

**Some commutator properties of the
generalised wreath product.**

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

In the name of Allah, most gracious, most merciful.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Aminur Rahman Miah)

Abstract

The generalised wreath product of permutation groups, due to Dixon, Fournelle and Silcock, is studied in this thesis. Under nice conditions this turns out to be a good generalisation of the permutational wreath product. We will explain precisely what we mean by nice. The centre of the generalised wreath product is determined and we look at the centraliser of certain elements of the group. The remainder of the thesis is concerned with looking to answer the question: given a class \mathcal{X} , can we find necessary and sufficient conditions for the generalised wreath product to lie in \mathcal{X} ? We consider the class of abelian groups; nilpotent groups; locally nilpotent groups; ZA groups; residually nilpotent groups; locally boundedly nilpotent groups; bounded Engel groups; soluble groups; and locally soluble groups.

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Chapter 1

Introduction

“Situation number one; it’s the one that’s just begun.” *Jack Johnson*

Let A be group and X be a set. We say that (A, X) is a permutation group if A is a group of permutations acting on the set X .

Definition 1.0.1. Let (A, X) and (B, Y) be non trivial permutation groups. For each $y \in Y$, let $A^y := A$. Define $A^{(Y)} := \prod_{y \in Y} A^y$, the restricted direct product of copies of A indexed by the elements of Y . We consider an element $f \in A^{(Y)}$ as a function $f : Y \rightarrow A$. We define the permutational wreath product, denoted $A wr_Y B$, as the set of pairs fb , with $f \in A^{(Y)}$ and $b \in B$, acting on the set $X \times Y$ by $(x, y)fb := (xf(y), yb)$ for each $(x, y) \in X \times Y$. And the action is faithful.

Remark. It follows that $b^{-1}fb(y) := f(yb^{-1})$. This gives rise to a semidirect product $A^{(Y)}B$, which is in fact the wreath product $A wr_Y B$.

If we take $X := A$ and $Y := B$ in Definition 1.0.1, then the permutational wreath product $A wr_Y B$ is just the standard wreath product of A and B , denoted $A wr B$, using the right regular representation. Most of the research into wreath products has been done for the standard wreath product. For example, Hartley in [11] describes the residual nilpotence of the standard wreath product. However, the more natural version is the permutational wreath product and Meldrum in [16] deals, almost exclusively, with the permutational wreath product. Under nice conditions, the generalised wreath product is a generalisation of the permutational wreath product. We shall write wreath product to mean the permutational wreath product.

In this thesis we are mainly interested in extending some commutator results known about the wreath product to the generalised wreath product. The first examples of a generalised wreath product were obtained using towers of wreath products with suitable embeddings. If (G_1, X_1) , (G_2, X_2) and (G_3, X_3) are non trivial permutation groups, then $(G_1 \text{ wr}_{X_2} G_2) \text{ wr}_{X_3} G_3$ and $G_1 \text{ wr}_{X_2 \times X_3} (G_2 \text{ wr}_{X_3} G_3)$ are isomorphic. A proof of this can be found in Meldrum [16]. In light of this result, if $(G_1, X_1), \dots, (G_n, X_n)$ are non trivial permutation groups, we may form the iterated (permutational) wreath product, denoted $G_1 \text{ wr}_{X_2} \dots \text{ wr}_{X_n} G_n$, in the natural way. In fact, if p is prime then the Sylow p subgroups of $\text{Sym}(p^n)$ appear as iterated wreath products.

In 1962, P. Hall in [10] gave a generalisation which involved sets of groups indexed by a totally ordered set. P. Hall used this construction to construct a countably infinite characteristically simple group which is also a locally finite p group and has trivial Baer radical for some prime p . In 1977, Silcock in [25] and later, in 1984, Dixon and Fournelle in [4] generalised this construction to the case where the index set is a partially ordered set. Before we present the generalisation due to Dixon, Fournelle and Silcock we first define a connected partially ordered set and introduce some notation.

Definition 1.0.2. A partial order is a binary relation \leq over a set Λ which is reflexive, antisymmetric and transitive. If a set Λ is equipped with a partial order \leq , we say (Λ, \leq) is a partially ordered set. If in addition either $\lambda \leq \mu$ or $\mu \leq \lambda$ for each $\lambda, \mu \in \Lambda$, then we say \leq is a total order and (Λ, \leq) is a totally ordered set.

Definition 1.0.3. Let (Λ, \leq) be a partially ordered set. We say (Λ, \leq) is connected if for any $\lambda, \mu \in \Lambda$ there exists $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_1 = \lambda$, $\lambda_n = \mu$ and either $\lambda_i \leq \lambda_{i+1}$ or $\lambda_{i+1} \leq \lambda_i$ for each $i = 1, \dots, n - 1$.

Throughout the thesis, (Λ, \leq) will denote a connected partially ordered set containing at least two elements unless otherwise stated. With an abuse of notation, we may sometimes write just Λ instead of (Λ, \leq) .

Notation. Let G be a group. If $H = \{h_i : i \in I\} \subseteq G$ is a subset of G , then we write $\langle H \rangle$ or $\langle h_i : i \in I \rangle$ to denote the subgroup of G generated by H . If H is empty, we take $\langle H \rangle := \{1\}$.

We now introduce the generalised wreath product.

Definition 1.0.4. Let (Λ, \leq) be a (connected) partially ordered set and let (G_λ, X_λ) be a non trivial permutation group for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, choose $\iota_\lambda \in X_\lambda$ and write $\iota = (\iota_\lambda)_{\lambda \in \Lambda}$. We call ι the distinguished element. Let $X := \{x \in \prod_{\lambda \in \Lambda} X_\lambda : \sigma(x) \text{ is finite}\}$ where $\sigma(x) := \{\lambda \in \Lambda : x_\lambda \neq \iota_\lambda\}$. Fix $\mu \in \Lambda$ and let $x \in X$ and $g_\mu \in G_\mu$. Define $\bar{g}_\mu \in \text{Sym}X$ as follows

$$x\bar{g}_\mu := \begin{cases} x & \text{if } x_\lambda \neq \iota_\lambda \text{ for some } \lambda > \mu \\ y & \text{if } x_\lambda = \iota_\lambda \text{ for each } \lambda > \mu \end{cases}$$

where

$$y_\lambda := \begin{cases} x_\lambda & \text{if } \lambda \neq \mu \\ x_\mu g_\mu & \text{if } \lambda = \mu. \end{cases}$$

We write $\bar{G}_\mu := \langle \bar{g}_\mu : g_\mu \in G_\mu \rangle$, the group generated by the set $\{\bar{g}_\mu : g_\mu \in G_\mu\}$. We define the generalised (restricted) wreath product of (G_λ, X_λ) , $\lambda \in \Lambda$, by $wr_\iota\{G_\lambda : \lambda \in \Lambda\} := \langle \bar{G}_\lambda : \lambda \in \Lambda \rangle$.

Lemma 1.0.5. $\bar{G}_\lambda \cong G_\lambda$ as abstract groups for each $\lambda \in \Lambda$.

This is just Lemma 7.2.2 in [16]. Now we write G_λ instead of \bar{G}_λ where no confusion will arise.

Throughout the thesis (G_λ, X_λ) will denote non trivial permutation groups unless otherwise stated. We will normally denote the generalised wreath product $wr_\iota\{G_\lambda : \lambda \in \Lambda\}$ by W .

In chapter 2 we develop some basic structure results about the generalised wreath product. We form a canonical way to write an element of W . We explain how the generalised wreath product can be seen to be a generalisation of the wreath product. We find that we require the permutation groups (G_λ, X_λ) , $\lambda \in \Lambda$, to be transitive in order to obtain a reasonable generalisation. This is an undesirable aspect of the definition and makes dealing with the generalised wreath product when the groups (G_λ, X_λ) , $\lambda \in \Lambda$, are not transitive more difficult. We introduce the notion of constituents of a permutation group to overcome this difficulty. It will become evident that there is a clear dichotomy between the case where the groups (G_λ, X_λ) , $\lambda \in \Lambda$, are transitive and the case where they need not be transitive. The approach we adopt in the thesis reflects this. We will normally consider the two cases separately or at least mention the difference between the two cases. Throughout the thesis the groups (G_λ, X_λ) , $\lambda \in \Lambda$, will be assumed to be not necessarily transitive unless explicitly stated otherwise.

In the first part of chapter 3 we determine the centre of W and state some

immediate consequences of the result. In the second part we look at the centraliser of the element \bar{g}_μ as defined in Definition 1.0.4.

In chapter 4 we are concerned with the question: given a class of groups \mathcal{X} , can we find necessary and sufficient conditions for the generalised wreath product W to lie in \mathcal{X} ? We consider the class of abelian groups; nilpotent groups; locally nilpotent groups; ZA groups; residually nilpotent groups; locally boundedly nilpotent groups; bounded Engel groups; soluble groups; and locally soluble groups. The main aim of this chapter is to extend known results about the wreath product to the generalised wreath product. Work done on residually nilpotent wreath products has only been done for the standard wreath product by Hartley. So section 4.5 starts with a translation of the work done by Hartley in [11] to the case of the permutational wreath product. In section 4.6, we start by finding necessary and sufficient conditions for the wreath product of two non trivial permutation groups (A, X) and (B, Y) to be locally boundedly nilpotent. We extend this result to the generalised wreath product. In section 4.7 we consider bounded Engel wreath products. We are only able to find partial results to characterise the bounded Engel wreath product. As a result, we do not extend these results to the generalised wreath product.

In the case of abelian groups; nilpotent groups; locally nilpotent groups; soluble groups; and locally soluble groups we obtain full characterisations. In the other cases we obtain positive results when we impose certain conditions on the partially ordered set Λ or the permutation groups (G_λ, X_λ) , $\lambda \in \Lambda$.

1.1 Preliminary definitions and notation

Throughout the thesis (Λ, \leq) will denote a connected partially ordered set containing at least two elements and (G_λ, X_λ) will denote a non trivial permutation group for each $\lambda \in \Lambda$, except where explicitly stated otherwise. Below we list some definitions and notation which will be used in the thesis.

Definition 1.1.1. Let (A, X) be a permutation group and let $x \in X$. We define the orbit of X containing x to be the set $xA := \{xa : a \in A\}$.

Definition 1.1.2. Let (A, X) be a permutation group. We say (A, X) is transitive if for any $x, y \in X$, there exists $a \in A$ such that $xa = y$.

Remark. If (A, X) is transitive, then $xA = X$ for each $x \in X$.

Definition 1.1.3. Let (A, X) be a non trivial transitive permutation group. (A, X) is said to be regular if for any $x, y \in X$, there exists precisely one element $a \in A$ such that $xa = y$.

Definition 1.1.4. Let (A, X) be a permutation group and let $Y \subseteq X$. We say Y is a fixed block of A if $ya \in Y$ for each $y \in Y, a \in A$. Then each $a \in A$ induces a permutation on Y which we denote by $a|_Y$. We call the totality of $a|_Y$'s formed for all $a \in A$ the constituent $A(Y)$ of A on Y .

Remark. Notice that if (A, X) is a permutation group, then any orbit $Y \subseteq X$ of A is a fixed block and the permutation group $(A(Y), Y)$ is transitive. And we note that the mapping $\psi : A \rightarrow A(Y), a \mapsto a|_Y$ is an epimorphism and $\ker \psi = \{a \in A : ya = y \text{ for each } y \in Y\}$. Furthermore, the constituent $A(Y)$ is isomorphic to a quotient of A , namely $A(Y) \cong A/\ker \psi$.

Definition 1.1.5. $\Sigma \subseteq \Lambda$ is said to be a chain if Σ is a totally ordered subset of Λ .

Definition 1.1.6. $\gamma \in \Lambda$ is said to be a minimal element of Λ if $\lambda \in \Lambda$ and $\lambda \leq \gamma$ imply $\lambda = \gamma$. And $\omega \in \Lambda$ is said to be a maximal element of Λ if $\lambda \in \Lambda$ and $\lambda \geq \omega$ imply $\lambda = \omega$.

Notation. Throughout the thesis Γ will denote the set of all minimal elements of Λ and Ω will denote the set of all maximal elements of Λ .

Definition 1.1.7. We say $\omega \in \Lambda$ is 1-maximal if it is maximal. We denote the set of all 1-maximal elements of Λ by Ω_1 . Now if $i \in \mathbb{N}$, we say $\omega \in \Lambda$ is i -maximal if $\omega \leq \lambda$ implies $\lambda = \omega$ or $\lambda \in \Omega_{i-1}$, where Ω_{i-1} is the set of all $(i-1)$ -maximal elements.

Remark. We note that if $\mu \in \Omega_i$, then any chain which has μ as its minimal element contains at most i elements.

Definition 1.1.8. Let G be a group and let $g, h \in G$. We write $[g, h] := g^{-1}h^{-1}gh$ and we call this the commutator of g and h . Further, if $g_1, \dots, g_n \in G$, we write $[g_1, \dots, g_n] := [[g_1, \dots, g_{n-1}], g_n]$.

Notation. Let G be a group and let $g, h \in G$. We write $g^h := h^{-1}gh$.

Remark. If G is a group and $g, h, k \in G$, then $[g, hk] = [g, k][g, h]^k$ and $[gh, k] = [g, k]^h[h, k]$. These appear in Hall [9] and throughout this thesis we will refer to these identities as Hall's identities.

Notation. Let G be a group. Let $K, L \subseteq G$ be subgroups of G . Define $[K, L] := \langle [k, l] : k \in K, l \in L \rangle$ and define $K^L := \langle k^l : k \in K, l \in L \rangle$.

Notation. Let H be a group and let $K \subseteq H$ be a subgroup of H . If $h \in H$, we define $h^{-1}Kh := \langle h^{-1}kh : k \in K \rangle$.

Definition 1.1.9. Let G be group. We say that G has finite exponent if there exists $n \in \mathbb{N}$ such that $g^n = 1$ for each $g \in G$. The smallest such n is known as the exponent of G .

Definition 1.1.10. If G is a group, then the centre of G is $Z(G) := \{g \in G : [g, h] = 1 \text{ for each } h \in G\}$.

Definition 1.1.11. Let G be a group and let $g \in G$. We define the centraliser of g in G to be the subgroup $C_G(g) := \{h \in G : [g, h] = 1\}$.

Definition 1.1.12. A group G is said to be abelian if $[g, h] = 1$ for each $g, h \in G$.

Definition 1.1.13. A group G is said to be nilpotent if there exists $c \in \mathbb{N}$ such that $[g_0, \dots, g_c] = 1$ for any $g_0, \dots, g_c \in G$. The smallest such c is known as the nilpotency class of G .

Definition 1.1.14. Let G be a group. Let $\gamma_0(G) := G$ and define inductively $\gamma_n G := [G, \gamma_{n-1}(G)]$. Then, for each $n \in \mathbb{N} \cup \{0\}$, $\gamma_n(G)$ is a normal subgroup of G and the series $G = \gamma_0(G) \supseteq \dots \supseteq \gamma_n(G) \supseteq \dots$ is known as the lower central series of G .

Remark. If G is a group, then an equivalent definition for G to be nilpotent is: G is nilpotent if and only if G has a finite lower central series. That is to say, there exists $n \in \mathbb{N}$ such that $\gamma_n(G) = \{1\}$.

Definition 1.1.15. A group G is said to be locally nilpotent if every finitely generated subgroup of G is nilpotent.

Definition 1.1.16. Let G be a group. We say a set of normal subgroups $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ of G is an ascending central series if $G_\alpha \subseteq G_{\alpha+1}$ with $G_{\alpha+1}/G_\alpha \subseteq Z(G/G_\alpha)$ for each $\alpha \in \mathcal{A}$ and $G_\beta = \cup_{\alpha < \beta} G_\alpha$ if $\beta \in \mathcal{A}$ is a limit ordinal.

Definition 1.1.17. A group G is said to be a ZA group if there exists an ascending central series, $\{G_c\}_{c \in \mathcal{C}}$, terminating at G .

Remark. We note that if $g \in G_c$ for some ordinal c and $h \in G$, then there exists an ordinal c' with $c' < c$ or $c' = 0$ such that $[g, h] \in G_{c'}$.

Definition 1.1.18. We say that a group G satisfies the sequence property if for each sequence $(g_i)_{i \in \mathbb{N}} \subseteq G$ there exists an integer $n \in \mathbb{N}$ such that $[g_1, \dots, g_n] = 1$.

Remark. Definition 1.1.17 and Definition 1.1.18 are equivalent. This will be shown in Chapter 4.4.

Definition 1.1.19. Let \mathcal{X} be a class of groups. A group G is said to be residually a \mathcal{X} group if for each $g \in G \setminus \{1\}$, there exists a normal subgroup N of G such that $g \notin N$ and the quotient group G/N is a \mathcal{X} group.

Remark. Clearly G is residually nilpotent if and only if $\bigcap_{n \in \mathbb{N}} \gamma_n(G) = \{1\}$, as is mentioned in Hartley [11]. Here $\gamma_n(G) := \langle [g_0, \dots, g_n] : g_0, \dots, g_n \in G \rangle$.

Definition 1.1.20. We say a group G is locally boundedly nilpotent if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every n -generator subgroup of G is nilpotent of nilpotency class at most $f(n)$. We call f an lbn function for G .

Remark. We note that the lbn function is not unique. For example, if G is locally boundedly nilpotent with lbn function f then f' is also a lbn function for G , where $f' : \mathbb{N} \rightarrow \mathbb{N}$ with $f'(n) \geq f(n)$ for each $n \in \mathbb{N}$.

Notation. Let G be a group and let $g, h \in G$. We define $[g, h] := [g, h]$ and inductively we define $[g, n h] := [[g, n-1 h], h]$:

Definition 1.1.21. We say a group G is a bounded Engel group if there exists $n \in \mathbb{N}$ such that $[g, n h] = 1$ for each $g, h \in G$. The smallest such n is known as the Engel class of G .

Definition 1.1.22. Let $n \in \mathbb{N}$. We say a group G is n Engel if $[g, n h] = 1$ for each $g, h \in G$. In particular G is 1 Engel if and only if G is abelian.

Notation. Let G be a group. We define subgroups of G by the rule $\delta_0(G) := G$ and inductively we define $\delta_n(G) := [\delta_{n-1}(G), \delta_{n-1}(G)]$.

Definition 1.1.23. A group G is said to be soluble if there exists $n \in \mathbb{N}$ such that $\delta_n(G) = \{1\}$. The smallest such n is known as the solubility class of G .

Definition 1.1.24. A group G is said to be locally soluble if every finitely generated subgroup of G is soluble.

Definition 1.1.25. We introduce an alphabet \mathcal{A} of letters x_1, x_2, \dots and denote by A_∞ the free group freely generated by \mathcal{A} .

1. A word is an element of A_∞ .
2. If G is a group and $\psi := \mathcal{A} \rightarrow \mathcal{A}\psi \subseteq G$ is a mapping of the free generators of A_∞ into G , then we can extend ψ uniquely to a homomorphism $\psi : A_\infty \rightarrow G$. The word w is a law in G if $w\psi = 1$ for each homomorphism ψ from A_∞ to G .

3. A variety of groups is the class of all groups satisfying each one of a given (finite) set of laws.

Remark. In this thesis we assume a variety is defined by a finite set of laws. We note that the class of abelian groups is a variety. This class is defined by the law $[x_1, x_2]$. Moreover, the class of nilpotent groups with nilpotency class at most n is defined by the law $[x_1, \dots, x_{n+1}]$ and is hence a variety. Also, the class of soluble groups with solubility class at most m is a variety.

Chapter 2

Some basic properties

“Situation number two; it’s the only chance for you.” *Jack Johnson*

In this chapter, we develop some basic structure results about the generalised wreath product W . In the first part of this chapter we form a canonical way of expressing an element of W and consider some special normal subgroups of W . We show that, if p is a fixed prime, the generalised wreath product of finitely many p groups of finite exponent is again a p group of finite exponent. In the second part, we explain how W can be seen to be a good generalisation of the wreath product.

2.1 Some structure results

First we state a useful lemma about partially ordered sets.

Lemma 2.1.1. *Every partial order on a set can be extended to a total order.*

This is an unpublished result of Banach, Kuratowski and Tarski, which appears in Szpilrajn [27]. We now state a couple of useful theorems which gives us a nice way to express an element of W .

Theorem 2.1.2. *Let λ, μ be two unrelated elements of a partially ordered set Λ . Then $[G_\lambda, G_\mu] = \{1\}$.*

For a proof see Meldrum [16] 7.4.9.

Theorem 2.1.3. *Let (Λ, \leq) be a partially ordered set and let (G_λ, X_λ) be non trivial permutation groups for each $\lambda \in \Lambda$. We can extend, using Lemma 2.1.1, (Λ, \leq) to a total order (Λ, \preceq) . Then every element $g \in W$ can be expressed uniquely in the form $g = g_1 \dots g_n$ where $g_i \in G_{\lambda_i}^{(G_\lambda: \lambda > \lambda_i)}$ for each $i = 1, \dots, n$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_1 < \dots < \lambda_n$.*

For a proof see Meldrum [16] 7.4.14. Now fix $\mu \in \Lambda$. We define an equivalence relation \sim_μ on X by setting $x \sim_\mu y$ if and only if $x_\lambda = y_\lambda$ for each $\lambda > \mu$. It is clear that this is an equivalence relation. We denote the equivalence class of \sim_μ containing x by $[x]_\mu$. That is $y \in [x]_\mu$ if and only if $x_\lambda = y_\lambda$ for each $\lambda > \mu$. We note that $[x]_\mu = X$ if μ is maximal. And we denote the set of all equivalence classes of \sim_μ by X_{\sim_μ} .

Lemma 2.1.4.

1. If $\mu \in \Lambda$, $g \in G_\mu$ and $h, k \in \langle G_\lambda : \lambda > \mu \rangle$ with $\iota h = \iota k$, then $g^h = g^k$.
2. If $\mu \in \Lambda \setminus \Omega$, then $[h_1^{-1}G_\mu h_1, h_2^{-1}G_\mu h_2] = \{1\}$ where $h_1, h_2 \in \langle G_\lambda : \lambda > \mu \rangle$ with $\iota h_1 \neq \iota h_2$.
3. If $\mu_1, \mu_2 \in \Lambda$ with μ_1 and μ_2 unrelated, then $[G_{\mu_1}^{(G_\lambda: \lambda > \mu_1)}, G_{\mu_2}^{(G_\lambda: \lambda > \mu_2)}] = \{1\}$.

Proof. 1. Suppose $\iota h = x$. Let $y \in X$. Suppose $y \in [x]_\mu$. Fix $\omega > \mu$, then $x_\lambda = y_\lambda$ for each $\lambda \geq \omega$. So $(yh^{-1})_\omega = (xh^{-1})_\omega = \iota_\omega$ and also $(yk^{-1})_\omega = (xk^{-1})_\omega = \iota_\omega$. Thus

$$\begin{aligned}
yh^{-1}gh &= \begin{pmatrix} \iota_\lambda & \text{if } \lambda > \mu \\ y_\mu & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix} gh \\
&= \begin{pmatrix} \iota_\lambda & \text{if } \lambda > \mu \\ y_\mu g & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix} h \\
&= \begin{pmatrix} y_\lambda & \text{if } \lambda > \mu \\ y_\mu g & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix} \\
&= \begin{pmatrix} \iota_\lambda & \text{if } \lambda > \mu \\ y_\mu g & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix} k \\
&= \begin{pmatrix} \iota_\lambda & \text{if } \lambda > \mu \\ y_\mu & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix} gk \\
&= yk^{-1}gk.
\end{aligned}$$

Here $\begin{pmatrix} \iota_\lambda & \text{if } \lambda > \mu \\ y_\mu & \text{if } \lambda = \mu \\ y_\lambda & \text{otherwise} \end{pmatrix}$ denotes the element $z \in X$ such that $z_\lambda = \iota_\lambda$ if $\lambda > \mu$ and $z_\lambda = y_\lambda$ if $\lambda \not> \mu$. This notation will be used throughout the thesis.

Now suppose $y \notin [x]_\mu$. That is to say $y_\lambda \neq x_\lambda$ for some $\lambda > \mu$. Since $h \in \langle G_\lambda : \lambda > \mu \rangle$, it follows that $(yh^{-1})_\lambda \neq \iota_\lambda$ for some $\lambda > \mu$, otherwise $zh^{-1} = \iota = xh^{-1}$ where $z_\lambda := y_\lambda$ if $\lambda > \mu$ and $z_\lambda = \iota_\lambda$ if $\lambda \not> \mu$. Thus $(yh^{-1})g = yh^{-1}$. Similarly, $(yk^{-1})_\lambda \neq \iota_\lambda$ for some $\lambda > \mu$ and $(yk^{-1})g = yk^{-1}$. Hence

$$yh^{-1}gh = ((yh^{-1})g)h = (yh^{-1})h = y$$

and

$$yk^{-1}gk = ((yk^{-1})g)k = (yk^{-1})k = y.$$

This completes the proof.

2. Let $g_1, g_2 \in G_\mu$. We show that $[g_1^{h_1}, g_2^{h_2}] = 1$. Let $x \in X$. We note that $g_1^{h_1}$, $g_2^{h_2}$ and their inverses only affect the μ component of x non trivially. It follows that $x \in [\iota h_i]_\mu \Leftrightarrow x(g_j^{h_j})^{\pm 1} \in [\iota h_i]_\mu$ for $i, j \in \{1, 2\}$. Also, since $\iota h_1 \neq \iota h_2$ and $h_1, h_2 \in \langle G_\lambda : \lambda > \mu \rangle$ we have that $[\iota h_1]_\mu \neq [\iota h_2]_\mu$. We have three cases.

(i) If $x \in [\iota h_1]_\mu$ and $x \notin [\iota h_2]_\mu$, then

$$\begin{aligned} x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= \begin{pmatrix} x_\mu g_1^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} (g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\ &= \begin{pmatrix} x_\mu g_1^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} g_1^{h_1}g_2^{h_2} \\ &= xg_2^{h_2} \\ &= x. \end{aligned}$$

(ii) If $x \in [\iota h_2]_\mu$ and $x \notin [\iota h_1]_\mu$, then

$$\begin{aligned} x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= x(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\ &= \begin{pmatrix} x_\mu g_2^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} g_1^{h_1}g_2^{h_2} \\ &= \begin{pmatrix} x_\mu g_2^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} g_2^{h_2} \\ &= x. \end{aligned}$$

(iii) If $x \in X \setminus ([\iota h_1]_\mu \cup [\iota h_2]_\mu)$, then

$$\begin{aligned} x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= x(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\ &= xg_1^{h_1}g_2^{h_2} \\ &= xg_2^{h_2} \\ &= x. \end{aligned}$$

And the result follows.

3. Let $g_1 \in G_{\mu_1}$, $h_1 \in \langle G_\lambda : \lambda > \mu_1 \rangle$, $g_2 \in G_{\mu_2}$ and $h_2 \in \langle G_\lambda : \lambda > \mu_2 \rangle$. We shall show $[g_1^{h_1}, g_2^{h_2}] = 1$. Firstly we show

(i) $x \in [\iota h_1]_{\mu_1} \Leftrightarrow x g_1^{h_1} \in [\iota h_1]_{\mu_1}$; and

(ii) $x \in [\iota h_1]_{\mu_1} \Leftrightarrow x g_2^{h_2} \in [\iota h_1]_{\mu_1}$.

(i) We note that $g_1^{h_1}$ only affects the μ_1 component of x non trivially. Thus $x \in [\iota h_1]_{\mu_1} \Rightarrow x g_1^{h_1} \in [\iota h_1]_{\mu_1}$. For the reverse implication, we note that $(g_1^{h_1})^{-1}$ only affects the μ_1 component of x non trivially. Thus $x g_1^{h_1} \in [\iota h_1]_{\mu_1} \Rightarrow x = x g_1^{h_1} (g_1^{h_1})^{-1} \in [\iota h_1]_{\mu_1}$

(ii) We note that $g_2^{h_2}$ only affects the μ_2 component of x non trivially. Since μ_1, μ_2 are unrelated, and more specifically $\mu_1 \not\prec \mu_2$, it follows that $x \in [\iota h_1]_{\mu_1} \Rightarrow x g_2^{h_2} \in [\iota h_1]_{\mu_1}$. For the reverse implication, we note that $(g_2^{h_2})^{-1}$ only affects the μ_2 component of x non trivially. Since μ_1, μ_2 are unrelated, and more specifically $\mu_1 \not\prec \mu_2$, it follows that $x g_2^{h_2} \in [\iota h_1]_{\mu_1} \Rightarrow x = x g_2^{h_2} (g_2^{h_2})^{-1} \in [\iota h_1]_{\mu_1}$.

By symmetry, we may interchange 1 and 2 in statements (i) and (ii) to get

(i)' $x \in [\iota h_2]_{\mu_2} \Leftrightarrow x g_2^{h_2} \in [\iota h_2]_{\mu_2}$; and

(ii)' $x \in [\iota h_2]_{\mu_2} \Leftrightarrow x g_1^{h_1} \in [\iota h_2]_{\mu_2}$.

Now let $x \in X$. We use (i), (ii), (i)' and (ii)' to show that $x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} = x$. We have four cases.

(a) If $x \in [\iota h_1]_{\mu_1} \cap [\iota h_2]_{\mu_2}$, then

$$\begin{aligned} x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= \begin{pmatrix} x_{\mu_1}g_1^{-1} & \text{if } \lambda = \mu_1 \\ x_\lambda & \text{if } \lambda \neq \mu_1 \end{pmatrix} (g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\ &= \begin{pmatrix} x_{\mu_1}g_1^{-1} & \text{if } \lambda = \mu_1 \\ x_{\mu_2}g_2^{-1} & \text{if } \lambda = \mu_2 \\ x_\lambda & \text{if } \lambda \neq \mu_1, \mu_2 \end{pmatrix} g_1^{h_1}g_2^{h_2} \\ &= \begin{pmatrix} x_{\mu_2}g_2^{-1} & \text{if } \lambda = \mu_2 \\ x_\lambda & \text{if } \lambda \neq \mu_1, \mu_2 \end{pmatrix} g_2^{h_2} \\ &= x. \end{aligned}$$

(b) If $x \in [\iota h_1]_{\mu_1}$ and $x \notin [\iota h_2]_{\mu_2}$, then

$$\begin{aligned}
x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= \begin{pmatrix} x_{\mu_1}g_1^{-1} & \text{if } \lambda = \mu_1 \\ x_\lambda & \text{if } \lambda \neq \mu_1 \end{pmatrix} (g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\
&= \begin{pmatrix} x_{\mu_1}g_1^{-1} & \text{if } \lambda = \mu_1 \\ x_\lambda & \text{if } \lambda \neq \mu_1 \end{pmatrix} g_1^{h_1}g_2^{h_2} \\
&= xg_2^{h_2} \\
&= x.
\end{aligned}$$

(c) If $x \notin [\iota h_1]_{\mu_1}$ and $x \in [\iota h_2]_{\mu_2}$, then

$$\begin{aligned}
x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= x(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\
&= \begin{pmatrix} x_{\mu_2}g_2^{-1} & \text{if } \lambda = \mu_2 \\ x_\lambda & \text{if } \lambda \neq \mu_2 \end{pmatrix} g_1^{h_1}g_2^{h_2} \\
&= \begin{pmatrix} x_{\mu_2}g_2^{-1} & \text{if } \lambda = \mu_2 \\ x_\lambda & \text{if } \lambda \neq \mu_1, \mu_2 \end{pmatrix} g_2^{h_2} \\
&= x.
\end{aligned}$$

(d) If $x \notin [\iota h_1]_{\mu_1} \cup [\iota h_2]_{\mu_2}$, then

$$\begin{aligned}
x(g_1^{h_1})^{-1}(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} &= x(g_2^{h_2})^{-1}g_1^{h_1}g_2^{h_2} \\
&= xg_1^{h_1}g_2^{h_2} \\
&= xg_2^{h_2} \\
&= x.
\end{aligned}$$

The result follows. \square

Fix $\mu \in \Lambda$. For $x \in X$ with $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$, let $h([x]_\mu) \in \langle G_\lambda : \lambda > \mu \rangle$ such that $(xh([x]_\mu)^{-1})_\lambda = \iota_\lambda$ for each $\lambda > \mu$. In particular, if $y \in [x]_\mu$ then $(yh([x]_\mu)^{-1})_\lambda = \iota_\lambda$ for each $\lambda > \mu$. For $x \in X$ with $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, put $h([x]_\mu) := 1$. We formulate the following theorem which gives a canonical way to write an element of W which improves on Theorem 2.1.3 by explicitly writing the element.

Theorem 2.1.5. *By Lemma 2.1.1 we can extend (Λ, \leq) to a total order (Λ, \preceq) . Let $h \in W$. For each $\mu \in \Lambda$ and for each $[x]_\mu \in X_{\sim \mu}$ there exists $h([x]_\mu)_\mu \in \overline{G}_\mu$, where all but finitely many are trivial and $h([x]_\mu)_\mu = 1$ if $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, such that*

$$h = \prod_{\mu \in \Lambda} \left(\prod_{[x]_\mu \in X_{\sim \mu}} (h([x]_\mu)_\mu)^{h([x]_\mu)} \right)$$

where the first direct product is ordered by \prec and the second direct product is well defined by Lemma 2.1.4 part 2.

Proof. By Theorem 2.1.3, $h = g_1 \dots g_n$ where $g_i \in G_{\lambda_i}^{(G_\lambda : \lambda > \lambda_i)}$ for each $i = 1, \dots, n$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_1 < \dots < \lambda_n$. For each $i = 1, \dots, n$, there exist $h_{i_1}, \dots, h_{i_{j(i)}} \in G_{\lambda_i}$ and $k_{i_1}, \dots, k_{i_{j(i)}} \in \langle G_\lambda : \lambda > \lambda_i \rangle$ such that $g_i = h_{i_1}^{k_{i_1}} \dots h_{i_{j(i)}}^{k_{i_{j(i)}}$. By Lemma 2.1.4 parts 1 and 2, we find that if $\iota k_{i_m} \neq \iota k_{i_l}$ then $[h_{i_m}^{k_{i_m}}, h_{i_l}^{k_{i_l}}] = 1$ and if $\iota k_{i_m} = \iota k_{i_l}$ then $h_{i_m}^{k_{i_m}} h_{i_l}^{k_{i_l}} = (h_{i_m} h_{i_l})^{k_{i_m}}$. So we may assume, without loss of generality, that $\iota k_{i_m} \neq \iota k_{i_l}$ for any $m, l = 1, \dots, j(i)$. For each $m = 1, \dots, j(i)$ there exists $x_{i_m} \in X$ such that $x_{i_m} k_{i_m}^{-1} \sim_{\lambda_i} \iota$. Note $(x_{i_m})_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \lambda_i$. By Lemma 2.1.4 part 1, $h_{i_m}^{k_{i_m}} = h_{i_m}^{h([x_{i_m}]_{\lambda_i})}$ for each $m = 1, \dots, j(i)$. Now, if we write $h([x_{i_l}]_{\lambda_i})_{\lambda_i} := h_{i_l}$ for each $l = 1, \dots, j(i)$ and each $i = 1, \dots, n$, then

$$\begin{aligned} h &= g_1 \dots g_n \\ &= h_{1_1}^{k_{1_1}} \dots h_{1_{j(1)}}^{k_{1_{j(1)}}} \dots h_{n_1}^{k_{n_1}} \dots h_{n_{j(n)}}^{k_{n_{j(n)}}} \\ &= \prod_{i=1}^n h([x_{i_1}]_{\lambda_i})_{\lambda_i}^{h([x_{i_1}]_{\lambda_i})} \dots h([x_{i_{j(i)}}]_{\lambda_i})_{\lambda_i}^{h([x_{i_{j(i)}}]_{\lambda_i})} \\ &= \prod_{i=1}^n \left(\prod_{l=1}^{j(i)} h([x_{i_l}]_{\lambda_i})_{\lambda_i}^{h([x_{i_l}]_{\lambda_i})} \right). \end{aligned}$$

And this is in the required form. \square

Remark. We note that if $h = \prod_{\mu \in \Lambda} \left(\prod_{[x]_\mu \in X_{\sim \mu}} (h([x]_\mu)_\mu)^{h([x]_\mu)} \right)$ as in Theorem 2.1.5, then $(xh)_\lambda = x_\lambda h([x]_\lambda)_\lambda$ for any $x \in X$ and any $\lambda \in \Lambda$.

We let W act on $X_{\sim \mu}$ by setting $[x]_\mu h := [xh]_\mu$ for each $x \in X$ and each $h \in W$. Fix $h \in W$ and write $h = \prod_{\mu \in \Lambda} \left(\prod_{[x]_\mu \in X_{\sim \mu}} (h([x]_\mu)_\mu)^{h([x]_\mu)} \right)$, as in Theorem 2.1.5. Fix $x \in X$ and let $y \in [x]_\mu$. That is to say that $x_\lambda = y_\lambda$ for each $\lambda > \mu$. In particular, $[x]_\lambda = [y]_\lambda$ for each $\lambda > \mu$. So if $\lambda > \mu$, then

$$(xh)_\lambda = x_\lambda h([x]_\lambda)_\lambda = y_\lambda h([y]_\lambda)_\lambda = (yh)_\lambda.$$

We have shown that if $[x]_\mu = [y]_\mu$, then $[x]_\mu h = [y]_\mu h$ and hence the action is well defined.

Now for each $\gamma \in \Gamma$, we define an equivalence relation \approx_γ on W by the rule $g \approx_\gamma h$ if and only if $[g]_\gamma = [h]_\gamma$. It is easy to see that \approx_γ is an equivalence relation. So W is partitioned into equivalence classes. Let $S_\gamma \subseteq W$ be a set of representatives for all the equivalence classes.

Note that if $g \in G$ we can write $g = g_{\lambda_1}^{h_{\lambda_1}} \dots g_{\lambda_n}^{h_{\lambda_n}}$ with $g_{\lambda_i} \in G_{\lambda_i}$ and $h_{\lambda_i} \in \langle G_\lambda : \lambda > \lambda_i \rangle$ with $\lambda_1 < \dots < \lambda_n$, where (Λ, \preceq) is an extension of (Λ, \leq) to a total order using Lemma 2.1.1. Let $\Sigma := \{\lambda_1, \dots, \lambda_n\} \cap \{\lambda \in \Lambda : \lambda > \gamma\}$. If Σ is empty,

then it is clear that $[\iota]_\gamma g = [\iota]_\gamma 1$. If Σ is non empty, then write $\Sigma = \{\sigma_1 \dots, \sigma_m\}$ with $\sigma_1 \prec \dots \prec \sigma_m$ and it is easy to verify that $[\iota]_\gamma g = [\iota]_\gamma g^{\sigma_1} \dots g^{\sigma_m}$. It follows that we can take S_γ to be a subset of $\langle G_\lambda : \lambda > \gamma \rangle$.

Proposition 2.1.6.

$$\langle g^{-1} \langle G_\gamma : \gamma \in \Gamma \rangle g : g \in W \rangle = \prod_{\gamma \in \Gamma} \prod_{g \in S_\gamma} g^{-1} G_\gamma g.$$

Proof. Firstly, we will

1. show $[g^{-1} G_\gamma g, h^{-1} G_\gamma h] = \{1\}$ if $\gamma \in \Gamma$ and $g, h \in S_\gamma$ with $g \neq h$;
2. show $[g^{-1} G_\gamma g, h^{-1} G_\mu h] = \{1\}$ if $\gamma, \mu \in \Gamma$ with $\gamma \neq \mu$, $g \in S_\gamma$ and $h \in S_\mu$;
3. show $g^{-1} G_\gamma g \cap h^{-1} G_\gamma h = \{1\}$ if $\gamma \in \Gamma$ and $g, h \in S_\gamma$ with $g \neq h$; and
4. show $g^{-1} G_\gamma g \cap h^{-1} G_\mu h = \{1\}$ if $\gamma, \mu \in \Gamma$ with $\gamma \neq \mu$, $g \in S_\gamma$ and $h \in S_\mu$.

1. This follows immediately from Lemma 2.1.4 part 2.

2. This follows immediately from Lemma 2.1.4 part 3.

3. Suppose, for a contradiction, that $g^{-1} G_\gamma g \cap h^{-1} G_\gamma h \neq \{1\}$. Let $l \in (g^{-1} G_\gamma g \cap h^{-1} G_\gamma h) \setminus \{1\}$. Write $l = g^{-1} g_\gamma g = h^{-1} h_\gamma h$ for some $g_\gamma, h_\gamma \in G_\gamma \setminus \{1\}$. Choose $y_\gamma \in X_\gamma$ with $y_\gamma g_\gamma \neq y_\gamma$. Choose $x \in [\iota g]_\gamma$ with $x_\gamma = y_\gamma$. Then

$$\begin{aligned} xl &= xg^{-1}g_\gamma g \\ &= \begin{pmatrix} y_\gamma g_\gamma & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \\ &\neq \begin{pmatrix} y_\gamma & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \\ &= xh^{-1}h_\gamma h \\ &= xl. \end{aligned}$$

The penultimate equality holds since $[\iota]_\gamma = [\iota g]_\gamma g^{-1} \neq [\iota g]_\gamma h^{-1}$. A contradiction. Thus $g^{-1} G_\gamma g \cap h^{-1} G_\gamma h = \{1\}$.

4. Let $g^{-1} g_\gamma g \in g^{-1} G_\gamma g$ and $h^{-1} h_\mu h \in h^{-1} G_\mu h$. Let $x \in X$. We note that $g^{-1} g_\gamma g$ acts trivially on x or changes only the γ component of x and $h^{-1} h_\mu h$ acts trivially on x or changes only the μ component of x . It follows that $g^{-1} G_\gamma g \cap h^{-1} G_\mu h = \{1\}$. This completes the proof of part 4.

And it follows that $\prod_{\gamma \in \Gamma} \prod_{g \in S_\gamma} g^{-1} G_\gamma g$ is well defined.

It is clear that $\prod_{\gamma \in \Gamma} \prod_{g \in S_\gamma} g^{-1} G_\gamma g \subseteq \langle g^{-1} \langle G_\gamma : \gamma \in \Gamma \rangle g : g \in W \rangle$. Now let $\gamma, \mu \in \Gamma, g \in W$ and $g_\gamma \in G_\gamma$.

We now

1. find $k \in \langle G_\lambda : \lambda \in \Lambda \setminus \{\gamma\} \rangle$ and $k_\gamma \in G_\gamma$ such that $g^{-1} g_\gamma g = k^{-1} k_\gamma k$; and
2. show $h^{-1} k_\gamma h = k^{-1} k_\gamma k$ for some $h \in S_\gamma$ with $h \approx_\gamma k$.

And the result will follow.

1. By Lemma 2.1.1, we can extend (Λ, \leq) to a totally ordered set (Λ, \preceq) and we may assume, without loss of generality, that $\gamma \prec \lambda$ for each $\lambda \in \Lambda \setminus \{\gamma\}$. Write $g = g_1^{h_1} \dots g_n^{h_n}$, where $g_i \in G_{\lambda_i}$ and $h_i \in \langle G_\lambda : \lambda > \lambda_i \rangle$ for each $i = 1, \dots, n$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_1 \preceq \dots \preceq \lambda_n$. By Lemma 2.1.4 part 1 we may assume, without loss of generality, that if $h_i \neq 1$ then $\iota h_i \neq \iota$.

If $\lambda_1 \neq \gamma$, then we can take $k := g$ and $k_\gamma := g_\gamma$ and we are done.

Suppose $\lambda_1 = \gamma$. Then take $k' := g_2^{h_2} \dots g_n^{h_n}$ and $k'_\gamma := g_1^{-h_1} g_\gamma g_1^{h_1}$. And hence

$$g^{-1} g_\gamma g = (g_n^{-h_n} \dots g_2^{-h_2}) (g_1^{-h_1} g_\gamma g_1^{h_1}) (g_2^{h_2} \dots g_n^{h_n}) = (k')^{-1} k'_\gamma k'$$

Now if $h_1 = 1$, then $k'_\gamma = g_1^{-1} g_\gamma g_1 \in G_\gamma$. If $h_1 \neq 1$, we find that g_γ and $g_1^{h_1}$ commute. This follows from Lemma 2.1.4 part 2. Hence $k'_\gamma = g_\gamma \in G_\gamma$. If $\lambda_2 \neq \gamma$, then take $k := k'$ and $k_\gamma := k'_\gamma$ and we are done. Otherwise, we repeat the above process. We can only repeat this process at most n times. Hence we can find $k \in \langle G_\lambda : \lambda \in \Lambda \setminus \{\gamma\} \rangle$ and $k_\gamma \in G_\gamma$ such that $g^{-1} g_\gamma g = k^{-1} k_\gamma k$ as required.

2. Note that

$$\begin{aligned} h \approx_\gamma k &\Rightarrow [\iota h]_\gamma = [\iota]_\gamma h = [\iota]_\gamma k = [\iota k]_\gamma \\ &\Rightarrow [\iota]_\gamma = [\iota]_\gamma k k^{-1} = [\iota k]_\gamma k^{-1} = [\iota h]_\gamma k^{-1} \\ &\Rightarrow x k^{-1} \in [\iota]_\gamma \text{ for each } x \in [\iota h]_\gamma. \end{aligned}$$

Let $x \in X$. If $x \in [\iota h]_\gamma$, then $x h^{-1}, x k^{-1} \in [\iota]_\gamma$. Hence

$$(x h^{-1} k_\gamma h)_\lambda = \begin{cases} x_\gamma k_\gamma & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{cases}$$

and

$$(x k^{-1} k_\gamma k)_\lambda = \begin{cases} x_\gamma k_\gamma & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma. \end{cases}$$

So $xh^{-1}k_\gamma h = xk^{-1}k_\gamma k$.

If $x \notin [uh]_\gamma$, then $xh^{-1}, xk^{-1} \notin [v]_\gamma$, this holds since h^{-1} and k^{-1} act faithfully on the set of equivalence classes of \sim_γ . Hence $h^{-1}k_\gamma h$ and $k^{-1}k_\gamma k$ act trivially on x . In particular, $xh^{-1}k_\gamma h = xk^{-1}k_\gamma k$. We have shown $h^{-1}k_\gamma h = k^{-1}k_\gamma k$.

□

Proposition 2.1.7. $W/\langle g^{-1}\langle G_\gamma : \gamma \in \Gamma \rangle g : g \in W \rangle$ is isomorphic, as abstract groups, to $\langle G_\lambda : \lambda \in \Lambda \setminus \Gamma \rangle$.

Proof. By Lemma 2.1.1, we can extend (Λ, \leq) to a total order (Λ, \preceq) . And we may assume, without loss of generality, that $\gamma \prec \lambda$ for each $\gamma \in \Gamma$ and each $\lambda \in \Lambda \setminus \Gamma$. Define

$$\psi : W \rightarrow \langle G_\lambda : \lambda \in \Lambda \setminus \Gamma \rangle, \quad g = h_1^{k_1} \dots h_n^{k_n} \mapsto h_{j+1}^{k_{j+1}} \dots h_n^{k_n}$$

where $h_i \in G_{\lambda_i}$ and $k_i \in \langle G_\lambda : \lambda \succ \lambda_i \rangle$ for each $i = 1, \dots, n$, $\lambda_1, \dots, \lambda_j \in \Gamma$ and $\lambda_{j+1} \dots \lambda_n \notin \Gamma$.

Clearly ψ is surjective. Let $g, h \in W$. We may write $g = g_1 \dots g_n$ and $h = h_1 \dots h_n$ with $g_i, h_i \in G_{\lambda_i}^{(\langle G_\lambda : \lambda \succ \lambda_i \rangle)}$; $\lambda_i \prec \lambda_k$ if $i < k$; $\lambda_1, \dots, \lambda_j \in \Gamma$; and $\lambda_{j+1}, \dots, \lambda_n \notin \Gamma$. Then

$$\begin{aligned} gh\psi &= (g_1 \dots g_n h_1 \dots h_n) \psi \\ &= (g_1 h_1^{(g_2 \dots g_n)^{-1}} \dots g_j h_j^{(g_{j+1} \dots g_n)^{-1}} g_{j+1} h_{j+1}^{(g_{j+2} \dots g_n)^{-1}} \dots g_n h_n) \psi \\ &= g_{j+1} h_{j+1}^{(g_{j+2} \dots g_n)^{-1}} \dots g_n h_n \\ &= g_{j+1} (g_{j+2} \dots g_n h_{j+1} g_n^{-1} \dots g_{j+2}^{-1}) g_{j+2} (g_{j+3} \dots g_n h_{j+2} g_n^{-1} \dots g_{j+3}^{-1}) \dots g_n h_n \\ &= g_{j+1} \dots g_n h_{j+1} \dots h_n \\ &= g \psi h \psi. \end{aligned}$$

Hence ψ is a homomorphism

Now let $\gamma_1, \gamma_2 \in \Gamma$ and $g_{\gamma_i} \in G_{\gamma_i}$ for $i = 1, 2$. If $\gamma_1 \neq \gamma_2$, then $g_{\gamma_2}^{-1} g_{\gamma_1} g_{\gamma_2} = g_{\gamma_1}$. This follows from Theorem 2.1.2. In particular, $g_{\gamma_2}^{-1} g_{\gamma_1} g_{\gamma_2} \in G_{\gamma_1}$. And hence, adopting the above notation,

$$\begin{aligned} \ker \psi &= \{h_1^{k_1} \dots h_j^{k_j} : \lambda_1, \dots, \lambda_j \in \Gamma\} \\ &= \{g^{-1}\langle G_\gamma : \gamma \in \Gamma \rangle g : g \in \langle G_\lambda : \lambda \in \Lambda \setminus \Gamma \rangle\} \\ &= \{g^{-1}\langle G_\gamma : \gamma \in \Gamma \rangle g : g \in W\} \end{aligned}$$

and the result follows. □

We now show that the generalised wreath product of finitely many p groups of finite exponent is a p group of finite exponent.

Theorem 2.1.8. *Let Λ be a finite partially ordered set and let p be prime. If G_λ is a p group of finite exponent for each $\lambda \in \Lambda$, then W is a p group of finite exponent. Moreover, W has exponent dividing $\prod_{\lambda \in \Lambda} \exp(G_\lambda)$ where $\exp(G_\lambda)$ denotes the exponent of G_λ .*

Proof. By Lemma 2.1.1 we can extend (Λ, \leq) to a total order (Λ, \preceq) . We write $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \prec \dots \prec \lambda_n$. Let $g \in W$. There exists $g_i \in G_{\lambda_i}^{(G_{\lambda_j}: j > i)}$ for $i = 1, \dots, n$ such that $g = g_1 \dots g_n$. Let $x \in X$. Notice that for any $j \in \mathbb{N}$, we have $(xg^j)_{\lambda_n} = x_{\lambda_n} g_n^j$. In particular, $(xg^{\exp(G_{\lambda_n})})_{\lambda_n} = x_{\lambda_n} g_n^{\exp(G_{\lambda_n})} = x_{\lambda_n}$ where $\exp(G_{\lambda_n})$ denotes the exponent of G_{λ_n} . So we may write $g^{\exp(G_{\lambda_n})} = h_1 \dots h_{n-1}$ for some $h_i \in G_{\lambda_i}^{(G_{\lambda_j}: j > i)}$ for $i = 1, \dots, n-1$. As above, $(g^{\exp(G_{\lambda_n})})^{\exp(G_{\lambda_{n-1}})}$ fixes the λ_{n-1} (and λ_n) component of x . We proceed in this manner to find that $(xg^{\prod_{i=1}^n \exp(G_{\lambda_i})})_\lambda = x_\lambda$ for each $\lambda \in \Lambda$. In particular, $g^{\prod_{i=1}^n \exp(G_{\lambda_i})} = 1$. By assumption, there exists $m_i \in \mathbb{N}$ such that $\exp(G_{\lambda_i}) = p^{m_i}$ for each $i = 1, \dots, n$. So $\prod_{i=1}^n \exp(G_{\lambda_i}) = p^{\sum_{i=1}^n m_i}$ and g is a p element. Furthermore, W has exponent dividing $\prod_{i=1}^n \exp(G_{\lambda_i})$. \square

2.2 The generalised wreath product is a good generalisation of the wreath product

The following Theorem describes how the generalised wreath product can be seen to be a generalisation of the wreath product.

Theorem 2.2.1. *If (G_λ, X_λ) and (G_μ, X_μ) are transitive, then*

$$\langle G_\lambda, G_\mu \rangle \cong \begin{cases} G_\lambda \times G_\mu & \text{if } \lambda, \mu \text{ unrelated} \\ G_\lambda \text{ wr}_{X_\mu} G_\mu & \text{if } \lambda < \mu \\ G_\mu \text{ wr}_{X_\lambda} G_\lambda & \text{if } \mu < \lambda. \end{cases}$$

Proof. Suppose λ and μ are unrelated. The result now follows from Theorem 2.1.2.

Suppose $\lambda < \mu$. Since (G_μ, X_μ) is transitive, for each $x_\mu \in X_\mu$ there exists $g_\mu \in G_\mu$ such that $\iota_\mu g_\mu = x_\mu$. Let $K \subseteq W$ be a set of such elements g_μ such that if $g_\mu, h_\mu \in K$ with $g_\mu \neq h_\mu$ then $\iota_\mu g_\mu \neq \iota_\mu h_\mu$ and for each $x_\mu \in X_\mu$ there exists $g_\mu \in K$ such that $\iota_\mu g_\mu = x_\mu$.

Let $g \in \langle G_\lambda, G_\mu \rangle$. We can write $g = \prod_{i \in I} g_{\lambda_i}^{g_{\mu_i}}$ for some $g_{\lambda_i} \in G_\lambda$, $g_{\mu_i} \in G_\mu$ and $g_{\mu_i} \in K$ and some finite set I . Define

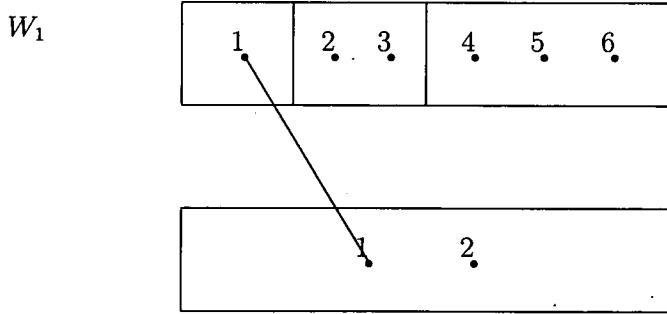
$$\psi : \langle G_\lambda, G_\mu \rangle \rightarrow G_\lambda wr_{X_\mu} G_\mu, g = \prod_{i \in I} g_{\lambda_i}^{g_{\mu_i}} \mapsto fg_\mu$$

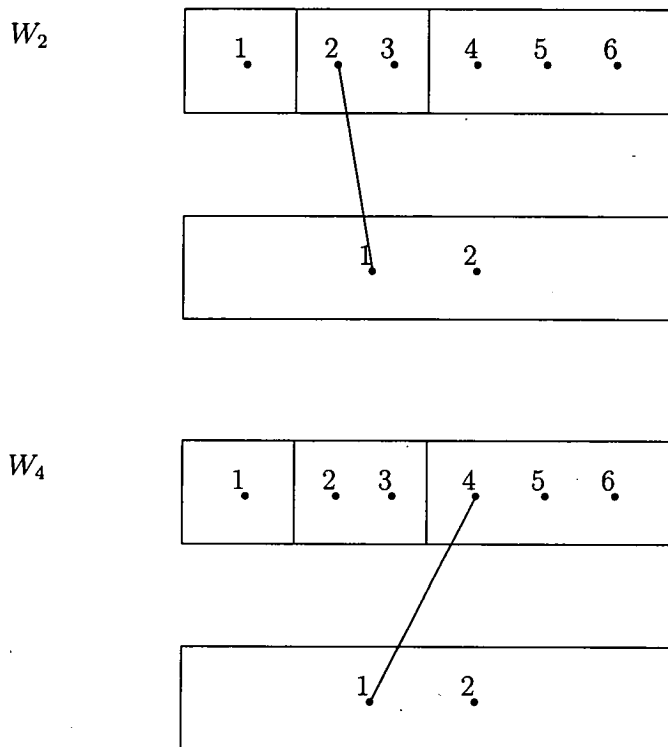
where $f(\iota_\mu g_{\mu_i}) = g_{\lambda_i}$ for each $i \in I$ and $f(x_\mu) = 1$ if $x_\mu \neq \iota_\mu g_{\mu_i}$ for any $i \in I$. It is easy to check that ψ is an isomorphism.

If $\mu < \lambda$, then the result holds by interchanging λ and μ in the above argument. □

We note that if $\lambda < \mu$ we require the permutation groups (G_λ, X_λ) and (G_μ, X_μ) to be transitive to obtain the isomorphism. This is an undesirable consequence of the definition. However, we can construct generalised wreath products having interesting properties. Below we construct the generalised wreath product of a cyclic group of order 2 and a cyclic group of order 6. By choosing the distinguished element ι carefully we obtain groups with differing properties.

Example 2.2.2. Let $\Lambda := \{\gamma, \mu\}$ with $\gamma < \mu$. Let $X_\gamma := \{1, 2\}$ and let $G_\gamma := \langle \tau \rangle \subseteq \text{Sym}\{1, 2\}$ where $\tau := (12)$. Let $X_\mu := \{1, 2, 3, 4, 5, 6\}$ and let $G_\mu := \langle \sigma \rangle \subseteq \text{Sym}\{1, 2, 3, 4, 5, 6\}$ where $\sigma := (23)(456)$. We consider the generalised wreath products $W_1 := wr_{(1,1)}\{G_\gamma, G_\mu\}$, $W_2 := wr_{(1,2)}\{G_\gamma, G_\mu\}$ and $W_4 := wr_{(1,4)}\{G_\gamma, G_\mu\}$.





Then

1. Elements of W_1 are of the form $\tau^i \sigma^j$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4, 5, 6\}$. Hence $|W_1| = 12$. Moreover, by Theorem 4.1.2, W_1 is abelian.
2. Elements of W_2 are of the form $\tau^{i_1}(\tau^{i_2})^\sigma \sigma^j$ for $i_1, i_2 \in \{1, 2\}$ and $j \in \{1, 2, 3, 4, 5, 6\}$. Hence $|W_2| = 24$. Moreover, by Theorem 4.1.2 and Theorem 4.2.10, W_2 is nilpotent but not abelian.
3. Elements of W_4 are of the form $\tau^{i_1}(\tau^{i_2})^\sigma(\tau^{i_3})^{\sigma^2} \sigma^j$ for $i_1, i_2, i_3 \in \{1, 2\}$ and $j \in \{1, 2, 3, 4, 5, 6\}$. Hence $|W_4| = 48$. Moreover, by Theorem 4.2.10 and Theorem 4.8.8, W_4 is soluble but not nilpotent.

In the case when the groups (G_λ, X_λ) are not transitive for each $\lambda \in \Lambda$, then the isomorphism in Theorem 2.2.1 cannot be used, as has been demonstrated in Example 2.2.2. Consequently, results known about the permutational wreath product cannot be extended easily to the generalised wreath product. We deal with overcoming this difficulty. Recall the constituent, as defined in Definition 1.1.4, of a permutation group.

Now fix $\mu \in \Lambda$. Let $y \in \prod_{\lambda \geq \mu} X_\lambda$ and let $h \in \langle G_\lambda(y_\lambda G_\lambda) : \lambda \geq \mu \rangle$. Write $h = h_1 \dots h_n$ where $h_i \in G_{\lambda_i}(y_{\lambda_i} G_{\lambda_i})$ for some $\lambda_i \geq \mu$. For each $i = 1, \dots, n$, there exists an epimorphism $\psi_i : G_{\lambda_i} \rightarrow G_{\lambda_i}(y_{\lambda_i} G_{\lambda_i})$, $g_{\lambda_i} \mapsto g_{\lambda_i}|_{y_{\lambda_i} G_{\lambda_i}}$ as defined in Definition 1.1.4. For each $i = 1, \dots, n$, there exists $g_i \in G_{\lambda_i}$ such that $g_i \psi_i = h_i$. Write $g = g_1 \dots g_n$. Let $z \in \prod_{\lambda \geq \mu} y_\lambda G_\lambda$. With an abuse of notation, we can view z as an element of X by setting $z_\lambda = \iota_\lambda$ if $\lambda < \mu$ or λ and μ are unrelated. We view $\langle G_\lambda(y_\lambda G_\lambda) : \lambda \geq \mu \rangle$ as a permutation group on $\prod_{\lambda \geq \mu} y_\lambda G_\lambda$ by setting $(zh)_\lambda := (zg)_\lambda$ for each $\lambda \geq \mu$.

We note that the choice of g_i is not unique. If we choose $g'_i \in G_{\lambda_i}$ such that $g'_i \psi_i = h_i$, then $x_{\lambda_i} g_i = x_{\lambda_i} g'_i$ for each $x_{\lambda_i} \in y_{\lambda_i} G_{\lambda_i}$. Let $g' = g'_1 \dots g'_n$. Since $z_\lambda \in y_\lambda G_\lambda$ for each $\lambda \geq \mu$, it follows that $(zg)_\lambda = (zg')_\lambda$ for each $\lambda \geq \mu$ for each $z \in \prod_{\lambda \geq \mu} y_\lambda G_\lambda$. And hence the action of $\langle G_\lambda(y_\lambda G_\lambda) : \lambda \geq \mu \rangle$ on $\prod_{\lambda \geq \mu} y_\lambda G_\lambda$ is well defined.

Theorem 2.2.3. *Let \mathcal{X} be a variety. Then $W \in \mathcal{X}$ if and only if $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle \in \mathcal{X}$ for each $\mu \in \Lambda$ and each $x \in X$.*

Proof. Let w be a law that all groups in \mathcal{X} satisfy.

Suppose $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle \in \mathcal{X}$ for each $\mu \in \Lambda$ and each $x \in X$. Let $g_1, \dots, g_m \in W$. For each $i = 1, \dots, m$, write $g_i = \bar{g}_{i_1} \dots \bar{g}_{i_k}$ where $\bar{g}_{i_j} \in \overline{G}_{\mu_{i_j}}$ for some $\mu_{i_j} \in \Lambda$. We write, with a slight abuse of notation, $w = w(g_1, \dots, g_m)$. Let $\mu \in \Lambda$ and let $x \in X$. We consider the μ component of x and xw . If $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, then it is clear that $(xw)_\mu = x_\mu$. Suppose $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$. Let $\psi_\lambda : G_\lambda \rightarrow G_\lambda(x_\lambda G_\lambda)$ be the natural map, as defined in Definition 1.1.4, for each $\lambda \in \Lambda$. By definition, if $\mu_{i_j} \geq \mu$ then $y \bar{g}_{i_j} = \overline{y(g_{i_j} \psi_{\mu_{i_j}})}$ for each $y \in X$ with $y_\lambda \in x_\lambda G_\lambda$ for each $\lambda \in \Lambda$. We may assume, without loss of generality that, $\mu_{i_j} \geq \mu$ for each i, j since $(xw)_\mu = (xw')_\mu$ where w' is obtained from w by simply omitting \bar{g}_{i_j} where it appears. This follows from Definition 1.0.4, since the action of $\bar{g}_\mu \in \overline{G}_\mu$ on $y \in X$ depends only on the λ component of y where $\lambda \geq \mu$. Now

$$\begin{aligned} (xw)_\mu &= (xw(\bar{g}_{1_1} \dots \bar{g}_{1_k}, \dots, \bar{g}_{m_1} \dots \bar{g}_{m_k}))_\mu \\ &= (xw(\overline{(g_{1_1} \psi_{\mu_{1_1}})} \dots \overline{(g_{1_k} \psi_{\mu_{1_k}})}, \dots, \overline{(g_{m_1} \psi_{\mu_{m_1}})} \dots \overline{(g_{m_k} \psi_{\mu_{m_k}})}))_\mu \\ &= x_\mu \end{aligned}$$

and the last equality holds since $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle \in \mathcal{X}$. It follows that $W \in \mathcal{X}$.

Conversely, suppose $W \in \mathcal{X}$. Fix $\mu \in \Lambda$ and let $x \in X$ with $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$. Let $g_1, \dots, g_m \in \langle G_\lambda : \lambda \geq \mu \rangle$. For each $i = 1, \dots, m$, write $g_i = \bar{g}_{i_1} \dots \bar{g}_{i_k}$ where $\bar{g}_{i_j} \in \bar{G}_{\mu_{i_j}}$ for some $\mu_{i_j} \in \Lambda$. Without loss of generality, we can take k as the same for each $i = 1, \dots, m$. Let $\psi_\lambda : G_\lambda \rightarrow G_\lambda(x_\lambda G_\lambda)$ be the natural map, as defined in Definition 1.1.4, for each $\lambda \in \Lambda$. Let $\lambda \in \Lambda$ with $\lambda \geq \mu$. We consider the λ component of x and $xw(\overline{(g_{1_1} \psi_{\mu_{1_1}})} \dots \overline{(g_{1_k} \psi_{\mu_{1_k}})}, \dots, \overline{(g_{m_1} \psi_{\mu_{m_1}})} \dots \overline{(g_{m_k} \psi_{\mu_{m_k}})})$. As above, we may assume, without loss of generality that, $\mu_{i_j} \geq \mu$ for each i, j . Now,

$$\begin{aligned} (xw(\overline{(g_{1_1} \psi_{\mu_{1_1}})} \dots \overline{(g_{1_k} \psi_{\mu_{1_k}})}, \dots, \overline{(g_{m_1} \psi_{\mu_{m_1}})} \dots \overline{(g_{m_k} \psi_{\mu_{m_k}})}))_\lambda &= (xw(g_1, \dots, g_m))_\lambda \\ &= x_\lambda. \end{aligned}$$

The last equality holds since $W \in \mathcal{X}$ and the result follows. \square

Chapter 3

Some central properties

“Situation number three; it’s the one that no one sees.” *Jack Johnson*

In the first part of this chapter we determine the centre of the generalised wreath product given any partially ordered set Λ , any permutation groups (G_λ, X_λ) and any distinguished element $\iota = (\iota_\lambda)_{\lambda \in \Lambda}$. This in itself is useful to know but it also provides an essential tool in the final chapter. For example, it is required in determining necessary and sufficient conditions for W to be nilpotent.

In the second part of this chapter we look at some commutative properties of certain elements of the generalised wreath product W . Let $\mu \in \Lambda$ and let $\bar{g}_\mu \in \bar{G}_\mu$ as defined in Definition 1.0.4. We determine the centraliser of \bar{g}_μ .

3.1 The centre of the generalised wreath product

The centre of the wreath product of $A wr_Y B$ is known.

Theorem 3.1.1. *Let (A, X) and (B, Y) be non trivial permutation groups. Then*

$$Z(A wr_Y B) = \langle \Delta_\pi(Z(A) wr_Y B) : \pi \in \Pi \rangle,$$

the subgroup of $A wr_Y B$ generated by the diagonal subgroups $\Delta_\pi(Z(A) wr_Y B)$ where $f \in \Delta_\pi(Z(A) wr_Y B)$ if and only if $f(y) \in Z(A)$ for each $y \in Y$, $f(x) = f(y)$ for each $x, y \in Y_\pi$ and $f(z) = 1$ if $z \in Y \setminus Y_\pi$ where $\{Y_\pi\}_{\pi \in \Pi}$ is the set of orbits of B on Y .

For a proof see Meldrum [16] 1.4.2. We give the natural analogue of the diagonal subgroups in the generalised wreath product.

Notation. Fix $\gamma \in \Gamma$. Let $g \in Z(G_\gamma)$ and define $\hat{g} \in \text{Sym}X$ by $(x\hat{g})_\lambda := x_\lambda$ for $\lambda \neq \gamma$ and

$$(x\hat{g})_\gamma := \begin{cases} x_\gamma g & \text{if } x_\lambda \in \iota_\lambda G_\lambda \text{ for each } \lambda > \gamma \\ x_\gamma & \text{if } x_\lambda \notin \iota_\lambda G_\lambda \text{ for some } \lambda > \gamma. \end{cases}$$

We let $\Delta(Z(G_\gamma))$ denote the set of all such \hat{g} that lie in W .

Remark. If (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$, then $h \in \Delta(Z(G_\gamma))$ if and only if $h = \prod_{[x]_\gamma \in X_{\sim\gamma}} h_\gamma^{h([x]_\gamma)}$ for some $h_\gamma \in Z(G_\gamma)$. Here we use the notation of Theorem 2.1.5.

It turns out that the subgroups $\Delta(Z(G_\gamma))$, $\gamma \in \Gamma$, lie in the centre of W .

Theorem 3.1.2. *If $\gamma \in \Gamma$, then $\Delta(Z(G_\gamma)) \subseteq Z(W)$.*

Proof. Let $\omega \in \Lambda$. Let $h \in G_\omega$ and define $\bar{h} \in \bar{G}_\omega$ as it is defined in Definition 1.0.4. Let $\hat{g} \in \Delta(Z(G_\gamma))$. Let $x \in X$. We consider all the cases.

1. If $\omega = \gamma$ and $x_\lambda = \iota_\lambda$ for each $\lambda > \gamma$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\gamma h^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma h^{-1}g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma h^{-1}g^{-1}h & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= \begin{pmatrix} x_\gamma h^{-1}g^{-1}hg & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \\ &= x. \end{aligned}$$

The last equality holds as $g \in Z(G_\gamma)$.

2. If $\omega = \gamma$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \gamma$; but $x_\lambda \neq \iota_\lambda$ for some $\lambda > \gamma$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= x. \end{aligned}$$

3. If $\omega = \gamma$ and $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \gamma$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

4. If $\omega > \gamma$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \gamma$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \omega, \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= x. \end{aligned}$$

5. If $\omega > \gamma$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \gamma$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= x. \end{aligned}$$

6. If $\omega > \gamma$; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\omega \geq \lambda > \gamma$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

7. If $\omega > \gamma$; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\omega \geq \lambda > \gamma$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

8. If $\omega > \gamma$ and $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

9. If ω, γ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \gamma$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \omega, \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= x. \end{aligned}$$

10. If ω, γ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \gamma$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \bar{h}\hat{g} \\ &= \begin{pmatrix} x_\gamma g^{-1} & \text{if } \lambda = \gamma \\ x_\lambda & \text{if } \lambda \neq \gamma \end{pmatrix} \hat{g} \\ &= x. \end{aligned}$$

11. If ω, γ are unrelated; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \gamma$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

12. If ω, γ are unrelated; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \gamma$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$,

then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= x\bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

And hence $[\bar{h}, \hat{g}] = 1$. Since $W = \langle \bar{G}_\lambda : \lambda \in \Lambda \rangle$ it follows that $\hat{g} \in Z(W)$. \square

If (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$, then we find that $Z(W) = \langle \Delta(Z(G_\gamma)) : \gamma \in \Gamma \rangle$. See Corollary 3.1.8: This is expected in light of Theorem 2.2.1. However this is not the case when the groups (G_λ, X_λ) are not transitive. We find that the subgroups $\Delta(H_\mu)$, as described below, also lie in $Z(W)$.

Notation. Fix $\mu \in \Lambda \setminus \Gamma$. Let $H_\mu := \{g \in Z(G_\mu) : yg = y \text{ for each } y \in \iota_\mu G_\mu\}$. Let $g \in H_\mu$ and define $\hat{g} \in \text{Sym}X$ by $(x\hat{g})_\lambda := x_\lambda$ for $\lambda \neq \mu$ and

$$(x\hat{g})_\mu := \begin{cases} x_\mu g & \text{if } x_\lambda \in \iota_\lambda G_\lambda \text{ for each } \lambda > \mu \\ x_\mu & \text{if } x_\lambda \notin \iota_\lambda G_\lambda \text{ for some } \lambda > \mu. \end{cases}$$

Let $\Delta(H_\mu)$ denote the set of all such \hat{g} that lie in W .

Remark. Fix $\mu \in \Lambda \setminus \Gamma$ and let $g \in G_\mu$. If (G_μ, X_μ) is transitive, then $yg = y$ for each $y \in \iota_\mu G_\mu = X_\mu$ implies $g = 1$. And hence $\Delta(H_\mu) = \{1\}$ if (G_μ, X_μ) is transitive.

Theorem 3.1.3. *Let $\mu \in \Lambda \setminus \Gamma$, then $\Delta(H_\mu) \subseteq Z(W)$.*

Proof. Let $\omega \in \Lambda$. Let $\hat{g} \in \Delta(H_\mu)$. Let $h \in G_\omega$ and define $\bar{h} \in \bar{G}_\omega$ as it is defined in Definition 1.0.4. Let $x \in X$. We consider all the cases.

1. If $\omega = \mu$; $x_\lambda = \iota_\lambda$ for each $\lambda > \mu$; and $x_\mu \in \iota_\mu G_\mu$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu h^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

2. If $\omega = \mu$; $x_\lambda = \iota_\lambda$ for each $\lambda > \mu$; and $x_\mu \notin \iota_\mu G_\mu$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\mu h^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu h^{-1}g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu h^{-1}g^{-1}h & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
&= \begin{pmatrix} x_\mu h^{-1}g^{-1}hg & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \\
&= x.
\end{aligned}$$

The last equality holds since $g \in Z(G_\mu)$.

3. If $\omega = \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \mu$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= x\bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

4. If $\omega = \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; $x_\mu \notin \iota_\mu G_\mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \mu$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
&= x.
\end{aligned}$$

5. If $\omega = \mu$ and $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= x\bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

6. If $\omega > \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\
 &= x\hat{g} \\
 &= x.
 \end{aligned}$$

7. If $\omega > \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; $x_\mu \notin \iota_\mu G_\mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \omega, \mu \end{pmatrix} \bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
 &= x.
 \end{aligned}$$

8. If $\omega > \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
 &= x\bar{h}\hat{g} \\
 &= x\hat{g} \\
 &= x.
 \end{aligned}$$

9. If $\omega > \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; $x_\mu \notin \iota_\mu G_\mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
 &= x.
 \end{aligned}$$

10. If $\omega > \mu$; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\omega \geq \lambda > \mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

11. If $\omega > \mu$; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\omega \geq \lambda > \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

12. If $\omega > \mu$ and $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

13. If $\omega < \mu$ and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\ &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

14. If $\omega < \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned} x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\ &= x\bar{h}\hat{g} \\ &= x\hat{g} \\ &= x. \end{aligned}$$

15. If $\omega < \mu$; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; and $x_\mu \notin \iota_\mu G_\mu$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
 &= x.
 \end{aligned}$$

16. If $\omega < \mu$ and $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
 &= x\bar{h}\hat{g} \\
 &= x\hat{g} \\
 &= x.
 \end{aligned}$$

17. If ω, μ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\
 &= x\hat{g} \\
 &= x.
 \end{aligned}$$

18. If ω, μ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; $x_\mu \notin \iota_\mu G_\mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned}
 x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \omega, \mu \end{pmatrix} \bar{h}\hat{g} \\
 &= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
 &= x.
 \end{aligned}$$

19. If ω, μ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda \geq \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$,

then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= x\bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

20. If ω, μ are unrelated; $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$; $x_\mu \notin \iota_\mu G_\mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \bar{h}\hat{g} \\
&= \begin{pmatrix} x_\mu g^{-1} & \text{if } \lambda = \mu \\ x_\lambda & \text{if } \lambda \neq \mu \end{pmatrix} \hat{g} \\
&= x.
\end{aligned}$$

21. If ω, μ are unrelated; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$; and $x_\lambda = \iota_\lambda$ for each $\lambda > \omega$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \hat{g}^{-1}\bar{h}\hat{g} \\
&= \begin{pmatrix} x_\omega h^{-1} & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega \end{pmatrix} \bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

22. If ω, μ are unrelated; $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$; and $x_\lambda \neq \iota_\lambda$ for some $\lambda > \omega$, then

$$\begin{aligned}
x\bar{h}^{-1}\hat{g}^{-1}\bar{h}\hat{g} &= x\hat{g}^{-1}\bar{h}\hat{g} \\
&= x\bar{h}\hat{g} \\
&= x\hat{g} \\
&= x.
\end{aligned}$$

And hence $[\bar{h}, \hat{g}] = 1$. Since $W = \langle G_\lambda : \lambda \in \Lambda \rangle$ it follows that $\hat{g} \in Z(W)$. □

Corollary 3.1.4. $\langle \Delta(H_\mu), \Delta(Z(G_\gamma)) : \mu \in \Lambda \setminus \Gamma, \gamma \in \Gamma \rangle \subseteq Z(W)$

Proof. This follows immediately from Theorem 3.1.2 and Theorem 3.1.3. □

We now consider the converse of Corollary 3.1.4.

Theorem 3.1.5. $Z(W) \subseteq \langle \Delta(H_\mu), \Delta(Z(G_\gamma)) : \mu \in \Lambda \setminus \Gamma, \gamma \in \Gamma \rangle$.

Proof. We note that, by Lemma 2.1.1, we can extend (Λ, \leq) to a total order (Λ, \preceq) . Let $\tilde{h} \in Z(W)$. We shall show the following.

1. If $\mu \in \Lambda \setminus \Gamma$, then $(x\tilde{h})_\mu = x_\mu$ for each $x \in X$ with $x_\mu \in \iota_\mu G_\mu$.
2. For fixed $\mu \in \Lambda$, if $x, y \in X$ with $x_\mu = y_\mu$ and $x_\lambda \in y_\lambda G_\lambda$ for each $\lambda > \mu$, then $(x\tilde{h})_\mu = (y\tilde{h})_\mu$.
3. For fixed $\gamma \in \Gamma$, if $Y_\gamma := \{x \in X : x_\lambda \in \iota_\lambda G_\lambda \text{ for each } \lambda > \gamma\}$, then there exists $h \in Z(G_\gamma)$ such that $(x\tilde{h})_\gamma = x_\gamma h$ for each $x \in Y_\gamma$ and $(x\tilde{h})_\gamma = x_\gamma$ for each $x \in X \setminus Y_\gamma$.
4. For fixed $\mu \in \Lambda \setminus \Gamma$, if $Y_\mu := \{x \in X : x_\lambda \in \iota_\lambda G_\lambda \text{ for each } \lambda > \mu\}$, then there exists $h \in H_\mu$ such that $(x\tilde{h})_\mu = x_\mu h$ for each $x \in Y_\mu$ and $(x\tilde{h})_\mu = x_\mu$ for each $x \in X \setminus Y_\mu$.

1. Let $x \in X$ with $x_\mu \in \iota_\mu G_\mu$. Suppose, for a contradiction, that $(x\tilde{h})_\mu \neq x_\mu$. So we have $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$ (otherwise \tilde{h} acts trivially on the μ component of x). Fix $\omega \in \Lambda$ with $\omega < \mu$. Define $y \in X$ by

$$y_\lambda := \begin{cases} x_\lambda & \text{if } \lambda \geq \mu \\ y_\omega & \text{if } \lambda = \omega \\ \iota_\lambda & \text{otherwise} \end{cases}$$

for some $y_\omega \in X_\omega$ with the orbit $y_\omega G_\omega$ consisting of more than one element. We note that $y_\mu = x_\mu \in \iota_\mu G_\mu$ and that $(y\tilde{h})_\mu = (x\tilde{h})_\mu \neq x_\mu = y_\mu$. Let $S := \sigma(y) \cap \{\lambda \in \Lambda : \omega < \lambda\}$. We note that, since $\sigma(y) \subseteq \sigma(x) \cup \{\omega\}$ is finite, S is finite. Write $S = \{\omega_1, \dots, \omega_k\}$ with $\omega_1 \prec \dots \prec \omega_k$. For each $i \in \{1, \dots, k\}$, choose $f_i \in G_{\omega_i}$ such that $y_{\omega_i} f_i = \iota_{\omega_i}$ and define $\bar{f}_i \in \bar{G}_{\omega_i}$ as it is defined in Definition 1.0.4. Now choose $f \in G_\omega$ such that $y_\omega f \neq y_\omega$ and define $\bar{f} \in \bar{G}_\omega$ as it is defined in Definition 1.0.4. Define $\tilde{g} := \bar{f}_k \dots \bar{f}_1 \bar{f} \bar{f}_1^{-1} \dots \bar{f}_k^{-1}$. That is,

$$z\tilde{g} = \begin{cases} z & \text{if } z_\lambda \neq y_\lambda \text{ for some } \lambda > \omega \\ w & \text{if } z_\lambda = y_\lambda \text{ for each } \lambda > \omega \end{cases}$$

where

$$w_\lambda = \begin{cases} z_\lambda & \text{if } \lambda \neq \omega \\ z_\omega f & \text{if } \lambda = \omega. \end{cases}$$

We note that $(y\tilde{g})_\omega = y_\omega f \neq y_\omega$ and in particular $y\tilde{g} \neq y$. Also $(y\tilde{h})\tilde{g} = y\tilde{h}$ since $(y\tilde{h})_\mu \neq y_\mu$ and $\mu > \omega$. So we have

$$y\tilde{g}\tilde{h} = (y\tilde{g})\tilde{h} \neq y\tilde{h} = (y\tilde{h})\tilde{g} = y\tilde{h}\tilde{g}.$$

A contradiction.

2. Let $x, y \in X$ with $x_\mu = y_\mu$ and $x_\lambda \in y_\lambda G_\lambda$ for each $\lambda > \mu$. Suppose, for a contradiction, that $(x\tilde{h})_\mu \neq (y\tilde{h})_\mu$. So we have $x_\lambda, y_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$ (otherwise \tilde{h} acts trivially on the μ component of x and y). Define $x', y' \in X$ by

$$(x')_\lambda = \begin{cases} x_\lambda & \text{if } \lambda \geq \mu \\ \iota_\lambda & \text{otherwise} \end{cases}$$

and

$$(y')_\lambda = \begin{cases} y_\lambda & \text{if } \lambda \geq \mu \\ \iota_\lambda & \text{otherwise.} \end{cases}$$

Notice that $x'_\mu = x_\mu = y_\mu = y'_\mu$ and $(x'\tilde{h})_\mu = (x\tilde{h})_\mu \neq (y\tilde{h})_\mu = (y'\tilde{h})_\mu$. Let $S := (\sigma(x') \cup \sigma(y')) \setminus \{\mu\}$. We note that, since $\sigma(x') \subseteq \sigma(x)$ and $\sigma(y') \subseteq \sigma(y)$ are finite, S is finite. So write $S = \{\nu_1, \dots, \nu_k\}$ with $\nu_1 < \dots < \nu_k$. For each $i \in \{1, \dots, k\}$, choose $f_i, g_i \in G_{\nu_i}$ with $x'_{\nu_i} f_i = \iota_{\nu_i}$ and $\iota_{\nu_i} g_i = y'_{\nu_i}$. And define $\bar{f}_i, \bar{g}_i \in \bar{G}_{\nu_i}$ as it is defined in Definition 1.0.4. Define $\tilde{g} := \bar{f}_k \dots \bar{f}_1 \bar{g}_1 \dots \bar{g}_k$. Then $x'\tilde{g} = y'$ and $(z\tilde{g})_\mu = z_\mu$ for any $z \in X$. Now

$$(x'\tilde{g}\tilde{h})_\mu = (y'\tilde{h})_\mu \neq (x'\tilde{h})_\mu = (x'\tilde{h}\tilde{g})_\mu.$$

A contradiction.

3. It is clear that $(x\tilde{h})_\gamma = x_\gamma$ for each $x \in X \setminus Y_\gamma$. It is also clear that there exists $h \in G_\gamma$ such that $(y\tilde{h})_\gamma = y_\gamma h$ for each $y \in [\iota]_\gamma \subseteq Y_\gamma$. Now let $x \in Y_\gamma$. Define $y \in X$ by

$$y_\lambda := \begin{cases} x_\gamma & \text{if } \lambda = \gamma \\ \iota_\lambda & \text{if } \lambda \neq \gamma. \end{cases}$$

Then $y \in [\iota]_\gamma \subseteq Y$ and $x_\gamma = y_\gamma$. By part 2, $(x\tilde{h})_\gamma = (y\tilde{h})_\gamma = y_\gamma h = x_\gamma h$. That is to say that there exists $h \in G_\gamma$ such that $(x\tilde{h})_\gamma = x_\gamma h$ for each $x \in Y_\gamma$. It remains to show $h \in Z(G_\gamma)$. Let $g \in G_\gamma$ and define $\bar{g} \in \bar{G}_\gamma \subseteq W$ as it is defined in Definition 1.0.4. Fix $x_\gamma \in X_\gamma$ and define $x \in Y_\gamma$ by

$$x_\lambda := \begin{cases} x_\gamma & \text{if } \lambda = \gamma \\ \iota_\gamma & \text{if } \lambda \neq \gamma. \end{cases}$$

Note that, by part 1, $(x\tilde{h})_\lambda = \iota_\lambda$ for each $\lambda > \gamma$. Thus

$$x_\gamma = (x[\bar{g}, \tilde{h}])_\gamma = x_\gamma[g, h].$$

This holds for each $x_\gamma \in X_\gamma$ and hence $h \in Z(G_\gamma)$.

4. In exactly the same way as in part 3 we have that $(x\tilde{h})_\mu = x_\mu$ for each $x \in X \setminus Y_\mu$ and there exists $h \in Z(G_\mu)$ such that $(x\tilde{h})_\mu = x_\mu h$ for each $x \in Y_\mu$. By part 1, we have that $x_\mu h = x_\mu$ for each $x_\mu \in \iota_\mu G_\mu$ and hence $h \in H_\mu$ as required. This completes the proof of part 4.

Now, there exists $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $\tilde{h} \in \langle \overline{G}_{\lambda_1}, \dots, \overline{G}_{\lambda_n} \rangle$. By part 3 and part 4, for each $i = 1, \dots, n$ there exists h_i in $Z(G_{\lambda_i})$ or H_{λ_i} depending on whether λ_i is a minimal element or not such that $\tilde{h} = \tilde{h}_1 \dots \tilde{h}_n$ where $\tilde{h}_i \in W$ defined by $(x\tilde{h}_i)_\lambda = x_\lambda$ for each $\lambda \neq \lambda_i$ and

$$(x\tilde{h}_i)_{\lambda_i} := \begin{cases} x_{\lambda_i} h_i & \text{if } x_\lambda \in \iota_\lambda G_\lambda \text{ for each } \lambda > \lambda_i \\ x_{\lambda_i} & \text{if } x_\lambda \notin \iota_\lambda G_\lambda \text{ for some } \lambda > \lambda_i \end{cases}$$

for each $i = 1, \dots, n$. We note that \tilde{h}_i lies in either $\Delta(Z(G_{\lambda_i}))$ or $\Delta(H_{\lambda_i})$ depending on whether λ_i is a minimal element or not. Thus $\tilde{h} = \tilde{h}_1 \dots \tilde{h}_n \in \langle \Delta(H_\mu), \Delta(Z(G_\gamma)) : \mu \in \Lambda \setminus \Gamma, \gamma \in \Gamma \rangle$ as required. This completes the proof. \square

This leads us to the main result of this chapter.

Corollary 3.1.6. $Z(W) = \langle \Delta(H_\mu), \Delta(Z(G_\gamma)) : \mu \in \Lambda \setminus \Gamma, \gamma \in \Gamma \rangle$.

Proof. This follows immediately from Corollary 3.1.4 and Theorem 3.1.5. \square

Example 3.1.7. Recall Example 2.2.2. We identify the subgroups $\Delta(H_\mu)$ and $\Delta(Z(G_\gamma))$ in W_1, W_2 and W_4 .

1. For W_1 , $\Delta(H_\mu) = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$ and $\Delta(Z(G_\gamma)) = \{1, \tau\}$. Hence $Z(W_1) = \{\tau^i \sigma^j : i = 1, 2 \text{ and } j = 1, \dots, 6\} = W_1$ and W_1 is abelian.
2. For W_2 , $\Delta(H_\mu) = \{1, \sigma^2, \sigma^4\}$ and $\Delta(Z(G_\gamma)) = \{1, \tau\tau^\sigma\}$. Hence $Z(W_2) = \{\tau^i (\tau^i)^\sigma \sigma^j : i = 1, 2 \text{ and } j = 2, 4, 6\} \subsetneq W_2$.
3. And for W_4 , $\Delta(H_\mu) = \{1, \sigma^3\}$ and $\Delta(Z(G_\gamma)) = \{1, \tau\tau^\sigma\tau^{\sigma^2}\}$. Hence $Z(W_4) = \{\tau^i (\tau^i)^\sigma (\tau^i)^{\sigma^2} \sigma^j : i = 1, 2 \text{ and } j = 3, 6\} \subsetneq W_4$.

Corollary 3.1.8. *If (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$, then*

$$Z(W) = \langle \Delta(Z(G_\gamma)) : \gamma \in \Gamma \rangle.$$

Proof. Firstly notice that $\Delta(H_\mu) = \{1\}$ if (G_μ, X_μ) is transitive. The result now follows from Corollary 3.1.6. \square

We find, as a Corollary of Corollary 3.1.8, necessary and sufficient conditions for the generalised wreath product W to have non trivial centre in the case where the permutation groups (G_λ, X_λ) , $\lambda \in \Lambda$, are transitive.

Corollary 3.1.9. *Suppose (G_λ, X_λ) is a non trivial transitive permutation group for each $\lambda \in \Lambda$. Then W has non trivial centre if and only if Γ is non empty and there exists $\gamma \in \Gamma$ such that G_γ has non trivial centre; $\{\lambda \in \Lambda : \lambda \geq \gamma\}$ is finite; and G_λ is finite for each $\lambda > \gamma$.*

Proof. We first note that, since (G_λ, X_λ) is transitive, G_λ is finite if and only if X_λ is finite.

Suppose Γ is non empty and there exists $\gamma \in \Gamma$ such that G_γ has non trivial centre; $\{\lambda \in \Lambda : \lambda \geq \gamma\}$ is finite; and G_λ is finite for each $\lambda > \gamma$. Let $h_\gamma \in Z(G_\gamma) \setminus \{1\}$. We note that, since $\{\lambda \in \Lambda : \lambda \geq \gamma\}$ is finite and G_λ is finite for each $\lambda > \gamma$, $X_{\sim\gamma}$ is finite. Let $h := \prod_{[x]_\gamma \in X_{\sim\gamma}} h_\gamma^{(h[x])_\gamma}$ as described in Theorem 2.1.5. By the remark to Theorem 2.1.5, it is easy to see that $(xh)_\gamma = x_\gamma h_\gamma$ for each $x \in X$. Hence $h \in \Delta(Z(G_\gamma))$. Corollary 3.1.8 implies $h \in Z(W)$ and W has non trivial centre.

Conversely, suppose W has non trivial centre. It is immediate from Corollary 3.1.8 that Γ is non empty and $Z(G_\gamma)$ is non trivial for some $\gamma \in \Gamma$. Let $\Sigma := \{\sigma \in \Gamma : Z(G_\sigma) \neq \{1\}\}$. Suppose, for a contradiction, that either $\{\lambda \in \Lambda : \lambda \geq \sigma\}$ is infinite or there exists $\lambda > \sigma$ with G_λ infinite for each $\sigma \in \Sigma$. That is to say $X_{\sim\sigma}$ is infinite for each $\sigma \in \Sigma$. Fix $\sigma \in \Sigma$. We note that if $h \in \Delta(Z(G_\sigma))$, then $h := \prod_{[x]_\sigma \in X_{\sim\sigma}} h_\sigma^{(h[x])_\sigma}$ for some $h_\sigma \in Z(G_\sigma)$. Since $X_{\sim\sigma}$ is infinite, it follows that $h_\sigma = 1$ and hence $h = 1$. Thus $\Delta(Z(G_\sigma)) = \{1\}$. This holds for each $\sigma \in \Sigma$. Now Corollary 3.1.8 implies that W has trivial centre, a contradiction. Hence there exists $\gamma \in \Gamma$ such that G_γ has non trivial centre; $\{\lambda \in \Lambda : \lambda \geq \gamma\}$ is finite; and G_λ is finite for each $\lambda > \gamma$. This completes the proof. \square

We use Corollary 3.1.8 to obtain necessary conditions on Λ and (G_λ, X_λ) for W to have the property that every non trivial subgroup of W has non trivial centre when (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$.

Theorem 3.1.10. *Suppose (G_λ, X_λ) is a non trivial transitive permutation group for each $\lambda \in \Lambda$. Suppose W has the property that every non trivial subgroup of W has non trivial centre, then*

1. G_λ has non trivial centre for each $\lambda \in \Lambda$;

2. every chain in Λ is finite;
3. $\{\lambda \in \Lambda : \lambda \geq \mu\}$ is finite for each $\mu \in \Lambda$; and
4. G_λ is finite for each $\lambda \in \Lambda \setminus \Gamma$.

Proof. 1. Since G_λ is isomorphic to a non trivial subgroup of W , G_λ has non trivial centre.

2. Let $\Sigma \subseteq \Lambda$ be a chain in Λ , let $H := \langle G_\sigma : \sigma \in \Sigma \rangle$ and let $Y := \prod_{\sigma \in \Sigma} X_\sigma$. By Corollary 3.1.8, $Z(H) = \langle \Delta(Z(G_\gamma)) : \gamma \in \Sigma \text{ is a minimal element of } \Sigma \rangle$ and, by assumption, $Z(H)$ is non trivial. So Σ has minimal elements. Since Σ is totally ordered, Σ must have a unique minimal element, γ say. Suppose, for a contradiction, that Σ is infinite. It follows that $Y_{\sim \gamma}$ is infinite. We note that if $h \in \Delta(Z(G_\gamma))$, then $h := \prod_{[y]_\gamma \in Y_{\sim \gamma}} h_\gamma^{h[y]_\gamma}$ for some $h_\gamma \in Z(G_\gamma)$. Since $Y_{\sim \gamma}$ is infinite, it follows that $h_\gamma = 1$ and hence $h = 1$. Thus, by Corollary 3.1.8, $Z(H) = \Delta(Z(G_\gamma)) = \{1\}$. A contradiction. Hence Σ is finite.

3. Fix $\mu \in \Lambda$ and let $\Sigma := \{\sigma \in \Lambda : \sigma \geq \mu\}$. Let $H := \langle G_\sigma : \sigma \in \Sigma \rangle$ and let $Y := \prod_{\sigma \in \Sigma} X_\sigma$. Suppose, for a contradiction, that Σ is infinite. Then $Y_{\sim \mu}$ is infinite. As in the proof of part 2, this yields a contradiction. Hence Σ is finite.

4. Let $\mu \in \Lambda \setminus \Gamma$. Fix $\omega \in \Lambda$ with $\omega < \mu$, let $H := \langle G_\lambda : \lambda \geq \omega \rangle$ and let $Y := \prod_{\lambda \geq \omega} X_\lambda$. Suppose, for a contradiction, that G_μ is infinite. Since (G_μ, X_μ) is transitive, X_μ is also infinite. And since $\omega < \mu$, it follows that $Y_{\sim \omega}$ is infinite. As in the proof of 2, this yields a contradiction. Hence G_μ is finite. \square

This result is particularly important in Chapter 4.2 to find necessary conditions for W to be nilpotent. We note that if G is a non trivial nilpotent group, then G satisfies the property that every non trivial subgroup of G has non trivial centre. Also, if G is a ZA group then G satisfies this property and Theorem 3.1.10 proves to be useful in Chapter 4.4.

3.2 Centralisers of certain elements

Theorem 3.2.1. *Let Λ be a partially ordered set and let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Fix $\mu \in \Lambda$. Let $g_\mu \in G_\mu \setminus \{1\}$ and define $\bar{g}_\mu \in \bar{G}_\mu$ as it is defined in Definition 1.0.4. Then $h \in C_G(\bar{g}_\mu)$ if and only if*

1. $\iota_\lambda h([l]_\lambda)_\lambda = \iota_\lambda$ for each $\lambda > \mu$;
2. $h([l]_\mu)_\mu \in C_{G_\mu}(g_\mu)$; and
3. if $\lambda < \mu$ then $h([x\bar{g}_\mu]_\lambda)_\lambda = h([x]_\lambda)_\lambda$ for each $x \in X$.

Here, by Theorem 2.1.5, we write $h = \prod_{\mu \in \Lambda} (\prod_{[x]_\mu \in X_{\sim \mu}} (h([x]_\mu)_\mu)^{h([x]_\mu)})$.

Proof. Suppose $h \in C_G(\bar{g}_\mu)$. We show conditions 1, 2 and 3 hold.

1. Since $\bar{g}_\mu \neq 1$ there exists $y_\mu \in X_\mu$ such that $y_\mu g_\mu \neq y_\mu$. Let $x \in [l]_\mu$ with $x_\mu = y_\mu$. In particular $x \in [l]_\lambda$ (and $[x]_\lambda = [l]_\lambda$) for each $\lambda \geq \mu$. And since \bar{g}_μ only affects the μ component of x non trivially, $x\bar{g}_\mu \in [l]_\lambda$ (and $[x\bar{g}_\mu]_\lambda = [l]_\lambda$) for each $\lambda \geq \mu$. Now

$$\begin{aligned}
\begin{pmatrix} y_\mu g_\mu h([l]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([l]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} &= \begin{pmatrix} y_\mu g_\mu h([x\bar{g}_\mu]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \\
&= \begin{pmatrix} y_\mu g_\mu & \text{if } \lambda = \mu \\ \iota_\lambda & \text{if } \lambda > \mu \\ x_\lambda & \text{otherwise} \end{pmatrix} h \\
&= x\bar{g}_\mu h \\
&= xh\bar{g}_\mu \\
&= \begin{pmatrix} y_\mu h([x]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([x]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \bar{g}_\mu \\
&= \begin{pmatrix} y_\mu h([l]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([l]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \bar{g}_\mu \\
&= \begin{pmatrix} z_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([l]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix}
\end{aligned}$$

where

$$z_\mu = \begin{cases} y_\mu h([l]_\mu)_\mu & \text{if } \iota_\lambda h([l]_\lambda)_\lambda \neq \iota_\lambda \text{ for some } \lambda > \mu \\ y_\mu h([l]_\mu)_\mu g_\mu & \text{if } \iota_\lambda h([l]_\lambda)_\lambda = \iota_\lambda \text{ for each } \lambda > \mu. \end{cases}$$

Now, $y_\mu g_\mu \neq y_\mu$ implies that $y_\mu g_\mu h([l]_\mu)_\mu \neq y_\mu h([l]_\mu)_\mu$. It follows that $z_\mu \neq y_\mu h([l]_\mu)_\mu$. Hence $z_\mu = y_\mu h([l]_\mu)_\mu g_\mu$ and $\iota_\lambda h([l]_\lambda)_\lambda = \iota_\lambda$ for each $\lambda > \mu$.

2. Let $x \in [l]_\mu$. As above, we have that $x_\mu g_\mu h([l]_\mu)_\mu = x_\mu h([l]_\mu)_\mu g_\mu$. Hence $h([l]_\mu)_\mu \in C_{G_\mu}(g_\mu)$.

3. Let $x \in X$ and let $\omega < \mu$. Let $y_\omega \in X_\omega$ and define $y \in X$ by

$$y_\lambda := \begin{cases} y_\omega & \text{if } \lambda = \omega \\ x_\lambda & \text{if } \lambda \neq \omega. \end{cases}$$

Then $[x]_\omega = [y]_\omega$ and $[x\bar{g}_\mu]_\omega = [y\bar{g}_\mu]_\omega$. Now

$$\begin{aligned}
y_\omega h([x\bar{g}_\mu]_\omega)_\omega &= y_\omega h([y\bar{g}_\mu]_\omega)_\omega \\
&= (y\bar{g}_\mu h)_\omega \\
&= (y h \bar{g}_\mu)_\omega \\
&= (y h)_\omega \\
&= y_\omega h([y]_\lambda)_\omega \\
&= y_\omega h([x]_\lambda)_\omega.
\end{aligned}$$

The fourth equality holds since \bar{g}_μ acts trivially on the ω component of y . This holds for each $y_\omega \in X_\omega$, hence $h([x\bar{g}_\mu]_\omega)_\omega = h([x]_\omega)_\omega$.

Conversely, suppose conditions 1,2 and 3 hold. Let $x \in X$. Suppose $x_\lambda = \iota_\lambda$ for each $\lambda > \mu$. Then

$$\begin{aligned}
x\bar{g}_\mu h &= \begin{pmatrix} x_\mu g_\mu & \text{if } \lambda = \mu \\ \iota_\lambda & \text{if } \lambda > \mu \\ x_\lambda & \text{otherwise} \end{pmatrix} h \\
&= \begin{pmatrix} x_\mu g_\mu h([x\bar{g}_\mu]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \\
&= \begin{pmatrix} x_\mu g_\mu h([\iota]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([\iota]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x\bar{g}_\mu]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \\
&= \begin{pmatrix} x_\mu h([\iota]_\mu)_\mu g_\mu & \text{if } \lambda = \mu \\ \iota_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \\
&= \begin{pmatrix} x_\mu h([\iota]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \bar{g}_\mu \\
&= \begin{pmatrix} x_\mu h([\iota]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([\iota]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \bar{g}_\mu \\
&= \begin{pmatrix} x_\mu h([x]_\mu)_\mu & \text{if } \lambda = \mu \\ \iota_\lambda h([x]_\lambda)_\lambda & \text{if } \lambda > \mu \\ x_\lambda h([x]_\lambda)_\lambda & \text{otherwise} \end{pmatrix} \bar{g}_\mu \\
&\stackrel{\text{3}}{=} x h \bar{g}_\mu.
\end{aligned}$$

The third equality holds since $(x\bar{g}_\mu)_\lambda = \iota_\lambda$ for each $\lambda > \mu$; the fourth equality holds by conditions 1,2 and 3; the sixth equality holds by condition 1; and the penultimate equality holds since $x_\lambda = \iota_\lambda$ for each $\lambda > \mu$.

Suppose $x_\lambda \neq \iota_\lambda$ for some $\lambda > \mu$. Then

$$\begin{aligned}
x\bar{g}_\mu h &= xh \\
&= (x_\lambda h([x]_\lambda)_\lambda)_{\lambda \in \Lambda} \\
&= (x_\lambda h([x]_\lambda)_\lambda)_{\lambda \in \Lambda} \bar{g}_\mu \\
&= xh\bar{g}_\mu
\end{aligned}$$

The third equality holds by condition 1. Hence $h \in C_G(\bar{g}_\mu)$. This completes the proof. \square

Corollary 3.2.2. *Let $\Lambda = \{\lambda, \mu\}$ with $\lambda < \mu$ and suppose (G_λ, X_λ) and (G_μ, X_μ) are non trivial transitive permutation groups. Let $g_\mu \in G_\mu \setminus \{1\}$. Then*

$$C_W(\bar{g}_\mu) = \{h \in W : h([x]_\mu)_\mu \in C_{G_\mu}(g_\mu) \text{ and } h([x\bar{g}_\mu]_\lambda)_\lambda = h([x]_\lambda)_\lambda \text{ for each } x \in X\}.$$

Proof. This follows immediately from Theorem 3.2.1. \square

Hence, as expected, there is a strong link between Corollary 3.2.2 and Meldrum [16] 2.2.1 which we state below.

Theorem 3.2.3. *Let (A, X) and (B, Y) be non trivial permutation groups. Let $b \in B$. Then*

$$C_{A \text{ wr } Y B}(b) = \{dc : c \in C_B(b) \text{ and } d \text{ is constant on all orbits of } \langle b \rangle\}.$$

Chapter 4

Classes of groups

“Situation number four; the one that left you wanting more.” *Jack Johnson*

In the remainder of the thesis we are concerned with the following question.

Question. Given a class of groups \mathcal{X} , can we find necessary and sufficient conditions for the generalised wreath product to lie in \mathcal{X} ?

We consider the class of abelian groups; nilpotent groups; locally nilpotent groups; ZA groups; residually nilpotent groups; locally boundedly nilpotent groups; bounded Engel groups; soluble groups; and locally soluble groups.

4.1 Abelian groups

Let $(A, X), (B, Y)$ be non trivial permutation groups. We consider the wreath product $Awr_Y B$. Since B is non trivial, there exists a B orbit Z of Y that contains more than one element. Let $x, y \in Z$ with $x \neq y$. There exists $b \in B$ such that $x = yb^{-1}$. Let $a \in A \setminus \{1\}$ and choose $f \in A^{(Y)}$ with $f(x) = 1$ and $f(y) = a$. Then

$$b^{-1}fb(y) = f(yb^{-1}) = f(x) = 1 \neq a = f(y).$$

In particular $b^{-1}fb \neq f$ and $Awr_Y B$ is not abelian. And so we obtain the following Theorem.

Theorem 4.1.1. *Let $(A, X), (B, Y)$ be non trivial permutation groups. Then $Awr_Y B$ is not abelian.*

However this is not the case in the generalised wreath product when the groups (G_λ, X_λ) need not be transitive. This is highlighted in Example 2.2.2. In fact, we can formulate the following Theorem.

Theorem 4.1.2. *W is abelian if and only if G_λ is abelian for each $\lambda \in \Lambda$ and $\iota_\lambda G_\lambda = \{\iota_\lambda\}$ for each $\lambda \in \Lambda \setminus \Gamma$.*

Proof. Suppose W is abelian. Since G_λ is isomorphic to a subgroup of W , it follows that G_λ is abelian for each $\lambda \in \Lambda$. Suppose, for a contradiction, that there exists $\mu \in \Lambda \setminus \Gamma$ such that $\iota_\mu G_\mu \neq \{\iota_\mu\}$. Choose $x_\mu \in \iota_\mu G_\mu$ with $x_\mu \neq \iota_\mu$. Choose $g_\mu \in G_\mu$ such that $x_\mu g_\mu = \iota_\mu$ and define $\bar{g}_\mu \in \bar{G}_\mu$ as defined in Definition 1.0.4. Since μ is not a minimal element, we can choose $\lambda \in \Lambda$ with $\lambda < \mu$. Since (G_λ, X_λ) is non trivial there exists $x_\lambda \in X_\lambda$ and $g_\lambda \in G_\lambda$ such that $x_\lambda g_\lambda \neq x_\lambda$. Define $\bar{g}_\lambda \in \bar{G}_\lambda$ as defined in Definition 1.0.4. Define $x \in X$ by

$$x_\omega := \begin{cases} x_\mu & \text{if } \omega = \mu \\ x_\lambda & \text{if } \omega = \lambda \\ \iota_\omega & \text{if } \omega \neq \mu, \lambda. \end{cases}$$

Then

$$\begin{aligned} \begin{pmatrix} \iota_\mu & \text{if } \omega = \mu \\ x_\lambda & \text{if } \omega = \lambda \\ \iota_\omega & \text{if } \omega \neq \mu, \lambda \end{pmatrix} &= x \bar{g}_\mu \\ &= x \bar{g}_\lambda \bar{g}_\mu \\ &= x \bar{g}_\mu \bar{g}_\lambda \\ &= \begin{pmatrix} \iota_\mu & \text{if } \omega = \mu \\ x_\lambda & \text{if } \omega = \lambda \\ \iota_\omega & \text{if } \omega \neq \mu, \lambda \end{pmatrix} \bar{g}_\lambda \\ &= \begin{pmatrix} \iota_\mu & \text{if } \omega = \mu \\ x_\lambda g_\lambda & \text{if } \omega = \lambda \\ \iota_\omega & \text{if } \omega \neq \mu, \lambda. \end{pmatrix} \end{aligned}$$

A contradiction since $x_\lambda g_\lambda \neq x_\lambda$. Hence we must have $\iota_\lambda G_\lambda = \{\iota_\lambda\}$ for each non minimal element λ of Λ as required.

Conversely, suppose G_λ is abelian for each $\lambda \in \Lambda$ and $\iota_\lambda G_\lambda = \{\iota_\lambda\}$ for each $\lambda \in \Lambda \setminus \Gamma$. Let $\lambda, \mu \in \Lambda$ and let $g_\lambda \in G_\lambda$ and $g_\mu \in G_\mu$. Define $\bar{g}_\lambda \in \bar{G}_\lambda$ and $\bar{g}_\mu \in \bar{G}_\mu$ as defined in Definition 1.0.4.

If $\lambda = \mu$, then $[\bar{g}_\lambda, \bar{g}_\mu] = 1$ since $\bar{G}_\lambda \cong G_\lambda$ is abelian (and the isomorphism follows from Lemma 1.0.5).

If λ and μ are unrelated, then $[\bar{g}_\lambda, \bar{g}_\mu] = 1$ by Theorem 2.1.2.

Suppose $\lambda < \mu$. In particular μ is not a minimal element and hence $\iota_\mu G_\mu = \{\iota_\mu\}$ and $[\iota]_\lambda \subseteq [\iota]_\mu$. Let $x \in X$. We have three cases.

1. Suppose $x \in [\iota]_\lambda$. So $x_\mu = 1_\mu$ and $(x\bar{g}_\mu)_\mu = x_\mu g_\mu = \iota_\mu g_\mu = \iota_\mu = x_\mu$. In particular \bar{g}_μ acts trivially on x . Also, $x\bar{g}_\lambda \in [\iota]_\lambda$ and in the same manner \bar{g}_μ acts trivially on $x\bar{g}_\lambda$. It follows that $x[\bar{g}_\lambda, \bar{g}_\mu] = x$.
2. Suppose $x \in [\iota]_\mu \setminus [\iota]_\lambda$. Since $\iota_\mu G_\mu = \{\iota_\mu\}$ and the fact that \bar{g}_μ only affects the μ component non trivially, it follows that $x\bar{g}_\mu \in [\iota]_\mu \setminus [\iota]_\lambda$. Hence \bar{g}_λ acts trivially on x and $x\bar{g}_\mu$ and it follows that $x[\bar{g}_\lambda, \bar{g}_\mu] = x$.
3. Suppose $x \in X \setminus [\iota]_\mu$. Then both \bar{g}_λ and \bar{g}_μ act trivially on x and it follows that $x[\bar{g}_\lambda, \bar{g}_\mu] = x$.

We have shown that $[\bar{g}_\lambda, \bar{g}_\mu] = 1$ for any $\lambda, \mu \in \Lambda$ and any $\bar{g}_\lambda \in \bar{G}_\lambda, \bar{g}_\mu \in \bar{G}_\mu$. Since $W = \langle \bar{G}_\lambda : \lambda \in \Lambda \rangle$, we have that W is abelian. This completes the proof. \square

We obtain the following Corollary. This is the expected extension, in light of Theorem 2.2.1, of the fact that the wreath product $A wr_Y B$ of two non trivial permutation groups $(A, X), (B, Y)$ is not abelian.

Corollary 4.1.3. *Let (G_λ, X_λ) be non trivial transitive permutation groups for each $\lambda \in \Lambda$. Then W is not abelian.*

Proof. Firstly note that $\Lambda \setminus \Gamma$ is non empty since Λ is connected and contains more than one element. Let $\lambda \in \Lambda \setminus \Gamma$. Since (G_λ, X_λ) is non trivial, X_λ contains more than one element. Hence $\iota_\lambda G_\lambda = X_\lambda \neq \{\iota_\lambda\}$, where the first equality follows since (G_λ, X_λ) is transitive. And Theorem 4.1.2 implies that W is not abelian. \square

4.2 Nilpotent groups

We have just seen in the previous section that the generalised wreath product of non trivial transitive permutation groups is never abelian. A generalisation of the abelian property is the nilpotent property. We see that we can find examples of the generalised wreath product being nilpotent. See Example 2.2.2 for an example. We can also find examples when the generalised wreath product of non trivial transitive permutation groups is nilpotent.

In this section we are concerned with developing necessary and sufficient conditions for the generalised wreath product to be nilpotent. Extending results known about the wreath product, we obtain a complete characterisation of the nilpotent generalised wreath product.

The nilpotent wreath product attracted the attention of Meldrum, Scott and Ljeskovacs. We state some known results, emanating from their work, regarding the nilpotency of the permutational wreath product. The proofs of these theorems can be found in Meldrum [16].

Theorem 4.2.1. *Let $(A, X), (B, Y)$ be non trivial permutation groups. If the wreath product $A wr_Y B$ is nilpotent then all the orbits of B on Y are finite.*

It follows immediately from Theorem 4.2.1 that if (B, Y) is transitive and $A wr_Y B$ is nilpotent, then B is finite.

Theorem 4.2.2. *Let $(A, X), (B, Y)$ be non trivial permutation groups, where B is finite and transitive on Y . Then $A wr_Y B$ is nilpotent if and only if for some prime p , A and B are p groups and A is nilpotent of finite exponent.*

Theorem 4.2.3. *Let $r \in \mathbb{N}$, $r \geq 2$. For $1 \leq i \leq r$, let $n_i \geq 1$. And let $|Y_i| = p^{n_i}$ for $i \geq 2$. Then $C_{p^{n_1}} wr_{Y_2} C_{p^{n_2}} wr_{Y_3} \dots wr_{Y_r} C_{p^{n_r}}$ is nilpotent and*

$$cl(C_{p^{n_1}} wr_{Y_2} C_{p^{n_2}} wr_{Y_3} \dots wr_{Y_r} C_{p^{n_r}}) = p^{\sum_{i=2}^r (n_i) - 1} (p + (p - 1)(n_1 - 1))$$

where C_{p^n} is the cyclic group of order p^n and $cl(C_{p^{n_1}} wr_{Y_2} C_{p^{n_2}} wr_{Y_3} \dots wr_{Y_r} C_{p^{n_r}})$ denotes the nilpotency class of $C_{p^{n_1}} wr_{Y_2} C_{p^{n_2}} wr_{Y_3} \dots wr_{Y_r} C_{p^{n_r}}$.

The following Lemma is a well known result. We provide a straightforward proof.

Lemma 4.2.4. *Let G be a group and let N be a normal subgroup of G such that N is contained in some group of a finite ascending central series for G and the quotient group G/N is nilpotent. Then G is nilpotent*

Proof. Suppose G/N is nilpotent of class c and suppose that $N \subseteq G_d$ for some $c, d \in \mathbb{N}$. Let $g_0, g_1, \dots, g_{c+d} \in G$. We have $1 = [g_0, g_1, \dots, g_c]N$, which implies that $[g_0, g_1, \dots, g_c] \in N \subseteq G_d$. So

$$[g_0, g_1, \dots, g_{c+d}] = [[g_0, g_1, \dots, g_c], g_{c+1}, \dots, g_{c+d}] \in G_0 = \{1\}$$

and hence G is nilpotent. □

The transitive case

We find a complete characterisation of the nilpotent generalised wreath product. We use the fact that a non trivial nilpotent group satisfies the property that any non trivial subgroup has non trivial centre. Theorem 3.1.10 suggests that we will require Λ to have finite chains; the set $\{\lambda \in \Lambda : \lambda > \gamma\}$ to be finite for each $\gamma \in \Gamma$; and G_λ to be finite for each $\lambda \in \Lambda \setminus \Gamma$ where Γ denotes the set of all minimal elements of Λ . It turns out that we do indeed need these conditions, along with a few more.

Theorem 4.2.5. *Suppose (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$. Suppose there exists a prime p such that*

1. every chain in Λ is finite and the lengths of all chains in Λ are bounded;
2. for each $\gamma \in \Gamma$, $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is a finite set and sizes of the sets $\{\lambda \in \Lambda : \gamma \leq \lambda\}$, $\gamma \in \Gamma$, are bounded (by m say);
3. G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$ and G_γ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$;
4. the nilpotency classes of G_λ , $\lambda \in \Lambda$, are bounded (by c say);
5. the exponents of G_λ , $\lambda \in \Lambda \setminus \Omega$, are bounded (by n say); and
6. the orders of G_λ , $\lambda \in \Lambda \setminus \Gamma$, are bounded (by r say).

Then W is nilpotent.

Proof. Suppose conditions 1 - 6 hold. Let $\{1\} = W_0 \subseteq W_1 \subseteq \dots \subseteq W_n \subseteq \dots$ be the upper central series of W . Fix $\gamma \in \Gamma$ and let $h \in (G_\gamma)^W$, the normal closure of G_γ in W , and let $\langle h \rangle$ denote the cyclic group generated by h . We define $H := \langle h, \{G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda\} \rangle$. We note that, by Meldrum [16] 7.4.3, $H \cong \langle h \rangle wr_Y \langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda \rangle$ where $Y = \prod_{\gamma < \lambda} X_\lambda$ the direct product. Thus, if $j := n^{r^m}(r^m)!$, then

$$\begin{aligned}
 |H| &= |\langle h \rangle|^{|Y|} |\langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda \rangle| \\
 &\leq \exp(G_\gamma)^{|Y|} |\langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda \rangle| \\
 &\leq \exp(G_\gamma)^{|\prod_{\gamma < \lambda} X_\lambda|} (\prod_{\gamma < \lambda} |X_\lambda|)! \\
 &\leq n^{r^m}(r^m)! \\
 &= j
 \end{aligned}$$

by conditions 2 – 6, noting $|\langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda \rangle| \leq (|\prod_{\gamma < \lambda} X_\lambda|)!$ since $\langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma < \lambda \rangle$ can be thought of as a permutation group on $\prod_{\gamma < \lambda} X_\lambda$. Now, by Theorem 2.1.8, H is a finite p group and is hence nilpotent of nilpotency class at most j which is independent of γ and h . That is to say $[g_0, \dots, g_j] = 1$ for any $g_0, \dots, g_j \in H$.

Let $K := \langle G_\lambda : \lambda \in \Lambda \text{ such that } \lambda \text{ and } \gamma \text{ unrelated} \rangle$ and let $L := \langle G_\lambda : \lambda \in \Lambda \text{ such that } \gamma \leq \lambda \rangle$. We note that $[\langle h \rangle^W, K] = \{1\}$ and, since γ is a minimal element, $W = \langle K, L \rangle$. So if $g \in W$, there exist $l_1, \dots, l_w \in L$ and $k_1, \dots, k_w \in K$ such that $g = l_1 k_1 \dots l_w k_w$. We recall Hall's identities; $[g, hk] = [g, k][g, h]^k$ and $[gh, k] = [g, k]^h [h, k]$ for all $g, h, k \in W$. Let $l \in \langle h \rangle^W$, $l_1, \dots, l_w \in L$ and let $k_1, \dots, k_w \in K$. Then, using Hall's identities repeatedly and the fact that $[\langle h \rangle^W, K] = \{1\}$, we have that $[l, l_1 k_1 \dots l_w k_w] = [l, l_w][l, l_{w-1}]^{l_w} \dots [l, l_1]^{l_2 \dots l_w}$. We claim that if $g_1, \dots, g_v \in W$ then $[h, g_1, \dots, g_v] = \prod_{i \in I} [f_{i_0}, \dots, f_{i_v}]^{f_i}$ for some finite set I and $f_i, f_{i_0}, \dots, f_{i_v} \in L$ for each $i \in I$. We proceed by induction on v . Write $g_1 = l_1 k_1 \dots l_w k_w$ for some $l_1, \dots, l_w \in L$ and some $k_1, \dots, k_w \in K$. Then $[h, g_1] = [h, l_w][h, l_{w-1}]^{l_w} \dots [h, l_1]^{l_2 \dots l_w}$, as required. Now let $g_1, \dots, g_v \in W$ and suppose $[h, g_1, \dots, g_v] = \prod_{i \in I} [f_{i_0}, \dots, f_{i_v}]^{f_i}$ for some finite set I and $f_i, f_{i_0}, \dots, f_{i_v} \in L$ for each $i \in I$. Write $l := [h, g_1, \dots, g_v] \in \langle h \rangle^W$ and $g_{v+1} = l_1 k_1 \dots l_w k_w$ for some $l_1, \dots, l_w \in L$ and some $k_1, \dots, k_w \in K$. Then

$$\begin{aligned} [h, g_1, \dots, g_v, g_{v+1}] &= [l, l_1 k_1 \dots l_w k_w] \\ &= [l, l_w][l, l_{w-1}]^{l_w} \dots [l, l_1]^{l_2 \dots l_w}. \end{aligned}$$

Noting that $l = \prod_{i \in I} [f_{i_0}, \dots, f_{i_v}]^{f_i}$ and using Hall's identities, this can easily be written in the desired form.

If $g_1, \dots, g_j \in \langle G_\lambda : \lambda \in \Lambda \setminus \{\gamma\} \rangle$ then $f_{i_0}, f_{i_1}, \dots, f_{i_j} \in H$ for each $i \in I$ and so $[f_{i_0}, f_{i_1}, \dots, f_{i_j}] = 1$ since H is nilpotent of class at most j and hence $[h, g_1, \dots, g_j] = 1$. We call this identity (*).

Suppose $h \in (Z(G_\gamma))^W$. Then, by Lemma 2.1.4 parts 1 and 2, $[\langle h \rangle^W, (G_\gamma)^W] = \{1\}$. Let $g_1, \dots, g_j \in W$. By Theorem 2.1.3, there exist $f_1, \dots, f_j \in (G_\gamma)^W$ and $h_1, \dots, h_j \in \langle G_\lambda : \lambda \in \Lambda \setminus \{\gamma\} \rangle$ such that $g_i = f_i h_i$ for each $i = 1, \dots, j$. We claim that $[h, g_1, \dots, g_j] = [h, h_1, \dots, h_j]$. We proceed by induction. We first note that

$$[h, g_1] = [h, f_1 h_1] = [h, h_1][h, f_1]^{h_1} = [h, h_1].$$

The second equality follows from Hall's identities and the final equality follows since $[\langle h \rangle^W, (G_\gamma)^W] = \{1\}$. Now suppose for some $r \in \{1, \dots, j-1\}$, we have

$[h, g_1, \dots, g_r] = [h, h_1, \dots, h_r]$. Then

$$\begin{aligned}
[h, g_1, \dots, g_{r+1}] &= [[h, g_1, \dots, g_r], g_{r+1}] \\
&= [[h, h_1, \dots, h_r], f_{r+1}h_{r+1}] \\
&= [[h, h_1, \dots, h_r], h_{r+1}][[h, h_1, \dots, h_r], f_{r+1}]^{h_{r+1}} \\
&= [[h, h_1, \dots, h_r], h_{r+1}] \\
&= [h, h_1, \dots, h_{r+1}].
\end{aligned}$$

The second equality follows by induction; the third equality follows from Hall's identities; and the fourth equality follows since $[\langle h \rangle^W, (G_\gamma)^W] = \{1\}$ noting that $[h, h_1, \dots, h_r] \in \langle h \rangle^W$. This completes the induction. And it follows, from (*), that $[h, g_1, \dots, g_j] = [h, h_1, \dots, h_j] = 1$.

Now fix $s \in \mathbb{N}$ and suppose $h \in (Z_{s+1}(G_\gamma))^W$. Let $g_1, \dots, g_j \in W$ and write $g_i = f_i h_i$ for each $i = 1, \dots, j$ for some $f_1, \dots, f_j \in (G_\gamma)^W$ and $h_1, \dots, h_j \in \langle G_\lambda : \lambda \in \Lambda \setminus \{\gamma\} \rangle$. Let $\psi : W \rightarrow W/(Z_s(G_\gamma))^W, g \mapsto g(Z_s(G_\gamma))^W$. We note that ψ is a homomorphism. By Lemma 2.1.4 parts 1 and 2, we have that $[(Z_{s+1}(G_\gamma))^W, (G_\gamma)^W] \subseteq (Z_s(G_\gamma))^W$. And hence $[(Z_{s+1}(G_\gamma))^W \psi, (G_\gamma)^W \psi] = [(Z_{s+1}(G_\gamma))^W, (G_\gamma)^W] \psi = 1$. Using induction in precisely the same manner as above we find that $[h, g_1, \dots, g_j] \psi = [g, h_1, \dots, h_j] \psi = 1$. The second equality follows from (*). Hence $[h, g_1, \dots, g_j] \in (Z_s(G_\gamma))^W$.

We claim that if $h \in (Z_s(G_\gamma))^W$, then $[h, g_1, \dots, g_{js}] = 1$ for any $g_1, \dots, g_{js} \in W$. We have shown that this is true for $s = 1$. We proceed by induction. Suppose that $[g, g_1, \dots, g_{js}] = 1$ for each $g \in (Z_s(G_\gamma))^W$ and any $g_1, \dots, g_{js} \in W$. Let $h \in (Z_{s+1}(G_\gamma))^W$ and let $g_1, \dots, g_{j(s+1)} \in W$. Then

$$[h, g_1, \dots, g_{j(s+1)}] = [[h, g_1, \dots, g_{js}], g_{js+1}, \dots, g_{j(s+1)}] = 1$$

where the last equality holds by induction noting that $[h, g_1, \dots, g_{js}] \in (Z_s(G_\gamma))^W$. Hence $[h, g_1, \dots, g_{\text{cl}(G_\gamma)j}] = 1$ for any $g_1, \dots, g_{\text{cl}(G_\gamma)j} \in W$ and any $h \in G_\gamma$. In particular, $[h, g_1, \dots, g_{cj}] = 1$ for any $g_1, \dots, g_{cj} \in W, h \in G_\gamma$. As cj is independent of the choice of $\gamma \in \Gamma$, it follows that $G_\gamma \subseteq W_{cj}$ for each $\gamma \in \Gamma$.

For each $i \in \mathbb{N}$, let Ω_i be the set of i maximal elements of Λ . Let $H_i = \langle G_\lambda : \lambda \in \Omega_i \rangle$. We show that H_i is nilpotent for each $i \in \mathbb{N}$. We proceed by induction on i . Firstly, we note that $H_1 = \prod_{\lambda \in \Omega_1} G_\lambda$, which is nilpotent being the direct product of nilpotent groups of bounded nilpotency class. Suppose that H_{i-1} is nilpotent. Let N_i be the smallest normal subgroup of H_i which contains $\langle G_\lambda : \lambda \in \Omega_i \setminus \Omega_{i-1} \rangle$. Now Proposition 2.1.7 implies $H_i/N_i \cong H_{i-1}$ which is nilpotent by assumption.

Since $\Omega_i \setminus \Omega_{i-1}$ is contained in the set of minimal elements of Ω_i , there exists $c_i \in \mathbb{N}$ such that $\langle G_\lambda : \lambda \in \Omega_i \setminus \Omega_{i-1} \rangle \subseteq (H_i)_{c_i}$, where $(H_i)_{c_i}$ is the c_i th term of the upper central series of H_i . Since $(H_i)_{c_i}$ is a normal subgroup of H_i , we have $N_i \subseteq (H_i)_{c_i}$. Hence, by Lemma 4.2.4, H_i is nilpotent. By condition 1, it follows that W is nilpotent. \square

Corollary 4.2.6. *Suppose (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$ and conditions 1 - 6 of Theorem 4.2.5 hold. Then W is nilpotent of nilpotency class at most $kc_n r^m (r^m)!$.*

Proof. By the proof of Theorem 4.2.5, we have that W is nilpotent of nilpotency class at most $\sum_{i=1}^k c_i$ where $c_1 = c$ and $c_i \leq cn r^m (r^m)!$ for each $i = 1, \dots, k$. The result follows. \square

Lemma 4.2.7. *Let Λ be a finite partially ordered set of size $n + 1$ with a unique minimal element, γ say, and let (G_λ, X_λ) be transitive for each $\lambda \in \Lambda$. Suppose there exists a prime p such that G_γ is a nilpotent p group of finite exponent and G_λ is a finite p group for each $\lambda \in \Lambda \setminus \{\gamma\}$ and let W denote the generalised wreath product. Then W is nilpotent of nilpotency class at least $n + 1$.*

Proof. W is nilpotent by Theorem 4.2.5. Let (Λ, \preceq) be an extension of (Λ, \leq) to a total order, this can be done by Lemma 2.1.1. Write $\Lambda = \{\lambda_1, \dots, \lambda_{n+1}\}$ where $\lambda_1 \prec \dots \prec \lambda_{n+1}$. For each $\lambda \in \Lambda$ choose $g_\lambda \in G_\lambda$ with the order of g_λ being p . Let H_i be the subgroup of W defined by $H_i := \langle g_{\lambda_k} : k = 1, \dots, i \rangle$ for $i = 1, \dots, n$. Since H_i is a subgroup of W for each $i = 1, \dots, n$, it is clear that H_i is nilpotent for each $i = 1, \dots, n$.

We show that the nilpotency class of H_{n+1} is at least $n + 1$. We proceed by induction. We note that H_1, H_2 are nilpotent groups of nilpotency class at least 1 and 2 respectively. Suppose that H_i is nilpotent of nilpotency class c , where $c \geq i$. We know that $Z(H_i) = \Delta_{H_i}(Z(G_{\lambda_i})) \cong \langle g_{\lambda_i} \rangle$ and this has size p . Let $\gamma_c(H_i)$ be the c th term of the lower central series. Now, $\{1\} \neq \gamma_c(H_i) \subseteq Z(H_i)$ and hence $\gamma_c(H_i) = Z(H_i)$. However $\Delta_{H_i}(Z(G_{\lambda_i}))$ and $\Delta_{H_{i+1}}(Z(G_{\lambda_i}))$ intersect trivially, ie $Z(H_i)$ and $Z(H_{i+1})$ intersect trivially. So there exists $h \in H_{i+1}$ such that $[h_1, \dots, h_c, h] \neq 1$ for some $h_1, \dots, h_c \in H_i$. And it follows that the nilpotency class of H_{i+1} is at least $c + 1$ and $c + 1 \geq i + 1$. Furthermore, since $H_{n+1} \subseteq W$, the nilpotency class of W is at least $n + 1$. \square

We now turn our attention to proving that conditions 1 - 6 in Theorem 4.2.5 are indeed necessary for W to be nilpotent.

Theorem 4.2.8. *Suppose (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$. If W is nilpotent then there exists a prime p such that*

1. *every chain in Λ is finite and the lengths of all chains in Λ are bounded;*
2. *for each $\gamma \in \Gamma$, $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is a finite set and the sizes of the sets $\{\lambda \in \Lambda : \gamma \leq \lambda\}$, $\gamma \in \Gamma$, are bounded;*
3. *G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$ and G_γ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$;*
4. *the nilpotency classes of G_λ , $\lambda \in \Lambda$, are bounded;*
5. *the exponents of G_λ , $\lambda \in \Lambda \setminus \Omega$, are bounded; and*
6. *the orders of G_λ , $\lambda \in \Lambda \setminus \Gamma$, is bounded.*

Proof. 1. Since W is nilpotent, W has the property that any non trivial subgroup of W is nilpotent and hence any non trivial subgroup of W has non trivial centre. Theorem 3.1.10 implies that every chain in Λ is finite. Now suppose, for a contradiction, that the lengths of chains in Λ are not bounded. For each $n \in \mathbb{N}$, there exists a chain $\Pi_n \subseteq \Lambda$ with n elements. Then, by Lemma 4.2.7, $\text{cl}(W) \geq \text{cl}(\langle G_\lambda : \lambda \in \Pi_n \rangle) \geq n$ for each $n \in \mathbb{N}$. A contradiction.

2. Theorem 3.1.10 implies that $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$. We now show that $|\{\lambda \in \Lambda : \gamma \leq \lambda\}|$ is bounded for $\gamma \in \Gamma$. Suppose, for a contradiction, that this is not the case. For each $n \in \mathbb{N}$, choose minimal elements γ_n in Γ with $|\{\lambda \in \Lambda : \gamma_n \leq \lambda\}| \geq n$. Let $\Pi_n := \{\lambda \in \Lambda : \gamma_n \leq \lambda\}$. Then, by Lemma 4.2.7, $\text{cl}(W) \geq \text{cl}(\langle G_\lambda : \lambda \in \Pi_n \rangle) \geq n$ for each $n \in \mathbb{N}$. A contradiction.

3. Fix $\gamma \in \Gamma$ and choose $\lambda \in \Lambda$ with $\gamma < \lambda$. Then $G_\gamma \text{ wr}_{X_\lambda} G_\lambda \cong \langle G_\gamma, G_\lambda \rangle \subseteq W$ which is nilpotent. By Theorem 4.2.1 and Theorem 4.2.2, there exists prime p such that G_γ is a nilpotent p group of finite exponent and G_λ is a finite p group. Since Λ is connected, G_γ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$ and G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$, for the same prime p .

4. Let $\lambda \in \Lambda$, then G_λ is isomorphic to a subgroup of W and hence $\text{cl}(G_\lambda) \leq \text{cl}(W)$.

5. Firstly note that, by Theorem 4.2.3, $C_{p^n} \text{ wr}_Y C_p$ is nilpotent and if $|Y| = p$ then $\text{cl}(C_{p^n} \text{ wr}_Y C_p) = p + (p-1)(n-1)$ for each $n \in \mathbb{N}$. Suppose, for a contradiction,

that the exponents of G_λ , $\lambda \in \Lambda \setminus \Omega$, are not bounded. Choose groups $G_{\lambda_{n_k}}$ with $\lambda_{n_k} \in \Lambda \setminus \Omega$ and $\exp(G_{\lambda_{n_k}}) = p^{n_k}$ where $(n_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence in \mathbb{N} . Choose $\mu_{n_k} \in \Lambda$ with $\lambda_{n_k} < \mu_{n_k}$. Now $C_{p^{n_k}} wr_{X_{\mu_{n_k}}} C_p$ is isomorphic to a subgroup of $G_{\lambda_{n_k}} wr_{X_{\mu_{n_k}}} G_{\mu_{n_k}}$ and hence

$$\begin{aligned} \text{cl}(W) &\geq \text{cl}(G_{\lambda_{n_k}} wr_{X_{\mu_{n_k}}} G_{\mu_{n_k}}) \\ &\geq \text{cl}(C_{p^{n_k}} wr_{X_{\mu_{n_k}}} C_p) \\ &\geq \text{cl}(C_{p^{n_k}} wr_Y C_p) \\ &= p + (p-1)(n_k - 1) \\ &\geq n_k \end{aligned}$$

where Y is a set of size p . This holds for each n_k , a contradiction.

6. We have already shown, in 3, that G_λ is finite for each $\lambda \in \Lambda \setminus \Gamma$. Suppose, for a contradiction, that $|G_\lambda|$, $\lambda \in \Lambda \setminus \Gamma$, is not bounded. Then there exists $\lambda_n \in \Lambda \setminus \Gamma$ with $|G_{\lambda_n}| \geq p^{n+1}$ for each $n \in \mathbb{N}$. There exist $\gamma_n \in \Gamma$ such that $\gamma_n < \lambda_n$ for each $n \in \mathbb{N}$. Then, by Scott [24] 3.3.7, $\text{cl}(W) \geq \text{cl}(G_{\gamma_n} wr_{X_{\lambda_n}} G_{\lambda_n}) \geq n$. A contradiction. This completes the proof. \square

Corollary 4.2.9. *Suppose (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$. Then W is nilpotent if and only if there exists a prime p such that*

1. every chain in Λ is finite and the lengths of all chains in Λ are bounded;
2. for each $\gamma \in \Gamma$, $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is a finite set and the sizes of the sets $\{\lambda \in \Lambda : \gamma \leq \lambda\}$, $\gamma \in \Gamma$, are bounded;
3. G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$ and G_γ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$;
4. the nilpotency classes of G_λ , $\lambda \in \Lambda$, are bounded;
5. the exponents of G_γ , $\gamma \in \Gamma$, are bounded; and
6. the orders of G_λ , $\lambda \in \Lambda \setminus \Gamma$, are bounded.

Proof. This follows immediately from Theorem 4.2.5 and Theorem 4.2.8. \square



The non transitive case

We note that the class of nilpotent groups of nilpotency class at most n is a variety. So we appeal to Theorem 2.2.3 to extend Corollary 4.2.9 to the case where the groups (G_λ, X_λ) need not be transitive. To avoid some perverse cases we will assume that $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for each $\lambda \in \Lambda \setminus \Gamma$.

Theorem 4.2.10. *Suppose $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for each $\lambda \in \Lambda \setminus \Gamma$. Then W is nilpotent if and only if there exists a prime p such that*

1. every chain in Λ is finite and the lengths of all chains in Λ are bounded (by k say);
2. for each $\gamma \in \Gamma$, $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is a finite set and the sizes of the sets $\{\lambda \in \Lambda : \gamma \leq \lambda\}$, $\gamma \in \Lambda$, are bounded (by m say);
3. $G_\gamma(x_\gamma G_\gamma)$ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$ and each $x \in X$; $G_\lambda(\iota_\lambda G_\lambda)$ is a finite p group for each $\lambda \in \Lambda \setminus (\Gamma \cup \Omega)$; $G_\lambda(x_\lambda G_\lambda)$ is a nilpotent p group of finite exponent for each $\lambda \in \Lambda \setminus (\Gamma \cup \Omega)$; $G_\omega(\iota_\omega G_\omega)$ is a finite p group for each $\omega \in \Omega$; and $G_\omega(x_\omega G_\omega)$ is nilpotent for each $\omega \in \Omega$ and each $x \in X$;
4. the nilpotency classes of G_λ , $\lambda \in \Lambda$, are bounded (by c say);
5. the exponents of $G_\lambda(x_\lambda G_\lambda)$, $\lambda \in \Lambda \setminus \Omega$, $x \in X$, are bounded (by n say); and
6. the orders of $G_\lambda(\iota_\lambda G_\lambda)$, $\lambda \in \Lambda \setminus \Gamma$, are bounded (by r say).

Proof. Suppose conditions 1 - 6 hold. Fix $\mu \in \Lambda$ and fix $x \in X$. Put $\Sigma := \{\lambda \in \Lambda : \lambda > \mu\}$ and define $\iota' \in (\prod_{\sigma \in \Sigma} \iota_\sigma G_\sigma) \times x_\mu G_\mu$ by setting $\iota'_\sigma := \iota_\sigma$ for each $\sigma \in \Sigma$ and $\iota'_\mu := x_\mu$. Notice

$$\begin{aligned} \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle &\cong \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Sigma\}, G_\mu(x_\mu G_\mu) \rangle \\ &\cong wr_{\iota'} \{ \{G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Sigma\}, G_\mu(x_\mu G_\mu) \} \end{aligned}$$

and each $G_\lambda(\iota_\lambda G_\lambda)$ and $G_\mu(x_\mu G_\mu)$ is transitive on the set it acts on. By Corollary 4.2.9, $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is nilpotent. And by Corollary 4.2.6, the nilpotency class of $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is at most $kcnr^m(r^m)!$. By Theorem 2.2.3, W is nilpotent.

Conversely, suppose W is nilpotent. We show conditions 1 - 6 hold.

1. Let $\Pi \subseteq \Lambda$ be a chain in Λ . Theorem 2.2.3 implies that $\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi \rangle$ is nilpotent. Furthermore $\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi \rangle$ satisfies the property that every non trivial subgroup has non trivial centre. Now, Theorem 3.1.10 implies Π is finite. Now suppose, for a contradiction, that the length of a chain in Λ is not bounded. For each $n \in \mathbb{N}$, there exists a chain $\Pi_n \subseteq \Lambda$ with n elements. Then, by Theorem 2.2.3 and Lemma 4.2.7 applied to $\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi_n \rangle$, $\text{cl}(W) \geq \text{cl}(\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi_n \rangle) \geq n$ for each $n \in \mathbb{N}$, a contradiction.

2. Fix $\gamma \in \Gamma$. Then Theorem 2.2.3 and Theorem 3.1.10 applied to $\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \geq \gamma \rangle$ imply that $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite. We now show that $|\{\lambda \in \Lambda : \gamma \leq \lambda\}|$ is bounded for $\gamma \in \Gamma$. Suppose, for a contradiction, that this is not the case. For each $n \in \mathbb{N}$, choose minimal elements γ_n in Γ with $|\{\lambda \in \Lambda : \gamma_n \leq \lambda\}| \geq n$. Let $\Pi_n := \{\lambda \in \Lambda : \gamma_n \leq \lambda\}$. Then, by Theorem 2.2.3 and Lemma 4.2.7 applied to $\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi_n \rangle$, $\text{cl}(W) \geq \text{cl}(\langle G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Pi_n \rangle) \geq n$ for each $n \in \mathbb{N}$, a contradiction.

3. Fix $\lambda, \mu \in \Lambda$ with $\lambda < \mu$ and let $x \in X$. If $G_\lambda(x_\lambda G_\lambda) = \{1\}$, it is clearly a nilpotent p group for any prime p . Suppose $G_\lambda(x_\lambda G_\lambda) \neq \{1\}$; this must happen for some $x \in X$ as $G_\lambda \neq \{1\}$. Then $G_\lambda(x_\lambda G_\lambda) \text{wr}_{\iota_\mu G_\mu} G_\mu(\iota_\mu G_\mu) \cong \langle G_\lambda(x_\lambda G_\lambda), G_\mu(\iota_\mu G_\mu) \rangle$ is nilpotent by Theorem 2.2.3. By Theorem 4.2.1 and Theorem 4.2.2, there exists a prime p such that $G_\lambda(x_\lambda G_\lambda)$ is a nilpotent p group of finite exponent and $G_\mu(\iota_\mu G_\mu)$ is a finite p group. Since Λ is connected, $G_\lambda(x_\lambda G_\lambda)$ is a nilpotent p group of finite exponent for each $\lambda \in \Lambda \setminus \Omega$ and $G_\mu(\iota_\mu G_\mu)$ is a finite p group for each $\mu \in \Lambda \setminus \Gamma$, for the same prime p . Finally, if $\omega \in \Omega$, then $G_\omega(x_\omega G_\omega)$ is nilpotent since it is isomorphic to a quotient of G_ω and G_ω is isomorphic to a subgroup of W .

4. Note that G_λ is isomorphic to a subgroup of W , hence $\text{cl}(G_\lambda) \leq \text{cl}(W)$.

5. Firstly note that, by Theorem 4.2.3, $C_{p^n} \text{wr}_Y C_p$ is nilpotent and if $|Y| = p$ then $\text{cl}(C_{p^n} \text{wr}_Y C_p) = p + (p - 1)(n - 1)$ for each $n \in \mathbb{N}$. Suppose, for a contradiction, that the exponents of $G_\lambda(x_\lambda G_\lambda)$, $\lambda \in \Lambda \setminus \Omega$, $x \in X$, are not bounded. Choose groups $G_{\lambda_{n_k}}(x_{\lambda_{n_k}} G_{\lambda_{n_k}})$ with $\lambda_{n_k} \in \Lambda \setminus \Omega$, $x_{\lambda_{n_k}} \in X_{\lambda_{n_k}}$ and $\exp(G_{\lambda_{n_k}}(x_{\lambda_{n_k}} G_{\lambda_{n_k}})) = p^{n_k}$ where $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ is a strictly increasing sequence. Choose $\mu_{n_k} \in \Lambda$ with $\lambda_{n_k} < \mu_{n_k}$. Now $C_{p^{n_k}} \text{wr}_{\iota_{\mu_{n_k}} G_{\mu_{n_k}}} C_p$ is isomorphic to a subgroup of $(G_{\lambda_{n_k}}(x_{\lambda_{n_k}} G_{\lambda_{n_k}})) \text{wr}_{\iota_{\mu_{n_k}} G_{\mu_{n_k}}} (G_{\mu_{n_k}}(\iota_{\mu_{n_k}} G_{\mu_{n_k}}))$ and hence, using Theorem

2.2.3,

$$\begin{aligned}
\text{cl}(W) &\geq \text{cl}((G_{\lambda_{n_k}}(x_{\lambda_{n_k}} G_{\lambda_{n_k}})) \text{wr}_{\iota_{\mu_{n_k}} G_{\mu_{n_k}}} (G_{\mu_{n_k}}(\iota_{\mu_{n_k}} G_{\mu_{n_k}}))) \\
&\geq \text{cl}(C_{p^{n_k}} \text{wr}_{\iota_{\mu_{n_k}} G_{\mu_{n_k}}} C_p) \\
&\geq \text{cl}(C_{p^{n_k}} \text{wr}_Y C_p) \\
&= p + (p - 1)(n_k - 1) \\
&\geq n_k
\end{aligned}$$

where Y is a set of size p . This holds for each n_k , a contradiction.

6. We have already shown, in 3, that $G_\lambda(\iota_\lambda G_\lambda)$ is finite for each $\lambda \in \Lambda \setminus \Gamma$. Suppose, for a contradiction, that $|G_\lambda(\iota_\lambda G_\lambda)|$, $\lambda \in \Lambda \setminus \Gamma$, is not bounded. Then there exists $\lambda_n \in \Lambda \setminus \Gamma$ with $|G_{\lambda_n}(\iota_{\lambda_n} G_{\lambda_n})| \geq p^{n+1}$ for each $n \in \mathbb{N}$. There exist $\gamma_n \in \Gamma$ with $\gamma_n < \lambda_n$ for each $n \in \mathbb{N}$. Then, by Theorem 2.2.3 and Scott [24] 3.3.7 applied to $G_{\gamma_n}(\iota_{\gamma_n} G_{\gamma_n}) \text{wr}_{\iota_{\lambda_n} G_{\lambda_n}} G_{\lambda_n}(\iota_{\lambda_n} G_{\lambda_n})$, we have

$$\begin{aligned}
\text{cl}(W) &\geq \text{cl}(\langle G_{\gamma_n}(\iota_{\gamma_n} G_{\gamma_n}), G_{\lambda_n}(\iota_{\lambda_n} G_{\lambda_n}) \rangle) \\
&= \text{cl}(G_{\gamma_n}(\iota_{\gamma_n} G_{\gamma_n}) \text{wr}_{\iota_{\lambda_n} G_{\lambda_n}} G_{\lambda_n}(\iota_{\lambda_n} G_{\lambda_n})) \\
&\geq n.
\end{aligned}$$

A contradiction. This completes the proof. \square

Remark. We note that if we did not impose the condition that $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for each $\lambda \in \Lambda$, using Theorem 4.1.2 we can easily construct an abelian generalised wreath product that does not satisfy the conditions of Theorem 4.2.10.

4.3 Locally nilpotent groups

The immediate generalisation of the nilpotent condition is the locally nilpotent condition. In Meldrum [16] 6.2.10 we see the following is true.

Theorem 4.3.1. *Let (A, X) and (B, Y) be non trivial permutation groups. Then $A \text{wr}_Y B$ is locally nilpotent if and only if there exists a prime p such that A is a locally nilpotent p group of finite exponent; B is locally nilpotent; and all orbits of finitely generated subgroups of B have orders which are powers of p .*

We use this result to find necessary and sufficient conditions for W to be locally nilpotent.

The transitive case

We first consider the case where the groups (G_λ, X_λ) , $\lambda \in \Lambda$, are all transitive.

Theorem 4.3.2. *Suppose (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$. Then W is locally nilpotent if and only if there exists a prime p such that*

1. G_λ is a locally nilpotent p group for each $\lambda \in \Lambda$;
2. G_λ is locally of finite exponent for each $\lambda \in \Lambda \setminus \Omega$; and
3. all orbits of finitely generated subgroups of G_λ have order a power of p for each $\lambda \in \Lambda \setminus \Gamma$.

Proof. Suppose there exists a prime p such that 1,2 and 3 hold. Let $g_1, \dots, g_m \in W$. Then there exist $\lambda_1, \dots, \lambda_n \in \Lambda$ and finitely generated subgroups H_{λ_i} of G_{λ_i} such that $\{g_1, \dots, g_m\} \subseteq \langle H_{\lambda_1}, \dots, H_{\lambda_n} \rangle$. Let $\Sigma := \{\lambda_1, \dots, \lambda_n\}$, $\Gamma_\Sigma \subseteq \Sigma$ be the set of minimal elements of Σ and let $\Omega_\Sigma \subseteq \Sigma$ be the set of maximal elements of Σ . And we assume, without loss of generality, that Σ is connected of size at least 2. We show that conditions 1 - 6 of Theorem 4.2.10 holds.

(i). Since Σ is finite, all chains in Σ are finite of length at most n .

(ii). Since Σ is finite, we have that $\{\sigma \in \Sigma : \lambda \leq \sigma\}$ is finite of size at most n for each $\lambda \in \Sigma$.

(iii). $H_\lambda(\iota_\lambda H_\lambda)$ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma_\Sigma$: If $\gamma \in \Gamma_\Sigma$ then $\gamma \notin \Omega$. So by 1 and 2, H_γ is a nilpotent p group of finite exponent and since $H_\gamma(\iota_\gamma H_\gamma)$ is isomorphic to a quotient of H_γ we have that $H_\gamma(\iota_\gamma H_\gamma)$ is a nilpotent p group of finite exponent.

$H_\lambda(\iota_\lambda H_\lambda)$ is a finite p group for each $\lambda \in \Sigma \setminus \Gamma_\Sigma$: If $\lambda \in \Sigma \setminus \Gamma_\Sigma$ then $\lambda \notin \Gamma$. So by 3, $H_\lambda(\iota_\lambda H_\lambda)$ is a finite p group.

$H_\omega(x_\omega H_\omega)$ is nilpotent for each $\omega \in \Omega_\Sigma$ and each $x \in X$: This follows immediately from 1 noting that $H_\omega(x_\omega H_\omega)$ is isomorphic to a quotient of H_ω .

$H_\lambda(x_\lambda H_\lambda)$ is a nilpotent p group of finite exponent for each $\lambda \in \Sigma \setminus \Omega_\Sigma$ and each $x \in X$: Let $x \in X$. If $\lambda \in \Sigma \setminus \Omega_\Sigma$ then $\lambda \notin \Omega$. So by 1 and 2, H_λ is a nilpotent p group of finite exponent and since $H_\lambda(x_\lambda H_\lambda)$ is isomorphic to a quotient of H_λ we have that $H_\lambda(x_\lambda H_\lambda)$ is a nilpotent p group of finite exponent.

(iv). $\max_{\lambda \in \Sigma} \text{cl}(H_\lambda)$ exists by 1 and using the fact that Σ is finite.

(v). Let $\lambda \in \Sigma \setminus \Omega_\Sigma$ and fix $x \in X$. Note $\Sigma \cap \Omega \subseteq \Omega_\Sigma$ and hence $\Sigma \setminus \Omega_\Sigma \subseteq \Sigma \setminus (\Sigma \cap \Omega)$. Now $H_\lambda(x_\lambda H_\lambda)$ is isomorphic to a quotient of H_λ , so $\exp(H_\lambda(x_\lambda H_\lambda)) \leq \exp(H_\lambda) \leq \max_{\lambda \in \Sigma \setminus (\Sigma \cap \Omega)} \exp(H_\lambda)$. And $\max_{\lambda \in \Sigma \setminus (\Sigma \cap \Omega)} \exp(H_\lambda)$ exists by 2 and using the fact that Σ is finite.

(vi). Let $\lambda \in \Sigma \setminus \Gamma_\Sigma$. By 3 we have that $H_\lambda(\iota_\lambda H_\lambda)$ is finite. Moreover, $|H_\lambda(\iota_\lambda H_\lambda)| \leq \max_{\lambda \in \Sigma \setminus \Gamma_\Sigma} |H_\lambda(\iota_\lambda H_\lambda)|$. And $\max_{\lambda \in \Sigma \setminus \Gamma_\Sigma} |H_\lambda(\iota_\lambda H_\lambda)|$ exists since Σ is finite.

Hence by Theorem 4.2.10 $\langle H_\lambda : \lambda \in \Sigma \rangle$ is nilpotent. Whence W is locally nilpotent.

Conversely, suppose W is locally nilpotent. Choose $\lambda, \mu \in \Lambda$ such that $\lambda < \mu$. Now $\langle G_\lambda, G_\mu \rangle \cong G_\lambda \text{wr}_{X_\mu} G_\mu$. By Theorem 4.3.1, there exists a prime p such that G_λ is a locally nilpotent p group of finite exponent and G_μ is locally nilpotent and all orbits of finitely generated subgroups of G_μ have order a power of p . Since Λ is connected, the result follows. \square

The non transitive case

We now consider the case where the groups (G_λ, X_λ) , $\lambda \in \Lambda$, need not be transitive.

Theorem 4.3.3. *For each $\lambda \in \Lambda$, let $H_\lambda \subseteq G_\lambda$ be a finitely generated subgroup of G_λ and suppose $\iota_\lambda H_\lambda \neq \{\iota_\lambda\}$ for any $\lambda \in \Lambda \setminus \Gamma$. Consider the conditions*

1. $H_\omega(\iota_\omega H_\omega)$ is a finite p group and $H_\omega(x_\omega H_\omega)$ is nilpotent for each $\omega \in \Omega$ and each $x \in X$;
2. $H_\gamma(x_\gamma H_\gamma)$ is a nilpotent p group of finite exponent for each $\gamma \in \Gamma$ and each $x \in X$;
3. $H_\lambda(\iota_\lambda H_\lambda)$ is finite p group and $H_\lambda(x_\lambda H_\lambda)$ is a nilpotent p group of finite exponent for each $\lambda \in \Lambda \setminus (\Gamma \cap \Omega)$ and each $x \in X$;
4. for each $\lambda \in \Lambda$ there exists $c_\lambda \in \mathbb{N}$ such that $\text{cl}(H_\lambda(x_\lambda H_\lambda)) \leq c_\lambda$ for each $x \in X$; and

5. for each $\lambda \in \Lambda \setminus \Omega$ there exists $n_\lambda \in \mathbb{N}$ such that $\exp(H_\lambda(x_\lambda H_\lambda)) \leq n_\lambda$ for each $x \in X$.

Then W is locally nilpotent if and only if there exists a prime p such that conditions 1 - 5 hold for each finitely generated subgroup H_λ of G_λ .

Proof. Suppose conditions 1 - 5 hold for each finitely generated subgroup H_λ of G_λ . Let $g_1, \dots, g_m \in W$. Then there exist $\lambda_1, \dots, \lambda_n \in \Lambda$ and finitely generated subgroups H_{λ_i} of G_{λ_i} such that $\{g_1, \dots, g_m\} \subseteq \langle H_{\lambda_1}, \dots, H_{\lambda_n} \rangle$. Let $\Sigma := \{\lambda_1, \dots, \lambda_n\}$. We assume, without loss of generality, that Σ is connected of size at least 2. We apply Theorem 4.2.10 to the group $\langle H_\lambda : \lambda \in \Sigma \rangle$. It is clear that conditions 1 - 6 of Theorem 4.2.10 holds and hence $\langle H_\lambda : \lambda \in \Sigma \rangle$ is nilpotent. Whence W is locally nilpotent.

Conversely, suppose W is locally nilpotent. Choose $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Let H_λ and H_μ be finitely generated subgroups of G_λ and G_μ respectively. Then $\langle H_\lambda, H_\mu \rangle$ is nilpotent. Now Theorem 4.2.10 implies $H_\lambda(x_\lambda H_\lambda)$ is a nilpotent p group of finite exponent for each $x \in X$, $H_\lambda(x_\lambda H_\lambda)$, $H_\mu(x_\mu H_\mu)$ is a finite p group, $H_\mu(x_\mu H_\mu)$ is nilpotent for each $x \in X$, $H_\mu(x_\mu H_\mu)$ and the exponent and nilpotency class of constituents are bounded. This completes the proof. \square

4.4 ZA groups

Note that if a group G is nilpotent, then it is automatically a ZA group. The ZA group property is a natural generalisation of the nilpotent property. In this chapter we aim to characterise when W is a ZA group. In the case where the groups (G_λ, X_λ) are transitive for each $\lambda \in \Lambda$, we obtain a complete characterisation. We note that the class of ZA groups is not a variety and hence we cannot appeal to Theorem 2.2.3 and consequently we do not find a complete characterisation when the groups (G_λ, X_λ) need not be transitive. However we do obtain some positive results if we impose extra conditions on Λ and (G_λ, X_λ) .

Recall Theorem 2.2.1. We find that, using Theorem 4.4.14, the following is true.

Theorem 4.4.1. *Let (A, X) and (B, Y) be non trivial transitive permutation groups. Then $A \text{ wr}_Y B$ is a ZA group if and only if there exists a prime p such that A is a ZA p group and B is a finite p group.*

We start this section by proving the equivalence of Definition 1.1.17 and Definition 1.1.18.

Equivalence of Definition 1.1.17 and Definition 1.1.18

It is immediate that if, in Definition 1.1.18, n can be chosen independently of the sequence $(g_i)_{i \in \mathbb{N}} \subseteq G$, then G is nilpotent. And the converse is also true. In fact, we show that a group G satisfies the sequence property if and only if G is a ZA group.

Lemma 4.4.2. *If G is a non trivial group that satisfies the sequence property, then G has non trivial centre.*

Proof. Suppose G has trivial centre. We construct a sequence of elements $(g_i)_{i \in \mathbb{N}} \subseteq G$ such that $[g_1, \dots, g_n] \neq 1$ for any $n \in \mathbb{N}$. Choose $g_1 \in G \setminus \{1\}$. Then g_1 does not lie in the centre of G , so there exists $g_2 \in G$ such that $[g_1, g_2] \neq 1$ and $[g_1, g_2] \notin \{1\} = Z(G)$. We proceed in this manner to get a sequence of elements $(g_i)_{i \in \mathbb{N}} \subseteq G$ such that $[g_1, \dots, g_n] \neq 1$ for any $n \in \mathbb{N}$, as required. The result follows. \square

Lemma 4.4.3. *If G is a non trivial group that satisfies the sequence property and N is a normal subgroup of G , then the quotient group G/N satisfies the sequence property.*

Proof. Let $(g_i N)_{i \in \mathbb{N}} \subseteq G/N$ be a sequence in G/N . Since G satisfies the sequence property, there exists $n \in \mathbb{N}$ such that $[g_1, \dots, g_n] = 1 \in N$. Now $[g_1 N, \dots, g_n N] = [g_1, \dots, g_n] N = 1$ and hence G/N satisfies the sequence property. \square

Theorem 4.4.4. *Let G be a group. Then G is a ZA group if and only if G satisfies the sequence property.*

Proof. Suppose G is a ZA group. Let $(g_i)_{i \in \mathbb{N}} \subseteq G$ be a sequence in G . There exists an ordinal c_1 such that $g_1 \in G_{c_1}$. We construct a non increasing sequence, $(c_i)_{i \in \mathbb{N}}$, of ordinals by putting $[g_1, \dots, g_i] \in G_{c_i}$. Further, by our remark to Definition 1.1.17, we insist that if $c_i \neq 0$ then $c_i < c_{i-1}$. We know that there is no infinite

decreasing sequence of ordinals and hence there exists $n \in \mathbb{N}$ such that $c_n = 0$. That is to say $[g_1, \dots, g_n] \in G_0 = \{1\}$.

Conversely, suppose G satisfies the sequence property. By Lemma 4.4.2, G has non trivial centre. By Lemma 4.4.3, the quotient group $G/Z(G)$ also satisfies the sequence condition. It follows that we have an ascending central series terminating at G and hence G is a ZA group. \square

Using Theorem 4.4.4 we state and prove a well known result about ZA groups.

Lemma 4.4.5. *If G is a group with normal subgroup N which is contained in some term of the upper central series and G/N is a ZA group, then G is a ZA group.*

Proof. Let $(g_i)_{i \in \mathbb{N}} \subseteq G$ be a sequence in G . Now G/N is a ZA group and, by Theorem 4.4.4, G/N satisfies the sequence property. So there exists $m \in \mathbb{N}$ such that $1 = [g_1N, \dots, g_mN] = [g_1, \dots, g_m]N$. In particular, $[g_1, \dots, g_m] \in N \subseteq G_c$ for some ordinal c . As in the proof of Theorem 4.4.4 we find that there exists $n \in \mathbb{N}$ such that $[g_1, \dots, g_{m+n}] = [[g_1, \dots, g_m], g_{m+1}, \dots, g_{m+n}] = 1$. Hence G satisfies the sequence property and Theorem 4.4.4 implies that G is a ZA group. \square

The transitive case

We find necessary and sufficient conditions for a generalised wreath product of transitive permutation groups to be a ZA group.

Notation. Let G be a group and let c be an ordinal. We write G_c to be the c th term of the upper central series of G .

Theorem 4.4.6. *Let (G_λ, X_λ) be non trivial transitive permutation groups for each $\lambda \in \Lambda$. Suppose there exists a prime p such that every chain in Λ is finite; the set $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$; G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$; and G_γ is a ZA p group for each $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$. Then there exists an ordinal d such that $G_\gamma \subseteq W_d$.*

Proof. We show the following.

1. $Z(G_\gamma) \subseteq W_{d_1}$ for some ordinal d_1 .

2. If there exist ordinals α and β such that $(G_\gamma)_\alpha \subseteq W_\beta$, then $(G_\gamma)_{\alpha+1} \subseteq W_{\beta+d_2}$ for some ordinal d_2 .
3. If α is a limit ordinal and for each $\beta < \alpha$, $(G_\gamma)_\beta \subseteq W_{d_3}$ for some ordinal d_3 , then $(G_\gamma)_\alpha \subseteq W_{d_3}$.

Then in this case the result follows immediately.

1. For $i \in \mathbb{N}$, define $H_i := \langle h \in Z(G_\gamma) : h^{p^i} = 1 \rangle$. Now $\langle H_i, \langle G_\lambda : \lambda > \gamma \rangle \rangle$ is nilpotent of nilpotency class c_i say, as it satisfies conditions 1 to 6 of Theorem 4.2.5. Now as in the proof of Theorem 4.2.5, we have that $H_i \subseteq W_{c_i}$. This holds for each $i \in \mathbb{N}$. Let $d_1 := \lim_i c_i$, then $Z(G_\gamma) \subseteq W_{d_1}$ as required.

2. For each $i \in \mathbb{N}$, let $c_i := p^{|\prod_{\gamma < \lambda} X_\lambda|} |\langle G_\lambda : \gamma < \lambda \rangle|$. We note, by the hypothesis, that $\prod_{\lambda > \gamma} X_\lambda$ is finite and since $\langle G_\lambda : \gamma < \lambda \rangle$ can be thought of as a permutation group on $\prod_{\lambda > \gamma} X_\lambda$, this group is finite. For each $i \in \mathbb{N}$, define subsets of $(G_\gamma)_{\alpha+1}$ by $H_i := \{g \in (G_\gamma)_{\alpha+1} : g^{p^i} = 1\}$. Fix $h \in H_i$ and consider the group $\langle \{h, \langle G_\lambda : \gamma < \lambda \rangle\} \rangle$. This is a finite p group of size at most

$$|\langle h \rangle|^{\prod_{\gamma < \lambda} |X_\lambda|} |\langle G_\lambda : \gamma < \lambda \rangle| \leq p^{i \prod_{\gamma < \lambda} |X_\lambda|} |\langle G_\lambda : \gamma < \lambda \rangle| = c_i.$$

And we note that c_i is independent of the choice of h . Now since $[(G_\gamma)_{\alpha+1}, G_\gamma] \subseteq W_\alpha$, it follows as in the proof of Theorem 4.2.5 that, for any $g_1, \dots, g_{c_i} \in W$ we have $[h, g_1, \dots, g_{c_i}] \in ((G_\gamma)_\alpha)^W \subseteq W_\beta$. And hence $h \in W_{\beta+c_i}$. This holds for each $h \in H_i$, hence $H_i \subseteq W_{\beta+c_i}$. But this holds for each $i \in \mathbb{N}$, thus $(G_\gamma)_{\alpha+1} \subseteq W_{\beta+d_2}$, where $d_2 := \lim_i c_i$, as required.

3. Let $g \in (G_\gamma)_\alpha = \cup_{\beta < \alpha} (G_\gamma)_\beta$. Then there exists an ordinal β with $\beta < \alpha$ such that $g \in W_\beta \subseteq W_{d_3}$. And hence $(G_\gamma)_\alpha \subseteq W_{d_3}$ as required. \square

Theorem 4.4.7. *Let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Suppose there exists a prime p such that every chain in Λ is finite; the set $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$; G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$; and G_γ is a ZA p group for each $\gamma \in \Gamma$. Then W is a ZA group.*

Proof. For each $i \in \mathbb{N}$, let Ω_i be the set of i maximal elements of Λ . Let $H_i = \langle G_\lambda : \lambda \in \Omega_i \rangle$. We show that H_i is a ZA group for each $i \in \mathbb{N}$. We proceed by induction on i . Firstly, we note that $H_1 = \prod_{\lambda \in \Omega_1} H_\lambda$ is a ZA group being the direct product of nilpotent groups. Suppose that H_{i-1} is a ZA group. Let N_i be the smallest normal subgroup of H_i which contains $\langle G_\lambda : \lambda \in \Omega_i \setminus \Omega_{i-1} \rangle$. Now $H_i/N_i \cong H_{i-1}$ which is a ZA group by assumption. Since $\Omega_i \setminus \Omega_{i-1}$ is contained

in the set of minimal elements of Ω_i , by Theorem 4.4.6, there exists an ordinal c_i such that $\langle G_\lambda : \lambda \in \Omega_i \setminus \Omega_{i-1} \rangle \subseteq (H_i)_{c_i}$. Since $(H_i)_{c_i}$ is a normal subgroup of H_i , we have $N_i \subseteq (H_i)_{c_i}$. Hence, by Lemma 4.4.5, H_i is a ZA group. It follows that W is a ZA group. \square

We now consider the converse of Theorem 4.4.7. We start with proving a few straightforward results.

Lemma 4.4.8. *Let $n \in \mathbb{N}$. Then $\sum_{i=0}^n \binom{n}{i} (-1)^i = 0$ where $\binom{n}{i} := \frac{n!}{(n-i)!i!}$*

Proof. It is clear that $\sum_{i=0}^n \binom{n}{i} (-1)^i = (1-1)^n = 0$. \square

Lemma 4.4.9. *Let $r_1, \dots, r_n \in \mathbb{R}$ be such that $\sum_{i=1}^n r_i = 0$, then either $r_1 = \dots = r_n = 0$ or there exists $i, j \in \{1, \dots, n\}$ such that $r_i \neq r_j$.*

Proof. Suppose $r_1 = \dots = r_n$. Then $nr_1 = \sum_{i=1}^n r_i = 0$ and hence $r_1 = 0$. This completes the proof. \square

Lemma 4.4.10. *Let $(A, X), (B, Y)$ be non trivial permutation groups. If A is abelian and $f \in A^{(Y)}$, $b \in B$ and $n \in \mathbb{N}$, then*

$$[f, {}_n b] = \prod_{k=0}^n \left(f^{b^k} \right)^{\binom{n}{k} (-1)^{n+k}}$$

where $\binom{n}{k} := \frac{n!}{(n-k)!k!}$ and $b^0 := 1$. The direct product is well defined as A is abelian.

Proof. Note

$$\begin{aligned} [f, b] &= f^{-1} b^{-1} f b \\ &= f^{-1} f^b \\ &= \prod_{k=0}^1 \left(f^{b^k} \right)^{\binom{1}{k} (-1)^{1+k}} \end{aligned}$$

So the result holds for $n = 1$.

We proceed by induction on n . Suppose $[f, {}_n b] = \prod_{k=0}^n (f^{b^k})^{\binom{n}{k}(-1)^{n+k}}$. Then

$$\begin{aligned}
[f, {}_{n+1} b] &= [[f, {}_n b], b] \\
&= [f, {}_n b]^{-1} [f, {}_n b]^b \\
&= \prod_{k=0}^n (f^{b^k})^{\binom{n}{k}(-1)^{n+k+1}} \prod_{k=0}^n (f^{b^{k+1}})^{\binom{n}{k}(-1)^{n+k}} \\
&= \prod_{k=0}^n (f^{b^k})^{\binom{n}{k}(-1)^{n+k+1}} \prod_{k=1}^{n+1} (f^{b^k})^{\binom{n}{k-1}(-1)^{n+k+1}} \\
&= f^{(-1)^{n+1}} \left(\prod_{k=1}^n (f^{b^k})^{\left(\binom{n}{k} + \binom{n}{k-1}\right)(-1)^{n+k+1}} \right) f^{b^{n+1}} \\
&= \prod_{k=0}^{n+1} (f^{b^k})^{\binom{n+1}{k}(-1)^{(n+1)+k}}
\end{aligned}$$

The third equality follows by induction and the penultimate equality follows since A is abelian. This completes the proof. \square

Theorem 4.4.11. *Let $(A, X), (B, Y)$ be non trivial permutation groups such that A is abelian and is not a torsion group and B is finite. Then $Awr_Y B$ is not a ZA group.*

Proof. Choose a non torsion element $a \in A$ and choose $b \in B$ of prime order p for some prime p . Fix $y \in Y$ such that $yb \neq y$. Let $f \in A^{(Y)}$ be the element defined by $f(y) = a$ and $f(z) = 1$ for each $z \in Y \setminus \{y\}$. For $n \in \mathbb{N}$, let $g_n := [f, {}_n b]$. Note that $g_1 = [f, b] = f^{-1}f^b \neq 1$. Suppose, for a contradiction, that there exists $n \in \mathbb{N}$ such that $g_n = 1$. Let m be the smallest integer such that $g_{m+1} = 1$. Note that $1 = g_{m+1} = [g_m, b] = (g_m)^{-1}(g_m)^b$. It follows that g_m is constant on the orbit $y\langle b \rangle$. Now

$$\begin{aligned}
1 &\neq g_m \\
&= [f, {}_m b] \\
&= \prod_{i=0}^m (f^{b^i})^{\binom{m}{i}(-1)^{m+i}} \\
&= \prod_{i=0}^{\infty} (f^{b^0})^{\binom{m}{ip}(-1)^{m+ip}} \dots \prod_{i=0}^{\infty} (f^{b^{p-1}})^{\binom{m}{ip+p-1}(-1)^{m+ip+p-1}} \\
&= (f^{b^0})^{\sum_{i=0}^{\infty} \binom{m}{ip}(-1)^{m+ip}} \dots (f^{b^{p-1}})^{\sum_{i=0}^{\infty} \binom{m}{ip+p-1}(-1)^{m+ip+p-1}}
\end{aligned}$$

where we set $\binom{n}{k} := 0$ if $n < k$. The second equality follows from Lemma 4.4.10 and the third equality follows since b has order p and A is abelian. Let $r_j :=$

$\sum_{i=0}^{\infty} \binom{m}{ip+j} (-1)^{m+ip+j}$ for $j = 0, \dots, p-1$. Then $\sum_{j=0}^{p-1} r_j = \sum_{i=0}^m \binom{m}{i} (-1)^{m+i} = 0$, the second equality following from Lemma 4.4.8. By Lemma 4.4.9, either $r_0 = \dots = r_{p-1} = 0$ or there exist $k, l \in \{1, \dots, p-1\}$ such that $r_k \neq r_l$. If $r_0 = \dots = r_{p-1} = 0$, then $g_m = f^0 \dots (f^{b^{p-1}})^0 = 1$. A contradiction. If there exist $k, l \in \{1, \dots, p-1\}$ such that $r_k \neq r_l$, then $a^{r_k} = g_m(yb^k) = g_m(yb^l) = a^{r_l}$. The second equality follows since g_m is constant on the orbit $y\langle b \rangle$. A contradiction since a is a non torsion element.

Hence, if we set $h_1 := f$ and $h_i := b$ for $i \geq 2$, then we have a sequence $(h_i)_{i \in \mathbb{N}}$ in $Awr_Y B$ such that $[h_1, \dots, h_n] \neq 1$ for any $n \in \mathbb{N}$. By Theorem 4.4.4, $Awr_Y B$ is not a ZA group, as required. \square

Theorem 4.4.12. *Let $(A, X), (B, Y)$ be non trivial transitive permutation groups. If $Awr_Y B$ is a ZA group, then there exists a prime p such that A and B are p groups.*

Proof. Let $a \in A \setminus \{1\}$, $b \in B \setminus \{1\}$ and choose $y \in Y$ such that $yb \neq y$. Note that $Awr_Y B$ has the property that every non trivial subgroup of $Awr_Y B$ has non trivial centre. By Theorem 2.2.1 and Theorem 3.1.10, B is finite. In particular, b has finite order. Now $\langle a \rangle wr_{y\langle b \rangle} \langle b \rangle$ is isomorphic to a subgroup of $Awr_Y B$ and is hence a ZA group. Theorem 4.4.11 applied to $\langle a \rangle wr_{y\langle b \rangle} \langle b \rangle$ implies a has finite order and hence $\langle a \rangle wr_{y\langle b \rangle} \langle b \rangle$ is finite. Thus $\langle a \rangle wr_{y\langle b \rangle} \langle b \rangle$ is nilpotent. By Theorem 4.2.2, there exists a prime p such that a and b are p elements. This completes the proof. \square

We now prove the converse of Theorem 4.4.7.

Theorem 4.4.13. *Let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Suppose W is a ZA group. Then there exists a prime p such that every chain in Λ is finite; the set $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$; G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$; and G_γ is a ZA p group for each $\gamma \in \Gamma$.*

Proof. Since G_λ is isomorphic to a subgroup of W for each $\lambda \in \Lambda$ and W is a ZA group, it follows that G_λ is a ZA group. Since W is a ZA group, W has the property that every non trivial subgroup of W has non trivial centre. Theorem 3.1.10 implies every chain in Λ is finite; the set $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$; and G_λ is finite for each $\lambda \in \Lambda \setminus \Gamma$. Choose $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Then $G_\lambda wr_{X_\mu} G_\mu \cong \langle G_\lambda, G_\mu \rangle \subseteq W$ which is a ZA group. By Theorem 4.4.12, there exists a prime p such that G_λ and G_μ are p groups. Since Λ is connected, the result follows. \square

Corollary 4.4.14. *Let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Then W is a ZA group if and only if there exists a prime p such that*

1. every chain in Λ is finite;
2. $\{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite for each $\gamma \in \Gamma$;
3. G_λ is a finite p group for each $\lambda \in \Lambda \setminus \Gamma$; and
4. G_γ is a ZA p group for each $\gamma \in \Gamma$.

Proof. This follows immediately from Theorem 4.4.7 and Theorem 4.4.13. \square

The non transitive case

Here to avoid any perverse cases, we will assume that $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for any $\lambda \in \Lambda \setminus \Gamma$.

Theorem 4.4.15. *Suppose Λ is finite and suppose $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for any $\lambda \in \Lambda \setminus \Gamma$. Let (G_λ, X_λ) be a non trivial permutation group with finitely many orbits for each $\lambda \in \Lambda$. Then W is a ZA group if and only if there exists a prime p such that*

1. $G_\omega(\iota_\omega G_\omega)$ is a finite p group and $G_\omega(x_\omega G_\omega)$ is a ZA group for each $\omega \in \Omega$ and each $x \in X$;
2. $G_\lambda(\iota_\lambda G_\lambda)$ is a finite p group and $G_\lambda(x_\lambda G_\lambda)$ is a ZA p group for each $\lambda \in \Lambda \setminus (\Omega \cup \Gamma)$ and each $x \in X$; and
3. $G_\gamma(x_\gamma G_\gamma)$ is ZA p group for each $\gamma \in \Gamma$ and each $x \in X$.

Proof. Suppose W is a ZA group. Fix $\mu \in \Lambda$ and fix $x \in X$. If $G_\mu(x_\mu G_\mu) = \{1\}$, then it is clearly a ZA p group for any prime p . Suppose $G_\mu(x_\mu G_\mu) \neq \{1\}$; this must happen for some $x \in X$ as $G_\mu \neq \{1\}$. If $\mu \in \Omega$, $G_\mu(x_\mu G_\mu)$ is isomorphic to a quotient of G_μ and is hence a ZA group. Suppose $\mu \in \Lambda \setminus \Omega$ and consider the group $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$. Let $(h_i)_{i \in \mathbb{N}} \subseteq \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ be a sequence in $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$. Then there exists a sequence $(g_i)_{i \in \mathbb{N}} \subseteq W$ such that for each $\lambda \geq \mu$ and each $i \in \mathbb{N}$ we have $(zh_i)_\lambda = (zg_i)_\lambda$ for each $z \in \prod_{\lambda \geq \mu} y_\lambda G_\lambda$ where $y_\mu := x_\mu$ and $y_\lambda := \iota_\lambda$ if $\lambda > \mu$. Since W is a

ZA group, by Theorem 4.4.4, there exists $n \in \mathbb{N}$ such that $[g_1, \dots, g_n] = 1$. In particular, $(z[h_1, \dots, h_n])_\lambda = (z[g_1, \dots, g_n])_\lambda = z_\lambda$ for each $z \in \prod_{\lambda \geq \mu} y_\lambda G_\lambda$ and for each $\lambda \geq \mu$. That is, $[h_1, \dots, h_n] = 1$. By Theorem 4.4.4, $(\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle)$ is a ZA group. Hence, by Corollary 4.4.14, $G_\lambda(\iota_\lambda G_\lambda)$ is a finite p group and $G_\mu(x_\mu G_\mu)$ is a ZA p group for some prime p . Since Λ is connected, conditions 1, 2 and 3 hold.

Conversely, suppose conditions 1, 2 and 3 hold. Let $(g_i)_{i \in \mathbb{N}} \subseteq W$ be a sequence in W and let $\mu \in \Lambda$. We find $n_\mu \in \mathbb{N}$ such that $(x[g_1, \dots, g_{n_\mu}])_\mu = x_\mu$ for each $x \in X$. Let $x \in X$. If $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, then it is clear that $(x[g_1, \dots, g_n])_\mu = x_\mu$ for any $n \in \mathbb{N}$. Suppose $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$. Then we can find $(h_i)_{i \in \mathbb{N}} \subseteq (\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle)$ such that $(y g_i)_\mu = (y h_i)_\mu$ for each $y \in \prod_{\lambda \geq \mu} x_\lambda G_\lambda$ and each $i \in \mathbb{N}$. By Corollary 4.4.14, $(\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle)$ is a ZA group. So, by Theorem 4.4.4, there exists $m_\mu \in \mathbb{N}$ such that $[h_1, \dots, h_{m_\mu}] = 1$. In particular, $(y[g_1, \dots, g_{m_\mu}])_\mu = (y[h_1, \dots, h_{m_\mu}])_\mu = y_\mu$ for each $y \in \prod_{\lambda \geq \mu} x_\lambda G_\lambda$. Let n_μ be the maximum of all such m_μ as we range over all orbits of G_μ on X_μ . Since there are finitely many orbits, n_μ exists. It follows that $(x[g_1, \dots, g_{n_\mu}])_\mu = x_\mu$ for each $x \in X$. Let $n = \max_{\lambda \in \Lambda} n_\lambda$, this exists since Λ is finite. Now, $(x[g_1, \dots, g_n])_\lambda = x_\lambda$ for each $\lambda \in \Lambda$ and each $x \in X$. That is, $[g_1, \dots, g_n] = 1$ and W satisfies the sequence property. By Theorem 4.4.4, W is a ZA group as required. \square

4.5 Residually nilpotent groups

Another generalisation of the nilpotent property is the residually nilpotent property. The residually nilpotent standard wreath product was studied by Hartley. See [11]. In this chapter we generalise some of the results of Hartley to the permutational wreath product. We find that not all the results of Hartley extend to the permutational wreath product. We identify precisely where the approach used by Hartley fails to generalise.

We finish the section by developing some simple extensions of these results to W . We first state and prove some straightforward results.

Lemma 4.5.1. *Let G be a residually nilpotent group and let H be a subgroup of G . Then H is residually nilpotent.*

Proof. Notice

$$\{1\} \subseteq \bigcap_{n=1}^{\infty} \gamma_n(H) \subseteq \bigcap_{n=1}^{\infty} (\gamma_n(G) \cap H) \subseteq \bigcap_{n=1}^{\infty} \gamma_n(G) = \{1\}.$$

Hence $\bigcap_{n=1}^{\infty} \gamma_n(H) = \{1\}$ and H is residually nilpotent. \square

Lemma 4.5.2. *Let p and q be primes and let H be a finite group. Suppose H is residually a finite p group and contains an element of order q . Then $p = q$.*

Proof. Let $h \in H$ be of order q . Now there exists a normal subgroup N of H such that $h \notin N$ and the quotient group H/N is a nontrivial finite p group. Note that $(hN)^q = h^q N = 1$. So $q = p^n$ for some $n \in \mathbb{N}$. Since p and q are prime it follows that $p = q$ as required. \square

Now, consider the following three mutually exclusive sets of conditions which a pair of groups A and B may satisfy.

- R1: For some prime p , B is a finite p group and A is residually a nilpotent p group of finite exponent.
- R2: B is infinite but not torsion free; A is abelian; and for some prime p , residually A and B are nilpotent p groups of finite exponent.
- R3: B is torsion free; $B \neq \{1\}$; A is abelian; and for all $p \in \pi(A)$, B is residually a nilpotent p group of finite exponent.

Here $\pi(A)$ denotes the set of primes p such that A contains an element of order p . Hartley in [11] finds some partial results for the standard wreath product of two groups A and B to be residually nilpotent. These are listed below.

Theorem 4.5.3. *Suppose that the standard wreath product $A wr B$ is residually nilpotent, where A and B are non trivial. Then one of conditions R1, R2 and R3 holds.*

Theorem 4.5.4. *If A and B satisfy R1 or R2, then the standard wreath product $A wr B$ is residually a nilpotent p group of finite exponent.*

In this section, we are concerned with generalising these results to the case of the permutational wreath product. We first remark that the proof of Lemma 9(ii) in [11] relies on the fact the action of B on B with the right regular representation

is a regular action. However, the action of B on Y need not be regular. We find that Lemma 9(ii) of [11] fails to generalise to the case of the permutational wreath product.

Theorem 4.5.5. *Let (A, X) and (B, Y) be non trivial transitive permutation groups. Suppose (B, Y) is such that there exist $x, y \in Y$ such that $B^{p^n} \gamma_n(B) \cap \{b \in B : xb = y\}$ is infinite for each $n \in \mathbb{N}$. Let $a \in A \setminus \{1\}$ and define $f \in A^{(Y)}$ by $f(x) := a$, $f(y) := a^{-1}$ and $f(z) := 1$ if $z \neq x, y$. Then the normal closure in $Awr_Y B$ of any normal subgroup N of B such that B/N is a nilpotent p group of finite exponent contains f .*

Proof. Let N be a normal subgroup of B such that B/N is a nilpotent p group of finite exponent. There exists $n \in \mathbb{N}$ such that $B^{p^n} \gamma_n(B) \subseteq N$. It follows that $N \cap \{b \in B : xb = y\} \supseteq B^{p^n} \gamma_n(B) \cap \{b \in B : xb = y\}$ is infinite. In particular, $N \cap \{b \in B : xb = y\}$ is nonempty. It follows that $f \in N[N, A^{(Y)}] = N^{Awr_Y B}$. The equality follows from Meldrum [16] 1.4.15. This completes the proof. \square

Consequently, the approach used by Hartley in [11] to prove Theorem 4.5.4 does not generalise to the permutational wreath product. In fact, the following theorem shows that Theorem 4.5.4 does not extend to the permutational wreath product.

Theorem 4.5.6. *Let (A, X) and (B, Y) be non trivial transitive permutation groups. Suppose condition R2 holds. If there exists $x, y \in Y$ such that $\gamma_n(B) \cap \{b \in B : xb = y\}$ is infinite for each $n \in \mathbb{N}$, then $Awr_Y B$ is not residually nilpotent.*

Proof. Suppose, for a contradiction, that $Awr_Y B$ is residually nilpotent. Define f as in Theorem 4.5.5. Then there exists a normal subgroup K of $Awr_Y B$ such that $f \notin K$ and $(Awr_Y B)/K$ is nilpotent. So there exists $n \in \mathbb{N}$ such that $\gamma_n(Awr_Y B) \subseteq K$. In particular, $\gamma_n(B) \subseteq \gamma_n(Awr_Y B) \cap B \subseteq K \cap B$. Let $N := K \cap B$. Then $N \cap \{b \in B : xb = y\}$ is infinite. In particular, $N \cap \{b \in B : xb = y\}$ is nonempty. It follows that $f \in N[N, A^{(Y)}] = N^{Awr_Y B} \subseteq K$. The equality follows from Meldrum [16] 1.4.15. This yields a contradiction. \square

We introduce another condition which a pair of permutation groups (A, X) and (B, Y) may satisfy.

S2: B is infinite but not torsion free; A is abelian; for some prime p , residually A and B are nilpotent p groups of finite exponent; and for each $x, y \in Y$, there exists $n \in \mathbb{N}$ such that $B^{p^n} \gamma_n(B) \cap \{b \in B : xb = y\}$ is finite.

We note that in the case where the action of B on Y is regular, condition S2 is precisely condition R2. And this is the case in the standard wreath product. We now prove the analogue of Theorem 4.5.3 and Theorem 4.5.4 with R2 replaced by S2. See Theorem 4.5.19, Theorem 4.5.25 and Theorem 4.5.26. Most of the proofs in this section are direct translations of Hartley's proofs. In Lemma 4.5.7 we introduce the notion of topological groups to justify the last line in the proof of Lemma 1 in [11].

Throughout this section (A, X) and (B, Y) are non trivial transitive permutation groups.

Lemma 4.5.7. *If B is infinite (and hence Y is infinite) and $A wr_Y B$ is residually nilpotent, then A is abelian.*

Proof. Let H be a group. Suppose H is residually nilpotent. Let H have the topology \mathcal{T} which has base $\gamma_n(H)$ and all its cosets for each $n \in \mathbb{N}$. This makes H a topological group. Let $g, h \in H$ with $g \neq h$. Suppose, for a contradiction, g and h lie in the same coset of $\gamma_n(H)$ for each $n \in \mathbb{N}$. Hence $gh^{-1} \in \gamma_n(H)$ for each $n \in \mathbb{N}$. That is, $gh^{-1} \in \bigcap_{n \in \mathbb{N}} \gamma_n(H) = \{1\}$. A contradiction since $g \neq h$. So there exists $m \in \mathbb{N}$ such that g and h lie in different cosets of $\gamma_m(H)$. Since these are open sets and different cosets of $\gamma_m(H)$ have empty intersection, \mathcal{T} is Hausdorff. Now for each $n \in \mathbb{N}$, let $f_n \in \gamma_n(H)$. Now let U be an element of the base for \mathcal{T} that contains 1. There exists $m \in \mathbb{N}$ such that $U = \gamma_m(H)$. For each $n \geq m$, we have that $f_n \in \gamma_n(H) \subseteq \gamma_m(H) = U$. Hence f_n converges to 1.

Now consider the wreath product $A wr_Y B$ and suppose $A wr_Y B$ is residually nilpotent. Let $A wr_Y B$ have topology \mathcal{T} as defined above. Let $\pi : A^{(Y)} \rightarrow A$, $f \mapsto f(\iota)$ be a projection. It follows that π is a homomorphism and is continuous. Suppose, for each $n \in \mathbb{N}$, there exists $f_n \in \gamma_n(A wr_Y B) \cap A^{(Y)}$ such that $f_n \pi = a$ for some non trivial element $a \in A$. We have shown above that f_n converges to 1. Since π is continuous, we have that $f_n \pi$ converges to $1\pi = 1$. But $f_n \pi = a$ for each $n \in \mathbb{N}$, so $f_n \pi$ converges to a . And since \mathcal{T}_A is Hausdorff, we know that a sequence in A has at most one limit in A . Here \mathcal{T}_A is the restriction of the topology \mathcal{T} to the set A . Hence $a = 1$ which yields a contradiction. So given that we can find elements $f_n \in \gamma_n(A wr_Y B) \cap A^{(Y)}$ with $f_n(\iota) = f_n \pi = a$ for some $a \in A \setminus \{1\}$, we have that $A wr_Y B$ cannot be residually nilpotent.

Now let $f \in A^{(Y)}$. Since Y is infinite there exists $y \in Y$ such that $f(y) = 1$. Since B is transitive there exists $b \in B$ such that $\iota b^{-1} = y$. Now $[f, b](\iota) = (f^{-1} f^b)(\iota) = f^{-1}(\iota) f(\iota b^{-1}) = f^{-1}(\iota)$. Suppose, for a contradiction, that A is not abelian.

Choose $a_1, a_2 \in A$ such that $[a_1, a_2] \neq 1$. Choose $g_1, g_2 \in A^{(Y)}$ such that $g_1(\iota) = a_1$ and $g_2(\iota) = a_2$. As above, we can choose an infinite sequence b_1, b_2, \dots of elements of B such that $[g_1, b_1, \dots, b_i](\iota) = a_1^{(-1)^i}$ for each $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $f_n := [[g_1, b_1, \dots, b_{2n}], g_2] \in \gamma_{2n+1}(A \text{ wr}_Y B) \cap A^{(Y)}$ and $f_n(\iota) = [a_1, a_2] \neq 1$. Hence $A \text{ wr}_Y B$ is not residually nilpotent. A contradiction. Hence A is abelian and this completes the proof. \square

Notation. Let H, K be groups. We define $\gamma^1(H, K) := [H, K]$ and inductively we write $\gamma^{n+1}(H, K) := [\gamma^n(H, K), K]$.

Lemma 4.5.8. *Suppose A is abelian. Then $A \text{ wr}_Y B$ is residually nilpotent if and only if*

$$\bigcap_{n \in \mathbb{N} \cup \{0\}} \gamma^n(A^{(Y)}, B) = \{1\}.$$

Proof. Clearly $\gamma^n(A^{(Y)}, B) \subseteq \gamma_{n+1}(A \text{ wr}_Y B)$ for $n \geq 0$. Thus, if $A \text{ wr}_Y B$ is residually nilpotent then $\bigcap_{n \in \mathbb{N} \cup \{0\}} \gamma^n(A^{(Y)}, B) \subseteq \bigcap_{n \in \mathbb{N} \cup \{0\}} \gamma_{n+1}(A \text{ wr}_Y B) = \{1\}$.

Conversely, suppose $\bigcap_{n \in \mathbb{N} \cup \{0\}} \gamma^n(A^{(Y)}, B) = \{1\}$. By Theorem 10.3.6 in [8], it follows that $[A^{(Y)}, \gamma_n(B)] \subseteq \gamma^n(A^{(Y)}, B)$ for $n \geq 1$. And hence

$$[A^{(Y)}, \bigcap_{n \in \mathbb{N}} \gamma_n(B)] \subseteq \bigcap_{n \in \mathbb{N}} \gamma^n(A^{(Y)}, B) = \{1\}.$$

Since A is non trivial, we have that $\bigcap_{n \in \mathbb{N}} \gamma_n(B) = \{1\}$. Furthermore, we have that $[W_n, A \text{ wr}_Y B] \subseteq W_{n+1}$ where $W_n := \gamma^{n-1}(A^{(Y)}, B) \gamma_n(B)$. Since $A \text{ wr}_Y B$ is the semidirect product $A^{(Y)}B$, we have that

$$\bigcap_{n \in \mathbb{N}} W_n = \left(\bigcap_{n \in \mathbb{N} \cup \{0\}} \gamma^n(A^{(Y)}, B) \right) \left(\bigcap_{n \in \mathbb{N}} \gamma_n(B) \right) = \{1\}.$$

Hence $A \text{ wr}_Y B$ is residually nilpotent as required. \square

We adopt the approach used by Hartley and we turn our attention to looking at rings. Let $f \in A^{(Y)}$. We define $f.b := b^{-1}fb = f^b$ and $f.n := f^n$ for $b \in B$ and $n \in \mathbb{Z}$ and extend linearly to get $A^{(Y)}$ as a $\mathbb{Z}B$ module. Let $\Delta := \sum_{b \in B} \mathbb{Z}(1 - b)$ be the difference ideal of $\mathbb{Z}B$. Δ consists, by definition, of all $\sum_{b \in B} n_b b \in \mathbb{Z}B$ such that $\sum_{b \in B} n_b = 0$.

Lemma 4.5.9. *Suppose A is abelian. Then $A \text{ wr}_Y B$ is residually nilpotent if and only if*

$$\bigcap_{n \in \mathbb{N} \cup \{0\}} \langle A^{(Y)} \Delta^n \rangle = \{1\}.$$

Proof. Notice, if $f \in A^{(Y)}$ and $b \in B$ then $f(-1 + b) := f^{-1}f^b = [f, b]$ and it follows that $\langle A^{(Y)}\Delta^n \rangle = \gamma^n(A^{(Y)}, B)$ for each $n \in \mathbb{N}$. The result follows from Lemma 4.5.8. \square

Now, if A is abelian it follows, from Theorem 10.3.6 in [8], that $[A^{(Y)}, \gamma_n(B)] \subseteq \gamma^n(A^{(Y)}, B)$. Noting, as in the proof of Lemma 4.5.9, $\langle A^{(Y)}\Delta^n \rangle = \gamma^n(A^{(Y)}, B)$ we can formulate the following Lemma.

Lemma 4.5.10. *Suppose A is abelian, then $[A^{(Y)}, \gamma_n(B)] \subseteq \langle A^{(Y)}\Delta^n \rangle$ for each $n \in \mathbb{N}$.*

Now suppose $A = C_p$, the cyclic group of order p for some prime p . Let $f \in A^{(Y)}$. We define $f.b := f^b$ and $f.n := f^n$ for $b \in B$ and $n \in \mathbb{Z}_p$ and extend linearly to get $A^{(Y)}$ as a $\mathbb{Z}_p B$ module. We let Δ_p be the difference ideal in $\mathbb{Z}_p B$.

Lemma 4.5.11. *Suppose $A = C_p$ and $Awr_Y B$ is residually nilpotent, then*

$$\bigcap_{n \in \mathbb{N}} \langle A^{(Y)}\Delta_p^{p^n} \rangle = \{1\}.$$

Proof. Note that $\langle A^{(Y)}\Delta_p^{p^n} \rangle = \gamma^{p^n}(A^{(Y)}, B) \subseteq \gamma_{p^n}(Awr_Y B)$. And since $Awr_Y B$ is residually nilpotent, we have that

$$\bigcap_{n \in \mathbb{N}} \langle A^{(Y)}\Delta_p^{p^n} \rangle \subseteq \bigcap_{n \in \mathbb{N}} \gamma_{p^n}(Awr_Y B) = \{1\}.$$

\square

Lemma 4.5.12. *Suppose $A = C_p$ and $Awr_Y B$ is residually nilpotent. Let $\psi_n : B \rightarrow \text{Aut}(A^{(Y)} / \langle A^{(Y)}\Delta_p^n \rangle)$ where $(f \langle A^{(Y)}\Delta_p^n \rangle)(b\psi_n) := f^b \langle A^{(Y)}\Delta_p^n \rangle$ for $b \in B$ and $f \in A^{(Y)}$. Then*

1. $B^{p^n} \subseteq \ker \psi_{p^n}$ for each $n \in \mathbb{N}$;
2. $\gamma_{p^n}(B) \subseteq \ker \psi_{p^n}$ for each $n \in \mathbb{N}$; and
3. $\bigcap_{n \in \mathbb{N}} \ker \psi_{p^n} = \{1\}$.

Proof. 1. Let $b \in B$ and let $f \in A^{(Y)}$. Then

$$ff^{-b^{p^n}} = f(1 - b^{p^n}) = f(1 - b)^{p^n} \in \langle A^{(Y)}\Delta_p^{p^n} \rangle$$

noting, using Lemma 4.7.2, $(1 - b^{p^n}) = (1 - b)^{p^n}$ in $\mathbb{Z}_p B$. Hence $b^{p^n} \in \ker \psi_{p^n}$.

2. Let $b \in \gamma_{p^n}(B)$ and let $f \in A^{(Y)}$. Then

$$f^{-1}f^b = [f, b] \in [A^{(Y)}, \gamma_{p^n}(B)] \subseteq \langle A^{(Y)} \Delta_p^{p^n} \rangle$$

where the last inclusion follows from Lemma 4.5.10. Hence $b \in \ker \psi_{p^n}$.

3.

$$\begin{aligned} b \in \bigcap_{n \in \mathbb{N}} \ker \psi_{p^n} &\Rightarrow b \in \ker \psi_{p^n} \text{ for each } n \in \mathbb{N} \\ &\Rightarrow f^{-1}f^b \in \langle A^{(Y)} \Delta_p^{p^n} \rangle \text{ for each } n \in \mathbb{N} \text{ and for each } f \in A^{(Y)} \\ &\Rightarrow f^{-1}f^b \in \bigcap_{n \in \mathbb{N}} \langle A^{(Y)} \Delta_p^{p^n} \rangle = \{1\} \text{ for each } f \in A^{(Y)} \\ &\Rightarrow f^{-1}f^b = 1 \text{ for each } f \in A^{(Y)} \\ &\Rightarrow b = 1. \end{aligned}$$

The third implication follows from Lemma 4.5.11 and the final implication follows since A is non trivial. \square

Lemma 4.5.13. *If $A = C_p$ and $A \text{ wr}_Y B$ is residually nilpotent, then B is residually a nilpotent p group of finite exponent.*

Proof. Let $B_n := B^{p^n} \gamma_{p^n}(B)$ for each $n \in \mathbb{N}$. By Lemma 4.5.12,

$$\bigcap_{n \in \mathbb{N}} B_n \subseteq \bigcap_{n \in \mathbb{N}} \ker \psi_{p^n} = \{1\}$$

where ψ_{p^n} is defined in Lemma 4.5.12. Also B/B_n is a nilpotent p group of finite exponent. Hence B is residually a nilpotent p group of finite exponent as required. \square

Lemma 4.5.14. *If $b \in B$ has order p , then $p(1 - b) \in \Delta^p$.*

Proof. Let $c = 1 - b$. Note $c \in \Delta$. Then $b = 1 - c$ and for $i = 2, \dots, p - 1$ there exists $v_i \in \mathbb{Z}$ such that

$$\begin{aligned} 1 &= b^p \\ &= (1 - c)^p \\ &= 1 + \sum_{i=1}^{p-1} \binom{p}{i} (-c)^i + (-c)^p \\ &= 1 - pc + p \sum_{i=2}^{p-1} (-v_i) c^i + (-1)^p c^p. \end{aligned}$$

The last equality follows from Lemma 4.7.2. Hence

$$pc = p \sum_{i=2}^{p-1} v_i c^i + (-1)^{p-1} c^p.$$

We use this identity to prove that $pc^i \in \Delta^p$ for each $i \geq 1$. We proceed by induction. This is clear if $i \geq p$ since $c \in \Delta$. Fix $k \in \{1, \dots, p-1\}$ and suppose $pc^i \in \Delta^p$ for each $i > k$. Then

$$pc^k = p \sum_{i=2}^{p-1} v_i c^{i+k-1} + (-1)^{p-1} c^{p+k-1} \in \Delta^p.$$

The last inclusion follows by induction. It follows $p(1-b) = pc \in \Delta^p$ as required. \square

Lemma 4.5.15. *If $b \in B$ has order p , then $(1-b)(1-d^{p^n}) \in \Delta^{n+2}$ for each $d \in B$ and for each $n \in \mathbb{N} \cup \{0\}$.*

Proof. We proceed by induction on n . If $n = 0$, then $(1-b)(1-d) \in \Delta^2$ since $1-b, 1-d \in \Delta$. Now suppose result holds for $n-1$. So, for some $\alpha \in \mathbb{Z}B$,

$$\begin{aligned} (1-b)(1-d^{p^n}) &= (1-b)(1-d^{p^{n-1}})(p + \sum_{i=0}^{p-1} d^{ip^{n-1}} - p) \\ &= p(1-b)(1-d^{p^{n-1}}) + (1-b)(1-d^{p^{n-1}}) \left(\sum_{i=0}^{p-1} d^{ip^{n-1}} - p \right) \\ &= (1-b)(1-d^{p^{n-1}}) \left(\alpha(1-b)^{p-1} + \left(\sum_{i=0}^{p-1} d^{ip^{n-1}} - p \right) \right) \\ &\in \Delta^{n+2}. \end{aligned}$$

The third equality follows from Lemma 4.5.14 and the last inclusion follows by induction and noting $\sum_{i=0}^{p-1} d^{ip^{n-1}} - p \in \Delta$ since the sum of the coefficients is zero. \square

Lemma 4.5.16. *If B is residually a nilpotent p group of finite exponent, then $b \in B$ is either a p element or has infinite order.*

Proof. Suppose $b \in B$ is a q element for some prime q . There exists a normal subgroup N of B with $b \notin N$ such that B/N is a nilpotent p group of finite exponent. Let $\psi : B \rightarrow B/N, c \mapsto cN$. Then ψ is a homomorphism and $b\psi \neq 1$. So $b\psi$ is a q element in B/N , which is a p group. So $p = q$. \square

Lemma 4.5.17. *If $b \in B$ has order p and $\bigcap_{n \in \mathbb{N}} \langle A^{(Y)} \Delta^n \rangle = \{1\}$, then B is residually a nilpotent p group of finite exponent.*

Proof. Firstly suppose that there exists $a \in A$ of order q for some prime q . Lemma 4.5.9, applied to $\langle a \rangle wr_Y B$, implies that $\langle a \rangle wr_Y B$ is residually nilpotent. Now Lemma 4.5.13, applied to $\langle a \rangle wr_Y B$, implies that B is residually a nilpotent q group of finite exponent. Lemma 4.5.16 implies B contains no element of order prime to q . Hence $p = q$ and the result is established in this case.

Now suppose A is torsion free. So A contains an infinite cyclic group. Without loss of generalisation, we assume A is the infinite cyclic group and $\bigcap_{n \in \mathbb{N}} \Delta^n = \{0\}$. For each $n \in \mathbb{N}$, let $B_n := B^{p^n} \gamma_{n+2}(B)$. Let $c \in \bigcap_{n \in \mathbb{N}} B_n$. Then, for each $n \in \mathbb{N}$, $c \in B_n$ and Lemma 4.5.10 and Lemma 4.5.15 imply that $(1-b)(1-c) \in \Delta^{n+2}$. Hence $(1-b)(1-c) \in \bigcap_{n \in \mathbb{N}} \Delta^{n+2} = \{0\}$. Hence $1-b-c+bc=0$. We note that $1, b, c$ and bc are not all distinct, otherwise this would contradict the linear independence of $1, b, c$ and bc in $\mathbb{Z}B$. And since $1, b, c$ and bc are non zero and $b \neq 1$, it follows that either $b = c$ or $b = bc$. However, if $b = c$ then this implies that $b = 1$. This is a contradiction. Hence $b = bc$ and so $c = 1$. Thus $\bigcap_{n \in \mathbb{N}} B_n = \{1\}$. This completes the proof since B/B_n is a nilpotent p group of finite exponent. \square

Lemma 4.5.18. *Let p be prime. If $A wr_Y B$ is residually nilpotent and $B = C_p$, then A is residually a nilpotent p group of finite exponent.*

Proof. We show that $A^{(Y)}$ is residually a nilpotent p group of finite exponent. Write $B = \langle b \rangle$. We first show that for each $n \in \mathbb{N}$, there exists $\alpha_n \in \mathbb{Z}B$ such that

$$p^n(1-b) = \alpha_n(1-b)^{n(p-1)+1}$$

in the group ring $\mathbb{Z}B$. We proceed by induction.

If $n = 1$, result follows from Lemma 4.5.14 noting Δ is generated by $1-b$. Now suppose the result holds for n . Then

$$\begin{aligned} p^{n+1}(1-b) &= \alpha_n(1-b)^{n(p-1)} p(1-b) \\ &= \alpha_n(1-b)^{n(p-1)} \alpha_1(1-b)^p \\ &= \alpha_{n+1}(1-b)^{(n+1)(p-1)+1} \end{aligned}$$

where $\alpha_{n+1} := \alpha_n \alpha_1$. In particular, $p^n(1-b) \in \Delta^{c(n)}$ where $c(n) := n(p-1) + 1$. Hence, rewriting this in multiplicative notation, we have

$$[f^{p^n}, b] \in \langle A^{(Y)} \Delta^{c(n)} \rangle \subseteq \gamma_{c(n)}(A wr_Y B)$$

for each $f \in A^{(Y)}$. Now let $A_n^{(Y)} := (A^{(Y)})^{p^n} (\gamma_{c(n)}(A \text{ wr}_Y B) \cap A^{(Y)})$. Now $[A_n^{(Y)}, b] \subseteq \gamma_{c(n)}(A \text{ wr}_Y B)$ for each $n \in \mathbb{N}$. And hence

$$[\bigcap_{n \in \mathbb{N}} A_n^{(Y)}, b] \subseteq \bigcap_{n \in \mathbb{N}} \gamma_{c(n)}(A \text{ wr}_Y B) = \{1\}.$$

Thus $\bigcap_{n \in \mathbb{N}} A_n^{(Y)} = \{1\}$. Since $A^{(Y)}/A_n^{(Y)}$ is a nilpotent p group of finite exponent, the result follows. \square

We are now in a position to give a generalisation of Theorem 4.5.3 to the case of permutational wreath product.

Theorem 4.5.19. *Let $(A, X), (B, Y)$ be non trivial transitive permutation groups. Suppose $A \text{ wr}_Y B$ is residually nilpotent then one of the conditions R1, S2 and R3 holds.*

Proof. Firstly suppose that B is finite. Since B is non trivial there exists $b \in B$ of order p for some prime p . Then Lemma 4.5.18, applied to $A \text{ wr}_Y \langle b \rangle$, implies A is residually a nilpotent p group of finite exponent. Let $a \in A \setminus \{1\}$. We apply Lemma 4.5.9 and Lemma 4.5.17 to the residually nilpotent group $\langle a \rangle \text{ wr}_Y B$ to conclude that B is residually a nilpotent p group of finite exponent. Hence, using Lemma 4.5.2, B is a finite p group. So R1 holds.

Now suppose B is infinite and not torsion free. Lemma 4.5.7 implies A is abelian. Since B is not torsion free then there exists $b \in B$ of order p . Then Lemma 4.5.9 and Lemma 4.5.17 imply B is residually a nilpotent p group of finite exponent and Lemma 4.5.18, applied to $A \text{ wr}_Y \langle b \rangle$, implies A is residually a nilpotent p group of finite exponent. Now, Theorem 4.5.6 implies that for each $x, y \in Y$ there exists $n \in \mathbb{N}$ such that $B^{p^n} \gamma_n(B) \cap \{b \in B : xb = y\}$ is finite. So S2 holds.

Finally suppose B is torsion free. Lemma 4.5.7 implies A is abelian. Let $p \in \pi(A)$. There exists $a \in A$ such that a has order p . Then Lemma 4.5.13, applied to $\langle a \rangle \text{ wr}_Y B$, implies B is residually a nilpotent p group of finite exponent. So R3 holds. This completes the proof. \square

We now consider the generalisation of Theorem 4.5.4 to the permutational wreath product.

Theorem 4.5.20. *Let \mathcal{X} be a class of groups. Suppose A is residually a \mathcal{X} group. If $fb \in A \text{ wr}_Y B \setminus \{1\}$, then there exists a normal subgroup N of A and a normal subgroup K of $A \text{ wr}_Y B$ such that*

1. A/N is a \mathcal{X} group;
2. $fb \notin K$; and
3. $(A \text{ wr}_Y B)/K \cong A/N \text{ wr}_Y B$.

Proof. Let $fb \in A \text{ wr}_Y B \setminus \{1\}$. If $b \neq 1$, then $fb \notin A^{(Y)}$ and $(A \text{ wr}_Y B)/A^{(Y)} \cong \{1\} \text{ wr}_Y B$. If $b = 1$, there exists $y \in Y$ such that $f(y) \neq 1$. Since A is residually a \mathcal{X} group, there exists a normal subgroup N of A such that $f(y) \notin N$ and A/N is a \mathcal{X} group. Now, $f \notin N^Y$ and $(A \text{ wr}_Y B)/N^Y \cong A/N \text{ wr}_Y B$. The isomorphism follows from Meldrum [16] 1.4.13. This completes the proof. \square

Theorem 4.5.21. *Suppose condition S2 holds. If $fb \in A \text{ wr}_Y B \setminus \{1\}$, then there exists a normal subgroup N of B and a normal subgroup K of $A \text{ wr}_Y B$ such that*

1. B/N is a nilpotent p group of finite exponent;
2. $fb \notin K$; and
3. $(A \text{ wr}_Y B)/K \cong A \text{ wr}_{\{yN : y \in Y\}}(B/N)$.

Proof. Let $fb \in A \text{ wr}_Y B \setminus \{1\}$. Suppose $b \neq 1$. There exists a normal subgroup N of B such that $b \notin N$ and B/N is a nilpotent p group of finite exponent. Now $fb \notin N[N, A^{(Y)}] = N^{A \text{ wr}_Y B}$. The equality follows from [16] 1.4.15. Further, $(A \text{ wr}_Y B)/N^{A \text{ wr}_Y B} \cong A \text{ wr}_{\{yN : y \in Y\}}(B/N)$.

Now suppose $b = 1$ and $\prod_{y \in Y} f(y) \neq 1$. Let N be a normal subgroup of B such that B/N is a nilpotent p group of finite exponent. We note that, since A is abelian, if $g \in [N, A^{(Y)}]$ then $\prod_{x \in yN} g(x) = 1$ for each $y \in Y$. In particular, $\prod_{y \in Y} g(y) = 1$ for each $g \in [N, A^{(Y)}]$. Hence $f \notin N[N, A^{(Y)}] = N^{A \text{ wr}_Y B}$. The equality follows from Meldrum [16] 1.4.15. Further, $(A \text{ wr}_Y B)/N^{A \text{ wr}_Y B} \cong A \text{ wr}_{\{yN : y \in Y\}}(B/N)$.

Finally suppose $b = 1$, $\prod_{y \in Y} f(y) = 1$ and $\sigma(f) = \{y_1, \dots, y_k\}$. Let N be a normal subgroup of B such that B/N is a nilpotent p group of finite exponent. There exists $n \in \mathbb{N}$ such that $B^{p^n} \gamma_n(B) \subseteq N$ and $B^{p^n} \gamma_n(B) \cap (\cup_{i=2}^k \{b \in B : y_i b = y_1\})$ is finite. Write $B^{p^n} \gamma_n(B) \cap (\cup_{i=2}^k \{b \in B : y_i b = y_1\}) = \{b_1, \dots, b_m\}$. For each $i = 1, \dots, m$, there exists a normal subgroup N_i of B such that $b_i \notin N_i$ and B/N_i is a nilpotent p group of finite exponent. Then $M := (\cap_{i=1}^m N_i) \cap B^{p^n} \gamma_n(B)$ is a normal subgroup of B such that M and $\cup_{i=2}^k \{b \in B : y_i b = y_1\}$ have empty intersection and B/M is a nilpotent p group of finite exponent. We note that,

since A is abelian, if $g \in [M, A^{(Y)}]$ then $\prod_{x \in yM} g(x) = 1$ for each $y \in Y$. It is easy to see that $\prod_{x \in y_1 M} f(x) = f(y_1) \neq 1$ and hence $f \notin M[M, A^{(Y)}] = M^{Awr_Y B}$. Furthermore, $(Awr_Y B)/M^{Awr_Y B} \cong Awr_{\{yM : y \in Y\}}(B/M)$. \square

Definition 4.5.22. Let H be a group and suppose H has a finite series of normal subgroups $\{1\} = H_{c+1} \subseteq H_c \subseteq \dots \subseteq H_1 = H$. We say that this is a strong central series if $[H_i, H_j] \subseteq H_{i+j}$ for each i, j and also each of the groups H_i/H_{i+1} is a direct product of (possibly infinitely many) cyclic groups which are either infinite or of order p^k , where p is some fixed prime and k is bounded by an integer N depending on H .

Theorem 4.5.23. *Let H be a group having a finite strong central series, each factor of which is the direct product of a free abelian group and a group of exponent dividing p^N , where p is a fixed prime and N is a fixed integer. Let R be a commutative ring with 1 satisfying $\bigcap_{n \in \mathbb{N}} p^n R = \{0\}$, and let Δ be the difference ideal of RH . Then $\bigcap_{n \in \mathbb{N}} \Delta^n = \{0\}$.*

For a proof of Theorem 4.5.23, see Theorem E in [11].

Corollary 4.5.24. *Let p be prime. If A is a cyclic p group and B is a nilpotent p group of finite exponent, then $Awr_Y B$ is residually nilpotent.*

Proof. Since B is a nilpotent p group of finite exponent, the lower central series of B is a strong central series with each of the factors an abelian p group of finite exponent. Theorem 4.5.23 now shows that if Δ is the difference ideal of $\mathbb{Z}B$, then $\bigcap_{n \in \mathbb{N}} \Delta^n = \{0\}$. Now, Lemma 4.5.9 implies that $Awr_Y B$ is residually nilpotent. \square

We now give the generalisation of Theorem 4.5.4 with R2 replaced by S2.

Theorem 4.5.25. *If condition R1 holds, then $Awr_Y B$ is residually a nilpotent p group of finite exponent. In particular, $Awr_Y B$ is residually nilpotent.*

Proof. Let $fb \in Awr_Y B \setminus \{1\}$. By Lemma 4.5.20, there exists a normal subgroup N of A and a normal subgroup K of $Awr_Y B$ such that A/N is a nilpotent p group of finite exponent, $fb \notin K$ and $(Awr_Y B)/K \cong A/N wr_Y B$. Now Theorem 2.1.8, Theorem 2.2.1 and Theorem 4.2.2 imply $A/N wr_Y B$ is a nilpotent p group of finite exponent. This completes the proof. \square

Theorem 4.5.26. *If condition S2 holds, then $Awr_Y B$ is residually nilpotent.*

Proof. Let $a \in A \setminus \{1\}$. There exists a normal subgroup N of A such that $a \notin N$ and A/N is an abelian p group of finite exponent. Thus A/N is a direct product of cyclic p groups. Hence A/N , and therefore A also, is residually a cyclic p group. By Theorem 4.5.20 and Theorem 4.5.21, we may assume that A is a cyclic p group and B is a nilpotent p group of finite exponent. Corollary 4.5.24 implies $A \text{ wr}_Y B$ is residually nilpotent as required. \square

The transitive case

We now look at the case where Λ is any totally ordered set. We obtain a complete characterisation of the residually nilpotent generalised wreath product when Λ is totally ordered and the groups (G_λ, X_λ) are transitive. We start with a result that gives some information about the structure of Λ .

Proposition 4.5.27. *Let Λ be a totally ordered set and let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Suppose W is residually nilpotent. Then the set $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is finite for each $\mu \in \Lambda$. In particular, Λ has a maximal element.*

Proof. Let $\mu \in \Lambda$. Suppose, for a contradiction, that $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is infinite. Choose elements $\mu_i \in \{\lambda \in \Lambda : \mu < \lambda\}$, $i \in \mathbb{N}$, with $\mu_i < \mu_j$ if $i < j$. Then $\langle G_{\mu_i} : i \geq 2 \rangle$ is infinite. Notice that $\langle G_\mu, G_{\mu_1} \rangle \text{ wr}_Y \langle G_{\mu_i} : i \geq 2 \rangle \cong \langle G_\mu, \{G_{\mu_i} : i \in \mathbb{N}\} \rangle \subseteq W$, for some set Y , and is hence residually nilpotent. By Theorem 4.5.7, we have that $G_\mu \text{ wr}_{X_{\mu_1}} G_{\mu_1} \cong \langle G_\mu, G_{\lambda_1} \rangle$ is abelian, which contradicts Theorem 4.1.1. Thus $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is finite. \square

Remark. Suppose Λ is totally ordered and satisfies the condition $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is finite for each $\mu \in \Lambda$. If Λ has a minimal element, γ say, then $\Lambda = \{\lambda \in \Lambda : \gamma \leq \lambda\}$ is finite. In particular, if Λ is infinite then Λ has no minimal element.

Theorem 4.5.28. *Let Λ be an infinite totally ordered set. Then W is residually nilpotent if and only if the set $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is finite for each $\mu \in \Lambda$ and there exists a prime p such that G_λ is a finite p group for each $\lambda \in \Lambda$.*

Proof. Suppose W is residually nilpotent. Fix $\mu \in \Lambda$. The set $\{\lambda \in \Lambda : \mu \leq \lambda\}$ is finite by Proposition 4.5.27. Suppose, for a contradiction, that G_μ is infinite. By the remark to Proposition 4.5.27, Λ does not have a minimal element. So there exists $\mu_1, \mu_2 \in \Lambda$ with $\mu_1 < \mu_2 < \mu$. Now $\langle G_{\mu_1}, G_{\mu_2} \rangle \text{ wr}_{X_\mu} G_\mu \cong \langle G_{\mu_1}, G_{\mu_2}, G_\mu \rangle \subseteq$

W is residually nilpotent. Now, Theorem 4.5.7 implies that $\langle G_{\mu_1}, G_{\mu_2} \rangle$ is abelian. This contradicts Theorem 4.1.1. Hence G_μ is finite. It follows from Theorem 4.5.19 that G_μ is a p_μ group for some prime p_μ . Now let $\omega \in \Lambda$ be the maximal element; ω exists by Proposition 4.5.27.

Since Λ is connected it follows, from Lemma 4.5.2 and Theorem 4.5.19, that G_μ is a p group for the same prime p for each $\mu \in \Lambda$.

Conversely, suppose that the set $\{\lambda \in \Lambda : \mu < \lambda\}$ is finite for each $\mu \in \Lambda$ and there exists a prime p such that G_λ is a finite p group for each $\lambda \in \Lambda$. Let $g \in W$. Then $g \in \langle G_{\lambda_1}, \dots, G_{\lambda_n} \rangle$ for some $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_1 < \dots < \lambda_n$. Now $g \in \langle G_{\lambda_1}, \dots, G_{\lambda_n} \rangle \subseteq \langle G_\lambda : \lambda_1 \leq \lambda \rangle$. Since $\{\lambda \in \Lambda : \lambda_1 \leq \lambda\}$ is finite and G_λ is a finite p group for each $\lambda \geq \lambda_1$ it follows, by Corollary 4.2.9, that $\langle G_\lambda : \lambda_1 \leq \lambda \rangle$ is nilpotent. Let N be the smallest normal subgroup of W to contain that group $\langle G_\lambda : \lambda < \lambda_1 \rangle$. It is easy to see that $g \notin N$ and the quotient group $W/N \cong \langle G_\lambda : \lambda_1 \leq \lambda \rangle$ which is nilpotent. Hence W is residually nilpotent as required. \square

Theorem 4.5.28 characterises fully the residually nilpotent generalised wreath product when Λ is a infinite totally ordered set and (G_λ, X_λ) is a non trivial transitive permutation group for each $\lambda \in \Lambda$. We now consider the case where Λ is finite.

Theorem 4.5.29. *Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 < \dots < \lambda_n$ and $n \geq 3$. Let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Then W is residually nilpotent if and only if there exists a prime p such that either*

1. G_{λ_1} is residually a nilpotent p group of finite exponent and G_{λ_i} is a finite p group for each $i = 2, \dots, n$; or
2. G_{λ_1} is abelian and residually a nilpotent p group of finite exponent; G_{λ_2} is infinite and residually a nilpotent p group of finite exponent; G_{λ_i} is a finite p group for each $i = 3, \dots, n$; and for each $x, y \in \prod_{i=2}^n X_{\lambda_i}$ there exists $n \in \mathbb{N}$ such that $H^{p^n} \gamma_n(H) \cap \{h \in H : xh = y\}$ is finite, where $H := \langle G_{\lambda_i} : i = 2, \dots, n \rangle$.

Proof. Suppose W is residually nilpotent. Fix $i \geq 3$ and suppose, for a contradiction, that G_{λ_i} is infinite. Then $\langle G_{\lambda_1}, G_{\lambda_2} \rangle wr_{X_{\lambda_i}} G_{\lambda_i} \cong \langle G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_i} \rangle \subseteq W$ is residually nilpotent. So, by Theorem 4.5.7, $G_{\lambda_1} wr_{X_{\lambda_2}} G_{\lambda_2} \cong \langle G_{\lambda_1}, G_{\lambda_2} \rangle$ is abelian

which contradicts Theorem 4.1.1. So G_{λ_i} is finite and, by Theorem 4.5.19, we have that G_{λ_i} is a p_i group for some prime p_i . Now fix $j \in \{3, \dots, n\}$, then $\langle G_{\lambda_j}, G_{\lambda_n} \rangle \subseteq W$ is residually nilpotent. Theorem 4.5.19 implies that G_{λ_j} is residually a nilpotent p_n group of finite exponent. However, we have already shown that G_{λ_j} is a finite p_j group, so Lemma 4.5.2 implies that $p_j = p_n$. It follows all the primes in the above statements are the same. Say $p_i = p$ for each $i = 1, \dots, n$. By Theorem 4.5.19 applied to the residually nilpotent subgroups $\langle G_{\lambda_1}, G_{\lambda_n} \rangle$ and $\langle G_{\lambda_2}, G_{\lambda_n} \rangle$, it follows that G_{λ_1} and G_{λ_2} are residually a nilpotent p group of finite exponent.

If G_{λ_2} is finite, then Lemma 4.5.2 implies that G_{λ_2} is a finite p group. So condition 1 holds.

If G_{λ_2} is infinite, then $\langle G_{\lambda_i} : i = 2, \dots, n \rangle$ is infinite but not torsion free. And Theorem 4.5.19 implies that G_{λ_1} is abelian and for each $x, y \in \prod_{i=2}^n X_{\lambda_i}$ there exists $n \in \mathbb{N}$ such that $H^{p^n} \gamma_n(H) \cap \{h \in H : xh = y\}$ is finite, where $H := \langle G_{\lambda_i} : i = 2, \dots, n \rangle$.

Conversely, suppose condition 1 holds. Notice that, by repeated use of Lemma 2.1.8, $\langle G_{\lambda_i} : i = 2, \dots, n \rangle$ is a finite p group. By Theorem 4.5.25, we have that $W = \langle G_{\lambda_1}, \langle G_{\lambda_i} : i = 2, \dots, n \rangle \rangle$ is residually nilpotent.

Now suppose condition 2 holds. Notice that, by repeated use of Lemma 2.1.8, $\langle G_{\lambda_i} : i = 3, \dots, n \rangle$ is a finite p group. Now, Theorem 4.5.25 implies that $\langle G_{\lambda_i} : i = 2, \dots, n \rangle$ is residually a nilpotent p group of finite exponent. Also, $\langle G_{\lambda_i} : i = 2, \dots, n \rangle$ is infinite but not torsion free and Theorem 4.5.26 implies that W is residually nilpotent. \square

4.6 Locally boundedly nilpotent groups

The next class of groups we consider is the class of locally boundedly nilpotent groups. This again generalises the nilpotent condition. We develop necessary and sufficient conditions for the wreath product $A wr_Y B$ to be locally boundedly nilpotent. We find $A wr_Y B$ is locally boundedly nilpotent if and only if A and B are both locally boundedly nilpotent p groups of bounded exponent, for some prime p . We generalise this result to W where Λ is finite.

We start by stating some known results.

Lemma 4.6.1. *Let H be a group. If H is locally boundedly nilpotent, then H is a bounded Engel group.*

Proof. Suppose H is locally boundedly nilpotent with lbn function f . Let $g, h \in H$. Then $\langle g, h \rangle \subseteq H$ is nilpotent of nilpotency class at most $f(2)$. Now $g, h \in \langle g, h \rangle$ and hence $[g,_{f(2)} h] = 1$. This completes the proof. \square

Theorem 4.6.2. *If $Awr_Y B$ is a bounded Engel group, then there exists a prime p such that A and B are bounded Engel p groups of bounded exponent.*

For a proof see Meldrum [16] 6.3.3.

Theorem 4.6.3. *Let H be a finitely generated nilpotent group with r generators, of exponent n and nilpotency class c . Then H is finite with order at most $r^{\frac{nc(1+c)}{2}}$.*

For a proof see Quintana Jr [19].

We come to the main result of this chapter.

Theorem 4.6.4. *$Awr_Y B$ is locally boundedly nilpotent if and only if there exists a prime p such that A and B are both locally boundedly nilpotent p groups of bounded exponent.*

Proof. Suppose $Awr_Y B$ is locally boundedly nilpotent. Since A and B are isomorphic to subgroups of $Awr_Y B$, A and B are both locally boundedly nilpotent. Moreover, by Lemma 4.6.1, $Awr_Y B$ is a bounded Engel group and, by Theorem 4.6.2, there exists a prime p such that A and B are both p groups of bounded exponent.

Conversely, suppose there exists a prime p such that A and B are locally boundedly nilpotent p groups of finite exponent. Let $f_A : \mathbb{N} \rightarrow \mathbb{N}$ and $f_B : \mathbb{N} \rightarrow \mathbb{N}$ be lbn functions for A and B respectively. Let $n \in \mathbb{N}$ and let $\{f_1 b_1, \dots, f_n b_n\} \subseteq Awr_Y B$ with $\{f_1, \dots, f_n\} \subseteq A^{(Y)}$ and $\{b_1, \dots, b_n\} \subseteq B$. Let $B^* := \langle b_1, \dots, b_n \rangle$. We show that $\langle f_1 b_1, \dots, f_n b_n \rangle$ is nilpotent of bounded nilpotency class, bounded by a function of n . By hypothesis, B^* is nilpotent of nilpotency class at most $f_B(n)$ and is also a p group of finite exponent. So, by Theorem 4.6.3, B^* is finite of size at most $k_n := n^{\frac{\exp(B)f_B(n)(1+f_B(n))}{2}}$ where $\exp(B)$ is the exponent of B . Thus the orbits of B^* have order a power of p . Now Theorem 4.3.1 implies that $Awr_Y B$ is locally nilpotent.

Let \mathcal{Z} be the set of orbits of B^* on Y . If $Z \in \mathcal{Z}$, then Z has order at most k_n . For each $Z \in \mathcal{Z}$, let $A_Z^* := \langle f_i(z) : i = 1, \dots, n, z \in Z \rangle \subseteq A$. Then A_Z^* has at most nk_n generators, is of exponent dividing $\exp(A)$ and has nilpotency class at most $f_A(nk_n)$, as A is a locally boundedly nilpotent p group of finite exponent with lbn function f_A . Thus, by Theorem 4.6.3, A_Z^* is finite of order at most $c_n := (nk_n)^{\frac{(\exp(A)f_A(nk_n)(1+f_A(nk_n)))}{2}}$. Let $Z \in \mathcal{Z}$ and let $\bar{A}_Z := \prod_{y \in Z} A_{Z,y}^* \subseteq A^{(Y)}$, where $A_{Z,y}^* := \{f \in A^Z : f(y) \in A_Z^* \text{ and } f(z) = 1 \text{ if } y \neq z\}$. Since Z is an orbit of B^* , it follows that $\bar{A}_Z^{B^*} = \bar{A}_Z$ and we have a semi direct product $\bar{A}_Z B^*$. Let $d_n := c_n k_n^2$. Now $\bar{A}_Z B^*$ is finite and

$$\begin{aligned} |\bar{A}_Z B^*| &= |\bar{A}_Z| |B^*| \\ &\leq |Z| \max_{y \in Z} |A_{Z,y}^*| |B^*| \\ &\leq |A_Z^*| |B^*|^2 \\ &\leq c_n k_n^2 \\ &= d_n. \end{aligned}$$

Also $\bar{A}_Z B^*$ is a p group and hence nilpotent of nilpotency class at most d_n . Thus, $Z_{d_n}(\bar{A}_Z B^*) \supseteq \bar{A}_Z$ for each $Z \in \mathcal{Z}$.

Let $\bar{A} := \prod_{Z \in \mathcal{Z}} \bar{A}_Z \subseteq A^{(Y)}$, the restricted direct product. Since \bar{A}_Z is normalised by B^* for each $Z \in \mathcal{Z}$, so is \bar{A} and we have a semi direct product $\bar{A} B^*$ containing $\{f_1 b_1, \dots, f_n b_n\}$, by the construction of \bar{A} . We show that $\bar{A} B^*$ is nilpotent of bounded nilpotency class. We claim that for each $i \in \mathbb{N}$, $Z_i(\bar{A} B^*) \supseteq \bar{A} \cap \prod_{Z \in \mathcal{Z}} Z_i(\bar{A}_Z B^*)$, where the direct product is restricted. We proceed by induction on i . Clearly, the result holds for $i = 0$. Suppose the claim holds for i and let $f \in \bar{A} \cap \prod_{Z \in \mathcal{Z}} Z_{i+1}(\bar{A}_Z B^*)$. Let $gb \in \bar{A} B^*$, where g is considered as a function on \mathcal{Z} with $g(Z) \in \bar{A}_Z$. Then $[f, gb] = f^{-1} b^{-1} g^{-1} f g b$. We consider the Z component of this element which lies in \bar{A} , $(f^{-1} b^{-1} g^{-1} f g b)(Z) = f^{-1}(Z) g^{-b}(Z) f^b(Z) g^b(Z)$. Now, since \bar{A}_Z is normalised by B^* , it follows that $f^b(Z)$ and $g^b(Z)$ lie in \bar{A}_Z . But this element is just $[f(Z), g(Z)b]$ with the elements being considered as lying in $\bar{A}_Z B^*$. Since $f(Z) \in Z_{i+1}(\bar{A}_Z B^*)$ it follows that $[f(Z), g(Z)b] \in Z_i(\bar{A}_Z B^*)$. This is true for each $Z \in \mathcal{Z}$. And hence, by induction, $[f, gb] \in \prod_{Z \in \mathcal{Z}} Z_i(\bar{A}_Z B^*) \subseteq Z_i(\bar{A} B^*)$. Thus $f \in Z_{i+1}(\bar{A} B^*)$ and we have proved our claim.

Thus $Z_{d_n}(\bar{A} B^*) \supseteq \bar{A}$ and since B^* is nilpotent of nilpotency class at most $f_B(n)$, it follows that $Z_{d_n + f_B(n)}(\bar{A} B^*) = \bar{A} B^*$. This completes the proof. \square

It easy to extend, by induction using Theorem 2.1.8, Theorem 4.6.4 to the iterated wreath product $A_1 \text{ wr}_{X_2} \dots \text{ wr}_{X_n} A_n$ for non trivial permutation groups

$(A_1, X_1), \dots, (A_n, X_n)$. We obtain the following Theorem.

Theorem 4.6.5. *Let $(A_1, X_1), \dots, (A_n, X_n)$ be non trivial permutation groups with $n \geq 2$. Then the iterated wreath product $A_1 wr_{X_2} \dots wr_{X_n} A_n$ is locally boundedly nilpotent if and only if there exists a prime p such that A_1, \dots, A_n are locally boundedly nilpotent p groups of finite exponent.*

Proof. Suppose there exists a prime p such that A_1, \dots, A_n are locally boundedly nilpotent p groups of finite exponent. By Theorem 2.1.8 and Theorem 4.6.4, $A_1 wr_{X_2} A_2$ is locally boundedly nilpotent p group of finite exponent. We proceed by induction. Suppose $A_1 wr_{X_2} \dots wr_{X_i} A_i$ is a locally boundedly nilpotent p group of finite exponent. Theorem 2.1.8 and Theorem 4.6.4 imply that $(A_1 wr_{X_2} \dots wr_{X_i} A_i) wr_{X_{i+1}} A_{i+1}$ is a locally boundedly nilpotent p group of finite exponent. It follows that $A_1 wr_{X_2} \dots wr_{X_n} A_n$ is locally boundedly nilpotent.

Conversely, suppose $A_1 wr_{X_2} \dots wr_{X_n} A_n$ is locally boundedly nilpotent. Fix $i \in \{2, \dots, n\}$. Then $A_1 wr_{X_i} A_i$ is isomorphic to a subgroup of $A_1 wr_{X_2} \dots wr_{X_n} A_n$ and is hence locally boundedly nilpotent. Theorem 4.6.4 implies that there exists a prime p such that A_1 and A_i are locally boundedly nilpotent p groups of finite exponent. \square

The transitive case

Note that if $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 < \dots < \lambda_n$ is a finite totally ordered set and (G_λ, X_λ) is transitive for each $\lambda \in \Lambda$, then W is locally boundedly nilpotent if and only if there exists a prime p such that G_λ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda$. This follows from Theorem 4.6.5 noting the isomorphism $W \cong G_{\lambda_1} wr_{X_{\lambda_2}} \dots wr_{X_{\lambda_n}} G_{\lambda_n}$. We extend this result to the case where Λ is a finite partially ordered set.

Theorem 4.6.6. *Suppose Λ is a finite partially ordered set and let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Then W is locally boundedly nilpotent if and only if there exists a prime p such that G_λ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda$.*

Proof. Suppose W is locally boundedly nilpotent. Let $\mu \in \Lambda$. Since Λ is connected, there exists $\lambda \in \Lambda$ with either $\lambda < \mu$ or $\mu < \lambda$. If $\lambda < \mu$, then $G_\lambda wr_{X_\mu} G_\mu \cong \langle G_\lambda, G_\mu \rangle \subseteq W$ and $G_\lambda wr_{X_\mu} G_\mu$ is locally boundedly nilpotent.

The isomorphism follows from Theorem 2.2.1. Theorem 4.6.4 implies that G_μ is a locally boundedly nilpotent p group of finite exponent. If $\mu < \lambda$, we find that $G_\mu \text{wr}_{X_\lambda} G_\lambda$ is locally boundedly nilpotent and Theorem 4.6.4 implies that G_μ is a locally boundedly nilpotent p group of finite exponent. Since Λ is connected we have G_μ is a p group for the same prime p for each $\mu \in \Lambda$.

Conversely, suppose there exists a prime p such that G_λ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda$. By Lemma 2.1.1, we can extend (Λ, \leq) to a total order (Λ, \preceq) . We write $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \prec \dots \prec \lambda_n$. Let V denote the generalised wreath product of the permutation groups (G_λ, X_λ) with Λ equipped with the total order \prec . We have shown above that V is locally boundedly nilpotent. Let $\psi : V \rightarrow W, k_1 \dots k_n \mapsto k_1 \dots k_n$ where $k_i \in G_{\lambda_i}^{(G_{\lambda_j}, j > i)}$ for each $i = 1, \dots, n$. Then ψ is a homomorphism and is surjective. It follows that W is locally boundedly nilpotent. \square

If (A, X) and (B, Y) are locally boundedly nilpotent p groups of finite exponent, then Theorem 4.6.4 implies that $A \text{wr}_Y B$ is locally boundedly nilpotent. Now if f_A and f_B are lbn functions for A and B respectively, and $d_{A,B} : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto d_n$ where d_n is defined in the proof of Theorem 4.6.4, then we find in the proof of Theorem 4.6.4 that $f_{A \text{wr}_Y B} := d_{A,B} + f_B$ is an lbn function for $A \text{wr}_Y B$. Now suppose Λ is a finite partially ordered and (G_λ, X_λ) is a transitive locally boundedly nilpotent p group of finite exponent. By Lemma 2.1.1 we extend (Λ, \leq) to a totally ordered set (Λ, \preceq) . We write $\Lambda := \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \prec \dots \prec \lambda_n$. By Theorem 2.1.8 and Theorem 4.6.5, $G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_n}} G_{\lambda_n}$ is a locally boundedly nilpotent p group of finite exponent. We note that for each $i = 2, \dots, n$ we have that $f_{G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_i}} G_{\lambda_i}} = d_{G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_{i-1}}} G_{\lambda_{i-1}}, G_{\lambda_i}} + f_{G_{\lambda_i}}$. Using this we can explicitly calculate an lbn function $f_{G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_n}} G_{\lambda_n}}$ for $G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_n}} G_{\lambda_n}$. We note that this expression is a function of the lbn functions and exponents of the groups $G_{\lambda_1}, \dots, G_{\lambda_n}$. Now, Theorem 2.1.8 and Theorem 4.6.6 imply that W is a locally boundedly nilpotent p group of finite exponent where W is the generalised wreath product of the groups $(G_\lambda, X_\lambda), \lambda \in \Lambda$, with Λ equipped with the partial order \leq . Since there exists an epimorphism $\psi : G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_n}} G_{\lambda_n} \rightarrow W$, it follows $f_W := f_{G_{\lambda_1} \text{wr}_{X_{\lambda_2}} \dots \text{wr}_{X_{\lambda_n}} G_{\lambda_n}}$ is an lbn function for W .

The non transitive case

We now turn our attention to the case where the groups (G_λ, X_λ) need not be transitive. Again to avoid any perverse cases, we will assume that $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for any $\lambda \in \Lambda \setminus \Gamma$.

Theorem 4.6.7. *Let Λ be a finite partially ordered set and let (G_λ, X_λ) be a non trivial permutation group with finitely many orbits for each $\lambda \in \Lambda$. Suppose $\iota_\lambda G_\lambda \neq \{\iota_\lambda\}$ for any $\lambda \in \Lambda \setminus \Gamma$. Then W is locally boundedly nilpotent if and only if there exists a prime p such that*

1. $G_\lambda(x_\lambda G_\lambda)$ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda \setminus \Omega$ and each $x \in X$;
2. $G_\omega(\iota_\omega G_\omega)$ is a locally boundedly nilpotent p group of finite exponent for each $\omega \in \Omega$; and
3. $G_\omega(x_\omega G_\omega)$ is locally boundedly nilpotent for each $\omega \in \Omega$ and each $x \in X$.

Proof. Suppose W is locally boundedly nilpotent with lbn function f . Let $\mu \in \Lambda$ and fix $x \in X$. If $G_\mu(x_\mu G_\mu) = \{1\}$, then it is clearly a locally boundedly nilpotent p group of finite exponent for any prime p . Suppose $G_\mu(x_\mu G_\mu) \neq \{1\}$; this must happen for some $x \in X$ as $G_\mu \neq \{1\}$. If $\mu \in \Omega$, then $G_\mu(x_\mu G_\mu)$ is isomorphic to a quotient of G_μ , which is isomorphic to a subgroup of W . Hence $G_\mu(x_\mu G_\mu)$ is locally boundedly nilpotent. Suppose $\mu \notin \Omega$. Consider the group $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$. Let $n \in \mathbb{N}$ and let $k_1, \dots, k_n \in \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$. Let $\psi : W \rightarrow \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ be the natural homomorphism. Note that ψ is surjective. There exist $g_1, \dots, g_n \in W$ such that $g_i \psi = k_i$ for each $i = 1, \dots, n$. Now $\langle g_1, \dots, g_n \rangle \subseteq W$ is a nilpotent subgroup of nilpotency class at most $f(n)$. Now if $l_0, l_1, \dots, l_{f(n)} \in \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$, there exist $h_0, h_1, \dots, h_{f(n)} \in \langle g_1, \dots, g_n \rangle$ such that $h_i \psi = l_i$ for each $i = 0, \dots, f(n)$. Now $[l_0, \dots, l_{f(n)}] = [h_0, \dots, h_{f(n)}] \psi = 1$ and $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is nilpotent of nilpotency class at most $f(n)$. That is to say that $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is locally boundedly nilpotent. Theorem 4.6.6 implies that there exists a prime p such that $G_\lambda(\iota_\lambda G_\lambda)$ for $\lambda > \mu$ and $G_\mu(x_\mu G_\mu)$ are locally boundedly nilpotent p groups of finite exponent. Since Λ is connected, we are done.

Conversely, suppose there exists a prime p such that conditions 1,2 and 3 hold. By the hypothesis there are finitely many groups of the form $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda >$

$\mu\}, G_\mu(x_\mu G_\mu))$ for all possible $\mu \in \Lambda$ and each $x \in X$. By Theorem 4.6.6, each of these groups are locally boundedly nilpotent. Since there are finitely many such groups, they have common lbn function, f say. Let $g_1, \dots, g_n \in W$ and let $H := \langle g_1, \dots, g_n \rangle$. Fix $h_0, h_1, \dots, h_{f(n)} \in H$ and let $h = [h_0, \dots, h_{f(n)}]$. We show that $h = 1$. Let $x \in X$ and fix $\mu \in \Lambda$. If $x_\lambda \notin \iota_\lambda G_\lambda$ for some $\lambda > \mu$, then it is clear that $(xh)_\mu = x_\mu$. Suppose $x_\lambda \in \iota_\lambda G_\lambda$ for each $\lambda > \mu$. Let $\psi : W \rightarrow \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ be the natural homomorphism. Then $H\psi \subseteq \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is a finitely generated subgroup of $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ with at most n generators, and is hence nilpotent of class at most $f(n)$. Thus $h\psi = [h_0\psi, \dots, h_{f(n)}\psi] = 1$. Further, $(xh)_\mu = (x(h\psi))_\mu = (x1)_\mu = x_\mu$. Here we view x as an element of $\prod_{\lambda \geq \mu} x_\lambda G_\lambda$ by truncation. It follows $h = 1$, and we are done. \square

Corollary 4.6.8. *Let Λ be a finite partially ordered set and let (G_λ, X_λ) be a non trivial permutation group for each $\lambda \in \Lambda$. Let p be prime. Then W is a locally boundedly nilpotent p group of finite exponent if and only if G_λ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda$.*

Proof. We note that by Lemma 2.1.1, we can extend (Λ, \leq) to a total order (Λ, \preceq) . We write $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \prec \dots \prec \lambda_n$.

Suppose W is a locally boundedly nilpotent p group of finite exponent. Then, since G_λ is isomorphic to a subgroup of W , G_λ is a locally boundedly nilpotent p group of finite exponent.

Conversely, suppose G_λ is a locally boundedly nilpotent p group of finite exponent for each $\lambda \in \Lambda$. Firstly, Theorem 2.1.8 implies that W is a p group of finite exponent. Now, fix $\mu \in \Lambda$ and let $x \in X$. As in the proof of Theorem 4.6.7 we find that $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is locally boundedly nilpotent. Further, $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ has lbn function $f_{G_{\lambda_1} wr_{X_{\lambda_2}} \dots wr_{X_{\lambda_n}} G_{\lambda_n}}$. It follows, as in the proof of Theorem 4.6.7, that W is locally boundedly nilpotent. \square

4.7 Bounded Engel groups

The final generalisation of nilpotent groups we consider are bounded Engel groups. Recall Theorem 4.6.2 which states that if $A wr_Y B$ is a bounded Engel group, then A and B are both bounded Engel p groups of finite exponent, for some prime p . However, it is not known whether these are sufficient conditions for $A wr_Y B$ to be

a bounded Engel group. But, it will be shown that if, in addition, A is nilpotent then $A wr_Y B$ is indeed a bounded Engel group. We should mention here that May, in [14], shows that $A wr B$, the standard restricted wreath product, is a bounded Engel group if, in addition, A is soluble.

The 2 Engel wreath product

Before we consider the bounded Engel wreath product, we first have a look at 2 Engel wreath products. The Theorem demonstrates the restrictive nature of this condition.

Theorem 4.7.1. *Let $(A, X), (B, Y)$ be non trivial permutation groups. Then $A wr_Y B$ is a 2-Engel group if and only if A and B are abelian groups of exponent 2 and the orbits of B on Y have size at most 2.*

Proof. Suppose $A wr_Y B$ is a 2-Engel group. Notice that for $f \in A^{(Y)}$ and $b \in B$, we have $1 = [f, {}_2 b] = f^{-b} f f^{-b} f b^2$

Let $b \in B$. Suppose, for a contradiction, that there exists $y \in Y$ with y, yb^{-1} and yb^{-2} all distinct. If $a_1, a_2, a_3 \in A$ we can choose $f \in A^{(Y)}$ with $f(y) = a_1$, $f(yb^{-1}) = a_2^{-1}$ and $f(yb^{-2}) = a_3$. Then $1 = [f, {}_2 b](y) = a_2 a_1 a_2 a_3$. This holds for all $a_1, a_2, a_3 \in A$. In particular, if $a_2 = a_3 = 1$, then for all $a_1 \in A$, $a_1 = 1$. That is, $A = \{1\}$, a contradiction. So the orbits of $\langle b \rangle$ on Y have size at most 2. That is, $yb^2 = y$ for each $y \in Y$. i.e $b^2 = 1$ and B has exponent 2.

Let $b \in B \setminus \{1\}$. Then there exists $y \in Y$ with $y \neq yb^{-1}$. If $a_1, a_2 \in A$, we can choose $f \in A^{(Y)}$ with $f(y) = a_1$ and $f(yb^{-1}) = a_2^{-1}$. Then, noting $y = yb^{-2}$, we have $1 = [f, {}_2 b](y) = a_2 a_1 a_2 a_1$. This holds for all $a_1, a_2 \in A$. In particular, if $a_2 = 1$, then for all $a_1 \in A$ we have $a_1^2 = 1$. That is, A has exponent 2.

As A and B have exponent 2, they are abelian. It follows that $A^{(Y)}$ is also abelian of exponent 2.

Now let $fb, gc \in A wr_Y B$ with $f, g \in A^{(Y)}$ and $b, c \in B$. Then, noting $A^{(Y)}$ and B are abelian of exponent 2, we have $1 = [fb, {}_2 gc] = gg^b g^c g^{bc}$. Now suppose, for a contradiction, that there exists an orbit of B on Y with at least three elements. So there exists $y \in Y$, and $b, c \in B \setminus \{1\}$ with y, yb and yc all distinct. But since $y \neq yb$ it follows that $yc \neq ybc$ and similarly $ybc \neq yb$ and $ybc \neq y$. So y, yb, yc and ybc are all distinct. If $a_1, a_2, a_3, a_4 \in A$, we can choose $g \in A^{(Y)}$ with $g(y) = a_1$,

$g(yb^{-1}) = a_2$, $g(yc^{-1}) = a_3$ and $g(y(bc)^{-1}) = a_4$. Then $1 = [b, {}_2gc](y) = a_1a_2a_3a_4$. This holds for all $a_1, a_2, a_3, a_4 \in A$. In particular, if $a_1 = a_2 = a_3 = 1$, we have for all $a_1 \in A$, $a_1 = 1$. That is $A = \{1\}$, a contradiction. Thus, the orbits of B on Y have size at most 2, as required.

Conversely, suppose that A and B are abelian groups of exponent 2 and the orbits of B on Y have size at most 2. Note that $A^{(Y)}$ is also abelian of exponent 2. Let $fb, gc \in A wr_Y B$ with $f, g \in A^{(Y)}$ and $b, c \in B$. So, as above, we have $[fb, {}_2gc] = g^{bc}g^b g^c g$. Let $y \in Y$. We have five cases

1. $y = yb = yc$;
2. $y \neq yb$ and $y = yc$;
3. $y \neq yb$ and $yb = yc$;
4. $y \neq yc$ and $y = yb$; and
5. y, yb, yc all distinct.

1. Notice that $y = yb$ if and only if $yc = ybc$ and hence $y = yb = yc = ybc$ and $[fb, {}_2gc](y) = (g^{bc}g^b g^c g)(y) = (g(y))^4 = 1$ since $A^{(Y)}$ has exponent 2.

2. Notice $y = yc$ if and only if $yb = ybc$ and hence $yc = y \neq yb = ybc$ and $[fb, {}_2gc](y) = g(y)^2 g(yb)^2 = 1$ since $A^{(Y)}$ is abelian of exponent 2.

3. Notice $yb = yc$ if and only if $y = ybc$ and hence $yc = yb \neq y = ybc$ and $[fb, {}_2gc](y) = g(yb)^2 g(y)^2 = 1$ since $A^{(Y)}$ is abelian of exponent 2.

4. Similar to case 2.

5. This contradicts the hypothesis that the orbits of B on Y have size at most 2.

Thus we have $[fb, {}_2gc] = 1$ for all $fb, gc \in A wr_Y B$ and $A wr_Y B$ is 2-Engel. This completes the proof. \square

The Bounded Engel wreath product

We now consider the bounded Engel wreath product. We start with a few straightforward results.

Lemma 4.7.2. *If p is prime and $n \in \mathbb{N}$, then $p \mid \binom{p^n}{k}$ for each $k \in \{1, \dots, p^n - 1\}$ where $\binom{p^n}{k} := \frac{p^n!}{(p^n - k)!k!}$.*

Proof. First note that

$$\binom{p^n}{k} = \frac{p^n(p^n - 1) \dots (p^n - (k - 1))}{1 \dots (k - 1)k} = \frac{p^n}{k} \prod_{i=1}^{k-1} \frac{p^n - i}{i}.$$

Now fix $i \in \{1, \dots, k - 1\}$. If $p^j \mid i$ for some $j \in \mathbb{N}$, then $p^j \mid p^n - i$ and since $k < p^n$ and p is prime, the result follows. \square

Lemma 4.7.3. *Let $(A, X), (B, Y)$ be non trivial permutation groups. If $H \subseteq A$ and $f \in H^{(Y)}$, then $f^b \in H^{(Y)}$ for each $b \in B$. Here $H^{(Y)} := \{f \in A^{(Y)} : f(y) \in H \text{ for each } y \in Y\}$.*

Proof. Let $f \in H^{(Y)}$ and let $b \in B$. For each $y \in Y$, $f^b(y) = f(yb^{-1}) \in H$. Hence $f^b \in H^{(Y)}$. \square

Lemma 4.7.4. *Let $(A, X), (B, Y)$ be non trivial permutation groups. Suppose $A = A_0 \supseteq \dots \supseteq A_n = \{1\}$ is a central series for A . Let $f \in A_i^{(Y)}$ for some $i \in \{0, \dots, n\}$. Then for each $m \in \mathbb{N}$, $gc \in A \text{ wr}_Y B$ with $g \in A^{(Y)}$ and $c \in B$, we have $[f, {}_m gc] = [f, {}_m c] \pmod{A_{i+1}^{(Y)}}$.*

Proof. Let $\psi : A \text{ wr}_Y B \rightarrow (A \text{ wr}_Y B)/A_{i+1}^{(Y)}, fb \mapsto fbA_{i+1}^{(Y)}$. Since $A_i^{(Y)}/A_{i+1}^{(Y)} \subseteq Z(A^{(Y)}/A_{i+1}^{(Y)})$ it follows that $[f, g]^c \in A_{i+1}^{(Y)}$. Now, by Hall's identities, $[f, gc] = [f, c][f, g]^c$ and result holds for $m = 1$. We proceed by induction. Suppose $[f, {}_m gc] = [f, {}_m c] \pmod{A_{i+1}^{(Y)}}$. Now,

$$\begin{aligned} [f, {}_{m+1} gc]\psi &= [[f, {}_m gc], gc]\psi \\ &= [[f, {}_m gc]\psi, gc\psi] \\ &= [[f, {}_m c]\psi, g\psi c\psi] \\ &= [[f, {}_m c]\psi, c\psi][[f, {}_m c]\psi, g\psi]^{c\psi} \\ &= [[f, {}_m c], c]\psi[[f, {}_m c], g]^c\psi \\ &= [f, {}_{m+1} c]\psi. \end{aligned}$$

The third equality follows by induction; the fourth equality follows from Hall's identities; and the final equality follows since $[f, {}_m c] \in A_i^{(Y)}$ and hence $[[f, {}_m c], g] \in A_{i+1}^{(Y)}$. This completes the proof. \square

We now come to the main result of this section.

Theorem 4.7.5. *Let $(A, X), (B, Y)$ be non trivial permutation groups. If A is a nilpotent p group of finite exponent and B is a bounded Engel p group of finite exponent, for some prime p , then $A wr_Y B$ is a bounded Engel group.*

Proof. Firstly suppose that A is abelian of exponent p . Let $\exp(B) = p^n$, for some $n \in \mathbb{N}$. Let $k \in A^{(Y)}, c \in B$, then

$$\begin{aligned} [k, {}_{p^n}c] &= \prod_{i=0}^{p^n} (k^{c^i})^{(p_i^n)(-1)^{p^n+i}} \\ &= k^{(-1)^{p^n} k^{c^{p^n}}} \\ &= k^{(-1)^{p^n} k} \\ &= 1 \end{aligned}$$

by Lemma 4.4.10; Lemma 4.7.2; and using the fact that exponent A is p and exponent of B is p^n . Now if m is the Engel class of B and $fb, gc \in A wr_Y B$, then $[fb, {}_m gc] = k[b, {}_m c] = k$ for some $k \in A^{(Y)}$. Since A is abelian, it follows by induction, $[fb, {}_{m+i} gc] = [k, {}_i c]$ for each $i \in \mathbb{N}$. In particular, $[fb, {}_{m+p^n} gc] = [k, {}_{p^n} c] = 1$ as required.

Now suppose A is abelian of exponent p^n for some $n \in \mathbb{N}$. For $i \in \{0, \dots, n\}$ define $A_i := \langle a^{p^i} : a \in A \rangle$. Since A is abelian we have a normal series

$$A = A_0 \supseteq A_1 \supseteq \dots \supseteq A_n = \{1\}$$

such that A_i/A_{i+1} is an abelian p group of exponent p , for each $i \in \{0, \dots, n-1\}$. And so $(A_i/A_{i+1}) wr_Y B$ is a bounded Engel p group of Engel class at most $l := \exp(B) + \text{cl}(B)$. Since $(A wr_Y B)/A_1^{(Y)} \cong (A/A_1) wr_Y B$ we have, for $f \in A^{(Y)}$ and $b \in B$, $[f, {}_l b] \in A_1^{(Y)}$. The isomorphism follows from Meldrum [16] 1.4.13. Now if $j \in \{1, \dots, n-1\}$, $g \in A_j^{(Y)}$ and $c \in B$ we have $[g, {}_l c] \in A_{j+1}^{(Y)}$, since $(A_j wr_Y B)/A_{j+1}^{(Y)} \cong (A_j/A_{j+1}) wr_Y B$. Thus, by induction, $[f, {}_{ml} b] \in A_n^{(Y)} = \{1\}$. Now as A is abelian, it follows as above that $A wr_Y B$ is a bounded Engel group of Engel class at most $\log_p(\exp(A))(\exp(B) + \text{cl}(B)) + \text{cl}(B)$.

Finally suppose that A is a nilpotent p group of finite exponent. Let m be the Engel class of B and let $fb, gc \in A wr_Y B$, with $f, g \in A^{(Y)}$ and $b, c \in B$. Then $[fb, {}_m gc] = l[b, {}_m c] = l$ for some $l \in A^{(Y)}$. Now let $A = A_1 \supseteq \dots \supseteq A_n = \{1\}$ be a central series for A . For each $i \in \{1, \dots, n-1\}$, $A_i/A_{i+1} wr_Y B$ is a bounded Engel group of Engel class m_i , say, since A_i/A_{i+1} is an Abelian p group of finite exponent. We note that, by Lemma 4.7.4, if $k \in A_i^{(Y)}$ for some $i \in \{1, \dots, n-1\}$, then $[k, {}_{m_i} gc] = [k, {}_{m_i} c] \pmod{A_{i+1}^{(Y)}}$. And since $(A_i wr_Y B)/A_{i+1}^{(Y)} \cong A_i/A_{i+1} wr_Y B$

and this is a bounded Engel group of Engel class m_i , we have $[k, m_i c] \in A_{i+1}^{(Y)}$. In particular, $[k, m_i gc] \in A_{i+1}^{(Y)}$. And hence, by induction, we have $[fb, m + \sum_{i=1}^{n-1} m_i gc] \in A_n^{(Y)} = \{1\}$ as required. \square

4.8 Soluble groups

In this section we develop necessary and sufficient conditions for a generalised wreath product to be soluble. Firstly we deal with the case where the permutation groups (G_λ, X_λ) are transitive. We note that the class of soluble groups of solubility class at most n , for some $n \in \mathbb{N}$, is a variety and so we appeal to Theorem 2.2.3 to obtain a complete characterisation of soluble generalised wreath products when the groups (G_λ, X_λ) need not be transitive.

A full characterisation of the soluble wreath product is known. The proof is straightforward and a version is given below.

Theorem 4.8.1. *Let $(A, X), (B, Y)$ be non trivial permutation groups. Then $A wr_Y B$ is soluble if and only if A and B are soluble.*

Proof. Suppose $A wr_Y B$ is soluble. Then A and B are soluble since A and B are isomorphic to subgroups of $A wr_Y B$.

Conversely, suppose A and B are soluble. There exist $m, n \in \mathbb{N}$ such that $\delta_m(A) = \{1\}$ and $\delta_n(B) = \{1\}$. It is easy to see that $\delta_i(A wr_Y B) \subseteq A^{(Y)} \delta_i(B)$ for each $i \in \mathbb{N}$. In particular, $\delta_n(A wr_Y B) \subseteq A^{(Y)}$ and hence $\delta_{n+m}(A wr_Y B) \subseteq \delta_m(A^{(Y)}) = \{1\}$. \square

Before we proceed to the generalised wreath product, we state a well known result about soluble groups. For a proof see Rotman [23] 6.13.

Lemma 4.8.2. *Let G be a group and let N be a normal subgroup of G . If N and G/N are soluble, then G is soluble and $cl(G) \leq cl(G/N) + cl(N)$ where $cl(G)$, $cl(G/N)$ and $cl(N)$ denote the solubility class of G , G/N and N respectively.*

The transitive case

We find that W is soluble if and only if G_λ is soluble of bounded solubility class and all chains in Λ are finite of bounded length.

Theorem 4.8.3. *Let Λ be a partially ordered set with all chains finite of length at most n and let (G_λ, X_λ) be transitive permutation groups for each $\lambda \in \Lambda$. Suppose G_λ is soluble of solubility class at most c for each $\lambda \in \Lambda$. Then W is soluble of solubility class at most nc .*

Proof. Let Ω_i be the set of all i -maximal elements of Λ . Let $H_i := \langle G_\lambda : \lambda \in \Omega_i \rangle$. Notice that $H_n = W$. Now $H_1 = \prod_{\lambda \in \Omega_1} H_\lambda$ is soluble of solubility class at most c , being the direct product of soluble groups of solubility class at most c . We proceed by induction. Suppose H_k is soluble of class at most kc . Let N_{k+1} be the smallest normal subgroup in H_{k+1} containing $\langle G_\lambda : \lambda \in \Omega_{k+1} \setminus \Omega_k \rangle$. By Proposition 2.1.6, since $\Omega_{k+1} \setminus \Omega_k$ is contained in the set of minimal elements of Ω_{k+1} , N_{k+1} is the direct sum of soluble groups of solubility class at most c and is hence soluble of solubility class at most c . By Proposition 2.1.7, we have that $H_k \cong H_{k+1}/N_{k+1}$ and, by Lemma 4.8.2, it follows that H_{k+1} is soluble of solubility class at most $(k+1)c$. And hence $W = H_n$ is soluble of solubility class at most nc , as required. \square

We now turn our attention to proving the converse of Theorem 4.8.3.

Theorem 4.8.4. *Let $(A, X), (B, Y)$ be non trivial soluble transitive permutation groups. Then $A wr_Y B$ is soluble and $cl(A) < cl(A wr_Y B)$ where $cl(A)$ and $cl(A wr_Y B)$ denote the solubility class of A and $A wr_Y B$ respectively.*

Proof. Theorem 4.8.1 implies $A wr_Y B$ is soluble. Since (B, Y) is a non trivial permutation group, there exist $x, y \in Y$ with $x \neq y$. For each $a \in A$, define $f_a \in A^{(Y)}$ by

$$f_a(z) = \begin{cases} a & \text{if } z = x \\ a^{-1} & \text{if } z = y \\ 1 & \text{if } z \neq x, y. \end{cases}$$

Now $\langle f_a : a \in A \rangle \subseteq \delta_1(A wr_Y B)$ by Meldrum [16] 1.4.9. Consider the map $\pi_x : \langle f_a : a \in A \rangle \rightarrow A, f \mapsto f(x)$. This is a homomorphism and is surjective. Thus $A \cong \langle f_a : a \in A \rangle / \ker \pi_x$ and $cl(A) \leq cl(\langle f_a : a \in A \rangle) \leq cl(\delta_1(A wr_Y B)) < cl(A wr_Y B)$. This completes the proof. \square

Theorem 4.8.5. *Let $(A_1, X_1), \dots, (A_n, X_n)$ be non trivial soluble transitive permutations groups. Then $A_1 wr_{Y_2} A_2 wr_{Y_3} \dots wr_{Y_n} A_n$ is soluble of class at least n .*

Proof. $A_1 wr_{Y_2} A_2 wr_{Y_3} \dots wr_{Y_n} A_n$ is soluble by repeated use of Theorem 4.8.1. By a simple induction, using Theorem 4.8.4, the result follows. \square

Theorem 4.8.6. *Let Λ be a partially ordered set and let (G_λ, X_λ) be a non trivial transitive permutation group for each $\lambda \in \Lambda$. Suppose W is soluble. Then G_λ is soluble of bounded solubility class for each $\lambda \in \Lambda$ and all chains in Λ are finite of bounded length.*

Proof. Since G_λ is isomorphic to a subgroup of W for each $\lambda \in \Lambda$, we have G_λ is soluble of solubility class at most the solubility class of W . Let $\Sigma \subseteq \Lambda$ be a chain in Λ . Suppose, for a contradiction, that Σ is not finite. For each $k \in \mathbb{N}$, let Σ_k be a subset of Σ of size k . Write $\Sigma_k := \{\sigma_1, \dots, \sigma_k\}$ with $\sigma_1 < \dots < \sigma_k$. Then $G_{\sigma_1} wr_{X_{\sigma_2}} \dots wr_{X_{\sigma_k}} G_{\sigma_k} \cong \langle G_{\sigma_1}, \dots, G_{\sigma_k} \rangle \subseteq W$ and is soluble. Furthermore, by Theorem 4.8.5, $G_{\sigma_1} wr_{X_{\sigma_2}} \dots wr_{X_{\sigma_k}} G_{\sigma_k}$ is soluble of solubility class at least k . This holds for each $k \in \mathbb{N}$, which contradicts the solubility of W . Hence Σ is finite. Write $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ with $\sigma_1 < \dots < \sigma_n$. Since (G_σ, X_σ) is transitive for each $\sigma \in \Sigma$, we have that $\langle G_\sigma : \sigma \in \Sigma \rangle \cong G_{\sigma_1} wr_{X_{\sigma_2}} \dots wr_{X_{\sigma_n}} G_{\sigma_n}$. And hence, by Theorem 4.8.5, $\text{cl}(W) \geq \text{cl}(\langle G_\sigma : \sigma \in \Sigma \rangle) = \text{cl}(G_{\sigma_1} wr_{X_{\sigma_2}} \dots wr_{X_{\sigma_n}} G_{\sigma_n}) \geq n = |\Sigma|$, as required. \square

Corollary 4.8.7. *Suppose (G_λ, X_λ) is a non trivial transitive permutation group for each $\lambda \in \Lambda$. Then W is soluble if and only if G_λ is soluble of bounded solubility class for each $\lambda \in \Lambda$ and all chains in Λ are finite of bounded length.*

Proof. This follows immediately from Theorem 4.8.3 and 4.8.6. \square

The non transitive case

We note that the class of soluble groups of solubility class at most n , for some $n \in \mathbb{N}$, is a variety. We use this fact and Theorem 2.2.3 to extend Corollary 4.8.7 to the case where the groups (G_λ, X_λ) need not be transitive.

Theorem 4.8.8. *Let $\Sigma := \{\sigma \in \Lambda : \iota_\sigma G_\sigma \text{ is infinite or } |\iota_\sigma G_\sigma| > 1\}$. Then W is soluble if and only if every chain in Σ is finite of bounded length and G_λ is soluble of bounded solubility class for each $\lambda \in \Lambda$. Further if the length of each chain in Σ is at most k and the solubility class of G_λ is at most c for each $\lambda \in \Lambda$, then the solubility class of W is at most kc .*

Proof. Suppose W is soluble. Note G_λ is isomorphic to a subgroup of W and is hence soluble of solubility class at most $\text{cl}(W)$. Now let $\Omega \subseteq \Sigma$ be a chain in Σ . By Theorem 4.8.5, $\text{cl}(W) \geq \text{cl}(\langle G_\omega(\iota_\omega G_\omega) : \omega \in \Omega \rangle) \geq |\Omega|$ as required.

Conversely, suppose every chain in Σ is finite of bounded length and G_λ is soluble of bounded solubility class for each $\lambda \in \Lambda$. Let k be the maximum length of any chain in Σ and let c be the bound on the solubility class of G_λ , $\lambda \in \Lambda$. Fix $\mu \in \Lambda$ and fix $x \in X$. Note $G_\mu(x_\mu G_\mu)$ is isomorphic to a quotient of G_μ and is hence soluble of solubility class at most c . Note $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle \cong \langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Sigma \text{ and } \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$. By Theorem 4.8.3, we have that $\langle \{G_\lambda(\iota_\lambda G_\lambda) : \lambda \in \Sigma \text{ and } \lambda > \mu\}, G_\mu(x_\mu G_\mu) \rangle$ is soluble of solubility class at most kc . By Theorem 2.2.3, W is soluble. \square

Notice that Theorem 4.8.8 implies W is soluble of solubility class at most kc if every chain in Σ is of length at most k and G_λ is soluble of class at most c . We find an example when W attains the maximal solubility class.

Example 4.8.9. Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a finite totally ordered set with $\lambda_1 < \dots < \lambda_n$. Let (G_λ, X_λ) be a non trivial transitive abelian permutation group for each $\lambda \in \Lambda$. Then W is soluble by Theorem 4.8.8. We claim that $\text{cl}(W) = n$.

Note that $\langle G_{\lambda_1}, G_{\lambda_2} \rangle$ is soluble and $\text{cl}(\langle G_{\lambda_1}, G_{\lambda_2} \rangle) \leq 2$ by Theorem 4.8.8. However $\langle G_{\lambda_1}, G_{\lambda_2} \rangle \cong G_{\lambda_1} \text{wr}_{X_{\lambda_2}} G_{\lambda_2}$ is not abelian. Hence $\text{cl}(\langle G_{\lambda_1}, G_{\lambda_2} \rangle) = 2$. Now let $k \leq n - 1$ and suppose $\text{cl}(\langle G_{\lambda_1}, \dots, G_{\lambda_k} \rangle) = k$. Now Theorem 4.8.4 and Theorem 4.8.8 implies $\langle G_{\lambda_1}, \dots, G_{\lambda_k}, G_{\lambda_{k+1}} \rangle$ is soluble with

$$\begin{aligned} k + 1 &\geq \text{cl}(\langle G_{\lambda_1}, \dots, G_{\lambda_{k+1}} \rangle) \\ &= \text{cl}(\langle G_{\lambda_1}, \dots, G_{\lambda_k} \rangle \text{wr}_{X_{\lambda_{k+1}}} G_{\lambda_{k+1}}) \\ &> \text{cl}(\langle G_{\lambda_1}, \dots, G_{\lambda_k} \rangle) \\ &= k. \end{aligned}$$

That is to say $\text{cl}(\langle G_{\lambda_1}, \dots, G_{\lambda_{k+1}} \rangle) = k + 1$. And hence $\text{cl}(W) = n$, the maximum it can be by Theorem 4.8.8.

4.9 Locally soluble groups

The immediate generalisation of the soluble condition is the locally soluble condition. In light of Theorem 4.8.8 it is easy to give a complete characterisation of the locally soluble generalised wreath product.

Theorem 4.9.1. *W is locally soluble if and only if G_λ is locally soluble for each $\lambda \in \Lambda$.*

Proof. Suppose W is locally soluble. Since G_λ is isomorphic to a subgroup of W , it follows that G_λ is locally soluble for each $\lambda \in \Lambda$.

Conversely, suppose G_λ is locally soluble for each $\lambda \in \Lambda$. Let $g_1, \dots, g_m \in W$. Then there exist $\lambda_1, \dots, \lambda_n \in \Lambda$ and finitely generated subgroups $H_{\lambda_i} \subseteq G_{\lambda_i}$ such that $\{g_1, \dots, g_m\} \subseteq \langle H_{\lambda_1}, \dots, H_{\lambda_n} \rangle$. Let $\Sigma := \{\lambda_1, \dots, \lambda_n\}$. Without loss of generality, we may assume Σ is connected. Now H_σ is soluble of solubility class at most $\max\{\text{cl}(H_{\lambda_1}), \dots, \text{cl}(H_{\lambda_n})\}$ for each $\sigma \in \Sigma$, where $\text{cl}(H_{\lambda_i})$ denotes the solubility class of H_{λ_i} . Moreover, $H_\sigma(x_\sigma H_\sigma)$ is soluble of solubility class at most $\max\{\text{cl}(H_{\lambda_1}), \dots, \text{cl}(H_{\lambda_n})\}$ for each $x \in X$ and each $\sigma \in \Sigma$. As Σ is finite, it follows from Theorem 4.8.8 that $\langle H_{\lambda_1}, \dots, H_{\lambda_n} \rangle$ is soluble. This completes the proof.

□

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