THE MOTION OF AN AIRCRAFT BOMB IN THE ATTENUATED ATMOSPHERE.

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INTRODUCTION.

This paper is now in two parts. In the first part a new exact solution of the plane flight of a particle is discussed, by which it is possible to take a very long first arc straight from tables of functions defined in the solution. Fairly extensive tables of these functions have been computed (42 sets of each function:- a total of I,900 entries).

Having devised a method of computation, the writer employed another to do this work, and accordingly omits them from this edition, every part of which, to the best of his belief, is original and unaided. A series solution for the lower arcs is also discussed in Part I.

In Part II, solutions in finite terms of the vertical fall of a particle, an approximate solution of the plane flight, and of the effect of changing winds are given.

PART I.

THE FLIGHT OF A PARTICLE IN AN ATMOSPHERE OF VARYING DENSITY, UNDER THE QUADRATIC DRAG LAW.

The first step in solving the flight of a projectile, whether bullet, shell, or aircraft bomb, is taken by treating it as a heavy particle. The results of this treatment are subsequently corrected for the effects of angular motion; for the complete problem is found to be too complex to admit of practical analysis in one stage. The treatment would, of course, be exact provided the projectile remained during its flight with its axis of figure parallel to the direction of motion of its centre of gravity, so that the air reaction would necessarily act exactly in opposition to that direction. By fitting a vane to the tail of a bomb it is possible to make it fall approximately in this manner. We may expect, therefore, that the results of the particle analysis will constitute closer approximations to its motion than to the motion of a shell, which, as the researches of Fowler, Gallop and Lock have shown, may yaw through a large angle near the apex of its trajectory.

The solution of even this simplified problem for a shell is incapable of an analytical solution, because the variation of Drag with projectile speed becomes very complicated for speeds in the neighbourhood, and in excess of, the speed of sound, where it cannot be represented continuously by any simple formula. Aircraft bombs, on the other hand, falling under gravity, seldom acquire speeds much beyond 900 feet per second. A considerable volume of evidence is now available to show that at such speeds the simple quadratic Drag law is obeyed with considerable accuracy, especially for bombs of the modern streamline shape. The bomb problem is thus essentially simpler than its artillery counterpart. It is only with reluctance, therefore, that computers will remain content to apply to the former the elaborate numerical methods which have been devised to deal with the latter. (see Text Book of Anti-Aircraft Gunnery, Vol.1.

published by H.M.Stationery Office).

The only alternative which has hitherto been proposed is to apply Bernoulli's famous solution of the flight of a particle in a uniform atmosphere, in which he assumed a Drag Law of the form Kv^n. Tables of the functions defined therein have been computed by Prof. Karl Pearson, and others, for n = 2. In so doing, however, it is necessary to split the atmosphere into a series of layers, attributing to each a uniform density equal to that actually occurring in the middle. The process of fitting these necessarily short arcs together is found to be nearly, if not quite, as laborious as are the methods used for shell.

The present paper is based on the discovery that the equations of motion admit of an exact solution, provided that the ratio of the air density at any point to that at a point y feet higher, is expressed as $\frac{1}{1-cy}$, c being a constant.

This form is not in exact agreement with either of the mathematical forms which have been proposed to represent the mean meteorological readings. But with an appropriate choice of c the discrepancy is very small over depths of 5000 - 7000 feet.(It is indeed smaller than the daily variation). Furthermore, the ratio of the two factors is a function which varies very slowly with height, is always of the same sign, and, in a drop of any stated altitude, reaches a small maximum value which, together with the order of its effects on the motion, may be readily determined. In computing the fall of a bomb dropped from a very high altitude it may still be necessary to split the trajectory into arcs; but the depth of each may be made very much greater than formerly , and the labour of fitting them together correspondingly reduced. The method, moreover, is particularly suited to the calculation of the trajectories of bombs released from very fast flying aircraft. Modern aircraft are already capable of airspeeds which place the fundamental constant of Bernoulli's solution outside the range considered in the existing tables.

The "Tenuity Function". The name is introduced to distinguish it from the "Tenuity Factor", which is usually taken to mean the ratio of the air density at any point to that at mean sea level. Let ρ_H be the density H feet above datum, ρ the density at a level y feet lower, (i.e. H - y feet above datum). Then the tenuity function R(y) is defined by the identity

$$R(y) = \rho/\rho_H \, (1 - cy) - 1 \qquad \ldots\ldots \quad (1)$$

c being a constant as yet to be defined. R(y) evidently vanishes for y = 0, and it will vanish for a second value y = h provided

$$1 = \frac{\rho_{H-h}}{\rho_H}(1 - ch)$$

$$\text{i.e. if} \qquad c = \frac{1 - \dfrac{\rho_H}{\rho_{H-h}}}{h} \qquad \ldots \quad (2)$$

Assuming R(y) is continuous with a continuous first derivative then $R'(y_1)$ must vanish, where y_1 is some value intermediate between 0 and h, at which point $|R(y)|$ takes on its maximum value for the interval $0 \leqslant y < h$. Let this value be $\epsilon = |R(y_1)|$ Then, for a given form for ρ/ρ_H, ϵ will be a function of h. In the following analysis, h will be taken as the exact depth of the arc considered.

ρ/ρ_H will be written $\dfrac{1 + R(y)}{1 - cy}$ from (1), c being defined by (2).

This definition holds for any continuous function ρ/ρ_H which has a continuous first derivative, and the following method is therefore applicable to any "atmosphere" whatever, whether the density is expressed as a mathematical formula, or merely as a (smoothed) table of values. Particular interest, however, attaches to its meaning when the tenuity factor is identified with the reciprocal of the Standard Density Function, used exclusively (up to 35,000 feet) by the Ballistic Office. We then have

$$\rho/\rho_0 = e^{-a(H-y)}$$

and $\rho/\rho_H = e^{ay}$ and is independant of H. (This formula is said to agree with the mean Meteorological readings more closely than any other simple formula, a having the value $3{\cdot}24665 \times 10^{-5}$ (feet)$^{-1}$)

Here $R(y) = e^{ay}(1-cy) - 1$

with $$c = \frac{1-e^{-ah}}{h}$$

Hence $$R'(y_1) = e^{ay_1}(a - c - acy_1) = 0$$

$$\therefore \quad y_1 = \frac{1}{c} - \frac{1}{a}$$

$$\therefore \quad \epsilon = R(y_1) = \tfrac{c}{a}\, e^{a/c - 1} - 1 \qquad (3)$$

$$= \frac{(1-e^{-ah})}{ah}\, e^{\frac{ah}{1-e^{-ah}} - 1} - 1 \quad \ldots\ (4)$$

A table of ϵ as defined by (4) is appended. Using the exponential tenuity factor $R(y)$ is always positive in the interval, and it will be assumed for the sake of brevity that $R(y)$ will always turn out so, even if actual meteorological readings were taken for ρ/ρ_H.

The equations of motion.

These are, of course, well known.

The horizontal component of force is the resolved part of the Drag only, hence, using the quadratic drag law :-

$$\ddot{x} = -\, {}^{g}/_{Z^2} \cdot {}^{\rho}/_{\rho_H} \cdot \dot{x}^2 \sqrt{1+p^2} \quad \ldots\ldots \quad (5)$$

Where g is the acceleration of gravity, Z the terminal velocity in a uniform atmosphere of density ρ_H , $p = dy/dx$ = tangent of downward inclination of flight path to the horizon.

Again, since the Drag acts in opposition to the direction of motion, the component of acceleration at right angles to this direction is the resolved part of gravity only :-

$$v\dot{\phi} \quad = \quad g\cos\phi \qquad (6)$$

where v is the speed of the particle in the flight path and $\phi = \tan^{-1} p$, the downward inclination of the motion to the horizon.

From (6) we obtain the well known relations :-

$$\sec^2\phi\,\dot{\phi} \quad = \quad \frac{dp}{dt} \quad = \quad \frac{g}{v\cos\phi} \quad = \quad \frac{g}{\dot{x}} \qquad (7)$$

$$\frac{dp}{dt} \quad = \quad \frac{dp}{dy}\cdot\frac{dy}{dx}\cdot\dot{x} \quad = \quad \dot{x}\,p\,\frac{dp}{dy}$$

$$\therefore \quad \frac{g}{\dot{x}^2}\,\frac{dy}{dp} \quad = \quad p \quad \ldots\ldots \qquad (8)$$

Also,
$$\frac{1}{g}\,\frac{dx}{dp} \quad = \quad \frac{1}{g}\cdot\cancel{\frac{dp}{dt}}\,\frac{dt}{dp}\cdot\dot{x} \quad = \quad \frac{\dot{x}^2}{g^2}$$

$$\therefore \quad \frac{dt}{dp} \quad = \quad \sqrt{\frac{1}{g}\,\frac{dx}{dp}} \quad \ldots\ldots \qquad (9)$$

Now
$$\ddot{x} \quad = \quad \frac{d\dot{x}}{dp}\cdot\frac{dp}{dt} \quad = \quad \frac{g}{\dot{x}}\,\frac{d\dot{x}}{dp}$$

Substituting this value ~~for~~ of $\ddot{x}$, namely

$$\frac{g}{\dot{x}}\,\frac{d\dot{x}}{dp} \qquad \textit{in equation (5)}$$

also $\dfrac{1 + R(y)}{1 - cy}$ for ρ/ρ_H and multiplying throughout by $-\dfrac{2(1-cy)}{\dot{x}^2}$, we have:-

$$-\frac{2g(1-cy)}{\dot{x}^3}\,\frac{d\dot{x}}{dp} \quad = \quad \frac{2g}{Z^2}\left(1 + R(y)\right)\sqrt{1+p^2}$$

Adding to this:-

$$-\frac{cg}{\dot{x}^2}\,\frac{dy}{dp} \quad = \quad -cp \qquad (\textit{from } (8))$$

we have:-

$$\frac{d}{dp}\left\{\frac{g(1-cy)}{\dot{x}^2}\right\} \quad = \quad \frac{2g}{Z^2}\left(1 + R(y)\right)\sqrt{1+p^2} - cp \quad \ldots\ldots \qquad (10)$$

Integrating between limits p_o and p :-

$$\frac{g(1-cy)}{\dot{x}^2} - \frac{g}{\dot{x}_o^2} = \frac{2g}{Z^2}\int_{p_o}^{p}(1+R(y))\,d\eta(p) - \tfrac{1}{2}cp^2 + \tfrac{1}{2}cp_o^2 \qquad \ldots\ldots (11)$$

Where $\eta(p)$ is written for

$$\int_0^p \sqrt{1+p^2}\,dp = \tfrac{1}{2}\left\{p\sqrt{1+p^2} + \log\left(p+\sqrt{1+p^2}\right)\right\}$$

Now provided p does not exceed values which render $y > h$, both $R(y)$ and $\eta(p)$ are positive in the range of integration. Hence

$$0 < \int_{p_o}^{p} R(y)\,d\eta(p) < \epsilon\left\{\eta(p) - \eta(p_o)\right\}$$

It follows, therefore, that at any point of the trajectory at which the tangent of the downward inclination is p the true value of the L.H.S. of (11) is intermediate between

$$\frac{2g}{Z^2}\left(\eta(p) - \eta(p_o)\right) - \tfrac{1}{2}c\left(p^2 - p_o^2\right)$$

and

$$\frac{2g}{Z^2}(1+\epsilon)\left(\eta(p) - \eta(p_o)\right) - \tfrac{1}{2}c\left(p^2 - p_o^2\right)$$

Hence it follows that all the co-ordinates of the motion, which are monotonic functions of p, are intermediate between those which may be calculated from two equations of the type

$$\frac{g(1-cy)}{c\dot{x}^2} = \frac{g}{c\dot{x}_o^2} + 2\beta\left\{\eta(p) - \eta(p_o)\right\} - \tfrac{1}{2}p^2 + \tfrac{1}{2}p_o^2 \qquad \ldots\ldots (12)$$

in which β is put equal to $\frac{g}{cZ^2}$ and $\frac{g(1+\epsilon)}{cZ^2}$ respectively. The effect, in other words, of finite values of $R(y)$ is less than that of varying $1/Z^2$ by $100\epsilon\%$. Equation (12) admits of complete solution in terms of integrals similar to those discovered by Bernoulli,(to which,of course, they reduce if c is made to vanish).

If values of these are computed (and they lend themselves readily to numerical integration) we may actually find between what limits the true co-ordinates lie, or, if the value of ϵ is small, we may take as a sufficiently close approximation the values given by the solution of (12) in which β is equated to

$$\frac{g}{cZ^2}\left(1+\tfrac{1}{2}\epsilon\right) \qquad \text{or} \qquad \frac{g\sqrt{1+\epsilon}}{cZ^2}$$

whichever may be more convenient.

Proceeding, then, with the solution of (12) and writing

$$\alpha = \frac{g}{c\dot{x}_0^2} - 2\beta\eta(p_0) + \tfrac{1}{2}p_0^2$$

we have :-

$$g\frac{(1-cy)}{c\dot{x}^2} = \alpha + 2\beta\eta(p) - \tfrac{1}{2}p^2$$

$$\therefore \frac{\frac{cp\dot{x}^2}{g}}{(1-cy)} = \frac{p}{\alpha + 2\beta\eta(p) - \tfrac{1}{2}p^2}$$

$$\text{i.e.} \quad \frac{c\frac{dy}{dp}}{(1-cy)} = \frac{p}{\alpha + 2\beta\eta(p) - \tfrac{1}{2}p^2} \qquad (\text{by } (8))$$

Whence, integrating

$$-\log(1-cy) = \int_{p_0}^{p} \frac{p\,dp}{\alpha + 2\beta\eta(p) - \tfrac{1}{2}p^2} \qquad \ldots \quad (13)$$

For convenience of calculation we prefer to divide throughout by $\log_e 10$, giving

$$-\log_{10}(1-cy) = \int_{p_0}^{p} \frac{p\,dp}{A^2 + 2B\eta(p) - \tfrac{1}{2}p^2\log_e 10} \qquad (14)$$

Where $\quad A^2 = \log_e 10\,\alpha$

$\quad\quad\quad\;\; B = \log_e 10\,\beta$

We may now define a function

$$U_{A,B}(p) = \int_0^p \frac{p\,dp}{A^2 + 2B\eta(p) - \frac{1}{2}\log_e 10\, p^2} \quad \ldots (15)$$

Then

$$1 - cy = 10^{U_{A,B}(p_0) - U_{A,B}(p)} \quad (16)$$

Whence

$$c\frac{dy}{dp} = \log_e 10\, U' \, 10^{U_{AB}(p_0) - U_{AB}(p)}$$

$$\therefore \quad c \log_{10} e \frac{dx}{dp} = 10^{U_{AB}(p_0)} \frac{10^{-U_{AB}(p)}\, U'}{p}$$

and $$\sqrt{cg\log_{10} e}\,\frac{dt}{dp} = 10^{\frac{1}{2}U_{AB}(p_0)} \frac{10^{-\frac{1}{2}U_{AB}(p)}\, U'^{\frac{1}{2}}}{p^{\frac{1}{2}}}$$

Hence if we define the functions:-

$$X_{A,B}(p) = \int_0^p \frac{10^{-U_{A,B}(p)}\, dp}{A^2 + 2B\eta(p) - \frac{1}{2}p^2 \log_e 10} \quad (17)$$

and

$$T_{A,B}(p) = \int_0^p \frac{10^{-\frac{1}{2}U_{A,B}(p)}\, dp}{\sqrt{A^2 + 2B\eta(p) - \frac{1}{2}p^2 \log_e 10}} \quad (18)$$

We shall have :-

$$c \log_{10} e\, x = 10^{U_{A,B}(p_0)} \left\{ X_{AB}(p) - X_{AB}(p_0) \right\} \quad (19)$$

and $$\sqrt{cg\log_{10} e}\; t = 10^{\frac{1}{2}U_{AB}(p_0)} \left\{ T_{AB}(p) - T_{AB}(p_0) \right\} \quad (20)$$

with the aid of (9)

Tables of these functions ~~are appended~~ have been computed, for values of A by units from 5 to 11, and of B by ·5 from 1 to 3·5. (These steps are found to be rather large for easy interpolation, and further sub-tabulation might be undertaken with advantage). We may now employ these tables to exhibit the scope and accuracy of the method. Without concerning ourselves with interpolation to fit a stated problem, let us simply select some entries and find what can be learnt from them. Taking an extreme case (from the point of view of the effect of air resistance), A = 5, B = 3·5, and p = 4 we find

$$U_{5,3\cdot 5}(4) = \cdot 16669$$

$$X_{5,3\cdot 5}(4) = \cdot 08990$$

$$T_{5,3\cdot 5}(4) = \cdot 5873$$

Let us use these results to find, in one stage, the height lost, horizontal distance traversed, and time of fall of some bomb. Since bombs are released from horizontal flight p = 0 for all first arcs, and U (0) = 0. ~~Neglecting~~ Approximating to the effect of R(y), ~~to begin with~~ as explained, we shall have :-

$$A^2 = \frac{g}{c \log_{10} e \; V^2} = 25$$

$$B = \frac{g\sqrt{1+\epsilon}}{c \log_{10} e \; Z^2} = 3\cdot 5$$

Where V is the air-speed of the releasing aircraft. To find c however, we must know h, the height of the arc. Now from (16)

$$-\log_{10}(1-ch) = U = \cdot 16669$$

But by (2A), $1-ch = e^{-ah}$

$$\therefore -\log_{10}(1-ch) = ah \log_{10} e = 1\cdot 41 \times 10^{-5} \times h$$

(using the standard value of a)

hence $\quad h = \dfrac{10^5\, U}{1.41} = \dfrac{16669}{1.41} = $ 11,822 feet.

This is a general method of finding h from U in the first arc. From the tables of c we readily interpolate for this altitude, namely •000026962

$$\therefore\ V^2 = \frac{32.191}{.000026962 \times 25 \times \log_{10} e} \times \left(\frac{15}{22}\right)^2 \quad \therefore\ V = 226{\cdot}1 \text{ m.p.h.}$$

and $\quad Z_H^2 = Z_o^2\, e^{ah}$ $\quad$ by quadratic Law.

$$= Z_o^2\, 10^U$$

$$\therefore\ Z_o^2 = \frac{32.191\,\sqrt{1.0185}}{.000026962 \times 3.5 \times \log_{10} e \times 10^{.16669}}$$

i.e. $\quad Z_o = \cancel{731{\cdot}5}\ 734.8$ feet/sec.

(taking g as 32•191 feet/sec^2)

Next $\quad x = \dfrac{X}{c \log_{10} e}$ $\quad$ by (19)

$$= \frac{.08990}{c \log_{10} e}$$

$$= 7678 \text{ feet.}$$

and $\quad t = \dfrac{T}{\sqrt{g c \log_{10} e}}$

$$= \frac{.5873}{\sqrt{g c \log_{10} e}}$$

$$= 30{\cdot}25 \text{ secs.}$$

~~Now these results are upper limits. We know that the true values of h, x, and t for V = 226.1 m.p.h. and Z = 731.5 feet/sec~~

APPENDIX.A.

TABLE OF C AND ϵ FOR EXPONENTIAL ATMOSPHERE.

h (feet)	c (feet)$^{-1}$	Δ	Δ^2	ϵ	Δ	Δ^2
0	.000032467			.0000		
		- 521			1	
1000	31946		10	.0001		3
		- 511			4	
2000	31435		11	.0005		3
		- 500			7	
3000	30935		11	.0012		2
		- 489			9	
4000	30446		11	.0021		3
		- 478			12	
5000	29968		10	.0033		3
		- 468			15	
6000	29500		10	.0048		2
		- 458			17	
7000	29042		9	.0065		3
		- 449			20	
8000	28593		9	.0085		2
		- 440			22	
9000	28153		10	.0107		3
		- 430			25	
10000	27723		9	.0132		3
		- 421			28	
11000	27302		8	.0160		3
		- 413			31	
12000	26889		9	.0191		3
		- 404			34	
13000	26485		8	.0225		2
		- 396			36	
14000	26089		9	.0261		3
		- 377			39	
15000	.000025702			.0300		

It will be noticed that for a single arc of a depth of even 15,000 feet, the maximum deviation of the density representation is only 3%.

If this is corrected by putting $B = \dfrac{g\sqrt{1+\epsilon}}{cZ_H^2 \log_{10} e}$

the results should have real accuracy of quite as high an order as the Meteorological data; that is to say, that they will be as nearly in agreement with the practical results as our knowledge of the density of the atmosphere permits. (No density function professes to agree with the mean Meteorological readings to more than two significant figures. See Chap. XI Text Book of Anti-aircraft Gunnery).

(1) p	(2) $25 + 3\eta(p) - \frac{1}{2}\log_e 10 p^2$	(3) $\log p - \log (2)$	(4) U' = antilog(3)	(5) $\Delta U'$	(6) $\Delta^2 U'$	(7) $\Delta^3 U'$	(8) $\underline{\Delta U}$	(9) $\underline{U}$	(10) X' = antilog $(-U \log (2))$	(11) $\Delta X'$	(12) $\Delta^2 X'$	(13) $\Delta^3 X'$	4)	(15) $\underline{X}$	(16) T' $(= \sqrt{(10)})$	(17) $\Delta T'$	(18) $\Delta^2 T'$	(19) $\Delta^3 T'$	(20) ΔT	(21) T
0	25		•0					•00000	•040000					•00000	•20000					•00000
				1903			•004822			- 2357			416			- 598			098524	
$\frac{1}{2}$	26•2725	69897 41950 27947	•01903		- 142			•00482	•037643		- 197			•01942	19402		- 72			•09852
				1761		25	013972			- 2554		- 158	199			- 670		- 52	09538	
1	27•2920	0 43604 56396	•03664		- 117			•01879	•035089		- 355			•03762	18732		- 124			•1939
				1644		- 2	022479			- 2909		52	335			- 794		0	09174	
$1\frac{1}{2}$	28•2580	17609 45114 72495	•05308		- 119			•04127	•032180		- 303			•05445	17938		- 124			•2856
				1525		- 8	030404			- 3212		141	298			- 918		20	08745	
2	29•2684	30103 46640 83463	•06833		- 127			•07168	•028968		- 162			•06975	17020		- 104			•3731
				1398		- 7	037715			- 3374		154	645			- 1022		32	08259	
$2\frac{1}{2}$	30•3724	39794 48248 91546	•08231		- 134			•10939	•025594		- 8			•08339	15998		- 72			•4557
				1264		- 5	044373			- 3382		148	948			- 1094		34	07728	
3	31·5963	47712 49964 97748	·09495					·15377	·022212		135									

APPENDIX.B. METHOD OF COMPUTING THE FUNCTIONS.

A specimen schedule for A = 5, B = 1·5 is appended.

The increments ΔV, ΔX, and ΔT are calculated by the formulae

$$\int_a^{a+\omega} f(z)\,dz = \omega\left\{ f(a) + \tfrac{1}{2}\Delta f(a) - \tfrac{1}{12}\Delta^2 f(a) + \tfrac{1}{24}\Delta^3 f(a) - \cdots \right\}$$

for the first increments,

and

$$\int_a^{a+\omega} f(z)\,dz = \omega\left\{ \tfrac{1}{2}\left(f(a) + f(a+\omega)\right) - \tfrac{1}{24}\left(\Delta^2 f(a) + \Delta^2 f(a-\omega)\right) + \cdots \right\}$$

for the remainder.

U^1 means $\frac{dU}{dp}$, etc.

It is as well to carry out the divisions of columns (3) - (4) by logs, since log(2) is required in column (10).

The integrations are checked in groups of six by Weddle's formula.

$$\int_a^{a+6\omega} f(z)\,dz = \frac{3\omega}{10}\left\{ f(a) + 5f(a+\omega) + f(a+2\omega) + 6f(a+3\omega) + f(a+4\omega) + 5f(a+5\omega) + f(a+6\omega) \right\}$$

Which is very rapid in use, and is accurate where sixth differences are negligible.

Thus :-

U(3) should be given by

0	1903	5308
3664	8231	6
6833	10134	31848
9495	5	
19992	50670	
50670		
31848		
1.02510		
.3		
2).307530		
.153765		

Which is precisely the sum of ΔU's.

Again U(5) - U(2) should agree by the same method, (and does).

APPENDIX.C. CALCULATION OF THE SECOND AND SUBSEQUENT ARCS.

If the tables are to be used, new values of A and B must be **takem.** These are readily calculated as follows.

Let A_n, B_n and h_n be the values of the constants and the height lost in the n^{th} arc, $\dot{x}_n$, p_n and Z_n the values of the horizontal component of velocity, gradient, and T.V., at the boundary between the n^{th} and $(n+1)$st. arcs.

Then $$A_{n+1}^2 = \frac{g\log_e 10}{c_{n+1}\dot{x}_n^2} - \frac{2g\log_e 10\,\eta(p_n)}{c_{n+1}Z_n^2} + \tfrac{1}{2}\log_e 10\, p_n^2$$

and $$A_n^2 = \frac{g\log_e 10}{c_n\dot{x}_{n-1}^2} - \frac{2g\log_e 10\,\eta(p_{n-1})}{c_n Z_{n-1}^2} + \tfrac{1}{2}\log_e 10\, p_{n-1}^2$$

Also $$\frac{g\log_e 10\,(1-c_n h_n)}{c_n\dot{x}_n^2} = A_n^2 + \frac{2g\log_e 10\,\eta(p_n)}{c_n Z_{n-1}^2} - \tfrac{1}{2}\log_e 10\, p_n^2 \quad \ldots\ldots (21)$$

But $(1-c h_n) = e^{-a h_n}$

and $Z_n^2 = Z_{n-1}^2 e^{-a h_n}$

So provided $c_{n+1} = c_n$ (i.e. $h_{n+1} = h_n$, or else small)

$$A_{n+1}^2 = \frac{A_n^2 - \frac{1}{2}\log_e 10\, c h_n p_n^2}{1 - c h_n}$$

and

$$B_{n+1} = \frac{B_n}{1 - c h_n}$$

Notice if $A_{n+1} = A_n$, then

$$A_n^2 = \tfrac{1}{2}\log_e 10\, p_n^2$$

$$\text{i.e.} \quad \tfrac{1}{2}(p_n^2 - p_{n-1}^2) = \frac{g}{c\dot{x}_{n-1}^2} - \frac{\partial_\delta \eta(p_{n-1})}{c Z_{n-1}^2}$$

This relation may be taken advantage of, provided Z is sufficiently ~~great~~ small (so that the height lost may not be too great) to save interpolation for the A's as well as the B's. Thus if it were applied to the case already considered, namely

A = 5 B = 3·5

$$p_2 = \frac{\sqrt{2} \times 5}{\sqrt{\log_e 10}} = 4{\cdot}6599$$

So that the height lost would not be much greater than that stated.

(In order to save interpolation for p at this stage, it would be useful, in recalculating these functions, to take for A integer multiples of $\sqrt{\frac{\log_e 10}{2}}\;\omega$, where ω is the step in p. This involves no additional labour, since A is additive in the first column.)

The step from 1st arc to 2nd arc might then be accomplished with but one interpolation, ~~for~~ namely for B).

Such a method is useful when we wish to arrive as rapidly as possible to the final values of the co-ordinates after a considerable fall. The normal requirements of a bomb aimer demand that the elements of the trajectory should be tabulated at constant (short) intervals of height. If this is to be accomplished without elaborate interpolation, it is essential that a method should be developed in which y assumes the role of independant variable. This is clearly only possible with the aid of a :-

SERIES SOLUTION FOR THE LOWER ARCS WITH y AS INDEPENDANT VARIABLE.

This is impossible for the first arc, because the differential co-efficients (with regard to y) of all the elements become infinite (beyond a certain order) at the vertex. Fortunately, however, an intimate knowledge of the behaviour of a bomb during its first few thousand feet of fall is rarely required by the aimer. The effects of air resistance during this part of the flight, although exercising a vital influence on the subsequent motion, are in themselves as yet slight; so that an aimer bombing a target only such a short distance below him will be content with a rough approximation ~~of~~ to their values.
We shall accordingly suppose that the first arc (a few thousand feet in depth) is calculated directly from the Tables. ~~To save backward interpolation for p, we may, if we so desire, complete the exact integer multiple of the constant depth in the second ar arc.~~ The subsequent arcs, each of the constant depth, may then be calculated as follows.

Equation (5) may be rewritten as

$$\frac{\dfrac{d^2p^2}{dy^2}}{\dfrac{dp^2}{dy}} = \frac{\alpha g}{Z^2}\,\frac{\operatorname{cosec}\phi}{1-cy}$$

Differentiating logarithmically we have

$$\frac{\frac{d^3p^2}{dy^3}}{\frac{d^2p^2}{dy^2}} - \frac{\frac{d^2p^2}{dy^2}}{\frac{dp^2}{dy}} = -\frac{g\cot^2\phi\cos^2\phi}{\dot{x}^2} + \frac{c}{1-cy}$$

since $\frac{d\phi}{dy} = \frac{g\cos^2\phi\cot\phi}{\dot{x}^2} = \frac{g\cot\phi}{v^2}$

Hence

$$\frac{dp^2}{dy} = \frac{2g}{\dot{x}^2} \qquad \text{(Formula ( ))}$$

$$\frac{d^2p^2}{dy^2} = \frac{4g^2}{\dot{x}^2Z^2}\frac{\operatorname{cosec}\phi}{1-cy}$$

$$\frac{d^3p^2}{dy^3} = \frac{4g^2}{\dot{x}^2Z^2}\frac{\operatorname{cosec}\phi}{(1-cy)}\left\{\frac{2g\operatorname{cosec}\phi}{Z^2(1-cy)} - \frac{g\cot^2\phi\cos^2\phi}{\dot{x}^2} + \frac{c}{1-cy}\right\}$$

$\frac{d^4p^2}{dy^4}$ will evidently be very complicated, so we shall do without it.

Putting $\phi = \phi_0$, $\dot{x} = \dot{x}_0$, $y = 0$ we have the Mc Laurin series.

$$p^2 = p_0^2 + \frac{2gy}{\dot{x}_0^2} + \frac{2g^2\operatorname{cosec}\phi_0}{\dot{x}_0^2Z^2}y^2 + \frac{2g^2\operatorname{cosec}\phi_0}{3\dot{x}_0^2Z^2}\left\{\frac{2g\operatorname{cosec}\phi_0}{Z^2} - \frac{g\cot^2\phi_0\cos^2\phi_0}{\dot{x}_0^2} + c\right\}y^3$$

If we assume $Z^2(1-by) = Z'^2$ and substitute Z' for Z the new term in y^3 will be

$$-\frac{2g^2\operatorname{cosec}\phi_0\, b\, y^3}{\dot{x}_0^2Z'^2}$$

And (neglecting terms involving the fourth and higher powers of y) this will cancel out with the term involving c explicitly provided :-

$$b = 1/3 \; c$$

Hence for arcs short enough to permit the termination of the Mc Laurin Series at the fourth term, a uniform atmosphere may be assumed the density of which is equal to that at $\frac{1}{3}$ of the depth of the arc measured from the top; i.e. 2/3rds of the arc measured from the bottom. This agrees with H.E.Wimperis's rule. For practical computation, the series may be re-written putting y = h (the constant depth step) and using previous notation.

$$p_{n+1}^2 - p_n^2 = \frac{2gh}{\dot{x}_n^2}\left[1 + \frac{g \operatorname{cosec}\phi_n h}{N_n \dot{x}_n^2}\left\{1 + \frac{g \operatorname{cosec}\phi_n h}{3\dot{x}_n^2}\left(\frac{2}{N_n} - \cot\phi_n \cos^3\phi_n\right)\right\}\right]$$

$$\text{Where } N_n = \frac{Z_1^2}{\dot{x}_n^2} = \frac{Z_n^2(1 - \frac{1}{3}ch)}{\dot{x}_n^2}$$

The depth of h must be chosen that the third term of this series affects only the last significant digit required. It will evidently converge the more rapidly as p_n increases, so that the greater the depth of the first arc (taken from the tables) the easier will be the subsequent computation.
The new value of $\dot{x}_{n+1}$ may be calculated by formula (21), and so the values of p and $\dot{x}$ obtained at constant intervals h. The remaining co-ordinates may then be found by forming the quantities

$$\frac{1}{p_n} \quad \text{and} \quad \frac{1}{p_n \dot{x}_n}$$

at every stage. These may be differenced and integrated (as in Appendix B) with regard to y, to give x and t.

PART II.

THE VERTICAL FALL OF A PARTICLE IN THE ATTENUATED ATMOSPHERE.

Much useful information may be derived from the study of the purely vertical fall of a bomb (treated as a particle) released without a horizontal component of velocity, as from an airship with the engines shut off. It is possible to exhibit solutions of this problem in finite terms, involving only functions already tabulated. In the first place it is to be noticed that the speed of fall after a long drop is practically independent of the initial horizontal velocity. Now it is known that the quadratic drag law, upon which most methods of solution are based, ceases to be obeyed after speeds in the neighbourhood of 900 feet per second are attained. A computer, therefore, will desire to know in advance whether this critical speed will be reached in the trajectory under consideration. An ~~immeid~~ immediate answer is provided by the following.

The equation of motion determining a purely vertical fall, (from rest) is:-

$$\ddot{y} = g - \frac{g}{Z_H^2}\, {}^{\rho}/_{\rho_H}\, \dot{y}^2 \quad \ldots\ldots (1)$$

Putting ${}^{\rho}/_{\rho_H} = e^{ay}$ this takes the form:-

$$\frac{d}{dy}(\dot{y}^2) + \frac{2g}{Z_H^2} e^{ay} \dot{y}^2 = 2g \quad \ldots\ldots (2)$$

A linear equation of the first order whose solution is seen to be:-

$$\dot{y}^2 = \frac{2g}{a} e^{-\frac{2g}{aZ_H^2} e^{ay}} \left\{ Ei\left(\frac{2g}{aZ_H^2} e^{ay}\right) - Ei\left(\frac{2g}{aZ_H^2}\right) \right\} \quad \ldots\ldots (3)$$

Remembering that $\dot{y}_0 = 0$

Where $Ei\,(z) = \int_{-\infty}^{z} \frac{e^t\,dt}{t}$.

The Exponential Integral, which is tabulated already.

Now, putting $y = H$, the height of attack, we notice that

$$Z_H^2 e^{-aH} = Z_0^2$$

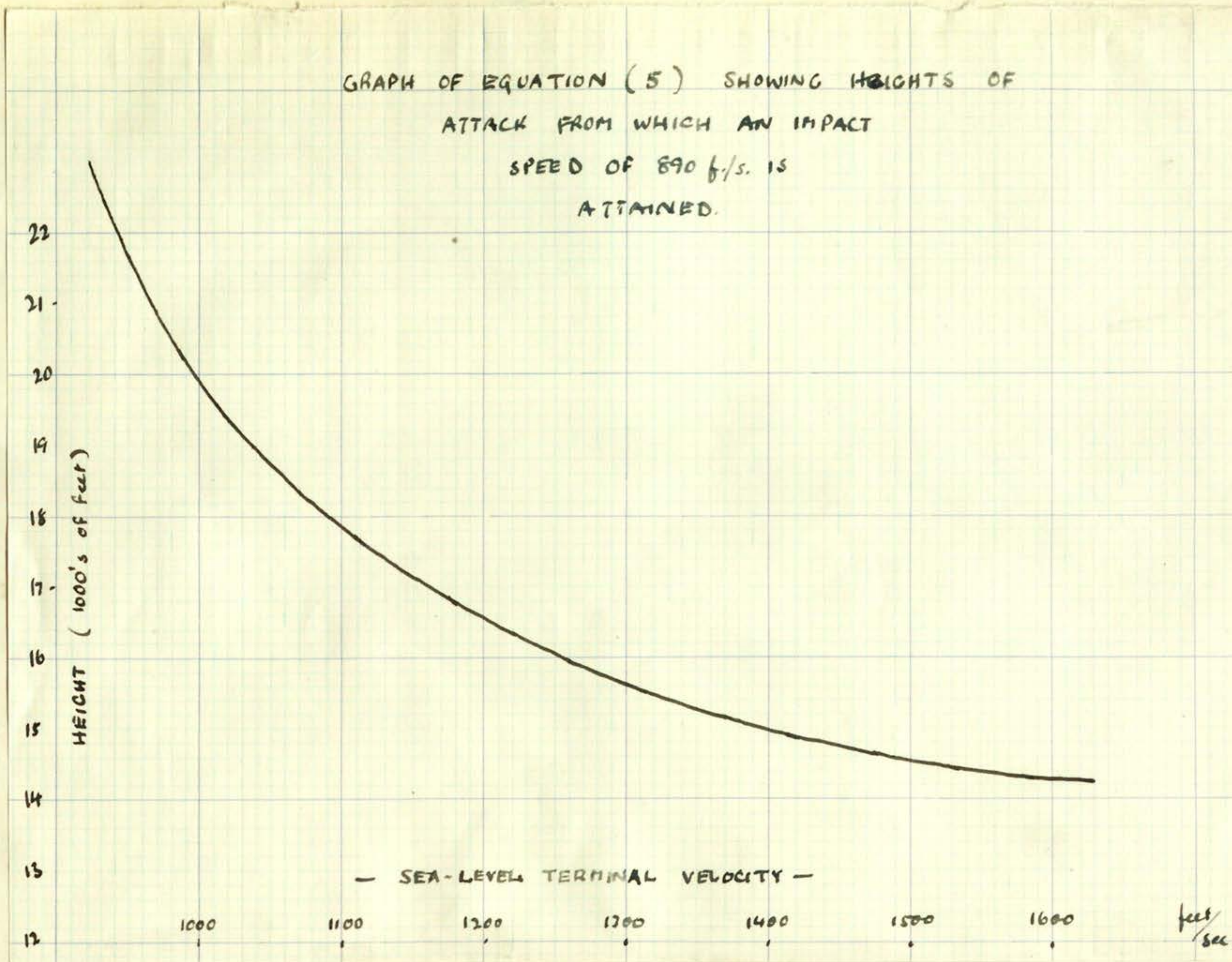
The terminal velocity at sea-level density. Hence:-

$$\dot{y}_H^2 = \frac{2g}{a} e^{-\frac{2g}{aZ_0^2}} \left\{ Ei\left(\frac{2g}{aZ_0^2}\right) - Ei\left(\frac{2g}{aZ_H^2}\right) \right\}$$

or, solving for Z_H

$$Ei\left(\frac{2g}{aZ_H^2}\right) = Ei\left(\frac{2g}{aZ_0^2}\right) - \frac{a\dot{y}_H^2}{2g} e^{\frac{2g}{aZ_0^2}} \quad \ldots\ldots (4)$$

Giving a and g their values here, and putting $\dot{y}$ = 890 feet/sec this reduces to:-


GRAPH OF EQUATION (5) SHOWING HEIGHTS OF
ATTACK FROM WHICH AN IMPACT
SPEED OF 890 f./s. IS
ATTAINED.
HEIGHT (1000's of feet)
22
21
20
19
18
17
16
15
14
13
12
— SEA-LEVEL TERMINAL VELOCITY —
1000
1100
1200
1300
1400
1500
1600
feet/sec.

$$Ei\left\{\frac{1983000}{Z_H^2}\right\} = Ei\left\{\frac{1983000}{Z_0^2}\right\} - .4\times 10^{\frac{861200}{Z_0^2}} \quad \ldots\ldots (5)$$

From which Z_H, therefore H can be calculated. This equation determines the maximum altitude from which the impact speed will not exceed 890 feet/sec. A graph of H against Z_H is appended.

Now, clearly, the maximum speed which the particle can attain will occur at the point at which it has reached its local Terminal Velocity; for, thereafter, it will be falling into denser air, and the drag will become greater than its weight, so that it must decelerate. (Or see equation (2). Put $\frac{d}{dy}(\dot{y}^2) = 0$ for maximum, and $\dot{y}^2 = Z_H^2 \, \rho_H/\rho$). If, then, $Z_0 > 890$, the condition laid down by (5) is sufficient, for the bomb has not yet started to decelerate. If $Z_0 < 890$ ft/sec then, dropped from the altitude given by (5), a greater speed must have been reached higher up. This observation is, however, of little practical importance, since the sea-level terminal velocities of most bombs exceed 900 feet/sec. (This assertion bears, of course, no relation to actual maximum speeds, since the quadratic drag law ceases to be obeyed. Terminal velocities in excess of 900 feet/sec must be regarded only as labels which place bombs in different categories).

Equation (4) may also be served up as an answer to the vexed question "when does a bomb attain its terminal velocity?" (This question, it must be again be stated, has an answer only when the T.V. is less than the critical speed). Putting $\dot{y} = Z_0$ we find:-

$$Ei\left(\frac{2g}{aZ_H^2}\right) = Ei\left(\frac{2g}{aZ_0^2}\right) - \frac{aZ_0^2}{2g}\, e^{\frac{2g}{aZ_0^2}} \quad \ldots (6)$$

Equation (3) does not admit of further integration in terms of known functions. In order to exhibit Formulae for times of fall, we have recourse to the form:-

$$\rho/\rho_H = \frac{1+R(y)}{1-cy}$$

Then we have

$$\frac{d}{dy}(\dot{y}^2) + \frac{2g}{Z_H^2}\frac{(1+R(y))}{1-cy}\,\dot{y}^2 = 2g \quad \ldots (7)$$

Now

$$\int_0^y \frac{(1+R(y))\,dy}{1-cy} \qquad (y < H)$$

is again intermediate between:-

$$-\frac{1}{c}\log(1-cy) \quad \text{and} \quad -\frac{(1+\epsilon)}{c}\log(1-cy)$$

So once more we find that the true value of the time of vertical fall will be intermediate between those calculable from solutions of (7) in which R(y) is omitted, and Z_H is given, alternately, its true value, and the value $Z_H/\sqrt{1+\epsilon}$.

The solution of (7), neglecting R(y), is:-

$$\dot{y}^2 = \frac{2g/c}{\left(\frac{2g}{cZ_H^2} - 1\right)}(1-cy)\left\{1 - (1-cy)^{\frac{2g}{cZ_H^2}-1}\right\} \quad \ldots (8)$$

provided $\dot{y}_o = 0$.

We are now in a position to test the accuracy of the author's method of allowing for the Tenuity function R (y). Equation (4) is exact.

Worked Example.

In (4) put

g	=	32.I9I	feet/(sec)^2
a	=	3.2667 $\times 10^{-5}$	(feet)$^{-1}$
Z_o	=	I000	feet/sec
H	=	I0,000	feet.

$$\dot{y}_H^2 = \frac{2g}{a} e^{-\frac{2g}{aZ_o^2}} \left\{ Ei\left(\frac{2g}{aZ_o^2}\right) - Ei\left(\frac{2g}{aZ_H^2}\right) \right\}$$

Log $2g$ ..	I.80876
Log aZ_o^2 ..	I.5II44
	.29732
	.I4I
	.I5632

$\frac{2g}{aZ_o^2}$ = I.9830

$\frac{2g}{aZ_H^2}$ = I.4332

From Dales Tables:-

u	Ei (u)	Δ	Δ^2	Δ^3
I.9	4.5937	36I5	I85	I9
2.0	4.9542	3790	204	
2.I	5.3332	3994		
2.2	5.7326			
I.4	3.0072	294I	99	I8
I.5	3.30I3	3040	II7	
I.6	3.6053	3I57		
I.7	3.92I0			

Ei (I.983) = 4.5937 + .83 X 36I5 - .07I X I85 + .028 X I9

= 4.8925

Ei (I.4332) = 3.0072 + .332 X 294I - .III X 99 + .062 X I8

= 3.I039. Difference=I.7886.

log $\frac{2g}{aZ_o^2}$ ..	.29732
log log e ..	$\bar{I}$.63778
	$\bar{I}$.93510
	(.86II9)
log $^{2g}/_a$	6.29732
	5.436I3
log I.7886	.2525I
	2/5.68864
	2.84432 (698.75).

Hence, according to (4) $\dot{y}_H$ = 698.8 feet/sec.

Now in (8) we must put:-

$c = 2.7723 \times 10^{-5}$ (its value at 10,000 feet)

$$Z_H^2 = \frac{10^6 \times 10^{.141}}{\sqrt{1+\epsilon}}$$

$$= \frac{10^{6.141}}{\sqrt{1.0132}} \quad \text{(from the Table of } \epsilon \text{ in Part I.)}$$

and $(1 - cy) = e^{-aH} = 10^{-.141}$

log 2g . .	1.80876	log $^{2g}/_c$ ...	6.36592
log c . .	$\overline{5}.44284$	log .6895 ...	$\overline{1}.83853$
	6.36592		6.52739
	6.141	log e^{aH} ...	$.141$
	$.22492$		6.38639
$\frac{1}{2}$ log 1.0132	$.00285$		
	$.22777$ (1.6895)		(24.344×10^5)

log .141	$\overline{1}.14922$	
log $\frac{2g\sqrt{1+\epsilon}}{cZ_H^2}$	$.22777$	
	$\overline{1}.37699$ (n)	
	(- .23823	$\overline{1}.76177$
		6.52739
		6.28916
		(19.461×10^5).

Thus $\dot{y}_H^2 = (24.344 - 19.461) \times 10^5$.

$= 488300$

$\dot{y}_H = 698.8$ feet/sec.

The agreement is quite remarkable, and should establish confidence in the reasoning which led us to expect that the substitution of $\frac{\sqrt{1+\epsilon}}{Z_H^2(1-cy)}$ for $\frac{e^{ay}}{Z_H^2}$ would give accuracy as great as is required.

To integrate (8) further we put:-

$$\frac{2g}{cZ_H^2} = \frac{N+2}{N} \qquad \text{.............(9)}$$

and

$$1 - cy = \cos^N\theta \qquad \text{...........(10).}$$

Then $$\dot{y} = \frac{N}{c}\cos^{N-1}\theta\,\sin\theta\,\dot{\theta} \qquad \text{...........(IOA)}$$

and $$\Big[_0^H y = \Big[_0^{\cos^{-1}(1-cH)^{\frac{1}{N}}} \theta = \Big[_c^{\cos^{-1}e^{-\frac{aH}{N}}} \theta.$$

by the definition of c. (See part I).

So (8) becomes:-

$$\frac{N}{c}\cos^{N-2}\theta\,\dot{\theta}^2 = g$$

or

$$t = \sqrt{\frac{N}{gc}} \int_0^{\cos^{-1}e^{-\frac{aH}{N}}} \cos^{N/2-1}\theta\, d\theta.$$

.............(II).

Integrals of this type are of frequent occurrence in Ballistics, and have, in fact, been made the subject of a special notation, viz:-

$$\xi_m(\theta) = \int_0^\theta \sec^{m+1}\theta\, d\theta$$

$$\eta_m(\theta) = \int_0^p (1+p^2)^{\frac{m-1}{2}} dp$$

$$(p = \tan\theta)$$

It is to be regretted that full tables of neither function, for values of m between 0 and 2 appear to have been computed. In "Ballistic Tables" issued by H.M.Stationary Office, 5 figure tables

of $\xi_{1.5}(\theta)$, $\xi_{1.6}(\theta)$, $\xi_{1.67}(\theta)$, $\xi_{1.8}(\theta)$, $\xi_3(\theta)$ & $\xi_{6.45}(\theta)$

are, however, given.

(II) may be evaluated in terms of ordinary tabulated functions for all integer values of N, and certain fractional ones. Much information can be derived from these formulae since the values of Z_H given by (9)

$$Z_H = \sqrt{\frac{2gN}{(N+2)c}} \qquad \text{............(I2).}$$

for small positive and negative values of N, together with the value N = ∞ cover the range of practical terminal velocities with sufficient density for ordinary purposes.

Before exhibiting the series of formulae afforded, however, we shall transform (II) in various ways.

For negative values of N, put $N = -M$ and $\cos\theta = \sec\psi$ giving:-

$$t_M = \sqrt{\frac{M}{gc}} \int_0^{\cos^{-1} e^{-\frac{aH}{M}}} \cos^{\frac{M}{2}}\psi \, d\psi \qquad \text{.................(13)}$$

with $Z_H = \sqrt{\dfrac{2gM}{(M-2)\,c}}$ ....(14)

Again we have:-

$$\left(1-\frac{N}{2}\right)\int_0^{\alpha} \cos^{\frac{N}{2}-1}\theta \, d\theta$$

$$= \left(2-\frac{N}{2}\right)\int_0^{\alpha} \sec^{3-\frac{N}{2}}\theta \, d\theta - \cos^{\frac{N}{2}-1}\alpha \tan\alpha \qquad \text{.............(15)}$$

$$\therefore \quad t_N = \sqrt{\frac{N}{gc}} \cdot \frac{1}{(2-N)} \left\{ (4-N)\, \xi_{2-\frac{N}{2}}(\alpha) - 2\sec^{1-\frac{N}{2}}\alpha \tan\alpha \right\}$$

$$\text{with} \quad \cos\alpha = e^{-\frac{aH}{N}} \qquad \text{.................(16)}$$

Consider now the range covered by the tables of referred to, for m = 1.5, 1.6, 1.67, and 1.8. Using formula (15) as it stands:-

$$m = 2 - \frac{N}{2} \quad ; \quad N = 4 - 2m$$

Hence the corresponding values of N are 1, .8, .66, and .4. Starting with the lowest, we have:-

$N = \cdot 4, \quad Z_H = \sqrt{\dfrac{g}{3c}}$ and

$$t = \sqrt{\frac{\cdot 4}{gc}} \left\{ \tfrac{9}{4}\, \xi_{1\cdot 8}(\alpha) - \tfrac{5}{4} \sec^{\cdot 8}\alpha \tan\alpha \right\}$$

$$\text{with} \quad \cos\alpha = e^{-\frac{5aH}{2}} \qquad \text{.................(17)}$$

$N = \cdot 66, \quad Z_H = \sqrt{\dfrac{66g}{133c}}$, and

$$t = \sqrt{\frac{\cdot 66}{gc}} \left\{ \frac{167}{67}\, \xi_{1\cdot 67}(\alpha) - \frac{100}{67} \sec^{\cdot 67}\alpha \tan\alpha \right\} \qquad \text{.................(18).}$$

$$\text{with} \quad \cos\alpha = e^{-\frac{50aH}{33}}$$

$$N = \cdot 8\,, \quad z_H = \sqrt{\frac{4g}{7c}}$$

$$t = \frac{1}{3}\sqrt{\frac{\cdot 8}{gc}}\left\{ 8\, \xi_{1.6}(\alpha) - 5 \sec^{\cdot 6}\alpha \tan\alpha \right\}$$

$$\text{with} \quad \cos\alpha = e^{-\frac{5aH}{4}} \quad \text{................(I9)}$$

$N = 1\,, \quad z_H = \sqrt{\frac{2g}{3c}}$, and

the integral is more easily evaluated in terms of Elliptic Functions:-

$$t = \sqrt{\frac{1}{gc}} \int_0^{\cos^{-1} e^{-aH}} \frac{d\theta}{\sqrt{\cos\theta}}$$

Putting $\cos\theta = \cos^2\psi$ this becomes:-

$$t = \sqrt{\frac{2}{gc}}\, F(45^\circ, \alpha)$$

$$\text{with} \quad \cos\alpha = e^{-\frac{1}{2}aH} \quad \text{................(20)}$$

and F is Legendre's Elliptic function.

$$N = 4/3\,, \quad z_H = \sqrt{\frac{4g}{5c}}$$

$$\text{and} \quad t = \sqrt{\frac{4}{3gc}} \int_0^{\cos^{-1} e^{-\frac{3aH}{4}}} \frac{d\theta}{\cos^{1/3}\theta}$$

By the successive substitutions, convenient for calculation:-

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$$

$$u = \frac{\sec^{1/3}\theta \cos\psi}{2\sqrt{2}}$$

$$\cos\phi = \frac{\cos\frac{\pi}{12} - u}{\sin\frac{\pi}{12} + u}$$

the integral reduces to:-

$$t = 3^{3/4}\sqrt{\frac{1}{gc}} \int_0^{\phi_H} \frac{d\phi}{\left(1 - \sin^2\frac{\pi}{12}\sin^2\phi\right)^{\frac{1}{2}}}$$

$$= 3^{3/4}\sqrt{\frac{1}{gc}}\, F\left\{15^\circ, \phi_H\right\}$$

$$\text{where} \quad \cos\alpha = e^{-\frac{3aH}{4}}, \quad \log\tan\left(\frac{\pi}{4} + \frac{1}{2}\psi_H\right) = \frac{1}{3}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

$$u_H = \frac{e^{\frac{aH}{4}} \cos \psi_H}{2\sqrt{2}}$$

$$\cos \phi_H = \frac{\cos 15^\circ - u_H}{\sin 15^\circ + u_H} \quad \text{................(21)}$$

N = 2,

$$Z_H = \sqrt{g/c}$$

$$t = \sqrt{\frac{2}{gc}} \cos^{-1} e^{-\frac{aH}{2}} \quad \text{................(22)}$$

N = 3 ,

$$Z_H = \sqrt{\frac{6g}{5c}}$$

$$t = \sqrt{\frac{3}{gc}} \int_0^{\cos^{-1} e^{-\frac{aH}{3}}} \cos^{\frac{1}{2}}\theta \, d\theta.$$

Putting $\cos\theta = \cos^2\psi$ as before.

$$t = \sqrt{\frac{6}{gc}} \left\{ 2E(45^\circ, \alpha) - F(45^\circ, \alpha) \right\}$$

with

$$\cos\alpha = e^{-\frac{aH}{6}} \quad \text{................(23).}$$

N = 4, $Z_H = \sqrt{\frac{4g}{3c}}$ and

$$t = 2\sqrt{\left(\frac{1 - e^{-\frac{1}{2}aH}}{gc}\right)} \quad \text{................(24)}$$

N = 6, $Z_H = \sqrt{\frac{3g}{2c}}$

$$t = \sqrt{\frac{3}{2gc}} \left\{ \alpha + \tfrac{1}{2} \sin 2\alpha \right\}$$

$$\cos\alpha = e^{-\frac{aH}{6}} \quad \text{................(25)}$$

N = 8, $Z_H = \sqrt{\frac{8g}{5c}}$

$$t = \sqrt{\frac{8}{gc}} \left(\sin\alpha - \tfrac{1}{3} \sin^3\alpha \right)$$

$$\cos\alpha = e^{-\frac{aH}{8}} \quad \text{................(26)}$$

These formulae may evidently be produced indefinitely. The step in Z_H ,however, is now becoming smaller, so we proceed to the value $N = \infty$, $Z_H = \sqrt{\frac{2g}{c}}$

The previous formulae become indeterminate but for $\dot{y}$ we have:-

$$\dot{y}^2 = -\frac{2g}{c}(1-cy)\log(1-cy)$$

Putting $(1 - cy) = e^{-2u^2}$

we have:-

$$t = \frac{2}{\sqrt{gc}} \int_0^{u_H} e^{-u^2} du.$$

But $e^{-2u_H^2} = (1-cH) = e^{-aH}$

$$\therefore \quad u_H = \sqrt{\frac{aH}{2}}$$

So finally:-

$N = \infty \qquad Z = \sqrt{\frac{2g}{c}}$

$$t = \sqrt{\frac{\pi}{gc}} \; Erf\left(\sqrt{\frac{aH}{2}}\right) \qquad \text{.............(27)}$$

Where Erf (x) $= \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-u^2} du$

Passing now to negative values of N (positive values of M) we have, if $M = 2 \quad Z_H = \infty$ and

$$t = \sqrt{\frac{2}{gc}} \int_0^{\cos^{-1} e^{-\frac{aH}{2}}} \cos\psi \, d\psi$$

$$= \sqrt{\frac{2}{gc}} \sqrt{1-e^{-aH}}$$

But $e^{-aH} = 1-cH$

$$\therefore \quad t = \sqrt{\frac{2}{gc}} \sqrt{cH} = \sqrt{\frac{2H}{g}} \qquad \text{.............(28)}$$

as for an unresisted bomb, as we should expect.

$M = 4 \qquad Z_H = 2\sqrt{g/c}$ and

$$t = \frac{1}{\sqrt{gc}} \left\{ \alpha + \tfrac{1}{2} \sin 2\alpha \right\}$$

$$\text{with } \cos\alpha = e^{-\frac{aH}{4}} \qquad \text{..............(29)}$$

~~(B5.245)~~).

$M = 6 \qquad Z_H = \sqrt{2g/c}$

$$t = \sqrt{\frac{6}{gc}} \left\{ \sin\alpha - \tfrac{1}{3} \sin^3\alpha \right\}$$

$$\cos\alpha = e^{-\frac{aH}{6}} \qquad \text{...............(30)}$$

The above formulae give sufficient points on the "time-lag" - terminal velocity curves at various heights to make graphical interpolation easy. (or the use of Legrange's Interpolation formula).

In applying these formulae, C must be given its value at the height of attack selected, and the results associated with $Z_H^2\sqrt{1+\epsilon}$ instead of with Z_H^2 itself where ϵ is also given its appropriate value. (See tables in Part I).

These times of fall are close approximations to times in the plane trajectories with moderate air speeds, and make a valuable check for step-by-step methods. The vertical times of fall are, of course, slightly less than the plane trajectory times.

Another useful application of this analysis is to be found in the following Section.

AN APPROXIMATE SOLUTION OF THE PLANE FLIGHT OF A PARTICLE IN THE ATTENUATED ATMOSPHERE IN TERMS OF FUNCTIONS ALREADY TABULATED.

The approximation introduced is identical with that proposed by H.E.Wimperis in seeking a similar solution in an atmosphere of constant density. We assume that the vertical component of velocity in the plane trajectory will be the same as in a vertical fall from the same altitude. When it is considered that the motion is practically vertical for the majority of the flight path- in the important part where the speed is greatest- the relatively small effects of air resistance which may be calculable from such an approximation should be reasonably close to those actually encountered. These effects will, of course, be slightly less than the true ones, and the approximation may thus be used as a check on step-by-step methods. More important, however, is the application of the approximation to perturbations in the motion caused by such factors as changing winds. For,clearly, in calculating such small displacements, ~~one~~ the accuracy of the approximation will be sufficient.

In all resisted motion in which the Drag acts in opposition to the direction of motion we have:-

$$\frac{d}{dy}(\log p) = \frac{g}{\dot{y}^2} \qquad \text{.................(3I)}$$

where $p = \frac{dy}{dx}$

$$\therefore \frac{d}{d\theta}(\log p) = \frac{g}{\dot{\theta}^2}\frac{d\theta}{dy}$$

$$= \frac{\sec^2\theta}{\tan\theta} \qquad \text{.................(32)}$$

by (9) (IO) and (IOA).

$$\therefore p = k\tan\theta \qquad \text{.................(33)}$$

To evaluate K, note $\dot{x} = \dot{y}/p$

$$\therefore \dot{x}^2 = \frac{\dot{\theta}^2}{p^2}\left(\frac{dy}{d\theta}\right)^2 = \frac{Ng}{ck^2}\cos^{N+2}\theta$$

by (33) and (IOA).

Now when $\theta = 0$, $\underbrace{\dot{x} = V}$ the airspeed of the releasing craft.

$$\therefore \quad V^2 = \frac{Ng}{ck^2}$$

So

$$\dot{x}^2 = V^2 \cos^{N+2}\theta \quad \ldots\ldots\ldots\ldots\ldots(34)$$

and

$$p = \frac{1}{V}\sqrt{\frac{Ng}{c}}\tan\theta = \frac{dy}{dx}$$

$$\frac{dx}{d\theta} = \frac{dx}{dy}\cdot\frac{dy}{d\theta}$$

$$= V\sqrt{\frac{N}{cg}}\cos^N\theta$$

Whence

$$x/V = \sqrt{\frac{N}{gc}}\int_0^{\cos^{-1}e^{-\frac{aH}{N}}} \cos^N\theta\, d\theta \ldots\ldots\ldots\ldots\ldots(35)$$

This is transformable as before.
For N = -M we have:-

$$x_M = V\sqrt{\frac{M}{gc}}\int_0^{\cos^{-1}e^{-\frac{aH}{M}}} \cos^{M-1}\theta\, d\theta \quad \ldots\ldots\ldots\ldots\ldots(36)$$

The formula of reduction need not be used. A sufficient number of points are obtainable by half odd integer and integer values of N and M.

$N = \frac{1}{2}$ $\quad Z_H = \sqrt{\frac{2g}{5c}}$

$$x/V = \frac{1}{\sqrt{2gc}}\int_0^{\cos^{-1}e^{-2aH}} \cos^{\frac{1}{2}}\theta\, d\theta$$

$$= \sqrt{\frac{1}{gc}}\left\{2E(45^\circ, \alpha) - F(45^\circ, \alpha)\right\}$$

$$\text{with } \cos\alpha = e^{-aH} \quad \ldots\ldots\ldots\ldots\ldots(37).$$

$N = I$

$$Z_H = \sqrt{\frac{2g}{3c}}$$

$$x/V = \frac{1}{\sqrt{gc}}\int_0^{\cos^{-1}e^{-aH}} \cos\theta\, d\theta.$$

$$= \frac{1}{\sqrt{gc}}\sqrt{1-e^{-2aH}} \quad \ldots\ldots\ldots\ldots\ldots(38).$$

$N = \frac{3}{2}$ $\qquad Z_H = \sqrt{\frac{6g}{7c}}$

$$\frac{x}{V} = \sqrt{\frac{2}{3gc}} \int_0^{\cos^{-1} e^{-\frac{2aH}{3}}} \cos^{\frac{3}{2}} \theta \, d\theta.$$

$$= \sqrt{\frac{2}{3gc}} \left\{ \tfrac{2}{3} e^{-\frac{aH}{3}} \sqrt{1 - e^{-\frac{4aH}{3}}} + \tfrac{2}{3} F(45^{\circ}, \alpha) \right\}$$

$$\cos\alpha = e^{-\frac{aH}{3}} \qquad \text{..................(39).}$$

$N = 2$ $\qquad Z_H = \sqrt{g/c}$

$$\frac{x}{V} = \sqrt{\frac{2}{gc}} \int_0^{c^{-1} e^{-\frac{1}{2}aH}} \cos^2\theta \, d\theta$$

$$= \frac{1}{\sqrt{2gc}} \left\{ \alpha + \tfrac{1}{2} \sin 2\alpha \right\}; \quad \cos\alpha = e^{-\frac{1}{2}aH} \qquad \text{..............(40).}$$

$N = 3$

$$Z_H = \sqrt{\frac{6g}{5c}}$$

$$\frac{x}{V} = \sqrt{\frac{3}{gc}} \left\{ \sin\alpha - \tfrac{1}{3} \sin^3\alpha \right\}$$

with $\cos\alpha = e^{-\frac{1}{3}aH}$ ....(4I).

and so on.

$N = \infty$ $\qquad Z_H = \sqrt{2g/c}$ and we find.

$$\frac{x}{V} = \sqrt{\frac{\pi}{2gc}} \, \mathrm{Erf}\left(\sqrt{aH}\right) \text{..................(42).}$$

Next for negative values of N, we have:-

$M = 2 \qquad Z_H = \infty$

and $x = Vt$ as expected ....(43).

$M = 3 \qquad Z_H = \sqrt{\frac{6g}{c}}$

$$\frac{x}{V} = \frac{1}{2}\sqrt{\frac{3}{gc}}\left\{\alpha + \frac{1}{2}\sin 2\alpha\right\}$$

with $\cos\alpha = e^{-\frac{1}{3}aH}$ ....(44)

$M = 4, \qquad Z_H = 2\sqrt{g/c}$

$$\cos\alpha = e^{-\frac{1}{4}aH}.$$

$$\frac{x}{V} = 2\sqrt{\frac{1}{gc}}\left\{\sin\alpha - \frac{1}{3}\sin^3\alpha\right\} \text{......(45)}$$

$M = 6 \qquad Z_H = \sqrt{\frac{3g}{c}}$

$$\frac{x}{V} = \sqrt{\frac{6}{gc}}\left\{\sin\alpha - \frac{2}{3}\sin^3\alpha + \frac{1}{5}\sin^5\alpha\right\}$$

$\cos\alpha = e^{-\frac{1}{6}aH.}$ ....(46)

and so on.

THE EFFECT OF CHANGING WINDS ON THE FLIGHT OF A PARTICLE IN THE ATTENUATED ATMOSPHERE

This investigation is most briefly conducted in the notation of Vector Analysis. Vectors will be denoted by placing a bar across the symbol employed. (Thus $\bar{v}$ is the vector velocity, v the scalar speed).

Let $\bar{v}$ be the vector velocity of the particle during flight in air undisturbed by changing winds, and let $\delta\bar{v}$ be the vector perturbation due to the change of air velocity with height lost. Then ~~v and~~ $\bar{v}+\delta\bar{v}$ is the actual velocity, and the equation of motion is

$$\frac{d}{dt}(\bar{v}+\delta\bar{v}) = g\bar{j} - \frac{g}{Z_H^2}\,{}^{e}/_{e_H}\,v'(\bar{v}+\delta\bar{v}-\bar{W}) \quad \ldots\ldots (1)$$

Where $\bar{j}$ is a unit vertical vector, $\bar{W}$ is the vector wind velocity relative to that in the stratum of release, and v' is the square root of the scalar product.

$$(\bar{v}+\delta\bar{v}-\bar{W})(\bar{v}+\delta\bar{v}-\bar{W})$$

Because the Drag acts in opposition to the direction of motion relative to the air.

For the majority of a long fall (and it is only in such that the perturbation need be considered) v will be great in comparison with $\delta\bar{v}$ or W, so it will be sufficiently accurate to place

$$v' = v$$

When $\bar{W}$ is zero, as in unperturbed motion,

$$\frac{d\bar{V}}{dt} = g\bar{j} - {}^{g}/_{Z_H^2}\cdot{}^{e}/_{e_H}\,v\bar{v} \quad \ldots (2)$$

Subtracting (2) from (1), (with $v'=v$)

$$\frac{d}{dt}(\delta\bar{V}) = -\,{}^{g}/_{Z_H^2}\cdot{}^{e}/_{e_H}\,v(\delta\bar{V}-\bar{W}) \quad \ldots (3)$$

Now (2) contains the component:-

$$\ddot{x} = -\,{}^{g}/_{Z_H^2}\cdot{}^{e}/_{e_H}\cdot v\dot{x},$$ hence (3) may be written:-

$$\frac{d}{dt}(\delta\bar{V}) = \frac{\ddot{x}}{\dot{x}}(\delta\bar{V}-\bar{W})$$

$$\therefore \quad \frac{1}{\dot{x}}\frac{d}{dt}(\delta\bar{V}) - \frac{\ddot{x}}{\dot{x}^2}\,\delta\bar{V} = -\frac{\ddot{x}}{\dot{x}^2}\,\bar{W}$$

i.e. $$\frac{d}{dt}\left(\frac{\delta\bar{V}}{\dot{x}}\right) = \bar{W}\,\frac{d}{dt}\left(\frac{1}{\dot{x}}\right)$$

or, simply $$d\left(\frac{\delta\bar{V}}{\dot{x}}\right) = \bar{W}\,d\left(\frac{1}{\dot{x}}\right) \quad \ldots\ldots \quad (4).$$

We may obtain a practical solution as follows. The mean wind veers at an almost constant rate with altitude. Assume, then, that

$$\bar{W} = \frac{(\bar{W}_0 - \bar{W}_H)}{H}\,y \quad \ldots\ldots \quad (5)$$

Where $\bar{W}_0$ is the wind relative to the ground at ground level, & $\bar{W}_H$ that in the stratum of release. (Both may be measured by an observer in the aircraft)

Then (4) may be written:-

$$\frac{d}{d\theta}\left(\frac{\delta\bar{V}}{\dot{x}}\right) = \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\,(1-\cos^N\theta)\,\frac{d}{d\theta}\left\{\frac{1}{V}\cos^{-\frac{N}{2}-1}\theta\right\}$$

by (10) and (34) of part II.

$$\therefore \quad \frac{\delta\bar{V}}{\dot{x}} = \frac{(\bar{W}_0 - \bar{W}_H)}{cVH}\left\{\sec^{\frac{N}{2}+1}\theta - \frac{2+N}{2-N}\cos^{\frac{N}{2}-1}\theta + \frac{2N}{2-N}\right\}$$

remembering that $\delta\bar{V} = 0$ when $\theta = 0$.

But $\delta\bar{V} = \frac{d}{dt}\delta\bar{x}$ where $\delta\bar{x}$ is the displacement.

and $$\dot{x} = V\cos^{N/2+1}\theta \quad \text{by } (34)$$

Thus $$\frac{d}{d\theta}(\delta\bar{x}) = \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\frac{dt}{d\theta}\left\{1 - \frac{2+N}{2-N}\cos^N\theta + \frac{2N}{2-N}\cos^{\frac{N}{2}+1}\theta\right\}$$

$$= \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\sqrt{\frac{N}{gc}}\left\{\cos^{\frac{N}{2}-1}\theta - \frac{2+N}{2-N}\cos^{\frac{3N}{2}-1}\theta + \frac{2N}{2-N}\cos^N\theta\right\}$$

$$\therefore\ \delta\bar{x} = \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\left\{t + \frac{2N}{2-N}x - \frac{(2+N)}{(2-N)}\sqrt{\frac{N}{gc}}\int_0^{\cos^{-1}e^{-\frac{aH}{N}}}\cos^{\frac{3N}{2}-1}\theta\, d\theta\right\} \quad \ldots\ldots (6)$$

The integral is tractable for all integral values of N, except 2.

Thus suppose $N = 1$, $H = 10{,}000$ feet. $\therefore Z_H = \sqrt{\frac{2g}{3c}}$

$$\delta\bar{x} = \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\left\{\sqrt{\frac{2}{gc}}\,F(45^\circ, \sec^{-1}e^{\frac{1}{2}aH}) + 2\frac{1}{\sqrt{gc}}\sqrt{1-e^{-2aH}} - 3\sqrt{\frac{1}{gc}}\int_0^{\cos^{-1}e^{-\frac{1}{2}aH}}\cos^{\frac{1}{2}}\theta\, d\theta\right\}.$$

The ~~[illegible]~~ integral is $-3\sqrt{\frac{2}{gc}}\left\{2E(45^\circ,\alpha) - F(45^\circ,\alpha)\right\}$

$$\cos\alpha = e^{-\frac{1}{2}aH}.$$

$$\delta\bar{x} = \frac{(\bar{W}_0 - \bar{W}_H)}{cH}\sqrt{\frac{2}{gc}}\left\{4F - 6E + \sqrt{2(1-e^{-2aH})}\right\}$$

$$\log\sec\alpha = \tfrac{1}{2}\cdot 141 = \cdot 0705$$

$$\alpha = 31^\circ 46' = 31\cdot 767$$

From Dale's Tables.

α	$F(45^\circ, \alpha)$			$E(45^\circ, \alpha)$		
31°	·55432	1878	9	·52834	1622	−8
32°	·57310	1887	9	·54456	1614	−7
33°	·59197	1896		·56070	1607	
34°	·61093			·57677		

$$F(45^\circ, 31^\circ\!\cdot 767) = \cdot 55432 + \cdot 767 \times 1878 - \cdot 09 \times 9$$
$$= \cdot 56872$$
$$E(45^\circ, 31^\circ\!\cdot 767) = \cdot 52834 + \cdot 767 \times 1614 + \cdot 09 \times 8$$
$$= \cdot 54073$$

$$4 \times \cdot 56872 = 2\cdot 27488$$
$$6 \times \cdot 54073 = 3\cdot 24438$$

$$\log e^{-2aH} = -2 \times 1\cdot 41 \times 10^{-1} = -\cdot 282 = \bar{1}\cdot 71800$$

$$e^{-2aH} = \cdot 52240$$
$$2(1 - e^{-2aH}) = \cdot 95520, \quad \sqrt{\cdot 95520} = \cdot 97734$$

Thus quantity within brackets is :—

$$2\cdot 27488 - 3\cdot 24438 + \cdot 97734 = \cdot 00784$$

At 10,000, $c = 2\cdot 7723 \times 10^{-5}$

log g ...	1·50773	~~log cH =~~	~~$\bar{1}$·44284~~
log c	$\bar{5}$·44284	log ·00784	$\bar{3}$·89432
	$\bar{4}$·95057		2·23239
log 2	·30103		·12671
½ colog 2	3·35046		[1·34]
	1·67523		
log cH	$\bar{1}$·44284		
	2·23239		

Thus :-

$$\delta\bar{x} = 1.34\ (\bar{W}_0 - \bar{W}_H)$$

Now $Z_H = \sqrt{\frac{2g}{3c}}\,(1+\epsilon)^{\frac{1}{2}} = \sqrt{\frac{2g}{3c}}\,(1.0132)^{\frac{1}{2}}.$

Log $2g/c$ ...	6.36592
log 3	·47712
	5.88880
log e^{+aH}	·141
log Z_0^2 = 2	5.74780
	2.87390 [748].

Thus if Z_0 = 748 feet/sec , H = 10,000 feet.

$$\delta x = 1.34\ (\bar{w}_0 - \bar{w}_H)$$

and if the scalar value of $\bar{W}_0 - \bar{W}_H$ is, say, 30 m.p.h. (a not improbable value.)

$$\delta x = 1.34 \times 30 \times \frac{22}{15} \text{ feet.}$$

$$= 59 \text{ feet.}$$

By no means a negligible quantity.