Fourier-Mukai transforms for surfaces and moduli spaces of stable sheaves

Tom Bridgeland

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Abstract

In this thesis we study Fourier-Mukai transforms for complex projective surfaces. Extending work of A.I. Bondal and D.O. Orlov, we prove a theorem giving necessary and sufficient conditions for a functor between the derived categories of sheaves on two smooth projective varieties to be an equivalence of categories, and use it to construct examples of Fourier-Mukai transforms for surfaces. In particular we construct new transforms for elliptic surfaces and quotient surfaces. This enables us to identify all pairs of complex projective surfaces having equivalent derived categories of sheaves. We also derive some general properties of Fourier-Mukai transforms, and give examples of their use. The main applications are to the study of moduli spaces of stable sheaves. In particular we identify many such moduli spaces on elliptic surfaces, generalising results of R. Friedman.

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Other sources of inspiration have been the papers of S. Mukai, A.I. Bondal and D.O. Orlov, and conversations with Martijn Dekker and Fabio Pioli.

Finally I would like to thank my friends, flatmates and climbing partners. Without them, this thesis might well have been much bigger and better, but I would not have had half as much fun whilst writing it.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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(*Tom Bridgeland*)

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Introduction

1.1 The main idea

Many problems in algebraic geometry are concerned with classes of sheaves on projective varieties. The idea behind the theory of FM transforms is to replace a problem about sheaves on one variety X with a different (hopefully easier) problem involving sheaves on a second variety Y. The most successful applications so far have been to the study of moduli spaces of sheaves on smooth projective varieties.

Moduli spaces of sheaves are important for several reasons. Firstly, the class of sheaves on a variety is an intrinsically interesting object which contains a great deal of geometric information. Secondly, moduli spaces of sheaves allow one to define new invariants of varieties. Thirdly, physicists have interpreted various moduli spaces as solution spaces to physically interesting differential equations.

Throughout we shall work in the category of schemes of finite type over a fixed algebraically closed field k of characteristic zero. In Part II we shall always take $k = \mathbb{C}$. By a sheaf on a scheme X we mean a coherent $\mathcal{O}_X$-module. A *family of sheaves* on a variety X, parameterised by a scheme S, is a sheaf $\mathcal{E}$ on $S \times X$, flat over S. For each closed point $s \in S$, $\mathcal{E}_s$ denotes the sheaf on X obtained by restricting $\mathcal{E}$ to the subvariety $\{s\} \times X$. We often write the family $\mathcal{E}$ as a set $\{\mathcal{E}_s : s \in S\}$. Two families $\mathcal{E}$ and $\mathcal{F}$ on $S \times X$ are said be equivalent if $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* L$ for some line bundle L on S. Note that this implies that $\mathcal{E}_s \cong \mathcal{F}_s$ for all $s \in S$.

Let $\mathcal{A}$ be a class of sheaves on a variety X. A *universal family* for the class $\mathcal{A}$ is a family $\{\mathcal{P}_m : m \in \mathcal{M}\}$ of sheaves in $\mathcal{A}$, parameterised by a scheme $\mathcal{M}$, such that whenever $\{\mathcal{E}_s : s \in S\}$ is a family of sheaves in $\mathcal{A}$, there is a unique morphism $f : S \to \mathcal{M}$ such that the families $\mathcal{E}$ and $(f \times 1_X)^*(\mathcal{P})$ are equivalent. If such a universal family exists then $\mathcal{M} = \mathcal{M}(\mathcal{A})$ is called a *fine moduli space* for the class $\mathcal{A}$. Note that the closed points of $\mathcal{M}$ correspond to isomorphism classes of sheaves in $\mathcal{A}$ in a natural way.

By no means all classes of sheaves on a given variety will have moduli spaces, but as we shall see later, one can define a notion of *stability* for sheaves on a projective variety, such that the class of stable sheaves with certain numerical invariants always has a fine moduli space. It is these moduli spaces which we wish to study.

The application of FM transforms to the problem of computing moduli spaces is based on the following simple observation. Suppose we can find a bijection Φ between two classes of sheaves $\mathcal{A}$ and $\mathcal{B}$, on varieties X and Y respectively. Suppose further that Φ preserves families, in the sense that whenever $\{\mathcal{E}_s : s \in S\}$ is a family of sheaves in $\mathcal{A}$, then $\{\Phi(\mathcal{E}_s) : s \in S\}$ is a family of sheaves in $\mathcal{B}$. Then the moduli problems for $\mathcal{A}$ and $\mathcal{B}$ are identical, and in particular if fine moduli spaces $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{B})$ exist then they are isomorphic.

In Chapter 3 we shall prove the following theorem.

Theorem 1.1.1. *Let X and Y be smooth projective varieties of dimension n over an algebraically closed field k of characteristic zero, and let $\mathcal{P}$ be a locally free sheaf on $Y \times X$. Let π_X and π_Y be the projection maps $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$. Suppose that for any closed point $y \in Y$,*

$$\mathrm{Hom}_X(\mathcal{P}_y, \mathcal{P}_y) = k \text{ and } \mathcal{P}_y \otimes \omega_X = \mathcal{P}_y,$$

and for any pair of distinct closed points $y_1, y_2 \in Y$,

$$\mathrm{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0 \quad \forall i \in \mathbb{Z}.$$

Then, for any integer i, there is a family-preserving bijection Φ^i between the class of coherent sheaves F on Y satisfying

$$\mathbf{R}^j\pi_{X,*}(\mathcal{P} \otimes \pi_Y^* F) = 0 \text{ unless } j = i, \tag{1.1}$$

and the class of coherent sheaves E on X satisfying

$$\mathbf{R}^j\pi_{Y,*}(\mathcal{P}^\vee \otimes \pi_X^* E) = 0 \text{ unless } j = n - i,$$

given by the formula $\Phi^i(F) = \mathbf{R}^i\pi_{X,*}(\mathcal{P} \otimes \pi_Y^* F)$.

The bijection Φ^i of the theorem is (roughly speaking) an FM transform. There are certain formal analogies with the usual Fourier transform for real-valued functions. In particular, the family $\{\mathcal{P}_y : y \in Y\}$ of the theorem could be thought of as an 'orthonormal set' of sheaves on X.

The sheaves F on Y satisfying condition (1.1) are said to be WIT with respect to the transform Φ. Here WIT stands for 'weak index theorem'. As we shall see, the sheaves of interest in moduli problems are often WIT with respect to a suitably chosen transform.

Let $\mathrm{D}(X)$ denote the bounded derived category of coherent sheaves on a variety X. This is the category obtained by adding morphisms to the homotopic category of bounded complexes of coherent sheaves on X in such a way that any morphism of complexes which induces isomorphisms in cohomology becomes an isomorphism. See Chapter 2 for more details.[1]

Given a pair of smooth varieties X and Y, and an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$, one can define an *integral functor*

$$\Phi^{\mathcal{P}}_{Y \to X} : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

by the formula

$$\Phi^{\mathcal{P}}_{Y \to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)). \tag{1.2}$$

Note that we must use the derived tensor product functor $\overset{\mathbf{L}}{\otimes}$ when we allow arbitrary objects $\mathcal{P}$ of $\mathrm{D}(Y \times X)$. This is not necessary if $\mathcal{P}$ is a Y-flat sheaf on $Y \times X$, as is the case in most of our applications.

It turns out that if we take X, Y and $\mathcal{P}$ as in Theorem 1.1.1, the functor $\Phi^{\mathcal{P}}_{Y \to X}$ is an equivalence of categories. The fact that the maps Φ^i are bijections is a simple consequence of this. In general, a Fourier-Mukai transform is defined to be an equivalence

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

between the bounded derived categories of coherent sheaves on two projective varieties, which is isomorphic to a functor of the form $\Phi^{\mathcal{P}}_{Y \to X}$ for some object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$.

Although rather abstract, derived categories of sheaves provide the natural context for the theory of FM transforms. Indeed, it is hard to see how one could prove Theorem 1.1.1 without them. In applications, however, one often thinks of an FM transform as a set of correspondences between certain classes of sheaves, as in Theorem 1.1.1.

1.2 Moduli spaces of stable sheaves

The concept of a stable sheaf was introduced by D. Mumford in the early 1960's [Mum1]. If X is a smooth curve, and E is a vector bundle on X, then E is said

[1] In this introductory chapter, and throughout Part II, $\mathrm{D}(X)$ will denote the bounded derived category of coherent sheaves on X. Note, however, that in Part I, this category will be denoted $\mathrm{D}^b_c(X)$, as in [Ha1], and the notation $\mathrm{D}(X)$ will be reserved for the (unbounded) derived category of (arbitrary) $\mathcal{O}_X$-modules.

to be *semistable* if for all proper sub-bundles $0 \neq A \subsetneq E$ one has

$$\frac{\mathrm{d}(A)}{\mathrm{r}(A)} \leq \frac{\mathrm{d}(E)}{\mathrm{r}(E)},$$

where $\mathrm{d}(E)$ and $\mathrm{r}(E)$ denote the degree and rank of the bundle E respectively. If the inequality is always strict, E is said to be *stable.*

Mumford used his geometric invariant theory [Mum2] to show that for any pair of integers (r, d) with $r > 0$, the class of semistable bundles of rank r and degree d on a smooth curve X has a coarse moduli space $\mathcal{M}_X(r, d)$. This is a weaker condition than the existence of a fine moduli space (as defined above), and in general the points of $\mathcal{M}_X(r, d)$ represent isomorphism classes of semistable bundles modulo a non-trivial equivalence relation. However one can show that $\mathcal{M}_X(r, d)$ is a fine moduli space whenever r and d are coprime.

It follows from Mumford's construction that $\mathcal{M}_X(r, d)$ is a projective scheme. Much work has been done on the geometric properties of these spaces. We refer to [Le] for details. As an example of what is known, we mention the fact that when the curve X has genus $g > 1$, $\mathcal{M}_X(r, d)$ is an (irreducible) variety of dimension $r^2(g-1)+1$, and is smooth whenever r and d are coprime. Note that bundles on smooth curves of genus 0 and 1 were already completely classified in the 1950s by A. Grothendieck [Gr1] and M.F. Atiyah [At] respectively.

In the 1970's D. Gieseker, M. Maruyama and F. Takemoto attempted to generalise Mumford's construction so as to include bundles on higher-dimensional projective varieties, particularly smooth surfaces [Gi], [Mar], [Ta]. Two new ideas were required. Firstly, to obtain a compact moduli space one has to include torsion-free sheaves which are not locally free (a torsion-free sheaf on a smooth curve is necessarily locally free). Secondly to define stability of sheaves on a higher-dimensional variety X, one must first choose a polarisation of X (i.e. a numerical equivalence class of ample line bundles).

The problem of calculating moduli spaces of sheaves on smooth surfaces was given added impetus in the 1980's by the work of S.K. Donaldson on real four-manifolds [DK]. He showed that a vector bundle on a complex algebraic surface is stable (in the sense of Mumford-Takemoto) if and only if the corresponding bundle on the underlying four-manifold admits an irreducible Hermitian-Einstein connection [Do1]. This allowed certain moduli spaces of stable bundles to be interpreted as solution spaces to the anti-self-dual Yang-Mills equations (so-called *instanton spaces*). Furthermore, Donaldson defined new invariants for four-manifolds [Do2], which in the case of complex algebraic surfaces were most easily computed by studying the corresponding moduli spaces of stable sheaves [OG1], [Li1].

The general definition of stability of sheaves is due to C. Simpson [Si]. Let X

be a complex projective scheme and choose a polarisation ℓ on X. A sheaf E on X is said to have *pure dimension* d if the support of any non-zero subsheaf of E has dimension d. For example, a sheaf on a smooth surface has pure dimension 2 precisely when it is torsion-free. A sheaf E on X is said to be *semistable* if it has pure dimension and if for all proper subsheaves $0 \neq A \subsetneq E$, one has

$$\wp_A(n) \leq \wp_E(n) \qquad \forall n \gg 0. \tag{1.3}$$

Here $\wp_E$ denotes the normalised Hilbert polynomial of the sheaf E with respect to the projective embedding of X determined by ℓ. If the inequality (1.3) is always strict for large n, E is said to be *stable*. Simpson showed that the class of semistable sheaves with fixed numerical invariants on a complex projective scheme X (with respect to a given polarisation) always has a coarse moduli space which is a projective scheme. See Chapter 5 for more details.

Very little is known about these spaces in general, but in the case when X is a smooth complex surface they have been the focus of a great deal of research in recent years. If (X, ℓ) is a smooth polarised complex surface, we shall use the notation $\mathcal{M}_X^\ell(r, \Delta, c)$ to denote the coarse moduli space of sheaves E on X with rank r and Chern classes $c_1(E) = \Delta$, $c_2(E) = c$, which are semistable with respect to the polarisation ℓ. The main thrust of this thesis is to use Fourier-Mukai transforms to study these moduli spaces.

We shall review some of the known facts concerning moduli spaces of semistable sheaves on surfaces. Throughout we use the word surface to mean a smooth complex projective variety of dimension 2. We write $\mathrm{NS}(X)$ for the Neron-Severi group of such a surface (i.e. the subgroup of $\mathrm{H}^2(X, \mathbb{Z})$ consisting of first Chern classes of line bundles on X). Recall that the Hilbert scheme $\mathrm{Hilb}^n X$ parameterises zero-dimensional subschemes of X of length n, and is a smooth projective variety birationally equivalent to the symmetric product $\mathrm{Sym}^n X$.

The first class of results is concerned with the asymptotic behaviour of the spaces $\mathcal{M}_X^\ell(r, \Delta, c)$ for large c. These are due to Gieseker working with J. Li [GL], and independently K.G. O'Grady [OG3].

Theorem 1.2.1. (Gieseker-Li, O'Grady) *Let (X, ℓ) be a smooth, polarised surface, fix $r \in \mathbb{N}$ and $\Delta \in \mathrm{NS}(X)$. Then there is a constant N such that whenever $c > N$, the projective scheme $\mathcal{M}_X^\ell(r, \Delta, c)$ is a variety (i.e. is reduced and irreducible), of dimension*

$$2rc - (r-1)\Delta^2 - (r^2-1)\chi(\mathcal{O}_X) + \dim_{\mathbb{C}} \mathrm{H}^1(X, \mathcal{O}_X).$$

Furthermore, if ℓ' is another polarisation of X, there is a constant M such that for $c > M$ the space $\mathcal{M}_X^{\ell'}(r, \Delta, c)$ is birationally equivalent to $\mathcal{M}_X^\ell(r, \Delta, c)$. □

In the rank 2 case even more is known. For example, for large c, the Picard group of the space $\mathcal{M}_X^\ell(2,\Delta,c)$ is known (up to finite index) [Li4], as are its first two Betti numbers [Li3]. It seems likely that these results can be generalised to include higher rank moduli.

Other researchers have focused on calculating moduli spaces on certain special surfaces X. For surfaces X of general type no moduli spaces have been explicitly calculated, although a result of Li [Li2] states that for general X most of the spaces $\mathcal{M}_X^\ell(r,\Delta,c)$ are of general type for large c. However for surfaces of lower Kodaira dimension many results are now known.

Any surface X of Kodaira dimension 1 is an elliptic surface, that is to say there is a smooth curve C and a morphism $\pi : X \to C$ whose general fibre is an elliptic curve. Semistable sheaves of rank 2 on such surfaces have been studied by various authors [Ba], [Fr], [FM], [LO]. The aim was to compute Donaldson invariants of the underlying real four-manifolds. This project was highly successful and led to a complete understanding of the relationship between diffeomorphism and deformation equivalence for complex elliptic surfaces.

One of Friedman's main results was the following theorem [Fr, Part III].

Theorem 1.2.2. (Friedman) *Let $\pi : X \to C$ be a simply-connected, relatively minimal, nodal elliptic surface with at most two multiple fibres, and let $\Delta \in \mathrm{NS}(X)$ be such that $\Delta \cdot f = 2b-1$ is odd, where f is the cohomology class of a fibre of π. Then for any integer c, such that*

$$2t = 4c - \Delta^2 - 3\chi(\mathcal{O}_X) \geq 0,$$

and any suitable polarisation ℓ of X, the moduli space $\mathcal{M}_X^\ell(2,\Delta,c)$ is a smooth variety, which is birationally equivalent to $\mathrm{Hilb}^t(\mathrm{J}_{X/C}(b))$, where $\mathrm{J}_{X/C}(b)$ is the elliptic surface over C whose fibre over a general point $p \in C$ is the space of line bundles of degree b on the elliptic curve $\pi^{-1}(p)$.

Here a *nodal* elliptic surface is one whose singular fibres are all either multiple fibres whose reductions are smooth, or reduced and irreducible with a single ordinary double point. A *suitable* polarisation is one for which a sheaf with the given numerical invariants is stable precisely when its restriction to the general fibre of π is stable.

Other rank 2 moduli spaces on elliptic surfaces have also been studied. In general these spaces are non-fine and non-reduced.

In Chapter 8 we shall use FM transforms to give a generalisation of Theorem 1.2.2 to sheaves of arbitrary rank.

Theorem 1.2.3. *Let $\pi : X \to C$ be a simply-connected, relatively minimal elliptic surface and take a triple*

$$(r, \Delta, c) \in \mathbb{N} \times \mathrm{NS}(X) \times \mathbb{Z},$$

such that r is coprime to $\Delta \cdot f$, and

$$2t = 2rc - (r-1)\Delta^2 - (r^2-1)\chi(\mathcal{O}_X) \geq 0.$$

Then for any suitable polarisation ℓ of X, the moduli space $\mathcal{M}_X^\ell(r, \Delta, c)$ is a smooth variety, which is birationally equivalent to $\mathrm{Hilb}^t(\mathrm{J}_{X/C}(b))$, where b is an integer such that $br \equiv 1$ modulo $\Delta \cdot f$.

Note that we do not need to impose any conditions on the elliptic surface $\pi : X \to C$ beyond relative minimality. A similar result has been proved by K. Yoshioka [Yo1] using different methods.

Turning now to the case when X has Kodaira dimension 0, note that any such surface has a non-branched cover by either a K3 surface or an Abelian surface. Sheaves on quotient surfaces, i.e. Enriques and bielliptic (or hyperelliptic) surfaces, are best studied by looking at invariant sheaves on the cover [Nai], or by using the elliptic fibrations which these surfaces always have, so let us assume that X has trivial canonical bundle. Let ϵ be 0 or 1, depending on whether X is an Abelian surface or a K3 surface, respectively.

S. Mukai [Muk5] introduced a lattice structure (the *Mukai lattice* of X) on the group $\mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}$, by putting

$$\langle (r_1, \Delta_1, k_1), (r_2, \Delta_2, k_2) \rangle = \Delta_1 \cdot \Delta_2 - r_1 k_2 - r_2 k_1.$$

For any pair of sheaves E and F on X the Riemann-Roch theorem asserts that

$$\chi(E, F) := \sum_{i=0}^{2} \dim_{\mathbb{C}} \mathrm{Ext}^i_X(E, F) = -\langle \mathrm{v}(E), \mathrm{v}(F) \rangle.$$

where

$$\mathrm{v}(E) = (\mathrm{r}(E), \mathrm{c}_1(E), \frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E) - \epsilon\, \mathrm{r}(E))$$

is the *Mukai vector* of the sheaf E. In [Muk3] Mukai proves

Theorem 1.2.4. (Mukai) *Let (X, ℓ) be a polarised Abelian or K3 surface, and take an element v of the Mukai lattice of X. Suppose that all sheaves with Mukai vector v which are semistable (with respect to ℓ) are stable. Then the moduli space $\mathcal{M}_X^\ell(v)$ of stable sheaves (with respect to ℓ) with Mukai vector v is either empty or a finite disjoint union of smooth projective complex symplectic varieties of dimension $\langle v, v \rangle + 2$.* □

For details on complex symplectic manifolds see [Be], [Hu]. Theorem 1.2.4 aroused great interest because it provided new examples of compact hyperkähler manifolds, which are of importance in mathematical physics. Mukai also asserted [Muk6, Thm 5.15], that when X was a K3 surface, each space $\mathcal{M}_X^\ell(v)$ of rank 2 sheaves with dimension larger than 2 was non-empty and irreducible, with weight 2 Hodge structure given by the form $\langle\ ,\ \rangle$ restricted to the subspace $v^\perp$ of the Mukai lattice. These statements were later verified by O' Grady [OG2] using a deformation argument, after making some extra assumptions on v. If the global Torelli conjecture [Hu, 10.1], for complex symplectic manifolds is shown to be true, this will be enough to determine the birational types of the moduli spaces.

In the case when X is an Abelian surface analagous results should hold. Recently, M. Dekker [De] has used an argument similar to O' Grady's to show that many of the moduli spaces of stable sheaves on Abelian surfaces are indeed irreducible.

Many particular examples of moduli spaces of stable sheaves on K3 surfaces and Abelian surfaces have been computed using FM transforms [BM], [Mac1], [Mac4], [Mac5], [Muk2], [Muk4]; indeed it was for this purpose that FM transforms were first introduced. Other methods have also proved succesful, particularly in the rank 2 case [Nak2], [Qi3], [Yo2], [Zuo]. In Chapter 7, as an example of the use of Mukai's original transform, we shall prove

Theorem 1.2.5. *Let (X, ℓ) be a principally polarized Abelian surface, and let $n \leq m$ be positive integers, with m even. Then there is a component of the moduli space $\mathcal{M}_X^\ell(mn+1, n\ell, 0)$ which is birationally equivalent to $X \times \mathrm{Hilb}^{n^2} X$.*

The remaining cases are when X has Kodaira dimension $-\infty$. Here FM transforms are less useful, although Yoshioka [Yo1] used Theorem 1.2.3 to prove that certain moduli spaces of sheaves on $\mathbb{P}^2$ were rational. Here we content ourselves with giving some recent references. Stable sheaves on $\mathbb{P}^2$ have been studied by many authors. We refer to [Le] and [OSS] for details. Briefly, it is known that each moduli space $\mathcal{M}_{\mathbb{P}^2}(r, c_1, c_2)$ is either empty or irreducible, and there is a simple numerical condition [Le, Thm. 16.2.1] for distinguishing the two cases. Even less is known about stable sheaves on ruled surfaces. See [Bro], [Bu], [Qi2], [Wa].

There are two final topics which we should mention. Firstly, most of the results discussed above are concerned with sheaves on minimal surfaces. Stable sheaves on blown-up surfaces have been studied in [Bru] and [Nak1]. Secondly, it is an interesting problem to understand how the structure of a given moduli space of stable sheaves on a surface X changes when one varies the polarisation of X. There has been recent progress in this area. See [EG], [Qi1], [Qi4] for details.

1.3 Fourier-Mukai transforms

Recall that a Fourier-Mukai transform is an equivalence between the derived categories of sheaves on two projective varieties. The first such equivalence was introduced by Mukai in 1981 [Muk2], although its use is implicit in the earlier paper [Muk1].

Theorem 1.3.1. (Mukai) *Let X be an Abelian variety, $\hat{X}$ its dual, and $\mathcal{P}$ the Poincaré line bundle on $\hat{X} \times X$. Then the functor*

$$\mathcal{F} = \Phi^{\mathcal{P}}_{\hat{X}\to X} : \mathrm{D}(\hat{X}) \longrightarrow \mathrm{D}(X)$$

defined by the formula (1.2) is an equivalence of categories.

Mukai called $\mathcal{F}$ the *Fourier functor*, and went on to use it to study various moduli spaces of sheaves on Abelian varieties. In particular he showed [Muk4, Thm. 0.3, Cor. 4.5], that if (X, ℓ) is a principally polarised Abelian surface, not a product of elliptic curves, then for any integer $r \geq 1$,

$$\mathcal{M}^{\ell}_X(r, \ell, 2) \cong X \times \mathrm{Hilb}^{r+1} X, \qquad \mathcal{M}^{\ell}_X(r, 0, 1) \cong X \times \mathrm{Hilb}^{r} X.$$

Following Mukai's work, various authors used the functor $\mathcal{F}$ to study further moduli problems on Abelian surfaces [BMT], [Mac1], [Mac3]. In particular, there was much interest in applications to moduli spaces of instanton bundles, and the relation between Mukai's functor and its analytic analogue, the Nahm transform [BvB]. The functor was also used by A. Maciocia to study enumerative problems on principally polarised Abelian surfaces [Mac6].

In view of the success of the functor $\mathcal{F}$ in solving moduli problems, it was natural for researchers to look for similar transforms for sheaves on other (non-Abelian) varieties. Mukai had already made some progress in this direction in his study of moduli spaces of sheaves on K3 surfaces [Muk5]. Indeed the reflection functor [Muk5, p. 362] is an FM transform.

In general Mukai showed that if X is a K3 surface, Y is a complete, 2-dimensional fine moduli space of stable sheaves on X, and $\mathcal{P}$ is a universal sheaf on $Y \times X$, then Y is also a K3 surface, and the functor $\Phi^{\mathcal{P}}_{Y\to X}$ is fully faithful [Muk5, Prop. 4.10]. Unfortunately, he was not able to show that the functor was an equivalence, although he proved that the induced map on cohomology was an isomorphism of Hodge structures. This was enough for him to determine the K3 surface Y, using the global Torelli theorem [Muk5, Thm. 1.5].

The functors $\Phi^{\mathcal{P}}_{Y\to X}$ were proved to be equivalences in a special case by C. Bartocci, U. Bruzzo and D. Hernández Ruipérez [BBH], and this enabled Bruzzo and Maciocia [BM] to show that various moduli spaces of stable sheaves on K3 surfaces were Hilbert schemes. In fact, as we shall prove in Chapter 6 (see also [Bri2]), the functors $\Phi^{\mathcal{P}}_{Y\to X}$ are all equivalences:

Theorem 1.3.2. *Let X be a K3 surface, Y a complete, 2-dimensional fine moduli space of stable sheaves on X and $\mathcal{P}$ a universal sheaf on $Y \times X$. Then Y is a K3 surface, and the functor $\Phi^{\mathcal{P}}_{Y\to X}$ defined above is an equivalence of categories.*

The new equivalences of Bartocci, Bruzzo *et al* were called (generalised) Fourier-Mukai transforms. Maciocia [Mac5] gave further examples of such transforms for Abelian surfaces, and these led to new results on moduli spaces of sheaves on Abelian surfaces [Mac4], [Mac5]. Clearly some sort of classification of FM transforms (at least for surfaces) was in order.

In [Mac4], Maciocia considered a pair of smooth projective varieties X and Y and a Y-flat sheaf $\mathcal{P}$ on $Y \times X$, and looked for conditions for the resulting functor

$$\Phi^{\mathcal{P}}_{Y\to X} : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

to be an equivalence of categories. He showed that the family $\{\mathcal{P}_y : y \in Y\}$ had to be a complete family of simple sheaves of the same dimension as X. Unfortunately he mistakenly asserted that if the rank r of $\mathcal{P}$ was positive, then the canonical line bundle ω_X had to be trivial. In fact the correct condition is $\omega_X^{\otimes r} \cong \mathcal{O}_X$.

Nonetheless it was clear that the existence of an FM transform $\Phi^{\mathcal{P}}_{Y\to X}$ related to the triviality of the canonical sheaf ω_X. This observation led to the idea of relative FM transforms for elliptic surfaces [Bri1]. These are studied in Chapter 8. In particular we shall prove

Theorem 1.3.3. *Let $X \xrightarrow{\pi} C$ be a relatively minimal elliptic surface, let $f \in \mathrm{NS}(X)$ be the divisor class of a fibre of π, and take a polarisation ℓ of X. Let $a > 0$ and b be integers such that $a\ell \cdot f$ is coprime to b, put $Y = \mathcal{M}^{\ell}_X(0, af, -b)$ and let $\mathcal{P}$ be a universal sheaf on $Y \times X$. Then Y is isomorphic to the elliptic surface $\mathrm{J}_{X/C}(b)$ (see Theorem 1.2.2), and the functor $\Phi^{\mathcal{P}}_{Y\to X}$ is an equivalence of categories.*

Note that Y parameterises stable sheaves supported on the fibres of π. The resulting relative transforms lead to a simple proof of Theorem 1.2.3. In the case when X is an elliptic K3 surface these transforms have been related to ideas in string theory [BBHM].

If X is a general elliptic surface of Kodaira dimension 1, one can show that all FM transforms for X are given by Theorem 1.3.3. However if X has Kodaira dimension 0 it may be an elliptic surface in more than one way. Composing the equivalences of Theorem 1.3.3 corresponding to different elliptic fibrations leads to a large number of FM transforms. In particular this approach yields new transforms for Enriques and bielliptic surfaces. An FM transform for Enriques surfaces corresponding to Mukai's reflection functor for K3 surfaces had already been constructed by S. Zube [Zub]. We shall discuss transforms on quotient surfaces in detail in Chapter 9.

Simultaneously with this research, a group of Russian mathematicians including A.I. Bondal, M.M. Kapranov, and D.O. Orlov were undertaking a more abstract study of the derived category of sheaves on a projective variety [Bo], [BK], [Ka], [Or1], [Ru]. In the course of this work they looked at functors of the form (1.2) in which $\mathcal{P}$ was allowed to be any object of $\mathrm{D}(Y \times X)$. In 1995, following the original ideas of Mukai, Bondal and Orlov [BO1] gave the following necessary and sufficient conditions for such a functor to be fully faithful.

Theorem 1.3.4. (Bondal-Orlov) *Suppose X and Y are smooth projective varieties over an algebraically closed field k and $\mathcal{P}$ is an object of $\mathrm{D}(Y \times X)$. Then the functor $F = \Phi^{\mathcal{P}}_{Y \to X}$ is fully faithful if, and only if, for each point $y \in Y$,*

$$\mathrm{Hom}_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = k,$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i,

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0,$$

unless $y_1 = y_2$ and $0 \leq i \leq \dim Y$.

In Chapter 4 we shall prove

Theorem 1.3.5. *Suppose X and Y are smooth projective varieties over an algebraically closed field k, and*

$$F : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

is a fully faithful exact functor. Then F is an equivalence of categories precisely when $F\mathcal{O}_y \otimes \omega_X \cong F\mathcal{O}_y$ for every point $y \in Y$.

Together these theorems solve the problem of determining when a triple $(X, Y, \mathcal{P})$ consisting of a pair of smooth projective varieties (X, Y) together with an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$ yields a Fourier-Mukai transform.

Orlov also proved the following remarkable result [Or2]

Theorem 1.3.6. (Orlov) *Let X and Y be smooth projective varieties over an algebraically closed field k, and let*

$$F : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

be an exact equivalence of categories. Then there is an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$*, and an isomorphism of functors* $F \cong \Phi^{\mathcal{P}}_{Y\to X}$. □

Thus for smooth projective varieties, all exact equivalences of derived categories are FM transforms. The problem of classifying such transforms splits naturally into two parts. Given a smooth projective variety X these are:

(1) Find the set of varieties Y such that the derived categories $\mathrm{D}(Y)$ and $\mathrm{D}(X)$ are equivalent as triangulated categories. We have called these varieties the *Fourier-Mukai partners* of X.

(2) Find the group of auto-equivalences of $\mathrm{D}(X)$, i.e. the group of exact equivalences $\mathrm{D}(X) \longrightarrow \mathrm{D}(X)$, up to isomorphism of functors.

Note that for any variety X there is a group of auto-equivalences of $\mathrm{D}(X)$ generated by the translation functor on $\mathrm{D}(X)$, together with pull-backs by automorphisms of X and twists by elements of $\mathrm{Pic}(X)$. These transforms will be called *trivial.*

It was proved by Bondal and Orlov [BO1], [BO2] that if X has either ample or anti-ample canonical class, then the only FM partner of X is X itself, and the group of auto-equivalences of $\mathrm{D}(X)$ is exactly the group of trivial transforms.

If X has dimension 1, the only other case is when X is an elliptic curve. Here too, X is its only FM partner, but this time the group of auto-equivalences is generated by the trivial transforms together with the original Fourier functor $\mathcal{F}$ of Theorem 1.3.1.

For higher-dimensional varieties, much less is known. The set of FM partners of a K3 surface was described by Orlov [Or2][2], and problems (1) and (2) have both recently been solved for Abelian varieties, again by Orlov [Or3]. In Chapter 10 we shall give a complete solution to problem (1) when X is a minimal surface.

The thesis is split roughly into two parts. The first part (consisting of Chapters 2,3 and 4) is rather abstract and is concerned with giving conditions for functors of the form (1.2) to be equivalences of categories. The main aim is to prove Theorems 1.3.4 and 1.3.5.

The second part is more geometrical and contains constructions of various FM transforms for projective surfaces, together with applications to computations of moduli spaces.

For further details consult the summaries on pages 16 and 50 respectively.

[2]Lemma 3.12 of [Or2] is incorrect, but once Theorem 1.3.5 is known, the proof of [Or2, Theorem 3.12] is immediate.

Part I

Complexes of sheaves and integral functors

Part I

In Part I we shall study various properties of integral functors. These will be essential for our study of Fourier-Mukai transforms in Part II. Our main aim is to prove Theorems 1.3.4 and 1.3.5. Although Theorem 1.3.4 was proved by Bondal and Orlov, the proof we give is somewhat simpler and seems sufficiently different to be worth including.

We start in Chapter 2 with a summary of the construction of the derived category of an Abelian category, and list some properties of derived categories of sheaves on schemes. These categories will appear frequently in subsequent chapters.

Chapter 3 contains some deeper properties of complexes of coherent sheaves, and in particular we prove a result relating the homological dimension of a complex of sheaves to the depths of the supports of its cohomology objects. These results are used in Chapter 4 where we prove the main inversion theorems and derive some further useful properties of integral functors.

The contents of Chapters 3 and 4 were very much inspired by the papers of Bondal and Orlov, and the original papers of Mukai. With the exception of Theorem 1.3.5, most of the results are already known in some form or other, but many of our statements are more general, and most of our proofs are new.

Notation

Throughout we shall fix an algebraically closed field k of characteristic zero, and work in the category of schemes of finite type over k. Thus all schemes are assumed to be of finite type over k, and all morphisms of schemes are k-morphisms. If X is a scheme, a point x of X, written $x \in X$, will always mean a *closed* or *geometric* point. Thus the points of X correspond to morphisms from $\mathrm{Spec}(k)$ to X.

Recall the following definitions [Ha2, p. 203]. A *complex* X of objects of an Abelian category $\mathcal{A}$ is a sequence $(X^i : i \in \mathbb{Z})$, of objects of $\mathcal{A}$, and a sequence of morphisms $d^i_X : X^i \to X^{i+1}$, called the *differentials* of X, such that $d^{i+1}_X \circ d^i_X = 0$ for all i. The ith *cohomology object* $\mathrm{H}^i(X)$ of such a complex is the quotient $\ker(d^i_X)/\operatorname{im}(d^{i-1}_X)$. We shall also use the notation $H_i(X)$ for the ith *homology* object of X; by definitions this is the $(-i)$th cohomology object $\mathrm{H}^{-i}(X)$ of X.

A *morphism of complexes* $f : X \to Y$ is a sequence of morphisms $f^i : X^i \to Y^i$ such that $d^i_Y \circ f^i = f^{i+1} \circ d^i_X$ for all $i \in Z$. Any such morphism f induces morphisms $\mathrm{H}^i(f) : \mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ on cohomology; if these are all isomorphisms f is said to be a *quasi-isomorphism.* Two morphisms of complexes $f, g : X \to Y$ are called *homotopic* if there are morphisms $h^i : X^i \to Y^{i-1}$ such that for all $i \in \mathbb{Z}$,

$$d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i = f^i - g^i.$$

Homotopic morphisms of complexes induce the same morphisms on cohomology.

Chapter 2

Derived categories

This chapter contains the essential definitions and results concerning derived categories which we shall need in our study of Fourier-Mukai transforms. Our main reference is [Ha1], but [Ve1], [KS] and [GM] are also useful.

2.1 Motivation

Let X be a projective variety over our algebraically closed field k. In the usual procedure for computing the cohomology groups of an $\mathcal{O}_X$-module, one applies the global sections functor $\Gamma(X,-)$ to an injective resolution

$$0 \to E \to I^0 \to I^1 \to I^2 \to \cdots$$

of the $\mathcal{O}_X$-module E. The required cohomology groups $\mathrm{H}^i(X,E)$ then appear as the cohomology of the resulting complex of Abelian groups

$$0 \to \Gamma(X,I^0) \to \Gamma(X,I^1) \to \Gamma(X,I^2) \to \cdots . \tag{2.1}$$

The groups $\mathrm{H}^i(X,E)$ are well-defined because any two injective resolutions of E are homotopy equivalent. Thus if one takes a different injective resolution

$$0 \to E \to J^0 \to J^1 \to J^2 \to \cdots ,$$

the resulting complex of groups

$$0 \to \Gamma(X,J^0) \to \Gamma(X,J^1) \to \Gamma(X,J^2) \to \cdots ,$$

will be homotopy equivalent to (2.1), and so will have the same cohomology.

Derived categories arise when one tries to make this construction more functorial. In particular one would like the operation of replacing a sheaf by an injective resolution to be a functor of some kind. Now a morphism of sheaves does not uniquely define a morphism of injective resolutions, but only a homotopy equivalence class of such morphisms. Thus it is natural to define the category $\mathrm{K}(X)$, whose objects are complexes of $\mathcal{O}_X$-modules, and whose morphisms are homotopy equivalence classes of morphisms of complexes. One can then define a functor from the category of $\mathcal{O}_X$-modules, to $\mathrm{K}(X)$, by choosing an injective resolution for each $\mathcal{O}_X$-module, and the cohomology functor $\mathrm{H}^*(X, -)$ is then obtained by composing this functor with the global sections functor and taking the cohomology of the resulting complex of groups.

This idea is not entirely satisfactory however, because the resolutions one uses to compute cohomology groups in practice are often not injective. Thus one might consider flasque or Čech resolutions of a sheaf E. Similarly, when one is computing the left-derived functors of certain right-exact functors (such as $\otimes$), one must use flat resolutions, since projective resolutions do not usually exist. It is no longer true that two resolutions of these more general types are homotopy equivalent.

Ideally one would like to have a category in which an $\mathcal{O}_X$-module E is isomorphic to any resolution of E one might wish to use. Now a resolution of E is just a quasi-isomorphism between E and a complex of sheaves of some type, so such a category can be obtained by adding morphisms to $\mathrm{K}(X)$ in such a way that all quasi-isomorphisms becomes isomorphisms. The derived category $\mathrm{D}(X)$ is the category obtained by formally inverting all quasi-isomorphisms in $\mathrm{K}(X)$ in this way. Inside $\mathrm{D}(X)$ one may identify an $\mathcal{O}_X$-module with any of its resolutions. Furthermore, the usual functors of algebraic geometry, such as direct images, pull-backs, tensor products etc, induce derived functors between the relevant derived categories. For example, the global sections functor $\Gamma(X, -)$ induces a derived functor

$$\mathbf{R}\Gamma(X, -) : \mathrm{D}(X) \longrightarrow \mathrm{D}(\mathrm{Spec}(k)),$$

and the cohomology groups $\mathrm{H}^i(X, E)$ of a sheaf E are the cohomology objects of $\mathbf{R}\Gamma(X, E)$.

Although the derived category is a rather abstract object, its introduction greatly simplifies many results and proofs which involve homological algebra. For example, results which used to involve spectral sequences (e.g. the Leray spectral sequence), often become simple statements about compositions of derived functors. Furthermore, the derived category allows one to use the ideas of category theory to provide a framework for various results involving cohomology of sheaves. For example, in Grothendieck's duality theory, one is constructing a right adjoint to the derived direct image functor $\mathbf{R}f_* : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y)$ of a proper morphism of schemes $f : X \to Y$. In fact it was in this context that J.- L. Verdier, guided by Grothendieck, first introduced derived categories.

2.2 Triangulated categories

Let $\mathcal{A}$ be an Abelian category.

Definition 2.2.1. Let $\mathrm{K}(\mathcal{A})$ denote the category whose objects are complexes of objects of $\mathcal{A}$, and whose morphisms are homotopy equivalence classes of morphisms of complexes.

One also defines full subcategories $\mathrm{K}^+(\mathcal{A})$, $\mathrm{K}^-(\mathcal{A})$ and $\mathrm{K}^b(\mathcal{A})$ whose objects consist of complexes bounded below, above and on both sides respectively.

Note that $\mathrm{K}(\mathcal{A})$ is not in general Abelian [Ve1, II.1.3.6]. It does however have various special properties, as we now explain.

Firstly, one has a *translation functor* $T : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ which shifts all complexes to the left by one place, and changes the signs of the differentials. Thus for any object X of $\mathrm{K}(\mathcal{A})$, let

$$T(X)^i = X^{i+1}, \qquad d^i_{T(X)} = -d^{i+1}_X,$$

and for any morphism of complexes $u : X \to Y$ representing a morphism $[u]$ of $\mathrm{K}(\mathcal{A})$, let $T([u])$ be the homotopy equivalence class of the morphism of complexes $T(u)$, where

$$T(u)^i = u^{i+1}.$$

Secondly, one has the notion of a *mapping cone*. The mapping cone M_u of a morphism of complexes $u : X \to Y$ is the complex $T(X) \oplus Y$, with differential given by the matrix

$$\begin{pmatrix} d_{T(X)} & 0 \\ T(u) & d_Y \end{pmatrix}.$$

There are morphisms of complexes

$$v_u : Y \to M_u, \qquad w_u : M_u \to T(X),$$

which give a long exact sequence in cohomology

$$\cdots \xrightarrow{\mathrm{H}^{i-1}(w_u)} \mathrm{H}^i(X) \xrightarrow{\mathrm{H}^i(u)} \mathrm{H}^i(Y) \xrightarrow{\mathrm{H}^i(v_u)} \mathrm{H}^i(M_u) \xrightarrow{\mathrm{H}^i(w_u)} \cdots .$$

Definition 2.2.2. Let $\mathcal{C}$ be an additive category with an automorphism $T : \mathcal{C} \to \mathcal{C}$. A *triangle* in $\mathcal{C}$ is a sextuple (X, Y, Z, u, v, w), where X, Y, Z are objects of $\mathcal{C}$ and $u : X \to Y$, $v : Y \to Z$ and $w : Z \to T(X)$ are morphisms of $\mathcal{C}$. A *morphism of triangles*

$$(X_1, Y_1, Z_1, u_1, v_1, w_1) \longrightarrow (X_2, Y_2, Z_2, u_2, v_2, w_2)$$

in $\mathcal{C}$ is a triple (f, g, h) of morphisms of $\mathcal{C}$ making the following diagram commute

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & T(X_1) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & T(X_2). \end{array}$$

Such a triple is an isomorphism if f, g and h are all isomorphisms of $\mathcal{C}$.

The mapping cones of morphisms of complexes provide a natural collection of triangles in $\mathrm{K}(\mathcal{A})$. Thus one says that a triangle in $\mathrm{K}(\mathcal{A})$ is *distinguished* if it is isomorphic to a triangle of the form

$$(X, Y, M_u, [u], [v_u], [w_u])$$

for some morphism of complexes u. This collection of distinguished triangles satisfies the following four axioms (see [KS, I.1.4] for proofs).

(TR1) For any object X the triangle $(X, X, 0, 1_X, 0, 0)$ is distinguished. Every triangle isomorphic to a distinguished triangle is distinguished. Every morphism $u : X \to Y$ can be embedded in a distinguished triangle (X, Y, Z, u, v, w).

(TR2) The triangle (X, Y, Z, u, v, w) is distinguished iff so is the triangle

$$(Y, Z, T(X), v, w, -T(u)).$$

(TR3) Suppose there are two distinguished triangles

$$(X, Y, Z, u, v, w), \qquad (X', Y', Z', u', v', w'),$$

and morphisms $f : X \to X'$, $g : Y \to Y'$ such that $g \circ u = u' \circ f$. Then there is a morphism $h : Z \to Z'$ such that the triple (f, g, h) is a morphism of the first triangle into the second.

(TR4) Suppose there are three distinguished triangles

$$(X_1, X_2, Z_3, u_3, v_3, w_3), \qquad (X_2, X_3, Z_1, u_1, v_1, w_1), \qquad (X_1, X_3, Z_2, u_2, v_2, w_2),$$

such that $u_2 = u_1 \circ u_3$. Then there are two morphisms $m_1 : Z_3 \to Z_2$, $m_3 : Z_2 \to Z_1$ such that $(1_{X_1}, u_1, m_1)$ and $(u_3, 1_{X_3}, m_3)$ are morphisms of triangles, and such that the triangle

$$(Z_3, Z_2, Z_1, m_1, m_3, T(v_3) \circ w_1)$$

is distinguished.

Abstracting these properties of $\mathrm{K}(\mathcal{A})$ gives the notion of a triangulated category.

Definition 2.2.3. A *triangulated category* is an additive category $\mathcal{C}$ together with an automorphism $T_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (called the *translation functor*) and a collection of triangles of $\mathcal{C}$ (called *distinguished triangles*) satisfying the axioms (TR1) - (TR4) above.

Notes. (i) An automorphism of a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ with a genuine inverse, i.e. a functor $T^{-1} : \mathcal{C} \to \mathcal{C}$ such that $T \circ T^{-1} = T^{-1} \circ T = 1_{\mathcal{C}}$.

(ii) In fact none of the results concerning triangulated categories that we shall come across require the octohedral axiom (TR4). We just include it to conform with the standard definition. See [Ve1, II.2.2.12] for more details.

Functors between triangulated categories which preserve the relevant structure are called exact. More precisely

Definition 2.2.4. An additive functor $F : \mathcal{C} \to \mathcal{D}$ between triangulated categories is called *exact* (or a *δ-functor*) if there is an isomorphism of functors $\zeta : F \circ T_{\mathcal{C}} \to T_{\mathcal{D}} \circ F$, such that whenever (X, Y, Z, u, v, w) is a distinguished triangle of $\mathcal{C}$,

$$(F(X), F(Y), F(Z), F(u), F(v), \zeta(Z) \circ F(w))$$

is a distinguished triangle of $\mathcal{D}$.

A third feature of the category $\mathrm{K}(\mathcal{A})$ is the cohomology functor $H : \mathrm{K}(\mathcal{A}) \to \mathcal{A}$ which takes a complex X to its 0th cohomology group, i.e. the quotient

$$\ker d_X^0 / \operatorname{im} d_X^{-1}.$$

For arbitrary triangulated categories, one makes the

Definition 2.2.5. Let $H : \mathcal{C} \to \mathcal{B}$ be an additive functor from a triangulated category to an Abelian category, and write H^i for $H \circ T_{\mathcal{C}}^i$. One says that H is a *cohomological functor* if for any distinguished triangle (X, Y, Z, u, v, w) the long sequence

$$\cdots \xrightarrow{\mathrm{H}^{i-1}(w)} \mathrm{H}^i(X) \xrightarrow{\mathrm{H}^i(u)} \mathrm{H}^i(Y) \xrightarrow{\mathrm{H}^i(v)} \mathrm{H}^i(Z) \xrightarrow{\mathrm{H}^i(w)} \cdots$$

is exact in $\mathcal{B}$.

2.3 Derived categories

Let us fix a triangulated category $\mathcal{C}$, an Abelian category $\mathcal{B}$ and a cohomological functor $H : \mathcal{C} \longrightarrow \mathcal{B}$.

Definition 2.3.1. A morphism $s : x \to y$ in $\mathcal{C}$ is called a *quasi-isomorphism* with respect to H if $H(T_{\mathcal{C}}^i s)$ is an isomorphism for all $i \in \mathbb{Z}$.

Let S denote the collection of all H-quasi-isomorphisms in $\mathcal{C}$. The following proposition [Ha1, I §§3-4], allows one to formally invert the elements of $\mathcal{C}$.

Proposition 2.3.2. *There exists a triangulated category $\mathcal{C}_S$ and an exact functor*

$$Q : \mathcal{C} \longrightarrow \mathcal{C}_S$$

called the localisation functor *with the following universal property*

(a) $Q(s)$ is an isomorphism for all $s \in S$,

(b) if $Q' : \mathcal{C} \to \mathrm{D}$ is an exact functor such that $Q'(s)$ is an isomorphism for all $s \in S$, then there is an exact functor $F : \mathcal{C}_S \to D$, unique up to isomorphism, such that $Q' \cong F \circ Q$. $\square$

Clearly the category $\mathcal{C}_S$ is unique up to equivalence of categories. It is constructed as follows. Firstly one takes the objects of $\mathcal{C}_S$ to be the same as the objects of $\mathcal{C}$. Then for two such objects x and y, one takes $\mathrm{Hom}_{\mathcal{C}_S}(x, y)$ to be the set of equivalence classes of diagrams in $\mathcal{C}$ of the form

$$x \xleftarrow{s_1} z_1 \xrightarrow{a_1} y, \qquad s_1 \in S, \tag{2.2}$$

where another such diagram

$$x \xleftarrow{s_2} z_2 \xrightarrow{a_2} y, \qquad s_2 \in S,$$

is in the same equivalence class as (2.2) if there is an object w and morphisms $f_i : w \to z_i$, $i = 1, 2$, such that $s_1 \circ f_1 = s_2 \circ f_2 \in S$ and $a_1 \circ f_1 = a_2 \circ f_2$. Finally one puts the structure of a triangulated category on $\mathcal{C}_S$ by defining a triangle of $\mathcal{C}_S$ to be distinguished precisely when it is isomorphic to the image under the natural functor $Q : \mathcal{C} \to \mathcal{C}_S$ of a distinguished triangle of $\mathcal{C}$.

Definition 2.3.3. Let $\mathcal{A}$ be an Abelian category. The derived category $\mathrm{D}(\mathcal{A})$ is defined to be the category $\mathrm{K}(\mathcal{A})_S$ where S is the class of H-quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$, and H is the cohomology functor defined in the last section.

One also defines full triangulated subcategories $\mathrm{D}^+(\mathcal{A})$, $\mathrm{D}^-(\mathcal{A})$ and $\mathrm{D}^b(\mathcal{A})$ of $\mathrm{D}(\mathcal{A})$ consisting of objects of $\mathrm{D}(X)$ whose cohomology objects are bounded below, above and on both sides, respectively. There are natural localisation functors $Q : \mathrm{K}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{A})$, where $*$ represents any of the symbols $+$, $-$ or b.

Notes. (i) If

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

is a short exact sequence of complexes of objects of $\mathcal{A}$, then there is a morphism $w : Z \to T(X)$ in $\mathrm{D}(\mathcal{A})$, such that (X, Y, Z, u, v, w) is a distinguished triangle in $\mathrm{D}(\mathcal{A})$ (where we have identified the morphisms of complexes u, v with the corresponding morphisms in $\mathrm{D}(\mathcal{A})$). For a proof see [Hal, I.6.1]

(ii) The full subcategory of $\mathrm{D}(\mathcal{A})$ consisting of objects X satisfying $\mathrm{H}^i(X) = 0$ for $i \neq 0$, is equivalent to the original Abelian category $\mathcal{A}$. We often identify these two categories, by confusing an object X of $\mathcal{A}$ with the complex (X^i) where $X^0 = X$ and $X^i = 0$ for $i \neq 0$.

2.4 Derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. F extends to a functor $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ in the obvious way. The right-derived functor $\mathbf{R}F$ of F (if it exists), is the exact functor which 'comes closest' to making the diagram

$$\begin{array}{ccc} \mathrm{K}^+(\mathcal{A}) & \xrightarrow{F} & \mathrm{K}^+(\mathcal{B}) \\ {\scriptstyle Q}\downarrow & & {\scriptstyle Q}\downarrow \\ \mathrm{D}^+(\mathcal{A}) & \xrightarrow{\mathbf{R}F} & \mathrm{D}^+(\mathcal{B}) \end{array}$$

commute. More precisely one has

Definition 2.4.1. The exact functor $\mathbf{R}F : \mathrm{D}^+(\mathcal{A}) \longrightarrow \mathrm{D}^+(\mathcal{B})$ is called a *right-derived functor* of F if there is a morphism of functors

$$\xi : Q \circ F \longrightarrow \mathbf{R}F \circ Q,$$

such that for any other exact functor $G : \mathrm{D}^+(\mathcal{A}) \longrightarrow \mathrm{D}^+(\mathcal{B})$, and morphism of functors

$$\zeta : Q \circ F \longrightarrow G \circ Q,$$

there is a unique morphism of functors $\eta : \mathbf{R}F \to G$, such that $\zeta = (\eta \circ Q) \circ \xi$.

Clearly, if $\mathbf{R}F$ exists, it is unique up to isomorphism of functors. Analogously, one has

Definition 2.4.2. The exact functor $\mathbf{L}F : \mathrm{D}^-(\mathcal{A}) \longrightarrow \mathrm{D}^-(\mathcal{B})$ is called a *left-derived functor* of F if there is a morphism of functors

$$\xi : \mathbf{L}F \circ Q \longrightarrow Q \circ F,$$

such that for any other exact functor $G : \mathrm{D}^-(\mathcal{A}) \longrightarrow \mathrm{D}^-(\mathcal{B})$, and morphism of functors

$$\zeta : G \circ Q \longrightarrow Q \circ F,$$

there is a unique morphism of functors $\eta : G \to \mathbf{L}F$, such that $\zeta = \xi \circ (\eta \circ Q)$.

The following result [GM, III.6.8], allows one to construct derived functors

Proposition 2.4.3. *Assume that F is left- (respectively, right-) exact. Suppose there is a class $\mathcal{P}$ of objects of $\mathcal{A}$, closed under finite direct sums, such that*

(i) Every object of $\mathcal{A}$ admits an injection into (respectively, a surjection from) an object of $\mathcal{P}$.

(ii) If X is a complex of objects of $\mathcal{P}$ which is acyclic (i.e. has no cohomology), then $F(X)$ is also acyclic.

Then F has a right-derived functor $\mathbf{R}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ (respectively, a left-derived functor $\mathbf{L}\,F : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$). □

We shall sketch the construction of the derived functor in the case when F is left-exact. First of all one considers the full subcategory $\mathrm{K}^+(\mathcal{P})$ of $\mathrm{K}^+(\mathcal{A})$ consisting of complexes of objects of $\mathcal{P}$. This subcategory is triangulated because $\mathcal{P}$ is closed under direct sums. Now let $\mathrm{K}^+(\mathcal{P})_{quis}$ be the triangulated category obtained by inverting all quasi-isomorphisms in $\mathrm{K}^+(\mathcal{P})$. There is a natural fully faithful and exact functor

$$R : \mathrm{K}^+(\mathcal{P})_{quis} \longrightarrow \mathrm{D}^+(\mathcal{A}),$$

which by assumption (i) is an equivalence of categories. Assumption (ii) implies that the functor F passes to the quotient to give a functor

$$\bar{F} : \mathrm{K}^+(\mathcal{P})_{quis} \longrightarrow \mathrm{D}(\mathcal{B}).$$

Composing with a quasi-inverse of R one obtains the right-derived functor $\mathbf{R}F$.

Thus, given an object E of $\mathcal{A}$, one obtains $\mathbf{R}F(E)$ by replacing E by a quasi-isomorphic complex of F-acyclic objects, and applying F to the result. In the case when $\mathcal{A}$ has enough injectives, one can always take $\mathcal{P}$ to be the class of injective objects of $\mathcal{A}$, so that in this case, any left-exact functor F has a right-derived functor, and the cohomology objects $\mathrm{H}^i(\mathbf{R}F(E))$, are just the usual derived functors $\mathbf{R}^iF(E)$, as defined in [Ha2, III, §§1-3].

Similar remarks apply to left-derived functors of right-exact functors.

Note. If the functor $F : \mathcal{A} \to \mathcal{B}$ is exact (i.e. takes short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$), then the corresponding functor $F : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ preserves quasi-isomorphisms, and hence descends to a functor $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ which we also denote by F. The restrictions of this functor to $\mathrm{D}^+(\mathcal{A})$ and $\mathrm{D}^-(\mathcal{A})$ are right- and left-derived functors of F respectively.

For computational purposes the following result ([Ve1, III.4.4.6]) is extremely useful. We shall often apply it in the case when $F = H \circ G$, and $G : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ is an exact functor.

Proposition 2.4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories, and let $F : \mathrm{D}^*(\mathcal{A}) \to \mathcal{B}$ be a cohomological functor, where $*$ represents $+$, $-$ or b. Given an integer p, put $F^p = F \circ T^p$. Then for any object X of $\mathrm{D}^*(\mathcal{A})$ there is a spectral sequence with*

$$E_2^{p,q} = F^p(\mathrm{H}^q(X)),$$

which converges to $F^{p+q}(X)$. $\square$

2.5 Derived categories of sheaves

Recall that all our schemes are assumed to be of finite type over k. Given a scheme X, the Abelian category of $\mathcal{O}_X$-modules on X is denoted $\mathrm{Mod}(X)$. The derived category $\mathrm{D}(\mathrm{Mod}(X))$ will be denoted simply by $\mathrm{D}(X)$, and the full triangulated subcategory of $\mathrm{D}(X)$ consisting of objects whose cohomology sheaves are all coherent by $\mathrm{D}_c(X)$. If $*$ is one of the symbols $+$, $-$, b, define full subcategories of $\mathrm{D}(X)$

$$\mathrm{D}^*(X) = \mathrm{D}^*(\mathrm{Mod}(X)), \qquad \mathrm{D}^*_c(X) = \mathrm{D}^*(X) \cap \mathrm{D}_c(X).$$

We write $[n]$ for $T^n_{\mathrm{D}(X)}$, the nth iterate of the translation functor of $\mathrm{D}(X)$. Thus if E is an object of $\mathrm{D}(X)$ then for all i,

$$\mathrm{H}^i(E[n]) = \mathrm{H}^{i+n}(E).$$

As is well known [Ha2, III.2], $\mathrm{Mod}(X)$ has enough injectives, so all left-exact functors on $\mathrm{Mod}(X)$ have right-derived functors. Furthermore, every element of $\mathrm{Mod}(X)$ is a quotient of a flat $\mathcal{O}_X$-module [Ha1, II.1.2], so any right-exact functor on $\mathrm{Mod}(X)$ which takes acyclic complexes of flat $\mathcal{O}_X$-modules to acyclic complexes has a left-derived functor.

2.5.1 Direct image

Let $f : X \to Y$ be a morphism of schemes. The functor $f_* : \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ has a right-derived functor

$$\mathbf{R}f_* : \mathrm{D}^+(X) \longrightarrow \mathrm{D}^+(Y).$$

Notes. (i) If $g : Y \to Z$ is another morphism of schemes then [Ha1, II.5.1], there is an isomorphism of functors

$$\mathbf{R}(g \circ f)_* \cong \mathbf{R}g_* \circ \mathbf{R}f_*.$$

(ii) If f is proper then [GD, III.3.2.1], $\mathbf{R}f_*$ takes $\mathrm{D}^b_c(X)$ into $\mathrm{D}^b_c(Y)$.

2.5.2 Pull-back

Let $f : X \to Y$ be a morphism of schemes. The right-exact functor $f^* : \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ has a left-derived functor

$$\mathbf{L}\, f^* : \mathrm{D}^-(Y) \longrightarrow \mathrm{D}^-(X).$$

Notes. (i) $\mathbf{L}\, f^*$ takes $\mathrm{D}^-_c(Y)$ into $\mathrm{D}^-_c(X)$ [Ha1, II.4.4].

(ii) If $g : Y \to Z$ is another morphism of schemes, then [Ha1, II.5.4],

$$\mathbf{L}(g \circ f)^* \cong \mathbf{L}\, f^* \circ \mathbf{L}\, g^*.$$

(iii) If f is a flat morphism, then f^* is an exact functor, so $\mathbf{L}\, f^*$ is just f^* applied to complexes. In particular $\mathbf{L}\, f^*$ takes $\mathrm{D}^b(Y)$ into $\mathrm{D}^b(X)$.

2.5.3 Global Hom

Let X be a scheme. If E and F are objects of $\mathrm{D}(X)$ one defines the Abelian groups

$$\mathrm{Hom}^i_X(E, F) = \mathrm{Hom}_{\mathrm{D}(X)}(E, F[i]).$$

If E and F are $\mathcal{O}_X$-modules, then

$$\mathrm{Hom}^i_X(E, F) \cong \mathrm{Ext}^i_X(E, F).$$

2.5.4 Local Hom

Let X be a scheme. Given objects E of $\mathrm{D}(X)$ and F of $\mathrm{D}^+(X)$, one defines an object $\mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(E, F)$ by first replacing F by an isomorphic complex of injective $\mathcal{O}_X$-modules, and then defining $\mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(E, F)$ to be the complex

$$\bigoplus_i \mathcal{H}om_{\mathcal{O}_X}(E^i, F^{i+n}), \qquad n \in \mathbb{Z},$$

with differentials $d^n = \bigoplus_i (d^i_E + (-1)^{n+1} d^{i+n}_F)$. This construction gives rise to a functor

$$\mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(-,-) : \mathrm{D}(X)^\circ \times \mathrm{D}^+(X) \longrightarrow \mathrm{D}(X),$$

exact in both arguments, such that if E is an $\mathcal{O}_X$-module, $\mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(E, -)$ is the right-derived functor of the left-exact functor $\mathcal{H}om_{\mathcal{O}_X}(E, -) : \mathrm{Mod}(X) \to \mathrm{Mod}(X)$. Thus, for any other sheaf $\mathcal{O}_X$-module,

$$\mathbf{R}\,\mathcal{H}om^i_{\mathcal{O}_X}(E, F) \cong \mathcal{E}xt^i_{\mathcal{O}_X}(E, F).$$

See [Ha1, II.3] for details. For any object E of $\mathrm{D}(X)$ we define the *derived dual*,

$$E^\vee = \mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X).$$

2.5.5 Tensor product

Let X be a scheme. Given objects E and F of $\mathrm{D}^-(X)$, one defines an object $E \overset{\mathbf{L}}{\otimes} F$ of $\mathrm{D}^-(X)$ by first replacing E by an isomorphic complex of flat $\mathcal{O}_X$-modules, and then defining $E \overset{\mathbf{L}}{\otimes} F$ to be the complex

$$\bigoplus_i (E^i \otimes F^{n-i}), \qquad n \in \mathbb{Z},$$

with differentials $d^n = \bigoplus_i (d_E^i + (-1)^n d_F^{n-i})$. This construction gives rise to a functor

$$(- \overset{\mathbf{L}}{\otimes} -) : \mathrm{D}^-(X) \times \mathrm{D}^-(X) \longrightarrow \mathrm{D}^-(X),$$

which is exact in both arguments, such that for any $\mathcal{O}_X$-module F, the functor $- \overset{\mathbf{L}}{\otimes} F$ is the left-derived functor of the right-exact functor $- \otimes F$. See [Ha1, II.4] for details.

Notes. (i) If E and F have coherent cohomology, then so does $E \overset{\mathbf{L}}{\otimes} F$.

(ii) The derived tensor product is symmetric and associative [Ha1, II.5.13], i.e. for objects E, F and G of $\mathrm{D}^-(X)$ there are natural functorial isomorphisms

$$E \overset{\mathbf{L}}{\otimes} F \cong F \overset{\mathbf{L}}{\otimes} E, \qquad E \overset{\mathbf{L}}{\otimes} (F \overset{\mathbf{L}}{\otimes} G) \cong (E \overset{\mathbf{L}}{\otimes} F) \overset{\mathbf{L}}{\otimes} G.$$

(iii) If $f : X \to Y$ is a morphism of schemes, and E and F are objects of $\mathrm{D}^-(X)$, there is a natural functorial isomorphism [Ha1, II.5.9],

$$\mathbf{L} f^*(E \overset{\mathbf{L}}{\otimes} F) \cong \mathbf{L} f^*(E) \overset{\mathbf{L}}{\otimes} \mathbf{L} f^*(F).$$

(iv) If E and F are $\mathcal{O}_X$-modules and x is a point of X, then for any integer p,

$$H_p(E \overset{\mathbf{L}}{\otimes} F)_x \cong \mathrm{Tor}_p^{\mathcal{O}_{X,x}}(E_x, F_x).$$

2.5.6 Grothendieck-Verdier duality

Let $f : X \to Y$ be a proper morphism of schemes. Then [Ve2] there exists an exact functor

$$f^! : \mathrm{D}_c^+(Y) \longrightarrow \mathrm{D}_c^+(X),$$

which is a right adjoint to the functor

$$\mathbf{R}f_* : \mathrm{D}_c^+(X) \longrightarrow \mathrm{D}_c^+(Y).$$

Thus for objects E and F of $\mathrm{D}_c^+(X)$ and $\mathrm{D}_c^+(Y)$ respectively, there are natural functorial isomorphisms

$$\mathrm{Hom}_{\mathrm{D}(X)}(E, f^!(F)) \cong \mathrm{Hom}_{\mathrm{D}(Y)}(\mathbf{R}f_*(E), F).$$

We call $f^!$ the *twisted inverse image functor.*

Notes. (i) If f is flat then [Ve2, Thm. 2], for any object $F \in \mathrm{D}^b_c(Y)$ there is a canonical isomorphism

$$f^!(F) \cong f^!(\mathcal{O}_Y) \otimes f^*(F).$$

(i) If f is smooth of relative dimension n, then, [Ve2, Thm. 4],

$$f^!(\mathcal{O}_Y) \cong \omega_{X/Y}[n]$$

where $\omega_{X/Y}$ is the sheaf of relative differentials of f.

Chapter 3

Complexes of sheaves

This chapter contains various technical results which we shall need in Chapter 4. The first two sections deal with certain properties of complexes of sheaves, the third contains a base-change result, and the last section is concerned with Kodaira-Spencer maps associated to families of sheaves.

3.1 Tor-dimension of complexes

The aim of this section is to prove Proposition 3.1.5 below, which we shall need in several places in Chapter 4.

Throughout we shall fix a flat morphism of schemes $\pi : X \to S$. For each point $s \in S$ we let $j_s : X_s \hookrightarrow X$ denote the inclusion of the fibre $X_s = \pi^{-1}(s)$.

We start with two lemmas

Lemma 3.1.1. *Let E be a coherent $\mathcal{O}_X$-module and take a point $s \in S$. Then*

(a) $\mathbf{L}_0\, j_s^*(E) = 0$ if and only if, the support of E does not meet X_s,

(b) $\mathbf{L}_1\, j_s^*(E) = 0$ if and only if, E is flat over S at all points of X_s.

In either case $\mathbf{L}_p\, j_s^*(E) = 0$ for all $p > 0$.

Proof. The projection formula [Ha1, II.5.6] implies that

$$j_{s,*}(\mathbf{L}\, j_s^*(E)) \cong E \overset{\mathbf{L}}{\otimes} \pi^*(\mathcal{O}_s). \tag{3.1}$$

so by 2.5.5 (iv), the stalk of $\mathbf{L}_p\, j_s^*(E)$ at the point $x \in X_s$ is given by

$$\operatorname{Tor}_p^A(E_x, A/m),$$

where (A, m) is the local ring $(\mathcal{O}_{S,s}, m_{S,s})$. The result follows from Nakayama's lemma and the local criterion for flatness [Ei, Thm. 6.8]. □

Lemma 3.1.2. *Let E be an object of $\mathrm{D}_c^-(X)$ and take $s \in S$. Suppose*

$$\mathbf{L}_p\, j_s^*(E) = 0 \textit{ unless } p = 0, \tag{3.2}$$

Then for any $i \neq 0$, the support of $\mathrm{H}_i(E)$ does not meet X_s. Furthermore the sheaf $\mathrm{H}_0(E)$ is flat over S at all points of X_s. In particular, if (3.2) holds for all $s \in S$, then E is quasi-isomorphic to an S-flat $\mathcal{O}_X$-module.

Proof. Consider the spectral sequence

$$E_2^{p,q} = \mathbf{L}_{-p}\, j_s^*(\mathrm{H}_{-q}(E)) \implies \mathbf{L}_{-(p+q)}\, j_s^*(E).$$

Note that by Lemma 3.1.1(a), the support of $\mathrm{H}_{-i}(E)$ meets X_s if, and only if, $E_2^{0,i} \neq 0$. Suppose i is the largest integer such that the support of $\mathrm{H}_{-i}(E)$ meets X_s. Then $E_2^{0,i} \neq 0$, but $E_2^{p,q} = 0$ for all $q > i$. Hence $E_2^{0,i}$ survives to infinity and $\mathbf{L}_i\, j_s^*(E)$ is non-zero. By hypothesis we must have $i = 0$, and so $E_2^{p,q} = 0$ unless $p \leq 0$, $q \leq 0$.

Now $E_2^{-1,0}$ survives to infinity so must be zero. By Lemma 3.1.1(b), this implies that $\mathrm{H}_0(E)$ is flat over S at all points of X_s, and $E_2^{p,0} = 0$ for all $p < 0$. In turn this implies that $E_2^{0,-1}$ survives to infinity so is zero. Hence the support of $\mathrm{H}_1(E)$ does not meet X_s, and $E_2^{p,-1} = 0$ for all p. Continuing in this way we see that $E^{p,q} = 0$ unless $p = q = 0$. □

Definition 3.1.3. Let $\pi : X \to S$ be a flat morphism of schemes. Let E be a non-zero object of $\mathrm{D}_c(X)$. The *tor-dimension* of E over S, written $\mathrm{td}_S(E)$, is defined to be the smallest non-negative integer n such that E is quasi-isomorphic to a complex of S-flat $\mathcal{O}_X$-modules of length n. If there is no such integer we put $\mathrm{td}_S(E) = \infty$.

To simplify the statement of our result we also make the

Definition 3.1.4. Let X be a scheme. An object E of $\mathrm{D}(X)$ will be called *left-centred* if $\mathrm{H}^0(E) \neq 0$, and $\mathrm{H}^i(E) = 0$ for all $i > 0$.

The main result of this section is

Proposition 3.1.5. *Let $\pi : X \to S$ be a flat morphism of schemes, let E be a left-centred object of $\mathrm{D}(X)$, and let n be a non-negative integer. Then $\mathrm{td}_S(E) \leq n$ if and only if, for all $s \in S$,*

$$\mathbf{L}_p\, j_s^*(E) = 0 \qquad \textit{for all } p > n. \tag{3.3}$$

Proof. Note first that for any object E of $\mathrm{D}^-(X)$, and any point $s \in S$, we can compute $\mathbf{L}\, j_s^*(E)$ by applying the functor j_s^* to a resolution of E by S-flat $\mathcal{O}_X$-modules. It follows immediately that if E is left-centred with tor-dimension n then $\mathbf{L}_p\, j_s^*(E) = 0$ unless $0 \leq p \leq n$.

For the converse we assume that condition (3.3) holds for each $s \in S$, and show that E has tor-dimension at most n. If $n = 0$ the result follows from Lemma 3.1.2, so assume n positive. By [Hal, I.4.6], [Hal, II.1.2] we can assume that E is given by a complex L of flat $\mathcal{O}_X$-modules of the form

$$\cdots \longrightarrow L_i \xrightarrow{d_i} L_{i-1} \xrightarrow{d_{i-1}} L_{i-2} \longrightarrow \cdots \xrightarrow{d_1} L_0 \longrightarrow 0. \tag{3.4}$$

Consider the object F of $\mathrm{D}(X)$ defined by the truncated complex

$$\cdots \longrightarrow L_{n+3} \xrightarrow{d_{n+3}} L_{n+2} \xrightarrow{d_{n+2}} L_{n+1} \xrightarrow{d_{n+1}} L_n \longrightarrow 0.$$

Applying the functor j_s^* and comparing with E, one sees that for any $s \in S$, $\mathbf{L}_p j_s^*(F) = 0$ for $p > n$ and $p < n$. Applying Lemma 3.1.2 again, this implies that F has tor-dimension 0 over S. It follows that $\mathrm{H}_i(E) = 0$ for all $i > n$, and that $L_n/\operatorname{im}(d_{n+1})$ is S-flat.

Finally consider the short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow L \longrightarrow B \longrightarrow 0,$$

where A is the complex

$$\cdots \longrightarrow L_{n+2} \xrightarrow{d_{n+2}} L_{n+1} \longrightarrow \operatorname{im}(d_{n+1}) \longrightarrow 0,$$

and B is the length n complex of S-flat sheaves

$$0 \longrightarrow L_n/\operatorname{im}(d_{n+1}) \longrightarrow L_{n-1} \xrightarrow{d_{n-1}} L_{n-2} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow 0.$$

Since B has tor-dimension n over S, it is enough to show that A is quasi-isomorphic to zero. But this is clear since $\mathrm{H}_i(A) = \mathrm{H}_i(L) = 0$ for $i > n$. □

Another application of Lemma 3.1.1 we shall need is

Lemma 3.1.6. *Let $\pi : X \to S$ be a flat and proper morphism of schemes, and let E be an object of $\mathrm{D}^b_c(X)$. Then the set V of points $s \in S$ such that*

$$\mathbf{L}_p j_s^*(E) = 0 \qquad \forall p \neq 0,$$

is the set of points of a (possibly empty) open subset of S.

Proof. Let U be the open subset of X which is the intersection of the complements of the supports of the sheaves $\mathrm{H}_i(E)$, intersected with the open subset of points at which $\mathrm{H}_0(E)$ is S-flat ([GD, IV_3 11.1.1]). By Lemma 3.1.2, a point $s \in S$ lies in V precisely when the fibre X_s is contained in U. Since π is proper, V is an open subset of S. □

We shall apply Proposition 3.1.5 in particular to the case when the morphism $\pi : X \longrightarrow S$ is the identity map on X. Let us make the

Definition 3.1.7. Let X be a scheme, and let E be a non-zero object of $\mathrm{D}_c(X)$. Then the homological dimension of E, written $\mathrm{hd}(E)$, is equal to the smallest integer n such that E is quasi-isomorphic to a complex of locally free $\mathcal{O}_X$-modules of length n. If no such integer exists we put $\mathrm{hd}(E) = \infty$.

Then one has

Lemma 3.1.8. *Let X be a quasi-projective scheme. Let E be a non-zero object of $\mathrm{D}_c(X)$. Then the homological dimension of E is equal to the tor-dimension of E over X.*

Proof. Since any locally free $\mathcal{O}_X$-module is flat, it is obvious that $\mathrm{td}_X(E) \leq \mathrm{hd}(E)$. For the reverse inequality, suppose $\mathrm{td}_X(E) = n < \infty$ and apply the proof of Proposition 3.1.5, taking (3.4) to be a complex of locally free $\mathcal{O}_X$-modules. This is possible since any coherent $\mathcal{O}_X$-module on the quasi-projective scheme X is a quotient of a locally free $\mathcal{O}_X$-module [Ha2, III.8.8]. Then $L_n/\operatorname{im}(d_{n+1})$ is a flat, coherent $\mathcal{O}_X$-module, hence is locally free [Ha2, III.9.2], and B is a length n complex of locally free $\mathcal{O}_X$-modules. $\square$

3.2 Algebraic codimension of homology objects of complexes

In this section we prove a result which relates the homological dimension of a complex of sheaves to the algebraic codimension of its homology objects.

Definition 3.2.1. Let X be a scheme, and E a coherent $\mathcal{O}_X$-module. Then the *algebraic codimension* of E, written $\mathrm{cd}(E)$ is the smallest non-negative integer p such that

$$\mathcal{E}\mathrm{xt}^p_{\mathcal{O}_X}(E, \mathcal{O}_X) \neq 0.$$

Lemma 3.2.2. *Let X be a projective scheme. Then the algebraic codimension of a coherent $\mathcal{O}_X$-module E is the smallest non-negative integer i such that there is a locally free $\mathcal{O}_X$-module F with*

$$\mathrm{Ext}^i_X(E, F) \neq 0.$$

Proof. Let p be the algebraic codimension of E. Then [Ha2, III.6.7], for any integer $q < p$, and any locally free $\mathcal{O}_X$-module F,

$$\mathcal{E}\mathrm{xt}^q_{\mathcal{O}_X}(E, F) = \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes F = 0,$$

so the spectral sequence [Ha1, II.5.3]

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_X}(E,F)) \implies \mathrm{Ext}^{p+q}_X(E,F)$$

implies that $\mathrm{Ext}^i_X(E,F) = 0$ for all $i < p$. Furthermore

$$\mathrm{Ext}^p_X(E,F) = \mathrm{H}^0(X, \mathcal{E}\mathrm{xt}^p_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes F),$$

which is non-zero if we take F to be a sufficiently ample line bundle on X. □

If X is a Cohen-Macaulay variety then, as one would expect, the notions of algebraic codimension and codimension coincide:

Lemma 3.2.3. *Let X be a Cohen-Macaulay variety, and let E be a coherent $\mathcal{O}_X$-module. Then the algebraic codimension of E is equal to the codimension of the support of E.*

Proof. By [Ha2, III.6.8], $\mathrm{cd}(E)$ is the smallest non-negative integer i such that there is a point $x \in X$ with

$$\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(E_x, \mathcal{O}_{X,x}) \neq 0.$$

By [Ei, Prop. 18.4] this is equal to the infimum over $x \in X$, of the algebraic codimension of the ideal $\mathrm{ann}(E_x)$ of the local ring $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, algebraic codimension is the same as codimension, so $\mathrm{cd}(E)$ is the infimum over points $x \in X$ of the codimension of the ideal $\mathrm{ann}(E_x)$ of $\mathcal{O}_{X,x}$. This is just the codimension of the support of the sheaf E. □

We now prove the main result of this section.

Proposition 3.2.4. *Let X be a projective scheme, and let E be a left-centred object of $\mathrm{D}_c(X)$ of homological dimension $n < \infty$. Let p be a non-negative integer such that each homology object $\mathrm{H}_i(E)$ has algebraic codimension at least p. Then $\mathrm{H}_i(E) = 0$ for $i > n - p$. In particular, if $p = n$ then E is concentrated in degree 0, and if $p > n$ then $E \cong 0$.*

Proof. Since E has homological dimension n, it is isomorphic to a complex of the form

$$0 \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} L_{n-2} \longrightarrow \cdots \xrightarrow{d_1} L_0 \longrightarrow 0$$

with each L_i locally free. In particular , one has $\mathrm{H}_i(E) = 0$ for $i > n$. Thus we can assume that $p > 0$ and take j to be the greatest integer such that $\mathrm{H}_j(E) \neq 0$. We must show that $j \leq n - p$.

There are short exact sequences

$$0 \longrightarrow \mathrm{im}(d_{j+1}) \longrightarrow \ker(d_j) \longrightarrow \mathrm{H}_j(E) \longrightarrow 0, \tag{3.5}$$

and

$$0 \longrightarrow \mathrm{im}(d_{i+1}) \longrightarrow L_i \longrightarrow \mathrm{im}(d_i) \longrightarrow 0, \tag{3.6}$$

for all $i > j$. There are no non-zero morphisms from $\mathrm{H}_j(E)$ to $\ker(d_j)$, because the latter sheaf is a subsheaf of the locally free sheaf L_j, and $\mathrm{H}_j(E)$ has algebraic codimension at least 1. Thus (3.5) defines a non-trivial element of

$$\mathrm{Ext}^1_X(\mathrm{H}_j(E), \mathrm{im}(d_{j+1})).$$

Applying the long exact sequence in $\mathrm{Hom}_X(\mathrm{H}_j(E), -)$ to each of the sequences (3.6), and noting that

$$\mathrm{Ext}^i_X(\mathrm{H}_j(E), L) = 0,$$

for any locally free sheaf L, and any integer $i < p$, one obtains a non-trivial element of

$$\mathrm{Ext}^{p-1}_X(\mathrm{H}_j(E), \mathrm{im}(d_{j+p-1})).$$

Now if $j > n-p$ then $j+p-1 \geq n$ and $\mathrm{im}(d_{j+p-1})$ is locally free, which contradicts the fact that $\mathrm{H}_j(E)$ has algebraic codimension at least p. □

We shall use make use of this result in Chapter 4 via the following corollary.

Corollary 3.2.5. *Let X be a projective Cohen-Macaulay variety of dimension n. Let E be an object of $\mathrm{D}^-_c(X)$, and x_0 a point of X such that given a point $x \in X$,*

$$\mathrm{Hom}^p_{\mathrm{D}(X)}(E, \mathcal{O}_x) = 0,$$

unless $x = x_0$ and $0 \leq p \leq n$. Then E is concentrated in degree 0, and is supported at x_0.

Proof. First note that for any $x \in X$, there is an isomorphism of vector spaces [Ha1, II.5.11]

$$\mathrm{Hom}^p_{\mathrm{D}(X)}(E, \mathcal{O}_x) = \mathbf{L}_p j_x^*(E),$$

where $j_x : \{x\} \hookrightarrow X$ is a the embedding of the point x in X. Proposition 3.1.5 shows that E has homological dimension at most n. Restricting E to the open subscheme $X - \{x_0\}$ of X, and applying Proposition 3.1.5 again we see that each homology sheaf of E is supported at the point x_0. By Lemma 3.2.3 any such sheaf has algebraic codimension n. Proposition 3.2.4 now gives the result. □

It would be interesting to know whether this result holds without the condition that X be Cohen-Macaulay.

3.3 A base-change result

The following base-change lemma is a slight generalisation of a result of Bondal and Orlov [BO1, Lemma 1.3].

Lemma 3.3.1. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ T & \xrightarrow{f} & S. \end{array}$$

be a commutative diagram of morphisms of schemes, such that the induced morphism $Y \to X \times_S T$ is an isomorphism. Suppose that one of the following assumptions holds:

(a) p is smooth,

(b) there is a projective scheme Z over k such that $X = Z \times S$ and p is the projection $Z \times S \to S$.

Then for any object E of $\mathrm{D}_c(S \times X)$ of finite tor-dimension over S, there is an isomorphism

$$\mathbf{L}\, f^*(\mathbf{R}p_*(E)) \cong \mathbf{R}q_*(\mathbf{L}\, g^*(E))$$

Proof. This is based on Bondal and Orlov's proof of [Bo1, Lemma 1.3]. Firstly I claim that there is an isomorphism

$$\mathbf{L}\, g^*(p^!(\mathcal{O}_S)) \cong q^!(\mathbf{L}\, f^*(\mathcal{O}_S)). \tag{3.7}$$

If p is smooth this follows from [Ve2, Thm. 3] and [Ha1, Rmk. 2, p. 141]. In case (b), (3.7) is proved by applying [Ve2, Thm. 2] with g the projection from S or T to $\mathrm{Spec}(k)$.

Now by the theorem on flat base change [Ha1, II.5.12], there is an isomorphism of functors

$$p^* \circ \mathbf{R}f_* \cong \mathbf{R}g_* \circ q^*.$$

Using [Ve2, Cor. 2], the projection formula [Ha1, II.5.6], and (3.7), one has natural functorial isomorphisms

$$\begin{aligned} p^!(\mathbf{R}f_*(-)) &\cong p^*(\mathbf{R}f_*(-)) \overset{\mathbf{L}}{\otimes} p^!(\mathcal{O}_S) \\ &\cong \mathbf{R}g_*(q^*(-)) \overset{\mathbf{L}}{\otimes} p^!(\mathcal{O}_S) \\ &\cong \mathbf{R}g_*(q^*(-) \overset{\mathbf{L}}{\otimes} \mathbf{L}\, g^*(p^!(\mathcal{O}_S)) \cong \mathbf{R}g_*(q^!(-)). \end{aligned}$$

If E is an object of $\mathrm{D}^b_c(X)$ of finite tor-dimension over S (so that $\mathbf{L}\, g^*(E)$ is an object of $\mathrm{D}^b_c(Y)$), and F is an object of $\mathrm{D}^b_c(T)$, there are natural functorial isomorphisms [Ha1, 5.11] and [Ve2, Theorem 1]

$$\begin{aligned}\operatorname{Hom}_{\mathrm{D}(T)}(\mathbf{L}\, f^*(\mathbf{R}p_*(E)), F) &\cong \operatorname{Hom}_{\mathrm{D}(X)}(E, p^!(\mathbf{R}f_*(F))) \\ &\cong \operatorname{Hom}_{\mathrm{D}(X)}(E, \mathbf{R}g_*(q^!(F))) \\ &\cong \operatorname{Hom}_{\mathrm{D}(T)}(\mathbf{R}q_*(\mathbf{L}\, g^*(E)), F).\end{aligned}$$

Put $A = \mathbf{L}\, f^*(\mathbf{R}p_*(E))$ and $B = \mathbf{R}q_*(\mathbf{L}\, g^*(E))$. Since B is an object of $\mathrm{D}^b_c(T)$, the above isomorphisms imply that there is a morphism $i : A \to B$ such that for any object F of $\mathrm{D}^b_c(T)$ the induced map

$$i^* : \operatorname{Hom}_{\mathrm{D}(T)}(B, F) \longrightarrow \operatorname{Hom}_{\mathrm{D}(T)}(A, F)$$

is an isomorphism. If we complete the morphism i to a triangle

$$A \xrightarrow{i} B \longrightarrow C \longrightarrow A[1],$$

in $\mathrm{D}^-_c(T)$ then this implies that $\operatorname{Hom}_{\mathrm{D}(T)}(C, F) = 0$ for all objects F of $\mathrm{D}^b_c(T)$. It follows that $C \cong 0$ (see for example Lemma 4.1.4), and i is an isomorphism.

$\square$

3.4 Kodaira-Spencer maps

In this section we introduce the Kodaira-Spencer map (KS map) associated to a family of sheaves, and give a very weak condition for it to be injective.

Let X and S be schemes, and let $\mathcal{E}$ be a sheaf on $S \times X$, flat over S. For each point $s \in S$, the *Kodaira-Spencer map* for the family $\mathcal{E}$ at the point s is a linear map

$$\theta_s(\mathcal{E}) : \mathrm{T}_s(S) \longrightarrow \operatorname{Ext}^1_X(\mathcal{E}_s, \mathcal{E}_s).$$

We recall its definition. Firstly, let us define

$$D = \operatorname{Spec} k[\epsilon]/(\epsilon^2).$$

Then an element $v \in \mathrm{T}_s(S)$ is just a morphism of schemes $v : D \to S$ which takes the unique (closed) point $0 \in D$ to s. Pulling the family $\mathcal{E}$ back via v one obtains a sheaf $\mathcal{F}$ on $D \times X$, flat over D, such that the restriction of $\mathcal{F}$ to $\{0\} \times X$ is $\mathcal{E}_s$. Tensoring the structure sequence of $0 \in D$,

$$0 \longrightarrow k \longrightarrow \mathcal{O}_D \longrightarrow k \longrightarrow 0,$$

with $\mathcal{F}$ gives a sequence of $\mathcal{O}_{D\times X}$-modules

$$0 \longrightarrow \mathcal{E}_s \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_s \longrightarrow 0,$$

which viewed as a sequence of $\mathcal{O}_X$-modules gives an element of $\operatorname{Ext}^1_X(\mathcal{E}_s, \mathcal{E}_s)$.

Lemma 3.4.1. *Let S and Y be varieties with Y projective. Let $\mathcal{E}$ be a sheaf on $S \times Y$, flat over S, such that for each $s \in S$, $\mathcal{E}_s$ is the structure sheaf of a zero-dimensional closed subscheme of Y. Suppose also that for all pairs of points $s_1, s_2 \in S$*

$$\mathcal{E}_{s_1} \cong \mathcal{E}_{s_2} \implies s_1 = s_2. \tag{3.8}$$

Then there exists a point $s \in S$ such that the Kodaira-Spencer map $\theta_s(\mathcal{E})$ for the family $\mathcal{E}$ at s is injective.

Proof. We may assume that S is affine. Let $\pi : S \times Y \to S$ be the projection map. By the theorem on cohomology and base-change [Ha2, III.12.11], the natural map

$$\pi_*(\mathcal{E}) \otimes \mathcal{O}_s \to \mathrm{H}^0(Y, \mathcal{E}_s)$$

is surjective, so we can find a section $g : \mathcal{O}_{S\times Y} \to \mathcal{E}$ such that the restriction $g_s : \mathcal{O}_Y \to \mathcal{E}_s$ is surjective. Passing to an open subset of S we can assume that g is surjective, so that $\mathcal{E}$ is the structure sheaf of a closed subscheme of $S \times Y$.

By the general existence theorem for Hilbert schemes [Gr2], there is a scheme $\mathrm{Hilb}(Y)$ representing the functor which assigns to a scheme S the set of S-flat quotients $\mathcal{Q}$ of $\mathcal{O}_{S\times Y}$. Let $\mathcal{Q}$ be the universal quotient on $\mathrm{Hilb}(Y) \times Y$. Then there is a morphism $f : S \to \mathrm{Hilb}(Y)$ such that $\mathcal{E} = (f \times 1_Y)^*(\mathcal{Q})$, and our hypothesis implies that f is injective on points. The KS map for the family $\mathcal{E}$ at $s \in S$ is obtained by composing the KS map for the family $\mathcal{Q}$ at $f(s)$ (which is an isomorphism by universality) with the differential

$$\mathrm{T}_s(f) : \mathrm{T}_s S \longrightarrow \mathrm{T}_{f(s)} \mathrm{Hilb}(Y).$$

Let S' denote the scheme-theoretic image of f. Passing to an open subset of S we can assume that S and S' are non-singular, and $f' : S \to S'$ is smooth [Ha2, III.10.7]. Since f is injective on points, f' has relative dimension 0, so $T_s(f)$ is injective for all $s \in S$. $\square$

Chapter 4

The inversion theorem for integral functors

In this chapter we study integral functors, and prove a powerful inversion theorem.

Definition 4.0.2. Let $(X, Y, \mathcal{P})$ be a triple consisting of a pair of projective varieties X and Y, and an object $\mathcal{P}$ of $\mathrm{D}_c(Y \times X)$ which is quasi-isomorphic to a bounded complex of locally free sheaves. Then $\Phi^{\mathcal{P}}_{Y\to X}$ will denote the exact functor $\mathrm{D}^b_c(Y) \longrightarrow \mathrm{D}^b_c(X)$ defined by the formula

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi^*_Y(-)), \tag{4.1}$$

where π_X and π_Y are the projection maps $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$.

Theorem 4.0.3. *Let $(X, Y, \mathcal{P})$ be a triple as in Definition 4.0.2, and suppose that X is smooth and Y is Cohen-Macaulay. Then the functor $F = \Phi^{\mathcal{P}}_{Y\to X}$ is an equivalence of categories if, and only if, for each point $y \in Y$,*

$$\mathrm{Hom}_{\mathrm{D}(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = k \text{ and } F\mathcal{O}_y \otimes \omega_X = F\mathcal{O}_y,$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i,

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0 \text{ unless } y_1 = y_2 \text{ and } 0 \le i \le \dim Y.$$

Furthermore, if these conditions are satisfied then Y is also smooth.

Note that the condition on $\mathcal{P}$ in Definition 4.0.2 is automatically satisfied if X and Y are smooth, or if X is smooth and $\mathcal{P}$ is a Y-flat sheaf on $Y \times X$. It is these cases which are of most interest to us. Indeed, in most applications Y is a moduli space of stable sheaves on a smooth projective variety X, and $\mathcal{P}$ is a universal sheaf on $Y \times X$. For this reason I have tried to assume as little as possible about the the space Y. Unfortunately, we shall have to assume Y Cohen-Macaulay in order to apply Corollary 3.2.5, but if this were shown to be true without the Cohen-Macaulay assumption, then Theorem 4.0.3 would hold for arbitrary projective schemes Y.

These considerations are only important for possible future applications of FM transforms to threefolds and other higher-dimensional varieties; if X is a smooth surface it is easy to check that the relevant moduli spaces Y are smooth directly.

Following Mukai, we shall refer to functors of the form $\Phi^{\mathcal{P}}_{Y\to X}$ as *integral functors.* When such a functor is an equivalence of categories it is called a *Fourier-Mukai transform.* These are the transforms which we shall use in later chapters to study moduli spaces of sheaves.

The chapter is split into three parts. The first part is purely category theoretic, and is devoted to proving a general condition for a fully faithful exact functor between triangulated categories to be an equivalence. In Section 2 we prove various properties of integral functors, and in Section 3 we give a proof of Theorem 4.0.3.

4.1 Equivalences of triangulated categories

In this section we give a condition for a fully faithful exact functor between triangulated categories to be an equivalence (Theorem 4.1.6 below).

Definition 4.1.2 below, captures a type of 'connectedness' common to the bounded derived categories of all connected schemes. If a_1 and a_2 are objects of a triangulated category $\mathcal{A}$ we put

$$\mathrm{Hom}^i_{\mathcal{A}}(a_1,a_2)=\mathrm{Hom}_{\mathcal{A}}(a_1,T^i_{\mathcal{A}}(a_2)),\qquad \forall i\in\mathbb{Z}.$$

Definition 4.1.1. An additive category $\mathcal{A}$ will be called *trivial* if every object of $\mathcal{A}$ is a zero object of $\mathcal{A}$.

Definition 4.1.2. A triangulated category $\mathcal{A}$ will be called *indecomposable* if there do not exist two non-trivial, full subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ such that

(a) for any pair of objects $a_j\in\mathrm{Ob}(\mathcal{A}_j)$,

$$\mathrm{Hom}^i_{\mathcal{A}}(a_1,a_2)=\mathrm{Hom}^i_{\mathcal{A}}(a_2,a_1)=0\quad\forall i\in\mathbb{Z},$$

(b) for every object a of $\mathcal{A}$ there exist objects $a_j\in\mathrm{Ob}(\mathcal{A}_j)$ such that a is a biproduct of a_1 and a_2.

Proposition 4.1.3. *Let X be a scheme. Then $\mathrm{D}^b_c(X)$ is indecomposable if and only if X is connected.*

Proof. If X is the disjoint union of two open subschemes U_1 and U_2, then putting $\mathcal{A}_i=\mathrm{D}^b_c(U_i)$, it is easy to see that $D^b_c(X)$ is not indecomposable. For the converse we shall suppose that X is connected and prove that $\mathcal{A} = \mathrm{D}^b_c(X)$ is indecomposable.

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are full subcategories of $\mathcal{A}$ satisfying conditions (a) and (b) of Definition 4.1.2. For any integral closed subscheme Y of X, the sheaf $\mathcal{O}_Y$ is indecomposable, and is therefore isomorphic to some object of $\mathcal{A}_j$, $j = 1$ or 2. For any point $y \in Y$ we must then have that $\mathcal{O}_y$ is also isomorphic to an object of $\mathcal{A}_j$, since otherwise (a) would imply that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_Y, \mathcal{O}_y) = 0$, which is not the case. Let X_j be the union of those Y such that $\mathcal{O}_Y$ is isomorphic to an object of $\mathcal{A}_j$. Then X_1 and X_2 are closed subsets of X (since X is Noetherian) and $X = X_1 \cup X_2$. If a point $x \in X$ lies in X_1 and X_2 then $\mathcal{O}_x$ is isomorphic to an object of $\mathcal{A}_1$ and to an object of $\mathcal{A}_2$. This contradicts (a). Thus the union is disjoint, and the fact that X is connected implies that one of the X_j (without loss of generality X_2) is empty. But then (a) implies that for any object a of $\mathcal{A}_2$ one has

$$\mathrm{Hom}^i_{\mathcal{A}}(a, \mathcal{O}_x) = 0 \quad \forall i \in \mathbb{Z} \quad \forall x \in X,$$

and hence, by the lemma below, $a \cong 0$. This completes the proof. □

Lemma 4.1.4. *Let X be a scheme and E an object of $\mathrm{D}^-(X)$ such that*

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(E, \mathcal{O}_x) = 0, \quad \forall x \in X \quad \forall i \in \mathbb{Z}.$$

Then $E \cong 0$.

Proof. Suppose that E is non-zero and let q_0 be maximal amongst integers q satisfying $\mathrm{H}^q(E) \neq 0$. Let x be a point in the support of $\mathrm{H}^{q_0}(E)$. There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathrm{H}^{-q}(E), \mathcal{O}_x) \implies \mathrm{Hom}^{p+q}_{\mathrm{D}(X)}(E, \mathcal{O}_x).$$

Any non-zero element of $E^{0,-q_0}$ survives to give an element of $\mathrm{Hom}^{-q_0}_{\mathrm{D}(X)}(E, \mathcal{O}_x)$. □

Remark 4.1.5. Let us define the support of an object E of $D^b_c(X)$ to be the union of the supports of the cohomology sheaves of E. Then the proof of the lemma shows that a point $x \in X$ lies in the support of E if and only if

$$\mathrm{Hom}^{\bullet}_{D(X)}(E, \mathcal{O}_x) \neq 0.$$

We can now prove the main theorem of this section. First we make an elementary remark concerning adjoint functors. See [ML, §IV.1] for more details.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. Suppose F has a left adjoint functor $G : \mathcal{B} \longrightarrow \mathcal{A}$, and let

$$\zeta : 1_{\mathcal{B}} \longrightarrow F \circ G, \qquad \delta : G \circ F \longrightarrow 1_{\mathcal{A}},$$

be the unit and counit of $G \dashv F$, respectively. Then for any pair of objects a_1 and a_2 of $\mathcal{A}$, there is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(a_1, a_2) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{A}}(Fa_1, Fa_2) \\ {\scriptstyle \delta(a_1)^*}\big\downarrow & & \big\uparrow{\scriptstyle \zeta(Fa_1)^*} \\ \mathrm{Hom}_{\mathcal{A}}(GFa_1, a_2) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{B}}(FGFa_1, Fa_2). \end{array} \tag{4.2}$$

in which the composition $\zeta(Fa_1)^* \circ F$ is an isomorphism.

Suppose F is fully faithful. Then all the maps in (4.2) are isomorphisms, and so δ is an isomorphism of functors. Since for any object b of $\mathcal{B}$,

$$\delta(Gb) \circ G(\zeta(b)) = 1_{Gb},$$

it follows that $G(\zeta(b))$ is an isomorphism for all $b \in \mathrm{Ob}(\mathcal{B})$.

Theorem 4.1.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories, with $\mathcal{B}$ indecomposable, and $\mathcal{A}$ non-trivial. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful exact functor. Then F is an equivalence of categories if, and only if, F has a left adjoint G and a right adjoint H such that for any object b of $\mathcal{B}$,*

$$Gb \cong 0 \implies Hb \cong 0.$$

Proof. If F is an equivalence then any quasi-inverse of F is both a left and right adjoint for F.

For the converse note first that the functors G and H are exact [Or]. Take an object b of $\mathcal{B}$ and (with notation as above) embed the morphism $\zeta(b)$ in a triangle of $\mathcal{B}$:

$$c \longrightarrow b \xrightarrow{\zeta(b)} FGb \longrightarrow T_{\mathcal{B}}(c).$$

Applying G one sees that $Gc \cong 0$, because as we noted above, the fact that F is fully faithful implies that the morphism $G(\zeta(b))$ is an isomorphism. Define full subcategories $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ consisting of objects satisfying $FGb \cong b$ and $Gb \cong 0$ respectively. Now our hypothesis implies that

$$\mathrm{Hom}^i_{\mathcal{B}}(b_1, b_2) = \mathrm{Hom}^i_{\mathcal{B}}(b_2, b_1) = 0 \quad \forall i \in \mathbb{Z},$$

whenever $b_j \in \mathcal{B}_j$. Furthermore, the lemma below applied to the triangle above shows that every object of $\mathcal{B}$ is a biproduct $b_1 \oplus b_2$. Since $\mathcal{B}$ is indecomposable we must have either

$$\begin{array}{rlll} & Gc \cong 0 & \implies c \cong 0 & \forall c \in \mathrm{Ob}(\mathcal{B}), \\ \text{or} & FGc \cong c & \implies c \cong 0 & \forall c \in \mathrm{Ob}(\mathcal{B}). \end{array}$$

But $FGF \cong F$, and F is fully faithful, so the second possibility implies that $\mathcal{A}$ is trivial, which we have forbidden. Thus we conclude that for any object b of $\mathcal{B}$, the morphism $\zeta(b)$ is an isomorphism, so $FG \cong 1_{\mathcal{B}}$ and F is an equivalence. $\square$

Lemma 4.1.7. *Let $\mathcal{A}$ be a triangulated category and let*

$$a_1 \xrightarrow{i_1} b \xrightarrow{p_2} a_2 \xrightarrow{0} Ta_1,$$

be a triangle of $\mathcal{A}$. Then b is a biproduct of a_1 and a_2 in $\mathcal{A}$.

Proof. Applying the functors $\mathrm{Hom}_{\mathcal{A}}(-, a_1)$ and $\mathrm{Hom}_{\mathcal{A}}(a_2, -)$, one obtains morphisms $p_1 : b \to a_1$ and $i_2 : a_2 \to b$, such that $p_1 \circ i_1 = 1_{a_1}$ and $p_2 \circ i_2 = 1_{a_2}$. The composition $p_2 \circ i_1$ is always 0 and replacing i_2 by $i_2 - i_1 \circ p_1 \circ i_2$, we can assume that $p_1 \circ i_2 = 0$. Then [ML, VIII.2] it is enough to check that the endomorphism of b given by

$$\phi = 1_b - i_1 \circ p_1 - i_2 \circ p_2$$

is the zero map. But this follows from the fact that $p_1 \circ \phi = p_2 \circ \phi = 0$. $\square$

4.2 Properties of integral functors

Fix a triple $(X, Y, \mathcal{P})$ consisting of a pair of projective varieties X, Y and an object $\mathcal{P}$ of $\mathrm{D}_c(Y \times X)$ of finite homological dimension, as in Definition 4.0.2. Let $F = \Phi^{\mathcal{P}}_{Y \to X}$ be the corresponding integral functor $\mathrm{D}^b_c(Y) \to \mathrm{D}^b_c(X)$ defined by the formula (4.1).

Given a scheme S, one can define a relative version of F over S. This is the functor

$$F_S : \mathrm{D}^b_c(S \times Y) \longrightarrow \mathrm{D}^b_c(S \times X),$$

given by the formula

$$F_S(-) = \mathbf{R}\pi_{S\times Y,*}(\mathcal{P}_S \overset{\mathbf{L}}{\otimes} \pi^*_{S\times X}(-)),$$

where $S \times X \xleftarrow{\pi_{S\times X}} S \times X \times Y \xrightarrow{\pi_{S\times Y}} S \times Y$ are projection maps, and $\mathcal{P}_S$ is the pull-back of $\mathcal{P}$ to $S \times X \times Y$.

The following result is similar to [Muk4, Prop. 1.3].

Lemma 4.2.1. *Let $g : T \to S$ be a morphism of schemes, and let E be an object of $\mathrm{D}_c(S \times Y)$, of finite tor-dimension over S. Then there is an isomorphism*

$$F_T \circ \mathbf{L}(g \times 1_Y)^*(E) \cong \mathbf{L}(g \times 1_X)^* \circ F_S(E).$$

Proof. This is just a matter of pulling and pushing complexes around in the obvious way. See [Muk4, Prop. 1.3] for details. One needs to base-change around the diagram

$$\begin{array}{ccc} T \times X \times Y & \xrightarrow{(g \times 1_{X \times Y})} & S \times X \times Y \\ \big\downarrow{\scriptstyle \pi_{T \times X}} & & \big\downarrow{\scriptstyle \pi_{S \times X}} \\ T \times X & \xrightarrow{(g \times 1_X)} & S \times X \end{array}$$

This is justified by Lemma 3.3.1. □

We can now show that integral functors preserve families of sheaves. It is this property which makes them so useful for studying moduli problems.

Proposition 4.2.2. *Let S be a scheme, and $\mathcal{E}$ a sheaf on $S \times Y$, flat over S. Suppose that for each $s \in S$, $F(\mathcal{E}_s)$ is concentrated in degree 0. Then there is an S-flat $\mathcal{O}_{S\times X}$-module $\hat{\mathcal{E}}$, such that for every $s \in S$, $\hat{\mathcal{E}}_s = F(\mathcal{E}_s)$.*

Proof. Let $\hat{\mathcal{E}} = F_S(\mathcal{E})$, and take a point $s \in S$. Applying Lemma 4.2.1 with $T = \{s\}$, we see that

$$\mathbf{L}\, j_s^*(\hat{\mathcal{E}}) = F(\mathcal{E}_s),$$

where $j_s : \{s\} \times X \hookrightarrow S \times X$, is the inclusion morphism. Applying Lemma 3.1.2 shows that $\hat{\mathcal{E}}$ is concentrated in degree 0, and is flat over S. □

In the next section we shall need

Lemma 4.2.3. *Suppose that $\mathcal{P}$ is a sheaf on $X \times Y$, flat over Y. Then the homomorphism*

$$\mathrm{Hom}^1_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^1_{\mathrm{D}(X)}(\mathcal{P}_y, \mathcal{P}_y), \tag{4.3}$$

induced by the functor F is the Kodaira-Spencer map for the family $\mathcal{P}$, if we identify the first space with the tangent space to Y at y in the usual way.

Proof. Let $D = \operatorname{Spec} k[\epsilon]/\epsilon^2$ denote the double point.

Note that we can identify the domain of (4.3) with the set of deformations of $\mathcal{O}_y$ over D, and the image with the set of deformations of $\mathcal{P}_y$ over D. If we do this, it is easy to see that the map F is just given by applying the functor F_D.

Given an element $f : D \to Y$ of $\mathrm{T}_y Y$, the corresponding deformation of $\mathcal{O}_y$ over D is obtained by pulling-back the family $\mathcal{O}_\Delta$ on $Y \times Y$ to $D \times Y$ using f (here Δ denotes the diagonal in $Y \times Y$). By Lemma 4.2.1, if we then apply F_D, we get the same result as if we first applied F_Y, which gives the sheaf $\mathcal{P}$ on $X \times Y$, and then pulled-back via f. But this is the Kodaira-Spencer map for the family $\mathcal{P}$. □

We finish this section with two easy lemmas.

Lemma 4.2.4. *Suppose X is smooth. Then the functor*

$$G = \Phi_{X \to Y}^{\mathcal{P}^\vee \otimes \pi_X^* \omega_X [\dim X]}$$

is a left adjoint for F. If Y is also smooth then

$$H = \Phi_{X \to Y}^{\mathcal{P}^\vee \otimes \pi_Y^* \omega_Y [\dim Y]}$$

is a right adjoint for F.

Proof. This follows immediately from Grothendieck-Verdier duality together with the adjunctions [Ha1, II.5.11] and [Ha1, II.5.16]. See [BO, Lemma 1.3] for more details. $\square$

Lemma 4.2.5. *Suppose Z is a third projective variety, and $\mathcal{Q}$ is an object of $\mathrm{D}_c(Y \times Z)$ of finite homological dimension. Then*

$$\Phi_{Y \to X}^{\mathcal{P}} \circ \Phi_{Z \to Y}^{\mathcal{Q}} \cong \Phi_{Z \to X}^{\mathcal{R}},$$

where $\mathcal{R}$ is the object of $\mathrm{D}_c^b(X \times Z)$ given by

$$\mathbf{R}\pi_{XZ,*}(\pi_{XY}^*(\mathcal{P}) \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*(\mathcal{Q})),$$

and $\pi_{XZ}, \pi_{XY}, \pi_{YZ}$ are the projections of $X \times Y \times Z$ onto the factors $X \times Z$, $X \times Y$ and $Y \times Z$ respectively.

Proof. This is entirely mechanical. See [Muk2, Prop. 1]. $\square$

4.3 Invertibility of integral functors

As in the last section, fix projective varieties X and Y, an object $\mathcal{P}$ of $\mathrm{D}_c(Y \times X)$ of finite homological dimension, and let F be the functor $\Phi_{Y \to X}^{\mathcal{P}}$ defined by (4.1). In this section we prove Theorem 4.0.3 in two parts. The first part was essentially proved by Bondal and Orlov [BO].

Theorem 4.3.1. *Suppose X is smooth, and Y is Cohen-Macaulay. Then the functor F is fully faithful if and only if, for each point $y \in Y$,*

$$\mathrm{Hom}_{\mathrm{D}(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = k,$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i,

$$\mathrm{Hom}^i_{\mathrm{D}(X)}(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0 \textit{ unless } y_1 = y_2 \textit{ and } 0 \le i \le \dim Y.$$

In this case Y is also smooth.

Proof. It is clear that the given conditions are necessary for F to be fully faithful; we prove sufficiency. Let G be the left adjoint of F given by Lemma 4.2.4. By Lemma 4.2.5,

$$G \circ F \cong \Phi^{\mathcal{Q}}_{Y \to Y},$$

for some object $\mathcal{Q}$ of $\mathrm{D}^b_c(Y \times Y)$. If $j_y : Y \hookrightarrow Y \times Y$ is the inclusion of the fibre $\{y\} \times Y$, then for all $y \in Y$,

$$GF\mathcal{O}_y \cong \mathbf{L}\, j_y^*(\mathcal{Q})$$

Take a point $y \in Y$. For any point $z \in Y$ there are isomorphisms of vector spaces

$$\mathrm{Hom}^p_{\mathrm{D}(Y)}(GF\mathcal{O}_y, \mathcal{O}_z) \cong \mathrm{Hom}^p_{\mathrm{D}(X)}(F\mathcal{O}_y, F\mathcal{O}_z)$$

coming from the adjunction $G \dashv F$. By Corollary 3.2.5, $GF\mathcal{O}_y$ is concentrated in degree 0 and is supported at the point y, and so by Lemma 3.1.2, $\mathcal{Q}$ is a Y-flat $\mathcal{O}_{Y\times Y}$-module. Furthermore, there is a unique morphism $GF\mathcal{O}_y \to \mathcal{O}_y$. If K_y is the kernel of this morphism, one has a short exact sequence

$$0 \longrightarrow K_y \longrightarrow GF\mathcal{O}_y \xrightarrow{\delta(\mathcal{O}_y)} \mathcal{O}_y \longrightarrow 0.$$

I claim that $K_y = 0$. Note that it will be enough to prove this for one point $y \in Y$, since $\chi(\mathcal{Q}_y)$ is constant as y varies. Applying the functor $\mathrm{Hom}_{\mathrm{D}(Y)}(-, \mathcal{O}_y)$, and using the diagram (4.2), it will be enough to show that the homomorphism

$$F : \mathrm{Hom}^1_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^1_{\mathrm{D}(X)}(F\mathcal{O}_y, F\mathcal{O}_y), \tag{4.4}$$

is injective. By Lemma 4.2.3, the map

$$\mathrm{Hom}^1_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}^1_{\mathrm{D}(Y)}(GF\mathcal{O}_y, GF\mathcal{O}_y),$$

induced by the functor GF is given by the Kodaira-Spencer map for the family $\mathcal{Q}$. By Lemma 3.4.1 together with Lemma 4.3.2 below, there is a point $y \in Y$, such that this map is injective. Clearly the map (4.4) must then also be injective, and the claim follows.

For any point $y \in Y$, $\mathcal{Q}_y = GF\mathcal{O}_y = \mathcal{O}_y$. It follows that $\mathcal{Q} = \Delta_*(L)$, where $\Delta : Y \hookrightarrow Y \times Y$ is the diagonal, and L is some line bundle on Y. Then $G \circ F \cong (L \otimes -)$, which implies that F is fully faithful. Finally Y is smooth by [AK, III.5.9, III.5.15] because for any $y \in Y$, only finitely many of the groups $\mathrm{Ext}^i_Y(\mathcal{O}_y, \mathcal{O}_y)$ can be non-zero. □

Lemma 4.3.2. *Let Y be a projective variety over k, and let Q be a sheaf on Y supported at a point $y \in Y$. Suppose that*

$$\mathrm{Hom}_Y(Q, \mathcal{O}_y) = k.$$

Then Q is the structure sheaf of a zero-dimensional closed subscheme of Y.

Proof. There exists a short exact sequence

$$0 \longrightarrow P \longrightarrow Q \xrightarrow{g} \mathcal{O}_y \longrightarrow 0.$$

Suppose $f : \mathcal{O}_Y \to Q$ is a non-surjective morphism of sheaves. Considering the cokernel of f shows that there is a non-zero morphism $h : Q \to \mathcal{O}_y$ such that $h \circ f = 0$. But by hypothesis h must be a multiple of g, so one must have $g \circ f = 0$, hence f comes from a morphism $\mathcal{O}_Y \to P$. Now

$$\dim_k \mathrm{H}^0(Y, P) = \chi(P) < \chi(Q) = \dim_k \mathrm{H}^0(Y, Q),$$

so there must be a morphism $\mathcal{O}_Y \to Q$ which is surjective. $\square$

Theorem 4.3.3. *Suppose X is smooth and F is fully faithful. Then F is an equivalence if, and only if, for every point $y \in Y$,*

$$F\mathcal{O}_y \otimes \omega_X \cong F\mathcal{O}_y. \tag{4.5}$$

Proof. Let G and H denote the left and right adjoint functors of F respectively. Suppose first that F is an equivalence. Then G and H are both quasi-inverses for F, so for any $y \in Y$,

$$G(F\mathcal{O}_y) \cong H(F\mathcal{O}_y) \cong \mathcal{O}_y.$$

From the formulas for G and H given in Lemma 4.2.4,

$$G(F\mathcal{O}_y) \cong G(F\mathcal{O}_y) \otimes \omega_Y \cong H(F\mathcal{O}_y \otimes \omega_X)[\dim X - \dim Y].$$

But G is an equivalence, so one concludes that X and Y have the same dimension, and there is an isomorphism (4.5).

For the converse, let X have dimension n, and suppose that (4.5) holds for all $y \in Y$. Take an object b of $\mathrm{D}(X)$ such that $Hb \cong 0$. For any point $y \in Y$, and any integer i,

$$\begin{aligned}\mathrm{Hom}^i_{\mathrm{D}(Y)}(Gb, \mathcal{O}_y) &= \mathrm{Hom}^i_{\mathrm{D}(X)}(b, F\mathcal{O}_y) = \mathrm{Hom}^i_{\mathrm{D}(X)}(b, F\mathcal{O}_y \otimes \omega_X) \\ &= \mathrm{Hom}^{n-i}_{\mathrm{D}(X)}(F\mathcal{O}_y, b)^\vee = \mathrm{Hom}^{n-i}_{\mathrm{D}(Y)}(\mathcal{O}_y, Hb)^\vee = 0,\end{aligned}$$

so by Lemma 4.1.4, $Gb \cong 0$. Applying Theorem 4.1.6 completes the proof. $\square$

Remark 4.3.4. Theorem 4.0.3 now follows from Theorems 4.3.1 and 4.3.3.

We finish this chapter with the following lemma.

Lemma 4.3.5. *Let X and Y be smooth projective varieties, and suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two objects of $\mathrm{D}^b_c(Y \times X)$ such that the two functors $\Phi_i = \Phi^{\mathcal{P}_i}_{Y \to X}$ are isomorphic equivalences of categories. Then there is an isomorphism $\mathcal{P}_1 \cong \mathcal{P}_2$ in $\mathrm{D}^b_c(Y \times X)$.*

Proof. Let Ψ_1 be a quasi-inverse to Φ_1. Then $\Phi_2 \circ \Psi_1 \cong 1_{\mathrm{D}^b_c(X)}$ so by the argument given in the proof of Theorem 4.3.1, and Lemma 4.2.5,

$$\mathbf{R}\pi_{13*}(\pi_{12}^*(\mathcal{P}_1^\vee) \overset{\mathbf{L}}{\otimes} \pi_{23}^*(\mathcal{P}_2)) \cong \mathcal{O}_\Delta,$$

where π_{ij} denotes the projection of $X \times Y \times X$ onto the product of its ith and jth factors, and $\Delta \subset X \times X$ is the diagonal. Restricting to the subset of $X \times Y \times X$ given by $\pi_{12} = \pi_{23}$, we see that

$$\mathbf{R}\pi_{X,*}(\mathbf{R}\,\mathcal{H}om_{\mathcal{O}_{Y\times X}}(\mathcal{P}_1, \mathcal{P}_2)) = \mathcal{O}_X.$$

In particular, up to scalar multiples, there exists a unique non-zero morphism $f : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$. Embed f in a triangle

$$\mathcal{P}_1 \xrightarrow{f} \mathcal{P}_2 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{P}_1[1],$$

of $\mathrm{D}(X \times Y)$. Given a point $y \in Y$, let $i_y : X \hookrightarrow Y \times X$ be the inclusion of the fibre $\{y\} \times X$. Then for any point $y \in Y$ there is a triangle

$$\Phi_1(\mathcal{O}_y) \xrightarrow{\mathbf{L}\,i_y^*(f)} \Phi_2(\mathcal{O}_y) \longrightarrow \mathbf{L}\,i_y^*(\mathcal{Q}) \longrightarrow \Phi_1(\mathcal{O}_y)[1].$$

Since $E = \Phi_1(\mathcal{O}_y) \cong \Phi_2(\mathcal{O}_y)$, and

$$\mathrm{Hom}_{\mathrm{D}(X)}(E, E) = \mathrm{Hom}_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) = k,$$

the morphism $\mathbf{L}\,i_y^*(f)$ is an isomorphism whenever it is non-zero, and this happens if, and only if, $\mathbf{L}\,i_y^*(\mathcal{Q}) = 0$. Since f is non-zero there is a non-empty open subset U of Y such that $\mathbf{L}\,i_y^*(f)$ is an isomorphism for $y \in U$.

Repeating the argument, up to scalar multiples, there is a unique non-zero morphism $g : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$, and a non-empty open subset V of Y such that $\mathbf{L}\,i_y^*(g)$ is an isomorphism for $y \in V$. Now Y is irreducible, so U and V have non-zero intersection, and the morphisms $g \circ f$ and $f \circ g$ are non-zero. But the argument above shows that (up to scalar multiples) there is a unique non-zero morphism $\mathcal{P}_i \to \mathcal{P}_i$. Thus, rescaling g, one has $g \circ f \cong 1_{\mathcal{P}_1}$ and $f \circ g \cong 1_{\mathcal{P}_2}$. $\square$

Part II

Fourier-Mukai transforms and stable sheaves

Part II

Part II is concerned with constructing examples of Fourier-Mukai transforms and applying them to moduli problems. We shall work throughout in the category of schemes of finite type over $\mathbb{C}$.

In Chapter 5 we give a summary of the simple properties of stable sheaves on projective schemes, and quote Simpson's result which guarantees the existence of moduli spaces of semistable sheaves. We give particular emphasis to the properties of moduli spaces of sheaves on surfaces.

In Chapter 6 we introduce FM transforms. Various properties common to all such transforms are listed and we give some simple examples. In particular we construct a large number of transforms for K3 surfaces. As an example of the use of FM transforms in solving moduli problems we give a derivation of Atiyah's well-known classification of stable bundles on an elliptic curve. We also classify all FM transforms for smooth curves.

As an extended example of the use of FM transforms in solving moduli problems, Chapter 7 contains a novel application of Mukai's original transform for Abelian surfaces to the computation of moduli spaces of sheaves.

In Chapter 8 we develop a new class of FM transforms for elliptic surfaces. These are essentially relative transforms, obtained by gluing FM transforms on each elliptic fibre. It is hard to see how to perform this gluing directly, since a general elliptic surface has singular fibres, so we use Simpson's relative moduli schemes, which parameterise stable sheaves on the fibres of a morphism. As an application of our transforms we prove Theorem 1.2.3.

Chapter 9 is concerned with FM transforms on quotient surfaces, i.e. Enriques and bielliptic surfaces. We obtain a result which shows that FM transforms between such surfaces correspond to invariant FM transforms between certain unbranched covering surfaces. We thus obtain many new examples of FM transforms for quotient surfaces.

Finally in Chapter 10 we address ourselves to the problem of classifying FM transforms for surfaces. This is a difficult problem, due to the existence of many transforms for K3 surfaces which are numerically trivial. These transforms are related to the existence of rigid sheaves on K3 surfaces, and in general, the group of auto-equivalences of the derived category of sheaves on a K3 seems to be rather complicated. We do manage, however, to give a complete answer to the following question : given a minimal surface X, for which surfaces Y are the derived categories of X and Y equivalent?

Notation

All schemes X will be of finite type over $\mathbb{C}$, and all morphisms will be $\mathbb{C}$-morphisms. A point $x \in X$ will always mean a closed point, and a sheaf will mean a coherent $\mathcal{O}_X$-module. A surface will mean a smooth projective variety of dimension 2, but a curve will just mean a variety of dimension 1: if we mean a smooth curve we will say so. The canonical divisor of a surface is denoted by $\mathcal{K}_X$. A sheaf E on a variety X is *simple* if $\operatorname{End}_X(E) = \mathbb{C}$.

Given a scheme X, $\mathrm{D}(X)$ will denote the bounded derived category of coherent sheaves on X. This is what was referred to as $D^b_c(X)$ in Part I. Given a morphism of schemes $f : X \to Y$ we can define the derived functor

$$\mathbf{R}f_* : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y),$$

whenever f is proper, and the derived functor

$$\mathbf{L}\, f^* : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

whenever f is flat (or more generally of finite tor-dimension), or when Y is a smooth projective variety.

Chapter 5

Moduli spaces of sheaves

One of the most important applications of the theory of FM transforms is to the study of stable sheaves on projective varieties. Stable sheaves are important because they move in well-behaved moduli spaces. In this chapter we define stable sheaves and review some of their well-known properties. The main theorem is Theorem 5.2.4 below, which guarantees the existence of moduli spaces of semistable sheaves on projective schemes. In the form we state it, it is due to C. Simpson, although less general versions were proved earlier by D. Gieseker and M. Maruyama.

5.1 Stable sheaves on projective schemes

Let us fix a projective scheme X together with a projective embedding

$$i : X \hookrightarrow \mathbb{P}^N(\mathbb{C}).$$

Let $\mathcal{O}_X(1)$ be the corresponding very ample line bundle on X. Given a sheaf E on X, and an integer n, put $E(n) = E \otimes \mathcal{O}_X(1)^{\otimes n}$.

Definition 5.1.1. Let E be a sheaf on X. The *Hilbert polynomial* of E is the unique polynomial $\mathrm{P}_E \in \mathbb{Q}[t]$, such that

$$\mathrm{P}_E(n) = \dim_{\mathbb{C}} \mathrm{H}^0(X, E(n)) \text{ for all } n \gg 0.$$

The degree of P_E is called the *dimension* of E, and is equal to the dimension of the support of E. The *normalised Hilbert polynomial* of E, denoted $\wp_E$, is the unique rational multiple of P_E which is monic.

We order the elements of $\mathbb{Q}[t]$ lexicographically, thus for two polynomials $\mathrm{P}_1, \mathrm{P}_2 \in \mathbb{Q}[t]$,

$$\mathrm{P}_1 \leq \mathrm{P}_2 \quad \Longleftrightarrow \quad \mathrm{P}_1(n) \leq \mathrm{P}_2(n) \quad \forall n \gg 0.$$

We can now define what it means for a sheaf on X to be stable.

Definition 5.1.2. A non-zero sheaf E on X is said to have *pure dimension* if all non-zero subsheaves $0 \neq A \subseteq E$ have the same dimension.

A non-zero sheaf E on X is said to be *(semi)stable* if E has pure dimension, and if for any proper subsheaf $0 \neq A \subsetneq E$ one has $\wp_A < \wp_E$ (respectively $\wp_A \leq \wp_E$).

Notes. (i) The notions of stability and semistability of sheaves on X depend on the projective embedding $i : X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ chosen for X.

(ii) If $j : Y \hookrightarrow X$ is a closed subscheme of X, and $\mathcal{O}_Y(1) = j^*\mathcal{O}_X(1)$, then a sheaf E on Y is stable if and only if j_*E is stable on X.

(iii) One can equally well define stability in terms of quotients : a sheaf E on X of pure dimension is (semi)stable precisely when any non-zero, proper quotient B of E satisfies $\wp_E < \wp_B$ (respectively $\wp_E \leq \wp_B$).

The following proposition summarises some well-known properties of stable sheaves.

Proposition 5.1.3. *(a) Every pure dimension sheaf E on X has a unique filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$

called the Harder-Narasimhan filtration, *such that for each $1 \leq i \leq n$, the factor $F_i = E_i/E_{i-1}$ is semistable, and $\wp_{F_1} > \wp_{F_2} > \cdots > \wp_{F_n}$.*

(b) Every semistable sheaf E on X has a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$

called a Jordan-Hölder filtration *such that for each $1 \leq i \leq n$, the factor $F_i = E_i/E_{i-1}$ is stable, and $\wp_{F_i} = \wp_E$. While such a filtration may not be unique, the sheaf*

$$\mathrm{gr}(E) = \bigoplus_{1 \leq i \leq n} F_i,$$

is well-defined. Two semistable sheaves A and B on X are called gr-equivalent *if the sheaves* $\mathrm{gr}(A)$ *and* $\mathrm{gr}(B)$ *are isomorphic.*

(c) If A and B are semistable sheaves on X with $\wp_A > \wp_B$, then there are no non-zero morphisms $A \to B$.

(d) If A and B are stable sheaves on X and $\wp_A = \wp_B$, then any non-zero morphism $A \to B$ is an isomorphism.

(e) Any stable sheaf on X is simple.

(f) If S is a scheme and $\mathcal{E}$ is a sheaf on $S \times X$, flat over S, then the set of points of $s \in S$ for which the sheaf $\mathcal{E}_s$ on X is (semi)stable is the set of points of an open subscheme of S.

(g) If E is a non-zero sheaf on X which is not semistable, there exists a non-trivial short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

with $\mathrm{Hom}_X(A, B) = 0$.

Proof. Parts (a) - (e) appear on [Si], pages 55-56 and part (f) is a special case of [Si], Lemma 3.7. For (g), if E does not have pure dimension, take A to be the maximal subsheaf of E which has dimension less than the dimension of E. If E does have pure dimension consider the Harder-Narasimhan filtration and put $A = E_{n-1}$ and $B = F_n$. $\square$

Remark 5.1.4. Given a pure dimension sheaf E on X, a *destabilising sequence* for E, is a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

with $\wp_A \geq \wp_E \geq \wp_B$. If E is unstable, then by taking A to be a subsheaf of E with smallest dimension and maximal normalized Hilbert polynomial we can always find such a sequence with A semistable. Similarly we can always find a destabilising sequence with B semistable.

5.2 Families of sheaves and moduli spaces

First we review some notions from category theory. Given a category $\mathcal{C}$, and an object X of $\mathcal{C}$, h_X will denote the functor

$$\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^\circ \longrightarrow \mathrm{Sets}\,.$$

If $f : X \to Y$ is a morphism of $\mathcal{C}$, $f_* : h_X \to h_Y$ will denote the morphism of functors given by composition with f.

Definition 5.2.1. Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and

$$F : \mathcal{C}^\circ \longrightarrow \mathrm{Sets}$$

a functor. The object X is said to *represent* F if there is an isomorphism of functors $F \cong h_X$. One says that X *corepresents* F if there is a morphism of functors

$$\alpha : F \longrightarrow h_X,$$

such that for any other morphism of functors

$$\beta : F \to h_Y,$$

with Y an object of $\mathcal{C}$, there is a unique morphism $f : X \to Y$ in $\mathcal{C}$, such that $\beta = f_* \circ \alpha$.

Note that if an object of $\mathcal{C}$ represents F then it corepresents F, and if two objects of $\mathcal{C}$ corepresent F then they are isomorphic.

Returning to geometry, let us fix a scheme X.

Definition 5.2.2. A *family of sheaves* on X, parameterised by a scheme S, is a sheaf $\mathcal{E}$ on $S \times X$, flat over S. Two families of sheaves $\mathcal{E}$, $\mathcal{F}$ on X parameterised by S are called *equivalent families* if there is a line bundle L on S such that $\mathcal{E} = \mathcal{F} \otimes \pi^*(L)$.

Notes. (a) If $\mathcal{E}$ is a family of sheaves on X parameterised by S, and s is a point of S, the restriction of $\mathcal{E}$ to $\{s\} \times X$ is denoted $\mathcal{E}_s$.

(b) If $\mathcal{E}$ and $\mathcal{F}$ are equivalent families on X parameterised by S then $\mathcal{E}_s \cong \mathcal{F}_s$ for all $s \in S$.

(c) If $\mathcal{E}$ is a family of sheaves on X parameterised by S, and $f : T \to S$ is a morphism of schemes, then $(f \times 1_X)^*(\mathcal{E})$ is a family of sheaves on X parameterised by T. We refer to it as the family $\mathcal{E}$ pulled-back via f.

Let us fix a class $\mathcal{A}$ of sheaves on X. If S is a scheme, let $F_{\mathcal{A}}(S)$ be the set of equivalence classes of families $\mathcal{E}$ of sheaves on X parameterised by S, such that $\mathcal{E}_s \in \mathcal{A}$ for all $s \in S$. If $f : T \to S$ is a morphism of schemes, then $(f \times 1_X)^*$ defines a map of sets $F_{\mathcal{A}}(S) \to F_{\mathcal{A}}(T)$. Thus $F_{\mathcal{A}}$ defines a functor

$$F_{\mathcal{A}} : (\mathrm{Sch})^{\circ} \longrightarrow \mathrm{Sets}\,.$$

Definition 5.2.3. A scheme $\mathcal{M}$ is a *(coarse) moduli space* for $\mathcal{A}$-sheaves if it corepresents the functor $F_{\mathcal{A}}$. The moduli space is called *fine* if $\mathcal{M}$ represents $F_{\mathcal{A}}$.

Notes. (a) The moduli space of $\mathcal{A}$-sheaves, if it exists, is unique up to isomorphism.

(b) To say that $\mathcal{M}$ is a fine moduli space of $\mathcal{A}$-sheaves is to say that there is a family $\mathcal{P}$ of $\mathcal{A}$-sheaves parameterised by $\mathcal{M}$, such that for any scheme S, and any family $\mathcal{E}$ of $\mathcal{A}$-sheaves parameterised by S, there is a unique map $f : S \to \mathcal{M}$ such that $\mathcal{E}$ is equivalent to the family $\mathcal{P}$ pulled-back via f. In particular, there is a bijection between the points of $\mathcal{M}$ and the isomorphism classes of sheaves in $\mathcal{A}$. We call the sheaf $\mathcal{P}$ on $\mathcal{M} \times X$ a *universal sheaf.*

Examples show that the class of all sheaves on a projective scheme is in general too large to be parameterised by a scheme, and also exhibits unpleasant topological behaviour ('jumping', see [New, Ch. 5, §1]). Thus to obtain well-behaved moduli spaces one must restrict the class of sheaves $\mathcal{A}$ which are to be parameterised. The following theorem ([Si], Theorem 1.21) shows that if one chooses $\mathcal{A}$ to be the class of semistable sheaves one always obtains a nice moduli space.

Theorem 5.2.4. *Let X be a projective scheme and fix an embedding $i : X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ and a polynomial $P \in \mathbb{Q}[t]$. Then if the class of semistable sheaves on X with Hilbert polynomial P is non-empty, it has a (coarse) moduli space $\mathcal{M}_X^P$ which is projective over $\mathbb{C}$.* $\square$

Simpson also proved that the points of $\mathcal{M}_X^P$ correspond to gr-equivalence classes of semistable sheaves with Hilbert polynomial P (see Proposition 5.1.3).

Following ideas of Maruyama, Mukai gave the following criterion for the moduli space $\mathcal{M}_X^P$ to be fine.

Theorem 5.2.5. *Let X and P be as in Theorem 5.2.4. Suppose that 1 is the highest common factor of the integers $P(n)$, $n \in \mathbb{Z}$. Then all semistable sheaves with Hilbert polynomial P are actually stable, and the moduli space $\mathcal{M}_X^P$ is fine.*

Proof. If E is a sheaf on X which is semistable but not stable, then considering a Jordan-Hölder filtration shows that there is a sheaf A on X and an integer $m > 1$ such that $\mathrm{P}_E = m\,\mathrm{P}_A$. It follows that m divides $\mathrm{P}_E(n)$ for all n. This proves the first statement. The second statement follows from the results of [Muk3, App. 2]. $\square$

5.3 Relative moduli schemes

In our treatment of FM transforms on elliptic surfaces we shall need a relative version of Theorem 5.2.4.

Let us fix a projective morphism of schemes $\pi : X \to S$, and an embedding $i : X \hookrightarrow S \times \mathbb{P}^N(\mathbb{C})$ commuting with the projections to S. This determines a projective embedding of each fibre $X_s = X \times_S \{s\}$ of f. Fix also a polynomial $P \in \mathbb{Q}[t]$. The idea is to construct a relative moduli space

$$\mathcal{M}_{X/S}^P \longrightarrow S$$

whose fibre over each point $s \in S$ is the moduli space $\mathcal{M}_{X_s}^P$.

Definition 5.3.1. A (semi)stable sheaf on X/S with Hilbert polynomial P is defined to be a sheaf $\mathcal{E}$ on X, flat over S, such that for each $s \in S$ the restriction of $\mathcal{E}$ to the fibre X_s is (semi)stable with Hilbert polynomial P.

Two such sheaves $\mathcal{E}$ and $\mathcal{F}$ are equivalent if there is a line bundle L on S such that $\mathcal{E} \cong \mathcal{F} \otimes \pi^*(L)$.

Now define a functor

$$F_{X/S}^P : (\mathrm{Sch}\,/S)^\circ \longrightarrow \mathrm{Sets},$$

by letting $F^P_{X/S}(T)$ be the set of equivalence classes of semistable sheaves on $X \times_S T/T$ with Hilbert polynomial P. We shall assume that there exist S-schemes T such that $F^P_{X/S}(T)$ is not empty.

Theorem 5.3.2. *There is a projective S-scheme $\mathcal{M}^P_{X/S}$ which corepresents the functor $F^P_{X/S}$. Moreover, if $\gcd\{P(n) : n \in \mathbb{Z}\} = 1$, $\mathcal{M}^P_{X/S}$ represents $F^P_{X/S}$.*

Proof. The first statement is [Si], Theorem 1.21. The second part is proved using the same argument as for Proposition 5.2.5. □

The following result is rather obvious, but we include a proof for completeness.

Proposition 5.3.3. *Suppose $\mathcal{M}^P_{X/S}$ represents $F^P_{X/S}$. Then the fibre of the projective morphism $\mathcal{M}^P_{X/S} \to S$ over a point $s \in S$ is isomorphic to $\mathcal{M}^P_{X_s}$. Moreover if $\mathcal{P}$ is a universal sheaf on $\mathcal{M}^P_{X/S} \times_S X$, then the restriction of $\mathcal{P}$ to the fibre $\mathcal{M}^P_{X_s} \times X_s$ is a universal sheaf for the functor $F^P_{X_s}$.*

Proof. To say that $\mathcal{M} = \mathcal{M}^P_{X/S}$ represents $F = F_{X/S}(P)$ is to say that there is a sheaf $\mathcal{P} \in F(\mathcal{M})$, such that for any S-scheme T, and any sheaf $\mathcal{E} \in F(T)$, there is a unique S-morphism $f : T \to \mathcal{M}$ such that $\mathcal{E}$ is equivalent to $(f \times_S 1_X)^*(\mathcal{P})$.

Fix $s \in S$ and let $\mathcal{P}_s$ be the restriction of $\mathcal{P}$ to the fibre $\mathcal{M}_s \times X_s$ of $\mathcal{M} \times_S X$ over s. Let T be a scheme and let $\mathcal{E}$ be a sheaf on $T \times X_s$ representing an element of $F^P_{X_s}(T)$. Making T into an S-scheme via the composite $T \to \{s\} \to S$, $\mathcal{E}$ defines an element of $F(T)$. Hence there is a unique S-morphism $f : T \to \mathcal{M}$ such that $\mathcal{E}$ is equivalent to $(f \times_S 1_X)^*(\mathcal{P})$. But an S-morphism $T \to \mathcal{M}$ is just a morphism $T \to \mathcal{M}$ which factors through the fibre $\mathcal{M}_s$. Thus $\mathcal{M}_s$ represents $F^P_{X_s}$, and $\mathcal{P}_s$ is a universal sheaf. □

5.4 Stable sheaves on surfaces

Let X be a surface (smooth and projective as always). Given a sheaf E on X, we write its Chern class as a triple

$$(\mathrm{r}(E), \mathrm{c}_1(E), c_2(E)) \in \mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}.$$

Here $\mathrm{NS}(X)$ is the Neron-Severi group of X, i.e. the subgroup of $\mathrm{H}^2(X, \mathbb{Z})$ consisting of the first Chern classes of line bundles on X. We also put

$$\mathrm{ch}_2(E) = \frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E).$$

Let ℓ be a polarisation on X. Thus ℓ is an element of $\mathrm{H}^2(X, \mathbb{Z})$ which is the first Chern class of an ample line bundle L on X. Now some power $L^{\otimes p}$ of L is very ample and defines a projective embedding of X. It is easy to check using the Riemann-Roch formula (see Lemma 5.4.7 below) that the notion of (semi)stability of sheaves is independent of the value of p chosen, so one can speak of sheaves on X as being (semi)stable with respect to the polarisation ℓ.

If $\mathcal{E}$ is a family of sheaves on a surface X, parameterised by a connected scheme S, the Chern classes of $\mathcal{E}_s$ are constant for all $s \in S$. Thus the moduli space of semistable sheaves on X splits into disjoint components corresponding to the different Chern classes.

Definition 5.4.1. Let (X, ℓ) be a polarised surface, and take a triple

$$(r, \mathrm{c}_1, \mathrm{c}_2) \in \mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}.$$

Then $\mathcal{M}^{\ell}_X(r, \mathrm{c}_1, \mathrm{c}_2)$ is the union of those components of the moduli space of semistable sheaves on X with respect to ℓ, which parameterise sheaves of the given Chern class.

In the case of sheaves on surfaces the definition of (semi)stability simplifies.

Definition 5.4.2. Let (X, ℓ) be a polarised surface. The *degree* of E is the integer $\mathrm{d}(E) = \mathrm{c}_1(E) \cdot \ell$, and the *slope* of E is the quotient $\mu(E) = \mathrm{d}(E)/\,\mathrm{r}(E)$.

Lemma 5.4.3. *Let (X, ℓ) be a polarised surface.*

(a) A sheaf E on X has pure dimension 2 on X precisely when it is torsion-free.

(b) A torsion-free sheaf E on X is (semi)stable (with respect to ℓ) iff for all proper subsheaves $0 \neq A \subsetneq E$ one has $\mu(A) \leq \mu(E)$, and if $\mu(A) = \mu(E)$ one has

$$\frac{\chi(A)}{\mathrm{r}(A)} < \frac{\chi(E)}{\mathrm{r}(E)}, \qquad \textit{(respectively } \leq).$$

(c) A pure dimension 1 sheaf E on X is (semi)stable (with respect to ℓ) iff for all proper subsheaves $0 \neq A \subsetneq E$ one has

$$\frac{\chi(A)}{\mathrm{d}(A)} < \frac{\chi(E)}{\mathrm{d}(E)}, \qquad \textit{(respectively } \leq).$$

(d) A pure dimension 0 sheaf on X is just a sheaf supported at a finite set of points. Any such sheaf is semistable. The only stable pure dimension 0 sheaves are the structure sheaves of single (closed) points. □

Remark 5.4.4. A torsion-free sheaf E on a polarised surface (X, ℓ) is called μ-(semi)stable (with respect to ℓ) if for all proper subsheaves $0 \neq A \subsetneq E$,

$$\mu(A) < \mu(E), \qquad \text{(respectively } \leq).$$

Thus there is a chain of implications

$$E \ \mu\text{-stable} \implies E \text{ stable} \implies E \text{ semistable} \implies E \ \mu\text{-semistable}.$$

In later chapters we shall apply FM transforms to the problem of computing some of the spaces of Definition 5.4.1. The following well-known result describes their local structure.

Proposition 5.4.5. *Let (X, ℓ) be a polarised surface, and let E be a torsion-free stable sheaf on X with Hilbert polynomial P. Then*

(a) the tangent space to $\mathcal{M}_X^P$ at the point corresponding to E is naturally identified with the vector space $\mathrm{Ext}_X^1(E, E)$.

(b) if the vector spaces $\mathrm{Ext}_X^2(E, E)$ and $\mathrm{H}^2(X, \mathcal{O}_X)$ have the same dimension, the moduli space $\mathcal{M}_X^P$ is smooth at the point corresponding to E.

Proof. See [Ar1]. □

We finish this chapter by quoting a form of the Riemann-Roch theorem for smooth surfaces.

Definition 5.4.6. If (E, F) is a pair of sheaves on X let

$$\chi(E, F) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}_X^i(E, F).$$

Note that $\chi(E) = \chi(\mathcal{O}_X, E)$.

Proposition 5.4.7. (Riemann-Roch formula) *For any pair of sheaves (E, F) on a surface X,*

$$\begin{aligned}\chi(E, F) = \mathrm{r}(E)\,\mathrm{ch}_2(F) - \mathrm{c}_1(E) \cdot \mathrm{c}_1(F) + \mathrm{r}(F)\,\mathrm{ch}_2(E) \\ + \frac{1}{2}(\mathrm{r}(F)\,\mathrm{c}_1(E) - \mathrm{r}(E)\,\mathrm{c}_1(F)) \cdot \mathcal{K}_X + \mathrm{r}(E)\,\mathrm{r}(F)\,\chi(\mathcal{O}_X),\end{aligned}$$

where $\mathcal{K}_X$ is the first Chern class of the canonical line bundle ω_X. □

Remark 5.4.8. More generally, given an object E of $\mathrm{D}(X)$ one defines its Chern classes to be the alternating sums

$$\mathrm{c}_r(E) = \sum_i (-1)^i \mathrm{c}_r(\mathrm{H}^i(E)).$$

Given two objects E and F of $\mathrm{D}(X)$ the Riemann-Roch formula still holds providing one makes the definition

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{D}(X)}(E, F[i]).$$

Chapter 6

Fourier-Mukai transforms

In this chapter we define Fourier-Mukai transforms, list some of their general properties, and study some examples. To demonstrate their use we give a simple derivation of the well-known classification of stable bundles on elliptic curves. We also classify all FM transforms for smooth curves.

6.1 Definitions and simple properties

Recall that an *integral functor* is a functor $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ between the bounded derived categories of sheaves on two projective varieties X and Y, which is isomorphic to the functor $\Phi^{\mathcal{P}}_{Y\to X}$ defined by the formula

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)), \tag{6.1}$$

where π_X and π_Y are the projection maps $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$, and $\mathcal{P}$ has finite homological dimension.

Definition 6.1.1. A *Fourier-Mukai transform* is an integral functor which is an equivalence of categories.

Remark 6.1.2. If X and Y are smooth, Theorem 1.3.6 implies that any exact equivalence of categories $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is an FM transform. We shall not need this result.

It will also be useful to have the following definition.

Definition 6.1.3. A *Fourier-Mukai triple* is a triple $(X, Y, \mathcal{P})$ consisting of a pair of projective varieties (X, Y) and an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$ of finite homological dimension, such that the corresponding integral functor $\Phi^{\mathcal{P}}_{Y\to X}$ defined by (6.1) is an equivalence of categories.

Clearly an FM triple determines an FM transform. Lemma 4.3.5 allows us to go the other way:

Definition 6.1.4. Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform with X and Y smooth. Then the *kernel* of Φ is the unique object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$, up to isomorphism, such that $\Phi \cong \Phi^{\mathcal{P}}_{Y \to X}$.

Proposition 4.2.4 gives

Lemma 6.1.5. *Let X and Y be smooth projective varieties of dimension n, and let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform with kernel $\mathcal{P}$. Let*

$$\mathcal{Q} = \mathcal{P}^\vee \otimes \pi_X^* \omega_X[n].$$

Then $\Psi = \Phi^{\mathcal{Q}}_{X \to Y}$ is a quasi-inverse for Φ. □

Theorem 4.0.3 gives necessary and sufficient conditions for an integral functor to be an equivalence of categories. For our purposes a less general statement will be more useful. First we need a definition.

Definition 6.1.6. Let X be a projective variety, and $\mathcal{E}$ a family of sheaves on X parameterised by a scheme S. Then $\mathcal{E}$ is called *strongly simple* over S if each sheaf $\mathcal{E}_s$ is simple, and if for any two distinct points s_1, s_2 of S, and any integer i, one has

$$\mathrm{Ext}^i_X(\mathcal{E}_{s_1}, \mathcal{E}_{s_2}) = 0.$$

The following theorem is an immediate consequence of Theorem 4.0.3.

Theorem 6.1.7. *Let X and Y be smooth projective varieties of the same dimension, and let $\mathcal{P}$ be a sheaf on $Y \times X$, flat over Y. Then $(X, Y, \mathcal{P})$ is an FM triple if and only if $\mathcal{P}$ is strongly simple over Y, and for all $y \in Y$, $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$.* □

The FM transforms of Theorem 6.1.7 are all *sheaf transforms*, in that they take the structure sheaves of points on Y to sheaves on X. These are the transforms in which we are most interested.

Definition 6.1.8. Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform. Then Φ will be called a *sheaf transform* if for all $y \in Y$, the object $\Phi(\mathcal{O}_y)$ is concentrated in degree 0. If, moreover, each sheaf $\Phi(\mathcal{O}_y)$ is locally free, Φ will be called a *bundle transform.*

The following is an immediate consequence of Lemma 3.1.2.

Lemma 6.1.9. *An FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is a sheaf transform if and only if its kernel $\mathcal{P}$ is concentrated in degree 0, and is flat over Y. In this case, Φ is a bundle transform precisely when $\mathcal{P}$ is locally free.* □

For the rest of this section let us fix smooth projective varieties X and Y and an FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$. Given an object A of $\mathrm{D}(Y)$, and an integer i, we shall write $\Phi^i(A)$ for $\mathrm{H}^i(\Phi(A))$. We shall often use the following obvious lemma

Lemma 6.1.10. *Suppose one has a short exact sequence of sheaves on Y*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then there is a long exact sequence

$$\begin{aligned} \cdots \quad \longrightarrow \quad & \Phi^0(A) \longrightarrow \Phi^0(B) \longrightarrow \Phi^0(C) \\ \longrightarrow \quad & \Phi^1(A) \longrightarrow \Phi^1(B) \longrightarrow \Phi^1(C) \longrightarrow \cdots . \end{aligned} \tag{6.2}$$

of sheaves on X. □

We now define WIT sheaves. Here WIT stands for weak index theorem.

Definition 6.1.11. A sheaf A on Y is said to be Φ-WIT_i if $\Phi^j(A) = 0$ for $j \neq i$. One says that a sheaf A on Y is Φ-WIT if it is Φ-WIT_i for some i, and one then writes $\hat{A}$ for $\Phi^i(A)$, and refers to $\hat{A}$ as the *transform* of A.

6.1.1 Parseval theorem

The following simple but useful result is sometimes referred to as the Parseval theorem.

Lemma 6.1.12. *Let A and B be sheaves on Y, with A Φ-WIT_a and B Φ-WIT_b. Then for all i, one has*

$$\mathrm{Ext}^i_Y(A, B) = \mathrm{Ext}^{i+a-b}_X(\hat{A}, \hat{B}).$$

Proof. The fact that Φ is an equivalence implies that

$$\mathrm{Hom}_{\mathrm{D}(Y)}(A, B[i]) = \mathrm{Hom}_{\mathrm{D}(X)}(\hat{A}[-a], \hat{B}[i-b]),$$

so the result follows. □

As a special case of the lemma, note that if A is a simple Φ-WIT sheaf, then the transform $\hat{A}$ is also simple.

6.1.2 Preservation of families

The following two important results are immediate consequences of Lemma 3.1.6 together with the proof of Proposition 4.2.2.

Proposition 6.1.13. *Let $\{\mathcal{E}_s : s \in S\}$ be a family of sheaves on Y parameterised by a scheme S, and let i be an integer. Then the set of points $s \in S$ for which $\mathcal{E}_s$ is Φ-WIT_i is the set of points of a (possibly empty) open subscheme of S.* $\square$

Proposition 6.1.14. *Let $\{\mathcal{E}_s : s \in S\}$ be a family of Φ-WIT_i sheaves on Y parameterised by a scheme S. Then $\{\hat{\mathcal{E}}_s : s \in S\}$ is a family of sheaves on X.* $\square$

6.1.3 Chern character formula

By Grothendieck's Riemann-Roch theorem [Ha2, A.5.3], there is a linear map of vector spaces ϕ making the following diagram commute

$$\begin{array}{ccc} \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ \mathrm{H}^*(Y,\mathbb{Q}) & \xrightarrow{\phi} & \mathrm{H}^*(X,\mathbb{Q}) \end{array}$$

It is given by

$$\phi(y) = \pi_{X,*}(p \cdot \pi_Y^* y),$$

where $p = \mathrm{ch}(\mathcal{P}) \cdot \pi_Y^*(\mathrm{td}_Y)$ and td_Y is the Todd class of Y. We shall often denote ϕ by $\mathrm{ch}(\Phi)$.

6.1.4 Mukai spectral sequence

Let $\Psi : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y)$ be a functor such that

$$\Psi \circ \Phi \cong 1_{\mathrm{D}(Y)}[-d], \qquad \Phi \circ \Psi \cong 1_{\mathrm{D}(X)}[-d], \tag{6.3}$$

for some integer d. Thus Ψ is just a shift of a quasi-inverse of Φ. By Lemma 6.1.5 Ψ is also an FM transform with kernel given by the object

$$\mathcal{Q} = \mathcal{P}^\vee \otimes \pi_X^* \omega_X[n-d].$$

We may wish to take the integer d to be non-zero so that the object $\mathcal{Q}$ becomes a sheaf. For example, if $\mathcal{P}$ is a vector bundle on $X \times Y$ and $d = n$, $\mathcal{Q}$ is also a vector bundle.

The following lemma is an obvious consequence of (6.3).

Lemma 6.1.15. *Let A be a Φ-WIT_i sheaf on Y. Then $\hat{A}$ is a Ψ-WIT_{d-i} sheaf on X and $\hat{\hat{A}} = A$.* $\square$

For a general sheaf one obtains the Mukai spectral sequence:

Lemma 6.1.16. *Let A be a sheaf on Y. Then there is a spectral sequence*

$$E_2^{p,q} = \Psi^p(\Phi^q(A)) \implies \begin{cases} A & \text{if } p+q=d \\ 0 & \text{otherwise} \end{cases}$$

□

6.1.5 IT sheaves

Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be a bundle transform, with kernel $\mathcal{P}$. Let n be the common dimension of X and Y. Note that for any sheaf A on Y, $\Phi^i(A) = 0$ unless $0 \leq i \leq n$. The theorem on cohomology and base-change [Ha2, III.12.11] gives the following result

Lemma 6.1.17. *Let A be a sheaf on Y. If $\Phi^{i+1}(A)$ is locally free then the fibre of $\Phi^i(A)$ at the point $x \in X$ is given by the vector space $\mathrm{H}^i(Y, A \otimes \mathcal{P}_x)$.* □

Definition 6.1.18. Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be a bundle transform, with kernel $\mathcal{P}$. Then a sheaf A on Y is said to be Φ-IT_i, if for all $x \in X$,

$$\mathrm{H}^j(Y, A \otimes \mathcal{P}_x) = 0 \text{ unless } j = i.$$

The next two results are easy consequences of Lemma 6.1.17.

Lemma 6.1.19. *Let A be a sheaf on Y and i an integer. Then A is Φ-IT_i precisely when A is Φ-WIT_i and $\hat{A}$ is locally free.* □

Lemma 6.1.20. *Let A be a sheaf on Y.*

(a) If A is Φ-WIT_0 then it is Φ-IT_0.

(b) If A is Φ-WIT_n then it is locally free.

Proof. Lemma 6.1.17 clearly implies (a). For (b), note that by Lemma 6.1.15, $\hat{A}$ is Ψ-WIT_0 where $\Psi : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y)$ is the bundle transform with kernel $\mathcal{P}^\vee \otimes \pi_X^* \omega_X$. Thus, by (a), $\hat{A}$ is Ψ-IT_0, and $A = \hat{\hat{A}}$ is locally free. □

6.2 Examples

We give some simple examples of FM transforms. Many more will follow in later chapters.

Example 6.2.1. Let X be a smooth projective variety. If we take an automorphism f of X, a line bundle L on X and an integer n, the functor

$$\Phi(-) = L \otimes f_*(-)[n],$$

is an equivalence of categories. It is isomorphic to $\Phi^{\mathcal{P}}_{X\to X}$ where

$$\mathcal{P} = \pi_X^*(L) \otimes \mathcal{O}_{\Delta_f}[n],$$

and $\Delta_f \subset X \times X$ is the graph of the automorphism f. We call such transforms *trivial.*

Conversely suppose that $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is an FM transform such that for each point $y \in Y$, there is a point $f(y) \in X$ and an integer i_y such that

$$\Phi(\mathcal{O}_y) \cong \mathcal{O}_{f(y)}[i_y].$$

Since Y is connected, Proposition 6.1.13 implies that $i_y = n$ is the same for all points $y \in Y$. Then Lemma 6.1.9 implies that if the kernel of Φ is $\mathcal{P}[n]$, then $\mathcal{P}$ is a sheaf on $Y \times X$, flat over Y, such that $\mathcal{P}_y \cong \mathcal{O}_{f(y)}$ for all $y \in Y$. It follows from this that f is a morphism, and $\mathcal{P}$ is isomorphic to $\pi_Y^*(L) \otimes \mathcal{O}_{\Delta_f}$ for some line bundle L on Y. Finally, since Φ is an equivalence, f is an isomorphism, and identifying Y with X via f,

$$\Phi(-) \cong L \otimes f_*(-)[n].$$

Example 6.2.2. Let X be an Abelian variety. Then [Mum4, p. 125] there exists a dual Abelian variety $\hat{X}$ and a line bundle $\mathcal{P}$ on $\hat{X} \times X$ (called the *Poincaré bundle*) with the following properties

(a) Given a point $\hat{x} \in \hat{X}$, $\mathcal{P}_{\hat{x}} \cong \mathcal{O}_X \iff \hat{x} = 0$.

(b) Given two points $\hat{x}, \hat{y} \in \hat{X}$, $\mathcal{P}_{\hat{x}} \otimes \mathcal{P}_{\hat{y}} \cong \mathcal{P}_{\hat{x}+\hat{y}}$.

Now for any two points $\hat{x}, \hat{y} \in \hat{X}$, and any integer i, [Mum3, p. 76] implies that

$$\mathrm{Ext}^i_X(\mathcal{P}_{\hat{x}}, \mathcal{P}_{\hat{y}}) = \mathrm{H}^i(X, \mathcal{P}_{\hat{y}-\hat{x}}) = 0,$$

unless $\hat{x} = \hat{y}$. Thus $\mathcal{P}$ is strongly simple over $\hat{X}$, and since the canonical sheaf of any Abelian variety is trivial, Theorem 6.1.7 implies that the functor $\mathcal{F} = \Phi^{\mathcal{P}}_{Y\to X}$ is an equivalence of categories.

This functor $\mathcal{F}$ was the first example of a non-trivial FM transform. We shall refer to it as the *Fourier functor*, or the *original* FM transform. Its properties are described in [Muk2]. We shall consider the cases when X has dimension 1 (elliptic curves) and 2 (Abelian surfaces) in more detail later.

Example 6.2.3. We can use the Fourier functor of Example 6.2.2 to show that there are no rigid sheaves on an Abelian surface. Indeed, let X be an Abelian surface, and let E be a sheaf on X. Twisting by a sufficiently ample line bundle we can assume that E is $\mathcal{F}$-IT_0. Applying $\mathcal{F}$, we obtain a vector bundle $\hat{E}$ on the dual Abelian surface $\hat{X}$. Now $\mathcal{O}_{\hat{X}}$ is a direct summand of $\mathcal{E}nd_{\mathcal{O}_{\hat{X}}}(\hat{E})$ (see for example [Ar1, Prop. 2]), so

$$\mathrm{Ext}^1_X(E,E) = \mathrm{Ext}^1_{\hat{X}}(\hat{E},\hat{E}) = \mathrm{H}^1(\hat{X}, \mathcal{E}nd_{\mathcal{O}_{\hat{X}}}(\hat{E}))$$

has dimension at least 2.

Example 6.2.4. The first example of an FM transform for a K3 surface was the reflection functor of [Muk4], although Mukai never explicitly mentions the fact that it is an equivalence of derived categories.

To construct it, take a K3 surface X and let $\mathcal{P}$ be the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal in $X \times X$. For any $x \in X$, $\mathcal{P}_x \cong \mathcal{I}_x$, so for any pair of distinct points $x, y \in X$,

$$\mathrm{Ext}^2_X(\mathcal{P}_x, \mathcal{P}_y) = \mathrm{Hom}_X(\mathcal{P}_y, \mathcal{P}_x) = 0.$$

Riemann-Roch implies that $\chi(\mathcal{I}_x, \mathcal{I}_y) = 0$, so this is enough to show that $\mathcal{P}$ is strongly simple over each factor. It follows from Theorem 6.1.7 that $\Phi^{\mathcal{P}}_{X\to X}$ is an equivalence of categories. Note that the kernel of the inverse of this transform is not concentrated in any degree.

Example 6.2.5. Theorem 1.3.2 gives many more examples of FM transforms for K3 surfaces. To prove it apply the argument of Example 6.2.4. The assumption that Y is 2-dimensional implies that $\chi(\mathcal{P}_y, \mathcal{P}_y) = 0$ [Muk5, Cor. 2.5], and the fact that Y is a K3 surface is [Muk5, Thm. 1.4].

Example 6.2.6. Suppose (X, ℓ) is a polarised surface, and $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is a sheaf transform, with kernel $\mathcal{P}$, such that for each $y \in Y$, the sheaf $\mathcal{P}_y$ is stable with respect to ℓ. Let E be a sheaf on X which is stable with respect to ℓ and which has the same Chern class as $\mathcal{P}_y$. Then I claim that E is isomorphic to $\mathcal{P}_y$ for some point $y \in Y$.

Suppose not. Then for each $y \in Y$, by Proposition 5.1.3 (d),

$$\mathrm{Hom}_X(\mathcal{P}_y, E) = \mathrm{Hom}_X(E, \mathcal{P}_y) = 0.$$

By Serre duality, and the fact that $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$, one has $\mathrm{Ext}^2_X(E, \mathcal{P}_y) = 0$. Now since E has the same Chern class as $\mathcal{P}_y$, by Riemann-Roch,

$$\chi(E, \mathcal{P}_y) = \chi(\mathcal{P}_y, \mathcal{P}_y) = \chi(\mathcal{O}_y, \mathcal{O}_y) = 0,$$

so it follows that $\mathrm{Ext}^1_X(E, \mathcal{P}_y) = 0$. If Ψ is a quasi-inverse to Φ,

$$\mathrm{Hom}^\bullet_{\mathrm{D}(Y)}(\Psi(E), \mathcal{O}_y) \cong \mathrm{Hom}^\bullet_{\mathrm{D}(X)}(E, \mathcal{P}_y) = 0,$$

so by Lemma 4.1.4, $\Psi(E) \cong 0$, contradicting the fact that Φ is an equivalence.

It follows from the claim that Y is equal to a component of the moduli space of stable sheaves on X (with respect to ℓ), and $\mathcal{P}$ is a universal sheaf on $Y \times X$.

6.3 Stable sheaves on elliptic curves

Let X be an elliptic curve. If E is a sheaf on X we write its Chern class as a pair $(\mathrm{r}(E), \mathrm{d}(E))$. Given a pair of integers (r, d), with r positive, we write $\mathcal{M}_X(r, d)$ for the moduli space of semistable bundles on X with Chern class (r, d). In this section we use Fourier-Mukai transforms to give a simple proof of the following result, which is essentially due to Atiyah.

Theorem 6.3.1. *For any coprime pair of integers (r, d), with r positive, there is an isomorphism $\mathcal{M}_X(r, d) \cong X$.*

Lemma 6.3.2. *Let E be a sheaf on X. Then E is simple if, and only if, E is stable.*

Proof. One implication is Proposition 5.1.3 (e). For the converse suppose E is simple. If E is not semistable then by Proposition 5.1.3 (g), there is a non-trivial short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0, \tag{6.4}$$

with $\operatorname{Hom}_X(A, B) = 0$. But then by Serre duality $\operatorname{Ext}^1_X(B, A) = 0$, so $E = A \oplus B$. Since E is simple, it is indecomposable, so this is a contradiction.

Now suppose E is semistable but not stable. Then the Jordan-Hölder filtration gives a non-trivial sequence (6.4) with $\mu(A) = \mu(B)$. By Riemann-Roch, $\chi(B, A) = \chi(A, A) = 0$, so since $\operatorname{Ext}^1(B, A) \neq 0$, there must be a non-trivial morphism $B \to A$. Again this contradicts simplicity of E. $\square$

Fix a point $x_0 \in X$, let $\Delta \subset X \times X$ be the diagonal and put

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^*(\mathcal{O}_X(x_0)) \otimes \pi_2^*(\mathcal{O}_X(x_0)).$$

Then $\mathcal{P}$ is a vector bundle on $X \times X$, and for any point $x \in X$, the restriction $\mathcal{P}|_{\{x\} \times X}$ is the degree 0 line bundle $\mathcal{O}_X(x_0 - x)$.

Lemma 6.3.3. *The triple $(X, X, \mathcal{P})$ is an FM triple.*

Proof. By Theorem 6.1.7 and Serre duality, the only thing to check is that if x_1, x_2 are distinct points of X,

$$\operatorname{Hom}_X(\mathcal{O}_X(x_0 - x_1), \mathcal{O}_X(x_0 - x_2)) = 0.$$

This is easy. $\square$

Clearly $\mathcal{P}$ is isomorphic to the Poincaré line-bundle on $X \times X$. Let $\mathcal{F}$ be the FM transform $\Phi^{\mathcal{P}}_{X \to X}$. If we put $\widehat{\mathcal{F}} = \Phi^{\mathcal{P}^\vee}_{X \to X}$, then

$$\widehat{\mathcal{F}} \circ \mathcal{F} \cong 1_{\mathrm{D}(X)}[-1], \qquad \mathcal{F} \circ \widehat{\mathcal{F}} \cong 1_{\mathrm{D}(X)}[-1]. \tag{6.5}$$

Lemma 6.3.4. *If E is a sheaf of Chern class (r, d) on X, then $\mathcal{F}(E)$ has Chern class $(d, -r)$.*

Proof. By linearity (see Section 6.1.3) it will be enough to check the result when E has Chern class $(0,1)$ or $(1,0)$. So suppose that $E = \mathcal{O}_x$ for some point $x \in X$. Then $\mathcal{F}(E) = \mathcal{O}_X(x_0 - x)$ has Chern class $(1,0)$ as required. Similarly, $\widehat{\mathcal{F}}(E) = \mathcal{O}_X(x - x_0)$ has Chern class $(1,0)$. But by (6.5), $\mathcal{F}(\widehat{\mathcal{F}}(E)) = E[-1]$ has Chern class $(0,-1)$, so this is enough. $\square$

Lemma 6.3.5. *Let E be a simple sheaf on X. Then E is $\mathcal{F}$-WIT and $\hat{E}$ is also simple.*

Proof. Consider the Mukai spectral sequence. It gives a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

where $A = \widehat{\mathcal{F}}^1(\mathcal{F}^0(E))$, and $B = \widehat{\mathcal{F}}^0(\mathcal{F}^1(E))$, together with the information that $\mathcal{F}^1(E)$ is $\widehat{\mathcal{F}}$-WIT$_0$, and $\mathcal{F}^0(E)$ is $\widehat{\mathcal{F}}$-WIT$_1$. This implies that A is $\mathcal{F}$-WIT$_0$ and B is $\mathcal{F}$-WIT$_1$. By the Parseval theorem $\mathrm{Hom}_X(A, B) = 0$, so by Serre duality $\mathrm{Ext}^1_X(B, A) = 0$. Since E is indecomposable, one of the sheaves A or B is zero, so E is $\mathcal{F}$-WIT. By the Parseval theorem $\hat{E}$ is simple. $\square$

Now it follows from Proposition 6.1.14, that for any pair of coprime integers (r, d), with $r > 0$, the FM transform Φ gives an isomorphism of moduli spaces

$$\mathcal{M}_X(r, d) \cong \mathcal{M}_X(d, -r).$$

Twisting by a degree 1 line bundle on X gives an isomorphism

$$\mathcal{M}_X(r, d) \cong \mathcal{M}_X(r, d + r).$$

Applying these two isomorphisms repeatedly, and using Euclid's algorithm we obtain an isomorphism

$$\mathcal{M}_X(r, d) \cong \mathcal{M}_X(1, 0) \cong X.$$

Thus we have proved the theorem. As a corollary of the proof note that

Lemma 6.3.6. *If E is a stable sheaf on X with Chern class (r, d), then r is coprime to d.*

Proof. Let h be the highest common factor of r and d. Applying the algorithm above gives a simple (hence stable) sheaf E on X of Chern class $(0, h)$. If $x \in X$ lies in the support of E there is a non-zero morphism $E \to \mathcal{O}_x$, which, by Proposition 5.1.3 (d), must be an isomorphism. Hence $h = 1$. $\square$

6.4 Classification of transforms on curves

Let X be an elliptic curve and take a pair of coprime integers (a, b) with $a > 0$. Let $Y = \mathcal{M}_X(a, b)$. By Theorem 6.3.1, Y is isomorphic to X; we preserve the distinction for clarity. By Theorem 5.2.5 there is a universal bundle $\mathcal{P}$ on $Y \times X$; in fact this follows from the proof of Theorem 6.3.1.

Lemma 6.4.1. *The triple $(X, Y, \mathcal{P})$ is an FM triple.*

Proof. For any pair of distinct points $y_1, y_2 \in Y$, one has

$$\mathrm{Ext}^1_X(\mathcal{P}_{y_2}, \mathcal{P}_{y_1}) = \mathrm{Hom}_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0,$$

by Serre duality and Proposition 5.1.3 (d). This is enough by Theorem 6.1.7. $\square$

Let $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$ and let $\Psi = \Phi^{\mathcal{P}^\vee}_{X \to Y}$. By the results of section 6.1.3 there are integers c, d such that for any object E of $\mathrm{D}(Y)$,

$$\begin{pmatrix} \mathrm{r}(\Phi(E)) \\ \mathrm{d}(\Phi(E)) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix}.$$

Since $\Psi \circ \Phi \cong 1_{\mathrm{D}(Y)}[-1]$, we must have $bc - ad = \pm 1$ and

$$\begin{pmatrix} \mathrm{r}(\Psi(E)) \\ \mathrm{d}(\Psi(E)) \end{pmatrix} = \pm \begin{pmatrix} -b & a \\ d & -c \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix},$$

for any object E of $\mathrm{D}(X)$. But $\Psi(\mathcal{O}_x) = \mathcal{P}^\vee_x$ is a sheaf, so we must take the positive sign, i.e. $bc - ad = 1$. This relation does not define c and d uniquely: we may replace them by $c + na$ and $d + nb$ for any integer n. This corresponds to twisting $\mathcal{P}$ by the pull-back of a line bundle of degree n on Y. By varying n we obtain all possible values of c and d.

Theorem 6.4.2. *Let X be an elliptic curve and take an element*

$$A = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z}),$$

such that $a > 0$. Then there exist vector bundles on $X \times X$ which are strongly simple over both factors, and which restrict to give stable bundles of Chern class (a, c) on the first factor and (a, b) on the second. For any such bundle $\mathcal{P}$, the resulting functor $\Phi = \Phi^{\mathcal{P}}_{X \to X}$ is an FM transform, and satisfies

$$\begin{pmatrix} \mathrm{r}(\Phi E) \\ \mathrm{d}(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix},$$

for all objects E of $\mathrm{D}(X)$. $\square$

We shall now prove a theorem which classifies all FM transforms for smooth curves.

Theorem 6.4.3. *Let X be a smooth curve and let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM tranform. Then either Φ is a trivial transform, as in Example 6.2.1, or X is an elliptic curve and there exists a pair of coprime integers (a, b), with $a > 0$, and an integer n, such that $Y = \mathcal{M}_X(a, b)$, and the kernel of $\Phi[n]$ is a universal bundle on $Y \times X$. In particular X and Y are isomorphic.*

Proof. For each point $y \in Y$ there is a spectral sequence [BO1, Prop. 4.2]

$$E_2^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^p_X(\mathrm{H}^i(\Phi(\mathcal{O}_y)), \mathrm{H}^{i+q}(\Phi(\mathcal{O}_y))) \implies \mathrm{Hom}^{p+q}_{\mathrm{D}(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)).$$

Also, since Φ is an equivalence

$$\mathrm{Hom}^0_{\mathrm{D}(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \mathbb{C}.$$

Since X has dimension 1 the spectral sequence degenerates, so the vector space

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{End}_X(\mathrm{H}^i(\Phi(\mathcal{O}_y)))$$

has dimension at most 1. This implies that for some integer i_y, $\Phi(\mathcal{O}_y)[i_y]$ is concentrated in degree 0. Since Y is connected, Proposition 6.1.13 implies that $i_y = n$ is the same for all $y \in Y$, and Lemma 6.1.9 then implies that the kernel $\mathcal{P}$ of $\Phi[n]$ is a sheaf on $X \times Y$, flat over Y.

By Theorem 4.0.3, for each $y \in Y$, $\mathcal{P}_y$ is a simple sheaf on X satisfying $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$. If X is not an elliptic curve this implies that $\mathcal{P}_y$ is the structure sheaf of a point of X, and the argument of Example 6.2.1 shows that Φ is trivial. If X is an elliptic curve, and $\mathcal{P}_y$ is supported in dimension 1, then by Lemma 6.3.2, $\mathcal{P}_y$ is a stable bundle for all $y \in Y$, of Chern class (a, b), say. Then a and b are coprime (by Lemma 6.3.6), and there is a natural map $Y \to \mathcal{M}_X(a, b)$. Since this is injective on points, it is an isomorphism. □

Chapter 7

Stable sheaves on Abelian surfaces

Throughout this chapter (X, ℓ) will be a fixed principally polarised Abelian surface. Thus X is a complex Abelian variety of dimension 2, and ℓ is the Chern class of an ample line bundle L on X which satisfies $\mathrm{c}_1(L)^2 = 2$. The dual Abelian variety $\hat{X}$ can be identified with X via the isomorphism ϕ_L [Mum4, p. 60, 150], and we shall always do this.

Given a sheaf E on X, we write its Chern character as a triple

$$(\mathrm{r}(E), \mathrm{c}_1(E), \chi(E)),$$

where by Riemann-Roch,

$$\chi(E) = \mathrm{ch}_2(E) = \frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E).$$

By the degree of E we always mean $\mathrm{d}(E) = \mathrm{c}_1(E) \cdot \ell$, and we shall always use the polarisation ℓ to define stability. We use the notation $\mathcal{M}_X(r, \Delta, k)$ to denote the moduli space of sheaves on X with the given Chern character which are semistable with respect to ℓ. Note that this is a slightly different convention from in other parts of the thesis, where we have used the second Chern class c_2 instead of ch_2.

Some moduli spaces of stable sheaves on a principally polarised Abelian surfaces were computed by Mukai using the original FM transform [Muk2], [Muk4], and other moduli spaces were computed in [Mac1], [Mac2]. In this chapter we use similar techniques to prove the following two theorems

Theorem 7.0.4. *Let r and n be positive integers, with $r > n^2$. Then there is an isomorphism*

$$\mathcal{M}_X(r, n\ell, 0) \cong \mathcal{M}_X(0, n\ell, -r).$$

Theorem 7.0.5. *Let $n \leq m$ be positive integers, with m even. Then there is a component of $\mathcal{M}_X(mn+1, n\ell, 0)$ which is birationally equivalent to $X \times \mathrm{Hilb}^{n^2} X$.*

7.1 The original FM transform

In this section we prove some useful properties of the Fourier functor. We shall need the following well-known result, usually known as *Bogomolov's inequality.*

Lemma 7.1.1. *Let E be a semistable sheaf on X. Then*

$$\mathrm{d}(E)^2 \geq 4\,\mathrm{r}(E)\,\chi(E).$$

Proof. First suppose that E is stable. Then E is simple, and since the canonical bundle of X is trivial, Serre duality implies that $\mathrm{Ext}^2_X(E,E) = \mathbb{C}$. Using the result of Example 6.2.3, Riemann-Roch now gives

$$c_1(E)^2 - 2\,\mathrm{r}(E)\,\chi(E) = -\,\chi(E,E) = \dim_{\mathbb{C}} \mathrm{Ext}^1_X(E,E) - 2 \geq 0.$$

The Hodge index theorem implies that

$$2c_1(E)^2 \leq (c_1(E)\cdot \ell)^2,$$

so the given inequality follows. The case when E is semistable follows easily by considering Jordan-Hölder filtrations. □

The Poincaré line bundle $\mathcal{P}$ on $X \times X$ gives rise to an FM transform $\mathcal{F}$: $\mathrm{D}(X) \longrightarrow \mathrm{D}(X)$. Its properties were studied in detail by Mukai. In particular he proved:

Proposition 7.1.2. *(a) There is an isomorphism of functors*

$$\mathcal{F} \circ \mathcal{F} \cong (-1)^*_X[-2],$$

where $(-1)_X : X \longrightarrow X$ is the map which sends an element of the Abelian group X to its inverse.

(b) If E is an object of $\mathrm{D}(X)$ with Chern character (r,Δ,k) then $\mathcal{F}(E)$ has Chern character $(k,-\Delta,r)$. □

Proof. For (a) note that by the see-saw theorem

$$(1_X \times (-1)_X)^*(\mathcal{P}) \cong \mathcal{P}^\vee,$$

so the result follows from Lemma 6.1.5. The proof of (b) is more difficult. We refer to [Muk4, Prop. 1.17]. □

In this chapter $\mathcal{F}$ will be the only FM transform occurring, so when we speak of a sheaf E being WIT or IT, we shall always mean with respect to the transform $\mathcal{F}$. Two points which occurs frequently are

Lemma 7.1.3. *Let E be a sheaf on X of dimension d. Then $\mathcal{F}^i(E) = 0$ unless $0 \leq i \leq d$. In particular, any dimension 0 sheaf on X is IT_0.*

Proof. Since $H^i(X, E \otimes \mathcal{P}_x) = 0$ for all $x \in X$ and all $i > d$, this follows from Lemma 6.1.17. □

Lemma 7.1.4. *Let E be a semistable sheaf E on X with positive degree. Then for all $x \in X$, $\mathrm{H}^2(X, E \otimes \mathcal{P}_x) = 0$, so $\mathcal{F}^2(E) = 0$.*

Proof. By Serre duality and Proposition 5.1.3 (c),

$$\mathrm{Ext}^2_X(\mathcal{O}_X, E \otimes \mathcal{P}_x) = \mathrm{Hom}_X(E, \mathcal{P}_{-x}) = 0.$$

The second statement follows from Lemma 6.1.17. □

We shall also need the following lemma, which follows directly from the Mukai spectral sequence. See [Mac3, §2] for more details.

Lemma 7.1.5. *Let E be a sheaf on X with $\mathcal{F}^2(E) = 0$. Then $\mathcal{F}^0(E)$ is WIT_2, and there is a long exact sequence*

$$0 \longrightarrow \mathcal{F}^0(\mathcal{F}^1(E)) \longrightarrow \mathcal{F}^2(\mathcal{F}^0(E)) \longrightarrow (-1)^*_X E \longrightarrow \mathcal{F}^1(\mathcal{F}^1(E)) \longrightarrow 0.$$

Furthermore $\mathcal{F}^2(\mathcal{F}^0(E))$ is IT_0. □

The following useful result was first proved by A. Maciocia [Mac3, §3], who referred to it as the fundamental lemma.

Proposition 7.1.6. *Let E be a sheaf on X.*
(a) If E is IT_0 then $\mathrm{d}(E) \geq 0$.
(b) If E is WIT_2 then $\mathrm{d}(E) \leq 0$.

Proof. Part (b) follows from (a) on taking transforms. For (a) it is convenient to prove more, as in the next lemma. □

Lemma 7.1.7. *Let E be a sheaf on X satisfying $\mathrm{H}^2(X, E \otimes \mathcal{P}_x) = 0$ for all $x \in X$. Suppose also that the set*

$$\{y \in X \ : \ \mathrm{H}^1(X, E \otimes \mathcal{P}_y) \neq 0\}$$

is finite. Then $\mathrm{d}(E) \geq 0$, and furthermore equality holds if and only if E has dimension 0.

Proof. If $\mathrm{r}(E) = 0$ the result follows because E is a torsion sheaf, so assume $\mathrm{r}(E) \geq 1$. We can suppose that E is torsion-free since if E had torsion T then

E/T would also satisfy the hypotheses and $\mathrm{d}(E) \geq \mathrm{d}(E/T)$. Now there must be some $x \in X$ such that $\mathrm{H}^0(X, E \otimes \mathcal{P}_{-x}) \neq 0$, so we can take a sequence

$$0 \longrightarrow \mathcal{P}_{\hat{x}} \longrightarrow E \longrightarrow F \longrightarrow 0.$$

The sheaf F satisfies our hypotheses and so by induction on the rank of E, $\mathrm{d}(E) = \mathrm{d}(F) \geq 0$.

To prove the last statement, we may assume that E has rank 1. Then by the Hodge index theorem, $\mathrm{d}(E) = 0$ implies that $\chi(E) \leq 0$, which is impossible. □

We can strengthen this result as follows:

Lemma 7.1.8. *If E is an IT_0 sheaf on X then* $\mathrm{d}(E)^2 \geq 4\,\mathrm{r}(E)$.

Proof. Note first that if $\mathrm{r}(E) > 0$, the inequality can be written as $\mu(E)\,\mathrm{d}(E) \geq 4$. Observe also that we can assume E torsion-free. Now if a sheaf E is IT_0, we have $\chi(E) \geq 1$, so in the case when E is semistable, the lemma follows at once from Lemma 7.1.1. For a general torsion-free sheaf E let $n \geq 1$ be the length of its Harder-Narasimhan filtration. Then there is a sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

with A semistable and $\mu(A) \geq \mu(E) \geq \mu(B)$, such that the length of the Harder-Narasimhan filtration of B is $n-1$. By Lemma 7.1.7, $\mathrm{d}(E) > 0$, so $\mu(A) > 0$, and hence by Lemma 7.1.4, $\mathrm{H}^2(X, A \otimes \mathcal{P}_x) = 0$ for all $x \in X$. Since E is IT_0 it follows from the long exact sequence in cohomology that B is IT_0, so by induction on n, the given inequality holds for B. But $\mu(E)\,\mathrm{d}(E) \geq \mu(B)\,\mathrm{d}(B)$, so the inequality holds for E too. □

Corollary 7.1.9. *If E is a semistable torsion-free sheaf with* $\mathrm{d}(E) > 0$ *and* $\mathrm{d}(E)^2 < 4\,\mathrm{r}(E)$ *then E is WIT_1.*

Proof. By Lemma 7.1.4 and Lemma 7.1.5, there is a long exact sequence

$$0 \longrightarrow \mathcal{F}^0(\mathcal{F}^1(E)) \longrightarrow \mathcal{F}^2(\mathcal{F}^0(E)) \longrightarrow (-1)_X^* E \xrightarrow{f} \mathcal{F}^1(\mathcal{F}^1(E)) \longrightarrow 0.$$

By Proposition 7.1.6, $\mathrm{d}(\mathcal{F}^0(\mathcal{F}^1(E))) \leq 0$. Let P be the kernel of f. We must have $P = 0$ since otherwise

$$\mu(\mathcal{F}^2(\mathcal{F}^0(E)))\,\mathrm{d}(\mathcal{F}^2(\mathcal{F}^0(E))) \leq \mu(P)\,\mathrm{d}(P) \leq \mu(E)\,\mathrm{d}(E) < 4, \tag{7.1}$$

contradicting Lemma 7.1.8. Hence the sequence splits and we have $\mathcal{F}^0(\mathcal{F}^1(E)) = \mathcal{F}^2(\mathcal{F}^0(E))$. Since these two sheaves are WIT_2 and WIT_0 respectively, they must both be zero. This implies that $\mathcal{F}^0(E) = 0$, so E is WIT_1. □

Example 7.1.10. We can use the fundamental lemma to prove that for all $r \geq 1$, L^{-r} is IT_2 with μ-stable transform.

Indeed, L^{-r} is IT_2 by the vanishing theorem [Mum4, p. 150]. Put $E = \mathcal{F}^2(L^{-r})$. Then E is locally free with Chern character $(r^2, r\ell, 1)$, and is IT_0. Suppose we have a destabilising sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with A semistable and $\mu(A) \geq \mu(E) \geq \mu(B)$. Then by Lemma 7.1.4, $\mathrm{H}^2(X, A \otimes \mathcal{P}_x) = 0$ for all $x \in X$, and it follows that B is IT_0. Now

$$\mu(B)\,\mathrm{d}(B) < \mu(E)\,\mathrm{d}(E) = 4.$$

But this is impossible since B is IT_0.

7.2 An application of the transform

Theorem 7.0.4 is a consequence of the following more general result:

Theorem 7.2.1. *Let Δ be the Chern class of an effective divisor on X, and take an integer $r > \frac{1}{4}(\Delta \cdot \ell)^2$. Then there is an isomorphism*

$$\mathcal{M}_X(r, \Delta, 0) \cong \mathcal{M}_X(0, \Delta, -r).$$

Proof. Given a semistable torsion-free sheaf E on X of character $(r, \Delta, 0)$, Corollary 7.1.9 shows that E is WIT_1. Then $F = \hat{E}$ has character $(0, \Delta, -r)$ and is WIT_1 so has no dimension 0 subsheaves, by Lemma 6.1.10 and Lemma 7.1.3. If F is not semistable there is a destabilising sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0,$$

where we can assume that B is semistable. Then A must be WIT_1, by Lemma 6.1.10 and Lemma 7.1.3. Let B have character $(0, C, -s)$. By Lemma 5.4.3, we must have

$$\frac{-r}{\Delta \cdot \ell} \geq \frac{-s}{C \cdot \ell}$$

so by Lemma 7.2.2 below, B is WIT_1 also. Taking transforms gives a destabilising sequence for E, and hence a contradiction. This shows that F is semistable.

Conversely, given a semistable sheaf F of character $(0, \Delta, -r)$, Lemma 7.2.2 shows that F is IT_1, so $E = \hat{F}$ is locally free of character $(r, \Delta, 0)$. If

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

is a destabilising sequence, with B torsion-free and semistable, then the long exact sequence in cohomology implies that $\mathrm{H}^2(X, B \otimes \mathcal{P}_x) = 0$ for all $x \in X$, and we can apply the argument of Corollary 7.1.9 to conclude that B is WIT_1. Then by the first part $\hat{B}$ is semistable, so the induced morphism $\hat{E} \to \hat{B}$ is zero. This means that $\hat{B} \cong \mathcal{F}^2(A)$, which is impossible since the latter sheaf is WIT_0 ([Muk3, §2.2]). This gives a contradiction. $\square$

Lemma 7.2.2. *Let B be a semistable sheaf of character $(0, C, -s)$, with $s > \frac{1}{4}(C \cdot \ell)^2$. Then B is IT_1.*

Proof. Since B is supported in dimension 1, $\mathrm{H}^2(X, B \otimes \mathcal{P}_x) = 0$ for all $x \in X$. If B is not IT_1, then replacing B by $B \otimes \mathcal{P}_x$ we can assume there is a non-zero morphism $\mathcal{O}_X \to B$. Let the cokernel K of this morphism have first Chern class D. Then since K is a subsheaf of B, $C \cdot \ell \geq D \cdot \ell$, and since K is a pure dimension 1 quotient of $\mathcal{O}_X$, $\chi(K) = -\frac{1}{2}D^2$. By semistability this implies that

$$\frac{-D^2}{2(D \cdot \ell)} \leq \frac{-s}{C \cdot \ell},$$

so $D^2 \geq \frac{1}{2}(C \cdot \ell)(D \cdot \ell)$. But the Hodge index theorem implies that $D^2 \leq \frac{1}{2}(D \cdot \ell)^2$, so this gives a contradiction. $\square$

Let us fix an element $\Delta \in \mathrm{NS}(X)$ which is the Chern class of an effective divisor. We show that a component of $\mathcal{M}_X(0, \Delta, -1)$ is birational to $X \times \mathrm{Hilb}^{\frac{1}{2}\Delta^2} X$. To do this we use the Fourier-Mukai transform to construct a birational equivalence between $\mathcal{M}_X(1, \Delta, 0)$ and $\mathcal{M}_X(0, \Delta, -1)$. Note that by Proposition 5.2.5, both of these moduli spaces are fine. We identify their points with the stable sheaves they represent.

If we let $\mathcal{U}$ be the open subset of $\mathcal{M}_X(1, \Delta, 0)$ consisting of WIT_1 sheaves whose transforms are stable, and $\mathcal{V}$ the open subset of $\mathcal{M}_X(0, \Delta, -1)$ consisting of WIT_1 sheaves whose transforms are torsion-free, the Fourier-Mukai transform clearly gives an isomorphism $\mathcal{U} \cong \mathcal{V}$. The following lemma shows that both sets are non-empty.

Lemma 7.2.3. *Fix a line bundle M with $c_1(M) = \Delta$. Then for generic $X \in \mathrm{Hilb}^{\frac{1}{2}\Delta^2} X$, the sheaf $M \otimes \mathcal{I}_X$ is WIT_1, and the transformed sheaf is stable of pure dimension 1.*

Proof. Given $X \in \mathrm{Hilb}^{\frac{1}{2}\Delta^2} X$, let $E = M \otimes \mathcal{I}_X$. By the vanishing theorem [Mum4, p. 150], and Riemann-Roch, $\mathrm{H}^0(X, M)$ has dimension $\frac{1}{2}\Delta^2$, so for generic X we will have $\mathrm{H}^0(X, E) = 0$. Note that $\mathcal{F}^2(E) = 0$ by Lemma 7.1.4. Since $\chi(E) = 0$, $\mathrm{H}^1(X, E) = 0$ and therefore $\mathcal{F}^1(E)$ is a torsion sheaf. But $\mathcal{F}^0(E)$ and $\mathcal{F}^1(E)$ have the same rank, and $\mathcal{F}^0(E)$ is WIT_2, hence by Lemma 6.1.20, locally free, so must vanish. This shows that E is WIT_1.

Now take $X \in \mathrm{Hilb}^{\frac{1}{2}\Delta^2} X$ with $E = M \otimes \mathcal{I}_X$ WIT_1. Suppose there is a destabilising sequence

$$0 \longrightarrow B \longrightarrow \hat{E} \longrightarrow C \longrightarrow 0.$$

Since $\chi(\hat{E}) = -1$ we must have $\chi(B) \geq 0$. But by Lemma 6.1.10 and Lemma 7.1.4, B is WIT_1 so $\chi(B) = 0$, and $\hat{B}$ is a torsion sheaf. Upon taking transforms the map $\hat{B} \to E$ must be zero (because E is torsion-free) so $\hat{B} \cong \mathcal{F}^0(C)$. But these sheaves are WIT_1 and WIT_2 (by Lemma 7.1.5) respectively, so must both be zero. This gives a contradiction. $\square$

For any integer m, twisting by the line bundle L^{-m} gives an isomorphism

$$\mathcal{M}_X(0, \Delta, -1) \cong \mathcal{M}_X(0, \Delta, -1 - m(\Delta \cdot \ell)).$$

Thus, by Theorem 7.2.1, we obtain the following theorem, which reduces to Theorem 7.0.5 in the case when $\Delta = n\ell$.

Theorem 7.2.4. *For any integer $m \geq \frac{1}{4}(\Delta \cdot \ell)$, there is a component of the moduli space $\mathcal{M}_X(1 + m(\Delta \cdot \ell), \Delta, 0)$ which is birationally equivalent to $X \times \mathrm{Hilb}^{\frac{1}{2}\Delta^2} X$.* $\square$

Chapter 8

Fourier-Mukai transforms for elliptic surfaces

In this chapter we introduce a new class of FM transforms for elliptic surfaces. The main theorem is Theorem 8.3.4 below. As an application we prove the following more precise version of Theorem 1.2.3. For the definition of a *suitable polarisation* see Proposition 8.5.1.

Theorem 8.0.5. *Let $\pi : X \to C$ be a relatively minimal elliptic surface, and take a triple*

$$(r, \Delta, k) \in \mathbb{N} \times \mathrm{NS}(X) \times \mathbb{Z}$$

such that r is coprime to $\Delta \cdot f$, and

$$2t = 2rk - (r-1)\Delta^2 - (r^2-1)\,\chi(\mathcal{O}_X) \geq 0.$$

Let a, b be the unique pair of integers satisfying $br - a(\Delta \cdot f) = 1$ and $0 < a < r$. Then for any suitable polarisation ℓ of X, the moduli space $\mathcal{M}^{\ell}_X(r, \Delta, k)$ is a smooth projective variety, birationally equivalent to

$$\mathrm{Pic}^{\circ}(\mathrm{J}_{X/C}(b)) \times \mathrm{Hilb}^t(\mathrm{J}_{X/C}(b)).$$

Furthermore, if $r >$ at the birational equivalence extends to give an isomorphism of varieties.

Recall that for any integer b, the elliptic surface $\mathrm{J}_{X/C}(b)$ is the relative Picard scheme parameterising degree b divisors on the smooth fibres of X. This is the same as Friedman's $J^b(X)$ (see [Fr, §1.1]). We shall give a more precise definition in Section 2 below.

8.1 Elliptic surfaces

Here we collect some pertinent facts regarding elliptic surfaces. Good references are [FrMo] and [BPV]. We shall only consider relatively minimal, algebraic, elliptic surfaces over $\mathbb{C}$.

Definition 8.1.1. By an elliptic surface we shall mean a surface X together with a smooth curve C and a morphism $X \xrightarrow{\pi} C$ whose general fibre is an elliptic curve, such that none of the fibres of π contain a (-1)-curve.

If $X \xrightarrow{\pi} C$ is an elliptic surface, the underlying surface X may have Kodaira dimension $-\infty$, 0 or 1. Examples are the product of $\mathbb{P}^1$ with an elliptic curve, the product of two elliptic curves, and the product of an elliptic curve with a curve of genus $g \geq 2$ respectively. Surfaces of Kodaira dimension 0 may have infinitely many different elliptic fibration structures [FrMo, p. 51]. In contrast one has [FrMo, Prop. I.3.24],

Theorem 8.1.2. *Let X be a minimal projective surface of Kodaira dimension 1. Then there is a unique curve C and a unique morphism $X \xrightarrow{\pi} C$ making X into an elliptic surface.* □

In general, not all fibres of an elliptic surface $X \xrightarrow{\pi} C$ are smooth. The various possibilities were listed by Kodaira (see [BPV, §V.7]). In particular, some of the fibres may be non-reduced.

Definition 8.1.3. Let $X \xrightarrow{\pi} C$ be an elliptic surface, and take a point $p \in C$. Then the fibre $\pi^{-1}(p)$ is said to be *multiple* if there is a divisor D on X with $\pi^{-1}(p) = mD$ for some integer $m > 1$.

The canonical class of an elliptic surface is described by the following formula [BPV, Thm. V.12.1].

Proposition 8.1.4. *Let $X \xrightarrow{\pi} C$ be an elliptic surface. Then*

$$\omega_X = \pi^*(L) \otimes \mathcal{O}_X(\sum_i (m_i - 1) f_i),$$

where L is a line bundle on C and $m_1 f_1, \cdots, m_k f_k$ are the multiple fibres of π. □

Let us make the following

Definition 8.1.5. Let $X \xrightarrow{\pi} C$ be an elliptic surface, and let $f \in \mathrm{NS}(X)$ denote the algebraic equivalence class of a fibre of π. Then for any sheaf E on X, the *fibre degree* of E (with respect to the fibration π) is defined to be

$$\mathrm{d}(E) = c_1(E) \cdot f.$$

Note that if $X \xrightarrow{\pi} C$ is an elliptic surface, then the restriction of a sheaf E on X to the general fibre of π has rank $\mathrm{r}(E)$ and degree $\mathrm{d}(E)$.

Definition 8.1.6. Let $X \xrightarrow{\pi} C$ be an elliptic surface. A sheaf E on X is a *fibre sheaf* if $\mathrm{r}(E) = \mathrm{d}(E) = 0$, or equivalently if the support of E is contained in the union of finitely many fibres of π.

Finally we define an important invariant of an elliptic surface

Definition 8.1.7. If $X \xrightarrow{\pi} C$ is an elliptic surface, let $\lambda_{X/C}$ denote the highest common factor of the fibre degrees of sheaves on X. Equivalently $\lambda_{X/C}$ is the smallest positive integer such that there is a divisor σ on X with $\sigma \cdot f = \lambda_{X/C}$.

Remark 8.1.8. By Riemann-Roch, given a divisor of positive fibre degree on an elliptic surface X, we can add a large multiple of f and obtain an effective divisor of the same fibre degree.

8.2 FM partners of elliptic surfaces

Let $X \xrightarrow{\pi} C$ be an elliptic surface and let $a > 0$ and b be integers such that $a\,\lambda_{X/C}$ is coprime to b. Let ℓ be a polarisation of X whose fibre degree is coprime to b. We can always find such a polarisation because, by the remark above, there exist effective divisors on X with fibre degree $\lambda_{X/C}$.

Let P be the polynomial $a(\ell \cdot f)t + b$. This is the Hilbert polynomial with respect to ℓ of a rank a, degree b bundle supported on a smooth fibre of π. By Theorem 5.3.2, and the coprimality assumption we made, there is a fine relative moduli scheme $\mathcal{M}^P_{X/C} \to C$, whose points represent stable sheaves supported on the fibres of π.

Definition 8.2.1. Let $\mathrm{J}_{X/C}(a, b)$ be the union of those components of $\mathcal{M}^P_{X/C}$ which contain a point representing a rank a, degree b vector bundle on a non-singular fibre of π. Let $\hat{\pi}$ denote the natural map $\hat{\pi} : \mathrm{J}_{X/C}(a, b) \to C$. Also write $\mathrm{J}_{X/C}(b)$ for $\mathrm{J}_{X/C}(1, b)$.

Since the moduli space $\mathcal{M}^P_{X/C}$ is fine, there is a universal sheaf $\mathcal{P}$ on $Y \times_C X$, such that for each point $y \in Y$, the stable sheaf corresponding to y is given by $\mathcal{P}_y$, the restriction of $\mathcal{P}$ to $\{y\} \times X_{\hat{\pi}(y)}$.

Let U be the (open) set of points $p \in C$ such that the fibre X_p is non-singular. The fibre of $\hat{\pi}$ over a point $p \in U$ is the moduli space of rank a, degree b stable sheaves on X_p, which, from Theorem 6.3.1, is isomorphic to X_p. Thus $\hat{\pi}$ is an elliptic fibration. Clearly $\hat{\pi}$ is dominant, hence surjective, so there is some component of Y which contains sheaves supported on every fibre of π. Any other component of Y must contain a sheaf supported on a non-singular fibre, but the fibre of $\hat{\pi}$ over every point of U is connected. It follows that Y is connected.

Now let $Z = \mathcal{M}^{\ell}_X(0, af, -b)$. There is a natural 'extension by zero' morphism $i : Y \longrightarrow Z$, which maps a point $y \in Y$ representing the stable sheaf $\mathcal{P}_y$ on the fibre $X_{\hat{\pi}(y)}$, to the point $z \in Z$ representing the stable sheaf on X obtained by extending $\mathcal{P}_y$ by zero. This morphism i induces an injection on points. I claim that Z is a disjoint union of non-singular projective surfaces; it will follow from this that i is an embedding and that Y is an elliptic surface over C.

Given a point $y \in Y$ we shall identify the sheaf $\mathcal{P}_y$ with its extension by zero on X. If $y \in Y$ is such that $\mathcal{P}_y$ is supported on a non-singular fibre of π, then

$$\mathcal{P}_y = \mathcal{P}_y \otimes \omega_X, \tag{8.1}$$

because the restriction of ω_X to any non-singular fibre of π is trivial. By [DG, III.7.7.8] the dimension of the space

$$\mathrm{Hom}_X(\mathcal{P}_y, \mathcal{P}_y \otimes \omega_X)$$

is upper semi-continuous on Y, so for all $y \in Y$ there is a non-zero morphism $\mathcal{P}_y \to \mathcal{P}_y \otimes \omega_X$. But both these sheaves are stable with the same Chern class, so they are isomorphic and (8.1) holds for all $y \in Y$.

The Riemann-Roch formula gives

$$\chi(\mathcal{P}_y, \mathcal{P}_y) = -(af)^2 = 0,$$

so by Proposition 5.4.5, the Zariski tangent space to Z at a point $i(y)$, always has dimension 2. Now Y fibres over C with elliptic fibres, so has dimension at least 2, and it follows that Z is a non-singular projective surface as claimed.

Extending our universal sheaf $\mathcal{P}$ on $Y \times_C X$ by zero, we obtain a sheaf on $Y \times X$ which we shall also denote by $\mathcal{P}$, such that for each point $y \in Y$, $\mathcal{P}_y$ is a stable sheaf of Chern class $(0, af, -b)$ on X.

For any two distinct points y_1, y_2 of Y, Serre duality implies that

$$\mathrm{Ext}^2_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = \mathrm{Hom}_X(\mathcal{P}_{y_2}, \mathcal{P}_{y_1})^{\vee} = 0,$$

and since $\chi(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$, this is enough to show that $\mathcal{P}$ is strongly simple over Y. By Theorem 6.1.7, the functor $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$ is an equivalence of categories.

Proposition 8.2.2. *The scheme $Y = J_{X/C}(a, b)$ is an elliptic surface over C, and $(Y, X, \mathcal{P})$ is an FM triple.*

Proof. It only remains to show that Y is relatively minimal over C. Suppose not, i.e. that there exists a (-1)-curve D contained in a fibre of $\hat{\pi}$. Then by the Riemann-Roch theorem on Y,

$$\chi(\mathcal{O}_D, \mathcal{O}_Y) - \chi(\mathcal{O}_Y, \mathcal{O}_D) = D \cdot \mathcal{K}_Y = -1.$$

Since Φ is an equivalence this implies that $\chi(E,F) - \chi(F,E) = -1$, where $E = \Phi(\mathcal{O}_D)$ and $F = \Phi(\mathcal{O}_Y)$. But for each i, $\mathrm{H}^i(E)$ is a fibre sheaf (because $\mathcal{O}_D$ is), so $\mathrm{c}_1(E) \cdot \mathcal{K}_X = 0$, and hence this contradicts the Riemann-Roch theorem on X.

□

Remark 8.2.3. Note that in fact, by the argument of Example 6.2.6, the embedding $i : Y \hookrightarrow Z$ considered above is an isomorphism.

Remark 8.2.4. Suppose we use two different polarisations of X to define elliptic surfaces $\mathrm{J}_{X/C}(a,b)$ and $\mathrm{J}'_{X/C}(a,b)$ over C. Then, since the stability of a sheaf on a smooth curve does not depend on a choice of polarisation, the two spaces will be isomorphic over the open subset U considered above, and hence birational. Since both are relatively minimal over C, [BPV, III.8.4] implies that they are isomorphic as elliptic surfaces over C.

Proposition 8.2.5. *There is an isomorphism of elliptic surfaces*

$$\mathrm{J}_{X/C}(a,b) \cong \mathrm{J}_{X/C}(b).$$

Proof. Let $Y = \mathrm{J}_{X/C}(a,b)$ and let U be the set of points p of C such that the fibre X_p is non-singular. Put $Y_U = Y \times_C U$ and let $\mathcal{P}_U$ be the restriction of the universal sheaf $\mathcal{P}$ to the open subset $Y_U \times_C X$ of $Y \times_C X$. Then $\mathcal{P}_U$ is locally free because $\mathcal{P}_y$ is a bundle on $X_{\pi(y)}$ for each $y \in Y_U$, so $\mathcal{E} = \det(\mathcal{P}_U)$ is a line bundle on $Y_U \times_C X$ such that for each point $y \in Y_U$, the restriction $\mathcal{E}_y$ is a line bundle of degree b on the fibre $X_{\pi(y)}$. The argument of Remark 8.2.4 completes the proof. □

8.3 FM transforms for elliptic surfaces

Let $X \xrightarrow{\pi} C$ be an elliptic surface, and fix integers $a > 0$ and b with $a\,\lambda_{X/C}$ coprime to b. As in the last section, put $Y = \mathrm{J}_{X/C}(a,b)$ and take a universal sheaf $\mathcal{P}$ on $Y \times X$, supported on $Y \times_C X$. Let $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ be the corresponding FM transform.

By Lemma 6.1.5, an inverse transform of Φ is $\Psi = \Phi^{\mathcal{Q}[1]}_{X\to Y}$ where $\mathcal{Q}$ is the object $\mathcal{P}^\vee \otimes \pi_X^*\omega_X[1]$ of $\mathrm{D}(Y \times X)$. The next lemma shows that $\mathcal{Q}$ is in fact a sheaf on $Y \times X$.

Lemma 8.3.1. *The object $\mathcal{Q}$ is concentrated in degree* 0. *Moreover, $\mathcal{P}$ and $\mathcal{Q}$ are both flat over X and Y.*

Proof. For each point $(y,x) \in Y \times X$, consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C} & \xrightarrow{j_x} & X \\ {\scriptstyle j_y}\downarrow & & \downarrow{\scriptstyle i_y} \\ Y & \xrightarrow{i_x} & Y \times X \end{array}$$

where j_x and i_x are the inclusions of $\{x\}$ in X and $Y \times \{x\}$ in $Y \times X$ respectively. Similarly for j_y and i_y.

First note that by an argument of Mukai [Muk4, p. 105], any pure dimension 1 sheaf on any surface has a two-step locally free resolution. This implies that for all $y \in Y$, $(\mathcal{P}_y)^\vee[1]$ is concentrated in degree 0. But by [Ha1, III.8.8],

$$\mathbf{L}\, i_y^*(\mathcal{P}^\vee[1]) = (\mathbf{L}\, i_y^*(\mathcal{P}))^\vee[1] = (\mathcal{P}_y)^\vee[1],$$

so by Lemma 3.1.2, $\mathcal{P}^\vee[1]$ is concentrated in degree 0 and is flat over Y.

To show that $\mathcal{P}$ is flat over X consider the spectral sequence

$$E_2^{p,q} = \mathbf{L}_{-p}\, j_y^*(\mathbf{L}_{-q}\, i_x^*(\mathcal{P})) \implies \mathbf{L}_{-(p+q)}\, j_x^*(\mathcal{P}_y).$$

Since $\mathcal{P}_y$ has a two-step resolution, the right-hand side is non-zero only if $p+q = 0$ or 1, so one concludes that

$$\mathbf{L}_1\, j_y^*(\mathbf{L}_1\, i_x^*(\mathcal{P})) = 0,$$

for all $y \in Y$. By Lemma 3.1.1, this implies that $\mathbf{L}_1\, i_x^*(\mathcal{P})$ is locally free on Y. But for any $x \in X$ one can find $y \in Y$ such that (y,x) does not lie in the support of $\mathcal{P}$, so $\mathbf{L}_1\, i_x^*(\mathcal{P}) = 0$ for all $x \in X$, and $\mathcal{P}$ is flat over X.

Finally, the isomorphism

$$\mathbf{L}\, i_x^*(\mathcal{P}^\vee[1]) \cong (\mathcal{P}_x)^\vee[1],$$

implies that both sides are concentrated in degree 0, so by Lemma 3.1.2 again, $\mathcal{P}^\vee[1]$ is flat over X. $\square$

The next lemma shows that the relationship between X and Y is entirely symmetrical.

Lemma 8.3.2. *There exists an integer c such that $X \cong \mathrm{J}_{Y/C}(a,c)$.*

Proof. If X_p is a non-singular fibre of π then the restriction of $\mathcal{P}$ to $Y_p \times X_p$ is a universal bundle parameterising stable bundles on X_p. It follows from the results of Section 6.3, that for any point $x \in X$ lying on a non-singular fibre of π, the sheaf $\mathcal{P}_x$ is a stable sheaf on Y. Let its Chern class be $(0, af, -c)$. I claim that

c is coprime to $a\,\lambda_{Y/C}$, so that $\mathrm{J}_{Y/C}(a,c)$ is well-defined. Assuming this for the moment, note that as in Remark 8.2.4, the two elliptic surfaces X and $J_{Y/C}(a,c)$ over C are isomorphic away from the singular fibres, so are isomorphic.

Since the object $\mathcal{Q}_x$ of $\mathrm{D}(Y)$ has Chern class $(0,af,c)$, to prove the claim it will be enough to exhibit an object E of $\mathrm{D}(Y)$ such that $\chi(\mathcal{Q}_x,E)=1$. But this is possible, since

$$\chi(\mathcal{Q}_x,E) = -\,\chi(\mathcal{O}_x,\Phi E) = -\,\mathrm{r}(\Phi E),$$

for any object E of $\mathrm{D}(Y)$, and Φ is an equivalence. $\square$

The following lemma shows that the restriction of Φ to a non-singular fibre of π yields one of the transforms of Theorem 6.4.2.

Lemma 8.3.3. *Let $p\in C$ be such that the fibre $\pi^{-1}(p)$ is smooth. Let*

$$i_p : X_p \hookrightarrow X, \qquad j_p : Y_p \hookrightarrow Y,$$

be the inclusion of the non-singular fibres X_p and Y_p, and let $\mathcal{P}_p$ be the restriction of $\mathcal{P}$ to $Y_p\times X_p$. Then $\mathcal{P}_p$ is locally free and there is an isomorphism of functors

$$\mathbf{L}\,i_p^* \circ \Phi \cong \Phi_p \circ \mathbf{L}\,j_p^*.$$

where Φ_p is the FM transform $\Phi^{\mathcal{P}_p}_{Y_p\to X_p}$.

Proof. This is very similar to the proof of Lemma 4.2.1. $\square$

Now the functor Φ_p of the lemma coincides with one of the transforms of Theorem 6.4.2. In particular there is a matrix

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

such that for all objects E of $\mathrm{D}(Y)$,

$$\begin{pmatrix} \mathrm{r}(\Phi E) \\ \mathrm{d}(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix}. \tag{8.2}$$

We can now prove

Theorem 8.3.4. *Let $X \xrightarrow{\pi} C$ be an elliptic surface and take an element*

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z}),$$

such that $\lambda_{X/C}$ divides d and $a>0$. Let Y be the elliptic surface $\mathrm{J}_{X/C}(a,b)$ over C. Then there exist sheaves $\mathcal{P}$ on $Y\times X$, supported on $Y\times_C X$, which are flat and strongly simple over both factors such that for any point $(y,x)\in Y\times X$, $\mathcal{P}_y$ has Chern class $(0,af,-b)$ on X and $\mathcal{P}_x$ has Chern class $(0,af,-c)$ on Y.

For any such sheaf $\mathcal{P}$, the resulting functor $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ is an FM transform and satisfies

$$\begin{pmatrix} \mathrm{r}(\Phi E) \\ \mathrm{d}(\Phi E) \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix}, \tag{8.3}$$

for all objects E of $\mathrm{D}(Y)$.

Proof. Take a universal sheaf $\mathcal{P}$ on $Y \times_C X$ and put $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$. As we showed above, Φ is an equivalence and there exist integers c and d such that (8.3) holds. Now $\lambda_{X/C}$ divides $\mathrm{d}(\Phi E)$ for any object E of $\mathrm{D}(Y)$, so $\lambda_{X/C}$ divides $\lambda_{Y/C}$ and d. By symmetry $\lambda_{X/C} = \lambda_{Y/C}$. As in Section 6.4, c and d are not uniquely defined: we can replace them by $c + n\,\lambda_{X/C}\,a$ and $d + n\,\lambda_{X/C}\,b$ by twisting $\mathcal{P}$ by the pull-back of a line bundle of fibre degree $n\,\lambda_{X/C}$ on Y. □

Remark 8.3.5. As a corollary of the proof of Theorem 8.3.4, note that we can always choose $\mathcal{P}$ so that for some polarisation of X, $\mathcal{P}_y$ is stable for all $y \in Y$. By Lemma 8.3.2, we could also view X as a moduli space of sheaves on Y, and take $\mathcal{P}$ such that for some polarisation of Y, $\mathcal{Q}_x = \mathcal{P}^\vee_x[1]$ is stable for all $x \in X$.

8.4 Properties of the transforms

Let $X \xrightarrow{\pi} C$ be an elliptic surface, fix an element

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

with $a > 0$ and $\lambda_{X/C}$ dividing d, let Y be the elliptic surface $\mathrm{J}_{X/C}(a,b)$ and take a sheaf $\mathcal{P}$ on $Y \times X$ as in Theorem 8.3.4. As in the last section, define the sheaf $\mathcal{Q} = \mathcal{P}^\vee \otimes \pi_X^*\omega_X[1]$, and the functors

$$\Phi = \Phi^{\mathcal{P}}_{Y\to X} : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X), \qquad \Psi = \Phi^{\mathcal{Q}}_{X\to Y} : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y).$$

By Lemma 6.1.5,

$$\Psi \circ \Phi \cong 1_{\mathrm{D}(Y)}[-1], \qquad \Phi \circ \Psi \cong 1_{\mathrm{D}(X)}[-1]. \tag{8.4}$$

In this section we give some properties of the transforms which will be useful later. Because of the symmetry of the situation, for each result we give here, there will be another result obtained by exchanging Φ and Ψ, and X and Y.

Note first that because the morphism $Y \times_C X \to Y$ has relative dimension 1, for any sheaf E on X, the sheaves $\Psi^i(E)$ are only non-zero if $i = 0$ or 1. Thus, as in Lemma 6.3.5, the Mukai spectral sequence yields a short exact sequence

$$0 \longrightarrow \Phi^1(\Psi^0(E)) \longrightarrow E \longrightarrow \Phi^0(\Psi^1(E)) \longrightarrow 0,$$

together with the information that $\Psi^0(E)$ is Φ-WIT_1 and $\Psi^1(E)$ is Φ-WIT_0. Note that by Lemma 6.1.15, $\Phi^1(\Psi^0(E))$ is Ψ-WIT_0 and $\Phi^0(\Psi^1(E))$ is Ψ-WIT_1.

Lemma 8.4.1. *For any sheaf E on X, $\Psi^0(E)$ is Φ-WIT_1 and $\Psi^1(E)$ is Φ-WIT_0. Furthermore there is a unique short exact sequence*

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

such that A is Ψ-WIT_0 and B is Ψ-WIT_1.

Proof. For the uniqueness, suppose there is another such sequence

$$0 \longrightarrow A' \longrightarrow E \longrightarrow B' \longrightarrow 0.$$

Then, since A' is Ψ-WIT_0 and B is Ψ-WIT_1, the Parseval theorem implies that there is no non-zero map $A' \to B$, so the inclusion of A' in E factors through A. By symmetry $A = A'$. $\square$

Given a torsion-free sheaf E on X, put $\mu(E) = \mathrm{d}(E)/\,\mathrm{r}(E)$.

Lemma 8.4.2. *Let E be a torsion-free sheaf on X. If E is Ψ-WIT_0 then $\mu(E) \geq b/a$. Similarly if E is Ψ-WIT_1 then $\mu(E) \leq b/a$.*

Proof. If E is Ψ-WIT_1 then $\Psi(E)[1]$ is a sheaf so $\mathrm{r}(\Psi E) \leq 0$. Similarly, if E is Ψ-WIT_0 then $\mathrm{r}(\Psi E) \geq 0$. Since $\Psi[1]$ is the inverse of Φ, one has

$$\begin{pmatrix} \mathrm{r}(\Psi E) \\ \mathrm{d}(\Psi E) \end{pmatrix} = \begin{pmatrix} -b & a \\ d & -c \end{pmatrix} \begin{pmatrix} \mathrm{r}(E) \\ \mathrm{d}(E) \end{pmatrix},$$

for any object E of $\mathrm{D}(X)$. The result follows. $\square$

A similar argument gives

Lemma 8.4.3. *Let T be a Ψ-WIT_1 torsion sheaf on X. Then T is a fibre sheaf.* $\square$

Combining Lemma 8.4.2 with Lemma 8.4.1 we obtain

Lemma 8.4.4. *Let E be a torsion-free sheaf on X such that the restriction of E to the general fibre of π is stable. Suppose $\mu(E) < b/a$. Then E is Ψ-WIT_1.*

Proof. Consider the short exact sequence of Lemma 8.4.1. If A is non-zero, it is torsion-free and one has $\mu(A) \geq b/a > \mu(E)$. Restricting to the general fibre of π this gives a contradiction. Hence $A = 0$ and E is Ψ-WIT_1. $\square$

The final result we shall need is

Lemma 8.4.5. *A sheaf F on Y is Φ-WIT_0 iff*

$$\operatorname{Hom}_Y(F, \mathcal{Q}_x) = 0 \qquad \forall x \in X.$$

Proof. First note that $\mathcal{Q}_x = \Psi(\mathcal{O}_x)$ is Φ-WIT_1. If F is Φ-WIT_0, then the Parseval theorem implies that there are no non-zero maps $F \to \mathcal{Q}_x$.

Conversely, if F is not Φ-WIT_0, then by the argument of Lemma 8.4.1, there is a surjection $F \to B$ with B a Φ-WIT_1 sheaf. Applying the Parseval theorem again gives

$$\operatorname{Hom}_Y(B, \mathcal{Q}_x) = \operatorname{Hom}_X(\hat{B}, \mathcal{O}_x).$$

Since $\hat{B}$ is non-zero, there exists an $x \in X$ and a non-zero map $B \to \mathcal{Q}_x$, hence a non-zero map $F \to \mathcal{Q}_x$. $\square$

8.5 Application to moduli of stable sheaves

In this section we use the relative transforms we have developed to prove Theorem 8.0.5.

The proof of the following result is entirely analogous to the rank 2 case [Fr, Thm I.3.3] so we omit it (see also [OG, Prop. I.1.6]).

Proposition 8.5.1. *Let $X \xrightarrow{\pi} C$ be an elliptic surface and take a triple*

$$(r, \Delta, k) \in \mathbb{N} \times \mathrm{NS}(X) \times \mathbb{Z},$$

such that r is coprime to $d = \Delta \cdot f$. Then there exist polarisations ℓ of X with respect to which a torsion-free sheaf E on X with Chern class (r, Δ, k) is μ-stable whenever it is μ-semi-stable, and this is the case iff the restriction of E to all but finitely many fibres of π is stable. Such polarisations will be called suitable *for the triple (r, Δ, k).* $\square$

Let us fix an elliptic surface $X \xrightarrow{\pi} C$ and a triple

$$(r, \Delta, k) \in \mathbb{N} \times \mathrm{NS}(X) \times \mathbb{Z},$$

such that r is coprime to $d = \Delta \cdot f$. Put $\mathcal{M} = \mathcal{M}_X^{\ell}(r, \Delta, k)$, where ℓ is suitable with respect to the triple (r, Δ, k). Note that by [Muk4, App. 2], $\mathcal{M}$ is a fine moduli space. We identify the closed points of $\mathcal{M}$ with the stable sheaves which they represent. As in the rank 2 case [Fr, Lemma III.3.6], one shows that for any $E \in \mathcal{M}$,

$$\operatorname{Ext}^2_X(E, E) = \mathrm{H}^2(X, \mathcal{O}_X).$$

It then follows from Proposition 5.4.5, that $\mathcal{M}$, if non-empty, is smooth of dimension $\dim(\mathrm{Pic}^\circ(X)) + 2t$, where

$$2t = 2rk - (r-1)\Delta^2 - (r^2-1)\chi(\mathcal{O}_X),$$

and that if $t < 0$ then $\mathcal{M}$ is empty. In what follows we take $r > 1$ and assume that t is non-negative.

Let a and b be the unique pair of integers satisfying $br - ad = 1$, with $0 < a < r$. Let $\hat{\pi} : Y \to C$ be the elliptic surface $J_{X/C}(a, b)$ and put

$$\mathcal{N} = \mathcal{M}_Y(1, 0, t) = \mathrm{Pic}^\circ(Y) \times \mathrm{Hilb}^t(Y).$$

We shall show that $\mathcal{M}$ is birationally equivalent to $\mathcal{N}$.

Let $\mathcal{P}$ be a sheaf on $Y \times X$ as in Theorem 8.3.4, with matrix

$$\begin{pmatrix} r & a \\ d & b \end{pmatrix},$$

and define equivalences of categories Φ and Ψ as in the last section. As we noted in Remark 8.3.5, we can assume that we have chosen $\mathcal{P}$, and a polarisation of Y, so that $\mathcal{Q}_x$ is a stable sheaf for all $x \in X$.

Take a sheaf E on X of Chern class (r, Δ, k). The formula given in the proof of Lemma 8.4.2 shows that $\Psi(E)$ has rank 1 and fibre degree 0. Twisting $\mathcal{P}$ by the pull-back of a line bundle on Y we can assume that $c_1(\Psi E) = 0$, and Riemann-Roch together with the Parseval theorem then implies that $\Psi(E)$ has Chern class $(1, 0, t)$.

Note that, by Lemma 8.4.4, any element E of $\mathcal{M}$ is Ψ-WIT_1. Define

$$\mathcal{U} = \{E \in \mathcal{M} : \hat{E} \text{ is torsion-free}\}.$$

Then $\mathcal{U}$ is an open subscheme of $\mathcal{M}$. Also define the open subscheme

$$\mathcal{V} = \{F \in \mathcal{N} : F \text{ is } \Phi\text{-}\mathrm{WIT}_0\}.$$

Lemma 8.5.2. *The transform Φ gives an isomorphism between the two schemes $\mathcal{U}$ and $\mathcal{V}$.*

Proof. For any point $E \in \mathcal{U}$, E is Ψ-WIT_1 and $\hat{E} \in \mathcal{V}$. Suppose now that $F \in \mathcal{V}$ and put $E = \hat{F}$. Claim that $E \in \mathcal{U}$. By Lemma 8.3.3, the restriction of E to the general fibre of X is simple, hence by Lemma 6.3.2, stable, so it is only necessary to check that E is torsion-free. Suppose E has a torsion subsheaf T. Then since E is Ψ-WIT_1, Lemma 6.1.10 implies that T is Ψ-WIT_1 also, hence, by Lemma 8.4.3, a fibre sheaf. Applying Ψ gives a sequence

$$0 \longrightarrow \Psi^0(E/T) \xrightarrow{f} \hat{T} \longrightarrow F \longrightarrow \Psi^1(E/T) \longrightarrow 0.$$

Since F is torsion-free and $\hat{T}$ is a fibre sheaf, f must be an isomorphism. But, by Lemma 8.4.1, $\Psi^0(E/T)$ is Φ-WIT$_1$, and $\hat{T}$ is Φ-WIT$_0$, so both sheaves are zero, $T = 0$ and E is torsion-free. $\square$

Clearly, we need to show that $\mathcal{U}$ and $\mathcal{V}$ are non-empty. Take $F \in \mathcal{N}$. Then $F = L \otimes \mathcal{I}_Z$, with $L \in \mathrm{Pic}^\circ(Y)$ and Z a zero-dimensional subscheme of Y of length t. By Lemma 8.4.5, F is Φ-WIT$_0$ precisely when there is no non-zero map $F \to \mathcal{Q}_x$ for any $x \in X$. Since $\mathcal{Q}_x$ is supported on the fibre $Y_{\pi(x)}$ of $\hat{\pi}$, any map $F \to \mathcal{Q}_x$ factors via $F|_{Y_{\pi(x)}}$, and hence via a stable, pure dimension 1 sheaf on Y of Chern class $(0, f, s)$, where s is the number of points of Z lying on the fibre $Y_{\pi(x)}$. Now $\mathcal{Q}_x$ has Chern class $(0, af, r)$, and is stable, so if $s < r/a$, any map $F \to \mathcal{Q}_x$ is zero. This argument, and the fact that $r > a$, gives the following results.

Lemma 8.5.3. *Let $F = L \otimes \mathcal{I}_Z$, with $L \in \mathrm{Pic}^\circ(Y)$ and Z a set of t points $\{y_1, \cdots, y_n\}$ lying on distinct fibres of $\pi : Y \to C$. Then F is an element of $\mathcal{V}$.* $\square$

Lemma 8.5.4. *If $r > at$ then $\mathcal{V} = \mathcal{N}$.* $\square$

Remark 8.5.5. Applying Φ to the short exact sequence

$$0 \longrightarrow F \longrightarrow L \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

gives a sequence

$$0 \longrightarrow \hat{F} \longrightarrow \hat{L} \longrightarrow \bigoplus \mathcal{P}_{y_i} \longrightarrow 0.$$

Let us assume for simplicity that X is simply-connected. Then we see that an open subset of $\mathcal{M}$ is obtained from the fixed bundle $\widehat{\mathcal{O}_Y}$ by taking t distinct non-singular fibres $\{f_1, \cdots, f_t\}$ of π and stable bundles P_i of rank a and degree b on f_i, and taking the kernel of the unique morphism

$$\widehat{\mathcal{O}_Y} \longrightarrow \oplus P_i.$$

Furthermore, when X is nodal, the proof of [Fr, Prop. III.3.11] shows that $\widehat{\mathcal{O}_Y}$ is the unique sheaf up to twists on X whose restriction to every reduction of a fibre of π is stable.

In the rank 2 case, this corresponds to Friedman's method of constructing bundles using elementary modifications [Fr, §III.3].

To complete the proof of Theorem 8.0.5 we must show that $\mathcal{M}$ is irreducible, i.e. that $\mathcal{M}$ has only one connected component. Let us suppose, for contradiction, that there is a connected component $\mathcal{W}$ of $\mathcal{M}$ which does not meet $\mathcal{U}$.

Let E be a point of $\mathcal{W}$. Then E is Ψ-WIT$_1$, and the transform $\hat{E}$ is a sheaf of Chern class $(1,0,t)$ on Y, with a non-zero torsion sheaf. By the argument of Lemma 8.5.2, the restriction of $\hat{E}$ to the general fibre of $\hat{\pi}$ is simple, hence stable.

Lemma 8.5.6. *Let $n \geq 1$ be an integer. Then for a general zero-dimensional subscheme $Z \in \mathrm{Hilb}^{rn}(Y)$, there is a unique morphism $\hat{E} \to \mathcal{O}_Z$. Furthermore, for general Z, this morphism surjects and the kernel K is Φ-WIT$_0$. The transform $\hat{K}$ is then an element of the moduli space*

$$\widetilde{\mathcal{M}} = \mathcal{M}_X^{\ell'}(r, \Delta - (rna)f, k + rnb - rna(\Delta \cdot f)),$$

where ℓ' is a suitable polarisation.

Proof. We may suppose that Z consists of rn points lying on distinct non-singular fibres $f_1, \cdots, f_{rn}$ of $\hat{\pi}$. We can also suppose that $\hat{E}$ is locally-free at each of the points of Z. Then there is a unique morphism $\hat{E} \to \mathcal{O}_Z$ and this map surjects, giving an exact sequence

$$0 \longrightarrow K \longrightarrow \hat{E} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

By Lemma 8.4.5, to prove that K is Φ-WIT$_0$, we must show that there are no non-zero morphisms $K \to \mathcal{Q}_x$ for any $x \in X$. We only need to check this when $\mathcal{Q}_x$ is supported on one of the fibres $f_1, \cdots, f_{rn}$ since the restrictions of $\hat{E}$ and K to any other fibre are identical, and $\hat{E}$ is Φ-WIT$_0$. But we can always take Z so that the restriction of $\hat{E}$ to each of the fibres f_i is a degree 0 line bundle. This will be enough since $\mathcal{Q}_x$ is stable of degree $-r$. $\square$

Twisting by $\mathcal{O}_X(anf)$ gives an isomorphism between the spaces $\mathcal{M}_X(r, \Delta, k + n)$ and $\widetilde{\mathcal{M}}$, so Theorem 1.2.1 implies that for large enough n, $\widetilde{\mathcal{M}}$ is irreducible. It follows from what we proved above that the general element of $\widetilde{\mathcal{M}}$ has torsion-free transform. Now the construction of the lemma gives a rational map

$$\theta : \mathcal{W} \times \mathrm{Hilb}^{rn}(Y) \dashrightarrow \widetilde{\mathcal{M}},$$

and since all points in the image of θ have non-torsion-free transforms, θ cannot be dominant. But we shall show below that the general fibre of θ is zero-dimensional. Since θ is a map between two varieties of the same dimension, this will give a contradiction.

Take an element of $\widetilde{\mathcal{M}}$, and let K be its transform. We must show that there are only finitely many pairs

$$(E, Z) \in \mathcal{W} \times \mathrm{Hilb}^{rn}(Y),$$

such that Z consists of rn distinct points at which $\hat{E}$ is locally free, and $K = \hat{E} \otimes \mathcal{I}_Z$.

Given such a pair, note that Z does not meet the support of the torsion subsheaf of $\hat{E}$, so the torsion subsheaves T of $\hat{E}$ and $\hat{E} \otimes \mathcal{I}_Z = K$ are equal. Thus Z is a subset of the finite set of points at which K/T is not locally free. This implies that the number of possible choices of Z is finite.

Finally, if we have two pairs (E_1, Z), and (E_2, Z) then $E_1 = E_2$, because there is only one extension of K by $\mathcal{O}_Z$ which is locally free at each of the points of Z. This completes the proof.

Chapter 9

Fourier-Mukai transforms for quotient surfaces

9.1 Canonical covers

In this chapter we wish to study FM transforms on Enriques and bielliptic surfaces. Collectively we shall call these surfaces *quotient surfaces.*

All quotient surfaces have elliptic fibrations, indeed any bielliptic surface has exactly two distinct elliptic fibrations, and the general Enriques surface is an elliptic surface in infinitely many different ways [FrMo, p. 51]. Thus the methods of Chapter 8 yield many examples of FM transforms for quotient surfaces. In this chapter we adopt a different approach, by describing the relationship between invariant FM transforms between surfaces with a group action, and FM transforms between the quotient surfaces.

We shall only consider quotients of a very restricted type. Recall that if a finite group G acts freely by automorphisms on a smooth projective variety $\widetilde{X}$ the quotient $X = \widetilde{X}/G$ is also a smooth projective variety.

Proposition 9.1.1. *Let X be a smooth projective variety whose canonical bundle has finite order n. Then there is a smooth projective variety $\widetilde{X}$ with trivial canonical bundle, and an unbranched cover $p : \widetilde{X} \to X$ of degree n, such that*

$$p_* \mathcal{O}_{\widetilde{X}} \cong \bigoplus_{i=0}^{n-1} \omega_X^i. \tag{9.1}$$

Furthermore, $\widetilde{X}$ is uniquely defined up to isomorphism, and there is a free action of the cyclic group $G = \mathbb{Z}/(n)$ on $\widetilde{X}$ such that $p : \widetilde{X} \to X = \widetilde{X}/G$ is the quotient morphism.

Proof. By the results of [BPV, §I.17], there exists a smooth projective variety $\widetilde{X}$ and a degree n unbranched cover satisfying (9.1). Furthermore $\omega_{\widetilde{X}} = p^* \omega_X = \mathcal{O}_{\widetilde{X}}$.

By [Ha2, Ex. II.5.17], $\widetilde{X}$ is isomorphic to $\operatorname{Spec}(\mathcal{A})$, where

$$\mathcal{A} = \bigoplus_{i=0}^{n-1} \omega_X^i,$$

which proves uniqueness. The action of G is generated by the automorphism $\otimes\omega_X$ of $\mathcal{A}$, and clearly $X = \widetilde{X}/G$. $\square$

Definition 9.1.2. Let X be a smooth projective variety whose canonical bundle has finite order n. By the *canonical cover* of X we shall mean the unique smooth projective variety $\widetilde{X}$ of Proposition 9.1.1, together with the natural quotient morphism $p_X : \widetilde{X} \longrightarrow X$.

Examples 9.1.3. (a) An Enriques surface is a surface X with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ whose canonical bundle has order 2. The canonical cover of such a surface is a K3 surface $\widetilde{X}$, and X is the quotient of $\widetilde{X}$ by the group generated by a fixed-point-free automorphism of order 2. See [BPV, Ch. VIII].

(b) A bielliptic surface is a surface X with $\mathrm{H}^1(X, \mathcal{O}_X) = \mathbb{C}^2$ whose canonical bundle has finite order $n > 1$. The possible values of n are $2, 3, 4$ and 6. The canonical cover of such a surface is an Abelian surface $\widetilde{X}$, and X is the quotient of $\widetilde{X}$ by a free action of a cyclic group of automorphisms of order n. See [BPV, §V.5].

9.2 Complexes of sheaves on quotient surfaces

Let X be a smooth projective variety whose canonical bundle has finite order n, and let

$$p : \widetilde{X} \longrightarrow X$$

be the canonical cover of X. Thus X is the quotient of $\widetilde{X}$ by a free action of $G = \mathbb{Z}/(n)$. We let g denote a generator of G, and for any integer i we put $g_i = g^i$. Thus

$$G = \{1, g, g_2, g_3, \cdots, g_{n-1}\}.$$

The following obvious-looking result is surprisingly hard to prove.

Proposition 9.2.1. *(a) Let $\widetilde{E}$ be an object of $\mathrm{D}(\widetilde{X})$. Then there is an object E of $\mathrm{D}(X)$ such that $p^*E \cong \widetilde{E}$ if and only if there is an isomorphism $g^*\widetilde{E} \cong \widetilde{E}$.*

(b) Let E be an object of $\mathrm{D}(X)$. Then there is an object $\widetilde{E}$ of $\mathrm{D}(\widetilde{X})$ such that $p_*\widetilde{E} \cong E$ if and only if there is an isomorphism $E \otimes \omega_X \cong E$.

Proof. We start with (a). One implication is easy, so let us assume that there is an isomorphism $s : \widetilde{E} \to g^*\widetilde{E}$, and find E such that $p^*E \cong \widetilde{E}$. Note that in the case when $\widetilde{E}$ is concentrated in degree 0, the result is well-known [Mum4, §7, Prop. 2]. To prove the general case we use induction on the number r of non-zero cohomology sheaves of $\widetilde{E}$. Suppose $r > 1$. We may as well assume that $\widetilde{E}$ is left-centred. Note that $\mathrm{H}^0(\widetilde{E})$ is G-invariant, so $\mathrm{H}^0(\widetilde{E}) = p^*M$ for some $\mathcal{O}_X$-module M. There is a canonical morphism $\widetilde{E} \to \mathrm{H}^0(\widetilde{E})$, and hence a triangle

$$\widetilde{E} \longrightarrow \mathrm{H}^0(\widetilde{E}) \xrightarrow{\tilde{f}} \widetilde{F} \longrightarrow \widetilde{E}[1],$$

in $\mathrm{D}(\widetilde{X})$, where $\widetilde{F}$ has $r-1$ non-zero cohomology objects. Applying g^* we obtain an isomorphic triangle, because there is a commutative diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r] \arrow[d, "s"'] & \mathrm{H}^0(\widetilde{E}) \arrow[d, "\mathrm{H}^0(s)"'] \\ g^*\widetilde{E} \arrow[r] & \mathrm{H}^0(g^*\widetilde{E}) \end{tikzcd}$$

Hence $g^*\widetilde{F} \cong \widetilde{F}$. By induction $\widetilde{F} \cong p^*F$ for some object F of $\mathrm{D}(X)$. The lemma below implies that $\tilde{f} = p^*(f)$ for some morphism $f : M \to F$ of $\mathrm{D}(X)$. Thus there is an object E of $\mathrm{D}(X)$ and a triangle

$$E \longrightarrow M \xrightarrow{f} F \longrightarrow E[1].$$

Applying p^* one sees that $p^*E \cong \widetilde{E}$.

For part (b), first note that if $E \cong p_*\widetilde{E}$ then by the projection formula

$$E \otimes \omega_X \cong p_*(\widetilde{E} \otimes p^*(\omega_X)) = p_*\widetilde{E} \cong E.$$

For the converse suppose that $E \otimes \omega_X \cong E$, and assume first that E is concentrated in degree 0. Then E is a $p_*(\mathcal{O}_{\widetilde{X}})$-module, and hence isomorphic to $p_*(\widetilde{E})$ for some $\mathcal{O}_{\widetilde{X}}$-module $\widetilde{E}$. The general case is then proved in exactly the same way as part (a). □

Lemma 9.2.2. *Let M be an $\mathcal{O}_X$-module, and let F be an object of* $\mathrm{D}(X)$. *Let $\tilde{f} : p^*M \to p^*F$ be a morphism of* $\mathrm{D}(\widetilde{X})$ *such that $g^*(\tilde{f}) = \tilde{f}$. Then $\tilde{f} = p^*(f)$ for some morphism $f : M \to F$ of* $\mathrm{D}(X)$.

Proof. Replace F by an injective resolution

$$\cdots \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots,$$

as in [Ha1, Lemma I.4.6]. Then $\tilde{f}$ is represented by a morphism of $\mathcal{O}_{\widetilde{X}}$-modules $s : p^*M \to p^*I^0$.

If V is a finite-dimensional vector space on which G acts, define operators A and B by

$$A = \sum_{i=0}^{n-1} g_i^*, \qquad B = 1 - g^*.$$

Then since $AB = BA = 0$, it is easy to check that $\ker A = \operatorname{im} B$.

Take V to be the image of the map

$$p^*(d^{-1})_* : \operatorname{Hom}_{\widetilde{X}}(p^*M, p^*I^{-1}) \longrightarrow \operatorname{Hom}_{\widetilde{X}}(p^*M, p^*I^0).$$

The fact that $\tilde{f}$ is G-invariant means that Bs is an element of V. Since $A(Bs) = 0$, there is an element k of V with $Bk = Bs$. Now $t = s - k \in \operatorname{Hom}_{\widetilde{X}}(p^*M, p^*I^0)$ also represents $\tilde{f}$, and since $Bt = 0$, is equal to $p^*(u)$ for some $u \in \operatorname{Hom}_X(M, I^0)$ (by [Mum4, §7, Prop. 2]). The result follows. □

We shall also need

Lemma 9.2.3. *Let $\widetilde{E}$ be an object of $\mathrm{D}(\widetilde{X})$ and let $1 < d \leq n$ be a factor of n. If $g^*_{n/d}(\widetilde{E}) \cong \widetilde{E}$, then d divides $\chi(p_*\widetilde{E}, B)$ for all objects B of $\mathrm{D}(X)$.*

Proof. The object

$$\widetilde{F} = \bigoplus_{i=0}^{n/d-1} g_i^*\widetilde{E},$$

is G-invariant, so there exists an object F of $\mathrm{D}(X)$ such that $p^*F \cong \widetilde{F}$. Then

$$\bigoplus_{i=0}^{n/d-1} p_*\widetilde{E} \cong p_*\widetilde{F} \cong F \otimes p_*(p^*\mathcal{O}_X) \cong \bigoplus_{i=0}^{n} F \otimes \omega_X^i.$$

Now by Riemann-Roch, for any object B of $\mathrm{D}(X)$, $\chi(F, B) = \chi(F, B \otimes \omega_X)$, and the result follows. □

9.3 Lifts of FM transforms

Definition 9.3.1. Let X and Y be surfaces with canonical bundles of finite order, and let $p_X : \widetilde{X} \longrightarrow X$, $p_Y : \widetilde{Y} \longrightarrow Y$ be the canonical covers. Then given a functor $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$, a *lift* of Φ is a functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ such that the following diagram of functors commutes up to isomorphism

$$\begin{array}{ccc} \mathrm{D}(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{D}(\widetilde{X}) \\ p_Y^* \uparrow \downarrow p_{Y,*} & & p_X^* \uparrow \downarrow p_{X,*} \\ \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X), \end{array}$$

i.e. such that there are isomorphisms of functors

$$p_{X,*} \circ \widetilde{\Phi} \cong \Phi \circ p_{Y,*}, \qquad p_X^* \circ \Phi \cong \widetilde{\Phi} \circ p_Y^*. \tag{9.2}$$

We also say that $\widetilde{\Phi}$ *descends* to give the functor Φ.

Throughout this section we fix a pair of smooth projective surfaces X and Y with canonical bundles of finite order n_X and n_Y respectively, and denote the canonical covers by $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$. Thus X and Y are the quotients of $\widetilde{X}$ and $\widetilde{Y}$ by free actions of the groups $G = \mathbb{Z}/(n_X)$ and $H = \mathbb{Z}/(n_Y)$ respectively.

Lemma 9.3.2. *Suppose the integral functor $\widetilde{\Phi}$ is a lift of the identity functor $1_{\mathrm{D}(X)}$. Then $\widetilde{\Phi} \cong g^*$ for some element $g \in G$.*

Proof. Take a point $\tilde{x} \in \widetilde{X}$, and put $x = p_X(\tilde{x})$. Then $E = \widetilde{\Phi}(\mathcal{O}_{\tilde{x}})$ satisfies $p_{X,*}(E) = \mathcal{O}_x$, so $E = \mathcal{O}_{f(\tilde{x})}$ for some point $f(\tilde{x})$ in the fibre $p^{-1}(x)$. By the argument of Example 6.2.1, f is an endomorphism of $\widetilde{X}$ and $\widetilde{\Phi} = f_*$. It follows that $f \cong g^{-1}$ for some $g \in G$. ☐

Lemma 9.3.3. *Let $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ be objects of $\mathrm{D}(\widetilde{Y} \times \widetilde{X})$ and $\mathrm{D}(Y \times X)$ respectively, such that*

$$(p_Y \times 1_X)^*(\mathcal{P}) \cong (1_{\widetilde{Y}} \times p_X)_*(\widetilde{\mathcal{P}}). \tag{9.3}$$

Then $\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}}$ is a lift of $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$.

Proof. Put

$$f = (1_{\widetilde{Y}} \times p_X), \qquad h = (p_Y \times 1_X),$$

and consider the commutative diagram

$$\begin{tikzcd}
\widetilde{Y} \arrow[d, equal] & \widetilde{Y} \times \widetilde{X} \arrow[l, "\pi_{\widetilde{Y}}"'] \arrow[r, "\pi_{\widetilde{X}}"] \arrow[d, "f"] & \widetilde{X} \arrow[d, "p_X"] \\
\widetilde{Y} \arrow[d, "p_Y"'] & \widetilde{Y} \times X \arrow[l, "j"'] \arrow[r, "k"] \arrow[d, "h"] & X \arrow[d, equal] \\
Y & Y \times X \arrow[l, "\pi_Y"'] \arrow[r, "\pi_X"] & X.
\end{tikzcd}$$

Let E be an object of $\mathrm{D}(\widetilde{Y})$. By [Hal, II.5.6, II.5.12] there are natural isomorphisms

$$\begin{aligned}
p_{X,*}(\widetilde{\Phi}(E)) &= p_{X,*}\mathbf{R}\pi_{\widetilde{X},*}(\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} \pi^*_{\widetilde{Y}}E) \\
&\cong \mathbf{R}k_*(f_*(\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} f^*j^*E)) \cong \mathbf{R}k_*(f_*\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} j^*E) \\
&\cong \mathbf{R}\pi_{X,*}(h_*(h^*\mathcal{P} \overset{\mathbf{L}}{\otimes} j^*E)) \cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} h_*j^*E) \\
&\cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi^*_Y(p_{Y,*}E)) = \Phi(p_{Y,*}(E)).
\end{aligned}$$

The second isomorphism of (9.2) can be proved in the same way, or by taking adjoints. ☐

The next result shows that all FM transforms downstairs lift to FM transforms upstairs.

Definition 9.3.4. Let X and Y be surfaces with canonical bundles of finite order n. Then a functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ will be called $\mathbb{Z}/(n)$-*equivariant* if there is an isomorphism of functors

$$g^* \circ \widetilde{\Phi} \cong \widetilde{\Phi} \circ g^*,$$

for each $g \in G$.

Proposition 9.3.5. *Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform. Then Φ has a lift $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$. Moreover the canonical bundles of X and Y have the same order n, $\widetilde{\Phi}$ is $G = \mathbb{Z}/(n)$-equivariant and if $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ are two lifts of Φ then there is an isomorphism*

$$\widetilde{\Phi}_2 \cong g^* \circ \widetilde{\Phi}_1,$$

for some $g \in G$.

Proof. Let $\mathcal{P}$ be the kernel of Φ. By Lemma 4.2.4 we have an isomorphism

$$\mathcal{P} \otimes \pi_X^* \omega_X \cong \mathcal{P} \otimes \pi_Y^* \omega_Y.$$

Let $\mathcal{Q} = (p_Y \times 1_X)^*(\mathcal{P})$. Then $\mathcal{Q} \otimes \omega_{\widetilde{Y} \times X} \cong \mathcal{Q}$ so there is an object $\widetilde{\mathcal{P}}$ of $\mathrm{D}(\widetilde{Y} \times \widetilde{X})$ satisfying (9.3). Define $\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}}$. Then by Lemma 9.3.3, $\widetilde{\Phi}$ is a lift of Φ.

Let Ψ be a quasi-inverse for Φ. Then Ψ is also an FM transform and hence lifts to a functor $\widetilde{\Psi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ by the same argument. Now it is easy to check that $\widetilde{\Psi} \circ \widetilde{\Phi}$ is a lift of $\Psi \circ \Phi \cong 1_{\mathrm{D}(Y)}$. Hence, by Lemma 9.3.2, $\widetilde{\Psi} \circ \widetilde{\Phi}$ is an equivalence of categories. Similarly, $\widetilde{\Phi} \circ \widetilde{\Psi}$ is an equivalence, so $\widetilde{\Phi}$ is an equivalence, hence a FM transform.

Now take $g \in G$ and consider the FM transform $g^* \circ \widetilde{\Phi}$. This is also a lift of Φ, so $\widetilde{\Psi} \circ g^* \circ \widetilde{\Phi}$ is a lift of $1_{\mathrm{D}(Y)}$. By Lemma 9.3.2 again, there is an element $\mu(g) \in H$ such that

$$g^* \circ \widetilde{\Phi} \cong \widetilde{\Phi} \circ (\mu(g))^*.$$

Clearly, μ must be injective, and so by symmetry it is an isomorphism. Identifying H with G via μ we conclude that $\widetilde{\Phi}$ is G-equivariant.

Finally, if $\widetilde{\Phi}'$ is another lift of Φ, then $\widetilde{\Phi} \circ \widetilde{\Psi}$ is a lift of $1_{\mathrm{D}(X)}$, so by Lemma 9.3.2 there is a $g \in G$ such that $\widetilde{\Phi}' \cong g^* \circ \widetilde{\Phi}$. □

Conversely, invariant FM transforms descend to give FM transforms downstairs:

Proposition 9.3.6. *Suppose the canonical bundles of X and Y have the same order n, say, and let $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ be a G-invariant FM transform. Then there is a FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ such that $\widetilde{\Phi}$ lifts Φ. If Φ' is another such transform, then $\Phi' \cong \omega_X^i \otimes \Phi$, for some integer i.*

Proof. To prove the uniqueness statement, note that for any point $\tilde{y} \in \widetilde{Y}$ with $p_Y(\tilde{y}) = y$ we must have

$$\Phi(\mathcal{O}_y) \cong p_{X,*}(\widetilde{\Phi}(\mathcal{O}_{\tilde{y}})) \cong \Phi'(\mathcal{O}_y).$$

Hence $\Phi' \cong L \otimes \Phi$ for some line bundle L on X. But $1_{\mathrm{D}(\widetilde{Y})}$ lifts $\Phi^{-1} \circ \Phi'$, and it follows that $L = \omega_X^i$.

For existence, let the kernel of $\widetilde{\Phi}$ be $\widetilde{\mathcal{P}}$. It is easy to check that the G-invariance of $\widetilde{\Phi}$ is equivalent to the condition

$$(g \times 1_{\widetilde{X}})^*(\widetilde{\mathcal{P}}) \cong (1_{\widetilde{Y}} \times g)^*(\widetilde{\mathcal{P}}) \quad \forall g \in G.$$

It follows that $(1_{\widetilde{Y}} \times p_X)_*(\widetilde{\mathcal{P}})$ is G-invariant so that

$$(p_Y \times 1_X)^*(\mathcal{P}) \cong (1_{\widetilde{Y}} \times p_X)_*(\widetilde{\mathcal{P}})$$

for some object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$. Hence by Lemma 9.3.3 $\widetilde{\Phi}$ lifts $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$.

We must show that Φ is an equivalence of categories. Let $\widetilde{\Psi}$ be a quasi-inverse of $\widetilde{\Phi}$. Then $\widetilde{\Psi}$ is G-invariant and hence is the lift of some integral functor $\Psi : \mathrm{D}(X) \to \mathrm{D}(X)$. But then $\widetilde{\Psi} \circ \widetilde{\Phi} \cong 1_{\mathrm{D}(\widetilde{Y})}$ lifts $\Psi \circ \Phi$, so by the uniqueness statement, twisting Ψ by some power of ω_X, $\Psi \circ \Phi \cong 1_{\mathrm{D}(Y)}$. Similarly $\Phi \circ \Psi \cong 1_{\mathrm{D}(X)}$.

□

9.4 Examples

The examples we shall give are based on the following result, which gives a purely numerical criterion for determining when FM transforms given by stable sheaves descend to quotients.

Let X be a surface whose canonical sheaf has order $n > 1$, let $p_X : \widetilde{X} \to X$ be the universal cover, and let $G = \mathbb{Z}/(n)$ be the corresponding group of automorphisms of $\widetilde{X}$ as in the last section.

Proposition 9.4.1. *Suppose that*

$$\Phi = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

is an FM transform such that for each $\tilde{y} \in \widetilde{Y}$, the object $\widetilde{\mathcal{P}}_{\tilde{y}} = \Phi(\mathcal{O}_{\tilde{y}})$ is a stable sheaf on X. Suppose also that there is an object B of $\mathrm{D}(X)$ such that for one *(and hence any) point $\tilde{y} \in \widetilde{Y}$, $\chi(\widetilde{\mathcal{P}}_{\tilde{y}}, p_X^* B) = 1$. Then there is a unique free action of G on $\widetilde{Y}$ such that*

$$\widetilde{\mathcal{P}}_{g(\tilde{y})} \cong g^*(\widetilde{\mathcal{P}}_{\tilde{y}}), \tag{9.4}$$

the quotient $p_Y : \widetilde{Y} \to Y = \widetilde{Y}/G$ is a canonical cover, and the functor $\widetilde{\Phi}$ descends to an FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$.

Proof. By Lemma 9.2.3, if $g \neq 1$, $g^*(\widetilde{\mathcal{P}}_{\tilde{y}})$ is never isomorphic to $\widetilde{\mathcal{P}}_{\tilde{y}}$. But by Example 6.2.6, we know that $g^*(\widetilde{\mathcal{P}}_{\tilde{y}}) \cong \widetilde{\mathcal{P}}_{g(\tilde{y})}$ for some $g(\tilde{y}) \in \widetilde{Y}$. Hence we can define a free G-action on $\widetilde{Y}$ so that (9.4) holds. This action is algebraic because $g^* = \Phi^{-1} \circ g^* \circ \Phi$. The argument of Proposition 9.3.6 shows that $\widetilde{\Phi}$ descends to an FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$, and it is then easy to check that $p_Y : \widetilde{Y} \to Y$ is a canonical cover. $\square$

Example 9.4.2. Let X be an Enriques surface. Then there is a K3 surface $\widetilde{X}$ with an automorphism σ of order 2 such that X is the quotient of $\widetilde{X}$ by the 2-element group generated by σ. For any point $\tilde{x} \in \widetilde{X}$ one clearly has

$$\sigma^*(\mathcal{I}_{\tilde{x}}) = \mathcal{I}_{\sigma(\tilde{x})}.$$

Thus the reflection functor of Example 6.2.4 descends to give an FM transform

$$\Phi : \mathrm{D}(X) \longrightarrow \mathrm{D}(X),$$

This has the property that for each $x \in X$ one has an exact sequence

$$0 \longrightarrow \Phi(\mathcal{O}_x) \longrightarrow \mathcal{O}_X \oplus \omega_X \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

It is this transform which was studied in [Zu, §3.7].

Example 9.4.3. Let X be a bielliptic surface whose fundamental group is cyclic of order n. Then X is the quotient of a product of elliptic curves $\widetilde{X} = C_1 \times C_2$ by $G = \mathbb{Z}/(n)$.

The original Fourier-Mukai functor never descends because the sheaf $\mathcal{O}_{\widetilde{X}} = \mathcal{F}(\mathcal{O}_0)$ is G-invariant.

Consider instead the moduli space $\widetilde{Y}$ of stable sheaves on $\widetilde{X}$ of Chern character $(4, 2\ell, 1)$, where $\ell = C_1 + C_2$ is a principal polarisation. It is easy to prove that $\widetilde{Y} \cong \widetilde{X}$ and that the universal sheaf on $\widetilde{Y} \times \widetilde{X}$ gives an FM transform $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ (see [Mac5, Thm. 0.1]). By Proposition 9.4.1, $\widetilde{\Phi}$ descends to give an FM transform

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

such that $\Phi(\mathcal{O}_y)$ is a locally free sheaf of rank $4n$ for all $y \in Y$.

Chapter 10

Fourier-Mukai partners of minimal surfaces

In this chapter we address the following question : for a given minimal surface X, which projective varieties have equivalent derived categories?

Definition 10.0.4. Let X be a projective variety. A *FM partner* for X is a projective variety Y such that there is an FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$.

Note. Any FM partner of a smooth projective variety X is necessarily smooth of the same dimension as X.

We shall prove the following theorem.

Theorem 10.0.5. *Let X be a minimal surface. Then any FM partner of X is isomorphic to X, apart from in the following cases:*

(a) if the Kodaira dimension of X is 1, then X has a unique elliptic fibration $X \xrightarrow{\pi} C$, and the FM partners of X are the relative Picard schemes $\mathrm{J}_{X/C}(b)$ of Chapter 8, with b running through integers coprime to $\lambda_{X/C}$.

(b) If X is a K3 surface (respectively an Abelian surface) the FM partners of X are those K3 surfaces (respectively Abelian surfaces) Y with isometric extended Hodge lattices.

For the definition of extended Hodge lattices see Section 10.2 below.

10.1 Basic results

We start our classification of FM transforms with the following simple observation. We use the notation $D \equiv 0$ to signify that the divisor D is numerically equivalent to zero.

Lemma 10.1.1. *Let X be a surface. Then $\mathcal{K}_X \equiv 0$ if and only if, for all pairs of objects E and F of* $\mathrm{D}(X)$*, one has*

$$\chi(E,F) = \chi(F,E).$$

Proof. By the Riemann-Roch theorem, for any two objects E and F of $\mathrm{D}(X)$,

$$\chi(E,F) - \chi(F,E) = (\mathrm{r}(F)\,\mathrm{c}_1(E) - \mathrm{r}(E)\,\mathrm{c}_1(F)) \cdot \mathcal{K}_X,$$

so the given equality holds for all objects E and F precisely when $\mathcal{K}_X$ is numerically equivalent to zero. $\square$

Corollary 10.1.2. *Let X be a surface with $\mathcal{K}_X \equiv 0$, and let Y be an FM partner of X. Then $\mathcal{K}_Y \equiv 0$.* $\square$

Let us make the following definition:

Definition 10.1.3. Let E be a sheaf on a surface X. Then E is *special* if $E \otimes \omega_X \cong E$.

The following result is our basic tool for classifying surface transforms.

Lemma 10.1.4. *Let X and Y be surfaces, let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform and take a point $y \in Y$. Then there is an inequality*

$$\sum_i \dim_{\mathbb{C}} \mathrm{Ext}^1_X(\Phi^i(\mathcal{O}_y), \Phi^i(\mathcal{O}_y)) \leq 2,$$

and moreover, each of the sheaves $\Phi^i(\mathcal{O}_y)$ is special.

Proof. The second statement is immediate from Theorem 4.0.3. For the first part consider the spectral sequence [BO1, Prop. 4.2]

$$E_2^{p,q} = \bigoplus_i \mathrm{Ext}^p_X(\Phi^i(\mathcal{O}_y), \Phi^{i+q}(\mathcal{O}_y)) \implies \mathrm{Hom}^{p+q}_{\mathrm{D}(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)).$$

The $E_2^{1,0}$ term survives to infinity, and

$$\mathrm{Hom}^1_{\mathrm{D}(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \mathrm{Hom}^1_{\mathrm{D}(Y)}(\mathcal{O}_y, \mathcal{O}_y) = \mathbb{C}^2,$$

so the result follows. $\square$

Corollary 10.1.5. *Let X be a surface and let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be a FM transform. Suppose that all special sheaves E on X satisfy*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E,E) \geq 2.$$

Then there is some integer n such that $\Phi[n]$ is a sheaf transform. $\square$

A more detailed analysis gives

Proposition 10.1.6. *Let X be a surface of non-zero Kodaira dimension, let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform and take a point $y \in Y$. Then either the support of $\Phi(\mathcal{O}_y)$ contains an integral curve D satisfying*

$$D \cdot \mathcal{K}_X = 0 \text{ and } D^2 \leq 0, \tag{10.1}$$

or for some integer i and some point $x \in X$,

$$\Phi(\mathcal{O}_y)[i] \cong \mathcal{O}_x.$$

In the second case, X and Y are birational, and if X is minimal and $\kappa(X) > 0$, X and Y are isomorphic.

Proof. Note that every cohomology sheaf of $\Phi(\mathcal{O}_y)$ is supported in codimension 1. Suppose first that every non-zero cohomology sheaf of $\Phi(\mathcal{O}_y)$ has dimension 0. Let E be one such. By Riemann-Roch $\chi(E,E) = 0$, so the dimension of the space $\mathrm{Ext}^1_X(E,E)$ is at least 2. Lemma 10.1.4 then implies that for some integer i, $\Phi(\mathcal{O}_y) \cong E[i]$, and so

$$\mathrm{Hom}_X(E,E) = \mathrm{Hom}_Y(\mathcal{O}_y, \mathcal{O}_y) = \mathbb{C},$$

and it follows that E is the structure sheaf of a point $f(y)$ of X. Now the argument of Example 6.2.1 shows that f defines a birational equivalence between Y and X. If X is minimal with $\kappa(X) > 0$ then Y must be isomorphic to the blow-up of X at $n \geq 0$ points. But we must have $n = 0$ because Φ induces an isomorphism

$$\mathrm{ch}(\Phi) : H^{2*}(Y, \mathbb{Q}) \longrightarrow H^{2*}(X, \mathbb{Q}),$$

preserving the Hodge decomposition. Thus $\mathrm{NS}(Y)$ and $\mathrm{NS}(X)$ have the same rank.

Suppose now that E is a cohomology sheaf of $\Phi(\mathcal{O}_y)$ supported in dimension 1. Suppose

$$\mathrm{c}_1(E) = \sum_i n_i D_i, \quad n_i \in \mathbb{N},$$

with each D_i an integral subscheme of X of dimension 1. Then $D_i \cdot \mathcal{K}_X = 0$ for all i, because otherwise twisting the surjective morphism $E \to E|_{D_i}$ by powers of ω_X would yield quotients of E of arbitrarily high Euler character, which is impossible. By Lemma 10.1.4 and Riemann-Roch,

$$\chi(E,E) = -\,\mathrm{c}_1(E)^2 \geq 0,$$

so we must have $D_i^2 \leq 0$ for some i. $\square$

We shall use this result in conjunction with

Lemma 10.1.7. *Let X and Y be surfaces, with $\mathcal{K}_X \not\equiv 0$, and let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ an FM transform. Let $S \subset X$ be a finite set of points of X. Then for a general point $y \in Y$, the support of $\Phi(\mathcal{O}_y)$ is disjoint from S.*

Proof. Let Ψ be an inverse transform for Φ, and take a pair of points $x \in X$, $y \in Y$. Then x lies in the support of $\Phi(\mathcal{O}_y)$ precisely when y lies in the support of $\Psi(\mathcal{O}_x)$. This follows immediately from the isomorphism

$$\mathrm{Hom}^{\bullet}_{\mathrm{D}(Y)}(\Psi(\mathcal{O}_x), \mathcal{O}_y) \cong \mathrm{Hom}^{\bullet}_{\mathrm{D}(X)}(\mathcal{O}_x, \Phi(\mathcal{O}_y)) \cong \mathrm{Hom}^{\bullet}_{\mathrm{D}(X)}(\Phi(\mathcal{O}_y), \mathcal{O}_x)^{\vee}[2],$$

together with Remark 4.1.5. Now if every point of y lies in the union over $x \in S$ of the supports of the objects $\Psi(\mathcal{O}_x)$, then for some $x \in S$, the support of $\Psi(\mathcal{O}_x)$ is the whole of Y. It follows that $\mathcal{K}_Y \equiv 0$, which contradicts Corollary 10.1.2. □

We can now start the proof of Theorem 10.0.5.

Proposition 10.1.8. *Let X be a minimal surface of non-zero Kodaira dimension with an elliptic fibration structure $X \xrightarrow{\pi} C$. If Y is an FM partner of X then Y is isomorphic to the relative Picard scheme $\mathrm{J}_{X/C}(b)$ for some integer b coprime to $\lambda_{X/C}$.*

Proof. Let f be the cohomology class of a fibre of π. Then there exist non-zero integers p, q such that $p\mathcal{K}_X = qf$. Take $x \in X$ lying on a smooth fibre of π, and take a point $y \in Y$ such that the support of the object $E = \Phi(\mathcal{O}_y)$ contains x. Each sheaf $\mathrm{H}^i(E)$ is a special sheaf, hence supported on a fibre of π. Since $\mathrm{Hom}_{\mathrm{D}(X)}(E, E) = \mathbb{C}$, the support of E is connected, hence equal to the fibre of π containing x. Since this fibre is smooth, each cohomology sheaf of E is non-rigid, so Lemma 10.1.4 implies that E has only one non-zero cohomology sheaf. Now the Chern class of E must be $(0, af, -b)$ for some integers a and b, and since

$$\chi(E, \Phi(\mathcal{O}_Y)) = \chi(\mathcal{O}_y, \mathcal{O}_Y) = 1,$$

one must have $\lambda_{X/C}$ coprime to b. Furthermore a shift of E is a simple sheaf on a smooth elliptic curve, hence stable. Now there is a transform $\Psi : \mathrm{D}(\mathrm{J}_{X/C}(b)) \longrightarrow \mathrm{D}(X)$ which takes the structure sheaf of some point of $\mathrm{J}_{X/C}(b)$ to E. Applying the argument of Proposition 10.1.6 to the transform $\Psi^{-1} \circ \Phi$ shows that Y is isomorphic to $\mathrm{J}_{X/C}(b)$. □

Proposition 10.1.9. *Let X be a minimal surface of Kodaira dimension 2 or $-\infty$. Then the only FM partner of X is X itself.*

Proof. If X has Kodaira dimension 2 [BPV, VII.2.3, VII.2.5] implies that X has only finitely many irreducible curves D satisfying condition (10.1) above. The result then follows by Proposition 10.1.6 and Lemma 10.1.7.

If $X = \mathbb{P}^2$ then $-\mathcal{K}_X$ is ample, so all special sheaves are zero-dimensional, and the result follows from Corollary 10.1.5 and the argument of Example 6.2.1.

The remaining case is when X is a ruled surface over a curve C of genus g. We freely use notation and results from [Ha2, IV.2]. Let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be a FM transform and suppose Y is not isomorphic to X. By Proposition 10.1.6 there is an integral curve D on X satisfying (10.1). Then [Ha2, V.2.10] implies that D cannot be a fibre of π, so the induced morphism from D to C is a finite morphism of curves. Since D has genus at most 1, it follows from [Ha2, IV.2.5.4] that $g \leq 1$.

First suppose $g = 0$. By [Ha2, IV.2.20], $D = C_0$ must be a section of π, and $D \cdot \mathcal{K}_X = 0$ implies that the invariant e of X is 2. Since there is only one section C_0, Y must be birational to X, and hence also a ruled surface over $\mathbb{P}^1$. By symmetry the invariant e of Y is also 2, so we must have $X \cong Y$, a contradiction.

Now suppose $g = 1$, so $\mathcal{K}_X^2 = 0$. We shall assume that X is not isomorphic to $C \times \mathbb{P}^1$ since this case is covered by the last proposition. If E is a special sheaf on X then E has Chern class $(0, \Delta, s)$ for some effective divisor Δ satisfying $\Delta \cdot \mathcal{K}_X = 0$. The Hodge index theorem then implies that $\Delta^2 = 0$, so by Riemann-Roch,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E, E) \geq 2.$$

Corollary 10.1.5 now implies that (up to shifts) Φ is a sheaf transform. Since X and Y are non-isomorphic, Φ cannot be trivial, so for each point $x \in X$, there is an integral curve D passing through x with $D^2 = D \cdot \mathcal{K}_X = 0$. Applying [Ha2, IV.2.20, IV.2.21] gives a contradiction. □

10.2 Abelian and K3 surfaces

By a *lattice* we shall mean a free Abelian group equipped with a symmetric bilinear form. The following definitions are standard.

Definition 10.2.1. Let Λ be a lattice. An element $v \in \Lambda$ is *primitive* if $\Lambda/\langle v \rangle$ is torsion-free, or equivalently if there is an element $w \in \Lambda$ such that $\langle v, w \rangle = 1$. An element $v \in \Lambda$ is *isotropic* if $\langle v, v \rangle = 0$.

Definition 10.2.2. Let X be an Abelian or K3 surface. The *extended Hodge lattice* $\mathrm{H}^{2*}(X, \mathbb{Z})$ of X is the group $\mathbb{Z} \oplus \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathbb{Z}$ with inner product given by

$$\langle (r_1, \Delta_1, k_1), (r_2, \Delta_2, k_2) \rangle = \Delta_1 \cdot \Delta_2 - r_1 k_2 - r_2 k_1.$$

Definition 10.2.3. Let X be an Abelian or K3 surface, and let E be an object of $\mathrm{D}(X)$. Then $v(E)$, the *Mukai vector* of E, is an element of the extended Hodge lattice of X. If X is Abelian it is given by the Chern character $(\mathrm{r}(E), \mathrm{c}_1(E), \frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E))$ of E. If X is a K3 surface it is the vector $(\mathrm{r}(E), \mathrm{c}_1(E), \frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E) - \mathrm{r}(E))$.

Lemma 10.2.4. *Let X be an Abelian or K3 surface, and let E and F be objects of $\mathrm{D}(X)$. Then*

$$\chi(E, F) = -\langle \mathrm{ch}(E), \mathrm{ch}(F) \rangle.$$

Proof. Immediate from the Riemann-Roch theorem. □

Definition 10.2.5. Let X and Y be Abelian or K3 surfaces. An isometry of lattices

$$\phi : \mathrm{H}^{2*}(Y, \mathbb{Z}) \longrightarrow \mathrm{H}^{2*}(X, \mathbb{Z}),$$

is called a *Hodge isometry* if its $\mathbb{C}$-linear extension takes the one-dimensional subspace $\mathrm{H}^{0,2}(Y, \mathbb{C})$ of $\mathrm{H}^2(Y, \mathbb{C})$ onto $\mathrm{H}^{0,2}(X, \mathbb{C})$.

We now consider the case of Abelian surfaces separately. The following consequence of Example 6.2.3 and Corollary 10.1.5 has no analogue for K3 surfaces.

Lemma 10.2.6. *Let X be an Abelian surface, and let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be an FM transform. Then for some integer n, $\Phi[n]$ is a sheaf transform.* □

Proposition 10.2.7. *Let X be an Abelian surface and let E be a simple sheaf on X such that $v(E)$ is primitive and isotropic. Then E is stable with respect to any polarisation of X.*

Proof. Let ℓ be a polarisation of X. If E is not semistable with respect to ℓ, then by Proposition 5.1.3 (g), there is a short exact sequence,

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

with $\mathrm{Hom}_X(A, B) = 0$. Now [Muk7, Prop. 2.8] implies that the sum of the dimensions of the spaces $\mathrm{Ext}^1_X(A, A)$ and $\mathrm{Ext}^1_X(B, B)$ is at most the dimension of $\mathrm{Ext}^1_X(E, E)$, which is 2. But then Example 6.2.3 implies that one of A and B is zero. Hence E is semistable. Let

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$

be a Jordan-Hölder filtration for E, and put $F_i = E_i/E_{i-1}$. Each sheaf F_i is stable, so by Example 6.2.3, $\chi(F_i, F_i) \leq 0$.

If E has dimension 2, [Muk5, Prop. 2.19] implies that each F_i has the same normalised Mukai vector as E. Since $v(E)$ is primitive, it follows that E is stable.

If E has dimension 1, note that since $D_1 \cdot D_2 \geq 0$ for any two effective divisors on X, we must have

$$\mathrm{c}_1(F_i)^2 = \mathrm{c}_1(F_i) \cdot \mathrm{c}_1(E) = \mathrm{c}_1(E)^2 = 0$$

for all i. It follows from the Hodge index theorem that $\mathrm{c}_1(E)$ is a multiple of $\mathrm{c}_1(F_i)$ for each i. Since $v(E)$ is primitive, E must be stable. □

This result should be compared with [Muk1, Prop. 6.16]. For any primitive isotropic element of the extended Hodge lattice of an Abelian surface X we can now define the space $\mathcal{M}_X(v)$ to be the moduli space of stable sheaves with respect to any polarisation of X.

Lemma 10.2.8. *Let X be an Abelian surface which is the product of elliptic curves, and let v be a primitive isotropic element of $\mathrm{H}^{2*}(X, \mathbb{Z})$ which is the Mukai vector of a sheaf on X. Then $\mathcal{M}_X(v)$ is non-empty.*

Proof. By the argument of Lemma 10.2.6, it will be enough to show that there is an object E of $\mathrm{D}(X)$ satisfying $\mathrm{Hom}_X(E, E) = \mathbb{C}$ whose Chern character is v. Twisting by $\mathcal{O}_X(mE_1)$ allows us to replace $v = (r, aE_1 + bE_2, k)$ by

$$(r, (a + rm)E_1 + bE_2, k + bm).$$

Since X is principally polarised, it is isomorphic to its dual, so using the Fourier functor we can replace this vector by

$$(k + bm, -(a + rm)E_1 - bE_2, r).$$

Since v is primitive, there is no common factor of (r, a, b, k), so if we choose m correctly, $k + bm$ will be coprime to $a + rm$, so by Theorem 8.0.5, there is a stable sheaf of this character. □

Theorem 10.2.9. *Let X be an Abelian surface, and let v be a primitive isotropic element of $\mathrm{H}^{2*}(X, \mathbb{Z})$ which is the Mukai vector of a sheaf on X. Then $Y = \mathcal{M}_X(v)$ is a (non-empty) Abelian surface, and there is a universal sheaf $\mathcal{P}$ on $Y \times X$. The resulting functor $\Phi^{\mathcal{P}}_{Y \to X}$ is a FM transform.*

Conversely, suppose that X and Y are Abelian surfaces, and $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is an FM transform, and let v be the Mukai vector of $\Phi(\mathcal{O}_y)$ for a point $y \in Y$. Then v is primitive and isotropic, Y is isomorphic to the space $\mathcal{M}_X(v)$ and for some integer n, the kernel of $\Phi[n]$ is a universal sheaf on $Y \times X$.

Proof. Suppose $Y = \mathcal{M}_X(v)$ is non-empty. Then it is smooth by Proposition 5.4.5, and by [Muk5, App. 2], there is a universal sheaf on $Y \times X$. By Theorem 6.1.7, the resulting functor $\Phi^{\mathcal{P}}_{Y \to X}$ is an FM transform. By Corollary 10.1.2, $\mathcal{K}_Y \equiv 0$, and since $\mathrm{ch}(\Phi)$ gives an isomorphism

$$\mathrm{H}^{2*}(Y, \mathbb{Q}) \longrightarrow \mathrm{H}^{2*}(X, \mathbb{Q}),$$

the only possibility is that Y is an Abelian surface.

To show that Y is non-empty we use a deformation argument of Mukai's. See also [Muk1, Thm. 7.11]. Firstly, twisting by a sufficiently ample line bundle, we may suppose that $\mathrm{c}_1(v) = H$ is an ample line bundle on X. Now let $(\mathcal{X}, \mathcal{H})$ be a smooth deformation of (X, H) over a base scheme S. Thus $\mathcal{H}$ is a line bundle on the scheme $\mathcal{X}$, there is a smooth morphism

$$\pi : \mathcal{X} \longrightarrow S,$$

and a point $0 \in S$, such that $(\mathcal{X}_0, \mathcal{H}|_{\mathcal{X}_0}) = (X, H)$. Then by Theorem 5.3.2, there is a projective morphism $\mathcal{M}_{\mathcal{X}/S}(v) \to S$ whose fibre over a point $s \in S$ is the moduli space $\mathcal{M}_{\mathcal{X}_s}(v)$. By [Muk3, Thm. 1.17], if this morphism is non-zero, it is smooth, and hence surjective. Thus it will be enough to show that $\mathcal{M}_X(v)$ is non-empty for some deformation (X', H') of (X, H). But any polarised Abelian surface deforms to a product of elliptic curves, so Lemma 10.2.8 suffices.

For the converse, note that $\Phi \cong \Phi^{\mathcal{P}}_{Y \to X}$ for some sheaf $\mathcal{P}$ on $Y \times X$, flat over Y. Furthermore, for each $y \in Y$, $\mathcal{P}_y$ is simple with Mukai vector v. Now

$$\chi(\mathcal{P}_y, \mathcal{P}_y) = \chi(\mathcal{O}_y, \mathcal{O}_y) = 0, \quad \chi(\Phi(\mathcal{O}_Y), \mathcal{P}_y) = \chi(\mathcal{O}_Y, \mathcal{O}_y) = 1,$$

so v is primitive and isotropic. The rest follows from Example 6.2.6. □

The fact that all FM transforms for an Abelian surface X are shifts of sheaf transforms is a consequence of the fact that there are no sheaves on X satisfying $\mathrm{Ext}^1_X(E, E) = 0$. Such sheaves are called *rigid*. On a K3 surface X, there are many rigid sheaves, the structure sheaf $\mathcal{O}_X$ being a good example, and as we have seen, any K3 surface admits FM transforms which are not shifts of sheaf transforms. Thus we cannot expect a result as simple as Theorem 10.2.9 to hold for K3 surfaces. However the following theorem is true.

Theorem 10.2.10. *Let X and Y be Abelian (respectively K3) surfaces. Then $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ are equivalent as triangulated categories if, and only if, the extended Hodge lattices $\mathrm{H}^{2*}(X, \mathbb{Z})$ and $\mathrm{H}^{2*}(Y, \mathbb{Z})$ are isometric.*

Proof. We shall assume that X and Y are Abelian surfaces. The case when X and Y are K3 surfaces is very similar and is covered in [Bo]. Firstly, if $\Phi : D(Y) \longrightarrow D(X)$ is an FM transform, then

$$\mathrm{ch}(\Phi) : \mathrm{H}^{2*}(Y, \mathbb{Q}) \longrightarrow \mathrm{H}^{2*}(X, \mathbb{Q})$$

is an isomorphism, which preserves the integral lattices and the Hodge decomposition. Conversely, suppose

$$\phi : \mathrm{H}^{2*}(Y, \mathbb{Z}) \longrightarrow \mathrm{H}^{2*}(X, \mathbb{Z})$$

is a Hodge isometry. Then $v = \phi(0,0,1)$ is a primitive, isotropic Mukai vector, and changing the sign of ϕ if necessary, we may assume that v is the Mukai vector of a sheaf on X. By Theorem 10.2.9 there is an FM transform $\Psi : \mathrm{D}(\mathcal{M}_X(v)) \longrightarrow \mathrm{D}(X)$ such that $\mathrm{ch}(\Psi)(0,0,1) = v$. Now $\Phi^{-1} \circ \Psi$ is a sheaf transform, so by the argument of Example 6.2.1 must be trivial. Hence $Y \cong \mathcal{M}_X(v)$, and Y and X are FM partners. □

10.3 Quotient surfaces

To complete our classification of FM transforms for minimal surfaces we shall prove the following theorem.

Theorem 10.3.1. *Let X be an Enriques or bielliptic surface. Then the only FM partner of X is X itself.*

We shall treat the two cases in rather different ways. Assume first that X is an Enriques surface, let $p_X : \widetilde{X} \to X$ be the canonical cover, and put $G = \mathbb{Z}/(2)$. Thus X is the quotient of $\widetilde{X}$ by a free action of G. Let g be the generator of G.

Define the G-invariant sublattice

$$\mathrm{H}^{2*}(\widetilde{X}, \mathbb{Z})_G = \{\theta \in \mathrm{H}^{2*}(\widetilde{X}, \mathbb{Z}) : g^*(\theta) = \theta\},$$

and let $M_X = \mathrm{H}^{2*}(\widetilde{X}, \mathbb{Z})_G^{\perp}$ be its orthogonal complement inside $\mathrm{H}^{2*}(\widetilde{X}, \mathbb{Z})$. Note that $M_{\widetilde{X}}$ is in fact a subspace of $\mathrm{H}^2(\widetilde{X}, \mathbb{Z})$. Furthermore I claim that $\mathrm{H}^{0,2}(\widetilde{X}) \subset M_{\widetilde{X}} \otimes \mathbb{C}$. Indeed, if θ is a non-zero element of the one-dimensional complex vector space $\mathrm{H}^{0,2}(\widetilde{X})$, then since this space is preserved by g, we must have $g^*(\theta) = \pm\theta$. But if θ is G-invariant it descends to give a 2-form on X which is impossible, so $g^*(\theta) = -\theta$ and it follows that θ is orthogonal to all G-invariant elements of $\mathrm{H}^2(\widetilde{X}, \mathbb{Z})$.

Suppose $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is an FM transform. By Corollary 10.1.2, Y has Kodaira dimension 0. Let $p_Y : \widetilde{Y} \to Y$ be a universal cover. By Proposition 9.3.5,

ω_Y also has order 2, and Φ lifts to a G-invariant transform $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$. This induces a G-invariant Hodge isometry, between $\mathrm{H}^{2*}(\widetilde{Y}, \mathbb{Z})$ and $\mathrm{H}^{2*}(\widetilde{X}, \mathbb{Z})$, and hence gives a G-invariant isometry $f : M_{\widetilde{Y}} \to M_{\widetilde{X}}$, taking the subspace $\mathrm{H}^{0,2}(\widetilde{Y})$ onto $\mathrm{H}^{0,2}(\widetilde{X})$. I claim that f extends to an isometry

$$f : \mathrm{H}^2(\widetilde{Y}, \mathbb{Z}) \longrightarrow \mathrm{H}^2(\widetilde{X}, \mathbb{Z}).$$

Assuming this for the moment, note that f is then a G-invariant Hodge isometry, so by the Torelli theorem for Enriques surfaces, [BPV, VIII.21.2], X and Y are isomorphic.

To prove the claim we use the methods of V.V. Nikulin. If h is a G-invariant polarisation on X, then the triple $(\mathrm{H}^2(\widetilde{X}, \mathbb{Z}), -g^*, h)$ is a polarised integral involution ([Ni, Defn. 3.1.1]). Hence the orthogonal complement of $M_{\widetilde{X}}$ in $\mathrm{H}^2(\widetilde{X}, \mathbb{Z})$, which is equal to $\mathrm{H}^2(\widetilde{X}, \mathbb{Z})_G$, is even, 2-elementary ([Ni, Defn. 3.6.1]) and indefinite. The claim then follows from Prop. 1.14.1, Prop. 3.6.2 and Thm. 3.6.3 of [Ni].

For the second case let X be a bielliptic surface. Any such surface has Picard number 2, so $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its two elliptic pencils f_1 and f_2. There is an unbranched cover of X by a product of two elliptic curves, and it is easy to check that the degree m of this cover is equal to $f_1 \cdot f_2$. Note that m divides the intersection number of any two divisors on X.

Let n be the order of ω_X. Consulting the table on [BPV, p. 148] one sees that m and n always have the same set of prime factors.

Let $\Phi : D(Y) \longrightarrow D(X)$ be an FM transform. By Lemma 10.2.6 and Proposition 9.3.5, we can assume that Φ is a sheaf transform. For any point $y \in Y$, $\Phi(\mathcal{O}_y)$ is special, so has rank divisible by n. Let $\Phi(\mathcal{O}_Y)$ have Chern character $(r, af_1 + bf_2, s)$. Since

$$\chi(\Phi(\mathcal{O}_Y), \Phi(\mathcal{O}_y)) = \chi(\mathcal{O}_Y, \mathcal{O}_y) = 1, \tag{10.2}$$

r must be coprime to n and m. Let $h = \gcd(r, b)$ and take integers c, d with $cr - dmb = h$. Composing Φ with one of the transforms of Theorem 8.3.4 with matrix

$$\begin{pmatrix} c & d \\ m(b/h) & r/h \end{pmatrix},$$

we see that we can assume $b = 0$ and $r = h > 0$. But since $\chi(\Phi(\mathcal{O}_Y), \chi(\Phi(\mathcal{O}_Y)) = 0$, Riemann-Roch then implies that $s = 0$. Now (10.2) shows that r is coprime to am, so applying Theorem 8.3.4 again, we can assume that $r = 1$. Twisting by a line bundle on X we can assume that $\Phi(\mathcal{O}_Y) \cong \mathcal{O}_X$.

Let $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ be a lift of Φ. Then $\widetilde{\Phi}(\mathcal{O}_{\widetilde{Y}}) \cong \mathcal{O}_{\widetilde{X}}$. If $\mathcal{F}_{\widetilde{X}}$ and $\mathcal{F}_{\widetilde{Y}}$ denote the original FM transforms on $\widetilde{X}$ and $\widetilde{Y}$ respectively, then the composite transform

$$\mathcal{F}_{\widetilde{X}} \circ \widetilde{\Phi} \circ \mathcal{F}_{\widetilde{Y}} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X}),$$

takes $\mathcal{O}_0$ to $\mathcal{O}_0$. It follows that $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic.

If v is the element $(1,0,0)$ of $\mathrm{H}^{2*}(\widetilde{Y},\mathbb{Z})$, then the lattice $v^{\perp}/v$ is isomorphic to $\mathrm{H}^2(\widetilde{Y},\mathbb{Z})$. It follows that there is a G-invariant Hodge isometry

$$f : \mathrm{H}^2(\widetilde{Y},\mathbb{Z}) \longrightarrow \mathrm{H}^2(\widetilde{X},\mathbb{Z}),$$

which by [Sh, Thm. 1, Thm. 2, Lemma 1] (and the fact that $\widetilde{X} \cong \widetilde{Y}$), induces a G-invariant isomorphism $\widetilde{Y} \to \widetilde{X}$. This descends to give an isomorphism $Y \to X$.

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