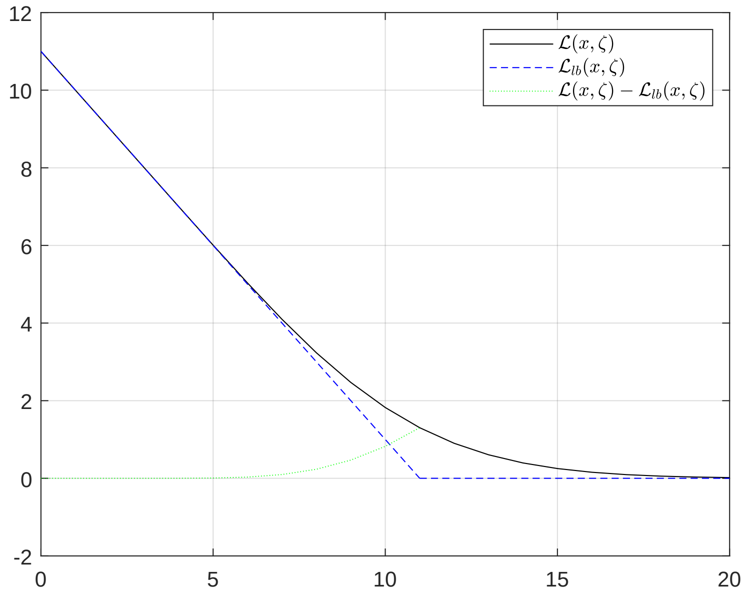


Figure 1.13: Loss function $\mathcal{L}(x, \zeta)$ with its two-segment approximation in lower bound $\mathcal{L}_{lb}(x, \zeta)$.

1.2.7 Mixed integer linear programming

To solve a broad class of stochastic inventory optimisation problems with non-linear expected total costs (Section 1.2.3) , this thesis applies the piecewise linear approximation (Section 1.2.6) in the frame of Mixed Integer Linear Programming (MILP) in Chapters 2, 3 and 4. This subsection presents the general MILP form and its solution methodologies.

The general formulation of MILP takes the following form as (Wolsey, 1998)

$$\min \quad c_1^T \vec{x} + c_2^T \vec{y}, \tag{1.41}$$

$$\text{s.t.} \quad A\vec{x} + B\vec{y} \leq b, \tag{1.42}$$

$$\vec{x} \geq 0, \tag{1.43}$$

$$x_i \geq 0, y_j \in \mathbb{N}, \forall i \in \mathcal{I} \text{ and } j \in \mathcal{J}, \tag{1.44}$$

where x and y are vectors of decision variables with $|\mathcal{I}|$ and $|\mathcal{J}|$ dimensions, re-

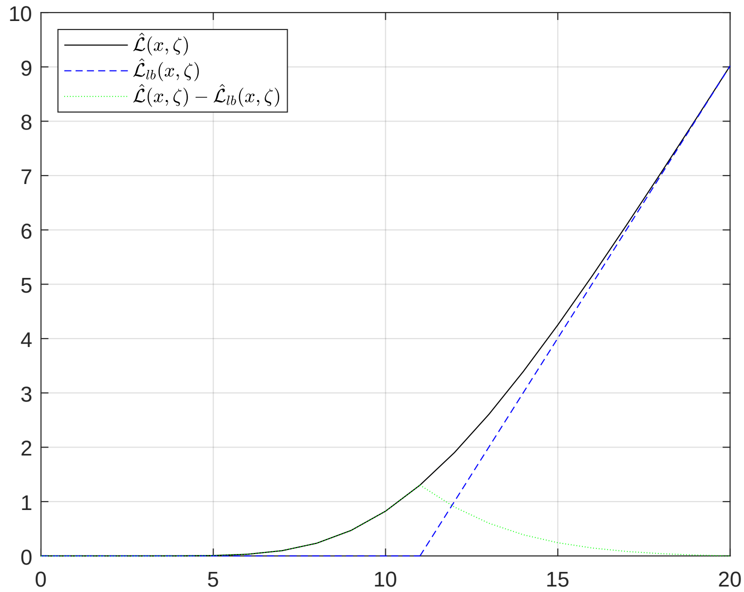


Figure 1.14: Complementary of the loss function $\hat{\mathcal{L}}(x, \zeta)$ with its two-segment approximation in lower bound $\hat{\mathcal{L}}_{lb}(x, \zeta)$.

spectively; c_1, c_2 and b are parameters and A and B are matrices. The objective function and all constraints are linear, and decision variables can be integers (or binaries).

MILP is extensively employed due to the broad availability of effective and state-of-the-art MILP solvers such as Xpress, CPLEX and GUROBI with common solution methods such as branch-and-bound (Land and Doig, 1960) and branch-and-cut (Padberg and Rinaldi, 1991). This thesis applies MILP to formulate stochastic inventory control problems and develops MILP-based heuristics to obtain near-optimal policies in Chapters 2, 3 and 4.

1.2.8 Shortest path problem

Based on MILP models, this thesis also discusses the efficient heuristic method in Chapter 3 developed from the shortest path problem, which is introduced in this subsection.

Definition 1.2.1. (Bertsimas and Tsitsiklis, 1997) Given a set $\mathcal{V}$ of nodes and a set $\mathcal{E}$ of directed arcs, $G = (\mathcal{V}, \mathcal{E})$ is defined as a *directed graph*, where a *directed arc* is an ordered pair (i, j) of distinct nodes.

This definition allows for both (i, j) and (j, i) to be elements of the arc set $\mathcal{E}$, but self-arcs such as (i, i) are not allowed.

Definition 1.2.2. (Bertsimas and Tsitsiklis, 1997) In a directed graph, a *walk* is defined as a sequence $i_1, \ldots, i_t$ of nodes, together with an associated sequence $a_1, \ldots, a_{t-1}$ of arcs such that for $k = 1, \ldots, t-1$.

We have either $a_k = (i_k, i_{k+1})$, in which a_k is a *forward arc*, or $a_k = (i_{k+1}, i_k)$, in which a_k is a *backward arc.* Note that if i_k and i_{k+1} are consecutive nodes in a walk and if (i_k, i_{k+1}) are both arcs of the underlying directed graph, then either arc can be used in the walk.

Definition 1.2.3. (Bertsimas and Tsitsiklis, 1997) A walk is defined to be a *path* if all of its nodes $i_1, \ldots, i_t$ are distinct, and a *cycle* if the nodes $i_1, \ldots, i_{t-1}$ are distinct and $i_t = i_1$. *A walk or path* is defined as *directed* if it only contains forward arcs. The graph containing no cycle is defined as an *acyclic* graph.

Given a directed graph $G = (\mathcal{V}, \mathcal{E})$. For each arc $(i, j) \in \mathcal{E}$, we associate a non-negative cost c_{ij}. A path from a source node s to a target node t is said to be the shortest if it has the minimum cost among all possible paths with the same origin and destination. The cost c_{ij} can be interpreted, for instance, as the length of the arc (i, j). Then the minimum cost path from node s to t is the shortest path that connects these two nodes.

1.2.8.1 Dijkstra's algorithm

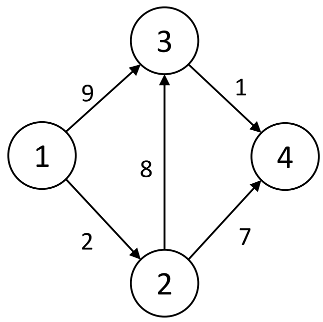
Classic solution approaches the shortest-path problems include the Bellman-Ford algorithm, Label correcting methods and Dijkstra's algorithm. Bertsimas and Tsitsiklis (1997) show that Dijkstra's algorithm is an alternative to the other two and is more efficient with a computation time $\mathcal{O}(|V|^2)$. The key idea of Dijkstra's algorithm is to identify the nodes in the order of their respective shortest path lengths, beginning with the node with the smallest path length.

The objective of Dijkstra's algorithm is to find the shortest path between nodes in a graph. Its procedure is presented as follows.

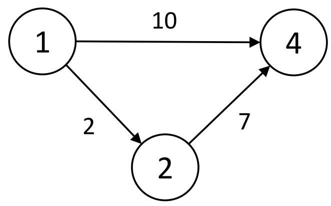
Dijkstra's Algorithm

1. Assume $c_{i,j}$ is assigned to every pair (i, j) of distinct nodes with $i \neq n$; this may be equal to infinity for some pairs.
2. Find a node $l \neq n$ such that $c_{l,n} < c_{i,n}$ for all $i \neq n$.
3. Set $p_l^* = c_{l,n}$.
4. For every node $i \neq l, n$, set
$$c_{i,n} := \min\{c_{in}, c_{i,l} + c_{l,n}\}. \tag{1.45}$$
5. Remove node l and all edges connected to l from the graph and apply the same steps to the new graph.
6. Apply steps 2 to 5 to the new graph. Each iteration evaluates the shortest path length for one more node and, therefore, after $n-1$ iterations, the algorithm terminates.

Example 1.2.5. Apply Dijkstra's algorithm to the graph shown in Figure 1.15a with node $n = 4$ as the destination. The procedure first defines $l = 3$ and $p_3^* = 1$. The following arc lengths are $c_{14} = \min\{\infty, 9+1\} = 10$ and $c_{24} = \min\{7, 8+1\} = 7$; then eliminates node 3 and obtains the graph shown in Figure 1.15b. $l = 2$ and $p_2^* = 7$ and $c_{14} = \min\{10, 2+7\} = 9$ are obtained, from which node 2 is eliminated. Since node 1 is the only non-terminal node left, p_1^* is then equal to the current value $c_{14} = 9$.



(a) Applying Dijkstra's Algorithm in the graph. Assume the arcs that are not shown have infinite length.



(b) The graph obtained after one iteration of Dijkstra's Algorithm.

The typical iteration starts by comparing the coefficients $c_{i,n}$ and this takes $\mathcal{O}(n)$ time. Having determined l, the procedure needs to update c_{in} for each node i. It can be concluded that there are only $\mathcal{O}(n)$ arithmetic operations per iteration. The overall complexity is $\mathcal{O}(n^2)$. For a dense graph with $\Omega(n^2)$ arcs, any shortest path algorithm needs $\Omega(n^2)$ arithmetic operations because, in general, every arc has to be examined at least once. Thus, for dense graphs, Dijkstra's algorithm is the best possible.

Dijkstra's algorithm requires that the lengths of all arcs are non-negative. This thesis will apply this algorithm in Chapter 3 to improve the computation efficiency.

1.3 Literature Review

This section discusses related research in single- and multi-location stochastic inventory control and particular on inventory policies according to uncertainty strategies of Bookbinder and Tan (1988) in Section 1.3.1 and for lateral transshipment in Section 1.3.2. Research gaps and challenges are discussed in Section 1.3.3.

For a comprehensive overview of inventory control literature, readers may refer to Aggarwal (1974); Yano and Lee (1995); Ullah and Parveen (2010); Glock et al. (2014); Bushuev et al. (2015), and of the lateral transshipment to Paterson et al.

(2011).

1.3.1 Stochastic inventory control and uncertainty strategies

The literature review in this subsection is part of a published work as declared at the beginning of the submission:

Ma, X., Rossi, R., and Archibald, T. W. (2019). Stochastic inventory control: A literature review. *IFAC-PaperOnline*, 52(13):1490-1495.

The simplest case for stochastic lot sizing is the Newsvendor problem addressed by Edgeworth (1888), which is concerned with controlling a single item over a single time period. In the early sixties, Wagner and Whitin's work (1958) on dynamic lot sizing with deterministic demand was extended with stochastic settings. To deal with the uncertainty of demands, three uncertainty control strategies were discussed and adopted by Bookbinder and Tan (1988): the "static," the "static-dynamic" and the "dynamic uncertainty." The rest of the literature review on single-location inventory models is based on these strategies regarding inventory policies. Table 1.1 summarises the literature on a single policy (rather than a combination of two or more policies) discussed in this subsection on single-item single-location stochastic inventory control.

Table 1.1: Literature on single-item single-location stochastic lot-sizing problem

Literature	Policy	Demand	Back-order	Lost sale	Service level	Lead time	Ordering cost	Penalty cost
Sox (1997)	(R,Q)	non-stationary	✓			zero	fixed, non-stationary	unit, non-stationary
Vargas (2009)	(R,Q)	non-stationary	✓			fixed	fixed	unit, non-stationary
Galliher et al. (1959)	(s,Q)	stationary	✓			stochastic	fixed, unit	unit
Hadley and Whitin (1962)	(s,Q)	stationary	✓			fixed	fixed, unit	unit
Browne and Zipkin (1991)	(s,Q)	stationary	✓			fixed	fixed, unit	unit
Federgruen and Zheng (1992)	(s,Q)	stationary	✓			fixed	fixed	unit
Rosling (1999)	(s,Q)	stationary	✓			fixed	fixed	unit
Johansen and Thorstenson (1996)	(s,Q)	stationary		✓		fixed/exponential	fixed	unit
Gallego (1998)	(s,Q)	stationary	✓			fixed	fixed/zero	unit
Shenas et al. (2009)	(s,Q)	stationary	✓			fixed	fixed, unit	unit
Tarim and Kingsman (2004)	(R,S)	non-stationary			α	zero	fixed	
Tarim and Kingsman (2006)	(R,S)	non-stationary			α	zero	fixed, unit	
Tarim et al. (2011)	(R,S)	non-stationary			α		fixed	
Özen et al. (2012)	(R,S)	non-stationary	✓		α	zero		unit, non-stationary
Rossi et al. (2015)	(R,S)	non-stationary	✓		α, β^{cyc}	zero	fixed, unit	unit
Tunc et al. (2014)	(R,S)	non-stationary			α	zero	fixed, unit	
Tunc et al. (2018)	(R,S)	non-stationary	✓		α	zero	fixed, unit	
Rossi et al. (2011)	(R,S)	non-stationary			α	zero	fixed, unit	
Scarf (1960)	(s,S)	stationary	✓			fixed, zero	fixed, unit	unit
Iglehart (1963)	(s,S)	stationary	✓			fixed, zero	fixed, unit	unit
Veinott Jr and Wagner (1965)	(s,S)	stationary	✓			fixed	fixed, unit	unit

Richards (1975)	(s, S)	stationary	✓			zero		
Sahin (1982)	(s, S)	stationary	✓			fixed	fixed, unit	unit
Archibald and Silver (1978)	(s, S)	stationary	✓			fixed	fixed, unit	unit
Federgruen and Zipkin (1984)	(s, S)	stationary	✓			fixed	fixed	unit
Zheng and Federgruen (1991)	(s, S)	stationary	✓			fixed	unit	unit
Feng and Xiao (2000)	(s, S)	stationary	✓			zero	fixed	unit
Silver (1978)	(s, S)	stationary	not allowed			zero	fixed	unit
Askin (1981)	(s, S)	non-stationary	✓			fixed	fixed, unit	unit
Bollapragada and Morton (1999)	(s, S)	non-stationary	✓			zero	fixed, unit	unit
Xiang et al. (2018)	(s, S)	non-stationary	✓			zero	fixed, unit	unit
Tunc et al. (2011)	(s, S)	non-stationary	✓			zero	fixed	unit

1.3.1.1 Static uncertainty strategy

Sox (1997) studies the dynamic lot sizing problem with a known cumulative demand function in each period. In this setting, costs are non-stationary, and back-ordering is allowed. A mixed integer non-linear program is applied to formulate this problem. The model describes the immediate cost incurred at the end of each period by the loss function in inventory theory, which is convex but non-linear. By substituting cumulative order quantity into the immediate cost function, inventory variables are eliminated so that the objective function is separable in respect of cumulative order quantity variables. Given this separability, a solution algorithm is derived based on Wagner-Whitin's algorithm (1958) for deterministic problems by adding additional feasibility constraints. The algorithm transforms the objective cost function into a series of multi-period newsvendor problems with various constraints by decomposition and conducts rolling-horizon implementation to obtain the optimal solution.

This policy is also investigated by Vargas (2009) to determine the optimal solution over the entire finite planning horizon for dynamic lot sizing problem, where demand is of known demand density. The model applies assumptions of Wagner and Whitin (1958) and introduces a penalty cost for back-ordering. This model is shown to be equivalent to the shortest path problem in a specified acyclic network using stochastic dynamic programming. Vargas (2009) also provides an optimisation algorithm with a rolling horizon with two stages: (1) to determine optimal replenishment quantities for any sequence of replenishment points, and (2) to sort out the optimal sequence of replenishment from (1).

1.3.1.2 Static-dynamic uncertainty strategy

Research on static-dynamic strategy comprises (s, Q) and (R, S) policies.

(s, Q) policy.

The research on (s, Q) systems where Q, the order quantity, is fixed for all periods — often referred to as (r, Q) systems — started with Galliher et al. (1959), which compares two systems with an arbitrary form stationary probability distribution and with the Poisson distribution. They find that the value of s should be increased along with the increase of Q in the variance of replenishment time to maintain the solution's optimality.

Hadley and Whitin (1962) present an exact solution to the problem where penalty cost is applied for back-ordering. This work also derives a heuristic algorithm that ignores the possibility that demand exceeds the quantity ordered and stock-out during the lead time. Browne and Zipkin (1991) discusses the (s, Q) system when demand is discrete and (multivariate) diffusion based on the discussion in Hadley and Whitin (1962).

Federgruen and Zheng (1992) presents an exact algorithm for the discrete cases. In this method, the optimal parameters are found by incrementally enlarging the interval of base-stock policies over which the average is taken until the total cost stops decreasing. This method's high computational complexity is improved by Rosling (1999) through providing a revised version: Q is initially set to be a lower bound, and a discrete square root algorithm obtains the corresponding s. The number of iterations required in Rosling (1999) method is still significant, and the upper bound is not tight, which means that it is still pseudo-polynomial.

After Hadley and Whitin (1962), several works discussed exact models and heuristic solution methods for the stationary problem under various assumptions.

Johansen and Thorstenson (1996) consider an (s, Q) system with Poisson demand and lost sales. They formulate an exact model and design a policy-iteration algorithm for discounted cases. Gallego (1998) derives a distribution-free solution and provides upper bounds on the optimal long-run average cost and on the optimal batch size. Lau and Lau (2002) propose a method using a spreadsheet's direct optimisation to solve an (s, Q) system with back-ordering. Shenas et al. (2009) propose a recursive procedure for determining the exact policy costs for (s, Q) policy with Poisson demand and constant lead time. Bright and Rossetti (2013) provide a comparison among algorithms for the unconstrained (s, Q) inventory system by evaluating computational performance and solution accuracy of the algorithms for a series of randomly generated instances.

(R, S) policy.

In the view of non-stationary stochastic demand, Tarim and Kingsman (2004) formulate the problem as a mixed integer programming on non-stationary (R, S) policy. They model the total expected cost by minimising the summation of holding and ordering costs under service-level constraints. This formulation allows the simultaneous determination of reorder point and size. Tarim and Kingsman (2006) provide another MIP formulation where the objective function is obtained by the mean of piecewise linear approximation. The accuracy of the approximation can be adjusted by introducing new breakpoints.

Tarim et al. (2011) provide an efficient computational approach to solve the MIP model presented in Tarim and Kingsman (2004). The algorithm converts the relaxation of the original MIP model to the shortest path problem implemented by branch-and-bound procedures. This algorithm also considers the case of infeasibility, where the solution generates a tight lower bound for the optimal cost, and it can be modified to obtain a feasible solution to generate an upper bound. This approach is refined by Rossi et al. (2011) through a filtering and augmenting procedure. Starting from a relaxed state space graph, the method proposed by Rossi et al. (2011) eliminates provably sub-optimal arcs and states (filtering). It then efficiently builds up (augmenting) a reduced state space graph representing the original problem.

Özen et al. (2012) consider both penalty cost and service level, proving that the optimal policy is the base stock policy for both penalty and service-level constrained models and for the capacity limitations and minimum order quantity requirements. Based on the analytical results in piecewise approximation discussed by Rossi et al. (2014), Rossi et al. (2015) extended the model in Tarim and Kingsman (2006) to a mixed integer linear programming formulation for non-stationary stochastic demand and present the constraints of expected holding and back-ordered/lost-sale inventory level under the approximation. The model first applies the loss function and its complementary function to describe the total cost. A piecewise linear approximation approach is utilised to convert the cost function from non-linear to linear. This research also considers several service level measures (α service level on each period, β^{cyc} service level independently for each replenishment cycle and the classic β service level) by adding various constraints. Moreover, the model considers penalty cost for back-ordering, and it can be adapted into a lost sale scheme by introducing a parameter that presents the selling price per product to take into account the associated opportunity cost, which is related to the demand that is not immediately satisfied under the control policy as a prerequisite.

On the other hand, Tarim and Kingsman's model on non-stationary (R, S) policy is reformulated by Tunc et al. (2014) through disconnecting consecutive replenishment periods based on a MIP formulation via the network flow structure of the problem. This formulation has a tighter linear relaxation and superior computational performance. Tunc et al. (2018) generalise this model to develop a dynamic cut generation approach, which can deploy the piecewise model with no prior approximation of the cost function while guaranteeing an arbitrary level of precision. The proposed approach follows the idea of dynamically adding cuts into a relaxed version of the original (R, S) problem for each approximated loss value that exceeds the real loss function value. As such, it can be regarded as a means to establish a piecewise linear approximation on the fly while solving the problem.

1.3.1.3 Dynamic uncertainty strategy

Given the complexity of computing parameters for (s, S) policy, there are many works developing efficient algorithms for stationary (s, S) policy. This stream of research started with Iglehart (1963), which gives bounds for the sequences of $\{s_n\}$ and $\{S_n\}$. It also investigates the limiting behaviours of $\{s_n\}$ and $\{S_n\}$ over the infinite time horizon under back-ordering and lead-time settings. It is proved that the sequences $\{s_n\}$ and $\{S_n\}$ contain convergent subsequences, and every limit point of the sequence $\{S_n\}$ is a minimum for the cost function. Finally, if the cost function has a unique optimum, $\{S_n\}$ will converge.

After Iglehart (1963), Veinott Jr and Wagner (1965) derive a computational approach from renewal theory and stationary analysis and generalise it for the unit interval range of value for discount factor. A resolution is found to guarantee the optimum of (s, S) policy is determined by computation.

Richards (1975) examines the condition under which the inventory position is uniformly distributed, if and only if demands are of unit size. It is also proven that this condition is independent of the lead time distribution or demand distribution. Sahin (1982) also gives mathematical results on (s, S) inventory models based on renewal function. The author also proves that the total cost function is pseudo-convex if the underlying renewal function is concave. This conclusion guarantees that every local minimum is a global minimum of total cost and permits the efficient computation of the optimal policy parameters through a one-dimensional search routine.

To solve the problem efficiently, Archibald and Silver (1978) develop a series of formulae to calculate the cost for (s, S) policies recursively when the inventory system is continuously reviewed with discrete compound Poisson demand. The algorithm starts by finding the optimal s for a given n, where $n = S - s - 1$; then S is determined by finding an optimal n (local optimum). It finds that the value n increases until a local minimum pair of (s, n) is found.

Inspired by Archibald and Silver (1978), Federgruen and Zipkin (1984) propose an algorithm to compute (s, S) policy starting with any arbitrary parameter pair. The algorithm is based on an adaption of the general policy-iteration method for solving a Markov decision problem, where the special structure of (s, S) policies was exploited in several ways. However, this algorithm is easily trapped in local optima due to the quasi-convex nature of the cost function. It is then improved by Zheng and Federgruen (1991) based on properties of the cost function of (s, S) system and the tight lower and upper bounds for two parameters, which are iteratively and easily updated and converge monotonically. Zheng and Federgruen (1991) also exploit a characterisation of the cost function to allow fast update by only altering the value of s. Being different from Zheng and Federgruen (1991), Feng and Xiao (2000) introduce a dummy cost factor and an auxiliary function to search for the optimal cost value. The algorithm revises the dummy cost based on the sign of the auxiliary function and identifies the non-prospective set of S to reduce search efforts. The numerical test shows that this algorithm saves more than 30% evaluation effort compared with the algorithm in Zheng and Federgruen (1991).

Computing (s, S) policy parameters under non-stationary demand is a challenging task. The classic Silver and Meal heuristic algorithm (Silver and Meal, 1973) for deterministic demand was extended by Silver (1978) and Askin (1981) to study the lot-sizing problem under non-stationary stochastic demand.

The algorithm presented in Silver is a stochastic version of (Silver and Meal, 1973). It uses a deterministic model to calculate the number of periods that each order must cover; when this replenishment plan was known, the associated safety stocks were then myopically determined. Askin (1981) explicitly includes the cost effects of probabilistic demand in the choice of the number of periods for which to order (Askin, 1981). In contrast to Silver (1978), Askin (1981) uses the least cost per unit time approach to determine the number of periods the immediate replenishment must cover.

Bollapragada and Morton (1999) approximate the non-stationary problem via a series of stationary problems based on the method developed by Zheng and Federgruen (1991). Parameters were determined by equating the cumulative mean demand of stationary and non-stationary problems over the expected reorder cycle.

Dural-Selcuk et al. (2020) implement computational experiments of Askin (1981) and Bollapragada and Morton (1999) with a rolling horizon approach on a new common test bed. For both approaches, they observed relatively large optimality gaps of 3.9% and 4.9%, respectively.

To overcome these shortcomings, Xiang et al. (2018) introduce a mixed integer non-linear programming formulation for (s, S) system applying the piecewise linear approximation method proposed by Rossi et al. (2015). Xiang et al. (2018) also derive a heuristic algorithm with binary search. Both solution methods outperform the previous heuristics in computational efficiency for short and long-time horizon tests. The comparison between the two proposed algorithms finds that the binary search requires significantly less time than the MINLP.

Finally, Kilic and Tarim (2011) provide the grounds for measuring system nervousness[6] in (R, S) and (s, S) systems when demand is non-stationary. Tunc et al. (2013) conduct a numerical study to compare these policies for cost-effectiveness. Tunc et al. (2011) apply the stationary policy as the approximation to the optimal non-stationary system and find that stationary policies may be an efficient approximation to the optimal non-stationary system when demand information is of high uncertainty with a low penalty cost.

1.3.2 Inventory control with lateral transshipment

This subsection of the literature survey concentrates on the product flows among suppliers and various retailers in multi-location inventory systems, in which the lot-sizing problem takes a critical role in researching the timing and the quantity of replenishment as well as the lateral transshipment.

Inventory systems apply lateral transshipment according to various aspects such as the products' natures, the structure of logistic routes, capital management policies, and so forth. If we assume that the transshipment is applied, the transshipment policy can be categorised as *hybrid* policies with the consideration of the regular replenishment from the warehouse and the lateral transshipment, and *stand-alone* policies, where the demand of a location does not exhaust the local inventory stock. Paterson et al. (2011) review the transshipment papers before 2010 and categorise transshipment into proactive and reactive types; the distinction is that the proactive transshipment takes place at a predefined time to reduce the risk of stock-outs due to future demand, and the reactive transshipment is a recourse action to deal with existing stock-outs or demand (also considered as an emergency replenishment). This thesis refers to the way that the system applies both proactive and reactive transshipment as a *joint* policy. Among vast transshipment literature, this thesis focuses on multi-location inventory systems with stochastic demand for a single item, where the system applies proactive transshipment. Figure 1.16 illustrates the classification referred to in this thesis.

The early research in this scope started from Gross (1963), showing that the

[6] In stochastic inventory systems, unfolding uncertainties in demand lead to the revision of earlier replenishment plans, resulting in instability or so-called *system nervousness*.

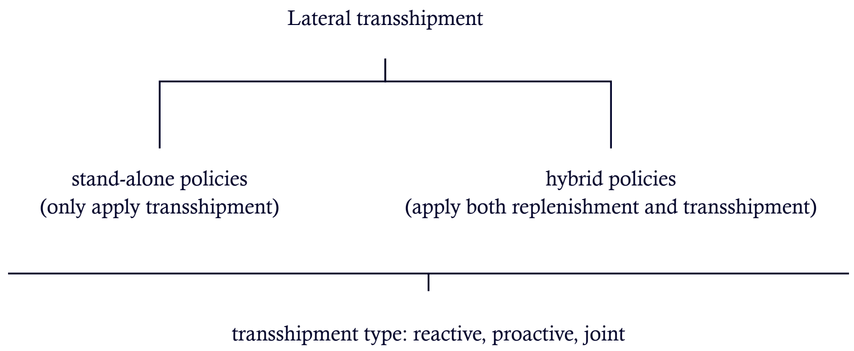


Figure 1.16: Lateral transshipment classification referred to in this thesis.

optimal hybrid policy of a two-location single-period problem depends on the starting inventory level and cost parameters if the lead time is negligible, where the starting inventory levels divide the plane into six regions and each region has a corresponding optimal policy according to different cost parameters; the corresponding multi-location case is considered by Karmarkar and Patel (1977) with robust linear programming. For a special case when the transshipment takes place at the end of the order cycle once only for non-identical locations, Bertrand and Bookbinder (1998) show that it results in a single-period transshipment problem. The multi-period cases are found to have similar characteristics as Gross in the optimal solutions by Karmarkar (1981).

Hoadley and Heyman (1977) state that applying proactive transshipment statically at the beginning of a period is beneficial as an additional opportunity is given to managing stock-outs. In general, the transshipment timing has a significant influence on the transshipment policy's performance, as Tagaras and Vlachos (2002) explained, that the transshipment timing has a large influence on transshipment policy's performance, but it depends on system characteristics. However, due to the increased complexity, most research is accomplished only when the proactive transshipment is set to static points. Under this "static policy", referred by Agrawal et al. (2004), the redistribution of stock can take place at the beginning of a period (Allen, 1958, 1961, 1962) or at a predefined point of a period (Das, 1975). However, a dynamic policy developed by Agrawal et al. (2004) outperforms the static in terms of costs, where real-time demand information is embedded in the model to schedule transshipment with dynamic programming and solve near-optimal timing of transshipment and new stocking levels at stocking locations via a heuristic based on dynamic programming.

To further discuss the timing, in replenishment models, the start or end (or any other point) of an order period provides a 'natural' opportunity to mitigate the mismatch between supply and demand by redistributing the stock over all locations. Paterson et al. (2011) argue that this explains the reason why the majority of research on proactive transshipment is based on a periodic review. Most research on proactive lateral transshipment is accomplished in a periodic review setting. Seidscher and Minner (2013) is one case in which a proactive transshipment policy is introduced in a continuous review setting, where a proactive transshipment policy is introduced in a continuous review setting that is based on redirection of customer demand.

In the line of proactive transshipment, Diks and De Kok (1996) consider the lead time and propose the Consistent Appropriate Share (CAS) rationing policy to balance the system stock to keep each location's constant fraction. Later, Diks and De Kok (1998) propose the Balanced Stock (BS) rationing policy as a general form of CAS and show that it outperforms CAS. They also show that the system benefited most from BS when a large number of retailers, a high service level, or a long replenishment lead time is involved. Abouee-Mehrizi et al. (2015) provide a thorough analytical study and provide an 'order-up-to curve' policy formed by four switching curves that divide the plane, which is mathematically proved to be optimal.

This class of problems are also addressed by the approach of approximate dynamic programming. Archibald et al. (2010) implement approximate dynamic programming that applies a dynamic programming policy improvement step to a static policy, which is shown to be close to the optimal policy for a small number of locations in a numerical study. Powell (2016) applies an approximation technique for the first time, which is extended by Meissner and Senicheva (2018) for stand-alone proactive transshipment.

Comparison between proactive and simple reactive transshipment is made by Banerjee et al. (2003) with a base-stock policy for replenishment actions. They propose two transshipment policies: Transshipment Inventory Equalisation (TIE) based on proactive transshipment and Transshipment Based on Availability (TBA) triggered by shortages in a reactive way. Two policies are compared by simulation and concluded that TBA is slightly more effective in preventing shortage, and TIE achieves a lower cost for the system in general. This comparison is extended by Lee et al. (2007) with a proactive Service Level Adjustment (SLA) policy, which is illustrated to be a better policy than TIE and TBA in terms of cost performance when transportation costs are sufficiently low.

More recently, Paterson et al. (2012) introduced a joint transshipment under a continuous review replenishment policy. Van der Heide and Roodbergen (2013) propose a hybrid transshipment policy for a multi-location system along with two heuristics, where one considers clustering of locations, and another evaluates periods before determining the transshipment decisions. Van Wijk et al. (2019) investigate the optimal policy for a two-location system for multi-classes of demand via the Markov decision process and show that the optimal policy is a threshold type policy. They show the ordering of the threshold and give conditions under which the obtained policy is optimal. The benefit of hybrid transshipment policy is investigated by Glazebrook et al. (2015) for a multi-item system with periodic replenishment under non-stationary demand.

Table 1.2 summarises the literature in proactive transshipment that we referred to.

Table 1.2: Key literatures on proactive transshipment.

Literature	**Location**	**Back-order**	**Lost sale**	**Hybrid transshipment**	**Joint replenishment**	**stand-alone transshipment**
Allen (1958)	N, non-identical		✓	✓		
Allen (1961)	N, non-identical		✓	✓		
Allen (1962)	N, non-identical		✓	✓		
Gross (1963)	2, non-identical	✓		✓		
Hoadley and Heyman (1977)	N, identical	✓		✓		
Karmarkar and Patel (1977)	N, non-identical	✓		✓		
Karmarkar (1981)	N, non-identical	✓		✓		
Diks and De Kok (1996)	N, non-identical		✓	✓		
Diks and De Kok (1998)	N, non-identical		✓	✓		
Bertrand and Bookbinder (1998)	N, non-identical	✓		✓		
Tagaras and Vlachos (2002)	2, non-identical	✓		✓		
Banerjee et al. (2003)	N, non-identical	✓			✓	
Agrawal et al. (2004)	N, identical	✓		✓		
Lee et al. (2007)	N, non-identical	✓		✓		
Paterson et al. (2012)	N, non-identical	✓			✓	
Seidscher and Minner (2013)	N, non-identical		✓		✓	
Van der Heide and Roodbergen (2013)	N, non-identical		✓		✓	
Abouee-Mehrizi et al. (2015)	2, non-identical		✓	✓		
Glazebrook et al. (2015)	N, non-identical	✓	✓		✓	
Meissner and Senicheva (2018)	N, non-identical		✓	✓		✓
Van Wijk et al. (2019)	2, non-identical		✓		✓	

1.3.3 Research gaps and Challenges

Based on the literature survey discussed above, this section summarises the research gaps in the following three aspects.

- **Lack of research on (s, Q)-type policies with non-stationary stochastic demand.**

A gap is noticed in the study of non-stationary demand: no literature discusses or investigates the static-dynamic uncertainty strategy in the form of an (s, Q) policy, while another static-dynamic inventory policy, (R, S) policy, is discussed in (Bookbinder and Tan, 1988; Tarim and Kingsman, 2004; Rossi et al., 2015).

To fill the gap, in Chapter 2, this thesis will propose a new control strategy for the stochastic lot-sizing problem under non-stationary demand. Under this strategy, the reorder points s_t vary with time with two types of order quantities Q and Q_t, which leads to (s_t, Q_t) and the (s_t, Q) policies. These policies require values for s_t and Q_t (or Q) to be determined at the beginning of the planning horizon. A further comparison between two static-dynamic inventory policies will also be conducted in the computational analysis to solidify a proposition of the optimal form under the static-dynamic strategy.

- **Low efficiency on near-optimal policy computation with the penalty-cost scheme.**

This discussion continues building on the static-dynamic strategy for the non-stationary single-location problems, where the mathematical formulations or heuristics on (R_t, S_t) policy discuss more the service levels than the shortage/penalty cost. According to the definition of service levels, these problems (Tunc et al., 2014, 2018) formulate lower bounds of the expected order quantity and upper bounds of the expected total cost and further produce more compact formulations than Tarim and Kingsman (2004) and efficient model-based algorithms. Under the penalty cost scheme, back-ordered items or lost sales are added to the trade-off between the expected total cost and the order quantity, and this complicates the modelling.

In Chapter 3, this thesis provides an approximation for (R_t, S_t) policy under a penalty cost scheme with a shortest-path formulation and introduces an efficient approach to generate a near-optimal solution feasible to the original problem.

- **Lack of research on multi-location non-stationary stochastic inventory control considering fixed costs and non-stationarity.**

In multi-location inventory systems, decisions must account for both the source and quantity of transshipments, which complicates inventory optimisation and makes it challenging to achieve optimality. Consequently, existing studies have primarily focused on either two-location systems or single-period problems (or both) to analyse and identify optimal policies.

These simplifications cannot adequately capture the complexities of real-world situations, where demand is often characterised by uncertainty and non-stationary. Indeed, in most practical industry circumstances, demand is not only stochastic but also non-stationary. Despite this, there is limited research addressing the challenges associated with non-stationary demand in multi-location inventory systems.

Moreover, the importance of transshipment timing has been highlighted by Agrawal et al. (2004) for stand-alone proactive transshipment and by Tagaras and Vlachos (2002) for hybrid policy, noting that it is often treated as a static point. Within this context, [7]existing research on analysing and computing optimal policies heavily relies on either excluding fixed costs (for replenishment and transshipment) (Diks and De Kok, 1996, 1998; Abouee-Mehrizi et al., 2015). However, fixed costs can have a significant impact on inventory decisions. For example, high fixed ordering costs discourage frequent replenishment and encourage large order quantities, and the variation of the fixed transshipment cost can influence transshipment frequency among retailers and further affect the need for replenishment from the warehouse.

Therefore, there is a gap in the literature in that few researchers conduct study on the computation and analysis for inventory systems with proactive lateral trans-

[7]Other research in this scope employs approximate dynamic programming (Archibald et al., 2010; Meissner and Senicheva, 2018). This thesis does not extend the discussion on approximate dynamic programming.

shipment considering the non-stationary demand and the fixed costs, which will be filled by this thesis in Chapter 4.

1.4 Thesis statement

This section summarises works described in this dissertation in Section 1.4.1 and main contributions in Section 1.4.2. Sections 1.4.3 to 1.4.5 elaborate respective summaries for each of the following chapters.

1.4.1 Summary

This thesis investigates near-optimal inventory policies in the area of stochastic inventory control. Hybrid techniques integrating mathematical programming and heuristic algorithms are applied from Operational Research to improve decision-making.

A significant challenge is to determine near-optimal inventory control policies under non-stationary stochastic demand for the classical situation as described by Scarf (1960). This is a noteworthy problem in practice and research, as mentioned in Sections 1.2 and 1.3 and has become more prevalent in recent years with the globalisation and the development of large-scale supply chains. Under the modelling assumptions of single- and multi-location, uncertainty strategies, penalty-cost schemes and hybrid replenishment settings, this thesis examines various existing formulations of this problem, for which three issues and research gaps are identified in the existing methods.

Firstly, for single-location problems, no literature discusses or investigates the static-dynamic uncertainty strategy in the form of an (s, Q)-type policy; on the other hand, (R_t, S_t) policy has been widely explored (Tarim and Kingsman, 2004, 2006; Tarim et al., 2011; Özen et al., 2012; Rossi et al., 2015) with mathematical programming techniques. It follows that there is a lack of a comparative analysis of both policies regarding applicability and optimality.

Secondly, for multi-location problems, the existing computational approaches are difficult to generalise due to the strong reliance on system characteristics without the fixed costs (Diks and De Kok, 1996, 1998; Abouee-Mehrizi et al., 2015).In addition, most of the existing research for multi-location lot-sizing problems with lateral transshipment are conducted upon stationary stochastic demand. Although the non-stationary uncertainty complexes the problem, it fits more in real-world instances.

Thirdly, for both aspects of lot-sizing problems, due to the combinatorial nature of stochastic inventory control, numerical approaches discussed in the existing literature for determining near-optimal policy parameters are complex and can hardly be implemented efficiently, involving considerable efforts in coding. Although some easy-to-implement approaches are presented as Rossi et al. (2015); Tunc et al. (2018), they are constrained by specific problem characteristics.

The following chapters describe unique, near-optimal, easily implementable, and broadly applicable mathematical programming heuristics to address the shortcomings of the literature's existing techniques. These methods can relax the strict assumptions made in the literature on stochastic inventory control, allowing them to be applied in previously unrecognised contexts.

These models that are provided hereinafter fill these gaps and answer classical questions for the lot-sizing problems summarised by Silver (1981) under non-stationary demand and various location assumptions. They address near-optimal inventory policies under the classical categorisation on the uncertainty strategies by Bookbinder and Tan (1988) and discuss hybrid replenishment policy with lateral transshipment under the importance of transshipment timing by Agrawal et al. (2004) and Tagaras and Vlachos (2002). Contrary to other ways suggested in the literature, the methods in this dissertation can be applied and solved using straightforward CPLEX and MATLAB modules; no laborious coding is needed.

Comprehensive numerical analyses are conducted on methods presented in the thesis. Results show tight optimality gaps for a single-location problem (2.31%) with (s_t, Q_t) policy and a two-location problem (0.55%) with a hybrid replenishment policy. The computational efficiency is improved into polynomial time for the single-location cases. Overall, these models produce competitive optimality gaps and computational efficiency relative to existing approaches in the literature.

1.4.2 Contributions

By developing these cutting-edge, near-optimal and simple-to-implement models to solve a variety of problems, this thesis contributes to the literature on stochastic inventory control.

- **Contribution 1.**

  Chapter 2 addresses the single-item single-location non-stationary stochastic lot-sizing problem under a reorder point – order quantity control strategy. This first research discusses non-stationary (s, Q)-type policies and provides a MILP-based binary search algorithm. An analytic proof shows the optimality of (s_t, Q_t) policy and other static-dynamic inventory policies. This chapter also applies the method introduced to solve large-scale problems. Numerical experiments demonstrate that this method computes near-optimal policy parameters for a broad class of problem instances with an average optimality gap of 2.31% on average.

- **Contribution 2.**

  Chapter 3 addresses an efficient approximation to the single-item single-location lot-sizing problem with non-stationary stochastic demand, where any unmet demand is considered back-order with a penalty cost. This chapter develops an approximation for (R_t, S_t) policy under a shortest-path formulation based on a relaxation of the original problem, whose infeasibility on negative replenishment orders is checked against the non-relaxed problem and resolved by a graph-based augmenting procedure. This is the first paper to apply the shortest-path formulation to non-stationary lot-sizing problems with a penalty cost scheme. The numerical experiment demonstrates that near-optimal (R_t, S_t) policy parameters can be approximated on average in 2.41 seconds for 25-period instances.

- **Contribution 3.**

Chapter 4 introduces a hybrid replenishment policy for single-item multi-location problems with proactive lateral transshipment. This chapter presents stochastic dynamic programming and two-stage dynamic programming for the problem, which is approximated by a MILP-based static uncertainty strategy under a rolling-horizon framework. A computational study shows that the model well-approximates a near-optimal policy with an average optimality gap of 0.55%.

1.4.3 Chapter 2 (Paper I): Approximations for non-stationary stochastic lot-sizing under (s, Q)-type policy

This chapter is joint work with Roberto Rossi and Thomas W. Archibald.

Bookbinder and Tan's control strategies (static, static-dynamic and dynamic) of uncertainty represent different approaches for determining the replenishment timing and quantity and are captured by various inventory policies.

Under the static-dynamic uncertainty strategy, which is captured by (R_t, S_t) and (s, Q)-type policies, a wide range of research is conducted for the non-stationary demand on the (R_t, S_t) policy since Tarim and Kingsman (2004) formulate the problem as a mixed integer programme and followed by various development conducted by Tarim and Kingsman (2006), Tarim et al. (2011), Özen et al. (2012), Rossi et al. (2015) and Tunc et al. (2018). However, no study discusses the non-stationary (s, Q)-type policies.

To formulate the non-stationary lot-sizing problem under (s, Q)-type policies, this paper considers two cases of order quantities: (s_t, Q_t) and (s_t, Q) policies. This chapter separately presents stochastic dynamic programmes for both policies against decision order and order quantity Q (or quantities Q_t), followed by the approach to determining the reorder points s_t accordingly. This chapter shows that the optimal policy results in the (s_t, Q_t) form.

To avoid unnecessary efforts of computing stochastic dynamic programmes, this chapter introduces a mixed integer non-linear programme, which leverages a key block of piecewise linear approximation by Rossi et al. (2014) (presented in Section 1.2.6). This technique is combined with a binary search heuristic to implement large-scale instances.

A comprehensive numerical analysis is carried out on 600 instances with 6-period (60) and 25-period (540) test sets to demonstrate the computational performance of the proposed MINLP-based heuristics. First, this chapter investigates the performance of optimal (s, Q)-type policies against that of optimal non-stationary (s, S) policy and evaluates the difference between optimal and heuristic (s, Q)-type policies. We observe average optimality gaps 1.91% and 3.61% for (s_t, Q_t) SDP and heuristics, and 2.76% and 4.66% for (s_t, Q) SDP and heuristics. Second, this chapter investigates the performance of (s, Q)-type heuristics versus that of the optimal (s_t, S_t) policy. The results are also compared between (s, Q)-type heuristics and another existing static-dynamic uncertainty heuristic, namely the (R_t, S_t) policy discussed by Rossi et al. (2015). We observe average optimality gaps 2.31% for (s_t, Q_t) heuristics, which is slightly better than (R_t, S_t) heuristics with gaps of 2.90%.

In this work, my contribution can be summarised as follows.

1. I surveyed the literature related to the long-standing problems of single-item single-location lot-sizing problems with stochastic demand regarding uncertainty control strategies.
2. I implemented the stochastic dynamic programme of the (s_t, S_t) policy in JAVA.
3. I formulated the stochastic dynamic programmes of (s_t, Q_t) and (s_t, Q) policies and solved them in JAVA.
4. I developed the approach to determine the reorder points under (s_t, Q_t) and (s_t, Q) policies with the optimal Q_t or Q pre-solved.
5. I adapted the piecewise linear approximation introduced by Rossi et al. (2014) and applied in Rossi et al. (2015) and Xiang et al. (2018) to implement MINLP models for both policy parameters; I implemented MINLP-based binary search in JAVA.
6. I conducted numerical studies on 60 small instances (non-stationary Poisson

demand) and 540 large instances (non-stationary Normal demand) with pre-computed results by Rossi et al. (2014), for which I conducted (R_t, S_t) MILP-based heuristic developed by Rossi et al. (2015) to facilitate comparison.

7. I organised all materials and wrote the paper.

1.4.4 Chapter 3 (Paper II): An efficient computation approach to the non-stationary lot-sizing problem under static-dynamic strategy with penalty scheme

This chapter is joint work with Roberto Rossi and Thomas W. Archibald.

This chapter extends Chapter 2 (paper I) to improve the computational efficiency for non-stationary lot-sizing problems under static-dynamic uncertainty and penalty cost schemes. This long-standing problem of static-dynamic control has been studied by Tarim and Kingsman (2004), Tarim and Kingsman (2006), Tempelmeier (2007), Özen et al. (2012) and Rossi et al. (2015) based on the assumption that no negative replenishment order is allowed. On the other hand, Tunc et al. (2014) reformulates the problem based on the network flow structure, and Tunc et al. (2018) generalises the model to a dynamic cut generation approach combining with piecewise linear approximation by Rossi et al. (2014). However, all discussions above are in the scope of the service-level constraints.

This chapter develops an efficient approach to approximate (s_t, S_t) and (R_t, S_t) policy parameters for non-stationary stochastic lot-sizing problems. Unlike existing literature, which addresses a service-level oriented formulation, where a lower bound is naturally produced through the service-level constraints, this method is developed under a penalty cost scheme with unmet demand back-ordered.

We then leverage the existing mathematical programming formulations and develop a shortest-path formulation by relaxing the original problem. An augmenting procedure is proposed to resolve the infeasibility. This procedure classifies infeasible scenarios in each period, introduces new node(s) and augments the graph by re-directing, re-computing and duplicating based on the original.

Computational experiments apply the same 25-period test instances as used in Chapter 2. The performance of the approximation and computational efficiency is investigated. The results show that the average optimality gap obtained from the proposed method is 0.616% for (s_t, S_t) policy and 3.34% for (R_t, S_t) policy. The approximation of (s_t, Q_t) policy by $Q_t = S_t - s_t$ produces an average optimality gap 6.69%. Further it demonstrates that the newly introduced approach can approximate (s_t, S_t) and (R_t, S_t) policy in 73.2 and 2.41 seconds, respectively.

In this work, my contribution can be summarised as follows.

1. I surveyed the literature related to the non-stationary lot-sizing problem with (R_t, S_t) policy and the computation of near-optimal policy parameters
2. I implemented the stochastic dynamic programme of the (R_t, S_t) policy with piecewise linear approximation in JAVA and CPLEX.
3. I formulated the shortest path problem and introduced the determination of reorder points based on the graph. I implemented the shortest-path formulation in Matlab.
4. I leveraged the filtering and augmenting graph originally applied in Rossi et al. (2011) and developed the augmenting procedure for the penalty cost.
5. I implemented the augmenting procedure on (R_t, S_t) policy in Matlab.
6. I conducted numerical studies on 540 large instances which follow non-stationary Normal demand. I approximated (s_t, S_t), (R_t, S_t) and (s_t, Q_t) policies with shortest-path formulation and augmenting procedure and compared results with existing approaches.
7. I organised all materials and wrote the paper.

1.4.5 Chapter 4 (Paper III): A hybrid inventory policy for the non-stationary lot-sizing problem with lateral transshipment

This chapter is joint work with Roberto Rossi and Thomas W. Archibald.

This chapter extends Chapter 2 (paper I) to a multi-location stochastic inventory system under lateral transshipment, which has received increasing attention over the last decades since Gross (1963) initially proposed the problem. Hoadley and Heyman (1977), Diks and De Kok (1996), Diks and De Kok (1998), Agrawal et al. (2004), Tagaras and Vlachos (2002) and Abouee-Mehrizi et al. (2015) consider different types of transshipment and replenishment policies and provide computational approaches and analytical investigation. However, these discussions are mainly based on stationary demand.

This chapter considers the non-stationary stochastic hybrid replenishment policy with proactive lateral transshipment for a two-location inventory system. In our approach, we revise the feasible action space that integrates transshipment and replenishment. Inspired by Agrawal et al. (2004), we design the solution spaces as two splits from the original one according to the independence between transshipment and replenishment. Based on the separated solution spaces, we re-formulate the problem through a two-stage stochastic dynamic programming, where, for one period, the system first solves the order quantity and then the transshipment-related decisions including the direction and the quantity.

To obtain the near-optimal solution that approximates the optimal expected total cost, we develop a heuristic approach under a rolling-horizon framework, where we use an approximation technique different from Meissner and Senicheva (2018) for approximate dynamic programming.

We access the computation performance of the proposed two-stage formulation and the LP-based rolling-horizon heuristic in two test sets. We predefine constraints $K > R$ (K: fixed ordering cost, R: fixed transshipping cost) to assume that the transshipping takes predecease over the ordering and $K \leq 2R$ to ensure the system would not order only once for the planning horizon without any other transshipment. We also set $v < b$ (v: unit transshipping cost, b: penalty cost) to assume that the system would not leave unmet demand back-ordered even though the transshipment is reasonably worthwhile.

The numerical study on a 4-period Poisson-distributed demand test set accesses the performance of the two-stage formulation. Due to the computational complexity of the stochastic dynamic programme, this test set also applies Latin hypercube sampling introduced by McKay et al. (1979) to generate a near-random sample of demand patterns. The results demonstrate an average optimality gap of 0.202%, from which we conclude that the average accuracy of the two-stage formulation can remain at a small value to facilitate further experiments.

Further experiments on a 10-period Normal-distributed demand test set are carried out to evaluate the LP-based rolling-horizon heuristic's performance, where the two-stage formulation results are applied as benchmarks. This small set of instances reveals an average optimality gap of 0.552%.

In this work, my contribution can be summarised as follows.

1. I surveyed the literature related to lot-sizing problems with lateral transshipment for two- and multi-location and different cost settings.
2. I implemented the stochastic dynamic programme of the hybrid replenishment policy in JAVA.
3. I proposed the two-stage stochastic programme to approximate the dynamic programming and implemented it in JAVA.
4. I leveraged the piecewise linear approximation approach introduced by Rossi et al. (2014) and developed the linear-programming-based static heuristic in the rolling horizon framework to approximate near-optimal policy parameters of the hybrid replenishment policy.
5. I conducted numerical studies on 4-period Poisson-distributed and 10-period Normal-distributed problem instance sets and applied Latin hypercube sampling to generate near-random demand patterns. All experiments were implemented in JAVA.
6. I organised all materials and wrote the paper.

1.5 Future work

This section summarises open questions for each of the following chapters and then describes potential research areas where mathematical programming can be successfully applied. All of the mathematical programming-based models presented are innovative, near-optimal and simple to use, but most significantly, they all share a similar modelling approach. A relevant research topic is applying this modelling approach to additional stochastic inventory control problems.

Chapter 2 (Paper I). This chapter presents the first MINLP-based heuristic for near-optimal (s, Q)-type policies with non-stationary stochastic demand and single-item single-location. This method provides the best optimality gaps in the literature for (s, Q)-type policies and can be easily implemented.

One direction of future research is based on the current working paper about the capacitated lot-sizing problem under (s_t, Q_t) policy with non-stationary stochastic demand since it is noticed that capacity of the inventory is easily modelled for the (s_t, Q_t) policy. The single-item, periodic review production and inventory system have been extensively studied in the literature. In the context of single-level production planning, with a finite planning horizon and known stochastic demand, the classical capacitated lot-sizing problem still consists of determining the amount and the timing of the production of products in the planning horizon. The single-item capacitated lot-sizing problem has been shown by Florian et al. (1980) and Bitran and Yanasse (1982) to be NP-hard for the deterministic demand. Therefore, developing effective heuristics has been a profitable research area for a long time.

Another direction of future research is the analytical investigation of the (s, Q)-type policy. The optimality of (s, S) policy has been shown mathematically by Scarf (1960), from which the reorder point is determined by $s_t = \inf\{I_t : G_t(I_t) < G_t(S_t^*) + c(S_t^*)\}$, where $G_t(I_t)$ is the expected total cost over horizon (t, T) with no order in period t. In the (s, Q)-type policies, the shape of numerical tests form similar convexities as (s, S) does, while the $K-$convexity does not hold and the reorder points are retrieved by $s_t = \inf\{I_t : G_t(I_t) - G_t(I_t + Q_t^*) < c(Q_t^*)\}$ due to the order quantity not shifting with inventory level I_t. An analytical convexity analysis can be conducted to investigate this difference, and consequently, a gap of (s, Q)-type policies can be formed against the optimal (s, S) policy systematically.

Chapter 3 (Paper II). This paper investigated the single-item single-location lot-sizing problem under the penalty cost. Further research from this study is two-fold.

First, improvement of the augmenting procedures. We observed a difference between (R_t, S_t)-Piecewise and the proposed augmenting procedure; a reason behind this is that optimality is compromised when we deal with negative orders. In the current procedure, for a period incurring a negative order, its opening inventory level is directly raised to the expected closing inventory of the last period, which means that the current period is merged with the previous replenishment cycle. One possible solution is to apply a repetitive step of updating the opening inventory level and calculating the expected total cost of the updated complete policy of the horizon until no better solution is found.

Second, an extension of the (s_t, Q_t) policy. Determining order quantity Q_t for an (s_t, Q_t) policy, using either an exact optimum from the stochastic dynamic programme or a near-optimum from piecewise linear approximation, is a massive computation due to its generality to all feasible opening inventory levels of each period. Unlike (R_t, S_t) policy, an order quantity can be obtained through the order-up-to level and the mean of demand, and it solely responds to the replenishment cycle, which can also be easily retrieved from the shortest path for the (R_t, S_t) since the order-up-to levels can be solved from the graph. However, the determination of (s_t, Q_t) is more complicated since the order quantity is not directly included in the graph as a decision variable. The determination of reorder points associated with order quantities may also rely on $G_t(I_t)$ since $s_t = \inf\{I_t : G_t(I_t) - G_t(I_t + Q_t^*) < c(Q_t^*)\}$.

Chapter 4 (Paper III). This chapter formulates the 2-location, multi-period transshipment problem with back-order in cases of stock-outs. Future research from this paper originates from the current development or further extension of the approaches. Research can be conducted on the following aspects.

The connection between stochastic dynamic programmes and the two-stage formulation. The two-stage formulation is developed by treating transshipment and replenishment as independent activities. From the numerical study, we see that the difference between the two formulations roughly follows a Normal distribution, whose average can remain in small value. A study on this relation can reveal where the difference originates.

The analytical study on the transshipment feasible space. A line search is necessary for all feasible options to obtain the optimal transshipment quantity, which is time-consuming even for a 10-period instance. An analytical study on the structure of the feasible space could simplify the computation, which varies with the opening inventory at the current stage.

More generally, the study can be extended to involve more realistic assumptions; for example, (non-)identical lead times of replenishment and transshipment to different locations, the lost-sale scheme to deal with unmet demand, and capacity imposed on either inventory storage or two modes of transportation.

Other structures of the inventory system. This chapter considers a two-echelon two-location transshipment problem. This simple structure can be extended to a more complex network to involve more echelons and connections, a system with N warehouses and M stocking locations, where N and $M \in \mathcal{N}$, to exploit the transshipment policy in a general network.

1.6 Conclusions

This thesis focuses on the classical questions in the field of lot-sizing problems, when are the times to reorder, and what are the order quantities to optimally control the expected total cost of inventory systems that involve single and multiple stocking locations. In particular, this thesis considers the demand as non-stationary stochastic for a single item. For both location assumptions, this thesis introduces replenishment policies under various uncertainty control strategies and develops heuristic models based on mathematical programmes and computation techniques to solve near-optimal policy parameters.

Under the single-location settings, Chapter 2 (Paper I) presents the first stochastic dynamic programmes for the (s, Q)-type policies and shows that the resulting optimal policy takes the non-stationary (s, Q) form. The answers to order timing and size comprise the policy parameters, respectively, determined by an MINLP-based heuristic algorithm designed for large-scale instances. A comprehensive computation study demonstrates that regarding the optimal (s_t, S_t) policy, this algorithm yields an average optimality gap 2.31% by the near-optimal (s_t, Q_t) policy, which is slightly better than the (R_t, S_t) policy through the piecewise linear approximation approach developed by Rossi et al. (2015).

In this regard, static-dynamic near-optimal policies are introduced in Rossi et al. (2015) and chapter 2 for non-stationary stochastic demand with a single location under the penalty cost scheme; however, their computation efficiency needs to be improved. Chapter 3 (Paper II) introduces a shortest-path formulation by relaxing unacceptable negative replenishment orders from the original problem, followed by an augmenting procedure on the graph to resolve the infeasibility. The answers to order timing and size are directly retrieved by (R_t, S_t) policy parameters from the shortest path of the augmented graph. The computational analysis presents an average optimality gap of 3.34% for approximated (R_t, S_t) policy and an average computation time of 2.41 seconds. This approach overcomes tedious computation on expectations and probabilities of each partition when the piecewise linear approximation is involved. Instead, it is established based on simple numerical integrations of the first order loss function and its complementary function.

Under the multi-location problems, one more dimension is added to the answer to indicate the source of replenishment. Combined with the timing and quantity, these questions are answered in Chapter 4 (Paper III) by a hybrid replenishment policy involving lateral transshipment. Chapter 4 presents a stochastic dynamic programme for lot-sizing problems with proactive transshipment before receiving replenishment from the warehouse and satisfying customers' demand. This formulation is approximated by decoupling the solution space through a two-stage stochastic formulation with an average optimality gap of 0.202% in a numerical study. This chapter also develops a rolling-horizon heuristic based on a static approximation model to solve large-scale instances, which produces a tight optimality gap of 0.552% on average against the two-stage formulation.

In summary, this thesis stands in one of the active research areas in Operational Research and discusses the core questions of stochastic inventory control regarding different assumptions on locations. Each chapter contributes to the literature on non-stationary lot-sizing problems with near-optimal replenishment policies based on well-developed mathematical programmes and easy-to-implement heuristic techniques. The further contribution may be achieved by improving the approximation methods to give lower optimality gaps and applicability to more general assumptions (such as inventory capacity) or conducting systematical analysis for a particular uncertainty control strategy.

Overall, this thesis presents contributions as one piece of the vast literature on inventory control and lot-sizing problems over the last century, as what was anticipated by Harris (1913) as the first research in this field, quoted as *"The general theory ... will be found to give good results"*.

Chapter 2

Paper I: Approximations for non-stationary stochastic lot-sizing under (s, Q)-type policy

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Abstract

This paper addresses the single-item single-stocking location non-stationary stochastic lot-sizing problem under a reorder point – order quantity control strategy. The reorder points and order quantities are chosen at the beginning of the planning horizon. The reorder points are allowed to vary with time and we consider order quantities either to be a series of time-dependent constants or a fixed value; this leads to two variants of the policy: the (s_t, Q_t) and the (s_t, Q) policies, respectively. For both policies, we present stochastic dynamic programs (SDP) to determine optimal policy parameters and introduce mixed integer non-linear programming (MINLP) heuristics that leverage piecewise linear approximation of the cost function. Numerical experiments demonstrate that our solution method effi-

ciently computes near-optimal parameters for a broad class of problem instances.

Keywords Inventory, (s,Q) policy, stochastic lot-sizing, non-stationary demand

2.1 Introduction

The non-stationary stochastic lot-sing problem is an extension of the well-known dynamic lot-sizing problem (Wagner and Whitin, 1958). In this problem, one considers a single-item single-stocking-location inventory system under a finite planning horizon and periodic review; the demand is stochastic and non-stationary. To deal with the uncertainty inherent in a stochastic lot-sizing problem, Bookbinder and Tan (1988) introduce three control strategies: the "static uncertainty", the "static-dynamic uncertainty", and the "dynamic uncertainty", which represent different approaches for determining the timing and size of orders.

Bookbinder and Tan's control strategies are captured by various policies. The (R, Q) policy determines the inventory review schedule R and the order quantity Q before the system operates; this is the static uncertainty strategy. The (s, S) policy is the dynamic uncertainty strategy, in which the timing and size of orders are decided as late as possible, in a wait-and-see fashion, by leveraging the reorder point s, and the order-up-to level S. Scarf (1960) shows that if the holding and shortage costs are convex, the optimal policy in each period is of (s, S) type. In a static-dynamic uncertainty strategy, one either fixes the order schedule at the outset, and computes the exact order quantity only when orders are issued, via suitable order-up-to-levels; or fixes the order quantities at the outset, and decides when orders are issued in a wait-and-see fashion, by relying on a reorder threshold. This leads to the (R, S) policy and (s, Q) policy (also referred to as the (r, Q) policy), respectively.

Compared to stationary demand, there are relatively few studies in the literature that consider non-stationary demand. However, in the majority of practical circumstances, demand is not only stochastic but also non-stationary.

The following works investigate the static uncertainty strategy under non-stationary demand. Sox (1997) proposes a mixed integer non-linear programming (MINLP) formulation of the dynamic lot-sizing problem with dynamic costs and develops a solution algorithm that resembles the Wagner-Whitin algorithm. This strategy is also investigated by Vargas (2009), who develops a stochastic dynamic programming model which is equivalent to the shortest path problem in a specified acyclic network. Vargas also provides a rolling horizon optimisation algorithm comprising two stages: (1) to determine optimal replenishment quantities for any sequence of replenishment points, and (2) to identify the optimal sequence of replenishment points.

For the static-dynamic uncertainty strategy, research under non-stationary demand mostly considers the (R, S) policy. Tarim and Kingsman (2004) formulates the problem as a mixed integer program (MIP). They model the total expected cost by minimising the sum of holding and ordering costs under a constraint on the probability of the closing inventory being non-negative in each time period. A method to solve this model efficiently is introduced in (Tarim et al., 2011), where the relaxation of the original MIP model is converted to a shortest path problem and implemented by branch-and-bound procedures. Tarim and Kingsman (2006) provide another MIP formulation where the objective function is obtained by the mean of a piecewise linear approximation. The accuracy of the approximation can be adjusted ad libitum by introducing new breakpoints. Özen et al. (2012) consider both penalty cost and service level and prove that the optimal policy is a base stock policy for both penalty and service-level constrained models, and also for capacity limitations and minimum order quantity requirements. More recently, Rossi et al. (2015) consider several service level measures — α service level on each period, β^{cyc} service level independently for each replenishment cycle, and the classic β service level — by adding suitable constraints that leverage the loss function and its complementary function to describe the expected total holding and penalty cost. A piecewise linear approximation approach is utilized to convert the cost function from non-linear to linear form. Tunc et al. (2018) presents an efficient MIP reformulation along with a dynamic cut generation approach that progressively refines the piecewise linear approximation to achieve a prescribed linearisation error.

Computing (s, S) policy parameters under non-stationary demand is a challenging task. The classic Silver and Meal heuristic algorithm (Silver and Meal, 1973) for deterministic demand has been extended by Silver (1978) and Askin (1981). Silver's algorithm uses a deterministic model to calculate the number of periods that each order must cover; when this replenishment plan is known, the associated safety stocks are then determined myopically. Askin (1981) explicitly includes the cost effects of probabilistic demand in the choice of the number of periods in which to order. Bollapragada and Morton (1999) approximate the non-stationary problem via a series of stationary problems based on the method developed by Zheng and Federgruen (1991). Parameters are determined by equating the cumulative mean demand of stationary and non-stationary problems over the expected reorder cycle. Xiang et al. (2018) introduce a MINLP formulation for an (s, S) policy by applying the piecewise linear approximation proposed by Rossi et al. (2015). Xiang et al. (2018) also derives a heuristic algorithm with binary search. Both solution methods outperform the previous heuristics in computational efficiency in tests involving short and long planning horizons. The comparison of the two proposed algorithms shows that binary search requires significantly less time than the MINLP. Visentin et al. (2021) propose a hybrid of branch-and-bound and stochastic dynamic programming model to compute optimal (R, s, S) policy parameters.

Based on this literature survey, we note a gap in the study of non-stationary demand: no literature discusses or investigates the static-dynamic uncertainty strategy in the form of an (s, Q) policy. In this paper, we propose a new control strategy for the stochastic lot-sizing problem under non-stationary demand. Under this strategy, the reorder points s_t vary with time, and we consider two cases for the order quantities: one in which the order quantity varies with time (Q_t) and another in which the order quantity is constant (Q). This leads to two (s, Q)-type policies: the (s_t, Q_t) policy and the (s_t, Q) policy. These policies require values for s_t and Q_t (or Q) to be determined at the beginning of the planning horizon. Compared to the optimal policy introduced by Scarf (1960), which allows the order quantity to vary with inventory level and time period, the order quantity in an (s_t, Q_t) policy is only affected by the time period and applies to all inventory levels, while the order quantity in an (s_t, Q) policy is a constant value for the entire planning horizon, and does not vary with inventory level or time period.

We make the following contributions to the stochastic lot-sizing literature.

- We model the non-stationary stochastic lot-sizing problem under a static-dynamic uncertainty policy in which order quantities are determined "statically", at the beginning of the planning horizon, while reordering decisions are determined "dynamically", in a wait-and-see-fashion. *We prove that the resulting optimal policy takes the non-stationary (s, Q) form.*
- We develop a new heuristic algorithm to efficiently determine near-optimal policy parameters of the proposed (s_t, Q_t) and (s_t, Q) policies. The algorithm is composed of two steps. The first step uses the (s, S)-policy heuristic introduced in Xiang et al. (2018) to determine the order quantities, and the second step is based on a newly developed MILP model that applies the piecewise linear approximation approach discussed in Rossi et al. (2014) to determine the order-up-to levels.
- In a comprehensive numerical study, we show that optimality gaps for the (s_t, Q_t) policy obtained via our heuristic are tighter than those of a near-optimal (R_t, S_t) policy obtained via the approach in Rossi et al. (2015). We also observe that an (s_t, Q) policy lacks flexibility and leads to substantial optimality gaps.

The rest of this paper is structured as follows. In Section 2.2 we introduce the problem settings and present a stochastic dynamic programming (SDP) formulation. Section 2.3 discusses the stochastic dynamic programming formulation of the (s_t, Q_t) and (s_t, Q) policies. We also show that the resulting optimal policies take the non-stationary (s_t, Q_t) and (s_t, Q) forms through the uniqueness of reorder points. In Section 2.4, we develop a heuristic algorithm to compute near-optimal policy parameters for the (s_t, Q_t) policy and discuss the application of this algorithm to the (s_t, Q) policy. A computational analysis is presented in Section 2.5; finally, we draw conclusions in Section 2.6.

2.2 Problem description

We consider a single-item single-location non-stationary stochastic lot-sizing problem over a planning horizon of T periods. Replenishment orders are placed and instantaneously delivered at the beginning of each time period. Each replenishment order incurs an ordering cost $c(\cdot)$ comprising a fixed ordering cost K and a linear ordering cost z proportional to the non-negative order quantity Q, where

$$c(Q) \triangleq \begin{cases} K + z \cdot Q, & Q > 0; \\ 0, & Q = 0. \end{cases} \tag{2.1}$$

The periods' demands d_t, for $t = 1, \cdots, T$, are independent random variables with known probability density functions $g_t(\cdot)$. Any unmet demand at the end of the period is back-ordered. At the end of each period, a linear holding cost h is incurred for each unit carried from one period to the next, and a linear penalty cost b is charged on each unit back-ordered. The expected immediate holding and penalty cost at the end of period t is expressed as

$$L_t(y) \triangleq \mathbb{E}[h \max(y - d_t, 0) + b \max(d_t - y, 0)], \tag{2.2}$$

where y denotes the inventory level after receiving the replenishment and $\mathbb{E}[\cdot]$ denotes the expectation operator.

Let $C_t(x)$ represent the expected total cost of an optimal policy over periods $t, \ldots, T$ with opening inventory level x; then the problem can be modelled as a stochastic dynamic program (Bellman, 1957)

$$C_t(x) \triangleq \min_{y \geq x} \{c(y - x) + L_t(y) + \mathbb{E}[C_{t+1}(y - d_t)]\}, \tag{2.3}$$

where $C_{T+1}(x) \triangleq 0$, is the boundary condition.

Scarf (1960) showed that if $L_t(y)$ is convex, the optimal policy of the dynamic inventory problem is of an (s, S) type, where the inventory system places a replenishment to reach the order-up-to level S when the stock is found to be below the reorder point at a review point. This conclusion is based on a study of the function $G_t(y) + zy$, where

$$G_t(y) \triangleq L_t(y) + \mathbb{E}[C_{t+1}(y - d_t)], \tag{2.4}$$

and $G_t(y)$ represents the expected total cost over periods t to T when the opening inventory is y and no order is placed in period t. Table 2.8 in Appendix 2.A summarises the notation used in this paper.

In the rest of this paper, we conduct the discussion assuming $L_t(y)$ is convex. In fact, as the holding and penalty costs used in this paper are linear, $L_t(y)$ is a weighted sum of two convex functions and hence convex. A detailed proof can be found in (Rossi et al., 2014, page 490).

Example 2.2.1. Consider a 4–period stochastic lot-sizing problem under Poisson-distributed demand with rates $d_t = \langle 20, 40, 60, 40 \rangle$. The cost parameters are $K = 100$, $z = 0$, $h = 1$ and $b = 10$. Fig. 2.1 illustrates the variation of $G_t(I_0)$ with $I_0 \in [0, 200]$ and no replenishment order placed in period 1, where $G_1(0) = 481$.

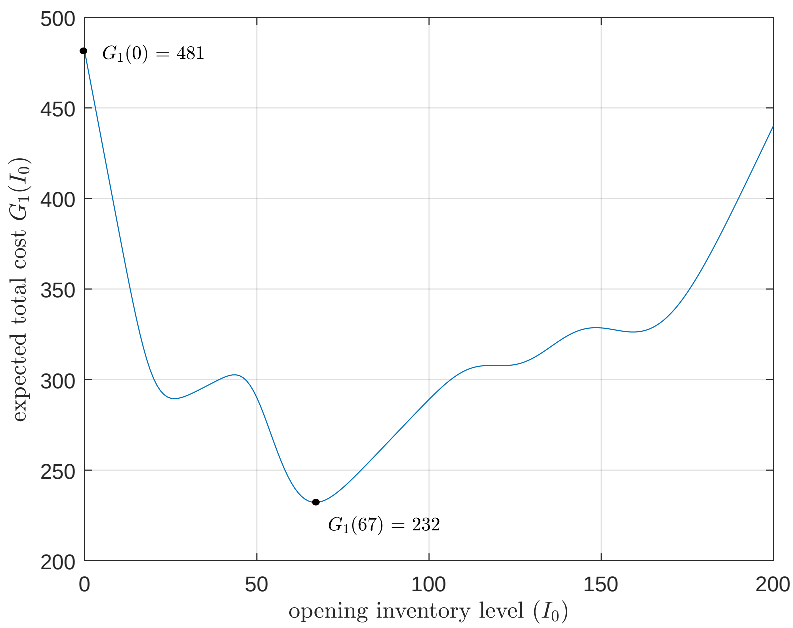

Figure 2.1: Plot of $G_1(I_0)$

2.3 Stochastic dynamic programs for the (s_t,Q_t) and (s_t,Q) policies

This section introduces the stochastic dynamic programming formulations of the stochastic lot-sizing problem under the (s_t, Q_t) policy and the (s_t, Q) policy in Section 2.3.1 and Section 2.3.2, respectively.

2.3.1 A stochastic dynamic program for the (s_t, Q_t) policy

An (s_t, Q_t) policy places a replenishment order of size Q_t at the beginning of period t if the inventory level is below the reorder point s_t, and does not place any order otherwise (Silver et al., 1998). The optimal expected total cost of the system controlled under an (s_t, Q_t) policy can be determined by computing all feasible combinations of reorder quantities Q_t, for $t = 1, \ldots, T$. Let $\vec{q}_t = \langle Q_t, \ldots, Q_T \rangle$ denote a $(T - t + 1)$-dimensional vector representing order quantities $Q_t, \ldots, Q_T$ and $\mathcal{Q}_t$ be the vector space representing all combinations of order quantities $\vec{q}_t$. For any $\vec{q}_t \in \mathcal{Q}_t$, the expected total cost over periods t to T when the opening inventory level is x is denoted as

$$V_t(x, \vec{q}_t) \triangleq \min_{\delta \in \{0,1\}} \{c(\delta Q_t) + L_t(x + \delta Q_t) + \mathbb{E}[V_{t+1}(x + \delta Q_t - d_t, \vec{q}_{t+1})]\}, \quad t < T, \tag{2.5}$$

where δ is a binary variable that represents the reordering decision in period t when the initial inventory level is x; finally,

$$V_T(x, \vec{q}_T) \triangleq \min_{\delta \in \{0,1\}} \{c(\delta Q_T) + L_T(x + \delta Q_T)\} \tag{2.6}$$

is the boundary condition. Therefore, considering all combinations, the optimal expected total cost when the inventory level at the beginning of the planning horizon is x can be defined as

$$V_0(x) \triangleq \min_{\vec{q}_1 \in \mathcal{Q}_1} \{V_1(x, \vec{q}_1)\}. \tag{2.7}$$

Let the optimal order quantity be represented by the vector $\vec{q}_t^* \triangleq \langle Q_t^*, \ldots, Q_T^* \rangle$.

Next, we show that the policy found by the formulation in Section 2.3.1 is of an (s_t, Q_t) form. The following discussion is inspired by the work of Gallego and

Toktay (2004) on all-or-nothing ordering policies under a capacity constraint. For any opening inventory level x and a vector of order quantities $\vec{q_t}$, let $J_t(x, \vec{q_t})$ and $\hat{J}_t(x, \vec{q_t})$ denote the expected total cost when the decision in period t is not to order ($\delta = 0$) and to order ($\delta = 1$) respectively, it follows that

$$J_t(x, \vec{q_t}) \triangleq L_t(x) + \mathbb{E}[V_{t+1}(x - d_t, \vec{q}_{t+1})] \tag{2.8}$$

and

$$\hat{J}_t(x, \vec{q_t}) \triangleq c(Q_t) + L_t(x + Q_t) + \mathbb{E}[V_{t+1}(x + Q_t - d_t, \vec{q}_{t+1})]. \tag{2.9}$$

Recall that Eq. (2.5) optimises the system over the reorder decision $\delta \in \{0, 1\}$ and is equivalent to

$$\begin{aligned} V_t(x, \vec{q_t}) &= \min\{\hat{J}_t(x, \vec{q_t}),\ J_t(x, \vec{q_t})\} \\ &= \min\{K + zQ_t + L_t(x + Q_t) + \mathbb{E}[V_{t+1}(x + Q_t - d_t, \vec{q}_{t+1})], \\ &\qquad\qquad L_t(x) + \mathbb{E}[V_{t+1}(x - d_t, \vec{q}_{t+1})]\} \\ &= \min\{K + zQ_t + J_t(x + Q_t, \vec{q_t}),\ J_t(x, \vec{q_t})\} \\ &= J_t(x, \vec{q_t}) + \min\{K + zQ_t - \Delta J_t(x, \vec{q_t}),\ 0\}, \end{aligned} \tag{2.10}$$

where we define

$$\Delta J_t(x, \vec{q_t}) \triangleq J_t(x, \vec{q_t}) - J_t(x + Q_t, \vec{q_t}). \tag{2.11}$$

From Eq. (2.10), it is optimal to reorder in period t with opening inventory x when $\Delta J_t(x, \vec{q_t}) > K + zQ_t$ and not to reorder otherwise. If we choose not to reorder when $\Delta J_t(x, \vec{q_t}) = K + zQ_t$, then the range of opening inventory level x for which it is optimal to reorder can be expressed as

$$\{x : \Delta J_t(x, \vec{q_t}) > K + zQ_t\}. \tag{2.12}$$

If $\Delta J_t(x, \vec{q_t})$ is non-increasing in x for given order quantities $\vec{q}_t^*$, then either there exits an s_t such that it is optimal to order in period t when $x < s_t$ and not otherwise, or it is never optimal to order in period t; and it hence leads to the (s_t, Q_t) policy. In the following, for any given $\vec{q_t}$, we show the monotonicity of $\Delta J_t(x, \vec{q_t})$ in x.

Lemma 2.3.1. $L_t(y) - L_t(y+a)$ is non-increasing in y for any $a > 0$ and $t = 1, \ldots, T$.

Proof. Since $L_t(y)$ is convex, its derivative $L_t'(y)$ is non-decreasing by the definition of convexity. For any $a > 0$ and any $t = 1, \ldots, T$, $[L_t(y) - L_t(y+a)]' = L_t'(y) - L_t'(y+a) \leq 0$; therefore, $L_t(y) - L_t(y+a)$ is non-increasing in y. □

Lemma 2.3.2. For a given $\vec{q_t}$, the function $\Delta J_t(x, \vec{q_t})$ is non-increasing with respect to the opening inventory level x for any $t = 1, \ldots, T$.

Proof. We prove this by induction. For period T,

$$\Delta J_T(x, \vec{q_T}) = J_T(x, \vec{q_T}) - J_T(x + Q_T, \vec{q_T}) = L_T(x) - L_T(x + Q_T)$$

is non-increasing by Lemma 2.3.1. Assuming that $\Delta J_t(x, \vec{q_t})$ is non-increasing in x, we want to show that $\Delta J_{t-1}(x, \vec{q}_{t-1})$ is non-increasing in x. We find that

$$\begin{aligned}
& K + zQ_t + V_t(x + Q_t, \vec{q_t}) - V_t(x, \vec{q_t}) \\
= \; & K + zQ_t + J_t(x + Q_t, \vec{q_t}) - J_t(x, \vec{q_t}) + \min\{0, K + zQ_t - \Delta J_t(x + Q_t, \vec{q_t})\} \\
& \qquad - \min\{0, K + zQ_t - \Delta J_t(x, \vec{q_t})\} \\
= \; & K + zQ_t - \Delta J_t(x, \vec{q_t}) + \min\{0, K + zQ_t - \Delta J_t(x + Q_t, \vec{q_t})\} \\
& \qquad - \min\{0, K + zQ_t - \Delta J_t(x, \vec{q_t})\} \\
= \; & \max\{0, K + zQ_t - \Delta J_t(x, \vec{q_t})\} + \min\{0, K + zQ_t - \Delta J_t(x + Q_t, \vec{q_t})\}
\end{aligned}$$

is the sum of two non-decreasing functions because $\Delta J_t(x, \vec{q_t})$ is assumed to be non-increasing. It follows that $V_t(x, \vec{q_t}) - V_t(x + Q_t, \vec{q_t})$ is non-increasing. Consequently, since $L_{t-1}(x) - L_{t-1}(x + Q_{t-1})$ is non-increasing in x,

$$\begin{aligned}
\Delta J_{t-1}(x, \vec{q_t}) &= J_{t-1}(x, \vec{q}_{t-1}) - J_{t-1}(x + Q_{t-1}, \vec{q}_{t-1}) \\
&= L_{t-1}(x) - L_{t-1}(x + Q_{t-1}) \\
&\qquad + \mathbb{E}[V_t(x - d_{t-1}, \vec{q_t}) - V_t(x + Q_t - d_{t-1}, \vec{q_t})]
\end{aligned} \tag{2.13}$$

is the sum of two non-increasing functions; therefore, $\Delta J_{t-1}(x, \vec{q_t})$ is non-increasing in x. This completes the proof by induction. □

For a given $\vec{q_t}$, the monotonicity of $\Delta J_t(x, \vec{q_t})$ in x assures the unique existence of the reorder point s_t, which defines the region of opening inventory $x < s_t$ for which it is optimal to reorder, where s_t can be denoted as

$$s_t = \inf\{x : \Delta J_t(x, \vec{q_t}) < K + zQ_t\}; \tag{2.14}$$

if the inventory levels are discrete, then s_t is the minimum value of x such that $\Delta J_t(x, \vec{q_t}) < K + zQ_t$, where Q_t is the first argument of the order quantities $\vec{q_t}$. The reorder points associated with the optimal order quantities $\vec{q_t^*}$ hence can be denoted as $\vec{s_t^*} \triangleq \langle s_t^*, \dots, s_T^* \rangle$.

Example 2.3.1. Consider a 4–period stochastic lot-sizing problem under Poisson-distributed demand with rates $d_t = \langle 2, 1, 5, 3 \rangle$. The cost parameters are $K = 5$, $z = 0$, $h = 1$ and $b = 3$. The maximum order quantity is set to 9. After exhaustive enumeration of all order quantity vectors, we obtain $\vec{q_1^*} = \langle 3, 3, 8, 5 \rangle$ and the associated reorder points $\vec{s_1^*} = \langle 1, 0, 4, 1 \rangle$. The expected total cost of the optimal (s_t, Q_t) policy is 22.5 when the initial inventory is 0. Under discrete inventory levels with Poisson demand, Fig. 2.2 and Fig. 2.3 illustrate determining s_1^* by scatter plots. In Fig. 2.2, $s_1^* = 1$ is selected as the minimum value such that $\Delta J_1(I_0, \vec{q_1^*}) < K$, which is equivalent to $J_1(I_0, \vec{q}^*) > \hat{J}_1(I_0, \vec{q}^*)$ when $I_0 \leq 0$, suggesting it is optimal to order; and $J_1(I_0, \vec{q}^*) < \hat{J}_1(I_0, \vec{q}^*)$ when $I_0 \geq 1$, suggesting it is optimal not to order, as Fig. 2.3 shows.

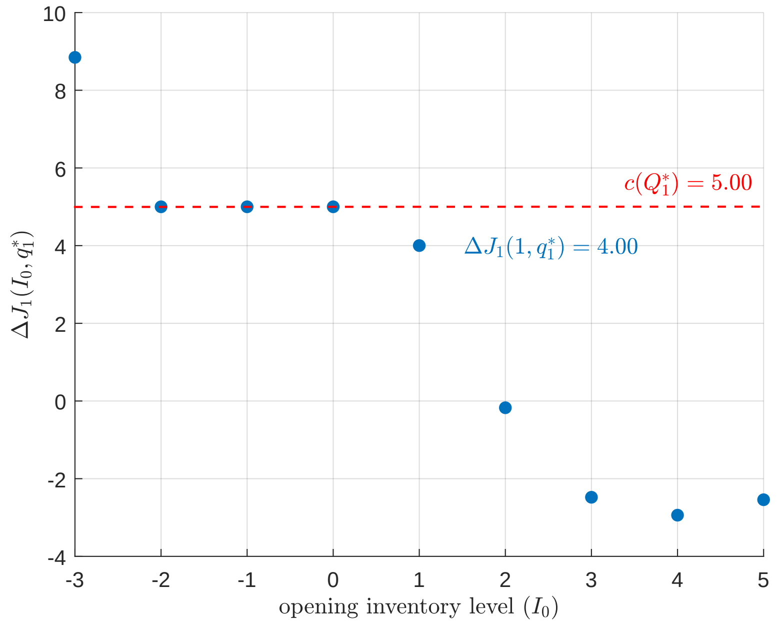

Figure 2.2: $s_1^* = 1$ determined by comparing $\Delta J_1(I_0, \vec{q}_1^*)$ and $c(Q_1^*)$.

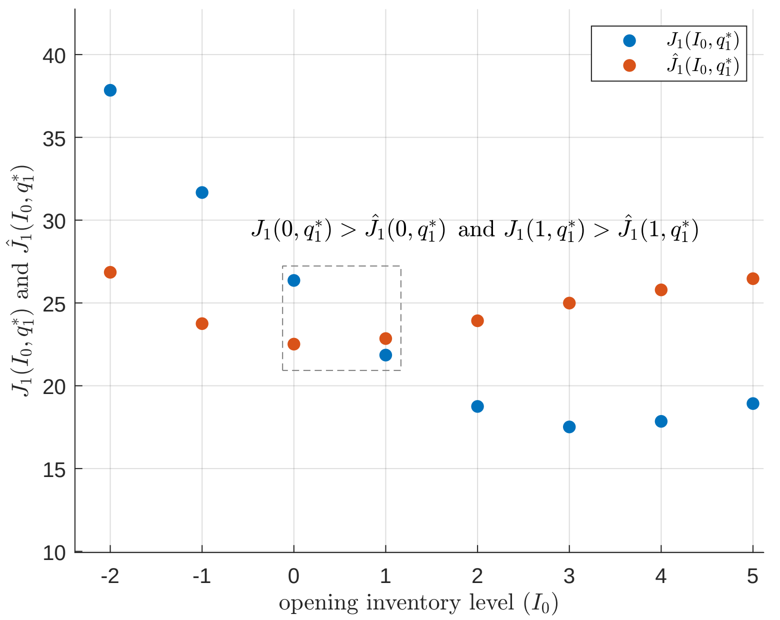

Figure 2.3: $s_1^* = 1$ determined by comparing $J_1(I_0, \vec{q}_1^*)$ and $\hat{J}_1(I_0, \vec{q}_1^*)$.

2.3.2 A stochastic dynamic program for the (s_t, Q) policy

An (s_t, Q) policy places a replenishment order of size Q if the inventory level falls below the reorder point s_t and does not place an order otherwise. It is therefore a special case of (s_t, Q_t) in which all Q_t's are equal. We modify the vector space $\mathcal{Q}_t$ introduced in section 2.3.1 to explore the (s_t, Q) policy.

Let $\dot{\vec{q}_t} \triangleq \langle Q, \ldots, Q \rangle$ be a $(T - t + 1)$-dimensional vector of reorder quantities for the (s_t, Q) policy and $\dot{\mathcal{Q}}_t$ be the vector space containing all combinations of order quantities $\dot{\vec{q}_t}$. It follows that $\dot{\mathcal{Q}}_t$ is a subspace of $\mathcal{Q}_t$. For a given $\dot{\vec{q}_t} \in \dot{\mathcal{Q}}_t$, the expected total cost over periods t to T when the opening inventory level is x is

$$V_t(x, \dot{\vec{q}_t}) = \min_{\delta \in \{0,1\}} \{c(\delta Q) + L_t(x + \delta Q) + \mathbb{E}[V_{t+1}(x + \delta Q - d_t, \dot{\vec{q}}_{t+1})]\}, \quad t < T, \tag{2.15}$$

and

$$V_T(x, \dot{\vec{q}_T}) = \min_{\delta \in \{0,1\}} \{c(\delta Q) + L_T(x + \delta Q)\} \tag{2.16}$$

is the boundary condition. The optimal expected total cost under the (s_t, Q) policy with opening inventory level x can be defined as

$$V_0(x) = \min_{\dot{\vec{q}_1} \in \dot{\mathcal{Q}}_1} \{V_1(x, \dot{\vec{q}_1})\}. \tag{2.17}$$

We let the optimal order quantity vector be $\dot{\vec{q}}^*_t \triangleq \langle Q^*, \ldots, Q^* \rangle$. Since $\dot{\mathcal{Q}}_t$ is a subspace of $\mathcal{Q}_t$, Lemma 2.3.2 holds for any $\dot{\vec{q}_t} \in \dot{\mathcal{Q}}_t$. The determination of reorder points under the (s_t, Q) policy follows in the same fashion as for the (s_t, Q_t) policy by Eq. (2.14). We denote the reorder points associated with $\dot{\vec{q}}^*_t$ as $\dot{\vec{s}}^*_t \triangleq \langle \dot{s}^*_t, \ldots, \dot{s}^*_T \rangle$.

Example 2.2.1 (Continued). Recall the 4–period stochastic lot-sizing problem under Poisson-distributed demand with rates $d_t = \langle 20, 40, 60, 40 \rangle$. Under the (s_t, Q) policy, the optimal order quantity is $Q^* = 83$ as illustrated by Fig. 2.4. The reorder points associated with $\dot{\vec{q}}^*_1$ are determined as $\dot{\vec{s}}^*_1 = \langle 13, 33, 54, 24 \rangle$. Fig. 2.5 and Fig. 2.6 illustrate determining $\dot{s}^*_1 = 13$. Note that we apply curves to show the trend of expected costs, while the system is in fact discrete. In Fig. 2.6, a unique sign change in $[\Delta J_1(I_0, \dot{\vec{q}}^*_1) - c(Q^*)]$ is detected between $I_0 = 12$ and 13 and so, by Eq. (2.14), $I_0 = 13$ is chosen as $\dot{s}^*_1$.

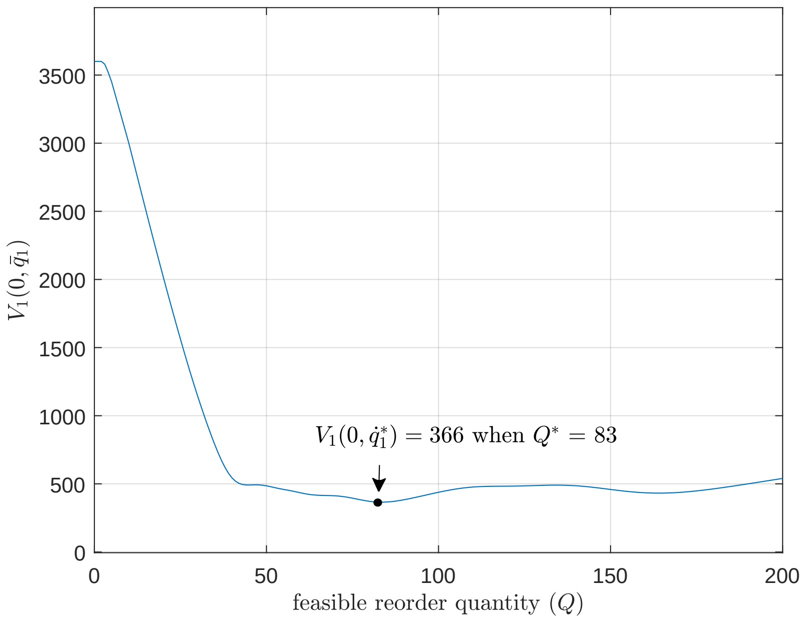

Figure 2.4: $Q^* = 83$ under (s_t, Q) policy for Example 2.2.1.

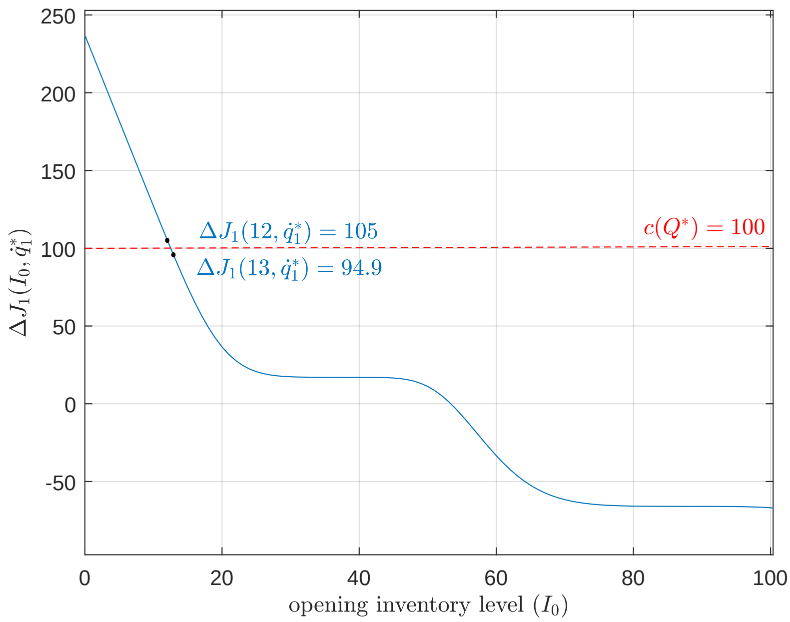

Figure 2.5: $\dot{s}_1^* = 13$ determined by comparing $\Delta J_1(I_0, \dot{q}_1^*)$ and $c(Q^*)$.

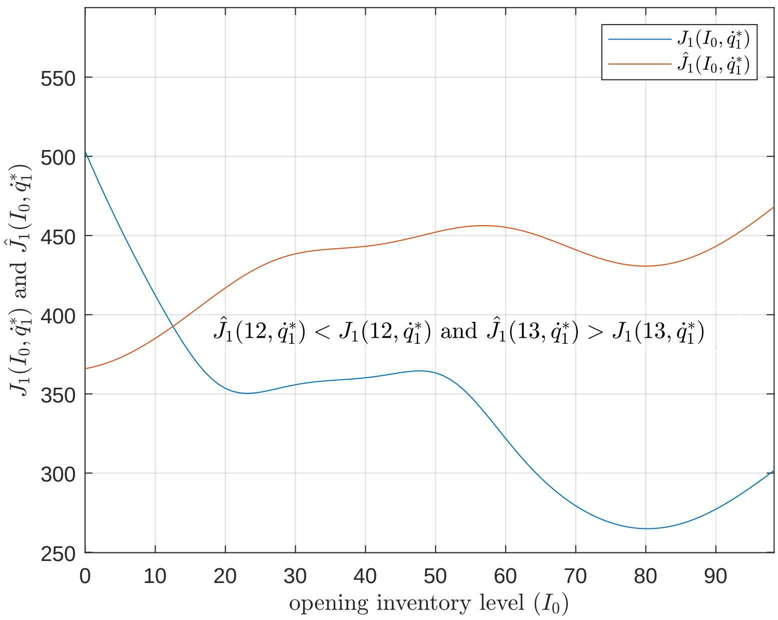

Figure 2.6: $\dot{s}_1^* = 13$ determined by comparing $J_1(I_0, \dot{\tilde{q}}_1^*)$ and $\hat{J}_1(I_0, \dot{\tilde{q}}_1^*)$.

2.4 A MINLP-based heuristic for the (s_t, Q_t) policy

Optimal (s_t, Q_t) and (s_t, Q) policies can be obtained by enumerating all possible order quantities and using the stochastic dynamic programming formulations presented in Section 2.3. However, as the length of planning horizon increases, the enumeration increases exponentially and it becomes impractical to use this method. In this section, we therefore introduce an effective heuristic to compute near-optimal (s_t, Q_t) and (s_t, Q) policy parameters in reasonable time. Our heuristic leverages a MINLP approximation of $V_t(\cdot)$ and, similarly to Bookbinder and Tan (1988), it comprises two steps: in the first step, we determine a set of near-optimal order quantities; in the second step, we compute the associated reorder points.

2.4.1 Step I: Order quantity Q_t of the (s_t, Q_t) policy

We first aim to derive a vector of near-optimal order quantities $\hat{\vec{q_t}} \triangleq \langle \hat{Q}_1 \ldots, \hat{Q}_T \rangle$ for our heuristic (s_t, Q_t) policy. The reader should note that we seek a policy that is near-optimal in terms of expected total cost, not in terms of how close the policy parameters obtained are to the true optimal ones. Therefore, our approximated order quantities and reorder points do not need to be close to the true optimal ones for the (s_t, Q_t) policy, as long as the expected total cost they provide is close enough to the expected total cost of an optimal policy.

Note that if an order is placed in period t under the (s_t, S_t) policy, the order quantity is at least $S_t - s_t$; in fact, if the opening inventory level $I_{t-1} < s_t$ in period t, a further $s_t - I_{t-1}$ items will be ordered to ensure the order-up-to level is reached. In our heuristic (s_t, Q_t) policy, we define $\hat{Q}_t \triangleq S_t - s_t$ to be our approximate order quantity in period t; and we will denote the vector of approximate order quantities as $\hat{\vec{q_t}} \triangleq \langle \hat{Q}_t, \ldots, \hat{Q}_T \rangle$. While these $\hat{Q}_t$'s may not be optimal, we will compensate for this in Section 2.4.2, by computing suitable reorder points that are tailored for these approximate order quantities.

Of course, to compute $\hat{Q}_t$, we need optimal or near-optimal values of parameters s_t and S_t of the (s_t, S_t) policy. We use the approach introduced by Xiang et al. (2018) to compute near-optimal s_t and S_t values. For completeness, the model used is presented in Appendix B.

2.4.2 Step II: Reorder point s_t of the (s_t, Q_t) policy

Since approximate order quantities $\hat{\vec{q_t}}$ are a lower bound for order quantities observed under an (s_t, S_t) policy, we cannot directly use the reorder points from the optimal (s_t, S_t) policy as the reorder points for a heuristic (s_t, Q_t) policy. To compensate for the under-estimation in the order quantities, we need higher reorder points.

For a given vector $\hat{\vec{q_t}}$ of approximate order quantities, we may compute the associated optimal reorder points by using an SDP formulation. This would be relatively straightforward for Poisson demand, but would require a discretisation step for continuous demand distributions. In order to provide a framework that can be applied to Poisson, normal, and possibly other continuous demand distributions, we modify the model in Xiang et al. (2018) to capture the characteristics of an (s_t, Q_t) and provide an approximation $\mathcal{J}_t(x, \hat{\vec{q}_t})$ of $J_t(x, \hat{\vec{q}_t})$ that can be used in Eq. (2.14) to compute near-optimal reorder points. Let $\mathcal{J}_t(x, \hat{\vec{q}_t})$ be our approximation of $J_t(x, \hat{\vec{q}_t})$ for the set of near-optimal order quantities $\hat{\vec{q}_t}$ computed in Section 2.4.1.

$$
\begin{aligned}
\mathcal{J}_t(x, \hat{\vec{q}_t}) = \min \quad & h\tilde{H}_t + b\tilde{B}_t + \sum_{k=t+1}^{T} [h\tilde{H}_k + b\tilde{B}_k + c(\delta_k \hat{Q}_k)], && && (2.18)\\
\text{s.t.} \quad & \delta_t = 0, && && (2.19)\\
& \tilde{I}_t + \tilde{d}_t = \tilde{I}_{t-1}, && && (2.20)\\
& \delta_k = 0 \rightarrow \tilde{I}_k + \tilde{d}_k - \tilde{I}_{k-1} = 0, && k = t+1, \ldots, T, && (2.21)\\
& \delta_k = 1 \rightarrow \tilde{I}_k + \tilde{d}_k - \tilde{I}_{k-1} = \hat{Q}_k, && k = t+1, \ldots, T, && (2.22)\\
& P_{jk} \geq \delta_j - \sum_{r=j+1}^{k} \delta_r, && k = t, \cdots, T \text{ and } j = t, \ldots, k, && (2.23)\\
& \textstyle\sum_{j=t}^{k} P_{jk} = 1, && k = t+1, \ldots, T, && (2.24)\\
& P_{jk} = 1 \rightarrow \tilde{H}_k = \hat{\mathcal{L}}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), && k = t, \ldots, T \text{ and } j = t, \ldots, k, && (2.25)\\
& P_{jk} = 1 \rightarrow \tilde{B}_k = \mathcal{L}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), && k = t, \ldots, T \text{ and } j = t, \ldots, k, && (2.26)\\
& \tilde{H}_k, \tilde{B}_k \geq 0, P_{jk}, \delta_k \in \{0, 1\}, && k = t, \ldots, T \text{ and } j = t, \ldots, k. && (2.27)
\end{aligned}
$$

Let $\tilde{H}_k$ and $\tilde{B}_k$ denote the expected positive inventory and back-order levels at the end of period k, respectively; their values are computed by following the piecewise linear approximation strategy in Rossi et al. (2015), which is based on the first-order loss function $\mathcal{L}$ and its complement $\hat{\mathcal{L}}$. We discuss in detail the loss function and its piecewise linear approximation under non-stationary Poisson demand in Appendix 2.C.

In line with (Tarim and Kingsman, 2006), we introduce the binary decision variable δ_k that takes value 1 if and only if an order is placed in period k. In the model above, the objective function $\mathcal{J}_t(x, \hat{\vec{q}}_t)$ approximates the expected total cost over horizon (t, T) with no order in period t. Constraint (2.19) indicates that no order is placed in period t and leads to the flow balance equation for period t as constraint (2.20). Constraints (2.21) and (2.22) are indicator constraints[1] representing the flow balance equations and reorder conditions under the (s_t, Q_t) policy that applies order quantities $\hat{\vec{q}}_{t+1}$ over horizon $(t+1, T)$.

We introduce a binary variable P_{jk} to properly account for demand variance while computing the first-order loss function. Let P_{jk} $(j \leq k)$ take value of 1 if the last order before period k (including period k itself) is placed at the beginning of period j. Note that the combination of constraints (2.23) and (2.24) ensures that demand variance is properly accounted even when no order takes place within the horizon (j, k). Constraints (2.25) and (2.26) are indicator constraints modelling end of period k expected excess inventory and back-orders by means of the first order loss function (Xiang et al., 2018).

Since $J_t(x, \vec{q}_t)$ is approximated as $\mathcal{J}_t(x, \hat{\vec{q}}_t)$, the near-optimal reorder point $\hat{s}_t$ can be determined, following Eq. (2.14), as

$$\hat{s}_t = \inf\{x : \Delta\mathcal{J}_t(x, \hat{\vec{q}}_t) < K + z\hat{Q}_t\}, \tag{2.28}$$

or as the minimum value of x such that

$$\Delta\mathcal{J}_t(x, \hat{\vec{q}}_t) < K + z\hat{Q}_t \tag{2.29}$$

for discrete inventory levels, where $\Delta\mathcal{J}_t(x, \hat{\vec{q}}_t) \triangleq \mathcal{J}_t(x, \hat{\vec{q}}_t) - \mathcal{J}_t(x + \hat{Q}_t, \hat{\vec{q}}_t)$. Note that $\Delta\mathcal{J}_t(x, \hat{\vec{q}}_t)$ is not necessarily monotonic in x, since the piecewise linear approximation produces errors; our model applies the optimal partitioning strategy to maintain a minimum error (Rossi et al., 2014, Thm. 11). We denote the vector of near-optimal reorder points associated with $\hat{\vec{q}}_t$ as $\hat{\vec{s}}_t \triangleq \langle \hat{s}_t, \ldots, \hat{s}_T \rangle$.

[1] An indicator constraint (denoted by $\rightarrow$), see e.g. (Belotti et al., 2016), is a way to express relationships among variables by identifying a binary variable to control whether or not a specified constraint is active. Indicator constraints are standard constructs that are nowadays implemented in most off-the-shelf MILP solvers.

2.4.3 A binary search approach to approximate the reorder points s_t

A line search for $\hat{s}_t$ following Eq. (2.28) may be too time-consuming for large-scale instances. This subsection introduces a heuristic algorithm to approximate $\hat{s}_t$ and reduce computational complexity.

The algorithm applies a binary search on $\Delta\mathcal{J}_t(x, \hat{\vec{q}_t})$ with $\hat{\vec{q}_t}$ known as an input. For any period t, the opening inventory level x_0 and given step-size w ($w > 0$) define an interval of inventory level $[x_0,\ x_0 + w]$, which maps to $[\Delta\mathcal{J}_t(x_0 + w, \hat{\vec{q}_t}),\ \Delta\mathcal{J}_t(x_0, \hat{\vec{q}_t})]$. The binary search halves the length of the interval in each iteration until $\hat{s}_t$ is detected according to Eq. (2.28). If the initial interval does not span the point at which the sign of $\Delta\mathcal{J}_t(x, \hat{\vec{q}_t}) - K - z\hat{Q}_t$ changes, we renew $[x_0,\ x_0 + w]$ by panning it w units to the left if $\Delta\mathcal{J}_t(x_0 + w, \hat{\vec{q}_t}) < K + z\hat{Q}_t$ or to the right, otherwise; and then proceed with the binary search.

We present the following algorithm for integer inventory levels. One can extend it to discrete systems with any interval between two adjacent inventory levels. For integer inventory levels, the algorithm terminates if a pair of inventory levels x and $x + 1$ are found such that $\Delta\mathcal{J}_t(x, \hat{\vec{q}_t}) \leq K + z\hat{Q}_t \leq \Delta\mathcal{J}_t(x + 1, \hat{\vec{q}_t})$, and then $\hat{s}_t = x + 1$. The procedure in detail is as follows.

Algorithm 1 Computing the reorder points $\hat{s}_t$ associated with $\vec{\hat{q}_t}$.

1: **Input:** demand rates $\tilde{d}_t$; cost parameters (K, z, h, b); the step-size w; an opening inventory x_0; order quantities $\vec{\hat{q}_t}$.
2: **Output:** reorder point $\hat{s}_t$ associated with $\vec{\hat{q}_t}$.
3: **for** $t = 1 \to T$ **do**
4: Compute the ordering cost of placing an order $\mathcal{J}_0 = K + z\hat{Q}_t$;
5: $x_l = x_0$ and $x_r = x_0 + w$;
6: compute $\mathcal{J}_l = \Delta\mathcal{J}_t(x_l, \vec{\hat{q}_t})$ and $\mathcal{J}_r = \Delta\mathcal{J}_t(x_r, \vec{\hat{q}_t})$ with $\hat{Q}_t$;
7: **if** $\mathcal{J}_l > \mathcal{J}_0 > \mathcal{J}_r$ **then**
8:   $x_m = \lfloor \frac{x_l + x_r}{2} \rfloor$ and $\mathcal{J}_m = \Delta\mathcal{J}_t(x_m, \vec{\hat{q}_t})$;
9:   **if** $\mathcal{J}_m > \mathcal{J}_0$ **then**
10:    **if** $\Delta\mathcal{J}_t(x_m + 1, \vec{\hat{q}_t}) < \mathcal{J}_0$ **then**
11:     output $\hat{s}_t = x_m$;
12:    **else** $x_l = x_m$, $x_r = x_r$, and repeat lines 6 – 20;
13:    **end if**
14:   **else**
15:    **if** $\Delta\mathcal{J}_t(x_m - 1, \vec{\hat{q}_t}) > \mathcal{J}_0$ **then**
16:     output $\hat{s}_t = x_m - 1$;
17:    **else** $x_l = x_l$, $x_r = x_m$, and repeat lines 6 – 20;
18:    **end if**
19:   **end if**
20: **end if**
21: **end for**

Example 2.3.1 (Continued). Recall the 4–period stochastic lot-sizing problem under Poisson-distributed demand with rates $d_t = \langle 2, 1, 5, 3 \rangle$. Applying 20 partitions in the piecewise linear approximation, $\vec{\hat{q}_1} = \langle 3, 4, 9, 5 \rangle$ approximates $J_1(I_0, \vec{q}_1^*)$ as shown in Fig. 2.7 for $I_0 \in [-4, 14]$. The curves are plotted to demonstrate the difference between $J_1(I_0, \vec{q}_1^*)$ and $\mathcal{J}_t(I_0)$, while the system is in fact discrete. Table 2.1 compares the (s_t, Q_t) policy parameters obtained by the SDP in Section 2.3.1 and the heuristic for a zero initial inventory level.

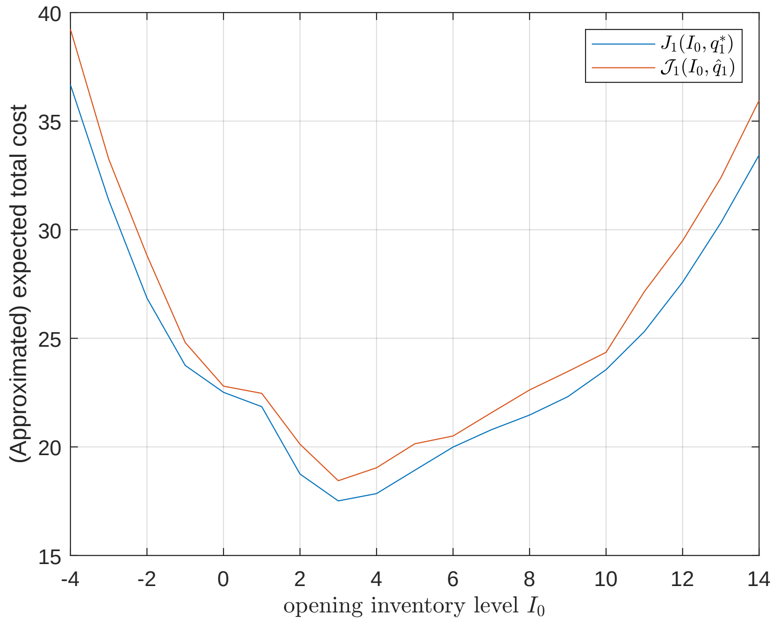

Figure 2.7: Plot of $J_1(I_0, \vec{q}_1^*)$ and $\mathcal{J}_1(I_0, \hat{\vec{q}}_1)$.

Table 2.1: Parameters of the (s_t, Q_t) policy for Example 2.3.1 computed by SDP and the heuristic.

	$\hat{Q}_t$				$\hat{s}_t$			
t	1	2	3	4	1	2	3	4
SDP	3	3	8	5	1	0	4	1
Heuristic	3	4	9	5	1	-2	4	0

Taking $G_1(0) = 21.8$ as a benchmark, the optimality gaps of the (s_t, Q_t) policy determined by SDP and our heuristic, relative to the (s_t, S_t) policy are shown in Table 2.2. We note that the (s_t, Q_t) policy produces large optimality gaps in Example 2.3.1, where values of the expected total cost are small, while the approximation accuracy of the heuristic $(23.1 - 22.5)/22.5 \times 100\% = 2.67\%$ is acceptable. We will extend our computational study in Section 2.5 to investigate how the (s_t, Q_t) policy performs on several problem instances.

Table 2.2: Expected total cost (ETC) and optimality gap (OG) of SDP and the heuristic for Example 2.3.1 under the (s_t, Q_t) policy.

	ETC	OG(%)
SDP	22.5	3.33
Heuristic	23.1	5.93

2.4.4 Approximation of the (s_t,Q) policy parameters

For the (s_t, Q) policy, a direct way to approximate the order quantity is to simplify the model in Appendix 2.B by replacing Q_t with Q and then following the steps in Sections 2.4.1 and 2.4.2; however, this is found to produce large optimality gaps in terms of the expected total cost.

Following the line of reasoning illustrated in Section 2.4.1 for (s_t, Q_t), one can derive a single order quantity in period 1 as $S_1 - I_0$ for a known opening inventory I_0. However, a high value for I_0 may result in a low order quantity imposed over a long period. In our heuristic (s_t, Q) policy, we define $\hat{Q} \triangleq S_1$ to be our approximate order quantity for horizon $(1, T)$; and we denote the vector of approximate order quantities as $\hat{\vec{q}} \triangleq \langle \hat{Q}, \dots, \hat{Q} \rangle$. The reorder points are adjusted to compensate for the overestimation of cases with high opening inventory levels.

The computation of reorder points $\hat{\vec{s}}_t$ associated with order quantity $\hat{Q}$ follows the same procedure proposed in Section 2.4.2 for (s_t, Q_t). We apply Model 2.4.2 with $\hat{Q}$ to obtain the approximated expected cost over the horizon (t, T) when no order is placed in period t, denoted as $\mathcal{J}_t(x, \hat{\vec{q}})$, and we apply our previously introduced heuristic algorithm on the function $\Delta\mathcal{J}_t(x, \hat{\vec{q}})$ to determine $\hat{s}_t$.

Example 2.2.1 (**Continued**)**.** Applying 20 partitions in the piecewise linear approximation, Fig. 2.8 approximates $J_1(I_0, \dot{\vec{q}}_1^*)$ by $\mathcal{J}_1(I_0, \hat{\vec{q}})$. While the inventory system is discrete, we plot interpolated curves for the sake of clarity. For a zero initial inventory level, Table 2.3 compares the parameters of the (s_t, Q) policy computed by SDP and the approximation. Taking $G_1(0) = 481$ as the benchmark, Table 2.4 summarises the optimality gaps of the (s_t, Q) policy computed by SDP and the heuristic. The approximation accuracy $(505 - 503)/503 \times 100\% = 0.398\%$

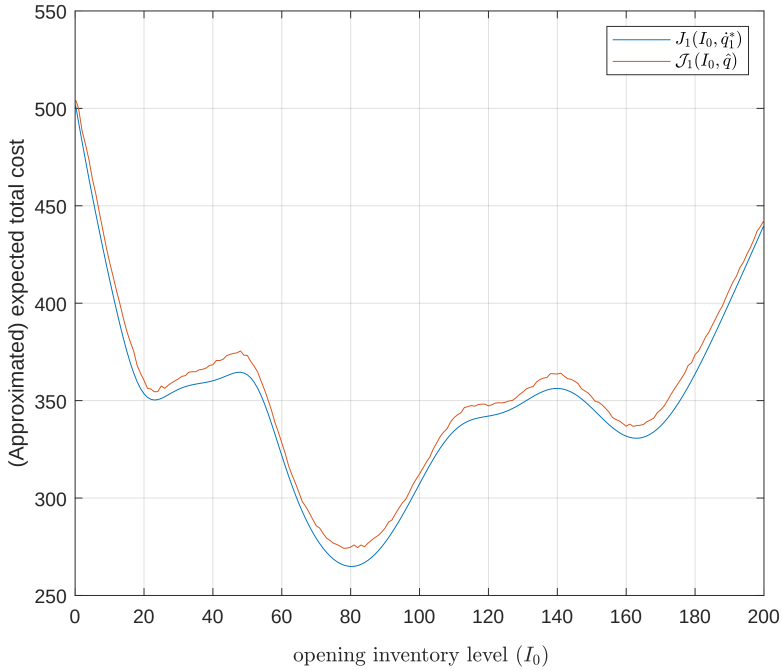

Figure 2.8: Plot of $J_1(I_0, \dot{\vec{q}}_1^*)$ and $\mathcal{J}_1(I_0, \hat{\vec{q}})$.

is high. We discuss the performance of the (s_t, Q) policy in detail in the next section.

2.5 Computational analysis

This section presents a computational analysis to evaluate (s, Q)-type policies under non-stationary stochastic demand. The analysis considers both the stochastic dynamic programming formulations and our heuristics for the (s_t, Q_t) and (s_t, Q) policies. In Section 2.5.1, we consider a test set comprising small problem instances with 6 periods; we investigate the performance of optimal (s, Q)-type policies against that of optimal non-stationary (s, S) policies, and we evaluate the difference between optimal (s, Q) and heuristic (s, Q) policies. In Section 2.5.2, we consider a test set comprising large problem instances with 25 periods; we investigate the performance of (s, Q)-type heuristics versus that of the optimal

Table 2.3: Parameters of the (s_t, Q) policy for Example 2.2.1 computed by SDP and the heuristic.

	$\hat{Q}$	$\hat{s}_t$			
t	–	1	2	3	4
SDP	83	13	33	54	24
Heuristic	84	14	34	55	24

Table 2.4: Expected total cost (ETC) and optimality gap (OG) of SDP and the heuristic for Example 2.2.1 under the (s_t, Q) policy

	ETC	OG(%)
SDP	503	4.57
Heuristic	505	4.99

.

non-stationary (s, S) policy; we also compare the performance between our (s, Q)-type heuristics and another existing static-dynamic uncertainty heuristic, namely the (R_t, S_t) policy discussed in (Rossi et al., 2015).

We name the optimal policy for the stochastic lot-sizing problem, which takes an (s, S) form, (s_t, S_t)-SDP. In our experiment we consider two variants of the (s, Q) policy: the (s_t, Q_t) policy, and the (s_t, Q) policy; presented in Section 2.3.1 and Section 2.3.2, respectively. For each variant, we discuss results for the optimal SDP formulation, named (s_t, Q_t)-SDP and (s_t, Q)-SDP, respectively; and results for our MINLP heuristics formulations presented in Section 2.4, named (s_t, Q_t)-Heuristic and (s_t, Q)-Heuristic, respectively. We apply 10 partitions in the piecewise linear approximation for both heuristics to ensure a good approximation according to Rossi et al. (2014). We simulate each test instance with the policy parameters obtained from the heuristics and derive the average total cost of 500,000 simulation runs.

For each approach, we always use the optimal (s, S) policy as a benchmark. Approaches are compared in terms of their expected total cost (ETC) using the percent optimality gap computed as $100 \times (\text{ETC}_2 - \text{ETC}_1)/\text{ETC}_1$, where ETC_1 is the expected total cost of the optimal non-stationary (s, S) policy, and ETC_2 is the expected total cost of the other approach benchmarked. We set a zero initial inventory for all test instances and test the robustness of heuristics for (s, Q)-type policies.

In our numerical study, we consider ten expected demand patterns: two life cycle patterns, one moves from the launch stage to maturity via a growth (LCY1) and the other moves from the growth stage through maturity and into decline (LCY2); two sinusoidal patterns, one with stronger (SIN1) and the other with weaker (SIN2) oscillations; a stationary demand pattern (STAT); a random demand pattern (RAND); and lastly, 4 empirical patterns derived according to (Strijbosch et al., 2011).

All computations are performed by a 4.0 (1.90+2.11) gigahertz Intel(R) Core(TM) i7-8650U CPU with 16.0 gigabytes of RAM in JAVA 1.8.0_201.

2.5.1 A test set with 6-period Poisson-distributed demand

The first test set involves 60 instances over a 6-period planning horizon in which the demand follows a non-stationary Poisson distribution. Our aim is twofold: first, we aim to investigate the performances of optimal (s, Q)-type policies obtained via SDP against the optimal non-stationary (s, S) policy; second we aim to evaluate the difference between the optimal and heuristic (s, Q) policies.

We assume the maximum order quantity is 9, which allows us to enumerate all combinations of order quantities for the (s_t, Q_t) policy by stochastic dynamic programming. The problems in this test set are designed with very small mean demands λ_t, as illustrated in Fig. 2.9. The values of λ_t are set to be between 1 and 7 in all cases which allows variation in the optimal values of Q_t and ensures that the optimal order quantity is never as high as 9 in any period. The problem coefficients are considered over $z \in \{0, 1\}$ and the three sets of K and b shown in Table 2.5 with different ratios of K to b. Holding cost is set as $h = 1$ for all instances. The step size in the proposed algorithm is set as 4.

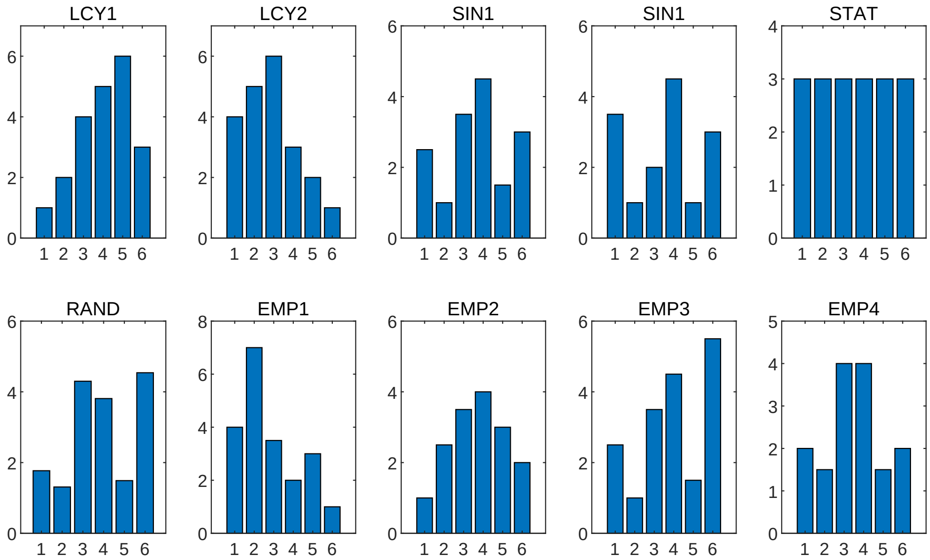

Figure 2.9: Demand patterns of 6-period instances.

Table 2.5: parameter groups of fixed ordering cost (K) and penalty cost (p)

set	K	b	ratio
1	5	3	1.67
2	10	3	2.00
3	10	7	1.43

For each approach considered, Table 2.6 reports the optimality gaps observed relative to the optimal (s_t, S_t) policy. The results for (s_t, Q_t)-SDP and (s_t, Q)-SDP give the exact optimality gaps for these policies against optimal (s_t, S_t) policy, which are on average 1.91% and 3.61% respectively. In detail, (s_t, Q_t)-SDP performs better than (s_t, Q)-SDP in every individual demand pattern; and (s_t, Q)-SDP is dominated by (s_t, Q_t)-SDP even in the case of a stationary demand pattern. In view of cost parameters, there is no obvious relation between optimality gaps and the variation in demand patterns or in the ratio of K to b. Optimality gaps also decrease when the unit cost increases. On the other hand, the increase in penalty cost results in a small increase in the optimality gap for both (s_t, Q_t)-SDP

(1.43% to 1.49%) and (s_t, Q)-SDP (2.92% to 3.22%).

For (s_t, Q_t)-Heuristic and (s_t, Q)-Heuristic we found that the optimality gaps increase by an average of 0.85% and 1.05%, respectively. The largest average increases arise under demand pattern EMP3 (1.04%) for (s_t, Q_t)-Heuristic and RAND (1.75%) for (s_t, Q)-Heuristic. We conclude that the difference between SDP and the heuristic approach is generally low.

Table 2.6: Average percent optimality gap over our 6-period test set under different demand patterns and pivoting parameters.

Problem Settings	(s_t, Q_t)-SDP	(s_t, Q_t)-Heuristic	(s_t, Q)-SDP	(s_t, Q)-Heuristic
demand pattern				
LCY1	1.96	2.60	2.55	3.30
LCY2	2.70	3.60	5.37	6.11
SIN1	1.95	2.89	3.96	4.80
SIN2	2.13	3.04	3.18	4.75
STAT	1.54	2.41	2.45	4.00
RAND	1.17	2.02	3.12	4.86
EMP1	1.98	2.87	3.98	5.33
EMP2	2.32	2.94	3.56	4.44
EMP3	1.13	2.17	3.11	3.66
EMP4	2.21	3.11	4.80	5.39
unit cost				
0	2.03	2.93	3.83	5.15
1	1.79	2.59	3.38	4.18
set				
1	2.81	3.76	4.67	5.76
2	1.43	2.29	2.92	3.96
3	1.49	2.23	3.22	4.28
Average	1.91	2.76	3.61	4.66

2.5.2 A test set with 25-period Normally-distributed demand

We extend the planning horizon to 25 periods. The purpose of implementing this test set is twofold. First we aim to investigate the performance of (s, Q)-type heuristics versus that of the optimal non-stationary (s, S) policy for larger instances; second, we aim to compare the performances of (s, Q)-type heuristics and the non-stationary (R, S) policy introduced in (Rossi et al., 2015), which we name (R_t, S_t)-Heuristic.

Since the computation of piecewise linear approximation parameters consumes a large amount of computation time for large non-stationary demand following a Poisson distribution, in what follows we will focus on normally distributed demand patterns, for which Rossi et al. (2014) present precomputed optimal partitioning coefficients.

We refer to the 25-period instances in Xiang et al. (2018). The demand d_t in each period t is assumed to be a normally distributed random variable with known mean $\tilde{d}_t$ and standard deviation $\sigma_t = \rho \cdot \tilde{d}_t$, where ρ denotes the coefficient of variation of the demand, which remains fixed over time as prescribed in Bollapragada and Morton (1999); demands are assumed to be independent of each other. We allow the standard deviation parameter ρ to vary over $\rho \in \{0.1, 0.2, 0.3\}$. Demand patterns are illustrated in Figure 2.10. Other problem parameters are $K \in \{500, 1000, 1500\}$; $b \in \{5, 10, 20\}$; $z \in \{0, 1\}$; and $h = 1$. The step size in the proposed algorithm is set as 32.

The reader should note that, since stochastic dynamic programming is pseudo-polynomial, an increase in the average value of the demand or of its standard deviation will lead to a dramatic increase in the state space and hence of computational times (Dural-Selcuk et al., 2020). The (s_t, Q)-SDP can be implemented by bounding the inventory level, while it is no longer possible to compute (s_t, Q_t)-SDP within a reasonable time for normal demand or large planning horizons such as 25.

Table 2.7 reports average optimality gaps for our 25-period instances. For the (s_t, Q_t)-Heuristic, the average optimality gap is 2.31%, which is similar to the

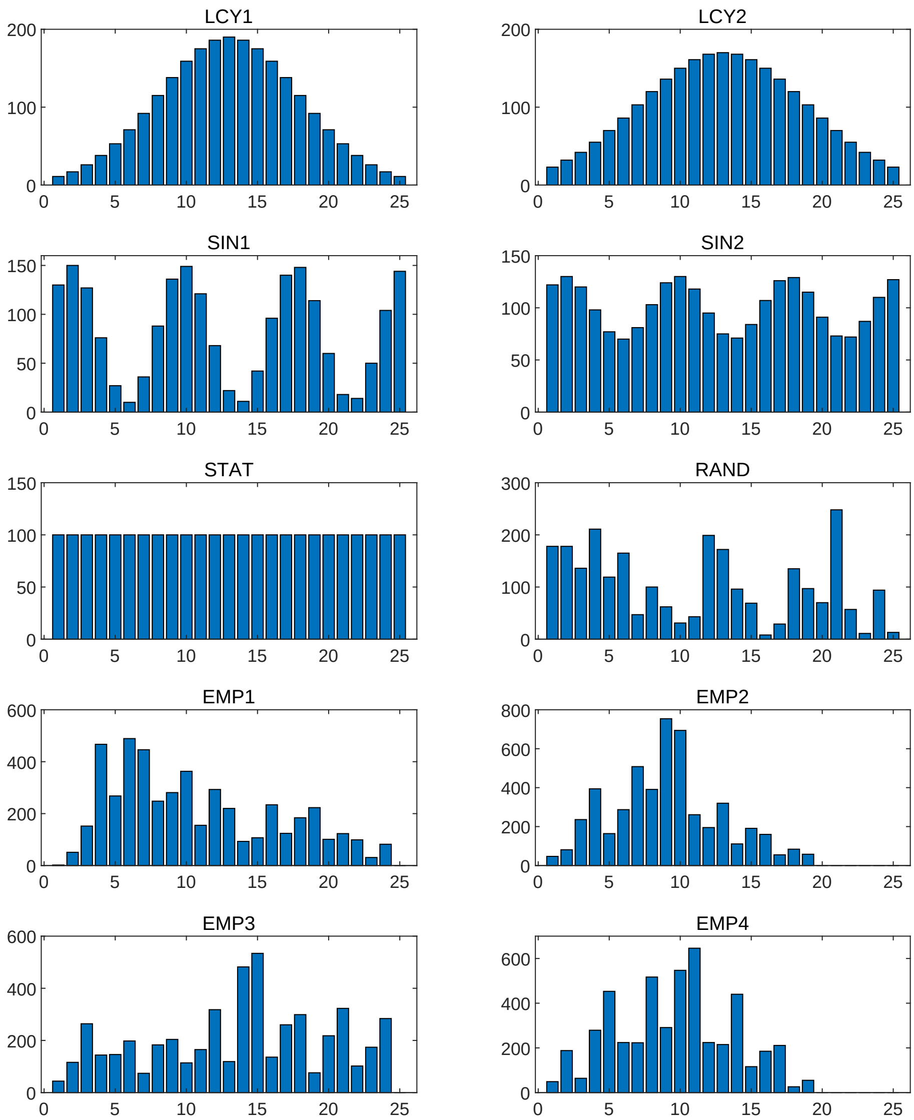

Figure 2.10: Demand patterns of 25-period instances.

result obtained for the 6-period test problems. The optimality gap also exhibits similar trends with the penalty cost and the unit cost, while the gap increases with the penalty cost and decreases when the unit cost is increased. For the normal distribution, the increase of the standard deviation parameter ρ reduces the optimality gap, which suggests the (s_t, Q_t) policy performs slightly better when the standard deviation of demand is higher.

For (s_t, Q)-Heuristic, once more, as with the 6-period test set, we observe that the (s_t, Q)-Heuristic is not satisfactory. The average optimality gap now increases up to 11.5%; and for an individual demand pattern, the optimality gap reaches 25.9%. We also cross-validated results against optimal (s_t, Q) parameters obtained via SDP, to ensure the accuracy of the result, but found that the optimality gap remained as large as 10.5% on average. This confirms that it is not just the approximation, but the policy itself that performs poorly. We believe that under non-stationary demand, when the length of the planning horizon increases, the single order quantity Q in the (s_t, Q) policy cannot properly hedge against demand, and thus it produces substantially higher expected cost than other policies that provide more flexibility. It should be noted that the maximum optimality gaps observed for (s_t, Q)-SDP (24.9% and 23.8%) concern empirical demand patterns with a series of 0 demand. A single order quantity for all periods causes either a large amount of holding cost for 0-demand periods or penalty cost for large-demand periods. Despite the unsatisfactory performance of the (s_t, Q) policy, it is worth noting that the results show the same trends with respect to ρ, b and z as the (s_t, Q_t) policy.

The optimality gaps for the (R_t, S_t)-Heuristic are 2.90% on average, which is larger than the optimality gap observed for the (s_t, Q_t)-Heuristic over all demand patterns and pivoting parameters. As a result, we conclude that in the context of our test set the (s_t, Q_t) is better than (R_t, S_t) policy in terms of expected cost.

2.6 Conclusion

This paper investigated (s, Q)-type policies for the non-stationary stochastic lot-sizing problem.

By adopting a variant of the Bookbinder and Tan (1988) static-dynamic uncertainty strategy in which order quantities are fixed once and for all at the beginning of the planning horizon, we derived a stochastic dynamic formulation for the problem and proved that the associated optimal policy must take the (s, Q) form.

To compute optimal policy parameters, we enumerated all possible order quantity configurations to determine an optimal one and then used a dynamic programming recursion to determine associated reorder points. Since this brute force approach is not scalable, we introduce MINLP-based heuristics to tackle large-size problems under (s, Q)-type policies. Our heuristics leverage the MINLP approaches introduced in Xiang et al. (2018) for the non-stationary (s, S) policy, in which the non-linearity is dealt with via a piecewise linear approximation of the cost function.

We carried out extensive computational experiments on a test set of small problems with short (6-period) planning horizons and a test set of large problems with long (25-period) planning horizons. Both test sets include 10 demand patterns and various coefficient settings. In the numerical study on small problems, our results show that the average optimality gaps for the (s_t, Q_t) policy and the (s_t, Q) policy versus the optimal (s_t, S_t)-SDP are 1.91% and 3.61%, respectively; and the optimality gaps associated with (s_t, Q_t)-Heuristic and (s_t, Q)-Heuristic (2.76% and 4.66%, respectively) are close to those of the corresponding SDP.

In the numerical study on large problems, we found that the average optimality gaps of the (s_t, Q_t)-Heuristic remained small (2.31%); while the optimality gap of the (s_t, Q)-Heuristic remained unsatisfactory (11.5%). Our comparison against the (R_t, S_t)-Heuristic showed that the optimality gap of the (s_t, Q_t)-Heuristic was slightly better than that of the (R_t, S_t)-Heuristic (2.90%).

Our investigation demonstrates the effectiveness of (s, Q)-type policies for the non-stationary stochastic lot-sizing problem. The (s_t, Q_t) policy can be well approximated by a heuristic that provides satisfactory results in a reasonable time. (Near-)optimal parameters for the (s_t, Q) policy can be found in a reasonable time using either SDP or a heuristic, but it produces larger optimality gaps than the (s_t, Q_t) policy.

Future research from this study is three-folded.

- One direction of future research is based on the current working paper about the capacitated lot-sizing problem under (s_t, Q_t) policy with non-stationary stochastic demand, since it is noticed that capacity of the inventory is easily modelled for the (s_t, Q_t) policy. The single-item, periodic review production and inventory system has been extensively studied in the literature. In the context of single-level production planning, with a finite planning horizon and known stochastic demand, the classical capacitated lot-sizing problem still consists of determining the amount and the timing of the production of products in the planning horizon. The single-item capacitated lot-sizing problem has been shown by Florian et al. (1980) and Bitran and Yanasse (1982) to be NP-hard for the deterministic demand. Therefore, developing effective heuristics has been a profitable research area for a long time.

- Another direction of future research is the analytical investigation of the (s, Q)-type policy. The optimality of (s, S) policy has been shown mathematically by Scarf (1960), from which the reorder point is determined by $s_t = \inf\{I_t : G_t(I_t) < G_t(S_t^*) + c(S_t^*)\}$, where $G_t(I_t)$ is the expected total cost over horizon (t, T) with no order in period t. In the (s, Q)-type policies, the shape of numerical tests form similar convexities as (s, S) does, while the $K-$convexity does not hold and the reorder points are retrieved by $s_t = \inf\{I_t : G_t(I_t) - G_t(I_t + Q_t^*) < c(Q_t^*)\}$ due to the order quantity not shifting with inventory level I_t. An analytical convexity analysis can be conducted to investigate this difference, and consequently, a gap of (s, Q)-type policies can be formed against the optimal (s, S) policy systematically. In conjunction with this point, in the heuristic method, the reorder point can be more easily obtained, leading to more efficient computation and potentially better inventory decisions.

- One potential improvement that can be made to the heuristic method is in the determination of order quantities for each period. In Chapter 2, the order quantities are approximated based on the (s, S) policy's method introduced in Xiang et al. (2018), upon which the reorder points are subsequently determined. Investigating alternative approaches for approximating order quantities could result in better adaptation to changing demand patterns and further enhance the performance of the heuristic method.

Table 2.7: Average percent optimality gap over our 25-period test set under different demand patterns and pivoting parameters.

Problem Settings	(s_t, Q_t)-Heuristic	(R, S)-Heuristic	(s_t, Q)-SDP	(s_t, Q)-Heuristic
demand pattern				
LCY1	2.38	2.50	9.56	10.5
LCY2	2.20	2.20	7.06	7.60
SIN1	2.52	2.87	6.25	8.06
SIN2	2.00	2.03	3.29	3.79
STA	1.45	1.50	1.91	2.25
RAND	2.58	2.99	7.24	8.98
EMP1	2.62	3.19	12.5	13.3
EMP2	2.50	4.22	24.9	25.9
EMP3	2.19	2.79	8.73	9.49
EMP4	2.70	4.71	23.8	25.3
std parameter				
0.1	2.52	2.68	10.3	11.4
0.2	2.48	2.50	11.0	11.9
0.3	1.94	3.53	10.3	11.3
fixed ordering cost				
500	2.71	3.36	13.8	14.7
1000	1.86	2.61	9.97	10.8
1500	2.35	2.69	7.70	8.90
penalty cost				
5	2.15	2.37	8.79	9.93
10	2.17	2.97	10.8	11.6
20	2.62	3.37	12.0	13.0
unit cost				
0	2.53	2.47	11.7	12.7
1	2.10	3.33	9.32	10.3
Average	2.31	2.90	10.5	11.5

Appendix 2.A Notations

Table 2.8: Notations of important functions

Functions	Explaination
$c(Q)$	cost of an order of size Q
$L_t(y)$	expected immediate holding and penalty cost when the inventory level after replenishment is y at period t
$C_t(x)$	expected total cost of the optimal policy over periods t to T when the opening inventory level is x
$G_t(y)$	expected total cost over periods t to T when the opening inventory level is y and no order is placed in period t
$V_t(x, \vec{q_t})$	expected total cost with a combination of reorder quantities $\vec{q_t} \in \mathcal{Q}_t$ when the opening inventory level is x
$V_0(x)$	minimum expected total cost over $\mathcal{Q}$, the set of possible order quantities, when opening inventory level is x
$J_t(x, \vec{q_t})$	expected total cost with a combination of reorder quantities $\vec{q_t} \in \mathcal{Q}_t$ when no order is placed for opening inventory level x in period t
$\hat{J}_t(x, \vec{q_t})$	expected total cost with a combination of reorder quantities $\vec{q_t} \in \mathcal{Q}_t$ when an order is placed for opening inventory level x in period t
$\Delta J_t(x, \vec{q_t})$	$= J_t(x, \vec{q_t}) - J_t(x + Q_t, \vec{q_t})$, the difference between expected total costs with opening inventory levels x and $x + Q_t$
$\mathcal{J}_t(x, \hat{\vec{q}})$	an approximation of $J_t(x, \vec{q_t}^*)$ by MINLP

Appendix 2.B MINLP model to compute S_t

This appendix section presents the MINLP model introduced in (Xiang et al., 2018) to compute the order-up-to level S_t of the (s_t, S_t) policy. To properly account for the proportional ordering cost z, we modify the objective function in line with Tarim and Kingsman (2006).

We apply a superscript 'S' to distinguish decision variables from other formulations.

$$
\begin{aligned}
\min \quad & z(\tilde{I}_T^S + \tilde{d}_{tT}) + \sum_{k=t}^{T} (K\delta_k^S + Q_k^S + h \cdot \tilde{H}_k + b \cdot \tilde{B}_k), && \\
\text{s.t.} \quad & \delta_t^S = 1, && (2.30) \\
& \tilde{I}_t^S + \tilde{d}_t = S_t, && (2.31) \\
& \delta_k^S = 0 \to \tilde{I}_k^S + \tilde{d}_k = \tilde{I}_{k-1}^S, \quad k = t+1, \ldots, T, && (2.32) \\
& \delta_k^S = 1 \to \tilde{I}_k^S + \tilde{d}_k = \tilde{I}_{k-1}^S + Q_k^S, \quad k = t+1, \ldots, T, && (2.33) \\
& \sum\nolimits_{j=t}^{k} P_{jk}^S = 1, \quad k = t+1, \ldots, T, && (2.34) \\
& P_{jk}^S \geq \delta_j^S - \sum_{r=j+1}^{k} \delta_r^S, \quad k = t, \ldots, T \text{ and } j = t, \ldots, k, && (2.35) \\
& P_{jk}^S = 1 \to \tilde{H}_k = \hat{\mathcal{L}}(\tilde{I}_k^S + \tilde{d}_{jk}, d_{jk}), \quad k = t, \ldots, T \text{ and } j = t, \ldots, k, && (2.36) \\
& P_{jk}^S = 1 \to \tilde{B}_k = \mathcal{L}(\tilde{I}_k^S + \tilde{d}_{jk}, d_{jk}), \quad k = t, \cdots, T \text{ and } j = t, \ldots, k, && (2.37) \\
& Q_k^S, \tilde{H}_k, \tilde{B}_k \geq 0, \quad k = t, \ldots, T, && (2.38) \\
& P_{jk}^S, \delta_k^S \in \{0, 1\}, \quad k = t, \ldots, T \text{ and } j = t, \ldots, k. && (2.39)
\end{aligned}
$$

We add constraints (2.30) and (2.31) to force the system to place an order in the first period of the horizon (t, T) in order to approximate S_t. The other constraints remain as in Xiang et al. (2018). Constraints (2.32) and (2.33) capture the inventory flow balance equations and reorder conditions. Constraint (2.35) forces $P_{jk}^S = 1$ if the most recent replenishment before period k in horizon (t, k) is placed in period j; constraint (2.34) ensures $P_{jk}^S = 0$ otherwise. Constraints (2.36) and (2.37) model the expected inventory and back-order levels at the end of period k through first order loss functions.

Appendix 2.C Piecewise linear approximation with non-stationary Poisson demand

Consider a random variable ω and a scalar variable x, the first order loss function is defined as $\mathcal{L}(x, \omega) = \mathbb{E}[\max(\omega - x, 0)]$ and its complement as $\hat{\mathcal{L}}(x, \omega) = \mathbb{E}[\max(x - \omega, 0)]$.

Decision variables $\tilde{H}_t \geq 0$ and $\tilde{B}_t \geq 0$ denote the expected inventory and back-order levels at the end of period t.

Rossi et al. (2014) presented the approach with bounding techniques to generate piecewise linear lower and upper bounds and discussed the implementation on the standard normal distribution. Instances in this paper involve non-stationary Poisson demand to enable the computation analysis on problems with small means of demand. Therefore, we extend the results of Rossi et al. to the Poisson distribution.

To minimise the expected inventory and back-order levels at the end of each period with a lower bounding piecewise linear approximation, $\tilde{H}_t$ is constrained by

$$\tilde{H}_t \geq (\tilde{I}_t + \sum_{j=1}^{t} \tilde{d}_{jt} P_{jt}) \sum_{k=1}^{i} p_k + \sum_{j=1}^{t} (\sum_{k=1}^{i} p_k \mathbb{E}[d_{jt}|\Omega_{jt}]) P_{jt}, \tag{2.40}$$

and $\tilde{B}_t$ by

$$\tilde{B}_t \geq -\tilde{I}_t + (\tilde{I}_t + \sum_{j=1}^{t} \tilde{d}_{jt} P_{jt}) \sum_{k=1}^{i} p_k + \sum_{j=1}^{t} (\sum_{k=1}^{i} p_k \mathbb{E}[d_{jt}|\Omega_{jt}]) P_{jt}. \tag{2.41}$$

where d_{jt} follows the notation in section 2.4.1 denoting the convolution of d_j to d_t, demand d_t is a random variable that is of a Poisson distribution with mean λ_t, and its domain $\mathbb{R}^+$ is partitioned into N disjoint adjacent subregions $\Omega_1, \Omega_2, \cdots, \Omega_N$.

According to the technique in (Rossi et al., 2014), $\Omega_1 = [0, a_1]$, $\Omega_i = [a_{i-1}, a_i]$ for $i = 2, \cdots, N-1$ and $\Omega_N = [a_{N-1}, \infty]$. Let the probability density function of d_t be $g_{\lambda_{\lambda_t}}(k) = e^k/\lambda_t!$ with its cumulative function $G_{\lambda_t}(k)$, $g_{\lambda_t}^{-1}$ and $G_{\lambda_t}^{-1}$ be their inverse functions, which returns the value of k satisfying $g_{\lambda_t}(k) = p$, then

$$a_i = \lceil G_{\lambda_t}^{-1}(\frac{i}{N}) \rceil, \tag{2.42}$$

and the probability p_i that a realisation of the Poisson random variable d_t (i.e. a value of demand d_t) locates within the subregion i is

$$p_i = \Pr\{d_t \in \Omega_i\} = \sum_{\Omega_i} g_{\lambda_t}(u) \ \mathrm{d}u, \tag{2.43}$$

and

$$\mathbb{E}[d_t|\Omega_i] = \frac{N}{i} \sum_{\Omega_i} u g_{\lambda_t}(u) \ \mathrm{d}u, \tag{2.44}$$

where $i = 1, 2, \cdots, N$.

Chapter 3

Paper II: An efficient computation approach to the non-stationary lot-sizing problem under static-dynamic strategy with penalty scheme

Abstract

This paper addresses an efficient approximation to the single-item single-location lot-sizing problem with non-stationary stochastic demand, where any unmet demand is considered back-order with a penalty cost. We leverage the mixed integer linear programming of the original problem under static-dynamic uncertainty strategy and introduce a relaxation to form a weighted directed acyclic graph for determining the shortest path; we also introduce a method to obtain reorder points from the graph, and this constructs an approximation of (s_t, S_t) policy. As an intermediate, this shortest-path formulation's infeasibility on negative replenishment orders is checked against the non-relaxed problem and resolved by an augmenting procedure. This procedure classifies infeasible scenarios in each period, introduces new node(s) and augments the graph by re-directing, re-computing and duplicat-

ing based on the original. In the end, the (R_t, S_t) policy parameters are retrieved from the shortest path of the augmented graph. In the numerical experiments, we approximate the (s_t, Q_t) policy by (s_t, S_t) and (R_t, S_t) policy parameters obtained from proposed graph-based methods. The results demonstrate that our approach can quickly solve a broad class of problem instances with reasonable optimality gaps.

Keywords inventory control, lot-sizing problem, non-stationary demand, uncertainty strategy, shortest path problem.

3.1 Introduction

The study of lot-sizing problems starts with Harris (1913) and expands to incorporate more realistic assumptions about product demand into inventory models as pointed out by Graves (1999). In industrial settings, demand is typically erratic and difficult to predict. Numerous demand histories exhibit the characteristics of random walks as they fluctuate through time, frequently changing their orientations and rates of rising or falling. A class of research considers non-stationary stochastic demand as Wemmerlöv (1989) illustrates that lot-sizing studies need to be undertaken on stochastic dynamic environments with at least a modicum of resemblance to reality.

Planning for future lot sizes will always include accounting for forecast errors for the difficulty of the prediction. As Silver (1978) raises, probabilistic modelling is more appropriate when significant uncertainty exists instead of deterministic lot-sizing rules. For this point, Silver (1978) proposes a heuristic method based on the assumption that the prediction errors are normally distributed. In contrast, Askin (1981) uses the least cost per unit time approach to determine the number of periods the immediate replenishment must cover and explicitly includes the cost effects of probabilistic demand in the choice of the number of periods for which to order. However, Bollapragada and Morton (1999) points out that this approach relies on the computation of demand convolution and is prohibitively expensive.

From another perspective of planning, Bookbinder and Tan (1988) presents three uncertainty strategies and develops a heuristic for the static-dynamic strategy. The static-dynamic strategy statically determines one of the parameters between order quantity and order timing at the beginning of the planning horizon and dynamically determines another in a wait-and-see fashion. This uncertainty strategy is adapted into (R_t, S_t) and (s_t, Q_t) policies for non-stationary stochastic demand.

This study focuses on developing computationally efficient approaches for computing near-optimal policy parameters for the single-item single-location non-stationary stochastic lot-sizing problem with a penalty cost scheme. We develop an efficient dynamic programming approach that, starting from a shortest-path formulation, leverages an augmenting procedure to resolve infeasibility concerning the original problem.

We begin with the existing theoretical literature and outline the normative theory, which includes uncertainty strategies, inventory policies, the stochastic dynamic programme for the optimal policy, and the approximation approach with the first-order loss function. We formulate a shortest path problem based on a relaxation problem and obtain an approximation of (s_t, S_t) policy. The infeasibility against the original problem is resolved by an augmenting procedure, from which (R_t, S_t) policy parameters are approximated directly from the shortest path. In short, this paper models the non-stationary stochastic lot-sizing problems under the static-dynamic uncertainty strategy through shortest-path heuristics and analyses the optimality gap and the computation efficiency for the proposed approaches.

3.2 Literature Review

Our work contributes to the literature on non-stationary stochastic lot-sizing problems and heuristics under static-dynamic uncertainty strategy. This section first reviews the uncertainty strategy and corresponding inventory policies and then demonstrates the gap in the literature that our work fills.

3.2.1 Uncertainty strategies and inventory policies

In the early Sixties, the work of Wagner and Whitin (1958) on dynamic lot sizing is extended into the stochastic lot-sizing problem considering single-item single-location inventory control problems with multiple time periods. To deal with the uncertainty of demands, Bookbinder and Tan (1988) propose uncertainty control strategies of three types: the “static,” the “dynamic uncertainty”, and the “static-dynamic”.

- The static uncertainty strategy determines the replenishment timing (R) and quantity (Q) before the system operates and is captured by the (R, Q) policy. The research under this policy with non-stationary demand is conducted with dynamic programming. Sox (1997) develops a solution algorithm for equivalent deterministic problems by adding additional feasibility constraints. And Vargas (2009) applies the stochastic dynamic programming and shows that the proposed model is equivalent to the shortest path problem in an acyclic network, solved by an optimisation algorithm with a rolling horizon.

- The (s, S) policy follows a dynamic uncertainty strategy, where the system places a replenishment order to restore the current stock to the level S if the current stock decreases to a reorder point s. Scarf (1960) shows that if the holding and shortage costs are linear, the optimal policy in each period is of (s, S) type. Due to the computational complexity of non-stationary (s, S) policy, Silver (1978) and Askin (1981) extend Silver and Meal (1973) deterministic algorithm, Bollapragada and Morton (1999) and Xiang et al. (2018) provide model-based heuristics with approximation techniques to obtain near-optimal policy parameters.

- The static-dynamic uncertainty is captured by (R, S) and (s, Q) policies in two ways. The (R, S) policy applies a fixed reordering timing, at which the replenishment quantity is determined dynamically by setting the order-up-to level in advance, while the actual order quantity is decided only when the order is issued. An alternative to the (R, S) policy is the (s, Q) policy, an order with quantity Q is issued when inventory falls below or at the reorder threshold s. The literature for this strategy is discussed in the next subsection.

3.2.2 Literature in non-stationary static-dynamic heuristics

In the scope of non-stationary stochastic demand, Tarim and Kingsman (2004) present a mixed integer programming (MIP) formulation to determine the order schedule and order-up-to levels of (R^n, S^n) policy simultaneously in a single step for α service level problem, where n denotes the n-th replenishment cycle. This formulation is adapted by Tempelmeier (2007) for α_c service level and by Rossi et al. (2015) for β and β^{cyc} service levels as well as the penalty cost scheme with back-orders. Apart from modelling service levels, Tarim and Kingsman (2006) provides another MIP formulation to account for the shortage costs, where the objective function is obtained by the mean of piecewise linear approximation. The accuracy of the approximation can be adjusted by introducing new breakpoints. This piecewise linear approximation is elaborated by (Rossi et al., 2014) with segmentation parameters for Normal Distribution provided.

Tarim and Kingsman's model on (R_t, S_t) policy is reformulated by Tunc et al. (2014) through disjointing consecutive replenishment periods based on a MIP formulation via the network flow structure of the problem. This formulation has a tighter linear relaxation and, in turn, has superior computational performance. Tunc et al. (2018) generalise this model and develop a dynamic cut generation approach combining with piecewise linear approximation by Rossi et al. (2015); this formulation enjoys the computation efficiency from the reformulation of Tunc et al. (2014) and the modelling flexibility of Rossi et al. (2015); however, it is designed solely for problems characterised by service levels. Özen et al. (2012) considers both penalty cost and service level and proves that the optimal policy under static-dynamic uncertainty strategy is the base stock policy for both penalty and service-level constrained models for the capacity limitations and minimum order quantity requirements.

We notice that these formulations operate under the assumption that negative order is not allowed; in other words, the retailers cannot sell back the excess stock to the warehouse and should carry the stock forward to the next period. Tarim et al. (2011) relax this constraint and propose a computationally-efficient relaxation MIP which provides an optimal solution most of the time. They also show that an infeasible solution yields a tight lower bound for the optimal cost and that the infeasible solution can be modified easily to obtain a feasible solution, which yields an upper bound. In the algorithm developed by Tarim et al. (2011), an optimal solution is obtained by iteratively solving a relaxed version of the problem that is formulated and solved as a shortest-path problem. In case of infeasibility, the relaxation approach is implemented at each node of the search tree in a branch-and-bound procedure to search for an optimal solution efficiently. This approach is refined by Rossi et al. (2011) through a filtering and augmenting procedure. Starting from a relaxed state space graph, the method proposed by Rossi et al. (2011) eliminates provably sub-optimal arcs and states (filtering) and then efficiently builds up (augmenting) a reduced state space graph representing the original problem.

3.2.3 Challenges and research gaps

Based on the literature survey above, we notice the mathematical formulations or heuristics on (R_t, S_t) policy discuss more the service levels than the shortage/penalty cost. According to the definition of service levels, these problems (Tunc et al., 2014, 2018) formulate lower bounds of the expected order quantity and upper bounds of the expected total cost and further produce more compact formulation and efficient model-based algorithms. Under the penalty cost scheme, back-ordered items or lost sales are added to the trade-off between the expected total cost and the order quantity, and this complicates the modelling.

This paper aims to investigate the static-dynamic uncertainty strategy under a penalty cost scheme, where the unmet demand is considered back-ordered. Our research contributes to the literature in three aspects as follows.

- We develop efficient approaches to approximate (s_t, S_t) and (R_t, S_t) policy parameters for non-stationary stochastic lot-sizing problem. Unlike other existing literature, which addresses a service-level oriented formulation, this method is developed under a penalty cost scheme with unmet demand back-ordered.

- We leverage the existing mathematical programming formulations and develop a shortest-path formulation through a relaxation of the original problem. An augmenting procedure is proposed to resolve the infeasibility.
- We demonstrate in a comprehensive numerical study that proposed approaches based on the shortest path are easy to implement through simple numerical integration based on non-stationary demand distributions.

The rest of the paper is organised as follows. Section 3.3 introduces the problem description followed by a mixed integer linear programme for (R_t, S_t) policy presented in Section 3.4. Section 3.5.1 presents a relaxation of the original problem and Section 3.5.2 formulates it to solve a shortest path. The determination of reorder points is presented in Section 3.5.3 to approximate an (s_t, S_t) policy. Section 3.6 introduces an augmenting procedure to resolve the infeasibility with respect to the original problem for an (R_t, S_t) policy. Section 3.7 illustrates the computation efficiency of the proposed method through a set of 25-period instances against the existing approaches.

3.3 Problem description

We consider a single-item single-location non-stationary stochastic lot-sizing problem over a planning horizon of T periods. Replenishment orders are placed and instantaneously delivered at the beginning of each time period. Each replenishment order incurs an ordering cost $c(\cdot)$ comprising a fixed ordering cost K and a linear ordering cost z proportional to the *non-negative order quantity* Q, i.e. $c(Q) \triangleq K + zQ$ for $Q \geq 0$. The non-negativity of Q indicates that the received items cannot be returned to the warehouse.

The periods' demands d_t, for $t = 1, \cdots, T$, are independent random variables with known probability density functions $g_t(\cdot)$. We introduce the penalty cost scheme for any unmet demand at the end of the period, which is back-ordered. At the end of each period, a linear holding cost h is incurred for each unit carried from one period to the next, and a linear penalty cost b is charged on each unit back-ordered. Then the expected immediate cost at the end of period t can be expressed as

$$L_t(y) = \mathbb{E}[h \max(y - d_t, 0) + b \max(d_t - y, 0)], \tag{3.1}$$

where y denotes the inventory level after receiving the replenishment and $\mathbb{E}$ denotes taking the expectation. In what follows, $\tilde{H}_t$ and $\tilde{B}_t$ are used to denote expected inventories, to be consistent with $\tilde{I}_t$ in sections 3.4 and 3.5.

Let the notation $\tilde{\cdot}$ denote the expectation of a random variable. For period t, we denote $\tilde{H}_t(y) = \mathbb{E}[\max(y - d_t, 0)]$ as the expected inventory carried from one period to the next and $\tilde{B}_t(y) = \mathbb{E}[\max(d_t - y, 0)]$ as the expected back-ordered inventory level, then we rewrite Eq.(3.1) as

$$L_t(y) = h\tilde{H}_t(y) + b\tilde{B}_t(y), \tag{3.2}$$

and the expected closing inventory level at the end of period t is $\tilde{I}_t(y) = \tilde{H}_t(y) - \tilde{B}_t(y)$ with the replenished inventory y at the beginning of period t.

Let $C_t(x)$ be the expected total cost of an optimal policy over periods t to T with the opening inventory level x, then the problem can be modelled as a stochastic dynamic programme (Bellman, 1957)

$$C_t(x) = \min_{y \geq x} \{c(y - x) + L_t(y) + \mathbb{E}[C_{t+1}(y - d_t)]\}, \tag{3.3}$$

where $C_{T+1}(x) = 0$ is the boundary condition.

We note that the optimisation works on the constraint $y \geq x$, which represents a non-negative replenishment. When a replenishment is placed with zero order size, the system takes the chance to review the inventory level. This is a feasible action in an optimal solution to reduce the nervousness (De Kok and Inderfurth, 1997; Rossi et al., 2011) and better design the replenishment cycle. We apply a running instance to demonstrate this optimal policy and other approaches in the following sections.

Example 3.3.1. We consider the same instance as Rossi et al. (2011, sec.7) do with 95% α service level, while we refer to the definition $\frac{b}{b+h} = \alpha = 0.95$ to have $b = 19$ when $h = 1$. Demand in each period is independent and follows the Normal distribution with mean $\tilde{d}_t = \{100, 125, 25, 40, 30\}$ and $\sigma_t/\tilde{d}_t = 0.3$. We apply a fixed ordering cost $K = 50$ and unit ordering cost $z = 0$. This instance is solved with the expected total cost $C_1(I_0) = 404$ under an optimal policy $\vec{S}^* = \{149, 186, 37, 82, 45\}$ and $\vec{s}^* = \{119, 154, 24, 45, 30\}$.

3.4 An MILP approximating (R, S) policy

Scarf (1960) shows that the optimal policy of the dynamic inventory problem is of an (s, S) type, if functions of holding and penalty costs are convex. However, in practice, a common issue for supply chain organisations is that their clients frequently alter their timetables of ordering and the number of purchasing. A model that is able to develop a plan for the timing of orders in advance for stochastic demand is an effort of practical interest, if there is to be more cooperation and coordination in supply chains.

This leads to a static-dynamic uncertainty strategy introduced by Bookbinder and Tan (1988) and (R, S) and (s, Q) policies, and here we focus on (R_t, S_t) policy for non-stationary stochastic demand. Rossi et al. (2015) introduce a mixed integer linear programme (MILP) to approximate the expected total cost for a single-item single-location inventory system with non-stationary stochastic demand as follows.

$$
\begin{aligned}
\min \quad & \sum_{k=t}^{T}(\delta_k \cdot K + z \cdot Q_k + h \cdot \tilde{H}_k + b \cdot \tilde{B}_k), && && (3.4)\\
\text{s.t.} \quad & \delta_k = 0 \rightarrow \tilde{I}_k + \tilde{d}_k - \tilde{I}_{k-1} = 0, && k = t, \ldots, T, && (3.5)\\
& \delta_k = 1 \rightarrow \tilde{I}_k + \tilde{d}_k - \tilde{I}_{k-1} = Q_k, && k = t, \ldots, T, && (3.6)\\
& P_{jk} \geq \delta_j - \sum_{r=j+1}^{k} \delta_r, && k = t, \ldots, T,\ j = t, \ldots, k, && (3.7)\\
& \textstyle\sum_{j=t}^{k} P_{jk} = 1, && k = t, \ldots, T, && (3.8)\\
& P_{jk} = 1 \rightarrow \tilde{H}_k = \hat{\mathcal{L}}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), && k = t, \ldots, T,\ j = t, \ldots, k, && (3.9)\\
& P_{jk} = 1 \rightarrow \tilde{B}_k = \mathcal{L}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), && k = t, \ldots, T,\ j = t, \ldots, k, && (3.10)\\
& \tilde{H}_k, \tilde{B}_k \geq 0, \quad P_{jk}, \delta_k \in \{0, 1\}, && k = t, \ldots, T,\ j = t, \ldots, k. && (3.11)
\end{aligned}
$$

The objective function denotes the expected total cost over horizon (t, T). Constraints (3.5) and (3.6) apply the indicator constraints (Belotti et al., 2016) to demonstrate the inventory flow balance, where δ_k is a binary variable denoting order decision in period k.

This model introduces a binary variable P_{jk} to properly account for demand variance while computing the first-order loss function. Let P_{jk} $(j \leq k)$ take the value of 1 if the last order before period k (including period k itself) is placed at the beginning of period j. Note that the combination of constraints (3.7) and (3.8) ensures that demand variance is properly accounted for even when no order takes place within the horizon (j, k).

We define a replenishment cycle $R(i, j)$ as periods $i, i+1, \ldots, j$ covered by one replenishment order placed at period i, where $\delta_i = 1$ and $\delta_{i+1}, \ldots, \delta_j = 0$, as illustrated in Figure 3.1.

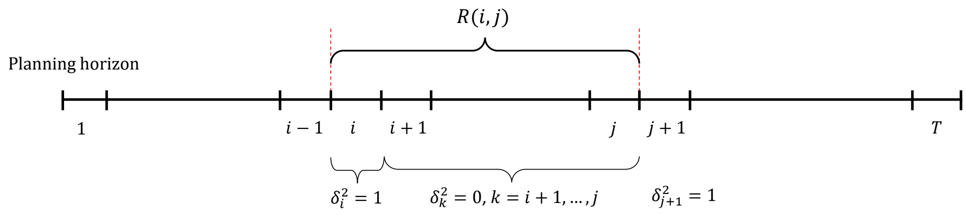

Figure 3.1: A replenishment cycle $R(i, j)$ over period i to j.

For any period in the replenishment cycle $R(j, k)$ ($j \leq k$), the expected closing inventory level is modelled by constraints (3.9) and (3.10) by means of the first order loss function $\mathcal{L}$ and its complementary $\hat{\mathcal{L}}$, where $\tilde{I}_k + \tilde{d}_{jk}$ is the opening inventory of replenishment cycle $R(j, k)$ and d_{jk} is the cumulative demand over the horizon (j, k). We discuss in detail the loss function and its piecewise linear approximation under non-stationary stochastic demand in Appendix 3.A.

Example 3.3.1 (Continued). Recall the 5-period instance previously considered. We apply 12 partitions to approximate the upper bound; according to Rossi et al. (2014) for Normal distribution (Table 2 Row 11), we approximate the optimal policy parameters as $\vec{\delta}^* = \{1, 1, 1, 0, 1\}$ and $\vec{S}^* = \{142, 177, 85, -, 43\}$. The simulated cost of this policy is 453 with 500,000 simulations. The expected outstanding and back-ordered inventory ($\tilde{H}_t$ and $\tilde{B}_t$) at the end of each period are summarised in Table 3.1, and inventory system with this policy is illustrated in Figure 3.2.

Table 3.1: Expected closing inventory levels solved by (R_t, S_t) policy with piecewise linear approximation.

t	1	2	3	4	5
d_t	100	125	25	40	30
(R_t, S_t) policy with piecewise linear approximation					
$\tilde{H}_t$	43.0	53.8	59.8	20.3	12.9
$\tilde{B}_t$	1.11	1.38	0.0441	0.522	0.332
$\tilde{I}_t$	41.9	52.4	59.8	19.8	12.6

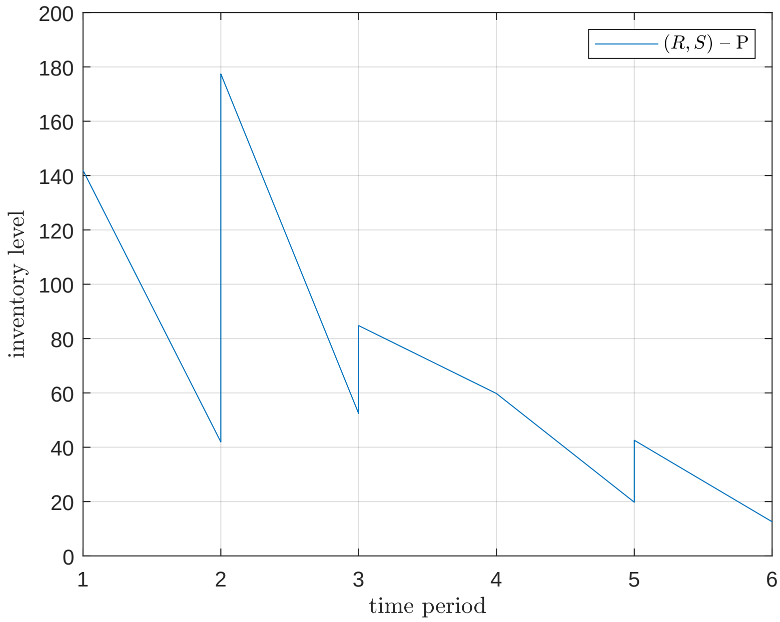

Figure 3.2: Expected inventory levels by (R_t, S_t) policy with piecewise linear approximation.

This piecewise linear approximation technique has been extended to (s_t, S_t) policy (Xiang et al., 2018) and (s, Q)-type policies (Ma et al., 2022) to determine near-optimal policy parameters. However, this technique cannot efficiently solve problems due to the computation of piecewise parameters. For example, for the reorder point s_t under an (s_t, Q_t) policy, one needs to compute a $(T-t+1)\times(T-t+1)\times W$ matrix of demand probability in segments, where W is the number of partitions; in total $(T-t+1)\times(T-t+1)\times W\times T$ for a T-period problem, and the process will be significantly decelerated if it allows large demand in sampling or integration. (Please see more detail for Poisson piecewise linear approximation in Appendix A.) Therefore, in the next sections, we present an approach based on the shortest path problem, which improves the computation efficiency and is applicable to the problem with any reasonable finite planning horizon.

3.5 A relaxation of (R_t, S_t) policy

This section presents a relaxation of the non-stationary (R, S) policy and reformulates the problem in a shortest-path framework. This relaxation is originally introduced by Tarim et al. (2011), where they also show that an infeasible solution yields a tight lower bound for the optimal cost and that the infeasible solution can be modified easily to obtain a feasible solution, which yields an upper bound. This method is refined by Rossi et al. (2011) through a filtering and augmenting procedure to reduce the number of states, while it can only be implemented under service-level context. In a further section, this relaxation is formulated as a shortest path problem for the penalty cost scheme and augmenting approach is introduced to solve the infeasibility brought about by the relaxation.

3.5.1 A relaxation of flow balance constraints

The model in Section 3.4 solves a set of periods i and j $(i \leq j)$ $\delta_i = 1$, $\delta_{j+1} = 1$, and for any k, $i < k \leq j$, $\delta_k = 0$. From the perspective of replenishment cycles, the lot-sizing problem under (R, S) policy can now be transformed to minimise the expected total cost of the sum of all replenishment cycles as follows.

$$\min \quad \sum_{k=t}^{T} [\delta_k \cdot K + z(S_k - \tilde{I}_{k-1}) + h \cdot \tilde{H}_k + b \cdot \tilde{B}_k], \tag{3.12}$$

$$\text{s.t.} \quad \delta_k = 0 \rightarrow \tilde{I}_k + \tilde{d}_k - \tilde{I}_{k-1} = 0, \qquad k = t, \ldots, T, \tag{3.13}$$

$$P_{jk} \geq \delta_j - \sum_{r=j+1}^{k} \delta_r, \qquad k = t, \ldots, T,\ j = t, \ldots, k, \tag{3.14}$$

$$\sum\nolimits_{j=t}^{k} P_{jk} = 1, \qquad k = t, \ldots, T, \tag{3.15}$$

$$P_{jk} = 1 \rightarrow \tilde{H}_k = \hat{\mathcal{L}}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), \qquad k = t, \ldots, T,\ j = t, \ldots, k, \tag{3.16}$$

$$P_{jk} = 1 \rightarrow \tilde{B}_k = \mathcal{L}(\tilde{I}_k + \tilde{d}_{jk}, d_{jk}), \qquad k = t, \ldots, T,\ j = t, \ldots, k, \tag{3.17}$$

$$\tilde{H}_k, \tilde{B}_k \geq 0, \quad P_{jk}, \delta_k \in \{0, 1\}, \qquad k = t, \ldots, T,\ j = t, \ldots, k. \tag{3.18}$$

Recall flow balance constraints (3.5) and (3.6), this model relaxes (3.6) and only requires a zero-order quantity when $\delta_t = 0$ by constraint (3.12); in this case, *it* *is possible to have a negative order taken place.* To be more specific, all feasible scenarios with order decisions and order quantities are as follows.

$$\delta_t = 0 \quad \rightarrow \quad \tilde{S}_t = \tilde{I}_{t-1} \quad \Rightarrow \quad Q_t = 0. \tag{3.19}$$

$$\delta_t = 1 \quad \rightarrow \quad \tilde{S}_t \geq \tilde{I}_{t-1} \quad \Rightarrow \quad Q_t \geq 0, \tag{3.20}$$

$$\delta_t = 1 \quad \rightarrow \quad \tilde{S}_t < \tilde{I}_{t-1} \quad \Rightarrow \quad Q_t < 0, \tag{3.21}$$

For the scenario in Eq.(3.21), we expect a "replenishment" order with negative order quantity at the beginning of period i, as illustrated in Figure 3.3.

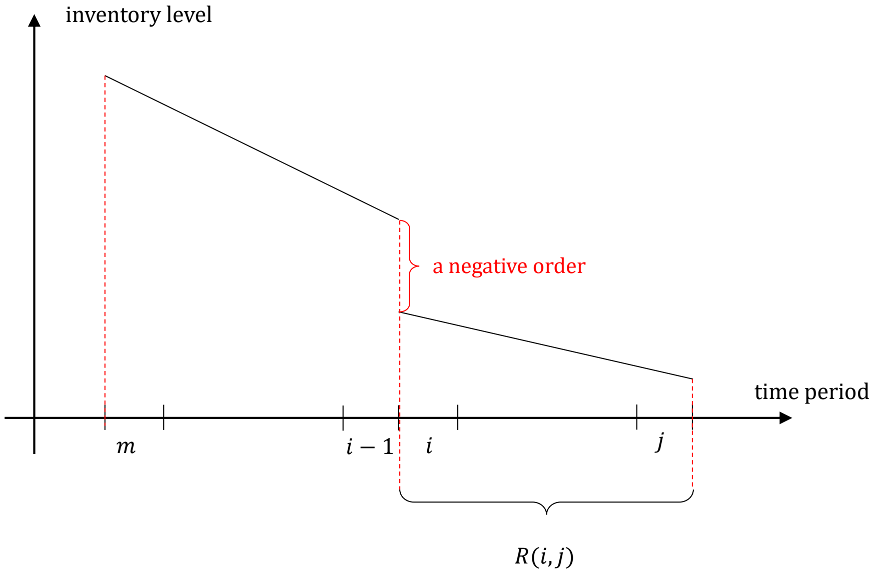

Figure 3.3: A negative order takes place in period j.

Example 3.3.1 (Continued). Recall the 5-period instance previously considered with 11 segments. Removing a flow balance constraint to allow negative orders, we solve a relaxed (R, S) policy with $\vec{\delta}^* = \{1, 1, 1, 1, 0\}$ and $\vec{S}^* = \{142, 177, 35, 91, -\}$. Without allowing any negative order, this policy resolves an expected total cost 465 by 500,000 simulations.The expected outstanding and back-ordered inventory ($\tilde{H}_t$ and $\tilde{B}_t$) at the end of each period extend Table 3.1 and are compared in Table 3.2, and an inventory system with this policy is illustrated in Figure 3.4. We notice a negative order with quantity -16 at the beginning of period 3.

Table 3.2: Expected closing inventory solved by (R_t, S_t) and relaxed (R_t, S_t) policies with piecewise linear approximation.

t	1	2	3	4	5
d_t	100	125	25	40	30
(R_t, S_t) policy with piecewise linear approximation					
$\tilde{H}_t$	43.0	53.8	59.8	20.3	12.9
$\tilde{B}_t$	1.11	1.38	0.0441	0.522	0.332
$\tilde{I}_t$	41.9	52.4	59.8	19.8	12.6
Relaxed (R_t, S_t) policy with piecewise linear approximation					
$\tilde{H}_t$	43.0	53.8	10.8	51.0	21.5
$\tilde{B}_t$	1.11	1.38	0.276	0.0706	0.553
$\tilde{I}_t$	41.9	52.4	10.5	51.0	21.0

With the sum of the expected cost of various replenishment cycles, the problem is relaxed to be a multi-stage newsvendor problem with non-stationary stochastic demand (Petruzzi and Dada, 1999). However, the piecewise linear approximation works less efficiently in large-scale instances with non-normally distributed demand. The next subsection presents a shortest path formulation to this problem and then discusses how to deal with the negative order in the optimal solution.

Besides, Özen et al. (2012) propose a heuristic algorithm based on the relaxation of non-negative order quantities, where the authors design experiments with high coefficients of variation with no negative demand impediment; focusing on the feasibility of the relaxed model, the result shows 15 instances out of 270 lead to infeasibility if the non-negative order quantity is relaxed. In this thesis, the primary focus of the approximation lies in proposing an augmenting procedure that combines the penalty scheme and the shortest-path formulation, which will be elaborated on in the next subsection.

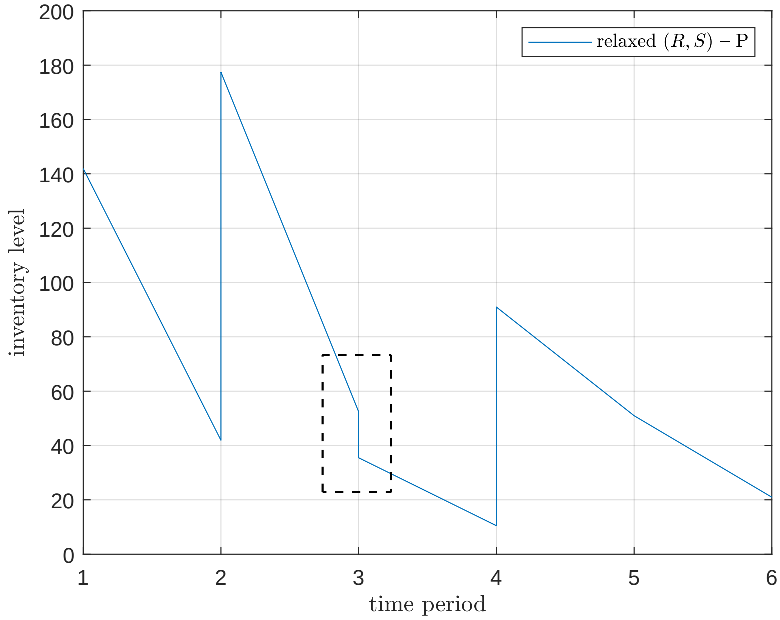

Figure 3.4: Expected inventory levels by **relaxed** (R_t, S_t) policy with piecewise linear approximation. A negative order quantity is observed at period 3.

3.5.2 A shortest path problem

Tarim (1996) show that the shortest path problem provides a valid lower bound for the non-relaxed formulation as a solution of the relaxed problem presented in Section 3.5.1. The shortest path problem can be efficiently solved by Dijkstra's algorithm (Cormen et al., 2001, pp. 595-–601) in $\mathcal{O}(n^2)$ time, where n is the number of nodes in the shortest-path graph.

Consider an acyclic network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V} = \{1, 2, \ldots, T+1\}$ denoting periods, where T is the length of the planning horizon and the dummy $T+1$ is introduced as the final node (boundary condition) to indicate the end of the planning horizon. Arc $(i, j) \in \mathcal{E}$ connecting all pairs of nodes with $i < j$. Each arc $(i, j) \in \mathcal{E}$ has an associated cost $c_{i,j}$ to denote the expected cost of replenishment cycle $R(i, j-1)$, where

$$c_{i,j} = c(y - x) + \sum_{k=i}^{j-1} L_k(y), \tag{3.22}$$

and x is the opening inventory level of cycle $R(i, j-1)$, y is the inventory level after replenishment, and

$$L_k(y) \quad = \quad h \times \underbrace{\mathbb{E}[\max(y - d_{i,k}, 0)]}_{\text{complementary first order loss function}} \quad + \quad b \times \underbrace{\mathbb{E}[\max(d_{i,k} - y, 0)]}_{\text{first order loss function}}. \tag{3.23}$$

Therefore, entries $c_{i,j}$ formulate a connection matrix with the expected cost of replenishment cycles $R(i, j-1)$, and the problem can be modelled as a one-way temporal feasibility problem (Wagner and Whitin, 1958), illustrated in Figure 3.5. We refer this problem as "ESP $\mathcal{G} = (\mathcal{V}, \mathcal{E})$". Let $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the

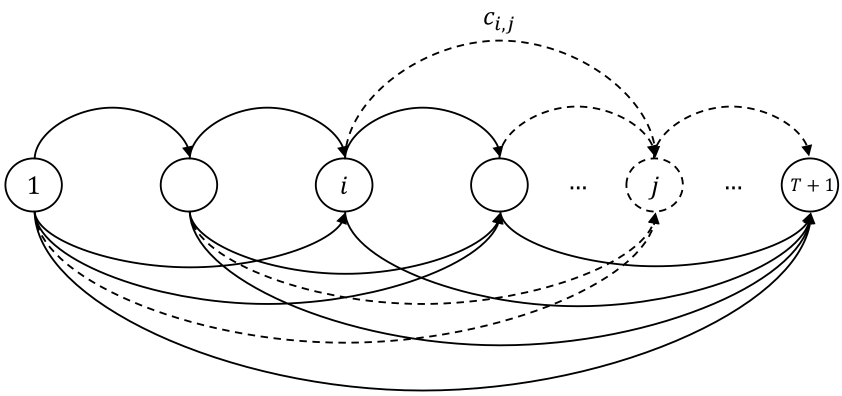

Figure 3.5: Shortest path problem.

optimal solution of this shortest path problem with $\frac{T(T+1)}{2}$ possible combination of replenishment cycles and associated costs. $\mathcal{G}^*$ indicates a set of consecutive disjoint replenishment cycles that satisfies the demand in the planning horizon with a minimum expected total cost. For any $t \in \mathcal{V}^*$ we have $\delta_t = 1$, otherwise $\delta_t = 0$; and the selected replenishment cycle $R(i, j-1)$ is indicated by arcs $(i, j) \in \mathcal{E}^*$.

In the end, we force *no* negative order should be assigned to any replenishment cycle; this should be checked by comparing the order-up-to level S^*_t and $\tilde{I}^*_{t-1}$ for every $t \in \mathcal{V}^*$, where replenishment cycle covers periods $(m, t-1)$, and

$$\tilde{I}^*_{t-1} = \mathbb{E}[\max(S^*_m - d_{m,t-1}, 0) + \max(d_{m,t-1} - S^*_m, 0)]. \tag{3.24}$$

Example 3.3.1 (Continued). We recall the 5-period instance with Normally distributed demand $\tilde{d}_t = \{100, 125, 25, 40, 30\}$ and the same 11 segments as did in Section 3.4. We solve the connection matrix with the expected costs of replenishment cycles in matrix C, where

$$C = \begin{bmatrix} 112 & 343 & 470 & 643 & 828 \\ & 127 & 215 & 346 & 490 \\ & & 65.5 & 140 & 228 \\ & & & 74.7 & 133 \\ & & & & 68.5 \end{bmatrix} \tag{3.25}$$

Figure 3.6 demonstrates the connection matrix with the expected costs of replenishment cycles and Figure 3.7 the optimal path. The optimal path $1 \to 2 \to 3 \to 4 \to 6$ generates an (R, S) policy with $\vec{S}^* = \{149, 187, 37, 89, -\}$ and $\vec{\delta}^* = \{1, 1, 1, 1, 0\}$. This policy resolves the expected total cost as 468 by 500,000 simulations.

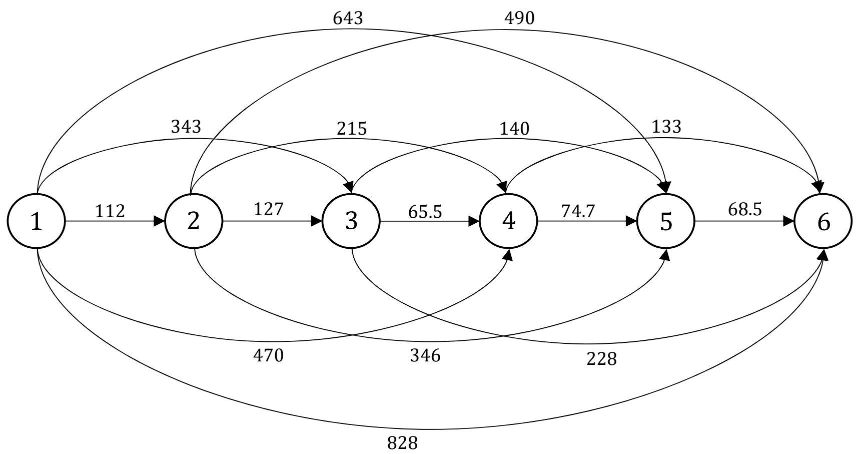

Figure 3.6: Solving 5-period instance by the shortest path with expected costs of replenishment cycles.

Figure 3.8 illustrates the (R_t, S_t) policies solved by piecewise linear approximation and shortest path, where a negative order can be noticed at the beginning of period 3 by both methods.

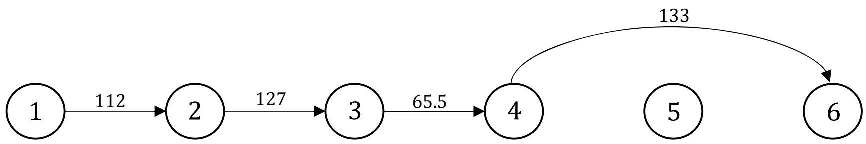

Figure 3.7: The optimal path for 5-period instance with expected costs of replenishment cycles.

3.5.3 Determining reorder points from the graph

When the order-up-to levels $\vec{S}^*$ are obtained from the graph, one can determine the reorder points $\vec{s}^*$ for $\vec{S}^*$. Let $G_t(y)$ denote the expected total cost over horizon (t, T) starting with inventory level y and no order is placed in the first-leading period t, where

$$G_t(y) \triangleq L_t(y) + \mathbb{E}[C_{t+1}(y - d_t)]. \tag{3.26}$$

As illustrated in Figure 3.9, $G_t(I_t)$ can be solved as the shortest path from node t to node $T+1$, where the weights on arcs that outbound from node t, $(t, t+1)$, $(t+1, t+2)$, $\cdots$, $(T, T+1)$, are replaced by

$$\bar{c}_{t,k} = L_k(I_t) \tag{3.27}$$

for $k = i, \cdots, T$. Then the reorder point for period t given S_t^* is obtained at inventory level I_t as

$$s_t^* = \inf\{I_t : G_t(I_t) \leq G_t(S_t) + c(S_t^* - I_t)\}. \tag{3.28}$$

These pairs of s_t^* and S_t^* approximate an (s_t, S_t) policy.

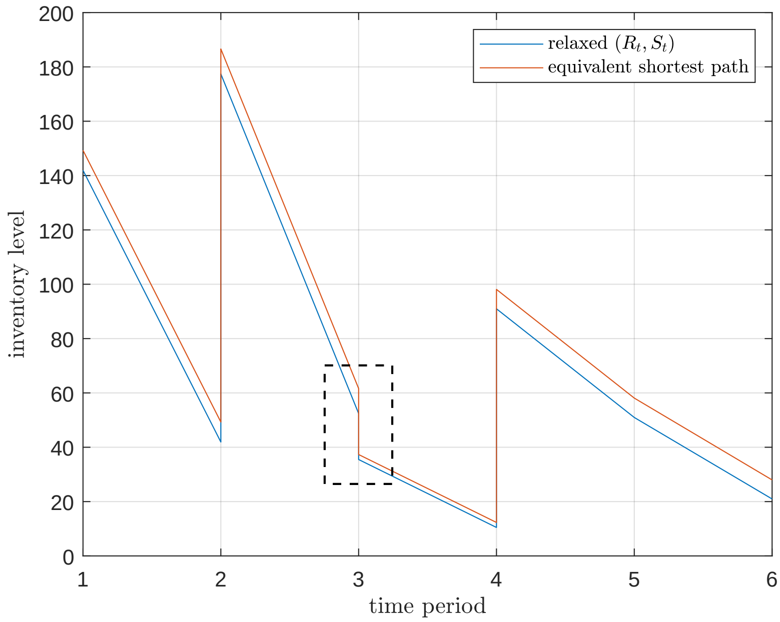

Figure 3.8: Expected inventory levels by relaxed (R_t, S_t) policy with piecewise linear approximation and shortest path formulation.

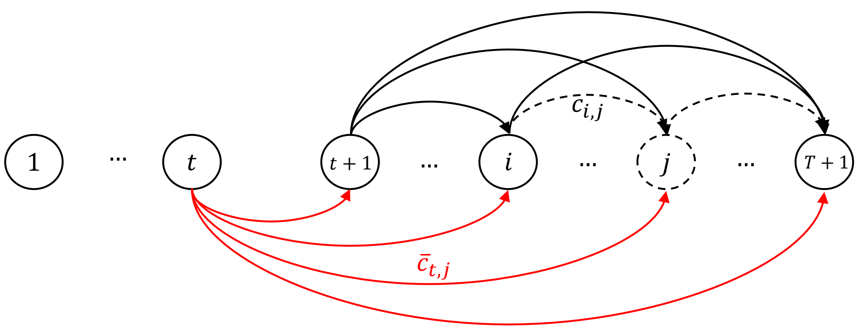

Figure 3.9: Solving $G_t(I_t)$ by updating weights $\bar{c}_{t,j}$ by $c_{t,j}$.

Example 3.3.1 (Continued). Recall the 5-period instance previously considered. We approximate the policy by $\vec{S}^* = \{149, 187, 37, 89, 45\}$ and $\vec{s}^* = \{121, 153, 25, 45, 31\}$ and a simulated cost 406 with an optimality gap 0.495%. Figure 3.10 presents the determination of reorder point at period 1.

Table 3.3: Expected closing inventory solved by (R_t, S_t) and relaxed (R_t, S_t) policies with piecewise linear approximation and the shortest path formulation.

t	1	2	3	4	5
d_t	100	125	25	40	30
(R_t, S_t) policy with piecewise linear approximation					
$\tilde{H}_t$	43.0	53.8	59.8	20.3	12.9
$\tilde{B}_t$	1.11	1.38	0.0441	0.522	0.332
$\tilde{I}_t$	41.9	52.4	59.8	19.8	12.6
Relaxed (R_t, S_t) policy with piecewise linear approximation					
$\tilde{H}_t$	43.0	53.8	10.8	51.0	21.5
$\tilde{B}_t$	1.11	1.38	0.276	0.0706	0.553
$\tilde{I}_t$	41.9	52.4	10.5	51.0	21.0
Shortest path formulation					
$\tilde{H}_t$	49.9	62.4	12.5	58.1	28.7
$\tilde{B}_t$	0.627	0.783	0.157	0.00	0.67
$\tilde{I}_t$	49.28	61.6	12.3	58.1	28.00

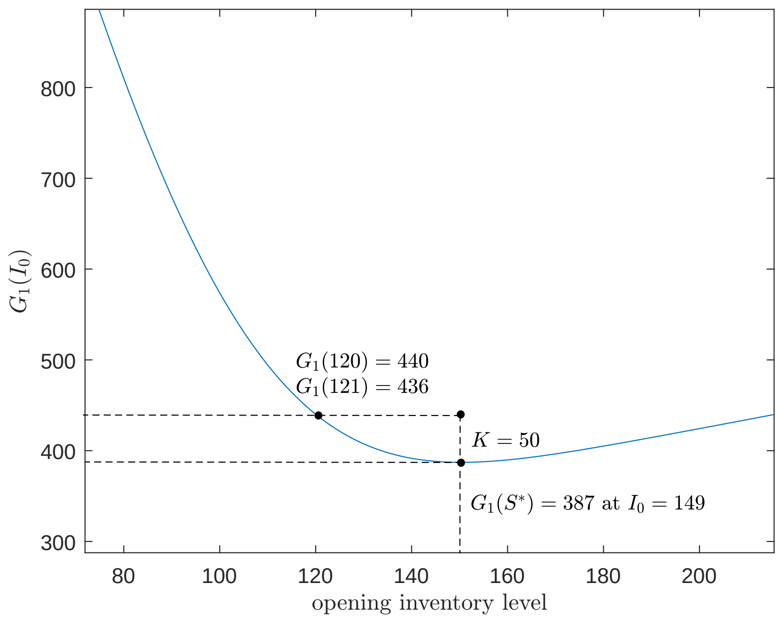

Figure 3.10: Determining reorder point $s_1 = 121$.

3.6 An augmenting procedure to eliminate negative order quantity for (R_t, S_t) policy

This section introduces an augmenting procedure to amend the infeasibility in the shortest path formulation caused by negative orders.

When the shortest path introduced above is formulated and solved, it is simple to determine whether the optimal solution meets every relaxed constraint. If no expected negative order quantity is planned in the optimal replenishment plan, then the solution determines a feasible and optimal replenishment plan for the original problem.

If, on the other hand, the solution is not feasible as it schedules negative replenishment quantities, we adopt a fast convex optimisation to raise the expected opening inventory level of the second replenishment cycle to the expected closing inventory of the first replenishment cycle (Figure 3.11 and 3.12). We augment the graph with additional nodes and arcs such that the shortest path on the augmented graph is guaranteed to offer a feasible and optimal solution for the original problem. In what follows, we will demonstrate how to augment the graph and efficiently calculate the optimal solution to the original problem.

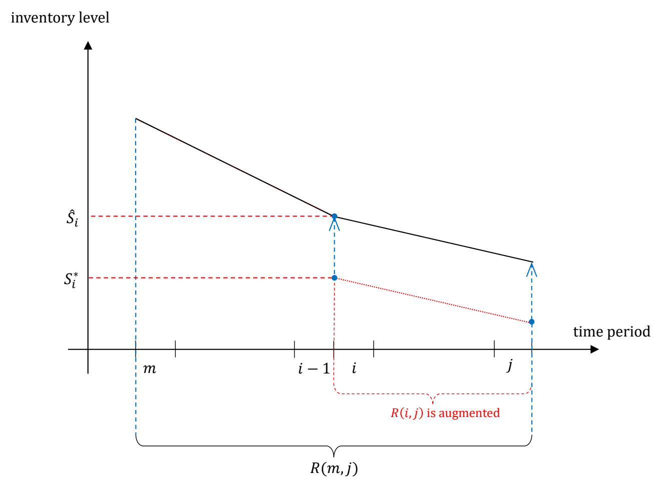

Figure 3.11: Increasing the opening inventory level of period i to eliminate the negative order: inventory-level view.

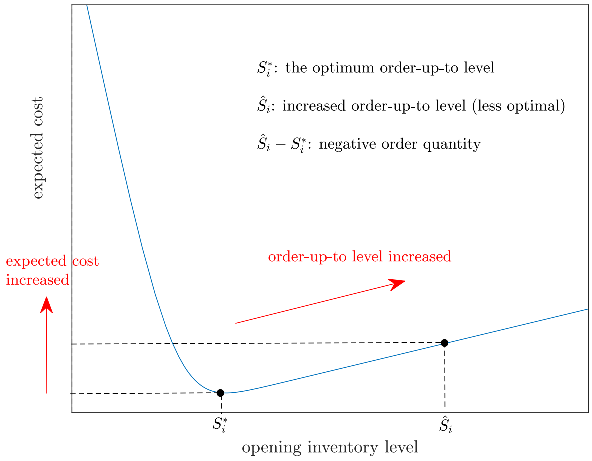

Figure 3.12: Increasing the opening inventory level of period i to eliminate the negative order: expected-cost view.

Since the shortest path can be easily selected according to the expected cost of replenishment cycles which does not affect augmenting the graph, in the following, instead of presenting the expected total cost in the graph, we associate the expected opening and closing inventory of a replenishment cycle by a pair. For any replenishment cycle $R(i,j)$ (illustrated in graph as node i to $j+1$), $(\tilde{S}(i,j), \tilde{I}_c(i,j))$ denotes the expected opening inventory level, $\tilde{S}(i,j)$ and the expected closing inventory at the end of period j, $\tilde{I}_c(i,j)$. The superscript $*$ denotes the optimal value by shortest path formulation.

Let $ESP(\mathcal{V},\mathcal{E})$ be a shortest path problem, where $\mathcal{V}$ denotes the set of nodes and $\mathcal{E}$ the set of arcs in the graph. The pseudo-code for the proposed augmenting procedure is presented in Algorithm 2. This procedure eventually generates an augmented graph $ASP(\mathcal{V}',\mathcal{E}')$, where $\mathcal{V}'$ and $\mathcal{E}'$ denote the set of nodes and arcs in the augmented graph, respectively.

Algorithm 1 initially duplicates $ESP(\mathcal{V},\mathcal{E})$ for $ASP(\mathcal{V}',\mathcal{E}')$. Then it considers each node in $\mathcal{V}'$ in the ascending order till node T. Node $T+1$ is a dummy node to denote the completion of the horizon with no outbound arc, so we do not consider it. This procedure is repeated for each node, so we will only explain it on a single node.

For every node $i \in \mathcal{V}'$, we consider each inbound arc and implement the following steps. Given an inbound arc (p, i) with the expected closing inventory $\tilde{I}_{i-1}$, for each outbound arc (i, j) with $(\tilde{S}^*(i, j-1), \tilde{I}_c(i, j-1))$, we check that $\tilde{I}_{i-1} \leq \tilde{S}^*(i, j-1)$. If it holds for every outbound arc, then we preserve the inbound arc (p, i) at node i (Figure 3.13); otherwise, if $\tilde{I}_{i-1} > \tilde{S}^*(i, j-1)$, a negative order is placed, and we perform the following transformation to resolve the infeasibility (Figure 3.14).

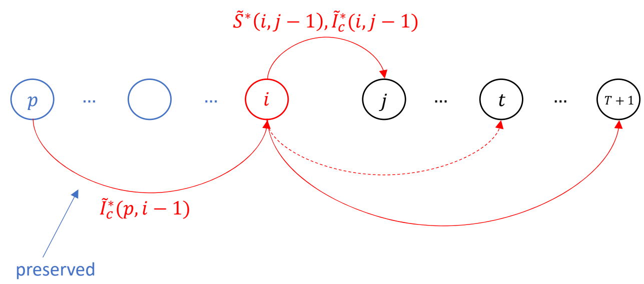

Figure 3.13: Feasible scenarios.

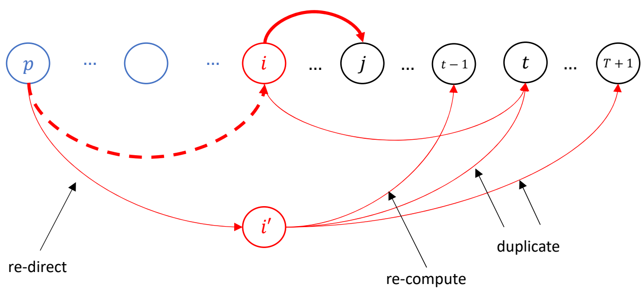

Figure 3.14: Infeasible scenarios.

- *re-direct*. We introduce a new node i' in the augmented graph and a new arc (p, i') to connect node p with node i' with associated closing inventory $\tilde{I}_{i-1}$. Thus, arc (p, i) is re-direct and then deleted.

- *Duplicate*. Let $t > i$ be the *minimum* index such that $\tilde{I}_c^*(p, i-1) \leq \tilde{S}^*(i, t-1) \leq \tilde{S}^*(i, t) \leq \cdots \leq \tilde{S}^*(i, T)$. We duplicate arcs $(i', t), \cdots, (i', T+1)$ with inventory pairs $\tilde{I}_c^*(i, k)$ and $\tilde{S}^*(i, k)$ for $k = t-1, \ldots, T+1$.

- *Re-compute.* Let $t-1>i$ be the *maximum* index such that $\tilde{I}_c^*(p,i)>\cdots>\tilde{S}(i,t-2)$. We introduce arc $(i',t-1)$ with the closing inventory level $\tilde{I}_c^*(p,i'-1)$ and re-compute the expected closing inventory as

  $$\hat{I}_c(i,t-2)=\mathbb{E}\{\max[\hat{S}(i,t-2)-d_{i,t-2},0]+\max[d_{i,t-2}-\hat{S}(i,t-2),0]\}. \quad (3.29)$$

  We can see that the inventory carries on from the previous replenishment cycle is adequate to cover periods up to $t-1$, then we do not introduce suboptimal arcs $(i',t-2)$, $(i',t-3)$, $\cdots$, $(i',i+1)$.

When the feasibility is checked for all nodes (excluding node $T+1$), we delete the nodes that associate with no outbound or inbound arcs; then the augmented graph $ASP(\mathcal{V}',\mathcal{E}')$ is obtained. We further solve the associated shortest path by constructing the expected costs, and this path is the optimal solution to the original problem with every possible negative order quantity scenario that has been considered and resolved.

Algorithm 2 Arc augment procedure to compute the optimal replenishment for the non-stationary lot-sizing problem under (R_t, S_t) policy.

Input: the shortest path problem $ESP(\mathcal{V}, \mathcal{E})$.
Output: an augmented graph $ASP(\mathcal{V}', \mathcal{E}')$, a set of arcs combination $\mathcal{M}$.
$ASP(\mathcal{V}', \mathcal{E}') \leftarrow ESP(\mathcal{V}, \mathcal{E})$ and $\mathcal{M} \leftarrow \varnothing$.
for node $i = 1, \ldots, T \in \mathcal{V}'$ **do**
  for arc $(p, i) \in \mathcal{E}'$ **do**
    for arc $(i, j) \in \mathcal{E}'$ **do**
      if $\tilde{I}_{i-1} > \tilde{S}(i, j)$ **then**
        create node i' by $\mathcal{V}' \leftarrow \mathcal{E}' \bigcup \{i'\}$;
        create arc (p, i') by $\mathcal{E}' \leftarrow \mathcal{E}' \bigcup \{(p, i')\}$ with expected closing inventory level $\tilde{I}_{i-1}$;
        update the set $\mathcal{M} \leftarrow \mathcal{M} \bigcup \{\{(p, i), (i, j)\}\}$;
        create arc (i', t) with expected closing inventory $\tilde{I}_c(i, t-1)$ by $\mathcal{E}' \leftarrow \mathcal{E}' \bigcup \{(i', t)\}$, where $t > i$ is the minimum index such that $\tilde{I}_{i-1} \leq \tilde{S}(i, t-1) \leq \tilde{S}(i, t) \leq \cdots \leq \tilde{S}(i, T)$;
        for arc $(i, k) \in \mathcal{E}'$, $k = t+1, \ldots, T+1$ **do**
          create arc (i', k) with expected closing inventory $\tilde{I}_c(i, k-1)$ by $\mathcal{E}' \leftarrow \mathcal{E}' \bigcup \{(i, k-1)\}$;
        end forintroduce arc $(i', t-1)$ with expected closing inventory $\tilde{I}_{i-1} - \sum_{k=1}^{t-1} \tilde{d}_k$ by $\mathcal{E}' \leftarrow \mathcal{E}' \bigcup \{(i', t-1)\}$, where $t - 1 > i$ is the maximum index such that $\tilde{I}_{i-1} > \cdots > \tilde{S}(i, t-2)$.
      end if
    end for
  end for
end for

We revisit the running instance to demonstrate how this procedure implements.

Example 3.3.1 (Continued). Recall the 5-period instance previously discussed. According to the augmenting approach presented in Algorithm 2, we obtain the augmented graph as Figure 3.15 with the optimal path $1 \rightarrow 2 \rightarrow 3'' \rightarrow 5 \rightarrow 6$. This path incurs the policy of $\vec{S}^* = \{149, 187, 83, -, 45\}$ and $\vec{\delta}^* = \{1, 1, 1, 0, 1\}$ and resolves an expected total cost of 459 by 500,000 simulations. Details of implementing the augmented procedure can be found in Appendix 3.B.

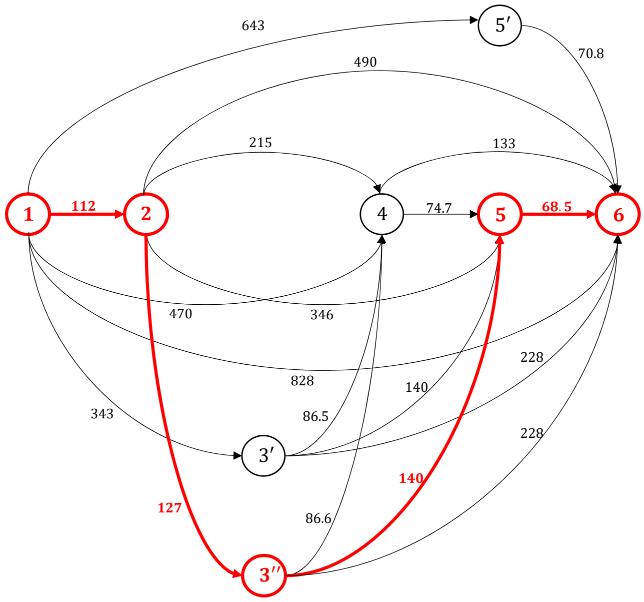

Figure 3.15: Optimal path of Example 1.

Overall, Table 3.4 summarises the obtained optimal policies with corresponding expected total cost (simulated). We observe that the expected total cost of the augmented shortest path well approximates the (R_t, S_t) policy with a small difference from (R_t, S_t) with piecewise linear approximation.

Table 3.4: Optimal policies and expected total costs (simulated) by methods discussed.

Methods	order-up-to levels	order decisions	Expected total cost
optimal (s_t, S_t) policy	{149, 186, 37, 82, 45}	{119, 154, 24, 45, 30}	404
(s_t, S_t) policy by shortest path	{149, 187, 37, 89, 45}	{121, 153, 25, 45, 31}	406
(R_t, S_t) policy with piecewise	{142, 177, 85, -, 43}	{1, 1, 1, 0, 1}	453
relaxed (R_t, S_t) with piecewise	{142, 177, 35, 91, -}	{1, 1, 1, 1, 0}	465
shortest path	{149, 187, 37, 89, -}	{1, 1, 1, 1, 0}	468
augmented shortest path	{149, 187, 83, -, 45}	{1, 1, 1, 0, 1}	459

3.7 Computational study

This section presents a computational analysis to evaluate proposed algorithms for reorder points discussed in Section 3.5.3 and augmenting algorithms discussed in Section 3.6 for non-stationary stochastic demand under a penalty cost scheme. We analyse the performance regarding to the optimality gap and the computation efficiency of the (s_t, S_t) policy solved by shortest paths, near-optimal (s_t, Q_t) policy defined by s_t and $S_t = Q_t - s_t$ and near-optimal (R_t, S_t) policy solved by the augmenting procedures.

This analysis applies the same 25-period test set as the one in (Ma et al., 2022), which considers ten expected demand patterns: two life cycle patterns, one raises larger on expectations from growth stage to decline (LCY1) and the other smaller (LCY2); two sinusoidal patterns, one with stronger (SIN1) and the other with weaker (SIN2) oscillations; a stationary demand pattern (STAT); a random demand pattern (RAND); and lastly, 4 empirical patterns derived according to (Strijbosch et al., 2011). The demand d_t in each period t is assumed to be a normally distributed random variable with known mean $\tilde{d}_t$ and standard deviation $\sigma_t = \rho{\cdot}\tilde{d}_t$, where ρ denotes the coefficient of variation of the demand, which remains fixed over time as prescribed in (Bollapragada and Morton, 1999); demands are assumed to be independent among each other. We allow the standard deviation parameter ρ to vary over $\rho \in \{0.1, 0.2, 0.3\}$. Demand patterns are illustrated in Figure 3.16. Other problem parameters are $K \in \{500, 1000, 1500\}$; $b \in \{5, 10, 20\}$; $z \in \{0, 1\}$; and $h = 1$.

For each instance, we solve the near-optimal order-up-to levels (S_t) and corresponding reorder points (s_t) for (s_t, S_t) policy through shortest path formulation and apply the s_t and $Q_t = S_t - s_t$ as policy parameters to the (s_t, S_t) policy (see Section 4.1 in Ma et al. (2022)). We also implement the augmenting procedures on the shortest path problem to determine the near-optimal (R_t, S_t) policy parameters. Apart from the optimal (s_t, S_t) policy, the expected total cost of each experiment is obtained by averaging 500,000 simulation runs substituting approximated policy parameters.

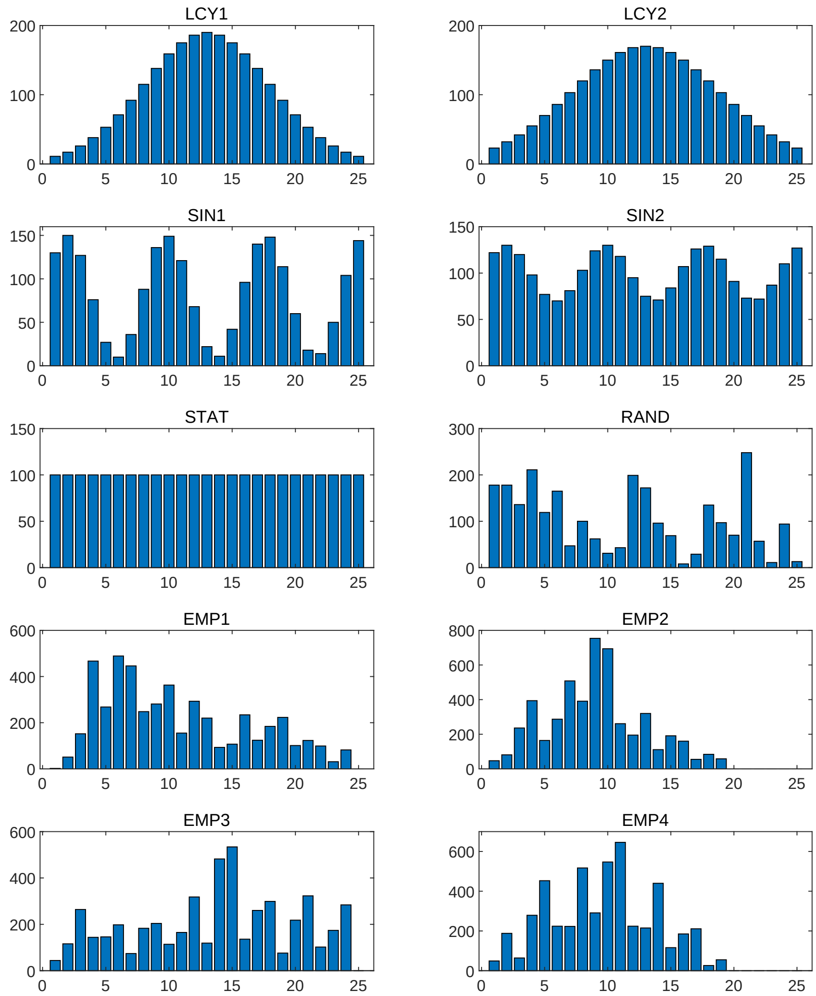

Figure 3.16: Demand patterns of 25-period instances.

We use the optimal (s_t, S_t) policy as the benchmark to compare the percentage optimality gap of expected total costs (ETCs) among approaches, computed as $100 \times (\text{ETC}_2 - \text{ETC}_1)/\text{ETC}_1$, where ETC_1 is always the expected total cost of the optimal (s_t, S_t) policy, and ETC_2 is the expected total cost of the other approach benchmarked. We also compare the (R_t, S_t) policy by augmenting procedures against ones solved by piecewise linear approximation with 10 partitions. For computation efficiency, we record the computation time in seconds for every experiment and compare the average time according to pivoting parameters. For simplicity reasons, we denote the (s_t, S_t) policy by the shortest path approach as "(s_t, S_t)-ESP" and the (R_t, S_t) policy by the augmenting procedures as "(R_t, S_t)-ASP", which is comparing with "(R_t, S_t)-Piecewise".

Since only simple integral operations are required, approaches based on the shortest path are computed in Matlab (R2019a) performed by an Intel(R) Core(TM) i5-10500 3.1G + 1.9G 32G RAM. Other computations are performed in JAVA 1.8.0_201 by 4.0 (1.90+2.11) gigahertz Intel(R) Core(TM) i7-8650U CPU with 16.0 gigabytes of RAM. Note that the computation time used for efficiency comparison does not include simulation time for each approach.

Table 3.5 reports the average percentage optimality gap obtained by different approaches and policies discussed above. We can observe that the average optimality gap of (s_t, S_t) policy solved by the shortest path problem, 0.616%, is sufficiently small compared with the optimal policy by the stochastic dynamic programming. The (R_t, S_t)-ASP produces an average optimality gap of 3.34%. The variation of (R_t, S_t)-ASP gaps pivoting demand patterns is hard to generally remark, but we observe the patterns with little variation in demand means have close optimality gaps compared to the ones obtained from (R_t, S_t)-Piecewise (LCY1, LCY2 and STA). When pivoting system parameters, the optimality gaps present a rising trend when the standard deviation parameter and unit cost increase independently. The approximated (s_t, Q_t) policy with S_t and $Q_t = S_t - s_t$ exhibits a relatively large optimality gap with an average 6.69%, while this application provides an approximation for the (s_t, Q_t) policy and is easily solved for generic distribution.

Table 3.6 reports the computation time of our methods on (s_t, S_t) and (R_t, S_t) policies for different pivoting parameters. Note that for each method, we apply numerical integration to determine S_t on the shortest path. Please see more details in Table 3.7 and Appendix 3.C. The time for (s_t, S_t)-ESP comprises (1) solving the shortest path and (2) line searching for the reorder points, and the time for (R_t, S_t)-ASP comprises (1) solving the optimal order-up-to level $\tilde{S}^*(i, j-1)$ and the expected closing inventory levels $\tilde{I}_c^*(i, j-1)$ for any node in the graph $(1 \leq i < j \leq T+1)$ and (2) implementing augmented procedure for infeasible scenarios in each period. Overall, the computation time of the shortest path for (s_t, S_t) parameters and augmenting procedure for (R_t, S_t) parameters are stable for different problem settings, on average 73.2 and 2.41 seconds. We observe the cost parameters have no significant effect on (s_t, S_t)-ESP but lead to increases in (R_t, S_t)-ASP when the standard deviation parameters and unit cost increase.

3.8 Conclusion

This paper investigated the single-item single-location lot-sizing problem under the penalty cost scheme. We developed a shortest-path formulation that approximated the optimal order-up-to levels. For each period t, reorder point is determined accordingly by updating the expected cost of arcs outgoing from node t; and these pairs of parameters approximate the (s_t, S_t) policy with the penalty cost of back-orders. We also tackled the infeasibility of the shortest-path formulation due to negative replenishment orders. Based on the graph, we proposed an augmenting procedure to modify the graph for infeasible scenarios by introducing new nodes with re-directing, re-computing and duplicating the costs of new arcs on the original graph. The approximated (R_t, S_t) policy parameters are then efficiently retrieved directly from the shortest path of the augmented graph.

For (s_t, S_t) and (R_t, S_t) policies approximated by our approaches, we conducted an extensive computation experiment on a test set of problems with 25-period planning horizons. This test set includes 10 demand patterns and various parameter settings. The results showed that the average optimality gap obtained from the proposed method is 0.616% for (s_t, S_t) policy and 3.34% for (R_t, S_t). We approximated the order quantities for (s_t, Q_t) policy by $Q_t = S_t - s_t$ and obtained an average optimality gap of 6.69%. More importantly, our investigation demonstrates the efficiency of the shortest path formulations with the augmenting procedure in solving the non-stationary stochastic lot-sizing problem. The numerical study showed that our approaches can approximate an (s_t, S_t) policy in 73.2 seconds and an (R_t, S_t) policy in 2.41 seconds on average.

Further research from this paper is two-fold.

- First, improvement of the augmenting procedures. We observed a difference between (R_t, S_t)-Piecewise and the proposed augmenting procedure; a reason behind this is that optimality is compromised when we deal with negative orders. In the current procedure, for a period incurring a negative order, its opening inventory level is directly raised to the expected closing inventory of the last period, which means that the current period is merged with the previous replenishment cycle. One possible solution is to apply a repetitive step of updating the opening inventory level and calculating the expected total cost of the updated complete policy of the horizon until no better solution is found.

- Second, extension of the (s_t, Q_t) policy. Determining order quantity Q_t for an (s_t, Q_t) policy, using either an exact optimum from the stochastic dynamic programme or a near-optimum from piecewise linear approximation, is a massive computation due to its generality to all feasible opening inventory levels of each period. Unlike (R_t, S_t) policy, an order quantity can be obtained through the order-up-to level and the mean of demand, and it solely responds to the replenishment cycle, which can also be easily retrieved from the shortest path for the (R_t, S_t) since the order-up-to levels can be solved from the graph. However, the determination of (s_t, Q_t) is more complicated since the order quantity is not directly included in the graph as a decision variable. The determination of reorder points associated with order quantities may also rely on $G_t(I_t)$ since $s_t = \inf\{I_t : G_t(I_t) - G_t(I_t + Q_t^*) < c(Q_t^*)\}$.
- The solution's feasibility can be guaranteed based on the original problem proposed in Tarim et al. (2011). In Chapter 3, this model is adapted to incorporate back-ordering for the penalty cost scheme and further relaxed to allow negative orders. These modifications do not reduce the feasible solutions of the original model; rather, they expand the solution space. After obtaining a solution for the relaxed model, the augmentation procedure begins to check for and address any infeasibilities. This process involves introducing new nodes and arcs based on the current solution without requiring the resolution of any additional models. Future research will explore evaluating the quality of the approximation, providing further insight into the effectiveness of the proposed approach.

Table 3.5: Average percent optimality gap over our 25-period test set under different demand patterns and pivoting parameters.

Problem Settings	(s_t, S_t)-ESP	(R_t, S_t)-ASP	(R_t, S_t)-Piecewise	(s_t, Q_t)
demand pattern				
LCY1	0.296	2.50	2.50	5.60
LCY2	0.279	2.20	2.20	5.75
SIN1	0.896	3.45	2.87	6.87
SIN2	0.621	2.05	2.03	6.63
STA	0.142	1.50	1.50	4.26
RAND	1.418	3.72	2.99	8.14
EMP1	0.435	3.92	3.19	6.82
EMP2	1.064	4.88	4.22	8.11
EMP3	0.597	3.56	2.79	7.60
EMP4	0.415	5.59	4.71	7.14
std paramater				
0.1	0.659	2.68	2.68	7.69
0.2	0.506	2.88	2.50	6.36
0.3	0.647	4.46	3.53	6.03
fixed ordering cost				
500	0.528	3.51	2.37	7.67
1000	0.652	3.13	2.97	6.20
1500	0.668	3.38	3.37	6.21
penalty cost				
5	0.283	4.34	3.36	6.08
10	0.448	2.97	2.61	6.64
20	0.552	2.71	2.69	7.36
unit cost				
0	0.699	2.91	2.47	7.08
1	0.534	3.77	3.33	6.30
Average	0.616	3.34	2.90	6.69

Table 3.6: Average computation time over our 25-period test set under different demand patterns and pivoting parameters.

Problem Settings	(s_t, S_t)-ESP	(R_t, S_t)-ASP
demand pattern		
LCY1	66.9	2.02
LCY2	69.0	2.08
SIN1	70.6	2.41
SIN2	71.8	2.14
STA	79.5	2.14
RAND	83.0	2.67
EMP1	78.0	2.62
EMP2	79.5	2.53
EMP3	78.7	2.69
EMP4	53.7	2.78
std parameter		
0.1	72.3	2.40
0.2	71.1	2.40
0.3	75.3	2.43
penalty cost		
5	72.3	2.43
10	73.1	2.41
20	75.3	2.38
fixed ordering cost		
500	73.6	2.41
1000	73.1	2.41
1500	73.7	2.41
unit cost		
0	73.5	2.42
1	72.8	2.39
Average	73.2	2.41

Appendix 3.A Piecewise linear approximation with non-stationary Poisson demand

Consider a random variable ω and a scalar variable x, the first order loss function is defined as $\mathcal{L}(x,\omega) = \mathbb{E}[\max(\omega - x, 0)]$ and its complement as $\hat{\mathcal{L}}(x,\omega) = \mathbb{E}[\max(x - \omega, 0)]$. Rossi et al. (2014) present the approach with bounding techniques to generate piecewise linear lower and upper bounds and discussed the implementation on the standard Normal distribution. To minimise the expected inventory and back-order levels at the end of each period with a lower bounding piecewise linear approximation, $\tilde{H}_t$ is constrained by

$$\tilde{H}_t \geq (\tilde{I}_t + \sum_{j=1}^{t} \tilde{d}_{jt} P_{jt}) \sum_{k=1}^{i} p_k + \sum_{j=1}^{t} (\sum_{k=1}^{i} p_k \mathbb{E}[d_{jt} | \Omega_{jt}]) P_{jt}, \qquad (3.30)$$

and $\tilde{B}_t$ by

$$\tilde{B}_t \geq -\tilde{I}_t + (\tilde{I}_t + \sum_{j=1}^{t} \tilde{d}_{jt} P_{jt}) \sum_{k=1}^{i} p_k + \sum_{j=1}^{t} (\sum_{k=1}^{i} p_k \mathbb{E}[d_{jt} | \Omega_{jt}]) P_{jt}. \qquad (3.31)$$

where $\Omega_1, \Omega_2, \cdots, \Omega_N$ denote N disjoint adjacent subregions that partition the domain of Normal demands ($\mathbb{R}$) and $p_i = \Pr\{d_t \in \Omega_i\}$, $i = 1, \ldots, N$.

Appendix 3.B Implementing augmenting procedure on Example 1 by periods.

Figure 3.17 presents the optimal order-up-to levels and corresponding expected closing inventory levels between each node.

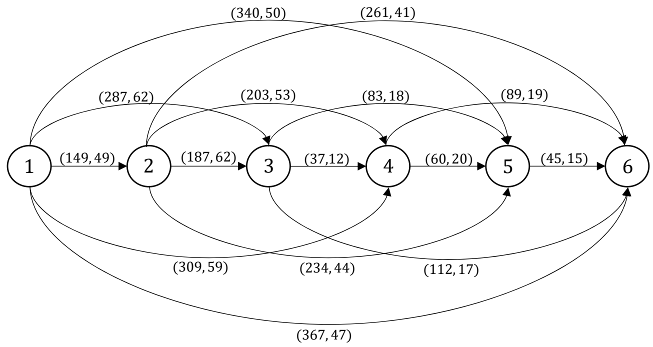

Figure 3.17: Paths constructed by equivalent shortest path to the relaxed problem.

Then we move on to check its feasibility to the original problem, where the flow balance constraints are not relaxed. According to the algorithm, for every node from 1 to 5, we check inbound arcs with the expected closing inventory level and the order-up-to levels.

We set the initial inventory level as 0 and assume that an order is always placed at the beginning of period 1. So all outbound arcs are preserved for node 1 at this stage.

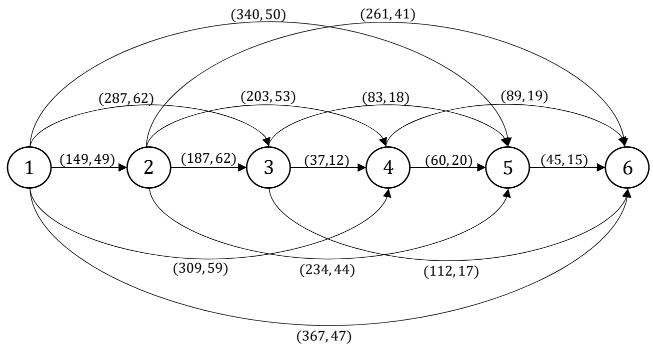

Figure 3.18: Augmenting procedure for node 1.

For node 2, one inbound arc is observed: (1,2) with the expected closing inventory 49, and four outbound arcs are observed:
$(2, 3)$ with the order-up-to level $187 > 49$,
$(2, 4)$ with the order-up-to level $203 > 49$,

(2, 5) with the order-up-to level 234 > 49,

(2, 6) with the order-up-to level 261 > 49,

Feasibility holds for all. So these arcs are all preserved.

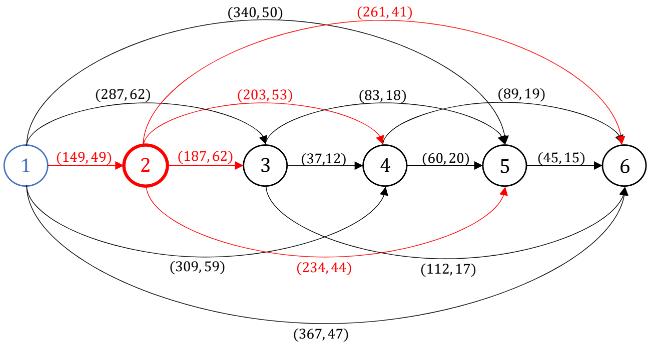

Figure 3.19: Augmenting procedure for node 2.

For node 3, two inbound arcs are observed:

(1,3) with the expected closing inventory 62;

(2,3) with the expected closing inventory 62,

and three outbound arcs are observed:

(3,4) with order-up-to level 37

(3,5) with order-up-to level 83

(3,6) with order-up-to level 112.

It is found that outbound (3,4) with order-up-to level 37 violates both two inbound arcs on the feasibility check. Other combinations will pass. So arcs (3,5) and (3,6) are preserved. We start the augmentation procedure on arc (3,4).

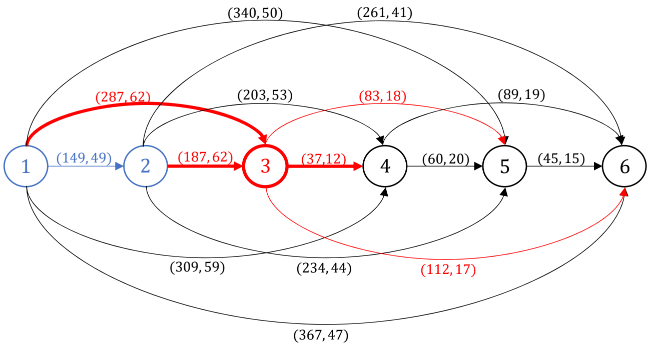

Figure 3.20: Augmenting procedure for node 3.

For inbound arc (1,3) and outbound arc (3,4), we create a new node $3'$ and an arc $(1, 3')$, then for $t = 3$, search the minimum index t for $t > 3$ such that $\tilde{I}_{i-1} \leq \tilde{S}(i, t-1) \leq \tilde{S}(i, t) \leq \cdots \leq \tilde{S}(i, T)$. We solve $t = 5$ with the opening inventory level 83. We create a new arc $(3', 5)$ with the expected closing inventory level of cycle -3, which starts from the opening inventory 62. We also create arc $(3', 6)$ with the expected closing inventory -33. Let $t - 1 > i$ be the maximum index for which $\tilde{I}_{i-1} > \cdots > \tilde{S}(i, t-2)$. for $i = 3$, we observe that $t - 1$ is 4. We introduce arc $(3', 4)$ with the expected closing inventory $62 - \tilde{d}_3 = 37$.

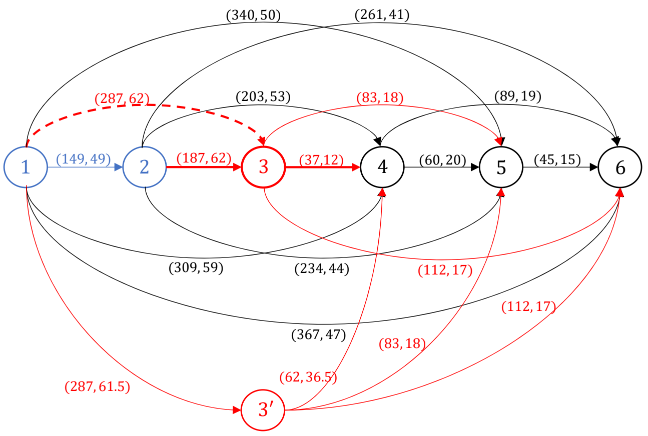

Figure 3.21: Augmenting procedure on arc (1,3) and (3,4).

Similarly, for the inbound arc (2,3) and outbound arc (3,4), we create another new node $3''$ and connect node 2 with $3''$, and this leads to an expected closing inventory at the end of period 2 as 62. We create arcs $(3'', 4)$, $('', 5)$ and $(3'', 6)$ and calculate the expected closing inventory levels as shown. We also update the set $\mathcal{M} = \{\{(1,3)(3,4)\}, \{(2,3)(3,4)\}\}$.

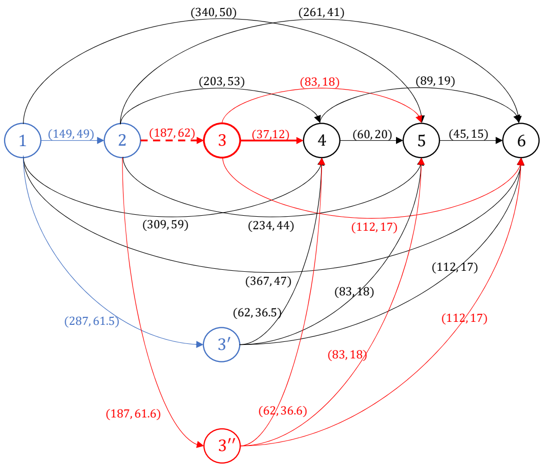

Figure 3.22: Augmenting procedure on arc (2,3) and (3,4).

All actions done for node 3 and we obtain the following figure.

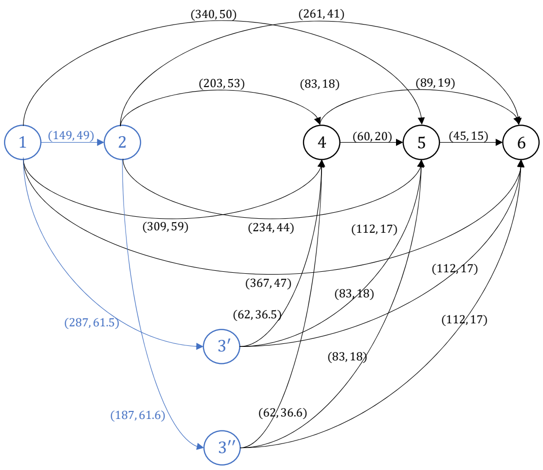

Figure 3.23: Augmenting procedure on arc (2,3) and (3,4).

For node 4, three inbound arcs are observed: (1,4), (2,4) and (3,4), and two outbound arcs are observed: (4,5) and (4,6). Feasibility holds for all combinations. These arcs are all preserved.

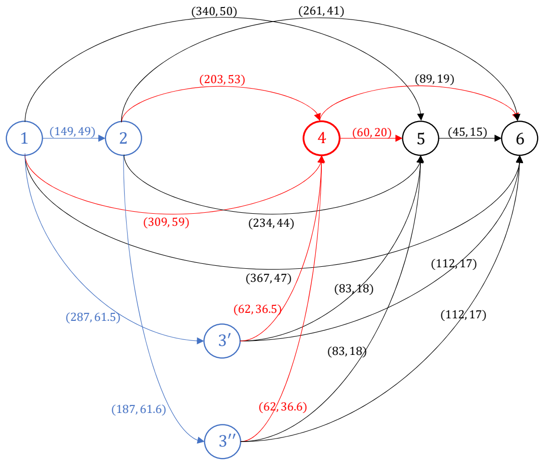

Figure 3.24: Augmenting procedure on node 4.

For node 5 with the single outbound arc (5,6), we found arc (1,5) and (5,6) violate the feasibility with the closing inventory 50 and opening inventory 45. Since arc (1,5) does not pair with other outbound arcs from node 5, so we delete arc (1,5) and create arc $(1, 5')$ and $(5', 6)$ with the new node $5'$. The expected closing inventory associated with arc $(5', 6)$ is $50 - \tilde{d}_5 = 20$.

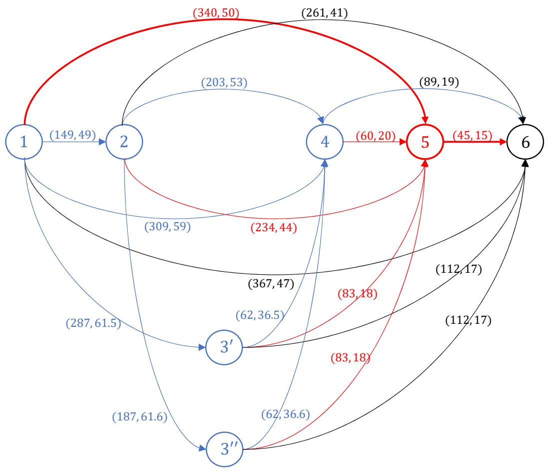

Figure 3.25: Augmenting procedure on node 5.

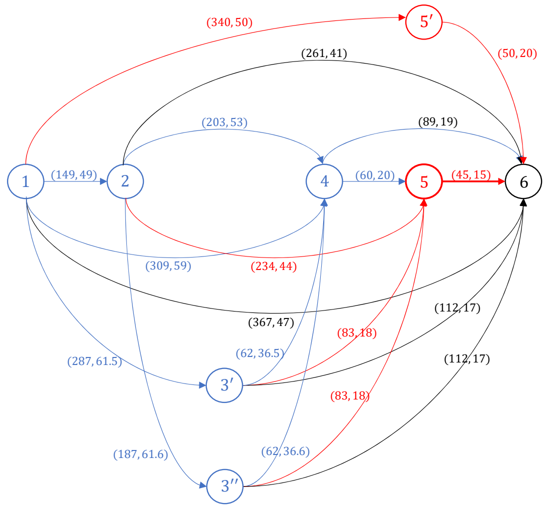

Figure 3.26: Augmenting procedure on arc (1,5) and (5,6).

When node 5 is checked, we can terminate the iteration since node 6 is a dummy set to denote the end of the planning horizon. Figure 3.27 presents the pair of the optimal opening inventory levels and the expected total costs on each arc. We observe that all pairs satisfy the constraints of the original problem.

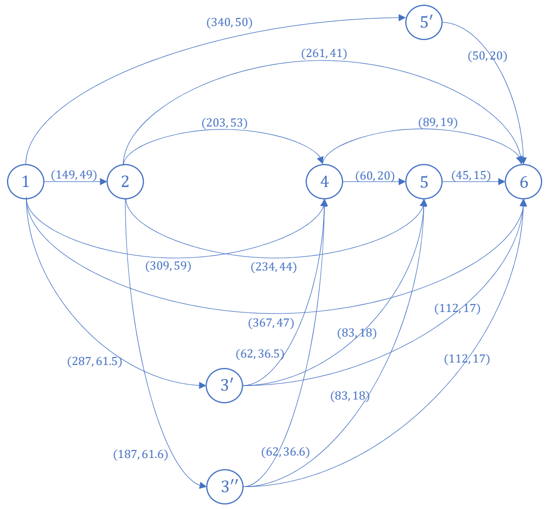

Figure 3.27: Augmented graph with the optimal opening inventory levels and the expected total cost.

Figure 3.28 presents the expected cycle costs between each pair of nodes in the graph. The optimal path can be selected to minimise the expected total cost and to further derive the policy parameters.

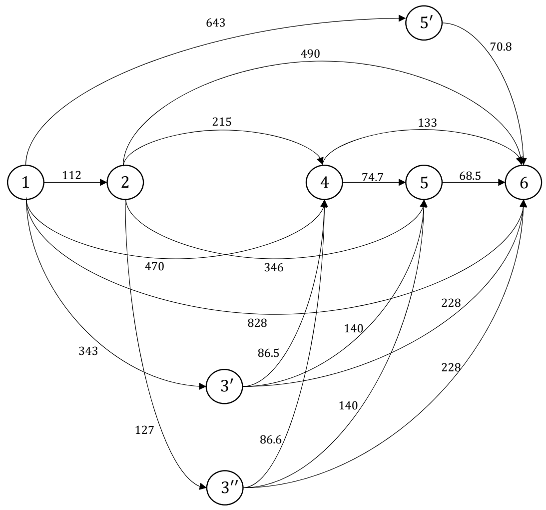

Figure 3.28: Augmented graph with expected cycle costs.

Appendix 3.C Pre-computation time for numerical integration on 25-period test set.

Recall Equation (3.22) and (3.23), the weight on each arc (i, j) represents the expected cost of the period $(i, j-1)$, where only a single replenishment order is placed in period i. In other words, each arc (i, j) indicates a $(j-i)$-period Newsvendor problem and the optima S_i of each convex function can be precisely located after numerical integrations. Note that this convex function has parameters ρ, K, z, b and h. On the other hand, observing the procedure of determining a reorder point s_t, arcs outgoing from node t are replaced by $\bar{c}_{t,k}$ according to Equation (3.27), where only h and b are involved. To avoid repetitive computation, we implement numerical integrations without ordering cost for $\bar{c}_{i,j}$ with only ρ, h

and b (saved as symbolic functions in MATLAB), and then call them and add ordering cost with K and z when computing S_i.

Therefore, in the computation analysis on a 25-period test set, where each instance is computed for 54 groups parameters, we only do numerical integration for 9 groups for $\rho = 0.1, 0.2, 0.3$ and $b = 5, 10, 20$. Table 3.7 summarises the computation time of integration on each instance.

Table 3.7: Computation time on numerical integration for the 25-period test set.

ρ	b	LCY1	LCY2	SIN1	SIN2	STA	RAND	EMP1	EMP2	EMP3	EMP4
0.1	5	394	490	477	655	25	922	603	848	374	500
	10	422	486	516	657	923	100	660	900	415	484
	20	433	489	516	659	123	971	713	833	421	481
0.2	5	440	522	512	727	891	102	684	1031	463	519
	10	430	569	511	778	86	1091	703	1084	459	479
	20	440	578	518	737	1039	116	687	1035	443	491
0.3	5	451	575	558	735	94	1384	820	993	492	545
	10	459	570	628	742	991	138	881	1039	535	507
	20	478	577	609	744	92	1346	893	998	505	502

Chapter 4

Paper III: A hybrid inventory policy for the non-stationary lot-sizing problem with lateral transshipment

Abstract

This paper addresses a single-item non-stationary stochastic lot-sizing problem with two stocking locations. The inventory level at each location is reviewed periodically. Items can be reordered from a common central warehouse and can also be transshipped laterally from the other location.

Lateral transshipment is used proactively to re-distribute the stock between the two locations prior to ordering from the central warehouse. Therefore, the order of actions in each period is transshipping (if necessary), reordering (if necessary) and satisfying the demand at each location. Costs are imposed on transshipping, ordering, holding stock, and back-ordering. The key issues in such a system are to determine the direction and the quantity of the lateral transshipment between locations and the order quantities from the warehouse for both locations.

We formulate the problem of minimising the expected total cost via stochastic dynamic programming. Sincethe number of actions increases exponentially as the feasible quantities of transshipment and replenishment grow, we develop a two-stage dynamic programming formulation to improve computational efficiency. We propose a heuristic algorithm to determine a near-optimal policy relative to this two-stage formulation based on a mixed integer linear programming and a rolling-horizon approach. Numerical experiments are implemented to demonstrate the performance of the two-stage model and the heuristic algorithm.

Keywords Inventory, lateral transshipment, stochastic lot-sizing, non-stationary demand

4.1 Introduction

A supply chain focuses on the core activities within organisations required to convert raw materials or component parts into finished products or services. We concentrate on the product flows among suppliers and various retailers, in which the lot-sizing problem takes a critical role in researching the timing and the quantity of replenishment (Silver, 1981). In multi-location systems, the flow of item(s) takes place between warehouses and retailers as regular replenishment, as well as between retailers and retailers. The latter movements among stocking locations in the same echelon are called lateral transshipments.

Inventory systems apply lateral transshipment according to various aspects such as the products' natures, the structure of logistic routes, capital management policies, and so forth. If assume that the transshipment is always applied, the transshipment policy can be categorised as joint, hybrid and stand-alone with the consideration of regular replenishment from the warehouse. Paterson et al. (2011) review the transshipment papers before 2010 and categorise transshipment into two types: proactive and reactive; the distinction is that proactive transshipment is to reduce the risk of stock-outs due to future demand while reactive transshipment is a recourse action to deal with existing stock-outs or demand (also considered as an emergency replenishment). A classification is illustrated in Fig. 4.1, where we refer to the way that applies both proactive and reactive transshipment as a 'joint' policy and that one applies both transshipments (in either type) and regular replenishment as a 'hybrid' policy. Other criteria of classification are mentioned in (Paterson et al., 2011), for example, complete or partial pooling, and centralised or decentralised systems. Among vast transshipment literature, we focus on multi-location inventory systems with stochastic demand for a single item, where the system applies proactive transshipment plus regular replenishment, as highlighted in Fig. 4.1.

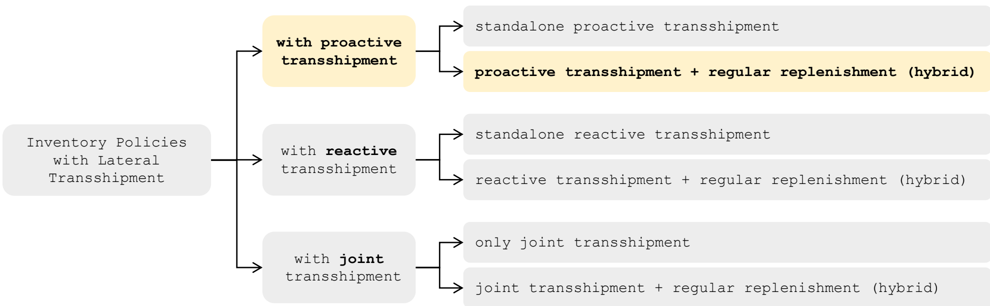

Figure 4.1: A classification of transshipment inventory policies.

The early research in this scope started from Gross (1963), showing that the optimal hybrid policy of a two-location single-period problem depends on the starting inventory level and cost parameters if the lead time is negligible, where the starting inventory levels divide the plane into six regions, and each region has a corresponding optimal policy according to different cost parameters; the corresponding multi-location case is considered by Karmarkar and Patel (1977) with robust linear programming. For a special case when the transshipment takes place at the end of the order cycle once only for non-identical locations, Bertrand and Bookbinder (1998) show that it results in a single-period transshipment problem. The multi-period cases are found to have similar characteristics as Gross in the optimal solutions by Karmarkar (1981).

Hoadley and Heyman (1977) state that applying proactive transshipment statically at the beginning of a period is beneficial as an additional opportunity is given to managing stock-outs. In general, the transshipment timing has a significant influence on the transshipment policy's performance, as Tagaras and Vlachos (2002) explained, that the transshipment timing has a large influence on transshipment policy's performance, but it depends on system characteristics. Most of the research on proactive transshipment is accomplished with or without regular replenishment by setting it at 'static' points, referred to as Agrawal et al. (2004), where the re-distributions take place at the beginning of a period or a predefined point of a period. A dynamic policy developed by Agrawal et al. (2004) outperforms the static regarding costs. They combine the real-time demand information to schedule transshipment with dynamic programming (DP) and solve near-optimal timing of transshipment and new stocking levels at stocking locations via a heuristic regarding the DP.

In replenishment models, the start or end (or any other point) of an order period provides a 'natural' opportunity to mitigate the mismatch between supply and demand by redistributing the stock over all locations. Paterson et al. (2011) argue that this explains the reason why the majority of research on proactive transshipment is based on a periodic review.

In this line of research, Abouee-Mehrizi et al. (2015) provides a thorough analytical study and provides an 'order-up-to curve' policy formed by four switching curves that divide the plane, which is mathematically proved to be optimal.

Considering other assumptions, Diks and De Kok (1996) consider the lead time and propose the Consistent Appropriate Share (CAS) rationing policy to balance the system stock to keep each location's constant fraction. Later, Diks and De Kok (1998) propose the Balanced Stock (BS) rationing policy as a general form of CAS and show that it outperforms CAS. They also show that the system benefited most from BS when a large number of retailers, a high service level, or a long replenishment lead time is involved. The transshipment problem is also studied by Archibald et al. (2010), Powell (2016) and Meissner and Senicheva (2018) through approximate dynamic programming.

From the survey above, we notice that the influence of transshiping timing is emphasised by Agrawal et al. (2004) for standalone proactive transshipment and by Tagaras and Vlachos (2002) for hybrid policy, claiming that it is often set as a static point. Meanwhile, the existing computation for hybrid policy in multi-period problems is either based on transshipment policies (Diks and De Kok, 1996, 1998) that with strong assumption on system characteristics such as lead time, or initiated from systematical analysis (Abouee-Mehrizi et al., 2015) and apply approximate dynamic programming (Archibald et al., 2010; Meissner and Senicheva, 2018). There is a gap in the literature in that few research develops solution approach to non-stationary stochastic inventory control with lateral transshipment.

In this paper, we propose a new optimisation method for two-location stochastic lot-sizing problems with lateral transshipment under non-stationary demand. In our approach, we revise the feasible action space that integrates transshipment and replenishment. Inspired by Agrawal et al. (2004), we design the solution spaces as two splits from the original one according to the independence between transshipment and replenishment. Based on the separated solution spaces, we reformulate the problem as a two-stage stochastic dynamic programming, where, for one period, the system first solves the transshipping direction and quantity, and then the order quantity. To obtain the near-optimal solution that approximates the optimal expected total cost, we develop a heuristic solution approach in a rolling-horizon framework, where we use an approximation technique that is different from Meissner and Senicheva for approximate dynamic programming. We make the following contributions to the stochastic lot-sizing literature.

- We model the non-stationary stochastic lot-sizing problem with a hybrid policy combining proactive transshipment and regular replenishment by a two-stage stochastic dynamic formulation. The transshipping direction and quantity are determined in the first stage, and the order decisions and quantities are determined in the second stage as a single-location problem by considering the future cost that incorporates the whole system.

- We formulate the problem under a static uncertainty strategy as a mixed integer linear programming and develop a new heuristic algorithm to efficiently determine near-optimal policy parameters of the problem in the framework of the rolling horizon.

- In a comprehensive numerical study, we show that two-stage formulation can well approximate the original dynamic programming and that the rolling-horizon heuristic leads to tight optimality gaps.

The rest of this paper is structured as follows. Section 4.2 describes the problem and formulates the problem as a stochastic dynamic program. Section 4.3.2 re-formulates the problem and proposes a two-stage formulation based on the one introduced in Section 4.3.1. In Section 4.4, a heuristic algorithm is developed to compute near-optimal policy for the 2-location problems with lateral transshipment. A computational analysis is presented in Section 4.5. Future research areas are indicated in a discussion in Section 4.6.

4.2 Problem Description

We consider an inventory network of two stocking locations, each of which faces demands that occur randomly according to independent non-stationary stochastic distributions over a planning horizon of T periods. The periods' demands d_t^j, for $t = 1, \cdots, T$ and $j = 1, 2$, are independent random variables with known probability density functions $g_t^j(\cdot)$. We assume that each location has unlimited stocking capacity. Any demand that cannot be satisfied immediately is back-ordered.

The inventory level at each stocking location is reviewed at the beginning of each period. The stock at a location can be reordered and replenished from a common central warehouse and also via lateral transshipment from the other location in the network. We specify that only actual commodities (positive inventory level at one location) can be transshipped, and so no back-order can be transferred. The two locations follow the same review periods and so locations are assumed to be replenished simultaneously if replenishment is applicable. The warehouse is assumed to have sufficient inventory capacity to satisfy orders from stocking locations in full. We assume that both supplement replenishment orders via transshipment and from the warehouse are issued and received instantaneously with negligible lead time.

At the beginning of a period of t, the inventory levels at both locations are reviewed. The system then makes decisions on transshipment and replenishment sequentially, including the transshipment direction, transshipment quantity and regular order quantities. The cost incurred in every period of the planning horizon consists of

- a transshipment cost $u(x) \triangleq R + vx$ for $x > 0$, with $u(0) = 0$, where x is the number of units transshipped, R is the fixed cost per transshipment, and v is the cost per unit transshipped;
- an ordering cost $c(Q) \triangleq K + zQ$ for $Q > 0$ with $c(0) = 0$, where Q is the order quantity, K is the fixed ordering cost per order from the warehouse, and z is the cost per unit ordered (Scarf, 1960);
- a linear holding cost h charged on every unit carried from one period to the next or at the end of the planning horizon;
- and a linear penalty cost b charged on every unit back-ordered at the end of each period.

The objective of this study is to identify the optimal policy that integrates both transshipment and replenishment to minimise the expected total cost over a finite planning horizon and develop an effective approach for an arbitrary non-stationary stochastic demand series. We start by formulating the problem as a stochastic dynamic program (Bellman, 1957) in the next section. Table 4.A in Appendix 4.3 summarises all notations used.

4.3 Stochastic dynamic programs

This section presents a stochastic dynamic program directly generated from the problem description. This model is then developed into a two-stage formulation to efficiently account for both transshipping and replenishing decisions.

4.3.1 The stochastic dynamic programming formulation

We model the problem as a stochastic dynamic program as follows.

We assume a finite horizon consisting of T periods and periodic review. The stage of the formulation $t \in \{1, \ldots, T\}$ corresponds to the time period. The state of the system is observed at the beginning of a period and is described by two factors, i^1 and i^2, the inventory levels in stocking locations 1 and 2, respectively. We denote $\boldsymbol{i} \triangleq \langle i^1, i^2 \rangle \in \mathcal{I}$, where $\mathcal{I}$ is the state space for all periods with no limitation.

An action $\boldsymbol{a}_t \triangleq \langle W_t, Q_t^1, Q_t^2 \rangle \in \mathcal{A}_t$ indicates a lateral transshipment of $|W_t|$ units from location 1 to location 2 if $|W_t| > 0$ and from location 2 to location 1 otherwise followed by replenishment orders of Q_t^1 and Q_t^2 ($Q_t^1, Q_t^2 \in \mathbb{N}$) units for locations 1 and 2, respectively, at the beginning of stage t. The transshipment quantity is limited by the stock available at the locations so a transshipment will only be deployed from location j if $i^j > 0$ $\min\{0, -i^2\} \leq W_t \leq \max\{0, i^1\}$. It follows that the probability the system makes a transition from state $\boldsymbol{i} \in \mathcal{I}$ to state $\langle r, k \rangle$ at stage $t+1$ under action $\boldsymbol{a}_t \in \mathcal{A}_t$ is given by

$$\Pr(i^1 = r, i^2 = k | \boldsymbol{i}, \boldsymbol{a}_t) = \begin{cases} g_t^1(i^1 + Q_t^1 - r) \times g_t^2(i^2 + Q_t^2 - k), & i^1, i^2 \leq 0, \\ g_t^1(i^1 - W_t + Q_t^1 - r) \times g_t^2(i^2 + W_t + Q_t^2 - k), & \text{otherwise.} \end{cases}$$

The immediate cost consisting of holding and penalty costs, $f_t(\boldsymbol{i}, \boldsymbol{a})$, can be derived as

$$f_t(\boldsymbol{i}, \boldsymbol{a}_t) \triangleq \sum\nolimits_{j=1}^{2} \mathbb{E}[h \max(0, i^j \mp W_t + Q_t^j - d_t^j) + b \max(0, d_t^j - i^j \pm W_t - Q_t^j)], \quad (4.1)$$

where the upper operations of '$\mp$' and '$\pm$' are applied for $j = 1$ and the lower operations are applied for $j = 2$.

Let $\boldsymbol{i'}$ represent the state after choosing action $\boldsymbol{a}_t$ in state $\boldsymbol{i}_t \in \mathcal{I}_t$, where $i'^1 = i^1 - W_t + Q_t^1 - d_t^1$ and $i'^2 = i^2 + W_t + Q_t^2 - d_t^2$. The expected total cost over periods $t, \ldots, T$ stating in state $\boldsymbol{i} \in \mathcal{I}_t$ at the beginning of stage t can be represented as $C_t(\boldsymbol{i})$, where

$$C_t(\boldsymbol{i}) = \min_{\boldsymbol{a}_t \in \mathcal{A}_t} \{u(|W_t|) + \sum_{j=1}^{2} c(Q_t^j) + f_t(\boldsymbol{i}, \boldsymbol{a}_t) + \mathbb{E}[C_{t+1}(\boldsymbol{i'})]\}, \quad (4.2)$$

and

$$C_{T+1}(\boldsymbol{i}) = 0 \quad (4.3)$$

is the boundary condition. The optimisation problem therefore can be modelled as a stochastic dynamic program to solve $C_1(\boldsymbol{i})$ with $\boldsymbol{i} \in \mathcal{I}_1$. For the convenience of notation, we denote this formulation as 'SDP-1'.

4.3.2 A two-stage stochastic dynamic programming formulation

Even though an exact optimal policy could be provided by solving $C_1(\boldsymbol{i})$ according to Eq. (4.2), the exponential growth of action space, which relates to the calculation with redundancy is extremely time-consuming, which makes it impractical for realistic settings with long planning horizons. This section exploits the formulation introduced in Section 4.3.1 and develops a two-stage formulation to minimise the expected total cost, where the action space is decoupled by separating the transshipment actions from the ordering actions.

The formulation continues to use the notation of the stage t and state $\boldsymbol{i}$. The differences between the new and the former formulation are as follows.

- We decouple the action space $\mathcal{A}_t = \mathcal{R}_t \times \mathcal{Q}_t$ for any period t, where $\mathcal{R}_t \subset \mathbb{Z}$ represents the space of feasible transshipping direction and quantity W_t, and $\mathcal{Q}_t \subset \mathbb{N}$ represents the space of feasible reorder quantity $\boldsymbol{q}_t \triangleq \langle Q_t^1, Q_t^2 \rangle$ for depot 1 and 2, respectively.
- We introduce a modified expected immediate cost $\tilde{f}_t$ of $\boldsymbol{q}_t$ and the state after transshipments as

$$\tilde{f}_t(\tilde{\boldsymbol{i}}, \boldsymbol{q}_t) \triangleq \sum_{j=1}^{2} \mathbb{E}[h \max(0, \tilde{i}^j + Q_t^j - d_t^j) + b \max(0, d_t^j - \tilde{i}^j - Q_t^j)]; \quad (4.4)$$

  where, for the clarity, $\tilde{\boldsymbol{i}} \in \widetilde{\mathcal{I}}_t$ denotes the state after transshipments and $\widetilde{\mathcal{I}}_t$ denotes the state space when the transshipment has completed.

Therefore, the expected total cost over periods $t, \ldots, T$ starting from state $\boldsymbol{i} \in \mathcal{I}_t$ can be described as

$$\widetilde{C}_t(\boldsymbol{i}) = \min_{W_t \in \mathcal{R}_t} \{u(|W_t|) + \widetilde{\widetilde{C}}(\langle i^1 - W_t, i^2 + W_t \rangle)\}, \quad (4.5)$$

where

$$\tilde{\tilde{C}}(\tilde{\boldsymbol{i}}) = \min_{\boldsymbol{q}_t \in \mathcal{Q}_t} \{\sum_{j=1}^{2} c(Q_t^j) + \tilde{f}_t(\tilde{\boldsymbol{i}}, \boldsymbol{q}_t) + \mathbb{E}[\widetilde{C}_{t+1}(\langle \tilde{i}^1 + Q_t^1 - d_t^1, \tilde{i}^2 + Q_t^2 - d_t^2 \rangle)]\}, \quad (4.6)$$

and

$$\widetilde{C}_{T+1}(\boldsymbol{i}) = 0 \quad (4.7)$$

is the boundary condition. The optimisation problem, therefore, can be modelled as a stochastic dynamic program to solve $\widetilde{C}_1(\boldsymbol{i})$ with $\boldsymbol{i} \in \mathcal{I}_1$. Let $G_t(\boldsymbol{i})$ denote the expected cost over the horizon (t, T) with no action taken in the first-leading period t. We denote this formulation as "SDP-2".

4.4 An LP-based algorithm for proactive lateral transshipment problems under rolling horizon

Although the optimal hybrid policy in this paper's scope can be obtained by enumerating all possible order and transshipping quantities as presented in Section 4.3.2, the computation complexity increases exponentially as the planning horizon expands, and it becomes impractical to apply dynamic programming. Instead of an exact solution, this section introduces an algorithm to solve a near-optimal policy for this transshipment problem.

According to the problem description, the objective function minimises the expected total cost that comprises four types of costs, where ordering and transshipping costs are associated with decisions while holding and penalty costs are based on closing inventories dependent on the given stochastic demand distribution(s).

This brings the difficulty of modelling stochastic lot-sizing problems twofold. The first challenge occurs in modelling the cost for the closing inventories of a period. Here we introduce the first-order loss function and its complementary to capture penalty and holding costs, respectively, which play key roles in inventory control (Silver et al., 1998). Recent applications of loss functions in inventory control involve works by Rossi et al. (2014), Rossi et al. (2015) and Xiang et al. (2018). And based on these works, the loss function and its complementary can be approximated by two piecewise linear functions, and it leads to a linear program (LP) for this problem. The second challenge is to incorporate the demand uncertainty when building up the LP model. Dural-Selcuk et al. (2020) clearly indicate that both static and static-dynamic uncertainty feature very competitive optimality gaps and fully dominate the dynamic strategy under the rolling horizon control (Bookbinder and Tan, 1988); this conclusion also applies to our problem. Therefore, we focus on featuring a more straightforward structure with a static strategy, which results in easier to implement in practice, rather than complexing an optimisation model and introducing it to a rolling horizon framework to solve and improve the solution by setting reasonable terminal conditions.

The development of the LP model and generic procedures of the algorithm are introduced in detail hereafter.

4.4.1 LP model under a static uncertainty strategy

Under a static uncertainty strategy, decisions on transship and order quantities over the planning horizon, W_t and Q_t^j for $t = 1, \dots, T$ and $j = 1, 2$, are all predefined according to demands and the closing inventories of the last period. We introduce I_{t-1}^j to denote the closing inventory level at location j in period $t-1$ and W_t the transship quantity between location 1 and 2 at period t.

Consider a random variable ω and a scalar variable x. The first order loss function is defined as $\mathcal{L}(x, \omega) = \mathbb{E}[\max(\omega - x, 0)]$ and its complementary as $\widehat{\mathcal{L}}(x, \omega) = \mathbb{E}[\max(x - \omega, 0)]$. Let $\bar{H}_t^j$ and $\bar{B}_t^j$ represent the expected outstanding and back-ordering inventory at the end of period t at location j respectively, then

$$
\begin{aligned}
\bar{H}_t^j &= \widehat{L}(\bar{X}_t^j, d_{1\dots t}^j), \\
\bar{B}_t^j &= L(\bar{X}_t^j, d_{1\dots t}^j),
\end{aligned}
$$

where $d^j_{1\dots t}$ denotes the convolution of random variable d^j_k, and

$$\bar{X}^j_t \triangleq \bar{I}^j_0 + \sum_{k=1}^{t}(Q^j_k \mp W^j_k) \tag{4.8}$$

indicates the expected cumulative inventory available to satisfy demand up to period t at location j. In this way, the expected total cost can be described as

$$\sum_{t=1}^{T}\{u(|W_t|) + \sum_{j=1}^{2}[h\bar{H}^j_t + b\bar{B}^j_t + c(Q^j_t)]\}$$

to comprise ordering, transshipping, holding and penalty costs, where $u(|W_t|)$ and $c(Q^j_t)$ denote the transshipping cost and ordering cost at location j in period t, respectively.

The constraints for the problem involve the flow balance equations and domains for decision variables. By taking expectations (denoted as $\bar{\cdot}$), the optimal policy under a static uncertainty strategy can be obtained by solving the following model, denoted as "LP-1".

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T}\{u(|W_t|) + \sum_{j=1}^{2}[h\bar{H}^j_t + b\bar{B}^j_t + c(Q^j_t)]\}, & (4.9)\\
\text{s.t.} \quad & \text{for } t = 1\dots,T \text{ and } j = 1,2, & \\
& \bar{H}^j_t = \widehat{L}(\bar{X}^j_t,\, d^j_{1\dots t}), & (4.10)\\
& \bar{B}^j_t = L(\bar{X}^j_t,\, d^j_{1\dots t}), & (4.11)\\
& \bar{I}^j_{t-1} \mp W_t + Q^j_t - \bar{d}^j_t = \bar{I}^j_t, & (4.12)\\
& W_t \le \max\{0, \bar{I}^1_{t-1}\}, & (4.13)\\
& W_t \ge \min\{0, -\bar{I}^2_{t-1}\}, & (4.14)\\
& \bar{H}^j_t, \bar{B}^j_t \ge 0. & (4.15)
\end{aligned}$$

Rossi et al. (2014) presents the piecewise linear upper and lower bounds for the first order loss function that can be immediately embedded in programming models. According to Lemma 3, 10 and 11 in (Rossi et al., 2014), we generate the bounds for $\bar{H}^j_t$ and $\bar{B}^j_t$ under the static uncertainty strategy as

$$\bar{H}^j_t \ge \bar{X}^j_t \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k \mathbb{E}[d_{1\dots t}|\Omega_k] \tag{4.16}$$

and

$$\bar{B}_t^j \geq \bar{X}_t^j \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k \mathbb{E}[d_{1\ldots t}|\Omega_k] - X_t^j + \bar{d}_{1\ldots t}, \tag{4.17}$$

where N disjoint adjacent subregions $\Omega_1, \Omega_2, \cdots, \Omega_N$ partition the domain of random variables d_t^j. These inequalities are included in the model to enable the computation for linear programming.

4.4.2 A heuristic approach for transshipment problem based on static uncertainty strategy

Model LP-1, as above, does not stand alone in the solving process but is embedded in a rolling horizon framework. This subsection introduces a heuristic algorithm for the two-location lot-sizing problem with lateral transshipment with the following procedures.

In Algorithm 3, the LP-1 model is integrated into a rolling horizon framework to efficiently compute a near-optimal solution for the two-location lot-sizing problem with lateral transshipments. The algorithm begins by generating random demand series $\hat{d}_t^j$ for locations 1 and 2, based on the given demand rates d_t^j for $t = 1, \ldots, T$.

Subsequently, the rolling horizon approach is applied, starting from the first period $t = 1$, to iteratively solve the LP-1 model and obtain solutions $\hat{W}_t$ and $\hat{Q}_k^j$ for $k = t, \ldots, T$, given the opening inventory $\boldsymbol{i}_{k-1}$. Utilising the current solutions $\hat{W}_t$ and $\hat{Q}_k^j$, the algorithm generates feasible transshipment quantities and computes the optimal transshipment quantity based on Eq.(4.6). Here, the variables $Q_k^j, \ldots, Q_T^j$, $W_{k+1}, \ldots, W_T$ are replaced by the current solutions $\hat{Q}_k^j, \ldots, \hat{Q}_T^j$, $\hat{W}_{k+1}, \ldots, \hat{W}_T$, and the demand variables $d_k^j, \ldots, d_T^j$ are substituted with the randomly generated demands $\hat{d}_k^j, \ldots, \hat{d}_T^j$. This procedure is repeated until all periods have been considered, resulting in an effective and near-optimal solution for the transshipment problem.

We simulate the control policy (both transshipping and ordering) obtained by implementing the aforementioned approach. The simulation iterates on the realised demand and implements a stopping rule so as to achieve a 0.95 confidence probability (e.g. see (Dural-Selcuk et al., 2020)).

Algorithm 3 Computing near-optimal policy for non-stationary lot-sizing problem with lateral transshipment.

1: **Input:** demand rates d_t^j; cost parameters (K, z, R, v, h, b); an opening state of inventory $\boldsymbol{i}_0$; a natural number $n = 0$.

2: **Output:** a hybrid policy with transship and order quantities $\tilde{W}_t$ and $\tilde{Q}_t^j$ for $t = 1, \ldots, T$ and $j = 1, 2$.

3: **do**

4: Randomly generate two series $\hat{d}_t^j$ as demands at locations 1 and 2 according to demand rates d_t^j for $t = 1, \ldots, T$;

5: **for** $k = 1 \rightarrow T$ **do**

6: Solve model LP-1 and obtain solutions $\hat{W}_t$ and $\hat{Q}_k^j$ for $k = t, \ldots, T$ with opening inventory $\boldsymbol{i}_{k-1}$ given, and update $\tilde{Q}_k^j = \hat{Q}_k^j$;

7: Generate the transship quantity's domain $\mathcal{W}_k = \{W_k | \min\{0, -i_{k-1}^2\} \leq W_k \leq \max\{0, i_{k-1}^1\}\}$ with opening inventory $\boldsymbol{i}_{k-1}$, compute

$$\tilde{W}_k = \arg\min_{W_k \in \mathcal{W}_k} \tilde{\tilde{\tilde{C}}}(\boldsymbol{i}_{k-1}) \tag{4.18}$$

by Eq.(4.5) – (4.6), where $Q_k^j, \ldots, Q_T^j$, $W_{k+1}, \ldots, W_T$ are substituted by $\hat{Q}_k^j, \ldots, \hat{Q}_T^j$, $\hat{W}_{k+1}, \ldots, \hat{W}_T$ from LP-1, and $d_k^j, \ldots, d_T^j$ by $\hat{d}_k^j, \ldots, \hat{d}_T^j$;

8: Compute inventory $\boldsymbol{i}_k$ with static demand $\hat{d}_k^j$ and decisions $\tilde{W}_k$ and $\tilde{Q}_k^j$;

9: **end for**

10: Update the number of experiments $n = n + 1$;

11: **while**

12: The number of experiments n is not sufficient large to contribute an $\alpha \times 100\%$ confidence interval.

4.5 Computational experiments

This section presents a computational analysis to evaluate the accuracy of the proposed two-stage formulation in Section 4.3.2 and the performance of the MILP-based rolling-horizon heuristic in Section 4.4. In Section 4.5.1, we consider a test set comprising small instances with 4 periods with Poisson demands and investigate the performance of the two-stage model against the optimal SDP in Section 4.3.1. In Section 4.5.2, we extend the planning horizon to 10 periods and alter the demands to Normal distributions; we compare the effectiveness of the heuristic against the two-stage SDP formulation. Note that in this experiment, two locations are not necessarily identical.

We refer to the percentage optimality gap of the expected total cost (ETC) as the measure of the comparison, which is computed according to $100 \times (\text{ETC}_2 - \text{ETC}_1)/\text{ETC}_1$. We name three optimising approaches aforementioned "optimal-SDP", "two-stage-SDP", and "heuristic". In the 4-period test set, ETC_1 and ETC_2 represents results by "optimal-SDP" and "two-stage-SDP", respectively. And in the 10-period test set, they stand for results by "two-stage-SDP" and "heuristic", respectively.

For each test set, we set the initial inventory level as zero and consider three major parameters: fixed ordering cost K, fixed transshipping cost R, and penalty cost b. The other cost parameters are designed according to the scale of planning horizon lengths and demands and kept consistent within one set of experiments. However, we predefine constraints $K > R$ to assume that the transshipping takes predecease over the ordering and $K \leq 2R$ to ensure the system would not order only once for the planning horizon without any other transshipment. Besides, we set $v < b$ to assume that the system would not leave unmet demand back-ordered even though the transshipment is reasonably worthwhile. These parameters will be elaborated in each subsection as below.

All computations are performed by a 4.0 (1.90+2.11) gigahertz Intel(R) Core(TM) i7 − 8650U CPU with 16.0 gigabytes of RAM in JAVA 1.8.0_201.

4.5.1 4-period test set with Poisson-distributed demand

This test set is designed to investigate the accuracy of the two-stage formulation against the optimal SDP. We compare the difference between the expected total cost computed by the original SDP in Section 4.3.1 and by the two-stage SDP in Section 4.3.2 regarding percentage optimality gaps.

Due to the heavy computation for SDP approaches, we constrain that the demand means for Poisson distribution are no more than 20 as in Fig. 4.2. And we apply Latin hypercube sampling, introduced by McKay et al. (1979), to generate near-random samples of demand patterns in a multidimensional way.

We consider cost parameters including $K = 10, 20, 30$, $R = 5, 10, 20$ and $b = 3, 5$, which are implemented with two groups of unit ordering and transshipping pairs: $z = 2$, $v = 1$ and $z = 1$, $v = 0.5$ and $h = 1$ for all experiments in this subsection. We also consider ten demand patterns: two life cycle patterns, one moves from the launch stage to maturity via a growth (LCY1) and the other moves from the growth stage through maturity and into decline (LCY2); two sinusoidal patterns, one with stronger (SIN1) and the other with weaker (SIN2) oscillations; a stationary demand pattern (STAT); a random demand pattern (RAND); and lastly, 4 empirical patterns derived according to Strijbosch et al. (2011). Since the arbitrariness of locations, for one group of system parameters, two locations choose one demand out of ten sequentially without repetition of previous experiments, and demand patterns of two locations are not necessarily to be non-identical.

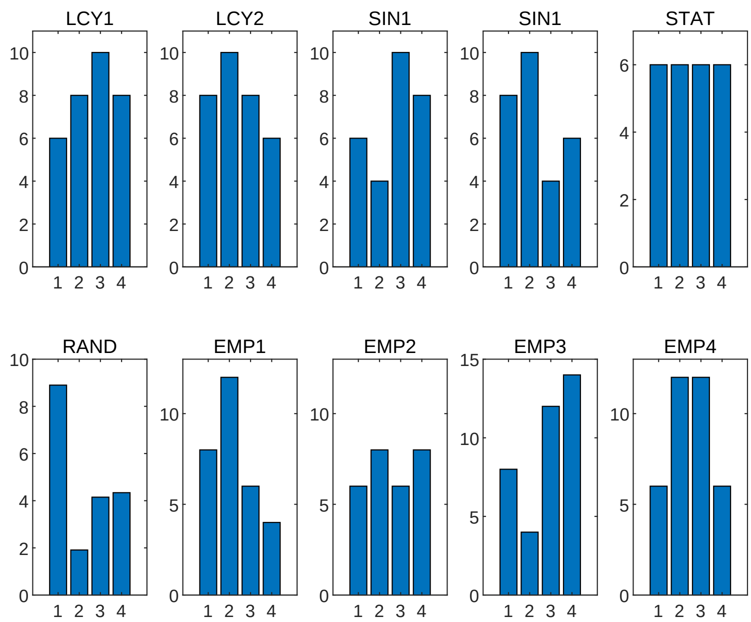

Figure 4.2: Demand patterns for 4-period instances.

For the two-stage SDP approach, Table 4.1 reports the optimality gaps observed relative to the optimal SDP, pivoting demand patterns and various cost parameters. It should be noted that the two locations are of arbitrariness, therefore, we reformulate the results by pivoting the demand pattern of location 1 for the analysis. The results for two-stage-SDP give the exact optimality gaps for these policies against the optimal SDP, which is, on average 0.2924%. There is no obvious relation between optimality gaps and the variation in demand patterns or in cost parameters. The largest average arises under demand pattern LCY1 (0.3194%), while it still does not deviate from the average. We conclude that the two-stage SDP can nicely approximate the optimal SDP. We then proceed with the results by two-stage formulation as the benchmark in evaluating the heuristic approach's performance.

Table 4.1: Average percent optimality gap over the 4-period test set under pivoting parameters (%).

Pivoting parameters	two-stage-SDP
demand patterns	
LCY1	0.3194
LCY2	0.3076
SIN1	0.2930
SIN2	0.2863
STAT	0.2815
RAND	0.2886
EMP1	0.2890
EMP2	0.2850
EMP3	0.2896
EMP4	0.2840
fixed ordering cost *K*	
10	0.2891
20	0.2836
30	0.3045
fixed transshipping cost *R*	
5	0.2875
10	0.3054
20	0.2829
penalty cost *b*	
3	0.2807
5	0.2829
Average	0.2924

Focusing on characteristics of results inside, Fig. 4.3 and 4.4 illustrate the distribution of optimality gaps against demand patterns and cost parameters, where all box plots present a Normal-distribution shape. More specifically, in the view of demand patterns, we observe that the mediums shift remarkably when changing demand patterns, while they can all be ranged in the percentage of [0.2,0.4]. The long lower and upper whiskers indicate that the optimality gaps vary amongst each demand pattern. In view of cost parameters, we observe that box plots are of similar mediums at 0.30% and distributed slightly differently. More outliers are noticed compared with Fig. 4.3 for demand patterns. Considering an average optimality gap 0.2924% from Table 4.1, we conclude that the average accuracy of two-stage SDP can remain in a small value to facilitate further experiments.

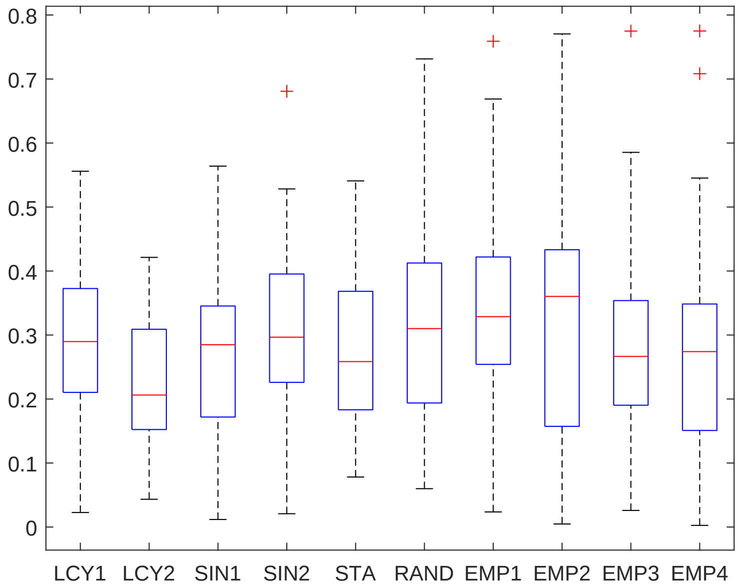

Figure 4.3: Box plots on the optimality gap regarding demand patterns.

4.5.2 10-period test set with Normally-distributed demand

This subsection extends the planning horizon to 10 periods. The purpose of implementing this test set is to evaluate the effectiveness of the proposed heuristic against the two-stage formulation. Since the computation of piecewise linear approximation parameters consumes a large computation time for long-horizon Poisson demand, this test set focuses on Normal demand, for which Rossi et al. (2014) present pre-computed optimal partitioning coefficients. Note that we apply ten partitions in piecewise linear approximation.

For demands in Normal distribution, we introduce the standard deviation $\sigma_t = \rho \cdot d_t$, where d_t is the mean of a Normal distribution, and ρ denotes the

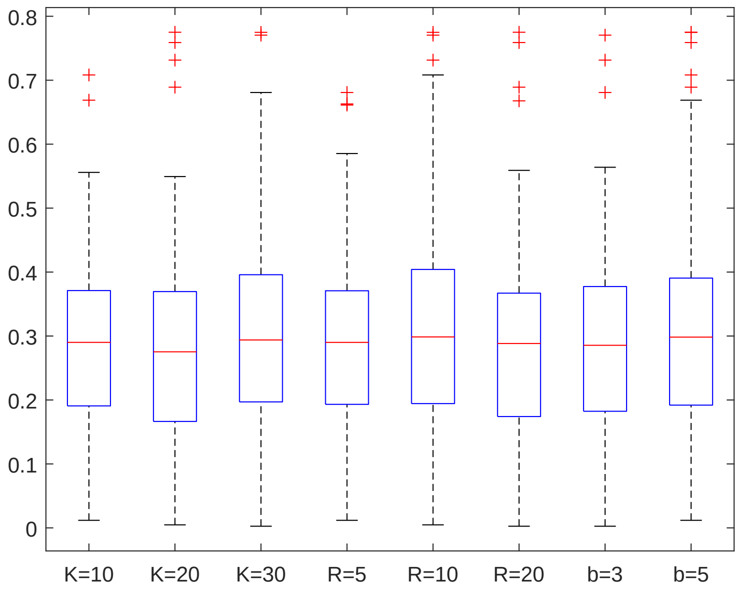

Figure 4.4: Box plots on the optimality gap regarding cost parameters.

coefficient of variation of the demand, which remains fixed over time as prescribed by Bollapragada and Morton (1999), and here we set $\rho = 0.1$.

The Normal-distributed demand will be implemented on fixed ordering cost $K = 10, 20$, fixed transshipping cost $R = 5, 10$, the penalty cost $b = 3, 5$, holding cost $h = 1$, unit ordering cost $z = 0.5$ and unit transshipping cost $v = 1$, which construct eight parameter groups. Due to the computation complexity, in this set, we downsize the number of demand patterns to only include life cycle (LCY), sinusoidal (SIN), stationary (STA), random (RAND), empirical (EMP), and they can still cover majority realistic demand situations. With arbitrary combinations of these patterns to each location, in total, one group of parameters will be applied on 25 pairs of demand patterns that two locations do not necessarily take the same, and 200 instances are tested in this set.

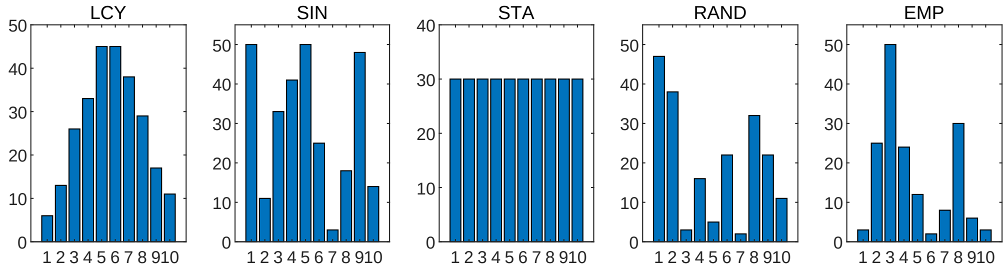

Figure 4.5: Demand patterns for 10-period instances.

Table 4.2 reports the average percentage optimality gaps against expected total cost computed by the two-stage SDP approach over 10-period instances pivoting five demand patterns and cost parameters. Similarly, we only pivot the demand pattern of location 1 for the analysis.

This small set of instances reveals an average optimality gap of 0.5515%. In the view of the demand pattern, we observe that the stationary pattern presents the minimum gap in 0.5140%, followed by the random pattern (0.5405%) and life cycle pattern (0.5425%); sinusoidal pattern's average gap (0.5557%) is slightly greater, and empirical pattern demonstrates the maximum (0.6048%). Considering the shape of demand means in Fig. 4.5, we find that the performance of this rolling-horizon heuristic deteriorates when demand is heavily non-stationary. In view of cost parameters, the average gap pivoting fixed ordering cost K varies more evidently than pivoting fixed transshipping cost R and penalty cost b. And the gap increases when K and b independently rise but decreases when R increases on the contrast.

4.6 Discussion

In this paper, we formulated and modelled a non-stationary 2-location, multi-period transshipment problem with back-order in cases of stock-outs. We set the proactive transshipment takes place at the beginning of every period if applicable and established stochastic dynamic programming to model the problem. However, the size of state and decision spaces makes it impossible to find the optimal policy

Table 4.2: Average percent optimality gap over the 10-period test set under pivoting parameters (%).

Pivoting parameters	MILP-based rolling-horizon heuristic
demand patterns	
LCY	0.5425
SIN	0.5557
STAT	0.5140
RAND	0.5405
EMP	0.6048
fixed ordering cost *K*	
10	0.5373
20	0.5656
fixed transshipping cost *R*	
5	0.5592
10	0.5438
penalty cost *b*	
3	0.5506
5	0.5524
Average	0.5515

for real-life sized problems; hence we introduced a two-stage stochastic dynamic programming, where the proactive transshipping quantity and order quantity are sequentially determined.

Although the computation efficiency has been improved via the two-stage model, it is still hard to be applied. To obtain a near-optimal policy, we built a mixed integer linear programming under a static uncertainty strategy, which features a simpler structure and is increased by the rolling-horizon framework, leading to a tight optimality gap shown in the experimental study.

Future research from this paper is originated from the current development or further extension of the approaches. They can be conducted on the following aspects.

- The connection between stochastic dynamic programmes and the two-stage formulation. The two-stage formulation is developed by treating transshipment and replenishment as independent activities. From the numerical study, we see that the difference between the two formulations roughly follows a Normal distribution, whose average can remain in small value. A study on this relation can reveal where the difference originates.

- The analytical study on the feasible transshipment space. A line search is necessary for all feasible options to obtain the optimal transshipment quantity, which is time-consuming even for a 10-period instance. An analytical study on the structure of the feasible space could simplify the computation, which varies with the opening inventory at the current stage.
- More generally, the study can be extended to involve more realistic assumptions; for example, (non-)identical lead times of replenishment and transshipment to different locations, the lost-sale scheme to deal with unmet demand and capacity imposed on either inventory storage or two modes of transportation.
- Other structures of the inventory system. This chapter considers a two-echelon two-location transshipment problem. This simple structure can be extended to a more complex network to involve more echelons and connections, a system with N warehouses and M stocking locations, where N and $M \in \mathcal{N}$, to exploit the transshipment policy in a general network.
- The generation on multi-location problems, where the challenge is two-folded. First, as the number of locations increases, the number of possible inventory states and transshipment options grows exponentially, leading to a significant increase in the solution space size. This makes it computationally challenging to find the optimal solution within a reasonable time frame, especially for large-scale problems with multiple items and locations. Second, in a multi-location problem with rolling horizon planning, each decision stage requires solving an optimisation problem that considers the interactions among multiple locations over the entire planning horizon. As the number of locations increases, the time complexity of solving each stage increases, making it difficult to solve the problem within the rolling horizon framework efficiently.

To overcome the computational challenges, exploring alternative solution approaches, such as decomposition techniques or heuristics, may be necessary, which can provide near-optimal solutions in a reasonable amount of time. And this generation will be considered in future research.

Appendix 4.A Notations

Table 4.3: Notations of important functions/parameters

Functions	Explanation
d_j^t	demand for location j in period t, $t = 1, \dots, T$;
$g_t^j(\cdot)$	probability density function of demand d_j^t;
$u(x)$	transshipping cost, $u(x) = R + v(x)$ for $x > 0$, $u(x) = 0$ if $x \leq 0$;
$c(Q)$	ordering cost, $c(Q) = K + zQ$ for $Q > 0$, $c(Q) = 0$ if $Q \leq 0$.
$\boldsymbol{i}$	a state in the stochastic dynamic programming that presents the inventory levels of two locations, $\boldsymbol{i} = \langle i^1, i^2 \rangle \in \mathcal{I}_t$;
$\boldsymbol{a}_t$	a feasible action for state $\boldsymbol{i}$ in the stochastic dynamic programming that integrates transshipping quantity and order quantities, $\boldsymbol{a}_t = \langle W_t, Q_t^1, Q_t^2 \rangle \in \mathcal{A}_t$ and $W_t \in \{W_t \mid \min\{0, -i^2\} \leq W_t \leq \max\{0, i^1\}\}$;
$f_t(\boldsymbol{i}, \boldsymbol{a}_t)$	the expected holding and penalty cost of two locations that start from state $\boldsymbol{i}$ applying action $\boldsymbol{a}_t$;
$C_t(\boldsymbol{i})$	the expected total cost over periods $t, \dots, T$ with state $\boldsymbol{i} \in \mathcal{I}_t$ at the beginning of stage t;
$G_t(\boldsymbol{i})$	the expected cost over horizon (t, T) with no action taken in the first-leading period t.
$\tilde{f}_t$	the modified expected immediate cost of $\boldsymbol{q}_t$ and the state after transshipment;
$\widetilde{C}_t(\boldsymbol{i})$	the expected total cost over periods t to T starting from state Algorithm $\boldsymbol{i} \in \mathcal{I}_t$;
$\tilde{\tilde{C}}(\boldsymbol{i})$	the minimised expected cost, including the current ordering cost and immediate cost, and the future cost from period t to T.

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