Mathematical Continua

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The Intuitive Idea of Continuity

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I declare that the thesis is the result of my own work, has been composed by myself, and has not been submitted for any degree or professional qualification other than the degree for which I am now a candidate.

Date 05/11/2005 Signature.

Abstract

How does philosophy understand the concept of continuity? The intuitive idea of continuity is about perceptual smoothness; but what *looks* smooth may *be* discontinuous, meaning that phenomenal continuity does not constitute a reliable definition. Metaphysics speaks of continuants with respect to temporal parts, but does not provide a definition of continuity. When properly defined, it is then associated with a minimal change divided into infinitesimal parts, which is an implicit reference to Leibniz's law of continuity such that a continuous change pertains to a geometric graph differentiable at arbitrary points. Yet, does it make sense to define continuity by means of discontinuous points?

We must view Leibniz's definition as a transitory stage between two contradictory concepts, i.e. geometric and arithmetical continua. While Aristotle shows that a continuous line is infinitely divisible into lines, Dedekind defines an arithmetical continuum (or real line) as a complete domain of real numbers. This distinction opposes the intuitive idea of a smooth extension to a discontinuous and extensionless sequence of numbers, meaning that algebraic formalisms do not solve Zeno's geometric paradoxes but make them irrelevant. The consequences for physical continuity are such that an Aristotelian time is a smooth temporal interval devoid of indivisible parts; namely, instants of time are abstract limits and not physical durations. Arithmetical continuity defines a continuous time as isomorphic to a set of real numbers, but the measure of this extensionless structure is physically meaningless, and there is no physical argument to claim that a continuous time is a better model than a discrete time.

Arithmetical continuity is omnipresent in modern mathematics; yet, it is fraught with difficulties in relation to the infinite. Cantor distinguishes an infinitely *countable* set of natural (or rational) numbers from an infinitely *uncountable* continuum. These infinite cardinalities imply the 'axiom' of choice, such that it is always possible to choose a unique element in a set over an infinite collection of disjoint, non-empty sets. Brouwer rejects this postulate because based on the unjustified idea that the infinite has a same ordering as the finite. He then claims that only infinitely incomplete sequences can be generated, since the nature of the infinite is to be merely potential. Others directly challenge arithmetic. C.S. Peirce suggests a topological geometry devoid of discrete numbers; however, it is clear that modern topology rests on an arithmetical ordering of real numbers and cannot be defined as pure geometry. More recently, J.L. Bell rejects the intuitive discontinuity of algebraic structures by defending an axiomatic system of smooth infinitesimals; yet, the identification of axiomatic smoothness with intuition neglects the necessity for any axiomatic property to belong to the axioms alone.

Still, the construction of an axiomatic system can help us defend arithmetical continuity. Hilbert shows that a Euclidean model of geometry is isomorphic to an algebraic model, such that the axiom of continuity is satisfiable in either model. As for the absolute consistency of the axiomatic system, it requires a metamathematics, which aims to demonstrate the arithmetical infinite on finite logical grounds. First-order logic fails to define a continuum as a concrete object, since the *uncountable* set of all countable subsets is independent of any logic whose models have only *countable* domains (Löwenheim-Skolem theorem). By contrast, second-order logic makes sense of a continuum as an abstract set, which means that arithmetical continuity is nothing more than an ideal, hypothetical abstraction.

With acute terseness Poincaré once confronted Tolstoy, who had said that it was unwise to demand 'science for the sake of science'. 'As we choose our quests', Tolstoy asked, 'should we allow ourselves to be led by the moods of our greed for knowledge? Would it not be better to make the decision according to its usefulness, that is, according to our practical and moral needs?' Strange, that it should be Tolstoy whom we mathematicians must reject as a tedious realist and a narrow-minded utilitarian... When the well-known Fourier once said that the chief purpose of mathematics is the explanation of natural phenomena, it was Jacobi who rebuked him with all the enthusiasm of his character. A philosopher like Fourier ought to have known, Jacobi exclaimed, that the sole aim of all science is the honour of the human mind, and that from this viewpoint a problem of pure number theory is as equally valuable as a problem with practical applications.

Hilbert, 'Naturerkennen und Logik' (1930, §§ 24-5)

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Introduction

Science speaks of continuous motion, time and space; yet, it is difficult to make sense of what continuity means. We seem to have an easy justification through our perceptions, such that motion, time, and space are continuous because perceived as smooth and unbroken. It is obvious that such a commonsensical reply cannot be taken seriously, since *looking* continuous is not the same as *being* continuous. The sun looks in motion with a size more or less equal to the moon; yet, such true perceptions are scientific nonsense. Likewise, our true perceptions of continuity could be mere illusion, and this uncertainty should make us cautious about phenomenal continuity. In this sense, physics resorts to mathematics in order to define an abstract property of continuity. Does it mean that continuity and continua are physical concepts? Einstein writes in his *Über die spezielle und die allgemeine Relativätstheorie* (1916):

> The surface of a marble table is spread out in front of me. I can get from any one point on this table to any other point by passing continuously from one point to a "neighbouring" one, and repeating this process a (large) number of times, or, in other words, by going from point to point without executing "jumps". I am sure the reader will appreciate with sufficient clearness what I mean here by "neighbouring" and by "jumps" (if he is not too pedantic). We express this property of the surface by describing the latter as a continuum. (p. 83)

I am afraid of being too pedantic, but Einstein's definition of a physical continuum is somehow misleading. His interpretation pertains to Leibniz's law of continuity based on a differential calculus, such that a series of minimal changes, divided into neighbouring points, is defined as continuous. However, we shall see that Leibniz's law of continuity is above all a mathematical principle, meaning that a continuum is a mathematical concept. In this sense, the absence of "jumps" between "neighbouring" points is a deceptive empirical metaphor which has no role to play in the definition of a continuum in Euclidean and Non-Euclidean algebraic geometries. In other words, Einstein starts with a physical metaphor but ends up with a mathematical definition of continuity, leading to the conclusion that a scientific concept of continuity is nothing but mathematical.

Mathematical continua are not the same if defined before or after the algebraic revolution. Aristotle understands a geometric line as composed of infinitely divisible lines. If indivisible points were parts of a line, they would interrupt its geometric continuity. With Descartes and Leibniz, a line is no longer purely geometric since translatable into a linear algebraic equation, such that Cartesian coordinates attribute numerical values to each point of the line. Then, Cantor and Dedekind construct a mathematical continuum based on an arithmetic domain of numbers independently of any geometric concepts; and we shall see that Dedekind's *real* line is nothing more that the continuous system R of all *real* numbers, namely an arithmetic infinite interval $[-\infty, \infty]$ devoid of geometric meaning. Eventually, we deal with two kinds of continuity. On the one hand, a geometric continuity rests on the intuitive idea of continuity understood as smoothness. On the other, an arithmetic continuity pertains to an abstract property of completeness in such a way that a set of real numbers is both arithmetically continuous and intuitively discontinuous.

I intend to make sense of these two conflicting mathematical continua, and to grasp their respective definitions in ancient and modern foundational mathematics. Yet, three caveats can help the reader understand my project. First, this thesis does not intend to construct the objective history of mathematical continuity. Although I have paid particular attention to original texts, I have not respected a chronological order in the sense that my outline is purely thematic; besides, my study is far from exhaustive. Second, the use of mathematical formalism has been reduced to its minimum. The danger of lacking mathematical rigour (that I hope to have avoided) is greatly compensated by the advantage of not repeating the same canonical expressions (present in any textbook) that only mathematicians and logicians would have understood. It is necessary not to shy away from the complex mathematical definitions, but philosophy of mathematics is not mathematics in the sense that a philosophical interpretation must demonstrate the meaning and relevance of the mathematical complexity. Suppose the trivial (if not simplistic) example of a multiplication table, whose abstraction is obvious for young children. If we multiply 9 times 4, we learn that the mathematical solution is 36. However, we philosophically understand its mathematical role only when we grasp that multiplication is a shortcut for addition, such that 36 corresponds to the 'four' additions of the number 'nine', i.e. $9 \times 4 = 9 + 9 + 9 + 9 = 36$.

As well, division is a shortcut for subtraction, such that 36 ÷ 9 = 4 is reinterpreted as the 'four' subtractions of the number 'nine', i.e. 36 – 9 – 9 – 9 – 9 = 0. My purpose in this thesis is to do the same with respect to mathematical continua, namely to explain the philosophical motivations implied by mathematical abstractions. A final caveat pertains to a metaphysical commitment on my part, in the sense that the positions that I shall hold will be an implicit and quiet objection to Quine's (1976, 1980, 1981) and Putnam's (1979) indispensability argument, which claims that the applicability of mathematics to empirical science should commit us to believe in the existence of mathematical entities. In contrast, I shall show that modern mathematics does not need to be 'applied mathematics' in order to be indispensable, and that mathematical concepts are perfectly consistent without postulating an ontological commitment to individual entities.

Let me first sketch the six chapters of my thesis. Chapter 1 will examine the meaning of continuity as implied by contemporary philosophy. The concept of a continuant plays a huge role in modern metaphysics, but does not seem to be justified through a precise definition of continuity. We shall see as well that the unreliable concept of phenomenal continuity must not be confused with an abstract definition of continuity. Thus, if a continuum is a change infinitely divided into infinitesimal parts (Hirsch 1982), we should not speak of a phenomenal change. Indeed, this definition is similar to Leibniz's (1687) law of continuity, namely a mathematical law based on infinitesimal calculus, and a continuous change is a mathematical concept whose geometric curve is divided into algebraic derivatives (slopes of tangents) at infinitesimal points. In this sense, a mathematically continuous change does not conflict with Leibniz's definition of discrete matter in the physical world.

Chapter 2 will show that Leibniz's definition of continuity constitutes a transitory stage between two opposite concepts, i.e. a purely geometric continuum as defined by Aristotle and the arithmetic continuum of the nineteenth-century foundational mathematics. The two continua must be distinguished from each other (contra Bostock 1991), insofar as Aristotle's continuous line is infinitely divisible into potential line, and does not make sense of Dedekind's (1872) real line understood as an arithmetic domain of real numbers. I shall stress that an Aristotelian continuum implies the intuitive idea of a smooth *extension*, unlike an arithmetic continuum based on an *extensionless* algebraic sequence of numbers. We will then refute the traditional claim that algebraic formalisms solve Zeno's geometric paradoxes; on the contrary, what algebra does is to make them merely irrelevant. The only way to reintroduce a paradox is to postulate an *ad hoc* correspondence between the intrinsically consistent algebraic formalism and an empirical concept of motion. Such paradoxes should be understood as Zeno-like algebraic paradoxes distinct from Zeno's genuinely geometric paradoxes.

Chapter 3 will draw the physical consequences of the distinction between the two mathematical continua. Aristotle defines a physical concept of continuous time whose temporal intervals are infinitely divisible and thereby devoid of indivisible instants; thus, instants are indivisible limits of time and not divisible parts. Continuous time is derived from a continuous motion, but seems to contradict Aristotle's definition of time as the number of a motion, i.e. a discontinuous plurality of the 'before' and 'after'. Yet, we shall see that the contradiction is merely apparent by distinguishing a continuous time measuring an *actual* motion from the numbered times, counted by the mind, of the *potential* divisions (*qua* the 'before' and 'after') of a motion. By contrast, if arithmetic continuity is applied to time, then temporal intervals become isomorphic (i.e. identical in form) to a continuous domain of real numbers. I shall show that this definition is devoid of physical meaning, since the mathematical measure theory defines the measure of a continuous time as the arithmetic length of a continuous interval. In other words, no physical argument can make sense of a continuous time distinct from a discrete time. These set-theoretic abstractions deny any relevance to the intuitive idea of a physical measure applied to an empirical extension.

Chapter 4 will deal with the infinite cardinality of an arithmetic continuum. A cardinal number defines an infinite set as an actual infinite (or Dedekind infinite), such that a set is infinite when in one-one correspondence with one of its proper subset. Thus, Peano's arithmetic and Zermelo-Fränkel set theory define the infinite set N of *all* natural numbers through a complete induction, whose countable cardinality is $\aleph_0$ (*aleph-0*). Yet, Cantor demonstrates through a *reductio ad absurdum* that the continuum has an uncountable cardinality, and the continuum hypothesis claims that this higher cardinality is $\aleph_1$ (*aleph-1*). This is explained by the mathematical fact that a continuum implies second-level real numbers (i.e. Dedekind cuts) which are abstract properties irreducible to individual numbers (unlike natural number or rational numbers). To make sense of infinite cardinalities, we must postulate the axiom of choice, namely the possibility to choose a unique element in each subset of an infinite set. We are unable to prove the existence of such singleton subsets, but their existence must be postulated in order to define each infinite set as well ordered, i.e. a set with a least or first element (cf. Zermelo's well-ordering theorem, 1904). However, not all mathematicians accept this postulate, and we shall see that Brouwer's mathematical intuitionism promotes the potential infinite which transforms an arithmetic continuum of real numbers into incomplete (law-like and free choice) sequences of rational numbers; then a continuum is a process in becoming.

Chapter 5 will present three objections to the arithmetic definition of a continuum, and their common criticism purports to reject the relevance of real analysis (in this sense, they diverge from Brouwer's position). First, Weyl's *Das Kontinuum* (1918) defends a predicative continuity by criticising second-level real numbers (Dedekind cuts) whose definition implies a vicious circle; yet, Weyl does not succeed in presenting a sound mathematical alternative, and will join Brouwer's intuitionism from 1920. Second, C.S. Peirce defends a true continuum, which aims to eliminate the irreducible gap between geometric and arithmetic continua; he speaks of a topological structure as a pure geometry devoid of discrete numbers. However, his conception of topology is indefensible from a modern mathematical point of view. I shall show that modern topology rests on an arithmetic ordering of real numbers, in the sense that a non-metrical topology does not imply the disappearance of the set-theoretic structure. Finally, J.L. Bell (1998) criticises arithmetic continuity for the sake of a basic smooth infinitesimal analysis (BSIA). He defines a consistent axiomatic system, and identifies axiomatic smooth infinitesimals with a non-axiomatic conception of traditional infinitesimals as implied by Leibniz's differential calculus. In other words, he neglects the axiomatic requirement that the consistency of BSIA rests on the axioms alone, and cannot apply to non-axiomatic infinitesimals, whatever they are.

Chapter 6 will conclude that axiomatic mathematics is the right structure to understand the full scope of arithmetic continuity. Hilbert (1899) transforms Euclid's geometry into a Euclidean model of axioms, and defines it as isomorphic to a Cartesian algebraic model.

This isomorphism is made possible through the axiom of completeness (categoricity) defining the maximality of elements for each model, such that a geometric system of points is not different from an algebraic system of numbers. We shall see that Frege rejects Hilbert's axiomatic system and defends a traditional conception of mathematics, such that any true concept must be a first-level predicate, whose Fregean meaning is a mathematical object. In other words, Frege denies any relevance to the axiomatic concepts of consistence and existence. Hilbert defends his own view by separating semantics from syntax, such that the second-level mathematical models are distinguished from a first-level metamathematical axiomatisation (also called proof theory). Gödel tells us that the consistency proof of a consistent axiom system requires a larger system, but this does not constitute a refutation of Hilbert's axiomatisation. Indeed, Hilbert's purpose is to reduce the infinite semantics of mathematics to a finitary syntax, so that the axiom system makes arithmetic continuity syntactically consistent. Yet, the second-level idealisation of an arithmetic continuum prevents it from being defined as an individual mathematical entity. The Löwenheim-Skolem theorem (1920) confirms that any first-level logic or set theory cannot make sense of the *uncountable* semantics of a continuum, since its models have only an infinitely *countable* domain. Therefore, axiomatic continuity is a purely abstract property devoid of the intuitive idea of continuity.

Note that a part of chapter 5 has been accepted for publication in *Transactions of the Charles S. Peirce Society* under the title 'Peirce's Potential Continuity and Pure Geometry'. Moreover, the writing of my thesis has benefited from the presentations of several papers at the following conferences: Logica 2004 (Czech Republic, June 2004), the First International Conference on the Ontology of Space-Time (Montréal, May 2004), the Joint session of the Aristotelian Society and the Mind Association (Belfast, July 2003), the annual conferences of the Canadian Society for the History and Philosophy of Science (Halifax, May 2003) and the British Society for the Philosophy of Science (Belfast, July 2003; Glasgow, July 2002). These conferences helped me both defend my views and improve my arguments through questions from the audience. Finally, I would like to thank my supervisors, Prof. Theodore Scaltsas and Dr Peter Milne, for their help and guidance. Of course, all mistakes are my own.

Chapter 1
Continuity, Intuition and Mathematics

References to continuity are multiple in philosophy and even crucial in the particular context of some metaphysical concepts, such as temporal parts and continuant. Yet, nobody seems to be willing to provide a clear definition of continuity, as though the definition were too obvious to be made explicit. It is true that phenomenal continuity does not seem to deserve any lengthy explanation, since it is easy to grasp the intuitive synonymy between perceptual continuity and smoothness. However, we shall see that the intuitive idea of continuity is unable to provide a reliable definition, and as such, must be replaced with theoretical arguments, whether provided by set theory or mathematical analysis. The pernicious consequence is that the intrinsic meaning of a theoretic concept of continuity conflicts with the intuitive idea of continuity. The perfect example is Leibniz's mathematical law of continuity based on the infinitesimal divisions of a continuous change into intuitively discontinuous points.

1.1 Intuitive continuants and mathematical abstractions

The notion of a continuant in relation to a changing object in time illustrates the importance of continuity in contemporary metaphysics. What is known as the theory of endurance (or three-dimensionalism) explains change through the changing properties of a persisting thing; namely, you sit at a time t_1 and stand at a distinct time t_2, such that 'yourself' is enduring a change by persisting over t_1 and t_2 (cf. section 1.3). In contrast, the theory of perdurance (or four-dimensionalism) defines change in time through temporal parts; that is, time is a fourth-dimensional extension, and each physical object is composed of temporal parts. For instance, the temporal parts of a person are such that yesterday's person is different from today's person, and the succession of such temporal parts defines distinct person stages. Thus, 'you sit' and 'you stand' are two temporal parts about two successive 'yourselfs'.

The obvious difficulty pertains to the nature of this succession which is often referred as 'continuants', despite the appearance and disappearance of instantaneous temporal parts.

David Lewis (1983, 1986, 1988, 2002) defends perdurance, and defines a continuant through two relations (1983, pp. 58-59).[1] First, an R-relation pertains to a mental continuity relating person stages to each other; for instance, your own future is made out of a stage of 'you-later' that is R-related to the stage of 'you-now'. Second, an I-relation corresponds to an identity relation that holds among several stages defined as continuants. Lewis defines continuity as a principle of connectedness, and the degree of connectedness between two interrelated stages depends upon a mathematical scale from zero to one (p. 69). Every number within this interval is called a delineation, and refers to the location of an arbitrary boundary between two interrelated stages. Two stages are interrelated if the degree of an R-relation is a delienation between 0 and 0.9; if the delineation is higher than 0.9 up to 1, then the two stages are *not* interrelated. By contrast, an R-relation with a delineation equal to zero is an I-relation. Suppose the person stages S_1, S_2, S_3, etc. with the continuant person C_1 of whom S_1 is a stage, the continuant person C_2 of whom S_2 is a stage, and so on, then the person stages S_1 and S_2 are I-related if and only if the corresponding continuant persons C_1 and C_2 are identical. Note that an I-relation is at least an R-relation, since any identity relation among stages (or temporal parts) implies a relation of continuity. Thus, Lewis defines a continuant person as "an aggregate of person stages", such that each stage "is I-related to all the rest (and to itself)" (1983, *ibid.*). In other words, identical continuants form an aggregate of identical stages.

We must now provide the precise meaning of an aggregate. Yet, Lewis (1983) does not think of this question as crucial; he writes:

> It does not matter what sort of 'aggregate'. I prefer a mereological sum, so that the stages are literally parts of the continuant. But a class of stages would do as well, or a sequence or ordering of stages, or a suitable function from moments or stretches of times to stages. (1983, p. 59, footnote 4)

Lewis is keen to defend his conception of person stages, but does not mind about the

[1] In favour of perdurance (or four-dimensionalism), see also Armstrong 1980 and Sider 2000, 2001, whereas Mellor 1981, Hirsch 1982, Forbes 1987, Johnston 1987 and Haslanger 1989 defend endurance (or three-dimensionalism).

mathematical meaning of an aggregate. Nevertheless, it is very difficult to understand how stages can be aggregated, while they are defined as instantaneous temporal parts, such that each stage "begins to exist abruptly, and it abruptly ceases to exist soon after" (1983, p. 76). We may wonder how an aggregate can be formed, since temporal parts are successive to each other; and to claim that an aggregate is a sum, class, sequence, ordering, or function does not seem to solve the problem. The only thing to grasp from this enumeration of mathematical terms is that an aggregate is a second-level concept or property, namely an abstract principle defining stages, not as first-level individual entities, but as a set of second-level elements satisfying a particular property. Lewis (1983) does not provide any further clarification about the nature of an aggregate, except that he mentions an analogy with a train understood as a maximal aggregate of cars "interrelated by the ancestral relation of being coupled together" (p. 77). Consequently, a person continuant is based on the aggregate of instantaneous but identical person stages.

Lewis speaks of an aggregate of identical parts, and not of a whole subsuming identical parts, because temporal parts are not static; they are always successive and instantaneous in such a way that two parts can never coexist. That is the reason why Lewis (2002) defines a continuant or perduring thing as an entity distinct from a temporal part. Unlike Sider (2000, 2001), he rejects the possibility for a temporal part to be a continuant, since a continuant persists while a temporal part does not. Lewis writes, "A persisting thing is like a parade: first, one part of it shows up, and then another. (Except that most persisting things are much more continuous than most parades)." (2002, p. 1). Thus, perdurance cannot define a persisting thing independently of its temporal parts, such that the continuous parade of temporal parts gives the impression of a persisting thing. I do not know whether Lewis would agree with my saying, "giving the impression", but it seems difficult to interpret his statement in another way. Indeed, a continuous parade (as Lewis calls it) is not a thing, but an event or process. It is not a thing because we do not know which kind of substratum may persist over the succession of temporal parts. In *this* sense, the notion of continuant is unclear, and Lewis does not say more, although he admits that a persisting thing is much more 'continuous' than a parade could never be.

However, I want to show that Lewis's definition of a continuant as a continuous parade or aggregate of temporal parts conflicts with the idea of a mereological sum, a class, a sequence, or even a function. Lewis's analogy of a train defined as a maximal aggregate of cars, i.e. an aggregate of I-interrelated (identical) stages, is wholly intuitive, such that a continuant is thought to be a smooth parade of instantaneous, successive parts. The problem is that Lewis's mathematical concepts of mereological sum, class, sequence and function do not satisfy the intuitive idea of smoothness. We shall see that a mereological sum (or class) is based on overlapping subsets that contradict the intuitively successive, non-overlapping temporal parts (or stages) of an aggregate; as well, a function (or sequence) implies elements whose arithmetic continuity cannot mirror the intuitively continuous parade of the I-interrelated temporal parts.

Mereology is the formal theory of part-whole relations, such that a whole is identical with the sum of its parts. Lewis (1991, 1999) offers a variant of mereology by defining an arithmetic set as the mereological sum of its singleton subsets; that is, the parthood relation of a mereological whole is identified with the membership relation of a set. [2] Lewis's "mereologized arithmetic" is based on the principle of extensionality, namely each sum or set is exclusively defined by its elements; and elements are subsets of a set, unless they are explicitly claimed to be urelements. [3] Lewis then defines three possible relations:

> At one end we find the complete identity of a thing with itself: it and itself are entirely identical, not at all distinct. At the opposite end we find the case of two things that are entirely distinct: they have no parts in common. In between we find all the cases of partial overlap: things with parts in common and other parts not in common... The things are not entirely identical, not entirely distinct, but some of each. They are partially identical, partially distinct. (1999, p. 177)

Lewis applies the set-theoretic rule that elements of sets are subsets, such that subsets overlap. This statement is counterintuitive, because we tend to think of a whole as composed of non-overlapping parts. Suppose the whole {A, B, C}, then A, B, and C are defined as the non-overlapping parts of the whole, namely the disjoint singleton subsets {A}, {B} and {C}. However, set theory defines eight subsets for the set {A, B, C}, such as:

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[2] For a detailed study of mereology and of its different variants, see Tarski 1956 and Simons 1987.

[3] An urelement is an element which is not a set; in other words, it cannot contain any other elements or be identical with the empty set (cf. Moore 1982 and Rubin 1967). Urelements are sometimes called 'individuals' or 'atoms' (note that the prefix 'ur' in German pertains to the idea of something primitive). In standard set theory, elements are always sets and never urelements.

{∅}, {A}, {B}, {C}, {A, B}, {B, C}, {A, C}, {A, B, C}.

The seven non-empty subsets overlap each other. Thus, Lewis claims that mereological sums (or classes) are fusions of singletons, such that singletons are parts of a class and every class is identical to the union of its singleton subclasses (1991, pp. 16, 97, 115). Yet, set theory enables him to add that all parts overlap in a mereological sum (p. 73), which would not be case if the mereology were not set-theoretic. This means that for a set {A, B, C}, the part or subset {A, B} overlaps {B, C}, and the part {B, C} overlaps {A, C}. Likewise, the singleton parts {A}, {B} and {C} overlap the parts {A, B}, {B, C} and {A, C}. Finally, every non-empty part overlap the part {A, B, C}.

If we apply the definition of mereological sum to an aggregate of temporal parts, this at first seems quite satisfactory. Indeed, a mereological sum implies singleton subsets, which can be successfully compared with a train composed of 'singleton' cars. However, the notion of overlap implied by Lewis's mereology is too strong a property with respect to temporal parts, in the sense that the latter are fundamentally separate through distinct temporal locations. When a 'red traffic light' succeeds to a 'red traffic light', we deal with the succession of two identical yet instantaneous stages without the presence of a *persisting* (enduring) traffic light. In contrast, the mereological sum is not just a fusion of singleton subsets, but implies other parts that are not singleton subsets. The idea of overlap between the parts of a mereological sum is due to the non-singleton subsets, such as {A, B} or {A, C}. This set-theoretic property is absent from the aggregate of temporal parts, since there is no two temporal parts which may be defined in a same subset. In other words, temporal parts are based on the intuitive idea that the aggregate {A, B, C} is composed of the three instantaneous temporal parts {A}, {B} and {C}; there is nothing in the theory of perdurance which could interpret the subsets {A, B}, {B, C} and {A, C} as temporal parts of a continuant. In other words, the abstraction of overlapping subsets based on a mereologised set theory does not fit the intuitive idea that the successive stages of an aggregate have distinct temporal locations.

A similar objection pertains to Lewis's definition of an aggregate as a sequence or function. A sequence is mathematically defined as a function whose domain equals a set of numbers. Thus, a finite sequence of natural numbers is a function from the set N of natural numbers into some given finite set S, such as:

$$S = \{1, 2, 3, \ldots, n - 1, n\}$$

Each number is a term of the sequence, such that n is the n-th term. Unlike a mereological sum, a sequence perfectly describes the instantaneous succession of temporal parts, since the elements of a sequence are distinct and do not overlap; but can we claim that the numbers of a sequence are identical? Indeed, we must interpret the fact that temporal parts can be identical, despite their instantaneous succession. We could convince ourselves that abstract numbers, whatever they are, are ontologically defined as identical objects. Yet the fact that a sequence $\{1, 2, 3, \ldots\}$ of natural numbers jumps from one number to another seems to imply that the aggregate or parade of temporal parts is not continuous at all (as though the cars of a train were not connected to each other). A possible reply is to replace the discrete sequence of natural numbers with a dense sequence of rational numbers, or even a continuous sequence of real numbers. Denseness means that there is always a rational number between any two rational numbers, so that no rational number can be next to another one (since a third one always pops up between any two). As for arithmetic continuity, it implies that real numbers are limits of the dense sequences of rational numbers (cf. chapter 2). This roughly means that there is no gap left in the sequence since even irrational numbers (e.g. the square root of two) are taken into account by being defined as real numbers, i.e. limits of sequences of rational numbers. May the denseness or continuity of a sequence correspond to a continuous parade or aggregate of temporal parts? The answer is negative, since denseness and continuity are arbitrary properties that do not mirror our intuition of continuity.

The instance illustrating my objection is Lewis's definition of a continuant as a "suitable function" from instants of time to stages (1983, p. 59). This implies the definition of a continuous function f, such that:

$$\text{If} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

This means that if each positive value of x differs from a by less than a positive real number δ, then the value of $f(x)$ differs from the value of the limit L by less than an arbitrarily small positive real number ε. In brief, a continuous function f implies that all numbers close to x map numbers close to $f(x)$. This constitutes Weiertrass's delta-epsilon definition of a continuous function (cf. section 1.2). Yet, continuity is derived from a set of real numbers, and thereby is an arithmetic (or set-theoretic) property. Applied to a continuant or aggregate, we may claim that there is a continuous function from instants of time to temporal parts, but this merely means that instants and temporal parts are defined by a mathematically continuous function which corresponds to a convergent infinite sequence of real numbers. The sequence is convergent, since real numbers always imply that the sequence has a limit L. The abstract property of convergence is based on the arithmetic continuity of a domain of real numbers and does not say anything about the intuitive continuity of an aggregate of temporal parts. In other words, we deal with two notions of continuity: an abstract principle derived from a set of real numbers and an intuitive idea defined as a smooth change of temporal parts; but there are neither proofs nor sound justifications to convince us that the former mirrors the latter.

Accordingly, the concept of a continuant defined as a continuous aggregate of I-interrelated temporal parts has an intuitive meaning that cannot be identified with the abstract concepts of mereological sum, sequence and function. We have the insight of a smooth continuant but no mathematical argument helps us justify our intuition. In this sense, Lewis's definition of a continuant is a merely intuitive view with no justification through mathematical abstraction.

1.2 Phenomenal smoothness and mathematical continuity

We may wonder whether there is a need for abstraction in order to justify the continuity of physical change. A change seems to be continuous only because we perceive it as such. The weakness of phenomenal continuity, i.e. perceptual smoothness, is that a change may *look* continuous without *being* continuous. Suppose a change of colour between the two ends of a continuous spectrum. If the change is perceived as continuous, then all adjacent parts of the colour spectrum look homogeneous. Yet, there is an obvious paradox, namely: how a change of colour may occur if all adjacent parts look the same? If the ends of the spectrum do no look the same, then the adjacent parts should not look the same. That is, if the end A is distinct from the end C, then A must be distinct from the adjacent B, and B distinct from the adjacent C, which amounts to defining transitivity. Yet, phenomenal continuity is based on the paradox of non-transitivity, since the adjacent parts A and B, along with B and C, look the same despite the fact that A and C do not look the same. Poincaré (1893) and Russell (1897) acknowledge this paradox, and explain it through the limited nature of our perceptions that are unable to distinguish infinitesimal changes among the adjacent parts of a colour spectrum. In other words, this paradox is purely phenomenal, and merely results from our perceptual inability to locate the change of colour in a continuous spectrum.

However, Delia Graff's 'Phenomenal Continua and the Sorites' (2001) defends the possibility for phenomenal continuity to be transitive. Her argument is that we cannot be sure of what a continuum looks like, and in this sense, we may doubt the perceptual homogeneity of its adjacent parts. Graff rejects the homogeneity thesis that any phenomenally continuous (i.e. smooth) change necessarily implies homogeneous parts. The absence of homogenous parts then makes transitivity possible, since if the adjacent parts A and B, and B and C, do not look the same, then the extremities A and C do not look the same. Yet, the question is whether a change composed of non-homogeneous parts may still be a phenomenal continuum, i.e. a smooth change. Graff believes so, and justifies her position by identifying a phenomenal continuum with a continuous function, such that the numerical values of a continuous change are defined at distinct, indivisible points which are by definition non-homogeneous parts. In other words, a phenomenal continuum does not exclude the possibility of non-homogeneous parts. I shall show that Graff's conception of phenomenal continuity goes beyond the mere reference to a perceived phenomenon, and implies an ontological definition of continuity based on the structure of a continuous function. Thus, she deals with a mathematical abstraction that cannot correspond to what we intuitively define as phenomenal continuity, namely perceptual smoothness.

Graff's phenomenal continuity goes beyond the definition of a perceived phenomenon, when she compares two kinds of cursor on a computer screen: the first looks to move discontinuously from one character position to the next, while the second looks to move continuously but jerkily (p. 923). Although she postulates that the discontinuous-looking change *is* distinct from the continuous-looking one, she claims that both are *perceived* as continuous. In brief, a phenomenon may *look* continuous, even though it *is* properly discontinuous-looking; she writes, "We should not be misled into thinking that just because it may be convenient to describe a change as apparently continuous, that it really is that way" (*ibid.*). Graff's statement is puzzling, insofar as it implies an unclear distinction between *being* looking-continuous and *looking* looking-continuous. In other words, the appearance of a perceptual change is distinct from its ontological status independently of perception. This view sounds contradictory since phenomenal continuity, by definition, refers to something merely perceptual. Thus, Graff ambiguously speaks of "continuous-but-very-jerky-looking motion", as though it were possible to define a continuous-looking motion that looks jerky as well; but if a motion looks continuous, it cannot look jerky (and inversely) since both statements involve contradictory predicates. It is obvious that Graff is aware of this, so that we must understand the expression "continuous-but-very-jerky-looking motion" as pertaining merely to a continuous motion that looks very jerky. If so, are we still dealing with a phenomenally continuous change?

Graff's answer is affirmative, but her conception of phenomenal continuity allows the parts of a continuous change to be non-homogeneous. She writes:

> I have found that some people take the homogeneity thesis to be true because they take it just to state a necessary fact about what's involved in something's being a phenomenal continuum. The idea here would be that it just is a necessary condition of a change in colour looking continuous across a spectrum that narrow enough regions of the spectrum look homogeneous in colour... This idea is not right—it reflects a misunderstanding of the nature of continua. (2001, p. 924)

Her statement indicates that she regards phenomenal continuity, not as a mere perceptual smoothness, but as something more which does not rely on our perceptions. Indeed, she identifies phenomenal continuity with the purely abstract concept of mathematical continuity. She writes, "What is required for motion to *look* continuous, then, is presumably that for any distance ε, no matter how small, there is a positive amount of time δ which is small enough so that during any time-span shorter than δ, the object *looks* to move less than ε in that time-span" (p. 925, original emphases). In other words, a motion 'looks' continuous when an object 'looks' to cover a small distance less than ε in a small time-span shorter than δ. Yet, this assertion is misleading, since it combines purely phenomenal expressions, such as "looks continuous" and "an object looks to move", with a purely mathematical definition, i.e. Weierstrass's delta-epsilon definition of a continuous function (cf. section 1.2). We have seen that a function f is continuous according to the following inference: if each positive value of x differs from a by less than a real number δ, then each value of $f(x)$ differs from the value of the limit L by less than an arbitrarily small number ε, i.e. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Graff reinterprets this purely mathematical definition in physical terms, such that the real numbers ε and δ are respectively defined as distance and temporal instants. Thus, the distance ε goes smaller in relation to a smaller time δ, and in any time less than δ a given motion is calculated in a traversed distance less than ε. The problem is that the physical interpretation of the delta-epsilon definition is merely *ad hoc*, insofar as no sound justification can defend the claim that the phenomenal continuity of a motion (or a change of colour) is intrinsically defined by the arithmetic continuity of an analytic function. Yet, it is exactly the inference that Graff draws when she writes:

> Just as we may think of the motion of the object as a function which, given a time as argument, yields the location of the object at that time as value, we may think of change in colour along the spectrum as a function which, given a point on the spectrum as argument, yields the colour of the spectrum at that point as value. *The continuity of either motion or change in colour can then be identified with what it is for these respective functions to be continuous.* (p. 924, footnote 16, my emphasis)

Her view defines some equivalence between a continuous function and a continuous motion, such that each value of a function for a given argument is interpreted as either a definite distance *at* a given time or a definite colour *at* a given point of the spectrum. A claim of this kind has some pedagogical purposes since this physical metaphor helps us illustrate the mathematical definition. In this sense, Graff follows "the at-at theory of motion" (Salmon 1970) defining the motion of an object through a functional relation from instants of time to spatial points. However, physical interpretations are not parts of the intrinsic definition of a continuous function, meaning that there is no justification to infer physical continuity from the mathematical definition of a continuous function.

Accordingly, Graf's position implies a radical shift in the meaning of continuity. A change of colour is *phenomenally* continuous because perceived as smooth, whereas an algebraic function is *mathematically* continuous through the arithmetic definition of a continuous domain of real numbers. As Graf identifies a mathematical model with its empirical illustration, she distorts the mathematical meaning of Weierstrass's continuous function based on the 'static' and abstract definition of a purely algebraic system. The continuity of the function does not even derive from its geometric graph. Thus, Weierstrass rejects Cauchy's formulation of a continuous function $f(x)$ as *approaching* a limit L when x approaches a number a, for it implies a 'dynamic' conception of numerical values "approaching" the limit, as though numbers could be in motion. [4] Weierstrass's aim is to distinguish the intrinsic, static, mathematical formalism from an extrinsic, dynamic, physical description. Note that Bolzano and Weierstrass provide the ultimate proof that a continuous function does not mirror the intuitive idea of continuity, when they demonstrate that a function can be continuous without being differentiable. This means that a function is continuous at a point, although there is no tangent at this point, in such a way that a continuous function may have a broken and intuitively discontinuous geometric graph. This reinforces the objection that phenomenal continuity must not be identified with a continuous function.

Consequently, Graff is able to defend the transitivity of phenomenal continuity, only because she relates the possibility of non-homogeneous parts to the definition of continuous functions at indivisible, non-homogeneous points. If A, B and C are non-homogeneous parts, then A and C are non-homogeneous ends, and the idea of a paradox becomes irrelevant. However, Graff's mistake is to identify phenomenal continuity, i.e. perceptually smooth phenomena, with the abstract continuity of an algebraic function, while both kinds of continuity have incommensurable meanings. Eventually, it would have been better to distinguish, as Poincaré (1893) and Russell (1897) do, mathematical transitivity from phenomenal non-transitivity.

1.3 Phenomenal change and mathematical differentiability

Since what *looks* continuous may *be* discontinuous, it is clear that phenomenal continuity cannot provide us with a reliable definition. Now the question is whether there is a *philosophical* concept of continuity distinct from both phenomenal continuity and mathematical continuity. Some philosophers have tried to answer this question; in particular, Hirsh (1982) suggests several philosophical definitions of continuity, but I want to show that

[4] See Dugac 1973 and 1976.

they amount to defining a mathematical abstraction based on the intuitively discontinuous differentiation of the parts of change.

Hirsch defines a succession of object-stages with respect to a single persisting object, such that the identity through time of material objects pertains to the theory of endurance (or three-dimensionalism) and is thereby distinct from Lewis's perdurance (cf. section 1.1). It is a matter of defining the "unity-making relationship" of the spatial and temporal parts belonging to a single persisting object, and the aim is to explain how the change of parts does not threaten the identity of the object. Hirsch applies two kinds of continuity to the change of object-stages, namely a qualitative continuity and a spatiotemporal continuity. Qualitative continuity describes a small change from a qualitative state to a distinct but very similar state, while spatiotemporal continuity is about the small motion of an object from one place p_1 at a time t_1 to a distinct but very similar place p_2 at a distinct but very similar time t_2. Hirsch suggests two possible readings of these definitions. The strong reading of continuity means that a continuous change between two states or places is divided into an infinite series of changes, contrary to the weak reading which allows a small jump between the two states or places. Then Hirsch claims: "I am not concerned with the question 'What is the ultimate nature of a change which is continuous in the strong sense?' but rather with the question 'Should we require of an object that its changes should be continuous in the strong sense?' " (p. 12). He answers this question by asserting that strong continuity is not always necessary, insofar as continuity may imply a small discontinuous jump. He takes the instance of a tree losing a branch at a time t_1, such that its volume is 30 cubic feet before t_1 but 28 after t_1 (pp. 13-14). It is obvious that there is a qualitative jump between the two states, and no infinite series of changes can apply to them. Following Hirsch, we can still speak of continuity through the weak reading of a continuous qualitative change, providing that the jump from one state to another is minimal.

Yet, Hirsch speaks of the 'Simple Continuity Analysis' owing to the incomplete nature of such definitions (1982, p. 8). He criticises it because the analysis is unable to explain how a succession of object-stages can form a single and persisting material object. He takes the instance of a car crushed in an accident in such a way that the car becomes a block of scrap metal. The weakly continuous principle applies to the succession from a car-stage to a block-stage; yet, this does not mean that a car and the block of metal constitute a unique persisting object. Thus, the 'Simple Continuity Analysis' is still relevant with respect to object-stages of different kinds, in the sense that the weak reading of continuity is independent of the principle of identity relative to a persisting object. Hirsch then supplements the property of continuity with a compositional criterion, such that x is identical to y on condition that all of x's parts are identical with y's parts; and this principle holds whether or not the definition of continuity is applied. Eventually, Hirsch wants to show that continuity on its own is unable to make sense of the concept of identity. [5]

Hirsch's concept of continuity implies its definition through differentiation, and it is particularly relevant to his weak, strong, and moderate criteria of spatiotemporal continuity, which amount to explaining how two places can be different despite being very similar. In other words, places are close enough to overlap each other, but the degree of overlap depends upon their intrinsic differences. Hirsch provides three definitions of continuity based on the partial overlap/non-overlap of different places (1982, pp. 16-19):

1. *Weak* criterion of spatiotemporal continuity: For any time t_1 of a moving object x, there is a temporal interval around t_1 such that for any t_2 in that interval the place that x occupies at t_1 *overlaps* the place that x occupies at t_2.
2. *Strong* criterion of spatiotemporal continuity: For any time t_1 of a moving object x and *for any small positive number n*, there is a temporal interval around t_1 such that for any t_2 in that interval *the extent of non-overlap* between the place that x occupies at t_1 and the place that x occupies at t_2 *is less than n*.
3. *Moderate* criterion of spatiotemporal continuity: For any time t_1 of a moving object x, there is a temporal interval around t_1 such that for any t_2 in that interval *the extent of overlap* between the place that x occupies at t_1 and the place that x occupies at t_2 *is greater than their extent of non-overlap.*

The weak criterion means that a place coinciding with an object x at a given time t must

[5] Hirsch's notion of compositional criterion is taken from Quinton's *The Nature of Things* (1973). Besides, Scaltsas (1981) claims that the identity of a changing object can be preserved through the spatiotemporal continuity of its actual form and its potential matter (or parts). These Aristotelian concepts provide a metaphysical framework to the concept of identity; yet, the intrinsic meaning of spatiotemporal continuity remains undefined.

overlap the places of x at times surrounding t. This is the minimal criterion of continuity based on a vague notion of overlap; vague, because no numerical value applies to it. By contrast, the strong criterion defines a small positive number n which measures the extent of non-overlap. The degree of non-overlap is reducible at will, so that the jump from one place to another is reduced to an ideal limit. We end up with the infinite sequence of positive real numbers, whose convergence implies an abstract mathematical limit which is not physically reachable. For instance, the S-sequence

$$\{1, 1/2, 1/4, 1/8, 1/16, \ldots\}$$

is a convergent infinite sequence of real numbers corresponding to the decreasing extent of non-overlap, whose limit is equal to zero, i.e. $lim\ S_n = 0$ with $n \to \infty$. The mathematical limit is ideal, since to reach it would imply the completion of an infinite sequence of numbers (cf. section 2.6 about supertask). This makes mathematically sense of completing an infinite, but physically speaking this means that it is impossible to fix a definitive value to the extent of non-overlap; namely, it will always be less than a given number n. Finally, Hirsch's moderate criterion represents an intermediary position between the weak and strong positions. Unlike a weak continuity, the extent of non-overlap is explicitly defined such that it is less than the extent of overlap; unlike a strong continuity, no mathematical formalism applies to the decreasing extent of non-overlap.

Hirsch's spatiotemporal continuity is based on the overlap of places, and the more two places overlap the more the change between them is continuous. However, this conception is ambiguous, insofar as Hirsch attributes three criteria of continuity to a conception of change that is intuitively discontinuous. Indeed, any change implies a partial overlap, whether it is a spatiotemporal change from a place p_1 to a place p_2 or a qualitative change from a grain of wheat to a heap of grains (Sorites paradox). A Sorites paradox implies that we can never know where the change takes places between the two contrary states of being a grain and a heap.[6] That is, when we add up two, three, etc. grains, we are unable to pinpoint the number from which the few grains become a heap. Likewise, we have seen that we cannot locate a change of colour in a continuous spectrum (cf. section 1.2). In other words, the so-called

[6] The name 'Sorites' comes from the ancient Greek word *soros*, meaning 'pile' or 'heap'.

continuity of change is merely derived from phenomenal vagueness due to the weak capacities of our senses, and if we go beyond such appearances, we quickly realise that any change implies differentiations between places or states. Eventually, change is a matter of replacing something with something else, such that a place p_1 becomes p_2, ten grains becomes eleven grains, and the colour yellow becomes an orange-yellow tending towards red. Therefore, Hirsch should conclude that any qualitative/spatiotemporal continuity, whether weak, strong, or moderate, implies a small discontinuous jump; and the intuitive idea of a continuous change is merely due to the vagueness of phenomenal continuity, itself derived from the weakness of our perceptions. By realising that the principle of intuitive continuity in a physical change has no proper justification, we should then conclude that only the mathematical idea of continuity, defined as a counterintuitive abstraction, can make sense of a continuous change. This means that change becomes a mathematical concept, and its abstract (non-intuitive) continuity pertains to the algebraic differentiation of a non-overlap.

Consequently, the strong criterion of continuity defines an intuitively discontinuous change, and implies a mathematical formalism which provides an abstract definition to a counterintuitive concept of continuity. Unfortunately, Hirsch does not try to define the mathematical framework through which we can explain why a mathematical concept of continuous change is definable, despite its intuitive discontinuity. It is a matter of understanding the transition from the physical instance of a qualitative or spatiotemporal change to its mathematical definition. In this sense, Leibniz's differential calculus will enable us to make the crucial distinction between a phenomenally continuous change derived from the intuitive idea of continuity and a mathematically continuous change defined through the abstraction of intuitively discontinuous differences.

1.4 Leibniz's mathematical law of a continuous change

My task is now to explain the hidden assumption of Hirsch's strong criterion of continuity, so that a mathematical framework can make sense of the abstract concept of continuity independently of phenomenal or empirical continuity. It is a matter of understanding what is 'continuous' in the infinite division of change, and in this sense, Leibniz is one the first mathematicians to systematise the mathematical concept of a continuous change. His aim is to provide a mathematical law that can describe and predict the motions of physical bodies. Leibniz writes in 1702, "The actual phenomena of nature are arranged, and must be, in such a way that nothing ever happens which violates the law of continuity... or any of the other most exact rules of mathematics" (G. IV. 554-571; L. 583). [7] The law of continuity is mathematical, since it pertains to the division of a difference or non-overlap between two mathematical states. Leibniz defines his mathematical law of continuity in Bayle's journal *Nouvelles de la république des lettres* (1687), in which he writes:

> When the difference between two instances in a given series or that which is presupposed can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be diminished or become less than any given quantity whatever. Or to put it more commonly, when two instances or data approach each other continuously, so that one at least passes over into the other, it is necessary for their consequences or results (or the unknown) to do so also. This depends on a more general principle: that, as the data are ordered, so the consequences are ordered also [*Datis ordinatis etiam quaesita sunt ordinata*]. (L. 351, G. III. 51)

The principle of continuity is the consequence of an order derived from numerical values. Suppose two opposite states, such as being at rest and being in motion. These two *physical* states are contradictory; yet, mathematics reduces the extent of the difference through an infinite series of *mathematical* states, such that there is a continuous change between any two infinitesimal states. This mathematical formalism has nothing to do with the smoothness of perceived phenomena, in the sense that all mathematical concepts at stake express an intuitive discontinuity, such as change, division, series, or infinitesimal points. Yet, all these concepts are parts of the law of mathematical continuity, in the sense that a change between two infinitesimal states is both mathematical and continuous. When Leibniz speaks of data approaching each other continuously, he means that the extent of non-overlap between two states is infinitely decreasing numerically, such that the infinite series of mathematical changes expresses its continuity.

[7] Leibniz adds, "Things can be rendered intelligible only by these rules, for they alone are capable, along with the rules of harmony or perfection which the true metaphysics provides, of leading us to the reasons and intentions of the Author of things" (*ibid.*).

Defining the law of continuity implies that we translate all physical concepts into mathematical ones. Thus, to claim that rest is the mathematical end of a motion means that rest is mathematically interpreted as an ideal extremity, such that the infinite sequence of infinitesimal values defines infinitely decreasing states of motions. No absolute rest can exist, since it is mathematically understood as a very slow motion whose velocity is infinitesimally near, but not equal to, zero. If rest terminates a motion, it is only through the definition of a mathematical limit, so that two physically incompatible states become mathematically compatible through the abstraction of infinitesimal changes. Indeed, there is no longer contradiction between rest and motion, since rest is mathematically defined as an infinitely slow motion. Likewise, equality does not contradict inequality, since the former is mathematically defined as the ideal limit of the latter; for instance, the inequality between zero and one implies a mathematically continuous change towards the ideal equality of 'a half', i.e. an ideal limit. While the state 'zero' endlessly changes through the infinite *increasing* sequence

$$\{0, 1/4, 3/8, 7/16, 15/32, \ldots\},$$

the state 'one' endlessly changes through the infinite *decreasing* sequence

$$\{1, 3/4, 5/8, 9/16, 17/32, \ldots\}.$$

Both infinite sequences have the ideal limit of 1/2, and in this sense, it is possible to *idealise* a concept of mathematical equality despite the actual inequality between zero and one. It is in this mathematical context that we can speak of a continuous change from inequality to equality or from motion to rest, insofar as the transition between two incompatible states is mathematically defined as a series of infinitesimal changes. Leibniz uses also the instance of a circle and a polygon to explain that there is a continuously mathematical change between these two opposite geometric figures. Leibniz writes in 1702, "Since we can move from polygons to a circle by a continuous change and without making a leap, it is also necessary not to make a leap in passing from the properties of polygons to those of a circle" (GM. IV. 104-06, L. 546). A circle eventually is a regular polygon with an infinite number of sides, and we must understand this definition as ideal, implying the ideal completion of an infinite sequence of changes. An ideal limit means that it cannot be included in the sequence, since it would contradict the infinite nature of the sequence. In other words, a circle is the ideal limit of a polygon, rest is the ideal limit of a motion, and equality is the ideal limit of inequality (and many other instances could be defined). The infinitesimal gradation of changes then prevents Leibniz from defining a leap between two incompatible states. Again, this does not mean that change is smooth, i.e. intuitively continuous, since each change is a passage from one indivisible point to another one, and however small a change is, it will always takes place between two broken points. Thus, a mathematical change corresponds to a change in numerical values, and whatever the smallness of infinitesimal values, numbers are intuitively discontinuous elements as long as they are individualised from each other (which is always the case, since Leibniz's infinitesimals are not smooth infinitesimals; cf. section 5.4).

In a modern mathematical language, we deal with a functional relation assigning a unique number, i.e. the value of the function, to a given number, i.e. the argument of the function. For instance, if an object is dropped from a tall building, then the time taken by the object to reach the ground is a function of the height of the building. The function is such that for any height of the building, i.e. any argument of the function, the amount of seconds will be the value of the function. We may illustrate this functional relationship with $f(x) = y$ assigning to any argument x the function value y. By applying Leibniz's law of continuity, we may then assert the following inference:

> If a function $f(x) = y$ and if two values x_1 and x_2 are such that $x_2 - x_1$ is less than any given small difference, then the corresponding values $y_2 - y_1$ are less than any given small difference.

The concept of function is anachronistic with respect to Leibniz, since we must wait for the eighteenth century with Euler to see its use generalised. Moreover, Leibniz provides an analytic definition of continuity, but he does not think of algebra as separated from, or even superior to, geometry. This means that geometric quantities have no algebraic autonomy, contrary to Bolzano's (1817) and Cauchy's (1821) definitions of independent algebraic variables, which make the concept of limit wholly independent of the notion of infinitesimals. Likewise, we cannot compare Leibniz's infinitesimal calculus with Weierstrass's continuous functions (cf. section 1.2), since the concept of real number receives a consistent definition only from the second half of the nineteenth century. Despite all these historical caveats, Leibniz clearly understands the concept of differentiable functions. [8] He deals with a mathematical relationship between two quantities, such that the infinitesimal change of one leads to the infinitesimal change of the other; and we may then define the rate at which the value of the function is changing. In other words, a differentiable function or derivative is a rate of change, i.e. the calculation of the slope of a tangent at every point x of the graph of the function $f(x)$. The algebraic variable is then defined as the mathematical rate of a continuous change; it is called a velocity, which is not a physical property and should not be confused with the physical concept of speed. Indeed, speed is a scalar quantity, namely a positive number describing how fast an object is moving. By contrast, velocity is a vector quantity, i.e. a series of numbers describing a rate at which an object changes its position and direction. In other words, velocities can be negative (as negative rates of change) with respect to a system of coordinate axis, and we may define a physical object as speeding up regardless of a negative velocity.

This shows the fundamental distinction between mathematical abstractions defined through Cartesian coordinates, and genuinely physical properties. Continuous change and velocity are purely mathematical concepts, so that their definitions do not refer to anything intrinsically physical. Velocity is the derivative of $f(x)$, written as either $f'(x)$ or $d/dx\, f(x)$ if we follow Leibniz's notation with d understood as the infinitesimal operator, and a continuous change is calculated by a derivative, i.e. the slope of the tangent for each infinitesimal point of the functional graph. In other words, the continuity of a mathematical change is related to its infinitesimal division into broken points, which contradicts the intuitive idea of continuity.

1.5 Infinitesimal points and qualitative states

Russell (1900, p. 111) claims that continuity is not a fundamental concept in Leibniz's philosophy, and he is right in the sense that Leibniz does not deal with the idea of physical continuity. We shall see that matter is discrete, and that separate physical bodies are in

[8] Leibniz's views should not be interpreted in the form of point-set or combinatorial topology (cf. Arthur 1986, Levey 1998); this argument implies the knowledge of real numbers without which modern topology cannot be consistently defined. I shall come back to this topic in sections 5.2 and 5.3 regarding Peirce's topological definition of a pure geometry.

relation to each other without assuming the existence of a continuous physical space. Yet, I would like to insist on the perfect compatibility between a discrete physical world and the mathematical law of continuity, such that it is possible to claim without contradiction that nature makes no leap, although matter is discrete and monads indivisible.

Leibniz defines three levels of existence. The metaphysical level pertains to monads or simple substances defined as non-spatial, extentionless beings. A second level is physical or phenomenal, such that physical bodies are derived from (but not composed of) the infinite aggregate of indivisible monads; hence any change in matter is physical or phenomenal. Leibniz explains physical phenomena through the perception of monads, in such a way that no Kantian division applies between a phenomenal thing related to an empirical concept and a thing in itself that we are unable to know. Thus, the knowledge of things in themselves, i.e. Leibniz's monads, are accessible through the perception of physical bodies (G. II. 450-52; L. 604).[9] In other words, physical phenomena pertain to the perceptions of discrete objects derived from extensionless, indivisible monads. A third and final level of existence defines mathematical entities, such as space, time, and change. They are mathematical abstractions unrelated to physical bodies and monads; more precisely, they are ideal continua, i.e. abstract objects of thought defined as potentially infinite. We have then a direct opposition between a discrete physical matter and a continuous mathematical space. Already in 1676, Leibniz writes, "It is clear… how great the difference between space and matter. Matter alone can be explained by a plurality without continuity." (Jag. 28-40; L. 158). Matter is infinitely divided in actuality, such that physical bodies are physically contiguous but not continuous. The transition from matter to space describes the passage from a real existence to an ideal abstraction, such that space is merely an object of the mind. Therefore, space and time are ideal orderings abstracted from not only physical matter but also metaphysical monads. In a letter to De Volder (1705), Leibniz writes:

[9] See Leibniz's *Theodicy* in the preface (G. VI. 29) and sections 348, 349, 384 (G. VI. 321, 343). See also McGuire 1976 and Hartz & Cover 1988.

> Matter is not continuous but discrete, and actually infinitely divided, though no assignable part of space is without matter. But space, like time, is something not substantial, but ideal, and consists in possibilities, or in an order of coexistents that is in some way possible. And thus there are no divisions in it but such as are made by the mind, and the part is posterior to the whole. In real things, on the contrary, units are prior to the multitude, and multitudes exist only through units (the same holds of changes, which are not really continuous). (G. II. 278-9; Russell 1900, p. 245)

Following Leibniz, the physical idea of an empty space (or of an empty time) is absurd, since this would imply for space to be substantial, i.e. composed of substances; and if so, space would not be empty but full of physical bodies derived from the multiplicity of discrete monads. Hence, if space is devoid of matter, it cannot be physical but is merely an ideal object of thought defined as a potential entity without actual (real) determination. Leibniz distinguishes spatial parts from material ones, in the sense that an ideal space is a prior whole whose existence of posterior parts directly depends upon the whole itself; and this saves the continuity of the whole. By contrast, the parts of matter are prior units derived from monads such that matter is the posterior collection of prior parts, and matter is discrete since defined as a multiplicity of distinct units. Note the last sentence into brackets in the above quotation, in which Leibniz claims that the same holds of change, insofar as changes are not continuous. There is nothing surprising with this argument, since Leibniz deals, in this case, with a physical change in matter; therefore, we must make the distinction between a physically discontinuous change in discrete matter and a mathematically continuous change derived from the infinitesimal calculus. There is no contradiction at all, since *real* discreteness cannot conflict with *ideal* continuity.

Infinitesimal points are ideal mathematical entities, in the sense that they are defined by a mathematical formalism (cf. section 1.4). Yet, points also exist with respect to the other two levels of existence. On the one hand, metaphysical points are the points of monadic substances and are real things. On the other, phenomenal or physical points are real and indivisible, but only with respect to our own perceptions since physical bodies derive from the perceptions of monads (G. IV. 482; L. 456-457). In other words, discreteness in physical matter implies determinate actual points, while continuity in mathematical space defines indeterminate potential points. Likewise, physical change in matter produces a discontinuous jump from a determinate actual point to another actual one, whereas ideal points are mere modalities, i.e. objects of thought whose existence is non-material and fictive. [10] In this sense, infinitesimal points are related to a mathematical space, and are by definition potential and ideal. This does not mean that infinitesimal points are vaguely defined, since infinitesimal calculus defines them through determinate numerical values; but it is their ideal existence that is indeterminate, which explains why infinitesimals do not exist in the physical world, and can thereby be discarded when the results of mathematical calculations are verified through the values of empirical measurements.

Does it mean that infinitesimals are parts of a continuous space? When the mathematical law of continuity defines a continuous change between infinitesimal points, points are indivisible and broken entities. If infinitesimals are parts of a continuous change, they are not parts of a continuous space. If this were the case, this would imply that infinitesimal parts are prior to the continuous whole, and would contradict Leibniz's principle that a continuum is always prior to its parts. Therefore, infinitesimal points are related to a continuous space by being defined, not as parts, but as indivisible limits or termini. In 'First Truths' (1680-84), the young Leibniz writes, "A continuum is not divided into points, nor is it divided in all possible ways. It is not divided into points, because points are not parts but limits." (OF. 518-23; L. 270). The mature Leibniz does not change his mind, and defines the same principle in his *New Essays on Human Understanding* (1705), "Strictly speaking, points and instants are not parts of time or space, and do not have parts either. They are only termini" (NE. 152). Thus, infinitesimal points are the indivisible limits of a continuum whose parts are infinitely divisible. They do not increase or decrease a continuous extension, since they are incorporeal, extensionless entities. Leibniz's conception of a continuous space excluding indivisible entities is close in meaning to Aristotle's purely geometric continuity (cf. chapter 2), and this constitutes the intrinsic weakness of Leibniz's position which denies the existence of infinitesimals as independent algebraic operators located within a purely

[10] Leibniz writes to De Volder (1706), "In actual body, there is only a discrete quantity, that is, a multitude of monads or of simple substances… But a continuous quantity is something ideal which pertains to possibles and to actualities only insofar as they are possible. A continuum, that is, involves indeterminate parts, while on the other hand, there is nothing indefinite in actual things, in which every division is made that can be made." (G. II. 282, L. 539). See also G. IV. 491, G. VII. 564, and Hartz and Cover 1988, p. 508.

algebraic Cartesian space. [11] Each infinitesimal point has a quantitative algebraic value, which is subsumed under the geometric (and qualitative) interpretation of infinitesimals. Leibniz does not define independent algebraic variables, unlike modern calculus whose algebraic operators are quantitative values devoid of geometric interpretation. The reference to a geometric graph is no longer required in the modern definition of a continuous function (cf. section 1.1). Likewise, the definition of a derivative $f'(x)$ is purely quantitative, since it is defined through a unique algebraic expression without reference to infinitesimal geometric points, as implied by Leibniz's derivative $d/dx\, f(x)$ (cf. section 1.4).

Consequently, Leibniz's contrast between a continuum (of space or time) and a limit (of points or instants) is not between the actual and the ideal (since both are ideal mathematical entities), but between the geometrically quantitative and the geometrically qualitative. This distinction matters since Leibniz has a geometric interpretation of algebraic values, such that infinitesimal points are qualitative limits (and not quantitative parts) of a continuous geometric space. In such a context, we may then understand Leibniz's claim in a letter to Des Bosses (1716): "There is continuous extension whenever points are assumed to be so situated that there are no two between which there is not an intermediate point" (G. II. 515; Russell 1900, p. 247). This sounds like the arithmetic definition of denseness applied to a domain of rational numbers, such that there is always a rational number between any two rational numbers. Yet, it would be misleading to follow this anachronistic interpretation since Leibniz has no conception of arithmetic properties applied to algebraic domains of numbers. Therefore, his statement merely means that infinitesimal points are indivisible divisions of a continuous extension, and the infinite divisibility of a continuum implies an infinite number of infinitesimals points. This is perfectly consistent with Leibniz's following claim: "Points are neither large nor small, and no leap is needed to pass them. The continuum, however, though it has such indivisibles everywhere, is not composed of them."

[11] In a letter to Christian Huygens (1679), Leibniz writes, "To reduce geometric problem to algebra, i.e., to reduce problems determined by figures to equations, is often a rather prolonged affair, and further complications and difficulties are necessary to return from the equation to the construction, from algebra back to geometry. Often, too, the constructions produced in this way are not entirely appropriate, unless we are lucky enough to stumble upon unforeseen postulates and assumptions. This Descartes himself tacitly admitted in solving a certain problem of Pappus in Book III of his *Geometry*." (GM. V 178-83, L. 254). See also OF. pp. 396-7.

(G. I. 416; *ibid.*). [12] That is, infinitesimals points are indivisible entities, whose extensionless nature makes the idea of a leap or jump irrelevant. In other words, Leibniz interprets geometric infinitesimals as *qualitative* states devoid of geometrically *quantitative* definitions. The mathematical law of continuity pertains to the definition of transitory states, and the qualitative abstraction of infinitesimals does not conflict with the quantitative definition of a geometric space. Continuity only means that we can move, say, "from polygons to a circle by a continuous change and without making a leap" (GM. IV. 107-6, L. 546). Consequently, a change is continuous for two reasons: first, there is always an intermediary point between any two points producing an infinite sequence of points; second, a change between infinitesimals is a qualitative abstraction which is not a quantitative part of a geometric extension.

1.6 The ideal, the actual, and the labyrinth of the continuum

If the fact that nature makes no leap is an obvious consequence of the mathematical law of continuity, this does not mean that physical motion is smooth. In his *New Essays* (1705), Leibniz defines continuity in nature through a direct reference to his mathematical law of continuity; he writes:

> Nothing takes place suddenly, and it is one of my great and best confirmed maxims that *nature never makes leap*. I called this the Law of Continuity when I discussed it formerly in the *Nouvelles de la république des letters* [cf. section 1.4]... There is much work for this law to do in natural science. It implies that any change from small to large, or vice versa, passes through something which is, in respect of degrees as well as of parts, in between; and that no motion ever springs immediately from a state of rest, or passes into one except through a lesser motion; just as one could never traverse a certain line or distance without first traversing a shorter one. (G. V. 49, NE. 56)

The law of continuity applies to natural science through the abstract system of differential calculus, namely the calculation of a derivative for each infinitesimal point of a geometric curve. Thus, it is always possible to define a third infinitesimal state of change with a given

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[12] Likewise, Leibniz writes (1709-1715), "Extension arises from situation, but it adds continuity to situation. Points have situation, but they neither have nor compose continuity, nor can they stand by themselves. So there is nothing to prevent an infinity of points from coming into being and perishing... without any increase or diminution of matter and extension, since they are only the modifications of matter and not its parts but its extremities." (G. II. 369-72, L. 598).

derivative (or velocity) between any two infinitesimal states. The division into infinitesimal changes is potentially infinite, and nature makes no leap only because infinitesimal differences are infinitely reducible without ever disappearing. Accordingly, Leibniz is able to combine the discrete with the continuous, the finite with the infinite, and the indivisible with the divisible without reaching any contradiction, insofar as the *real* existence of metaphysical monads as perceived through phenomenal bodies does not conflict with the *ideal* existence of mathematical abstraction. Metaphysics and physics pertain to real monads and discrete matter, while mathematics deals with a continuous change into ideal infinitesimals. The perfect coexistence between physics and mathematics is explained by the fact that the actual does not conflict with the ideal.

However, Leibniz is ready to admit that the real is governed by the ideal, such that mathematical abstraction defines our knowledge of continuity and of the infinite from which real discreteness and the finite are comprehensible. Leibniz writes in a letter to Varignon (1702), "One can say in general that though continuity is something ideal and there is never anything in nature with perfectly uniform parts, the real, in turn, never ceases to be governed perfectly by the ideal and the abstract and that the rules of the finite are found to succeed in the infinite." (GM. IV. 91-95, L. 544). Ideal continuity does not exist as such in nature, but it governs natural laws, such that mathematically continuous change is able to define physical events abstractly. This conception is radically modern, since it defines physical principles through mathematical concepts. In this sense, the law of continuity plays a fundamental role in physical laws. Leibniz writes to De Volder (1704), "For me nothing is permanent in things except the law itself which involves a continuous succession and which corresponds, in individual things, to that law which determines the whole world" (G. II. 262-65, L. 534). [13] Since geometry is defined as the mathematical structure of the physical world, there is no difficulty in applying infinitesimal calculus to our mathematical understanding of any

[13] Contrary to Leibniz, Aristotle defines the property of succession as opposed to the concept of continuity. He writes: "Now if the terms 'continuous', 'in contact', and 'in succession' are understood as defined above—things being continuous if their extremities are one, in contact if their extremities are together, and in succession if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles." (*Physics*, 231a21-24). We may explain the contrast between Aristotle and Leibniz through the Aristotelian definition of continuity based on a purely geometric magnitude which has no connection whatsoever with numbers (cf. sections 2.1, 2.2 and 2.3).

physical events. The whole of physics pertains to the mathematical law of continuous change as defined by the infinitesimal calculus. This constitutes a rupture with not only Aristotelian but also Cartesian physics. Unlike Aristotle, Leibniz does not define a purely *physical* concept of change understood as the actualisation in nature of something potential, qua potential. Unlike Descartes, he does not postulate a *metaphysical* change whose ultimate cause (i.e. force) resides in the idea of God. In other words, Leibniz has a mathematical conception of physical sciences, whose objects of knowledge are abstract constructions of the mind.

Leibniz intends to solve all difficulties in our understanding of a continuum by distinguishing the mathematical objects of thought, defining physics as a science, from the metaphysical substances and physical bodies. Leibniz writes, "The difficulties concerning the composition of the continuum will never be resolved, so long as extension is considered as making the substance of bodies" (G. II. 98, Russell 1900, p. 243). Extension is a mathematical concept, and its ideal existence cannot be assimilated to the substantial existence of phenomenal bodies based on real monads. Leibniz speaks of the labyrinth of the continuum in order to criticise the confusion between the ideal and the actual regarding the composition of the continuum. Indeed, it is meaningless to define actual parts in ideal things, i.e. physical bodies in space, or ideal parts in actual things, i.e. a continuous extension in matter. If we know that matter is real and that space is ideal, then we should not claim that space is a property of matter or that physical bodies are parts of space. In this sense, the careful distinction between both domains of existence, either real or ideal, makes the labyrinth of the continuum irrelevant. Leibniz writes to Des Bosses (1709-15), "In actuals, single terms are prior to aggregates, in ideals the whole is prior to the part. The neglect of this consideration has brought forth the labyrinth of the continuum." (G. II. 379; Russell 1900, p. 245). As a continuous space is an ideal concept, it is an abstract object of the mind, and we have seen that an ideal whole is always prior to its indeterminate parts. By contrast, matter is a collection of discrete parts, such that the parts are prior to the whole. Therefore, it is not the distinction between continuity and discreteness which matters, but rather its foundation into two levels of existence: an ideal whole is continuous because an abstract entity is always prior to its ideal parts, while an actual whole is discrete only because its actual parts are prior to the whole. [14]

The labyrinth of the continuum applies to the distinct definitions of point. Actual points are discrete parts of matter, since actual matter is composed of discrete units, themselves derived from a multiplicity of indivisible monads. By contrast, ideal infinitesimal points are objects of thought defined as the non-quantitative limits of a continuous space. To clarify the distinction between the two kinds of point, Leibniz uses an analogy with numbers. He writes in a letter to De Volder (1706), "Actual things are compounded as is a number out of unities, ideals as is a number out of fractions; the parts are actually in the real whole but not in the ideal whole" (G. II. 282, L. 539). That is, an actual thing is like a number whose parts are a plurality of units, e.g. '3' is a plurality of one *plus* one *plus* one; conversely, an ideal thing is like a number whose parts are fractions, e.g. '1' is composed of the fractions '1/2' *plus* '1/2'. Leibniz's analogy makes sense, insofar as any positive integer implies that its parts are prior to the whole, namely '3' results from the collection of three prior units. On the contrary, a positive integer, composed of fractions, is defined as a whole prior to parts, since '1' must first be given in order to make sense of its parts '1/2' and '1/2'. Thus, an actual number, i.e. a plurality of units, is opposed to an ideal number, i.e. a number composed of fractions. In his *New Essays* (1705), Leibniz uses this analogy to generalise the properties of numbers with respect to the continuum. He writes: "*Numbers in the broad sense*—comprising fractions, irrationals, transcendental numbers, and everything which can be found between two whole numbers—is analogous to a line, and does not admit of a minimum any more than the continuum does"; yet, "'number' as a multitude of units is appropriate only for whole numbers" (NE. 156, original emphasis). Thus, an abstract number is not at all different from an abstract line, in the sense that both are continua that are prior to their respective parts. Leibniz's position rejects the Aristotelian view that geometric continuity cannot be associated with numbers (cf. section 2.1). That is unsurprising since Leibniz takes into account Descartes' algebraic foundation of mathematics as systematized in his *Geometry*,

[14] Charlton (1991, p. 134) objects that Leibniz's position does not make sense. He claims that if parts are actual, then the whole is merely possible; by contrast, if the whole is actual, then the parts must be potential. This principle undoubtedly holds in an Aristotelian context, as we shall see in sections 2.1, 2.2, and 2.3. However, it cannot apply to Leibniz's position since the distinction between the actual (or the real) and the ideal pertains to two levels of existence, such that if a whole space is ideal, so are its spatial parts, and if material parts are actual, so is the whole matter.

such that each geometric line is translatable into an algebraic expression, i.e. a linear equation. Hence, a continuous line is divisible into indefinite lines whose exact limits are calculated through infinitesimal points in the same way that an abstract number is divisible into indefinite fractions whose exact limits are calculated through numerical values. By contrast, an actual (or real) line is discontinuous since defined as an aggregate of prior parts, i.e. a multiplicity of simple substances; likewise, an actual number is a plurality of prior and substantial units (G. IV. 491; Russell 1900, p. 246). This illustrates the two levels of existence, such that both lines and numbers are continuous if defined as ideal in any mathematical abstraction, but discrete if presented as actual in the physical world.

Consequently, the labyrinth of the continuum is avoidable providing that we do not confuse the ideal with the actual in its composition. In his *Remarks on the Objections of M. Foucher* (1695), Leibniz writes, "It is the confusion of the ideal and the actual which has embroiled everything and produced the labyrinth concerning the composition of the continuum" (G. IV. 491; Russell 1900, p. 246). He repeats the same argument in a letter to De Volder (1706): "But we confuse ideal with real substances when we seek for actual parts in the order of possibilities, and indeterminate parts in the aggregate of actual things, and so entangle ourselves in the labyrinth of the continuum and in contradictions that cannot be explained." (G. II. 282, L. 539). For Leibniz, the problem is easy to solve insofar as his metaphysics based on the multiplicity of indivisible monads implies a discrete physical world. In other words, anything actual is both physical and discrete, while anything ideal merely results from objects of thought. Thus, continuity pertains to the infinite sequence of infinitesimal points that divide (as limits) the geometric space. Leibniz concludes, "Space, time, extension, and motion are not things but well-founded modes of our consideration" (OF. 518-23; L. 270). That is, they are not actual things, but ideal objects of the mind defined as mathematical abstractions.

Leibniz solves the problem of continuity with a radical solution; but this makes it difficult to explain how objects of thought can have an influence on the physical world. If ideal continuity is wholly distinguished from real discreteness, then how can Leibniz conclude that the law of continuity has physical implication in nature? Indeed, we have seen that mathematical continuity implies that nature makes no leap, such that any physically discrete change is divided into a mathematical continuous change of infinitesimals. Leibniz admits the difficulty in his *Reply to the Thoughts on the System of Preestablished Harmony* (1702); he writes:

> It is true that perfectly uniform change, such as the mathematical idea of motion, is never found in nature any more then are actual figures which possess in full force the properties which we learn in geometry, because the actual world does not remain in this indifference of possibilities but arises from the actual divisions or pluralities whose results are the phenomena which are presented in practice and which differ from each other down to their smallest parts. (G. IV. 568-70, L. 583)

There is a clear separation between the ideal motion inferred from the infinitesimal calculus and the real motion of the physical world derived from the actual divisions of matter. Yet, Leibniz does not regard this argument as an objection to his system, insofar as the metaphysical divide between the ideal and the real does not prevent the former from influencing the latter. The fact that real matter and real change are physically discrete does not constitute an obstacle to the interpretation of the real through an idealisation from the mind. Thus, mathematics makes sense of the continuity in nature through the abstract definition of infinitesimal divisions, such that mathematics abstracts laws of physics through idealisation. This does not mean that there is a transformation from one world to another, since both worlds, either real or ideal, are wholly separate from each other. Yet, our knowledge of the physical world is not given by the metaphysics of monads which do not explain the interaction between discrete bodies. Only a mathematical conceptualisation through the law of continuity can explain and predict the behaviours of physical objects independently of their metaphysical foundations. In this sense, mathematical continuity is a much more fundamental concept than metaphysical discreteness. Leibniz (1702) writes:

> Although mathematical thinking is ideal... this does not diminish its utility, because actual things cannot escape its rule. In fact, we can say that the reality of phenomena, which distinguishes them from dreams, consists in this fact. However, mathematicians do not need all these metaphysical discussions, nor need then embarrass themselves about the real existence of points, indivisible, infinitesimals, and infinites in any rigorous sense. (G. IV. 569-571; L. 583-584)

Therefore, mathematicians only need to concentrate on their mathematics without dealing with metaphysical matters; yet, metaphysicians, let alone physicists, cannot ignore mathematical abstraction since mathematics is the only way to make a distinction between a dream and an object of knowledge. That is, metaphysicians can never be sure that their perceptions of discrete bodies are not dreams, as implied by Descartes's 'Dreaming Argument'. Hence, only mathematics can successfully fight against scepticism by providing an ideal abstraction to the construction of physical laws, abstraction that can confirm the true perceptions of physical bodies, themselves derived from the metaphysical existence of monads.

Consequently, Russell (1900) is right to claim that continuity is not a physical or metaphysical principle in Leibniz, since the discreteness of physical bodies is related to the indivisibility of monads. Yet, the real world is not so important, insofar as only ideal mathematics makes sense of extension, space, time and motion through the law of continuity. If there is a great divide between the real and the ideal, it is only through the distinction between metaphysics and epistemology. The real world of monads and phenomenal bodies exist independently of us and does not interact with us, while all continua are products of the mind and participate in the mathematical construction of our physical knowledge. Thus, Leibniz's mathematics idealises the world through abstractions, and the law of continuity is nothing more than the idealisation of the discrete real world.

Chapter 2
Geometric and Arithmetic Continua

Leibniz's continuous change expresses a combination of geometric infinitesimals with algebraic values, and constitutes an intermediary stage between two extreme definitions, namely a non-algebraic geometric continuity opposed to a non-geometric arithmetic continuity. This chapter will stress the absolute incompatibility of these two principles. I shall object to Bostock's (1991) position, which denies any relevance to an Aristotelian geometric continuity on the authority of arguments derived from arithmetic continuity. This view cannot be correct since it neglects the mathematical fact that Aristotelian geometry is devoid of numbers and cannot thereby make sense of the algebraic properties of an arithmetic continuum, even when this latter is defined as a real line. Then I shall show that a purely algebraic sequence of numbers is an extensionless formalism that cannot intrinsically pertain to the geometric concept of extension; that is, the correspondence between geometric points and algebraic numbers is not mathematically justified. In this sense, it is inexact to claim that algebra *solves* Zeno's paradoxes of motion; it only makes them *irrelevant* insofar as the algebraic sequences of numbers are unable to comprehend non-algebraic geometric paradoxes.

2.1 Continuous geometric magnitudes vs. continuous arithmetic intervals

The Cartesian algebraic line and the nineteenth-century arithmetic continuity have successfully challenged Aristotle's continuous geometric magnitude, as defined in Book VI of his *Physics*. [1] This means that modern mathematics denies any relevance to an Aristotelian continuum understood as a geometric magnitude excluding indivisible elements, such that the parts of a geometric line are divisible lines and not indivisible points. Aristotle

[1] For an introduction to Aristotle's continuity, see Ross 1936 (pp. 69-71) and Konstan 1989. As for some historical studies, see Kretzman 1982, Sorabji 1983, and White 1992.

opposes geometric continuity to arithmetical discontinuity. That is, a number (*arithmos*) is a plurality (*plêthos*) composed of discontinuous parts, while a geometric magnitude (*megethos*) is a quantity composed of continuous parts (*Categories*, 4b25-5a14, *Metaphysics*, 1020a7-14). Both quantities have incompatible structures, which explain why a geometric line excludes indivisibles from its composition. Thus, an Aristotelian line cannot be defined within Cartesian co-ordinates, since a Cartesian line is composed of indivisible points, whose numerical co-ordinates belong to the algebraic definition of the line as a linear equation. [2]

David Bostock (1991) criticises Aristotle's definition of continuity because of its dissimilarity with the modern conception. His main objection pertains to the Aristotelian exclusion of indivisibles from the composition of a continuum, which he defines through two premises: first, a continuum is divisible into at least two non-coincident parts; second, these two parts touch and share one and the same limit. I want to show that such premises cannot make sense of the Aristotelian conception of continuity, insofar as they rest on arithmetic properties. In contrast, I shall defend Aristotle's geometric continuity as a perfectly sound concept providing that we are aware that it cannot have algebraic or arithmetic properties. In other words, an Aristotelian continuity is irrelevant to modern mathematics only because Aristotle deals with a non-algebraic structure. The requirement is then to confine our interpretation of Aristotle's continuum to a pre-Cartesian geometry, in which geometric magnitudes are untranslatable into numbers.

Aristotle's geometric continuity rests on the intuitive idea of continuity, such that the parts of a continuum are as divisible as the continuum itself; and there is no end to its division because no indivisible interrupts its smooth composition. If points were parts of a line, then the line could not be continuous since composed of broken and separate parts. However, Bostock rejects this purely geometric approach, since his first premise defines a continuum as having at least two non-coincident parts. He writes:

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[2] Note that Leibniz attempts to reconcile Cartesian geometry with some Aristotelian geometric properties; in particular when he claims that infinitesimal points are not parts but limits of a geometric line or curve (cf. section 1.5). He still thinks of geometry, despite its algebraic properties, as the main field of mathematics.

> What must we take the definition of a continuum to be? First, it must evidently be given as a premiss that a continuum has at least two parts which do not coincide (231b4-6). For Aristotle clearly does not count a point as *itself* a continuum, and this seems to be the minimal premiss needed to rule out that suggestion. (1991, p. 182, original emphasis)

It is certain that Aristotle does not define a continuum as a point, and Bostock justifies his premise by quoting the passage 231b4-6 in *Physics*' Book VI, in which Aristotle writes: "What is continuous has one part here and another there, and is divided into these [parts], [which are] distinct and separate in place." (*to gar suneches echei to men allo to d' allo meros kai diaireitai eis houtôs hetera kai topôi kechôrismena*, 231b4-6).[3] We can infer from this statement that indivisibles cannot be continuous, insofar as such partless entities do not have parts here and there. However, 231b4-6, defined in its context, means much more than this: it relates a continuum to *potentially* divisible parts, which *may* be divided in actuality and *may* be located in distinct and separate places. Hence, a point cannot be continuous since its indivisibility negates the idea of potentially divisible parts. If Bostock's premise had wanted to assert such a principle, it would have been sufficient to claim that a continuum is infinitely divisible. In other words, it would not have been necessary to define a continuum as composed of two non-coincident parts. Yet, Bostock's premise is influenced by another conception of continuity, which contradicts Aristotle's definition. Indeed, if a continuum has two, three, four, etc. non-coincident parts, then its parts are *actually* distinct from each other; if so, they are no longer *potentially* defined. Nevertheless, Aristotle's *Physics* constantly repeats that a continuum has infinitely divisible parts, whose infinite divisibility requires potentiality (231b15-16, 232b24-25). That is, a continuous magnitude is infinitely divisible into continuous magnitudes (232a23-25), and what is infinitely divisible is in potentiality (*dunamis*, 263b5-6) and *not* in actuality (*energeia*, 206a16-17). Likewise, his *Metaphysics* (1020a10-11) stresses that a magnitude is "potentially divisible" (*diaireton dunamei*) into continuous parts, meaning that parts remain actually *indistinct* from each other. Therefore, an Aristotelian continuum cannot have non-coincident parts (whether two or one thousand), since non-coincidence implies the definition of actually *distinct* parts. Aristotle writes in Book VIII of his *Physics*:

[3] References to the Greek text are taken from Ross's (1936) and Carteron's (1926-31) editions. All translations are mine, unless specified otherwise.

> If we divide the continuous into two halves, we use one point as two; indeed, we regard [the point] both as a beginning [of a line] and as an end [of another line]. The same thing is made when we both number and divide into halves. In dividing in this manner, neither the line nor the motion will be continuous (*houtô de diairountos, ouk estai suneches outh' hê grammê outh' hê kinesis*); for a continuous motion is of something continuous, and in the continuous, there are infinite halves, but potentially (*dunamei*) and not actually (*ouk entelecheiai*). (263a23-29)

If a continuum were divisible into at least two halves, then its definition would have been the same as a number or plurality of indivisible parts, which cannot obviously be the case unless we are ready to contradict many statements in the whole Aristotelian Corpus. In other words, a continuous line cannot be composed of two actually distinct broken parts, since this property contradicts the infinite divisibility attached to each potential part of a geometric magnitude.

It is then unsurprising to realise that Bostock's premise rests on a mathematical principle distinct than Aristotle's geometric continuity, namely the definition of an arithmetic continuum to which Bostock refers as the "orthodox logic of parts". He writes:

> To avoid some complications I should like to strengthen this premiss a little to the following: a continuum may always be *divided* (without remainder) into two parts that do not coincide. It is true that, given a 'logic of parts' that is nowadays orthodox, the supposedly stronger version would follow from the original. For the now orthodox logic supposes that, given any proper part of a thing, there will always be some *one* further part of the thing, which comprises all the rest of it excluding the given part. Thus, the whole may be said to be divided into these two parts in the sense that the two do not overlap one another—i.e. they have no common part—and together they exhaust the original whole, insofar as every part of it must overlap at least one of these two. But it seems to me doubtful that Aristotle would accept the assumption on which this reasoning is based. (1991, pp. 182-183, original emphasis)

Bostock is right to claim that Aristotle would likely reject the assumption on which the first premise is based, but I want to reinforce the claim by adding that Aristotle would not even accept the first premise itself. The orthodox logic of parts rests on an arithmetic structure of continuity, such that parts pertain to either numbers or intervals of numbers. Arithmetic continuity defines a principle of completeness attributed to real numbers; as such, it has nothing to do with the intuitive idea of continuity. Dedekind defines arithmetical continuity by using the concept of a real line (i.e. a line of real numbers), which is composed of two disjoint parts such that only one point produces a cut between them. He writes in his *Stetigkeit und irrationale Zahlen* (1872):

> 'If all points of the straight line fall into two classes such that every point of the first class [*Klasse*] lies to the left of every point of the second class, then there exists one and only one point which produces this division [*Einteilung*] of all points into two classes, this severing [*Zerschneidung*] of the straight line into two portions [*Stücke*].' (1932, p. 322; 1996, p. 771)

Note that this statement is between quotes in Dedekind's original text, insofar as the real line and points are not geometric objects, but constitute geometric metaphors pertaining to an arithmetic domain of numbers. I insist on this claim, because the reference to a real line is sometimes viewed as geometric, and this leads to some confusion with respect to the definition of arithmetic continuity. In other words, the intrinsic definition of an arithmetic continuum pertains, not to points, but to real numbers, and not to a line, but to an unbounded open interval $(-\infty, \infty)$ defined as the continuous set R of all real numbers. [4] Dedekind's proper definition of arithmetic continuity is the following (to compare with the above quotation):

> The domain R possesses also continuity [*Stetigkeit*]; i.e., the following theorem is true: If the system R of all real numbers breaks up into two classes U_1, U_2, such that every number a_1 of the class U_1 is less than every number a_2 of the class U_2 then there exists one and only one number α by which this separation [*Zerlegung*] is produced. (1932, pp. 328-329; 1996, p. 776)

Dedekind attributes the arithmetic property of continuity to a system or structure R of all real numbers. This amounts to a principle of arithmetic completeness, such that the set of all real numbers exhausts every possible number in an arithmetic system. By contrast, the set Q of all rational numbers is merely dense, meaning that there is always a rational number between any two rational numbers. This set is not continuous, since it is arithmetically incomplete; indeed, it does not take into account irrational numbers (e.g. the square root of two or the number π).

Let us now define the logic of parts implied by Dedekind's arithmetic continuity. If we apply the above definition, we can claim that a continuous interval of real numbers is composed of two disjoint subintervals, whose one wholly precedes the other. If either the

[4] As a reminder: N is the set of all natural numbers, Z the set of all integers (i.e. N and the negative whole numbers), Q the set of rational numbers (i.e. N, Z, and the periodic decimal fractions), and R the set of all real numbers (i.e. Q and the non-periodic decimal fractions, i.e. irrational numbers). Note that an irrational number, in ancient geometry, corresponds to a geometric incommensurable, *intuitively* defined as the diagonal of a square.

first subinterval has a last term or the second subinterval has a first term, then this term is a Dedekind cut separating the two subintervals from each other. The notion of cut corresponds to a real number, i.e. either a rational or an irrational number; in this sense, the possibility for a Dedekind cut to be an irrational number implies the completeness (or continuity) of the arithmetic domain. Dedekind (1872) writes:

> The incompleteness or discontinuity of the domain Q of rational numbers consists in this property that not all cuts are produced by rational numbers. Thus, whenever we have a cut (A_1, A_2) produced by no rational number, we *create* a new number, an *irrational* number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as *different* or *unequal* if and only if they correspond to essentially different cuts (*verschiedenen Schnitten*). (1932, p. 325; 1996, p. 773, original emphases)

While two intervals of rational numbers are merely dense, i.e. arithmetically incomplete, the definition of Dedekind cuts as real numbers implies the abstract completeness of an arithmetic system. In other words, Dedekind replaces the *intuitive* idea of an indeterminate irrational number with the *abstract* conception of a real number understood as a well-defined entity, whether it is rational or irrational. Thus, Bostock speaks of an orthodox logic of parts, which is the mere reinterpretation of Dedekind's arithmetic continuity and does not rely on the *intuitive* concepts of parts and wholes. When Bostock's first premise implies the division of a continuous whole into at least two non-coincident parts, this means that the two parts "do not overlap one another", and since the two parts exhaust the whole, every part of the whole must overlap one of these two parts (1991, p. 183). Suppose the continuous interval [0, 1] belonging to the structure R of all real numbers, such that the partition of this interval produces two non-coincident parts, namely the half-open interval [0, 1) and the singleton {1}. [5] Then these two parts exhaust the whole [0, 1], and every part of [0, 1] must overlap either [0, 1) or {1}. Another property is that the singleton {1} corresponds to an indivisible part of the arithmetic continuum. This does not mean much in the context of arithmetic, but Bostock uses this anodyne property as an argument against Aristotle's geometric continuum, itself based on the exclusion of indivisibles. Bostock writes, "It is possible to divide a line

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5 We deal with a purely arithmetic structure, despite the fact that Bostock uses terms like 'the rest of the line' instead of 'half-open interval' and 'end-point' rather than 'singleton'.

into two parts, where *one* of these parts is a mere point, and hence not further divisible" (p. 184, original emphasis). If this statement pertains to an arithmetic continuum, it is true but completely trivial; if it applies to a geometric continuum, it is both false and anachronistic.

We face, therefore, two kinds of logic of parts. On the one hand, arithmetic continuity is about a continuous interval of real numbers, whose parts can be either divisible subintervals or indivisible singletons. On the other, Aristotle's continuity rests on a geometric magnitude whose potential parts are infinitely divisible. Bostock neglects the distinction between these two continua and concludes that Aristotle is wrong not to have defined a continuum in the form of Dedekind's real line He writes:

> He [Aristotle] does not also have, as we do, the numerical model of the real numbers, but that is scarcely a handicap. After all, one of our main ways of understanding the structure of the real numbers (namely via the notion of a 'Dedekind cut') is very naturally viewed as drawing upon our prior grasp of the structure of a geometrical line, with the real numbers understood as corresponding to the points on that line. (1991, p. 179)

This statement implies two mistaken postulates: first, arithmetic continuity is the only meaning of continuity available in the history of mathematics; second, Dedekind's real line has a geometric interpretation. The first postulate implicitly denies that geometry may exist independently of algebra; yet, Euclid defines geometric magnitudes without relying on algebraic numbers (cf. section 6.1), and Aristotle deals with a geometric property of continuity by opposing it to the discontinuity of numbers. A continuous line is devoid of indivisible points, because a point in a non-algebraic geometry has no role to play; indeed, it cannot correspond to an algebraic value, and has no reason for being a quantitative part of the continuum (cf. section 2.3). As for Bostock's second postulate, it makes us believe that arithmetic continuity has geometric properties because of Dedekind's metaphoric use of the real line. Yet, we have seen that a real line is an arithmetic concept, and Dedekind is keen to insist that a field of numbers (*Zahlkörper*) has no geometric meaning. He writes:

> The system [of rational numbers] forms a well-ordered [*wohlgeordnetes*] domain of one dimension extending to infinity on two opposite sides. What is meant by this is sufficiently indicated by my use of expressions borrowed from geometric ideas [*geometrischen Vorstellungen*]; but just for this reason it will be necessary to bring out clearly the corresponding purely arithmetic properties so as to avoid even the appearance that arithmetic is in need of such foreign ideas [*fremden Vorstellungen*]. (1932, p. 318; 1996, p. 768)

A real line *is* an arithmetic concept, and is nothing more than the set R of all real numbers, i.e. the arithmetic interval (– ∞, ∞). Note that this infinite interval is isomorphic to the interval [0, 1], such that the elements of both intervals are in one-one correspondence (bijection); and this implies the definition of the Dedekind infinite, meaning that an infinite set is always in one-one correspondence with one of its proper subsets (cf. section 4.1). [6] Thus, the continuum [0, 1] is not different from the continuum R of all real numbers, although the former is a proper subset of the latter. This counterintuitive principle amounts to defining a whole as equal to one of its parts, and it is perfectly consistent providing that we exclusively attribute arithmetic properties to the whole and its parts.

2.2 Dedekind cuts vs. Aristotelian numbers

Bostock suggests a second premise to the definition of an Aristotelian continuum, according to which the two non-coincident parts of a continuum must share a same limit. He writes:

> It is perfectly clear from the whole run of Aristotle's discussion that we must add a further premiss: any division of a continuum into two (non-coincident) parts must divide it into parts that are continuous with one another in the sense defined in Book V, i.e. the two parts must touch, and the limits where they touch must 'be one'. (1991, p. 183)

Like the first premise, Bostock attempts to combine Aristotle's geometric continuity with the definition of an arithmetic continuum. While he connects the first premise to 231b4-6 in Book VI, he now relates the second premise to 227a10-12 in Book V. This passage claims that what is in contact becomes continuous providing that "the limit of each thing becomes the same, i.e. one" (*tauto genêtai kai hen to hekaterou peras*); and Aristotle adds, "this is not possible if there are two extremities" (*touto d' ouch hoion te duoin ontoin einai toin eschatoin*, 227a13). If a continuum has one and only one limit, then its parts cannot have proper limits since this would make them distinct from each other, i.e. either in contact or far apart. Yet, Bostock's first premise claims that the two parts are non-coincident. This sounds contradictory insofar as two parts, in order to share a limit and to be defined as intuitively

[6] Note that the intervals of rational or real numbers are always infinite, since the property of arithmetic denseness tells us that there is always a rational (or real) number between any two rational (or real) numbers.

continuous, must be more than in a mere contact with each other; that is, they must coincide or overlap. We then face two alternatives: on the one hand, a unique limit shared by some parts implies that the parts themselves are not distinct from each other and must thereby coincide; on the other hand, if two parts are non-coincident, then they are distinct from each other in such a way that they are merely in contact. In other words, Bostock's first and second premises conflict with each other.

If we restrict our interpretation to Aristotle's text, there is no difficulty at all in combining 231b4-5 from Book VI with 227a11-13 from Book V. Indeed, the potentiality for a continuum to have divisible parts does not contradict the *actual* fact that its parts share the same limit, i.e. the limit of the continuum. Thus, the potentially divisible parts *are* (actuality) neither distinct nor separate, although we *may* (potentiality) define them in distinct and separate places. The potential property of divisibility does not conflict with the actual property of sharing a unique and same limit. A difficulty arises only when we take Bostock's first premise on board, namely the definition of two non-coincident parts. By using the concepts borrowed from set-theoretic continuity, he writes, "The end-point of the line, and the rest of the line, now do share a limit, namely the end-point itself. It is its own limit, and it must also be reckoned to be one of the limits of the rest of the line." (1991, p. 184). Through the instance of a continuous interval [0, 1], we may apply Bostock's view by stating that the 'end-point' {1} is not only the limit of the 'rest of the line' [0, 1), but also its own limit. Yet, this statement does not make sense. It is possible to interpret an end-point (singleton) and the rest of the line (interval) as the two arithmetic parts of a continuum. Nevertheless, this does not mean that a Dedekind cut can be confused with one of the two parts of the continuum, since a Dedekind cut is the limit of an interval and not a part of it. Thus, the parts [0, 1) and {1} of a continuum [0, 1] implies a Dedekind cut, i.e. the limit of the infinite interval [0, 1). This limit is equal to one; but, contrary to what Bostock claims, this does *not* mean that this algebraic limit is the end-point {1}, although their numerical values are the same. In modern mathematical terms, we call the Dedekind cut of the interval [0, 1) a least upper bound or supremum, such as:

$$\sup \{x \in \mathrm{R} : 0 < x \leq 1\} = 1$$

The supremum or least upper bound (equal to one) is the real number that is greater than any real number x in [0, 1), and this Dedekind cut produces the partition between the infinite interval [0, 1) and the singleton {1}. We may also represent the interval [0, 1) as the infinite *increasing* sequence S_n of real numbers, i.e.

$$S_n = \{0, 1/2, 3/4, 7/8, 15/16, \ldots\}$$

whose limit or least upper bound can be formalised as $lim\ S_n = 1$ with $n \to \infty$. If we are dealing with a *decreasing* infinite sequence, we define a Dedekind cut as a greatest lower bound or infininum (i.e. the mirror view of a least upper bound). In other words, the Dedekind cut is the limit of the part [0, 1) of a continuum [0, 1], and must not be confused with the other part {1} of the continuum. Bostock is likely misled by the fact that both correspond to the numerical value '1'; but a same *numeral* does not mean that we deal with a same *number*. Thus, the least upper bound 'sup (x) = 1' defines a second-level real number, while the singleton {1} corresponds to a first-level real number (cf. section 5.1). The second-level real number is the property of an interval or non-singleton subset, which is not reducible to a first-level real number or singleton subset. Therefore, we cannot accept Bostock's conclusion that the first-level real number {1} is a limit of the interval [0, 1], since only a second-level real number (i.e. a Dedekind cut) can be a limit.

The tension between Bostock's two premises results from the implicit combination of a geometric continuum with arithmetic properties. He writes: "These two premisses are, I imagine, the premisses that Aristotle regards his argument in this passage as depending upon, and as given by the (unstated) definition of what a continuum is" (1991, p. 183). It is certain that the definition of an arithmetic continuum is 'unstated' in Aristotle's *Physics*, and if Aristotle does not think of arithmetic continuity, it is *because* ancient geometry is devoid of arithmetic (i.e. algebraic) properties. However, Bostock is only concerned with two possible alternatives: either continuity is arithmetic or its definition is incomplete. In this sense, he concludes that Aristotle fails to define a proper concept of continuity:

> One [avenue] would be to note that I have merely conjectured his definition of a continuum (since he [Aristotle] fails to state any definition himself), and to seek for an alternative conjecture. Here one may remark that he quite often appears to take, as *definitive* of a continuum, the thesis that in this passage he is attempting to prove. That is, a continuum is defined as what is divisible (only?) into parts that are further divisible. (1991, p. 184, original emphasis)

This statement rejects the idea that a geometric continuity, based on the infinite divisibility of geometric magnitudes, may constitute a consistent mathematical principle. Bostock does not seem to realise that mathematics has not always used algebraic abstractions; thus, Euclid's geometry deals only with geometric magnitudes without translating them into numerical values. [7] Therefore, there is nothing inconsistent in Aristotle's following claim: "A quantity is a plurality if it is numerable, and a magnitude if it is measurable. Plurality means something potentially divisible into non-continuous [parts], and magnitude means something potentially divisible into continuous [parts]." (*Plêthos men oun poson ti ean arithmêton êi, megethos de an metrêton êi. Legetai de plêthos men to diaireton dunamei eis mê suneché, meghetos de to eis suneché.*) (*Metaphysics*, 1020a8-11). While the parts of geometric magnitudes are infinitely divisible, the parts of numerical pluralities are indivisible units; and the exclusion of indivisibles from the composition of a geometric continuum illustrates the contradicting properties between a geometric magnitude and an arithmetic plurality. In Book VI of his *Physics*, Aristotle repeats the obvious principle (at his time) that a geometric line is devoid of numerical properties, namely: "It is impossible that the continuous is made out of indivisibles, for instance a line out of points, if the line is continuous and the point indivisible." (*adunaton ex adiaretôn einai ti suneches, hoion grammên ek stigmôn, eiper hê grammê men suneches, hê stigmê de adiaireton*, 231a24-26; cf. 232b24-25). If the whole is a divisible line, then the principle of geometric continuity commands that its parts are divisible lines, since the intuitive idea of smoothness requires that the whole and its parts be of the same kind. In other words, the continuity of divisible magnitudes implies their infinite divisibility. Aristotle concludes: "Every magnitude is divisible into magnitudes (for we have shown that it is impossible for a continuum to be composed of indivisibles, and that every magnitude is continuous)." (*pan megethos eis megethê diaireton* (*dedeiktai gar hoti adunaton ex atomôn einai ti suneches, megethos d'estin hapan suneches*)) (232a23-25). If parts are indivisible while the whole is divisible, then the geometric continuity of the whole with its parts is impossible. In such a case, we

[7] From the fourteenth century onwards, Bradwardine, Buridan, Oresme, and many others started criticising the non-algebraic definition of geometry, insofar as the numerical values of empirical measurements could not be quantified within geometric models (unlike mechanical models). The systematised translations of geometric figures into algebraic structures have become a mathematical reality only since the seventeenth century.

deal with a discontinuous number defined as a divisible plurality of indivisible units. Consequently, a geometric continuum contradicts a discontinuous number, because the infinite divisibility of the former conflicts with the finite divisibility of the latter.

Bostock neglects the distinction between a magnitude and a number. He writes: "Elsewhere in *Phys.* VI, Aristotle is generally content to characterize a continuum as 'divisible into divisibles' (and hence 'infinitely divisible'), but he pays surprisingly little attention—either here or elsewhere—to the question of how much divisibility a continuum has" (1991, p. 186). From an Aristotelian point of view, the statement does not make sense, since to ask how much divisibility a continuum has means that a continuum is *finitely* divisible. Yet, Aristotle's Book VI repeatedly claims that a continuum is infinitely divisible; namely, a continuum is divisible into parts that are themselves divisible into parts and so on, such that the endless divisibility excludes indivisibles from its composition. Bostock seems to be unconcerned with the Aristotelian idea that the parts of a continuum must be potential, and that potentiality implies infinite divisibility. Indeed, he wonders why Aristotle does not use ratios in the composition of a continuum; he writes:

> One might have expected him [Aristotle] to stress that a continuum can always be divided in *any* desired ratio, not only those ratios that can be specified by the natural numbers, but also those irrational ratios that geometry forces upon us, e.g. the ratio of √2 to 1… But in fact there is no emphasis on this point, and indeed no mention of irrational ratios in the whole of Book vi. I can only comment that I find this surprising. (1991, p. 187, original emphasis)

The absence of the incommensurable ratio of √2 to 1 in the whole of Book VI is not surprising, insofar as Aristotle deals with the composition of a continuum through the definition of *potential* parts. While it makes sense to use ratios with respect to *actual* quantities, e.g. an actual line compared with another line or an actual time with another time (*Posterior Analytics*, 74a17-25), it is meaningless to mention them with respect to the *potential* (actually indistinct) parts of a continuous line or time. In other words, Aristotle's or Euclid's use of ratios applies to the actual parts of a divided line, and not to the *indistinct* parts of a continuous line. Consequently, the only way to make sense of Bostock's above claim is to define a plurality of actual lines, so that an actual line, composed of actual parts (and not *potential* ones), may be divided "in any desired ratio".

Let us summarise the two conflicting definitions of continuity. On the one hand, an

Aristotelian continuum is a purely geometric magnitude whose potential parts are infinitely divisible (*apeiron diaireton*, 200b18-20), and whose limit (*peras*), shared by all parts, is one (*hen*) and the same (227a10-12). On the other, an arithmetic continuum is an interval of real numbers whose parts are disjoint subsets of first-level real numbers (rational numbers), and whose limit is the Dedekind cut (second-level real number) of an infinite subset. Thus, the parts of an arithmetic continuum are distinct and separate, and its limit belongs to only one infinite part. Therefore, there is no agreement between either geometric and arithmetic parts or geometric and arithmetic limits.

2.3 Algebraic quantitative points vs. Aristotelian incorporeal points

The concept of point illustrates the incompatibilities between an algebraic and a non-algebraic geometry. Aristotle defines a point as the limit of a continuous line, namely an indivisible opposed to the divisible parts of a divisible line. The distinction between parts and limits pertains to the opposition between divisibility and indivisibility. In Book I of his *Physics*, Aristotle writes, "one is said to be either continuous or indivisible" (*legetai d' hen ê to suneches ê to adiaireton*, 185b7-8). If 'one' is said to be continuous, we may claim that "one is many" (*polla to hen*), since what is continuous is infinitely divisible (*eis apeiron gar diaireton to suneches*) (b10-11). By contrast, if 'one' is said to be indivisible, then it will have neither quantity nor quality (*outhen estai poson oude poion*) (b16-17). Aristotle concludes: "It is the limit which is indivisible, not the limited thing" (*to gar peras adiaireton, ou to peperasmenon*) (b18-19). In other words, a point is an indivisible limit, which contradicts the divisibility of a geometric line. Aristotle writes in the *De Anima*, "the point and all division and any indivisible in this way are made as manifest as privation" (*hê de stigmê kai pasa diairesis, kai to houtôs adiaireton, dêloutai hôsper hê sterêsis*, 430b21-23). [8] Thus, an Aristotelian point is a pure abstraction, namely an incorporeal limit or division that limits or divides a geometric line. As the *Metaphysics* (1022a5-10) tells us

[8] Berti (1978) writes: "Just as the notion of evil is inseparable from that of its contrary, good, and the notion of black is inseparable from that of its contrary, white; so the notion of point is inseparable from that of which it is the limit, and to which it is in that sense the contrary, i.e. the line; point in fact cannot be defined otherwise than as a limit of line." (p. 145).

that a limit is the form of a magnitude (*eidos megethous*), we must distinguish an indivisible point from a mathematical quantity, which is a compound of intelligible matter and form. A point has no individual existence, and as such, is always the limit of something else, i.e. of a limited quantity. If two lines meet one another, then the intersection is not located at a point but at another intersecting line, since incorporeal points do not have independent mathematical existence. Obviously, we may *think of* a point at the intersection of two lines, but the thought of a point is a pure abstraction (form) which is not a geometric object (i.e. a particular composed of form and intelligible matter). In brief, a point cannot exist as a freestanding entity. [9]

Aristotle's definition of a point contradicts the algebraic conception of modern mathematics, since the latter defines a point as corresponding to a numerical variable, namely an algebraic quantity defined as the part of an algebraic line. As Bostock is influenced by algebraic geometry, he defines a point as a quantitative part of a continuum. Which is more surprising is that he aims to show that his view is not very different from Aristotle's. To do so, he proceeds in two steps. First, he wants to convince us that a point has a limit; he writes: "We may surely assume that a finite line is a continuum, if anything is, so if Aristotle's definition is correct then the two parts into which we have divided it must share a limit. According to one of Aristotle's lines of thought, they do not, since he claims that a point has no limit. But in that case we must simply reject the definition as incorrect." (1991, p. 184). The so-called "Aristotle's definition" refers to Bostock's disputable premises, according to which two non-coincident parts share a limit. It is obvious that Aristotle's claim that a point is not a quantity and cannot therefore *have* a limit conflicts with Bostock's purpose to define a point as the indivisible part of a continuum. By attempting to defend an arithmetic continuum without betraying Aristotle's position, Bostock claims that Aristotle admits the existence of a contact between points, and since contact implies limits, this means that each point *is* its own limit. He writes, "If my interpretation is right, then he [Aristotle]

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9 Konstan (1988) writes, "Aristotle takes the theorem or proposition that a continuum is not composed of free-standing points or partless entities as equivalent to the proposition that division of a continuous stretch may in principle go on *ad infinitum*. For, if division were hypothesized as coming to an end or limit, that is, reaching a point at which all further division was impossible, then a continuum would, at least in theory, be broken down into indivisible or partless entities, and could thus be said to be composed of such entities." (p. 3).

also has another line of thought which allows points to be in contact with one another ('as wholes') despite their alleged lack of limits. This is most easily harmonized with his general position by allowing that a point may be said to be *its own* limit." (1991, p. 184, original emphasis). Bostock's mention of Aristotle's "another line of thought" pertains to the passage 231a29-b6 already used for the justification of his first premise. Yet, his interpretation misreads a *reductio ad absurdum*. Indeed, Aristotle starts with an initial assumption containing the following disjunction: "It is necessary that points, of which a continuum is composed, be either continuous or in contact with each other." (*d' anagkê êtoi sunecheis einai tas stigmas ê haptomenas allêlôn, ex hôn esti to suneches*, 231a29-31). He goes on by generalising the assumption: "the same argument also applies to all indivisibles" (*ho d' autos logos kai epi pantôn tôn adiairetôn*, 231a31-b1). Then Aristotle's aim is to show that this assumption leads to a contradiction, so that the contrary statement is (indirectly) proved to be true. Thus, he concludes a few lines further down, "but nothing of what is continuous is divisible into things without parts" (*all' outhen ên tôn sunechôn eis amerê diaireton*, 231b11-12). In other words, partless points cannot be the parts of a continuum.

Regarding a possible contact between points, Bostock takes advantage of the ambiguity of 231a29-b6, insofar as the passage does not clearly reject the possibility for points to be in contact with each other. We must wait for the lines 237a31-34 in which Aristotle explicitly denies a contact between two points. In this passage, he uses another *reductio ad absurdum* with the initial assumption that a continuous magnitude is indivisible, such that "something partless will be contiguous with something partless" (*ameres amerous estai echomenon,* 237a32). Yet, Aristotle immediately rejects this absurd hypothesis: "but since this is impossible, it is necessary that what is intermediary be a magnitude and be infinitely divisible" (*epei de touto adunaton, anagkê megethos einai to metaxu kai eis apeira diaireton*, 237a32-33). This means that only divisible quantities can be in contact with each other, and no incorporeal, indivisible points can make sense of a quantitative contact. His *Metaphysics* (1023b26-1024a10) provides us with the argument that indivisibles cannot be defined as wholes unless they are composed of parts. This constitutes another proof that the passage 231a29-b6 is a *reductio ad absurdum* starting with the impossible assumption that a contact between two indivisibles is necessarily a contact between two wholes (*anagkê holon holou* *haptesthai*, 231b3). Consequently, the claim that two points are in contact cannot be Aristotelian.

Nevertheless, Bostock has no other choice than defending this non-Aristotelian position, since this latter is directly related to the non-Aristotelian definition of a point as the indivisible part of an algebraic continuum. Bostock is even ready to assert that points or limits can be in the same place; he writes:

> I take it that Aristotle means what he says [at 226b21-23], and is (rightly?) supposing that when the face of one (perfect) cube touches the face of another the two faces are different things but are in *exactly* the same place... Admittedly, this perfectly proper use of 'same place' is somewhat difficult to reconcile with Aristotle's definition of 'place' in Book iv, which seems not to allow such things as limits to have places (1991, p. 181, footnote 5, original emphasis)

Yet, if we carefully study 226b21-23, we see no contradiction with Book IV, insofar as Aristotle claims nowhere in the *Physics* that a limit has a place. In this passage, Aristotle writes, "I say now that things according to place are together, when these are in one primary place" (*hama men oun legô taut' einai kata topon, hosa en heni topôi esti prôtôi*). The word *hama* (meaning 'together') applies to anything which can be in a place, i.e. not an extremity that is incorporeal by definition. To use Bostock's instance of a cube, this means that two (physical) cubes are together when both are in the same place. [10] Then Aristotle adds a second argument, such that things are "in contact whose extremities are together" (*haptesthai de hôn ta akra hama*, 226b23). In this case, the word *hama* pertains to the extremities, but contrary to what Bostock claims, this does not mean that such extremities are in the same place. Indeed, incorporeal extremities are together only when physical objects are in contact, i.e. in two distinct places. Therefore, we face two alternatives: first, two corporeal objects are together only when they are in the same place; second, two incorporeal limits are together only when they belong to two contiguous quantities in distinct places. In other words, the argument in 226b21-23 is not different from what Book IV tells us, namely all physical objects, unlike their incorporeal extremities, have places.

Consequently, Aristotle does not contradict his fundamental view that a geometric

[10] Note that only physical (movable) objects have physical (absolute) places; by contrast, geometric magnitudes have merely (relative) positions, i.e. locations with respect to the mathematician.For a detailed study of Aristotle's concept of place, see Algra 1995.

continuum has infinitely divisible parts, excluding indivisibles from its composition. Points are non-quantitative limits or divisions and do not correspond to numbers, i.e. algebraic quantities. Aristotle's continuous line must not be identified with Dedekind's real line, since the former, unlike the latter, is devoid of algebraic properties. The fundamental incompatibility between these two continua derives from the irreducible contradiction of two kinds of mathematical abstractions, insofar as the algebraic concept of extensionless sequence does not *define* a geometric concept of extension.

2.4 The actual finite and the potential infinite in Zeno's Dichotomy paradox

Geometric and arithmetic continua are incommensurable concepts. In order to justify further this claim, I shall concentrate on a particular case study in the form of Zeno's paradoxes of motion. There exists a huge literature on Zeno's paradoxes; yet, their interpretations are often confusing because based on concepts that Zeno could not have defined and understood. I shall focus on the paradox of the Dichotomy, describing a runner's race, whose infinitely divided motion makes the end of the run unreachable. [11] My main purpose is to show that the mathematical structure implied by Zeno's paradox is purely geometric, since it is based on the infinite division of a geometric extension. If we confine our interpretation to non-algebraic properties, we can conclude to the soundness of Aristotle's position, which avoids the Dichotomy by defining the infinite division of motion as a mathematical abstraction and not as a physical property. In this sense, the potential or mathematical infinite divisibility does not contradict the actual or physical finite motion.

The paradox of the Dichotomy, as explained by Aristotle's *Physics*, defines motion as contradictory, because what is moving must first arrive at the halfway point before reaching the end (*dia to proteron eis to hêmisu dein aphikesthai to pheromenon ê pros to telos*, 239b11-13). A dichotomy is a division into two equal parts, and an infinite dichotomy is a paradox insofar as it implies the infinite division of an extended motion. There are two ways

[11] We shall see that the paradox of Achilles and the tortoise is close in meaning to the Dichotomy, insofar as Achilles's infinitely divided motion prevents him from overtaking the tortoise or even reaching the starting point of its race. For a general study of Zeno's paradoxes, see Barnes 1979.

to understand the paradox. First, to move from a point A to a point B means to reach a halfway point B_1, and then to move from B_1 to B requires traversing the halfway point B_2, and so on. Thus, a moving object traverses an infinite succession of parts divided into an infinite collection of halfway points B_n. In other words, the moving object will never reach the end of its motion. There is also a second interpretation of the paradox, called the inverse form of the Dichotomy, such that if a moving object goes from A to B, it first traverses a halfway point A_1 between A and B, but to arrive at A_1, it must first reach a second halfway point A_2 between A and A_1, and so on. Understood in this way, the infinite division of motion prevents any object from starting its motion.

We may quickly show that Zeno's paradox of Achilles is equivalent to the Dichotomy, as noticed by Aristotle himself (239b18-19). Achilles will never catch up the tortoise, for the pursuer must first arrive at where the pursued started from, so that the slower should necessarily be some distance ahead (*emprosthen gar anagkaion elthein to diôkon, hothen hôrmêse to pheugon, hôst' aei ti proechein anagkaion to braduteron*, 239b14-18). If Achilles starts his race at a point A while the tortoise starts one metre ahead at a point B, then Achilles must first overtake the starting point of the tortoise's race. In order to reach the point B, he will first cross the halfway point B_1 between A and B, and then cross the halfway point B_2 between B_1 and B, and so on. The Achilles paradox follows from the Dichotomy, and the fact that Achilles is racing against the tortoise is purely anecdotic. The only crucial point is that the tortoise must start ahead of Achilles, so that its starting point constitutes an (ideal) end to reach. Achilles' faster speed is irrelevant, since the Dichotomy paradox related to the infinite division of his motion applies in all cases; he would not reach the end-point B of the tortoise's race, even though his speed were *slower* than the tortoise's. Likewise, the tortoise will never reach the end-point of its own motion. Therefore, it is inexact to claim that the tortoise will win the race, since both Achilles and the tortoise are victims of the same Dichotomy paradox related to their respective motions.

We may try to avoid the paradox by defining motion as discontinuous. Indeed, a discontinuous motion, unlike a continuous one, corresponds to a finite plurality of indivisible parts. This means that the division of motion is finite, and the paradox does no longer apply since any moving object reaches the end of its motion through a finite succession of divisions. Nonetheless, it is not a satisfactory solution since the idea of a discontinuous motion contradicts our intuition of motion. Moreover, this would be viewed as a vindication of Zeno's paradox proving that continuous motion, i.e. motion *simpliciter*, does not exist. Either motion is continuous but its infinite division leads to a paradox, or motion is free of paradox but implies the refutation of motion as intuitively perceived. In both cases, Zeno wins the argument.

Defining the underlying mathematical structure behind the Dichotomy is not an easy task, since the extant fragments of Zeno's writings do not offer a coherent view of his thoughts. However, we may decide to follow Aristotle's reply to Zeno; this will not provide a rendering of Zeno's arguments, but this will at least enable us to understand how a non-algebraic geometry reacts to such paradoxes. Mainly, this will provide us with an opportunity to avoid the usual algebraic formalism that redefines Zeno's paradoxes on its own terms instead of interpreting them genuinely. The aim is to grasp how a geometric process of infinite division can apply to the physical concept of an extended motion. We know that Aristotle's *Physics* defines a line as a geometric continuum whose parts are lines. That is, a line is divisible into lines that are themselves divisible into lines, such that the infinite divisibility of a line defines its continuity (231b15-16, 232b24-25). Thus, an Aristotelian extension is geometrically continuous owing to the complete absence of extensionless indivisibles in its composition, such that extension is infinitely divisible into extended parts.

This geometric principle is automatically applied to a physical extension for the simple reason that a geometric line is a physical line devoid of its sensible qualities. In Book II of his *Physics*, Aristotle distinguishes geometry from optics in such a way that geometry deals with physical lines, *qua* geometric, while optics studies geometric lines, *qua* physical (194a10-12). This means that geometry defines a line as a compound of form and intelligible matter, contrary to optics that reintroduces sensible matter into lines, so that physical lines are rays of light. This view may surprise modern minds, since it is quite odd to define geometric abstraction as a subclass of physical concepts. Yet, Aristotle aims to avoid drawing a Platonist line between the intelligible and sensible worlds, and the dependence of mathematical objects upon the physical world makes this possible (193b23-33). Geometric magnitudes do not belong to an abstract universe of mathematical objects, insofar as geometric properties are abstracted from physical objects. Thus, there are geometrically continuous magnitudes in all physical bodies. Aristotle writes in the *Metaphysics*:

> The mathematician studies about abstractions (*ta ex aphaireseôs*) for in his study he eliminates all sensible qualities (*panta ta aisthêta*), such as heaviness and lightness, hardness and its opposite, and also hot and cold, and the other sensible opposites; he leaves only quantity and continuity (*to poson kai suneches*), sometimes in one, sometimes in two and sometimes in three dimensions, and the attributes of these *qua* quantitative and continuous (*hêi posa kai suneché*); he does not consider them in relation to anything else, and examines the relative positions of some and the consequences of these, and the commensurability and incommensurability (*tas summetrias kai asummetrias*) of others, and the ratios (*tous logous*) of others; but of all these things, there is only one and the same science of geometry (*epistêmên tên geômetrikên*). (XI, 1061a28-b2; see also b20-25)

An Aristotelian abstraction is not different from a physical body, except for the suppression of its sensible qualities. That is why both physical and mathematical objects are compounds of form and matter, and the only distinction between such particulars is that a geometric object has an intelligible matter devoid of sensible properties. In this sense, Aristotle may consistently define the physical concepts of motion, time and distance through a geometric infinite divisibility, since this mathematical property is already contained in such physical concepts. Thus, motion, time and distance are physical concepts whose continuity implies an infinite divisibility of their parts, meaning that motion is composed of infinitely divisible motions, time of infinitely divisible times, and distance of infinitely divisible distances. [12] We shall see later that an Aristotelian continuum is as much geometric as physical (cf. sections 3.1. 3.2 and 3.3).

Yet, a crucial distinction must be introduced between infinite division and infinite divisibility, since the former is in actuality while the latter is in potentiality. This constitutes Aristotle's solution to Zeno's paradox of the Dichotomy, such that the infinite is no longer defined as actual but as potential. Potentiality refers to an abstract process in the intellect

[12] Some commentators are not always clear about this principle. For instance, Waterlow (1983) writes: "Every motion is infinitely divisible with respect to time. That is why Zeno was wrong to argue that motion is impossible because motion entails being in contact with an infinity of points in a finite time. For there is a moment of time for each spatial point, and this temporal infinity in a finite duration accommodates the infinity of positions." (pp. 144-5). Her statement is ambiguous, insofar as "moment of time" and "spatial point" seem to be understood as quantitative entities, while Aristotle defines them as mere incorporeals.

without existing in the physical world. If divisibility takes place in thought, then its infinite domain does not paradoxically conflict with actually finite processes. Aristotle generalises his argument by rejecting the actual infinite in physics, insofar as every physical (and thereby geometric) quantity is by definition limited (*Physics*, 206a3-8). Only the potential infinite is mathematically acceptable and definable as the following: an infinite is either by division (*kata diairesin*) when related to a continuous magnitude of infinitely divisible parts or by addition (*kata prosthesin*) when pertaining to a discontinuous plurality of indivisible parts (206a14-25, 207b1-15). This means that magnitudes and pluralities are actually finite quantities, whose property of being infinite is accessible as either a potential division of geometric parts or a potential addition of numerical parts. We face once again the contrast between a continuous geometric magnitude and a discontinuous number, such that both are potentially infinite through the application of two distinct abstract properties, i.e. either divisibility or additivity. [13]

Through the definition of a potential infinite, we may then grasp why Aristotle's answer to Zeno's paradox of the Dichotomy is not an easy fix, but a serious mathematical answer. Indeed, if motion is geometrically defined as infinitely divisible, it implies that time measuring motion must be infinitely divisible as well. This means that a potentially infinite motion cannot be combined with an actually finite time, since it is inconsistent to associate a geometrically infinite property with a physically finite concept. In other words, if motion is defined through its geometric infinite divisibility, then the concept of time applied to motion must be geometric as well; that is, both must be infinitely divisible. Aristotle writes:

> Zeno's argument implies a falsehood—that one cannot traverse infinities or touch each of the infinities in a finite time (*en peperasmenôi chronôi*). For both length and time, and in general everything continuous, are said to be infinite in two ways, either by division or as to their extremities (*êtoi kata diairesin ê tois eschatois*). Now one cannot touch things infinite in respect of quantity in a finite time, but one can so touch things infinite by division—for the time itself is infinite in this way (*kai gar autos ho chronos houtôs apeiros*). Hence it is in an infinite and not in a finite time that, as it turns out, one traverses the infinite, and one touches infinities in the infinite and not in the finite (*kai haptesthai tôn apeirôn tois apeirois ou tois peperasmenois*). (VI, 233a21-31)

[13] I remind the reader that an Aristotelian number is a plurality of indivisible units; *e.g.* the number three is the plurality (or addition) of one *plus* one *plus* one units. A number cannot refer to a geometric extension, since the indivisible parts of the former contradict the infinitely divisible parts of the latter.

Therefore, Zeno is wrong to suppose that motion is infinitely divided in a finite time. The infinite division of motion cannot be a physical, actual property, since the infinite is merely potential and defines motion through the geometric property of infinite divisibility. Moreover, to combine an infinitely divisible motion with a finite time blurs the distinction between a mathematical object of thought and a finite process in nature. If an object moves from A to B, its motion is actually finite and takes place in a finite distance and time; but if motion is geometrically defined as infinitely divisible, then its measuring time and its traversed distance must also be geometrically defined as infinitely divisible. Either a finite motion is physically measured by a finite time or an infinitely divisible motion defines an infinitely divisible time (cf. section 3.1). Since the infinite is a mathematical abstraction *in thought*, it has no effect on the physical definitions of finite time and motion *in nature*. Thus, a paradox merely results from neglecting the distinction between the mathematical infinite and the physical finitude. Consequently, Aristotle succeeds in avoiding Zeno's paradox of the Dichotomy by demonstrating that motion, time and distance are both actually finite in physics and potentially infinite in geometry.

2.5 Algebraic Z-sequences and set-theoretic concepts

Aristotle's reply to Zeno's paradoxes is often regarded as at best irrelevant and at worst inconsistent; this is because Aristotle's potential divisibility is no longer viewed as a sound mathematical property. This judgment results from the modern interpretation of Zeno's paradoxes based on an algebraic mathematics that cannot make sense of Aristotle's non-algebraic reasoning. The two algebraic revolutions, the first with Viète and Descartes (among others) in the seventeenth century and the second with Weierstrass, Dedekind and Cantor (among others) in the nineteenth century, completely changed the way the paradox of the Dichotomy has been understood. Yet, I want to show that such algebraic interpretations distort the paradoxical structure of Zeno's geometric stories. While the Dichotomy implies the infinite division of an extended motion, we shall see that an algebraic sequence of numbers is an extensionless abstraction which does not make sense of the intuitive idea of motion. In other words, an algebraic formalism makes Zeno's paradox irrelevant, since it cannot mirror the paradox of dividing a finite extension *ad infinitum*.

Vlastos (1975, p. 202) provides an interpretation of the Dichotomy paradox based on the algebraic concepts of 'sequence' and 'interval' with the three following conditions:

1. To reach G the runner must traverse the Z-sequence.
2. It is impossible to traverse infinitely many intervals.
3. Therefore, the runner cannot reach G.

Suppose that the runner intends to traverse a distance equal to the continuous interval [0, 1]. After starting the race, he first reaches the halfway point 1/2 of the interval [0, 1], then the halfway point 3/4 of the interval [1/2, 1], then the halfway point 7/8 of the interval [3/4, 1], and so on. Over the bounded interval [0, 1], we may then construct a bounded (also called Cauchy) infinite Z-sequence of halfway points {0, 1/2, 3/4, 7/8, 15/16, ...}. This Z-sequence is equivalent to the infinite sequence of partial sums {0, 0 + 1/2, 0 + 1/2 + 1/4, 0 + 1/2 + 1/4 +1/8, 0 + 1/2 + 1/4 +1/8 + 1/16, ...}. We may then decompose the infinite Z-sequence into finite Z-series (or sums), such that:

$$Z_1 = z_1 = 0 \qquad Z_2 = z_1 + z_2 = 0 + 1/2 = 1/2$$

$$Z_3 = z_1 + z_2 + z_3 = 0 + 1/2 + 1/4 = 3/4$$

$$Z_4 = z_1 + z_2 + z_3 + z_4 = 0 + 1/2 + 1/4 + 1/8 = 7/8$$

$$Z_5 = z_1 + z_2 + z_3 + z_4 + z_5 = 0 + 1/2 + 1/4 + 1/8 + 1/16 = 15/16$$

We end up with the infinite series Z_n with a sum of infinitely many terms:

$$Z_n = z_1 + z_2 + z_3 + \ldots + z_n = 0 + 1/2 + 1/4 + 1/8 + \ldots + 1/2^n + \ldots = lim\ Z_n.$$

The infinite series Z_n gets closer to the limit value *lim* Z_n when n gets larger, and if n tends to the infinite, then Z_n has a limit value equal to one; more formally, *lim* $Z_n = 1$ if $n \rightarrow \infty$. Thus, the infinite Z-sequence $\{Z_1, Z_2, Z_3 \ldots, Z_n\}$ pertains to the infinite sequence of partial sums:

$$\{ z_1, z_1 + z_2, z_1 + z_2 + z_3, \ldots, z_1 + z_2 + z_3 + \ldots + z_n, \ldots\}.$$ [14]

Obviously, we cannot know all the elements of the infinite Z-sequence, and the definition of a limit value means that we *only* know the elements of the sequence which are closer and

[14] Notice the following distinction: on the one hand, the infinite sequence of partial sums {0 + 1/2, 0 + 1/2 + 1/4, 0 + 1/2 + 1/4 + 1/8,...} is equivalent to the *increasing* infinite sequence {0, 1/2, 3/4, 7/8, ...} whose limit is equal to one. On the other, the infinite sequence of partial sums {1 + (−1/2), 1 + (−1/2) + (−1/4), 1 + (−1/2) + (−1/4) + (−1/8),...} is equivalent to the *decreasing* infinite sequence {1, 1/2, 1/4, 1/8, ...}whose limit is equal to zero.

closer to the limit. Since we can define a limit to the Z-sequence, we say that the sequence is convergent, unlike a divergent infinite sequence without limit, e.g. {1, -1, 1, -1, 1, ...}. Therefore, the paradox of the Dichotomy is algebraically interpreted through the convergent, bounded, increasing, infinite Z-sequence {0, 1/2, 3/4, 7/8, 15/16, ...}.

All algebraic concepts are parts of an abstract system, meaning that a sequence, interval, number, and limit are notions devoid of intuitive and empirical meanings. As we have seen earlier, the limit of an increasing infinite sequence is defined as a least upper bound or supremum (cf. section 2.2). This means that we are dealing with real numbers, namely numbers defined as the abstract elements of the set R of all real numbers. A theorem in mathematical analysis tells us that any bounded increasing (or decreasing) infinite sequence of real numbers is convergent, and thereby has a limit called a least upper bound (or a greatest lower bound). The property of convergence is equivalent to the notion of arithmetic continuity defining the completeness of the set R of all real numbers (cf. chapter 6). As the complete set R exhausts all possible numbers, an increasing or decreasing sequence of real numbers has always a limit. Thus, the algebraic (or analytic) concept of convergence has an abstract meaning which cannot be assimilated to the intuitive idea of attributing a limit to a sequence. The abstractness of continuity or convergence derives from set-theoretic properties. Likewise, numbers are not intuitively defined individuals, but elements of an abstract pre-defined domain. For instance, an interval $[a, b]$ of real numbers is a subset of the totally ordered set R, and c is an element of the interval if and only if $a \leq c \leq b$. In this sense, the Z-sequence of real numbers {0, 1/2, 3/4, 7/8, 15/16, ...} pertains to the bounded closed continuous interval [0, 1] with every element c such as $0 \leq c \leq 1$. [15] By contrast, the bounded open dense interval (0, 1) of rational numbers is a subset of the set Q of all rational numbers, such that every element c is defined as $0 < c < 1$. We may even generalise the abstract Z-sequence by defining it as a continuous function through which its arguments {1, 2, 3, 4, ...} are based on the set N of all natural numbers, while its values {0, 1/2, 3/4, 7/8, 15/16,...} rest on the set R of all real numbers. Consequently, this algebraic formalism does not contain any reference to the intuitive idea of extension, whether it is geometric or

[15] Note that the bounded half-open interval [0, 1) has an element c such as $0 \leq c < 1$, while the *unbounded* closed interval $[0, \infty)$ has an element c such as $c \geq 0$.

empirical. If we claim that the interval [0, 1] corresponds to a one-kilometre distance, we only describe an empirical metaphor about an abstract structure whose intrinsic meaning is purely algebraic. We shall later see that the properties of discreteness and continuity in relation to modern physical concepts are purely mathematical, devoid of empirical meaning (cf. sections 3.4, 3.5 and 3.6).

As Vlastos's reference to a Z-sequence implies a complex algebraic structure (i.e. a convergent, bounded, increasing, infinite sequence of real numbers definable as a continuous function), we may wonder whether this algebraic structure can make sense of Zeno's paradox of the Dichotomy. The answer is negative, since an algebraic sequence of numbers cannot intrinsically define the purely geometric properties of division, motion, extension, continuity, and the infinite. Zeno attributes an infinite division to a physical continuous motion, and his paradox results from dividing a finite motion into infinitely many extended parts. In contrast, an algebraic sequence is an abstract collection of numbers following the rules of real analysis and set theory. Thus, the Dichotomy is replaced with a univocal algebraic formalism whose consistency is unproblematic. In his *Principles of Mathematics* (1903), Russell is aware of the distinction between algebraic definitions and Zeno's geometric paradoxes, since he takes the precaution to claim that any talk about algebraic sequences excludes *de facto* the physical and geometric meanings of Zeno's concepts. He writes: "Zeno's arguments are specially concerned with motion, and are not therefore, as they stand, relevant to our present purpose. But it is instructive to translate them, so far as possible, into arithmetical language." (p. 348). [16] Likewise, when Vlastos translates the paradox of the Dichotomy in terms of a Z-sequence, he is aware that algebraic concepts are at odd with Zeno's own mathematical knowledge. He writes:

> It should be hardly necessary to add that this solution could scarcely have occurred to Zeno, since it is an application of the conception of the sum of an infinite series as the limit of the sequence of the partial sums of that series—a conception which was not even reached by the greatest of Greek mathematicians [Eudoxus and Euclid], centuries after Zeno. If we had explained it to him, he might have reproached us with resorting to another paradox in order to escape his. 'Not so', we would have demurred. 'Ours only *seems* paradoxical, because it flouts our intuition. Yours *is* paradoxical because it flouts

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[16] Russell adds in a footnote, "Not being a Greek scholar, I pretend to no first-hand authority as to what Zeno really did say or mean… These [algebraic] arguments are in any case well worthy of considerations, and… their historical correctness is of little importance." (*ibid.*).

our reason.' (1975, p. 211, original emphases)

Vlastos is obviously right to claim that Zeno could not have any knowledge of algebraic systems. As well, he is right to show that the algebraic interpretation of the Dichotomy is not paradoxical, but merely counterintuitive, since we have seen that an algebraic sequence of numbers does not correspond to the intuitive ideas of extension and continuity. Yet, Vlastos seems to imply that an algebraic Z-sequence is able to solve the Dichotomy paradox. It is certain that Zeno would have been wrong to claim that an algebraic interpretation of the Dichotomy constitutes a paradox of another kind; but he would have been right to say that an infinite sequence of numbers distorts his own paradox. Which is surprising is that Vlastos agrees to say that a modern algebraic structure is a consistent formalism (although counterintuitive) and that Zeno could not have conceived of such non-algebraic properties; but he does not draw the sound conclusion that the algebraic formalism cannot be the right mathematical tool to make sense of Zeno's geometric paradoxes. Since algebraic concepts have no relation whatsoever with the physical concepts of motion and extension, they cannot *intrinsically* define infinitely divisible motions and times. Unlike Aristotle's infinite divisibility, an algebraic interpretation does not solve the Dichotomy paradox; it makes it merely irrelevant through the definitions of new algebraic tools.

2.6 *Ad Hoc* supertasks and the Zeno-like algebraic paradoxes

Since an algebraic formalism is consistent and cannot by itself explain why Zeno's Dichotomy is paradoxical, the only way to speak of a paradox is to introduce an *ad hoc* correspondence between an algebraic sequence and a physical motion, in such a way that it is impossible to traverse a motion if divided into an infinite sequence of numbers. The combination of a sequence with a motion is *ad hoc* because no sound justification can connect an extended motion with an extentionless sequence, contrary to the geometric property of infinite divisibility whose object is a physical concept, whether motion, time, or distance. In this sense, I shall speak of the Zeno-like algebraic paradoxes as distinct from Zeno's genuinely geometric paradoxes. In order to provide an instance of algebraic paradox, I choose the thought experiment of a supertask and aim to explain why it constitutes a perfect Zeno-like paradox.

A supertask defines the physical completion of an infinite sequence of tasks and corresponds to the mathematical limit of a convergent infinite sequence of real numbers. Thomson (1954) is the first to coin the term 'supertask', and concludes to the physical impossibility of this thought experiment, insofar as an infinite number of tasks can never be completed in a finite time. He uses the instance of a lamp that is either on or off, and asks the following question: if we press the button of the lamp an infinite number of times for two minutes, will the lamp be either off or on at the end of the two minutes? He writes as a reply:

> It seems impossible to answer the question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction. (1970, p. 95)

This means that no final state can apply to the infinite sequence of pressing the button on and off successively. Each time that the lamp is on, it is switched off; and vice versa. We may reinforce Thomson's view by rejecting even the idea of a final state. Indeed, a convergent sequence of real numbers has a mathematical limit, but it is meaningless to pretend that this limit corresponds to the final state of an infinite sequence of tasks. Suppose a bounded close interval of time [0, 2] with the lamp 'on' at the instant zero. Then the lamp is 'off' at the halfway instant 1 of the interval [0, 2], 'on' at the halfway point 3/2 of the interval [1, 2], 'off' at the halfway point 7/4 of the interval [3/2, 2], and so on. We have a succession of halfway instants defined as the infinite T-sequence {0, 1, 3/2, 7/4, 15/8, …}. Similar to the previous Z-sequence, the T-sequence is the infinite sequence of partial sums {0, 0 + 1, 0 + 1 + 1/2, 0 + 1 + 1/2 +1/4, …} with the partial sums $T_1 < T_2 < T_3 < T_4 < \ldots$ such as

$$T_1 = t_1 = 0 \qquad T_2 = t_1 + t_2 = 0 + 1 = 1$$

$$T_3 = t_1 + t_2 + t_3 = 0 + 1 + 1/2 = 3/2$$

$$T_4 = t_1 + t_2 + t_3 + t_4 = 0 + 1 + 1/2 + 1/4 = 7/4.$$

We end up with the infinite series T_n whose sum of infinitely many terms is a limit:

$$T_n = t_1 + t_2 + t_3 + \ldots + t_n = 0 + 1 + 1/2 + 1/4 + 1/8 + \ldots + 1/2^n + \ldots = lim\ T_n$$

We have $lim\ T_n = 2$ if $n \to \infty$. The T-sequence is a convergent, bounded, increasing, infinite sequence of real numbers, its limit is a least upper bound (or supremum), and the interval [0, 2] corresponds to an infinite subset of the totally ordered set R of all real numbers. Thus, the infinite T-sequence is a continuous function whose arguments {1, 2, 3, 4, ...} are based on the set N of all natural numbers and whose values {0, 1, 3/2, 7/4, 15/8, ...} rest on the set R of all real numbers. Note that the least upper bound is *intrinsically* distinct from the singleton subset {2}, since the former is a Dedekind cut, i.e. a property that completes an infinite subset, while the latter is a finite subset. Yet, if we say that the mathematical limit, equal to two, corresponds to a final task, then the statement is false since the mathematical completeness, implied by a Dedekind cut, is not a physical completion. A supertask has no physically sound meaning, insofar as our understanding of an infinite sequence of tasks is either mathematically meaningful or physically absurd. Mathematically speaking, a supertask is definable as a limit implying the mathematical completion of an infinite sequence of tasks. As a supertask exhausts the infinite ordering of tasks, it corresponds to Cantor's first transfinite ordinal ω (cf. chapter 4). That is, a supertask is a ω-task which is greater than the infinite number of tasks, i.e. greater than the infinite set N of all natural numbers. It implies the completeness of the infinitely countable set N, such as:

$$1 < 2 < \ldots < n < n + 1 < \ldots < n + n < \ldots < n \cdot n < \ldots < n^n < \ldots < \omega < \omega + 1 < \ldots$$

The mathematical definition of a supertask is intrinsically consistent (except for mathematical intuitionism; cf. section 4.6), but the idea of a paradox appears as soon as we introduce an *ad hoc* physical interpretation to the mathematical concept of ω-task. This amounts to defining a Zeno-like algebraic paradox, such that it is absurd to think of the completion of the infinite through physical concepts. Therefore, it would have been much clearer to conclude that the paradox of the lamp merely results from the absurd combination of a final physical state with an infinite sequence of numbers. A supertask as the completion of an infinite sequence of tasks is a Zeno-like paradox, in the sense that the paradoxical element derives from the *ad hoc* correspondence between an algebraic formalism and an intuitive physical event.

However, Thomson (1954) surprisingly draws a different conclusion when dealing with the instance of a racecourse as opposed to the case of the lamp. He claims that a racecourse is an infinite sequence of tasks that can be physically completed; he writes:

> It is sometimes possible to complete an infinite number of tasks. For to complete a journey is to complete a task, the task of getting from somewhere to somewhere else. And a man who completes any journey completes an infinite number of journeys. If he travels from 0 to 1, he travels from 0 to 1/2, from 1/2 to 3/4, and so on *ad inf.*, so when he arrives at 1 he has completed an infinite number of tasks… I think it is true but does not contradict the impossibility of super-tasks. (1970, p. 97)

Thomson makes a distinction between the instance of the lamp whose infinite sequence of tasks is impossible to complete and the case of a racecourse whose completion is possible, insofar as he considers the racecourse as implying a tangible end, contrary to the successive tasks of switching the lamp on and off. His view makes intuitively sense, but it is mathematically misleading since it implies that the mathematical limit of the infinite sequence of numbers be identified with the final singleton subset {1} of the continuous interval [0, 1]. [17] In other words, the instances of the racecourse and the lamp are similar since both imply the same mathematical formalism, such that the completeness of an infinite sequence is a ω-task without physical meaning. Benacerraf (1962) wittily claims that the definition of a racecourse as an infinite sequence of physical tasks implies that "a running genie" must vanish at the limit of the sequence. The disappearance of the genie follows from the mathematical definition of a limit, in the sense that nothing physical can reach an actual infinite, i.e. a transfinite ordinal number ω located beyond the potential infinite. Eventually, Thomson (1970) changes his mind and explains why he first favoured the contrary view:

> I thought that the ω-task of occupying all of the points 0, 1/2, 3/4, … in that order along a racecourse was unproblematic, indeed was to be called an ω-task only as a kind of joke, because for a runner to complete it and for him to move from 0 to 1 are demonstrably the same thing. The resulting situation is simply that he occupies 1. (1970, p. 130)

Thomson was wrongly influenced by the empirical fact that a racecourse implies a finite distance, but this does not mean that the corresponding mathematical interval is finite as well. Indeed, the intrinsic meaning of the interval [0, 1] pertains to denseness and continuity, namely two arithmetic properties that have no translation into physical terms (cf. sections 3.4, 3.5, and 3.6). In other words, we end up with a mathematical definition of the infinite, such that the always-infinite (i.e. dense) interval [0, 1] has no connection whatsoever with an empirically finite extension. In this sense, I shall not follow Max Black's claim that an

[17] As seen previously, Bostock (1991) makes the same mistake of identifying the Dedekind cut with a singleton subset (cf. section 2.2).

infinite sequence of numbers is the mathematical *description* of a physical reality. [18] My view is rather that an algebraic sequence is an extensionless concept, which cannot make sense of, let alone describe, a physical extension. Obviously, there is an *ad hoc* correspondence between the mathematical calculations, resulting from a given formalism, and empirical measurements; but this does not mean that the algebraic tools explain or describe physical concepts. The convergence of an infinite sequence pertains to the arithmetical continuity of real numbers, but to construct such abstractions will not help us comprehend the so-called empirical continuity of motion. Likewise, the velocity of a moving object at a given time does not provide us with the physical knowledge of its motion but merely gives us access to some numerical information for making predictions about it.

Consequently, a supertask is a perfectly consistent concept providing that its intrinsic meaning is restricted to mathematical analysis. If we add an *ad hoc* physical interpretation, then paradoxical claims will lead to define a supertask as a Zeno-like algebraic paradox. The latter merely follows from our culpable willingness to interpret a mathematical formalism physically. On the contrary, Zeno's geometric paradox of the Dichotomy directly pertains to an infinite division of motion, whose *intrinsic* meaning is paradoxical; and this leads Aristotle to distinguish the geometric property of potentially infinite divisibility from the actually finite division of a physical concept.

[18] Black writes: "The class of what will then be called 'distances' will be a series of pairs of numbers, not an infinite series of spatio-temporal things. The infinity of this series is then a feature of one way in which we find it useful to *describe* the physical reality; to suppose that therefore Achilles has to *do* an infinite number of things would be as absurd as to suppose that because I can attach two numbers to an egg I must take some special effort to hold its halves together." (1970, p. 80, original emphases).

Chapter 3

Mathematical Continua and Physical Concepts

Geometric continua pertain to a smooth geometric extension, whose physical interpretation is unproblematic since geometric continuity is an intrinsic property of a physical continuum, whether it is an extended motion, time, or length. Yet, Aristotle's conception of physical continua is far from obvious. I shall concentrate my analysis on the Aristotelian concept of time in order to understand how time can be both a continuum and a number, while Aristotle specifies that numbers are discontinuous pluralities. We shall see that the contradiction is merely apparent by distinguishing a physical, actual continuum from its mathematical definition as a plurality of potential divisions. By contrast, arithmetic continua have no physical interpretations, insofar as their concepts are intrinsically mathematical; the concept of measure in algebraic analysis belongs to a mathematical measure theory which defines the length of an arithmetic interval. Thus, a set-theoretically continuous time is isomorphic (identical in form) to the set R^+ of all positive real numbers; likewise, a dense time is isomorphic to the dense set Q^+ of all positive rational numbers, and a discrete time is isomorphic to the discrete set Z^+ of all positive integers (i.e. the set N of all natural numbers). Through such structures, we have only defined set-theoretic properties, and we have said nothing about the empirical nature of time.

3.1 Aristotelian time: a continuum vs. a number

Aristotle's Books IV and VI in the *Physics* defines time as continuous, meaning that time (*chronos*) is infinitively divisible and cannot be composed of indivisible instants (218a6-8). An instant (*nun*, also translated as 'now') cannot be part of a continuous time, unless we deny the smoothness of an extended time (in the same way that points contradict a smooth extended line). Time is a physical concept, and is directly associated with change and physical magnitudes. Aristotle tells us that time belongs to motion (*tês kinêseôs ti*, 219a9-10) by being defined as the state (*hexis*) or affection (*pathos*) of a motion (223a18-19, 251b26-28). It discriminates (*krinein*) the more and the less in a motion (219b4-5) in such a way that we measure (*metroumen*) a motion by time and time by a motion (220b22-24). Aristotle adds that change (*metabolê*) and motion (*kinêsis*) are synonymous terms in relation to time (218b19-20). [1] If time is more related to corruption than to generation, it does not produce change since generation and corruption are in time *only* by accident (221a31-b2, 222b16-22, b25-27). This means that time is not a motion or change, and cannot cause a motion (218b9-20).

Then Aristotle provides a very precise definition: time is "the number of a motion according to the 'before' and 'after'" (*arithmos kinêseôs kata to proteron kai husteron*, 219b1-2). [2] A number has two meanings: it is either a counted plurality or that by which the plurality is counted (219b5-9, 220b10-12). For instance, 'one hundred' men and 'one hundred' horses are two distinct counted pluralities, but the number by which the plurality is counted is the same. Aristotle claims that time must be understood as a counted plurality (219b7-8, 220b8-9). Thus, time is a plurality of 'before' and 'after', which is counted in a motion. He confirms this definition through the reference to a phenomenal time, such that the perception (*aisthesis*) of the 'before' and 'after' in a motion provides the feeling that time elapses (219a23-25). The perception of time describes an awareness of change; thus, sleeping people do not perceive time because they are unaware of their changing soul (218b20-219a2). Change and time are inseparable; and time is the number of motion without implying any particular definition of motion, whether generation, alteration, increase/decrease or locomotion (223a29-b1). Motion is a generic concept which includes rest, itself defined as the ability (or potentiality) to move; therefore, time is the number not only of motion but also of rest (221b7-23). This argument is crucial, because it enables us to answer some objections to the Aristotelian concept of change. For instance, Shoemaker (1969) criticises Aristotle's view by defining an elapsing time for a changeless region to

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[1] Independently of the study of time in Book IV, Aristotle defines a change as pertaining to contradictory substances (e.g. the change from a non-house to a house interpreted as generation), whereas motion applies to contrary attributes or properties (e.g. the motion from a non-white thing to a white thing interpreted as alteration) (225a34-b5).

[2] Since the term *kinêseôs* is a singular genitive, the expression 'number of a motion' means 'number belonging to a motion'. Thus, it is not a matter of counting how many *motions* there are.

which a 'local freeze' applies. Aristotle could have replied that the concept of 'local freeze' is still measurable by time, since it is defined as a potentiality to change, i.e. an ability to emerge and vanish.

If an Aristotelian time derives from a motion, this does not mean that motion constitutes the ultimate foundation for time. Indeed, both motion and time depend upon what Aristotle's *Metaphysics* defines as 'something through which the object is moved' (*ho ekinêthê*) (1020a26-32). This expression refers to a physical magnitude (*megethos*), namely an infinitely divisible quantity defining the corporeal extension of bodies. In his commentary of the *Metaphysics*, Ross (1924, p. 324) understands the expression *ho ekinêthê* as a "space that is directly measurable". The reference to space is unfortunate, insofar as it seems to mean that a magnitude takes place in a pre-existing space (such as a Newtonian empty space). Yet, physical space is a meaningless concept in Aristotelian physics, in the sense that Aristotle deals only with the places (*qua* limits) of physical objects. Therefore, we should rather insist on defining magnitude as the three-dimensional quantity of a physical object, from which motion and time are derived. Indeed, Aristotle claims that continuous time follows (*akolouthei*) continuous motion, and continuous motion follows continuous magnitude (219a12-14, b15-16, and 220b24-29). This means that an infinitely divisible magnitude determines the infinite divisibility of motion in the same way that an infinitely divisible motion makes the infinite divisibility of time possible (207b21-27, 231b18-233a21). [3] If magnitudes were not continuous, motion and time would be discontinuous. Accordingly, the relationship of dependence between magnitude, motion, and time constitutes a fundamental principle of Aristotelian physics.

However, Aristotle's statement that time is both a continuum and a number is problematic, insofar as a number is associated to a plurality, and we know that a plurality is potentially divisible into discontinuous (*mê suneché*) parts, as opposed to the continuous (*suneché*) parts of a magnitude (*Metaphysics*, 1020a8-11). Does it mean that time is not a true continuum, or not a true number? If we cannot solve this question, then Aristotle's position does not make sense. Fortunately, there is a way to avoid this apparent dilemma.

[3] This argument constitutes the prime justification of Aristotle's answer to Zeno's paradox of the Dichotomy (cf. section 2.4).

On the one hand, physical time is continuous because physically derived from a continuous motion, such that both are infinitely divisible. On the other, time as a number is not a physical time but a counted plurality, namely an object of thought pertaining to the potential divisions of a motion into 'before' and 'after'. Aristotle makes this distinction, when he claims that a continuous time is either short or long, contrary to a numbered time that is either little or much (220a32-b4). The contrast aims to oppose a singular time to plural times, namely a singular time is a continuous interval defining a present motion, while plural times is a plurality of 'before' and 'after' counted as past and future motions (220b5-8). I shall show that no contradiction occurs between the two aspects of time, in the sense that a continuous (present) time is wholly compatible with numbered (past and future) times.

3.2 Continuous time and numbered times

Aristotle makes the distinction between an absolute number, i.e. a number in itself, and a particular number, i.e. a number of something. He then resorts to the concepts of line and time in order to define the absence or presence of a smallest number with respect to such continua. In Book IV of the *Physics*, Aristotle writes:

> The smallest number, in an absolute sense, is two. But defined as a particular number, in one sense it exists, in another it does not. For example, in the case of a line (*grammês*), the smallest number with respect to plurality (*plêthei*) is two or one; but there is none with respect to magnitude (*megethei*): for every line is always divisible. Hence, it is likewise with time: the smallest with respect to number is one or two, but there is none with respect to magnitude (*Hôsth' homoiôs kai ho chronos: elachistos gar kata men arithmon estin ho heis ê hoi duo, kata megethos d' ouk estin*). (220a27-32)

The distinction is between a magnitude and a particular number, i.e. a counted plurality, such that a line is consistently definable as either a magnitude or a counted plurality. If it is a magnitude, then we do not refer to any particular number; but if it is a plurality, we can speak of one, two, three, etc. lines. We can apply the same argument to time such that a time as a continuum has no smallest number; but as a plurality, its smallest number is one or two. Consequently, there is no contradiction to think of time as either a continuum or a number; it is just a matter of definitions. If we define one time with respect to another one, then it is obvious that we refer to a plurality of two times. By contrast, if we deal with the infinitely divisible parts of time, then we refer to a continuum, and the idea of a smallest number is irrelevant since all temporal parts are infinitely divisible. We then face two kinds of time: on the one hand, a continuous time is one and cannot be counted; on the other, a temporal number is a discontinuous plurality of *at least* two distinct times.

The two times composing the temporal plurality are the 'before' and 'after', which are indivisible units and not divisible parts; they are two instants (*duo ta nun*) containing a middle or intermediate (*metaxu*) (219a25-30). In other words, a divisible interval of time is surrounded by two indivisible instants. Aristotle uses the Greek word *nun* ('now') to speak of an instant of time, and defines it as a close future which is both the synonym of 'presently' and the opposite of 'lastly' (222a20-24, 222b7-15). Yet, its primary meaning is to be defined as an indivisible limit of time that is distinct from, but essential to, a continuous time (220a4-5, 222a10-13). We must combine such indivisible limits with the infinitely divisible intervals of a continuous time; namely, we need to explain how the counting of temporal limits can be compatible with the infinitely divisible content of a temporal continuum. Ross (1936) is aware of the difficulty:

> The description of time as that in change which is *counted* is unfortunate. For 'counting' suggests denumeration, counting to the end; and Aristotle's language arouses the suggestion that we can count the nows [instants], or else the indivisible periods of time, involved in a change. This, however, would be foreign to Aristotle's whole theory; he is absolutely consistent in maintaining the infinite divisibility of time and change. (p. 65, original emphasis)

The only way to avoid the problem is to draw the distinction between the two aspects of time, meaning that Aristotle succeeds in defining a discontinuous plurality of *incorporeal* limits, counted by the mind, without contradicting the infinite divisibility of *quantitative* temporal intervals. We must, therefore, show that the discontinuous counting of limits can never disrupt the continuity of time, insofar as the incorporeal has no effect on the corporeal.

We know that the infinite divisibility of time depends on the infinite divisibility of motion; likewise, if time is a number, the 'before' and 'after' pertain to a motion. More precisely, Aristotle states that the 'before' and 'after' are primarily in place (*en topôi prôton*), providing that 'in place' means 'in position' (*têi thesei*) (219a14-16). We are not dealing with a natural place, namely a place whose power (*dunamis*) causes the light object to move up and the heavy object to move down (208b8-25, *De Caelo* 305a24-26). Rather, a place *qua* position refers to an abstract entity, such that a geometric line is in a position relative to the mathematician without possessing a natural place in the physical universe. Thus, to say that the 'before' and 'after' are in positions means that they are mere abstract entities, whose existence need not be physical. As such, we must stress Aristotle's distinction between the two following claims. On the one hand, we are told that the 'before' and 'after' are in motion (*esti de to proteron kai usteron en têi kinêsei*, 219a19-20); and they are in magnitude and in time as well, since time derives from a motion, and motion from a magnitude (219a16-19). On the other hand, the essence of the 'before' and 'after' is distinct from a motion (*ho men pote ôn kinêsis estin*, 219a20-21), for they are abstract divisions whose indivisibility contradicts any divisible quantity, whether a motion, time, or magnitude. In other words, they are incorporeal divisions that pertain to the divided positions of a moving object, and divisions are *incorporeal* indivisibles which have no physical existence. We can remind ourselves of Aristotle's definition of a division (cf. section 2.3): "The point and all division and anything indivisible (*adiaireton*) in this way are made as manifest as privation (*sterêsis*)" (*De Anima*, 430b21-23). Therefore, the 'before' and 'after' potentially divide a motion, and understood as indivisible positions, they negate a quantitative motion in the same way that points negate a quantitative line (*Physics*, 220a18-20, 235a25-37). Likewise, indivisible instants are privations of time and cannot be the constitutive parts of an extended physical time (cf. *De Caelo*, 300a7-14). Hence, time, motion and magnitude observe the same rule of being composed of quantitative parts whose divisions are incorporeal, i.e. non-quantitative. Accordingly, instants are not the divisible parts of time, positions are not the divisible parts of motion, and points are not the divisible parts of a magnitude. This means that a physical object neither moves at indivisible instants of time nor starts moving at a 'first' (*prôton*) instant. Change and rest can never take place at indivisible instants (234a24-b9), such that something is not changing, but has changed, at a first instant (235b6-236a7). 'Something has changed' refers to an end of change. Aristotle distinguishes an end of change, i.e. an indivisible state at a 'first' instant, from a beginning of change, i.e. an infinitely divisible change in an infinitely divisible time (236a7-27). As a physical process is based on continuity, i.e. infinite divisibility, it excludes an end of change (*metabolês telos*) definable as an indivisible limit (*adiaireton peras*) (236a10-15, a26-27, and a34-36).

Therefore, first, second, third, etc. instants, positions, and points are incorporeal indivisibles which must be distinguished from the infinitely divisible intervals of quantitative time, motion, and magnitude.

Both the divisible and indivisible complement each other, since time is made infinitely divisible only through its divisions, *qua* instants; in this sense, an indivisible instant is the accident (*sumbebêken*) of an infinitely divisible time (220a4-5, a21-22, and 222a10-12). Aristotle introduces an interesting, yet difficult to comprehend, comparison between an instant and a moved object, such that time is the number of a motion while an instant or moved object is like the unit of a number; he writes:

> It is obvious that if time did not exist, the instant (*nun*) would not exist, and that if the instant did not exist, time would not exist. Indeed, just as the moved object (*to pheromenon*) and locomotion (*hê phora*) are together (*hama*), so are the number of the moved object and the one of locomotion (*houtôs kai ho arithmos ho tou pheromenou kai ho tês phoras*). For time is the number of locomotion, and the instant as well as the moved object, is like the unit of a number (*hoion monas arithmou*). (219b33-220a4)

Indivisible instants and infinitely divisible time are correlated concepts, since instants are the divisions that potentially divide a continuous time. Likewise, the positions of a moved object potentially divide a continuous locomotion. Aristotle tells us that the number of locomotion is combined with the number of the moved object. We already know that time is the number of a motion (whether it is generation/corruption, alteration, increase/decrease, or locomotion); but the number of the moved object (*ho arithmos ho tou pheromenou*) is an odd expression. It would be tempting to claim that an instant is the number of a moved object in the same way that time is the number of a motion. Yet, this statement is misleading since an indivisible instant is obviously not a divisible number or plurality; that is the reason why Aristotle claims that an instant is like the indivisible unit of a number. [4] If time as a number pertains to a divided motion, i.e. a plurality of positions defined as the 'before' and 'after', then an instant as a unit of number is connected to a division of motion. Thus, each division

[4] Regarding the passage 219b33-220a4, Annas (1975) wrongly associates the now (or instant) to a number, but rightly claims that this is quite incomprehensible. She writes, "The relation of time to the motion is that time is the number of motion. Correspondingly, the now must be *the number of the moving body*. But this is not very illuminating; we do not know how to understand this." (p. 110). Likewise, Inwood (1991) writes, "The now is *the number of the moving body* and time the number of the motion." (p. 166). Yet, he corrects himself in a footnote: "Time is the number of movement, but the now corresponds to the moving object, like *the unit of number*." (footnote 18, my emphases).

of time *counts* each division of motion; as well, each division of motion counts each position of a moved object. This latter claim refers to the Aristotelian fact that not only the continuity of time follows the continuity of motion, but also the continuity of motion follows the continuity of magnitude (219a12-14, b15-16, and 220b24-29). Consequently, we may draw three parallel relations: one between a time and its indivisible instants, one between a motion and its indivisible divisions, and one between a magnitude (*qua* moved object) and its indivisible positions.

The comparison of an instant with a moved object is central to Aristotle's position that time is the number of a motion. There are two ways of assessing the importance of this comparison. The least convincing interpretation is to claim that an instant is like a moved body, because both are conditions for the knowledge of time and motion. While motion is knowable through a persisting individual thing (*tode ti*), time is knowable through a persisting individual instant (219b28-32). [5] The problem is that the notion of a persisting instant (or 'now') contradicts its definition as an indivisible entity, since only a divisible temporal interval has duration. The second interpretation is more illuminating; indeed, Aristotle tells us that a moved object has the property of being both the same and not the same: it is the same as the subject (*ho pote on*) of motion, but is not the same since it is distinct by definition (*tôi logôi*) (219b18-21). [6] Likewise, an instant is the same as the subject (*ho pote on*) of time, but is not the same by definition (*tôi logôi*) (219b12-15). An instant or a moved object follows the same principle: it is a unique division whose definitions are distinct. A moved object divides motion and, as a division, is always the same; but the definitions of the moved object change according to its distinct positions. Aristotle provides the following instance (219b20-22): Coriscus is a subject of motion, which potentially divides his own motion; yet 'Coriscus in the Lyceum' and 'Coriscus in the

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[5] Ross (1936) follows this interpretation by identifying the now with a "felt presentness" and "a lapse of time". He writes: "As it is by attention to the moving body that we recognize movement, it is by attention to the now, i.e. to the felt presentness of objects of experience which could not be experienced together at one time (such as whiteness and blackness in the same body), that we recognize the lapse of time." (p. 67).

[6] Inwood (1991) is unconvinced by this passage; he writes: "It is unclear whether, in contrasting *ho pote on* with *tôi logôi* he [Aristotle] is contrasting the object as such with the object as moving or, rather the object as moving with some particular state of the object." (p. 167). It seems to me that Aristotle distinguishes the subject of motion, i.e. a moved object dividing motion, from its distinct definitions, i.e. its different positions as the 'before' and 'after'.

Agora' are two distinct positions as the 'before' and 'after' in motion. In other words, a moved object is both the same and not the same. Likewise, an instant is the same as *being* the division of time, but not the same as *defining* either the 'before' or the 'after'.

From the comparison between an instant and a moved object, we may now suggest that the instant is derived from the moved object. Indeed, a moved object is a division through which we *know* (*gnôrizomen*) the 'before' and 'after' in motion (219b16-18); and we *know* time as a number with respect to the 'before' and 'after' in motion. I emphasise the verb 'to know', because the 'before' and 'after' are divisions, namely incorporeal abstractions defined as objects of thoughts; they are not physical objects. Aristotle writes:

> The instant follows the moved object, just as time [follows] motion: for it is by the moved object that we know the 'before' and 'after' in motion; and as being numerable, the instant is the 'before' and 'after'. (*Tôi de pheromenôi akolouthei to nun, hôsper ho chronos têi kinêsei: tôi gar pheromenôi gnôrizomen to proteron kai husteron en kinêsei; hêi d' arithmêton to proteron kai husteron, to nun estin*). (219b22-25)

The statement that time follows motion, which itself follows magnitude, is repeated several times (219a10-13, b15-16, and 220b24-26). Hence, each division of time is either a 'before' or an 'after' in motion, pertaining to the relative position of a moved object. Aristotle makes clear that the nature of the moved body matters little in this context: it is like a point, a stone, or something of this kind (*ê stigmê gar ê lithos ê ti allo toiouton esti*, 219b19). The only important thing to know is that the moved object is the subject of motion dividing its own motion; and time is the number of motion with respect to the positions of this moved object, i.e. the 'before' and 'after' (219a14-16). Thus, any instant which divides time measures a moved object which divides motion; likewise, an instant, *qua* the 'before' and 'after', measures time (*to de nun ton chronon metrei, hêi proteron kai husteron*, 219b11-12). The verb *metrein* (to measure) is apparently controversial, since it is applied to an indivisible instant, while Aristotle tells us in 218a6-7 and 220b32-221a7 that only divisible parts can be the objects of a measure, such that a divisible time measures a divisible motion. [7] However, it seems to me that a measure may apply to instants, and the reason lies at 219a30-b2. Aristotle explains that a unique instant is not time, but potential instants (i.e. the 'before' and

[7] To avoid the contradiction, Ross 1936, unlike Carteron 1926-31, does not use *metrein* but follows Torstrik's correction by replacing it with *horizein* (to delimit).

'after') defining indivisible units of time. In this sense, the 'before' and 'after' measure time as a number; and this interpretation confirms the relevant use of *metrein* at 219b12.

Consequently, we deal with two definitions of time. On the one hand, physical time is as continuous as motion, for both refer to actually finite processes in nature. On the other hand, time and motion are related to incorporeal indivisibles, which are potential objects of thought without actual existence. Such abstractions from the physical world enable Aristotle to construct time as the number of motion, i.e. the counted plurality of the divisions of motion.

3.3 Physical continua and the potential infinite

This does not mean that we deal with two separate concepts of time. Time, as a continuum and a number, is divisible through the same potential divisions, which also apply to motion and magnitude. Thus, all continua are potentially divided according to the same and equal divisions (*tas autas kai tas isas diaireseis*) (233a10-17). All changeless divisions are identical, whether they are divisions of time, of motion, of the being-moved (in activity), of the moved object (in locomotion), of the quantity (in increase/decrease), or of the quality (in alteration) (235a13-18). All divisions of continua are *potential*, and are never *actualised*, so that they do not to disrupt the infinite divisibility of the quantitative parts. Therefore, physical continuity, whether in magnitude, motion or time, is irreducibly linked to the potential infinite. Aristotle writes:

> To someone asking whether it is possible or not to traverse infinites (*apeira*) either in time or in length, we must say that in one way it is, in another it is not. If they are in actuality (*entelecheiai*), it is not possible, but if they are in potentiality (*dunamei*), it is possible, for what is moved continuously (*sunechôs kinoumenos*) has traversed the infinites accidentally (*kata sumbebêkos*), but not absolutely (*haplôs d' ou*); for it turns out that there is an infinity of halves in the line accidentally, but its substance and being are different (*hê d' ousia estin hetera kai to einai*). (VIII, 263b3-9)

A geometrically continuous line is both essentially (actually) finite and accidentally (potentially) infinite. Likewise, time and length are physically finite continua, whose divisibility pertains to the potential infinite. If divisions were producing actual points, positions and instants, potentially infinite divisibility would be replaced with actually finite divisions, meaning that magnitude, motion and time could no longer be defined as physical continua. Moreover, *incorporeal* instants have no proper existence, and cannot constitute a temporal order insofar as they are wholly dependent upon *quantitative* temporal intervals (231b6-10); as such, they are not definable as successive entities, since a succession (*ephexês*) implies independent distinct quantities. As well, instants are not contiguous (*echomenos*) with each other, for they are separate by intermediates (*metaxu*), i.e. divisible temporal intervals (237a24-25, b7-9). Aristotle compares the absence of contact between instants with the non-contiguity of indivisible points (218a18-19). While divisible parts are quantitative continua united within a continuous whole, indivisible divisions are incorporeal abstractions; and Aristotle uses a *reductio ad absurdum* to show that indivisibles cannot be the parts of a changing continuum (240b17-241a6). The absurd assumption to contradict is the following: if time is defined as a series of indivisible instants, then motion is made out of completed moves (*ek kinêmatôn*). Aristotle explains that completed moves are indivisible ends of motion, and if a motion is composed of completed moves, then a body moves without moving insofar as indivisibles prevent motion from changing. As it is absurd to speak of a motionless motion, we must conclude that it is impossible for a physical continuum to be composed of indivisibles. Therefore, a motion is made out of movable divisible parts and not of motionless indivisible limits; as well, time is not made out of instants, or magnitude out of points (232a6-11, 241a2-6). Indivisibles can never be in the process of coming into being (*Physics*, 241a6-14, *Metaphysics*, 1002b5-11), meaning that infinite divisibility is required for any physical change in time (234b10-235b5, 237b23-238b22).

I would like to insist on the specificity of a continuous time with respect to a continuous magnitude. An indivisible instant is not completely similar to an indivisible point (*stigmê*). A point limits a continuous line, such that two limits pertain to two distinct lines that are at best contiguous but not continuous (cf. 227a10-13); likewise, an instant limits a continuous time. However, there is a distinction to make between potential instants and potential points: while potential points have a same definition, potential instants are defined as either the 'before' or the 'after'. Indeed, we think (*noêsômen*) of the two distinct instants as separated by a divisible temporal interval, such that the soul (*psuchê*) counts the 'before' and after' (219a25-30). The intellectual part of the soul makes instants distinct by definition (*tôi logôi*), which does not contradict the fact that all indivisibles, in the divisions of a physical continuum, are the same (cf. section 3.2). Aristotle uses the analogy of a circle, which is both concave and convex at a given point, in order to show that an instant, *ontologically* unique as a division, is *epistemologically* defined as either the 'before' (i.e. the end of the past) or the 'after' (i.e. the beginning of the future) (222a33-b7). This semantic distinction pertains to the two distinct perspectives of a same division of time in the same way that the concave and the convex at a point are the two distinct perspectives of a same circle. In Book VIII of the *Physics*, Aristotle sums up the argument by stating that an instant is not like a point (*stigmê*) of the line, insofar as a point (*sêmeion*) of time has the following property: "It is one and the same numerically; but not the same by definition" (*kai tauton kai hen arithmôi*; *logôi d' ou tauton*) (263b12-14). Whereas a point is univocally defined as the limit of a line, an instant is a temporal limit which is either a 'before' or an 'after'. Consequently, time is the number of a motion, namely the plurality of 'before' and 'after' pertaining to the potential divisions of a motion. Time as a number of past and future instants is the abstraction of a present continuous time, in the sense that the abstract counting of 'before' and 'after' motions derives from the continuous motion of the physical world. Thus, a continuous physical time, following a continuous motion, is actually present in nature, as opposed to its potential divisions, i.e. its past and future limits, which are counted by the mind. In other words, an Aristotelian time is nothing more than the temporal qualification of a motion, whether present as an actual continuum or past and future as a number of potential divisions.

My interpretation of Aristotle's concept of time rests upon the idea that time as the mathematical number of a motion is derived from time as a physical continuum. They are two aspects of a same concept, such that a finite physical time, defined as the measure of a finite physical motion, is potentially (infinitely) divided into the 'before' and 'after', which constitutes a number of motion. My main purpose has been to avoid combining Aristotle's purely geometric property of infinite divisibility with anachronistic conceptions from algebraic mathematics and modern physics. One of the great temptations is to think of Aristotelian instants as a mathematical series of algebraic points. For instance, Sorabji (1979) writes, "An instant will be not a time-atom, nor any kind of period, but rather the *boundary* of a period, itself having no duration. Instants, unlike time-atoms, cannot be next to each other... Between any two instants, there will be another, indeed, an infinity of others. This is what is involved in time being continuous and our problem will apply to the real world only if time is so." (p. 160). Sorabji's argument is influenced by the algebraic property of denseness defined for a set Q of rational numbers, such that rational numbers cannot be next to each other since there is always a rational number between any two rational numbers. This cannot constitute a sound justification of Aristotle's position, insofar as denseness defines indivisible elements as the *quantitative* parts of a sequence. In contrast, Aristotle's instants are *incorporeal* limits, and two indivisible instants are *not* next to each other *only* because they are separated by an intermediate, i.e. a divisible *quantitative* interval (219a25-30, 237a24-25, b7-9). In other words, time is continuous only because its quantitative parts are infinitely divisible *intervals* (and not indivisible instants). White (1992) is right to claim the following: "It is scarcely necessary to point out that this Aristotelian conception of time as a linear continuum is not the contemporary, point-set conception of a dense and linearly ordered set of *ta nun*, 'instants' or temporal points, satisfying the additional Dedekind continuity condition." (p. 87). Yet, he still refers to an algebraic concept of denseness when he claims: "If such a model [of an asymmetric time] is to come at all close to capturing Aristotle's idea of the 'developmental' character of time, we must envision the model as being viewed from a (dense and continuous) *succession* of nows along one linear (but not predetermined) branch of the model" (p. 95, original emphasis). Again, the denseness and continuity of a series of instants (or nows) are abstract properties which have nothing to do with the intuitive idea of continuity; do not forget that instants are intuitively discontinuous elements. It is certain that the resort to set-theoretic concepts is consistent in itself, but it is wholly anachronistic in regard to Aristotle's arguments. Likewise, Owen (1979, p. 151) admits the possibility of successive instants as well as Kretzmann (1976) who writes, "Instants in the temporal *continuum*, like points on a line, are successive, distinct, and *non*consecutive" (p. 114, footnote 22). This statement is ambiguous, since Aristotle claims the impossibility for any indivisible instants or points to be in succession (*ephexês*) (231b6-10). Moreover, points or instants are *incorporeal* limits whose definitions depend upon a limited quantity, i.e. a *corporeal* line or time. In other words, they cannot be defined as free-standing entities, and cannot, as such, be successive, distinct and non-consecutive, since only quantities (like the *quantitative* points of an algebraic order) can make sense of such properties.

Any interpretation influenced by a modern structure of mathematics is unable to make sense of Aristotle's concept of time. For instance, to oppose a temporal order to a spatial order is a purely anachronistic move. Corish (1976) writes, "It is only when we perceive a *proteron* ['before'] and *husteron* ['after'] in *kinêsis* [motion] that we apprehend time... We are left to suppose that the *proteron* and *husteron* in *kinêsis* are in fact simply the *proteron* and *husteron* of spatial order itself." (p. 249). Corish defines the spatial order of 'before' and 'after', as though Aristotelian physics were not different from Newtonian physics with the common postulate of a spatial quantitative order independent of physical bodies. As well, Waterlow (1984) writes, "He [Aristotle] gives an analysis of temporal before and after in thoroughly quantitative terms. In this, he follows his own line concerning spatial order, which is just what we should expect in view of his repeated assertions in *Physics* IV that temporal order corresponds to, and derives from, spatial order." (p. 113). Yet, Aristotle's text does not speak of temporal or spatial orders of instants or points, insofar as the concept of order defines *quantitative* elements, and directly contradicts the Aristotelian idea that instants (such as the 'before' and 'after') and points are *incorporeal* limits, whose existence depends upon *quantitative* divisible intervals. At least, Corish (1978) explicitly admits that his interpretation of Aristotle's position relies on Newtonian concepts: "We think of movement, as Newton does (and, it seems clear, as Aristotle does implicitly), as a derivation from (as indeed, in terms of the graph of movement, a combination of) space and time" (p. 69). Clark (1975) is one the rare commentators who reject the implicit use of Newtonian concepts in the interpretation of Aristotelian physics; he rightly claims: "If change is the realization of potential we no more need dimensional time than we need empty space." (p. 127). Indeed, the Aristotelian definition of change is purely physical, and does not require a mathematical physics as implied by the numerical orderings of space and time. Consequently, the use of modern mathematical concepts is anything but helpful for our understanding of Aristotelian physics.

Furthermore, the modern philosophical debates about time do not really make sense of Aristotle's position. For instance, McTaggart's (1908) divides time into the A- and B-series, such that the A-series implies a changing time defined through past, present, and future tenses, whereas the relations 'earlier than' and 'later than' of the B-series are both tenseless and changeless. Sorabji (1983, pp. 47-50) rightly states that the static/flowing distinction is not directly relevant to Aristotle's time, but we can further claim that the divide between tense and tenseless times is meaningless within an Aristotelian context. Indeed, Aristotle does not suggest a mathematical structure whose foundation is at odd with physical reality; in other words, a tenseless mathematical time cannot be opposed to a tensed perception of time. Likewise, Williams (1951) has initiated the criticism of a temporal passage, according to which the mathematical formalism of our modern physical laws must be radically distinguished from the intuitive idea of a moving time. His position does not apply to an Aristotelian time, for it implies a mathematical model that does not make sense of Aristotle's physics. Yet, Kretzmann (1976) writes, "I think he [Aristotle] takes time to be essentially the permanent ordering of events and that he considers the passage of time to be an attribute of, or the appearance of, that linear sequence." (p. 107). His position defines the passage of time in Aristotle's system as merely an appearance, contrary to Waterlow (1983) who adopts the very opposite view: "Whereas many modern accounts take for granted the independence of the mathematizable properties of time and the temporal (the 'primary' qualities), but question the reality of 'passage', with Aristotle it is the other way about" (p. 141, footnote 15). She is quite right, in the sense that the passage of time in Aristotelian physics derives from the definition of time as a physical continuum.

A similar confusion pertains to the concept of instantaneous velocity. This concept is not definable within Aristotle's physics, and that is not surprising since its algebraic abstraction is meaningless from a non-algebraic viewpoint. [8] It is always possible to speak of the

[8] Owen (1957) writes: "The possibility of talking about motion at a moment rests on the possibility of talking of motion over a period, the two uses of 'motion' are not the same… He [Aristotle] failed to grasp that the two senses of 'moving' are not identical but yet systematically connected; and this failure to see this connection between two common uses of a common word led him to rule out one use entirely in favour of the other." (pp. 160-1). Owen's remark is unfair, insofar as Aristotle's position is consistent within his own physics. Aristotle does not have a mathematical definition of motion as provided by velocities, because he does not need one. Vlastos (1966) sums up Aristotle's standpoint well: "To say that the arrow is moving in any instant would be (strictly speaking) senseless: it is non-moving and non-resting in the same way in which e.g. a point is non-straight and non-curved, non-convex and non-concave—the predicates are not falsely applicable, but inapplicable." (p. 192). In this sense, I agree with Lear's (1981) well-balanced conclusion: "This does not show that there is any mistake in Aristotle's argument, only that there is a use of the expressions

average speed of a moving body by calculating its traversed distance in a divisible interval of time; but to define an instantaneous velocity implies the construction of an algebraic fiction (e.g. Leibniz's infinitesimal points; cf. sections 1.5 and 1.6) which is not *physically* meaningful. Indeed, to associate a quantitative value with an indivisible point is an absurdity for Aristotle, insofar as incorporeal indivisibles contradict geometrically divisible quantities. Accordingly, Aristotle's views cannot make sense of modern physics, and this shows how the use of modern physical concepts is harmful to a genuine interpretation of Aristotle's physics.

3.4 Physical concepts and set-theoretic properties

The obvious contrast between Aristotle's physical time and a modern concept of physical time consists in the introduction of set-theoretic properties. Thus, discrete, dense, and continuous time pertain to set-theoretic models of discreteness, denseness, and continuity. More precisely, a model of discrete time is isomorphic (i.e. identical in form) to the discrete set N of all natural numbers with the instants t_1 and t_2 defined as positive integers. Likewise, a model of dense time is isomorphic to the dense set Q^+ of all positive rational numbers. As there is always a rational number between any two rational numbers, no instants are next to each other since there will always be a third one popping up between any two. Finally, the model of continuous time is isomorphic to the continuous set R^+ of all positive real numbers, such that a second-level real number or Dedekind cut is the limit of an infinite subset of first-level real numbers (i.e. rational numbers). Applied to time, this means that a temporal continuum is composed of dense subsets of rational instants. Therefore, the abstract models of discreteness, denseness and continuity mirror three distinct set-theoretic structures. Yet, we may wonder whether such definitions have some empirical meanings. Intuitively speaking, the expression 'discrete time' refers to a time composed of indivisible instants, such that the passage from one instant to another implies an irreducible jump. By contrast, if

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'instantaneous velocity' and 'moving at an instant' that he did not envisage, i.e. as designating the limit of velocities. Of course, modern dynamics surpasses Aristotelian dynamics in part due to the fact that we understand the concept of a limit much better than he did; but this admission differs significantly from the claim that Aristotle fallaciously argued that there cannot be motion in an instant." (p. 99).

time is intuitively dense or continuous, then we deal with a smooth, uninterrupted process between any instants. It is easy to realise that such intuitions are too vague to be strictly justifiable, and I want to show that such empirical intuitions can never match the concept of time implied by set-theoretic models.

First, if our aim is to define a scientific concept of time, we must give up the notion of a tensed time, dealing with a perception of time and not with time itself. I am sitting at my desk at a present time t_1 only because I perceive the instant t_1 as present with respect to past and future instants. Past, present and future tenses always refer to perceptions of time, contrary to a tenseless time whose instants are defined independently of our tensed perceptions of them. By modelling the concept of time through arithmetic sets, we are able to provide well-defined structures. However, we face the following questions: how is it possible to bridge the gap between a physical concept of time and its mathematical model? Can we claim that time is *physically* continuous and not dense or discrete through the mere justification of an *arithmetic* model of continuity? In *Our Knowledge of the External World* (1926), Russell claims that it makes no sense to distinguish continuity from denseness, except by means of purely arithmetic arguments. He writes:

> Mathematicians have distinguished different degrees of continuity, and have confined the word "continuous", for technical purposes to series having a certain high degree of continuity. But for *philosophical* purposes, all that is important in continuity is introduced by the lowest degree of continuity, which is called "compactness" [denseness]. (1926, p. 138, my emphasis)

Arithmetic continuity is distinct from denseness (or compactness) only because we face two set-theoretic structures which distinguish a continuous set of real numbers from a dense set of rational numbers. A philosophical interpretation of this distinction has no relevance, for there is no proper philosophical argument that can explain why continuity should be distinct from denseness. The definition of a real number as a Dedekind cut partitioning two subsets is a mathematical abstraction without philosophical or intuitive meaning. Yet, Russell's above statement is somehow ambiguous, insofar as it opposes "the lowest degree of continuity" defined as arithmetic denseness to "a high degree of continuity" identified with arithmetic continuity. It would be misleading to believe in some external principle of continuity that could connect these two set-theoretic properties. It would be tempting to imagine an empirical verification of denseness, but Russell rightly excludes this hypothesis; he writes, "Mathematical space and time... have this property of compactness [denseness], though whether actual space and time have it is a further question, dependent upon empirical evidence, and probably incapable of being answered with certainty" (1926, p. 139). Therefore, the only reason for thinking that denseness is the lowest degree of continuity merely derives from the arithmetic fact that the set Q of rational numbers is dense, as opposed to the continuum R of real numbers which is both dense and continuous, i.e. arithmetically complete (cf. section 2.1).

Russell's position has been criticised by Grünbaum, in 'The Resolution of Zeno's Metrical Paradox of Extension' (1957, 1973), which claims that it is possible to prove, on physical grounds, that arithmetic continuity is a better model than denseness. Thus, arithmetic continuity, unlike denseness, is defined as metrically consistent insofar as it avoids Zeno's paradox of extension. This means *contra* Russell that a *non-arithmetic* argument is able to demonstrate the philosophical primacy of arithmetic continuity at the expense of arithmetic denseness. Grünbaum writes:

> Russell neglected the essential contribution made by the cardinality and ordinal structure of the continuum toward the avoidance of Zeno's mathematical paradoxes... We know that the mere existence of the denseness property guarantees only a denumerably [countably] infinite point-set. Since a super-denumerably [uncountably] infinite point-set is required by the demands of metrical consistency, it follows that there are *philosophical* reasons for requiring a higher degree of continuity than is ensured by the denseness property alone. (1973, pp. 174-175; original emphasis)

The upshot of the argument claims that a continuum has an infinitely *uncountable* structure which is metrically consistent. The notion of 'metrical consistency' pertains to the measure theory, namely a purely mathematical theory applying a concept of measure or length to arithmetic sets. Borel (1898) defines measure theory through the following properties:

1. Non-negativity, i.e. a measure is always nonnegative.
2. Monotony, i.e. the measure of the difference of a set and a subset is always equal to the difference of their measures.
3. Countable additivity, i.e. the measure of the union of countable disjoint sets is always equal to the sum of their measures.

If we deal with a dense countable set Q^+ of positive rational numbers, the measure or length of such sets is equal to zero. We may generalise this statement by asserting that every countable set has a measure equal to zero, insofar as it is reducible to singleton subsets of zero length. In other words, an infinitely countable set is a *totally disconnected set*, such that all its singleton subsets are disjoint. Thus, for a set S composed of two disjoint subsets A and B, we have:

$$A \cup B = S \qquad A \cap B = \varnothing \qquad (A \cap S) \cup (B \cap S) = B \cup A = S.$$

Every disjoint singleton subset has zero length, and the sum of the measures of all the singleton subsets is obviously zero. Therefore, the set N of all natural numbers (or positive integers) and the set Q^+ of all positive rational numbers are of zero length. In this sense, the sets N and Q^+ have a same countable cardinality, such that each element of N is in one-one correspondence with each element of Q^+ (cf. chapter 4). By contrast, a positive continuum R^+ has infinite subsets of positive real numbers, which are not singleton subsets but infinite intervals. A set of real numbers with non-null measure is a *connected set*, since its subsets are infinite intervals connected to each other through Dedekind cuts. Thus, the union of all countable subsets over an uncountable continuum $[a, b]$ has a nonnegative measure equal to $|b - a|$. For instance, the measure of the union of countable subsets over the continuum [0, 1] is equal to one, and the measure of the union of all countable subsets over $(-\infty, \infty)$, i.e. the set R or real line, is infinite. The measure of the union of countable subsets over a continuous set is systematised by Lebesgue measures defined over Borel sets. The Lebesgue measure λ is a function from countable subsets to an uncountable set of real numbers, such that

1. $\lambda(S) \geq 0$ for an infinite interval S.
2. $\lambda(\{\varnothing\}) = 0$
3. λ is *countably additive* since $S_1, S_2, S_3, \ldots$ is a countable sequence of *disjoint* intervals, and the measure of their union is equal to the sum of the measures of all the intervals, i.e. $\lambda(\cup S_i) = \sum \lambda(S_i)$ with $i = 0 \rightarrow \infty$.

A Lebesgue measurable set is a nonempty collection of subsets closed under countable set operations. [9] If measures were defined as uncountable, they could not be additive which would make measure theory inapplicable to the additive mathematics of physical sciences. Accordingly, the fact that a continuum of real numbers is the union of countable subsets

[9] To learn more about real analysis, see Kolmogorov & Fomin 1970 and Johnsonbaugh & Pfaffenberger 2002.

whose measure may be non-null means that its cardinality cannot be the same as the cardinality of N or Q, whose countable subsets are always of zero length (cf. section 4.3). [10]

Grünbaum, in his criticism of Russell's argument, uses Borel's measure theory and Lebesgue measurable sets in order to claim that an arithmetic continuum is metrically consistent, unlike a dense set of rational numbers. That is, the countable subsets of a set of real numbers have a non-null metric that can be interpreted on physical grounds, contrary to a dense set of rational numbers whose zero metric has no physical interpretation. In his 'Space, Time and Falsifiability' (1973), Grünbaum writes:

> By contrast to the situation in denumerable dense space, a countably additive M-metric Δ on a continuous space does *not* run afoul of Zeno's paradox of extension upon the imposition of the requirements $\Delta(\{a\}) = 0$, $\Delta(A) \neq 0$... The requirements $\Delta(\{a\}) = 0$, $\Delta(A) \neq 0$ do generate a Zenonian inconsistency in the denumerable case, whereas they do not do so in the continuous case. (p. 533)

The interesting case with Grünbaum is that he starts with the postulates, or rather with the requirements, that $\Delta(\{a\}) = 0$ and $\Delta(A) \neq 0$. The metric Δ of the singleton subset {a} of a dense set is equal to zero, i.e. $\Delta(\{a\}) = 0$, whereas the metric Δ of the infinite interval A of a continuum is non-null, i.e. $\Delta(A) \neq 0$. Yet, if we restrict ourselves to a denumerable dense space, we can explain the requirement $\Delta(\{a\}) = 0$ but not the requirement $\Delta(A) \neq 0$, since a dense set of singleton subsets has always a measure equal to zero. Grünbaum concludes to a 'Zenonian inconsistency', such that a finitely extended interval is paradoxically composed of an infinite number of extensionless parts. With a dense set, we can explain why the parts are extensionless since its singleton subsets $\Delta(\{a\})$ have zero length; but we cannot explain why the extension is finite, insofar as the measure of a countable set of singleton subsets is equal to the sum of the measures of all its subsets, i.e. equal to zero (principle of additivity). As this does not satisfy the physical requirement $\Delta(A) \neq 0$, we are left with a Zenonian inconsistency. By contrast, if we deal with a continuous space, we can fulfil the two

[10] Not every uncountable set of real numbers has a non-null measure or length; for instance, the Cantor set of real numbers is of zero length, for it is a totally disconnected set. To define a Cantor set, start with the interval [0, 1] and remove the middle open interval (1/3, 2/3) such that the remaining Cantor set is the union of closed intervals [0, 1/3] ∪ [2/3, 1]. Then apply the same operation to these two intervals with the middle open intervals (1/9, 2/9) and (7/9, 8/9), such that the Cantor set is [0, 1/9] ∪ [2/9, 1/3] ∪ [2/3, 7/9] ∪ [8/9, 1]. And so on. We then deal with a union of disjoints subsets of zero length.

requirements. On the one hand, a continuum of real numbers is dense, such that a dense set satisfies the requirement $\Delta(\{a\}) = 0$; on the other hand, the countable measure over an uncountable continuum A implies an infinite interval with a non-null metric, and this complies with the physical requirement $\Delta(A) \neq 0$. Therefore, the metric of a continuous space avoids the Zenonian inconsistency. As Grünbaum's argument is far from obvious, we may summarise it by starting with a reminder of Zeno's geometric paradox of extension and Aristotle's solution of it:

1. *Zeno's geometric paradox of extension*: A finite extension is divided into infinitely many extensionless indivisibles.
2. *Aristotle's geometric solution*: A physical extension is an actually finite continuum whose incorporeal divisions potentially divide infinitely many quantitative parts.
3. *Grünbaum's Zenonian inconsistency of arithmetic denseness*: A dense space of rational numbers is metrically inconsistent; the countable metric Δ of a dense set A with a singleton subset {a} is $\Delta(\{a\}) = 0$ and $\Delta(A) = 0$, which contradicts the physical requirement $\Delta(A) \neq 0$.
4. *Grünbaum's metrical consistency of arithmetic continuity*: A continuous space of real numbers is metrically consistent; the countable metric Δ of a continuous set A with a dense subset {a} is $\Delta(\{a\}) = 0$ and $\Delta(A) \neq 0$, which satisfies the physical requirement $\Delta(A) \neq 0$.

3.5 Physical extended metric vs. arithmetic extensionless metric

Grünbaum's position is problematic, insofar as it correlates set-theoretic properties with the intuitive idea of an extended space. Yet, we can show that a physical extension is confronted to Zeno's paradox, independently of set-theoretic properties. As such, the requirements $\Delta(\{a\}) = 0$ and $\Delta(A) \neq 0$ define purely abstract metrics whose main property is to be extensionless; and they should not be identified with the intuitive metric of a physical extension. Yet, Grünbaum (1957, 1973) believes in the possibility of interpreting arithmetic concepts geometrically. He seems to be convinced that both Dedekind's real line and the Cantorian definition of an uncountable set of real numbers have geometric interpretations;

he writes:

> Zeno's mathematical paradoxes are avoided in the formal part of a geometry built on Cantorian foundations. The consistency of the metrical analysis which I have given depends crucially on the non-denumerability [uncountability] of the infinite point-sets constituting the intervals on the line. (1973, p. 172)

In other words, Grünbaum's metrical consistency of an extended continuous line is based on an uncountable metric, which avoids "Zeno's mathematical paradoxes". He defends a geometric interpretation of a continuum which is more than a mere illustration, insofar as the geometric concept of extension is an *intrinsic* part of his interpretation of arithmetic continuity. His aim is to define a philosophical argument which shows that continuity is a better model than denseness. In this sense, his argument cannot be purely arithmetic; that is why he resorts to geometric and physical properties, such that an arithmetic continuum is defined as a linear continuum of geometric points and identified with an extended physical body; he writes:

> We can provide a physical interpretation quite unencumbered by the intrusion of the irrelevancies of *visual* space, if we associate not the term "point" but the term "linear continuum of points" of our theory with an appropriate body in nature. By a point of this body we then mean nothing more or less than an element of it possessing the formal properties prescribed for points by the postulates of geometry. (1973, p. 174, original emphasis)

The problem with Grünbaum's interpretation is that it does not contain any proper justification to convince us that a set-theoretic continuum is intrinsically identifiable with either a geometrically extended line or a physically extended body. Grünbaum takes the (useless) precaution to exclude visual space, but his geometric and physical constructions rest on an intuitive conception of extension, which does not make sense of extensionless set-theoretic properties. His position conflicts with Dedekind's insistence that geometric concepts must be discarded for the sake of a purely arithmetic system. [11] Thus, his reinterpretation of an extensionless structure of numbers through a geometric or physical

[11] In his *Stetigkeit und irrationale Zahlen* (1872), Dedekind shows that a purely arithmetic foundation of continuity must replace any vague and unreliable geometric representations. He writes: "The statement is frequently made that the differential calculus deals with continuous quantities, yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but they either appeal more or less consciously to geometric representations or to representations suggested by geometry, or they depend upon theorems which are never established in a purely arithmetic manner." (1996, p. 767).

extension implies a distortion of set-theoretic denseness and continuity.

The intuitive concept of a geometric or physical distance has no bearing on the analytic definition of a metric. The abstract metric of an arithmetic interval is a distance function *d*, namely a nonnegative real-valued function with the following properties for all real numbers *x* and *y*:

1. $d(x, y) \geq 0$ (nonnegativity)
2. $d(x, x) = 0$ (reflexivity)
3. $d(x, y) = d(y, x)$ (symmetry)
4. If $d(x, y) = 0$, then $x = y$ (identity of indiscernibles)
5. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

A distance function is a purely algebraic concept defined over the set of real numbers. It may be pedagogically useful to *illustrate* this definition by referring to a geometric distance between two points *x* and *y*. Yet, such an illustration has no influence on the intrinsically arithmetic definition of a distance function. Likewise, the notion of a metric space pertains to an ordering set of real numbers without geometric content. Thus, we claim that the continuous space [0, 1] is complete, since every bounded (Cauchy) infinite sequence defined over this interval is convergent, i.e. has a limit (cf. section 2.5).[12] These properties are purely algebraic, such that the concept of extension (whether geometric or physical) is foreign to an extensionless sequence of numbers. Besides, we have seen that Zeno's geometric paradoxes must be distinguished from the Zeno-like algebraic paradoxes, insofar as the latter does not contain a genuine and intrinsic definition of extension (cf. section 2.6).

A possible solution to Grünbaum's unfortunate combination of arithmetic and physical concepts is to define distance, interval, and space as purely empirical concepts, and thereby rejecting the mathematical abstractions of real analysis. In 'An Examination of Grümbaum's Philosophy of Geometry' (1963), Hilary Putnam defends this solution in his criticism of Grünbaum's position. By defining a concept of 'additive measure' applied to finite coordinates of rational numbers, he claims that distance is a physical concept whose physical

[12] As a reminder, the continuous interval [0, 1] has the same uncountable cardinality as the continuum R of all real numbers, i.e. the interval $(-\infty, \infty)$. These two intervals are in bijection, and their elements are in one-one correspondence.

measure or length cannot obviously be equal to zero; he writes:

> Let us assume that only points with rational coordinates exist... and that the physically significant notion of distance can be taken to be given by the square root of the sum of the squares of the coordinate differences (Pythagorean theorem). Then this measure is only *finitely* additive in this denumerable space; but no serious reason exists for supposing that we would not use this measure for that reason. The "intrinsic measure" of every distance, according to Grünbaum, is *zero* in any denumerable space, but to accept this would be to abandon the notion of distance altogether. (p. 223, footnote; original emphasis)

Putnam aims to show that a measure needs only to be "finitely additive" in order to measure a physical concept of distance, and there is no reason for being concerned with Zeno's paradox of extension, since any empirical extension necessarily implies finitely extended parts. Putnam rejects Grünbaum's identification of a physical distance with an "intrinsic measure", i.e. the measure or metric applied to an arithmetic set. His criticism is relevant, insofar as it outlines the crucial problem in Grünbaum's reasoning, namely the mistaken combination of arithmetic properties with the intuitive concepts of physics and geometry. In this sense, Putnam makes a clear choice by confining his view to a spatial distance, whose physical measure is devoid of intrinsically mathematical meanings. Grünbaum attempts to reply to Putnam's objection by stressing that he deals with a countably (i.e. infinitely countable) additive measure, which cannot be defined as 'finitely additive' since this mathematical measure belongs to infinitely countable subsets. In his 'Space, Time and Falsifiability' (1973), he writes:

> When I discussed the status of dense denumerable space vis-à-vis Zeno's paradox of extension, I did so in the following context: The measure function Δ was required to be *countably additive* on physical grounds, and on the same grounds Δ was required to satisfy the pair of minimal nontriviality conditions $\Delta(\{a\}) = 0$, $\Delta(A) \neq 0$. (p. 533)

Grünbaum does not seem to understand Putnam's objection in the sense that he merely repeats his previous argument based on the combination of measure theory with a physical concept of extension. He claims twice that the measure function Δ is defined "on physical grounds", which contradicts the arithmetic fact that this function is countably additive owing to the *infinitely countable* subsets of rational numbers. In other words, Grünbaum does not answer Putnam's relevant worry that a measure, if defined as a physical concept, should always be finitely additive.

In order to avoid Grünbaum's difficulties, we end up with two opposite views: either we define purely arithmetic properties by rejecting a geometric or physical concept of extension, or we adopt Putnam's naturalistic definition of a physical measure related to an empirically finite distance. Still, Putnam's solution is bold from a mathematical point of view, insofar as it uses rational coordinates which are based on the dense set Q of rational numbers. As the concept of measure pertains to rational coordinates, its definition should be mathematical and not physical; and as it is based on the infinitely countable set Q, it should be infinitely additive. [13] Putnam seems to be aware of this, since his interpretation of rational coordinates is calculated through purely geometric ratios, such that a finite measure is geometrically calculated by applying the Pythagorean theorem to a right triangle. In other words, his position implies that the finitely additive measure of a distance pertains to a geometrically extended magnitude without reference to set-theoretic domains of numbers. Putnam's naturalistic position is philosophically consistent, but its non-arithmetic foundation makes it mathematically old-fashioned. In contrast, Grünbaum's view is inconsistent, for its reliance on physical extension distorts the intrinsic meanings of arithmetically extensionless properties. Therefore, he does not successfully challenge Russell's position that only arithmetic arguments can make sense of the distinction between denseness and continuity; that is, a continuous time is arithmetically distinct from a dense time, but no geometric or physical definition can assess and comprehend such a distinction.

3.6 Empirical duration of instants vs. temporal interval of numbers

Newton-Smith states in *The Structure of Time* (1980) that, on empirical grounds, a continuous (or dense) time is a better model than a discrete time. Like Grünbaum, he combines set-theoretic properties with empirical concepts, such that the belief in a continuous time is explained through the projection of a mathematical structure onto a physical world. He writes: "Our belief in the continuity of time does not arise from any argument relating to infinite divisibility [denseness], it arises from our projecting onto the

[13] As a reminder, a dense interval of rational numbers is never finite since its denseness makes it infinitely divided; that is, there is always a rational number between any two rational numbers.

world the richness that is present in the mathematical system which we have found to date to be essential to the construction of viable physical theories." (1980, p. 118). Newton-Smith indirectly criticises Russell's claim that denseness and continuity, although arithmetically distinct, are philosophically equivalent. While Grünbaum relates the metrical consistency of a continuum to a physical concept of extension, Newton-Smith evokes, more trivially, the mathematical richness of arithmetic continuity which leads him to apply this theoretic property to empirical concepts. Both positions suffer from the same weakness, namely the unjustified combination of extensionless algebraic properties with extended physical concepts.

More precisely, Newton-Smith aims to prove that a dense time exists on empirical grounds at the expense of a discrete time. His justification rests on the arithmetic definition of denseness combined with an empirical argument. He writes:

> While there might be... limits to how finely we can operationally specify temporal intervals or spatial distances, this does not mean that there are not smaller distances and shorter times. It means only that *certain* ways of referring to these smaller bits of space and time are not available to us. The smaller bits might still be there to be referred to if we choose to. If it is legitimate to refer to ever smaller parts of durations even though those parts may not be operationally specifiable, we are constrained to regard time as infinitely divisible and hence as not being discrete... The argument from divisibility does not purport to do more than to establish the density [denseness] of time. (1980, p. 117, original emphasis)

Arithmetic denseness implies infinite division, since if there is always a rational number between any two rational numbers in the bounded dense interval (0, 1), we may conclude that this interval is infinitely divided into rational numbers. Applied to time, this means that the temporal interval (0, 1) is divided into an infinite number of instants. Newton-Smith combines this set-theoretic property with some intuitively defined concepts, such as 'parts of durations' and 'bits of time'. The problem is that instants, as defined by the set-theoretic model of dense time, are not empirical concepts but elements isomorphic to rational numbers. In contrast, if we suppose that the parts of durations and the bits of time are *purely* empirical concepts, then we should not be able to refer to denseness, insofar as this property is meaningless without the background of a set-theoretic model. In this sense, a purely empirical division of instants leads to a merely epistemic argument, such that we may believe in the infinite division of time, but we are unable to specify the infinite division as ontologically existent. Newton-Smith uses this argument to show that time cannot be discrete; yet, his position is weak, for we may reverse the epistemic belief: nothing can prevent us from believing in discrete parts of time, even though we cannot specify them as ontologically existent. Eventually, the two opposite epistemic arguments are acceptable, insofar as there is no unique solution on purely empirical grounds. This shows the limited scope of any analysis based on empirical properties. By contrast, if we refer to a purely arithmetic property of denseness, then the choice between dense time and discrete time is easy to make, since the infinite division of an interval of rational numbers contradicts the discreteness of an interval of natural numbers. If we accept this solution, the requirement is that instants and intervals are purely arithmetic concepts. Consequently, Newton-Smith should have made a clear distinction between mathematical models and empirical concepts. Either his definition of time is purely empirical and must be devoid of arithmetic properties or it pertains to set-theoretic models that exclude any intuitive references to empirical concepts.

The objection to Newton-Smith's view is reinforced by the fact that he denies the physical existence of discrete time in the name of a paradox, similar to Zeno's paradox of extension defined by Grünbaum. Newton-Smith writes:

> There is something conceptually problematic in the idea that time might be discrete… For instants are extensionless parts of temporal intervals and if time were discrete any extended period of time would have a finite number of durationless parts of instants. And it is hard to see how adding up a finite number of unextended temporal instants could give an extended temporal magnitude. (1980, p. 114)

As we know, Zeno's paradox of extension implies that an extended interval be composed of infinitely many extensionless parts; yet, Newton-Smith should not claim that this paradox follows from the definition of a discrete time. If discrete time is an empirical concept, then it is an extension divided into broken parts; as we have stated earlier, we may believe in the possibility of discrete time, but we may never prove on purely empirical grounds that time is ontologically composed of discrete instants. In this sense, there is no way to demonstrate that we are confronted to Zeno's paradox of extension, for we are unable to define extensionless parts empirically. Indeed, empirical times are always about durations, and all durations are extended by definition. Take any instances in experience, and we realise that any *empirical* point or instant is *in fact* divisible into smaller points and instants. Only a theoretical abstraction can make sense of the claim that a point or instant *is* indivisible. Thus, if discrete time is defined as a set-theoretic model, its isomorphism to a discrete set of natural numbers leads us to avoid Zeno's paradox, since any interval of natural numbers is as extensionless as its elements; both are extensionless abstract entities devoid of empirical content. Accordingly, the only way to reach a paradox is to combine set-theoretic properties with an empirical concept of time. For instance, if discrete instants defined as natural numbers are combined with a discrete time, understood as a physical duration, we then end up with the absurd claim that a discrete time is an extended whole composed of extensionless parts. The paradox does not result from a genuine definition of discrete time, but from the inconsistent combination of an empirical concept with set-theoretic properties. Therefore, the two alternatives consist in defining time either as a set-theoretic concept or an empirical one, such that if time is a set-theoretic entity, then its parts are as *extensionless* as the whole; while if time is an empirical duration, then its parts are as *extended* as the whole.

Newton-Smith's neglect of a clear distinction between a set-theoretic model and an empirical concept has another unfortunate consequence: it leads him to provide a mistaken definition of the mathematical concept of measure. He writes:

> Even if it should turn out that it is best to treat the time of the world as discrete, we will obviously not actually measure the length of an interval by counting up its extensionless parts… It should be noted that this measure will not satisfy the conditions of a measure in the sense of measure theory. For a measure in measure theory must be countable additive. That is, the measure of the union of any set of sets must equal the sum of the measure of the sets forming the union. The set consisting of the union of all singleton sets of points of a given interval will not satisfy this condition. For the measure of each singleton set will be zero and hence the sum of the measure will be zero but the measure of the union of these sets will be the measure of the interval which is not zero. (1980, p. 120)

As we have see previously, Borel's measure theory defines a measure, length or metric as countably additive, and pertaining to infinitely countable subsets. Regarding singleton subsets, their measures are equal to zero, and the measure of the union of singleton subsets is equal to the sum of the measures of all singleton subsets, i.e. zero (principle of additivity). Hence, Newton-Smith is wrong to claim that the "measure of the union of these sets will be the measure of the interval which is not zero". His statement refers to the empirical fact that an extension cannot have a zero length, but he forgets that measure theory is a purely mathematical abstraction defining properties only for arithmetic sets and intervals. Thus, discrete time pertains to the discrete set N of natural numbers, whose arithmetic measure or length is always zero. Accordingly, Newton-Smith's paradox does not pertain to the arithmetic structure of a discrete time, insofar as the latter is wholly consistent with measure theory. [14] By contrast, it is clear from the above quotation that the combination of "the time of the world" with arithmetic properties implied by measure theory leads Newton-Smith to believe that the measure of a discrete interval of time cannot be equal to zero. Like Grünbaum, he seems unable to choose between purely set-theoretic models and purely empirical concepts (as defended by Putnam). Thus, discrete time is either an arithmetic model, i.e. a set of zero length whose singleton subsets have zero length, or an empirical concept, i.e. an extended duration composed of extended instants.

Newton-Smith defines a paradox *only* because he jumps from empirical concepts to theoretic ones without warning the reader that both kinds of concept have incommensurable definitions. If empirical and theoretic properties are carefully kept away from each other, the paradox disappears on its own. Therefore, Newton-Smith should not have constructed an empirical concept of time based on arithmetic properties, and Grünbaum should not have defined an arithmetic continuum with geometric and physical properties. As a set-theoretic model of extensionless time is not applicable to the intuitive idea of an extended time, this leads me to believe that scientific concepts, defined by rigorous mathematics, do not match the approximations of our empirical intuitions.

[14] The only way to challenge measure theory is to suppose the (hypothetical) existence of an *uncountable* measure based on the principle of non-additivity, such that the uncountable measure of a continuum is not equal to the sum of the measures of its countable dense subsets.

Chapter 4
Set-Theoretic Cardinalities and Actual Infinites

The foundational definition of an arithmetic continuum pertains to a set-theoretic cardinal number defining the number of elements for this set. Infinite cardinalities imply actually infinite sets, as abstractly constructed by Peano's arithmetic and Zermelo-Fränkel set theory. The actual infinite challenges Galileo's 'paradox' (although not a paradox) that the relations of equality and inequality (i.e. 'smaller than', 'greater than' and 'equal to') are meaningless when applied to the infinite. We shall see why a continuum of real numbers is infinitely uncountable unlike the infinitely countable sets of natural numbers and rational numbers. A distinction between two actual infinites implies that each infinite set have a proper order, as postulated by Zermelo's axiom of choice, such that it is always possible to choose a subset with one element (i.e. a singleton subset) for each subset of an infinite set. Yet, there is no proof that an infinite number of singleton subsets exist. In this sense, Brouwer's mathematical intuitionism criticises both the axiom of choice and the actual infinite in order to defend a continuum based on the intuitive idea of a potential infinite.

4.1 A second-level complete induction and the Dedekind infinite

In his *Dialogues Concerning Two New Sciences* (1638), Galileo wonders whether the infinity of points in a long line is greater than the infinity of points in a shorter line. [1] His answer is negative for the mere reason that it is impossible to order the infinite. Two finite quantities are either equal or unequal to each other because we can order the finite

[1] Galileo (1638) writes: "*Salviati*: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes 'equal', 'greater', and 'less', are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number." (1954, pp. 32-33).

succession of elements. Thus, the set {1, 3, 5} is smaller than the set {2, 4, 6, 8} because the cardinality three of the former is smaller than the cardinality four of the latter. May we define a cardinal number with respect to an infinite set? Let us follow Galileo's instance by comparing an infinite sequence of numbers that are perfect squares, e.g. {1, 4, 9, 16, 25, 36, ...}, with an infinite sequence of natural numbers, e.g. {1, 2, 3, 4, 5, 6,...}. Is the former infinite smaller than the latter? If we start counting the elements, we shall quickly understand that the question is intuitively meaningless. As each natural number has only one square, and each square number has only one square root, we shall never be able to complete both sequences, so that no answer can be given to the above question. In this intuitive sense, it is false to claim that the first set has as infinitely many numbers as the second set, since his statement implies a notion of equality which cannot apply to incomplete sequences. Galileo concludes that only a complete sequence has a cardinal number, so that an infinite cardinality is intuitively meaningless.

Bolzano (1851), Dedekind (1872, 1888), and Cantor (1874) define a new conception of the infinite that challenges Galileo's conclusion, namely a set is said to be infinite if it is equal to one of its proper subsets. [2] This means that each element of an infinite set can be paired with each element of one of its proper subset in such a way that no elements in either set are left over. This amounts to defining a bijection between the two infinite sets with a one-one correspondence between their elements. In other words, both infinite sets have equal cardinal numbers, namely the same infinite cardinality (or same power). This new conception of the infinite is called 'the Dedekind infinite', such that the finite is defined with respect to the Dedekind infinite; thus, a finite set is not equal to, but greater than, one of its proper subsets. The set N of all natural numbers and the set Q of all rational numbers rest on the definition of a Dedekind infinite, meaning that both sets, along with each of their proper subsets, have an infinitely countable cardinality. The counterintuitive Dedekind infinite contradicts the intuitive Galileo infinite, since the complete set of the former is opposed to

[2] Cantor (1883) makes a distinction between the 'improper infinite' (*Uneigentlich-Unendliche*) and the 'proper infinite' (*Eigentlich-Unendliche*). The 'improper infinite' corresponds to the potential infinite, and is improper in the sense that it always refers to an *actually* finite number. Through the definition of well-ordered sets, Cantor shows that the actual infinite is consistently definable in the form of an *actually* infinite number, and, as such, must be called the 'proper infinite'.

the incomplete sequence of the latter. Thus, the Dedekind infinite postulates the actual infinite as both determinate and complete, in the sense that there is no difficulty in defining equal cardinalities between two infinite sets.

The actual infinite implies the completion of the potential infinite. The actually *finite* sequence $\{1, 2, 3,\ldots, n\}$ of natural numbers is potentially infinite, for it is always possible to add some new elements to the sequence. An intuitive definition of mathematical induction means that for each natural number other than zero or one, we may define its immediate successor. This mathematical induction is incomplete since there will always be a successor for each new number, which is not the case with the complete induction defined by Peano's (1889) abstract arithmetic. A complete induction means that we first define, not natural numbers, but a set N such that natural numbers are already pre-defined elements of N. Thus, induction is complete because the set N defines induction for *all* natural numbers. Peano's arithmetic is composed of the following five axioms:

Axiom 1: Suppose a natural number 0 in the set N, such that N is a non-empty set with one element.

Axiom 2: If $x = 0$ is in N, then its successor x' is in N; that is, for each x in N, there is exactly one natural number x' in N called the successor of x.

Axiom 3: There is no x in N such that $x' = 0$; namely there is no natural number whose successor is zero.

Axiom 4: If x and y are in N and $x' = y'$, then $x = y$; that is, two natural numbers of which the respective successors are equal are themselves equal.

Axiom 5: If S is a subset of N, if 0 is in S and if x' is the successor of any natural number x in S, then S = N; that is, the set S containing both a natural number zero and the property of successor corresponds to the set N of all natural numbers. This axiom is called the induction axiom. [3]

The fifth axiom constitutes the core of the abstract arithmetic of natural numbers, since it enables one to postulate the existence of all natural numbers without constructing them

[3] We may also define the induction axiom in the form of an ordered triple $\langle N, x, f\rangle$, such that N is a set, x is an element of N, and f is a function from N to itself. It follows three conditions: if f is an injective function, if S is a subset of the set N and $x \in S$, and if $a \in S$ with $f(a) \in N$, then S = N.

individually.[4] This consists in the transformation of an intuitive principle into a set-theoretic abstraction. Thus, we start with an intuitive mathematical induction such that:

the successor of x is x' if and only if $x \geq 0$, $x < x'$ and $x' = x + 1$.

Natural numbers are individually defined, in the sense that they are first-level concepts whose individual existence is proved through a concrete mathematical method. By contrast, the definition of a complete induction implies that natural numbers become second-level concepts; that is, they are no longer individually constructed objects, but elements of an abstractly defined system. In other words, natural numbers are second-level subsets of a set S, such that a complete induction follows from the following logical inference:

$$\forall S\, ((S0 \wedge \forall x\, (Sx \rightarrow Sx')) \rightarrow \forall x\, Sx).$$

The inference claims that if, for all subsets S, S contains both the number 0 and the number x with its successor x' for all x, then S contains all numbers x. This amounts to replacing the incomplete induction of first-level numbers with the complete induction of second-level numbers, such that the complete set S (including 0, x and x') is isomorphic to the infinite set N of all natural numbers.

A complete induction is a second-level (or second-order) axiom, since it deals with natural numbers defined, not as first-level individuals (or objects), but as second-level subsets (or properties) of a set. It is possible to define the induction axiom as a first-level principle, but if we do so, we must refer to an axiom schema, namely a sentential formula consisting of infinitely many axioms, and each axiom pertains to an arbitrary first-level formula F composed of a single variable x and the constants 0, 1, + such as:

$$(F0 \wedge \forall x\, Fx \rightarrow F(x + 1)) \rightarrow \forall x\, Fx.$$

The inference claims that if a formula is true for a number 0 and true for all x such that it is true for $x + 1$, then the formula is true for all x. Although we deal with a first-level language of formulae, natural numbers are still second-level concepts of the set N. In other words, we cannot define a complete induction for first-level objects since it is impossible to complete

[4] Historically, Dedekind was the first to present a second-level definition of natural numbers. In his *Was sind und was sollen die Zahlen*? (1888, §§4-6), he defines a set K as a chain relative to a mapping ϕ, and the image of K under ϕ is a subset of K. Thus, the natural numbers correspond to an infinite system N, such that N is the chain $\phi_0(1)$ of a basic number 1 which is itself not contained in $\phi(N)$; indeed, 1 is not the successor of any number (in such a case, 0 is not defined as a natural number). This principle was the source of Peano's axioms (as Peano 1889 admits it in his preface).

the infinite succession of individually defined objects. Thus, the infinite of first-level concepts is a potential infinite, i.e. an incomplete sequence of objects which cannot match the abstract property of the Dedekind infinite, i.e. a second-level infinite understood as the bijection of two isomorphic sets. Consequently, only a second-order induction can be complete and implies the definition of an infinite cardinality for the complete set N of all natural numbers.

Likewise, Zermelo-Fränkel (ZF) set theory defines a complete induction for second-level elements, such that elements of a set are abstract subsets and not urelements (cf. section 1.2, footnote 3). Although it is possible to express such second-level elements through a first-level language, I shall deal with second-level axioms, thereby avoiding the construction of first-level axiom schemas. [5] We may define ZF set theory with the following axioms:

1. *Axiom of extensionality*:

   If two sets are the same, they have the same elements.

2. *Axiom of the null set*:

   There exists a set with no element, namely the empty set.

3. *Axiom of pairing*:

   If S and T are sets, then the unordered pair ⟨S, T⟩ is a set containing S and T as its only elements.

4. *Axiom of union:*

   If S is a non-empty set, there is a set T that is the union of all the elements of S.

5. *Axiom of the power set*:

   Every set has a power set meaning that, for any set S, there is a set P that is the (power) set of all subsets of S.

6. *Axiom of regularity:*

   Every non-empty set S contains some element T, such that S and T are disjoint subsets.

[5] The expression 'second-level' is synonym with 'second-order'. Yet, the former expression will be preferred insofar as this thesis focuses on the meanings of such axioms independently of their formal canonical definitions.

7. *(Second-level) axiom of separation*:

   Given any set S and a second-level property P, there is a subset T of S containing the elements x that satisfy the property P.

8. *(Second-level) axiom of replacement*:

   Given any set S and any mapping defining a second-level property P, such that $P(x, y)$ and $P(x, z)$ imply $y = z$, there is a set T that contains the images of the elements of S. That is, the set T is the image of the set S under a mapping P.

9. *Axiom of infinity*:

   Suppose a set S with the empty set in S. Then for every element x in S, the union of the element x and its singleton subset $\{x\}$ (i.e. the second-level property of successor) is in S.

Note that if we want to define a first-level ZF set theory, we replace the property P of the second-level axioms of separation and of replacement with variables defined over a first-level predicate in a sentential formula; and an axiom schema represents infinitely many individual axioms, in which each axiom is expressed by a given formula.

The axiom, important for our present topic, is the axiom of infinity defining an inductive set S through the formal inference:

$$\exists S\, (\{\varnothing\} \in S \wedge (\forall x \in S) \rightarrow (x \cup \{x\} \in S)).$$

The successor x' of x is defined as the union of the element x with its singleton subset $\{x\}$. Thus, $x \cup \{x\} \in S$ in ZF set theory corresponds to $x' = x + 1$ in Peano's arithmetic. This is confirmed by the axiom of pairing (axiom 3), which tells us that the pairing of an element x with its singleton $\{x\}$ constitutes a set, namely the successor x' containing x and $\{x\}$ as its only elements. Besides, the axiom of union (axiom 4) defines the union $\cup$ of all elements as a set. The aim of the axiom of infinity is to postulate a complete induction so that it is possible to claim the existence of the set N of *all* natural numbers. In this sense, Peano's arithmetic of natural numbers is reinterpreted within ZF set theory. We shall see in the next section that the set N is definable by not only a transfinite limit ordinal ω (small *omega)*, i.e. the limit of the infinite sequence of *all* natural numbers, but also the infinite cardinality $\aleph_0$ (*aleph-zero*), i.e. the cardinal number counting the infinite elements of N.

4.2 Transfinite orderings and actual infinites

A complete induction implies a complete infinite sequence of all natural numbers, namely a limit beyond which we are no longer dealing with natural numbers. By applying the axiom of infinity from ZF set theory, we construct the sequence of natural numbers out of zero and the successor of zero, such as:

$1 = 0 \cup \{0\} = \{0\},\ 2 = 1 \cup \{1\} = \{0, 1\},\ 3 = 2 \cup \{2\} = \{0, 1, 2\}$ and so on....

Each natural number is equal to the set of all preceding natural numbers, such that a natural number implies a well-ordered set, and corresponds to an ordinal defined as the order type of a well-ordered set. An order on a set S implies a binary relation $\leq$ on S to itself, such that the relation is, for all a, b, and c in S, *reflexive*, i.e. $a \leq a$, *antisymmetric*, i.e. if $a \leq b$ and $b \leq a$, then $a = b$, and *transitive*, i.e. if $a \leq b$ and $b \leq c$, then $a \leq c$. If we include each natural number in a sequence, we obtain an order composed of ordinal numbers, and two sets with a same order type or same ordinal are defined as order isomorphic. Notice that cardinals coincide with ordinals in a finite sequence $\{1, 2, 3..., n\}$ of natural numbers, such that a cardinal number corresponds to the maximal and limit number n. Yet, cardinals and ordinals are not the same when referred to an infinite domain, since a cardinal number defines the 'size' of an infinite set, while an ordinal number pertains to an ordered infinite sequence.

Cantor (1883, 1897) is the first to have systematised the concepts of infinite ordinals and cardinal numbers. If we define the ordinals of the order sequence of all natural numbers, we imply an infinite limit corresponding to the smallest transfinite ordinal ω (defined as a supertask in section 2.6). The transfinite ordinal ω is the order type of the set of all natural numbers. [6] This ordinal limit is not an element of the infinite sequence, since its definition implies the completeness of the set N, such as:

$$\omega = \{0, 1, 2, \ldots, n, n + 1, \ldots, n + n, \ldots, n \cdot n, \ldots, n^n, \ldots\}$$

The limit ordinal ω implies that the union of the infinite set of all natural numbers has no maximum element; this means that ω is an ideal limit which is meaningful only if we define the set N as an actual infinite. If we follow the intuitive idea of a potential infinite, then the

[6] To learn more about this, see Dauben 1990, Moore 1982, and Rudin 1967.

transfinite ordinal ω is not only unreachable, but also incomprehensible. Thus, the limit ordinal ω defines the abstract principle that the complete infinite is countable up to a limit. Note that a transfinite ordinal is a limit and not a successor, in the sense that it completes mathematical induction. Cantor will then define a principle of transfinite induction, such that the property of successor may apply to transfinite classes of ordinals. Thus, the smallest transfinite ordinal ω corresponds to the infinite class of all natural numbers, and its transfinite successor will be the transfinite ordinal $\omega + 1$. We may then define the infinite order of transfinite ordinals, i.e.

$$0 < 1 < \ldots < \omega < \omega + 1 < \ldots < \omega + n < \ldots < \omega + \omega < \ldots < \omega \cdot \omega < \ldots < \omega^n < \ldots < \omega^\omega < \ldots$$

with $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$

$\omega + n = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega + (n-1)\}$

$\omega + \omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega + n, \ldots\}$

and so on, such as:

$$\varepsilon_0 = \{0, 1, 2, \ldots, \omega, \ldots, \omega + n, \ldots. \omega + \omega, \ \ldots, \omega \cdot n, \ldots, \omega \cdot \omega, \ldots. \omega^n, \ \ldots, \omega^\omega, \ldots\}$$

The transfinite ordinals ω, $\omega + \omega$, $\omega \cdot \omega$, ω^ω and ε_0 are limit ordinals which are not definable as immediate successors. Thus, the limit ordinal ε_0 implies the completeness of the sequence of all transfinite ordinals of order type ω (or ω_0) in the same way that the limit ordinal ω implies the completeness of the infinite sequence N of all natural numbers. [7] This means that a transfinite limit ordinal is never the immediate successor of another transfinite limit ordinal. Only finite ordinals can be the immediate successor of a transfinite ordinal, such as:

$$\omega < \omega + 1 \quad \text{distinct from} \quad \omega = 1 + \omega.$$

The expression $\omega = 1 + \omega$ means that the transfinite ordinal ω is equal to itself independently of any natural numbers (e.g. $\omega = n^n + \omega$), since it already contains all natural numbers. In contrast, $\omega + 1$ defines the infinite sequence of all natural numbers *plus* the finite ordinal one, such that the transfinite ordinal $\omega + 1$ is greater than ω. In other words, ω is a subset of $\omega + 1$. The axiom of separation (axiom 7 in section 4.1) in ZF set theory tells us that given a set S and a property P, there is a subset T whose elements of S satisfy P. Thus, the subset ω pertains to the elements of $\omega + 1$, which satisfy the property of being an element of N (which

[7] Note that the transfinite ordinal ε_0 is still countable; we shall see that the first *uncountable* transfinite ordinal is ω_1.

is not the case for the ordinal 1 in $\omega + 1$). Moreover, the axiom of extensionality (axiom 1) claims that two sets are the same if they have the same elements. Thus, the set ω corresponds to the set N, whose immediate successor is the set $\omega + 1$ in the infinitely countable order of transfinite ordinals.

While ordinals define a 'place' in an order of sets, cardinals describe the 'size' of each set by counting the number of its elements. The smaller infinite cardinality applied to the set N of all natural numbers is $\aleph_0$. A well-known illustration of this infinite cardinality is the so-called Hilbert paradox of the Grand Hotel. A special hotel has an infinite number of rooms, but *all* rooms are occupied. As a new guest arrives, the manager of the hotel asks the guest of room 1 to move into room 2; yet, the guest of room 2 must move into room 3, the guest of room 3 into room 4, and so on. The apparent paradox is that all the guests will be able to find a room, despite the fact that all the rooms were occupied before the arrival of the new guest. It is not a paradox, but merely the counterintuitive effect of the application of the bijection between two infinite sets, such that the infinite set {1, 2, 3, 4, ...} of rooms before the arrival of the new guest is in one-one correspondence with the infinite set {2, 3, 4, 5, ...} of rooms after his arrival. Both sets have the same infinite number of elements, i.e. the same infinite cardinality $\aleph_0$. [8] As {2, 3, 4, 5, ...} is a proper subset of {1, 2, 3, 4, ...}, we can verify the definition of the Dedekind infinite, such that an infinite set is in one-one correspondence with one of its proper subsets; both sets are equinumerous, equipotent or equipollent to each other. We may also imagine an infinite number of new guests arriving at the hotel, i.e. {1, 2, 3, 4, …}, so that the current guests are asked to occupy the even-numbered rooms, i.e. {2, 4, 6, 8,…}. Since both infinite sets are in one-one correspondence, we can conclude that both are infinitely equal with a same infinite cardinality. We could imagine many counterintuitive stories of this kind illustrating the mathematical property of an infinite cardinality based on the definition of a Dedekind infinite or complete induction. The only condition is that all infinite sets be isomorphic to the set N of all natural numbers with the infinite cardinality $\aleph_0$.

Unlike the transfinite ordinal ω, the infinite cardinal $\aleph_0$ does not depend on the ordering

[8] Note that the 'paradox' of the Grand Hotel is interpretable as a ω-task or supertask, i.e. the ideal completion of an infinite number of tasks in a finite time (cf. section 2.6).

of elements; that is, a cardinality never changes under the operations of addition and multiplication, such as:

$$\aleph_0 + n = \aleph_0 \qquad \aleph_0 \cdot n = \aleph_0 \qquad \aleph_0 + \aleph_0 = \aleph_0 \quad \text{and} \quad \aleph_0 \cdot \aleph_0 = \aleph_0.$$

A cardinality applies not only to elements defined as singleton subsets but also to the non-singleton subsets of a set. Thus, ZF set theory defines the axiom of the power set (axiom 5), such that for any set S there is a set P that is the power set of all subsets of S. This means that any set of elements with a given cardinality will have a larger cardinality for the (power) set of all its subsets. For instance, the set {1, 2, 3} has a finite cardinality of three, while the (power) set of all its subsets has a finite cardinality of eight, i.e.

$$\{\varnothing\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}.$$

We may generalise this instance by asserting that the cardinality of the power set P of a set S constitutes the successor cardinal of S, such that the successor cardinal x' of a cardinal x is equal to 2^x. In the above example, the finite cardinality of {1, 2, 3} is equal to three, while its successor cardinal is equal to eight, i.e. $2^3 = 8$. Likewise, the cardinality $\aleph_0$ cannot exhaust the actual infinite, since the power set of the set N of all natural numbers has an infinite cardinality higher than $\aleph_0$, i.e. $2^{\aleph 0}$ defined as $\aleph_1$. We can then define the higher cardinalities of the power sets, i.e. $\aleph_2$, $\aleph_3$, $\aleph_4$, etc., by referring to the original sets with lower cardinalities, i.e. $\aleph_1$, $\aleph_2$, $\aleph_3$, etc.. This amounts to defining distinct actual infinites, and we shall examine whether such abstractions are sound.

To sum up, a finite number, say five, is both a cardinal number defined as the counting of five elements in a set and an ordinal number understood as the fifth element of a sequence. The meanings of ordinality and cardinality do not change when applied to an infinite domain of numbers, since the infinite cardinality $\aleph_0$ still supposes the complete counting of elements, and the transfinite ordinal ω still implies the complete ordering of a sequence. The only difference is that such principles are intuitive and obvious in a finite domain, but wholly abstract and counterintuitive in the Dedekind infinite.

4.3 Infinite cardinalities and the continuum hypothesis

So far, we have been dealing with the set N of natural numbers which constitutes the foundation of number theory. Yet, its scope must be extended to real analysis, i.e. the mathematics of real numbers.[9] The extension of Peano's arithmetic to rational numbers and real numbers implies the definition of the set Q of all rational numbers and the set R of all real numbers. The set Q has the same infinite cardinality as the set N; this arithmetic fact is intuitively surprising but merely derives from the definition of the Dedekind infinite. We already know that the infinite set of all even numbers has the same cardinality as the set N of all natural numbers, since the Dedekind infinite defines an infinite set as equal to any of its proper subsets. Although intuition tells us that there are more natural numbers than even numbers, the natural numbers of the set N are in one-one correspondence with the even numbers of its proper subset. Likewise, the rational numbers of the set Q are in one-one correspondence with the natural numbers of its proper subset N, or with the integers of its proper subset Z. In other words, the sets N, Z and Q have the same infinite cardinality $\aleph_0$.

Can we apply the same reasoning to the set R of all real numbers? Cantor (1874, 1891) shows that the cardinality of the set R is higher than the cardinality of the sets N, Z, and Q, i.e. card (R) > $\aleph_0$. Indeed, a continuum R has an uncountable cardinality **c**, such as $\mathbf{c} = \aleph_1$, which cannot be equal to the countable cardinalities of N, Z and Q. Cantor provides two proofs: the proof of 1874 rests on Dedekind's arithmetic continuity, whereas the proof of 1891 refers to the well-known diagonal argument. I would like to reconstruct these two proofs with my own words in order to show their importance in our understanding of Cantor's and Russell paradoxes (cf. section 4.4).

The first proof of 1874 is based on the initial assumption that the set R of all real numbers is countable.[10] In other words, Cantor aims to demonstrate the uncountability of a continuum through a *reductio ad absurdum*. The continuum R implies the following arithmetic properties:

[9] Textbooks in foundational mathematics explain how the system of real numbers and their operations rest on Peano's axioms. For elementary accounts, see Klein 1932 and Wilder 1965; for a more advanced study, see Hobson 1927.

[10] See Ewald 1996 (pp. 839-43) for a translation of Cantor 1874.

1. R is a linearly ordered set with a relation that is reflexive ($a \leq a$), antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$), and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$).
2. R is an unbounded set since it has no smallest or largest elements, i.e. $(-\infty, +\infty)$.
3. R is a dense set, such that there is always a real number between any two real numbers (it is a property derived from the dense set Q of all rational numbers).
4. R is a continuous set; namely, if R is partitioned into two nonempty, disjoint intervals A and B, such that all real numbers of A are less than all real numbers of B, then there is a boundary number or cut c, such that every real number less than c is in A and every real number greater than c is in B. This amounts to defining arithmetic continuity by means of Dedekind cuts.

Cantor describes the properties of the continuum R by reminding us of Dedekind's construction of an arithmetic real line. If we suppose that an infinite sequence of real numbers is countable, then the set R is not different from a countable dense set; meaning that the cut c must be an *element* of the countable interval A or B. However, this assumption is not verified, for we may show that there will always be an element of the sequence of real numbers that is not an element of the dense interval, whether A or B:

If $a_i \in A \quad b_i \in B$ for some index $i = \{1, 2, 3, \ldots\}$ with $a_1 < b_1$

such as $a_1 < a_2 < a_3 < \ldots < a_n < a_{n+1} < \ldots < b_{n+1} < b_n < \ldots < b_3 < b_2 < b_1$

then there will always be a x which is an element of the sequence of real numbers without being an element a_i of A or b_i of B; that is:

either $a_{n+1} < x \leq b_{n+1}$ for a least upper bound x

or $a_{n+1} \leq x < b_{n+1}$ for a greatest lower bound x.

Therefore, a real number x of the continuous sequence R is a Dedekind cut which is not listed as an element of the dense interval A or B. In other words, the Dedekind cut x is the limit of one of the two dense intervals, namely either the least upper bound of the convergent *increasing* infinite sequence $\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots\}$ or the greatest lower bound of the convergent *decreasing* infinite sequence $\{b_1, b_2, b_3, \ldots, b_n, b_{n+1}, \ldots\}$. Since a Dedekind cut is an element of a continuum without being an element of a countable set, the continuum cannot be isomorphic to a countable set in the sense that their respective elements are not in one-one correspondence. This contradicts Cantor's initial assumption that R is a countable set, which was the aim of this *reductio ad absurdum*. Thus, a continuum is uncountable and has a cardinality higher than $\aleph_0$.

Cantor's first proof is convincing on condition that we believe in the soundness of Dedekind's arithmetic continuity. Indeed, the uncountability of the set R is demonstrated through the construction of a Dedekind cut out of two dense intervals of real numbers. In this sense, the abstract definitions of Dedekind cuts as second-level real numbers, i.e. limits of an infinite interval of first-level real numbers, already imply that the set R of all real numbers cannot be in bijection with a countable set of first-level real numbers (cf. Weyl's criticism in section 5.1). Cantor was undoubtedly aware of this restriction, since his second proof (1891) of an uncountable continuum is based on the countable set N of all natural numbers. Suppose a unit interval [0, 1] of real numbers, and let us assume through a *reductio ad absurdum* that this set is countable and definable through the countable sequence $\{E_1, E_2, E_3, E_4, E_5, \ldots\}$.[11] We may then list the real numbers in [0, 1] through the sequence E such as (numbers chosen in a random way):

$$
\begin{array}{lcl}
E_1 & = & 0.\ \mathbf{2}\ 4\ 0\ 7\ 9\ 1\ 3\ \ldots \\
E_2 & = & 0.\ 8\ \mathbf{0}\ 4\ 2\ 0\ 3\ 5\ \ldots \\
E_3 & = & 0.\ 5\ 9\ \mathbf{6}\ 5\ 2\ 8\ 1\ \ldots \\
E_4 & = & 0.\ 0\ 2\ 8\ \mathbf{4}\ 1\ 6\ 4\ \ldots \\
E_5 & = & 0.\ 6\ 1\ 0\ 9\ \mathbf{0}\ 7\ 2\ \ldots \\
E_6 & = & 0.\ 4\ 3\ 7\ 1\ 5\ \mathbf{0}\ 9\ \ldots \\
E_7 & = & 0.\ 1\ 7\ 5\ 8\ 4\ 6\ \mathbf{1}\ \ldots \\
E_n & = & \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \boldsymbol{\ldots}
\end{array}
$$

The diagonal argument implies that we choose the digits on the diagonal of the above quadrangle (see bold digits) in order to construct a number k according to the following rule: if the digit of the diagonal is 0, then the digit in the number k will be 1; conversely, if the digit of the diagonal is distinct from 0, then the digit in k will be 0. With respect to the above diagonal, we then obtain the number:

$$k = 0.\ \mathbf{0\ 1\ 0\ 0\ 1\ 1\ 0\ \ldots}$$

[11] See Ewald 1996 (pp. 920-922) for a translation of Cantor 1891.

The number k is a real number included in the continuous interval [0, 1] of real numbers. Yet, it will always differ from any numbers listed in the countable sequence E since there will always be a digit in the decimal expansion of the number k which will be distinct from the digits of the decimal expansion listed in the sequence E. In other words, the countable sequence does not correspond to a complete enumeration of all real numbers in [0, 1], which makes its cardinality uncountable, i.e. distinct from the countable cardinality of the sequence E.

The efficiency of this proof derives from its intuitive appeal, insofar as we have the impression to grasp why the infinite cardinality of a countable set cannot make sense of the infinite cardinality of a continuum. Besides, the indeterminacy of irrational numbers provides an intuitive argument to believe that a continuum R is distinct from the sets N, Z and Q, whose elements are all well-defined. However, Cantor deals with proofs based on a *reductio ad absurdum*, namely a proof that infers the consistency of a mathematical principle only through the contradiction of its contrary assumption. Note that any proof of uncountability cannot be other than a *reductio ad absurdum*, since no direct proofs can make sense of the existence of *all* natural numbers in the set N or of *all* real numbers in the set R. This is related to the absence of direct proof for the Dedekind infinite; indeed, a Dedekind infinite set *is* infinite if and only if the axiom of choice is postulated (note that mathematical intuitionism rejects this postulate; cf. sections 4.5 and 4.6).

Therefore, to claim that a continuum has an uncountable cardinality **c**, such as $\mathbf{c} = \aleph_1$, cannot be justified on the fact that the cardinality of R is distinct from the countable cardinality of N. Cantor is aware that his two proofs cannot establish that the infinite cardinality $\aleph_1$ is the successor cardinal of $\aleph_0$. We only deal with a hypothesis, namely the continuum hypothesis (CH) according to which no cardinal number exists between the cardinality $\aleph_0$ of a countable set and the cardinality $\mathbf{c} = \aleph_1$ of an uncountable continuum. At the Second International Congress of Mathematicians in Paris in 1900, Hilbert (1900b) draws a list of unsolved problems, in which the continuum hypothesis is presented as the first problem in need of a direct proof. Gödel (1939) shows that CH does not contradict ZF set theory; then, Cohen (1963) adds that the negation of CH does not contradict ZF either. In other words, CH is an undecidable assumption which is independent of ZF set theory. This is not surprising, since there is no way to prove the truth or falsehood of a transfinite order of infinite cardinalities, such that $\aleph_1$ is the successor cardinal of $\aleph_0$.

Hausdorff (1908) defines a generalised continuum hypothesis (GCH), which claims that no cardinal number lies between the cardinality $\aleph_0$ of a countable set S and the uncountable cardinality $\aleph_1$ of its power set P(S). While CH is restricted to the uncountable cardinality of a continuum, GCH is applied to the uncountable cardinality of any power set of a countable set. We can draw the following equalities:

$$\text{card}(R) = \mathbf{c} = \aleph_1 = \text{card}(P(N)) = \text{card}(P(Z) = \text{card}(P(Q))$$

The countable sets N, Z, and Q have uncountable power sets, i.e. P(N), P(Z) and P(Q). Indeed, an infinite set of all singleton subsets cannot have the same cardinality as the infinite power set of all singleton and non-singleton subsets in the same way that a dense set of all first-level real numbers (rational numbers) cannot have the same cardinality as a continuum of all first-level and second-level real numbers. We may then summarise CH and GCH:

$$\text{CH:} \quad \text{card}(N) = \text{card}(Z) = \text{card}(Q) = \aleph_0 < \aleph_1 = \text{card}(R)$$

$$\text{GCH:} \quad \text{card}(N) = \text{card}(Q) = \aleph_0 < 2^{\aleph 0} = \aleph_1 = \text{card}(P(N)) = \text{card}(P(Q))$$

Since these two hypotheses are independent of ZF set theory, nothing prevents one from defining higher infinite cardinalities, such as the cardinality of the power set of a continuum R:

$$\text{card}(R) = \aleph_1 < 2^{\aleph 1} = \aleph_2 = \text{card}(P(R))$$

We may even define the transfinite sequence of infinite cardinalities by starting with the cardinality of the set N, such as:

$$\{\text{card}(N), \text{card}(P(N)), \text{card}(P(P(N))), \ldots\} = \{\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots\}.$$

$$\text{with} \quad \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots$$

By ordering infinite cardinalities, we may interpret them in relation to transfinite ordinals. The finite ordinals belong to a first class, and the transfinite ordinals of order type ω or ω_0 belong to a second class ε_0 whose all subsets are countable with a cardinality equal to $\aleph_0$. By contrast, the third class ε_1 of transfinite ordinals of order type ω_1 pertains to uncountable sets of cardinality $\aleph_1$. Thus, the sequence of the distinct classes of transfinite ordinals, i.e.

$$\varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots$$

corresponds to the sequence of distinct infinite cardinalities, i.e.

$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots.$$

These higher and higher infinite cardinals and transfinite ordinals express Cantor's paradise of the actual infinites, and despite their hypothetical existences, we shall see that they *necessarily* derive from the definition of an actual infinite.

4.4 Infinite totalities and Cantor's paradox

A power set, which is the set of all infinitely *countable* subsets, must be uncountable in order to avoid an obvious paradox. Indeed, if the power set of all countable subsets were countable, then it would be a subset of itself and could not be the set of all countable subsets. If we say that the power set is uncountable, we avoid this paradox but we encounter a much serious one, i.e. Cantor's paradox, which claims that the power set is an uncountable subset of a countable set, meaning that no actual infinite can be an absolute totality. Therefore, Cantor's paradox is a direct consequence of his proof (1891) that the power set P(S) of a set S has a higher cardinality than the cardinality of S. Cantor uses algebraic terms, when he claims that a function $f(x) = \{x\}$ defined from a domain S to a co-domain P(S) is *injective*. [12] An injective function is neither surjective nor bijective, and means that every element x of a domain S maps a subset $f(x)$ of a co-domain P(S), but not every subset of P(S) is mapped by an element of S. This implies that the domain S is smaller than the co-domain P(S), so that there are more subsets in P(S) than there are elements in S; namely, card(S) < card(P(S)). In order to prove that the function is an injection, Cantor once again uses a *reductio ad aburdum* with the initial assumption that the function f: S → P(S) is a *surjection*, i.e. the very opposite of an injection. This means that all subsets $f(x)$ of P(S) are mapped by the elements x of S, but not every element x of S maps a subset $f(x)$ of P(S), since there are more elements in S than there are subsets in P(S), i.e. card(S) > card(P(S)). The hypothesis unsurprisingly leads to two contradictions, namely:

[12] For the details of the proof, see Stoll 1979 and Takeuti & Zaring 1982.

1. If all subsets of P(S) are mapped by the elements of S, then an element x of S maps not only a subset B $= f(x)$ of P(S) such as $x \in f(x)$ but also the negation of B such as $x \notin f(x)$. Yet, to claim both $x \in f(x)$ and $x \notin f(x)$ is contradictory.
2. If we say that x does not map the negation of B, then the subset ¬B of P(S) is not correlated to any element of S; this means that not all subsets $f(x)$ of P(S) are mapped by the elements x of S, which contradicts the initial assumption that the function is subjective.

Since we obtain a contradiction either way, the initial assumption that the function $f(x) = \{x\}$ from S to P(S) is a surjection must be rejected. In other words, every element x of S maps a subset $f(x)$ of P(S), but not every subset of P(S) is mapped by an element of S, meaning that the cardinality of the domain S is less than the cardinality of the co-domain P(S), i.e. card(S) < card(P(S)).

Cantor's proof is nowadays known as 'Cantor theorem'. [13] This confirms the principle that an infinite set of cardinality $\aleph_0$ has an infinite power set with the higher cardinality $2^{\aleph_0}$ (or $\aleph_1$). The paradox is that the set S is an actual infinite which is not complete in the sense that it contains a power set P(S) as its subset, and the cardinality of its subset is higher than its own cardinality; or put it differently, any set S has a power set P(S) which includes the negation of S. Thus, the set N of all natural numbers is infinitely countable, and its uncountable power set P(N) defines both a countable set N and its negation (i.e. the complement of N). The immediate effect is that there is no limit in the succession of infinite sets of higher cardinalities, which means that it is inconsistent to define the totality of *all* infinite sets. Cantor writes in a letter to Dedekind (28 July 1899):

> $\aleph_0$ means the cardinality of the sets "denumerable" [countable] in the usual sense, $\aleph_1$ is the next greater cardinal number, $\aleph_2$ is the next greater still, and so on; $\aleph_{\omega 0}$ is the one next following (that is, next greater than) all the $\aleph_\nu$ and equals
>
> $\lim \aleph_\nu$ when $\nu \to \omega_0$ [ω_0 or ω is the first transfinite ordinal number]
>
> and so on... If we start from the notion of a definite multiplicity [*Vielheit*] (a system, a totality) of things, it is necessary, as I discovered, to distinguish two kinds of multiplicity... For a multiplicity can be such that the assumption that *all* of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*. As we can readily see, the "totality of everything

[13] Zermelo (1908) canonises the expression 'Cantor's theorem' by reinterpreting Cantor's proof in purely set-theoretic terms.

> thinkable", for example, is such a multiplicity... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into "*one* thing", I call it a *consistent multiplicity* or a "set". (1967, pp. 113-4, original emphases)

Cantor makes a distinction between the consistent multiplicities that are the infinite sets of any given cardinality and the inconsistent multiplicities defined as infinite totalities. In this sense, a power set P(S), i.e. the set of all subsets of S, is not an infinite totality since it is a subset of S; likewise, S is not an infinite totality because of its power set of higher cardinality. Therefore, we can order the infinite sequence of *alephs* corresponding to the infinite cardinalities of consistent multiplicities, namely

$$\{\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_n, \aleph_{n+1}, \ldots, \aleph_\omega, \ldots, \aleph_{\omega+\omega}, \ldots, \aleph_{\omega\cdot\omega}, \ldots, \aleph_{\omega 1}, \ldots\}$$

By contrast, the set *Taw* defined as the totality of all infinite cardinals cannot be consistent; indeed, the power set of *Taw* has a higher cardinality than the own cardinality of *Taw*, which contradicts its definition as an infinite totality. If *Taw* has no cardinal number, then it is not defined as the property of an infinite set, and cannot thereby be an infinite multiplicity. Likewise, Cantor denies any consistency to the set Ω of all finite and transfinite ordinals, since every ordinal has a successor, such as:

$$\{0, 1, 2, \ldots, \omega, \ldots, \omega + \omega, \ldots, \omega \cdot \omega, \ldots, \omega^\omega, \ldots, \omega_1, \ldots\}$$

If Ω is an ordinal, it must have a successor, meaning that it cannot be an infinite totality. If Ω is not an ordinal, then it does not take place in the ordering sequence of ordinals, and cannot be set of all ordinals. Likewise, Burali-Forti (1897a) defines a paradox demonstrating that the set S of all ordinals is inconsistent. If S were such a set, it would have to be an ordinal; yet, it would be its own member, which would contradict the axiom of regularity in ZF set theory. This axiom states that every non-empty set S containing some element T must be such that S and T are disjoint subsets (cf. section 4.1). Hence, S cannot be the set of all finite and transfinite ordinals.

Russell's *Principles of Mathematics* (1903) reinterprets Cantor's (1891) proof that a function f from S to P(S) is injective, but it does so in logical terms. Russell defines a propositional function from x to φ_x, and claims that the negation of φ_x cannot be mapped by x. If it were the case, we will end up with the contradiction that $\neg\varphi_x$, which negates the correlation of x with φ_x, is correlated to x. Russell (1903) writes: "Suppose a correlation of all objects and some propositional functions to have been affected, and let φ_x be the correlate of x. Then 'not-$\varphi_x(x)$,' i.e. 'φ_x does not hold of x,' is a propositional function not contained in the correlation." (p. 367). Since $\neg\varphi_x$ is the negation of a correlate, it cannot be an *element* of the set of correlates, although it is a *subset* belonging to the power set of *all* subsets of the set of correlates. To illustrate the distinction between elements and subsets, we may refer to the empty set $\{\varnothing\}$ that is a subset but not an element. An empty subset is the negation of an element, and as such, belongs to the power set of a non-empty set. [14] In other words, the power set P(S) defines the set of all subsets by including the subsets which negate the elements (or singleton subsets) of S. Russell uses Cantor's proof to define his own paradox. Suppose the set S of all sets that are not elements of themselves, then we reach the following contradiction:

1. If S is not an element of itself, it satisfies the main property for which it is the set of all subsets; in this case, S is an element of itself, and this contradicts the initial assumption.
2. If S is an element of itself, then it does not satisfy the property for which it is the set of all subsets; meaning that S is not an element of itself, contrary to the initial assumption.

Either way, we have the contradiction that S is both an element of itself and *not* an element of itself. We may summarise Russell's paradox with the following inference:

$$\text{If } S = \{x : x \notin x\} \text{ then } S \in S \text{ and } S \notin S.$$

This paradoxical claim is not different from Cantor's paradox that *Taw* is the set of all infinite cardinalities or that Ω is the set of all ordinals. In this sense, Russell's paradox is a particular instance of the general rule that no set of all infinite sets can be an infinite totality. The only way to avoid the contradiction is to define this set as a power set, i.e. the subset of a primitive set whose infinite cardinality is less than the cardinality of its power set. Thus, the set S of *all subsets* that are not elements of themselves is a power set, and this means that it contains at least one subset which negates the property of 'not being an element of itself'.

[14] It is important to distinguish the singleton set $\{0\}$ from the empty set $\{\varnothing\}$; the former has a single element 0 defined as a number, while the latter is devoid of elements. Thus, the power set of $\{0\}$ is the set of two subsets, i.e. $\{\varnothing\}$ and $\{0\}$, while the power set of $\{\varnothing\}$ contains one subset, i.e. the empty set itself. The notation of the empty set as {} may help avoid any confusion.

That is why the set S is both *not* an element of itself and an element of itself, since it contains a subset and its complement.

Russell accepts Cantor's proof that a set has not the same cardinality as its power set, but he intends to use his own distinction between a set and a class in order to circumvent Cantor's paradox. He writes:

> It is necessary to examine more in detail the singular contradiction, already mentioned, with regard to predicates not predicable of themselves... I may mention that I was led to it in the endeavour to reconcile Cantor's proof that there can be no greatest cardinal number with the very plausible supposition that the class of all terms... has necessarily the greatest possible number of members. (1903, p. 101)

Thus, the proper class C of all sets that are not elements of themselves is a consistent class, since a proper class is not a set, and cannot thereby be a subset or an element of a set. However, Russell realises that his own solution does not work, in the sense that the concept of proper class and his doctrine of types are irrelevant to the problem implied by Cantor's paradox. Suppose the proper class C of all sets satisfying a property P, such that P is true for every set of the class C. If C is a proper class, C cannot be a subset or an element of P; but if C is not in P, then it cannot satisfy the property P, and cannot be the class of all sets satisfying the property P. By contrast, if the class C verifies the property P; then C must be a subset of P; but if C is a set, then it is not a proper class. In other words, the notion of proper class is powerless to avoid the inconsistent definition of an infinite totality of sets. The only way to avoid the inconsistency is to claim that the set of all sets is a power set, but this amounts to denying the existence of infinite totalities and accepting Cantor's paradox that all infinite set contain an infinite subset of higher cardinality. Russell himself agrees that his doctrine of types is unable to solve Cantor's paradox, since he writes as a conclusion to his *Principles of Mathematics* (1903):

> To sum up: it appears that the special contradiction [i.e. Russell's paradox] of chapter x is solved by the doctrine of types, but that there is at least one closely analogous contradiction [i.e. Cantor's paradox] which is probably not soluble by this doctrine. The totality of all logical objects, or of all propositions, involves, it would seem, a fundamental logical difficulty. What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic. (p. 528)

Russell seems to believe that his doctrine of types can solve his own paradox about predicates that are not predicable of themselves. Yet, if we understand Russell's paradox as a particular case of Cantor's paradox applying to any infinite set of all sets, we must conclude that the doctrine of types does not even solve Russell's paradox. In other words, the proper class of all sets that are not elements of themselves is as paradoxical as the proper class of all sets of ordinals. [15] We have seen that if a proper class shares a common property with its subsets, then it is a subset contradicting its definition as a proper class; likewise, if Ω is an ordinal, it has a successor and is the subset of another set, which contradicts its definition as the set of all ordinals. On the other, if a proper class does not share the common property of its subsets, it cannot be the class of all subsets satisfying this property; likewise, if Ω is *not* an ordinal, it does not satisfy the property for which it is the set of all subsets. Thus, Russell is confronted to the same problem encountered by Cantor in his foundational mathematics, and it is false to claim that Russell's paradox and his doctrine of types have corrected the paradoxes of the Cantorian 'naïve set theory'.

However, Russell's paradox remains relevant, and its relevance applies, not to Cantor's set theory, but to Frege's Basic Fifth Axiom. This axiom, as defined in Frege's *Grundgesetze der Arithmetik* (1893), relates an object O to each first-level concept (or predicate) P, such as $\forall x\ (Px \leftrightarrow x \in O)$. The extension of P is the object O. This conception is naïve, insofar as it assumes that any logical predicate *comprehensively* defines a set-theoretic object (axiom of comprehension). In this sense, Russell's definition of a proper class shows that a set-theoretic entity does not necessarily imply the construction of an ontological object. Note that Zermelo will avoid the inconsistency of Frege's Fifth Axiom in his set theory by defining an axiom of separation (*Axiom der Aussonderung*, cf. section 4.1) or subset axiom that confines the axiom of comprehension to particular subsets. Thus, for any set S and a property P, there is a subset of S containing the elements x for which Px holds; but the set of all subsets of S, i.e. its power set, is a purely set-theoretic property without extension.

[15] Russell is ready to admit it, since he writes in another passage that the concept of class does not *appear* to solve even his own paradox: "Consider now the whole class of propositions of the form 'every *m* is true', and having the property of not being members of their respective *m*'s. Let this class be *w*, and let *p* be the proposition 'every *w* is true'. If *p* is a *w*, it must possess the defining property of *w*; but this property demands that *p* should not be a *w*. On the other hand, if *p* be not a *w*, then *p* does possess the defining property of *w*, and therefore is a *w*. Thus the contradiction appears unavoidable." (1903; p. 527; see also pp. 367-68). Indeed, the contradiction *is* unavoidable.

4.5 Well-ordered sets and Zermelo's axiom of choice

The continuum hypothesis (CH) and the generalised continuum hypothesis (GCH) follow from Cantor's paradox, in the sense that both hypothetically postulate a succession of infinite cardinalities such that the uncountable cardinality $\aleph_1$ is the immediate successor cardinal of the countable cardinality $\aleph_0$. The only certainty that we may draw is that no actual infinite is complete since all infinite totalities are inconsistent, as it is demonstrated by Cantor's paradox and illustrated by Russell's paradox. Yet, the actual infinite rests on a postulate whose relevance is not accepted by all mathematicians. This postulate is called the axiom of choice (*Axiom der Auswahl*) and defines the infinite set of all disjoint singleton subsets of S. This means that it is always possible to choose one element for each subset of the infinite set S. We may provide a similar definition by using the language of functions, such that there is always a choice function $f(x) = B$ mapping a single element x of the set S to a disjoint subset B of S. The axiom of choice is not an axiom, and is not an intrinsic part of ZF set theory; but it is a required postulate for the arbitrary ordering of any infinite set. Thus, the soundness of the Dedekind infinite rests on the certainty that we will always be able to pick up a singleton subset for each subset of an infinite set. When Zermelo (1904) defines a well-ordering theorem, he resorts to the axiom of choice. The well-ordering of an infinite set S implies that S is well ordered with respect to a binary relation $\leq$, such that each subset of S has a least or first element. Zermelo's theorem makes sense providing that we postulate the axiom of choice; that is, the existence of *all* singleton subsets in S must be granted. [16]

There is an apparent distinction between two cases: on the one hand, the axiom of choice is related to the natural order of an infinite set; on the other, it forces the infinite set to be

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[16] Zermelo (1904) writes: "The present proof [about well-ordering sets] rests upon the assumption that coverings γ [well-ordered sets] actually do exist, hence upon the principle that even for an infinite totality of sets there are always mappings that associate with every set one of its elements, or, expressed formally, that the product of an infinite totality of sets, each containing at least one element, itself differs from zero. This logical principle cannot, to be sure, be reduced to a still simpler one, but it is applied without hesitation everywhere in mathematical deduction. For example, the validity of the proposition that the number of parts into which a set decomposes is less than or equal to the number of all of its elements cannot be proved except by associating with each of the parts in question one of its elements." (1967, p. 141).

well ordered. If we deal with an infinite set N of natural numbers, we imply the axiom of choice which is connected to the naturally well-ordered set N. Thus, N is defined through the order relation $\leq$ such that all non-empty subset have always a least or first element. Since the natural numbers are individual elements, each subset always contains a smallest element; and the choice function f (S) corresponds to the smallest element of each subset S. Yet, the situation seems to be different for the continuum R of real numbers. First, the denseness of this set means that there is always a rational number among two rational numbers, such that a dense subset (0, 1) of rational numbers is not well ordered, since defined as $\{x : 0 < x < 1\}$ it does not contain a least or first number. Fortunately, the continuity of R implies the definition of Dedekind cuts, corresponding to the first or least element for each infinite subset of real numbers. We may then define a choice function f (S) as a Dedekind cut for each dense subset S of the continuum. However, to claim that the continuum is well ordered is an abstract statement purely based on the axiom of choice, insofar as the continuum has no natural order which could have helped us understand its well-ordering intuitively. We are asked to believe in a continuum in which it is always possible to find singleton subsets, i.e. sets whose unique elements are Dedekind cuts; yet, we are unable to provide a concrete proof that it is possible. Consequently, we must make a distinction between the set N of natural numbers and the continuum of real numbers. The set N is naturally well ordered, such that the axiom of choice can always be justified by a concrete proof; that is, we can always prove that every subset of natural numbers has a least or first element. By contrast, a continuum is a well-ordered set only by blindly resorting to the axiom of choice, which tells us that a Dedekind cut is a least element for each subset. Since Dedekind cuts are limits of infinite intervals and not reducible to an individual number, we have no intuitive way to verify whether the axiom of choice is the right postulate to apply. In other words, our beliefs in the soundness of the axiom of choice and of the well-ordered continuum merely follow from our belief in the consistency of arithmetic continuity. If we are sure of the latter, we must accept the former. Russell (1906a, pp. 47-8) provides an analogy to illustrate the axiom of choice. Imagine an infinite number $\aleph_0$ of boots: if we consider the pair of boots as non-identical, such that there is a natural order which makes a left boot distinct from a right one, then the axiom of choice is easily acceptable. That is, we can always choose a left boot as opposed to a right one in the same way that it is always possible to choose a least or first element in a subset of the set N. On the contrary, if the pair of boots is identical (we may consider the 'more convincing' instance of identical socks, although Russell only speaks of identical boots), there is no natural order which enables one to distinguish them from each other. Thus, we must blindly trust the axiom of choice in order to discriminate each of them. Likewise, there is no natural order for a continuum, so that only the axiom of choice can make sense of the arbitrary well-ordering of a continuum.

Yet, I would like to insist on the fact that all sets with infinite cardinalities rest on the axiom of choice. Although it is intuitively easier to be convinced by the well-ordering of the set N than by the well-ordering of a continuum, both equally require the *arbitrary* postulate of the axiom of choice. We must resort to the axiom of choice as soon as we define an actual infinite, and no intuitive and concrete procedure exists to prove that actual infinites are well ordered. Only the potential infinite based on the iteration of a mathematical induction (without the postulate of a complete induction) can avoid the axiom of choice, since the successive and incomplete constructions of individual numbers is the direct proof that the potentially infinite (actually finite) set is well ordered. Therefore, there is a clear distinction to make between a finite set of natural numbers and the infinite set N of natural numbers, insofar the latter, unlike the former, requires the axiom of choice. The set N rests on the arbitrary postulate of a complete induction, whose property of the Dedekind infinite makes it impossible to order it intuitively (cf. section 4.1).

Zermelo's (1904) well-ordering theorem is directly connected to Zermelo's (1908) explicit definition of the axiom of choice, since both principles are equivalent. Zermelo (1908) writes, "A set S that can be decomposed into a set of disjoint parts A, B, C, ..., each containing at least one element, possesses at least one subset S_1 having exactly one element in common with each of the parts A, B, C,... considered." (1967, p. 186). Without postulating the well-ordering of the infinite set S, the above definition would be incomprehensible; thus, each singleton subset of S is in one-one correspondence with a unique element of the subsets A, B, C, ... of S. Since the Dedekind infinite has an arbitrary ordering, we can state the trichotomy law of infinite cardinalities, such that for any infinite cardinals κ and λ, we have:

$$\kappa < \lambda \quad \text{or} \quad \kappa = \lambda \quad \text{or} \quad \kappa > \lambda.$$

As well, the trichonomy law applies to the second-level real numbers (Dedekind cuts) of a continuum, such that for two Dedekind cuts x and y:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y.$$

Without the explicit postulate of the axiom of choice, it would be impossible to order such abstract numbers based on actually infinite sets. Another principle, known as Zorn's Lemma, is equivalent to the axiom of choice and is directly related to the definition of a continuum. Zorn's lemma claims that a non-empty partially ordered set, in which each totally ordered subset (or chain) has an upper bound, has at least one maximal element. Such concepts must first be explained:

1. A partially ordered set or 'poset' is a set P defined by a *partial* order relation $\leq$, such that the relation is reflexive ($a \leq a$), antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$), and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$) (cf. section 4.2).
2. A totally ordered set T implies the same properties as above *plus* a relation of comparability, such as either $a \leq b$ or $b \leq a$.
3. An element u of a partially ordered set P is an upper bound if and only if $a \leq u$ for any element a in P.
4. An element m of a 'poset' P is *maximal* if and only if $m \leq a$ means $m = a$ for an element a in P. In other words, a maximal element is a *least upper bound* in P that is smaller than, or equal to, an upper bound u, i.e. $m \leq u$.

Applied to a continuum, Zorn's lemma means that a continuous partially ordered set of real numbers as divided into two totally ordered subsets of rational numbers has a least upper bound (i.e. a maximal element) which is smaller than, or equal to, the upper bound of a totally ordered subset. For instance, suppose a continuous 'poset' [0, 1] of real numbers, whose totally ordered subsets are [0, 1) and {1}. Since the upper bounds of [0, 1] are greater than or equal to one, its maximal element or least upper bound (or supremum) is equal to one, such as:

$$\sup \{x \in \mathrm{R} : 0 < x \leq 1\} = 1$$

This maximal element is a Dedekind cut that partitions the two subsets [0, 1) and {1}; but a maximal element does not always mean a greatest element. For instance, a set R^{-} of negative real numbers has no greatest element, since there is always a greater element closer and closer to zero. Since the upper bounds of R^- are all real numbers greater than or equal to zero, zero will be both the least upper bound of R^- and the maximal element (but not the greatest negative real numbers). Consequently, Zorn's lemma amounts to postulating that a continuum is a well-ordered set through Dedekind cuts, and a maximal element is as arbitrary as a Dedekind cut.

Accordingly, Zorn's lemma, Zermelo's well-ordered sets, and the axiom of choice postulate the same principle, such that the actual infinite has an arbitrary order not very different from a finite domain. Thus, the equal and unequal are dual properties that indistinctly belong to finite and infinite sets. Note that the axiom of choice is neither true nor false when defined within Zermelo-Fränkel set theory (cf. Gödel 1939, Cohen 1963). In this sense, the axiom of choice is similar to the continuum hypothesis: both are independent claims whose relevance is much broader than set theory itself, for it is directly depending on hypothetical infinite cardinalities. As such, mathematical intuitionism rejects the continuum hypothesis, the axiom of choice, and infinite cardinalities.

4.6 A continuum in becoming and Brouwer's sequence of free choices

Brouwer is the main representative of mathematical intuitionism. As early as 1907, he criticises arithmetic continuity insofar as the abstract completeness of its infinite domain implies the absurd knowledge of all its elements. Brouwer (1907) writes, "The *continuum as a whole* was given to us by intuition; a construction for it, an action which would create from the mathematical intuition 'all' its points as individuals, is inconceivable and impossible" (1975, p. 45, original emphasis; cf. pp. 83-4). He is not the first to object to the Cantorian definition of a continuum. In 1904, Borel criticises Zermelo's axiom of choice for not belonging to mathematics, since it defines an arbitrary choice repeated uncountably many times (1972, pp. 1251-52). [17] In 'La philosophie mathématique et l'infini' (1914), he

[17] See the writings by Borel, Baire, Hadamard, and Lebesgue in 'Cinq lettres sur la théorie des ensembles' released in *Bulletin de la Société mathématique de France* in 1905 (cf. Borel 1972, pp. 1253-65). Apart from Hadamard supporting Zermelo's position, these French mathematicians (along with Poincaré, cf. section 5.1) are called the 'semi-intuitionists', insofar as they criticise the axiom of choice in foundational mathematics but do not renounce to its implicit use in the classical proofs of real analysis.

explains that there is no difficulty in defining a rational number as a periodic decimal fraction, in the sense that its decimal expansion is constructed through the immediate succession of distinct digits. Yet, a real number, other than a rational number, is impossible to define, insofar as an irrational number is a non-periodic decimal fraction whose decimal expansion is incomplete by definition. In other words, to define an irrational number as a definite real number amounts to supposing that we know all the arbitrary digits, and the axiom of choice makes this possible by postulating that the incomplete set of digits is well ordered. Borel rejects this postulate as wholly arbitrary, since no concrete proof has been put forward to verify the so-called complete determination of an irrational number. He describes the uncountable cardinality $\aleph_1$ as a pure symbol devoid of meaning, but still accepts the consistency of the countable cardinality $\aleph_0$ providing that a countable set is defined as an ideal object. In other words, a continuum is a potentially infinite domain, i.e. an endless enumeration of rational numbers (1914, p. 183).

In 'Intuitionism and Formalism' (1913), Brouwer presents the formalist definition of a continuum as opposed to an intuitionist view, which is still influenced by Borel's semi-intuitionist position. Brouwer writes:

> Let us consider the concept: "real number between 0 and 1". For the formalist this concept is equivalent to "elementary series of digits after the decimal point", for the intuitionist it means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations". And when the formalist creates the "set of all real numbers between 0 and 1", these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of the real numbers of the intuitionist, determined by finite laws of construction. (1975, pp. 133-4)

The contrast is between a set and a sequence. The formalist defines the interval [0, 1] as a set-theoretic continuum isomorphic to the infinitely uncountable set R of *all* real numbers. By contrast, the intuitionist defines the interval [0, 1] as an incomplete sequence of rational numbers interpretable as a potential infinite. This amounts to rejecting the existence of real numbers defined as either rationals or irrationals. Brouwer defines "the set of all real numbers" as a meaningless concept, because irrational numbers are incapable of being defined as determinate numbers. Yet, Brouwer (1918-19, 1927, 1928) defines a radical intuitionism distinct from Borel's semi-intuitionism, since he does not even accept the definition of countable subsets. He refers to species of incomplete sequences which cannot be assimilated to the species of complete subspecies. In other words, countable subsets are replaced with 'countably-unfinished' (*abzählbar-unfertige*) sequences. This implies the rejection of Dedekind cuts, since these latter are either least *upper* bounds or greatest *lower* bounds which complete and bound countable subsets either *above* or *below*. Thus, the intuitionist definition of continuity implies the construction of law-like sequences, namely sequences that are not freely generated but follow from the intuitive laws of number theory defining the finite constructions of natural numbers, integers, and rational numbers (i.e. finite decimal fractions). Brouwer uses the concept of a *reduced continuum* to deal with a continuum restricted to a convergent (potentially) infinite sequence of rational numbers, whose ideal limit is a rational number. Yet, a reduced continuum does not apply to the definition of an infinite sequence whose limit is an irrational number. Therefore, the law-like sequences of rational numbers pertain to only a part of the continuum, since irrational numbers are not contained in it. Brouwer then transforms the reduced continuum into a full continuum [*volle Kontinuum*] by adding up 'unfinished elements' to the 'finished elements' of the reduced continuum. This amounts to introducing an element of indeterminacy corresponding to the indeterminate irrational numbers. The aim is to add freely generated sequences to the already existing law-like sequences. They are generated by a spread (*Menge*), i.e. a finite law of construction defining an incomplete sequence of freely chosen numbers, also called an unrestricted sequence of free choices. Brouwer (1918-19) writes:

> A spread is a law on the basis of which whenever an arbitrary [natural] number is repeatedly chosen, each of these choices generates either a definite sign or nothing, or causes the restriction [*Hemmung*] of the process and the definitive destruction of its result; whereas, for each n, after each unrestricted sequence of $n - 1$ choices, at least one number can be given, which, if chosen as the nth number, does *not* cause the restriction of the process. Every sequence of signs generated in this way by the spread (which as a whole cannot be presented as complete) is called an *element of the spread*. (1975, p. 150; original emphases, my translation) [18]

A spread asserts the following property: the repeated choices of symbols (or signs) lead to

[18] Van Dalen 2000 has translated this difficult text, but his translation is both incomplete and inexact. He writes: "A spread is a law, according to which, if over and over again an arbitrary number of the sequence N is chosen, each of these choices either generate a certain sign or nothing, or checks the process and brings about the definite destruction of the result, where for each n after each unchecked sequence of n – 1 choices, at least one number is chosen, that does not check the process. Every sequence of signs that is generated in this way by the spread (which, thus can in general not be represented as finished) is called an element of the spread." (p. 136).

the construction of an unrestricted process, namely an incomplete sequence of n choices defined as the successor of an unrestricted sequence of $n - 1$ choices. Hence, if a given sequence belongs to a spread, so do all its predecessor sequences. A finite spread-construction pertains to an incomplete multiplicity of freely generated fractions which must be understood as indeterminate irrational numbers. In this sense, irrational numbers are only expressible through a finite construction rule, such that a potentially infinite (i.e. actually finite) sequence of freely generated rational numbers produces approximate values. For instance, the transcendental number $\pi = 3.141592654....$ is an irrational number whose approximate values are provided by the unrestricted *finite* (potentially infinite) sequence of free choices pertaining to the successive rational numbers:

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, ..., n\}$$

Such finite decimal fractions are freely generated, in the sense that there is no set-theoretic order or law-like principle which rules the choice of each element of the sequence. Consequently, Brouwer's definition of a continuum is of two kinds. On the one hand, the reduced continuum is produced by the finite sequences of rational numbers whose law-like definition implies the absence of free choices in the construction of their elements. On the other, the full continuum is composed of both law-like and freely generated sequences of rational numbers, such that the latter sequences are incomplete spreads defining indeterminate irrational numbers. Despite its name, the full continuum is incomplete by definition.

The idea of free choices in the construction of sequences is directly connected to Brouwer's (1948a, 1948b) conception of a creating subject identifiable with the human mind of an idealised mathematician. As mathematics is produced through the intellectual generation of mathematical objects, a full continuum is a potential infinite generated by an endless spread; indeed, the mind never ends producing free choice sequences for irrational numbers. Obviously, no axiom of choice is acceptable, since the arbitrary order of the actual infinite is incomprehensible for the mind. Suppose that a creating subject produces a freely generated sequence of rational numbers whose convergence defines a real number ρ (i.e. an irrational number). If we claim that $\rho > 0$, then we must conclude that $\rho < 0$ is a false statement, which implies the definition of an order for real numbers. Likewise, if we assert that $\rho < 0$, it is obvious that $\rho > 0$ is false; or if $\rho = 0$, then neither $\rho > 0$ nor $\rho < 0$ are true. It follows that any statement of order on real numbers merely depends upon supposedly true premises. In contrast, any relation between natural numbers, such as $2 > 1$, is proved to be true independently of the fact that $2 < 1$ is a false statement; we do not say that the former is true only because the latter is false, and vice-versa. Brouwer concludes that no order can be proved to exist between real numbers, since their indeterminateness makes the binary relations $<$, $>$, and $=$ irrelevant. To postulate an order is to make an unjustified statement about a thing that does not exist, and the only way to prove the truth of this postulate is to refer to a *reductio ad absurdum*, i.e. a pseudo-proof pertaining to the pseudo-falsehood of the contrary postulate. In other words, we face virtual orders about virtual real numbers, such that we are able to know that two indeterminate real numbers are not the same, but we cannot claim that the distinction between these two numbers results from a well-ordered set of all real numbers. Therefore, an infinite order, a definite continuum, and a complete induction are not constructively definable, since all depends on the cardinality of an actual infinite that is wholly incomprehensible for the creating subject.

Eventually, Brouwer's intuitionist continuum is a created concept in an endless becoming, such that the indeterminacy of its real numbers excludes the principle of the excluded middle. This means that mathematical intuitionism replaces the axiom of choice with another postulate, and the infinite is no longer actual and complete but incomplete and in becoming. This makes mathematical concepts accessible to intuition, since the potential infinite is always actually finite. However, this intuitionist postulate seems to be as unjustifiable as the axiom of choice. In other words, why should mathematical semantics be altered by the intrinsic limits of the mathematician's mind? We cannot intuitively comprehend the number 10^{857}, but does it mean that its intuitive inaccessibility make it irrelevant? The belief in the actual infinite enriches mathematical formalism through the definition of infinite cardinalities: is it not sufficient to believe in its consistency? We shall answer such questions in chapter 6, but we shall see before that Brouwer is not the only one to criticise the arithmetic continuum of real analysis.

Chapter 5
Intuitive Mathematics and First-Level Continua

Brouwer criticises arithmetic continuity for the sake of an intuitively defined potential infinite; by contrast, other positions base their rejection of an arithmetic continuum upon the meaning of continuity itself. First, Weyl defends a predicative continuity in *Das Kontinuum* (1918); yet, we shall see that he does not provide any mathematical alternative to his criticism of real analysis, which likely explains why he joins Brouwer's intuitionism from 1920 (cf. Weyl 1921). Second, C.S. Peirce defines a true continuum by means of a topological continuity devoid of discrete numbers; still, we shall conclude that his conception of topology as pure geometry is irrelevant on mathematical grounds, although acceptable from a metaphysical point of view. Finally, J.L. Bell rejects real analysis by replacing it with a system of axiom developing a smooth infinitesimals analysis; however, his position runs into problems insofar as it does not confine his axiomatic definitions to the axioms alone. In other words, he wrongly identifies his axiomatic smooth infinitesimals with the intuitive (and non-axiomatic) idea of smoothness, itself correlated with Leibniz's non-axiomatic differential calculus.

5.1 Predicative continuity and Weyl's criticism of Dedekind cuts

Weyl's *Das Kontinuum* rejects Peano's complete induction in the definition of the set N. If a sequence of natural numbers is consistently defined through the iterative property of successor, it cannot be completed in the sense that there is always an immediate successor to any given natural number. In other words, Weyl discards the property of infinite cardinality as non-predicative (or impredicative). This means that a set N of all natural numbers depends on a complete induction that is not predicated from, or definable through, the constructive definition of natural numbers as individual successors. This argument is not new, as it is shown by Russell and Poincaré. Russell (1906b) defines a proposition as predicative if and only if this latter ascribes a property (or predicate) to an object or extension. Thus, the set N is non-predicative if natural numbers are quantified as variables x over a set N independently of objects or extensions, i.e. individual natural numbers. Poincaré (1905, 1906, 1909) assimilates predicativity to definability, and uses Richard's (1905) paradox in order to make sense of this concept. Let us suppose a set N of all natural numbers definable by a sentence E of fifty words; then, we may paradoxically show that a natural number n less than fifty, i.e. included in the sentence E, does not belong to the set N. The justification is that the sentence F describing the number n is not contained in the sentence E. Richard is keen to conclude that the contradiction disappears, providing that the sentence E of the set N is *totally defined* through infinitely many words. [1] Poincaré reinterprets Richard's paradox for real numbers by saying that a totally defined set S of real numbers must include only well-defined numbers, i.e. rational ones; the exclusion of irrational numbers means that the set S cannot be the complete set of *all* real numbers. He wants to demonstrate that the requirement for definability implies the construction of a continuum as an unfinished sequence of well-defined rational numbers. Likewise, Poincaré objects to the completeness of induction defining a set N of all natural numbers, and criticises it in the name of a vicious circle, namely it is paradoxical to postulate an abstract set N whose property of complete induction implies the existence of all natural numbers, even though this property does not provide any concrete means to construct them individually. In other words, Poincaré's notion of predicativity understood as definability implies well-defined concepts referring to individually existing mathematical objects.

In the name of intuition, Weyl (1918) accepts Poincaré's rejection of the set N of all natural numbers based on the complete induction and the Dedekind infinite. He writes:

> I became firmly convinced (in agreement with Poincaré, whose philosophical position I share in so few other respects) that *the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought*—in spite of Dedekind's "theory of chains" [the Dedekind infinite subsets] which seeks to give a logical foundation for definition and inference by complete induction without employing our intuition of the natural numbers... Moreover, I must find the theory of chains guilty of a *circulus vitiosus*. (1987, p. 48, original emphasis)

[1] Peano (1906) disagrees with Richard's paradox; he writes, "Richard's example does not belong to mathematics, but to linguistics; an element that is fundamental in the definition of N cannot be defined in an exact way" (1967, p. 142). Peano stresses that the set N based on a complete induction defines natural numbers, not as individually existing first-level elements, but as second-level variables of N.

This constitutes the starting point on which Weyl founds his objection to real analysis. [2] Mathematics first pertains to natural numbers individually defined through a constant number 0 and the iterative property of succession, to which we apply primitive recursive functions, i.e. addition, multiplication, and exponentiation. We cannot quantify over the complete set of all natural numbers, since mathematical induction defines each natural number as having a successor, such that the sequence of numbers never ends. Weyl, like Poincaré, thinks of the complete induction as viciously circular, since it abstractly defines a set N without ever testing the individual existence of each of its natural numbers. Likewise, integers should not be postulated by a complete set Z, but should intuitively be constructed through natural numbers, such that each integer is the difference between two natural numbers. For instance, the negative integer –2 pertains to the pair of natural numbers (0, 2) such as 0 – 2, itself equivalent to (1, 3) such as 1 – 3, and so on. As well, rational numbers should not be the abstract elements of a set Q. They can rather be constructed through the quotients of two integers, such that the pairs of integers (a, b) with $b \neq 0$ constitutes the quotient or fraction a / b; thus, we construct the negative rational number –(1/2) with the pairs of integers (–1, 2). Such standard constructions imply that numbers are individually defined as mathematical objects prior to the collection to which they belong. In this sense, the respective sets N, Z and Q are nothing more than the natural numbers, the integers, and the rational numbers, all individually defined and reducible to primitive natural numbers.

The case for real numbers is more complex, as they include both rational and irrational numbers. Weyl (1918) starts by assessing Dedekind's concept of real number understood as a Dedekind cut with the following properties (1987, p. 31):

1. A real number c is a non-empty set of rational numbers, such that any rational number r is an element of c if and only if $r \subset c$ (but not every element of c is a rational number).

[2] Weyl writes in the Preface of *Das Kontinuum*: "It is not the purpose of this work to cover the 'firm rock' on which the house of [mathematical] analysis is founded with a fake wooden structure of formalism—a structure which can fool the reader and, ultimately, the author into believing that it is the true foundation. Rather, I shall show that this house is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded." (1987, p. 1).

2. For any rational number r defined as an element of the set c, there is always a rational number r' belonging to the set c such as $r' < r$.
3. For any rational number r defined as an element of the set c, there is always a rational number r^* belonging to the set c such as $r < r^*$.

The first property shows that a Dedekind cut c is a second-level real number defined as a set that bounds or limits an interval of rational numbers, such that any rational number r of this interval is included in c. The other two properties pertain to the denseness of an interval of rational numbers with no smallest element since, for each rational number r, we have $r' < r$, and no greatest element since, for each r, we have $r < r^*$. In other words, a second-level real number c is the limit of a dense interval of rational numbers. Weyl (1918) writes:

> Let M be a bounded set of first-level real numbers. In order to construct its *least upper bound*, we must form a set G of rational numbers to which a rational number r belongs if and only if *there is* a first-level real number belonging to M which is greater than r. This set G has properties *a*), *b*), *c*) [cf. above properties 1, 2, 3] and is therefore a real number, but one of the second level, since in its definition "there is" appears in connection with "a first-level real number" (i.e., "a set of first-level rational numbers" or "a primitive or derived first-level property"). (1987, pp. 31-32, original emphases)

Weyl's definition is strikingly clear, and his distinction between first- and second-level real numbers enables one to stress the abstraction of Dedekind cuts, in the sense that second-level real numbers are non-singleton subsets, i.e. convergent sequences of rational numbers, distinct from first-level real numbers understood as singleton subsets, i.e. rational numbers. Suppose a continuum [0, 1] of real numbers whose one of the subsets is the interval [0, 1). An upper bound u for [0, 1) is an element of [0, 1] which is greater than, or equal to, every element x of [0, 1), i.e. $u \geq x$. Then, a least upper bound of [0, 1) is an upper bound v which is smaller than, or equal to, every upper bound u of [0, 1], i.e. $v \leq u$. The least upper bound is equal to one, and bounds the interval [0, 1) of first-level real numbers (rational numbers), so that the continuum [0, 1] is a second-level real number in relation to first-level real numbers. Note the absolute distinction between the first-level real number {1}, which is a singleton subset of [0, 1], and the least upper bound equal to one, which is a second-level real number, i.e. the limit of [0, 1) defining the complete continuum [0, 1]. Although both have a same numerical value, the second-level limit of a non-singleton subset is incommensurable with a first-level singleton subset (cf. chapter 2).

Weyl stresses the divide between first- and second-level real numbers in order to doubt the soundness of Dedekind cuts. This property assumes that an infinite interval of first-level real numbers can be completed; the problem with this assertion is that a Dedekind cut already postulates that an infinite interval is the subset of a complete continuum. Weyl concludes to an obvious vicious circle, which should lead mathematicians to renounce to Dedekind cuts and arithmetic continuity in the construction of mathematical analysis; he writes:

> The *circulus vitiosus*, which is cloaked by the hazy nature of the usual concept of set and function, but which we reveal here, is surely not an easily dispatched formal defect in the construction of analysis... A "hierarchical" version of analysis is artificial and useless. It loses sight of its proper object, i.e., number. Clearly, we must take the other path—that is, we must restrict the existence concept to the basic categories (here the natural and rational numbers) and must not apply it in connection with the system of properties and relations (or the sets, real numbers, and so on, corresponding to them). In other words, the only natural strategy is *to abide by the narrower iteration procedure*... So a proposition, that every bounded set of real numbers has a least upper bound, must certainly be abandoned. But such sacrifices should keep the path ahead clear of confusion. (1987, p. 32, original emphases)

Weyl rejects all quantifications over infinite sets, which rest on infinite cardinalities that we cannot construct but can only postulate. The impossibility of proving the concrete existence of a complete continuum R implies the immediate irrelevance of Dedekind cuts, and first-level real numbers alone should be defined within a sound continuum. The reduction of *real* analysis to rational numbers, themselves inferred from integers and natural numbers, implies the replacement of the counterintuitive mathematical analysis with an intuitive mathematics based on an *incomplete* induction; and this mathematics is devoid of both arithmetic continuity and abstract real numbers. If Weyl speaks of a least upper bound, he defines it as the ideal limit of a convergent increasing infinite sequence of rational numbers; and there is no possibility to transform this incomplete sequence into a complete continuum. It is a return to Cauchy's (1821) pre-arithmetic definition of continuity, purely based on functions and sequences of rational numbers.

Consequently, Weyl's mathematical analysis is confined to predicative definitions, such that any sequence of numbers is a collection of prior individuals, namely first-level numbers individually defined through induction. Yet, Weyl does not provide any alternative definition of continuity in order to replace the apparent failure of real analysis. He rather favours intuition and seems to appeal to it as a *deus ex machina.* [3] He speaks of an "objectively presented time" whose mathematical and physical content cannot pertain to discrete points, since an individual point is "pure nothingness" and should merely be understood as a "point of transition" (1987, p. 92). Thus, the essence of time defines continuity and temporal points are indefinable owing to the approximate nature of time. We may wonder whether Weyl is really convinced by his metaphysical arguments about time; they may be viewed as a 'fire escape' providing a hasty solution to the mathematical dead end caused by his criticism of real analysis. The fact that he joins Brouwer's mathematical intuitionism in 1920 may retrospectively explain his discomfort to the lack of mathematical solution to his criticism of continuity in *Das Kontinuum* (1918).

5.2 Peirce's potential continuity: its Kanticity and Aristotelicity

C.S. Peirce is opposed to the classical definition of arithmetic continuity, insofar as he rejects the idea that a mathematical continuum may be composed of discrete elements. In this sense, his criticism is much more radical than Weyl's, for he denies that individual numbers can constitute a right foundation for the true principle of continuity. To understand his motivation, we may start with Russell's definition of arithmetic continuity. In his *Principles of Mathematics* (1903), he writes:

> The axiom of continuity itself may be stated in either of the two following forms. (1) All points on a line are limits of series of rational points, and all infinite series of rational points have limits; (2) if all points of a line be divided into two classes, of which one wholly precedes the other, then either the first class has a last term, or the last has a first term, but both do not happen. (p. 438)

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[3] In search for a possible solution to the principle of continuity, Weyl (1918) appeals to intuition or insight at the expense of abstract axiomatised systems. He writes, "Everyone agrees that the much lauded critique to which the nineteenth century subjected the foundations of classical analysis was justified. And, certainly, this critique is responsible for an immense advance in the rigor of thought. But what was positively erected in place of the old is, *if one's glance it directed to the ultimate principles*, even more unclear and assailable than what is replaced... As things now stand we must point out that, in spite of Dedekind, Cantor, and Weierstrass, the great task which has been facing us since the Pythagorean discovery of the irrationals remains today as unfinished as ever; that is, the *continuity* given to us immediately by intuition (in the flow of time and in motion) has yet to be grasped mathematically as a totality of discrete 'stages' in accordance with that part of its content which can be conceptualized in an 'exact' way. More or less arbitrarily axiomatized systems (be they ever so 'elegant' and 'fruitful') cannot further help us here. We must try to attain a solution which is based on objective insight." (1987, pp. 23-4).

We do not deal with a geometric construction *per se*; indeed, points and lines are nothing more than numbers and intervals. More precisely, Russell's statement (1) pertains to Cantor's (1872) definition that a continuum corresponds to a convergent infinite sequence of first-level real numbers (or rational numbers). [4] Convergence implies the definition of a limit, namely the least upper bound of an increasing sequence or the greatest lower bound of a decreasing sequence. As for Russell's statement (2), it refers to Dedekind's (1872) definition of continuity, such that if one interval of rational numbers totally precedes another one, then either the first interval has a least upper bound or the second one has a greatest lower bound. Dedekind's real line is equivalent to Cantor's convergent sequence of real numbers (cf. section 6.2). [5] Both imply the construction of a second-level real number that bounds an infinite interval or sequence, which amounts to attributing a property of completeness to an arithmetic continuum. In other words, a continuum is an infinite set of both first- and second-level real numbers.

Like Russell, Peirce objects to a purely metaphysical definition of continuity as implied by the dominant Hegelian philosophy at the end of the nineteenth-century. [6] He agrees with Russell's conclusion that an arithmetic continuum "may be called by any other name in or out of the dictionary" (1903, p. 353), since its counterintuitive definition makes it independent from the intuitive concepts of ordinary language. Yet, the crucial difference between Peirce and Russell is that the former find this conclusion highly problematic. Indeed, mathematical continuity, in the name of philosophical pragmatism, should be in agreement with the practical domains of intuition and perception. This does not lead Peirce to a mathematical intuitionism *à la* Brouwer, insofar as he accepts the consistency of arithmetic continuity; but its conflict with intuition means that it only expresses a pseudo-continuity (1931-35, CP 6.176). His dissatisfaction with arithmetic continuity is

[4] For a detailed study of Cantor 1872, see Dauben 1971.

[5] To learn more about this, see Courant and Robbins 1969 and Rudin 1976.

[6] Russell (1903) writes humorously: "The notion of continuity has been treated by philosophers, as a rule, as though it were incapable of analysis. They have said many things about it, including the Hegelian dictum that everything discrete is also continuous and *vice-versa*. This remark, as being an exemplification of Hegel's usual habit of combining opposites, has been tamely repeated by all his followers. But as to what they meant by continuity and discreteness, they preserved a discreet and continuous silence; only one thing was evident, that whatever they did mean could not be relevant to mathematics, or to the philosophy of space and time." (p. 287).

related to the intuitive discreteness of real numbers. [7] Peirce wants to reform arithmetic continua in order to make it compatible with the intuitive idea of continuity. It is a matter of suppressing Poincaré's (1893) distinction between an arithmetic continuum that is "nothing but a collection of individuals" and an intuitive continuum that implies "an intimate bond" (*lien intime*) between each of its parts (pp. 26-27). Poincaré means that an arithmetically continuous real line, composed of real numbers, is an intuitively discontinuous collection of prior individuals; by contrast, an intuitively continuous line is a prior whole with respect to indeterminate, unbroken parts. As real numbers contradict the intuitive idea of continuity, this conflicts with Peirce's pragmatist requirement, as defined in 'How To Make Your Ideas Clear' (1878), that mathematical concepts must be grasped in relation to their practical effects on our actions and perceptions. Accordingly, pragmatism should guide mathematicians in the definition of an intuitively continuous mathematical continuum. [8]

Unlike Brouwer and Weyl, Peirce is not interested in restricting mathematical analysis to constructive and intuitively defendable proofs; but he objects to the formalism of mathematical structures owing to their counterintuitive abstractions. Thus, he criticises Dedekind cuts, not because they imply the unjustified postulate of the actual infinite, but because they are indivisible elements contradicting the intuitive idea of continuity. Peirce writes:

> In the calculus and theory of functions it is assumed that between any two rational points... there are rational points and that further for every convergent series of such fractions (such as 3.1, 3.14, 3.141, 3.1415, 3.14159, etc.) there is just one limiting point; and such a collection of points is called *continuous*. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity. (1931-35, CP 6.168, original emphasis)

His criticism of arithmetic continuity relies on the fact that 'rational points' are perceived as discontinuous elements, meaning that the infinite division of a quantity into indivisible points cannot produce an intuitively true continuum. Peirce's analogy with grains of sand

[7] Putnam and Ketner (1992) write: "For Brouwer, it does not even make *mathematical* sense to imagine completing an infinite number of operations. Peirce had no such scruples. For Peirce, completed infinite processes are perfectly conceivable as long as the cardinal number of steps in the process is less than Peirce's ideal limiting cardinal." (p. 50; original emphasis).

[8] Dauben 1982, along with Putnam & Ketner 1992, rightly stresses that the intuitive idea of continuity is the ultimate guide for Peirce's true continuity.

illustrates the discontinuous perceptions of broken points within the arithmetic continuum; and the exclusion of such discrete points from its composition will constitute his main pragmatist claim. He writes: "I agree entirely with James, against Dedekind's view; and hold that there would be no actually existent points in an existent continuum, and that if a point were placed in a continuum it would constitute a breach of the continuity" (CP 6.182). [9] Yet, Peirce does not intend to return to an Aristotelian non-algebraic concept of continuity, such that a line is only composed of intuitively smooth lines. Rather, the pragmatist definition of arithmetic continuity implies that discontinuous points be transformed into smooth points, i.e. "completely merged" points (CP 4.219), so that their individuality or singularity is denied. This is made possible by defining points as potential in the sense that a continuum is a smooth collection of potential points (CP 3.568). The justification for potential points is not mathematical, but derives from the commonsensical argument that a smooth continuum has indeterminate parts. Thus, Peirce rejects the law of excluded middle in the name of intuition. He writes:

> If we are to accept the common sense idea of continuity... we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual... But places, being mere possibles without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity. (CP 6.168)

The law of excluded middle does not apply, since potential points are indistinct parts that are neither true nor false. As they do not exist 'classically', they cannot have classical truth-values. If existence is truth and non-existence falsehood, then potential existence is a middle way corresponding to indeterminacy. In other words, a true continuum is composed of "possibilities of determination", namely general conditions that "permit" the determination of distinct and separate points (CP 6.170, 6.185). Peirce opposes the incomplete, yet smooth, 'supermultitude' to the complete, yet broken, multitude of real numbers (1976, pp. 87-89). The rejection of numbers, whether natural, rational or real, is related to their intuitive discreteness, and a continuum of smooth points must be incomplete,

[9] For Peirce's interpretation of William James' *Principles of Psychology*, see Girel 2003.

since completeness implies the discrete individualisation of *all* points. [10] Peirce writes, "Cantor's definition of continuity is unsatisfactory as involving a vague reference to *all* the points, and one knows not what that may mean" (CP 6.165, original emphasis). The abstraction of 'all the points' pertains to the infinite ordering of all real numbers; yet, this abstract and counterintuitive view contradicts the intuitive idea of continuity.

Peirce defines his potential continuity through two properties, whose odd names are Kanticity and Aristotelicity. Being aware of Peirce's habits to coin his own concepts, we cannot be surprised to encounter such properties; rather, we must attempt to understand why he feels the need to refer to both Kant and Aristotle in order to justify his very personal definition of continuity. I want to show that his chief purpose is to combine arithmetic continuity with a purely geometric continuity. Peirce writes:

> I made a new definition, according to which continuity consists in *Kanticity* and *Aristotelicity*. The Kanticity is having a point between any two points. The Aristotelicity is having every point that is a limit to an infinite series of points that belong to the system. (CP 6.166)

An obvious mistake would be to define Kanticity and Aristotelicity through Kant's and Aristotle's geometric continuity. Indeed, Peirce speaks of points and infinite series of points, while Aristotle and Kant exclude *quantitative* points from their definitions. We have seen previously (cf. chapter 2) that Aristotle's *Physics* defines the parts of a continuous line as quantitative lines, and not as incorporeal points (231b15-16, 232a23-25, and 232b24-25). Kant provides a similar definition in his *Critique of Pure Reason* (1781) when he writes, "The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity" (A 169, B 211). Kant does not seem to take into account the algebraic properties of geometric figures as defined by Leibniz's differential calculus, such that a continuous curve is infinitely divided into infinitesimal points at which we calculate the slopes of tangents, i.e. the derivatives of a function (cf. sections 1.5 and 1.6). A possible explanation is that Kant merely deals with a physical concept of magnitude; indeed, he defines a continuous space as composed of infinitely

[10] Dauben (1982) writes, "The difficulty in describing the continuity of the real line, Peirce believed, reduced to the fact that numbers per se could never account for continuity... Numbers expressed nothing but the order, he believed, of discrete objects. Nothing discrete could possibly be multitudinous enough to account for the continuum." (p. 319).

divisible spatial parts, excluding indivisible points; likewise, a continuous time is composed of infinitely divisible temporal intervals, excluding indivisible instants. [11] Unlike Aristotle and Kant, Peirce admits the presence of points in the composition of a continuum. Although potential points are merged, they are still actualisable into discrete points corresponding to algebraic values; and this is not surprising since a true continuum postulates arithmetic continuity and reinterprets it in conformity with the intuitive idea of continuity. Consequently, Kanticity and Aristotelicity are definable as algebraic properties. If we read Peirce's above quotation carefully, we realise that Kanticity pertains to arithmetic denseness, while Aristotelicity describes the property of convergence. If Kanticity "is having a point between any two points", then it mirrors denseness such that there is always a rational number between any two rational numbers. Likewise, if Aristotelicity is "having every point that is a limit to an infinite series of points", then it matches the limit of a convergent infinite sequence of real numbers. Yet, if Peirce merely refers to denseness and convergence, we may wonder why he coins the strange concepts of Kanticity and Aristotelicity. The answer resides in the subtle (but disputable) combination of arithmetic properties with Kant's and Aristotle's geometric arguments.

The concept of Kanticity defines a property of infinite division, which combines the infinite divisibility of a geometric magnitude with the denseness of an arithmetic interval (CP 6.168). That is, we associate a finite line, infinitely divisible into potential lines, with a dense interval, infinitely divided into rational points; and it follows that Kanticity is the infinite divisibility of potential points. Infinite divisibility belongs to a true continuum, but it is not sufficient for defining true continuity; Kanticity is distinct from true continuity in the same way that denseness is distinct from arithmetic continuity. [12] However, we face an obvious difficulty, insofar as Kant's geometric continuum implies *infinitely divisible* parts,

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[11] Kant writes, "Space and time are *quanta continua*, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space or a time. Space therefore consists solely of spaces, time solely of times. Points and instants are only limits, that is, mere positions which limit space and time." (A 169, B 211, original emphasis).

[12] Peirce makes a distinction between infinite divisibility and continuity and notes that Kant neglects the distinction. He writes: "Kant's definition, that a continuum is that of which every part has itself parts of the same kind, seems to be correct. This must not be confounded (as Kant himself confounded it) with infinite divisibility, but implies that a line, for example, contains no points until the continuity is broken by marking the points." (CP 6.168).

i.e. parts that are themselves divisible into parts; on the other hand, arithmetic denseness defines an *infinite division* of parts, i.e. an infinite number of indivisible points. Thus, infinitely divisible geometric parts contradict indivisible arithmetic parts. Peirce tries to save the argument by blurring this distinction in such a way that potential points are neither arithmetic, i.e. not corresponding to distinct algebraic values, nor geometric, i.e. not being geometric magnitudes. Accordingly, Kanticity is a property combining two contradicting meanings, and the only way to make sense of this property is to believe in the existence of potential points which are neither geometric nor arithmetic.

Kanticity is a necessary but not sufficient condition for potential continuity, insofar as a true continuum implies both Kanticity and Aristotelicity (CP 4.121). Peirce defines Aristotelicity as the combination of Cantor's algebraic convergence with Aristotle's requirement that a geometric continuum has one and only one limit. We know that convergence defines a limit for an infinite interval of real numbers, such that all real numbers of the interval share the same limit. Peirce associates this property with Aristotle's claim in Book V of the *Physics* that the infinitely divisible parts of a geometric continuum share one and the same limit, i.e. the limit of the continuum itself (227a10-13). Then Aristotelicity can be summed up as the property for a true continuum to have a unique limit shared by all potential points. Peirce writes:

> What is required... is to state in non-metrical terms that if a series of points up to a limit is included in a continuum the limit is included. It may be remarked that this is the property of a continuum to which Aristotle's attention seems to have been directed when he defines a continuum as something whose parts have a common limit. (CP 6.122)

The expression 'non-metrical terms' is crucial to grasp that potential points do not correspond to numerical values defined within a two-dimensional metric of Cartesian coordinates. Yet, the property of Aristotelicity is problematic, because neither potential points nor the limit to the series of such potential points have a clear mathematical status. We know that Aristotle defines a point with respect to a continuous line as the potential and incorporeal division of a divisible quantity (cf. section 1.3). In contrast, the algebraic limit of an infinite increasing sequence is a least upper bound, i.e. a second-level real number defining convergence and continuity for the complete sequence. In this sense, Aristotelicity defines a limit which is neither purely geometric nor purely algebraic; and the relevance of this property depends upon the mathematical status of potential points.

Consequently, Kanticity and Aristotelicity are two concepts whose meanings derive from geometric and arithmetic properties. Kanticity is the compound of arithmetic denseness and geometric infinite divisibility, while Aristotelicity is the compound of algebraic convergence and the geometric sharing of a common limit. However, the mathematical consistency of Peirce's true continuum is at stake, since it amounts to combining two conflicting mathematical fields, i.e. real analysis and non-algebraic geometry. The only way to avoid the contradiction is to define an in-between mathematical structure which is neither purely algebraic nor purely geometric. It is exactly what Peirce does by defining a topological structure of smooth points.

5.3 Topological neighbourhoods vs. real numbers

The definition of potential points as non-metrical terms is a direct reference to topology. Peirce defines topology as the highest branch of mathematical geometry, meaning that it subsumes metrical geometry and projective geometry (CP 4.219). Topology must be understood as a non-metrical geometry, also called pure geometry. We may recall that a metric is a distance function *d* corresponding to a non-negative real number, and a metric in an algebraic geometry implies that each point of the metrical space is in one-one correspondence with a real number (cf. section 3.5). By constructing a non-metrical topology, Peirce wants to disconnect points from their algebraic properties, so that he may define them as the smooth and indeterminate parts of a true continuum. As such, they are topological neighbourhoods opposed to singular, discrete, and actual points (CP 4.125, 4.127). Thus, a true continuum of potential points is a smooth collection of topological neighbourhoods, and non-metrical topology is a particular field of mathematics which makes sense of the potentiality of points, such that they are both geometrically smooth and actualisable through algebraic numbers.

Peirce's conception of topology is historically justified, insofar as topology was viewed at his time as a new mathematical field relying more on geometric intuitions than on arithmetic definitions. Topology was perceived as a new kind of geometry which could escape from the formal rigidity of algebraic properties. In this sense, Kanticity and Aristotelicity are defined, not through the conflicting combinations of geometric and arithmetic properties, but through a sound topological structure subsuming algebraic numbers. The real numbers of a classical continuum constitute the mere approximations of potential points or topological neighbourhoods, so that the metrical pseudo-continuity of an arithmetic continuum approximates the non-metrical smoothness of a topological continuum. If topological continuity is intuitively continuous, it ceases to be so when its potential points are actualised through discrete numbers. Peirce concludes about topology:

> Topics [topology] is the study of the continuous connections and defects of continuity of loci which are free to be distorted in any way so long as the integrity of the connections and separations of all their parts is maintained. All strictly pure geometry, therefore, is topics. (CP 4.219)

The continuous connections of loci correspond to places in which potential points do not exist in actuality, as opposed to the actualisation of potential points producing singular points which constitute a breach of continuity. [13] The passage from smooth potential points to broken actual points constitutes the essential property of Peirce's non-metrical topology, insofar as it defines a new primacy for geometry at the expense of algebra. In other words, non-metrical topology defines a pure geometry devoid of actual algebra; indeed, algebra is merely potential. This means that pure geometry becomes algebraic and intuitively discontinuous as soon as the smooth potential points are actualised through discrete algebraic numbers.

However, Peirce's conception of topology is disputable, for it rests on geometric intuitions. I want to dispute the claim that topology is 'pure geometry', i.e. a geometry existing beyond real analysis and the mathematics of numbers. First, Peirce's topological continuity is impossible to generalise, insofar as his definition of a non-algebraic geometry is much too restrictive. When Peirce claims that non-metrical topology is a field on its own that lies on the highest branch of the mathematical tree, he ends us with a topological structure

[13] Peirce adds, "In a continuous locus no point has any individual entity, unless it be a topically singular point, that is, an isolating point, or either the extremity of a line, or a point from which three or more branches of a line, or two or more sheets of a surface extend" (CP 4.219).

that has no connection whatsoever with any other mathematical fields. By contrast, modern topology is not the highest branch, but likely the lowest (and largest) branch common to many mathematical fields. Thus, a topological space is definable as a geometric configuration, i.e. a topological invariance deforming a circle into, say, a triangle, a square or an ellipse, but mainly it may be generalised as a principle of *real* analysis, such that the geometric deformation is defined as a homeomorphism, i.e. a one-one continuous mapping f of one topological space onto another. In this sense, topological continuity is not different from real analysis, since both correspond to an ordering of real numbers. Likewise, the theorems of topology are applicable to the theorems of real analysis. For instance, the topological property of compactness is defined by the following Heine-Borel theorem:

1. A subspace S of a topological space R is compact if and only if a collection of open sets whose union contains S (open cover of S) has a finite subcollection of closed sets whose union also contains S (finite subcover of S).

This topological theorem is not limited to a topological structure, and is translatable into the concepts of real analysis, such that:

2. An interval S of a domain R of real numbers is compact if and only if S is a closed, bounded interval.

Both theorems are equivalent, in the sense that they derive from the same set-theoretic structure of real numbers, and the fact that the former theorem applies to topological subspaces has no effect on its equivalence to the latter theorem. Thus, a topological space is a continuum of real numbers, and topological subspaces are open intervals of rational numbers, e.g. the continuum [0, 1] with its open interval (0, 1). [14] Differential topology is non-metrical, unlike differential algebraic geometry; but this does not mean that it is devoid of algebraic properties. Both belong to general topology (called point-set topology), and are based on the complete structure R of real numbers, whether expressed metrically or non-metrically.

Accordingly, Peirce's true continuity is not mathematically satisfactory, since it rests on

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[14] Note the distinction between closed/open intervals and bounded/unbounded ones. The *closed* bounded interval [0, 1] of real numbers includes 0 and 1, while the *open* bounded interval (0, 1) of rational numbers does not. By contrast, an *unbounded* interval is both open and closed, as the unbounded interval $(-\infty, \infty)$ corresponding to the real line R.

a conception of topology which denies any primary role to the mathematics of numbers, i.e. algebra and mathematical analysis. Its intuitive justification is understandable, in the sense that the smooth parts of a true continuum cannot refer to definite numbers, i.e. distinct and separate elements. Yet, mathematical arguments should not be justified through intuition and perceptual appearance, because, if so, they are impossible to generalise. Thus, topological neighbourhoods, as applied to potential points, are untranslatable into general mathematics. They have no algebraic relevance, insofar as potential merged points are indeterminate and thereby reject the law of excluded middle; that is, their indeterminacy makes them both equal and not equal to a particular number. If equal to a given number, it corresponds to an actual broken point; if not equal to this number, it is actually distinct from it. In both cases, the smoothness of points is gone. Therefore, the intuitive idea of continuity constitutes the main obstacle to the definition of Peirce's true continuum as a foundational mathematical concept. Notice that Peirce has a conception of mathematics which rests on intuitive images and diagrams as opposed to abstract algebra, analysis, and set-theoretic continuity. In this sense, I am defending a position that Peirce condemns as sectarian. He writes:

> Remember it is by icons only that we really reason, and abstract statements are valueless in reasoning except so far as they aid us to construct diagrams. The sectaries of the opinion I am combating seem, on the contrary, to suppose that reasoning is performed with abstract "judgements", and that an icon is of use only as enabling me to frame abstract statements as premisses. (CP 4.127)

His statement is mathematically disputable, insofar as icons and diagrams are unreliable approximations. They are mere illustrations of purely abstract systems, and they can never replace mathematical abstractions. If a true continuum is metaphysically (and empirically) sound as a diagram, its mathematical foundation remains faulty. Thus, Peirce's belief that a collection of potential points is intuitively continuous has no mathematical relevance outside his *ad hoc* topological structure; this means that his concept of topological neighbourhoods belongs to a field of (metaphysical) knowledge that has no connection with usual mathematics. In other words, Peirce's views are defensible only from a metaphysical point of view in the sense that a true continuum pertains to the principle of 'synechism' (*sunechês* meaning 'the continuous'), itself defined as "the absence of ultimate parts in that which is divisible" (CP 6.173). Synechism then implies the metaphysical properties of Kantiticy, i.e.

the infinite divisibility of potential points, and Aristotelicity, i.e. the inclusion of a unique limit shared by all potential points. Such definitions are philosophically pragmatist, for they are directly and practically verified through perceptions. For instance, if a drop of black ink falls upon a white paper such that every part of the paper is perceived as either black or white, then we may say that the coloured surface of the paper have smooth parts definable as the potential points of a true continuum. In contrast, the boundary between the black and white parts of the surface is neither white nor black, since it is composed of actual broken points. Therefore, Peirce's definition of a true continuum verifies the intuitive claim that an indivisible boundary cannot share the smoothness of infinitely divisible parts.

5.4 Bell's smooth infinitesimals and axiomatic analysis

J.L. Bell's (1988, 1995, 1998) smooth infinitesimals are related to Peirce's potential points, insofar as both promote the intuitive idea of continuity. First, we may contrast smooth infinitesimals with Robinson's (1966) non-standard analysis. Robinson defines infinitesimals as non-standard real numbers. That is, the non-standard set R* is an extension of the standard set R of real numbers, such that R* includes infinitesimally small and large numbers. The great advantage of non-standard analysis is that infinitesimals have set-theoretic properties, and are compatible with real numbers. More precisely, the non-standard set R* is an algebraic structure, i.e. an ordered field whose subfield is the standard set R. Thus, the two sets R and R* have common properties, such as:

If $x \neq y$, then either $x < y$ or $y < x$ for $x, y \in \mathrm{R}$ and $x, y \in \mathrm{R}^*$,

and $x < x + 1$ for $x \in \mathrm{R}$ and $x \in \mathrm{R}^*$.

The only difference between R and R* is that no metric space can be defined over R*. The absence of metric is due to the non-standard property of infinitesimals, such that a non-standard real number is either infinitesimally small if less than any positive real number or infinitesimally large if greater than any negative real number. The distinction between real numbers and infinitesimals can be summed up in the following inference:

If x^* is a finite *non-standard* real number with $-n < x^* < n$ for all integers n, then there is a *standard* real number x such that $x^* - x$ is the infinitesimal part.

If we apply Robinson's non-standard analysis to Leibniz's notation *dx*, this means that *dx* is the infinitesimal part of the non-standard real number $x + dx$. The definition is clever, because it does not understand the infinitesimal *dx* as a special number but only as the difference between the non-standard part and the standard one, i.e. $x^* - x$. In other words, a non-standard real number contains a standard part, which means that the elimination of an infinitesimal does not amount to the mysterious suppression of a number, but merely to the reduction of a non-standard real number to a real number of the set R.

Unlike Robinson's non-standard analysis, Bell's smooth infinitesimals analysis is not reducible to real numbers, since it rejects the arithmetic structure of a continuum. [15] More precisely, *A Primer of Infinitesimal Analysis* (1998) defines an axiomatic system, called a 'basic smooth infinitesimal analysis' (BSIA), which rests on a smooth world S whose rules of inference correspond to a quantifier-free intuitionist logic, namely a quantifier-free classical logic devoid of the laws of excluded middle and double negation. BSIA defines a mathematical structure R (distinct from the classical continuum R of all real numbers) which is based on the following four axioms (1998, p. 101-3):

1. Axiom R_1: The structure R is an algebraic field composed of smooth (differentiable) maps and of the specified points 0 and 1.
2. Axiom R_2: The binary relation $<$ on R defines an ordered algebraic field through which the square roots of positive infinitesimals are calculated.
3. Axiom SIA_1 or *Principle of Microaffineness*:

   $$\text{If } \forall f \in R^{\Delta} \;\; \exists!\, a \in R \;\; \exists!\, b \in R \;\; \forall \varepsilon \in \Delta \quad \text{then} \quad f(\varepsilon) = a + b\varepsilon$$

   $$\text{with} \quad \exists!\, x\alpha(x) \equiv \exists x \forall y\, (\alpha(y) \leftrightarrow x = y) \quad \text{and} \quad \Delta = \{x \in R : x^2 = 0\}.$$

   This means that for any map $f: \Delta \to R$ (in the space R^{Δ} of all maps and from the micro-neighbourhood Δ to the structure R), we obtain $f(\varepsilon) = a + b\varepsilon$. This expression calculates, for each smooth infinitesimal ε in Δ, the smooth infinitesimal slope $b\varepsilon$ of the tangent of f at the point a. Microaffineness means that Δ is preserved under transformation.

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[15] Note that Moerdijk and Reyes (1991) present a combined version of Robinson's and Bell's systems, in the sense that they speak of smooth infinitesimals but define them as non-standard real numbers. This means that Moerdijk and Reyes's smooth infinitesimals are reducible to real numbers.

4. Axiom SIA_2 or *Constancy Principle*:

$$\forall f \in R^R \ (\forall x \in R \ \forall \varepsilon \in \Delta \ f(x+\varepsilon) = f(x) \ \rightarrow \ \forall x \in R \ \forall y \in R \ f(x) = f(y)).$$

This means that for any map f: $\Delta \rightarrow R$ (in the space R^R of all maps and from the micro-neighbourhood Δ to the structure R), if there is $f(x + \varepsilon) = f(x)$ for all x in R and all smooth infinitesimals ε in Δ, then $f(x) = f(y)$ for all x and y in R. Thus, if every point in the domain of f is a stationary point, i.e. devoid of smooth infinitesimals, then the map f is constant, i.e. not differentiable.

The first two axioms pertain to the definition of an algebraic structure, while the last two define the essential properties of BSIA, namely the differential map of smooth infinitesimals (axiom SIA_1) and the elimination of smooth infinitesimals for a constant map at a stationary point (axiom SIA_2). From these four axioms, we may derive the following properties for BSIA:

1. A microneighbourhood Δ is composed of smooth infinitesimals and is included in the closed interval [0, 0]; yet, it is not identical with {0}, so that Δ is said to be nondegenerate.
2. Every smooth infinitesimal ε of Δ is neither equal to, nor distinct from, zero; Thus, all ε are said to be indistinguishable from zero.
3. Δ satisfies the Principle of (Universal) Microcancellation, such that:

   If $\varepsilon a = \varepsilon b$ then $a = b$ for all smooth infinitesimals ε in Δ and all a, b in R.

   If $\varepsilon a = 0$ then $a = 0$ for all ε in Δ and all a in R.
4. In a smooth world S, all equations of the differential calculus are equal to:

   $f(x+\varepsilon) = f(x) + \varepsilon f'(x)$ for all x in R and all ε in Δ.

   The quantity $f'(x)$ is the derivative, i.e. the slope of the tangent at a point of the curve of the function f, and the microquantity $\varepsilon f'(x)$ corresponds to the change or exact increment in the value of f when passing from x to $x + \varepsilon$.
5. Every map f in a smooth world S has a derivative f', and the process of forming derivatives can be iterated indefinitely (with higher derivatives f'', f''', etc.).
6. Every map f defined on the structure R is called 'a smooth map' in the sense of possessing derivatives of all orders.

7. As a constant map f is devoid of smooth infinitesimals (cf. axiom SIA_2), the only discontinuous, decomposable parts of the structure R are the empty part ∅ (equal to zero) and the whole structure R itself (equal to one).

Such properties amount to defining smooth infinitesimals as the indistinguishable elements ε of a microneighbourhood Δ, and any differentiable map belonging to the smooth world S of a structure R involves smooth infinitesimals.

The crucial point to grasp is that we are dealing with an axiomatic system, in the sense that the basic smooth infinitesimal analysis (BSIA) is a system of four axioms, from which all theorems, proprieties and concepts are inferred. This implies the construction of axiomatic models, called smooth worlds, containing an axiomatic structure R, micro-neighbourhoods Δ, and smooth infinitesimals ε. The axiomatic consistency of BSIA follows from an absence of contradiction among the four axioms and their deriving theorems. Bell (1998) writes:

> The consistency of smooth infinitesimal analysis is established by the constructions of various *models* for it. Each model is a mathematical structure (a category) of a certain kind containing all the usual geometric objects such as the real line and Euclidean spaces, together with transformations or maps between them. Their key feature is that, within each, all maps between geometric objects are *smooth*, and *a fortiori* continuous. For this reason, any one of these models of smooth infinitesimal analysis will be referred to as a '*smooth world*'; we shall sometimes use the symbol S to denote an arbitrary smooth world. (pp. 4-5; original emphases)

A smooth world S is a category or topos of non-metrical manifolds, defined as a class of related objects between which the preserving maps are called morphisms. Mainly, a smooth world is an axiomatic model, through which smooth manifolds (whether non-metrical smooth infinitesimals or differentiable smooth maps) are axiomatic concepts whose meanings are restricted to the axiom system BSIA. In other words, Bell's reference to a mathematical structure containing "all the usual geometric objects such as the real line and Euclidean spaces" is misleading, insofar as it wrongly implies that an axiomatic model contains intuitively defined mathematical objects independent of the axioms. [16] This cannot be the case, because this would mean that a smooth world is not truly an axiomatic model. It

[16] We shall later see why Bell defines the real line and Euclidean spaces as usual geometric objects. Note that his statement contradicts Dedekind's claim that the concept of a *real* line pertains to a purely arithmetic set of *real* numbers (cf. chapter 2).

follows that if a real line and Euclidean spaces are defined within a model of BSIA, they must be strictly axiomatic concepts, whose definitions pertain to the axioms of BSIA alone. Smooth infinitesimals are indeterminate variables ε belonging to a smooth interval or microneighbourhood Δ, and their definitions in a model of smooth manifolds (called 'Man') only means that they are embedded in a smooth world S, i.e. a smooth topos E of the system BSIA. Bell writes:

> In constructing a smooth world we seek to embed 'Man' in a topos E which does not contain new maps between manifolds (so that all such maps in E are still smooth), yet does contain 'infinitesimal' objects: in particular, an object Δ which realizes the Principle of Microstraightness. Maps in E with domain Δ may then be identified with 'straight microsegments' of curves. (1998, p. 14)

The principle of Microstraightness is equivalent to the axiom SIA_1, i.e. the principle of Microaffineness. Both define smooth infinitesimals as the elements of a microneighbourhood or 'straight microsegment' Δ, which is itself a micro-part of the curve of a map *f*. This microsegment has a slope $b\varepsilon$ at the point a as defined by the smooth map $f(\varepsilon) = a + b\varepsilon$. Thus, a 'straight microsegment' is the microneighbourhood Δ of the axiomatic structure R, and an 'infinitesimal' object is the indistinguishable element ε of the axiomatic interval Δ.

However, Bell constantly combines the axiomatic properties of his system BSIA with intuitively defined concepts. In particular, he identifies the axiomatic consistency of his basic smooth infinitesimal analysis with the intuitive soundness of non-axiomatic infinitesimals methods. He writes:

> These facts [the four axioms] guarantee the consistency of smooth infinitesimal analysis, and so also the essential soundness of (many of) the infinitesimal methods employed by the mathematicians of the past. This is a striking achievement, since the conception of infinitesimal supporting these methods was vague and occasionally gave rise to outright inconsistencies. (p. 14)

Bell is dealing with two radically distinct mathematical objects, and he should not be able to claim that the consistency of axiomatic infinitesimals may justify the relevance of non-axiomatic infinitesimals, as intuitively defined by Leibniz and others in the seventeenth century. My aim is to stress the divide between the consistent axiomatic system BSIA and the non-axiomatic infinitesimals, such that the former has no effect or influence whatsoever on the latter. Smooth infinitesimals are mere variables (called ε, η, ξ, or ζ) ranging over a microneighbourhood Δ and defined by the Principles of Microaffineness and Constancy. By contrast, Leibniz's and Barrows' concepts of infinitesimals are embedded in non-axiomatic mathematical formalisms, and their intrinsic meanings are based on purely intuitive definitions. Nevertheless, Bell purports to show that the consistency of the axiomatic system BSIA leads to the soundness and relevance of infinitesimals *simpliciter*.

5.5 Non-axiomatic smoothness and the intuitive idea of continuity

Bell's claim that the consistency of axiomatic infinitesimals "guarantees" the soundness of historical infinitesimals leads him to expand the axiomatic definition of BSIA to algebraic, geometric and physical interpretations, themselves compatible with the seventeenth-century infinitesimal methods developed by Barrows, De l'Hospital and Leibniz. Bell (1998) writes:

> The role of infinitesimals will be seen to be twofolds. First, as straight microsegments of curves, they play a 'geometric' role, enabling each infinitesimal figure to be taken as rectilinear, and, as a result, ensuring that its area or volume, as the case may be, is a definite calculable quantity. And secondly, as nilsquare quantities, they play an 'algebraic' role in reducing the results of these calculations to a simple form, from which the desired result can be obtained by the principle of microcancellation. (p. 37)

In other words, the microneighbourhood or straight microsegment Δ of BSIA is either geometrically described as a non-null definite quantity or algebraically interpreted as a nilsquare quantity equal to zero, i.e. $\varepsilon^2 = 0$ (p. 9). We must understand Bell's argument as purely speculative, insofar as it is not at all inferred from the axiomatic system itself; yet, he wants to convince us that his axiomatic system has a direct application to non-axiomatic geometry. If we accept this move, we face some difficulties in the sense that the axiomatic concepts of BSIA are transformed into non-axiomatic concepts, whose meanings are no longer definable by the axioms of BSIA. As the consistency of smooth infinitesimals is purely axiomatic, we then lose this consistency when dealing with the non-axiomatic concepts of Leibniz's differential calculus. The consequence is then to be confronted, once again, to the same interpretative difficulties encountered by the traditional methods of infinitesimals. For instance, the elimination of infinitesimals is consistent when defined within the axioms of BSIA, since the principle of Microcancellation, derived from the *Constancy Principle* (axiom SIA_2), asserts that the cancellation of smooth infinitesimals ε results from the constancy of a map f, such that f is a constant function equal to either zero or one. On the contrary, if we deal with a geometric interpretation defining smooth infinitesimals as non-axiomatic quantities, the elimination of infinitesimals becomes as arbitrary as the suppression of infinitesimals in the traditional infinitesimal calculus. Suppose the derivative $f'(x)$ of the function $f(x) = x^2$ calculated with Leibniz's infinitesimal dx, such as:

$$\begin{aligned} f'(x) = (f(x+dx) - f(x)) / dx &= ((x+dx)^2 - x^2) / dx \\ &= (x^2 + 2x\,dx + dx^2 - x^2) / dx \\ &= 2x + dx = 2x \end{aligned}$$

The algebraic result is correct; but the interpretation of the infinitesimal is incoherent, insofar as dx is a non-null infinitesimal quantity which must be discarded in order to make sense of empirical results. That is, there is no mathematical justification for the elimination of infinitesimals. [17] Bell is confronted to the same dilemma as soon as he defines the smooth infinitesimals of BSIA as non-axiomatic geometric concepts. By appealing to geometric intuition, he makes smooth infinitesimals more intuitively accessible; unfortunately, he gets rid of their axiomatic consistency.

Consequently, the objection does not pertain to the axiomatic system BSIA, which happens to be perfectly consistent. There is nothing wrong to define smooth infinitesimals as indistinguishable variables ε ranging over the interval Δ of a structure R, itself defined within a smooth world S, i.e. a topos E. Through BSIA, we have a consistent and complete definition of smooth infinitesimals; and nothing beyond the axioms is required. We may complain that such abstractions are neither easily understandable nor intuitively meaningful, but such criticisms misinterpret the role played by an axiomatic system. Hilbert clearly spells out in his 'Grundlagen der Geometry' (1899) that an axiomatic system pertains to axiomatic elements (e.g. points, lines, planes), whose meanings are defined by the axioms alone independently of any intuitive definitions (cf. chapter 6). This requirement is unconditional,

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[17] We may now understand why infinitesimals have been so successfully replaced with a new formalism successively defined by Bolzano, Cauchy and Weierstrass. Indeed, the epsilon-delta definition of a limit replaces the infinitesimal dx with a real number interpreted as a limit value (cf. chapter 1). There is no need to suppress it, since the limit of a convergent sequence or continuous function is defined as an ideal entity whose consistency is purely mathematical.

insofar as any appeal to intuition implies that the axiom system is both incomplete and superfluous. In other words, we must choose between two alternatives. On the one hand, smooth infinitesimals are axiomatic concepts defined by the axioms of BSIA alone, meaning that no intuitive definitions can make sense of them. On the other hand, smooth infinitesimals have an intuitive content, and are identifiable with the traditional conception of geometric infinitesimals; yet, they cannot be defined through the consistency of BSIA, and their non-axiomatic meaning remains as ambiguous as traditional infinitesimals. It is certain that Bell is aware of such a distinction. Therefore, his aim is to defend an intuitive conception, so that he can conclude that smooth infinitesimals are mathematical objects, beyond their axiomatic definitions. BSIA only provides a logical framework to smooth infinitesimals, whose existence is justifiable through geometry and differential calculus. While BSIA should be regarded as a genuine axiomatic system merely defining axiomatic concepts, Bell interprets it as the formal justification for the non-axiomatic existence of smooth infinitesimals.

In particular, Bell defines a concept of smoothness, whose meaning is purely intuitive and is independent of BSIA. He writes, "The presence of nonpunctiform infinitesimals happily restores to the continuum concept Poincaré's 'intimate bond' between elements absent in arithmetical or set theoretic formulations" (1998, p. 15). We have seen that Poincaré makes a distinction between intuitive continuity and arithmetic continuity, such that the latter, unlike the former, is intuitively discontinuous (cf. section 5.2). Bell interprets the belief in smooth infinitesimals as a reconciliation of intuition with mathematics in the sense that smoothness negates the intuitive discreteness of a continuum, composed of distinct real numbers. Thus, his criticism of arithmetic continuity is similar to Peirce's; Bell writes:

> All mathematical entities [in set theory], being synthesized from collections of individuals, are ultimately of a *discrete* or *punctate* nature. This punctate character is possessed in particular by the set supporting the 'continuum' of real numbers – the 'arithmetical continuum'. As applied to the arithmetical continuum 'continuity' is accordingly not a property of the collection of real numbers *per se*. (1998, p. 2)

His view implies the rejection of real analysis, insofar as any real number is intuitively discrete, and represents a direct threat to an intuitively defined continuity. We may now understand why Bell claims that BSIA is a structure containing "all the usual geometric objects such as the real line and Euclidean spaces" (pp. 4-5). He deals with an old-fashioned mathematics, whose objects are primarily defined as geometric, as opposed to real analysis implying a Copernican revolution such that algebra defined over domains of numbers replaces geometry at the centre stage. Hence, to speak of a real line and Euclidean spaces as geometric objects constitutes a mathematical anachronism. Indeed, a Euclidean space is commonly defined as a n-dimensional topological space based on the set R^n of real numbers, along with a distance function d defined through the algebraic expression:

$d = \sqrt{\Sigma (x-y)^2}$ for the co-ordinates x and y of a two-dimensional set R^2.

Likewise, a real line pertains to a totally ordered set R of real numbers definable as an order topology. In other words, to think of a Euclidean space and a real line as geometric objects deny any mathematical relevance to real analysis.

Both Peirce's smooth points and Bell's smooth infinitesimals imply the rejection of the mathematics of numbers by reference to either a revolutionary topological geometry or a conservative infinitesimal calculus. The consequence, common to Bell and Peirce, is the rejection of the law of excluded middle (cf. section 5.2). Bell writes, "Assuming the law of excluded middle, each real number is either equal to 0 or unequal to 0, so that correlating 1 to 0 and 0 to each nonzero real number defines a function – the 'blip' function – on the real line which is obviously discontinuous" (1998, p. 5). As smooth infinitesimals are indistinguishable elements, they directly conflict with well-defined real numbers. Like Peirce, Bell applies a non-classical truth-condition of indeterminacy. [18] His objection that a 'blip' function of real numbers is "obviously discontinuous" is meaningful only *intuitively*, since it is easy to show that the blip function is *mathematically* continuous; indeed, it is defined over a continuous set of real numbers. We must remind ourselves that the continuity of real numbers rests on the abstract completeness of an arithmetic domain; yet, Bell does not accept this definition, for he defines the smoothness of a function only through its differentiability over smooth infinitesimals. That is, a smooth map is differentiable arbitrarily many times, meaning that its smooth curve has a tangent at any

[18] Bell adds, "The necessary failure in the models of the law of excluded middle suggests that it was the unqualified acceptance of the correctness of this law, rather than any inherent logical flaw in the concept of infinitesimal itself, which for so long prevented that concept from achieving mathematical respectability" (p. 15).

straight microsegments (pp. 5, 26-27). The smoothness of a map follows from the division of its smooth curve into smooth microsegments (and *not* into broken points), at which the slope of each tangent varies through smooth infinitesimals (cf. axiom SIA_1). This principle is consistent within BSIA, but is false if defined independently of the axiom system. Suppose the function $f(x) = |x|$ that is not differentiable at the point a when $a = 0$; this means that no tangent can be defined at the point zero, i.e. there is no derivative $f'(0)$. Bell would conclude that the map $f(x) = |x|$ is not smooth, since it is not differentiable at any arbitrary segment of its curve. Yet, modern mathematics replies that this map is *algebraically* continuous, in spite of not being *geometrically* smooth. Indeed, we can define a limit value for this function, such as:

$$lim\, f(x) = f(a) \quad \text{when} \quad a = 0.$$

The function is continuous, not because its geometric graph is a smooth curve differentiable at any arbitrary point, but because the variation of the argument x produces a variation of the value $f(x)$. More precisely, the function f is continuous at the point a if and only if, for any positive real numbers ε and δ, the value of $f(x)$ satisfies the following inequality:

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon \quad \text{with all } x \text{ such as} \quad a - \delta < x < a + \delta.$$

This claim derives from Weierstrass's definition of a continuous function (cf. sections 1.2 and 1.3), i.e.

$$\text{If} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - f(a)| < \varepsilon.$$

Therefore, if each positive value of x differs from an integer a by less than a number δ, then the value of the function $f(x)$ differs from a real number $f(a)$ by less than an arbitrarily small number ε. The continuity of the function f pertains to the delta-epsilon definition of the limit $f(a)$. The map $f(x) = |x|$ is not *geometrically* smooth, for it is not differentiable at any arbitrary points; yet, it is *algebraically* continuous providing that the map is defined over a continuous set of real numbers. In other words, the algebraic continuity of the function has nothing to do with the geometric smoothness of its graph. We may then conclude that Bell's resort to a smooth map *outside* BSIA is as intuitively vague as it was in the seventeenth century, and that is the reason why this intuitive and purely perceptual geometric criterion has been replaced by the algebraic definition of a continuous function.

5.6 Smooth infinitesimals *simpliciter* and Aristotelian continuity

Bell disregards the axiomatic consistency of BSIA for the sake of intuition, and looks for geometric and physical justifications. He writes, "The *Principle of Microstraightness*... – which will play a key role in smooth infinitesimal analysis – is closely related... to *Leibniz's Principle of Continuity*... Leibniz's principle, in essence, is the assertion that processes in nature occur continuously" (1998, p. 9, original emphases). The property of Microstraightness, derived from the axiom SIA_1, defines the microneighbourhood or straight microsegment Δ as invariant under a microaffine transformation. Yet, Bell intuitively identifies this axiomatic concept with Leibniz's law of continuity, such that the concepts of BSIA refer to non-axiomatic meanings. Bell is even more precise when he adds, "each such model [of smooth infinitesimals] may be thought of as embodying Leibniz's doctrine *natura non facit saltus* – nature makes no jump" (p. 5, footnote 4). In other words, he defines the principle of Microstraightness through a differential calculus; he writes:

> The principle of Microstraightness decisively solves the problem of assigning a quantitative meaning to the concept of *instantaneous rate of change*—the fundamental concept of the differential calculus. For, given a smooth curve representing a physical process, the instantaneous rate of change of the process at a point *P* on the curve is given simply by the slope of the straight microsegment *l* forming part of the curve at *P*: *l* is of course part of the tangent to the curve at P. (p. 10, original emphasis)

To claim that a smooth curve "represents a physical process" is to go even beyond Leibniz's own interpretation of his differential calculus. We have seen that Leibniz's law of continuity is a mathematical abstraction defined in geometric terms, and must thereby be understood as a mathematical law (cf. sections 1.4, 1.5 and 1.6). Bell misinterprets Leibniz's claim that nature makes no leap as a *physical* postulate. The fact that Bell disagrees with Leibniz is not surprising: the former aims to justify smooth infinitesimals on *empirical* grounds, while the latter defines continuity as a *mathematical* law. Thus, the smooth infinitesimals of a geometric curve correspond to the smooth parts of a motion. This means that the instantaneous rate of change has an immediate physical interpretation, such that the slope of a tangent at a straight microsegment is an instantaneous velocity, i.e. a physical *quantity* corroborating the empirical continuity of motion. This explains Bell's triumphant remark, in the above quotation, that the principle of Microstraightness "decisively solves the problem of assigning a quantitative meaning to the concept of instantaneous rate of change".

To accept Bell's interpretation amounts to defending a physicalist interpretation of continuity based on a purely geometric concept of smoothness. We then reject the mathematical abstractions from real analysis and Leibniz's differential calculus. In this sense, Bell's position is close in meaning to Aristotle's conception of geometric and physical continua. The first Aristotelian requirement is that indivisible points cannot be the parts of an infinitely divisible continuum. Bell writes, "We observe that the 'coherence' of a genuine continuum entails that any of its (connected) parts is also a continuum, and, accordingly, divisible. A point, on the other hand, is by its nature not divisible." (p. 3). Smooth infinitesimals are *nonpunctiform* parts of a straight microsegment. This claim is close to Aristotle's position in Book VI of his *Physics*, such that a continuous line is composed of continua, i.e. divisible lines and not indivisible points. The second Aristotelian requirement is that a continuum has potential parts which do not exist in actuality. Bell writes: "Nonzero infinitesimals can, and will, exist only in a 'potential' sense. Nevertheless... this potential existence will suffice for the development of 'infinitesimal' analysis in smooth world." (p. 7). The smoothness of the whole implies the potentiality of its infinitesimal parts, so that parts are not actually distinct from each other. Aristotle asserts that a line is continuous if and only if its parts are potentially divisible and not actually divided (263a23-29). Bell confirms this Aristotelian principle by forbidding the decomposition of a continuum into actually separated parts; he writes, "In a smooth world, a connected continuum X is 'continuous' in the strong sense that its only detachable parts are X itself and its empty parts." (p. 6). A strong sense of continuity pertains to the intuitive idea of smoothness, such that the only separate parts, which disrupt smoothness, are the continuum itself and the empty set. Finally, the third Aristotelian requirement is that geometric continuity underlies physical continuity. Bell writes:

> The principle of Microstraightness yields an intuitively satisfying account of *motion*. For it entails that infinitesimal parts of (the curve representing a) motion are not degenerate 'points' where, as Aristotle observed millenia ago, no motion is detectable (or indeed, even possible!), but are, rather, nondegenerate [nonpunctiform] spatial segments just large enough to make motion over each one palpable. (1998, p. 10, original emphasis)

Bell's explicit reference to Aristotle is quite relevant, since Aristotle claims that the parts of a motion are divisible motions. The principle of Microstraightness pertains to a motion, whose parts are motions, only because a geometric curve has parts which are smooth microsegments and not indivisible points. In other words, microsegment and motion are respectively geometric and physical continua, whose common foundation is the intuitive idea of smoothness. Yet, if it is certain that Aristotle defines an intuitive concept of purely geometric continuity that is relevant to the mathematics of his time, it is impossible to provide the same justification for Bell's position insofar as this latter deals with an axiomatic system. Take Hilbert's axioms of congruence in his definition of a Euclidean axiomatic geometry (cf. section 6.1). It would be misleading to suppose by analogy with Bell's above quotation that the axiomatic concept of congruence "yields an intuitively satisfying account" of the empirical equality of, say, two distances. The reason is that the concept of "physical equality" is purely intuitive and has no connection whatsoever with the axiomatic concept of congruence as defined by Hilbert's Euclidean system. Although the terms look the same, they are unrelated: congruence (or microstraightness) defined by the arbitrary definition of an axiom system does not mirror the physical concept of equality (or motion).

Bell's intuitive interpretation of his own axiomatic system BSIA leads to a dead end, and this is clearly illustrated by his non-axiomatic definition of a microsegment Δ. He writes, "This straight microsegment may be thought of as an infinitesimal 'rigid rod', just long enough to have a slope... but too short to bend. It is thus an entity possessing (location and) *direction without magnitude*, intermediate in nature between a point and a Euclidean straight line." (p. 10, original emphasis). It is impossible to make sense of Bell's intuitive definition. We are asked to believe in a microsegment with the following conflicting properties: it is a non-null quantity but not a magnitude, a straight microsegment but not a straight line, a part of a curve but not a bent part, and an infinitesimal but not an infinitesimal point. In other words, none of the difficulties encountered by the traditional conception of infinitesimals have been solved. We are still unable to explain how *intuitively* speaking a quantity may be both greater than zero and smaller than any definite magnitude. Therefore, we may wonder why bother defining a microsegment Δ intuitively, while BSIA provides us with a perfectly consistent concept. Yet, Bell seems to be interested in smooth infinitesimals only insofar as they correspond to infinitesimals *simpliciter*, i.e. non-axiomatic mathematical objects. If Bell has no problem to define the axiomatic consistency of BSIA, he wants to convince the reader of much more, namely of the sound existence of infinitesimals whose intuitive smoothness confirms the continuity of our physical concepts. Bell succeeds in defining smooth infinitesimals *simpliciter* outside BSIA, but his conception is as obscure as the intuitive definitions of the seventeenth-century mathematics. In other words, it is not exact to claim that smooth infinitesimals "guarantee the essential soundness of (many of) the infinitesimal methods employed by the mathematicians of the past" (p. 14). The only consistent definition is provided by the axiomatic system BSIA, whose smooth infinitesimals are axiomatic elements and not mathematical objects; that is, there is no one-one correspondence between the smooth infinitesimals of BSIA and the traditional infinitesimals of the old-style differential calculus. Eventually, Bell should have resisted the attractive but destructive prospect to make intuitively sense of axiomatic concepts. By doing so, he has endangered the relevance of BSIA, since the identification of BSIA with smooth infinitesimals *simpliciter* makes the axiomatic system redundant. It would have been better not to use an axiomatic system as a pretext to justify the existence of intuitively defined mathematical objects.

Chapter 6
Axiomatic Mathematics and Second-Level Continua

Axiomatic systems are not a recent creation, since the axiomatisation of mathematics goes back to the end of the nineteenth century. Hilbert (1899) was the first to systematise the axiomatisation of Euclid's geometry with some radical consequences for the consistency of mathematical concepts and the existence of mathematical objects. The rupture from intuition is complete since an axiom system provides abstract definitions whose consistency is relative to the system itself; and this amounts to constructing second-level mathematical models. Frege is one of the strongest opponents to Hilbert, but his defence of a traditional mathematics fails to assess the intrinsic value of a foundational axiomatisation, which transforms the *intuitive* concepts of Euclid's geometry into the *arbitrary* axioms of a Euclidean model, isomorphic to a Cartesian algebra. We shall see that arithmetic continua, interpreted in axiomatic terms, extend themselves to the whole of isomorphic mathematical models, and its consistency depends upon a finitary syntax, i.e. a first-level metamathematics. Although a meta-mathematical axiomatic system is not *logically* complete, it is *mathematically* so since it defines a complete second-level semantics, namely categorical mathematical models. Therefore, an axiomatic continuum is a consistent mathematical abstraction, but its second-level semantics defines it only as an ideal property, and *not* as a mathematical object embedded in a first-level logic or set theory.

6.1 Euclidean model of geometry vs. Euclid's intuitive geometry

In his *Grundlagen der Geometrie* (1899), Hilbert transforms Euclid's definitions, postulates and common notions into a system divided into five groups of axioms, such that each group is characterised by a particular relation, namely:

(I) A relation of belonging for the eight axioms of incidence.

(II) A relation of betweenness for the four axioms of order.

(III) A relation of equality for the five axioms of congruence.

(IV) A relation of parallelism for the single axiom of parallels.

(V) A relation of completeness for the two axioms of continuity.

Hilbert aims to show that mathematical concepts do not rely on a same foundation as the concepts of ordinary language, since the latter have intuitive meanings which are not well defined. This argument constitutes a radical criticism of the mathematical foundation of Euclid's geometry. Hilbert is dissatisfied with Euclid's so-called axioms, insofar as they are mere postulates derived from intuitive conceptual definitions. Indeed, Book I in Euclid's *Elements* presents a collection of twenty-three definitions pertaining to geometric concepts, and these definitions are explicit generalisations of what intuition, common sense and perception regard to be true; for instance (Heath 1908, pp. 153-4):

(1) A point is that which has no part (*sêmeion estin, hou meros outhen*).

(2) A line is a breadthless length (*grammê de mêkos aplates*).

(3) The extremities of a line are points (*grammês de perata sêmeia*).

(4) A straight (*eutheia*) line is a line which lies evenly with the points on itself.

(5) A surface (*epiphaneia*) is that which has length and breadth only.

(10) When a straight line set up on a straight line makes the adjacent angles (*tas ephechês gônias*) equal to one another, each of the equal angles is right (*orthê*) and the straight line standing on the other is called a perpendicular (*kathetos*) to that on which it stands.

(15) A *circle* (*kuklos*) is a plane figure contained by one line [circumference] such that all the straight lines falling from one point [centre], among those lying within the figure, are equal to one another.

(23) Parallels (*parallêloi*) are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Are these definitions true? Our intuition answers that they are, in the sense that we intuitively imagine a point as a partless entity, which is opposed to a quantitative line defined as a breadthless length. Are these concepts well defined? The question is difficult to answer, for we do not have a proper definition for each property. We know the intuitive meanings of a point and a line, but their respective definitions merely generalise what we perceive. In other words, point and line are respectively partless and breadthless, only because they look like so. Likewise, the definition of a perpendicular, a circle, or a parallel corroborates what we already know through perceptions, namely the intersection of two perpendiculars at a right angle, the roundness of a circle or the separation of two parallels. In this sense, we do not learn anything new, as the definitions are the mere repetitions of perceptual facts. It would be like defining a house as a three-dimensional rectangle with a door, two windows, and a roof; this apparently true definition does not make us learn anything new, which was not previously known through perceptions.

Consequently, Euclid's geometry rests upon the concrete constructions of drawings with a straightedge and compass, such that his so-called axioms (Heath 1908, pp. 154-5) are mere rules of construction, i.e.

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously (*kata to suneches*) in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Such postulates define some relations between concepts in the sense that they are basic rules required for the concrete drawings of lines, circles, right angles and parallels. [1] We successively learn how to draw a line out of points, how to extend a finite line continuously (in a potential infinite), how to generate a circle out of a centre and radius, that all right angles are equal, and that two lines are not parallel if a third one intersects them with two interior angles less than two right angles. Yet, if we do not know the definition of each geometric concept, we cannot make sense of such postulates; and the truth of the postulates rests on true definitions, which explains why the twenty-three definitions precede the five

[1] Half-century after Aristotle, Euclid feels no need to define the concept of continuity (*sunechês*); he mentions it once in the *Elements*, i.e. in the second postulate.

postulates. Thus, to draw a straight line "from any point to any point" (postulate 1) implies the definitions of a point as "that which has no part" (definition 1) and of a line as "a breadthless length" (definition 2). Likewise, the description of a circle with centre and radius (postulate 3) is based on the definition of a circle as a plane figure, such that all lines drawn from the centre of the circle to its circumference are equal to each other (definition 15). Therefore, Euclid's postulates depend upon concepts, whose definitions derive from intuition.

Hilbert provides a new mathematical foundation to geometry, insofar as it does not rely on perceptual descriptions. Axioms are abstract definitions, which have no connection whatsoever with intuitive conceptual definitions. This constitutes the crucial distinction between Hilbert's Euclidean geometry and Euclid's geometry. Hilbert (1899) writes:

> Consider three distinct systems of objects. Let the objects of the first system be called *points* and denoted A, B, C, ... ; let the objects of the second system be called *lines* and be denoted *a*, *b*, *c*,... ; let the objects of the third system be called planes α, β, γ,... The points, lines, and planes are considered to have certain mutual relations, denoted by words like "lie", "between", "congruent". The precise and mathematically complete description of these relations follows from the *axioms of geometry*. (1971, p. 3, original emphases)

While Euclid deals with intuitive concepts, Hilbert constructs an arbitrary system of axioms which defines relations between elements without referring to conceptual definitions outside the system. This systematic approach means that axioms replace any external foundation, whether Kant's *a priori* intuition (cf. section 6.3) or Mill's *a posteriori* experience. [2] Hilbert's axioms are sufficient to provide all the tools required to construct a Euclidean model of geometry, which is not the case of Euclid's postulates. In other words, concepts, relations, and objects have some meanings only insofar as they belong to the axioms. For instance, Hilbert's axioms of order define a relation of betweenness which does not mirror its intuitive concept; and we must apply this caveat to all concepts included in the following

[2] The distinction between Euclid's geometry and a Euclidean model is often neglected; and it is regrettable, insofar as this contrast makes us understand why the question whether Euclid's parallel postulate is deducible from his other postulates is no longer an interesting problem. Indeed, the Euclidean axiomatisation of geometry easily allows one to construct a Non-Euclidean model, i.e. a model that fails to satisfy Hilbert's axiom of Parallels (axiom IV-1). Thus, the ultimate objection to Kant's synthetic *a priori* intuition of space follows, not from the construction of Non-Euclidean geometries, but from the definition of Euclidean axiomatic models.

four axioms of order (1971, p. 5):

II-1 If a point B lies between a point A and a point C, then the points A, B, C are three distinct points of a line, and then B also lies between C and A.

II-2 For two points A and C, there is always at least one point B on the line AC such that C lies between A and B.

II-3 Of any three points on a line there is no more than one that lies between the other two.

II-4 Let A, B, C be three points that do not lie on a line and let *a* be a line in the plane ABC which does not meet any of the points A, B, C. If the line *a* passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.

The simplicity of such axioms is misleading, since it is tempting to verify such axioms by drawing points, lines, and planes on a blackboard. Yet, such mathematical concepts are only abstract elements of the system. That is, we know that A, B and C are all points only because they are elements of a same collection, but we do not know anything about the conceptual definition of a point; and this matters little, since the meaning of this concept is irrelevant. The only important thing to know is that a relation of order pertains to the axiomatic elements, such that (to compare with the above axioms):

II-1 If B is between AC, then B is between CA

II-2 If AC implies that C is between A and B, then AB implies AC and CB

II-3 If A, B and C belong to a same line, then either A is between BC or B is between AC or C is between AB.

II-4 If a line *a* is in a plane ABC and if *a* passes through AB, then *a* passes through either AC or BC.

We may conclude that the relation of betweenness about points and lines is symmetric (II-1), transitive (II-2), and exhaustive (II-3), and such properties also apply to the relation of betweenness regarding a line and plane triangle (II-4). These four axioms express a same relation based on three collections of objects, i.e. points, lines, and a plane; but such axiomatic meanings do not rely on the intuitive concepts of ‘between’, ‘point’, ‘line’, and 'plane'. Some speak of implicit definitions as opposed to explicit ones. [3] We may say that concepts are 'implicitly characterised' through axioms, in the sense that meanings are axiomatic and not intuitive; but this does not mean that Hilbert deals with a contrast between implicit and explicit definitions. [4] Indeed, Hilbert rejects the possibility of explicit definitions in foundational mathematics, for there is nothing beyond the axioms themselves. To speak of an implicit definition seems to admit that there are some explicit definitions for mathematical concepts, as though implicit definitions were incomplete. In other words, the claim that axiomatic definitions are implicit ones distorts Hilbert's view, since it amounts to postulating the possible existence of non-axiomatic mathematical objects whose definitions are explicit (as believed by Frege, cf. section 6.3). As Hilbert thinks of axioms as definitions *simpliciter*, he naturally concludes that no other kind of definition (whether intuitive or explicit) is required for the understanding of mathematical objects. Therefore, a Euclidean model defines axioms about points in such a way that the mathematical definition of point is *exclusively* axiomatic. This explains why axiomatic meanings transform Euclid's geometry, such that a well-defined axiomatic foundation replaces the vague references to intuition; and we shall fully understand the abstraction of a Euclidean model when it will be defined as isomorphic to an algebraic model.

6.2 Hilbert's axiomatic continuity and Archimedean completeness

In his *Über den Zahlbegriff* (*On the Concept of Number*, 1900a), Hilbert defines two possible methods in the construction of foundational mathematics. On the one hand, a genetic method pertains to a loose collection of axioms without the definition of a system, such that any new mathematical properties imply the adjunction of new axioms. This method

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[3] Shapiro (1997) identifies Hilbert's axiomatic definition with an implicit definition. He writes, "Hilbert (1899) does not contain the phrase 'implicit definition', but the book clearly delivers implicit definitions of geometric structures." (p. 158).

[4] Bernays (1922a) writes, "The words 'point', 'line', 'plane' serve only as names for three different sorts of objects, about which nothing else is assumed directly except that the objects of each sort constitute a fixed determinate system. Any further characterization is carried out only through the axioms. In the same way, expressions like 'the point A lies on the line *a*' or 'the point A lies between B and C' will not be associated with the usual intuitive meanings; rather these expressions will designate only certain, at first indeterminate, relations, which *are implicitly characterized* only through the axioms in which these expressions occur." (1998, p. 192, original emphasis).

is illustrated by Peano's arithmetic, in which five axioms define the complete induction of the set N of all natural numbers (cf. section 4.1). Yet, the definitions of the set Q of rational numbers and the set R of real numbers extend Peano's arithmetic through new axioms. Hilbert admits the pedagogical value of the genetic method through the gradual extension of a collection of axioms, but he rejects this method in the context of a foundational mathematics. He rather defends the axiomatic method based on a complete axiomatic system which cannot be extended. In this sense, a Euclidean geometry is a complete model of axioms, such that all axiomatic concepts are defined by the axioms themselves.

The notion of completeness is the central property of an axiomatic system, and is defined by the two axioms of continuity corresponding to the group V of axioms in Hilbert's Euclidean model (1971, p. 26), namely:

V-1 *Archimedes' axiom*: If AB and CD are any segments, then there exists a number n such that n segments CD constructed successively from A on, along the ray from A through B, will pass beyond the point B.

V-2 *Axiom of line completeness*: It is not possible to extend the system of points on a line with its order and congruence relations in such a way that the relations holding among the original elements, as well as the fundamental properties of line order and congruence following from Axioms I-III and V.1, are preserved. [5]

Archimedes' axiom defines two linear segments AB and CD, such that one exceeds the other by being a multiple of it. We have then a finite number of points, i.e.

$A_1, A_2, A_3, \ldots, A_n$ in a segment AB

so that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, and so on.

If we claim that CD is equal to AA_1, and that AA_1 is equal to A_1A_2, itself equal to A_2A_3, and so on until we reach the segment $A_{n-1}A_n$, then we may conclude that B lies between A and A_n. The *potentially infinite* extension of segments is *actually* finite, since the axiom of line completeness fixes a limit beyond which the axioms of incidence, order, congruence, along

[5] Archimedes' axiom is nowadays attributed to Eudoxus (cf. Boyer and Merzbach 1987). Besides, Hilbert's Axiom of Line Completeness is absent from the first German edition of *Grundlagen der Geometrie* (1899). It appears in the French translation (1902a), and is successively included to the English translation (1902b) and the second German edition (1903). See Moore 1988 and Hallett 1990.

with Archimedes' axiom, are no longer consistent. There is a clear distinction to make between the two axioms of continuity, insofar as the second axiom is much more important than the first one. Archimedes' axiom explains only the extension of a system of points embedded in line segments, while the axiom of line completeness defines a complete axiomatic model. Hilbert (1902a/b, 1903) writes, "The requirement of continuity has been decomposed into two essentially different parts, namely into Archimedes' axiom, whose role is to prepare the requirement of continuity, and the completeness axiom which *forms the cornerstone of the entire system of axioms*" (1971, p. 28, original emphasis). Thus, a complete axiom system implies an infinite number of isomorphic models, which all preserve the axiomatic relations between their elements. The well-known instance is the isomorphism between a Euclidean model of geometry and a Cartesian model of algebra, such that the one-one correspondence between the elements of each model preserves the relations of the axiomatic system. If we deal with Euclid's intuitive definitions of points and lines, we are unable to make sense of such equivalences. By contrast, an axiom system defines a set of elements, which is indistinctly definable as either a collection of points or a collection of numbers.

The completeness axiom demonstrates that a Euclidean geometry is *not* different from a Cartesian algebra, in the sense that both pertain to a complete domain of elements, semantically identified with the complete domain of real numbers, i.e. the definition of an arithmetical continuum. Hilbert (1902a/b, 1903) writes:

> The completeness axiom is not a consequence of Archimedes' Axiom. In fact, in order to show with the aid of Axioms I-IV that this geometry is identical to the ordinary analytic "Cartesian" geometry, Archimedes' Axiom by itself is not sufficient.... On the other hand, by invoking the completeness axiom... it is possible to prove the existence of a limit that corresponds to a Dedekind cut as well as the Bolzano-Weierstrass theorem concerning the existence of condensation points; hence this geometry turns out to be identical to Cartesian geometry. (1971, p. 28)

Hilbert's reference to the Bolzano-Weierstrass theorem is merely another way to define arithmetic continuity. This theorem, associated with the Heine-Borel theorem (cf. section 5.3), asserts that that every convergent bounded sequence of real numbers contains a convergent bounded subsequence. A subsequence is bounded if it has either a least upper bound if bounded above or a greatest lower bound if bounded below. Then, infinitely many bounded subsequences have infinitely many limit points (called condensation or accumulation points), such that a closed Cauchy (bounded) sequence of real numbers (or continuum) contains the limit points of all its convergent subsequences; and the continuum itself is convergent since all its subsequences converge towards the same limit. We may then show that Weierstrass's continuous function is nothing more than the definition of a convergent Cauchy sequence of real numbers x_n. That is, for every positive integer n less than M such as $a < b$, the increasing infinite sequence of real numbers defines a least upper bound L such as:

$$\text{If } \quad 0 < b - a < M \quad \text{then} \quad x_b - x_a < L \quad \text{with} \quad M > 0 \quad \text{and} \quad L > 0.$$

$$\text{or} \quad \text{if } \quad 0 < b - a < \delta \quad \text{then} \quad x_b - x_a < \varepsilon \quad \text{with} \quad \delta > 0 \quad \text{and} \quad \varepsilon > 0.$$

This is the epsilon-delta definition of a limit (cf. sections 1.1, 1.2 and 5.5), which enables one to construct a continuous metric function $d\,(x_a, x_b)$ (cf. section 3.5). The multiple references to Bolzano, Weierstrass, Cantor, and Dedekind do not mean that their respective definitions of continuity are essentially distinct; they are equivalent, insofar as they pertain to the same property of Dedekind-completeness applied to an arithmetic domain of real numbers. We may summarise such definitions in a chronological order:

1. *Bolzano's definition (1817)*: Every bounded sequence has bounded subsequences with lower and upper limit points (called condensation points).
2. *Weierstrass' definition (1861)*: Every infinite Cauchy (bounded) sequence of real numbers has either a least upper bound (if limited above) or a greatest lower bound (if limited below).
3. *Cantor's definition (1872)*: Every infinite Cauchy sequence of real numbers is a convergent sequence in the set R of all real numbers.
4. *Dedekind's definition (1872)*: For two disjoint, increasing subintervals A and B of the system R of all real numbers, there is a Dedekind cut c in R such as for all elements $a \in A$ and $b \in B$: either $a < c \leq b$ for a least upper bound c

   or $a \leq c < b$ for a greatest lower bound c.

These definitions are equivalent for the sake of mathematical generalisation, although, from a historical point of view, Bolzano, and at a less extent Weierstrass, did not have a concept of arithmetic continuity as elaborate as Cantor and Dedekind. We may synthesise these four definitions through the completeness of a continuous domain R of real numbers, whether defined as a set, sequence or interval. Completeness is equivalent with the properties of convergence, in the sense that a dense set Q of rational numbers is not complete since a Cauchy sequence of rational numbers is not convergent when its limit is an irrational number.

Hilbert's axiomatic completeness is not the same as Dedekind's completeness: the former pertains to a model, while the latter defines a domain of numbers. Yet, the completeness of arithmetic makes the completeness of a model possible. Axiomatic completeness is nowadays labelled 'categoricity', and pertains to the definition of two isomorphic models with maximal elements. The maximality of elements makes both the system of axioms complete and each model of the system isomorphic to each other. The one-one correspondence or bijection preserves all axiomatic relations so that models, such as Cartesian algebra and Euclidean geometry, are both equivalent and interchangeable. This means that something provable within a Euclidean model is automatically provable in an algebraic model; and if a contradiction pops up in one model, then any isomorphic model becomes contradictory as well, since all isomorphic models are derived from the same axioms. [6] In other words, points and lines are nothing more than numbers and intervals, and the consistency of a geometric model depends upon the consistency of an arithmetic one. In 'Mathematische Probleme' (1900b), Hilbert writes:

> In geometry, the proof of the consistency of the axioms can be affected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. Any contradiction in the deductions from the geometrical axioms must thereupon be recognizable in the arithmetic of this field of numbers. In this way the desired proof for consistency of the geometrical axioms is made to depend upon the theorem of consistency of the arithmetical axioms. (par. 39; 1996, p. 1104)

The axiomatic model of arithmetic is divided into four groups of axioms, namely: (I) Axioms of Composition, (II) Axioms of Calculating, (III) Axioms of Ordering, and (IV) Axioms of Continuity. These axioms replace Peano's arithmetic and are the algebraic versions of the Euclidean model of geometry, except for the axiom of parallels which has no application in algebra. Therefore, we face two axioms of continuity divided into an Archimedean axiom

[6] To learn more about categoricity, see Veblen 1904, Corcoran 1980, and Zach 1999.

and an axiom of completeness; Hilbert (1900a) writes (par. 7; 1996, p. 1094):

IV-1 *Archimedean axiom*: If $a > 0$ and $b > 0$ are two arbitrary numbers, then it is always possible to add a to itself so often that the resulting sum has the property that $a + a + \ldots + a > b$.

IV-2 *Axiom of completeness*: It is not possible to add to the system of numbers another system of things so that the axioms I, II, III, and IV-1 are also all satisfied in the combined system; in short the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.

The axiom of completeness defines the set R of real numbers as a complete system of element that cannot be extended further, meaning that the set R cannot be defined as the subsystem of another system of numbers. Thus, infinitesimals are non-standard elements that do not belong to the axiomatic system of arithmetic; they are non-Archimedean numbers since Archimedes' axiom does not apply to them.

While Dedekind's and Cantor's arithmetic continuity is confined to real analysis and set theory, Hilbert's axiomatic system reinterprets it as the cornerstone of foundational mathematics in the sense that it is applicable to any mathematical models. Thus, a Euclidean model of points, lines, and planes expresses arithmetic properties, whose consistency derives from a finite number of axioms. Hilbert (1900a) writes:

> The doubts which have been raised against the existence of the totality of all real numbers (and against the existence of infinite sets generally) lose all justification; for by the set of real numbers we do not have to imagine, say, the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as just described—a system of things whose mutual relations are given by the finite and closed system of axioms I-IV, and about which new statements are valid only if one can derive them from the axioms by means of a finite number of logical inferences. (par. 17; 1996, p. 1095)

The finite and closed system constitutes the axiomatic foundation for the consistency of the arithmetic continuum. In this sense, Hilbert solves the foundational problems encountered by Cantor and Dedekind; that is, numbers are no longer based on the vague and unreliable concept of freedom, whether applied to the mind or to the essence of mathematics. Dedekind (1888) writes, "In speaking of arithmetic (algebra, analysis) as merely a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time... Numbers are free creations of the human mind." (1996, pp. 790-1). [7] The lack of a well-defined foundation for numbers has dramatic consequences for foundational mathematics. The mind and the essence of mathematic are mysterious foundations, in the sense that their definitions are inaccessible. By contrast, Hilbert's axioms define numbers as the properties of a system, such that the foundation of numbers resides in the axiom system itself. This corresponds to a transition from a first-level concept to a second-level axiom. Dedekind and Cantor understand numbers as independent and individual mathematical entities generated by a creating power, whether the free mind or the free essence of mathematic. This amounts to identifying numbers with ordinary individual objects in the same way that the ultimate foundation for the perception of a table is related to the power of our senses. Hilbert rejects such solutions as obscure, insofar as numbers are (second-level) properties of a system and not (first-level) objects. In other words, axioms are the ultimate foundations for numbers, and define not only their consistency but also their existence. We shall see that Hilbert's position encounters Frege's strong objections according to which it does not make sense to define the truth and existence of mathematical objects through a consistent axiomatic system.

[7] Likewise, Cantor (1883) founds numbers on the freedom of mathematics, which constitutes its essence. He writes, "Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. In particular, in the introduction of new numbers it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real. I think this is the reason... why one must regard the rational, irrational, and complex numbers as being every bit as existent as the finite positive integers. It is not necessary, I believe, to fear, as many do, that these principles present any danger to science. For in the first place the designated conditions, under which alone the freedom to form numbers can be practised, are of such a kind as to allow only the narrowest scope for discretion. Moreover, every mathematical concept carries within itself the necessary corrective: if it is fruitless or unsuited to its purpose, then that appears very soon through its uselessness, and it will be abandoned for lack of success. But every superfluous constraint on the urge to mathematical investigation seems to me to bring with it a much greater danger, all the more serious because in fact absolutely no justification for such constraints can be advanced from the essence of the science—for the essence of mathematics lies precisely in its freedom." (1996, sect. 8, par. 4-5, p. 896).

6.3 Axiomatic consistency and logical truth

Both Hilbert and Frege are interested in foundational mathematics, but their aims diverge. Hilbert thinks of geometry and algebra as having common foundations, such that a Euclidean model of points is isomorphic to an algebraic model of numbers. As the meanings of mathematical concepts pertain to axioms, true definitions must be confined to the consistency of the axiom system. In contrast, Frege subsumes foundational mathematics under a philosophical thesis, namely, meaning and truth applied to mathematics must corroborate the logical principles derived from ordinary language. Furthermore, he defines distinct foundations for geometry and arithmetic: geometry rests on Kant's synthetic *a priori* intuition of space, while arithmetic has a purely logical foundation (distinct from Kant's synthetic *a priori* intuition of time), such that a number is a logical object defined as the extension of a concept. [8] Therefore, mathematical concepts are true if and only if they are related to independent objects, whether geometric or arithmetic. More precisely, Frege understands a concept as a predicate, to which he applies the three following relations:

(1) A relation of *subsumption* between an object and a first-level concept means that an object falls *under* a first-level concept; e.g., 'Edinburgh is a city', 'Points are joined by a line', and 'Two is a number'.

(2) A relation of *subordination* between two first-level concepts means that a first-level concept falls *under* a first-level concept; e.g., 'All whales are mammals', 'All squares are rectangles', and 'All even numbers are integers'.

(3) A relation of *quasi-subsumption* between a first- and second-level concepts means that a first-level concept falls *into* a second-level (existential or universal) concept; e.g., 'There is at least one God', 'For every two points, there is a line containing them', and 'There is a number two'.

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[8] In *Die Grundlagen der Arithmetik* (1884), Frege writes, "I consider Kant did great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic and *a priori*, he revealed their true nature. And this is still worth repeating, since even today it is often not recognized. If Kant was wrong about arithmetic, that does not seriously detract, in my opinion, from the value of his work. His point was, that there are such things as synthetic judgement a priori; whether they are to be found in geometry only, or in arithmetic as well, is of less importance." (§89; 1950, p. 101).

According to Frege, a mathematical concept is true or false if and only if it is definable as a first-level concept, which expresses a sense (or thought) and refers to an object (or meaning). For instance, the statement 'Two is a number' pertains to the object two falling under the first-level concept 'is a number'; the concept expresses a sense about a number, but its truth condition is the object itself (i.e. the extension of the concept 'is a number' is two). Likewise, the assertion 'Points are joined by a line' is *meaningful* providing that some objects, i.e. points, fall under the first-level concept 'are joined by a line'. On the other hand, if we claim, 'For every two points, there is a line containing them', we deal with a relation of quasi-subsumption between first-level concepts and second-level ones, such that the first-level concept 'two points' falls into the second-level universal concept 'every' as well as the first-level concept 'a line containing them' falls into the second-level existential concept 'there is'. In the absence of objects, this statement is *meaningless*, i.e. it is neither true nor false. Likewise, if we state, 'There is a number two', we deal with a mere relation between a first-level concept and a second-level existential concept, independently of the idea of truth or falsehood. In other words, Frege denies truth or falsehood in the absence of (mathematical) objects.

The obvious conclusion is that Frege cannot accept Hilbert's claim that axioms are definitions. He rather says that axiom and theorems are related to definitions only because they contain signs (or words), whose sense and meaning are already laid down. Thus, an axiom is true, only insofar as it includes meaningful signs with true definitions. Frege writes in a letter to Hilbert (27 December 1899):

> I have my doubts about the proposition that a precise and complete description of relations is given by the axioms of geometry... and that the concept 'between' is defined by axioms.... Here the axioms are made to carry a burden that belongs to definitions. To me this seems to obliterate the dividing line between definitions and axioms in a dubious manner.... Every definition contains a sign (an expression, a word) which had no meaning before and which is given meaning by the definition. Once this has been done, the definition can be turned into a self-evident proposition which can be used like an axiom. But we must not lose sight of the fact that a definition does not assert anything but lays down something. Thus, we must never present as a definition something that is in need of proof or of some other confirmation of its truth... The other propositions (axioms, fundamental laws, theorems) must not contain a word or sign whose sense and meaning, or whose contribution to the expression of a thought, was not already completely laid down, so that there is no doubt about the sense of the proposition and the thought it expresses.... Thus, axioms and theorems can never try to lay down the meaning of a sign or word that occurs in them, but it must already be laid down.... I call axioms propositions that are true but are not proved

> because our knowledge of them flows from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict to one another. There is no need for a further proof. (1980, pp. 35-7)

Frege's conception respects the traditional definition of axioms in the sense that geometric axioms are true providing that they are based on truly defined geometric terms. He agrees with the five axioms (postulates) of Euclid's geometry, whose truth relies on the twenty-three definitions (cf. section 6.1). The Fregean truth pertains to signs related to the geometric objects of Kant's synthetic *a priori* intuition of space. Axioms are merely assertive postulates, meaning that they are unable to lay down definitions, and let alone, to define a system. Moreover, the consistency of an axiom depends upon its truth, namely its consistency is the consequence of its truth and not a synonym for it. This conclusion seems trivial; nevertheless, it is at the core of the disagreement between Frege and Hilbert, insofar as their distinct conceptions of truth lead them to define two radically distinct foundations for mathematics. Indeed, Hilbert replies to Frege (29 December 1899; note the direct reference to the last lines of the above quotation):

> You write 'I call axioms propositions… From the truth of the axioms it follows that they do not contradict one another.' I found it very interesting to read this very sentence in your letter, for as long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence. (1980, pp. 39-40)

Hilbert defines truth as mere consistency, in the sense that an axiom is true only because it does not contradict any other axiom within the system. This conception of truth as relative consistency is not surprising, since axioms are definitions of relations applied to a collection of elements. As no external reference beyond the axiom system is relevant, it makes sense to claim that the truth of axioms is a matter of consistency between the axioms themselves. To distinguish Hilbert's position from Frege's, we may oppose three definitions of truth, so that Hilbert's definition (1) conflicts with the logical definitions (2) and (3), namely:

(1) *Truth as consistency*: an axiom A is *true* if and only if A does not contradict the rest of the axioms in at least one *mathematical* model of the axiomatic system (among an infinite number of models).

(2) *Truth by interpretation*: an axiom A is *true in a logical model* if and only if A is satisfiable in this unique model.

(3) *Truth as logical validity*: an axiom A is *logically valid* if and only if A is true in all the models of a given language, i.e. in a given logic.

Hilbert's conception of truth as consistency implies the absence of a logical model defining logical truth. In other words, true axioms require at least *two* axioms, whose lack of contradiction makes them true. Truth is relevant to the axioms alone, and not to a logical model of true axioms. Thus, truth as consistency contradicts truth by interpretation, namely a truth defined by a logical model (or interpretation). Hilbert's motivation is to show that truth is not definable outside the axiomatic system, which implies the rejection of the definitions (2) and (3), which pertains to a concept of truth derived from a logical model or a logical language. Indeed, if truth were logical, it would pre-exist the axioms themselves, such that *mathematical* semantics would depend upon a *logical* semantics. Therefore, Hilbert's conception of a model is purely mathematical, and neglects Frege's logical truth. Truth as consistency implies a notion of independence, such that an axiom A is independent if and only if A contradicts any axioms of a system in at least one mathematical model. We may define independence as axiomatic falsehood in the sense that it derives from a contradiction between at least *two* axioms, and is incompatible with a logical falsehood belonging to a logical model or given logic.

Hilbert's axiomatic consistency and independence, as opposed to Frege's logical truth and falsehood, illustrates a philosophical debate about two conceptions of foundational mathematics. Hilbert thinks of an axiomatic system as confined to foundational mathematics, and thereby inapplicable to the intuitive concepts of ordinary language. In contrast, Frege defines foundational mathematics for the sake of logic, such that mathematical propositions are not different from ordinary statements; all follow a same logic by expressing true thoughts in relation to objects. In *On the Foundations of Geometry* (1906), Frege writes:

> When one uses the phrase 'prove a proposition' in mathematics, then by the word 'proposition' one clearly means not a sequence of words or a group of signs, but a thought; something of which one can say that it is true. And similarly, when one is talking about the independence of propositions or axioms, this, too will be understood as being about the independence of thoughts. (p. 401; 1984, p. 332)

Frege's foundation of mathematical propositions is about logical thoughts, whose absolute truth and absolute independence are located outside mathematics. Yet, such a view is much too unreliable and intuitive to satisfy Hilbert, whose main concern pertains to the relative consistency of axiomatic models. Hence the foundation of mathematics must not be absolute, but relative to the axiomatic system itself. In other words, Hilbert is only interested in codifying mathematics through isomorphic models, independently of logical truth and falsehood which do not belong to foundational mathematics. As axiomatic truth and independence have no logical content, the logical status of the axiomatic system is purely hypothetical; hence, mathematical truth is not logical, and logical truth is not mathematical. [9] From a foundational point of view, Hilbert's position is wholly relevant insofar as his chief purpose is to create consistent mathematical models within an axiomatic system, such that a Euclidean geometry is isomorphic to a Cartesian algebra.

6.4 Axiomatic existence and second-level systems

The relative consistency of models defines the relative existence of concepts. This means that any collections of elements are interchangeable, providing that they are defined in isomorphic models which are both consistent and complete. Hilbert writes to Frege (29 December 1899):

> It is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney-sweep... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras's theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements. One only needs to apply a reversible one-one transformation and lay it down that the axioms shall be correspondingly the same for the transformed things. (1980, pp. 40-1) [10]

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9 Bernays (1922a) interprets Hilbert's position in the following: "The axioms are in no way judgments that can be said to be true or false; they have a sense only in the context of the whole axiom system. And even the axiom system as a whole does not constitute the statement of a truth; rather, the logical structure of axiomatic geometry in Hilbert's sense... is a purely hypothetical one." (1998, p. 192).

10 Already in 1891, Hilbert claims in a conversation that it must be possible to replace the words *point*, *line*, *plane* by *table*, *chair*, *mug* in all geometric statements (Weyl 1944, p. 153).

To speak of relative existence means that no axiomatic element may exist outside the axiom system. Any collections of objects are acceptable on condition that they preserve their mutual relations as defined by the axioms. In particular, the relation of completeness, included in the two axioms of continuity, defines each model as complete (categorical), which makes them isomorphic to each other. For instance, we may define the Pythagorean theorem in any isomorphic model, whatever the system of objects at stake:

1. In a *Euclidean model* of 'points, lines, planes', the Pythagorean theorem states that, in a right triangle ABC, the areas of the two small squares ABDE and BCFG are equal to the area of the large square ACHJ.
2. In a *Cartesian model* of 'real numbers', the Pythagorean theorem asserts the equality relation $a^2 + b^2 = c^2$ for all real numbers *a*, *b*, *c* such that $a < c$ and $b < c$ implies a right angle between *a* and *b*.
3. In a *pub model* of 'chairs, tables, mugs', the Pythagorean theorem claims that the volume of the large mug γ is equal to the sum of the two volumes of the small mugs α and β. Indeed, suppose three chairs A, B, C and three tables *a*, *b*, *c* such that *a* is between A and B, *b* is between B and C, and *c* is between C and A. The tables *a* and *b* form a right angle at B, meaning that *a* and *b* are smaller than the table *c*. Therefore, the addition of the two small mugs α and β, respectively at the small tables *a* and *b*, are equal to the large mug γ at the large table *c*.

The third model seems absurd; yet, it is as consistent as the two other models, insofar as all the relations of the axiom system are preserved, i.e. the relations of incidence, order, congruence, perpendicularity, and continuity. In other words, the choice of a collection of objects is purely arbitrary, and has no effect on the mathematical definitions directly derived from the axioms themselves.

Accordingly, consistency and completeness are the two conditions for the existence of axiomatic concepts. If an axiomatic concept has contradicting properties, its axiomatic existence is immediately compromised. Hilbert (1918) writes, "If contradictory attributes be assigned to a concept, I say, that *mathematically the concept does not exist* (1996, p. 1105, original emphasis). Axiomatic concepts exist as long as they depend upon consistent axioms; and if one concept has contradicting properties with respect to the axiom system, then it must be discarded. Frege unsurprisingly disagrees with Hilbert's conception of relative existence. More precisely, he accuses him not only of confusing the concrete existence of first-level concepts with the abstract existence of second-level ones, but also of neglecting the principle that only first-level concepts can be asserted though real axioms (cf. section 6.3). In other words, only points, lines, and planes defined as first-level concepts can be related to concrete geometric objects, and cannot constitute a second-level system interchangeable with 'chairs, tables, mugs' or 'love, law, chimney-sweep'. To make his point clear, Frege uses the following example in another letter to Hilbert (6 January 1900):

> What would you say about the following:
>
> Explanation: We imagine objects we call Gods.
> Axiom 1: All Gods are omnipotent
> Axiom 2: All Gods are omnipresent
> Axiom 3: There is at least one God
>
> Here we must consider my distinction between first- and second-level concepts... In the words 'there is' we have a second-level concept, which must not be combined with the first-level concepts *omnipotent* and *omnipresent* as a characteristic mark of a first-level concept. The characteristic marks you give in your axioms are apparently all higher than first-level; i.e. they do not answer to the question 'What properties must an object have in order to be a point (a line, plane, etc.)?', but they contain, e.g., second-level relations, e.g., between the concept *point* and the concept *line*. It seems to me that you really want to define second-level concepts but do not clearly distinguish them from the first-level ones. (1980, p. 46, original emphases)

As we have just seen, Frege makes the distinction between two kinds of relation. On the one hand, a relation of subsumption defines an object, i.e. God, falling *under* the first-level concept (predicate) 'are omnipotent' or 'are omnipresent'. On the other hand, the relation of quasi-subsumption pertains to the first-level concept 'at least one God' falling *into* the second-level existential concept 'there is'. In Fregean terms, the axiom 'There is at least one God' is formalised into $\exists x\ Gx$, such that the second-level concept of existence plays the role of a mathematical function which takes the first-level concept as its argument. Following Frege, Hilbert's lack of distinction between first- and second-level concepts makes the true definitions of geometric objects impossible; in contrast, real axioms, like in Euclid's intuitive geometry, contain only first-level concepts whose true thoughts are related to geometric objects. As Hilbert's axioms presuppose second-level concepts under which not objects fall, they do not include thoughts and truth-conditions; this leads Frege to conclude that they are neither genuine definitions nor proper axioms.

Let us quote Hilbert's first axiom of incidence (group I):

I-1 For every two points A, B there is a line *a* containing each of the points A, B.

According to Frege's interpretation, the points A and B are first-level concepts instantiated into a second-level universal quantifier, while the line *a* is a first-level concept instantiated into a second-level existential quantifier. According to Hilbert's interpretation, points and lines are second-level concepts defined by a second-level relation of incidence, i.e. a relation of belonging such that the two axiomatic points A and B belong to an axiomatic line *a*. Points and lines are mere elements of a system defined by the axioms alone. Frege criticises the fact that concepts (predicates) are not recognised as being independent of the axiomatic relation to which they belong, in the sense that the truth of concepts should refer to objects falling under them, and not to a second-level relation. From a Fregean point of view, it is absurd to claim that a second-level point is instantiated into a second-level existential quantifier, since second-level points are pure abstractions that do not refer to anything properly geometric. In other words, as soon as we speak of points, lines, and planes, we automatically refer to first-level concepts which are definable as either true or false, so that the consistency or independence of real axioms (unlike Hilbert's pseudo-axioms) are the consequence of the truth or falsehood of first-level concepts. [11]

Frege's opposition to Hilbert is ideological, insofar as he does not prove some inconsistency in Hilbert's position; yet, he tries to convince him that he is on a wrong logical path. In other words, the notion of axiomatic existence conflicts with Frege's philosophical faith that mathematical entities are as existent as any ordinary objects; and as second-level concepts are abstract elements of a system that are neither true nor false, they cannot pertain to the objects of mathematics. Frege's reaction shows his rather naïve conception of

[11] In *On the Foundations of Geometry: Second Series* (1906), Frege writes, "All mathematicians who think that Mr. Hilbert has proved the independence of the real axioms from one another have surely fallen into the same error. They do not see that in proving this independence, Mr. Hilbert is simply not using the word 'axiom' in Euclid's sense. The fault here lies in the double usage of the words 'point', 'straight line', etc., which on the one hand, like letters, are to lend generality to the whole theory, in which case they do not designate anything; and on the other hand have their traditional references in Euclid's axioms. In the former case his axioms are merely pseudo-axioms without sense, since only the whole… whose dependent parts they are, has a sense… In the other case real axioms do occur. But then these independent-proof are inappropriate, since it is impossible to substitute other concept-words for 'point', 'straight line', etc. But surely it is on this very possibility that such a proof depends." (1984, p. 333).

foundational mathematics, in the sense that any mathematical sign must mirror the concrete existence of a mathematical object. Mathematical propositions are composed of individual signs implying sense and meaning, such that the equation '3 + 1 = 4' is a second-level identity relation presupposing that we already know the respective sense and meaning of the first-level signs '+', '1', '3', and '4'. Likewise, the geometric statement 'There is a point B which lies between a point A and a point C' is a second-level proposition, whose truth pertains only to first-level concepts of points related to the geometric objects of Kant's spatial intuition. The consequence is that Frege has a rather archaic conception of the relationship between geometry and arithmetic. As a number and a point are two mathematical signs expressing two distinct thoughts related to two incompatible objects (with incommensurable foundations), it is then impossible for Euclid's geometry to be equivalent to Cartesian algebra. Unsurprisingly, Frege defends a conception of mathematics close in meaning to Leibniz's, such that points are geometric entities that are ontologically distinct from algebraic numbers. [12]

Accordingly, Frege accepts second-level relations; yet, the latter must always pertain to first-level concepts in order to make *sense* and be *meaningful*. If not, we end up with the following situation, as described in a letter to Hilbert (6 January 1900):

> Your system of definitions is like a system of equations with several unknown, where there remains a doubt whether the equations are soluble and, especially, whether the unknown quantities are uniquely determined. If they were uniquely determined, it would be better to give the solutions, i.e., to explain each of the expressions 'point', 'line', 'between' individually through something that was already known. Given your definitions, I do not know how to decide the question whether my pocket watch is a point. The very first axiom deals with two points; thus, if I wanted to know whether it held for my watch, I should first have to know of some other object that it was a point. But even if I knew this, e.g. of my penholder, I still could not decide whether my watch and my penholder determined a line, because I would not know what a line was. (1980, p. 45)

[12] Frege (1884) writes, "We shall do well in general not to overestimate the extent to which arithmetic is akin to geometry. I have already quoted a warning to this effect from Leibniz. One geometrical point, considered by itself, cannot be distinguished in any way from any other; the same applies to lines and planes. Only when several points, or lines or planes, are included together in a single intuition, do we distinguish them. In geometry, therefore, it is quite intelligible that general propositions should be derived from intuition; the points or lines or planes which we intuite are not really particular at all, which is what enables them to stand as representatives of the whole of their kind. But with the numbers it is different; each number has its own peculiarities. To what extent a given particular number can represent all the others, and at what point its own special character comes into play, cannot be laid down generally in advance." (§13; 1950, p. 19).

Frege compares Hilbert's axioms/definitions to a system of equations with several unknowns, namely a second-level equation devoid of first-level concepts. Suppose the equation $x + 3y = 5z$ composed of three unknowns x, y and z. If the only way to know that the equation is true is to lay down an explicit definition for each first-level concept x, y and z, then this means that $x + 3y = 5z$ is true if and only if we already know that $x = 1$, $y = 3$ and $z = 2$. Frege concludes that if we deal only with second-level concepts, we cannot know whether the equation is solvable. However, Hilbert would reply that this second-level equation, composed of second-level variables x, y, and z, is true in an algebraic model providing that all values other than $x = 1$, $y = 3$ and $z = 2$, are *independent* of the model. That is, the values defining the contradictory (independent) inequality $x + 3y \neq 5z$ cannot be part of the axiomatic model; and this can be done without any reference to first-level numbers. Therefore, when Frege claims that a point is not a pocket watch insofar as each geometric concept is "uniquely determined", Hilbert's answer would be to reject the relevance of the question as to whether or not a point is a pocket watch. Indeed, axiomatic elements have no intuitive definition, and are thereby unable to be confused with objects outside the axiom system. In other words, Frege asks the question only because he postulates that any objects, whether mathematical or not, fall under first-level concepts. As Hilbert rejects Frege's postulate as irrelevant to mathematics, he cannot deal with the question seriously; that is, a point is a second-level element, and as such, nothing prevents it from being called a pocket-watch. As well, there is no difficulty in supposing that a second-level Julius Caesar corresponds to a second-level number. This is even allowed by Frege's so-called Hume's principle (as named by Boolos) defining a second-level relation of equivalence between two second-level cardinal numbers, namely: the number of things for a concept F is equal to the number of things for a concept G if and only if there is a one-one correspondence between the set F and the set G. [13] Frege's (1893, 1903) failed attempt to prove that a number is logically distinct from Julius Caesar corroborates Hilbert's position. Indeed, his Basic Axiom V implies that a number is an extension falling under a first-level

[13] Parsons (1965) and Wright (1983) claims that Hume's Principle plus second-order logic enable one to derive Peano's arithmetic; this is true providing that we do not forget that Hume's principle is confined to second-level cardinal numbers that are properties of a set and not individual objects.

concept (predicate). As each concept has an extension, Frege concludes that there are as many extensions as first-level concepts. However, this principle is false for there are more concepts than extensions: for instance, the set of all extensions is a concept *under* which no extension falls (it is a second-level universal concept). Therefore, Frege's definition of numbers as first-level concepts is hampered by the fact that the definition of a number as an extension (i.e. a logical object) is inconsistent (cf. section 4.4). [14] In other words, we cannot be *logically* certain that Julius Caesar is not a number.

Hilbert has no such worries, as mathematical models only produce second-level concepts. He rejects first-level concepts related to mathematical objects, whose intuitive or logical definitions are mysterious at best, contradictory at worst. He writes to Frege (29 December 1899):

> This is apparently where the cardinal point of the misunderstanding lies. I do not want to assume anything as known in advance... If one is looking for other definitions of a 'point', e.g., through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there; and everything gets lost and becomes vague and tangled and degenerates into a game of hide-and-seek. (1980, p. 39)

To claim that a point is a partless entity is meaningless. It is *a priori* absurd since nobody is able to explain how an algebraic quantity can be related to an indivisible, non-quantitative geometric object; and it is *a posteriori* false since any empirical point drawn on a blackboard is a white dot composed of divisible parts, however tiny they may be. Which is surprising is that Frege never takes Hilbert's position seriously. He likely thinks of his own philosophical position as the unique answer to the problem raised by Hilbert; in other words, the foundation of mathematics is solved through the philosophy of language dealing with sense and meaning. As any referents or objects are truth-conditions of our ordinary logic, the same logical foundation must apply to the language of mathematics, and mathematical consistency is about a mathematical object. Frege replies to Hilbert (6 January 1900):

> Is there some other means of demonstrating lack of contradiction besides pointing out an object that has all the properties? But if we are given such an object, then there is no need to demonstrate in a roundabout way that there is such an object by first demonstrating lack of contradiction. (1980, p. 47)

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[14] Frege's Basic Axiom V wrongly claims that two concepts have the same extension if and only if they are extensionally equivalent (cf. Boolos 1986, 1993).

Is it sufficient to point out a mathematical object in order to derive the truth of a mathematical proposition? Hilbert regards this belief as a prejudice. Pointing out an object amounts to assuming the existence of an external foundation, which we are unable to grasp since we have not defined it. His aim, unlike Frege's, is to search, not for an absolute truth difficult to comprehend, but for well-defined principles constituting reliable foundations. Hilbert wants to break down the relationship between object and concept, insofar as nothing justifies the belief that concepts require objects. In his last letter to Frege (7 November 1903), he writes:

> As I see it, the most important gap in the traditional structure of logic is the assumption made by all logicians and mathematicians up to now that a concept is already there if one can state of any object whether or not it falls under it. This does not seem adequate to me. What is decisive is the recognition that the axioms that define the concept are free from contradiction. (1980, p. 52)

In other words, mathematics only needs second-level concepts defined by a system of axioms, and such concepts are consistent only because the axioms are so. Instead of looking for mysterious mathematical objects, it is sufficient to define axioms which are the foundations for truth and existence in mathematics. Some may object to the arbitrary construction of axioms, but that cannot constitute an objection as the very arbitrariness of such axioms is the property that makes them well defined. Indeed, arbitrariness is a sound criticism only if mathematical objects are believed to exist independently of the activity of constructing mathematics. Therefore, Hilbert's axiomatisation has the great advantage to avoid the metaphysical conundrum to refer to external mathematical objects that nobody is able to know properly.

6.5 Finitary metamathematical syntax and infinite mathematical semantics

So far, Hilbert's conceptions of truth and existence have been confined within the axiom system, which defines consistent and isomorphic models. The question is whether an axiomatic system may be true independently of the relative consistency of its second-level models. Hilbert regards the question as crucial since, in the Second International Congress of Mathematicians (Paris, 1900), he includes the consistency proof for arithmetic in a list of mathematical problems to solve. We shall see that the consistency proof of the axiom system is independent of the mathematical semantics, insofar as it is based on the syntax of the axioms themselves. In 'On the foundations of logic and arithmetic' (1904), Hilbert starts by defining a thought-object (*Gedankending*) as a formal concept denoted by a sign. He writes:

> In the axioms the arbitrary objects—taking the place of the notion "every" or "all" in ordinary logic—represent only those thought-objects and their mutual combinations that at this stage are taken as primitive or are to be newly defined. In the derivation of consequences from the axioms the arbitrary objects that occur in the axioms may therefore be replaced only by such thought-objects and their combinations. We must also duly note that, when a new thought-object is added and taken as primitive, the axioms previously assumed apply to a larger class of objects or must be suitably modified. (1967, p. 135)

Such thought-objects belong to the first-level formalisation of axioms without reference to the second-level mathematical models. Hilbert feels the need to deal with the syntax of axioms in order to avoid paradoxes, and in particular Cantor's and Russell's paradoxes (cf. section 4.4). He connects the manifestation of such paradoxes with Frege's unwise conception of first-level concepts (predicates or sets) related to mathematical objects. In 'Axiomatic Thought' (1918), Hilbert is first influenced by Zermelo's set theory in the construction of a syntax for the axiom system. [15] However, 'The New Grounding of Mathematics' (1922) and 'The Logical Foundations of Mathematics' (1923) neglect set theory and promote the distinction between a metamathematical syntax, called proof theory, and a mathematical semantics, called contentual mathematic. Metamathematics corresponds to a first-level formalisation (i.e. syntax plus deductive rules of inference) applicable to second-level mathematical models. It is a revolutionary step, since it amounts to defining the metamathematical consistency of (contentual) mathematical semantics through a purely deductive system of formulae. Hilbert (1923) writes:

[15] Hilbert (1918) writes, "The problem of the consistency of the axiom system for the *real numbers* can likewise be reduced by the use of set-theoretic concepts to the same problem for the integers; this is the merit of the theories of the irrational numbers developed by Weierstrass and Dedekind. In only two cases is this method of reduction to another special domain of knowledge clearly not available, namely, when it is a matter of the axioms for the *integers* themselves, and when it is a matter of the foundation of *set theory*; for here there is no other discipline besides logic which it would be possible to invoke." (par. 37-9; 1996, p. 1113, original emphases).

> The fundamental idea of my proof theory is as follows: Everything that previously made up mathematics is to be rigorously formalized, so that mathematics proper [*die eigentliche Mathematik*] or mathematics in the strict sense becomes a stock of formulae. These formulae are distinguished from the ordinary formulae of mathematics only by the fact that they contain logical signs in addition to the ordinary signs... A formula shall be called provable it if it either is an axiom, or results from an axiom by substitution, or is the end-formula of a proof. In addition to this formalized mathematics proper, we have a mathematics that is to some extent new: a metamathematics that is necessary for securing mathematics, and in which—in contrast to the purely formal modes of inference in mathematics proper—one applies contentual inference, but only to prove the consistency of the axioms. In this metamathematics we operate with the proofs of mathematics proper, and these proofs are themselves the objects of the contentual investigation. Thus, the development of mathematical science as a whole takes place in two ways that constantly alternate: on the one hand, we derive new probable formulae from the axioms by formal inference; on the other, we adjoin new axioms and proves their consistency by contentual inference. (par. 5-7; 1996, pp. 1137-8)

A first-level formalisation consists of signs, logical operations (such as implication and negation), and proofs based on inferences (such as *modus ponens*). Hilbert separates syntax from semantics, not for logical purposes, but for mathematical ones. He wants to subsume the semantics of mathematics under a purely syntactical formalisation, so that the axiom system of semantically (mathematically) consistent models can be proved to be syntactically (metamathematically) consistent. He follows this path because mathematical semantics pertains to infinite sets with countable and uncountable cardinalities. That is, the truth of an arithmetic continuum cannot rest on an infinite semantics, since any set of infinite cardinality correspond to an actual infinite, whose definition and ordering remain irreducibly hypothetical (cf. chapter 4). By contrast, the construction of an axiomatic system implies the *finitary* formalisation of syntactic formulae and deductive rules. [16]

In 'On the Infinite' (1926), Hilbert asserts his main thesis that the infinite semantics of mathematics is valid through a finitary syntactic proof. He makes the distinction between *semantic* propositions and *syntactic* formulae. Propositions are real if finite, and ideal if infinite, whereas formulae are always finitary. Ideal propositions are consistent relative to a second-level axiomatic model, but are merely hypothetical if defined in a first-level semantics. Yet, ideal propositions are syntactically valid if they are formalised into a finitary syntax of first-level formulae. For instance, the continuum R, semantically consistent in a

[16] A 'finitary' (*finit* in German) syntax is opposed to a 'finite' semantics, in the sense that a finitary metamathematical syntax is distinct from either a finite or infinite mathematical semantics.

second-level model of mathematics, becomes syntactically valid in a first-level finitary metamathematical formalisation. Therefore, a second-level infinite semantics is syntactically reduced to a first-level finitary syntax of formulae. Hilbert (1926) writes, "Mathematics become an inventory of formulae—first, formulae to which contentual communications of finitary propositions correspond and, second, further formulae that mean nothing in themselves and are the *ideal objects of our theory*." (1967, p. 380, original emphasis). [17] If both real and ideal propositions are syntactically transformed into finitary formulae, there is still a distinction to make between them. On the one hand, the finitary formulae of real propositions pertain to the finite semantics of contentual mathematics, such that the finitary formalisation of contentual mathematics has a first-level semantics combined with a first-level syntax. On the other, the finitary formulae of ideal propositions are meaningless, since the infinite semantics of a second-level model cannot apply to a first-level semantics. Therefore, the actual infinites, syntactically defined as finitary, do not corrupt the two Aristotelian logical laws, i.e. the law of non-contradiction and the principle of excluded middle. Hilbert (1926) uses the distinction between the intuitive number theory and the abstract real analysis in order to explain, by analogy, how the finitary syntax of metamathematics is abstractly detached from the infinite semantics of mathematics; he writes:

[17] In 'The new grounding of mathematics. First report' (1922), Hilbert explains why contentual mathematics cannot stand for the whole of mathematics. He writes, "We can of course make considerable further progress in number theory using the intuitive and contentual manner of treatment which we have depicted and applied. But we cannot conceive of the whole of mathematics in such a way. Already when we cross over into the higher arithmetic and algebra—for example, if we wish to make assertions about infinitely many numbers or functions—the contentual procedure breaks down... But we can achieve an analogous point of view if we move to a higher level of contemplation, from which the axioms, formulae, and proofs of the mathematical theory are themselves the objects of a contentual investigation. But for this purpose the usual contentual ideas of the mathematical theory must be replaced by formulae and rules, and imitated by formalisms. In other words, we need to have a strict formalization of the entire mathematical theory, inclusive of its proofs, so that—following the example of the logical calculus—the mathematical inferences and definitions become a formal part of the edifice of mathematics. The axioms, formulae, and proofs that make up this formal edifice are precisely what the number-signs were in the construction of elementary number theory...; and with them alone, as with the number-signs in number theory, contentual thought takes place—i.e. only with them is actual thought practised. In this way the contentual thoughts (which of course we can never wholly do without or eliminate) are removed elsewhere—to a higher plane, as it were; and at the same time it becomes possible to draw a sharp and systematic distinction in mathematics between the formulae and the formal proofs on the one hand, and the contentual ideas on the other." (par. 32-33; 1996, pp. 1123-24).

> In the logical calculus we possess a sign language that is capable of representing mathematical propositions in formulae and of expressing logical inference through formal processes. In a way that exactly corresponds to the transition from contentual number theory to formal algebra we regard the signs and operation symbols of the logical calculus as detached from their contentual meaning. In this way we now finally obtain, in place of the contentual mathematical science that is communicated by means of ordinary language, an inventory of formulae that are formed from mathematical and logical sign and follow each other according to definite rules. (1967, p. 381)

In other words, we are asked to abandon the intuitive language of mathematics as composed of a syntax and semantics, so that we may replace it with a purely syntactic formalisation only composed of finitary formulae. That is, we deal with a first-level finitary syntax related to a second-level infinite semantics. What is metamathematical is not mathematical, in the sense that the metamathematical foundation of mathematics pertains to the syntactic formalisation of axioms which is devoid of semantics. Hilbert (1922) writes: "The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding and communication—is this: *In the beginning was the sign*." (par. 25; 1996, pp. 1121-2). With this biblical-toned statement, Hilbert stresses that the certainty granted to mathematics and other sciences starts from a syntactic consistency derived from the construction of formulae. In mathematics, a formalisation is about number-signs or numerals (e.g. '1') and formulae composed of numerals (e.g. '1 + 1 = 2'); both are meaningless since they are devoid of semantics. They are formal signs related to the human mind, insofar as the mathematician constructs them; but they are objective since they take place in a syntactic language, and are concrete as well since they have shapes. [18] The syntax corresponds to a finitary string of symbols, which allows negation and the law of excluded middle to be applicable. Therefore, all mathematical semantics becomes syntactically consistent through its finitary metamathematical formalisation. Hilbert (1926) writes:

[18] Bernays (1923) writes about Hilbert's numerals: "The special shapes [*Formen*] '1' and '+' are inessential. If we disregarded the connection to habit, it would even be advisable, in order to emphasize the principle, to take as numerical signs figures of the type • • • • • • (which are thus constituted merely of points). And, of course, stars, vertical strokes, circles, and other shapes could just as well be chosen instead of points. One could also take a time sequence, say, of similar noises, instead of a spatial sequence. But it is essential that *specimens of equal shape be joined in the same sort of arrangement* [*Zusammensetzung*]." (1998, p. 224, original emphases).

> The problem of proving consistency arises wherever the axiomatic method is used. After all, in selecting, interpreting, and manipulating the axioms and rules we do not want to have to rely on good faith and pure confidence alone. In geometry and the physical theories the consistency proof is successfully carried out by means of a reduction to the consistency of the arithmetic axioms. This method obviously fails in the case of arithmetic itself. By making this important final step possible through the method of ideal elements, our proof theory forms the necessary keystone in the edifice of axiomatic theory. (1967, 383)

The foundation of mathematics is not semantic but syntactic, since the second-level infinite semantics deals with mathematical concepts that would be meaningless if interpreted in a first-level semantics; in other words, the actual infinites of an axiomatic system are semantically ideal elements. To understand what an ideal element is, we may use the analogy of ideal points in elementary geometry, such that an ideal element is the imaginary point at which two infinite parallel lines always intersect each other; this point cannot be individualised as a mathematical object since no finite parallel lines ever intersect. Thus, arithmetic continua, infinite cardinalities, and transfinite ordinals are ideal elements, which make sense only as (second-level) properties of an axiomatic system, since the (ideal) actual infinites are incompatible with a first-level finite mathematical semantics confined to the (real) potential infinite. Note that mathematics requires the axiomatisation of second-level models only because it deals with an infinite semantics. Hilbert's solution constitutes an intermediary step between Cantor's realism and Brouwer's intuitionism: it is a matter, not of renouncing to the actual infinite, but merely of admitting that its mathematical status cannot be the same as an actual finite domain, which amounts to distinguish a second-level infinite semantics from a first-level finite semantics. Yet, the ideal existence of the actual infinite is unacceptable for mathematical intuitionism, which equals mathematical existence to the finite construction of a mathematical object.

The *raison d' être* of metamathematics is to reduce the infinite semantics of contentual mathematics to a finitary syntax, so that proof theory defines the finitary consistency proof of an infinite mathematical semantics. This means that the infinite semantics proper to the axiom of choice must be formalised. We know that Zermelo's (1904, 1908) axiom of choice postulates that any set with infinite cardinality is well ordered, i.e. has a least or first element (cf. section 4.2). In 'The Logical Foundation of Mathematics' (1923), Hilbert defends the axiom of choice as "necessary and indispensable" to his metamathematics; he writes,

> Let us recall the axiom of choice in set theory, which Zermelo was the first to formulate, and on the basis of which he gave his ingenious proof of the well-ordering of the continuum. The objections that were raised against this proof... were essentially directed against the axiom of choice... I believe this opinion is false. Logical analysis of the sort studied in my proof theory shows that the essential thought underlying the principle of choice is a general logical principle which is necessary and indispensable even for the most elementary rudiments of mathematical inference. If we make these rudiments secure, we simultaneously establish the principle of choice: my proof theory does both. (par. 4; 1996, p. 1137)

Hilbert's proof theory requires the axiom of choice since it deals with sets of infinite cardinalities, although they are ideal elements belonging to a second-level semantic. If the axiom of choice were denied, then the infinitely countable set N or the infinitely uncountable continuum R would be indefinable, as it is the case with mathematical intuitionism. In other words, the axiom of choice is part of a transfinite semantics which must be proved to be syntactically consistent in a finitary formalised axiom. In 1923 (1996, p. 1140), Hilbert defines a transfinite function τ, such as:

$$A(\tau A) \rightarrow A(a).$$

This means that if a formula A syntactically expresses a transfinite function τ of all elements satisfying A (i.e. τA), then an element *a* is formalised into A. Thus, A(τA) corresponds to the first-level finitary syntax of a second-level transfinite semantics, such that the element *a* is syntactically provable in a formula A. In 1926 (1967, p. 382) and 1929 (1998, p. 229), Hilbert reformulates the transfinite function τ as the transfinite axiom ε applied to the infinite set N of all natural numbers, such as:

$$A(a) \rightarrow A(\varepsilon A).$$

This means that if a natural number *a* is formalised into the formula A, then *all* natural numbers satisfying A (i.e. εA) are formalised into A. This proves the complete induction (i.e. the Dedekind infinite) of the set N defining *all* natural numbers. The function ε is a universal choice function and is the equivalent of the axiom of choice, which claims that *all* subsets of an infinite set have a singleton subset. Note that if we deal with a unique transfinite axiom, the axiom is semantically defined as the property of a second-level infinite model. On the other, if transfinite axioms are in the plural, they are pure formalisation defined by a first-level syntax, such that the syntactic axiom (independently of second-level models) is repeated for each individual formula. Therefore, the transfinite ε-axiom is the source of all ideal elements of the infinite mathematical semantics, and its syntactic consistency is proved by the finitary metamathematical formalisation.

It is obvious that Hilbert's definition of a transfinite axiom is unacceptable from an intuitionist standpoint, since it implies well-ordered actual infinites which are not constructively provable. [19] Yet, Hilbert's formalist view is also objectionable from a realist viewpoint, since the ideal existence of infinite sets implies a syntactic formalisation without first-level semantics. Then the axiom of choice pertains to an ideal choice confined to second-level semantics. It is a purely formal choice which allows mathematicians to behave as though the choice were actual but without admitting its actual existence. Hilbert (1923) writes:

> Our thought is finite; when we think, a finite process takes place. This self-activating truth [*sich selbst betätigende Wahrheit*] is as it were used in my proof theory in such a manner that, if a contradiction were to emerge at any point, then, as soon as we recognize this contradiction, the relevant choice from the infinitely many things would also have to have been made. Accordingly, it is not asserted in my proof theory that we can always find an object from among the infinitely many objects, but rather that one can always act as though the choice had been made without risking an error. We can concede to Weyl the presence of a circle, but this circle is not vicious. Rather, the application of *tertium non datur* can never lead to danger. (par. 40; 1996, p. 1144)

Negation and the law of excluded middle (*tertium non datur*) safely apply to the first-level syntax, and the axiom of choice only belongs to a second-level model. Hilbert avoids Weyl's *circulus vitiosus*, namely the semi-intuitionist view that the complete induction of the actual infinite has an impredicative definition (cf. section 5.1). Following Poincaré, Weyl criticises the axiom of choice as postulating a vicious circle in arithmetic and real analysis. Indeed, the set N based on the axiom of choice already implies the existence of *all* natural numbers that

[19] In *The New Grounding of Mathematics* (1922), Hilbert defends his proof theory against Brouwer's and Weyl's intuitionism; he writes, "What Weyl and Brouwer do amounts in principle to following the erstwhile path of Kronecker: they seek to ground mathematics by throwing overboard all phenomena that make them uneasy and by establishing a dictatorship of prohibitions *à la* Kronecker. But this means to dismember and mutilate our science, and if we follow such reformers, we run the danger of losing a large number of our most valuable treasures… I believe that, just as Kronecker in his day was unable to get rid of the irrational numbers… so today Weyl and Brouwer will be unable to push their programme through. No: Brouwer is not, as Weyl believes, the revolution, but only a repetition, with the old tools, of an attempted coup that, in its day, was undertaken with more dash, but nevertheless failed completely; and now that the power of the state has been armed and strengthened by Frege, Dedekind, and Cantor, this coup is doomed to fail." (par. 10; 1996, p. 1119). Besides, Hilbert writes in 1931, "In those days [in 1888] we young mathematicians, Privatdozenten and students, played the game of transforming transfinite proofs of mathematical theorems into finite terms, in accordance with Kronecker's paradigm. Kronecker only made the mistake of declaring the transfinite mode of inference to be inadmissible." (par. 9; 1996, p. 1151).

the complete induction is supposed to define; likewise, the continuum R based on the axiom of choice already implies the existence of *all* real numbers that the construction of Dedekind cuts purports to define. Hilbert (1928; 1967, pp. 472-3) criticises Poincaré's rejection of the complete induction: Poincaré should not have proved a formal axiom through contentual mathematics, but should rather have made the following distinction. On the one hand, the iteration of each natural number through the intuitive procedure of an incomplete induction is a first-level principle of contentual mathematics based on a finite semantics. On the other, the complete induction of all natural numbers in the set N pertains to a second-level infinite semantics formalised through a first-level finitary syntax, i.e. the meta-mathematical axiom of the complete induction. Thus, the complete induction of the set N is syntactically consistent but not semantically true, since any semantics based on the actual infinite is meaningless in a first-level language. Hilbert admits that formalism does not make the circle disappear; but at least the circle is not vicious anymore, insofar as the existence of all natural numbers or of all real numbers is ideal and confined to second-level models.

6.6 Uncountable continua and second-level real numbers

A usual criticism against Hilbert's view is that formalism only pertains to a meaningless combination of symbols. This objection distorts his position, insofar as it only tells us half of the story. Meaning is still present but belongs to second-level semantic models, and a first-level formalisation of symbols is meaningless only insofar as its semantics is embedded, not in a first-level language, but in second-level models. Thus, the domain R of all real numbers defines an arithmetic completeness derived from second-level real number (Dedekind cuts). The circularity between completeness and Dedekind cuts is not vicious since confined to a second-level model. In other words, an arithmetic continuum is an ideal element syntactically formalised in a first-level axiomatic system through the two axioms of continuity, i.e. the Archimedean axiom and the completeness axiom. Hilbert (1922) writes:

> The continuum of real numbers is a system of things which are linked to one another by determinate relations, the so-called axioms. In place of the definition of real number by Dedekind cut, we have the two axioms of continuity, namely, the Archimedean axiom and the so-called completeness axiom. To be sure, the Dedekind cuts can then also be used to specify individual real numbers, but they do not provide the definition of the concept of real number. Rather, a real number is conceptually just a thing belonging to our system. (par. 6; 1996, p. 1118)

Therefore, real analysis is an infinite semantics belonging to a second-level model; and the arithmetic continuum, to sum up its second-level properties, is an algebraic field of real numbers to which we apply the operations of addition, subtraction, multiplication and division. This field is always infinite and has a total order defined by the relation $\geq$, such as:

1. If $a \geq b$, then $a + c \geq b + c$ for all $a, b, c \in \mathrm{R}$
2. If $a \geq 0$ and $b \geq 0$, then $a \cdot b \geq 0$ for all $a, b \in \mathrm{R}$.

Such properties postulate the axiom of choice, which allows one to claim that the order of the continuum is *Dedekind-complete*; this means that each of its infinite subsets has a Dedekind cut, either a least upper bound (supremum) or a greatest lower bound (infinum). A Dedekind cut is an ideal element, such that the interval [0, 1] of real numbers is the infinite increasing sequence {0, 1/2, 3/4, 7/8, 15/16,…}, whose Dedekind cut or least upper bound is equal to one; this sequence is a Cauchy sequence (since bounded), and is convergent (since having a limit). If the sequence were a first-level concept, then it would be actually finite and potentially infinite (incomplete) with intuitive numbers defined by iteration as individual objects. In contrast, the sequence, whose completeness derives from a second-level real number, has no real, concrete existence. A Dedekind cut or least upper bound is not even reducible to an intuitive number, since it is a pure abstract property which exhausts the countably infinite sequence, and, as such, is definable as the first transfinite ordinal ω (a ω-task or supertask; cf. section 2.6), i.e.

$$\{0, 1/2, 3/4, 7/8, 15/16, \ldots, 1 - 1/2^n, \ldots\} = \omega = \sup(S) = 1.$$

The inclusion of this supremum or least upper bound into a set transform the infinitely countable dense set [0, 1) into the infinitely uncountable continuum [0, 1].

The new argument brought by Hilbert is to define a first-level axiom of completeness, which formalises the second-level continuum through a finitary syntax. Continua are *Hilbert-complete* by being isomorphic models whose elements are in one-one correspondence. Isomorphic continua are Archimedean fields, whose subfields are the set N of natural numbers, the set Z of integers, and the set Q of rational numbers. Thus, Hilbert's proof theory defines the first-level syntactic consistency of second-level arithmetic continua. Consequently, Hilbert's first-level axioms of continuity and Dedekind's second-level arithmetic continuum complement each other, since a continuum is neither true nor false in a first-level semantics. The impossibility for a continuum to belong to a first-level semantic (in a first-level logic or set theory) is confirmed by Skolem (1920, 1922) through a reference to Löwenheim's (1915) theorem. He writes in 1922:

> Löwenheim's theorem reads as follows: If a counting proposition [*Zählaussage*] is satisfied in any domain at all, it is already satisfied in a denumerably infinite domain.... In a previous paper (1920), I gave a simplified proof of Löwenheim's theorem, along with some generalizations of it. One of these generalizations, which is of importance here, reads: Let there be given an infinite sequence U_1, U_2, ... of first-level propositions numbered with the integers; if, now, it is consistent to assume that all these propositions hold simultaneously, they can all be simultaneously satisfied in the infinite sequence of the positive integers, 1, 2, 3,... by a suitable determination of the class and relation symbols occurring in the propositions. (p. 140; 1967, p. 293)

A counting proposition belongs to a first-level language and is assimilated to Zermelo's 'definite proposition', i.e. a first-level set in Zermelo-Fränkel set theory. If a first-level proposition, composed of a syntax and semantics, is satisfiable in a first-level model, then the proposition is satisfiable in all infinitely countable models. This means that a first-level logic has a countable domain whose cardinality is $\aleph_0$. In other words, no first-level semantics can make sense of the uncountable cardinality of a continuum, because second-level real numbers (Dedekind cuts) are pure abstractions which are not reducible to countable elements. In contrast, the abstract sets N, Z, and Q define countable elements (natural numbers, integers, and rational numbers) which are reducible to individual natural numbers. Therefore, a continuum R, unlike a set N, Z or Q, has an uncountable cardinality in the sense that its impossible counting is due to second-level real numbers that are irreducible to rational numbers. This amounts to paraphrasing the so-called Skolem paradox according to which the set R of real numbers, as defined by a first-level set theory, has a *countable* semantics. It is not a paradox but a consistent claim, insofar as only the infinitely *countable* subfields of an uncountable field R are able to satisfy the first-level formulae of set theory. That is, only countable rational numbers constitute the first-level semantics of a continuum. Likewise, if we say that an algebraic equation has a real number as its solution, this result is only expressible into a first-level sentence with a first-level real number definable as a rational number. Thus, the intrinsic property of a continuum has no reality in a first-level logic or set theory, since the semantics of all *real* numbers make sense only through a second-level complete model. If this denies a logical existence to the continuum, it is harmless for its abstract mathematical meaning.

Is Hilbert's axiomatisation of mathematics complete? The answer depends on the way we understand the question. If we want to prove that a first-level axiomatisation is consistent for any of its formula, then we may conclude to the completeness of the axiomatic system. [20] Thus, Gödel's (1929) *completeness* theorem shows that any formula of a first-level axiomatic system S is syntactically deducible from S, providing that this formula hold for any semantic model satisfying S. In other words, Hilbert's first level axiomatisation of mathematical models is proved to be both consistent and complete. Now, if we want to deal with, not the consistency of the axiomatic system, but the consistency proof of the consistent axiom system, then we must conclude that the axiom system is *logically* incomplete. Indeed, the consistency proof of a consistent axiom system requires a larger system than the axiom system itself. This is proved by Gödel's (1931) two *incompleteness* theorems: the first claims that the consistency proof of a consistent axiomatic system implies undecidable propositions, while the second concludes that the consistency proof cannot be found in the consistent system itself. Yet, it would be wrong to claim that Gödel's incompleteness theorems constitute a direct refutation of Hilbert's position. Indeed, they do not contradict Hilbert's claim that only finitary proofs can prove the consistency of a consistent axiomatic system, insofar as Gödel accepts such finitary proofs providing that they are located outside the system. [21] It is certain that Gödel's incompleteness theorems are a complication for Hilbert's proof theory, since the consistency proof for the consistent axiom system of mathematics

[20] Hilbert and Ackermann (1928) and Hilbert (1929) define the syntactic completeness (without proof) of a system of axioms for number theory, such that if an unprovable formula of number theory is added to the axioms of the system, then a contradiction follows from the extended axiomatic system. This claim draws from Post 1921and Bernays 1926.

[21] Gödel (1931) makes it clear when he writes, "I wish to note expressly that Theorem XI (and the corresponding results for M and A) do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of P (or of M or A)." (1931, p. 197; 1986, p. 195, original emphasis).

requires a larger system. However, they do not challenge the finitary nature of the proof, which constitutes the cornerstone of Hilbert's proof theory. Therefore, he is quite right to assert the philosophical significance of his proof theory beyond its mathematical value, even though no system of rules is absolutely closed. Hilbert (1928) writes:

> The formula game that Brouwer so deprecates has, besides its mathematical value, an important general philosophical significance. For this formula is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. These rules form a closed system that can be discovered and definitively stated. The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. (1967, p. 475, original emphasis)

Thus, a finitary metamathematics is a collection C of numerals and formulae, such that a formula F proves the syntactic consistency of C; yet the formula F itself is proved to be syntactically consistent only if we deal with a system larger than C, since F is undecidable (unprovable) in the consistent collection C. Metamathematics is *logically* incomplete, and has no vocation to be an absolute logical principle; which matters is its definition as a consistent and complete foundation for mathematics. In other words, Hilbert's reduction of the infinite semantics of mathematics to the finitary syntax of metamathematics is both logically consistent and mathematically complete.

Conclusion

Should we be disappointed by the second-level definition of an arithmetic continuum which is devoid of logical truth and existence? All depend upon our metaphysical commitments about mathematical entities. Peano's arithmetic abstractly defines the complete induction of an actual infinite, such that natural numbers are second-level elements. Strictly speaking, they are abstract properties of the countable set N, but we intuitively know what natural numbers are because we are able to construct them individually through an intuitive mathematical induction. Thus, we may discard the set N without destroying the concept of natural numbers, and the same is true for integers with respect to Z and for rational numbers with respect to Q. By contrast, we do not have a similar safety net for real numbers, insofar as Dedekind cuts are pure abstractions irreducible to countable abstract elements, let alone individual numbers. That is why Brouwer and Weyl resort to rational numbers at the expense of real numbers, since second-level real numbers are, after all, not numbers at all.

Should we follow mathematical intuitionism by rejecting both infinite cardinalities and the actual infinites for the sake of intuition? The relevance of the potential infinite rests upon our ability to think of this concept intuitively. Yet, the promotion of intuition at the expense of mathematical abstraction is hazardous, since the mathematics of numbers rests upon counterintuitive properties. For instance, a Cartesian line of numbers rejects the intuition that a geometrically continuous line is composed of smooth lines and not of broken points. Then, the essential question is, why should intuition matter with respect to the infinite, while it is wholly ignored regarding algebraic properties? If intuition were accepted, a continuum should be nothing more than a smooth whole devoid of discrete numbers. In this sense, Peirce's mathematical pragmatism is philosophically more consistent than mathematical intuitionism, for he rejects all counterintuitive properties, i.e. not only the actual infinites but also the algebraic sequences of rational numbers. Algebraic formalism implies a counterintuitive definition of continuity in the same way that the actual infinites are counterintuitive abstractions replacing our intuitive idea of the potential infinite. Therefore, both algebraic and set-theoretic revolutions overcome the prejudice that only the intuitive idea of continuity and potential infinite are meaningful.

Some may object that no actual infinite is directly provable since the axiom of choice is a mere postulate, and that Cantor's proofs of the uncountability of a continuum are mere *reductio ad absurdum*. However, is there a direct proof that a geometric line *corresponds* to a succession of algebraic points within Cartesian coordinates? Mathematicians identify points with numbers, and we agree with this abstract system for the sake of its efficiency; yet, there is no constructive and direct proof demonstrating that a number *is* a point, or a point a number. This question is irrelevant (and inconsistent) for any axiomatic mathematics, but is not so for mathematical intuitionism which aims to prove each existent mathematical object through a direct and concrete proof. Thus, only a mysterious coincidence gently compels us to believe in the abstraction that points are numbers, and numbers are points. Likewise, the axiom of choice gently compels us to believe in the well-ordering of any infinite set. If we doubt the axiom of choice in the definition of an actual infinite, we must doubt as well that the points of geometry may correspond to the numbers of algebra. As well, we must doubt that the derivative of a function at a given point *is* the slope of a tangent line at that point, since we are unable to justify the equivalence between the algebraic derivative and the geometric slope; we merely postulate it. As these postulates are essential to the construction of mathematics, we should accept them whether applicable to algebraic geometry or set theory. Accordingly, there is a parallel to draw between the algebraic abstraction of a geometric extension into an extensionless sequence of numbers and the set-theoretic abstraction of a potential infinite into an actual infinite. Unless we are willing to contradict ourselves, we cannot change our beliefs in the middle of the road, rejecting intuition in the definition of a continuum of algebraic numbers but accepting it in the construction of an infinite domain of numbers.

Hilbert's axiomatic foundational mathematics constitutes a balanced view, insofar as all mathematical abstractions are acknowledged, although they are neither ontologically existent nor logically definable. We avoid applying the logical criteria of truth and existence, since second-level mathematical semantics rejects the awkward belief in first-level concepts under which mysterious objects fall. Thus, infinite cardinalities are properties of abstract sets as well as algebraic numbers are elements of abstract collections. Since the truth and existence of such abstractions are irrelevant questions, there is no need to believe in a first-level coincidence between geometric points and algebraic numbers or to compare the actual infinites with the first-level intuition of a potential infinite. All mathematical concepts have names, interchangeable with any terms of the dictionary. This amounts to recognising the full power of mathematical abstraction independently of intuition, and Hilbert's axiomatisation is the perfect tool, defining continuity through an axiomatic property of completeness. Dedekind's arithmetic continuum is somehow misleading when defined as a mathematical object constructed by the mind; it is nothing more than a purely mathematical abstraction, both semantically hypothetical and syntactically consistent. The metamathematical axiomatisation of the complete mathematical semantics succeeds in reducing the infinite semantics of an arithmetic continuum to a consistent finitary syntax. By being irreducible to a first-level semantics, an arithmetic continuum succeeds in being a consistent mathematical abstraction; it escapes from both the intuitive idea of continuity and the servitude of being a mysterious mathematical object.

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