CONTINUITY OF FUNCTIONALS AND COMPACT ACTION OF OPERATORS

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PREFACE.

This thesis deals with two topics in Functional Analysis. The first three chapters are concerned with positive linear functionals on Banach *-algebras, while the last two deal with operators that act compactly on an algebra of operators.

Chapter 1 is an account of the basic known results on Banach *-algebras with a bias towards positive linear functionals. Though this account is based on that in C.E. Rickart's book, "General Theory of Banach Algebras", considerable use is made of important new results proved since 1960.

In Chapter 2, the continuity and the representability of positive linear functionals are discussed. The work of Varopoulos concerning commutative Banach *-algebras A, with $A^2 = A$, is generalised: a new direct proof of Varopoulos's result, not demanding continuity of the involution, is provided. This proof also applies to certain special non-commutative Banach *-algebras A with the property that $A^2 = A$. Varopoulos's result for algebras with two-sided approximate identity is discussed and his factorisation technique is used to prove a new result on the representability of all positive linear functionals on such algebras.

The abundance of positive and representable linear functionals is discussed in Chapter 3. Grothendieck characterised B*-algebras as those Banach *-algebras with continuous involution , in which the dual space is spanned by the representable linear functionals. This is proved in Chapter 3 in a new way without assuming the involution continuous. An example of an algebra without non-zero positive linear functionals is provided and a link is pointed out between the abundance of positive linear functionals on a Banach *-algebra and the continuity of positive linear mappings into that algebra.

Chapter 4, on operators that act compactly on an algebra of operators, first presents some of the work of Bonsall, particularly his problem about operators that act compactly on their centralisers. A result of Vala about a different kind of compact action is brought to bear to solve Bonsall's problem for operators satisfying a polynomial identity.

In Chapter 5, the compact action of weighted shift operators on ℓ^1 on their centralisers is studied intensively. This leads to further positive results and a counterexample, which shows that operators on a Banach space that act compactly on their centralisers need not be compact.

The work contained in this thesis was done at the

University of Edinburgh as a research student under Professor F.F. Bonsall. I do wish to record my appreciation of the faithful way in which Professor Bonsall has discussed my work with me each week: in these discussions he offered me ideas, help and encouragement.

I have also had valuable conversations with my friends, Dr. B. Bollobás, Dr. T.A. Gillespie and Dr. A.L.T. Paterson and with Dr. G.R. Allan, who acted as my supervisor during the first term of session 1969 - 70, which I had the good fortune to spend in Cambridge.

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NOTATION.

Throughout this thesis we make the following conventions.

The complex conjugate of complex number α is denoted by $\bar{\alpha}$.

Let A, B, C be sets and let $f: B \to A$ and $g: C \to B$ be mappings; the composition of f and g is a mapping from C to A denoted by $f_{\circ} g$.

The closure of a set B in a topological space is denoted by $\bar{B}$.

Let X be a Banach space. Then X^* denotes the dual space of X. Let $x \in X$. Then $\hat{x}$ denotes the element of X^{**} defined by $\hat{x}(f) = f(x)$ for all $f \in X^*$. Let g be a linear functional on X. Then the restriction of g to subspace Y is denoted by $g|_Y$. The sum of two linear subspaces Y and Z is denoted by $Y + Z$.

Let A be a Banach algebra. Let $a \in A$. Then $\rho(a)$ denotes the spectral radius of a and sp(a) denotes the spectrum of a. A_1 denotes the algebra with identity element adjoined as follows. Let A_1 consist of all formal sums $a + \alpha$ where $a \in A$, $\alpha \in \mathbb{C}$. Define operations on A_1 by

$$(a + \alpha) + (b + \beta) = (a + b) + (\alpha + \beta)$$

$$\beta(a + \alpha) = \beta a + \beta\alpha \qquad (\beta \in \mathbb{C})$$

$$(a + \alpha)(b + \beta) = (ab + \alpha b + \beta a) + \alpha\beta .$$

Define the norm in A_1 by $\|a + \alpha\| = \|a\| + |\alpha|$. Then A_1 is a Banach algebra with identity element 1 and A_1 contains A as a subalgebra.

The Banach algebra of all bounded linear operators on the Banach space X is denoted by B(X). Let $T \in B(X)$. Then $|T|$ denotes the norm of T and Z(T) denotes the centraliser of T [defined in 4A]. Let $a \in X$, $f \in X^*$; then $(a \otimes f)$ denotes the rank 1 operator defined by $(a \otimes f)(x) = f(x)a$.

CHAPTER 1.

BANACH *-ALGEBRAS.

1A. Elementary properties.

In what follows we consider only complex Banach algebras — i.e. Banach algebras over the field of complex numbers. We assume neither that a Banach algebra has a unit element nor that a unit element, if present, has norm 1.

Definition. Let A be a complex Banach algebra. An involution on A is a mapping $x \rightarrow x^*$ from A onto itself such that for all $x, y \in A$ and for all $\alpha \in \mathbb{C}$

(1) $x^{**} = x$

(2) $(x + y)^* = x^* + y^*$

(3) $(\alpha x)^* = \bar{\alpha} x^*$

(4) $(xy)^* = y^* x^*$.

Definition. A Banach *-algebra is a Banach algebra with an involution *.

This chapter is an account of known results on Banach *-algebras bringing the basic theory, as in Rickart [21], up to date in the light of Johnson's theorem, Ford's lemma and Bonsall's work on B*-semi-norms. Note that we do not assume the involution on a Banach *-algebra continuous.

Let A be a complex Banach *-algebra. An element h of A such that $h^* = h$ is called self-adjoint. Every element x of A has a unique representation of the form $x = h + ik$ where h and k are self-adjoint. In fact, $h = \frac{1}{2}(x + x^*)$ and $k = \frac{1}{2i}(x - x^*)$. The set of all self-adjoint elements is a real linear subspace of A. This subspace is norm closed, and therefore a Banach space, if and only if the involution is continuous. This is an easy consequence of the closed graph theorem.

Should a complex Banach *-algebra A fail to contain an identity element, we can embed A in a larger Banach algebra A_1 with an identity element. [See Notation.] A_1 becomes a Banach *-algebra when an involution is defined by $(a + \alpha)^* = a^* + \bar{\alpha}$ where $a \in A$ and $\alpha \in \mathbb{C}$.

Definition. Let A, B be two Banach *-algebras. A *-homomorphism from A to B is a linear mapping β from A to B such that $\beta(ab) = \beta(a)\beta(b)$ and $\beta(a^*) = \beta(a)^*$ for all $a, b \in A$.

Definition. A *-ideal in a Banach *-algebra is an ideal closed under the involution.

The following square-root lemma, which will be of signal service to us later was first proved as it now stands by Ford [11]; the proof given here is elementary and is due to Bonsall and Stirling [6].

<u>LEMMA 1.1.</u> Let A be a Banach *-algebra. Let a be a self-adjoint element of A with $\rho(a) < 1$. Then there exists a self-adjoint element x in A such that $2x - x^2 = a$.

<u>Proof.</u> Since $\rho(a) = \inf |a|$, where the infimum is taken over all algebra norms $|.|$ equivalent to the given norm (by Holmes [15]), we may choose such a norm $|.|$ and a real number k with $|a| < k < 1$. Let C denote the least closed subalgebra containing a, and let $E = \{x ; x \in C, |x| \leq k\}$. Then T defined by

$$Tx = \frac{1}{2}(a + x^2)$$

is a contraction mapping of E into E, since by the commutativity of C

$$|Tx - Ty| = \frac{1}{2}|x^2 - y^2| \leq \frac{1}{2}|x - y||x + y| \leq k|x - y| \quad \text{for } x,y \in E.$$

Hence there exists $x \in E$ with $2x - x^2 = a$ and $\rho(x) \leq |x| < 1$.

Suppose now that $y \in A$, $2y - y^2 = a$ and $\rho(y) < 1$. Since y commutes with a and x is a limit of polynomials in a we have that y commutes with x. Therefore $\rho(x + y) < 2$ and we may again choose an equivalent algebra norm $|.|'$ with $|x + y|' < 2$. The inequality

$$|x - y|' = \frac{1}{2}|(a + x^2) - (a + y^2)|' \leq \frac{1}{2}|x + y|'|x - y|'$$

gives that $|x - y|' = 0$ and so $x = y$. Hence x is unique with the properties $\rho(x) < 1$ and $2x - x^2 = a$.

Now we show x self-adjoint. Since $sp(x^*) = \overline{sp(x)}$ we have that $\rho(x^*) < 1$. Furthermore we have

$a = a^* = (2x - x^2)^* = 2x^* - x^{*2}$. Hence by the uniqueness of x we find $x = x^*$.

COROLLARY 1.2. Let B be a Banach *-algebra with identity element 1 and let $b \in B$ with $b^* = b$ and $\varrho(1 - b) < 1$. Then there exists $u \in B$ such that $b = u^2$ and $u = u^*$.

Proof. By Lemma 1.1, there exists a self-adjoint element v such that $1 - b = 2v - v^2$. Then $(1 - v)^2 = b$ and $1 - v$ is self-adjoint since v is self-adjoint.

1B. Johnson's Theorem.

The following outstanding theorem was proved by Johnson [17] in 1967.

THEOREM 1.3. A semi-simple Banach algebra has a unique complete algebra-norm topology.

Proof. See Johnson [17].

COROLLARY 1.4. The involution on a semi-simple Banach *-algebra is continuous.

Proof. Define another norm $|\cdot|$ on A by $|a| = \| a^* \|$. This norm is clearly a complete norm. By Theorem 1.3, it must therefore be equivalent to $\|\cdot\|$. Hence we have $\| a^*\| \leq K\| a \|$ for all $a \in A$ and for some $K > 0$.

1C. B*-semi-norms.

Definition. A B*-semi-norm on a Banach *-algebra is an algebra semi-norm p on A such that $p(a^*a) = p(a)^2$ for all $a \in A$.

Recently Bonsall [5] has studied B*-semi-norms on a general Banach *-algebra, proving the following results.

LEMMA 1.5. Let p be an algebra semi-norm on a Banach algebra A. Then for all $a \in A$

$$\lim_{n\to\infty} p(a^n)^{1/n} \leq \rho(a).$$

Proof. Let $J = \{a ; a \in A, p(a) = 0\}$. Then J is a two-sided ideal of A and q given by $q([a]) = p(a)$ is a well defined algebra norm on A/J, where $[a]$ denotes the coset containing a.

Now there exists $\beta \in sp([a])$ with

$$|\beta| = \lim_{n\to\infty} q([a]^n)^{1/n} = \lim_{n\to\infty} p(a^n)^{1/n}.$$

So $\beta \in sp(a)$ and $|\beta| \leq \rho(a)$ and the result follows.

LEMMA 1.6. Let p be a B*-semi-norm on Banach *-algebra A. Then

(i) $p(a^*) = p(a) \quad (a \in A)$

(ii) $p(h) \leq \rho(h) \quad (h \in A, h^* = h)$

(iii) $p(a) \leq \rho(a^*a)^{1/2} \quad (a \in A)$.

Proof. (i) $p(a)^2 = p(a^*a) \leq p(a)p(a^*)$. So $p(a) \leq p(a^*)$ and vice versa.

(ii) $p(h^2) = p(h)^2$. So $p(h) = \lim_n p(u^{2^n})^{1/2^n} \leq \varrho(h)$ by Lemma 1.5.

(iii) Apply (ii) to a^*a.

THEOREM 1.7. There exists a positive number M such that $p(a) \leq M\| a \|$ for all B*-semi-norms p and for all $a \in A$.

Proof. Let p be a B*-semi-norm. Let R denote the radical of A. Let $[a]$ denote the coset in A/R containing a. Since $sp(a^*) = \overline{sp(a)}$, R is a *-ideal so that A/R has a natural involution defined by $[a]^* = [a^*]$. Thus A/R is a semi-simple Banach *-algebra and, by Corollary 1.4, there exists a constant $M^2 > 0$ such that $\| [a]^* \| \leq M^2 \| [a] \|$.

Given $r \in R$ we have $r^*r \in R$ and so $\varrho(r^*r) = 0$ and $p(r) = 0$ by Lemma 1.6(iii). Thus a B*-semi-norm q can be defined on A/R by $q([a]) = p(a)$. By Lemma 1.6 applied to the B*-semi-norm q on A/R we now have $q([a])^2 \leq \varrho([a]^*[a]) \leq \| [a]^*[a] \| \leq \| [a^*] \| \| [a] \| \leq M^2 \| [a] \|^2 \leq M^2 \| a \|^2$. So $p(a) \leq M\| a \|$.

THEOREM 1.8. There exists a greatest B*-semi-norm $m(.)$ on A defined by $m(a) = \sup p(a)$ where the supremum is taken over all B*-semi-norms on A.

Proof. This is clear. Theorem 1.7 shows that the supremum exists.

1D. Special types of *-algebra.

Definition. A C*-algebra is a norm closed *-subalgebra of the bounded linear operators on a Hilbert space.

Definition. A B*-algebra is a Banach *-algebra on which $\|b^*b\| = \|b\|^2$ for all elements b.

Definition. An A*-algebra is a Banach *-algebra on which there is a B*-semi- norm which is a norm.

THEOREM 1.9. Every *-homomorphism of a Banach *-algebra into a B*-algebra is continuous.

Proof. Given a *-homomorphism β of A into a B*-algebra, define p on A by $p(a) = \|\beta(a)\|$. Then p is a B*-semi-norm and so by Theorem 1.7, p is continuous. Hence β is continuous.

THEOREM 1.10. Let $(A, \|.\|)$ be an A*-algebra with auxiliary B*-norm $|.|$. Then there exists $K > 0$ such that $|a| \leq K\|a\|$ for all $a \in A$.

Proof. $|.|$, being a B*-semi-norm, is continuous.

1E. *-representations and positive functionals.

B*-algebras are extremely tractable and much is known about them. For example see Rickart [21] or Dixmier [9]. The more B*-structure within a general Banach *-algebra, the more tools we can apply to obtain information about the algebra. The presence of B*-semi-norms is one manifestation of B*-structure. Another is the presence of representable linear functionals, which will be discussed in 1F. Here we discuss positive linear functionals, which are somewhat more general.

Definition. A linear functional F on A is called positive if and only if $F(a^*a) \geq 0$ for all $a \in A$.

Definition. A linear functional F on A is called hermitian if and only if $F(a^*) = \overline{F(a)}$ for all $a \in A$.

From both B*-semi-norms and positive linear functionals on A, we can construct *-homomorphisms from A into B*-algebras. The two methods of construction are indicated in Bonsall [5] and Rickart [21] respectively. We discuss the latter here. First, however, we need some preliminaries.

Definition. A *-representation of a Banach *-algebra A on a Hilbert space H is a *-homomorphism of A into the algebra of all bounded linear operators on H.

*-representations are always continuous by Theorem 1.9.

We now examine positive linear functionals.

LEMMA 1.11. Let F be a positive linear functional on a Banach $*$-algebra A. Then, for all $x, y \in A$,

(i) $F(x^*y) = \overline{F(y^*x)}$

(ii) $|F(x^*y)|^2 \leq F(x^*x)F(y^*y)$ [Cauchy-Schwartz inequality]

(iii) If A has a 1, F is hermitian.

Proof. (i) is immediate from the identity

$$4x^*y = (y + x)^*(y + x) - (y - x)^*(y - x) + i(y + ix)^*(y + ix) - i(y - ix)^*(y - ix).$$

(ii) follows by noting that $F((x + \alpha y)^*(x + \alpha y)) \geq 0$ where $\alpha = -F(y^*x)/F(y^*y)$. [We can without loss assume that $F(y^*y) > 0$.]

(iii) follows from (i) with $y = 1$.

Given a hermitian positive linear functional F on A, we now construct a $*$-representation of A by the standard Gelfand-Naimark procedure.

Let $L_F = \{a ; a \in A, F(a^*a) = 0\}$. Let $X_F = A - L_F$. Define an inner product on X_F by $([x],[y]) = F(y^*x)$. Complete X_F with respect to this inner product norm, thereby obtaining a Hilbert space Y_F. We can define, for each $a \in A$, a linear operator $S_a : X_F \rightarrow X_F$ by $S_a[x] = [ax]$ and this operator can be extended to a bounded linear operator $T_a : Y_F \rightarrow Y_F$ provided that

$$\sup_{x \in A} \frac{F(x^*a^*ax)}{F(x^*x)} < \infty . \qquad (1)$$

In Theorem 1.12 we shall show that (1) is true for all positive linear functionals F. Then the mapping $\beta : A \rightarrow B(Y_F)$ defined by $\beta(a) = T_a$ is a *-representation of A on the Hilbert space Y_F. Note that p defined on A by $p(a) = |T_a|$ is a B*-semi-norm on A.

<u>THEOREM 1.12</u>. Let F be a positive linear functional on a Banach *-algebra A. Let $h \in A$ such that $h^* = h$. Then for all $u \in A$,

$$|F(u^*hu)| \leq F(u^*u)\rho(h).$$

<u>Proof</u>. Choose $\varepsilon > 0$. Let $v = (\rho(h) + \varepsilon)^{-1}h$. Note that $v^* = v$ and $\rho(v) < 1$. Then,

$$F(u^*u) - F(u^*vu) = F(u^*(1 - v)u).$$

Now, in A_1, the algebra with identity element adjoined, $\rho(1 - (1 - v)) < 1$. Hence, by Corollary 1.2, there is a self-adjoint element $w \in A_1$ such that $w^2 = 1 - v$. Hence $F(u^*u) - F(u^*vu) = F(u^*w^*wu) = F((wu)^*(wu)) \geq 0$ since F is positive and $wu \in A$. Hence for all $\varepsilon > 0$

$$F(u^*hu) \leq (\rho(h) + \varepsilon)F(u^*u).$$

By applying the same argument to $-h$ we find that for all $\varepsilon > 0$,

$$-F(u^*hu) \leq (\rho(h) + \varepsilon)F(u^*u).$$

Hence $|F(u^*hu)| \leq F(u^*u)\rho(h)$.

1F. Representable linear functionals.

We now consider an important subset of the positive linear functionals —— the representable ones. We define these in a purely algebraic way, though Rickart [21] does not do this.

Definition. A linear functional F on Banach *-algebra A is called representable if and only if F is hermitian and there exists a constant $K > 0$ such that for all $a \in A$, $|F(a)|^2 \leq KF(a^*a)$. [Note that a representable linear functional is positive.]

Examples. On the Banach *-algebra ℓ^1 with pointwise operations and the involution $(\alpha_n)^* = (\overline{\alpha_n})$, the functional $(\alpha_n) \rightarrow \alpha_1$ is representable, while the functional $(\alpha_n) \rightarrow \sum_{n=1}^{\infty} \alpha_n$ is not.

THEOREM 1.13. Let A be a Banach *-algebra. Let F be a linear functional on A. Then the following three conditions are equivalent:

(1) F is representable.

(2) F can be extended as a positive linear functional to A_1, the algebra with identity element adjoined.

(3) There exists a *-representation $a \rightarrow T_a$ on Hilbert space H and a vector y in H such that for all $a \in A$, $F(a) = (T_a y, y)$.

Proof. We show (1) implies (2), (2) implies (3) and (3) implies (1).

(1) $\rightarrow$ (2). Suppose $|F(a)|^2 \leq K^2F(a^*a)$ for all $a \in A$. Define G on A_1 by

$$G(a + \alpha 1) = F(a) + \alpha K^2.$$

Then

$$\begin{aligned} G((a + \alpha 1)^*(a + \alpha 1)) &= F(a^*a + \bar{\alpha}a + \alpha a^*) + |\alpha|^2K^2 \\ &\geq F(a^*a) - 2|\alpha||F(a)| + |\alpha|^2K^2 \\ &\geq K^{-2}|F(a)|^2 - 2|\alpha||F(a)| + |\alpha|^2K^2 \\ &= (K^{-1}|F(a)| - |\alpha|K)^2 \\ &\geq 0. \end{aligned}$$

So G is a positive extension of F to A_1.

(2) $\rightarrow$ (3). Let F be a positive linear functional on A which can be extended to a positive linear functional G on A_1. Then we can construct a *-represntation of A_1 on Hilbert space H as above. Then $F(a) = G(a) = (T_a[1], [1])$ for all $a \in A$. So the restriction of the *-representation of A_1 to A is a *-representation of A on H. The vector $[1]$ in H has the property that $F(a) = (T_a[1], [1])$ as required.

(3) $\rightarrow$ (1). When (3) holds, F is clearly hermitian. Also $|F(a)|^2 = |(T_ay, y)|^2 \leq |T_ay|^2|y|^2 = |y|^2(T_ay, T_ay) = |y|^2(T_a{}^*T_ay, y) = |y|^2(T_{a^*}T_ay, y) = |y|^2(T_{a^*a}y, y) = |y|^2F(a^*a)$. Hence F is representable.

<u>Definition.</u> To every representable linear functional on a Banach *-algebra A we assign a number μ_F defined by

$$\mu_F = \sup_{x \in A} \frac{|F(x)|^2}{F(x^*x)} .$$

<u>Definition.</u> We denote the set of representable linear functionals F with $\mu_F \leq 1$ on Banach *-algebra A by P_A.

<u>THEOREM 1.14.</u> Let F be a representable linear functional on Banach *-algebra A. Then for all $a \in A$, $|F(a)| \leq \mu_F m(a)$ where $m(.)$ denotes the greatest B*-semi-norm on A.

<u>Proof.</u> Extend F to G on A_1 by taking $G(1) = \mu_F$ as in Theorem 1.13. Then construct the *-representation of A_1 on Hilbert space corresponding to G in the usual way. Then for $a \in A$,
$|F(a)| = |(T_a[1], [1])| \leq |T_a||[1]|^2 \leq m(a)|([1], [1])|$
since $|T_a|$ is a B*-semi-norm on A dominated by $m(a)$. However, $|([1], [1])| = G(1) = \mu_F$. Hence we deduce that $|F(a)| \leq \mu_F m(a)$.

We now state an important corollary of Theorem 1.14 that will be the key to many results in Chapter 2.

COROLLARY 1.15. Let F be a representable linear functional on a Banach *-algebra A. Then F is continuous.

Proof. $|F(a)| \leq \mu_F m(a)$ by Theorem 1.14

$\leq M\mu_F \|a\|$ by Theorem 1.7 for some constant $M > 0$.

Hence F is continuous.

We can deduce another corollary which is an improvement of a result in Rickart [21].

COROLLARY 1.16. P_A is norm bounded.

Proof. For all $F \in P_A$, we have

$$|F(x)| \leq \mu_F m(x) \leq m(x) \quad \text{for each } x \in A.$$

Hence by the Principle of Uniform Boundedness there exists a constant $K > 0$ such that $\|F\| \leq K$ for all $F \in P_A$.

Actually Bonsall, in his work on B*-semi-norms, has shown that P_A is, in fact, the set of positive linear functionals F such that $|F(x)| \leq m(x)$.

CHAPTER 2.

CONTINUITY AND REPRESENTABILITY OF POSITIVE LINEAR FUNCTIONALS.

2A. Introduction.

A crucial part of Banach algebra theory is the interplay between the algebraic and topological structures. Johnson's theorem is an example of this. Another example is Theorem 1.9, in which continuity is a consequence of algebraic conditions. Positive linear functionals on Banach *-algebras are defined algebraically and, for certain Banach *-algebras, that a linear functional is positive is sufficient to force the functional to be continuous or even representable. In this chapter, we survey the present state of knowledge of these matters.

In this introductory section we discuss the state of knowledge at the time when Rickart wrote his book [21] in 1960. Theorems 2.1, 2.2 and 2.3 do appear in the book but the proofs given here take advantage of Johnson's theorem.

Remarks about the origins of various results concerning the continuity of positive linear functionals, particularly on group algebras, can be found in Hewitt and Ross [14]. Other results concerned more with abstract Banach *-algebras can be found in Gelfand and Naimark [12] and Yood [28].

THEOREM 2.1. Let A be a Banach *-algebra with unit element 1. Then all positive linear functionals on A are representable and therefore continuous.

Proof. By Lemma 1.11, every positive linear functional on A is hermitian and $|F(x)|^2 \leq F(1)F(x^*x)$. So F is representable. Hence, by Corollary 1.15, F is continuous.

For each positive linear functional F on Banach *-algebra A and for each $u \in A$ we can define a linear functional F_u by

$$F_u(x) = F(u^*xu).$$

F_u is positive. If A has a unit element, every positive linear functional is obviously of the form F_u. The following lemma shows that, in the general case, the functionals F_u are analagous to the positive linear functionals on an algebra with unit element.

LEMMA 2.2. Let F be a positive linear functional on Banach *-algebra A. Let $u \in A$. Then F_u is representable and therefore continuous.

Proof. $F(u^*xu) = \overline{F(u^*x^*u)}$ by Lemma 1.11(i). Hence F_u is hermitian. Also

$$|F_u(x)|^2 \leq F(u^*u)F_u(x^*x) \quad \text{by Lemma 1.11(ii).}$$

Hence F_u is representable and, by Corollary 1.15, therefore continuous.

For B*-algebras all positive linear functionals are continuous and representable. We record this result in Theorem 2.3 but do not prove it. The proof, which depends heavily on the very special properties of B*-algebras [continuity of the involution, functional calculus for normal elements and the existence of a self-adjoint approximate identity], can be found in Rickart [21].

THEOREM 2.3. All positive linear functionals on a B*-algebra are continuous and representable. For each positive linear functional F we have $\mu_F \leq \|F\|$.

Proof. See Rickart [21; Theorem 4.8.15].

2B. Algebras A with $A^2 = A$.

In this section we generalise some work of Varopoulos about continuity of positive linear functionals. Before proceeding we need a definition.

Definition. Given a complex Banach *-algebra A, for each $k \geq 2$, let

$$A^k = \left\{ \sum_{i=1}^{n} x_{i1}x_{i2}\ldots x_{ik} \; ; \; x_{ij} \in A, \; n \geq 1 \right\}.$$

Theorems 2.1 and 2.3 remained the only general results on continuity of positive linear functionals until

1964 when two papers of Varopoulos [24] and [25] appeared in rapid succession. [24] deals with positive linear functionals on certain algebras A where $A^2 = A$, while [25] applies to algebras with an approximate identity.

The main theorem in [24] asserts that, in a commutative Banach *-algebra A with $A^2 = A$ and with a continuous involution, all positive linear functionals are continuous. The proof given in [24] draws heavily on measure theory, partially ordered vector spaces and the commutative character of the algebra — the use of the carrier space is indispensable. The continuity of the involution is used twice — once because Ford's square root lemma was not available in 1964 and once to ensure that the set of all self-adjoint elements of the algebra is closed.

We do not give Varopoulos's proof here. Instead we prove in Theorem 2.4 a stronger result for a general Banach *-algebra A in which $A^2 = A$ and an additional condition holds. The method we adopt is direct and gives some hope that all positive linear functionals on a Banach *-algebra (not necessarily commutative) such that $A^2 = A$ are continuous. In Corollary 2.5 we prove the Varopoulos result without the assumption of a continuous involution. Both Theorem 2.4 and Corollary 2.5 appear in Murphy [19].

Definition. Let F, G be positive linear functionals. F dominates G if and only if F - G is positive.

THEOREM 2.4. Let A be a complex Banach *-algebra. Let $A^2 = A$ and on A let every non-zero positive linear functional dominate a continuous non-zero positive linear functional. Then all positive linear functionals on A are continuous.

Proof. Let F be a non-zero positive linear functional on A. Every $x \in A$ is expressible as $\sum_{i=1}^{n} a_i b_i$ and therefore as $\sum_{i=1}^{n} \alpha_i x_i^* x_i$ $(\alpha_i \in \mathbb{C})$ by the identity

$$4ab = (b + a^*)^*(b + a^*) - (b - a^*)^*(b - a^*) + i(b + ia^*)^*(b + ia^*) - i(b - ia^*)^*(b - ia^*).$$

Hence linear functionals that agree on all elements x^*x are identical. .... [1]

Let

$$S = \{ G \, ; \, G \neq 0, \, G \text{ continuous, linear, positive and dominated by } F \}.$$

By hypothesis S is not empty. Define a relation $>$ on S by $G_1 > G_2$ if and only if G_1 dominates G_2. $>$ is then a partial ordering on S, anti-symmetry being a consequence of [1] above.

If T is a totally ordered subset of S, T is directed by $>$. We note that, for all $y \in A$, $\lim_{>} G(y^*y)$ $[G \in T]$ exists since $G(y^*y) \leq F(y^*y)$ for all $G \in T$. Also, if $x \in A$, we have an expression $x = \sum_{i=1}^{n} \alpha_i x_i^* x_i$ where $\alpha_i \in \mathbb{C}$, $x_i \in A$.

We can therefore define

$$\phi(x) = \lim_{>} G(x) \quad (G \in T).$$

ϕ is clearly a positive linear functional dominated by F.

For all $G \in T$ and all $x \in A$

$$|G(x)| = |G(\sum_{i=1}^{n} \alpha_i x_i^* x_i)| \leq \sum_{i=1}^{n} |\alpha_i| G(x_i^* x_i) \leq \sum_{i=1}^{n} |\alpha_i| F(x_i^* x_i).$$

Hence by the Principle of Uniform Boundedness, there exists $M > 0$ such that $\|G\| \leq M$ for all $G \in T$. Hence

$$|\phi(x)| = \lim_{>} |G(x)| \quad (G \in T)$$

$$\leq M\|x\| .$$

Therefore ϕ is continuous, $\phi \in S$ and ϕ is an upper bound for T. By Zorn's Lemma it follows that S has a maximal element G_0.

If $F - G_0 \neq 0$, then, by hypothesis, there exists a continuous non-zero positive linear functional G_1 such that $F - G_0 - G_1$ is positive. Hence $G_0 + G_1 \in S$ and $G_0 + G_1 > G_0$. By the maximality of G_0 we conclude that $G_1 = 0$. Hence $F - G_0 = 0$, $F = G_0$ and so F is continuous.

COROLLARY 2.5. Let A be a complex commutative Banach *-algebra such that $A^2 = A$. Then every positive linear functional on A is continuous.

Proof. let F be a non-zero positive linear functional on A. Suppose $F_u = 0$ for all $u \in A$. Then for all $u, x, y \in A$

$$|F(u^*x^*y)|^2 \leq F(u^*x^*xu)F(y^*y)$$
$$= 0.$$

Hence $F(A^3) = 0$ which is false since $A^3 = A$. Therefore we can choose u such that $F_u \neq 0$ and suppose $\rho(u^*u) < 1$. This is possible since $F_{\alpha u} = |\alpha|^2 F_u$. Then

$$\begin{aligned} F_u(x^*x) &= F(u^*x^*xu) \\ &= F(x^*u^*ux) \qquad \ldots(B) \\ &\leq F(x^*x)\rho(u^*u) \qquad \text{by Th. 1.12.} \\ &\leq F(x^*x). \end{aligned}$$

Hence F dominates F_u. The conditions of Theorem 2.4 are satisfied and the result follows.

Remarks.

1. In proving the result of Corollary 2.5 the commutativity of the algebra is used only at stage (B) above. To prove the analogue of Corollary 2.5 in the non-commutative case only requires that for every non-zero positive linear functional we can exhibit a continuous non-zero positive linear functional dominated by it. The results of 2C show that the theorem is indeed true for

a wide class of non-commutative algebras A with $A^2 = A$. However the proof in this case proceeds in a different manner altogether and even with the knowledge that F is continuous we have failed, in general, to recover a functional F_u dominated by F.

2. If A is commutative and instead of assuming that $A^2 = A$, we merely assume that A^3 is closed in A, an examination of the proofs shows that the restriction of F to A^3 is continuous.

In particular, if A is commutative and A^3 is closed and of finite codimension in A, then every positive linear functional on A is continuous. This follows from the fact that $A = A^3 + B$ where B is a finite dimensional subspace. Then $F|_{A^3}$ and $F|_B$ are both continuous. A^3 and B are both closed so that application of Lemma 1 of Johnson [17] yields that F is continuous.
Varopoulos [24] proved these results under the additional assumption of a continuous involution.

3. It has been pointed out to me by Professor F. F. Bonsall and by Dr. A. M. Sinclair of the University of the Witwatersrand that, coupling together slightly modified versions of Theorem 2.4 and Corollary 2.5, we can obtain partial results for non-commutative algebras A such that $A^2 = A$. The strongest result of this type, due to Bonsall, is :

Let A be a complex Banach *-algebra such that $A^2 = A$. Let Z be the centre of A [the set of all elements that commute with every element of A] and let F be a positive linear functional on A. Then there exists a continuous positive linear functional G on A such that $F|_{ZA} = G|_{ZA}$.

To prove this, note that either $F|_{ZA} = 0$, in which case the result is trivial, or $F|_{ZA} \neq 0$, in which case $F(a^*bz) \neq 0$ for some $a, b \in A$ and $z \in Z$. Hence, as in Corollary 2.5, we have $F_z \neq 0$ and we can assume $\|z^*z\| < 1$. Then F_z is non-zero, continuous, positive and is dominated by F. The proof of Theorem 2.4 now applies, except that the maximal element G_0 agrees with F only on ZA.

4. A commutative Banach *-algebra with approximate identity [to be defined in 2C] has $A^2 = A$, in virtue of a lemma of P.J.Cohen [7], so that Corollary 2.5 applies to all such algebras.

5. Whether $A^2 = A$ implies representability of all positive linear functionals on A is not known though a special case is treated in 2C.

6. Every positive linear functional on a Banach *-algebra A with $A^2 = A$ is hermitian. For if $h^* = h$ and $h = \sum_{i=1}^{n} a_i b_i$, we have $h = 1/2.(h + h^*)$ $= 1/2.(\sum_{i=1}^{n} (a_i b_i + b_i^* a_i^*))$. The result follows from Lemma 1.11(i).

2C. Algebras with approximate identity.

Definition. A Banach algebra A is said to contain a left approximate identity if there exists in A a subset $\{ e_\alpha ; \alpha \in D \}$ where D is a directed set such that

(i) there exists $M > 0$ such that $\|e_\alpha\| \leq M$ for all $\alpha \in D$.

(ii) $\lim_\alpha e_\alpha a = a$ for all $a \in A$.

[Right and two-sided approximate identities are defined similarly.]

The following important theorem was proved in 1959 by P.J.Cohen [7].

THEOREM 2.6. Let A be a Banach algebra with left approximate identity. Then for each $a \in A$ and for each $\varepsilon > 0$, there exist elements $b, c \in A$ such that (i) $a = bc$ (ii) $\|c - a\| < \varepsilon$ (iii) c belongs to the closed left ideal generated by a.

Proof. See Cohen [7].

In his second paper [25], Varopoulos proved the following significantly more powerful version of the Cohen theorem. This result was proved independently by Johnson [16]. Since 1964 many further generalisations of Theorem 2.7 have appeared. For a list of these see Hewitt and Ross [14].

THEOREM 2.7. Let A be a Banach algebra with left approximate identity. Let $\{a_n\}$ be a sequence of elements of A with $a_n \to 0$ as $n \to \infty$. Then there exists an element b and a sequence $\{c_n\}$ of elements of A such that $c_n \to 0$ as $n \to \infty$ and $a_n = bc_n$.

Proof. See Varopoulos [25].

Varopoulos used Theorem 2.7 to establish continuity of positive linear functionals on Banach *-algebras with two-sided approximate identity. His result is Theorem 2.8.

THEOREM 2.8. Let A be a Banach *-algebra with two-sided approximate identity. Then every positive linear functional on A is continuous.

Proof. Let F be a positive linear functional on A. Let $x_n \to 0$ as $n \to \infty$. Then by Theorem 2.7, $x_n = ab_n$ where $b_n \to 0$. Again by Theorem 2.7 for right approximate identity $b_n = c_n d$ where $c_n \to 0$. So $x_n = ac_n d$ where $c_n \to 0$. By the identity used in Theorem 2.4 ,

$$\begin{aligned} 4F(x_n) &= 4F(ac_n d) \\ &= F_{d+a^*}(c_n) - F_{d-a^*}(c_n) + iF_{d+ia^*}(c_n) \\ &\qquad - iF_{d-ia^*}(c_n). \end{aligned}$$

So $4|F(x_n)| \leq (\|F_{d+a^*}\| + \|F_{d-a^*}\| + \|F_{d+ia^*}\| + \|F_{d-ia^*}\|)\|c_n\|$, by Theorem 2.2. Now $c_n \to 0$ so that $F(x_n) \to 0$ as $n \to \infty$. Hence F is continuous by the Closed Graph Theorem.

The question of the representability of positive linear functionals on an algebra with two-sided approximate identity was discussed by Ford in his thesis [10]. His conclusion was that on a Banach *-algebra with two-sided approximate identity where <u>either</u> the involution is continuous <u>or</u> the approximate identity consists of self-adjoint elements, all positive linear functionals are representable. We now show that neither of these extra conditions is necessary to prove the result.

<u>THEOREM 2.9</u>. Let A be a Banach *-algebra with two-sided approximate identity. Then every positive linear functional on A is representable.

<u>Proof</u>. Let $\{e_\alpha ; \alpha \in D\}$ be the approximate identity. Let F be a positive linear functional on A. Suppose there exists a sequence $\{e_{\alpha_n}\}$ $(\alpha_n \in D)$ with $F(e_{\alpha_n} e_{\alpha_n}{}^*) \geq n^2$ for all $n \geq 1$. Then we have $F([n^{-1}e_{\alpha_n}][n^{-1}e_{\alpha_n}]^*) \geq 1$ for all $n \geq 1$. By Theorem 2.7, since $[n^{-1}e_{\alpha_n}] \to 0$ as $n \to \infty$, we can write $[n^{-1}e_{\alpha_n}] = yz_n$ where $z_n \to 0$ as $n \to \infty$. So

$$\begin{aligned} F([n^{-1}e_{\alpha_n}][n^{-1}e_{\alpha_n}]^*) &= F(yz_n z_n{}^* y^*) \\ &= F_{y^*}(z_n z_n{}^*) \\ &\leq F(yy^*) m(z_n z_n{}^*) \end{aligned}$$

by Th. 1.14 and Lemma 2.2

$$= F(yy^*)m(z_n)^2$$

So $F([n^{-1}e_{\alpha_n}][n^{-1}e_{\alpha_n}]^*) \leq MF(yy^*)\|z_n\|^2$ by Theorem 1.7 where M is some constant > 0. The left-hand side of this inequality always exceeds 1 while the right-hand side tends to zero as $n \to \infty$. This is a contradiction. Hence there exists a constant $K > 0$ such that $F(e_\alpha e_\alpha^*) \leq K$ for all $\alpha \in D$. Then

$$|F(x)|^2 = \lim_\alpha |F(e_\alpha x)|^2 \quad \text{[by Th. 2.8]}$$
$$\leq \lim_\alpha F(e_\alpha e_\alpha^*)F(x^*x)$$
$$\leq KF(x^*x).$$

By Remark 6 of section 2B and Theorem 2.6, F is hermitian. Hence F is representable.

2D. Future progress.

The outstanding question left unanswered by the results of this chapter is:

(α) Let A be a Banach *-algebra such that $A^2 = A$. Are all positive linear functionals on A continuous?

The obvious line of attack is to attempt a generalisation of the proof of Corollary 2.5. Unfortunately, it seems that, in general, we cannot find a non-zero positive linear functional F_u dominated by F. Even in the case of an algebra with a two-sided approximate identity, where we know, by Theorem 2.8, that all positive linear functionals are continuous, we have failed to prove retrospectively the existence of a non-zero F_u dominated by F.

Any proof, it would seem, must introduce continuity into the arena at some point and the proofs of Theorems 2.4, 2.8 and Corollary 2.5 do this using the functionals F_u, which are automatically continuous. Furthermore, the proof of Theorem 2.4 uses the Principle of Uniform Boundedness and that of Theorem 2.8, the Closed Graph theorem; any proof of an answer to (α) could therefore reasonably be expected to make use of a theorem of this power and type.

On the other hand, the Cohen/Varopoulos factorisation techniques of Theorems 2.6 and 2.7 have nothing inherently to do with positive linear functionals and a proof of (α) might not require these at all.

There is one other possible direction in which fruitful work might be done: it may be that Theorem 2.8 contains Corollary 2.5 in the sense that all commutative Banach *-algebras A with $A^2 = A$ have an approximate identity. This question does not appear to have been published. Be this as it may, in practice it is often obvious that $A^2 = A$ and Corollary 2.5 gives an easy proof of continuity, avoiding the Varopoulos factorisation theorem, which is rather intricate.

In the non-commutative case $A^2 = A$ does not force the algebra to contain an approximate identity as the following example shows.

Example 2.10. Let A be the set of all ordered pairs of 2 by 2 matrices of the type

$$(P\ ,\ Q)\ =\ \left(\begin{bmatrix} a & b \\ a & b \end{bmatrix},\begin{bmatrix} c & c \\ d & d \end{bmatrix}\right)$$

where a, b, c, d are complex numbers.

We take $\|(P\ ,\ Q)\| = 2\max(|a|,|b|,|c|,|d|)$ and define operations by

$$(P\ ,\ Q) + (R\ ,\ S) = ((P + R\ ,\ Q + S)$$

$$\alpha(P\ ,\ Q) = (\alpha P\ ,\ \alpha Q) \qquad (\alpha \in \mathbb{C})$$

$$(P\ ,\ Q).(R\ ,\ S) = (PR\ ,\ QS).$$

We have $A^2 = A$ for

$$\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix},\begin{bmatrix} c & c \\ d & d \end{bmatrix}\right) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} c & c \\ d & d \end{bmatrix}\right).\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right).$$

However the algebra does not contain an approximate identity. This can be checked by seeking a constant $M > 0$ and, for each $n \geq 1$, an element e_n of the form

$$\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix},\begin{bmatrix} c & c \\ d & d \end{bmatrix}\right)$$

with norm not exceeding M and such that $\|e_n x_n - x_n\| < 1$ where x_n is the element

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} n+1 & n+1 \\ -n & -n \end{bmatrix}\right).$$

This algebra can be given the involution $(P\ ,\ Q)^* = (Q^*,\ P^*)$

where P* denotes the transposed complex conjugate of P. Unfortunately there are no non-zero positive linear functionals on this algebra. This situation is discussed in section 3C.

CHAPTER 3.

ABUNDANCE OF POSITIVE AND REPRESENTABLE LINEAR FUNCTIONALS.

3A. Introduction.

In this chapter we consider the supply of positive and representable linear functionals on a general Banach *-algebra. As mentioned in 1E, representable linear functionals are a manifestation of B*-structure within a general Banach *-algebra.

3B. B*-algebras.

We first examine B*-algebras and find them handsomely endowed with representable linear functionals and conversely. These results, it appears, are originally due to Grothendieck [13].

THEOREM 3.1. Every continuous linear functional on a B*-algebra is a linear combination of four representable linear functionals.

Proof. See Grothendieck [13] or Dixmier [9].

We now prove a converse of Theorem 3.1. The original proof by Grothendieck [13] assumed continuity of the involution on the Banach *-algebra. We give a new proof which does not require this assumption.

<u>THEOREM 3.2.</u> Let A be a Banach *-algebra such that every continuous linear functional is a linear combination of representable linear functionals. Then there exists a B*-norm on A equivalent to the original norm. [i.e. the topology on A can be defined by a B*-norm.]

<u>Proof.</u> Let $S = \{ \hat{x} ; \hat{x} \in A^{**}, m(x) \leq 1 \}$ where $m(.)$ is the greatest B*-semi-norm. For each $f \in A^*$ we have $f = \sum_{i=1}^{n} \alpha_i F_i$ where F_i are representable.

So
$$|f(x)| = |\sum_{i=1}^{n} \alpha_i F_i(x)|$$
$$\leq \sum_{i=1}^{n} |\alpha_i||F_i(x)|$$
$$\leq \sum_{i=1}^{n} |\alpha_i| \mu_{F_i} m(x) \quad \text{by Th. 1.14.}$$

So
$$|\hat{x}(f)| \leq \sum_{i=1}^{n} |\alpha_i| \mu_{F_i} \quad \text{for all } \hat{x} \in S.$$

By the Principle of Uniform Boundedness there exists $C > 0$ such that $\|\hat{x}\| \leq C$ for all $\hat{x} \in S$. So $\|x\| \leq C$ for all x with $m(x) \leq 1$. Hence $\|x\| \leq Cm(x)$ for all $x \in A$. Suppose $m(x) = 0$. Then $\|x\| = 0$ and so $x = 0$. So $m(.)$ is an auxilliary B*-norm and the algebra is an A*-algebra.

However, by Theorem 1.10 we can find $K > 0$ such that $m(x) \leq K\|x\|$ for all $x \in A$. So
$$\|x\| \leq Cm(x) \leq CK\|x\| \quad \text{for all } x \in A.$$
Hence $m(.)$ is a B*-norm equivalent to $\|.\|$.

Remark. In Theorem 3.2 it is not enough to demand just that every continuous linear functional be a linear combination of positive linear functionals. Let A be a Banach *-algebra. Define a new multiplication $_{o}$ in A by taking $a_o b = 0$ for all $a, b \in A$. Every continuous linear functional on the Banach *-algebra so obtained is positive but the algebra is not a B*-algebra.

The following theorem is similar to Theorem 3.2.

THEOREM 3.3. Let A be a Banach *-algebra. Suppose that for every $f \in A^*$, there exists a representable linear functional F and an element b of A such that $f(a) = F(ab)$ for all $a \in A$. Then there exists a B*-norm on A equivalent to the original norm.

Proof. Let $S = \{\hat{a} \,;\, \hat{a} \in A^{**},\, m(a) \leq 1\}$. Then for all $f \in A^*$,

$|\hat{a}(f)| = |f(a)| = |F(ab)| \leq F(aa^*)^{1/2}F(b^*b)^{1/2}$

$\leq \mu_F^{1/2}m(a)F(b^*b) \leq \mu_F^{1/2}F(b^*b)$ for all $\hat{a} \in S$ by Theorem 1.14. Hence by the Principle of Uniform Boundedness we deduce that $\|\hat{a}\| \leq C$ for all $\hat{a} \in S$. Hence $\|a\| \leq Cm(a)$ for all $a \in A$ and the proof is completed as in Theorem 3.2.

3C. Algebras with no non-zero positive functionals.

At the other extreme from B*-algebras are algebras devoid of non-zero positive linear functionals. An algebra of this type is that given in Example 2.10. Here we give a simpler example.

Example 3.4. Let c_o denote the Banach algebra of all sequences convergent to zero with the norm $\|(\alpha_n)\| = \sup_n |\alpha_n|$ and product $(\alpha_n).(\beta_n) = (\alpha_n\beta_n)$. Define an involution on c_o by

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\ldots)^* = (\overline{\alpha_2},\overline{\alpha_1},\overline{\alpha_4},\overline{\alpha_3},\overline{\alpha_6},\overline{\alpha_5},\ldots).$$

Then the resulting Banach *-algebra has no positive linear functionals other than the zero functional. To see this note that the algebra has a two-sided approximate identity consisting of the elements e_1, $e_1 + e_2$, $e_1 + e_2 + e_3$, $e_1 + e_2 + e_3 + e_4$,..... where e_k denotes the k-th basis vector of c_o. By Theorems 2.8 and 2.9, we deduce that all positive linear functionals are continuous and representable. Suppose therefore that F is positive and non-zero. For each basis vector e_k,

$$|F(e_k)|^2 \leq \mu_F F(e_k{}^*e_k) = 0.$$

So $F(e_k) = 0$ and by the continuity of F we find that F is zero on A.

3D. Algebras without involutions.

It is worth noting briefly that there exist Banach algebras which cannot carry an involution — that is, they cannot be made into Banach *-algebras. We describe one example.

Example 3.5. Consider the algebra of all 2 by 2 matrices of the type $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$.

This is a Banach algebra under the usual operations and the norm $\left\| \begin{bmatrix} a & b \\ a & b \end{bmatrix} \right\| = 2 \max (|a|, |b|)$.

This algebra has a left identity element E where

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Suppose † is an involution on A. Then $EB^\dagger = B^\dagger$ for all $B \in A$. So $BE^\dagger = B$ for all $B \in A$. So $E^\dagger$ is a right identity element in A. Suppose then that

$$E^\dagger = \begin{bmatrix} x & y \\ x & y \end{bmatrix}.$$

Then $$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \begin{bmatrix} x & y \\ x & y \end{bmatrix} = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \quad \text{for all } a, b \in \mathbb{C}.$$

Hence $(a + b)x = a$ and $(a + b)y = b$ for all $a, b \in \mathbb{C}$. This is clearly impossible and A cannot carry an involution.

Algebras of this type might bear investigation.

3E. Positive linear mappings.

We begin with two definitions.

Definition. Let A be a Banach *-algebra. The positive wedge of A is the set of all non-negative linear combinations of elements a^*a where $a \in A$.

Definition. Let A, B be two Banach *-algebras. A positive linear mapping from A to B is a linear mapping from A to B which maps the positive wedge of A into the positive wedge of B.

Positive linear mappings between two B*-algebras have been intensively studied. Størmer [22] gave a comprehensive account of this work and his paper provides further references. Yood [29] considered the following generalisation of a question studied in Chapter 2:

Question. Given two Banach *-algebras A and B, what conditions on A and B force all positive linear mappings from A to B to be continuous?

Among other things, Yood proved that if A has a unit element and B is a B*-algebra, then all positive linear mappings from A to B are continuous. We shall prove a generalisation of this result in Theorem 3.7. Our method bears no relation to that used by Yood. Our first step is to prove a lemma about normed spaces.

LEMMA 3.6. Let X, Y be normed linear spaces. Let $G : X \rightarrow Y$ be a linear mapping. Then G is continuous if and only if $f \circ G$ is continuous for all $f \in Y^*$.

Proof. If G is continuous, then $f \circ G$ is continuous for all $f \in Y^*$.

Conversely, suppose $f \circ G$ is continuous for all $f \in Y^*$. Then

$$|(f \circ G)(x)| \leq \|f \circ G\|\|x\|.$$

$$\text{i.e. } |f(G(x))| \leq \|f \circ G\|\|x\|.$$

$$\text{i.e. } |\widehat{G(x)}(f)| \leq \|f \circ G\|\|x\|.$$

Let $S = \left\{ \widehat{G(x)} \ ; \ \widehat{G(x)} \in Y^{**}, \ \|x\| \leq 1 \right\}$.

Then $|\widehat{G(x)}(f)| \leq \|f \circ G\|$ for all $\widehat{G(x)} \in S$.

Hence by the Principle of Uniform Boundedness, since Y^* is complete, we have that there exists $K > 0$ such that

$$\|\widehat{G(x)}\| \leq K \quad \text{for all } \widehat{G(x)} \in S.$$

Hence

$$\|G(x)\| \leq K\|x\| \quad \text{for all } x \in X.$$

THEOREM 3.7. Let A be a Banach *-algebra on which all positive linear functionals are continuous and let B be a B*-algebra. Then any positive linear mapping $G : A \rightarrow B$ is continuous.

Proof. By Lemma 3.6 the result will be proved if we show that $f \circ G$ is continuous for all $f \in B^*$. Let $f \in B^*$. Then $f = \sum_{i=1}^{4} \alpha_i F_i$ where F_i is a positive linear functional on B by Theorem 3.1. Now, $f \circ G = \sum_{i=1}^{4} \alpha_i F_i \circ G$ where $\alpha_i \in \mathbb{C}$. However, for each i, $F_i \circ G$ is a positive linear functional on A and so is continuous. Hence $f \circ G$ is continuous and the result follows.

Remark. The conclusion of Theorem 3.7 is still true under less restrictive conditions. B may be any Banach *-algebra with the property that every continuous linear functional is a linear combination of positive ones. A noteworthy example is the algebra ℓ^1 with pointwise operations and involution $(\alpha_n)^* = (\overline{\alpha_n})$. Other results may be obtained by dualising.

CHAPTER 4.

OPERATORS THAT ACT COMPACTLY ON ALGEBRAS OF OPERATORS.

4A. Introduction.

Throughout this chapter and the next let $B(X)$ denote the Banach algebra of all bounded linear operators on the Banach space X. Let $R \in B(X)$. Then denote the norm of R by $|R|$.

Definition. Let $R \in B(X)$ where X is a Banach space. The centraliser of $R = \{S ; S \in B(X), SR = RS\}$ and is denoted by $Z(R)$.

Definition. An element r of Banach algebra A is said to act compactly on A if and only if the mapping $a \rightarrow ra$ $(a \in A)$ is a compact operator on A.

In what follows we shall be concerned with bounded linear operators on a Banach space X that act compactly on various algebras of operators, and, in particular, operators that act compactly on their centralisers. Actually, these matters have been discussed by Bonsall [3] and [4]. In [3] he proved that a compact linear operator on a Banach space acts compactly on its centraliser and used this result to develop the Riesz-Schauder theory for compact operators. In [4] he discussed compact action in detail.

Those of Bonsall's results which form the background to the work in this thesis are presented in Theorems 4.1 to 4.7 below.

THEOREM 4.1. Let T be a compact linear operator on a Banach space X. Then T acts compactly on $Z(T)$.

Proof. Let X_1 denote the closed unit ball in X and let $E = \overline{TX_1}$. Then E is norm compact. Given $S \in Z(T)$ with $\|S\| \leq 1$ we have $STX_1 = TSX_1 \subseteq TX_1$. Hence $SE \subset E$. Let $\{S_n\} \subseteq Z(T)$ where $\|S_n\| \leq 1$ for $n = 1, 2, 3\ldots$. Then for each $x \in E$, the set $\{S_n x ; n = 1, 2, 3\ldots\}$ is contained in the compact subset E of the Banach space X. Also

$$\|S_n(x_1 - x_2)\| \leq \|x_1 - x_2\| \quad (x_1, x_2 \in E, \ n = 1, 2\ldots)$$

Hence the mappings $x \rightarrow S_n x$ $(x \in E, n = 1, 2, 3\ldots)$ form an equicontinuous sequence of mappings of the compact space E into X. By Ascoli's theorem [8 ; Th. 7.5.7], it follows that there exists a subsequence $\{S_{n_k}\}$ such that $\{S_{n_k} x\}$ converges uniformly for $x \in E$. So $\{S_{n_k}Tx\}$ converges [...]nly for $x \in X_1$. So $\{S_{n_k} T\}$ converges with respect to the operator norm. Since $\{S_{n_k} T\} \subseteq Z(T)$ and $Z(T)$ is norm closed, this shows that $\{S_{n_k} T\}$ converges in $Z(T)$. This completes the proof.

As a partial converse to Theorem 4.1 we have Theorem 4.2.

THEOREM 4.2. Let T be a bounded linear operator on a Banach space X such that T acts compactly on its centraliser. Then T is Riesz.

Proof. See [4].

For the definition and properties of Riesz operators see West [26] and [27].

In the light of these last two theorems, Bonsall [4] proposed two problems:

Problem 1. Given a Riesz operator $T \in B(X)$, does there exist a subalgebra C of $B(X)$ containing I and T and an algebra-norm $\|.\|$ making C a Banach algebra on which T acts compactly?

Problem 2. Given $T \in B(X)$ that acts compactly on its centraliser, is T a compact linear operator on X?

In [4] partial results for both these problems were obtained. However, Problem 1 is still unsolved. We do not discuss it here since we have nothing to add to the account given in [4].

On the other hand, the question asked in Problem 2 is answered negatively in Chapter 5 of this thesis. However, though the answer, in general, is negative, some positive results can be obtained. Those proved in [4] are detailed in Theorems 4.3 to 4.7 below.

In these five theorems we consider the following situation. Let $T \in B(X)$, where X is a Banach space and let T act compactly on its centraliser. Note that, since from Theorem 4.2, T is Riesz, we can let $\{\alpha_n\}$ denote the sequence of non-zero eigenvalues of T, P_n the spectral projection corresponding to α_n and $S_n = P_1 + P_2 + \dots + P_n$. The following theorems then hold:

<u>THEOREM 4.3.</u> If there exists a sequence of compact linear operators that commute with T and converge weakly to I, then T is compact.

<u>THEOREM 4.4.</u> If there exists a sequence of compact linear operators that commute with T and converge weakly to T, then T^2 is compact.

<u>THEOREM 4.5.</u> If $\{S_n\}$ is bounded, then T is a fully decomposable Riesz operator. [i.e. $T = C + Q$ where C is compact, Q is quasi-nilpotent and $CQ = QC = 0$.]

<u>THEOREM 4.6.</u> If $\{S_n\}$ is bounded and $\{TS_n\}$ converges weakly to T, then T is compact.

THEOREM 4.7. If X is a Hilbert space and T is normal, then T is compact.

In the remainder of this chapter we concentrate on proving the further positive result that a linear operator on a Banach space that satisfies a polynomial identity is compact if and only if it acts compactly on its centraliser. To prove this we first consider another type of compact action altogether. This we do in section 4B.

4B. Two-sided compact action on B(X).

Let X be a Banach space. Let $R, S \in B(X)$. Let $R \wedge S$ denote the element of $B(B(X))$ defined by

$$(R \wedge S)(A) = RAS \qquad (A \in B(X)).$$

The compactness of the operator $R \wedge S$ on B(X) was studied by Vala [23]. His main result is Theorem 4.8; he proved it using his own version of the Ascoli theorem. We shall not give his proof here but we substitute a direct proof due to Bonsall, which uses the standard Ascoli theorem and part of which appears in [3]. Dr. T.A. Gillespie has also rendered me some assistance with this proof.

THEOREM 4.8. Let $R, S \in B(X)$ where X is a Banach space. The operator $R \wedge S$ on B(X) is compact if and only if both R and S are compact.

Proof. Suppose first that R and S are compact.

Let
$$U = \{x \; ; \; x \in X, \; \|x\| \leq 1\},$$
$$V = \{A \; ; \; A \in B(X), \; |A| \leq 1\},$$
$$W = \{x \; ; \; x \in X, \; \|x\| \leq |S|\},$$
$$E = \overline{S(U)},$$
$$F = \overline{R(W)}.$$

Both E and F are compact subsets of X. For $x \in U$ and $A \in V$, $ASx \in W$. So $RASx \in F$. For each $A \in V$, define $G_A : E \to X$ by $G_A(z) = RAz$. Consider the set $H = \{G_A \; ; \; |A| \leq 1\}$. Then

$$\|G_A(z_1 - z_2)\| = \|RA(z_1 - z_2)\| \leq |R|\,\|z_1 - z_2\|.$$

So H is equicontinuous. ....(1)

Also, for each $z \in E$,

$$H(z) = \{G_A(z) \; ; \; |A| \leq 1\} = \{RAz \; ; \; |A| \leq 1\} \subseteq F$$ which is compact. ....(2)

By Ascoli's theorem [8 ; Th. 7.5.7] we find that H is a relatively compact set of mappings. Let $\{A_{n_k}\}$ be a bounded sequence of operators in B(X). Then for all $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\sup_{z \in E} \|(RA_{n_k} - RA_{m_k})(z)\| < \varepsilon \quad \text{for all } n_k, m_k \geq N.$$

i.e. $\sup_{x \in U} \|(RA_{n_k}S - RA_{m_k}S)(x)\| < \varepsilon$ for all $n_k, m_k \geq N$.

i.e. $|RA_{n_k}S - RA_{m_k}S| \to 0$ as $n_k, m_k \to \infty$.

Hence $R \wedge S$ is compact.

Conversely suppose that $R \wedge S$ is a compact operator. Since $R \neq 0$ there exists $g \in X^*$ such that $(R^*g)(a) = 1$, for some $a \in X$.

Let $\{f_n\}$ be a bounded sequence in X^*. Then $\{a \otimes f_n\}$ is a bounded sequence in $B(X)$. Hence $R(a \otimes f_{n_k})S \rightarrow U$ for some subsequence $\{n_k\}$ as $n_k \rightarrow \infty$. So $S^*(a \otimes f_{n_k})^*R^* \rightarrow U^*g$ as $n_k \rightarrow \infty$. Now,

$$
\begin{aligned}
[S^*(a \otimes f_{n_k})^*R^*g](x) &= [(R(a \otimes f_{n_k})S)^*g](x) \\
&= g([R(a \otimes f_{n_k})S](x)) \\
&= g[R(a \otimes f_{n_k})(Sx)] \\
&= f_{n_k}(Sx)g(Ra) \\
&= (S^*f_{n_k})(x).(R^*g)(a) \\
&= (S^*f_{n_k})(x).
\end{aligned}
$$

Hence $S^*f_{n_k} = S^*(a \otimes f_{n_k})^*R^*g \rightarrow U^*g$ as $n_k \rightarrow \infty$. Hence S^* is compact, so that S is compact. This ends the proof.

Remarks.

1. Let A be a Banach algebra. Let $t \in A$. consideration of the compactness of the mapping $a \rightarrow tat$ $(a \in A)$ is fundamental to the work of Alexander [1] on compact Banach algebras.

2. The types of compact action studied by Bonsall and Vala are not wholly unrelated as section 4C shows.

4C. Operators that satisfy a polynomial identity — compact action on the centraliser.

An operator T is said to satisfy a polynomial identity if and only if $p(T) = 0$ for some polynomial $p(T)$ in T. Nilpotent and idempotent operators clearly satisfy polynomial identities. The purpose of this section is to prove that a bounded linear operator on a Banach space that satisfies a polynomial identity acts compactly on its centraliser if and only if it is compact. We need only prove the necessity : sufficiency follows from Theorem 4.1.

In the proof below we bring Vala's result [Th.4.8] to bear on the problem. It turns out that we can easily find non-trivial families of operators lying in the centraliser. The proof proceeds by a sequence of lemmas, which are tortuous but not deep. The first of these is in the same vein as Theorem 4.8.

LEMMA 4.9. Let $R, S \in B(X)$ where X is a Banach space. Suppose $R_\wedge S + S_\wedge R$ is a compact operator on $B(X)$ and suppose that R is a compact operator on X. Then S is a compact operator on X.

Proof. Let $\{x_n\}$ be a sequence in X with $\|x_n\| \leq 1$ for all $n \geq 1$. Choose $z \in X$ such that $Rz \neq 0$. Let f be a continuous linear functional on X such that $f(Rz) = 1$. Then define operators B_n of rank one on X by $B_n(y) = f(y)x_n$. Note that $|B_n| \leq \|f\|$ for all $n \geq 1$. So, by hypothesis $\{RB_nS + SB_nR\}$ has a convergent subsequence. Without loss of generality assume that $\{RB_nS + SB_nR\}$ converges to B. Hence $\{(RB_nS + SB_nR)z\}$ converges to Bz. Hence $R(B_nSz) + f(Rz)Sx_n \to Bz$ as $n \to \infty$. However $\{B_nSz\}$ is a bounded sequence in X and R is compact. So a subsequence $\{RB_{n_k}Sz\}$ converges as $n_k \to \infty$. Hence $\{Sx_{n_k}\}$ converges as $n_k \to \infty$ and so S is compact.

We now apply this last result in the next two lemmas.

LEMMA 4.10. Let $T \in B(X)$ where X is a Banach space. Let

$$T^p(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \dots (T - \alpha_n I)^{q_n} = 0$$

where $p \geq 1$, $q_i \geq 0$ and $n \geq 0$. Let T act compactly on $Z(T)$. Then $T(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \dots (T - \alpha_n I)^{q_n}$ is compact.

Proof. Let $Q = (T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \dots (T - \alpha_n I)^{q_n}$. Suppose as induction hypothesis that we have already proved that T^pQ, $T^{p-1}Q$, ..., $T^{p-s+1}Q$ are compact $(1 \leq s \leq p-1)$. We now prove that $T^{p-s}Q$ is compact.

Given $A \in B(X)$, define an operator $G(A) \in B(X)$ by $G(A) = T^{p-1}QAQT^{p-s-1} + T^{p-2}QAQT^{p-s} + \dots + T^{p-s-1}QAQT^{p-1}$.

Now $TG(A) = T^{p-1}QAQT^{p-s} + \dots + T^{p-s}QAQT^{p-1} = G(A)T$. Hence $G(A) \in Z(T)$ for all $A \in B(X)$.

Let $\{A_n\}$ be a norm bounded sequence in $B(X)$. Then $\{G(A_n)\}$ is a norm bounded sequence in $Z(T)$ so that, by the compact action of T on $Z(T)$, we have that $\{TG(A_n)\}$ has a convergent subsequence. So for some subsequence $\{n_k\}$, $\{T^{p-1}QA_{n_k}QT^{p-s} + \dots + T^{p-s}QA_{n_k}QT^{p-1}\}$ converges as $n_k \to \infty$. ......(1)

In the case when $s = 1$, (1) says that $T^{p-1}Q \wedge T^{p-1}Q$ is a compact operator on $B(X)$ and by Theorem 4.8 we deduce that $T^{p-1}Q$ is compact as required.

In the case when $s \geq 2$, we note that by induction hypothesis $T^pQ, T^{p-1}Q, \dots, T^{p-s+1}Q$ are all compact. So, by (s-2) applications of Theorem 4.8, it follows that in (1) $\{T^{p-2}QA_{n_k}QT^{p-s+1} + \dots + T^{p-s+1}QA_{n_k}QT^{p-2}\}$ has a convergent subsequence. ......(2)

From (1) and (2), we deduce that $\{T^{p-1}QA_{n_k}QT^{p-s} + T^{p-s}QA_{n_k}QT^{p-1}\}$ has a convergent subsequence. Hence $T^{p-1}Q \wedge T^{p-s}Q + T^{p-s}Q \wedge T^{p-1}Q$ is a compact operator on $B(X)$. However, by induction hypothesis, in this case $[s \geq 2]$, $T^{p-1}Q$ is compact. Hence by Lemma 4.9, $T^{p-s}Q$ is compact. This completes the induction step.

Finally, since $T^pQ = 0$, T^pQ is trivially compact.

The result follows by induction which automatically terminates after the step with $s = p-1$. [In this last step, in the definition of $G(A)$, we make the convention that $T^0 = I$.]

We note an immediate corollary of Lemma 4.10.

<u>COROLLARY 4.11.</u> A nilpotent bounded linear operator on a Banach space acts compactly on its centraliser if and only if it is compact.

<u>Proof.</u> Use Theorem 4.1 for sufficiency. Use Lemma 4.10 for necessity.

We now prove a lemma, analagous to Lemma 4.10, in which we work on the factors $(T - \alpha_k I)^{q_k}$ of the polynomial.

<u>LEMMA 4.12.</u> Let $T \in B(X)$ where X is a Banach space. Let

$$T^p(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \dots (T - \alpha_n I)^{q_n} = 0$$

where $p \geq 0$, $q_i \geq 0$ and $n \geq 0$. Let T act compactly on $Z(T)$. Then

$$T^p(T - \alpha_2 I)^{q_2}(T - \alpha_3 I)^{q_3} \dots (T - \alpha_n I)^{q_n}$$

is compact.

[N.B. Here we permit $p = 0$.]

<u>Proof.</u> The proof is similar to that of Lemma 4.10. Let $R = T^p(T-\alpha_2 I)^{q_2}(T-\alpha_3 I)^{q_3}\ldots(T-\alpha_n I)^{q_n}$. As induction hypothesis suppose that $(T-\alpha_1 I)^{q_1}R$, $(T-\alpha_1 I)^{q_1-1}R, \ldots, (T-\alpha_1 I)^{q_1-s+1}R$ $[1 \leq s \leq q_1]$ are all compact.

Given $A \in B(X)$, define $J(A) \in B(X)$ by

$$J(A) = (T-\alpha_1 I)^{q_1-1}RAR(T-\alpha_1 I)^{q_1-s} + (T-\alpha_1 I)^{q_1-2}RAR(T-\alpha_1 I)^{q_1-s+1} + \ldots\ldots + (T-\alpha_1 I)^{q_1-s}RAR(T-\alpha_1 I)^{q_1-1}.$$

Now
$$(T-\alpha_1 I)J(A) = \sum_{k=1}^{s-1}(T-\alpha_1 I)^{q_1-k}RAR(T-\alpha_1 I)^{q_1-s+k} = J(A)(T-\alpha_1 I).$$

Hence $J(A) \in Z(T)$.

Let $\{A_n\}$ be a norm bounded sequence in $B(X)$. $\{J(A_n)\}$ is then a norm bounded sequence in $Z(T)$ and

$$TJ(A_n) = \alpha_1 J(A_n) + (T-\alpha_1 I)^{q_1-1}RA_nR(T-\alpha_1 I)^{q_1-s+1} + \ldots\ldots + (T-\alpha_1 I)^{q_1-s+1}RA_nR(T-\alpha_1 I)^{q_1-1}.$$

$$\ldots\ldots\ldots(1)$$

By induction hypothesis and s-1 applications of Theorem 4.8 we deduce from (1) that $\{TJ(A_n) - \alpha_1 J(A_n)\}$ has a convergent subsequence, $\{TJ(A_{n_k}) - \alpha_1 J(A_{n_k})\}$. By the compact action of T on Z(T), $\{TJ(A_{n_k})\}$ has a convergent subsequence. Hence $\{J(A_{n_k})\}$ has a convergent subsequence.

Now in the case where s = 1 this tells us that $(T - \alpha_1 I)^{q_1-1} R \wedge (T - \alpha_1 I)^{q_1-1} R$ is a compact operator on B(X). Hence, by Theorem 4.8, $(T - \alpha_1 I)^{q_1-1} R$ is compact.

In the case where $s \geq 2$ we have to work harder to complete the induction step. In this case,

$$J(A_{n_k}) = (T - \alpha_1 I)^{q_1-1} RA_{n_k} R(T - \alpha I)^{q_1-s}$$
$$+ (T - \alpha_1 I)^{q_1-2} RA_{n_k} R(T - \alpha_1 I)^{q_1-s+1}$$
$$+ \dots + (T - \alpha_1 I)^{q_1-s} RA_{n_k} R(T - \alpha_1 I)^{q_1-1}. \qquad \dots\dots\dots\dots(2)$$

By induction hypothesis and s-2 applications of

Theorem 4.8, we deduce that, on the right-hand side of (2), the sum of all terms other than the first and the last has a convergent subsequence. Hence

$$\left\{ (T - \alpha_1 I)^{q_1 - 1} RA_{n_k} R(T - \alpha_1 I)^{q_1 - s} + (T - \alpha_1 I)^{q_1 - s} RA_{n_k} R(T - \alpha_1 I)^{q_1 - 1} \right\} \text{ has}$$

a convergent subsequence, that is,

$$(T - \alpha_1 I)^{q_1 - 1} R \wedge (T - \alpha_1 I)^{q_1 - s} R + (T - \alpha_1 I)^{q_1 - s} R \wedge (T - \alpha_1 I)^{q_1 - 1} R$$

is a compact operator on $B(X)$. Since $(T - \alpha_1 I)^{q_1 - 1} R$ is compact by induction hypothesis, we deduce, by Lemma 4.9, that $(T - \alpha_1 I)^{q_1 - s} R$ is compact, thereby completing the induction step.

Finally, $(T - \alpha_1 I)^{q_1} R$ is the zero operator, which is compact.

The result follows by induction, which terminates after the step with $s = q_1$. [In this last step, in defining $J(A)$, we take $T^0 = I$.]

Remark. Note that in Lemma 4.12 we suceed in removing the last factor $(T - \alpha_1 I)$ from the polynomial $T^p(T - \alpha_1 I)^{q_1} \ldots (T - \alpha_n I)^{q_n}$, though we fail to remove the last factor T in Lemma 4.10.

We are now in a position to prove the result we have been seeking.

THEOREM 4.13. Let $T \in B(X)$ where X is a Banach space. Let

$$T^p(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \ldots (T - \alpha_n I)^{q_n} = 0$$

where $p \geq 0$, $q_i \geq 0$ and $n \geq 0$. Let T act compactly on $Z(T)$. Then T is compact.

Proof. Let

$$R_0 = T(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \ldots (T - \alpha_n I)^{q_n},$$

$$R_1 = T^p(T - \alpha_2 I)^{q_2}(T - \alpha_3 I)^{q_3} \ldots (T - \alpha_n I)^{q_n},$$

$$R_2 = T^p(T - \alpha_1 I)^{q_1}(T - \alpha_3 I)^{q_3} \ldots (T - \alpha_n I)^{q_n},$$

....

$$R_n = T^p(T - \alpha_1 I)^{q_1}(T - \alpha_2 I)^{q_2} \ldots (T - \alpha_{n-1} I)^{q_{n-1}}.$$

Let D be the greatest common divisor of all these polynomials. D is I or T according as $p = 0$ or $p \geq 1$.

By a standard theorem of algebra there exist polynomials $v_0(T)$, $v_1(T), \ldots,$ $v_n(T)$ in T such that

$$R_0 v_0(T) + R_1 v_1(T) + \ldots + R_n v_n(T) = D.$$

By Lemmas 4.10 and 4.12, R_i $(0 \leq i \leq n)$ are all compact. Hence $R_i v_i(T)$ $(0 \leq i \leq n)$ are all compact. So D is compact.

In the case where $p \geq 1$, this proves that T is compact. If $p = 0$, we deduce I is compact. Hence X is finite dimensional and T is automatically compact. The result is therefore proved.

CHAPTER 5.

COMPACT ACTION ON THE CENTRALISER FOR WEIGHTED SHIFT OPERATORS ON ℓ^1.

5A. Basic results.

In this chapter we study the compact action of weighted shift operators on the Banach space ℓ^1 on their centralisers. This will yield an example which answers negatively the question asked in Bonsall's Problem 2, which was stated in section 4A. Some other interesting results about compact action will emerge too.

The simplicity of the centraliser of a weighted shift on ℓ^1 is an enormous advantage in studying Problem 2 for these operators. For a weighted shift T, $Z(T)$ is the strong operator topology closure of the set of all polynomials in T. Even more is true as we shall see in section 5B.

In the Banach space ℓ^1 of all absolutely convergent sequences of complex numbers with the norm $\|(\xi_n)\| = \sum_{n=1}^{\infty} |\xi_n|$, we let e_k denote the basis vector $(0, 0, \ldots, 1, 0, \ldots)$ where the 1 occurs in the k-th place. Note that, for all operators $Q \in B(\ell^1)$ we have $|Q| = \sup_m \|Qe_m\|$.

From now on, let T denote the weighted shift operator on ℓ^1 defined by $Te_m = \alpha_m e_{m+1}$ $(m \geq 1)$ where

$\{\alpha_m\}$ is a bounded sequence of non-zero complex numbers. We denote this operator T by subdiag$(1; \alpha_1, \alpha_2, \alpha_3, \ldots)$ from consideration of its matrix representation. The 1 denotes that the weights lie in the first subdiagonal of the matrix. Note that, for each $k \geq 1$ and $m \geq 1$, we have $T^k e_m = \alpha_m \alpha_{m+1} \ldots \alpha_{m+k-1} e_{m+k}$. Hence for each $k \geq 1$,

$$|T^k| = \sup_m |\alpha_m \alpha_{m+1} \ldots \alpha_{m+k-1}| .$$

In 5B we shall compute the centraliser of T. Section 5C deals with the compact action of T and this leads to non-compact operators that act compactly on their centralisers in 5D. Finally, in 5E, we consider a class of weighted shifts on ℓ^1 for which compact action on the centraliser is linked to compactness of the operator in a very special way. Much of the material in this chapter appears in Murphy [20].

5B. The centraliser of a weighted shift on ℓ^1.

We prove that the centraliser of T is the strong operator topology closure of the set of polynomials in T. We show further that the polynomials can be selected in a particular way.

<u>THEOREM 5.1.</u> Let $T = \text{subdiag}(1; \alpha_1, \alpha_2, \alpha_3, \ldots)$ on ℓ^1. Let $S \in Z(T)$. Then there exists a sequence of polynomials $p_n(T)$ such that, for all x, $p_n(T)x \to Sx$ as $n \to \infty$ and $|p_n(T)| \leq |S|$ for all $n \geq 1$.

<u>Proof.</u> We can assume that $S \neq 0$.

Let $Se_1 = \sum_{i=1}^{\infty} \beta_i e_i$. Now, for all $n \geq 1$, let

$$p_n(T) = \beta_1 I + \sum_{k=1}^{n} \frac{\beta_{k+1} T^k}{\alpha_1 \alpha_2 \ldots \alpha_k} .$$

Then $p_n(T)e_1 = \sum_{k=1}^{n+1} \beta_k e_k \to Se_1$ as $n \to \infty$.

Hence, for all $k \geq 1$, $T^k p_n(T)e_1 \to T^k Se_1$ as $n \to \infty$

and so $p_n(T)T^k e_1 \to ST^k e_1$ as $n \to \infty$.

Hence $\alpha_1\alpha_2\ldots\alpha_k p_n(T)e_{k+1} \to \alpha_1\alpha_2\ldots\alpha_k Se_{k+1}$ as $n \to \infty$

and so $p_n(T)e_{k+1} \to Se_{k+1}$ as $n \to \infty$. So for all $m \geq 1$, $p_n(T)e_m \to Se_m$ as $n \to \infty$.

$$\|p_n(T)e_m\| = \left\| \left(\beta_1 I + \sum_{k=1}^{n} \frac{\beta_{k+1} T^k}{\alpha_1 \alpha_2 \ldots \alpha_k} \right) e_m \right\|$$

$$= |\beta_1| + \sum_{k=1}^{n} |\beta_{k+1}| \frac{|\alpha_m \alpha_{m+1} \ldots \alpha_{m+k-1}|}{|\alpha_1 \alpha_2 \ldots \alpha_k|} .$$

So $\|p_n(T)e_m\|$ increases with n. However,

$\|p_n(T)e_m\| \to \|Se_m\|$ as $n \to \infty$. Hence

$\|p_n(T)e_m\| \leq \|Se_m\|$ for all $n \geq 1$. This holds for

all basis vectors e_m. Hence

$$\sup_m \|p_n(T)e_m\| \leq \sup_m \|Se_m\| \quad \text{for all } n \geq 1.$$

Therefore $|p_n(T)| \leq |S|$ for all $n \geq 1$.

Let $\varepsilon > 0$ and let $x = \sum_{k=1}^{\infty} \xi_k e_k$ be a non-zero

element of ℓ^1. Choose $M \geq 1$ such that

$$\sum_{k=M+1}^{\infty} |\xi_k| \leq (4|S|)^{-1}\varepsilon.$$

Then choose $N \geq 1$ such that for all m with $1 \leq m \leq M$

and for all $n \geq N$ we have

$$\|(p_n(T) - S)e_m\| \leq (2\|x\|)^{-1}\varepsilon.$$

Then for all $n \geq N$,

$$\|(p_n(T) - S)x\| \leq \left\| \sum_{m=1}^{M} \xi_m (p_n(T) - S)e_m \right\| + \left\| \sum_{m=M+1}^{\infty} \xi_m (p_n(T) - S)e_m \right\|$$

$$\leq \sum_{m=1}^{M} |\xi_m| \|(p_n(T) - S)e_m\| + \sum_{m=M+1}^{\infty} |\xi_m| (|p_n(T)| + |S|)$$

$$\leq (2\|x\|)^{-1}\varepsilon \cdot \sum_{m=1}^{M} |\xi_m| + (4|S|)^{-1}\varepsilon \cdot 2|S|$$

$$\leq \varepsilon .$$

This completes the proof.

5C. Compact action of certain weighted shifts on ℓ^1.

In this section we prove that a weighted shift $T = \text{subdiag}(1; \alpha_1, \alpha_2, \alpha_3, \ldots)$ on ℓ^1, with $\sum_{n=1}^{\infty} |T^{n+1}|/|T^n|$ convergent, acts compactly on its centraliser.

First, however, it is worth digressing to prove a result that shows that, on a Banach space X, operators $Q \in B(X)$, with $\sum_{n=1}^{\infty} |Q^{n+1}|/|Q^n|$ convergent, are not without interest.

LEMMA 5.2. Let $Q \in B(X)$ where X is a Banach space. Let $\sum_{n=1}^{\infty} |Q^{n+1}|/|Q^n|$ be convergent to S. Then Q is quasi-nilpotent.

Proof. If Q is nilpotent, the result is immediate.

If Q is not nilpotent, the inequality of the arithmetic and geometric means of n positive numbers yields

$$\left(\frac{|Q^2|}{|Q|}\cdot\frac{|Q^3|}{|Q^2|}\cdots\frac{|Q^{n+1}|}{|Q^n|}\right)^{1/n} \le \frac{1}{n}\left(\frac{|Q^2|}{|Q|}+\frac{|Q^3|}{|Q^2|}+\cdots+\frac{|Q^{n+1}|}{|Q^n|}\right).$$

Hence $|Q^{n+1}|^{1/n+1} \le |Q|^{1/n+1}\,[n^{-1}S]^{n/n+1}$

Hence $\lim_{n\to\infty} |Q^n|^{1/n} = 0$ as required.

We now return to compact action. For $R \in B(X)$, let $A(I, R)$ denote the norm closure of the set of polynomials in R in $B(X)$.

<u>LEMMA 5.3.</u> Let $T = \text{subdiag}(1\,;\,\alpha_1, \alpha_2, \alpha_3, \ldots)$ on ℓ^1. Let $|T| \le 1$ and let $\sum_{n=1}^{\infty} |T^{n+1}|/|T^n|$ be convergent. Then T acts compactly on $A(I, T)$.

<u>Proof.</u> Choose $\varepsilon > 0$. Let

$$E = \left\{p(T)\,;\,p(T) \text{ a polynomial in } T,\ |p(T)| \le 1\right\}.$$

Let U denote the unit ball in $A(I, T)$. We show that $T(U)$ is a totally bounded set. Let $p(T) \in E$, where

$$p(T) = \beta_0 I + \beta_1\frac{T}{|T|} + \beta_2\frac{T^2}{|T^2|} + \cdots + \beta_n\frac{T^n}{|T^n|}.$$

Now

$$\|p(T)e_m\| = \left\|\beta_0 e_m + \sum_{k=1}^{n} \beta_k \frac{\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}}{|T^k|} e_{m+k}\right\|$$

$$= |\beta_0| + \sum_{k=1}^{n} |\beta_k| \frac{|\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}|}{|T^k|}$$

$$\leq 1.$$

So $|\beta_0| \leq 1$ and $|\beta_k| \dfrac{|\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}|}{|T^k|} \leq 1$ for all $m \geq 1$. However, $|T^k| = \sup\limits_m |\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}|$.

Hence $|\beta_k| \leq 1$ for all $k \geq 0$.

Now choose $N \geq 1$ such that $\displaystyle\sum_{n=N}^{\infty} \frac{|T^{n+1}|}{|T^n|} < \frac{1}{2}\varepsilon$.

Choose a finite set B of complex numbers such that for every $z \in \mathbb{C}$ with $|z| \leq 1$, there exists $b \in B$ such that $|z - b| \leq (2N)^{-1}\varepsilon$. Let

$$F = \left\{ b_0 T + b_1 \frac{T^2}{|T|} + \cdots + b_{N-1} \frac{T^N}{|T^{N-1}|} \; ; \; b_i \in B \right\}.$$

F is a finite set. Let $K = F + \varepsilon U$. K is closed.

We show that for every polynomial $p(T) \in E$ we have

$Tp(T) \in K$. Let

$$p(T) = \beta_0 I + \beta_1 \frac{T}{|T|} + \beta_2 \frac{T^2}{|T^2|} + \cdots + \beta_n \frac{T^n}{|T^n|} .$$

Then $|\beta_k| \leq 1$ for all $k \geq 0$ as above . Choose

$c_0, c_1, \ldots , c_{N-1} \in B$ such that

$$|\beta_k - c_k| \leq (2N)^{-1}\varepsilon \qquad (0 \leq k \leq N-1).$$

It then follows that

$$\begin{aligned}
&\left|Tp(T) - c_0 T - c_1 \frac{T^2}{|T|} - c_2 \frac{T^3}{|T^2|} - \cdots - c_{N-1} \frac{T^N}{|T^{N-1}|}\right| \\
&\quad \leq |(\beta_0 - c_0)T| + \left|(\beta_1 - c_1)\frac{T^2}{|T|}\right| + \cdots \\
&\qquad + \left|(\beta_{N-1} - c_{N-1})\frac{T^N}{|T^{N-1}|}\right| \\
&\qquad + \sum_{k=N}^{n} |\beta_k| \frac{|T^{k+1}|}{|T^k|} \\
&\quad \leq N.(2N)^{-1}\varepsilon + \sum_{k=N}^{\infty} \frac{|T^{k+1}|}{|T^k|} \\
&\quad \leq \varepsilon .
\end{aligned}$$

Hence $Tp(T) \in K$ and $T(E) \subseteq K$. So $T(U) \subseteq K$ since K is closed. Hence $T(U)$ is totally bounded and $\overline{T(U)}$ is compact by [18 ; Theorem 7.6]. This shows that T acts compactly on $A(I, T)$. This completes the proof.

We now extend Lemma 5.3 to show that all operators of this type act compactly on their centralisers.

<u>THEOREM 5.4</u>. Let $T = \text{subdiag}(1 ; \alpha_1, \alpha_2, \alpha_3, \ldots)$ on ℓ^1. Let $|T| \leq 1$ and let $\sum_{n=1}^{\infty} |T^{n+1}|/|T^n|$ be convergent. Then T acts compactly on $Z(T)$.

<u>Proof</u>. Let $\{S_k\}$ be a sequence in $Z(T)$ with $|S_k| \leq 1$. By Theorem 5.1, we can choose polynomials $p_{kn}(T)$ such that, for all $x \in X$ and for all $k \geq 1$,

$$p_{kn}(T)x \rightarrow S_k x \quad \text{as } n \rightarrow \infty$$

$$\underline{\text{and}} \quad |p_{kn}(T)| \leq |S_k| \quad \text{for all } n \geq 1.$$

For each fixed k, $\{p_{kn}(T)\}$ is therefore a norm bounded sequence in $A(I, T)$. Application of Lemma 5.3 yields $V_k \in A(I, T)$ and a subsequence $\{p_{kn_r}(T)\}$ such that $|Tp_{kn_r}(T) - V_k| \rightarrow 0$ as $n_r \rightarrow \infty$. Hence, for all x, we have $Tp_{kn_r}(T)x \rightarrow V_k x$ as $n_r \rightarrow \infty$. However, for all x, we have $Tp_{kn_r}(T)x \rightarrow TS_k x$ as $n_r \rightarrow \infty$.

We conclude that $TS_k = V_k$ so that

$|Tp_{kn_r}(T) - TS_k| \rightarrow 0$ as $n_r \rightarrow \infty$. For each $k \geq 1$,

we can therefore choose an integer $N(k)$ such that

$$|Tp_{kN(k)}(T) - TS_k| < 2^{-k}.$$

The sequence $\{p_{kN(k)}(T)\}$ in $A(I, T)$ is norm bounded since $\{S_k\}$ is norm bounded. By Lemma 5.3, there exists a subsequence $\{p_{k_rN(k_r)}(T)\}$ and an operator L such that

$$|Tp_{k_rN(k_r)}(T) - L| \longrightarrow 0 \text{ as } k_r \rightarrow \infty.$$

Now,

$$\begin{aligned} |TS_{k_r} - L| &\leq |TS_{k_r} - Tp_{k_rN(k_r)}(T)| + |Tp_{k_rN(k_r)}(T) - L| \\ &\leq 2^{-k_r} + |Tp_{k_rN(k_r)}(T) - L| \\ &\rightarrow 0 \text{ as } k_r \rightarrow \infty. \end{aligned}$$

Hence T acts compactly on $Z(T)$.

5D. Non-compact operators that act compactly on their centralisers.

The results of 5C will now be used to exhibit non-compact operators on a Banach space that act compactly on their centralisers. This answers negatively the question raised in Bonsall's Problem 2, stated in section 4A.

Let V be defined on ℓ^1 by

$$V = \text{subdiag}\,(1\ ;\ a^{2^4}, 1, a^{3^4}, a^{3^4}, 1, a,$$
$$a^{4^4}, a^{4^4}, a^{4^4}, 1, a^2, a^2,$$
$$a^{5^4}, a^{5^4}, a^{5^4}, a^{5^4}, 1, a^3, a^3, a^3,$$
$$a^{6^4}, \ldots\ldots\ldots)$$

where $0 < a < 1$.

For this operator the weights are chosen as follows. The weights can be grouped into successive batches, the k-th batch containing $2k$ terms, of which the first k are all $a^{(k+1)^4}$; we call these <u>small</u> weights. The remaining k are $1, a^{k-1}, a^{k-1}, \ldots, a^{k-1}$ in that order and we call these <u>large</u> weights.

<u>LEMMA 5.5</u>. Let V be defined as above. Then $|V^k| = a^{(k-1)^2}$.

<u>Proof</u>. Let $V = \text{subdiag}\,(1\ ;\ \alpha_1, \alpha_2, \alpha_3, \ldots)$.

For each $k \geq 1$, $|V^k| = \sup_m \|V^k e_m\| = \sup_m \alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}$

$\geq \alpha_{k^2+1} \alpha_{k^2+2} \cdots \alpha_{k^2+k} = 1.a^{k-1}.a^{k-1}. \cdots .a^{k-1} = a^{(k-1)^2}.$

We now show that all other products $\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}$

fail to exceed $\alpha_{k^2+1} \alpha_{k^2+2} \cdots \alpha_{k^2+k}$.

Clearly, for $m \geq k^2 + 2$, we have

$\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1} \leq \alpha_{k^2+1} \alpha_{k^2+2} \cdots \alpha_{k^2+k}$.

Consider a product $\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}$ with $1 \leq m \leq k^2$.

Let n be the unique integer such that

$n^2 + 1 \leq k \leq (n + 1)^2$. Since α_{k^2} is the last small

weight in the k-th batch of weights it follows that the

product $\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1}$ contains at least one small

weight. Furthermore, since the small weights in successive

batches decrease, we can choose this one small weight so

as not to exceed the least small weight in the product

$\alpha_1\alpha_2 \ldots \alpha_k$. The least small weight in the product $\alpha_1\alpha_2 \ldots \alpha_k$ is $\alpha_{(n+1)^2}$ and hence since all the weights are less than or equal to 1, we have

$$0 < \alpha_m\alpha_{m+1} \ldots \alpha_{m+k-1} \leq \alpha_{n^2} = a^{(n+1)^4} \leq a^{(k-1)^2} .$$

Hence $|V^k| = a^{(k-1)^2}$.

THEOREM 5.6. V is not compact but acts compactly on its centraliser.

Proof. Note that $|V| = 1$ and that $\{Ve_{k^2+1}\}$ has no convergent subsequence, so that V is not compact. Also $|V^{k+1}|/|V^k| = a^{2k-1}$ and so by Theorem 5.4, V acts compactly on its centraliser.

Remarks.

1. The condition that $\sum_{n=1}^{\infty} |T^{n+1}|/|T^n|$ should be convergent is not necessary for a non-compact weighted shift $T = \text{subdiag}(1 ; \alpha_1, \alpha_2, \alpha_3, \ldots)$ to act compactly on its centraliser. For example, if we construct a new operator W from V by demanding that $0 < a < 1/2$ and replacing the large weights in the k-th batch by $1, \frac{1}{2}, \frac{1}{3}, \ldots , \frac{1}{k}$ for all $k \geq 1$, the operator W still

acts compactly on its centraliser though $|W^k| = 1/(k-1)!$. This can be proved by modifying the proof of Lemma 5.3 to exploit the fact that, in this case, the sum $|\beta_0| + |\beta_1| + \dots + |\beta_n|$ does not exceed 1. This follows because $T, T^2, T^3, \dots, T^p$ <u>all</u> attain their norms at the points e_{k^2+1} where $k \geq p$.

2. An operator $G \in B(X)$ is called <u>quasi-compact</u> if some power of G is compact. A non-compact weighted shift on ℓ^1 that acts compactly on its centraliser need not be quasi-compact. No power of the operator W , constructed in Remark 1, is compact.

3. From the above results it would seem a difficult task to characterise operators $G \in B(X)$ that act compactly on their centralisers, merely by studying their action on X.

5E. Weighted shifts on ℓ^1 with centraliser isomorphic to ℓ^1 .

In this final section, we record a rather attractive special case of the previous results of this chapter.

Here we consider the weighted shift $T = \text{subdiag}(1 ; \alpha_1, \alpha_2, \alpha_3, \dots)$ on ℓ^1, subject to the restriction that, for all $n \geq 1$ <u>and</u> for all $k \geq 1$, we have

$$|\alpha_1\alpha_2 \dots \alpha_n| \geq |\alpha_k\alpha_{k+1} \dots \alpha_{k+n-1}|. \qquad \dots .(*)$$

These conditions ensure that for all $n \geq 1$

$$|T^n| = |\alpha_1\alpha_2 \dots \alpha_n| .$$

[Recall that $A(I, T)$ denotes the norm closure of the polynomials in T.] When (*) holds we show that $Z(T)$ and $A(I, T)$ coincide.

<u>THEOREM 5.7</u>. Let $T = \text{subdiag}\,(1 ; \alpha_1, \alpha_2, \alpha_3, \dots)$ on ℓ^1. Let T satisfy (*) above. Then $Z(T) = A(I, T)$.

<u>Proof</u>. Let $S \in Z(T)$, $S \neq 0$ and let

$$Se_1 = \sum_{k=1}^{\infty} \beta_k e_k .$$

Let
$$p_n(T) = \beta_1 I + \sum_{k=1}^{n} \frac{\beta_{k+1} T^k}{\alpha_1\alpha_2 \dots \alpha_k} .$$

$$|p_n(T) - S| = \sup_m \|(p_n(T) - S)e_m\| \qquad (m \geq 1)$$

$$= \sup_m \frac{\|T^{m-1}(p_n(T) - S)e_1\|}{|\alpha_1\alpha_2 \dots \alpha_{m-1}|}$$

$$[\text{taking } T^0 = I]$$

$$\leq \sup_m \frac{|T^{m-1}|}{|\alpha_1\alpha_2 \dots \alpha_{m-1}|} \cdot \|(p_n(T) - S)e_1\|$$

$$= \|(p_n(T) - S)e_1\|$$

$$= \|\beta_{n+2}e_{n+2} + \beta_{n+3}e_{n+3} + \dots\|$$

$$= |\beta_{n+2}| + |\beta_{n+3}| + \dots$$

$$\to 0 \text{ as } n \to \infty \text{ since } (\beta_k) \in \ell^1.$$

COROLLARY 5.8. In Theorem 5.7, we have that
$|S| = \sum_{k=1}^{\infty} |\beta_k|$.

Proof. $$|S| = \sup_m \|Se_m\| \qquad (m \geq 1).$$

$$= \sup_m \frac{\| T^{m-1} Se_1 \|}{|\alpha_1 \alpha_2 \ldots \alpha_{m-1}|}$$

[taking $T^0 = I$]

$$\leq \sup_m \|Se_1\|$$

$$= \sum_{k=1}^{\infty} |\beta_k|.$$

Also however, $|S| \geq \|Se_1\| = \sum_{k=1}^{\infty} |\beta_k|$.

Hence $|S| = \sum_{k=1}^{\infty} |\beta_k|$ as required.

Theorem 5.7 and Corollary 5.8 permit the construction of an isomorphism between ℓ^1 and $Z(T)$.

LEMMA 5.9. Let T be defined as in Theorem 5.7. Then the mapping $G : \ell^1 \rightarrow Z(T)$ defined by

$$G((\beta_n)) = \beta_1 I + \sum_{k=1}^{\infty} \frac{\beta_{k+1} T^k}{\alpha_1 \alpha_2 \cdots \alpha_k}$$

is a norm preserving topological isomorphism of ℓ^1 onto $Z(T)$ such that

$$G(Tx) = TG(x) \qquad \text{for all} \quad x \in \ell^1.$$

Proof. Clearly G is linear, onto and $|G(x)| = \|x\|$ by Theorem 5.7 and Corollary 5.8. By Banach's Isomorphism Theorem, we deduce that G is a topological isomorphism. Also,

$$\begin{aligned} G(Tx) &= G(T(\beta_n)) \\ &= G(0, \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \ldots) \\ &= \frac{\alpha_1\beta_1}{\alpha_1} T + \frac{\alpha_2\beta_2}{\alpha_1\alpha_2} T^2 + \frac{\alpha_3\beta_3}{\alpha_1\alpha_2\alpha_3} T^3 + \ldots. \\ &= T(\beta_1 I + \beta_2 \frac{T}{\alpha_1} + \beta_3 \frac{T^2}{\alpha_1\alpha_2} + \ldots.) \\ &= T(G(\beta_n)) \\ &= TG(x). \end{aligned}$$

Lemma 5.9 allows us to resolve the question of compact action for weighted shifts on ℓ^1 satisfying (*).

THEOREM 5.10. Let T be as in Theorem 5.7. Then T acts compactly on $Z(T)$ if and only if T is compact.

Proof. Sufficiency follows from Theorem 4.1.

To prove necessity, let $\{x_n\}$ be a sequence in ℓ^1 with $\|x_n\| \leq 1$. $\{G(x_n)\}$, as defined in Lemma 5.9, is then a bounded sequence of operators in $Z(T)$ with $|G(x_n)| \leq 1$. Since T acts compactly on $Z(T)$, $\{TG(x_n)\}$ has a subsequence convergent to $S = G(a)$, say. Choose $\varepsilon > 0$. Then there exists $N \geq 1$ such that for all $n_k \geq N$,

$$|TG(x_{n_k}) - G(a)| < \varepsilon$$

$$\text{i.e. } |G(Tx_{n_k}) - G(a)| < \varepsilon$$

and so $\| Tx_{n_k} - a \| < \varepsilon$ by Lemma 5.9. Hence $\{Tx_n\}$ has a convergent subsequence and T is compact.

Remarks.

1. The analagous result to Theorem 5.10 for superdiagonal operators on c_0 was proved by J.Duncan and appears in Bonsall [2]. His result can be obtained by dualising and using Theorem 5.10.

2. Professor Bonsall has pointed out that Theorem 5.10 still holds if we weaken the hypothesis (*) by demanding only that there exists $M > 0$ such that, for all $n \geq 1$ and for all $k \geq 1$, we have

$$|\alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1}| \leq M |\alpha_1 \alpha_2 \ldots \alpha_n| .$$

3. Theorem 5.10 applies to the operator subdiag $(1 ; 1, \beta, 1, \beta^2, 1, \beta^3, \ldots.)$ where $0 < |\beta| < 1$. This is an example of a non-compact, quasi-compact, quasi-nilpotent operator which fails to act compactly on its centraliser.

4. Modified versions of the results of this section can be proved for "block shift operators" on ℓ^1 ; these are defined by infinite matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot \\ \alpha_1 I_p & 0 & 0 & 0 & \cdot & \cdot \\ 0 & \alpha_2 I_p & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \alpha_3 I_p & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \alpha_4 I_p & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where each 0 denotes a p by p zero matrix, I_p denotes the identity p by p matrix.

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