Tauer masas in the hyperfinite II_1 factor

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To Blake,

What you've been through as a consequence of my fall, forms a debt I can never repay.

Abstract

In 1964, Tauer gave examples of countably many masas inside the hyperfinite II_1 von Neumann factor R. These masas were shown to be pairwise non-conjugate in R using a length invariant for the normalisers of semi-regular masas. A class of masas, the *Tauer masas*, is introduced consisting of all those masas obtained using her basic method of construction. The main body of this thesis is then concerned with examining the properties of these Tauer masas. In particular, the concepts of singularity, strong singularity and the weak asymptotic homomorphism property coincide for Tauer masas, and all Tauer masas have Pukánszky invariant $\{1\}$.

Modern methods for calculating von Neumann algebras generated by normalisers are used to examine Tauer's original examples, leading to shorter proofs of all of her results. Her initial example of a singular masa is studied in further detail. A generalisation of her semi-regular masas leads to the construction of an uncountable family of semi-regular masas of infinite length inside R. Examination of the Jones index of inclusions of the iterated normaliser algebras demonstrates that no pair of these masas can be conjugate by an automorphism of R.

Centralising sequences for R lying inside masas are examined, with examples given to show that singular masas can be found containing non-trivial centralising sequences. An invariant, $\Gamma(A)$, for a masa inside a II_1 factor is introduced as the size of a maximal cut-down for which the resulting masa contains non-trivial centralising sequences. This invariant is then used to exhibit a $d_{\infty,2}$-continuous path of uncountably many strongly singular masas in R with the same Pukánszky invariant, no pair of which is conjugate by an automorphism of R.

Various issues arising from these concepts are discussed, such as possible masas in R_ω and the relationship between A-valued centralising sequences and automorphisms of R fixing A pointwise. Possible connections between this relative automorphism group and the Pukánskzy invariant will also be touched upon.

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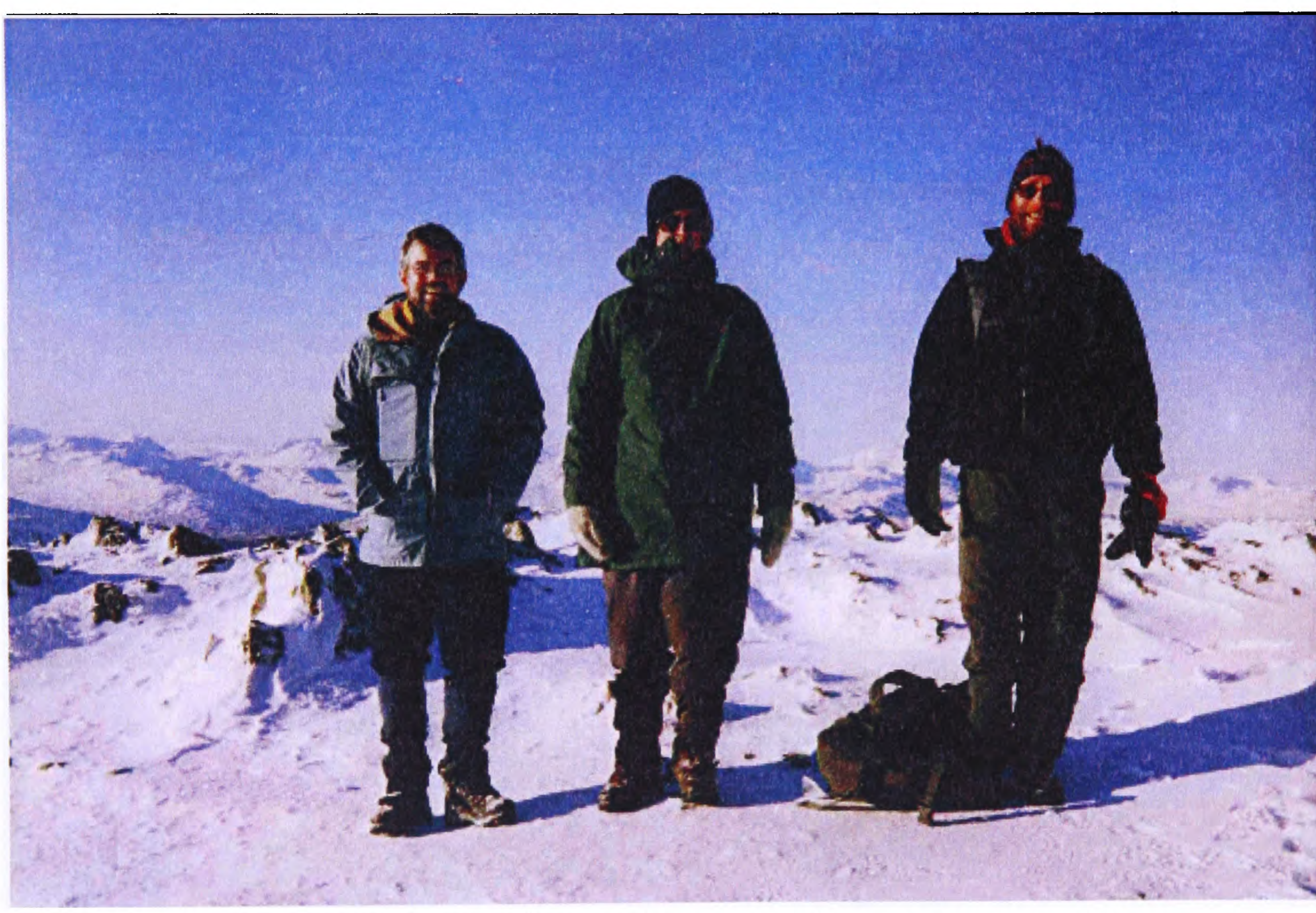
First and foremost, my thanks go to Allan Sinclair, who repeatedly offered me much more than one could expect from a PhD supervisor, continuing his role long after his well deserved retirement. Thanks also to all the staff and students at the School of Mathematics who have made these four years such a pleasurable experience. In particular, I would like to pick out Tony Carbery, Jim Wright and Toby Bailey for their advice, mathematical and otherwise. During my studies, I have enjoyed the opportunity to visit College Station, Texas on two occasions. My thanks to Roger Smith, for making this possible and for many helpful conversations both in the States and in Edinburgh.

Enjoying retirement



Allan Sinclair (middle), and friends on Stuc a' Chroin, 2005.

Photo: Unknown

I would like to thank my parents for their constant support over the last four years; they've always been there for me when I've needed it. They should also be congratulated for being brave enough to proof read this thesis! All my flat mates in Edinburgh, Gregor, Scott, Mike, Andy, Tessa, Blake, Tom and Hamish, have put up with various foibles. Their tolerance has been much appreciated. Escaping to the mountains has helped to keep me sane, cheers to Pete, Jon, Jonathan and Mark and all the yummicks I've been away with, for the good memories.

In their natural habitat

Mum and Dad, Bernese Oberland, 2005.

Photo: John Gay

Thanks to Tom Banister - without your capable actions the end result on March the 6th, 2005 would have much worse. The members of the Lochaber Mountain Rescue Team, RAF helicopter rescue squadron and Belford Hospital are literally lifesavers - what people would do without you I don't know. Finally thanks to Blake and countless other people for helping me to start to come to terms with things since then. You've all been great.

Edinburgh, June 27, 2005.

It was the best of times, it was the worst of times

Blake on Centre Post Direct (V 5), Creag Meagaidh, before the accident. 2005.

Photo: Saw

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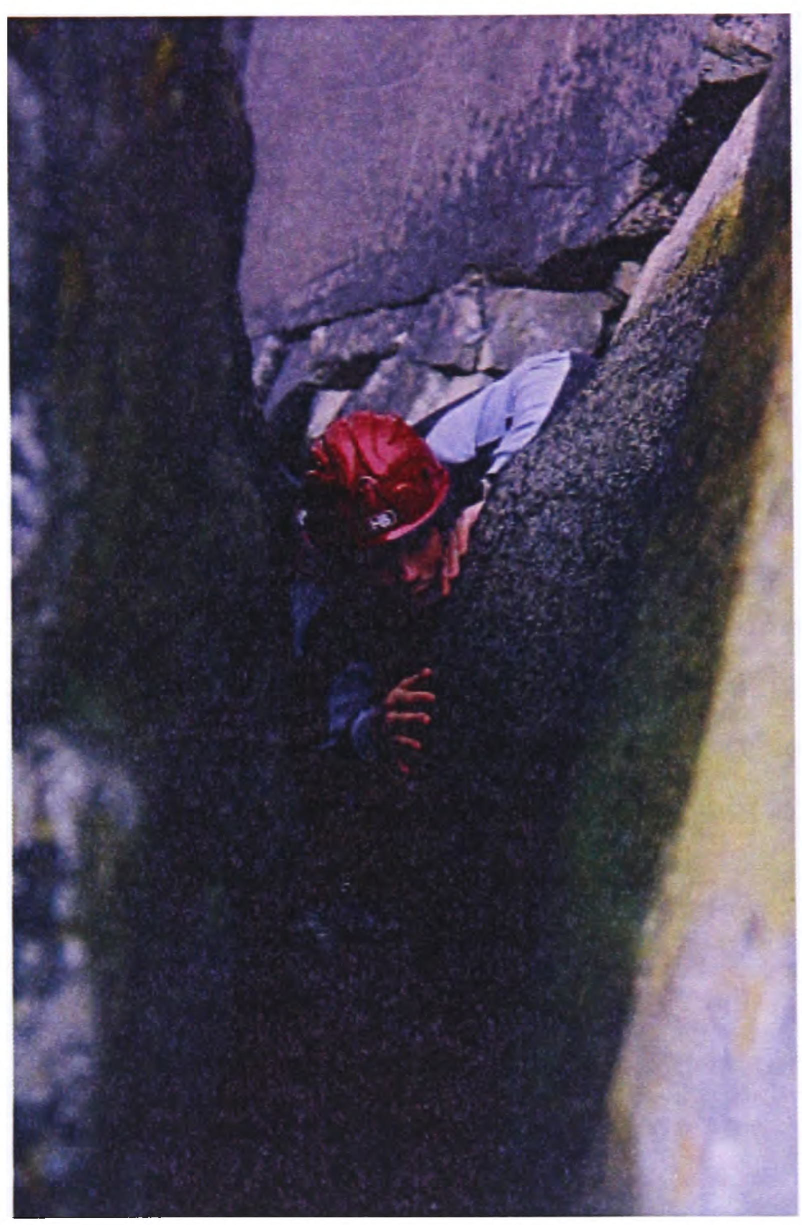
Chapter 1

Preliminaries

What to do?



Dulux Corner (HS 4b), Neist point, Skye, May 2002.

Photo: Jon Powell

1.1 Introduction

The study of weakly closed rings of operators acting on a Hilbert space dates back to Frank Murray and John von Neumann's series of papers [35, 36, 74, 37]. These rings are now called *von Neumann algebras* and have been extensively studied since. The basic building blocks of the subject are the *factors*, those von Neumann algebras with trivial centre, which Murray and von Neumann classified into types $\mathrm{I_n}$, $\mathrm{II_1}$, $\mathrm{II_\infty}$ and type III. Throughout we shall study only the $\mathrm{II_1}$ case, occasionally using finite type I_n factors[1] in our constructions. The focus of this thesis will be on masas, that is maximal abelian self-adjoint subalgebras, in $\mathrm{II_1}$ factors, usually the hyperfinite $\mathrm{II_1}$ factor. Extensive background material on masas is given in section 1.4, and various results in the theory of $\mathrm{II_1}$ factors are contained in sections 1.2 and 1.3 - we shall not repeat ourselves here.

The main motivation for this thesis is a fairly obscure paper, [70], due to Sister Rita Tauer in 1965, in which she gives examples of countably many different masas in the hyperfinite $\mathrm{II_1}$ factor. Another set of examples of such masas, due to Pukánszky ([51]), preceded Tauer's work, although the two approaches are very different. Pukánszky's examples come from groups, and are now much more well known than Tauer's examples which arise from elaborate matrix constructions. Chapter 2 is wholly concerned with this work of Tauer's. We introduce, in section 2.1, a class of masas, which we call *Tauer masas*, to be those that can be produced using Tauer's basic method. In sections 2.2 and 2.3 we give a modern account of Tauer's examples using tensor products rather than large matrix algebras. Up to date methods for calculating von Neumann algebras generated by normalisers allow us to give much shorter proofs of Tauer's results. Tauer's main idea for distinguishing between masas was to examine the iterated chain of normalising algebras, and she was able to give examples of masas where this chain reaches R in l finite steps. An interesting historical point is that the invariant Tauer used to show that this l did not completely determine (up to conjugacy) the original masa turns out to be an alternative characterisation[2] of an index 2 inclusion of subfactors. In section 2.4 we are able to use this index idea, and an extension of Tauer's examples, to present an explicit, elementary construction of uncountably many, pairwise non-conjugate semi-regular masas in the hyperfinite $\mathrm{II_1}$ factor. In contrast, the examples produced by Jones and Popa in [25] are non-constructive.

In chapter 3, we examine properties relating to the Pukánszky invariant; the idea Pukánszky used to distinguish between his original examples in [51]. Fol-

[1] An extravagant way of describing the $n \times n$ matrices over $\mathbb{C}$!

[2] At least under an additional finite index assumption, which I strongly suspect is unnecessary - see subsection 1.3.2.

lowing Popa's work, [46], this invariant has become a useful object in the study of masas, [52, 3, 38, 61, 17], not least because for a long time it was one of very few methods for showing that a masa is singular. We present the background and current state of play in subsections 3.1.1 and 3.1.2, while in subsection 3.1.3 we examine how the Pukánszky invariant behaves under $d_{\infty,2}$-limits of masas. In section 3.2, we show that the class of Tauer masas all have Pukánskzy invariant $\{1\}$, Theorem 3.2.1. After the Pukánskzy invariant, the presence or absence of non-trivial centralising sequences inside a masa has perhaps been the next most useful concept for showing the non-conjugacy of masas when we are unable to distinguish them using normalisers. A brief history of centralising sequences inside masas can be found in subsection 1.4.5. In section 3.3, we use centralising sequences and Tauer masas to give an uncountable family of pairwise non-conjugate singular masas in the hyperfinite II_1 factor with the same Pukánszky invariant. This extends a result of Størmer and Neshveyev ([38]) in which two non-conjugate singular masas with the same Pukánszky invariant appeared. Furthermore, the resulting family turns out to give us a continuous (with respect to the $d_{\infty,2}$-metric on masas) path from the unit interval into the singular masas with Pukánszky invariant $\{1\}$ in R, no two points on which are conjugate. In subsection 3.3.3, we introduce the concept of *transitivity* for masas, the idea being that two identically sized cut-downs of a transitive masa should look the same in the underlying II_1 factor. That is they should be conjugate.

In section 3.4, we look at centralising sequences further. The main objective is to determine which masas A in R give rise to a masa $A^\omega \cap R'$ in R_ω. We are unable to resolve this problem, but give some examples suggesting a connection between this property and the normalisers of A. Chapter 3 ends with a discussion, section 3.5, of a relative automorphism group of masas in II_1 factors. We state a translation of Connes' characterisation of approximately inner automorphisms [7, Theorem 3.1] in this context, Theorem 3.5.9, and ask some questions about the relationship between these automorphisms and the Pukánskzy invariant.

Chapter 4, begins by asking the question 'When is a Tauer masa singular?'. We resolve this completely in Theorem 4.1.3, giving a criterion for singularity of a Tauer masa in terms of the approximates used to construct it. Furthermore, we are able to show that singular Tauer masas have the weak-asymptotic homomorphism property and therefore are strongly singular. The question of whether all singular masas are strongly singular appears in [59, 50], and the Tauer masas are the first large class of masas for which this result is known. In section 4.2, we apply this criterion for singularity to a family of θ-masas introduced in section 2.2, and in section 4.3 we investigate these masas further. Tauer's original example of a singular masa falls into this subclass, and we are able to see it has some perhaps surprising properties such as being conjugate to its own infinite tensor product, Theorem 4.3.8, under an identification of R with $R^{\overline{\otimes}\infty}$. We would like to be able to show that this is an example of a transitive singular masa in R, as we only have examples of these in the free group factors at present. We present evidence in section 4.3 suggesting that this could be the case.

Finally we turn to the appendices. Appendix A gives an example of an inclusion of II_1 factors $M \subset R$ where M fails to have the relative weak asymptotic homomorphism property away from $\mathcal{N}(M)'' = N$, a concept defined in subsection 1.4.21. We are unable though to decide whether this M is *strongly normalised* by N. In Appendix B we do something very different and give a calculation, following [60], showing the strong singularity of the radial masas considered by Boca and Rădulescu, in [3]. The thesis ends with Appendix C, where we present a simple observation on automorphisms of group II_1 factors dating back to Kallman, [32].

We have now reached the end of the beginning of this thesis, and conclude this introduction by making some remarks about the notation used within. Capital Roman letters will in general refer to von Neumann algebras, by preference these will be N and M. We reserve R, S and occasionally T for hyperfinite factors, and A and B will usually denote abelian von Neumann algebras. However, J will be consistently used for the modular conjugation operator, and G and H will always be discrete groups. We shall occasionally refer to free groups, these will always have $k \geq 2$ generators and will be denoted $\mathbb{F}_k$. Small Roman letters, normally a, b, u, v, x, y, z, will be used for operators in von Neumann algebras, and Greek letters, often ξ and η, for elements of the Hilbert space they operate on. These Hilbert spaces, when not appearing as L^2 of some N, will be denoted $\mathcal{H}$. Script $\mathcal{A}$ (and occasionally $\mathcal{B}$) are not to be confused with Hilbert spaces - they will be used in Chapter 3 to denote the augmented algebra $(A \cup JAJ)''$ corresponding to a masa A.

A substantial number of footnotes can be found in the text. They mainly fall into one of three categories: giving citations and cross references that would otherwise disturb the flow; points of excessive mathematical pedantry; and remarks that could be found humorous. It is left to the reader to decide which is which.

Addendum 1.1.1. After the completion of this thesis, Questions 1.4.18 and 1.4.20 on whether all singular masas are strongly singular and have the weak asymptotic homomorphism property respectively, have now been answered positively which, as noted in the text, gives a partial positive result in the direction of Question 1.4.29. This result also completely supersedes Theorem 4.1.1. The proof can be found in [62], which is joint work between Allan Sinclair, Roger Smith, Alan Wiggins and myself.

1.2 Some topics in the theory of von Neumann algebras

We have endeavored to make this thesis relatively self contained, in that the vast majority of the definitions needed and statements of existing results used can be found in this chapter. One has to start somewhere though - we shall assume familiarity with the basic theory of von Neumann algebras as can be found in either Kadison and Ringrose's Magnum Opus [30, 31] or the first volume, [67], of Takesaki's extensive account of operator algebras, [67, 68, 69]. In this section we develop a few assorted topics from the theory, the selection not being quite as random as it appears. In the next section we discuss Vaughan Jones' index for subfactors and finally, in section 1.4, we develop the theory of masas in II_1 factors. We end these remarks with a note on the standard form.

Throughout the thesis, we shall only be interested in finite von Neumann algebras, usually factors, but if not they will always come equipped with a fixed faithful normal trace, tr. For x in such a von Neumann algebra N, we write $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ and then complete N with respect to this norm to obtain the Hilbert space $L^2(N)$. We shall follow the tradition in the subject of regarding N as a subset of $L^2(N)$ whenever it is convenient. The *standard form* is the representation $N \subset \mathbb{B}(L^2(N))$ obtained by letting each $x \in N$ act by left multplication[3] on $L^2(N)$. The *modular conjugation operator* is the conjugate linear map $J : L^2(N) \to L^2(N)$ obtained by extending the conjugation $x \mapsto x^*$ from N. Then JxJ is the operator of right multiplication on $L^2(N)$ by x^*, and the commutant N' of N in $\mathbb{B}(L^2(N))$ is precisely JNJ.

We shall use group von Neumann algebras occasionally, but not enough to justify giving a lengthy discussion. Suffice it to say that when G is a countable discrete I.C.C. group[4], we obtain a II_1 factor $\mathcal{L}(G)$ to be the von Neumann algebra generated by the left regular representation of G on $\ell^2(G)$. The trace is given by $\mathrm{tr}(x) = \langle x\delta_e, \delta_e\rangle$, where δ_e is the point mass in $\ell^2(G)$ corresponding to the identity. The left regular representation is already the standard form of $\mathcal{L}(G)$ in that $\ell^2(G) = L^2(\mathcal{L}(G))$. Finally, when g is an element of G we shall abuse notation and write g for the element of $\mathcal{L}(G)$ corresponding to the image of g

[3]extend $y \mapsto xy$ from N to $L^2(N)$ by $\|\cdot\|_2$-continuity.

[4]I.C.C. stands for infinite conjugacy class, the definition of which is that $\{ ghg^{-1} \mid g \in G \}$ is infinite, whenever $h \in G$ is not the identity.

under the left regular representation.[5]

Warning 1.2.1. The use of the term separable in this thesis differs from the standards in the literature. The appropriate concept of separability for von Neumann algebras is that of being *separably acting*, that is being faithfully represented as bounded operators on some separable Hilbert space, or equivalently having a *separable predual.* When we talk of a *separable* von Neumann algebra henceforth, it is these properties we mean and most definitively not that it is separable as a C^*-algebra - it almost surely won't be.

1.2.1 Tensor products

The theory of tensor products for C^*-algebras is fairly involved as there is not a unique C^*-norm on the algebraic tensor product of two C^*-algebras. In the von Neumann context, this problem does not arise.[6] Given two von Neumann algebras N_1, N_2 faithfully represented on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, form the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. An elementary tensor $x_1 \otimes x_2$ of operators from N_1 and N_2 respectively, acts on this Hilbert space in the obvious way; with $(x_1 \otimes x_2)(\xi_1 \otimes \xi_2) = (x_1\xi_1) \otimes (x_2\xi_2)$. The von Neumann tensor product, $N_1 \overline{\otimes} N_2$ of N_1 and N_2 is then generated by all these elementary tensors,

$$N_1 \overline{\otimes} N_2 = \{\, x_1 \otimes x_2 \,|\, x_i \in N_i \,\}'' \subset \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)\,.$$

This definition appears to depend on the choice of the faithful representations, but actually this is not the case. If we represent N_1 and N_2 faithfully on some other spaces then the resulting von Neumann tensor product is *-isomorphic to that obtained from our first choice of representation. In any event, the von Neumann algebras N considered within will all be finite with a given faithful normal trace, tr, and so we have a canonical representation - that on the standard form $L^2(N, \mathrm{tr})$. When the von Neumann algebras are finite dimensional, we shall write $\otimes$ rather than $\overline{\otimes}$, as no closure is involved.

We shall regularly work with inclusions of von Neumann algebras - obtaining further inclusions naturally from the tensor product. A key result here is the Tomita commutation theorem, for which we refer to [31, Theorem 11.2.16].

Theorem 1.2.2 (Tomita commutation theorem). *Let $M_1 \subset N_1$ and $M_2 \subset N_2$ be inclusions of von Neumann algebras. Working in $N_1 \overline{\otimes} N_2$, we have*

$$(M_1 \overline{\otimes} M_2)' \cap (N_1 \overline{\otimes} N_2) = (M_1' \cap N_1) \overline{\otimes} (M_2' \cap N_2).$$

[5] We reserve the right to further write g for the point mass δ_g in $\ell^2(G) \cong L^2(\mathcal{L}(G))$ corresponding to G in accordance with our policy of regarding finite von Neumann algebras N as being subsets of $L^2(N)$. Hopefully the location of g will be obvious from context!

[6] Of course von Neumann algebras are not in general nuclear as C^*-algebras, but we are not considering them as C^*-algebras.

We shall also need infinite tensor products of von Neumann algebras. Here we have to be slightly more careful, as the isomorphism class of the resulting von Neumann algebra is no longer independent of the choice of representations. We shall actually only work with finite factors, and follow [69, Section XIV.1]. Given finite factors $(M_n)_{n=1}^{\infty}$, denote by tr_n the unique faithful normal trace on M_n. Write $\bigotimes_{n=1}^{\infty} M_n$ for the algebraic tensor product of these M_n, that is finite linear combinations of elementary tensors $\bigotimes_{n=1}^{\infty} x_n$, where $x_n \in M_n$ and all but finitely many x_n are 1. We have the product state tr on $\bigotimes_{n=1}^{\infty} M_n$ defined on elementary tensors by

$$\mathrm{tr}(\bigotimes_{n=1}^{\infty} x_n) = \prod_{n=1}^{\infty} \mathrm{tr}(x_n).$$

Now let π be the representation of $\bigotimes_{n=1}^{\infty} M_n$ by left multiplication on the Hilbert space $L^2(\bigotimes_{n=1}^{\infty} M_n, \mathrm{tr})$ in the usual way. The infinite von Neumann tensor product of the M_n is then the weak-closure of the image of π. This is necessarily a finite factor, as it has a trace, namely the extension of tr, which is the unique normalised trace on $\bigotimes_{n=1}^{\infty} M_n$. We shall regularly denote this object as $(\bigotimes_{n=1}^{\infty} M_n)''$ in the sequel without reference to π. When all the M_n are identical to M say, we shall occasionally be lazy and write $M^{\overline{\otimes}\infty}$ for this infinite von Neumann tensor product. The Tomita commutation theorem remains true in this infinite setting.

1.2.2 The hyperfinite II_1 factor

Murray and von Neumann's defining property of hyperfiniteness was that of approximate finite dimensionality (AFD). Following [69, Section XIV.2], a II_1 factor N was said to be AFD when for any $x_1, \ldots, x_n \in N$ and strong neighbourhood V of 0 in N, a finite dimensional *-subalgebra M of N can be found with $x_i \in M+V$ for each i.

Examples are immediately apparent by taking infinite tensor products of matrix algebras. Let M_n be a algebra of matrices for each n, then the infinite tensor product $(\bigotimes_{n=1}^{\infty} M_n)''$ produced with respect to the unique normalised trace on each M_n is a II_1 factor, which is obviously AFD. In [37], Murray and von Neumann showed that up to isomorphism this is the unique way of obtaining an AFD II_1 factor and, in complete contrast with the C^* case, that the resulting object is independent of the size of the matrices involved.

Theorem 1.2.3 (Murray and von Neumann). *Let N be a separable II_1 factor. The following two conditions are equivalent:*

1. *N is isomorphic to $(\bigotimes_{n=1}^{\infty} \mathbb{M}at_2(\mathbb{C}))''$;*
2. *N is AFD.*

Henceforth, we use the term hyperfinite for AFD and denote the hyperfinite II_1 factor by R, and occasionally S and T. The proof of this result involves a careful approximation argument, which we are unable to circumvent entirely as in section 3.2.2 we shall deduce an approximation result for masas in the hyperfinite II_1 factor using these methods.[7] Here we state precisely the tools we shall need later, from Takesaki's account of Theorem 1.2.3, found in section XIV.2 of [69].

Lemma 1.2.4 ([69, Lemma XIV.2.1]). *If e and f are equivalent projections in a finite von Neumann algebra N then there exists a unitary $u \in N$ with*

$$|u-1| \leq \sqrt{2}\,|e-f| \quad \text{and } ueu^* = f.$$[8]

Lemma 1.2.5 ([69, Lemma XIV.2.2]). *Let N be a separable II_1 factor. If $h \in N$, $0 \leq h \leq 1$, satisfies the inequality*

$$\left\|h-h^2\right\|_2 = \delta < 1/4,$$

then the spectral projection e of h corresponding to the interval $[1-\sqrt{\delta}, 1]$ satisfies the estimate

$$\|e-h\|_2 \leq 2\sqrt{\delta}, \qquad \left\|h^{1/2}-e\right\|_2 \leq 2\delta^{1/4}.$$

In the next lemma we face a pedagogical problem - namely that some of the concepts appearing will be defined later. See section 1.3 for the conditional expectation map, $\mathbb{E}$, and Definition 1.4.10 for $\|.\|_{\infty,2}$. We have also chosen to replace the use of $\overset{\epsilon}{\subset}$ in the original, with infinity-two norm estimates.

Lemma 1.2.6 ([69, Lemma XIV.2.10]). *If V is a finite dimensional subspace of an AFD II_1 factor R_0 and N_1 is a subfactor of type I_{2^n}, then for any $\epsilon > 0$ there exists a type I_{2^p} subfactor N_2, for some large p, such that $N_1 \subset N_2$, and*

$$\|(I-\mathbb{E}_{N_2})\,\mathbb{E}_V\|_{\infty,2} \leq \epsilon.$$

Before moving on there is one more observation we should perhaps make.

Remark 1.2.7. When we work with R as the infinite von Neumann tensor product $(\bigotimes_{n=1}^{\infty} \mathbb{M}\mathrm{at}_2(\mathbb{C}))''$ with respect to the unique normalised trace, we shall often consider the finite dimensional approximates $N_n = \bigotimes_{m=1}^{n} \mathbb{M}\mathrm{at}_2(\mathbb{C}) \cong \mathbb{M}\mathrm{at}_{2^n}(\mathbb{C})$, whose union is by definition weakly dense in R. A priori, it might not be possible to use the Kaplanszky density theorem to weakly approximate unitaries in R

[7] Indeed, this is the only reason this discussion is present at all.

[8] The moduli in this equation are defined by $|x| = (x^*x)^{1/2}$. Our only use of these moduli in the sequel, will be the simple observation that $\||x|\|_2 = \|x\|_2$. We have stated the Lemma in the original form of [69] for easy reference.

by those in $\bigcup_{n=1}^{\infty} N_n$ - for this last object is not norm closed.[9] To approximate unitaries using the Kaplanszky density theorem, one expresses them in the form e^{ih} for some self-adjoint h, approximates the h by h_1 and takes e^{ih_1} - a process that requires norm-closure. In this direct limit context we are able to proceed as our h_1 will lie in $\bigcup_{n=1}^{\infty} N_n$ and so in some N_n, whence e^{ih_1} lies in N_n as this *is* a C^*-algebra.

1.2.3 Orthogonality of subalgebras

In [44], Sorin Popa introduced the concept of orthogonality for pairs of subalgebras in finite von Neumann algebras.

Definition 1.2.8. Let N be a finite von Neumann algebra with trace tr. Two von Neumann subalgebras M_1 and M_2 of N are said to be *orthogonal* if $M_1 \ominus \mathbb{C}1$ is orthogonal to $M_2 \ominus \mathbb{C}1$ in $L^2(N, \mathrm{tr})$. We shall write $M_1 \perp M_2$ when this is the case.

As noted in Lemma 2.1 of [44], there are many alternative formulations of this definition. We briefly highlight those we shall use later.

Proposition 1.2.9 ([44, 2.1]). *Let M_1 and M_2 be von Neumann subalgebras of a finite von Neumann algebra N with trace tr. The following conditions are then equivalent.*

1. *M_1 and M_2 are orthogonal von Neumann subalgebras of N;*
2. *$tr(x_1x_2) = tr(x_1)tr(x_2)$ whenever $x_i \in M_i$;*
3. *$\|x_1x_2\|_2 = \|x_1\|_2 \|x_2\|_2$ whenever $x_i \in M_i$;*
4. *$\mathbb{E}_{M_1}(\mathbb{E}_{M_2}(x)) = tr(x)1$, for all $x \in N$.*[10]

The main aim of [44] was to give a criterion for calculating the normalisers of certain masas, which we briefly discuss in subsection 1.4.2. This connection between orthogonality and singularity will appear later, both in our examination of Tauer's original examples in sections 2.2 and 2.3, and in constructing singular masas in section 3.3. In this second case, we shall need an abundance of mutually orthogonal masas in finite dimensional matrix algebras. Fortunately providence, in the form of Sorin Popa, has provided exactly what we need.

[9] Recall that the Kaplanszky density theorem allows the weak approximation of elements in the unit ball of a von Neumann algebra by elements in the unit ball of a weakly dense *-subalgebra, and of self-adjoint elements of this ball by self-adjoint elements in the dense ball.

[10] Here $\mathbb{E}_{M_i}$ is the unique trace-preserving conditional expectation from N onto M_i, which we eventually define in section 1.3.

Proposition 1.2.10 ([44, Theorem 3.2]). *For any prime p there exist a family of $p+1$ pairwise orthogonal masas in the algebra of $p \times p$ matrices.*

1.2.4 Property Γ and centralising sequences

In 1943, Frank Murray and John von Neumann, in the fourth part ([37]) of their series of work on rings of operators, demonstrated the existence of non-isomorphic II_1 factors. They introduced *property* Γ to show that none of factors $\mathcal{L}(\mathbb{F}_k)$ are hyperfinite, where $\mathbb{F}_k$ is the free group on $k \geq 2$ generators.

Definition 1.2.11 (Murray and von Neumann). A II_1 factor N has *property* Γ when, for all $\epsilon > 0$ and $x_1, \ldots, x_n \in N$, there exists a unitary $u \in N$ with $\mathrm{tr}(u) = 0$ and

$$\left\|ux_i - x_iu\right\|_2 < \epsilon,$$

for each $i = 1, \ldots, n$.

It comes as no surprise to learn, [37, Lemma 6.2.2], that it is impossible to find unitaries in $\mathcal{L}(\mathbb{F}_2)$ approximately commuting with the generators a and b of $\mathbb{F}_2$ so that $\mathcal{L}(\mathbb{F}_2)$, and in general the the free group factors $\mathcal{L}(\mathbb{F}_k)$, do not have property Γ. On the other hand, the hyperfinite II_1 factor does have property Γ ([37, Lemma 6.1.2]) as we can see immediately by writing R as an infinite von Neumann tensor product of matrix algebras $(\bigotimes_{n=1}^{\infty} \mathrm{Mat}_2(\mathbb{C}))''$. As this idea will appear frequently we shall spell it out. Elements $1^{\otimes n} \otimes u$, for some trace-free unitary $u \in \mathrm{Mat}_2(\mathbb{C})$, commute with $\bigotimes_{r=1}^{n} \mathrm{Mat}_2(\mathbb{C})$ and, as the union of all these sets is $\|.\|_2$-dense in R, the claim follows.

The concept of *central sequences*, introduced in [13], follows naturally from the idea of property Γ. A bounded sequence $(x_n)_{n=1}^{\infty}$ in a II_1 factor N is said to be a *central sequence* if

$$\lim_{n\to\infty} \|x_n y - y x_n\|_2 = 0,$$

for all $y \in N$. Two central sequences $(x_n^{(1)})_{n=1}^{\infty}$ and $(x_n^{(2)})_{n=1}^{\infty}$ are equivalent when $\left\|x_n^{(1)} - x_n^{(2)}\right\|_2 \to 0$ as $n \to \infty$, and a central sequence is called *trivial*, if it is equivalent to a central sequence for N lying in the scalars $\mathbb{C}1$. The equivalence classes of centralising sequences form a C^*-algebra, ensuring that a II_1 factor N has property Γ if and only if there exist non-trivial centralising sequences for N.

A separable II_1 factor N is called *strongly-stable* if it is isomorphic to $N \overline{\otimes} R$, where R is the hyperfinite II_1 factor. Dusa McDuff used centralising sequences to give a criterion, Theorem 3 of [34],[11] for strong-stability of a II_1 factor N. McDuff's result is that N is strongly-stable precisely when the centralising sequences of N give rise to a non-commutative C^*-algebra.

[11] See also Theorem 4.8 of Takesaki's book [69].

Given a subalgebra B of N, we say that B contains non-trivial centralising sequences for N if there is a non-trivial centralising sequence $(x_n)_{n=1}^{\infty}$ for N with each $x_n \in B$. In [1] and [2], Bisch investigated centralising sequences lying in subfactors. The first paper, [1], generalises Theorem 2.1 of Connes' injective factors paper, [7], and extends McDuff's result to give a criterion for the strong-stability of an inclusion of II_1 factors $M \subset N$,[12] in terms of centralising sequences for N lying in M. The second paper, [2], gives examples of finite index inclusions $M \subset N$ of hyperfinite II_1 factors, where M contains such non-trivial centralising sequences for N, and examples of such inclusions where all the centralising sequences for N lying in M are trivial.

To demonstrate the absence of non-trivial centralising sequences in a subfactor, Bisch used an idea which originally appears in Popa's orthogonality work, [44, Remark 5.4.2].[13]

Proposition 1.2.12 (Popa). *Let M be a subalgebra of a separable II_1 factor N. If there exists a unitary $u \in N$ with $uMu^* \perp M$ then any centralising sequence for N lying in M is trivial.*

Whether or not non-trivial centralising sequences can be found inside certain masas in II_1 factors has often been a usful tool. Unfortunately, this topic is perhaps less well known than it should be and so we shall collect together the work in this area in subsection 1.4.5.

1.2.5 The ultraproduct N^ω and central sequence algebras $N^\omega \cap N'$

In section 3.4 we shall have course to examine $R_\omega = R^\omega \cap R'$. The theory of ω-centralising sequences involved dates back to [13] and is developed in [34]; many alternative accounts also exist - see for example [69, Section XIV.4]. Here we content ourselves with a brief outline of the situation, for a fixed II_1 factor N.

Let ω be a non-principal ultrafilter on $\mathbb{N}$, that is an element of $\beta\mathbb{N} \setminus \mathbb{N}$.[14] We have a tracial state on the C^*-algebra $\ell^\infty(N)$ of all uniformly bounded sequences in N given by

$$\mathrm{tr}_\omega\left((x_n)\right) = \lim_{n\to\omega} \mathrm{tr}(x_n).$$

---

[12] Unsurprisingly, this is defined as there being an *-isomorphism between N and $N\overline{\otimes}R$ which takes M onto $M\overline{\otimes}R$.

[13] See also [2, Lemma 2.1].

[14] As is usual, $\beta\mathbb{N}$ denotes the Stone-Cêch compactification of $\mathbb{N}$ - an object of which we have no intention of developing the theory. Instead the reader is referred to Lemma XIV.4.2 in [69] and the preceding discussion, for the properties we need.

The GNS construction gives a representation π_ω of $\ell^\infty(N)$ on some Hilbert space $\mathcal{H}_\omega$ and a cyclic vector ξ_ω, such that $\mathrm{tr}_\omega((x_n)) = \langle \pi_\omega((x_n))\xi_\omega, \xi_\omega\rangle$. Let $N^\omega = \pi_\omega(\ell^\infty(N))''$ a finite von Neumann algebra in $\mathbb{B}(\mathcal{H}_\omega)$, which has trace $x \mapsto \langle x\xi_\omega, \xi_\omega\rangle$ which we continue to denote by tr_ω. In this way N^ω is a non separable type II_1 von Neumann algebra, called the *ultraproduct* of N.[15]

We have the natural inclusion of N into N^ω, given by taking $x \in N$ to the sequence (x_n) with each $x_n = x$ and then applying π_ω. Define $N_\omega = N^\omega \cap N'$, a finite von Neumann algebra. In [12], Dixmier showed this central sequence algebra is either trivial or diffuse.[16] This can also be found in [6], where the connection between the triviality of N_ω and automorphisms of N is developed. We should also note that referring to N_ω as a central sequence algebra is reasonable - elements of $N^\omega \cap N'$ *are* images under π_ω of sequences $(x_n)_{n=1}^\infty$ which are ω-centralising in that

$$\lim_{n\to\omega} \|x_n y - y x_n\|_2 = 0,$$

for all $y \in N$. This is not immediately obvious from the definition, since we have taken a weak closure in the formation of N^ω, but it is well known - see for example [69, Theorem XIV.4.6]. When two ω-centralising sequences give the same element in N_ω we shall call them *ω-equivalent.*

Given $M \subset N$ we regard M^ω as a subset of N^ω allowing us to define the central sequence algebra $M^\omega \cap N'$. For the following result, which is surely well known, it suffices to follow Takesaki's account of the original in [69, Theorem XIV.4.7] and check that elements can be chosen in M where necessary.

Proposition 1.2.13. *Let M be a diffuse von Neumann subalgebra of a II_1 factor N, then $M^\omega \cap N' \subset N_\omega$ is either trivial or diffuse. Elements of $M^\omega \cap N'$ are the (images under π_ω) of ω-centralising sequences lying in M. The latter case occurs precisely when M contains non-trivial centralising sequences for N. Furthermore, unitaries in $M^\omega \cap N'$ are the π_ω images of ω-centralising sequences of unitaries in M.*

Recently, Fang, Ge and Li have considered another object. In section 3 of [18] they examine $N^\omega \cap M'$ when $M \subset N$ is an irreducible inclusion of II_1 factors, showing that it too is either trivial or diffuse. This work also extends a result of Connes ([7]), which was also known to McDuff ([34]), that the von Neumann tensor product of two II_1 factors has property Γ if and only if at least one of the

[15] This is unfortunate, it would have been nice to prefix the entire thesis with a disclaimer asserting the separability of everything within.

[16] A diffuse von Neumann algebra is one with no minimal projections.

two factors does.[17] The result of [18] shows more.

Theorem 1.2.14 ([18, Theorem 4.7]). *Let ω be a free ultrafilter on $\mathbb{N}$. Suppose that M is a non-Γ factor of type II_1 and N is another type II_1 factor. Then $(M\overline{\otimes}N)_\omega$ is canonically isomorphic to N_ω.*

Unfortunately we need more still. Like the rest of this section everything is well behaved when we consider inclusions of diffuse von Neumann subalgebras. No essential changes are required to work in this situation, but[18] this time we give the details, as [18] is not yet readily available.

Theorem 1.2.15. *Let $M_1 \subset N_1$ and $M_2 \subset N_2$ be inclusions of diffuse von Neumann subalgebras in II_1 factors. Suppose that M_1 does not contain non-trivial centralising sequences for N_1, then $(M_1\overline{\otimes}M_2)^\omega \cap (N_1\overline{\otimes}N_2)'$ is canonically isomorphic to $M_2^\omega \cap N_2'$.*

Proof. Since M_1 does not contain centralising sequences for N_1, we can find $K > 0$ and unitaries $u_1, \dots, u_l \in N_1$ such that

$$\|x - \mathrm{tr}(x)1\|_2^2 \leq K \sum_{i=1}^{l} \|xu_i - u_ix\|_2^2, \tag{1.2.1}$$

for all $x \in M_1$.[19] Take an ω-centralising sequence $(z_n)_{n=1}^\infty$ in $M_1\overline{\otimes}M_2$ for $N_1\overline{\otimes}N_2$. By density we may assume that each z_n is a sum of elementary tensors: i.e.

$$z_n = \sum_{j=1}^{m_n} x_j^{(n)} \otimes y_j^{(n)},$$

with $x_j^{(n)} \in M_1$ and $y_j^{(n)} \in M_2$. Furthermore, for each n we can demand that the $y_j^{(n)}$ are orthogonal in $L^2(N_2)$ and have $\|y_j\|_2 = 1$. We write $\mathbb{E}_{N_2}$ for the conditional expectation onto N_2 (regarded as $\mathbb{C}1\otimes N_2$ - a von Neumann subalgebra of $N_1\overline{\otimes}N_2$), and note that

$$\mathbb{E}_{N_2}(z_n) = \sum_{j=1}^{m_n} \mathrm{tr}(x_j^{(n)})1 \otimes y_j^{(n)}.$$

[17] Compare with taking an infinite von Neumann tensor product of II_1 factors - this always produces a Γ factor.

[18] In contrast with Proposition 1.2.13.

[19] This is the diffuse von Neumann subalgebra version of Lemma 4.6 of [18] and, just as there, it follows directly from the definition.

We now estimate $\|z_n - \mathbb{E}_{N_2}(z_n)\|_2$, using (1.2.1):

$$\begin{aligned}\|z_n - \mathbb{E}_{N_2}(z_n)\|_2^2 &= \left\|\sum_{j=1}^{m_n}(x_j^{(n)} - \mathrm{tr}(x_j^{(n)})1)\otimes y_j^{(n)}\right\|_2^2 \\ &= \sum_{j=1}^{m_n}\left\|x_j^{(n)} - \mathrm{tr}(x_j^{(n)})1\right\|_2^2 \qquad (1.2.2)\\ &\leq K\sum_{j=1}^{m_n}\sum_{i=1}^{l}\left\|x_j^{(n)}u_i - u_i x_j^{(n)}\right\|_2^2 \qquad (1.2.3)\\ &= K\sum_{i=1}^{l}\|z_n(u_i\otimes 1) - (u_i\otimes 1)z_n\|_2^2. \qquad (1.2.4)\end{aligned}$$

Here (1.2.2) and (1.2.4) follow from our additional hypotheses on the form of the $y_j^{(n)}$, and (1.2.3) follows from (1.2.1). Since $(z_n)_{n=1}^\infty$ is ω-centralising for $N_1\overline{\otimes}N_2$, we see that

$$\lim_{n\to\omega}\|z_n - \mathbb{E}_{N_2}(z_n)\|_2 = 0,$$

which is the ω-equivalence of $(z_n)_{n=1}^\infty$ and $(\mathbb{E}_{N_2}(z_n))_{n=1}^\infty$, which lies in $(1\otimes M_2)^\omega$.

□

1.2.6 Automorphisms of II_1 factors

The group of automorphisms of a separable II_1 factor N is a well studied object. Equipped with the so called *u-topology*[20] of pointwise norm convergence on the predual N_*, as defined in [22], $\mathrm{Aut}(N)$ is a Polish space - that is a complete metric space - see [6]. We shall prefer to work with pointwise $\|.\|_2$-convergence, which gives the same topology on $\mathrm{Aut}(N)$. It is necessary though to ensure we genuinely work with this only on the automorphism group. Suppose we have automorphisms θ_n of N and some θ such that

$$\lim_{n\to\infty}\|\theta_n(x) - \theta(x)\|_2 = 0. \qquad (1.2.5)$$

Then θ is necessarily an injective *-homomorphism, but not necessarily an automorphism.[21] If we know that θ *is* an automorphism though, then (1.2.5) is equivalent to the convergence of θ_n to θ in the u-topology.

The normal subgroup of inner automorphisms, written here as $\mathrm{Inn}(N)$,[22] consist of all automorphisms of the form $\mathrm{Ad}\, u : x \mapsto uxu^*$ for some unitary $u \in N$. The quotient $\mathrm{Aut}(N)/\mathrm{Inn}(N)$ is the *outer automorphism* group of N, written $\mathrm{Out}(N)$.

[20] Formally, a net $(\theta_\alpha)_\alpha$ in $\mathrm{Aut}(N)$ converges to $\theta \in \mathrm{Aut}(N)$ if and only if $\|\phi\circ\theta_\alpha - \phi\circ\theta\| \to 0$ for all $\phi \in N_*$.

[21] Indeed examples can be given in the abelian von Neumann algebra $L^\infty[0,1]$ of this failure.

[22] Also denoted Int N in the literature, due to the French tradition in the subject.

The *approximately inner automorphisms*, namely the u-topology closure $\overline{\mathrm{Inn}\,(N)}$ of the inner automorphisms, was examined by Connes in [6]. He showed that the inner automorphisms of N are closed in $\mathrm{Aut}\,(N)$ if and only if N fails to have property Γ. In his classification of injective factors ([7, Theorem 3.1]) he went on to characterise the approximately inner automorphisms precisely, Theorem 1.2.16 below. He noted, see [69, Theorem XIV.2.16], that all automorphisms of the hyperfinite II_1 factor are approximately inner, and characterised hyperfiniteness [7, Theorem 5.1], amongst other ways, by the property that the swap automorphism on $N\overline{\otimes}N$ taking $x \otimes y$ to $y \otimes x$ is approximately inner. In section 3.5, we shall examine these ideas in the relative context of masas inside II_1 factors.

Theorem 1.2.16 (Connes). *Let N be a factor of type II_1 with separable predual acting on $L^2(N)$. Then the following conditions are equivalent for $\theta \in Aut(N)$:*

1. *$\theta \in \overline{Inn\,(N)}$;*

2. *There exists an automorphism of the C^*-algebra generated by N and N' in $L^2(N)$ which is θ on N and the identity on N';*

3. *For any unitary operators $u_1, \dots, u_n \in N$ and any $\epsilon > 0$ there is a $\xi \in L^2(N)$, with $\|\xi\|_2 = 1$ and $\|\theta(u_k)Ju_kJ\xi - \xi\|_2 < \epsilon$ for all $k = 1, \dots, n$;*

4. *There exists a bounded sequence $(x_n)_{n=1}^{\infty}$ in N, not converging strongly to 0, such that $x_ny - \theta(y)x_n$ converges to 0 strongly, for any $y \in N$.*

1.3 Conditional expectations, the basic construction and the Jones index

Throughout this thesis we will be examining a variety of inclusions $1 \in M \subset N$ of von Neumann algebras with the same unit. A key tool to study this situation is a conditional expectation operator, namely a norm 1, M-bimodule projection from N onto M. For general N and M, the existence of these conditional expectations is not guaranteed. Furthermore, the theory is also fairly involved - the enthusiastic reader will find a full account in [65]. Fortunately when N is a finite von Neumann algebra, as it is throughout, things are much easier. As usual in this situation we fix a normalised faithful normal trace tr on N, and now look only for conditional expectations preserving this trace.

In this case a trace preserving conditional expectation onto M not only exists, it is also unique and can be explicitly constructed. We write e_M for the orthogonal projection from $L^2(N, \mathrm{tr})$ onto $L^2(M, \mathrm{tr})$. If we regard N and M as subspaces of $L^2(N, \mathrm{tr})$ and $L^2(M, \mathrm{tr})$ respectively, then it can be easily checked that $e_M\,(N)$

is actually contained in M. We write $\mathbb{E}_M$ for the restriction of e_M to N, the (trace-preserving) *conditional expectation* from N onto M, which is the unique bounded linear map from N into M satisfying

- $\mathbb{E}_M(1) = 1$.
- M-bimodularity, i.e. $\mathbb{E}_M(m_1 x m_2) = m_1 \mathbb{E}_M(x) m_2$ for all $m_1, m_2 \in M$ and $x \in N$.
- $\mathrm{tr} \circ \mathbb{E}_M = \mathrm{tr}$

all of which can be easily verified from the construction. The uniqueness of the conditional expectation has many useful applications, for example it allows us to immediately deduce that

$$\mathbb{E}_{uMu^*}(x) = u\mathbb{E}_M(u^*xu)u^*, \tag{1.3.1}$$

for any unitary $u \in N$ and all $x \in N$. We shall also require a result of Christensen, for computing conditional expectations, which can be found in [4].

Proposition 1.3.1 (Christensen). *Let M be a von Neumann subalgebra of the II_1 factor N. For each $x \in N$, let $co_M(x)$ denote the convex hull of the set $\{ uxu^* \mid u$ a unitary in $M \}$. Let $E(x)$ denote the element of minimal $\|.\|_2$ in the $\|.\|_2$-closure of $co_M(x)$. This $E(x)$ lies in N, and has $E(x) = \mathbb{E}_{M' \cap N}(x)$.*

1.3.1 The basic construction and Jones index

The *basic construction*, which dates back to [64] and [4], and was developed extensively by Vaughan Jones in [28], associates to an inclusion $1 \in M \subset N$ of finite von Neumann algebras with fixed trace tr on N, the extension $\langle N, e_M \rangle$, defined to be the von Neumann algebra acting on $L^2(N, \mathrm{tr})$ generated by N and e_M. We shall not be heavily involved with the basic construction in the main body of this thesis, so content ourselves here by just stating the basic properties we shall need. A full discussion of these results can be found in a number of sources, including [28, Section 3.1] and [26, Chapter 3].

- $e_M x e_M = \mathbb{E}_M(x) e_M = e_M \mathbb{E}_M(x)$, for all $x \in N$.
- $\langle N, e_M \rangle = JM'J$ where, as usual, J denotes the modular conjugation operator on $L^2(N, \mathrm{tr})$.
- $\langle N, e_M \rangle$ is a factor if and only if M is a factor.
- The central support of e_M in $\langle N, e_M \rangle$ is 1.

The *Jones index*, $[N:M]$, for our inclusion $M \subset N$ can be defined using a variety of different methods: Murray and von Neumann's coupling constant ([36]) was used in Jones original paper [28]; other definitions involve examining the dimension of $L^2(N, \mathrm{tr})$ as an M-bimodule, which is the approach taken in Chapter 2 of [26]. We shall only be interested in the case where M and N are finite factors, so we are able to follow [75] and define the index, $[N:M]$, of M in N by

$$[N:M] = \begin{cases} \mathrm{Tr}(e_M)^{-1} & \langle N, e_M \rangle \text{ is a finite factor} \\ \infty & \langle N, e_M \rangle \text{ is an infinite factor} \end{cases}, \tag{1.3.2}$$

where in the first case, Tr denotes the unique normalised faithful trace on the finite factor $\langle N, e_M \rangle$. That this definition agrees with the more general non factor case is Proposition 3.1.7 of [28]. When the index is finite, we have the Markov property

$$\mathrm{Tr}(e_M x) = [N:M]^{-1}\mathrm{tr}(x), \tag{1.3.3}$$

for every $x \in N$. It is also important to note that, see for example [26, Corollary 2.3.6(b)], when we have a tower of inclusions of II_1 factors, $M \subset P \subset N$, the index is multiplicative in the sense that

$$[N:M] = [N:P][P:M].^{23} \tag{1.3.4}$$

The other elementary result we shall need in the sequel, is how the index interacts with tensor products. Let $M_1 \subset N_1$ and $M_2 \subset N_2$ be inclusions of II_1 factors. Working in the Hilbert space tensor product $L^2(N_1) \otimes L^2(N_2) \cong L^2(N_1 \overline{\otimes} N_2)$, the operator $e_{M_1 \overline{\otimes} M_2}$ of projection onto $L^2(M_1 \overline{\otimes} M_2)$ factorises as $e_{M_1} \otimes e_{M_2}$. In this way, $\langle N_1 \overline{\otimes} N_2, e_{M_1 \overline{\otimes} M_2} \rangle$ factorises as $\langle N_1, e_{M_1} \rangle \overline{\otimes} \langle N_2, e_{M_2} \rangle$ and then

$$\mathrm{Tr}_{\langle N_1 \overline{\otimes} N_2, e_{M_1 \overline{\otimes} M_2} \rangle}(e_{M_1 \overline{\otimes} M_2}) = \mathrm{Tr}_{\langle N_1, e_{M_1} \rangle}(e_{M_1}) \mathrm{Tr}_{\langle N_2, e_{M_2} \rangle}(e_{M_2}),$$

or alternatively

$$[N_1 \overline{\otimes} N_2 : M_1 \overline{\otimes} M_2] = [N_1 : M_1][N_2 : M_2]. \tag{1.3.5}$$

The main result of [28], is the striking observation that the index $[N:M]$ must take values in $\{\, 4\cos^2(\pi/k) \mid k = 3, 4, 5, \dots \,\} \cup [4, \infty]$. Jones also constructed examples showing all these values can be obtained when N is the hyperfinite II_1 factor. When $[N:M] < 4$, M is automatically irreducible, i.e. $M' \cap N = \mathbb{C}1$, see [28, Corollary 2.2.4]. All of Jones' initial examples with index greater than 4 are not irreducible; and the full range of all possible values for the index of irreducible subfactors is still unknown. In the sequel, we shall mainly be interested in the

[23] Here we have the usual convention that $x\infty = \infty$ for any x.

inclusion of a regular subfactor[24] – in this case Jones has shown the following additional restriction on the values of the index.

Proposition 1.3.2. *If* $1 \in M \subset N$ *is a unital inclusion of* II_1 *factors with* M *regular in* N*, then* $[N : M] \in \mathbb{N} \cup \{\infty\}$.

In the case when M is also assumed to be irreducible, which is all we shall use later, this can be found in [24]. The more general case is dealt with by the observation

> *A more refined analysis based on [27] shows that all regular subfactors have integer index.*

found on page 150 of Jones' book [21] with Goodman and de la Harpe.

1.3.2 Inclusions of index 2

Historically, the case of index 2 inclusions was the first to be considered. In 1960, Goldman characterised index 2 subfactors[25] as being those coming from cross products over $\mathbb{Z}_2$, ([20], see also [28, Corollary 3.4.3]). More precisely, if $1 \in M \subset N$ is a unital inclusion of II_1 factors with $[N : M] = 2$, then there exists a non-inner automorphism θ of M of order 2 such that N is *-isomorphic to $M \rtimes_\theta \mathbb{Z}_2$. Although we have not introduced cross products,[26] we shall briefly use this formulation. For our purposes, it is enough to regard $M \rtimes_\theta \mathbb{Z}_2$ as a *-subalgebra of the 2×2 matrices over M, by

$$N \cong M \rtimes_\theta \mathbb{Z}_2 = \left\{ \begin{pmatrix} x & y \\ \theta(y) & \theta(x) \end{pmatrix} \middle| x, y \in M \right\}. \tag{1.3.6}$$

Here M is included in $M \rtimes_\theta \mathbb{Z}_2$, by regarding M as isomorphic to the algebra of all matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix}.$$

Four years later, in her work [70] on semi-regular masas of varying lengths which we examine in section 2.3, Tauer distinguishes between subfactors of R using the following property for a subfactor M.

> *The invariant is the fact that the product of two operators in* $M^\perp$ *is always in* M.

[24] A pedagogical choice has left us with the undesirable situation of this concept not being specified until Definition 1.4.5 in the next section!

[25] Before the concept of index had been explicitly defined.

[26] and have no intention of doing so!

We have changed the notation slightly from the original statement, in the proof of Lemma 6.7 of [70]. The set $M^{\perp}$ consists, as one would expect, of all operators $x \in R$ orthogonal to M in the sense that $\mathbb{E}_M(x) = 0$.[27] We can see that any unital inclusion of subfactors $1 \in M \subset N$ of index 2 satisfies Tauer's property. Indeed, given such an inclusion, use Goldman's Theorem to write N in the form (1.3.6) for some $\theta \in \mathrm{Aut}(M)$ of order 2. Two elements of N orthogonal to M are then of the form

$$\begin{pmatrix} 0 & y_1 \\ \theta(y_1) & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & y_2 \\ \theta(y_2) & 0 \end{pmatrix}$$

for some $y_1, y_2 \in M$. These elements have product

$$\begin{pmatrix} y_1\theta(y_2) & 0 \\ 0 & \theta(y_1)y_2 \end{pmatrix},$$

which lies in M. In fact Tauer's property also essentially characterises inclusions of index 2 although, unlike Goldman who, as noted in [28], more or less explicitly defines the index by means of the M-dimension of $L^2(N)$, she was most likely unaware of this.

Proposition 1.3.3. *Let $1 \in M \subset N$ be a finite index unital inclusion of* II_1 *factors. The following two statements are equivalent:*

1. $[N : M] = 2$;

2. *If $x, y \in N$ have $\mathbb{E}_M(x) = \mathbb{E}_M(y) = 0$ then $xy^* \in M$.*

We have chosen to use the expression *essentially* characterises index two inclusions as it has proved awkward to extend this Proposition to show that no infinite index inclusion can satisfy condition 2 above. Let us first quickly see why, under the assumption of finite index, condition 2 guarantees that $[N : M] = 2$. The method we shall use here is the generic nature of the basic construction, [28, Corollary 3.1.9].

Proposition 1.3.4. *Let $1 \in M \subset N$ be a finite index, unital inclusion of* II_1 *factors. Then there is a subfactor $P \subset M$ with $[M : P] = [N : M]$ such that $\langle M, e_P \rangle \cong N$.*

We apply this by considering the element $x = 1 - [N : M]e_P \in \langle M, e_P \rangle$. Since $\mathbb{E}_M(e_P) = [N : M]^{-1}1$,[28] we have $\mathbb{E}_M(x) = 0$. On the other hand

$$xx^* = (1 - [N : M]e_P)^2 = 1 + \left([N : M]^2 - 2[N : M]\right) e_P,$$

[27] Formally, we should perhaps speak of orthogonality to $L^2(M)$ but, as we have previously noted, $\mathbb{E}_M(x) = e_M(x)$ for $x \in N$.

[28] We are regarding N as (isomorphic to) $\langle M, e_P \rangle$, so that e_P is an element of N and $\mathbb{E}_M$ is the conditional expectation from N onto the subfactor M.

so that $xx^* \in M$ if and only if $[N : M]^2 - 2[N : M] = 0$, from which we can deduce that $[N : M] = 2$. This concludes the proof of Proposition 1.3.3.

There is no hope of generalising the proof above to the case of an infinite index inclusion $M \subset N$. It is possible to give another, more involved proof, of the implication $2 \Rightarrow 1$ using Pimsner-Popa bases for the inclusion $M \subset N$. It is possible that this idea might generalise, for there is a notion of a generalised orthogonal basis for an infinite index inclusion ([47]), but this will consist of unbounded operators affiliated to M. An intricate analysis of these generalised orthogonal bases is needed, to determine whether these operators can be found in $L^2(M)$. In this thesis, we have been assiduous in our policy of only working with bounded operators - we leave the infinite index case as a conjecture.

Question 1.3.5. How do we extend Proposition 1.3.3 to the infinite index situation?

We shall briefly need these orthogonal bases in section 2.4. We will not develop the theory and just state exactly what we need. The result, which is well known, can easily be established by manipulating facts, from [39], about these Pimsner-Popa orthogonal bases. We give a proof which uses these facts implicitly.

Proposition 1.3.6. *Suppose that $M \subset N$ is an inclusion of II_1 factors, and that there are n unitaries $(u_i)_{i=1}^n$ in N with $\mathbb{E}_M\left(u_i u_j^*\right) = \delta_{i,j}1$ for all i, j, then $[N : M] \geq n$.*

Proof. We assume that $[N : M] < \infty$, otherwise the result is trivial. In $\langle N, e_M\rangle$, each $e_M u_i$ is a partial isometry with domain projection $u_i^* e_M u_i$ and range projection e_M. The hypothesis ensures that these domain projections are pairwise orthogonal, so that

$$\sum_{i=1}^{n} u_i e_M u_i^* \leq 1.$$

Take the trace of both sides to see that $n\mathrm{Tr}(e_M) \leq 1$, from which the result follows. □

1.3.3 Inclusions of finite dimensional C^*-algebras

Here, we give a brief exposition of the theory of inclusions of finite dimensional C^*-algebras, which are also necessarily weakly closed and so von Neumann algebras. All of this material can be found a variety of sources, such as [21, Section 2.3] and [26, Section 3.2]. Firstly, recall that every finite dimensional C^*-algebra is isomorphic to a direct sum of matrix algebras, see for example [10, Theorem

III.1.1]. Suppose that we have the identification

$$M \cong \mathbb{M}\mathrm{at}_{a_1}(\mathbb{C}) \oplus \mathbb{M}\mathrm{at}_{a_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}\mathrm{at}_{a_m}(\mathbb{C}),$$

where $\mathbb{M}\mathrm{at}_k(\mathbb{C})$ denotes the algebra of $k \times k$ matrices over $\mathbb{C}$, then

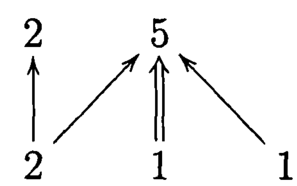
$$a = (a_1, \ldots, a_m)^{\mathrm{T}} \in \mathbb{N}^m,$$

is called the *dimension vector* of M. Similarly, let N have direct sum decomposition

$$N \cong \mathbb{M}\mathrm{at}_{b_1}(\mathbb{C}) \oplus \mathbb{M}\mathrm{at}_{b_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}\mathrm{at}_{b_n}(\mathbb{C}),$$

with corresponding dimension vector $b \in \mathbb{N}^n$. An inclusion $1 \in M \subset N$ or, more precisely, a unital injective *-homomorphism from M into N, is determined (up to unitary conjugation in N) by an $n \times m$ *inclusion matrix* Λ over $\mathbb{N}_0$, with $\Lambda a = b$.[29] The i,j-th entry $\lambda_{i,j}$ of Λ is naively defined to be the number of times the ith component $\mathbb{M}\mathrm{at}_{a_i}(\mathbb{C})$ of M is repeated in the jth summand $\mathbb{M}\mathrm{at}_{b_j}(\mathbb{C})$ of N.[30]

The data for a finite dimensional inclusion is also often contained in a *Bratteli diagram*, an concept best explained by example. The Bratteli diagram



represents the unital inclusion of M, with dimension vector $(2, 1, 1)^{\mathrm{T}}$, into N, with dimension vector $(2, 5)^{\mathrm{T}}$, by the inclusion matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Since the dimension vector of N is determined from that of M and Λ, the dimensions of N are often omitted from the diagram.

Since there is a unique normalised trace on any matrix algebra (as with any finite factor), a normalised trace tr on M is determined by a *trace vector*

$$s = (s_1, \ldots, s_m) \in \mathbb{R}_+^m,$$

with $sa = 1$. The components are given by $s_i = \mathrm{tr}(p_i)$, where p_i is a minimal projection in the ith summand $\mathbb{M}\mathrm{at}_{a_i}(\mathbb{C})$ of M, and tr is then a faithful trace

[29]There is no agreement in the literature about the orientation of the inclusion matrix. The convention we have chosen agrees with Davidson in [10] and Jones in [21], whereas Wenzl ([75]) and Jones (again!) in [28] and [26] prefer to work with row vectors and the transpose of this matrix.

[30]Formally we should examine the representation structure of M and N, defining $\lambda_{i,j}$ to be the number of times the ith irreducible representation of M occurs in the restriction to N of the jth irreducible representation of N but, as noted in [26], this does not provide additional illumination.

if each $s_i > 0$. If the trace on N is given by the trace vector $t \in \mathbb{R}^n_+$, then the restriction of tr to M has trace vector $s = t\Lambda$.

In this finite dimensional situation, the basic construction, $\langle N, e_M \rangle$ decomposes into m factors $\bigoplus_{i=1}^{m} \langle N, e_M \rangle_i$. A minimal projection for $\langle N, e_M \rangle_i$ can be found of the form $e_M p_i$, where p_i is a minimal projection in the ith summand of M. With this ordering of the decomposition of $\langle N, e_M \rangle$, the inclusion matrix of $1 \in N \subset \langle N, e_M \rangle$ is given by Λ^{T}. Since the inclusion matrix for a chain of finite dimensional inclusions is obtained by matrix multiplication[31] (e.g. [26, Corollary 2.3.2]), the inclusion matrix for $1 \in M \subset \langle N, e_M \rangle$ is $\Lambda^{\mathrm{T}}\Lambda$.

1.3.4 The Wenzl index formula

It is possible to construct an inclusion of factors by a chain of inclusions of finite dimensional C^*-algebras. The Wenzl index formula, Theorem 1.5 of [75], allows us to compute the resulting index for suitably periodic inclusions which will appear later. In fact we shall need a very slight extension of Wenzl's work which we formulate in this section. We begin with a definition of commuting squares [26, Definition 5.1.7], an idea originally due to Popa, who began the examinination of commutating conditional expectations in [43].

Definition 1.3.7. Let N_2 be a finite von Neumann algebra with fixed faithful trace tr. Consider the square of inclusions of von Neumann subalgebras of N_2

$$\begin{array}{ccc} N_1 & \hookrightarrow & N_2 \\ \uparrow & & \uparrow \\ M_1 & \hookrightarrow & M_2 \end{array}$$

with conditional expectations $\mathbb{E}_{M_1}$ from N_1 onto M_1, and $\mathbb{E}_{M_2}$ from N_2 onto M_2 obtained from the trace tr on N_2 in the usual way. The square is said to *commute* when $\mathbb{E}_{M_1}(x) = \mathbb{E}_{M_2}(x)$ for all $x \in N_1$ or, more formally, when

$$\begin{array}{ccc} N_1 & \hookrightarrow & N_2 \\ \downarrow{\scriptstyle \mathbb{E}_{M_1}} & & \downarrow{\scriptstyle \mathbb{E}_{M_2}} \\ M_1 & \hookrightarrow & M_2 \end{array}$$

is a commutative diagram of maps in the usual way. A diagram consisting of multiple (and possibly infinitely many) squares is a *commutative diagram* if each constituent square commutes.

We shall examine infinite commutative diagrams of inclusions of finite dimensional C^*-algebras M_n and N_n of the form of Figure 1.1, where Λ_n is the

[31]With our choice of notation, this happens in a contravariant way.

$$\begin{array}{ccccccccccccc}
N_1 & \hookrightarrow & N_2 & \hookrightarrow & \cdots & \hookrightarrow & N_n & \hookrightarrow & N_{n+1} & \hookrightarrow & \cdots & \hookrightarrow & N \\
\Lambda_1 \uparrow & & \Lambda_2 \uparrow & & & & \Lambda_n \uparrow & & \Lambda_{n+1} \uparrow & & & & \uparrow \\
M_1 & \hookrightarrow & M_2 & \hookrightarrow & \cdots & \hookrightarrow & M_n & \hookrightarrow & M_{n+1} & \hookrightarrow & \cdots & \hookrightarrow & M
\end{array}$$

Figure 1.1: The setup for the Wenzl index formula

inclusion matrix of $1 \in M_n \subset N_n$. We shall insist that the *-algebras $\bigcup_{n=1}^{\infty} M_n$ and $\bigcup_{n=1}^{\infty} N_n$ are infinite dimensional. Suppose that there is a *unique* normalised trace on the *-algebra $\bigcup_{n=1}^{\infty} N_n$, then N, defined to be the weak closure of the image of $\bigcup_{n=1}^{\infty} N_n$ under the GNS representation corresponding to tr, is a II_1 factor, which is hyperfinite by construction. We shall also require that the restriction of tr to $\bigcup_{n=1}^{\infty} M_n$ is the *unique* normalised trace on $\bigcup_{n=1}^{\infty} M_n$, so that M, defined to be the von Neumann subalgebra of N generated by $\bigcup_{n=1}^{\infty} M_n$, is also a hyperfinite II_1 factor. Actually, we only need to establish the uniqueness of these traces in the very limited situation covered by the next proposition.

Proposition 1.3.8. *Suppose that we have a chain of inclusions of finite dimensional C^*-algebras, with inclusion matrices Γ_n as indicated below.*

$$N_1 \xhookrightarrow{\Gamma_1} N_2 \hookrightarrow \cdots \hookrightarrow N_n \xhookrightarrow{\Gamma_n} N_{n+1} \hookrightarrow \cdots$$

If, for infinitely many n, every entry in the inclusion matrix Γ_n is identical, then there is at most one normalised trace on $\bigcup_{m=1}^{\infty} N_m$.

Proof. Take some n with the described property, so Γ_n has identical entries - say C_n. Consider a normalised trace tr on $\bigcup_{m=1}^{\infty} N_m$ with trace vector t on N_{n+1}. The trace vector s of N_n given by $s = t\Gamma_n$ has $s_i = C_n \sum_j t_j$, for each i. Hence, the restriction of tr to N_n is unique by normalisation. As there are infinitely many n for which the restriction of tr to N_n is unique, we see that there is at most one normalised trace on $\bigcup_{m=1}^{\infty} N_m$. □

If in addition we had supposed that all the inclusion matrices Γ_n in Proposition 1.3.8 had the described property, then an easy argument would also yield the existence of a normalised trace on $\bigcup_{n=1}^{\infty} N_n$. Despite this being exactly the situation occurring later, this is unnecessary as all of the algebras N_n and M_n will in fact lie in a larger hyperfinite II_1 factor, giving us the existence of a normalised trace by restriction.

The Wenzl index formula, Theorem 1.5 of [75], deals with diagrams of inclusions of the form of Figure 1.1 where we have additional periodicity requirements on the inclusions. We state the version of the formula from [21, Theorem 4.3.3], recalling that a matrix Λ (with entries in $\mathbb{R}_+$) is ***primitive*** when there is some $l \in \mathbb{N}$ such that all the entries of Λ^l are strictly positive.

Theorem 1.3.9 (Wenzl index formula). *Suppose we have finite dimensional C^*-algebras M_n and N_n as in the commutative diagram Figure 1.1. Suppose that there exists $n_0 \geq 1$ and $p \geq 1$ such that (with an appropriate ordering of the matrix algebras in the decompositions of M_n and N_n), we have for each $n \geq n_0$:*

1. *The inclusion matrix for $N_n \subset N_{n+1}$ is the same as that for $N_{n+p} \subset N_{n+p+1}$, and the inclusion matrix for $M_n \subset M_{n+1}$ is the same as that for $M_{n+p} \subset M_{n+p+1}$;*
2. *The inclusion matrices for $N_n \subset N_{n+p}$ and $M_n \subset M_{n+p}$ are primitive;*
3. *$\Lambda_n = \Lambda_{n+p}$.*

Then $M \subset N$ is an inclusion of II_1 factors with

$$[N : M] = \|\Lambda_n\|^2 ,$$

for every $n \geq n_0$.

As noted in [21] the periodicity data for the inclusions $N_n \subset N_{n+1}$ is only required to establish that N is a factor. While the index formula as stated will suffice to compute the index in one of the situations we shall require, in another case the size of the inclusion matrices Λ_n will increase with n, although they will retain the same structure so some periodicity will remain. We shall establish a version of the Wenzl index formula in this situation as a corollary of the next Theorem, in which Wenzl examines the structure of general extensions of the basic construction in finite dimensions.

Theorem 1.3.10 (Wenzl - [75, Theorem 1.1]). *Let $1 \in M \subset N$ be an inclusion of finite dimensional C^*-algebras acting on the Hilbert space $\mathcal{H}$, with dimension vector $a = (a_1, \dots, a_m)$ for M. Fix a normalised faithful trace tr on N, whose restriction to M has trace vector s. Suppose that e is a projection in $\mathbb{B}(H)$ such that*

- *$exe = e\mathbb{E}_M(x) = \mathbb{E}_M(x)e$ for all $x \in N$*
- *$eM \cong M \cong Me$,*

and let $\langle N, e\rangle$ be the (finite dimensional) C^*-algebra generated by N and e.

1. *$\langle N, e\rangle \cong \langle N, e_M\rangle \oplus K$ where K is isomorphic to a subalgebra of N.*

2. *The central projection z in $\langle N, e\rangle$ onto $\langle N, e_M\rangle$ (under the isomorphism of 1) coincides with the central support of e in $\langle N, e\rangle$.*

3. *Let Tr be a trace on $\langle N, e\rangle$ extending tr. Then $Tr(e) \geq d Tr(z)$, where $d = \min_{i=1,\dots,m} a_i/(\Lambda^T\Lambda a)_i$.*

4. *Let t be the trace vector of $Tr|_{\langle N, e_M\rangle}$ (under the isomorphism in 1), then $t\Lambda^T\Lambda \leq s$ pointwise.*

We now come to our well trailed corollary designed for application in section 2.4. It should be noted that the proof is a combination of Lemma 1.4 and Theorem 1.5(i) of [75] - we include it for completeness.

Corollary 1.3.11. *Suppose that in the situation of the commutative diagram of Figure 1.1 there is a unique faithful normalised trace on $\bigcup_{n=1}^{\infty} N_n$ and on $\bigcup_{n=1}^{\infty} M_n$. Suppose also that the dimension vectors $a^{(n)}$ for M_n are of the form*

$$a^{(n)} = A_n \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^T,$$

for some constants A_n, and that there exists an integer $\lambda \geq 2$ such that each of the inclusion matrices Λ_n takes the form

$$\Lambda_n = \overbrace{\begin{pmatrix} I & I & \dots & I \end{pmatrix}}^{\lambda},$$

where I is some identity matrix (whose size may vary with n).[32] *Then $M \subset N$ is an inclusion of II_1 factors with*

$$[N : M] = \lambda.$$

Proof. By hypothesis $1 \in M \subset N$ is an inclusion of II_1 factors. Let Tr be a (possibly semifinite) faithful normal trace on the basic construction $\langle N, e_M\rangle$. Since $e_M \langle M, e_N\rangle e_M = Me_M \cong M$ is finite, $\mathrm{Tr}(e_M) < \infty$.

Observe that the hypothesis on the form of the $a^{(n)}$ and Λ_n, guarantee that

$$\Lambda_n^{\mathrm{T}} \Lambda_n a^{(n)} = \lambda a^{(n)}. \tag{1.3.7}$$

The extensions $\langle N_n, e_M\rangle$ satisfy the hypothesis of Theorem 1.3.10, and so writing z_n for the central support of e_M in $\langle N_n, e_M\rangle$, part 3 of this Theorem and (1.3.7) gives the estimate

$$\mathrm{Tr}(z_n) \leq \lambda \mathrm{Tr}(e_M).$$

[32] Note that it is the possible change in size of these identity matrices which prevents us from using the usual Wenzl index formula in this situation.

Since $\cup_{n=1}^{\infty}\langle N_n, e_M\rangle$ is dense in $\langle N, e_M\rangle$, these z_n converge to 1, the central support of e_M in $\langle N, e_M\rangle$. In particular $\mathrm{Tr}(1) \leq \lambda\mathrm{Tr}(e_M) < \infty$, so Tr is a finite trace, and normalisation gives the estimate

$$[N : M] \leq \lambda.$$

For the reverse inequality, recall from subsection 1.3.3, that we can find a minimal projection in the ith summand of $\langle N_n, e_{M_n}\rangle$ of the form $p_i e_{M_n}$, for some minimal projection p_i in the ith summand of M_n. Let t be the trace vector of Tr on $\langle N_n, e_{M_n}\rangle$ and s be the corresponding trace vector on M_n. By the Markov property, (1.3.3), we have

$$t_i = \mathrm{Tr}(e_{M_n}p_i) = \mathrm{Tr}(e_M p_i) = \mathrm{Tr}(p_i)[N : M]^{-1} = s_i[N : M]^{-1}.$$

Part 4 of Theorem 1.3.10 then gives

$$t_i[N : M] \geq (t\Lambda^{\mathrm{T}}\Lambda)_i, \tag{1.3.8}$$

for every i. As each $(t\lambda^{\mathrm{T}}\Lambda)_i$ is the sum of λ elements t_j, we have $(t\Lambda^{\mathrm{T}}\Lambda)_i \geq \lambda \min_j t_j$, for each i. Plug this into (1.3.8) and minimise the left hand side over i to obtain

$$[N : M] \geq \lambda,$$

as required. □

1.4 Masas in II_1 factors

The study of maximal abelian self-adjoint *-subalgebras of II_1 factors, called *masas* hereafter, dates back to Murray and von Neumann's foundation of the subject in [35] and [36], where they appear naturally in a cross-product construction. By Zorn's lemma, these masas exist in abundance inside any II_1 factor, although for some considerable time the main emphasis has been on explicitly constructible masas.

In [11], Dixmier perhaps began the modern study of masas when he classified them by their normalisers, using this idea to define Cartan and singular masas - discussed later in subsection 1.4.2. Early in the 60's, Pukánszky ([51]) and Tauer ([70]) both gave countable families of pairwise non-conjugate masas in the hyperfinite II_1 factor. Both these classes of examples will appear within: the idea behind Tauer's construction makes up the main body of the thesis and her masas can be found in sections 2.2 and 2.3; Pukánszky's idea will be touched upon again after Definition 1.4.5 and developed fully in Chapter 3. More recently, masas have been used as tools in various areas of von Neumann algebra theory; Sinclair and Smith ([58]) showed that a separable II_1 factor containing a Cartan masa has vanishing continuous Hochschild cohomology - a concept we have no intention of allowing further discussion of here![33] Sorin Popa, in [49], uses the uniqueness of a certain Cartan masa[34] to demonstrate that $\mathcal{L}\left(\mathrm{SL}(2,\mathbb{Z}) \rtimes Z_2\right)$ has a trivial fundamental group. Over the last couple of years this amazing technique of Popa's has been used to resolve many major problems in the classification of II_1 factors, [40, 41, 23].

I would like to end these introductory remarks with a plug for Allan Sinclair and Roger Smiths' forthcoming book, [56], which will cover in full detail almost all the material in this section. Indeed, the concept of strong-singularity, see subsection 1.4.3, was initially developed to produce a cleaner proof of the singularity of certain masas for this work.

1.4.1 Basic properties of masas

As von Neumann algebras, masas inside separable II_1 factors are all the same. This result dates back to Murray and von Neumann, a proof can also be found in [57, Lemma 5.3.4].

Proposition 1.4.1. *Let A be a masa in a separable II_1 factor N. There is a *-isomorphism from A onto $L^\infty[0,1]$ which induces an isometry between $L^2(A,tr)$ and $L^2[0,1]$.*

We should study the inclusion $A \subset N$ of a masa inside a II_1 factor, rather than just the masa itself. The terminology in the literature is that of conjugacy.

Definition 1.4.2. Two masas A and B in a II_1 factor N are said to be *conjugate via an automorphism of* N, or sometimes lazily just *conjugate in* N, if there exists an automorphism θ of N with $\theta(A) = B$. They are *unitarily conjugate in* N if θ is an inner automorphism of N.

We wait until the next subsection to see some non-conjugate masas. In the remains of this subsection we collate some well known basic properties of masas. It is immediate that a von Neumann subalgebra A of a II_1 factor N is a masa if and only if it is its own relative commutant, that is $A = A' \cap N$. By thinking of A as $L^\infty[0,1]$, we can find a chain, $A_1 \subset A_2 \subset \dots$, of finite dimensional abelian C^*-algebras generating A as a von Neumann algebra. Popa connected these two observations to give the following criterion for determining when such a chain

[33] See [57] for an account of this cohomology theory.

[34] a so called HT masa.

generates a masa, which he used to good effect in [45]. A proof can also be found in [57, Lemma 5.3.2].

Proposition 1.4.3. *Given a chain $(A_n)_{n=1}^{\infty}$ of finite-dimensional abelian C^*-algebras in a* II_1 *factor N, let $A = (\bigcup_{n=1}^{\infty} A_n)''$ be the abelian von Neumann algebra it generates. Then A is a masa in N if and only if*

$$\lim_{n\to\infty} \left\|\mathbb{E}_{A_n}(x) - \mathbb{E}_{A_n'\cap N}(x)\right\|_2 = 0, \tag{1.4.1}$$

for all $x \in N$.

Both the conditional expectations appearing in (1.4.1) can easily be calculated. Suppose B is a finite dimensional abelian C^*-algebra with minimal projections $(e_i)_{i=1}^n$, contained in a II_1 factor N. Then $(e_i/\|e_i\|_2)_{i=1}^n$ is an orthonormal basis for $L^2(B)$ so that

$$\mathbb{E}_B(x) = \sum_{i=1}^n \frac{\mathrm{tr}(xe_i)e_i}{\|e_i\|_2^2},$$

for all $x \in N$. It is also well known that the conditional expectation onto the relative commutant is given by

$$\mathbb{E}_{B'\cap N}(x) = \sum_{i=1}^n e_i x e_i, \tag{1.4.2}$$

for all $x \in N$. This can be seen by checking that (1.4.2) defines a conditional expectation, then appealing to uniqueness.

In the case of von Neumann algebras coming from groups, we have a well known criterion for an inclusion of groups to give rise to a masa, which dates back to Diximer, [11].

Proposition 1.4.4. *Let H be an abelian subgroup of the countable discrete group G. Then $\mathcal{L}(H)$ is a masa in $\mathcal{L}(G)$ if and only if the set*

$$\left\{ hgh^{-1} \,\middle|\, h \in H \right\},$$

is infinite, whenever $g \in G \setminus H$.

1.4.2 Normalisers of masas

When $M \subset N$ is an inclusion of von Neumann algebras, we consider the group $\mathcal{N}(M)$ of all unitary normalisers of M in N given by

$$\mathcal{N}(M) = \left\{ u \in \mathcal{U}(N) \,|\, uMu^* = M \right\}.$$

When the larger von Neumann algebra is not obvious, we write $\mathcal{N}_N(M)$ for $\mathcal{N}(M)$. These groups where introduced by Dixmier ([11]) to classify masas by examining how many normalisers there are, with the Cartan masas at one extreme and the singular masas at the other.

Definition 1.4.5 (Dixmier). A von Neumann subalgebra M of a von Neumann algebra N is said to be *Cartan* or *regular* when $\mathcal{N}(M)'' = N$,[35] and *singular* when $\mathcal{N}(M)'' = M$. A masa A in N is called *semi-regular* when $\mathcal{N}(A)''$ is a proper subfactor of N.

These concepts gave the first examples, also in [11], of non-conjugate masas in a II_1 factor. One of the principal successes of this classification program is the remarkable result of Connes, Feldman and Weiss, [8] (see also [46]), Theorem 1.4.6 below, on the uniqueness of the Cartan masa in R. Examples of non-conjugate Cartan masas in a non-injective II_1 factor were later given in [9]. In [45], Popa showed that singular masas can always be found in any separable II_1 factor, on the other hand, Cartan masas are not always present; Voiculescu (in [73]) has shown that no Cartan masa exists in a free group factors, $\mathcal{L}(\mathbb{F}_k)$.

Theorem 1.4.6 (Connes, Feldman and Weiss). *Any two Cartan masas in the hyperfinite* II_1 *factor* R *are conjugate via an automorphism of* R.

Pukánszky's examples ([51]) gave countably many pairwise non-conjugate singular masas in the hyperfinite II_1 factor R. His method, for showing this non-conjugacy is now known as the *Pukánszky invariant* for a masa, an idea discussed at length in chapter 3, and which has recently been used by Størmer and Neshveyev ([38]) to give uncountably many pairwise non-conjugate singular masas in R. In section 3.3 we will give an alternative method of obtaining uncountably many pairwise distinct singular masas in R. It will not be possible to use Pukánszky's invariant to distinguish between these masas.

When a masa is Cartan, one can normally verify this be writing down a collection of normalisers which generate the underlying II_1 factor. Similarly in the case when A is a semi-regular masa in N with $\mathcal{N}(A)'' = M$ for some given subfactor M of N, the inclusion $M \subset \mathcal{N}(A)''$ should be easy to verify simply by exhibiting enough normalisers. To do this, it is often helpful to look at the *groupoid normaliser* of A, $\mathcal{GN}(A)$, consisting of all partial isometries $v \in N$ with initial and range projections in A and which normalise A, in the sense that $vAv^* = Avv^*$. This groupoid normaliser generates the same von Neumann algebra as the normalisers do, as (see [25, 46]) elements of $\mathcal{GN}(A)$ are precisely of the form ue for some $u \in \mathcal{N}(A)$ and projection $e \in A$.

However, the reverse inclusion, $\mathcal{N}(A)'' \subset M$ is in general much harder to establish. Proving that a given masa is singular is of a similar level of difficulty,

[35] Cartan is used in the context of masas, while other von Neumann algebras are called regular. Indeed, currently the expression Cartan subalgebra is used in the literature to mean a Cartan masa.

as the same style of inclusion has to be shown. Currently, the preferred method of establishing singularity is to use *strong singularity*, as the name suggests, a stronger concept than singularity which is often easier to verify in practice. We will discuss this idea further in subsections 1.4.3 and 1.4.4. Here we give a brief discussion of other methods for establishing upper bounds for $\mathcal{N}(A)$ that are either historically interesting or appear later, concentrating mainly on the singular case.

We have already mentioned, in subsection 1.2.3, that Popa used orthogonality to give a method for controlling the location of normalising unitaries. The main technical result is Corollary 2.6 of [44], which we state for completeness. We remind the reader that a diffuse von Neumann algebra is one with no minimal projections.

Proposition 1.4.7 (Popa). *Let M be a von Neumann subalgebra of the finite von Neumann algebra N and u be a unitary in N. If there exists a diffuse von Neumann subalgebra M_0 of M such that uM_0u^* is orthogonal to M, then u is orthogonal to $\mathcal{N}(M)''$.*

This proposition was particularly useful in the context of group von Neumann algebras, where it gives Proposition 4.1 of [44] below.

Corollary 1.4.8 (Popa). *Let $H \subset H_1 \subset G$ be an inclusion of infinite discrete groups. If $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H_1$, then whenever M is a diffuse von Neumann subalgebra of $\mathcal{L}(H)$, we have $\mathcal{N}(M)'' \subset \mathcal{L}(H_1)$.*

If we take $H = H_1$ in the above then we obtain the group theoretic condition of *malnormality*, namely that for every $g \in G \backslash \mathcal{H}$ we have $gHg^{-1} \cap H = \{1\}$. Suppose in addition that H is abelian, then $\mathcal{L}(H)$ is a masa in $\mathcal{L}(G)$ by Proposition 1.4.4[36] and so is diffuse. In this instance Corollary 1.4.8 gives the singularity of $\mathcal{L}(H)$ in $\mathcal{L}(G)$.

Popa is also responsible for two other methods of demonstrating the singularity of a masa. In [46], he gave a connection between the Pukánszky invariant of a masa and its normalisers - this is discussed in section 3.1, where the Pukánszky invariant is defined. The tool he used in [45], to show that singular masas exist in any separable II_1 factor was a δ-invariant. Formally when A is a masa in N, and $v \in N$ is a partial isometry in N such that v^*v and vv^* are mutually orthogonal projections in A define

$$\delta(v) = \sup_{x \in vAv^*, \|x\| \leq 1} \frac{\|x - \mathbb{E}_A(x)\|_2}{\|v^*v\|_2},$$

[36] It is easily checked that the malnormality of H in G implies that $\{ hgh^{-1} \mid h \in H \}$ is infinite for every $g \in G \setminus H$.

which takes values in $[0,1]$, and measures the distance between vAv^* and A.[37] When v is a *groupoid normaliser* of A (i.e. $vAv^* \subset A$), then $\delta(v) = 0$. Popa's δ-invariant is defined to be the infimum of these distances

$$\delta(A) = \inf\{\, \delta(v) \,|\, v^*v, vv^* \text{ are mutually orthogonal projections in } A\,\}.$$

If $\delta(A) > 0$, then A is necessarily singular, and indeed in [45] Popa obtains singular masas by showing that a masa with $\delta(A) > 10^{-4}$ can be found in any separable II_1 factor. More recently, in [48], Popa has gone on to show that any singular masa in a separable II_1 factor has $\delta(A) = 1$, so the δ-invariant only takes the values 0 or 1. We can view this result as a starting point for the perturbation work of Popa, Sinclair and Smith, [50], which we discuss in the next subsection. As noted in [45], if A is a masa in N with $\delta(A) > 0$, then A^ω is a singular masa in N^ω, whenever $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.[38] In particular, any singular masa A in N, must then give rise to a singular masa A^ω in N^ω. On the other hand, when D is the Cartan masa in R, D^ω is not Cartan in R^ω. In [44], Popa showed that if N is a separable II_1 factor then N^ω contains no Cartan masas. Recently, in [18, Corollary 6.2], it was observed that the same method can be used to see that there are no Cartan masas in $R^\omega \cap R'$.

We end this section by noting that in matrix algebras, a.k.a. finite type I factors, these concepts are moot. Here, by the elementary process of simultaneous diagonalisation, all masas are unitarily conjugate and Cartan.

Proposition 1.4.9. *Let N be a finite type* I *factor. Any two masas A and B in N are unitarily conjugate and*

$$\{\, u \in \mathcal{U}(N) \,|\, uAu^* = B\,\}'' = N.$$

1.4.3 A metric on masas and strong singularity

We begin by introducing a norm for bounded linear maps between II_1 factors, which naturally gives a metric on the set of all von Neumann subalgebras of a II_1 factor.

Definition 1.4.10 (Sinclair and Smith). Let M and N be II_1 factors. Given a bounded linear map $\Phi : M \to N$, we define $\|\Phi\|_{\infty,2}$ to be the norm of Φ regarded as an operator from M into $L^2(N)$. Formally, we have

$$\|\Phi\|_{\infty,2} = \sup_{x \in M, \|x\| \le 1} \|\Phi(x)\|_2 .$$

[37] More accurately, the distance between vAv^* and Avv^*.

[38] That a masa A in N gives rise to a masa A^ω in N^ω, in this situation is an easy calculation that can be found in [44], where Popa also shows that there are no Cartan masas in N^ω.

This norm gives rise to a metric $d_{\infty,2}$ on the von Neumann subalgebras M of a II_1 factor by

$$d_{\infty,2}(M_1, M_2) = \|\mathbb{E}_{M_1} - \mathbb{E}_{M_2}\|_{\infty,2}.$$

As noted in [50], the metric $d_{\infty,2}$ is equivalent to an older metric on this space defined by Erik Christensen in [4]. Although we do not define Christensen's metric here, preferring to work with the $d_{\infty,2}$-metric throughout, we shall collate various properties which appear in his work.

Proposition 1.4.11 (Christensen). *The $d_{\infty,2}$-metric makes the set of all von Neumann subalgebras of a II_1 factor N into a complete metric space. The map taking M to its relative commutant $M' \cap N$ is $d_{\infty,2}$-continuous, and the following sets are $d_{\infty,2}$-closed:*

1. *The set of all masas in N;*
2. *The set of all singular masas in N;*
3. *The set of all subfactors of N;*
4. *The set of all subfactors with trivial relative commutant in N.*

In [59] and [54] the concept of a *strongly singular* von Neumann subalgebra of a II_1 factor was introduced. The idea, based on Popa's δ-invariant (which we could have couched in terms of the $d_{\infty,2}$-metric) is to control the distance of a unitary u to the subalgebra M by the distance between M and uMu^*.

Definition 1.4.12 (Sinclair and Smith). Let $M \subset N$ be a von Neumann subalgebra of the II_1 factor N. For $\alpha \in (0,1]$, M is said to be *α-strongly singular* if for every unitary $u \in N$ we have

$$\alpha \|u - \mathbb{E}_M(u)\|_2 \leq \|\mathbb{E}_M - \mathbb{E}_{uMu^*}\|_{\infty,2} = d_{\infty,2}(M, uMu^*). \tag{1.4.3}$$

We write $\alpha(M)$ for the supremum of all such α for which (1.4.3) holds for every unitary $u \in N$, if such α exist, otherwise we take $\alpha(M) = 0$. If $\alpha(M) = 1$, then we say that M is *strongly singular.*

Indeed it is immediate that any α-strongly singular von Neumann subalgebra is singular, and any abelian α-strongly singular von Neumann subalgebra is a singular masa. Every singular masa A for which the α-invariant has been computed has turned out to be strongly singular: see [59] for generator masas coming from prime elements of hyperbolic groups; [60] for the Laplacian masa in a free group factor; and, in appendix B, we use these methods to give the analogous calculation for certain radial masas coming from free products of finite groups. In the spirit of Proposition 1.4.11, the appropriate closure result holds for strongly-singular masas.

Proposition 1.4.13 ([50, Corollary 6.7]). *For each $\alpha > 0$, the set of α-strongly singular masas in a separable* II_1 *factor N is $d_{\infty,2}$-closed.*

Motivated at least in part by Popa's work ([48]) on the distance between masas in II_1 factors, showing that the δ-invariant of a singular masa is 1, Popa, Sinclair and Smith have recently examined exactly how two von Neumann subalgebras of a II_1 factor can be close in the $d_{\infty,2}$-metric, [50]. This work applies in the general setting of all von Neumann subalgebras, but was originally done in the masa case which is all we shall give here. The starting point is to note that two masas are close in $d_{\infty,2}$-metric when there are large cutdowns of these masas which are unitarily conjugate via a unitary u close to 1. This is an easy estimate, versions of which can also be found in [59].

Proposition 1.4.14 ([50, Theorem 6.5(i)]). *Let A and B be masas in the separably acting* II_1 *factor N, such that there are projections $p \in A$ and $q \in B$ and a unitary in N with $u(Ap)u^* = Bq$. Then*

$$d_{\infty,2}(A,B) = \left\|\mathbb{E}_A - \mathbb{E}_B\right\|_{\infty,2} \leq 4\left\|u - \mathbb{E}_B(u)\right\|_2 + \left\|1-p\right\|_2 + \left\|1-q\right\|_2.$$

In their perturbation work ([50]) Popa, Sinclair and Smith, have shown that this is essentially the only way two masas can be close in the $d_{\infty,2}$-metric. More precisely, they show that if two masas are sufficiently close in the $d_{\infty,2}$-metric, then there are large cutdowns of these masas which are unitarily conjugate.

Theorem 1.4.15 ([50, Theorem 6.5(ii)]). *There are constants $0 < \delta < 1$ and K_1, K_2 such that, whenever A and B are masas in a separable* II_1 *factor N with $\left\|\mathbb{E}_A - \mathbb{E}_B\right\|_{\infty,2} = \epsilon < \delta$, there exist projections $p \in A$ and $q \in B$ and a unitary $u \in N$ such that:*

- $u(Ap)u^* = Bq$;
- $\left\|1-p\right\|_2 = \left\|1-q\right\|_2 \leq K_1\epsilon$;
- $\left\|u - \mathbb{E}_B(u)\right\|_2 \leq K_2\epsilon$.

We shall use this deep theorem to establish the $d_{\infty,2}$-continuity of the Pukánszky invariant and of a new Γ-invariant for masas in chapter 3. To this end, we state exactly what we need as a corollary.

Corollary 1.4.16. *There exists constants $0 < \delta < 1$ and K such that, whenever A and B are masas in a separable* II_1 *factor N with $\|\mathbb{E}_A - \mathbb{E}_B\|_{\infty,2} = \epsilon < \delta$, then a masa B_1 in N can be found with:*

- *$B_1 = uAu^*$ for some unitary u in N;*
- *$B_1p = Bp$ for a projection $p \in B_1 \cap B$ with $\|1 - p\|_2 \leq K\epsilon$.*

Proof. Take δ and K to be the constants δ and K_1 of Theorem 1.4.15. Let u be the unitary resulting from Theorem 1.4.15, p be the projection q appearing there, and take $B_1 = uAu^*$. □

Popa, Sinclair and Smith's perturbation work also gives a partial converse to the observation that every strongly singular masa is singular. They are able to show that every singular masa is α-strongly singular for some $\alpha > 0$, and in fact that $\alpha \geq \alpha_0$ for some absolute constant $\alpha_0 > 0$, which does not even depend on the underlying separable II_1 factor in which the masas live. The method used to obtain this result, is to bound from below the $d_{\infty,2}$ distance between A and a unitary perturbation uAu^* of A, by the distance between u and $\mathcal{N}(A)$. The best value of α_0 is not yet known, although it seems reasonable to hope, based on Popa's result [48] for the δ-invariant, that every singular masa will turn out to be strongly singular - we record this conjecture formally as Question 1.4.18 for later reference. The version of this result appearing as Theorem 6.4 of [50] gives $\alpha_0 \geq 1/90$, although we should observe that this is certainly not the best value of this constant. It is noted in [50], that using methods specific to masas[39] leads to the estimate $\alpha_0 \geq 1/31$. For the purposes of this thesis, the exact value of α_0 will not be important.

Theorem 1.4.17 (Popa, Sinclair, Smith). *There exists a constant $0 < \alpha_0 \leq 1$ such that whenever A is a masa in a separably acting type* II_1 *factor N and u is a unitary in N, we have*

$$\alpha_0 d_2(u, \mathcal{N}(A)) \leq \|(I - \mathbb{E}_{uAu^*})\mathbb{E}_A\|_{\infty,2} \leq d_{\infty,2}(A, uAu^*) \leq 4d_2(u, \mathcal{N}(A)).$$

In particular any singular masa A is α-strongly singular for some $\alpha \geq \alpha_0$.

Question 1.4.18. Is every singular masa in a separable II_1 factor necessarily strongly singular?[40]

[39] Recall that [50] treats the more general case of perturbations of von Neumann subalgebras of II_1 factors throughout.

[40] Since this thesis was written, this question has been answered. See Addendum 1.1.1 for further comments.

1.4.4 Establishing strong singularity: asymptotic homomorphism properties

To demonstrate the strong singularity of certain masas, the concept of an asymptotic homomorphism was introduced in [59] and [54]. The definition we give is the hypothesis of Lemma 2.1 of [54].

Definition 1.4.19. Let M be a von Neumann subalgebra of the II_1 factor N. We say that M has the *weak asymptotic homomorphism property* if and only if, for all $\epsilon > 0$ and $x_1, \ldots, x_m \in N$, we can find a unitary $v \in M$ with

$$\left\|\mathbb{E}_M\left(x_i v x_j^*\right) - \mathbb{E}_M\left(x_i\right) v \mathbb{E}_M\left(x_j^*\right)\right\|_2 < \epsilon, \tag{1.4.4}$$

for every i, j.

The original asymptotic homomorphism property,[41] defined in [59, Definition 4.1], required that we could find a unitary v in M such that

$$\lim_{n\to\infty} \|\mathbb{E}_M\left(x v^n y^*\right) - \mathbb{E}_M\left(x\right) v^n \mathbb{E}_M\left(y^*\right)\|_2 = 0,$$

for every $x, y \in N$. Both these concepts were introduced as they imply strong singularity, ([59, Theorem 4.7] and [54, Lemma 2.1]) the proof of which we use to obtain Lemma 1.4.24. Not all strongly singular masas have the asymptotic homomorphism property, [54, Remark 3.3]. At present though, no singular masa is known for which the weak asymptotic homomorphism property fails, and so accordingly it is this property we shall focus on in relation to attempting to determine the singularity of a masa henceforth.

Question 1.4.20. Does every singular masa A in a separable II_1 factor have the weak asymptotic homomorphism property?[40]

We can use these ideas to give a criterion for bounding the normalising algebra $\mathcal{N}(M)$ from above in the more general non-singular situation.

Definition 1.4.21. Let $M \subset B$ be von Neumann subalgebras of the II_1 factor N. We say that M has the *weak asymptotic homomorphism property away from B* if and only if, for all $\epsilon > 0$ and $x_1, \ldots, x_m \in N$ with $\mathbb{E}_B\left(x_i\right) = 0$ for all i, we can find a unitary $v \in M$ with

$$\left\|\mathbb{E}_M\left(x_i v x_j^*\right)\right\|_2 < \epsilon, \tag{1.4.5}$$

for every i and j.

[41] In which, by contrast with the weak version, something does appear to be asymptotically a homomorphism!

Observe that, with the notation of the definition above, if we take $x, y \in N$ and $v \in M$, we have

$$\begin{aligned}\mathbb{E}_M(xvy^*) &= \mathbb{E}_M\Big((x-\mathbb{E}_B(x))\,v\,(y^*-\mathbb{E}_B(y^*))\Big) + \mathbb{E}_M\Big(\mathbb{E}_B(x)\,v\,(y^*-\mathbb{E}_B(y^*))\Big)\\ &\quad + \mathbb{E}_M\Big((x-\mathbb{E}_B(x))\,v\mathbb{E}_B(y^*)\Big) + \mathbb{E}_M\Big(\mathbb{E}_B(x)\,v\mathbb{E}_B(y^*)\Big)\\ &= \mathbb{E}_M\Big((x-\mathbb{E}_B(x))\,v\,(y^*-\mathbb{E}_B(y^*))\Big) + \mathbb{E}_M\Big(\mathbb{E}_B(x)\,v\mathbb{E}_B(y^*)\Big),\end{aligned}$$

as

$$\mathbb{E}_M\Big(\mathbb{E}_B(x)\,v\,(y^*-\mathbb{E}_B(y^*))\Big) = \mathbb{E}_M\Big((x-\mathbb{E}_B(x))\,v\mathbb{E}_B(y^*)\Big) = 0,$$

since $M \subset B$, so $\mathbb{E}_M = \mathbb{E}_M\mathbb{E}_B$. This equality links the two notions of weak asymptotic homomorphism property, and will be repeatedly used in the sequel.

Proposition 1.4.22. *Let $M \subset B$ be von Neumann subalgebras of the II_1 factor N. For $x, y \in N$ and $v \in M$ we have*

$$\mathbb{E}_M(xvy^*) - \mathbb{E}_M\Big(\mathbb{E}_B(x)\,v\mathbb{E}_B(y^*)\Big) = \mathbb{E}_M\Big((x-\mathbb{E}_B(x))\,v\,(y^*-\mathbb{E}_B(y^*))\Big). \tag{1.4.6}$$

Hence, M has the weak asymptotic homomorphism property away from B if and only if, for all $\epsilon > 0$ and $x_1, \ldots, x_m \in M$ we can find a unitary $v \in M$ with

$$\left\|\mathbb{E}_M\left(x_i v x_j^*\right) - \mathbb{E}_M\left(\mathbb{E}_B(x_i)\,v\mathbb{E}_B\left(x_j^*\right)\right)\right\|_2 < \epsilon, \tag{1.4.7}$$

for every i and j. In particular, M has the weak asymptotic homomorphism property away from M precisely when M has the weak asymptotic homomorphism property as given in Definition 1.4.19.

In section 2, we shall use the weak asymptotic homomorphism criterion for singularity repeatedly in the context of direct limits of commuting squares, (see Definition 1.3.7). In this situation a density argument makes things slightly easier. We state this here for later use.

Lemma 1.4.23. *Let $M \subset B$ be von Neumann subalgebras of the II_1 factor N. Suppose that for each n we have von Neumann subalgebras $B_n \subset N_n$ with each $B_n \subset B_{n+1}$ and $N_n \subset N_{n+1}$. Suppose further that B and N are the direct limits of the B_n and N_n respectively and that Figure 1.2 is made of commutating squares.*

If, for each $n \geq 1$, $\epsilon > 0$ and $x_1, \ldots, x_m \in N_n$ with $\mathbb{E}_{B_n}(x_i) = 0$, we can find a unitary $v \in M$ with

$$\left\|\mathbb{E}_M\left(x_i v x_j^*\right)\right\|_2 < \epsilon,$$

for all i, j, then M has the weak asymptotic homomorphism property away from B.

$$
\begin{array}{ccccccccccc}
N_1 & \hookrightarrow & N_2 & \hookrightarrow & \cdots & \hookrightarrow & N_n & \hookrightarrow & N_{n+1} & \hookrightarrow \cdots \hookrightarrow & N\,, \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
B_1 & \hookrightarrow & B_2 & \hookrightarrow & \cdots & \hookrightarrow & B_n & \hookrightarrow & B_{n+1} & \hookrightarrow \cdots \hookrightarrow & B
\end{array}
$$

Figure 1.2: Commutative diagram for Lemma 1.4.23.

Proof. Since $(\cup_{n=1}^{\infty} N_n)'' = N$, the elements of $\cup_{n=1}^{\infty} N_n$ are 2-norm dense in N. By the preceding proposition, it is sufficient for the weak asymptotic homomorphism property away from B, to show that for given operators $y_1, \ldots, y_m$ in some N_n and $\epsilon > 0$, we can find a unitary $v \in M$ with

$$\left\|\mathbb{E}_M\left(x_i v x_j^*\right)\right\|_2 < \epsilon,$$

for all i and j, where we have taken $x_i = y_i - \mathbb{E}_B(y_i)$. Now note that, since Figure 1.2 commutes, $x_i = y_i - \mathbb{E}_{B_n}(y_i) \in N_n$, so the condition we are required to check reduces precisely to the hypothesis of the lemma. □

We are now in a position to state the technical result, generalising Lemma 2.1 of [54], which allows us to give an upper bound for the algebra generated by normalisers. It should be noted that very few modifications to the proof in [54] are required, although we give the details for completeness.

Lemma 1.4.24. *Let $M \subset B$ be von Neumann subalgebras of the II_1 factor N. If M has the weak asymptotic homomorphism property away from B, then we have*

$$\|u - \mathbb{E}_B(u)\|_2 \le \|(I - \mathbb{E}_{uMu^*})\,\mathbb{E}_M\|_{\infty,2} \le \|\mathbb{E}_{uMu^*} - \mathbb{E}_M\|_{\infty,2}\,, \tag{1.4.8}$$

for every unitary $u \in N$ and so $\mathcal{N}(M)'' \subset B$. In particular, if A is an abelain subalgebra of N with the weak asymptotic homomorphism property, then A is strongly singular and so a singular masa in N.

Proof. Let u be a unitary in the underlying II_1 factor N. Fix $\epsilon > 0$ and find, a unitary $v \in M$ such that

$$\|\mathbb{E}_M(u^* v u) - \mathbb{E}_M(\mathbb{E}_B(u^*)\, v \mathbb{E}_B(u))\|_2 < \epsilon.$$

Now compute

$$
\begin{aligned}
\|(I - \mathbb{E}_{uMu^*})\mathbb{E}_M\|_{\infty,2}^2 &\ge \|v - \mathbb{E}_{uMu^*}(v)\|_2^2 \\
&= 1 - \|\mathbb{E}_M(u^* v u)\|_2^2 \\
&\ge 1 - \left(\|\mathbb{E}_B(u^*)\, v \mathbb{E}_B(u)\|_2 + \epsilon\right)^2 \\
&\ge 1 - \|\mathbb{E}_B(u^*)\, v \mathbb{E}_B(u)\|_2^2 - (2\epsilon + \epsilon^2) \\
&\ge \|u - \mathbb{E}_B(u)\|_2^2 - (2\epsilon + \epsilon^2),
\end{aligned}
$$

and as $\epsilon > 0$ was arbitrary we have the desired first inequality of (1.4.8). The second inequality is immediate, and since (1.4.8) then implies that any unitary normaliser of M must lie in B, we have $\mathcal{N}(M)'' \subset B$, as required. □

Equation (1.4.8), in the proceeding lemma is the strong-singularity requirement when $B = M$. We can extend the definition of strongness from Definition 1.4.12 to non-singular algebras.

Definition 1.4.25. Let M be a von Neumann subalgebra of N. We say that M is *strongly normalised by* B when, (1.4.8) holds for every unitary $u \in N$.

When we work with subfactors, we fill not always get the weak asymptotic homomorphism property. The metric methods of Lemma 1.4.24 will still be used to control the location of normalisers by giving algebras B which strongly normalise M.

Lemma 1.4.26. *Let $M \subset B$ be von Neumann subalgebras of the II_1 factor N. Suppose that for every unitary $u \in N$ and $\epsilon > 0$, there exists a unitary $v \in M$ with*

$$\left\|\mathbb{E}_M\left(u^*vu\right)\right\|_2 \leq \left\|\mathbb{E}_B\left(u^*\right)\mathbb{E}_B\left(u\right)\right\|_2 + \epsilon, \tag{1.4.9}$$

then M is strongly normalised by B.

Proof. The deduction of (1.4.8) follows in the same way as the proof of Lemma 1.4.24. □

The weak asymptotic homomorphism property is easily applied in the context of group algebras. Here we have a sufficient condition for this property, and Lemma 1.4.24 gives the next result. Again, we should note that the strongly singular case below follows from [54, Lemma 2.1] and is stated explicitly in section 5 of [61].

Corollary 1.4.27. *Let $H \subset G_0 \subset G$ be inclusions of countable discrete groups. Suppose that for $g_1, \ldots, g_n \in G \setminus G_0$ there exists some $h \in H$ with $g_i h g_j^{-1} \notin H$ for all i, j, then $\mathcal{L}(H)$ is strongly normalised by $\mathcal{L}(G_0)$ in $\mathcal{L}(G)$ so $\mathcal{N}(\mathcal{L}(H))'' \subset \mathcal{L}(G_0)$. In particular, if in addition $H = G_0$, then $\mathcal{L}(H)$ is strongly singular in $\mathcal{L}(G)$.*

At present, all the calculations of normalisers of group von Neumann algebras I am aware of can be performed in this way. In particular, one can calculate the normalisers of Dixmier's original masas of [11] and Pukánszky's examples, [51], quickly with this tool.

We end this section by returning to tensor products. Observe that the weak asymptotic homomorphism property is preserved when taking tensor products.

Proposition 1.4.28. *Let $M_1 \subset B_1$ and $M_2 \subset B_2$ be von Neumann subalgebras of the* II_1 *factors N_1 and N_2 respectively. If each M_i has the weak asymptotic homomorphism property away from B_i in N_i then $M_1\overline{\otimes}M_2$ has the weak asymptotic homomorphism property away from $B_1\overline{\otimes}B_2$ in $N_1\overline{\otimes}N_2$.*

Proof. We use the formulation of Proposition 1.4.22. Observe that by linearity and $\|.\|_2$-density, it is enough to take finitely many elementary tensors $z_i = x_i \otimes y_i$ in $N_1\overline{\otimes}N_2$ and $\epsilon > 0$, then find a unitary $v \in M_1\overline{\otimes}M_2$ with

$$\left\|\mathbb{E}_{M_1\overline{\otimes}M_2}\left(z_i v z_j^*\right) - \mathbb{E}_{M_1\overline{\otimes}M_2}\left(\mathbb{E}_{B_1\overline{\otimes}B_2}\left(z_i\right) v \mathbb{E}_{B_1\overline{\otimes}B_2}\left(z_j^*\right)\right)\right\|_2 < \epsilon,$$

for all i and j. By applying the weak asymptotic homomorphism property, again in the form of Proposition 1.4.22, we are able to find unitaries $u_1 \in M_1$ and $u_2 \in M_2$ with

$$\left\|\mathbb{E}_{M_1}\left(x_i u_1 x_j^*\right) - \mathbb{E}_{M_1}\left(\mathbb{E}_{B_1}\left(x_i\right) u_1 \mathbb{E}_{B_1}\left(x_j^*\right)\right)\right\|_2 < \epsilon \frac{1}{2\sup_i \|y_i\|^2},$$

and

$$\left\|\mathbb{E}_{M_2}\left(y_i u_2 y_j^*\right) - \mathbb{E}_{M_2}\left(\mathbb{E}_{B_2}\left(y_i\right) u_2 \mathbb{E}_{B_2}\left(y_j^*\right)\right)\right\|_2 < \epsilon \frac{1}{2\sup_i \|x_i\|^2}.$$

Now take $v = u_1 \otimes u_2$ and estimate as follows:

$$\begin{aligned}
&\left\|\mathbb{E}_{M_1\overline{\otimes}M_2}\left(z_i v z_j^*\right) - \mathbb{E}_{M_1\overline{\otimes}M_2}\left(\mathbb{E}_{B_1\overline{\otimes}B_2}\left(z_i\right) v \mathbb{E}_{B_1\overline{\otimes}B_2}\left(z_j^*\right)\right)\right\|_2 \\
&= \Big\|\mathbb{E}_{M_1}\left(x_i u_1 x_j^*\right) \otimes \mathbb{E}_{M_2}\left(y_i u_2 y_j^*\right) \\
&\qquad - \mathbb{E}_{M_1}\left(\mathbb{E}_{B_1}\left(x_i\right) u_1 \mathbb{E}_{B_1}\left(x_j^*\right)\right) \otimes \mathbb{E}_{M_2}\left(\mathbb{E}_{B_2}\left(y_i\right) u_2 \mathbb{E}_{B_2}\left(y_j^*\right)\right)\Big\|_2 \\
&\leq \left\|\left(\mathbb{E}_{M_1}\left(x_i u_1 x_j^*\right) - \mathbb{E}_{M_1}\left(\mathbb{E}_{B_1}\left(x_i\right) u_1 \mathbb{E}_{B_1}\left(x_j^*\right)\right)\right) \otimes \mathbb{E}_{M_2}\left(y_i u_2 y_j^*\right)\right\|_2 \\
&\quad + \left\|\mathbb{E}_{M_1}\left(\mathbb{E}_{B_1}\left(x_i\right) u_1 \mathbb{E}_{B_1}\left(x_j^*\right)\right) \otimes \left(\mathbb{E}_{M_2}\left(y_i u_2 y_j^*\right) - \mathbb{E}_{M_2}\left(\mathbb{E}_{B_2}\left(y_i\right) u_2 \mathbb{E}_{B_2}\left(y_j^*\right)\right)\right)\right\|_2 \\
&< \epsilon \frac{\left\|\mathbb{E}_{M_2}\left(y_i u_2 y_j^*\right)\right\|_2}{2\sup_i \|y_i\|^2} + \epsilon \frac{\left\|\mathbb{E}_{M_1}\left(\mathbb{E}_{B_1}\left(x_i\right) u_2 \mathbb{E}_{B_1}\left(x_j^*\right)\right)\right\|_2}{2\sup_i \|x_i\|^2} \leq \epsilon.
\end{aligned}$$

□

Given masas A_1 and A_2 in the II_1 factors N_1 and N_2 respectively, consider the masa $A_1\overline{\otimes}A_2$ in $N_1\overline{\otimes}N_2$. Is it the case that if A_1 and A_2 are both singular, then so too is $A_1\overline{\otimes}A_2$? If A_1 and A_2 both have the weak asymptotic homomorphism property, then we have a positive answer to this question. The other two methods used to demonstrate singularity of an explicit masa also lead one to this conclusion. Remark 3.1.4 shows that, when we are able to use the Pukánszky invariant to show that two masas are singular, then their tensor product is also necessarily singular. It is also not difficult to check that when we are able to use the orthogonality method of Proposition 1.4.7 to exhibit two singular masas A

and B, then this result can also be used to see that $A\overline{\otimes}B$ is singular. On the other hand, this question seems not to have been addressed in the literature, and it seems difficult to apply Popa's result, of [48], on the δ-invariant, or Popa, Sinclair and Smith's perturbation work (Theorem 1.4.17) to achieve this aim. The latter work, for example, would allow us to control how far an elementary tensor $u_1 \otimes u_2$ of two unitaries $u_1 \in N_1$ and $u_2 \in N_2$ is from $\mathcal{N}_{N_1\overline{\otimes}N_2}(A_1\overline{\otimes}A_2)$. However, unlike the weak asymptotic homomorphism property, we are unable to handle linear combinations of such elementary tensors using this method. We state this problem formally, in the more general non-singular situation.

Question 1.4.29. Given masas A_1 and A_2 in the II_1 factors N_1 and N_2 respectively, do we have

$$\mathcal{N}_{N_1\overline{\otimes}N_2}(A_1\overline{\otimes}A_2)'' = \mathcal{N}_{N_1}(A_1)''\overline{\otimes}\mathcal{N}_{N_2}(A_2)''?$$

Note that here also, if each A_i has the weak asymptotic homomorphism property away from $\mathcal{N}_{N_i}(A_i)''$, then Proposition 1.4.28 gives a positive answer to this question. We should observe that if we were to state Question 1.4.29 for a general inclusion $M_i \subset N_i$, $(i = 1, 2)$, of von Neumann algebras, then we would not always have $\mathcal{N}_{N_1\overline{\otimes}N_2}(M_1\overline{\otimes}M_2)'' = \mathcal{N}_{N_1}(M_1)''\overline{\otimes}\mathcal{N}_{N_2}(M_2)''$.

Example 1.4.30. Let N be the algebra of 4×4 matrices, thought of as the tensor product $\mathbb{M}\mathrm{at}_2(\mathbb{C}) \otimes \mathbb{M}\mathrm{at}_2(\mathbb{C})$. We use matrix units $(e_{i,j})_{i,j=0,1}$ for $\mathbb{M}\mathrm{at}_2(\mathbb{C})$ and take $B = e_{0,0} \otimes \mathbb{M}\mathrm{at}_2(\mathbb{C}) \oplus e_{1,1} \otimes D_2$, where D_2 is the algebra of diagonal matrices in $\mathbb{M}\mathrm{at}_2(\mathbb{C})$. It is easy to check that $\mathcal{N}_N(B)'' = D_2 \otimes M_2$. Indeed pictorially, we have

$$B = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \qquad \mathcal{N}_N(B)'' = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

Consider the inclusion $B \otimes B \subset N \otimes N$. The automorphism of $N \otimes N$ given by $x \otimes y \mapsto y \otimes x$ is, like all automorphisms of finite dimensional factors, inner. Any unitary $u \in N \otimes N$ implementing this automorphism certainly normalises $B \otimes B$. We can find such a unitary u explicitly. Indeed, it is easily checked that

$$u = \sum_{i,j,k,l=0,1} e_{i,j} \otimes e_{k,l} \otimes e_{j,i} \otimes e_{l,k},$$

does exactly what we require. On the other hand, $u \notin \mathcal{N}_N(B)'' \otimes \mathcal{N}_N(B)'' = D_2 \otimes \mathbb{M}\mathrm{at}_2(\mathbb{C}) \otimes D_2 \otimes \mathbb{M}\mathrm{at}_2(\mathbb{C})$, so that $\mathcal{N}_{N\otimes N}(B \otimes B)'' \supsetneq \mathcal{N}_N(B)'' \otimes \mathcal{N}_N(B)''$.

This example might lead one to believe that there is no hope of a positive answer to Question 1.4.29, even in the restricted case of singular masas. However, even in finite dimensions we can not easily obtain a counter example in the manner of Example 1.4.30, for all masas in finite dimensions are Cartan. In the II_1 factor situation, Sakai's Theorem ([55]) ensures that the 'swap automorphism' of $N\overline{\otimes}N$ is not inner, so we can not obtain extra normalisers in this way. Furthermore, as part of his classification of injective factors, Connes has shown that the hyperfinite II_1 factor is the only II_1 factor for which this swap automorphism is even approximately inner, [7, Theorem 5.1: $1 \Leftrightarrow 3$]. It is not then unreasonable to hope for a positive answer to Question 1.4.29, at least in the singular situation. For singular Tauer masas, we are able to resolve this problem positively in Corollary 4.1.2.

1.4.5 Centralising sequences lying in masas

In a property Γ II_1 factor N, asking whether centralising sequences for N can be found inside a masa A, is often a useful technique for showing that two masas are non-conjugate. Despite the difficulty, in general, of showing non-conjugacy, this idea is perhaps under emphasised in the literature. We attempt here then, to give an account of where centralising sequences in masas have appeared previously. This will probably not prove definitive, a fact for which I can only apologise. To save us from constantly referring to non-trivial centralising sequences henceforth, we say that an inclusion $A \subset N$ is Γ when A contains non-trivial centralising sequences for N. Even more lazily, we shall also refer to Γ masas A in N in this case.

We start with an obvious piece of folklore. The Cartan masa D in the hyperfinite II_1 factor R is Γ. This is immediately seen by writing R as the von Neumann tensor product of infinitely many copies of the 2×2 matrices M, when the Cartan masa D is realised as the infinite von Neumann tensor product of copies of the diagonal matrices. Let $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, when $(1^{\otimes n} \otimes r)_{n=1}^{\infty}$ is patently a non-trivial centralising sequence for R lying in D. One could ask whether all Cartan masas in a property Γ II_1 factor N are automatically Γ? This is not the case as Connes and Jones in [9], use the existence of centralising sequences to give an example of a II_1 factor containing two non-conjugate Cartan masas - an interesting follow up to the Connes, Feldman Weiss uniqueness result, Theorem 1.4.6.

In his orthogonality paper ([44]) Popa notes that when G is an I.C.C. group and H a subgroup with $gHg^{-1} \cap H = \{1\}$, for some $g \in G \setminus H$, then $\mathcal{L}(H)$ can not contain any non-trivial centralising sequence of $\mathcal{L}(G)$. This observation has already appeared as Proposition 1.2.12, but here we apply it to see that the inclusions of group von Neumann algebras $\mathcal{L}(H) \subset \mathcal{L}(G)$ arising from Corollary 1.4.8 are not Γ.

The only other example I am are aware of is in [25]. Some background on the inclusion of masas in subfactors is in order. When A is a semi-regular masa in N, it is then contained in a proper subfactor $M = \mathcal{N}(A)''$ of N. The question of which subfactors M in N contain masas for N dates back to Kadison. Such a subfactor must be irreducible, as the following well known observation makes clear.

Proposition 1.4.31. *Let A be a masa in a II_1 factor N. Suppose that M is a subfactor of N containing A then*

$$M' \cap N = \mathbb{C}1.$$

Proof. Under the given hypotheses, we have $M' \cap N \subset A' \cap N = A \subset M$. Hence, $M' \cap N \subset M' \cap M = \mathbb{C}1$, as claimed. □

Popa showed that this is the only obstruction to a subfactor containing a masa. In [42], he demonstrated that any irreducible subfactor of a separable II_1 factor N contains a masa of N. Furthermore, he went on to show that, given an irreducible factor M of a separable II_1 factor N, then a singular masa A for N can be found inside M. In the continuation ([25]) of this work, Jones and Popa establish the next result, again using centralising sequences to see that two masas are non-conjugate.

Theorem 1.4.32 ([25, Theorem 3.1]). *Let R be the hyperfinite II_1 factor. Given an irreducible regular subfactor N of R, there exists masas A_1 and A_2 in R with $\mathcal{N}(A_1)'' = \mathcal{N}(A_2)'' = N$, such that A_1 is Γ while A_2 is not.*

As a consequence of this they are able, again in [25], to produce uncountably many pairwise non-conjugate semi-regular masas in R. This result requires lots of technical machinery, not least the previous theorem! We will give an elementary and explicit construction of such a family of masas in section 2.4, using Tauer's length ideas.

1.4.6 Invariants for singular masas

We end our initial discussion of masas by summerising the methods available for showing that two singular masas are not-conjugate via an automorphism of the underlying II_1 factor. These methods fit into two categories, those coming from 'yes/no' properties such as the asymptotic homomorphism property, and those coming from conjugacy invariants of singular masas. Of course, we could think of the first type as being $\mathbb{Z}_2$-valued invariants, but this seems somewhat perverse. However, as we shall later see, in section 3.3, there is a method of defining an invariant to be the size of a maximal cutdown on which a property holds.

Currently, all known invariants for singular masas are either naturally discrete or are not known to take values in a continuum. For example, the Pukánszky invariant, which we shall eventually formalise in Definition 3.1, associates to each masa A a subset of $\mathbb{N}\cup\{\infty\}$. While the power set $2^{\mathbb{N}\cup\{\infty\}}$ is uncountable, and indeed the Pukánszky invariant was used in [38] to give uncountably many pairwise non-conjugate singular masas in the hyperfinite II_1 factor, it is naturally topologised with the discrete topology. Furthermore, we shall establish a continuity result, Theorem 3.1.5, which shows (Corollary 3.1.8) that it is not possible to find a continuous map $t \mapsto A(t)$ from $[0,1]$ into the set of masas in a II_1 factor, equipped with the $d_{\infty,2}$-metric, for which each $A(t)$ has a different Pukánszky invariant.

Alternatively, Popa's δ-invariant naturally takes values in the continuous interval $[0,1]$ but, as we have already seen, only the values 0 and 1 can be attained. The α-invariant of Sinclair and Smith, also takes values naturally in the interval $[0,1]$, but here too not all values are possible, see Theorem 1.4.17, and it is not known whether α takes values other than 0 or 1. At present then we should not regard α as a continuous-valued invariant.

As there are uncountably many singular masas in R, no pair of which is conjugate by an automorphism of R, it would be desirable to have an invariant, which we will temporarily denote by ι, taking values in an interval I such that the map taking masas to the invariant is continuous with respect to the $d_{\infty,2}$ metric on masas and the Euclidian metric on I. Furthermore, it would also be desirable if we could exhibit a right inverse to ι, which will be a continuous map $t \mapsto A(t)$ with $\iota(A(t)) = t$ giving a continuous path of pairwise non-conjugate masas.[42] We shall call such an invariant a *genuinely continuous* invariant, and use centralising sequences to exhibit one in section 3.3.

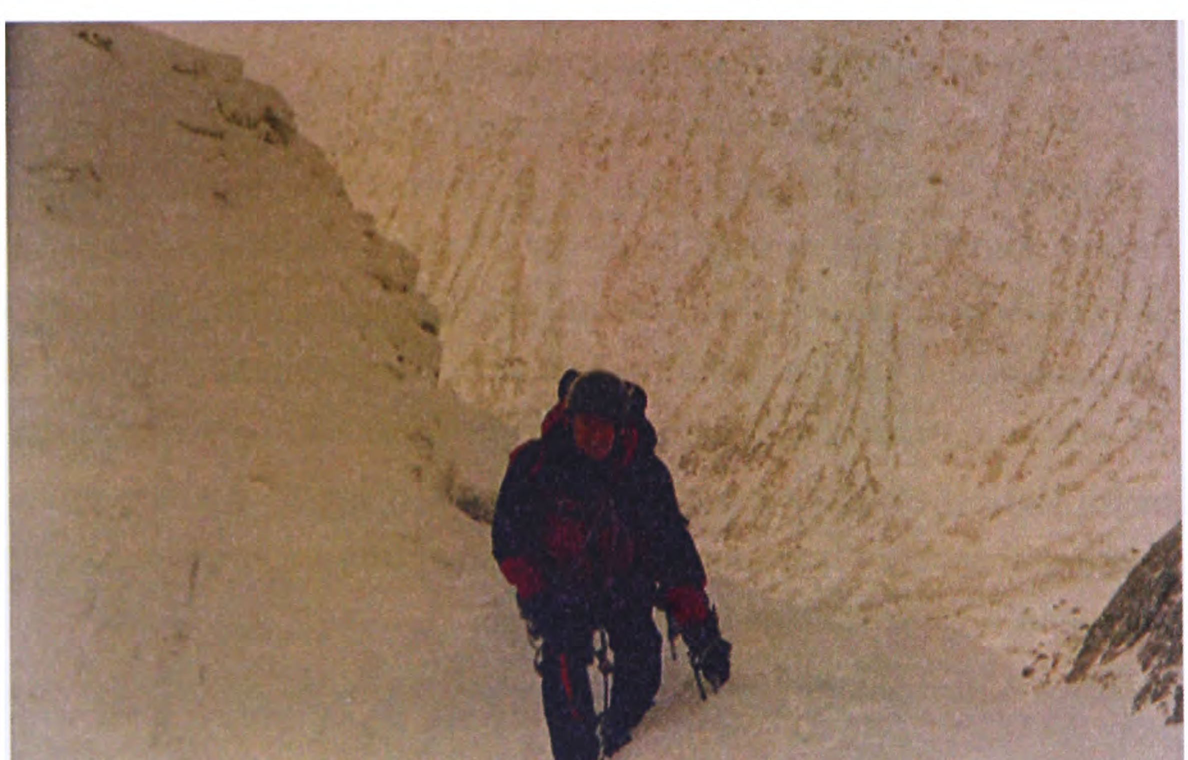
[42]Thereby excluding the δ-invariant from consideration.

Chapter 2

Tauer's original examples of masas

In this chapter we give a modern account of the masas constructed in 1965 by Sister Rita Tauer, [70]. The main thrust of Tauer's paper was to give countably many semi-regular masas in the hyperfinite II_1 factor R, no pair of which are conjugate via an automorphism of R. This was attained by introducing the concept of the *length* of a semi-regular masa and constructing semi-regular masas of all finite lengths using matrix techniques. We begin, in section 2.1, by examining Tauer's method for constructing masas in R and establishing some notation for use in the sequel. Parts of sections 2.1 and 2.2 are to appear in [77].

Round the towers



Andy Sole traversing great *tower* on Tower ridge (IV 3), Ben Nevis, Feb 2005.

Photo: Saw

2.1 Defining Tauer masas

Our starting point is the following observation, Theorem 2.5 of [70], which gives a method for constructing masas inside the hyperfinite II_1 factor. Accordingly, we shall call masas constructed using this method Tauer masas.

Proposition / Definition 2.1.1. Let $(N_n)_{n=1}^{\infty}$ be an increasing sequence of finite type I subfactors generating the hyperfinite II_1 factor R. Suppose we have masas A_n in N_n with $A_n \subset A_{n+1}$ for all n, then

$$A = (\cup_{n=1}^{\infty} A_n)'',$$

is a masa in R.

A masa A in the hyperfinite II_1 factor R is then said to be a *Tauer masa* if there exists such an increasing sequence $(N_n)_{n=1}^{\infty}$ of finite type I subfactors generating R such that $A \cap N_n$ is a masa in N_n for each n. In this case we shall write A_n for $A \cap N_n$, and say for emphasis that A is a *Tauer masa with respect to the subfactors* $(N_n)_{n=1}^{\infty}$.

Originally this was established by taking an element x of R commuting with A and approximating x inside the weakly dense subfactors N_n by $(a_n + x_n)$ with $a_n \in A_n$ and $x_n \in N_n \ominus A_n$. An ϵ-argument was then used to demonstrate that $\lim_{n\to\infty} \|x_n\|_2 = 0$, which established the result. In his brief discussion of Tauer's examples ([44, 5.2.2]) Sorin Popa observes that under the hypothesis of the proposition we have

$$\lim_{n\to\infty} \left\|\mathbb{E}_{A_n}(x) - \mathbb{E}_{A_n'\cap R}(x)\right\|_2 = 0, \tag{2.1.1}$$

for all $x \in R$, and so A is a masa in R by Proposition 1.4.3. For $x \in N_n$, it is immediate that $\mathbb{E}_{A_n}(x) = \mathbb{E}_{A_n'\cap R}(x)$, from which (2.1.1) follows by density. We shall often use this tool for calculating the conditional expectation onto a Tauer masa and so we record it formally.

Proposition 2.1.2. *Let A be a Tauer masa in R with respect to the subfactors $(N_n)_{n=1}^{\infty}$. For each $x \in N_n$, we have*

$$\mathbb{E}_{A_n}(x) = \mathbb{E}_A(x) = \mathbb{E}_{A_n'\cap R}(x).$$

In particular Figure 2.1, overleaf, is made up of commuting squares (in the sense of Definition 1.3.7).

Given a chain of finite type I subfactors $(N_n)_{n=1}^{\infty}$, we may write them as a tensor product. Namely we can find subfactors $(M_n)_{n=1}^{\infty}$ such that

$$N_n = \bigotimes_{r=1}^{n} M_r, \tag{2.1.2}$$

$$\begin{array}{ccccccccccc} N_1 & \hookrightarrow & N_2 & \hookrightarrow \cdots \hookrightarrow & N_n & \hookrightarrow & N_{n+1} & \hookrightarrow \cdots \hookrightarrow & R, \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A_1 & \hookrightarrow & A_2 & \hookrightarrow \cdots \hookrightarrow & A_n & \hookrightarrow & A_{n+1} & \hookrightarrow \cdots \hookrightarrow & A \end{array}$$

Figure 2.1: Inclusions of masas $A_n \subset N_n$ giving rise to a Tauer masa $A \subset R$.

for every n, with the inclusion $N_n \subset N_{n+1}$ given by the natural map $x \mapsto x \otimes 1$. In particular, if A is a Tauer masa with respect to the chain $(N_n)_{n=1}^{\infty}$, then for $m > n$ we can write the finite dimensional approximation A_m in the form

$$A_m = \bigoplus_{e \in \mathcal{P}_{\min}(A_n)} e \otimes A_{m,n}^{(e)}. \tag{2.1.3}$$

The direct sum in (2.1.3) is over the set $\mathcal{P}_{\min}(A_n)$ of all minimal projections e for A_n, and each $A_{m,n}^{(e)}$ is a masa in $\bigotimes_{r=n+1}^{m} M_r$.

Tauer produced her examples inside the copy of the hyperfinite factor obtained by repeatedly embedding the $2^n \times 2^n$ matrices inside the $2^{n+1} \times 2^{n+1}$ matrices. In the formulation above, we take each M_n to be a copy of the 2×2 matrices, and write D_n for the n-fold tensor product of the algebras of diagonal 2×2 matrices, which is a masa in N_n. Tauer's first example, Theorem 3.2 of [70], is that of *the* Cartan masa in the hyperfinite factor.[1] In this result, she demonstrates that the Tauer masa D generated by the approximates D_n in N_n is Cartan in R. We can see this immediately, as any unitary in N_n which normalises D_n actually normalises all of D. By Proposition 1.4.9, D_n is Cartan in N_n, so we have $N_n \subset \mathcal{N}(D)''$ for each n, and hence $\mathcal{N}(D)'' = R$.

To construct her other examples, Tauer used a sequence of unitaries u_n with each $u_n \in N_n \cap D_{n-1}'$. Define masas A_n in N_n by

$$A_n = u_1 \ldots u_n D_n u_n^* \ldots u_1^*, \tag{2.1.4}$$

and observe that, as u_{n+1} commutes with D_n, we have $A_n \subset A_{n+1}$ so a Tauer masa A is obtained from this sequence of unitaries. Properties of the masas resulting from certain sequences of unitaries were then deduced by intricate calculations, [70, Sections 4,5 and 6]. We note that given any Tauer masa A with respect to the subfactors $(N_n)_{n=1}^{\infty}$, one can find unitaries $u_n \in N_n \cap D_{n-1}'$ such that (2.1.4) holds by repeatedly appealing to the uniqueness up to unitary conjugation of masas in finite type I factors.

In examining Tauer's examples we shall consider both the formulations (2.1.3) and (2.1.4). The first presentation, combined with modern methods for examining

[1] Connes, Feldman and Weiss' result, Theorem 1.4.6, on the uniqueness of the Cartan masa up to conjugacy justifies referring to *the* Cartan masa in R.

the von Neumann algebra generated by normalisers will be used to demonstrate singularity and semi-regularity in the next two sections. The unitaries required for the second characterisation will sometimes be explicitly given, both for completeness and as they are naturally generalisable to give further interesting examples of masas.

Tensor products appear regularly in this thesis. We end this section by noting that the class of Tauer masas is closed under tensor products.

Proposition 2.1.3. *Let A and B be Tauer masas in the hyperfinite* II_1 *factor R. Then $A\overline{\otimes}B$ is a Tauer masa in $R\overline{\otimes}R$. More generally if, for each $m\in\mathbb{N}$, $A^{(m)}$ is a Tauer masa in R, then $(\bigotimes_{m=1}^{\infty}A^{(m)})''$ is a Tauer masa in $R^{\overline{\otimes}\infty}$.*

Proof. For the first statement, let $(M_n)_{n=1}^{\infty}$ and $(N_n)_{n=1}^{\infty}$ be chains of finite dimensional subfactors in R such that A and B are Tauer masas with respect to the subfactors $(M_n)_{n=1}^{\infty}$ and $(N_n)_{n=1}^{\infty}$ respectively. Write $A_n=A\cap M_n$ and $B_n=B\cap N_n$ for the finite dimensional approximates generating A and B. The chain $(M_n\otimes N_n)_{n=1}^{\infty}$ certainly generates $R\overline{\otimes}R$, and $A_n\otimes B_n$ is a masa in each $M_n\otimes N_n$ which is contained in $A_{n+1}\otimes B_{n+1}$. It is immediate that $(\bigcup_{n=1}^{\infty}A_n\otimes B_n)''=A\overline{\otimes}B$ in $R\overline{\otimes}R$, so that $A\overline{\otimes}B$ is a Tauer masa with respect to the chain of subfactors $(M_n\otimes N_n)_{n=1}^{\infty}$.

For the more general claim, we need a triangular argument to ensure everything remains suitably finite. Let $(N_n^{(m)})_{n=1}^{\infty}$ be a chain of finite dimensional subfactors, with respect to which $A^{(m)}$ is a Tauer masa with approximates denoted by $A_n^{(m)}=N_n^{(m)}\cap A^{(m)}$. Write $R^{(m)}$ for the copy of the hyperfinite II_1 factor obtained as the direct limit of the $N_n^{(m)}$. For each $m\in\mathbb{N}$, let

$$M_n=N_n^{(1)}\otimes N_{n-1}^{(2)}\otimes N_{n-2}^{(3)}\otimes\cdots\otimes N_2^{(n-1)}\otimes N_1^{(n)},$$

a finite dimensional subfactor of $R^{(1)}\overline{\otimes}R^{(2)}\overline{\otimes}\ldots\overline{\otimes}R^{(n)}$, which itself is included naturally in $(\bigotimes_{m=1}^{\infty}R^{(m)})''$. With the natural inclusion of M_n inside M_{n+1}, we have a chain $(M_n)_{n=1}^{\infty}$ which generates $(\bigotimes_{m=1}^{\infty}R^{(m)})''$. Furthermore,

$$A_n=A_n^{(1)}\otimes A_{n-1}^{(2)}\otimes A_{n-2}^{(3)}\otimes\cdots\otimes A_2^{(n-1)}\otimes A_1^{(n)}$$

is a masa in M_n for every n, and $A_n\subset A_{n+1}$. Hence, $(\bigcup_{n=1}^{\infty}A_n)''=(\bigotimes_{m=1}^{\infty}A^{(m)})''$ is a Tauer masa in $(\bigotimes_{m=1}^{\infty}R^{(m)})''$. □

2.2 Tauer's singular masa

In section 4 of [70], Tauer demonstrated the existence of a sequence of unitaries yielding a singular Tauer masa A from the approximates of (2.1.4). A complicated approximation argument was used in [70] to show directly that any element in $\mathcal{N}(A)''$ can be approximated by elements in the chain $(A_n)_{n=1}^\infty$ generating the Tauer masa. This method involves looking at infinitely many of the inclusions $A_n \subset N_n$ simultaneously. Alternatively, as noted in [44], it is possible to proceed using Popa's orthogonality method, Proposition 1.4.7. By verifying that Tauer's singular masa has the weak asymptotic homomorphism property of Definition 1.4.19, we are able to establish the singularity of A in a much more routine fashion below. In particular, we shall only ever need to look at one inclusion $A_n \subset N_n$ at once. To do this we set out how Tauer's singular masa appears when examined in the form (2.1.3), which we shall take as the definition.

Construction 2.2.1 (Tauer's Singular Masa). Let N_n be the n-fold tensor product of the I_2 factor of 2×2 matrices with normalised trace. Take A_1 to be the diagonal masa D_1 in N_1. Given the 2^t-th approximation A_{2^t}, index the minimal projections of A_{2^t} by

$$\mathcal{P}_{\min}(A_{2^t}) = \left\{ f_i^{(t)} \,\middle|\, i \in \{0,1\}^{2^t} \right\},$$

and take $A_{2^{t+1}}$ to be the masa in $N_{2^{t+1}}$ given by

$$A_{2^{t+1}} = \bigoplus_{i\in\{0,1\}^{2^t}} f_i^{(t)} \otimes \bigotimes_{r=1}^{2^t} (b^{i_r} D_1 b^{*i_r}). \tag{2.2.1}$$

Here D_1 is the algebra of diagonal 2×2 matrices and b denotes the self-adjoint unitary

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which conjugates D_1 onto bD_1b^* - an orthogonal masa to D_1 in N_1. In this way we obtain a Tauer masa A in the copy of the hyperfinite II_1 factor generated by the $(N_n)_{n=1}^\infty$, which we refer to[2] as *Tauer's singular masa*. Observe the lacunary nature of this construction, in that we are really working with respect to the chain $(N_{2^t})_{t=0}^\infty$ of type I_{2^t} factors generating R. In fact, the only reason for introducing n is to tie up our formulation with Tauer's original definition later.

2.2.1 Establishing singularity

We begin our proof that Tauer's singular masa is indeed singular by rephrasing Lemma 1.4.23, which deals with the weak asymptotic homomorphism property in a commuting squares context, to suit the Tauer masa situation.

[2]slightly ahead of the game

Proposition 2.2.2. *Let A be a Tauer masa with respect to the subfactors $(N_n)_{n=1}^{\infty}$. Suppose that, for infinitely many n, each minimal projection $e \in A_n$ and $\epsilon > 0$, we can find $m > n$ and a unitary in $w_e \in A_{m,n}^{(e)}$,*[3] *with*

$$\left\|\mathbb{E}_{A_{m,n}^{(f)}}(w_e)\right\|_2 \leq \epsilon, \tag{2.2.2}$$

for every minimal projection $f \neq e$ in A_n. Then A has the weak asymptotic homomorphism property, and so is a strongly singular masa in R.

Proof. Using the commutative diagram Figure 2.1 from Proposition 2.1.2, Lemma 1.4.23 demonstrates that it suffices to prove that, given operators $x_1, \ldots, x_m$ in some N_n with $\mathbb{E}_{A_n}(x_i) = 0$ and $\epsilon > 0$, we can find a unitary $v \in A$ with

$$\left\|\mathbb{E}_A\left(x_i v x_j^*\right)\right\|_2 < \epsilon,$$

for all i and j. By increasing n if necessary, we can assume that n belongs to the infinite set of the hypothesis.

We shall take $v = \sum_{e \in \mathcal{P}_{\min}(A_n)} e \otimes w_e$, where the w_e are the unitaries in $A_{m,n}^{(e)}$ satisfying (2.2.2) guaranteed by the hypothesis of the proposition. Suppose that u is a partial isometry in N_n with $uu^* = f \in \mathcal{P}_{\min}(A_n)$ and $u^*u = e \in \mathcal{P}_{\min}(A_n)$, with $e \neq f$. As $uvu^* = f \otimes w_e$, we have

$$\left\|\mathbb{E}_A\left(uvu^*\right)\right\|_2 = \|f\|_2 \left\|\mathbb{E}_{A_{m,n}^{(f)}}(w_e)\right\|_2 \leq \|f\|_2\, \epsilon.$$

If u' is another partial isometry with $u'u'^*, u'^*u' \in \mathcal{P}_{\min}(A)$ and one of $uu^* \neq f$ or $u^*u \neq e$, then it is immediate that

$$\mathbb{E}_A\left(uvu'^*\right) = \mathbb{E}_A\left(u'vu^*\right) = 0.$$

The result now follows, as by taking matrix units, any operator $x \in N_n$ with $\mathbb{E}_{A_n}(x) = 0$, can be written as a linear combination of partial isometries u in N_n with initial and range projections orthogonal minimal projections for A_n. Any two distinct partial isometries u and u' in this combination must have either $u^*u \neq u'^*u'$ or $uu^* \neq u'u'^*$. □

Theorem 2.2.3. *Tauer's singular masa A produced by Construction 2.2.1 above, has the weak asymptotic homomorphism property and so is indeed singular.*

Proof. Fix $t \geq 0$ and an index $i \in \{0,1\}^{2^t}$. We claim that the unitary u_i in $A_{2^{t+1},2^t}^{\left(f_i^{(t)}\right)}$ given by

$$u_i = \bigotimes_{r=1}^{2^t}\left(b^{i_r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(b^*)^{i_r}\right),$$

[3] Recall that $A_{m,n}^{(e)}$ is defined by equation (2.1.3).

has

$$\mathbb{E}_{A_{2^{t+1},2^t}^{\left(f_j^{(t)}\right)}}(u_i) = 0,$$

for every $j \in \{0,1\}^{2^t}$ with $j \neq i$. The theorem then follows from this claim and Proposition 2.2.2.

Using (1.3.1) for the second identity we see that

$$\begin{aligned}\mathbb{E}_{A_{2^{t+1}}^{(f_j)}}(u_i) &= \bigotimes_{r=1}^{2^t} \mathbb{E}_{b^{j_r} D_1 (b^*)^{j_r}} \left(b^{i_r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (b^*)^{i_r} \right) \\ &= \bigotimes_{r=1}^{2^t} b^{j_r} \left(\mathbb{E}_{D_1} \left(b^{i_r - j_r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (b^*)^{i_r - j_r} \right) \right) (b^*)^{j_r}. \qquad (2.2.3)\end{aligned}$$

As bD_1b^* and D_1 are orthogonal masas in the algebra of 2×2 matrices, we have

$$\mathbb{E}_{D_1} \left(b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} b^* \right) = 0.$$

The claim then follows by examining the r^{th} component of the tensor product in (2.2.3), for some r with $i_r \neq j_r$. □

The orthogonality used in the last step of Theorem 2.2.3 can be used, with a bit more work, to establish the hypothesis of Proposition 1.4.7 for an appropriate set of unitaries, giving the alternative proof of the singularity of Tauer's masa hinted at in [44]. We shall also see this orthogonality technique at work in section 3.3 in combination with Popa's observation, Proposition 1.2.10, that large families of pairwise orthogonal masas can be found in appropriately large matrix algebras, to produce singular Tauer masas which do not contain centralising sequences of R.

2.2.2 A reformulation and the θ-masas

Our next objective is to write down a sequence of unitaries $(u_n)_{n=1}^{\infty}$ giving Tauer's singular masa in the form (2.1.4). Fix matrix units $(e_{i,j})_{i,j=0}^{1}$ for the 2×2 matrices N_1. We first inductively index the minimal projections $\mathcal{P}_{\min}(A_{2^t})$, introduced in Construction 2.2.1. Namely we take $f_0^{(0)} = e_{0,0}$ and $f_1^{(0)} = e_{1,1}$. For an index $i \in \{0,1\}^{2^{t+1}}$, let $f_i^{(t+1)}$ be given by

$$f_i^{(t+1)} = f_{i|_{2^t}}^{(t)} \otimes \bigotimes_{r=1}^{2^t} \left(b^{i_r} e_{i_{2^t+r, 2^t+r}} b^{*i_r} \right), \qquad (2.2.4)$$

where we write $i|_{2^t} = (i_1, \ldots, i_{2^t})$, for the restriction of i to the first 2^t entries.

Set $u_1 = 1$ and assume inductively that we have found $u_2, \ldots, u_{2^t}$, for some $t \geq 0$, such that

$$u_1 \ldots u_{2^t} \left(\bigotimes_{r=1}^{2^t} e_{i_r,i_r} \right) u_{2^t}^* \ldots u_1^* = f_i^{(t)}, \tag{2.2.5}$$

for all $i \in \{0,1\}^{2^t}$. For $n = 2^t + r$, with $1 \leq r \leq 2^t$, define

$$u_n = 1^{\otimes(r-1)} \otimes \left(e_{0,0} \otimes 1^{\otimes(n-r)} + e_{1,1} \otimes 1^{(\otimes n-r-1)} \otimes b \right), \tag{2.2.6}$$

which is readily seen to lie in $N_n \cap D'_{n-1}$. Then

$$u_{2^t+1} u_{2^t+2} \ldots u_{2^{t+1}} = \sum_{i \in \{0,1\}^{2^t}} \left(\bigotimes_{r=1}^{2^t} e_{i_r,i_r} \otimes \bigotimes_{s=1}^{2^t} b^{i_s} \right),$$

which combines with the inductive hypothesis (2.2.5) and the inductive indexing of the $f_i^{(t)}$ to ensure that we have

$$u_1 \ldots u_{2^{t+1}} \left(\bigotimes_{r=1}^{2^{t+1}} e_{i_r,i_r} \right) u_{2^{t+1}}^* \ldots u_1^* = f_i^{(t+1)}.$$

The sequence of unitaries given by (2.2.6) then satisfies (2.1.4) and so represents the singular Tauer masa in this form. Careful examination of part 4.5 of [70] confirms that these unitaries are exactly those used by Tauer to give an example of a singular masa.

We can naturally generalise Tauer's singular masa by allowing the conjugating unitary b to depend on the stage n.

Construction 2.2.4. Given a sequence $(\theta_n)_{n=2}^{\infty}$ of real numbers in the interval $[0, \pi/4]$, we define a Tauer masa corresponding to this sequence with respect to the subfactors $(N_n)_{n=1}^{\infty}$ isomorphic to the n-fold tensor product of the 2×2 matrices with matrix units $(e_{i,j})_{i,j=0}^{1}$. As above, we write D_n for the n-fold tensor product of the diagonal 2×2 matrices - a masa in N_n.

Take $u_1 = 1$ and for $n > 1$, write $n = 2^t + r$ with $1 \leq r \leq 2^t$ and set

$$u_n = 1^{\otimes(r-1)} \otimes \left(e_{0,0} \otimes 1^{\otimes(n-r)} + e_{1,1} \otimes 1^{(\otimes n-r-1)} \otimes b_{\theta_n} \right),$$

where b_{θ_n} denotes the reflection matrix

$$b_{\theta_n} = \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ \sin\theta_n & -\cos\theta_n \end{pmatrix}. \tag{2.2.7}$$

This sequence has $u_n \in N_n \cap D'_{n-1}$ and so gives a Tauer masa A by

$$A_n = u_1 \ldots u_n D_n u_n^* \ldots u_1^*.$$

The calculations above show that Tauer's singular masa corresponds to taking each $\theta_n = \pi/4$ in this construction.

To express these Tauer masas in the form of (2.1.3) we reverse the calculation following the proof of Theorem 2.2.3. Write $D^{(0)}$ for the masa of diagonal matrices in N_1, with minimal projections

$$e_0^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } e_1^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Given $\theta \in [0, \pi/4]$, let $D^{(\theta)} = b_\theta D^{(0)} b_\theta^*$ - a masa in the algebra of 2×2 matrices with minimal projections

$$(e_i^{(\theta)} = b_\theta e_i^{(0)} b_\theta^*)_{i=0,1}.$$

The first approximation is given by $A_1 = D^{(0)}$, with minimal projections $f_i^{(0)} = e_i^{(0)}$ for $i = 0, 1$. Suppose that the set of minimal projections of A_{2^t} is $f_i^{(t)}$ indexed by $i \in \{0,1\}^{2^t}$, then the 2^{t+1}-th approximation is given by

$$A_{2^{t+1}} = \bigoplus_{i \in \{0,1\}^{2^t}} f_i^{(t)} \otimes \bigotimes_{r=1}^{2^t} b_{\theta_{2^t+r}}^{i_r} D^{(0)} {b_{\theta_{2^t+r}}^{i_r}}^* = \bigoplus_{i \in \{0,1\}^{2^t}} f_i^{(t)} \otimes \bigotimes_{r=1}^{2^t} D^{(\delta_{i_r,1}\theta_{2^t+r})}$$

Finally, for $i \in \{0,1\}^{2^{t+1}}$, let

$$f_i^{(t+1)} = f_{(i_1,\dots,i_{2^t})}^{(t)} \otimes \bigotimes_{r=1}^{2^t} e_{i_{2^t+r}}^{(\delta_{i_r,1}\theta_{2^t+r})},$$

giving us the minimal projections for $A_{2^{t+1}}$.[4]

Remark 2.2.5. Working with the reflection matrices of (2.2.7) was just an aesthetic choice based on the Tauer's extensive use of the reflection matrix b (of Construction 2.2.1) in constructing her singular masa, [70, Section 4]. We could have equally used rotation matrices for this construction, as these give rise to the same masas in the 2×2 matrices. More formally, let

$$c_{\theta_n} = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix}.$$

Then, for each n,

$$b_{\theta_n} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = c_{\theta_n},$$

so that $b_{\theta_n} D^{(0)} b_{\theta_n}^* = c_{\theta_n} D^{(0)} c_{\theta_n}^*$. Hence, if we replace all the b_{θ_n} in Construction 2.2.4 with the appropriate c_{θ_n}, then we obtain the same Tauer masa A.

[4]The δ appearing here and above, is the Kroneker δ-symbol. As usual $\delta_{i_r,1}$ is 1 when $i_r = 1$, and 0 otherwise.

Obviously not all the masas constructed above can be singular, for example if each $\theta_n = 0$, then we obtain the Cartan masa in R. We can, however characterise singularity of these masas as follows.

Theorem 2.2.6. *Let $(\theta_n)_{n=2}^{\infty}$ be a sequence of reals in the range $[0, \pi/4]$. The Tauer masa A obtained from Construction 2.2.4 corresponding to this sequence is singular if and only if, for every r, the sum*

$$\sum_{t \geq \lceil \log_2 r \rceil} \theta_{2^t+r}^2 \tag{2.2.8}$$

diverges.

We could prove this theorem now, using a much more involved calculation based on the proof of Theorem 2.2.3 to obtain the weak asymptotic homomorphism property and hence singularity of A when the divergence criterion is satisfied. A perturbation argument, using Theorem 1.4.17, can be used to establish the converse. In Theorem 4.1.3, we shall characterise singularity for any Tauer masa in terms of a convergence criterion, using these ideas, and so we prefer to deduce Theorem 2.2.6 from this result in section 4.2. These masas will also make an appearance in section 4.3, where additional properties will be investigated. For example, we show in Corollary 4.3.12, that all these θ-masas contain non-trivial centralising sequences for R.

2.3 Tauer's semi-regular masas

2.3.1 Tauer's length invariant

Tauer's *length* invariant for semi-regular masas, is the natural continuation of Diximer's original classification of masas into singular, semi-regular and Cartan masas by looking at the algebra generated by the normalisers. Her original concept, which we give below, was of finite length masas. In section 2.4, we will extend this idea to produce masas of infinite length.

Definition 2.3.1. Given a masa A in a von Neumann algebra N, we define for each integer $n \geq 0$, the *n-fold normalising algebra*, $\mathcal{N}^n(A)$, inductively by

$$\mathcal{N}^n(A) = \begin{cases} A & n = 0 \\ \mathcal{N}(\mathcal{N}^{n-1}(A))'' & n > 0 \end{cases}.$$

A masa A is then, not quite following Tauer[5], said to have *length l* if $\mathcal{N}^l(A) = N$ and $\mathcal{N}^{l-1}(A) \neq N$.

[5] Tauer defined a masa to be of length l when the $l+1$-fold normalising algebra is the first to generate N.

We briefly digress here to note it is enough to check that $\mathcal{N}^1(A)$ is a factor to see that all the normalising algebras of A are factors.

Proposition 2.3.2. *If A is a semi-regular masa in a type* II_1 *factor N, then each $\mathcal{N}^n(A)$ is an irreducible subfactor of N.*

Proof. Suppose inductively that $\mathcal{N}^n(A)$ is an irreducible subfactor of N. The inclusion

$$\mathcal{N}^{n+1}(A) \cap \mathcal{N}^{n+1}(A)' \subset N \cap \mathcal{N}^{n+1}(A)' \subset N \cap \mathcal{N}^n(A)' = \mathbb{C}1,$$

shows that $\mathcal{N}^{n+1}(A)$ is also an irreducible subfactor of N. □

In section 5 of [70], Tauer exhibited semi-regular masas in the hyperfinite II_1 factor R of all finite lengths greater than 1, while in section 6 she went on to give two such masas of length 2 which are not conjugate by an automorphism of R. The invariant she used to show this non-conjugacy, in Lemma 6.7 of [70], was condition 2 of Proposition 1.3.3. As that proposition shows, Tauer's idea was an early appearance of Jones' index for subfactors, for her condition is exactly the statement $[\mathcal{N}^2(A) : \mathcal{N}^1(A)] = 2$.[6]

Tauer's second example of a length 2 semi-regular masa A has $[\mathcal{N}^2(A) : \mathcal{N}^1(A)] = 4$. The Jones index $[\mathcal{N}^2(A) : \mathcal{N}^1(A)]$, gives an invariant for length 2 masas, which must take values in $\mathbb{N} \cup \{\infty\} \setminus \{1\}$ by Proposition 1.3.2. In subsection, 2.3.5, we indicate how to construct a Tauer masa of length 2 for each of these possible values.

More generally, the $(l-1)$-tuple $([\mathcal{N}^{m+1}(A) : \mathcal{N}^m(A)])_{m=1}^{l-1}$ gives an invariant for semi-regular masas of length l. By repeated application of Proposition 1.3.2 and the tower law (1.3.4), this $(l-1)$-tuple must consist of elements from $\mathbb{N} \cup \{\infty\} \setminus \{1\}$. In [71], Tauer followed up her work of [70] to indicate how one could construct 2^{l-1} pairwise non-conjugate semi-regular masas of length l in the hyperfinite II_1 factor. The *product-type* invariant she used to do this was really a determination of which m the iterated normaliser algebra $\mathcal{N}^m(A)$ has index 2 in $\mathcal{N}^{m+1}(A)$. We shall not give these examples, preferring in this section to give just the examples appearing in [70]. In the next section, 2.4, we shall examine infinite length masas, the definition of which is obvious but is nevertheless spelt out in full there. Using the methods there, it will be clear how to construct a Tauer masa of length l for which each $[\mathcal{N}^{m+1}(A) : \mathcal{N}^m(A)] = 2^{\mu_m}$ for some $\mu_m \in \mathbb{N}$.[7] This gives countably many pairwise non-conjugate masas of length l.

[6] Under the additional assumption that $[\mathcal{N}^2(A) : \mathcal{N}^1(A)] < \infty$.

[7] As we shall subsequently argue, there should be no intrinsic difficulty in obtaining any other value in $\mathbb{N} \cup \{\infty\} \setminus \{1\}$ for these $[\mathcal{N}^{m+1}(A) : \mathcal{N}^m(A)]$, merely a calculational inconvenience.

2.3.2 Tauer's examples of finite length masas: Setup

Our next objective will be to give an account of the masas constructed in section 5 of [70]. For most of this section, l will be a fixed integer with $l \geq 2$, the length of the masa we shall produce. We begin by giving an initial instalment of notation which will remain fixed for the remainder of this section.

Notation 2.3.3. Let M denote the algebra of 2×2 matrices and let $(e_{i,j}^{(0,0)})_{i,j=0}^{1}$ be the standard matrix units for M. Write $D^{(0)}$ for the masa in M consisting of the diagonal matrices, which then has minimal projections $e_{0,0}^{(0,0)}$ and $e_{1,1}^{(0,0)}$. As in the previous section, we write b for the unitary

$$b = b^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and $D^{(1)} = bD^{(0)}b^*$, a masa in M which is orthogonal to $D^{(0)}$. Given $i, i', j, j' \in \mathbb{Z}_2$, write

$$e_{i,j}^{(i',j')} = b^{i'} e_{i,j}^{(0,0)} b^{*j'},$$

a partial isometry in M with initial projection $e_{j,j}^{(j',j')}$ and range projection $e_{i,i}^{(i',i')}$. For brevity we adopt the convention that if only one index appears in the subscript or superscript it should be regarded as being repeated. In particular, we shall write $e_i^{(i')}$ for $e_{i,i}^{(i',i')}$. With this convention, the minimal projections of $D^{(i')}$ are $e_0^{(i')}$ and $e_1^{(i')}$.

Given indices $i_r^{(m)}$ and $j_r^{(m)}$ in $\mathbb{Z}_2$, for $1 \leq m \leq l-1$ and $1 \leq r \leq n$, we define equivalence relations $\overset{m}{\sim}$ by $i \overset{m}{\sim} j$ if and only if $\sum_{r=1}^{n}(i_r^{(m)} - j_r^{(m)}) = 0$ in $\mathbb{Z}_2$. Here and throughout, we write $i = (i_r^{(m)})_{r,m}$ and similarly for j. We write $f_{i,j}$ for the element of $M^{\otimes n(l-1)}$ given by

$$f_{i,j} = \bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n} e_{i_r^{(m)}, j_r^{(m)}}^{(\sum_{t=1}^{r-1} i_t^{(m-1)}, \sum_{t=1}^{r-1} j_t^{(m-1)})},$$

where the sums $\sum_{t=1}^{r-1} i_t^{(m-1)}$ and $\sum_{t=1}^{r-1} j_t^{(m-1)}$ are taken in $\mathbb{Z}_2$, and are defined to be zero when they make no sense (i.e. when $m = 1$ or $r = 1$). Note that these $f_{i,j}$ are matrix units for $M^{\otimes n(l-1)}$ and, following the precedent set earlier, we make the convention that $f_i = f_{i,i}$ here also.

Before setting up our Tauer masa, we record an easy feature of the $e_{i,j}^{(i',j')}$ which will be critical later in this section.

Proposition 2.3.4. *With the previous notation we have*

$$e_0^{(t)} - e_1^{(t)} = e_{0,1}^{(1+t)} + e_{1,0}^{(1+t)},$$

for $t \in \mathbb{Z}_2$.

Proof. This proposition can be readily checked by a simple 2×2 matrix calculation. □

We are now in a position to describe the Tauer masas we shall study in this section, which are an alternative formulation of those examined by Tauer.

Construction 2.3.5. Let $N_n = M^{\otimes n(l-1)} \otimes M^{\otimes n(l-1)}$. Write ι_n for the embedding of $M^{\otimes n}$ inside $M^{\otimes (n+1)}$ given by $\iota_n(x) = x \otimes 1$, so $\iota_n^{\otimes(l-1)} \otimes \iota_n^{\otimes(l-1)}$ gives an embedding $N_n \hookrightarrow N_{n+1}$. In this way, we obtain a chain of finite dimensional factors generating the hyperfinite II_1 factor R.

Let A_n be the masa in N_n given by

$$A_n = \bigoplus_{\substack{i_r^{(m)} \in \mathbb{Z}_2 \\ 1 \leq m \leq l-1 \\ 1 \leq r \leq n}} f_i \otimes \bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n} D^{(\sum_{t=1}^{r} i_t^{(m)})}. \tag{2.3.1}$$

Observe that when we embed N_n into N_{n+1}, we have $A_n \subset A_{n+1}$, giving rise to a Tauer masa A in R.

Our objective is to show that the masa constructed above has length l, this result being the main thrust of section 5 of [70]. The discussion in subsection 2.3.1, motivates the calculation of the index of each of the inclusions of the m-fold normalising algebras of these masas. Before getting underway, we note that we could use the same methods as in the previous section to give an explicit characterisation of a sequence of unitaries yielding the masa of Construction 2.3.5 in Tauer's form (2.1.4). The sequence of unitaries obtained in this instance, would then coincide with those which can be carefully extracted from 5.14 of [70], and so we are dealing with Tauer's original semi-regular masas here.

Theorem 2.3.6. *For each $l \geq 2$ the Tauer masa A produced by Construction 2.3.5 is a semi-regular masa of length l. Furthermore, $[\mathcal{N}^{m+1}(A) : \mathcal{N}^{m}(A)] = 2$ for $m = 1, \ldots, l-1$.*

In the next subsection, we shall embark upon the details for this Theorem. We end this subsection by outlining the plan of campaign.

1. Write down explicitly generators for a subfactor $\widetilde{S}$ of R (and check that it is indeed a subfactor).
2. Demonstrate that $\widetilde{S}$ is indeed the first normaliser algebra $\mathcal{N}(A)''$.
3. Describe explicitly a subfactor $\widetilde{T}$ of R containing $\widetilde{S}$ with $[\widetilde{T} : \widetilde{S}] = 2$.

4. Demonstrate that this $\widetilde{T}$ is generated, as a von Neumann algebra, by the normalisers of $\widetilde{S}$.

5. Repeatedly apply steps 3 and 4 to write down explicit formulations of all the normaliser algebras $\mathcal{N}^m(A)$.

2.3.3 Parts 1 and 2 of the proof of Theorem 2.3.6

The underlying hyperfinite II_1 factor, R, naturally factorises as the von Neumann tensor product $R_1 \overline{\otimes} R_2$, where R_1 and R_2 are hyperfinite factors, both generated as the direct limit of the chain (2.3.2).

$$M^{\otimes(l-1)} \overset{\iota_1^{\otimes(l-1)}}{\hookrightarrow} M^{\otimes 2(l-1)} \overset{\iota_2^{\otimes(l-1)}}{\hookrightarrow} \cdots \overset{\iota_{n-1}^{\otimes(l-1)}}{\hookrightarrow} M^{\otimes n(l-1)} \overset{\iota_n^{\otimes(l-1)}}{\hookrightarrow} M^{\otimes(n+1)(l-1)} \overset{\iota_{n+1}^{\otimes(l-1)}}{\hookrightarrow} \cdots \tag{2.3.2}$$

We shall obtain our target subfactor, $\widetilde{S}$, in the form $S\overline{\otimes}R_2$ - for an explicit subfactor S of R_1, which we now construct.

Write S_n for the sub*space* of $M^{\otimes n(l-1)}$ generated by those $f_{i,j}$ with $i \overset{m}{\sim} j$, in the sense of Notation 2.3.3, for each $m = 1, \dots, l-1$. Since this is an equivalence relation and the $(f_{i,j})$ are matrix units for $M^{\otimes n(l-1)}$, we see that S_n is a *-sub*algebra* of $M^{\otimes n(l-1)}$. Each S_n is included into S_{n+1} by $\iota_n^{\otimes(l-1)}$, so we can define S to be the von Neumann subalgebra of R_1 generated by the S_n.

To see that $\widetilde{S}$ is a subfactor of R, it is enough to check that S is a subfactor of R_1. We shall do this by examining the Bratteli diagram of the inclusion $S_n \subset S_{n+1}$. For each $k^{(1)}, \dots, k^{(l-1)} \in \mathbb{Z}_2$, the subspace of S_n generated by those $f_{i,j}$ with $\sum_{r=1}^n i_r^{(m)} = \sum_{r=1}^n j_r^{(m)} = k^{(m)}$ for all m, is a factor, which we temporarily denote by $S_n(k^{(1)}, \dots, k^{(l-1)})$. One can see this, as these $f_{i,j}$ are evidently matrix units for $S_n(k^{(1)}, \dots, k^{(l-1)})$.[8] By counting the matrix units, we see that each $S_n(k^{(1)}, \dots, k^{(l-1)})$ is a type $\mathrm{I}_{2^{(n-1)(l-1)}}$ factor. In conclusion we note that S_n decomposes into the direct sum of 2^{l-1} type $\mathrm{I}_{2^{(n-1)(l-1)}}$ factors:

$$S_n = \bigoplus_{k^{(1)}, \dots, k^{(l-1)} \in \mathbb{Z}_2} S_n(k^{(1)}, \dots, k^{(l-1)}).$$

The inclusion $\iota_n^{\otimes(l-1)}$ includes each factor $S_n(k^{(1)}, \dots, k^{(l-1)})$ in the decomposition of S_n precisely once into every factor $S_{n+1}(k'^{(1)}, \dots, k'^{(l-1)})$. The Bratelli diagram of the inclusion $S_n \subset S_{n+1}$ connects every element in each row once to every element of the next row, as indicated in the example corresponding to the case $l = 3$ in Figure 2.2. In this instance we can immediately deduce that S is a

[8]It is easy to find i_1, i_2, j_1 and j_2 with f_{i_1,j_1} and f_{i_2,j_2} in S_n, while $f_{i_1,j_2} \notin S_n$. In this way, all the $f_{i,j}$ with $i \overset{m}{\sim} j$, for each m, do not give matrix units for S_n.

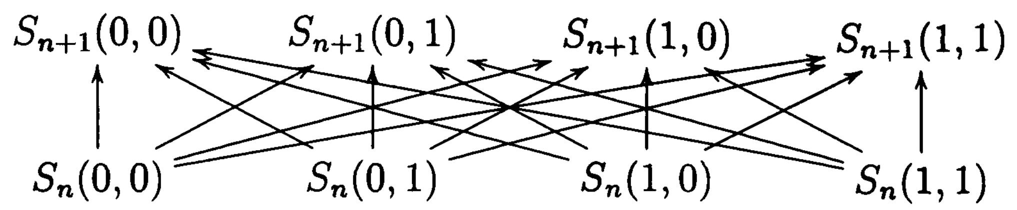


Figure 2.2: Bratteli diagram for the inclusion of $1 \in S_n \subset S_{n+1}$, with $l = 3$.

factor from Proposition 1.3.8.

For the next stage we demonstrate that $\widetilde{S} = \mathcal{N}(A)''$. As usual this splits into two parts.

Proposition 2.3.7. *With the notation above:*

1. $\widetilde{S} \subset \mathcal{GN}(A)'' = \mathcal{N}(A)''$;

2. *A has the weak asymptotic homomorphism property away from* $\widetilde{S}$.

Proof of 1. For each integer n, we shall demonstrate that $S_n \otimes M^{\otimes n(l-1)}$, regarded as a subalgebra of N_n, is contained within $\mathcal{GN}(A)''$.[9] For $1 \leq r \leq n$ and $1 \leq m \leq l-1$, fix indices $i_r^{(m)}, j_r^{(m)} \in \mathbb{Z}_2$ with $i \overset{m}{\sim} j$ for each m, and unitaries $x_r^{(m)}$ in M. Take

$$v = f_{i,j} \otimes \bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n} x_r^{(m)} \in S_n \otimes M^{\otimes n(l-1)}.$$

We consider $vA_{n_1}v^*$, for $n_1 > n$. Suppose we have indices $k_r^{(m)} \in \mathbb{Z}_2$ and elements $y_r^{(m)} \in D^{(\sum_{t=1}^{r} k_t^{(m)})}$ for $1 \leq r \leq n_1$ and $1 \leq m \leq l-1$, so that

$$y = f_k \otimes \bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n_1} y_r^{(m)},$$

is a typical generator of A_{n_1}. We have $vyv^* = 0$, unless $k_r^{(m)} = j_r^{(m)}$ for $1 \leq r \leq n$ and $1 \leq m \leq l-1$. In this case

$$vyv^* = f_{\overline{k}} \otimes \bigotimes_{m=1}^{l-1} \left(\bigotimes_{r=1}^{n} x_r^{(m)} y_r^{(m)} x_r^{(m)*} \otimes \bigotimes_{r=n+1}^{n_1} y_r^{(m)} \right),$$

where $\overline{k_r^{(m)}}$ is defined by

$$\overline{k_r^{(m)}} = \begin{cases} i_r^{(m)} & r \leq n \\ k_r^{(m)} & n < r \leq n_1 \end{cases}.$$

Under what conditions does vyv^* lie in A_{n_1}? Observe that, for $r > n$ and all m we have $y_r^{(m)} \in D^{(\sum_{t=1}^{r} \overline{k_t^{(m)}})}$, since $\sum_{r=1}^{n} i_r^{(m)} = \sum_{r=1}^{n} j_r^{(m)}$. To ensure that

[9]Recall from subsection 1.4.2 that the equality $\mathcal{GN}(A)'' = \mathcal{N}(A)''$ is well known for masas A.

$x_r^{(m)} y_r^{(m)} x_r^{(m)*} \in D^{(\sum_{t=1}^{r} i_t^{(m)})}$ for $1 \leq r \leq n$, we require that the unitaries $x_r^{(m)}$ satisfy

$$x_r^{(m)} D^{(\sum_{t=1}^{r} j_t^{(m)})} x_r^{(m)*} = D^{(\sum_{t=1}^{r} i_t^{(m)})}, \tag{2.3.3}$$

for all $1 \leq m \leq l-1$ and $1 \leq r \leq n$. Under these conditions v lies in the groupoid normaliser, $\mathcal{GN}(A)$.

Such v generate $S_n \otimes M^{\otimes(n-1)(l-1)}$, as the $x_r^{(m)}$ allowed by (2.3.3) generate M, by Proposition 1.4.9. Since n was arbitrary, it follows that $\widetilde{S} \subset \mathcal{GN}(A)''$ as required. □

Before proving part 2 of Proposition 2.3.7, we need to examine the conditional expectation onto S.

Proposition 2.3.8. *The square in diagram 2.3 commutes in the sense of Definition 1.3.7. For indices $i_r^{(m)}, j_r^{(m)}$ $(1 \leq r \leq n, 1 \leq m \leq l-1)$, with $i \not\stackrel{m}{\sim} j$ for some m, we have $\mathbb{E}_S(f_{i,j}) = 0$.*

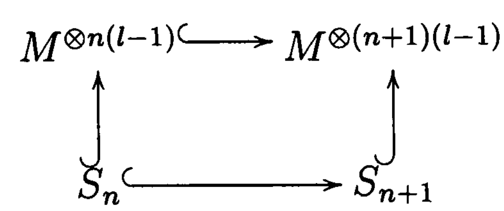


Figure 2.3: Commuting square

Proof. Since the $f_{i,j}$ are matrix units for $M^{\otimes n(l-1)}$, we note that

$$\mathrm{tr}(f_{i,j} f_{i',j'}^*) = 0,$$

whenever $i \not\stackrel{m}{\sim} j$ and $i' \stackrel{m}{\sim} j'$ for some $1 \leq m \leq l-1$. In this way, $\mathbb{E}_{S_n}(f_{i,j}) = 0$, whenever $i \not\stackrel{m}{\sim} j$ for some m. Once we have shown the commutativity of Figure 2.3, we will be able deduce that $\mathbb{E}_{S_{n'}}(f_{i,j}) = 0$ for all $n' \geq n$, and hence that $\mathbb{E}_S(f_{i,j}) = 0$.

To show that Figure 2.3 commutes, it suffices to check that $\iota_n^{\otimes(l-1)}(\mathbb{E}_{S_n}(f_{i,j})) = \mathbb{E}_{S_{n+1}}\left(\iota_n^{\otimes(l-1)}(f_{i,j})\right)$, for all indices i, j. When $i \stackrel{m}{\sim} j$ for all m, this is immediate from the fact that $\iota_n^{\otimes(l-1)}$ includes S_n into S_{n+1}. Suppose instead that there is some m with $i \not\stackrel{m}{\sim} k$, and let m_0 be the minimal m with this property. We must show that $\mathbb{E}_{S_{n+1}}\left(\iota_n^{\otimes(l-1)}(f_{i,j})\right) = 0$.

Now $\iota_n^{\otimes(l-1)}(f_{i,j})$ is an elementary tensor with the identity 1 in the $r = n+1$, $1 \leq m \leq m_0$ positions of the tensor product. In these positions, we have

$$1 = e_0^{\left(\sum_{t=1}^{n} i_t^{(m-1)}, \sum_{t=1}^{n} j_t^{(m-1)}\right)} + e_1^{\left(\sum_{t=1}^{n} i_t^{(m-1)}, \sum_{t=1}^{n} j_t^{(m-1)}\right)},$$

as $\sum_{t=1}^{n} i_t^{(m-1)} = \sum_{t=1}^{n} j_t^{(m-1)}$ by the minimality of m_0. In this way, we can write $\iota_n^{\otimes(l-1)}(f_{i,j})$ as a linear combination of $f_{i',j'}$ such that

$$i_r'^{(m)} = i_r^{(m)}, \quad j_r'^{(m)} = j_r^{(m)}$$

whenever $1 \leq m \leq l-1$ and $1 \leq r \leq n$ and

$$i_{n+1}'^{(m)} = j_{n+1}'^{(m)}$$

for $m \leq m_0$.[10] Such (i', j') inherit $i' \overset{m_0}{\not\sim} j'$ from the pair (i, j), so that $\iota_n^{\otimes(l-1)}(f_{i,j})$ is a linear combination of elements which are orthogonal to S_{n+1}. □

Proof of Proposition 2.3.7: Part 2. We shall work with a slightly different chain of finite approximations to R compared with the previous part. We approximate R_1, as before using the chain (2.3.2), however in the approximation of R_2, we start our approximating chain one stage behind. More precisely, we work in the chain $(M^{\otimes n(l-1)} \otimes \iota_{n-1}^{\otimes(l-1)}(M^{\otimes(n-1)(l-1)}))_{n=1}^{\infty}$ with repeated inclusion $\iota_n^{\otimes(l-1)} \otimes \iota_n^{\otimes(l-1)}$, which also generates R.[11] The factor $\widetilde{S}$ is then generated by the chain $(S_n \otimes \iota_{n-1}^{\otimes(l-1)}(M^{\otimes(n-1)(l-1)}))_{n=1}^{\infty}$. The square in Figure 2.4 commutes, as a consequence of Proposition 2.3.8.

$$\begin{array}{ccc}
M^{\otimes n(l-1)} \otimes \iota_{n-1}^{\otimes(l-1)}(M^{\otimes(n-1)(l-1)})) & \hookrightarrow & M^{\otimes(n+1)(l-1)} \otimes \iota_n^{\otimes(l-1)}(M^{\otimes n(l-1)})) \\
\uparrow & & \uparrow \\
S_n \otimes \iota_{n-1}^{\otimes(l-1)}(M^{\otimes(n-1)(l-1)}) & \hookrightarrow & S_{n+1} \otimes \iota_n^{\otimes(l-1)}(M^{\otimes n(l-1)})
\end{array}$$

Figure 2.4: Inclusions approximating $\widetilde{S}$ inside R

We are now in a good position to appeal to Lemma 1.4.23. Fix n and take indices $i_r^{(m)}, j_r^{(m)}, i_r'^{(m)}, j_r'^{(m)} \in \mathbb{Z}_2$, for $1 \leq r \leq n$ and $1 \leq m \leq l-1$ with $i \overset{m_1}{\not\sim} j$ and $i' \overset{m_2}{\not\sim} j'$, for some m_1 and m_2. We also take elements $x_r^{(m)}$ and $y_r^{(m)}$ of M and set

$$x = f_{i,j} \otimes \iota_{n-1}^{\otimes(l-1)} \left(\bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n-1} x_r^{(m)} \right) \text{ and } y = f_{i',j'} \otimes \iota_{n-1}^{\otimes(l-1)} \left(\bigotimes_{m=1}^{l-1} \bigotimes_{r=1}^{n-1} y_r^{(m)} \right).$$

Observe that such elements have linear span $\left(M^{\otimes n(l-1)} \ominus S_n\right) \otimes \iota_{n-1}^{\otimes(l-1)} \left(M^{\otimes(n-1)(l-1)}\right)$ and so it is enough to find a unitary $v \in A$, depending only on n (and explicitly not on x and y) such that $\mathbb{E}_A(xvy^*) = 0$.

[10] We can make no comments about the values of $i_{n+1}'^{(m)}$ and $j_{n+1}'^{(m)}$ for $m > m_0$.

[11] To be excessively precise, the repeated inclusion is actually the restriction of $\iota_n^{\otimes(l-1)} \otimes \iota_n^{\otimes(l-1)}$ to the domain.

Unsurprisingly, based on the calculation yielding Theorem 2.2.3, we take

$$v = \sum_{\substack{k_r^{(m)} \in \mathbb{Z}_2 \\ 1 \le m \le l-1 \\ 1 \le r \le n}} f_k \otimes \bigotimes_{m=1}^{l-1} \left(1^{\otimes(n-1)} \otimes (e_0^{(\sum_{r=1}^n k_r^{(m)})} - e_1^{(\sum_{r=1}^n k_r^{(m)})}) \right),$$

a unitary in A_n. As xvy^* lies in N_n, we can use the commuting squares of Proposition 2.1.2 to observe that $\mathbb{E}_A(xvy^*) = \mathbb{E}_{A_n}(xvy^*)$. It is apparent that $\mathbb{E}_{A_n}(xvy^*) = 0$ unless both $i = i'$ and $j = j'$. In this case we have

$$xvy^* = f_i \otimes \bigotimes_{m=1}^{l-1} \left(\bigotimes_{r=1}^{n-1} x_r^{(m)} y_r^{(m)*} \otimes (e_0^{(\sum_{r=1}^n j_r^{(m)})} - e_1^{(\sum_{r=1}^n j_r^{(m)})}) \right),$$

which is orthogonal to A_n. For this, recall that $\sum_{r=1}^n i_r^{(m_1)} \neq \sum_{r=1}^n j_r^{(m_1)}$, and then we have

$$\mathbb{E}_{D^{(\sum_{r=1}^n i_r^{(m_1)})}} \left(e_0^{(\sum_{r=1}^n j_r^{(m_1)})} - e_1^{(\sum_{r=1}^n j_r^{(m_1)})} \right) = 0,$$

as $D^{(0)}$ and $D^{(1)}$ are orthogonal masas in M. □

This completes the second step of the plan. Before progressing, we observe that for us to obtain the weak asymptotic homomorphism property for A away from $\widetilde{S}$ above, it was crucial that it was only necessary to work with the case $i = i'$ and $j = j'$. If we are unable to make this simplification then the weak asymptotic homomorphism property may turn out not to be available. At the risk of preempting matters, we point now to Appendix A, where a subfactor failing to have the weak asymptotic homomorphism property away from the algebra generated by its normalisers will be given. This subfactor is in fact the $S \subset R_1$ obtained by taking $l = 3$, and if we attempt to demonstrate the weak asymptotic homomorphism property, then we will not be able to make the same simplifying reduction that was possible for the masa A.

2.3.4 Parts $3 - 5$ of the proof of Theorem 2.3.6

Stage three calls for the explicit construction of a von Neumann subalgebra $\widetilde{T}$ of R which will turn out to be generated by the normalisers of $\widetilde{S}$ and so will automatically be an irreducible subfactor by Proposition 2.3.2. We will take $\widetilde{T} = T \overline{\otimes} R_2$, for a von Neumann subalgebra T of R_1, which of course will turn out to be that generated by the normalisers of S in R_1. Let T_n be generated as a subset of $M^{\otimes n(l-1)}$ by elements $f_{i,j}$ where $i_r^{(m)}, j_r^{(m)}$ are indices with $i \overset{m}{\sim} j$ for $m = 1, \dots, l-2$ and we impose no condition on the values of $i_r^{(l-1)}$ and $j_r^{(l-1)}$. Since this condition on the pair (i, j) is an equivalence relation, we have in fact defined a von Neumann subalgebra of $M^{\otimes n(l-1)}$ containing S_n. The embedding $\iota_n^{\otimes(l-1)}$ embeds T_n into T_{n+1}, and so we obtain a von Neumann subalgebra T of R_1 as the direct limit of these T_n. This T contains S.

At this point we jump to stage 5 and demonstrate why consideration of $\widetilde{S}$ and $\widetilde{T}$ suffice to establish Theorem 2.3.6. For the most part in this section, the length l has been fixed. The induction to follow will be over l, and so we shall need to briefly consider the factors S for varying values of l. We append the superscript ${}^{(l)}$ to one of our von Neumann algebras to explicitly refer to the value of l in question. We shall only use this when necessary, the rest of the time l should be considered fixed. The tensor product is associative so we are able to repeatedly rebracket $M^{\otimes n(l-1)}$ as

$$M^{\otimes n(l-1)} \cong M^{\otimes n(l-2)} \otimes M^{\otimes n}. \tag{2.3.4}$$

The inclusion also factorises as $\iota_n^{\otimes l-1} = \iota_n^{\otimes l-2} \otimes \iota_n$. Regarded inside $M^{\otimes n(l-2)} \otimes M^{\otimes n}$, T_n is then generated by $f_{i,j}^{(l-1)} \otimes M^{\otimes n}$, where $i \overset{m}{\sim} j$ for $1 \leq m \leq l-2$, and the indices i and j are indexed by $1 \leq m \leq l-2$ (and $1 \leq r \leq n$ as before), so

$$f_{i,j}^{(l-1)} = \bigotimes_{m=1}^{l-2} \bigotimes_{r=1}^{n} e_{i_r^{(m)}, j_r^{(m)}}^{\left(\sum_{t=1}^{r-1} i_t^{(m-1)}, \sum_{t=1}^{r-1} j_t^{(m-1)}\right)},$$

here. We can factorise R_1 as $R_{1,1}\overline{\otimes}R_{1,2}$, where $R_{1,1}$ is the direct limit of the chain

$$M^{\otimes(l-2)} \overset{\iota_1^{\otimes(l-2)}}{\hookrightarrow} M^{\otimes 2(l-2)} \overset{\iota_2^{\otimes(l-2)}}{\hookrightarrow} \cdots \overset{\iota_{n-1}^{\otimes(l-2)}}{\hookrightarrow} M^{\otimes n(l-2)} \overset{\iota_n^{\otimes(l-2)}}{\hookrightarrow} M^{\otimes(n+1)(l-2)} \overset{\iota_{n+1}^{\otimes(l-2)}}{\hookrightarrow} \cdots,$$

and $R_{1,2}$ the direct limit of

$$M \overset{\iota_1}{\hookrightarrow} M^{\otimes 2} \overset{\iota_2}{\hookrightarrow} \cdots \overset{\iota_{n-1}}{\hookrightarrow} M^{\otimes n} \overset{\iota_n}{\hookrightarrow} M^{\otimes(n+1)} \overset{\iota_{n+1}}{\hookrightarrow} \cdots.$$

The key point here is each that $T_n^{(l)}$ factorises as $S_n^{(l-1)} \otimes M^{\otimes n}$, and so $T^{(l)}$ factorises as $S^{(l-1)}\overline{\otimes}R_{1,2}$. Crucially, we have

$$\widetilde{T}^{(l)} = S^{(l-1)}\overline{\otimes}(R_{1,2}^{(l)}\overline{\otimes}R_2^{(l)}) \cong S^{(l-1)}\overline{\otimes}R_2^{(l-1)} = \widetilde{S}^{(l-1)},$$

as $R_{1,2}^{(l)}\overline{\otimes}R_2^{(l)}$ is isomorphic to $R_2^{(l-1)}$ as both are hyperfinite II_1 factors. Hence, if we able to show that $\mathcal{N}\left(\widetilde{S}\right)'' = \widetilde{T}$ then, on noting that $T^{(2)} = R_1^{(2)}$, we deduce that our original Tauer masa $A^{(l)}$ is of length l. Furthermore, to observe that $[\mathcal{N}^{m+1}(A) : \mathcal{N}^m(A)] = 2$ for each m, it is enough by (1.3.5), to check that $[\widetilde{T} : \widetilde{S}] = 2$. We record the conclusion of this discussion formally.

Lemma 2.3.9. *To complete the proof of Theorem 2.3.6, it suffices to show that* $\mathcal{N}\left(\widetilde{S}^{(l)}\right)'' = \widetilde{T^{(l)}}$ *and* $[\widetilde{T}^{(l)} : \widetilde{S}^{(l)}] = 2$, *for all* $l \geq 2$.

We now crack on with stages 3 and 4, verifying the hypothesis of the previous lemma.

Proposition 2.3.10. *With the notation above,* $[\widetilde{T} : \widetilde{S}] = [T : S] = 2$.

Proof. Equation (1.3.5) ensures that $[\widetilde{T} : \widetilde{S}] = [T : S]$, and we will calculate this second index using Wenzl's formula, Theorem 1.3.9. To do this we must examine the squares

$$\begin{array}{ccc} T_n & \hookrightarrow & T_{n+1} \\ \uparrow & & \uparrow \\ S_n & \hookrightarrow & S_{n+1} \end{array}$$

which commute, by restriction of the commutative diagram, Figure 2.4. To apply the Wenzl formula, we will check that the inclusion matrices of $S_n \subset S_{n+1}$, $T_n \subset T_{n+1}$ and $S_n \subset T_n$ are independent of n. We will then have $[T : S] = \|\Lambda\|^2$, where Λ is the inclusion matrix of $S_n \subset T_n$.

We have already seen that S_n decomposes as the direct sum of factors

$$S_n = \bigoplus_{k^{(1)},\ldots,k^{(l-1)} \in \mathbb{Z}^2} S_n(k^{(1)}, \ldots, k^{(l-1)}),$$

with each factor embedding once in each factor in the decomposition of S_{n+1}. The matrix for the inclusion is then the $2^{l-1} \times 2^{l-1}$ matrix with 1 in each position and certainly independent of n.

The factor decomposition of T_n, is also easy to come by. Given $k^{(1)}, \ldots, k^{(l-2)} \in \mathbb{Z}_2$, we define $T_n(k^{(1)}, \ldots, k^{(l-2)})$ to be the subspace of $M^{\otimes n(l-1)}$ generated by those $f_{i,j}$ with $\sum_{r=1}^{n} i_r^{(m)} = \sum_{r=1}^{n} j_r^{(m)} = k^{(m)}$ for $m = 1, \ldots, l-2$. Since such $f_{i,j}$ provide matrix units, we have a factor of type $\mathrm{I}_{2^{(n-1)(l-2)+n}}$. Furthermore, it is apparent that (for the same reasons as the S_n case) each of these factors embeds into each of the factors in the decomposition of T_{n+1}. In this case each embedding has multiplicity 2, leaving us again with an inclusion matrix independent of n.

We are now able to see that each $S_n(k^{(1)}, \ldots, k^{(l-2)}, k^{(l-1)})$ embeds precisely once into the corresponding $T_n(k^{(1)}, \ldots, k^{(l-2)})$. This gives us a $2^{l-2} \times 2^{l-1}$ inclusion matrix Λ which can be taken of the form

$$\Lambda = \begin{pmatrix} I & I \end{pmatrix},$$

where I is the identity $2^{l-2} \times 2^{l-2}$ matrix. This is certainly independent of n and so we can compute

$$[T : S] = \|\Lambda\|^2 = \left\|\Lambda\Lambda^{\mathrm{T}}\right\| = \|2I\| = 2. \qquad \square$$

We now reach stage four, where we must demonstrate that $\mathcal{N}\left(\widetilde{S}\right)'' = \widetilde{T}$. Once we have established the difficult inclusion $\mathcal{N}(S) \subset T$, it is enough to exhibit one non-trivial normaliser of S. We then ask, 'What is the index of the inclusion $\mathcal{N}(S)'' \subset T$?' Since $[T : S] = 2$, either $\mathcal{N}\left(\widetilde{S}\right)'' = T$ or $\mathcal{N}\left(\widetilde{S}\right)'' = S$, so rejecting the second possibility is enough for our purposes. We proceed like this, as we do not know that the groupoid normalisers and the unitary normalisers generate the same von Neumann algebra outwith the context of masas. We prefer to work mainly in the factor R_1 and extend the result to R later, by tensoring with R_2.

Proposition 2.3.11. *S is not a singular subfactor in R_1, so $\widetilde{S}$ is not singular in R.*

Proof. We give an normaliser in T_1 not lying in S. Consider the unitary

$$u = 1^{\otimes l-2} \otimes (e_{0,1}^{(0)} + e_{1,0}^{(0)}),$$

in $M^{\otimes 1(l-1)}$ which is orthogonal to S_1 and so orthogonal to S. Fix n and $i_r^{(m)}, j_r^{(m)} \in \mathbb{Z}_2$ for $1 \leq r \leq n$ and $1 \leq m \leq l-1$ with $i \overset{m}{\sim} j$ for each m. Then $uf_{i,j}u^* = f_{i',j'}$, where

$$i_r'^{(m)} = \begin{cases} 1 - i_1^{(l-1)} & m = l-1 \text{ and } r = 1 \\ i_r^{(m)} & \text{otherwise} \end{cases},$$

and

$$j_r'^{(m)} = \begin{cases} 1 - j_1^{(l-1)} & m = l-1 \text{ and } r = 1 \\ j_r^{(m)} & \text{otherwise} \end{cases}.$$

In particular, $i' \overset{m}{\sim} j'$ for each m, so that $uf_{i,j}u^*$ lies in S_n. Hence, u normalises S_n and, as n was arbitrary, u normalises S as required. □

It remains to demonstrate the more difficult inclusion $\mathcal{N}\left(\widetilde{S}\right) \subset \widetilde{T}$. We have already said that we will be unable to use the weak asymptotic homomorphism property - see appendix A for the details. Instead, we use Lemma 1.4.26 and an induction argument, based on [70, Part (ii) of the proof of lemma 5.30] in Tauer's original proof of the length statement in Theorem 2.3.6. First we need to introduce yet more von Neumann algebras into the equation. Define von Neumann subalgebras $U_n^{(s)}$ of $M^{\otimes n(l-1)}$, for $s = 0, \dots, l-2$ to be those generated by all the $f_{i,j}$ with $i_r^{(m)}, j_r^{(m)} \in \mathbb{Z}_2$ (for $1 \leq r \leq n$ and $1 \leq m \leq l-1$) satisfying $i \overset{m}{\sim} j$, for $1 \leq m \leq s$. Write $U^{(s)}$ for the von Neumann algebras (actually subfactors) resulting as the direct limit of these $U_n^{(s)}$ inside R_1. Observe that $U_n^{(l-2)} = T_n$ for each n, so $U^{(l-2)} = T$. We also have $U_n^{(0)} = M^{\otimes n(l-1)}$ for each n and so $U^{(0)} = R_1$. Finally we write $\widetilde{U^{(s)}} = U^{(s)} \overline{\otimes} R_2$, a von Neumann subalgebra of R. Our inductive plan is contained in the next lemma.

Lemma 2.3.12. *Let N be a finite factor. For each $n \in \mathbb{N}$ and $0 \leq s < l-2$, there is a unitary $v \in S_{n+1}$ such that*

$$\left\|\mathbb{E}_{S_{n+1}\otimes N}\left(u(v\otimes 1_N)u^*\right)\right\|_2 \leq \left\|\mathbb{E}_{U_n^{(s+1)}\otimes N}(u)\,\mathbb{E}_{U_n^{(s+1)}\otimes N}(u^*)\right\|_2, \tag{2.3.5}$$

whenever u is a unitary in $U_n^{(s)} \otimes N$.

Proof. Fix n and s. A typical unitary in $U_n^{(s)} \otimes N$ can be written as

$$u = \sum_{\substack{i_r^{(m)},j_r^{(m)}\in\mathbb{Z}_2\\ 1\leq m\leq l-1\\ 1\leq r\leq n\\ i\overset{m}{\sim}j \text{ for } 1\leq m\leq s}} f_{i,j}\otimes 1^{\otimes(l-1)}\otimes u_{i,j},$$

where the $u_{i,j}$ lie in N. The unitary nature of u yields the relation

$$\sum_{\substack{k\overset{m}{\sim}i\\ 1\leq m\leq s}} u_{i,k}u_{j,k}^* = \delta_{i,j}1, \tag{2.3.6}$$

for all indices i,j with $i \overset{m}{\sim} j$ for $1 \leq m \leq s$. We can easily compute the target estimate of the right hand side of (2.3.5) by using

$$\mathbb{E}_{U_n^{(s+1)}\otimes N}(u) = \sum_{\substack{i_r^{(m)},j_r^{(m)}\in\mathbb{Z}_2\\ 1\leq m\leq l-2\\ 1\leq r\leq n\\ i\overset{m}{\sim}j \text{ for } 1\leq m\leq s+1}} f_{i,j}\otimes 1^{\otimes(l-1)}\otimes u_{i,j}. \tag{2.3.7}$$

There will be no surprise in our choice of the unitary v. Namely, we take v to be the unitary in $S_{n+1}\otimes N$ given by

$$\sum_{\substack{k_r^{(m)}\in\mathbb{Z}_2\\ 1\leq m\leq s\\ 1\leq r\leq n}} f_{k,k}\otimes 1^{\otimes(s+1)}\otimes\left(e_0^{(\sum_{r=1}^n k_r^{(s+1)})} - e_1^{(\sum_{r=1}^n k_r^{(s+1)})}\right)\otimes 1^{\otimes(l-s-3)}\otimes 1_N,$$

the idea being that v has a Rachemacher style unitary in the $m = s+2$, $r = n+1$ term of the tensor product, and appropriate identity contributions elsewhere. We are able to evaluate uvu^* as

$$\sum_{\substack{i_r^{(m)},j_r^{(m)},k_r^{(m)}\in\mathbb{Z}_2\\ 1\leq m\leq l-2\\ 1\leq r\leq n\\ i\overset{m}{\sim}j\overset{m}{\sim}k \text{ for } 1\leq m\leq s}} f_{i,j}\otimes 1^{\otimes(s+1)}\otimes\left(e_0^{(\sum_{r=1}^n k_r^{(s+1)})} - e_1^{(\sum_{r=1}^n k_r^{(s+1)})}\right)\otimes 1^{\otimes(l-s-3)}\otimes u_{i,k}u_{j,k}^*. \tag{2.3.8}$$

We compute the conditional expectation onto $S_{n+1}\otimes N$ of (2.3.8) considering each term separately. Suppose i,j,k have $i \overset{m}{\sim} j \overset{m}{\sim} k$ for all $1 \leq m \leq s$. The term of (2.3.8) corresponding to i,j,k is orthogonal to $S_{n+1}\otimes N$, unless we also have

$i \overset{s+1}{\sim} j$. We now consider the two cases $i \overset{s+2}{\sim} j$ and $i \overset{s+2}{\not\sim} j$ separately. If $i \overset{s+2}{\sim} j$ and $k \overset{s+1}{\not\sim} i$, then Proposition 2.3.4 shows that the term is orthogonal to $S_{n+1} \otimes N$. If $i \overset{s+2}{\not\sim} j$ and $k \overset{s+1}{\sim} i$, then the term is also orthogonal to $S_{n+1} \otimes N$.

If $i \overset{s+2}{\not\sim} j$, then (2.3.6) gives

$$\sum_{\substack{k \overset{s+1}{\not\sim} i \\ k \overset{m}{\sim} i \\ 1 \leq m \leq s}} u_{i,k} u_{j,k}^* = - \sum_{\substack{k \overset{s+1}{\sim} i \\ k \overset{m}{\sim} i \\ 1 \leq m \leq s}} u_{i,k} u_{j,k}^*.$$

Using this, and the proceeding calculations, we see that

$$\left\| \mathbb{E}_{S_{n+1} \otimes N} (uvu^*) \right\|_2^2 \leq C \sum_{\substack{i \overset{m}{\sim} j \\ 1 \leq m \leq s+1}} \left\| \sum_{\substack{k \overset{m}{\sim} i \\ 1 \leq m \leq s+1}} u_{i,k} u_{j,k}^* \right\|_2^2 ,$$

where $C = \|f_{i,j}\|_2^2$ - a constant. Equation (2.3.7), shows that this is precisely the required estimate (2.3.5). $\square$

We are now able to complete the proof of Theorem 2.3.6. Given any unitary u_1 in some $U^{(s)} \otimes R_2$ (for some s with $0 \leq s < l-2$), and $\epsilon > 0$, we are able to find some n and a unitary $u \in U_n^{(s)} \otimes N$ with $\|u_1 - u\|_2 < \epsilon$. Lemma 2.3.12 gives a unitary $v \in S_n \otimes N$ with

$$\left\| \mathbb{E}_{S_{n+1} \otimes R_2} (u(v \otimes 1_{R_2}) u^*) \right\|_2 \leq \left\| \mathbb{E}_{U_n^{(s+1)} \otimes R_2} (u) \right\|_2 = \left\| \mathbb{E}_{U^{(s+1)} \otimes R_2} (u) \right\|_2$$

We deduce that

$$\| \mathbb{E}_{S \overline{\otimes} R_2} (u_1 (v \otimes 1_{R_2}) u_1^*) \|_2 \leq \left\| \mathbb{E}_{U^{(s+1)} \overline{\otimes} R_2} (u_1) \right\|_2 + 3\epsilon,$$

and then as ϵ is arbitary we obtain that $\mathcal{N}_{U^{(s)} \overline{\otimes} R_2} (S \overline{\otimes} R_2) \subset U^{(s+1)} \overline{\otimes} R_2$. Apply this $l-1$ times to obtain

$$\mathcal{N}_{U^{(0)} \overline{\otimes} R_2} (S \overline{\otimes} R_2) \subset U^{(l-2)} \overline{\otimes} R_2.$$

Since $U^{(0)} = R_1$, and $U^{(l-2)} = T$, this is precisely the inclusion $\mathcal{N}\left(\widetilde{S}\right) \subset \widetilde{T}$, as required.

2.3.5 Other semi-regular masas with length 2

In this subsection we discuss issues arising from Tauer's construction, Theorem 2.3.6, of semi-regular masas of finite length. As the title suggests we shall mainly be interested in length 2 masas. We should first note that the existence of these masas follows from Jones and Popa's work, Theorem 1.4.32, although Tauer's example of a length 2 masa, and those coming below are of a much more explicit nature. A semi-regular masa of length 2 in R with $[R : \mathcal{N}(A)''] = 2$, such as Tauer's example, has what we should consider as the largest possible set of normalisers for a semi-regular masa. Jones and Popa's construction shows that, unlike the Cartan masa in R, length 2, index 2 semi-regular masas are not unique up to conjugacy - for both Γ and non Γ length 2 masas can be found. We should then determine whether Tauer's masas have centralising sequences - fortunately this is straight forward.

Proposition 2.3.13. *For each integer $l \geq 2$, Tauer's masa of length l given in Construction 2.3.5 contains non-trivial centralising sequences for the hyperfinite II_1 factor R in which it lives.*

Proof. This works in the same way as the Cartan masa is shown to be Γ in R. For each n, take

$$x_n = \sum_{\substack{i_r^{(m)} \in \mathbb{Z}_2 \\ 1 \leq m \leq l-1 \\ 1 \leq r \leq n}} f_i \otimes \bigotimes_{m=1}^{l-1} \left(1^{\otimes(n-1)} \otimes r^{\left(\sum_{t=1}^{n} i_t^{(m)}\right)} \right),$$

for $r^{(q)} = e_0^{(q)} - e_1^{(q)}$. In this way each x_n is a unitary in A_n with $\mathrm{tr}(x_n) = 0$.[12] It is immediate that x_n commutes with elements of $M^{\otimes n(l-1)} \otimes \iota_{n-1}^{\otimes(l-1)}(M^{\otimes(n-1)(l-1)})$, and then, as the union of these algebras is weakly dense in R, the sequence $(x_n)_{n=1}^{\infty}$ is a non-trivial centralising sequence. ☐

We shall now give Tauer's second example of a length 2 masa from section 6 of [70] which has $[R : \mathcal{N}(A)''] = 4$, and then explain how to extend the elementary matrix methods of the previous subsections to obtain any other value in $\mathbb{N} \cup \{\infty\} \setminus \{1\}$ which, by Proposition 1.3.2, are the only possible values of $[R : \mathcal{N}(A)'']$ for a length 2 masa A. First up is Tauer's example - as with all her masas, this was originally expressed in terms of unitaries $(u_n)_{n=1}^{\infty}$ defining a Tauer masa as in equation (2.1.4). Actually, her example is much easier to describe than this. Let $A \subset R$ be the length 2 masa with $[R : \mathcal{N}(A)''] = 2$ given by Theorem 2.3.6, which of course was Tauer's original example of a length 2 masa in section 5 of [70]. Now form the masa $A\overline{\otimes}A$ in $R\overline{\otimes}R$. Part 2 of Proposition 2.3.7, shows that A has the weak asymptotic homomorphism property away from $\mathcal{N}(A)''$, which we will denote here by S. Proposition 1.4.28, then ensures that $A\overline{\otimes}A$ has the weak asymptotic homomorphism property away from $S\overline{\otimes}S$ and so

[12]We have seen this unitary before - it appeared as v in the proof of part 2 of Lemma 2.3.7, where it was used to establish that A has the weak asymptotic homomorphism property away from $\mathcal{N}(A)''$.

$\mathcal{N}_{R\overline{\otimes}R}(A\overline{\otimes}A)'' = S\overline{\otimes}S$. Equation (1.3.5) gives $[R\overline{\otimes}R : S\overline{\otimes}S] = 4$, and it is immediate that $\mathcal{N}_{R\overline{\otimes}R}(S\overline{\otimes}S)'' = R\overline{\otimes}R$, so that we have a masa of length 2. It is possible to give an explicit formulation of $A\overline{\otimes}A$ as a Tauer masa in $R\overline{\otimes}R$, and then carefully check that this agrees with the formulation given by the unitaries given in Definition 6.1 of [70], and so Tauer's second example of a length 2 semi-regular masa in the hyperfinite II_1 factor is (isomorphic to) the tensor product of her first example with itself. In particular, this example also contains non-trivial centralising sequences for the underlying hyperfinite II_1 factor. We shall also see, in section 4.3, that Tauer's singular masa given in section 2.2 also contains non-trivial centralising sequences so that all Tauer's original examples of masas have this property. In contrast, Pukánszky's examples ([51]) come from inclusions of groups $H \subset G$ with $gHg^{-1} \cap H = \{1\}$ for every $g \in G \setminus H$. We have already seen, in subsection 1.4.5, that masas arising in this way do not contain non-trivial centralising sequences.

To show that a length 2 Tauer masa A in R exists, for any possible value of $[R : \mathcal{N}(A)''] \in \mathbb{N} \cup \{\infty\} \setminus \{1\}$, it suffices to deal with the $[R : \mathcal{N}(A)''] = p$ case, where p is a prime. We will then be able to take finite tensor products as in the preceding paragraph to obtain any member of $\mathbb{N} \setminus \{1\}$ for $[R : \mathcal{N}(A)'']$; we obtain the index value ∞ by taking the infinite tensor product $A^{\overline{\otimes}\infty} \subset R^{\overline{\otimes}\infty}$ of Tauer's original example of a length 2 masa with itself.

Construction 2.3.14. Fix a prime $p > 2$, and let M be the algebra of $p \times p$ matrices, with matrix units $(e_{i,j})_{i,j \in \mathbb{Z}_p}$. Choose, by Proposition 1.2.10, a family of orthogonal masas $(D^{(i)})_{i \in \mathbb{Z}_p}$ in M. Take $N_n = M^{\otimes n} \otimes M^{\otimes n}$ with the natural inclusion $x \otimes y \mapsto (x \otimes 1) \otimes (y \otimes 1)$ of N_n into N_{n+1}. Let R be the hyperfinite II_1 factor obtained as the direct limit of the chain $(N_n)_{n=1}^{\infty}$.

Take A_n to be the masa

$$A_n = \bigoplus_{i_1,\dots,i_m \in \mathbb{Z}_p} \bigotimes_{r=1}^{n} e_{i_r,i_r} \otimes \bigotimes_{r=1}^{n} D^{(\sum_{t=1}^{r} i_t)}$$

in N_n, with the sums above taken in $\mathbb{Z}_p$. As A_n is included into A_{n+1}, we obtain a Tauer masa A inside R. As is becoming familiar, we factorise R as $R_1\overline{\otimes}R_2$ with R_1 and R_2 both being obtained as the direct limits of the natural inclusions $M^{\otimes n} \hookrightarrow M^{\otimes(n+1)}$. Let S_n be the subspace of $M^{\otimes n}$ generated by all $\bigotimes_{r=1}^{n} e_{i_r,j_r}$ satisfying

$$\sum_{r=1}^{n} i_r = \sum_{j=1}^{n} j_r$$

in $\mathbb{Z}_p$. The methods of the previous subsections apply here showing that each S_n is actually a *-subalgebra of $M^{\otimes n}$ and the direct limit is a subfactor S of R_1

with $[R_1 : S] = p$. Furthermore, the proof of Proposition 2.3.7, carries over to this situation and demonstrates that $\mathcal{N}(A)'' = S\overline{\otimes}R_2$ with A having the weak asymptotic homomorphism property away from $S\overline{\otimes}R_2$.

Since $[R : S\overline{\otimes}R_1] = p$, to complete our objective of explicitly producing a length 2 Tauer masa A with $[R : \mathcal{N}(A)''] = p$ we must show that S is Cartan in R_1. As $[\mathcal{N}(S)'' : S]$ must be an integer dividing p, primality tells that S is either regular or singular. Just as in the index 2 case, $u = \sum_{i\in\mathbb{Z}_p} e_{i,i+1} \in M$ is a non-trivial normaliser of each S_n, so that S is regular.

We are unable to use the exactly the same method to give other Tauer masas of length longer than 2. In demonstrating the difficult inclusion, namely that the normalisers of $\widetilde{S}$ are contained in $\widetilde{T}$, we appealed to Lemma 1.4.24 several times to deduce the containment. Unlike the calculation, Proposition 2.3.7, of the normaliser of A, we do not obtain the weak asymptotic homomorphism property for $\mathcal{N}^m(A)$ away from $\mathcal{N}^{m+1}(A)$ for any $1 \leq m < l-1$ from this calculation. Hence, in the absence of an answer to Question 1.4.29,[13] we are unable to tensor different masas together and retain control over the m-fold normalising algebras for $m \geq 2$. We do not know whether $\mathcal{N}^m(A)$ is strongly semi-regular in $\mathcal{N}^{m+1}(A)$ but, as claimed earlier, we can not get the weak asymptotic homomorphism property. The details are reserved for Appendix A, where we show this failure for Tauer's length 3 masa and $m = 1$. This is the first known failure of the weak asymptotic homomorphism property of a von Neumann algebra away from its normalisers.

2.4 Masas of Infinite Length

In this section we construct semi-regular masas in the hyperfinite II_1 factor, of *infinite length*, i.e. masas A for which the chain

$$A \subset \mathcal{N}^1(A) \subset \mathcal{N}^2(A) \subset \cdots \subset \mathcal{N}^n(A) \subset \mathcal{N}^{n+1}(N) \subset \dots$$

is strictly increasing. Motivated by our use of the Jones index in the previous section, associate to each masa of infinite length, the *index sequence*, $([\mathcal{N}^{n+1}(A) : \mathcal{N}^n(A)])_{n=1}^{\infty}$, which, by Proposition 1.3.2, takes values in $(\mathbb{N} \cup \{\infty\}) \setminus \{1\}$. Our objective will be to construct semi-regular masas with enough different index sequences to deduce the next result.

Corollary 2.4.1. *There exist uncountably many semi-regular Tauer masas in the hyperfinite* II_1 *factor* R*, no pair of which is conjugate by an automorphism of* R*.*

[13]More accurately an answer to the appropriate restatement of this question for general von Neumann subalgebras of II_1 factors rather than just masas.

It is almost surely[14] the case that any sequence in $\{2, 3, \dots\} \cup \{\infty\}$ can be obtained as an index sequence for some semi-regular masa in R; at present though, we are unable to achieve this. Here, our main objective is the next theorem, from which Corollary 2.4.1 follows immediately.

Theorem 2.4.2. *Given a sequence $(\mu_n)_{n=1}^{\infty}$ in $\mathbb{N}$, we can find a semi regular Tauer masa A in the hyperfinite* II_1 *factor R such that*

$$[\mathcal{N}^{n+1}(A) : \mathcal{N}^{n}(A)] = 2^{\mu_n},$$

for each n.

2.4.1 Defining the subfactors $\mathcal{N}^n(A)$

An attempt to generate infinite length Tauer masas by crudely generalising the examples of the previous section fails as we would end up taking an infinite tensor product of M to get our chain of finite dimensional subfactors - this is patently not possible. More care is required; a diagonal argument will be used to offset this difficulty. Unfortunately this, in addition to the extra details required to vary the values of the index sequence, lead to a significantly more unpleasant setup compared with section 2.3. We follow the same outline[15] as before, although we proceed more formally due to the extra complications.

Fix the sequence $(\mu_n)_{n=1}^{\infty}$ in $\mathbb{N}$ for the rest of this section. We will use the conventions of the first part of Notation 2.3.3, regarding the elements $e_{i,j}^{(i',j')}$ in the algebra M of 2×2 matrices throughout.

Definition 2.4.3. For integers $n \geq l \geq 1$, let $\triangle(n, l)$ be the finite set consisting of all quadruples (r, s, t_1, t_2) such that:

1. r and s are integers with $l \leq r$, $1 \leq s$ and $r + s \leq n + 1$;
2. t_1 and t_2 are integers with $1 \leq t_1 \leq \mu_r$ and $1 \leq t_2 \leq \mu_{r+1}$.

We shall work in the tensor product $M^{\otimes \triangle(n,l)}$. Given functions $i, j : \triangle(n, l) \to \mathbb{Z}_2$, we write

$$^{(n,l)}f_{i,j} = \bigotimes_{(r,s,t_1,t_2) \in \triangle(n,l)} e_{i(r,s,t_1,t_2),j(r,s,t_1,t_2)}^{\left(\sum_{u=1}^{s-2} \sum_{t_2'=1}^{\mu_{r+2}} i(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2} \sum_{t_2'=1}^{\mu_{r+2}} j(r+1,u,t_2,t_2')\right)}, \quad (2.4.1)$$

where, as before, the sums are taken in $\mathbb{Z}_2$ and defined to be zero when they would make no sense, i.e. when s is 1 or 2.

[14] In the non technical sense!
[15] Albeit in a different order.

Proposition 2.4.4. *The* ${}^{(n,l)}f_{i,j}$*, as defined in (2.4.1) give matrix units for* $M^{\otimes\triangle(n,l)}$.

Proof. Since it is apparent that ${}^{(n,l)}f_{i,j}^* = {}^{(n,l)}f_{j,i}$, it is only necessary to check that

$$ {}^{(n,l)}f_{i,j}{}^{(n,l)}f_{j',k} = \delta_{j,j'}{}^{(n,l)}f_{i,k}, \tag{2.4.2}$$

for all i, j, j', k. We perform a downwards induction on r. Suppose inductively that $j(r+1, s, t_1, t_2) = j'(r+1, s, t_1, t_2)$ for some r and all possible s, t_1, t_2. In the starting case, $r = n$, no assumption is made. Now ${}^{(n,l)}f_{i,j}{}^{(n,l)}f_{j',k}$ is an elementary tensor, with the element

$$\begin{aligned}
&e_{i(r,s,t_1,t_2),j(r,s,t_1,t_2)}^{\left(\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} i(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} j(r+1,u,t_2,t_2')\right)} \\
&\quad .e_{j'(r,s,t_1,t_2),k(r,s,t_1,t_2)}^{\left(\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} j'(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} k(r+1,u,t_2,t_2')\right)} \\
=&\delta_{j(r,s,t_1,t_2),j'(r,s,t_1,t_2)} e_{i(r,s,t_1,t_2),k(r,s,t_1,t_2)}^{\left(\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} i(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} k(r+1,u,t_2,t_2')\right)},
\end{aligned}$$

appearing in the (r, s, t_1, t_2) position, as the inductive hypothesis ensures that the inner sums (those over j and j') in the superscripts are identical. Hence (2.4.2) evaluates to zero unless $j(r, s, t_1, t_2) = j'(r, s, t_1, t_2)$ for all allowed (r, s, t_1, t_2). When $j = j'$, putting all the components above together gives the result. □

We are now in a position to define von Neumann algebras, which will turn out to give the iterated normalisers of the Tauer masa we shall eventually construct.

Definition 2.4.5. For integers $n \geq l \geq 1$, write $i \overset{n,l}{\sim} j$ when the functions $i, j : \triangle(n, l) \to \mathbb{Z}_2$, satisfy

$$\sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+1-r} (i(r, s, t_1, t_2) - j(r, s, t_1, t_2)) = 0,$$

for all $r = l, \ldots, n$ and $1 \leq t_1 \leq \mu_r$. Write ${}^{l}S_n$ for the subspace of $M^{\otimes\triangle(n,l)}$ generated by all the ${}^{(n,l)}f_{i,j}$ with $i \overset{n,l}{\sim} j$. Since $\overset{n,l}{\sim}$ is an equivalence relation and the ${}^{(n,l)}f_{i,j}$ are matrix units for $M^{\otimes\triangle(n,l)}$, ${}^{l}S_n$ is a *-subalgebra of $M^{\otimes\triangle(n,l)}$.

Think of $M^{\otimes\triangle(n+1,l)}$ as isomorphic to $M^{\otimes\triangle(n,l)} \otimes M^{\triangle(n+1,l)\setminus\triangle(n,l)}$. We then embed $M^{\otimes\triangle(n,l)}$ in $M^{\otimes\triangle(n+1,l)}$ in the natural way - by $x \mapsto x \otimes 1$. The direct limit of the chain

$$M^{\otimes\triangle(l,l)} \hookrightarrow M^{\otimes\triangle(l+1,l)} \hookrightarrow \ldots \hookrightarrow M^{\otimes\triangle(n,l)} \hookrightarrow M^{\otimes\triangle(n+1,l)} \hookrightarrow \ldots$$

is a copy of the hyperfinite II_1 factor, which we denote ${}^{l}R$.

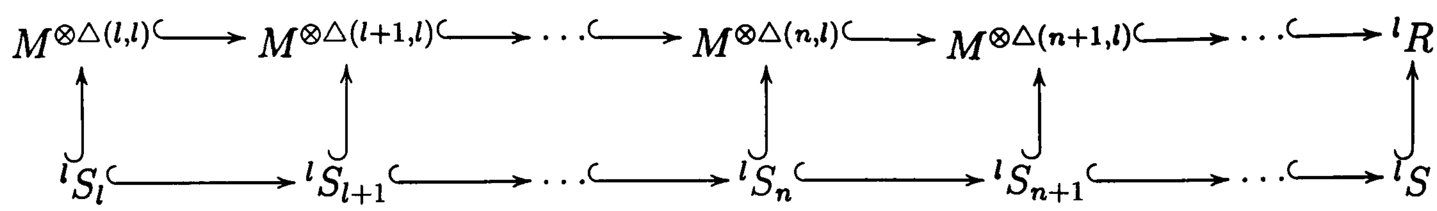


Figure 2.5: Diagram of inclusions defining ${}^lS$

Proposition / Definition 2.4.6. With the notation above, ${}^lS_n$ is included in ${}^lS_{n+1}$ by the inclusion $M^{\otimes\triangle(n,l)} \subset M^{\otimes\triangle(n+1,l)}$. We define ${}^lS$ to be the direct limit of these ${}^lS_n$ as in Figure 2.5, a von Neumann subalgebra of ${}^lR$, and then Figure 2.5 is a commutative diagram.

Proof. If $i,j : \triangle(n,l) \to \mathbb{Z}_2$ have $i \overset{n,l}{\sim} j$ then as an element of $M^{\otimes\triangle(n+1,l)}$, ${}^{(n,l)}f_{i,j} \otimes 1$ is the sum of all the elements ${}^{(n+1,l)}f_{i',j'}$, where i', j' are functions from $\triangle(n+1,l)$ into $\mathbb{Z}_2$ such that:

- i' and j' extend i and j respectively;
- i' and j' agree on $\triangle(n+1,l) \setminus \triangle(n,l)$.

Such (i',j') have $i' \overset{n+1,l}{\sim} j'$, so that ${}^{(n,l)}f_{i,j} \otimes 1$ is a linear combination of generators of ${}^lS_{n+1}$. In this way, we see that ${}^lS_n$ is included inside ${}^lS_{n+1}$.

If $i,j : \triangle(n,l) \to \mathbb{Z}_2$ have $i \overset{n,l}{\not\sim} j$, then $\mathbb{E}_{{}^lS_n}\left({}^{(n,l)}f_{i,j}\right) = 0$. To complete the proof, we must show that $\mathbb{E}_{{}^lS_{n+1}}\left({}^{(n,l)}f_{i,j}\right) = 0$ also. Let r_0 be maximal (in the range $l \le r_0 \le n$) such that there is some $1 \le t_1' \le \mu_{r_0}$ with

$$\sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+1-r_0} i(r_0, s, t_1', t_2) \neq \sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+1-r_0} j(r_0, s, t_1', t_2).$$

For $r \ge r_0$, we consider the $(r, n+2-r, t_1, t_2)$-positions of the tensor product $M^{\otimes\triangle(n+1,l)}$. The maximality of r_0 implies that

$$\sum_{u=1}^{n-r} \sum_{t_2'=1}^{\mu_{r+2}} i(r+1, u, t_2, t_2') = \sum_{u=1}^{n-r} \sum_{t_2'=1}^{\mu_{r+2}} j(r+1, u, t_2, t_2'),$$

for all $1 \le t_2 \le \mu_{r+1}$. In these positions, the elementary tensor ${}^{(n,l)}f_{i,j} \otimes 1$ has the identity, which we can write as the linear combination

$$1 = e_0^{\left(\sum_{u=1}^{n-r} \sum_{t_2'=1}^{\mu_{r+2}} i(r+1,u,t_2,t_2')\right)} + e_1^{\left(\sum_{u=1}^{n-r} \sum_{t_2'=1}^{\mu_{r+2}} i(r+1,u,t_2,t_2')\right)}.$$

Decomposing ${}^{(n,l)}f_{i,j} \otimes 1$ in this way, we can write it as a linear combination of ${}^{(n+1,l)}f_{i',j'}$ for pairs (i',j') of functions from $\triangle(n+1,l)$ into $\mathbb{Z}_2$ with

$$i'|_{\triangle(n,l)} = i, \quad j'|_{\triangle(n,l)} = j$$

and

$$i'(r, n+2-r, t_1, t_2) = j'(r, n+2-r, t_1, t_2)$$

for all $r \geq r_0$, $1 \leq t_1 \leq \mu_r$ and $1 \leq t_2 \leq \mu_{r+1}$.[16] These pairs inherit $i' \overset{n+1,l}{\not\sim} j'$ from $i \overset{n,l}{\not\sim} j$, so that they each have $\mathbb{E}_{S_{n+1}}\left({}^{(n+1,l)}f_{i',j'}\right) = 0$. Hence, $\mathbb{E}_{S_{n+1}}\left({}^{(n,l)}f_{i,j}\right) = 0$, as required. □

We wish to demonstrate that ${}^lS$ is a subfactor of ${}^lR$, for which we shall examine the inclusion data for ${}^lS_n \subset {}^lS_{n+1}$, then appeal to Proposition 1.3.8.

Proposition 2.4.7. *Fix $n > l \geq 1$. Let $I(n,l)$, denote the set of all pairs of integers (r, t_1) with $l \leq r \leq n$, and $1 \leq t_1 \leq \mu_r$. For a function $k : I(n,l) \to \mathbb{Z}_2$, define ${}^lS_n(k)$ to be the subspace of $M^{\otimes \triangle(n,l)}$ defined by those ${}^{(n,l)}f_{i,j}$ such that*

$$\sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+1-r} i(r,s,t_1,t_2) = \sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+1-r} j(r,s,t_1,t_2) = k(r,t_1), \tag{2.4.3}$$

for every $(r,t_1) \in I(n,l)$. Each ${}^lS_n(k)$ is a factor, and

$$ {}^lS_n = \bigoplus_{k:I(n,l)\to\mathbb{Z}_2} {}^lS_n(k). \tag{2.4.4}$$

Furthermore, in the inclusion $1 \in {}^lS_n \subset {}^lS_{n+1}$, every ${}^lS_n(k)$ is included exactly the same number of times into each ${}^lS_{n+1}(k')$.

Proof. Fix a function $k : I(n,l) \to \mathbb{Z}_2$. Since the condition (2.4.3) is an equivalence relation, Proposition 2.4.4 shows that ${}^lS_n(k)$ is a $*$-algebra with unit

$$\sum_{\substack{i:\triangle(n,l)\to\mathbb{Z}_2 \\ \sum_{t_2=1}^{\mu_{r+1}}\sum_{s=1}^{n+1-r} i(r,s,t_1,t_2)=k(r,t_1)}} {}^{(n,l)}f_{i,i}.$$

The collection of all ${}^{(n,l)}f_{i,j}$ generating ${}^lS_n(k)$ are matrix units for ${}^lS_n(k)$ and so ${}^lS_n(k)$ is a factor.[17] The direct summation (2.4.4) follows immediately.

A generator ${}^{(n,l)}f_{i,j}$ for ${}^lS_n(k)$ is included into ${}^lS_{n+1}$ as the sum

$$\sum_{\substack{i',j':\triangle(n+1,l)\to\mathbb{Z}_2 \\ i'|_{\triangle(n,l)}=i \text{ and } j'|_{\triangle(n,l)}=j \\ i'|_{\triangle(n+1,l)\setminus\triangle(n,l)}=j'|_{\triangle(n+1,l)\setminus\triangle(n,l)}}} {}^{(n+1,l)}f_{i',j'}. \tag{2.4.5}$$

[16] We can say nothing about the values of $i'(r, n+2-r, t_1, t_2)$ and $j'(r, n+2-r, t_1, t_2)$ for $r < r_0$.

[17] By contrast, the generators ${}^{(n,l)}f_{i,j}$ of ${}^lS_n$ are not matrix units for ${}^lS_n$ as they do not all lie in the same $\overset{n,l}{\sim}$-equivalence class. Compare with footnote 8 on page 61, which discusses this in the finite length case.

Each such ${}^{(n+1,l)}f_{i',j'}$ lies in ${}^{l}S_{n+1}(k')$, where k' is the function given on $I\,(n+1,l)$ by

$$k'(r,t_1) = \begin{cases} \sum_{t_2=1}^{\mu_{r+1}} \left(\sum_{s=1}^{n+1-r} i(r,s,t_1,t_2) + i'(r,n+2-r,t_1,t_2)\right) & l \le r \le n \\ \sum_{t_2=1}^{\mu_{n+2}} i'(n+1,1,t_1,t_2) & r = n+1 \end{cases}.$$

Now fix k'. For each r and t_1, there are $2^{\mu_{r+1}-1}$ choices of values $i'(r,n+2-r,t_1,t_2) = j'(r,n+2-r,t_1,t_2) \in \mathbb{Z}_2$, so that

$$\sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+2-r} i'(r,s,t_1,t_2) = \sum_{t_2=1}^{\mu_{r+1}} \sum_{s=1}^{n+2-r} j'(r,s,t_1,t_2) = k'(r,t_1).$$

${}^{(n+1,l)}f_{i',j'}$ lies in ${}^{l}S_{n+1}(k')$. In total there are $2^{\sum_{r=l}^{n}\mu_r(\mu_{r+1}-1)}$ choices of i' and j' for which ${}^{(n+1,l)}f_{i',j'}$ lies in ${}^{l}S_{n+1}(k')$. Hence, ${}^{l}S_n(k)$ is included $2^{\sum_{r=l}^{n}\mu_r(\mu_{r+1}-1)}$ times in ${}^{l}S_n(k')$ *independently* of the functions k and k', as required. □

Now Proposition 1.3.8, demonstrates that there is at most one normalised faithful trace on $\bigcup_{n=1}^{\infty} {}^{l}S_n$ and, since there is such a trace, obtained by restriction of that on ${}^{l}R$, we deduce that ${}^{l}S$ is a II_1 factor.

Corollary 2.4.8. *The direct limit ${}^{l}S$, defined by Figure 2.5, is a subfactor of ${}^{l}R$.*

By commutativity of the tensor product we have the identification

$$M^{\otimes\triangle(n,l)} \cong M^{\otimes\triangle(n,l+1)} \otimes M^{\otimes\triangle(n,l+1)\setminus\triangle(n,l)}. \tag{2.4.6}$$

Define ${}^{l}T_n$ to be the subalgebra of $M^{\otimes\triangle(n,l)}$ given, under the isomorphism above, by

$${}^{l}T_n = {}^{l+1}S_n \otimes M^{\otimes\triangle(n,l+1)\setminus\triangle(n,l)}. \tag{2.4.7}$$

Write ${}^{l}T$ for the von Neumann subalgebra of ${}^{l}R$ generated by these ${}^{l}T_n$, and observe that the commutativity of Figure 2.5 restricts to show that Figure 2.6 commutes.

$$\begin{array}{ccccccccccc} {}^{l}T_{l+1} & \hookrightarrow & {}^{l}T_{l+2} & \hookrightarrow \cdots \hookrightarrow & {}^{l}T_n & \hookrightarrow & {}^{l}T_{n+1} & \hookrightarrow \cdots \hookrightarrow & {}^{l}T \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ {}^{l}S_{l+1} & \hookrightarrow & {}^{l}S_{l+2} & \hookrightarrow \cdots \hookrightarrow & {}^{l}S_n & \hookrightarrow & {}^{l}S_{n+1} & \hookrightarrow \cdots \hookrightarrow & {}^{l}S_n \end{array}$$

Figure 2.6: Commuting diagram, giving rise to the inclusion $1 \in {}^{l}S \subset {}^{l}T$

The identification (2.4.6), can be repeated for each n, to see that ${}^{l}R$ naturally factorises as the von Neumann tensor product

$${}^{l}R = {}^{l+1}R \overline{\otimes} {}^{l}R_1, \tag{2.4.8}$$

say. Here ${}^lR_1$ is the hyperfinite II_1 factor obtained as the direct limit of the chain

$$M^{\otimes\triangle(l+1,l+1)\setminus\triangle(l+1,l)} \hookrightarrow \ldots \hookrightarrow M^{\otimes\triangle(n,l+1)\setminus\triangle(n,l)} \hookrightarrow M^{\otimes\triangle(n+1,l+1)\setminus\triangle(n+1,l)} \hookrightarrow \ldots,$$

with the natural inclusions. Under the factorisation (2.4.8), we have ${}^lT = {}^{l+1}S\overline{\otimes}{}^lR_1$ - a subfactor of ${}^lR$, as ${}^{l+1}S$ is a subfactor of ${}^{l+1}R$ by Corollary 2.4.8.

We can not use the Wenzl index formula to compute $[{}^lT : {}^lS]$ as the number of components in the decomposition of ${}^lS_n$ into subfactors goes to infinity with n, so there is no possibility of the inclusion matrices for $1 \in {}^lS_n \subset {}^lT_n$ satisfying the periodicity requirements of Theorem 1.3.9. Fortunately, we have prepared for this situation in subsection 1.3.4.

Theorem 2.4.9. *For each l, we have*

$$[{}^lT : {}^lS] = 2^{\mu_l}. \tag{2.4.9}$$

Proof. Proposition 2.4.7, shows that there are $2^{|I(n,l)|} = 2^{\mu_l+\mu_{l+1}+\cdots+\mu_n}$ subfactors in the direct sum decomposition of ${}^lS_n$, furthermore each of these factors has the same dimension. The decomposition of each ${}^lT_n$ into a direct sum of factors follows from the decomposition of the ${}^lS_n$ in Proposition 2.4.7, where the ${}^lS_n(k)$ are defined. Namely, we have

$${}^lT_n = \bigoplus_{k:I(n,l+1)\to\mathbb{Z}_2} {}^lT_n(k),$$

where, again under the identification of (2.4.6), we have

$${}^lT_n(k) = {}^{l+1}S_n(k) \otimes M^{\otimes\triangle(n,l)\setminus\triangle(n,l+1)}.$$

We see that there are then $2^{|I(n,l+1)|} = 2^{\mu_{l+1}+\cdots+\mu_n}$ factors in the direct sum decomposition of ${}^lT_n$.

Let ι denote the map including $I(n, l+1)$ into $I(n, l)$. For each function $k : I(n,l) \to \mathbb{Z}_2$, the summand ${}^lS_n(k)$, is included precisely once into ${}^lT_n(k \circ \iota)$. The inclusion matrix for $1 \in {}^lS_n \subset {}^lT_n$ is then a $2^{\mu_{l+1}+\cdots+\mu_n} \times 2^{\mu_l+\mu_{l+1}+\cdots+\mu_n}$ matrix consisting of 2^{μ_l} copies of the $2^{\mu_{l+1}+\cdots+\mu_n} \times 2^{\mu_{l+1}+\cdots+\mu_n}$ identity matrix I, as indicated below:

$$\overbrace{\begin{pmatrix} I & I & \ldots & I \end{pmatrix}}^{2^{\mu_l}}.$$

We have now verified all the conditions of Corollary 1.3.11, from which we immediately deduce (2.4.9). □

2.4.2 Computing the normaliser of S

Having constructed subfactors with the required Jones index, we now proceed to demonstrate that these subfactors give a tower of normalising algebras. The main statement we shall need is Theorem 2.4.10, which is the objective of this subsection. We choose to make the additional assumption that these normalisers are known to generate a subfactor. This is unnecessary and only included because we have only developed the theory of the index for subfactors rather than more general finite von Neumann algebras. This assumption causes no problems - the easy observation Proposition 2.3.2 will eventually be used to show that it is valid.

Theorem 2.4.10. *Let R be any hyperfinite* II_1 *factor. Fix l and suppose that $\mathcal{N}_{{}^lR}\left({}^lS\right)''\overline{\otimes}R$ is a subfactor of ${}^lR\overline{\otimes}R$. Then $\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right)'' = {}^lT\overline{\otimes}R$.*

As with the finite length case, we first show that there are enough normalisers to generate all of ${}^lT$, then go on to show that all the normalisers are contained in ${}^lT$.

Proposition 2.4.11. *For each l, there are 2^{μ_l} unitaries (u_k) in $\mathcal{N}\left({}^lS\right)$ with*

$$\mathbb{E}_{{}^lS}\left(u_{k_1}u_{k_2}^*\right) = \delta_{k_1,k_2}1. \tag{2.4.10}$$

Proof. We shall index these unitaries by $k = (k_1, \ldots, k_{\mu_l}) \in \mathbb{Z}_2^{\mu_l}$. Define

$$u_k = \bigotimes_{(r,s,t_1,t_2)\in\triangle(l+1,l)} x^{(k)}_{r,s,t_1,t_2},$$

where the $x^{(k)}_{r,s,t_1,t_2}$ are unitaries in M given by

$$x^{(k)}_{r,s,t_1,t_2} = \begin{cases} e^{(0)}_{0,1} + e^{(0)}_{1,0} & r = l, s = 1, t_2 = 1, k_{t_1} = 1 \\ 1 & \text{otherwise} \end{cases}.$$

We should check that these unitaries do normalise ${}^lS$. Fix n and take some

$${}^{(n,l)}f_{i,j} = \bigotimes_{(r,s,t_1,t_2)\in\triangle(n,l)} e^{\left(\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} i(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} j(r+1,u,t_2,t_2')\right)}_{i(r,s,t_1,t_2),j(r,s,t_1,t_2)}$$

with $i \overset{n,l}{\sim} j$ so that ${}^{(n,l)}f_{i,j} \in {}^lS_n$. Now $u_k({}^{(n,l)}f_{i,j})u_k^*$ is also an elementary tensor with

$$x^{(k)}_{r,s,t_1,t_2} e^{\left(\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} i(r+1,u,t_2,t_2'),\sum_{u=1}^{s-2}\sum_{t_2'=1}^{\mu_r+2} j(r+1,u,t_2,t_2')\right)}_{i(r,s,t_1,t_2),j(r,s,t_1,t_2)} {x^{(k)}_{r,s,t_1,t_2}}^* \tag{2.4.11}$$

in the (r,s,t_1,t_2) position, where we have extended the definition of $x^{(k)}_{r,s,t_1,t_2}$ to $\triangle(n,l)$ by taking $x_{r,s,t_1,t_2} = 1$ on $\triangle(n,l)\backslash\triangle(l+1,l)$. When $x^{(k)}_{r,s,t_1,t_2} \neq 1$, we must have $s=1$ so the sums appearing in the superscript on the $e_{i(r,s,t_1,t_2),j(r,s,t_1,t_2)}$ in (2.4.11) are zero by definition in this case. Furthermore, no terms $i(l,u,t_2,t_2')$ or $j(l,u,t_2,t_2')$ appear in any of these superscripts. Hence, $u_k\ {}^{(n,l)}f_{i,j}u_k^* = {}^{(n,l)}f_{i',j'}$, where

$$i'(r,s,t_1,t_2) = \begin{cases} 1-i(r,s,t_1,t_2) & r=l, s=1, t_2=1, k_{t_1}=1 \\ i(r,s,t_1,t_2) & \text{otherwise} \end{cases},$$

and

$$j'(r,s,t_1,t_2) = \begin{cases} 1-j(r,s,t_1,t_2) & r=l, s=1, t_2=1, k_{t_1}=1 \\ j(r,s,t_1,t_2) & \text{otherwise} \end{cases}.$$

The form of these i' and j' ensures that $i' \overset{n,l}{\sim} j'$ follows from $i \overset{n,l}{\sim} j$, showing that u_k normalises each ${}^lS_n$. As n was arbitrary, u_k normalises ${}^lS$ as claimed.

We now return to check (2.4.10). This is straightforward. Given $k^{(1)} \neq k^{(2)} \in \mathbb{Z}_2^{\mu_l}$, we find some $t_1 = 1,\ldots,\mu_l$ with $k^{(1)}_{t_1} \neq k^{(2)}_{t_1}$. Now $u_{k^{(1)}}u^*_{k^{(2)}}$ is an elementary tensor of unitaries, which has $e^{(0)}_{0,1}+e^{(0)}_{1,0}$ in the $(l,1,t_1,1)$ position, and 1 in the $(l,1,t_1,t_2)$ positions for $t_2\neq 1$. When we write $u_{k^{(1)}}u^*_{k^{(2)}}$ as a linear combination of ${}^{(n,l)}f_{i,j}$, these i,j will necessarily have

$$\sum_{s=1}^{1}\sum_{t_2=1}^{\mu_{l+1}} i(l,s,t_1,t_2) \neq \sum_{s=1}^{1}\sum_{t_2=1}^{\mu_{l+1}} j(l,s,t_1,t_2),$$

so that they do not have $i \overset{n,l}{\sim} j$. Hence, $x_{k^{(1)}}x^*_{k^{(2)}}$ is orthogonal to ${}^lS_{l+1}$, and so orthogonal to ${}^lS$ as claimed. $\square$

We now start work on the difficult inclusion of Theorem 2.4.10. The idea is to show that for any $n\geq l$ and $i,j:\triangle(n,l)\to\mathbb{Z}_2$ with $i \overset{n,l+1}{\not\sim} j$, the element ${}^{(n,l)}f_{i,j}$ is orthogonal to $\mathcal{N}_{{}^lS}\left({}^lR\right)$. From this we deduce that $\mathcal{N}_{{}^lS}\left({}^lR\right)\subset {}^lT$, as such ${}^{(n,l)}f_{i,j}$ span the orthogonal complement of ${}^lT$ in ${}^lR$. The stronger statement that $\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}S\right)\subset {}^lT\overline{\otimes}R$ for any II_1 factor R follows from showing that ${}^{(n,l)}f_{i,j}\otimes x$ is orthgonal to $\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right)$ - our actual objective. To establish this, we shall require some more machinery.

Fix $n>l$; r_0 with $l+1\leq r_0\leq n$ and some t_1' with $1\leq t_1'\leq\mu_{r_0}$. Write ${}^{(n,l)}U_{r_0,t_1'}$ for the unital *-subalgebra of $M^{\otimes\triangle(n,l)}$ generated by all ${}^{(n,l)}f_{i,j}$ where the functions $i,j:\triangle(n,l)\to\mathbb{Z}_2$ satisfy

$$\sum_{t_2=1}^{\mu_{r_0+1}}\sum_{s=1}^{n+1-r_0} i(r_0,s,t_1',t_2) = \sum_{t_2=1}^{\mu_{r_0+1}}\sum_{s=1}^{n+1-r_0} j(r_0,s,t_1',t_2).$$

This is an equivalence relation on pairs (i,j) so that Proposition 2.4.4 demonstrates that ${}^{(n,l)}U_{r_0,t_1'}$ is a *-subalgebra of $M^{\otimes\triangle(n,l)}$ spanned by these ${}^{(n,l)}f_{i,j}$.[18] We

[18] We will not be trying to include ${}^{(n,l)}U_{r_0,t_1'}$ into ${}^{(n+1,l)}U_{r_0,t_1}$ in the sequel. One good reason for this is that the inclusion of $M^{\otimes\triangle(n,l)}$ into $M^{\otimes\triangle(n+1,l)}$ does not map ${}^{(n,l)}U_{r_0,t_1'}$ into ${}^{(n+1,l)}U_{r_0,t_1'}$.

also decompose ${}^{l}S_n$ into the direct sum ${}^{l}S_n^{(r_0,t_1')}(0) \oplus {}^{l}S_n^{(r_0,t_1')}(1)$, where ${}^{l}S_n^{(r_0,t_1')}(k)$ is generated by those ${}^{(n,l)}f_{i,j}$ with $i \overset{n,l}{\sim} j$ and

$$\sum_{t_2=1}^{\mu_{r_0}+1} \sum_{s=1}^{n+1-r_0} i(r_0, s, t_1', t_2) = k. \tag{2.4.12}$$

Proposition 2.4.12. *With the notation above, and $k \in \mathbb{Z}_2$ define a unitary $y_k \in M^{\otimes(\triangle(n+1,l)\setminus\triangle(n,l))}$ to be the elementary tensor with $e_0^{(k)} - e_1^{(k)}$ in the $(r_0-1, n_0+3-r_0, \mu_{r_0-1}, t_1')$-position, and the identity in all other positions. Then $x \otimes y_k \in S_{n+1}$, whenever $x \in {}^{l}S_n^{(r_0,t_1')}(k)$; and $\mathbb{E}_{{}^{l}S}(x \otimes y_k) = 0$, whenever $x \in {}^{l}S_n^{(r_0,t_1')}(1-k)$.*

Proof. By linearity, it suffices to establish the claims when $x = {}^{(n,l)}f_{i,j}$. Suppose that $i, j : \triangle(n,l) \to \mathbb{Z}_2$ have $i \overset{n,l}{\sim} j$, and satisfy (2.4.12). In just the same way as Proposition 2.4.6, ${}^{(n,l)}f_{i,j} \otimes y_k$ is then a linear combination of elements ${}^{(n+1,l)}f_{i',j'}$ with

$$i'|_{\triangle(n,l)} = i, \quad j'|_{\triangle(n,l)} = j$$

and

$$i'(r, n+2-r, t_1, t_2) = j'(r, n+2-r, t_1, t_2)$$

for all $(r, n+2-r, t_1, t_2) \in \triangle(n+1,l) \setminus \triangle(n,l)$. All these pairs (i', j') have $i' \overset{n+1,l}{\sim} j'$ so that ${}^{(n,l)}f_{i,j} \otimes y_k \in {}^{l}S_{n+1}$.

Now suppose that i, j have $i \overset{n,l}{\sim} j$ but

$$\sum_{t_2=1}^{\mu_{r_0}+1} \sum_{s=1}^{n+1-r_0} i(r_0, s, t_1', t_2) = 1-k.$$

The methods of Proposition 2.4.6 combine with Proposition 2.3.4 to show that ${}^{(n,l)}f_{i,j} \otimes y_k$ is a linear combination of ${}^{(n+1,l)}f_{i',j'}$ satisfying:

- $i'|_{\triangle(n,l)} = i$, $j'|_{\triangle(n,l)} = j$;
- $i'(r_0-1, n+3-r_0, \mu_{r_0-1}, t_1') \neq j'(r_0-1, n+3-r_0, \mu_{r_0-1}, t_1')$;
- $i'(r, n+2-r, t_1, t_2) = j'(r, n+2-r, t_1, t_2)$, for all $(r, n+2-r, t_1, t_2) \in \triangle(n+1,l) \setminus (\triangle(n,l) \cup \{(r_0-1, n+3-r_0, \mu_{r_0-1}, t_1')\})$.

Such i', j' have $i' \overset{n+1,l}{\not\sim} j$, so that $\mathbb{E}_{S_{n+1}}\left({}^{(n,l)}f_{i,j} \otimes y_k\right) = 0$. The result then follows from Proposition 2.4.6. $\square$

Proposition 2.4.13. *Suppose that $x \in {}^{l}S_n^{(r_0,t_1')}(k)$, $y_1, y_2 \in {}^{n,l}U_{r_0,t_1'}$ and $z_1, z_2 \in M^{\otimes\triangle(n,l)}$ have $\mathbb{E}_{{}^{n,l}U_{r_0,t_1'}}(z_i) = 0$. Then*

$$\mathbb{E}_{{}^{l}S}(y_1 x y_2) \subset {}^{l}S_n^{(r_0,t_1')}(k), \tag{2.4.13}$$

and

$$\mathbb{E}_{{}^{l}S}(z_1 x z_2) \subset {}^{l}S_n^{(r_0,t_1')}(1-k). \tag{2.4.14}$$

Proof. By linearity, we may assume that $y_1 = {}^{(n,l)}f_{i_1,j_1}$, $x = {}^{(n,l)}f_{j_1,i_2}$ and $y_2 = {}^{(n,l)}f_{i_2,j_2}$ for appropriate functions $i_1, i_2, j_1, j_2 : \triangle(n,l) \to \mathbb{Z}_2$. In this way $y_1 x y_2 = {}^{(n,l)}f_{i_1,j_2}$, which is orthogonal to ${}^lS$ unless $i_1 \overset{n,l}{\sim} j_2$. Since ${}^{(n,l)}f_{j_1,i_2} \in {}^lS_n^{(r_0,t_1')}(k)$ and ${}^{(n,l)}f_{i_1,j_1} \in {}^{(n,l)}U_{r_0,t_1'}$ we have

$$\sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} i_1(r_0,s,t_1',t_2) = \sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} j_1(r_0,s,t_1',t_2) = k,$$

so that ${}^{(n,l)}f_{i_1,j_2} \in {}^lS_n^{(r_0,t_1')}(k)$ when $i_1 \overset{n,l}{\sim} j_2$, establishing (2.4.13).

We establish (2.4.14) in a similar way, taking $z_1 = {}^{(n,l)}f_{i_1,j_1}$, $x = {}^{(n,l)}f_{j_1,i_2}$ and $z_2 = {}^{(n,l)}f_{i_2,j_2}$ for functions with $j_1 \overset{n,l}{\sim} i_2$,

$$\sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} i_1(r_0,s,t_1',t_2) \neq \sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} j_1(r_0,s,t_1',t_2),$$

and

$$\sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} i_2(r_0,s,t_1',t_2) \neq \sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} j_2(r_0,s,t_1',t_2).$$

Again $z_1 x z_2 = {}^{(n,l)}f_{i_1,j_2}$, and if $i_1 \overset{n,l}{\sim} j_2$, we can check that

$$\sum_{t_2=1}^{\mu_{r_0}+1}\sum_{s=1}^{n+1-r_0} i_1(r_0,s,t_1',t_2) = 1-k,$$

as required. □

We are now in a position to give the technical lemma from which Theorem 2.4.10 will follow.

Lemma 2.4.14. *Fix a hyperfinite* II_1 *factor* R, $n \in \mathbb{N}$ *and* $\epsilon > 0$. *Let* u *be a unitary in* $M^{\otimes\triangle(n,l)} \otimes R$ *(regarded as a von Neumann subalgebra of* ${}^lR\overline{\otimes}R$*) such that*

$$\left\|(I - \mathbb{E}_{{}^lS\overline{\otimes}R})uxu^*\right\|_2 \leq \epsilon \left\|x\right\|, \tag{2.4.15}$$

for all $x \in {}^lS\overline{\otimes}R$. *For each* r_0 *and* t_1', *with* $l+1 \leq r_0 \leq n$ *and* $1 \leq t_1' \leq \mu_{r_0}$, *we have*

$$\left\|u - \mathbb{E}_{{}^{(n,l)}U_{r_0,t_1'}\otimes R}(u)\right\|_2 \leq 2\sqrt{\epsilon}.$$

Proof. Write $v_0 = \mathbb{E}_{{}^{(n,l)}U_{r_0,t_1'}\otimes R}(u)$ and $v_1 = u - \mathbb{E}_{{}^{(n,l)}U_{r_0,t_1'}\otimes R}(u)$. Since ${}^lS_n \subset {}^{(n,l)}U_{r_0,t_1'}$, we have

$$\mathbb{E}_{{}^lS_n\otimes R}(v_0 x v_1^*) = \mathbb{E}_{{}^lS_n\otimes R}(v_1 x v_0^*) = 0,$$

whenever $x \in {}^lS_n \otimes R$, by applying $\mathbb{E}_{{}^{(n,l)}U_{r_0,t_1'}\otimes R}$.

Now take $x \in {}^{l}S_n^{(r_0,t_1')}(k)$ for some $k \in \mathbb{Z}_2$, and consider $x \otimes y_k \otimes 1_R$, where y_k is the unitary of Proposition 2.4.12. The hypothesis (2.4.15) gives

$$\|u(x \otimes 1_R)u^* - \mathbb{E}_{S_n \otimes R}(v_0(x \otimes 1_R)v_0^* + v_1(x \otimes 1_R)v_1^*)\|_2 \leq \epsilon \|x\|$$

and

$$\left\|u(x \otimes y_k \otimes 1_R)u^* - \mathbb{E}_{S_{n+1} \otimes R}(u(x \otimes y_k \otimes 1_R)u^*)\right\|_2 \leq \epsilon \|x\| .$$

Putting these two equations together with the fact that u, v_0, v_1 allcommute with $(1_{M \otimes \triangle(n,l)} \otimes y_k \otimes 1_R$, we get

$$\begin{aligned}&\Big\|u(x \otimes y_k \otimes 1_R)u^* \\ &\quad - \mathbb{E}_{S_{n+1} \otimes R}(\mathbb{E}_{S_n \otimes R}(v_0(x \otimes 1_R)v_0^* + v_1(x \otimes 1_R)v_1^*)(1_{M \otimes \triangle(n,l)} \otimes y_k \otimes 1_R))\Big\|_2 \\ &\leq 2\epsilon \|x\| .\end{aligned} \tag{2.4.16}$$

Note that $\mathbb{E}_{S_n \otimes R}(v_1(x \otimes 1_R)v_1^*)$ lies in ${}^{l}S_n^{(r_0,t_1')}(1-k)$ by Proposition 2.4.13, so that the second case of Proposition 2.4.12 gives

$$\mathbb{E}_{S_{n+1} \otimes R}(\mathbb{E}_{S_n \otimes R}(v_1(x \otimes 1_R)v_1^*)(1_{M \otimes \triangle(n,l)} \otimes y_k \otimes 1_R)) = 0.$$

Similiarly, the first cases of Proposition 2.4.12 and 2.4.13 show that

$$\mathbb{E}_{S_n \otimes R}(v_0(x \otimes 1_R)v_0^*)(1_{M \otimes \triangle(n,l)} \otimes y_k \otimes 1_R) \in {}^{l}S_{n+1} \otimes R.$$

Substituting these identities into (2.4.16), we obtain

$$\|\mathbb{E}_{S_n \otimes R}(v_0(x \otimes 1_R)v_0^*)(1_{M \otimes \triangle(n,l)} \otimes y_k \otimes 1_R)\|_2 \geq \|u(x \otimes y_k \otimes 1_R)u^*\|_2 - 2\epsilon \|x\| ,$$

whenever $x \in {}^{l}S_n^{(r_0,t_1')}(k) \otimes R$, which simplfies to give

$$\|\mathbb{E}_{S_n \otimes R}(v_0(x \otimes 1_R)v_0^*)\|_2 \geq \|x\|_2 - 2\epsilon \|x\| .$$

Now take $x_k \in {}^{l}S_n^{(r_0,t_1')}(k)$, for $k \in \mathbb{Z}_2$. Since $\mathbb{E}_{S_n \otimes R}(v_0(x_k \otimes 1_R)v_0^*) \in {}^{l}S_n^{(r_0,t_1')}(k)$, we have

$$\begin{aligned}&\|\mathbb{E}_{S_n \otimes R}(v_0((x_0 + x_1) \otimes 1_R)v_0^*)\|_2^2 \\ &= \|\mathbb{E}_{S_n \otimes R}(v_0(x_0 \otimes 1_R)v_0^*)\|_2^2 + \|\mathbb{E}_{S_n \otimes R}(v_0(x_1 \otimes 1_R)v_0^*)\|_2^2 \\ &\geq (\|x_0\|_2 - 2\epsilon \|x_0\|)^2 + (\|x_1\|_2 - 2\epsilon \|x_1\|)^2 \\ &\geq \|x_0\|_2^2 + \|x_1\|_2^2 - 2\epsilon(\|x_0\|_2 \|x_0\| + \|x_1\|_2 \|x_1\|) \\ &= \|x_0 + x_1\|_2^2 - 2\epsilon(\|x_0\|_2 \|x_0\| + \|x_1\|_2 \|x_1\|)\end{aligned}$$

Take $x_k = \mathbb{E}_{{}^{l}S_n^{(r_0,t_1')}(k) \otimes R}(1)$, so that $x_0 + x_1 = 1$. In this way, we have

$$\|v_0\|_2^2 \geq \|\mathbb{E}_{S_n \otimes R}(v_0 v_0^*)\|_2^2 \geq 1 - 4\epsilon,$$

or alternatively

$$\left\|u - \mathbb{E}_{{}^{(n,l)}U_{r_0,t_1'}\otimes R}(u)\right\|_2 \leq 2\sqrt{\epsilon}.$$

□

Proof of Theorem 2.4.10. To establish $\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right) \subset {}^lT\overline{\otimes}R$, it suffices to show that, for any pair of functions $i, j : \triangle(n,l) \to \mathbb{Z}_2$ with $i \overset{n,l+1}{\not\sim} k$ and any $z \in R$, the element ${}^{(n,l)}f_{i,j} \otimes z$ is orthogonal to $\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right)$. Fix such i, j and let r_0 be maximal with

$$\sum_{s=1}^{n+1-r_0}\sum_{t_2=1}^{\mu_{r_0+1}} i(r_0,s,t_1',t_2) \neq \sum_{s=1}^{n+1-r_0}\sum_{t_2=1}^{\mu_{r_0+1}} j(r_0,s,t_1',t_2),$$

for some $1 \leq t_1' \leq \mu_{r_0}$. The maximality of r ensures that for $n_0 > n$, we can write ${}^{n,l}f_{i,j}$ as a linear combination of ${}^{(n_0,l)}f_{i',j'}$ with

$$\sum_{s=1}^{n_0+1-r_0}\sum_{t_2=1}^{\mu_{r_0+1}} i'(r_0,s,t_1',t_2) \neq \sum_{s=1}^{n_0+1-r_0}\sum_{t_2=1}^{\mu_{r_0+1}} j'(r_0,s,t_1',t_2).$$

Note that such ${}^{(n_0,l)}f_{i',j'}$ are orthogonal to ${}^{(n_0,l)}U_{r_0,t_1'}$ and hence so too is ${}^{(n,l)}f_{i,j}$.

Fix $\epsilon > 0$. Take a unitary $u \in {}^lR\overline{\otimes}R$ normalising ${}^lS\overline{\otimes}R$, and find by density $n_0 > n$ and a unitary $u_0 \in M^{\otimes\triangle(n_0,l)} \otimes R$ with $\|u - u_0\|_2 < \epsilon/4$. The estimates of Proposition 1.4.14 ensure that

$$\|(I - \mathbb{E}_{{}^lS\overline{\otimes}R})(u_0 x u_0^*)\|_2 \leq \epsilon\,\|x\|,$$

for all $x \in {}^lS\overline{\otimes}R$. Lemma 2.4.14 gives

$$\left\|u_0 - \mathbb{E}_{{}^{(n_0,l)}U_{r_0,t_1'}\otimes R}(u_0)\right\|_2 \leq 2\sqrt{\epsilon}.$$

Now

$$\begin{aligned}\left|\operatorname{tr}(\,{}^{(n,l)}f_{i,j}u^*)\right| &\leq \left|\operatorname{tr}(\,{}^{(n,l)}f_{i,j}u_0^*)\right| + \frac{1}{4}\epsilon\left\|{}^{(n,l)}f_{i,j}\right\|_2\\ &\leq \left(\frac{1}{4}\epsilon + 2\sqrt{\epsilon}\right)\left\|{}^{(n,l)}f_{i,j}\right\|_2 + \left|\operatorname{tr}(\,{}^{(n,l)}f_{i,j}\mathbb{E}_{{}^{(n_0,l)}U_{r_0,t_1'}}(u_0^*))\right|\\ &= \left(\frac{1}{4}\epsilon + 2\sqrt{\epsilon}\right)\left\|{}^{(n,l)}f_{i,j}\right\|_2,\end{aligned}$$

since ${}^{(n,l)}f_{i,j}$ is orthogonal to ${}^{(n_0,l)}U_{r_0,t_1'}$. As $\epsilon > 0$ is arbitary, we obtain $\operatorname{tr}(\,{}^{(n,l)}f_{i,j}u^*) = 0$, which we have already seen is enought to show that

$$\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right)'' \subset {}^lT\overline{\otimes}R. \tag{2.4.17}$$

We are assuming that $\mathcal{N}_{{}^lR}\left({}^lS\right)''\overline{\otimes}R$ is a subfactor of ${}^lR\overline{\otimes}R$. We can tensor the unitaries u_k in Proposition 2.4.11 by 1_R to give 2^{μ_l} unitaries v_k in $\mathcal{N}_{{}^lR}\left({}^lS\right)''\overline{\otimes}R$, with $\mathbb{E}_{S\overline{\otimes}R}\left(v_{k_1}v_{k_2}^*\right) = \delta_{k_1,k_2}1$. The observation on orthogonal bases, Proposition 1.3.6, applies to show us that $[\mathcal{N}_{{}^lR}\left({}^lS\right)''\overline{\otimes}R : {}^lS\overline{\otimes}R] \geq 2^{\mu_l}$. On the other hand Theorem 2.4.9, is that $[{}^lT : {}^lS] = 2^{\mu_l}$ and tensoring by R gives $[{}^lT\overline{\otimes}R : {}^lS\overline{\otimes}R] = 2^{\mu_l}$, see (1.3.5). Hence, using the inclusion (2.4.17), we must actually have

$$\mathcal{N}_{{}^lR\overline{\otimes}R}\left({}^lS\overline{\otimes}R\right)'' = {}^lT\overline{\otimes}R. \qquad \square$$

2.4.3 Defining the required infinite length masa

All the hard work for Theorem 2.4.2 is done. It remains only to construct a Tauer masa A inside ${}^1R\overline{\otimes}R$ for which

$$\mathcal{N}(A)'' = {}^1S\overline{\otimes}R,$$

for some hyperfinite II_1 factor R. In this case Proposition 2.3.2 ensures that $\mathcal{N}\left({}^1S\overline{\otimes}R\right)''$ is a subfactor, so Theorem 2.4.10 allows us to compute

$$\mathcal{N}^2(A) = {}^1T \overline{\otimes} R = {}^2S \overline{\otimes} {}^1R \overline{\otimes} R.$$

We can repeatedly apply Theorem 2.4.10 and the definitions, (2.4.7) of ${}^lT$ and (2.4.8) of ${}^lR_1$, to obtain

$$\begin{aligned}\mathcal{N}^{l+1}(A) &= {}^lT \overline{\otimes} \left({}^{l-1}R_1 \overline{\otimes} {}^{l-2}R_1 \overline{\otimes} \ldots \overline{\otimes} {}^1R_1 \overline{\otimes} R\right) \\ &= {}^{l+1}S \overline{\otimes} \left({}^lR_1 \overline{\otimes} {}^{l-1} R_1 \overline{\otimes} \ldots \overline{\otimes} {}^1R_1 \overline{\otimes} R\right),\end{aligned}$$

for all $l \geq 1$. The required hypothesis that $\mathcal{N}\left({}^lS\overline{\otimes} {}^{l-1}R\overline{\otimes} \ldots {}^1R\overline{\otimes}R\right)''$ is a factor comes from Proposition 2.3.2, just as in the $l = 1$ case above. Theorem 2.4.9 then gives

$$[\mathcal{N}^{l+1}(A) : \mathcal{N}^l(A)] = 2^{\mu_l},$$

for every l.

Looking back at the first two stages in the finite length case allows us to see how this masa should be constructed.

Construction 2.4.15. For each $n \geq 2$, define $J(n)$ to be set of all triples of integers (r, s, t_1) with $1 \leq r \leq n-1$, $1 \leq s \leq n-r$ and $1 \leq t_1 \leq \mu_r$. We have natural inclusions $J(n) \subset J(n+1) \subset \ldots$, giving rise to inclusions

$$M^{\otimes J(1)} \subset M^{\otimes J(n+1)}.$$

We let R be the II_1 factor obtained as the direct limit of these inclusions, with respect to the unique normalised trace, or alternatively as the von Neumann infinite tensor product of M over the set of all (r, s, t_1) with $s, r \in \mathbb{N}$ and $1 \leq t_1 \leq \mu_r$.[19]

The chain of finite dimensional factors

$$\cdots \subset M^{\otimes \triangle(n,1)} \otimes M^{\otimes J(n)} \subset M^{\triangle(n+1,1)} \otimes M^{\otimes J(n+1)} \subset \dots, \tag{2.4.18}$$

with the natural inclusions, generates the II_1 factor ${}^1R\overline{\otimes}R$. Let A_n be the direct sum

$$\bigoplus_{i:\triangle(n,1)\to\mathbb{Z}_2} {}^{(n,1)}f_{i,i} \otimes \bigotimes_{(r,s,t_1)\in J(n)} D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} i(r,u,t_1,t_2)\right)}, \tag{2.4.19}$$

a masa in $M^{\otimes\triangle(n,1)} \otimes M^{\otimes J(n)}$.[20] Under the inclusion (2.4.18) we have $A_n \subset A_{n+1}$ so the direct limit A *is* a Tauer masa in ${}^1R\overline{\otimes}R$.

It now remains to check that $\mathcal{N}_{{}^1R\overline{\otimes}R}(A)'' = {}^1S\overline{\otimes}R$. The reader must now be as tired as the author of the arguments necessary to demonstrate this – since we have proceeded formally so far in this section, we reluctantly include the details.

Proposition 2.4.16. *With the notation above,*

$${}^1S\overline{\otimes}R \subset \mathcal{N}(A)''.$$

Proof. Fix n, and consider $v \in M^{\otimes\triangle(n,1)} \otimes M^{\otimes J(n)}$ of the form

$${}^{(n,1)}f_{i,j} \otimes \bigotimes_{(r,s,t_1)\in I(n,1)} x_{r,s,t_1},$$

where:

- $i, j : \triangle(n,1) \to \mathbb{Z}_2$ with $i \overset{n,1}{\sim} j$;
- each x_{r,s,t_1} is a unitary in M with

$$x_{r,s,t_1} D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} j(r,u,t_1,t_2)\right)} x^*_{r,s,t_1} = D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} i(r,u,t_1,t_2)\right)}.$$

Such v are partial isometries generating ${}^1S_n \otimes M^{\otimes J(n)}$ as a *-algebra so, as $\mathcal{GN}(A)'' = \mathcal{N}(A)''$, it is sufficient to demonstrate that these v normalise A_{n_1} for all $n_1 \geq n$.

Take a typical generator

$$z = {}^{(n_1,1)}f_{k,k} \otimes \bigotimes_{(r,s,t_1)\in J(n_1)} y_{r,s,t_1},$$

[19] This tensor product is taken with respect to the unique trace.

[20] It is worth noting that the (r, u, t_1, t_2) appearing in the superscripts in (2.4.19) all lie in $\triangle(n-1, 1)$.

in A_{n_1}, where $k : \triangle(n_1, 1) \to \mathbb{Z}_2$ and the y_{r,s,t_1} lie in $D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2)\right)}$. Either by the usual induction procedure of Proposition 2.4.4, or by writing

$$^{(n,1)}f_{i,j} = \sum_{\substack{\tilde{i},\tilde{j}:\triangle(n_1,1)\to\mathbb{Z}_2 \\ \tilde{i}|_{\triangle(n,1)}=i \text{ and } \tilde{j}|_{\triangle(n,1)}=j \\ \tilde{i}=\tilde{j} \text{ away from } \triangle(n,1)}} {}^{(n_1,1)}f_{\tilde{i},\tilde{j}},$$

we see that $vzv^* = 0$, unless k is identical to j on $\triangle(n, 1)$. When this is the case, we have

$$vzv^* = {}^{(n_1,1)}f_{k',k'} \otimes \bigotimes_{(r,s,t_1)\in J(n)} x_{r,s,t_1} y_{r,s,t_1} x^*_{r,s,t_1} \otimes \bigotimes_{(r,s,t_1)\in J(n_1)\setminus J(n)} y_{r,s,t_1},$$

where $k' : \triangle(n_1, 1) \to \mathbb{Z}_2$ is defined so that $k'|_{\triangle(n,1)} = i$ and, away from $\triangle(n, l)$, k' coincides with k.

The definition of x_{r,s,t_1} ensures that the $x_{r,s,t_1} y_{r,s,t_1} x^*_{r,s,t_1}$ appearing above lie in $D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k'(r,u,t_1,t_2)\right)}$. For $(r, s, t_1) \in J(n_1) \setminus J(n)$, the remaining y_{r,s,t_1}, lie in $D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2)\right)}$ and we must check that this is $D^{\left(\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k'(r,u,t_1,t_2)\right)}$. Note that $s \geq n + 1 - r$, and split the sum appearing in the superscript at $u = n + 1 - r$ to obtain

$$\begin{aligned}
\sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2) &= \sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2) + \sum_{u=n+2-r}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2) \\
&= \sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_r+1} j(r,u,t_1,t_2) + \sum_{u=n+2-r}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2) \quad (2.4.20) \\
&= \sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_r+1} i(r,u,t_1,t_2) + \sum_{u=n+2-r}^{s}\sum_{t_2=1}^{\mu_r+1} k(r,u,t_1,t_2) \quad (2.4.21) \\
&= \sum_{u=1}^{s}\sum_{t_2=1}^{\mu_r+1} k'(r,u,t_1,t_2). \quad (2.4.22)
\end{aligned}$$

The identity (2.4.20) follows as all (r, u, t_1, t_2) with $1 \leq u \leq n + 1 - r$ lie in $\triangle(n, 1)$, on which k agrees with j. Then, as all (r, u, t_1, t_2) with $u > n + 1 - r$ lie in $\triangle(n_1, l) \setminus \triangle(n, l)$ where k and k' agree, the two sums of (2.4.21) can be reassembled to give (2.4.22), completing the proof. □

Proposition 2.4.17. *With the standing notation of this section, let A be the masa in ${}^1R\overline{\otimes}R$ produced in Construction 2.4.15. This A has the weak asymptotic homomorphism property away from ${}^1S\overline{\otimes}R$ of 1.4.21 and so*

$$\mathcal{N}(A)'' \subset {}^1S\overline{\otimes}R.$$

$$\begin{array}{ccccccc}
\cdots\hookrightarrow & M^{\otimes\triangle(n,1)}\otimes M^{\otimes J(n)} & \hookrightarrow & M^{\otimes\triangle(n+1,1)}\otimes M^{\otimes J(n+1)} & \hookrightarrow\cdots\hookrightarrow & {}^{1}R\overline{\otimes}R \\
 & \uparrow & & \uparrow & & \uparrow \\
\cdots\hookrightarrow & {}^{1}S_n\otimes M^{\otimes J(n)} & \hookrightarrow & {}^{1}S_{n+1}\otimes M^{\otimes J(n+1)} & \hookrightarrow\cdots\hookrightarrow & {}^{1}S\overline{\otimes}R
\end{array}$$

Figure 2.7: Commuting diagram giving rise to the inclusion of ${}^{1}S\overline{\otimes}R \subset {}^{1}R\overline{\otimes}R$.

Proof. We shall use Lemma 1.4.23, in the context of Figure 2.7 overleaf, which commutes as a consequence of the commutativity of Figure 2.5 in Proposition 2.4.6. Fixing n, we shall find a unitary v in A_{n+1} so that

$$\mathbb{E}_{A_{n+1}}\left(xvy^*\right) = 0,$$

whenever x and y take the form

$$x = {}^{(n,1)}f_{i,j} \otimes \bigotimes_{(r,s,t_1)\in J(n)} x_{r,s,t_1} \text{ and } y = {}^{(n,1)}f_{i',j'} \otimes \bigotimes_{(r,s,t_1)\in J(n)} y_{r,s,t_1},$$

where $i, i', j, j' : \triangle(n,1) \to \mathbb{Z}_2$ have $i \overset{n,1}{\not\sim} j$ and $i' \overset{n,1}{\not\sim} j'$, and the x_{r,s,t_1} and y_{r,s,t_1} are unitaries in M. This will be enough to satisfy the hypothesis of Lemma 1.4.23, as the linear span of all such x and y is $(M^{\otimes\triangle(n,1)} \otimes M^{\otimes J(n)}) \ominus ({}^{1}S_n \otimes M^{J(n)})$. Lemma 1.4.24 will then show that $\mathcal{N}(A)'' \subset {}^{1}S\overline{\otimes}R$, as needed.

Take

$$v = \sum_{k:\triangle(n,1)\to\mathbb{Z}_2} {}^{(n,1)}f_{k,k} \otimes \bigotimes_{\triangle(n+1,1)\setminus\triangle(n,1)} 1 \otimes \bigotimes_{J(n)} 1 \otimes v_k,$$

where v_k is the unitary in $M^{\otimes J(n+1)\setminus J(n)}$ given as an elementary tensor by

$$v_k = \bigotimes_{(r,n+1-r,t_1)\in J(n+1)\setminus J(n)} r^{\left(\sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_{r+1}} k(r,u,t_1,t_2)\right)},$$

with $r^{(q)}$ the Rademacher style unitary $e_0^{(q)} - e_1^{(q)}$. To check this makes sense, we observe that all (r,u,t_1,t_2) appearing in the sum above lie in $\triangle(n,1)$. We then examine xvy^* which, for familiar reasons, will be zero unless $j = j'$ and will be orthogonal to A_{n+1} unless $i = i'$ – assume both of these identities hold henceforth. Then

$$xvy^* = {}^{(n,1)}f_{i,i} \otimes \bigotimes_{\triangle(n+1,1)\setminus\triangle(n,1)} 1 \otimes \bigotimes_{J(n)} 1 \otimes v_j,$$

so, using the commuting squares property (Proposition 2.1.2) of Tauer masas, we have

$$
\begin{aligned}
&\mathbb{E}_A(xvy^*) = \mathbb{E}_{A_{n+1}}(xvy^*) \\
&= {}^{(n,1)}f_{i,i} \otimes \bigotimes_{\triangle(n+1,l)\setminus\triangle(n,l)} 1 \otimes \bigotimes_{J(n)} 1 \otimes \mathbb{E}_{A_{n+1,n}^{\left({}^{(n,1)}f_{i,i}\right)}}(v_j) \\
&= {}^{(n,1)}f_{i,i} \otimes \bigotimes_{\triangle(n+1,l)\setminus\triangle(n,l)} 1 \otimes \bigotimes_{J(n)} 1 \\
&\qquad \otimes \bigotimes_{(r,n+1-r,t_1)\in J(n+1)\setminus J(n)} \mathbb{E}_{D^{\left(\sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_{r+1}} i(r,u,t_1,t_2)\right)}}\left(r^{\left(\sum_{u=1}^{n+1-r}\sum_{t_2=1}^{\mu_{r+1}} i(r,u,t_1,t_2)\right)}\right)
\end{aligned}
\tag{2.4.23}
$$

As $i \not\sim^{n,1} j$, there is some r and t_1 so that

$$
\sum_{s=1}^{n+1-r}\sum_{t_2=1}^{\mu_{r+1}} i(r,s,t_1,t_2) \neq \sum_{s=1}^{n+1-r}\sum_{t_2=1}^{\mu_{r+1}} j(r,s,t_1,t_2).
$$

The $(r, n+1-r, t_1)$ component of the last tensor product in (2.4.23) must then be zero, as required, by the orthogonality of $D^{(0)}$ and $D^{(1)}$. □

2.4.4 Thoughts on infinite length masas

We first note that all the infinite length masas we have constructed are all Γ masas. The proof is the same as Proposition 2.3.13 and so is omitted.

Proposition 2.4.18. *The Tauer masas of infinite length exhibited to prove Theorem 2.4.2 all contain non-trivial centralising sequences for the hyperfinite* II_1 *factor in which they live.*

Recent work of Dykema, Sinclair and Smith ([17]) enables us to use the masas of Theorem 2.4.2 to produce uncountably many pairwise non-conjugate semi-regular masas inside II_1 factors arising from a free product with the hyperfinite II_1 factor. As free products and free group factors are not the main object of study here, economy dictates that we shall assume familiarity with this material - all of which can be found in [16]. Our starting point is the observation that no extra normalisers are obtained when we take an appropriate von Neumann free product.

Proposition 2.4.19. *Let N be a diffuse von Neumann subalgebra of the hyperfinite* II_1 *factor R and Q be a separable diffuse finite von Neumann algebra with fixed faithful normal trace tr_Q. Denote the unique faithful normal trace on R by tr_R, and form the von Neumann free product $R * Q$ with respect to tr_R and tr_Q. Under these hypothesis, $R * Q$ is a* II_1 *factor, which contains N and $\mathcal{N}_R(N)'' = \mathcal{N}_{R*Q}(N)''$. Finally, if A is a masa in R, then A is also a masa inside $R * Q$.*

Some comments on the proof. That $R * Q$ is a II_1 factor in this situation is well known, [17, Remark 2.6]. In our terminology, the key technical result - Lemma 2.2 of [17], ensures that any diffuse $N \subset R$ has the weak asymptotic homomorphism property in $R * Q$ away from R. The statement about the normalisers, [17, Theorem 2.3], then follows from Lemma 1.4.24. Finally, the statement about masas follows, as any unitary commuting with A normalises A and so lies in M. This too can be found in Theorem 2.3 of [17]. □

We will apply this result to our infinite length semi-regular masas in the hyperfinite II_1 factor R. Take a separable diffuse finite von Neumann algebra Q with fixed faithful normal traace. Proposition 2.4.19, ensures that when A is a semi-regular masa in R, we have

$$\mathcal{N}^l_R(A) = \mathcal{N}^l_{R*Q}(A),$$

for every l, as Proposition 2.3.2 ensures that each $\mathcal{N}^l_R(A)$ is a subfactor of R so diffuse. In particular, the uncountable family of pairwise non-conjugate semi-regular masas in R of Theorem 2.4.2 remain pairwise non-conjugate when viewed in $R*Q$; the index sequence $([\mathcal{N}^{l+1}_{R*Q}(A) : \mathcal{N}^l_{R*Q}(A)])_{l=1}^{\infty}$ is preserved as an invariant demonstrating this non-conjugacy. Finally, observe that taking $Q = L^{\infty}[0,1]$,[21] we have the isomorphism $R * Q \cong \mathcal{L}(F_2)$, see [16]. A further free product can be used to obtain uncountably many, pairwise non-conjugate semi-regular masas inside any free group factor.

Theorem 2.4.20. *Let Q be any finite diffuse von Neumann algebra with fixed faithful normal trace tr_Q. Let R be the hyperfinite II_1 factor and $M = R * Q$. There is an uncountable family of semi-regular masas in the II_1 factor M no pair of which is conjugate via an automorphism of M. In particular, for each $k = 2, 3, \ldots, \infty$, we can take $M = \mathcal{L}(\mathbb{F}_k)$ the II_1 factor corresponding to the free group with k generators.*

Currently, we have defined the iterated normaliser algebras $\mathcal{N}^n(A)$ of an inclusion $A \subset N$ for all $n \in \mathbb{N}$. The definition naturally extends to any ordinal α. Namely, for successor ordinals we define

$$\mathcal{N}^{\alpha+1}(A) = \mathcal{N}_N(\mathcal{N}^{\alpha}(A))'',$$

and, when α is a limit ordinal, take

$$\mathcal{N}^{\alpha}(A) = \left(\bigcup_{\beta<\alpha} \mathcal{N}^{\beta}(A)\right)''.$$

[21] Equipped with the canonical trace, $\mathrm{tr}(f) = \int_0^1 f(t)\mathrm{d}t$.

The infinite length Tauer masas constructed in this section have length ω in the sense that their ω-fold normaliser algebra is the underlying hyperfinite II_1 factor, and ω is minimal with this property. To see this we recall that this hyperfinite II_1 factor was realised as ${}^1R\overline{\otimes}R$, and the Tauer masa A had $\mathcal{N}^1(A) = {}^1S\overline{\otimes}R$. Now repeatedly use the factorisation (2.4.8) to see that

$$ {}^1R\,\overline{\otimes}\,R = {}^2R\,\overline{\otimes}\,{}^1R_1\,\overline{\otimes}\,R = \cdots = {}^{l+1}R\,\overline{\otimes}\left({}^lR_1\,\overline{\otimes}\,{}^{l-1}R_1\ldots\overline{\otimes}\,{}^1R_1\right)\overline{\otimes}\,R, $$

and so

$$ {}^1R\,\overline{\otimes}\,R = \left(\bigcup_{l=1}^{\infty} 1_{{}^{l+1}R}\otimes\left({}^lR_1\,\overline{\otimes}\,{}^{l-1}R_1\ldots\overline{\otimes}\,{}^1R_1\right)\overline{\otimes}\,R\right)''. $$

We have already seen at the beginning of subsection 2.4.3 that

$$ \mathcal{N}^{l+1}(A) = {}^{l+1}S\,\overline{\otimes}\left({}^lR_1\,\overline{\otimes}\,{}^{l-1}R_1\ldots\overline{\otimes}\,{}^1R_1\right)\overline{\otimes}\,R, $$

from which we can see that $\left(\bigcup_{l=1}^{\infty}\mathcal{N}^l(A)\right)'' = {}^1R_1\overline{\otimes}R$.

Proposition 2.4.21. *Let ω be the limit of all the finite ordinals, the first countable infinite ordinal. The Tauer masas $A \subset R$ constructed to establish Theorem 2.4.2 in subsections 2.4.1, 2.4.2 and 2.4.3 all have $\mathcal{N}^{\omega}(A) = R$.*

In the free product situation of Theorem 2.4.20, the semi-regular masas produced have $\mathcal{N}^{\omega}(A) = R \subset R * Q$, a singular subfactor of $R * Q$. At this point then the chain of normalising algebras terminates as any ordinal $\alpha > \omega$ has $\mathcal{N}^{\alpha}(A) = R$.

In the separable case, we only consider countable ordinals, as separability will ensure the chain of normalising algebras terminates, as described above, for uncountable ordinals. At the time of writing we do not have an example of a semi-regular masa of length $\omega + 1$, although it is easy to see what we should arrange to happen. We look for a unitary normalising $N = (\bigcup_{l=1}^{\infty}\mathcal{N}^l(A))''$ but not lying in N. Any u with $u\mathcal{N}^n(A)u^* \subset \mathcal{N}^{n+1}(A)$ for each n will give us a normaliser of N - we should aim to ensure that such a u can be found orthogonal to each $\mathcal{N}^n(A)$. We state this problem formally.

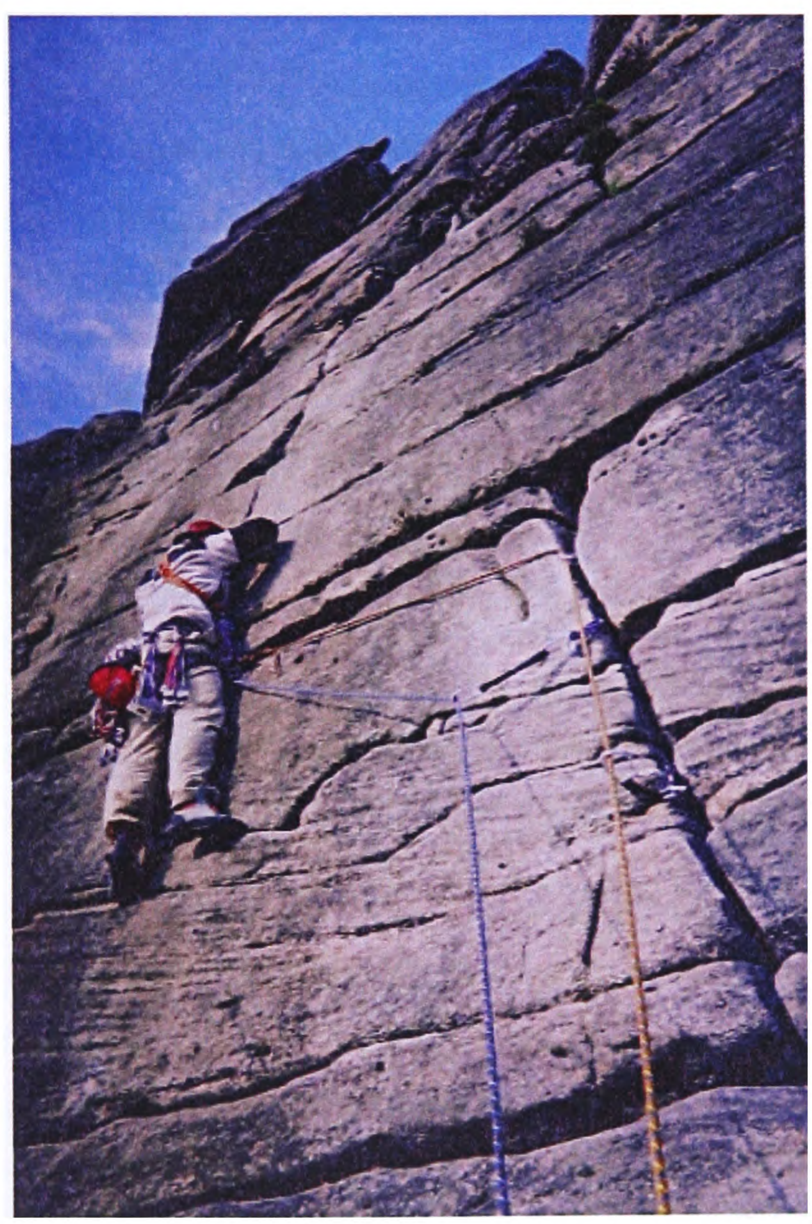
Question 2.4.22. For each countable ordinal α, does there exist a semi-regular masa A in the hyperfinite II_1 factor such that α is minimal with the property that $\mathcal{N}^{\alpha}(A) = R$?

Addendum 2.4.23. Since the completion of this thesis, we have been able to extend the methods in this chapter to show the existance of infinite semi-regular Tauer masas with all possible index sequences. In [76], we establish this result and give a positive answer to Question 2.4.22.

Chapter 3

The Pukánszky invariant and centralising sequences

Centralising



Central Trinity (VS 4c), Stanage Edge, 2003

Photo: Pete Dodd

3.1 Pukánszky's invariant: definition and background

In [51], Pukánszky exhibited countably many singular masas in R no pair of which are conjugate via an automorphism of R. The procedure used to demonstrate this non-conjugacy was to examine the abelian algebra $\mathcal{A}$ generated by the left and right actions of A on $L^2(R)$. The type decomposition of the commutant $\mathcal{A}'$ is then an invariant of the original masa A, which is now referred to as the *Pukánszky Invariant.*

More formally, given a masa A in a separably acting type II_1 factor N, we define $\mathcal{A}$ by

$$\mathcal{A} = (A \cup JAJ)'' \subset \mathbb{B}\left(L^2(N)\right),$$

where J denotes the usual modular conjugation operator on $L^2(N)$ given by extending the map $x \mapsto x^*$ from the dense subset N.[1] This $\mathcal{A}$ is then an abelian algebra, so has a type I commutant $\mathcal{A}'$ in $\mathbb{B}(L^2(N))$. We have seen, in subsection 1.4.1,[2] that the orthogonal projection e_A from $L^2(N)$ onto $L^2(A)$ can be written as a strong limit of projections of the form $\sum_i p_i J p_i J$, where the p_i are projections in A. In this way e_A lies in $\mathcal{A}$ - the centre of $\mathcal{A}'$. It is easy to see that $\mathcal{A}e_A = Ae_A$, a maximal abelian subalgebra of $\mathbb{B}(L^2(A))$. Hence, $\mathcal{A}'e_A = \mathcal{A}e_A$ is a non-zero type I_1 part of $\mathcal{A}'$ which is always present. As we wish to be able to distinguish between masas A where $\mathcal{A}'e_A$ is the only type I_1 part of $\mathcal{A}'$ and those with a larger type I_1 part, the Pukánskzy invariant of A is defined by examining the type decomposition of $\mathcal{A}'(1 - e_A)$.

Definition 3.1.1. Let A be a masa in the separably acting type II_1 factor N. With the notation above, we define the *Pukánszky invariant* of A to be the subset $\mathrm{Puk}\,(A)$ of $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ consisting of those n for which there exists a non-zero projection $p \leq 1 - e_{\mathcal{A}}$ in $\mathcal{A}$ for which $\mathcal{A}'p$ is type I_n.

It is not difficult to see that we have an isomorphism invariant here, in that if two masas A and B are conjugate by an automorphism of the underlying II_1 factor N, then they must necessarily have the same Pukánszky invariant - this is done formally in Pukánszky's original paper, [51].

Around the same time as Pukánszky, Ambrose and Singer also considered $\mathcal{A} = (A \cup JAJ)''$. In work, which unfortunately was unpublished, they introduced the concept of a masa A in N being *simple* when $\mathcal{A}$ is also a masa in $\mathbb{B}(L^2(N))$. The largest mention of this project is found in [29], Kadison's account of the life

[1] Note that $Jx^*J : y \mapsto yx$ is the operator of right multiplication by x.

[2] Using a combination of Proposition 1.4.3 and equation (1.4.2).

and work of Singer, where it is noted that all that remains is a one line abstract. In terms of the Pukánszky invariant, A is simple precisely when $\mathrm{Puk}\,(A) = \{1\}$. A formal definition of simplicity in this way is also given in [25].[3]

3.1.1 Known values of the Pukánszky invariant

The discussion preceding the definition, shows us that $\mathrm{Puk}\,(A)$ is always a non-empty subset of $\mathbb{N}_\infty$. It is not yet known which subsets of $\mathbb{N}_\infty$ are attainable as the Pukánszky invariant of some masa. Pukánszky's original examples used groups to produce a singular masa in the hyperfinite II_1 factor with invariant $\{n\}$, for each $n \in \mathbb{N}_\infty$. Recently, progress has been made by Størmer and Neshveyev in [38], who used the ergodic methods of [33] to show that any subset containing 1 arises as the Pukánszky invariant of some singular masa in the hyperfinite II_1 factor R. In particular, there exists uncountably many singular masas in R, no pair of which is conjugate by an automorphism of R. In [61], Sinclair and Smith noted that the invariant is well behaved under tensor products and so obtained sets of the form $\{m, n, nm\}$ as possible Pukánszky invariants of masas in R. More generally, for any $n_1, \dots, n_k \in \mathbb{N}$ they produce

$$\left\{ \prod_{j \in J} n_j \,\middle|\, \emptyset \neq J \subset \{1, 2, \dots, k\} \right\},$$

as the Pukánszky invariant of a masa in the hyperfinite II_1 factor, obtained by tensoring k copies of Pukánszky's examples together.[4]

Proposition 3.1.2 ([61, Theorem 2.1]). *Let A and B be masas in the II_1 factors M and N respectively. Then $A \overline{\otimes} B$ is a masa $M \overline{\otimes} N$ with*

$$Puk\,(A \overline{\otimes} B) = Puk\,(A) \cup Puk\,(B) \cup \{\, mn \,|\, m \in Puk\,(A)\,, n \in Puk\,(B) \,\}\,.$$

The main thrust of Sinclair and Smith's work in [61] is to give a method for calculating the Pukánszky invariant for masas arising from inclusions $H \subset G$, where H is an abelian subgroup of the discrete I.C.C. group G which satisfies the requirements of Proposition 1.4.4 for $\mathcal{L}\,(H)$ to be a masa in $\mathcal{L}\,(G)$. Using their machinery, they are able to produce examples of such groups yielding $\{2, 3, 12\}$, and more generally $\{n, m, knm\}$ for any $m, n \in \mathbb{N}$ and $k \in \mathbb{N}_\infty$, as possible Pukánszky invariants of masas in the hyperfinite II_1 factor.[5] It is still unknown whether the set $\{2, 3\}$ is attainable as the Pukánszky invariant of some masa in

[3] Takesaki ([66]) also has a definition of a simple masas A which, inconveniently, means something entirely different!

[4] Recall that $R \overline{\otimes} R \overline{\otimes} \dots \overline{\otimes} R$ is hyperfinite, so isomorphic to R.

[5] The groups for these examples are all amenable, so $\mathcal{L}\,(G)$ is injective.

R, or whether, if 2 and 3 are present in $\mathrm{Puk}(A)$, it is necessary to also have some common divisor[6] or common multiple of 2 and 3 in the Pukánszky invariant.

In the free group factor case, things are different. In [15], Dykema extended the result of Voiculescu ([73]) that no Cartan masa exists in a free group factor, to show that when A is a masa in $\mathcal{L}(\mathbb{F}_k)$ we have

$$\sup \mathrm{Puk}(A) = \infty.$$

It is, therefore, certainly not possible to obtain all non-empty subsets of $\mathbb{N}_\infty$ as Pukánszky invariants of masas in $\mathcal{L}(\mathbb{F}_k)$. Very recently, Dykema, Sinclair and Smith have used a free product construction, [17], to show that any subset of $\mathbb{N}_\infty$ containing ∞ can be obtained as the Pukánszky invariant of some masa in $\mathcal{L}(\mathbb{F}_k)$. At present it is not known whether ∞ must always be present in such a Pukánszky invariant, or whether it is possible to construct a masa in $\mathcal{L}(\mathbb{F}_k)$ with invariant $\mathbb{N}$, say.

3.1.2 The Pukánszky invariant and normalisers

As we have just hinted at, there is a strong connection between the Pukánszky invariant and the classification of masas by their normalisers into singulars, semi-regulars and Cartans. Indeed the next theorem, which can be assembled from Corollary 3.2 and Remark 3.4 of [46][7], makes this connection clear.

Theorem 3.1.3 (Popa). *Let A be a masa in a separable II_1 factor N.*

1. *If A is Cartan, then $Puk(A) = \{1\}$.*
2. *If $\mathcal{N}(A) \not\subset A$, then $1 \in Puk(A)$.*
3. *If $1 \notin Puk(A)$, then A is singular.*

The converse to part 3 is not true, as can be seen from Pukánszky's example of a singular masa with invariant $\{1\}$ in [51]. It is worth asking whether the first part of Theorem 3.1.3 extends to finite length masas, i.e. if a A is a masa of finite length in R, must we then have $\mathrm{Puk}(A) = \{1\}$, or is $\mathrm{Puk}(A)$ necessarily bounded? If either of these statements hold, then Dykema's result in the previous section would imply that no finite length masa exists in a free group factor. Popa's work reinvigorated interest in the Pukánszky invariant, with various authors (see for example [52], [3]) using it as a method of demonstrating singularity of masas.

[6]For example, $\{1,2,3\}$ is attainable using the ergodic methods of [38].

[7]Unfortunately there are two results in [46] numbered 3.4 – the remark of interest to us is the first of these.

In subsection 1.4.4, we indicated that the Pukánszky invariant gives a positive answer to Question 1.4.29 when Theorem 3.1.3 applies. It is only necessary then, to address Question 1.4.29 in the case when 1 lies in the Pukánszky invariant of A or B.

Remark 3.1.4. Let A and B be singular masas with $1 \notin \mathrm{Puk}\,(A)$ and $1 \notin \mathrm{Puk}\,(B)$. Proposition 3.1.2 ensures that $1 \notin \mathrm{Puk}\,(A\overline{\otimes}B)$, so that $A\overline{\otimes}B$ is singular by part 3 of Theorem 3.1.3.

How good is the Pukánszky invariant at distinguishing between different singular masas? The ergodic methods of Størmer and Neshveyev, [38] allowed them to construct two singular masas in R with Pukánszky invariant $\{1\}$ which are not conjugate by an automorphism of R. They used the existence of centralising sequences for R inside these masas to distinguish between them. In section 3.3 we shall extend this idea further, to give uncountably many non-conjugate singular masas with the same Pukánszky invariant.

As discussed in [61], the relationship between the Pukánszky invariant and standard constructions other than the tensor product is not yet well understood. In the sequel we shall touch upon the possibility of such a relationship for direct sums. Given two masas A and B in the same II_1 factor N, we can form the II_1 factor, $\mathbb{M}\mathrm{at}_2(N) \cong N\overline{\otimes}\mathbb{M}\mathrm{at}_2(\mathbb{C})$, of 2×2 matrices over N. Define

$$A \oplus B = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a \in A, b \in B \right\},$$

a masa in $\mathbb{M}\mathrm{at}_2(N)$. Recall that when N is hyperfinite, then $\mathbb{M}\mathrm{at}_2(N)$ is also hyperfinite so we can view $A \oplus B$ as being another masa in R. In [61] the question of how to relate $\mathrm{Puk}\,(A \oplus B)$ to $\mathrm{Puk}\,(A)$ and $\mathrm{Puk}\,(B)$ was asked. At present only the trivial case when A and B are unitarily conjugate appears to be known.

3.1.3 Continuity properties of the Pukánskzy invariant

We end this section by examining how the Pukánszky invariant behaves under taking limits. The result, which is deduced from the perturbation work [50] of Popa, Sinclair and Smith, is joint work with Allan Sinclair, and will be given in [63].

Theorem 3.1.5. *Let A_n be a sequence of masas in a separable II_1 factor N converging in the $d_{\infty,2}$-metric of Definition 1.4.10 to a von Neumann subalgebra B of N. In this case B is a masa in N and*

$$\mathit{Puk}\,(B) \subset \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \mathit{Puk}\,(A_n)\,. \tag{3.1.1}$$

Proof. That the set of masas is $d_{\infty,2}$-closed dates back to Christensen ([4]), see Proposition 1.4.11. Find, by Corollary 1.4.16, masas B_n in N and projections $p_n \in B \cap B_n$ such that:

- A_n and B_n are unitarily equivalent in N;
- $B_n p_n = B p_n$ for all such n;
- $\lim_{n\to\infty} \|1 - p_n\|_2 = 0$.

Note that Corollary 1.4.16 is only used for those large n making $d_{\infty,2}(A_n, B)$ sufficiently small.[8] For the finite initial segment of small n, take $B_n = A_n$ and $p_n = 0$.

For any $x \in N$,

$$\begin{aligned}\|p_n J p_n J x - x\|_2 = \|p_n x p_n - x\|_2 &\leq \|p_n x - x\|_2 + \|p_n(x p_n - x)\|_2 \\ &\leq \|p_n - 1\|_2 \left(\|x\| + \|p_n x\|\right) \\ &\leq 2\,\|x\|\,\|p_n - 1\|_2\,,\end{aligned}$$

so that the projections $p_n J p_n J$ in $\mathcal{B}_n \cap \mathcal{B}$ converge strongly to 1, by density of N in $L^2(N)$.

Given some $m \in \operatorname{Puk}(B)$, there must be a central projection $q \in \mathcal{B} = \mathcal{B}' \cap \mathcal{B}$ with $q \leq 1 - e_B$, such that $\mathcal{B}'q$ is of type $\mathrm{I_m}$. As $p_n J p_n J q$ converges strongly to q we must have $p_n J p_n J q \neq 0$ for all n sufficiently large, those with $n \geq n_1$, say. For these n,

$$\mathcal{B}_n' p_n J p_n J = \mathcal{B}' p_n J p_n J,$$

a type I von Neumann algebra with centre $\mathcal{B}_n p_n J p_n J = \mathcal{B} p_n J p_n J$. For $n \geq n_1$, $p_n J p_n J q \neq 0$ lies in this centre and $\mathcal{B}_n' p_n J p_n J q$ is then a central cutdown of $\mathcal{B}'q$, a type $\mathrm{I_m}$ von Neumann algebra.

Now observe that p_n and $J p_n J$ commute with both e_B and e_{B_n}, as $p_n \in B \cap B_n$. We also have $p_n e_{B_n} = p_n e_B$ and $J p_n J e_{B_n} = J p_n J e_B$, as $B_n p_n = B p_n$. In this way, $p_n J p_n J q \leq 1 - e_{B_n}$, so that $m \in \operatorname{Puk}(B_n)$, for $n \geq n_1$. As B_n and A_n are unitarily equivalent, we have $m \in \operatorname{Puk}(A_n)$ for all $n \geq n_1$, exactly as required. □

In the special case when the Pukánszky invariant of each A_n is $\{n\}$, the only possibility for the Pukánszky invariant of the limit masa B is also $\{n\}$.

Corollary 3.1.6. *Let N be a separable II_1 factor. For each $n \in \mathbb{N}_\infty$, the set of all masas with Pukánszky invariant $\{n\}$ is closed with respect to the metric $d_{\infty,2}$.*

[8] Less than the δ given in the corollary.

It is not hard to produce examples showing we cannot in general have equality in (3.1.1). The method used to do this is essentially that of direct sums - we will see more of these in section 3.3.

Example 3.1.7. Let A be a masa in R with Pukánskzy invariant $\{1\}$. Find projections $p_n \in A$ with $p_n \to 1$ strongly and $p_n \neq 1$, for each n. For each n, let B_n be a masa in $(1-p_n)R(1-p_n)$ with Pukánszky invariant $\{2\}$.[9] Define

$$A_n = \{\, ap_n + b \,|\, a \in A, b \in B_n \,\},$$

a masa in R. These A_n converge to A in the $d_{\infty,2}$-metric by Proposition 1.4.14. It is immediate that 1 and 2 lie in each $\mathrm{Puk}\,(A_n)$, although our lack of understanding of the relationship between the Pukánszky invariant and direct sums prevents us from determining fully the Pukánskzy invariant of the masas A_n.

Finally, we use Theorem 3.1.5 to show that the Pukánszky invariant is not a genuinely continuous invariant of singular masas, in the sense of subsection 1.4.6. To do this we should note that we could have stated Theorem 3.1.5 in terms of convergent nets of masas - no essential changes to the proof would be required.

Corollary 3.1.8. *There does not exist a continuous map $t \mapsto A(t)$ from $[0,1]$ into the set of masas in some II_1 factor, equipped with the $d_{\infty,2}$-metric, such that each $A(t)$ has a distinct Pukánszky invariant.*

Proof. Let $t \mapsto A(t)$ be a continuous map from $[0,1]$ into the masas in some II_1 factor N equipped with the $d_{\infty,2}$-metric. A 'net-version' of Theorem 3.1.5 ensures that, for each $t \in (0,1)$, we are able to find an open interval I, containing t, such that $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t))$, whenever $s \in I$.

Suppose, by way of obtaining a contradiction, that $t \mapsto \mathrm{Puk}\,(A(t))$ is injective. Fix $t_0 \in (0,1)$ and find an open interval I_0 containing t_0, such that $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t_0))$, whenever $s \in I_0$. Choose $t_1 \in I_0$, with $t_1 \neq t_0$, and find an open interval $I_1 \subset I_0$ containing t_1, such that $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t_1))$, whenever $s \in I$. Next we find $t_2 \in I_1$, strictly between t_0 and t_1, with $|t_2 - t_1| < 1/2$, and an open interval $I_2 \subset I_1$, containing t_2 and such that $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t_2))$, whenever $s \in I_2$. Continue in this way to find distinct points $(t_n)_{n=0}^{\infty}$ and nested open intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that:

- $t_n \in I_n$;
- $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t_n))$, whenever $s \in I_n$;

[9] Such a masa exists, for example using Pukánszky's original examples, in [51], as $(1-p_n)R(1-p_n)$ is a hyperfinite II_1 factor.

- $|t_n - t_{n-1}| < 2^{1-n}$, for each $n \geq 1$;
- For $m > n > 0$, t_m lies between t_n and t_{n-1}.

The second condition guarantees that $(t_n)_{n=0}^{\infty}$ is Cauchy, so convergent to some $t \in [0,1]$. For each n, the interval between t_n and t_{n-1} is a closed subinterval of I_{n-1}, so the third condition ensures that $t \in \bigcap_{n=0}^{\infty} I_n$. Now find some open interval I, containing t such that $\mathrm{Puk}\,(A(s)) \supset \mathrm{Puk}\,(A(t))$, whenever $s \in I$. There is some n_0 such that $t_n \in I$, for $n \geq n_0$, and so $\mathrm{Puk}\,(A(t_n)) \supset \mathrm{Puk}\,(A(t))$, for these n. On the other hand, $t \in \bigcap_{n=0}^{\infty} I_n$, so we also have $\mathrm{Puk}\,(A(t_n)) \subset \mathrm{Puk}\,(A(t))$, for every n. The injectivity of $t \mapsto \mathrm{Puk}\,(A(t))$ then implies that $t_n = t$, for every $n \geq n_0$ - contradicting the requirement that the $(t_n)_{n=0}^{\infty}$ are distinct points in $[0,1]$. □

3.2 The Pukánszky invariant of a Tauer masa

Let A be a Tauer masa in R with respect to the subfactors $(N_n)_{n=1}^{\infty}$. Consider the algebras $\mathcal{A}_n = (A_n \cup JA_nJ)''$ generated by the actions of the approximation A_n on the left and right of $L^2(R)$. By counting dimensions it is clear that $\mathcal{A}_n$ is a masa in $\mathbb{B}\,(L^2(N_n))$. In the same way that the approximates A_n generate A, we have that $\mathcal{A}$ is generated, as a von Neumann algebra, by the $\mathcal{A}_n$ and so it is reasonable to hope that $\mathcal{A}$ is then necessarily a masa in $\mathbb{B}\,(L^2(R))$, or equivalently that $\mathrm{Puk}\,(A) = \{1\}$. This indeed turns out to be the case. The next result and its proof, making up subsection 3.2.1, are to appear in [77].

Theorem 3.2.1. *Let A be a Tauer masa in R, then A has Pukánszky invariant $\{1\}$.*

3.2.1 Proof of Theorem 3.2.1

It is well known, see for example [10, Theorem II.2.2], that when $\mathcal{A}$ is an abelian von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, then $\mathcal{A}$ is a masa in $\mathbb{B}(\mathcal{H})$ if and only if it has a cyclic vector, by which we mean that there is some $\xi \in \mathcal{H}$ with $\overline{\mathcal{A}\xi} = \mathcal{H}$. We shall prove Theorem 3.2.1 by giving an algorithm for the construction of such a cyclic vector for $\mathcal{A}$ in $L^2(R)$, when A is a Tauer masa in R. We prefer this to making the brief discussion preceding the theorem rigorous, as a method for producing cyclic vectors is more likely to be of use elsewhere. We shall use the following easy observation repeatedly, so record it as a proposition.

Proposition 3.2.2. *Let A be a masa in a type I_k factor N. If X is a finite subset of N such that*

$$Span\,(AxA) \perp y, \tag{3.2.1}$$

whenever $x, y \in X$ with $x \neq y$, then given minimal projections e_1 and e_2 for A, there exists at most one $x \in X$ with

$$e_1 x e_2 \neq 0.$$

Proof. Take matrix units $(e_{i,j})_{i,j=1}^k$ for N such that $(e_{i,i})_{i=1}^k$ are the minimal projections of A. Given $x, y \in X$, write $x = \sum_{i,j=1}^k x_{i,j} e_{i,j}$ and $y = \sum_{i,j=1}^k y_{i,j} e_{i,j}$, for some scalars $x_{i,j}, y_{i,j} \in \mathbb{C}$. Note that for any i, j we have

$$0 = \mathrm{tr}(e_{i,i} x e_{j,j} y^*) = \|e_{i,i}\|_2^2 \, x_{i,j} \overline{y_{i,j}},$$

so one of $e_{i,i} x e_{j,j} = x_{i,j} e_{i,j}$, or $e_{i,i} y e_{j,j} = y_{i,j} e_{i,j}$, must be zero. □

The first use of this uniqueness is to demonstrate that we can test for the orthogonality of $\mathcal{A}$-cyclic subspaces generated by elements of the chain of subfactors $(N_n)_{n=1}^\infty$.

Proposition 3.2.3. *Let A be a Tauer masa with respect to the subfactors $(N_n)_{n=1}^\infty$. For $x, y \in N_n$ with*

$$Span\,(A_n x A_n) \perp y. \tag{3.2.2}$$

we have

$$\overline{\mathcal{A}x} \perp \overline{\mathcal{A}y}. \tag{3.2.3}$$

Proof. Suppose that x and y lie in N_n and satisfy (3.2.2). Now, given $m > n$ and minimal projections f_1, f_2 in A_m, find minimal projections e_1 and e_2 for A_n with $f_1 \leq e_1$ and $f_2 \leq e_2$. By Proposition 3.2.2, either $e_1 x e_2 = 0$ or $e_1 y e_2 = 0$. In either case

$$\mathrm{tr}(f_1 x f_2 y^*) = \mathrm{tr}(f_1 (e_1 x e_2) f_2 (e_1 y e_2)^* f_1) = 0,$$

and so by linearity

$$\mathrm{Span}\,(A_m x A_m) \perp y.$$

It is then clear that

$$\mathrm{Span}\,((\cup_{m=1}^\infty A_m) x (\cup_{m=1}^\infty A_m)) \perp y,$$

and taking the closure gives $\overline{\mathcal{A}x} \perp y$. If a_1, a_2 lie in $\mathcal{A} \subset \mathbb{B}(L^2(R))$ then

$$\langle a_1 x, a_2 y \rangle_{L^2(R)} = \langle a_2^* a_1 x, y \rangle_{L^2(R)} = 0,$$

establishing (3.2.3). □

We now start to construct our cyclic vector for $\mathcal{A}$, operating initially in the subfactors $(N_n)_{n=1}^\infty$.

Lemma 3.2.4. *Let A be a Tauer masa with respect to the subfactors $(N_n)_{n=1}^{\infty}$. There exists a sequence $(x_n)_{n=1}^{\infty}$ in R with the following properties:*

(i) $x_i \in N_i$, *for each* i;

(ii) $\|x_i\|_2 = 1$ *or* $x_i = 0$ *for each* i;

(iii) *Span* $(A_j x_i A_j) \perp x_j$, *whenever* $i < j$;

(iv) $\bigoplus_{i=1}^{l}$ *Span* $(A_l x_i A_l) = N_l$, *for each* l.

Proof. We proceed by induction. Take matrix units $(e_{i,j}^{(1)})_{i,j=1}^{k}$ for N_1 with

$$A_1 = \mathrm{Span}\left(e_{1,1}^{(1)}, \ldots, e_{k,k}^{(1)}\right),$$

then

$$x_1 = \frac{1}{k}\sum_{i,j=1}^{k} e_{i,j}^{(1)}$$

satisfies the requirements. Assume inductively that we have already found $x_1, \ldots, x_n$. Let X be a maximal family of elements of N_{n+1} with $\|x\|_2 = 1$ for each $x \in X$ such that

$$\mathrm{Span}\left(A_{n+1} x A_{n+1}\right) \perp x_i, \tag{3.2.4}$$

for $x \in X, i = 1, 2, \ldots n$ and

$$\mathrm{Span}\left(A_{n+1} x A_{n+1}\right) \perp y,$$

whenever x, y lie in X with $x \neq y$. If X is non-empty, we take

$$x_{n+1} = \frac{1}{\sqrt{|X|}}\sum_{x \in X} x,$$

otherwise we take $x_{n+1} = 0$. Either way x_{n+1} satisfies (i) and (ii). Condition (iii) follows from (3.2.4), so it remains to establish condition (iv) for $l = n+1$. By Proposition 3.2.3, we have

$$\mathrm{Span}\left(A_{n+1} x_i A_{n+1}\right) \perp \mathrm{Span}\left(A_{n+1} x_j A_{n+1}\right),$$

whenever $1 \leq i < j \leq n+1$, and so the direct sum

$$\bigoplus_{i=1}^{n+1} \mathrm{Span}\left(A_{n+1} x_i A_{n+1}\right)$$

exists. If this direct sum of is not the whole of N_{n+1}, then we would be able to find an element y in N_{n+1} orthogonal to it. In particular y is then orthogonal to each Span $(A_{n+1} x_i A_{n+1})$ for $i = 1, 2, \ldots, n$ and, using Proposition 3.2.2, orthogonal to each Span $(A_{n+1} x A_{n+1})$ for $x \in X$, which contradicts the maximality of X. $\square$

The following lemma now completes the proof of Theorem 3.2.1 by exhibiting the required cyclic vector.

Lemma 3.2.5. *Let A be a Tauer masa with respect to the type I_{k_n} subfactors $(N_n)_{n=1}^{\infty}$. Write $A_n = A \cap N_n$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in R satisfying the conditions of Lemma 3.2.4. Then*

$$\xi = \sum_{n=1}^{\infty} \frac{1}{k_n^3} x_n \in L^2(R)$$

is a cyclic vector for $\mathcal{A} = (A \cup JAJ)''$.

Proof. Since $k_n \geq 2^n$, the sum defining ξ converges. Fix $n \geq 1$ and matrix units $(e_{i,j})_{i,j=1}^{k_n}$ for N_n, such that the projections $(e_{i,i})_{i=1}^{k_n}$ generate A_n. We shall demonstrate that $e_{i,j}$ lies in the closure of $\mathcal{A}\xi$, for each i and j, the result then follows immediately from the density of $\cup_{n=1}^{\infty} N_n$ in $L^2(R)$.

Fix i and j, then Proposition 3.2.2 and property (iv) of the (x_n) combine to show that there exists a unique $l_0 \in \{1, \ldots, n\}$ with $e_{i,i} x_{l_0} e_{j,j} = \lambda e_{i,j}$ for some constant $\lambda \neq 0$. Let $m > n$, and write $(f_{i,s}^{(m)})_{s=1}^{k_m/k_n}$ for the minimal projections in A_m underneath $e_{i,i}$. Define $a_m \in \mathcal{A}$ by

$$a_m = \sum_{(s,t) \in S_m} f_{i,s}^{(m)} J f_{j,t}^{(m)} J,$$

where

$$S_m = \left\{ (s,t) \,\middle|\, 1 \leq s,t \leq k_m/k_n \text{ and } f_{i,s}^{(m)} x_{l_0} f_{j,t}^{(m)} \neq 0 \right\}.$$

Observe that $\|a_m\| \leq |S_m| \leq (k_m/k_n)^2$ and

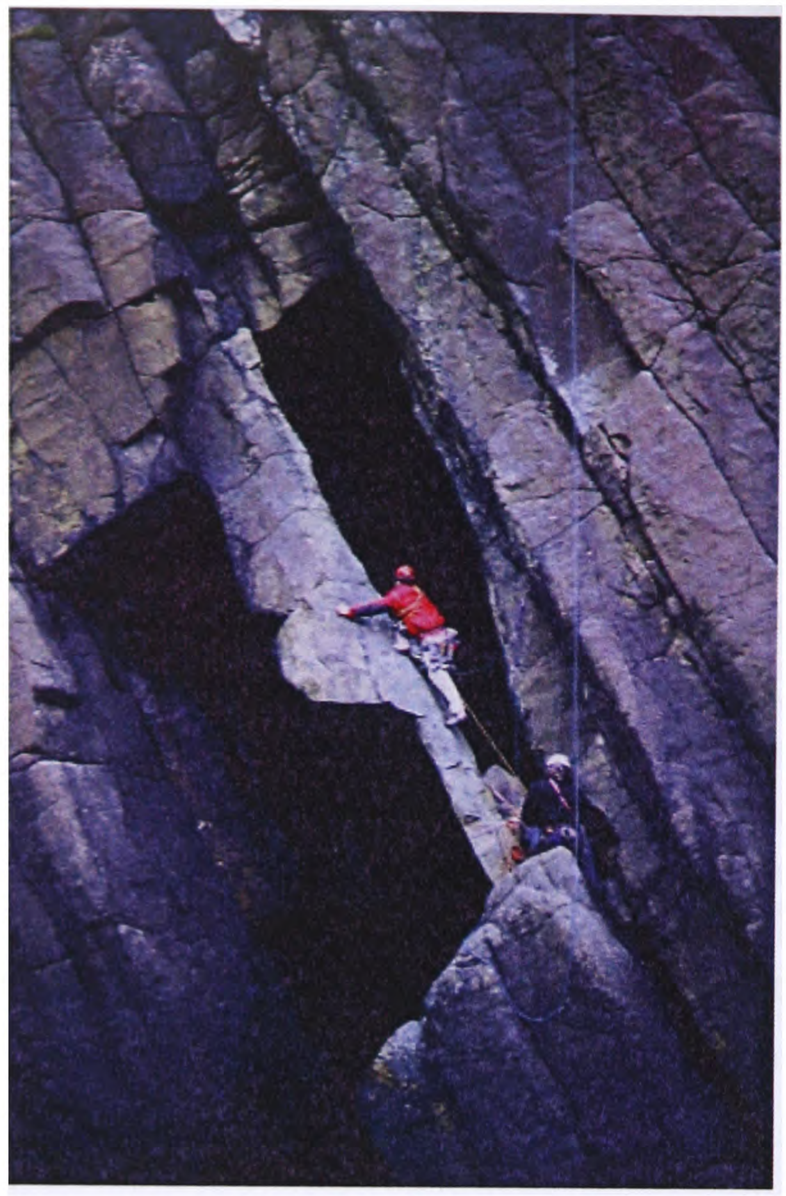
$$a_m \left(\sum_{l=1}^{m} k_m^{-3} x_m \right) = k_{l_0}^{-3} a_m(x_{l_0}) = k_{l_0}^{-3} e_{i,i} x_{l_0} e_{j,j} = \lambda k_{l_0}^{-3} e_{i,j},$$

by applying the uniqueness result of Proposition 3.2.2. In particular

$$\begin{aligned} \left\| a_m(\xi) - \lambda k_{l_0}^{-3} e_{i,j} \right\|_2 &\leq \|a_m\| \left\| \sum_{r=m+1}^{\infty} k_r^{-3} x_r \right\|_2 \\ &\leq \frac{k_m^2}{k_n^2} \sum_{r=m+1}^{\infty} k_r^{-3} \\ &\leq \frac{1}{k_n^2} \sum_{r=m+1}^{\infty} k_r^{-1} \end{aligned}$$

which converges to zero as $m \to \infty$ since $k_r \geq 2^r$. Hence, $e_{i,j}$ does indeed lie in $\overline{\mathcal{A}\xi}$, as required. $\square$

Calculating Spans



Spantastic (HVS 4c), Flodigarry, Skye, 2003.

Photo: Jon Powell

3.2.2 Approximating masas by matrices

The next very simple observation allows us to make a minute step towards understanding the Pukánszky invariant of direct sums of masas. The idea is, however, just what we shall need in section 3.3.

Remark 3.2.6. If A and B are Tauer masas in R *with respect to the same chain of subfactors* $(N_n)_{n=1}^{\infty}$ then $A \oplus B$ is a Tauer masa with respect to the chain $(N_n \otimes \mathbb{M}\mathrm{at}_2(\mathbb{C}))_{n=1}^{\infty}$ generating $R\overline{\otimes}\mathbb{M}\mathrm{at}_2(\mathbb{C})$. In particular, $A \oplus B$ has Pukánszky invariant $\{1\}$.

If we have two masas which are Tauer in R but with respect to different chains of subfactors, then we can not necessarily proceed in the crude manner of the preceding remark. Indeed, we do not know whether the direct sum of a Tauer masa constructed in the infinite tensor product of the 2×2 matrices and a Tauer masa constructed in the infinite tensor product of the 3×3 matrices is in general a Tauer masa, or whether it must have Pukánszky invariant $\{1\}$.

There is a related problem here. Given a Tauer masa A in R, which chains of generating subfactors for R make A Tauer? More formally, we define a *Glimm power* type invariant for Tauer masas.

Definition 3.2.7. Let $(N_n)_{n=1}^{\infty}$ be a chain of finite type I_{k_n} factors, with $N_n \subset N_{n+1}$ being a unital embedding. For a prime p, as $k_1|k_2|k_3|\ldots$, we can define $\epsilon_p \in \mathbb{N}_{\infty}$ by

$$\epsilon_p = \sup\{\, m \,|\, p^m \text{ divides } k_n \text{ for some } n \,\}.$$

The formal product

$$\mathrm{Glimm}\left((N_n)_{n=1}^{\infty}\right) = \prod_{p \text{ prime}} p^{\epsilon_p},$$

was shown by Glimm[10] in [19], to be a complete *-isomorphism invariant for the uniformly hyperfinite C^*-algebra obtained as the direct limit of $(N_n)_{n=1}^{\infty}$; see also [10, Theorem III.5.2] whose notation we have used.

Given a Tauer masa A in R, define $\mathrm{Glimm}(A)$ to be the set consisting of all $\mathrm{Glimm}\left((N_n)_{n=1}^{\infty}\right)$ whenever $(N_n)_{n=1}^{\infty}$ is a chain of type I factors generating R such that A is a Tauer masa with respect to $(N_n)_{n=1}^{\infty}$. This is evidentally a conjugacy invariant of A as a masa in R.

Our problem then becomes to determine $\mathrm{Glimm}(A)$ for a Tauer masa A. We only have an answer in the most trivial case, namely that of the Cartan masa D in R. Here, Connes, Feldman and Weiss' result on the uniqueness of the Cartan masa in R up to conjugacy by an automorphism ([8], see also [46]) allows us to see that $\mathrm{Glimm}(D)$ consists of the set of all allowed formal products.[11]

Motivated by Corollary 3.1.6, which shows that the set of all masas in R with Pukánszky invariant $\{1\}$ is closed in the $d_{\infty,2}$-metric, we ask, 'What about the Tauer masas?' It seems unlikely that these masas should be closed as the definition involves the existence of an appropriate chain of subfactors $(N_n)_{n=1}^{\infty}$. It might be possible to construct a sequence of Tauer masas which converges but the underlying generating chains change sufficiently that the limit masa can no longer be Tauer. Currently however, we have no methods other than examining the Pukánszky invariant for showing a masa is not Tauer, which suggests the following question.

Question 3.2.8. Working with the $d_{\infty,2}$-metric on von Neumann subalgebras of R, is the closure of the Tauer masas the set of all masas with Pukánszky invariant $\{1\}$?

[10] who interestingly was Tauer's PhD supervisor.

[11] i.e. those with $\prod_{p \text{ prime}} p^{\epsilon_p} = \infty$.

Not all masas in R are Tauer masas, but this does mean we can not hope to use matrices to get a handle on other masas. Let $(N_n)_{n=1}^{\infty}$ be a chain of matrix algebras generating the hyperfinite II_1 factor. Given a masa A_n we can take $A_n = A \cap N_n$, an increasing chain of finite dimensional subalgebras of A, which might be identically trivial. If A is not Tauer with respect to $(N_n)_{n=1}^{\infty}$, then we can not have $\mathbb{E}_{A_n}(x) = \mathbb{E}_{A_n' \cap R}(x)$ whenever $x \in N_n$, as having this condition for infinitely many n is enough to get a Tauer masa. We can hope though, to find $k_n > n$ such that $\mathbb{E}_{A_{k_n}}(x) = \mathbb{E}_{A_{k_n}' \cap R}(x)$ whenever $x \in N_n$ and, by omitting unnecessary stages, we may assume that $k_n = n + 1$ for each n. In this case $\mathbb{E}_{A_m}(x) = \mathbb{E}_{A_m' \cap R}(x) = \mathbb{E}_A(x)$, whenever x lies in N_n and $m > n$. By density we have

$$\lim_{n\to\infty} \left\| \mathbb{E}_{A_n}(x) - \mathbb{E}_{A_n' \cap R}(x) \right\|_2 = 0,$$

for all $x \in R$ which is precisely the condition appearing in Proposition 1.4.3 for the A_n to generate A. This situation is summerised in Figure 3.1.

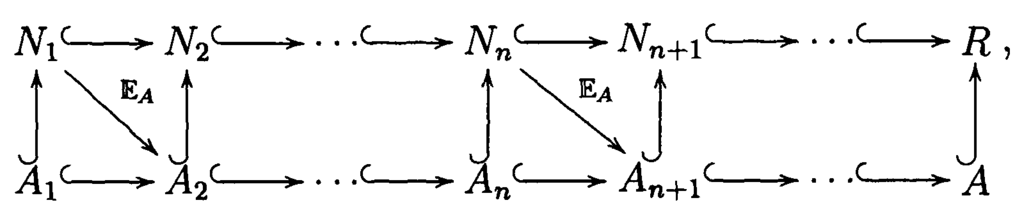


Figure 3.1: A masa A approximated in a chain $(N_n)_{n=1}^{\infty}$

Definition 3.2.9. Given a chain of matrix algebras $(N_n)_{n=1}^{\infty}$ generating the hyperfinite II_1 factor R, say that a masa A in R is *approximated in the* $(N_n)_{n=1}^{\infty}$ when we have $\mathbb{E}_{A\cap N_{n+1}}(x) = \mathbb{E}_{(A\cap N_{n+1})'\cap R}(x)$ for all $x \in N_n$. Say that A is *approximated by matrices* if there is some chain of matrix algebras $(N_n)_{n=1}^{\infty}$ generating R such that A is approximated in this chain. In this case, we write $A_n = N_n \cap A$.

A reasonable project for further investigation of the possible values of the Pukánszky invariant in R would be to calculate the Pukánszky invariant for some masas approximated by matrices. Time has not allowed this be done, but we can give a result showing that 'up to an ϵ' all masas in R are approximated by matrices. The ϵ involved arises as the projection of elements in N_n onto A_{n+1} is not guaranteed to agree with the projection onto A - only be a good approximation to it.

Theorem 3.2.10. *Let A be a masa in the hyperfinite* II_1 *factor R, and let $(\epsilon_n)_{n=1}^{\infty}$ be a sequence of strictly positive reals. There exists a chain $(M_n)_{n=1}^{\infty}$ of matrix algebras with*

$$\left(\bigcup_{n=1}^{\infty} M_n\right)'' = R,$$

such that, upon setting $A_n = M_n \cap A$, we have

$$\left\| \left(\mathbb{E}_{A_{n+1}} - \mathbb{E}_{A'_{n+1}\cap R} \right) \mathbb{E}_{M_n} \right\|_{\infty,2} \leq \epsilon_n,$$

for each n.

If one could control the Pukánszky invariant of masas approximated by matrices then one should be able to use this result to extend this control to determine the possible values of the invariant for masas in R. Another possible use of this theorem is to investigate Question 3.2.8 - if we also know that a masa has Pukánszky invariant $\{1\}$, can we deduce anything else from Theorem 3.2.10?

To prove Theorem 3.2.10, we use the methods of Murray and von Neumann developed ([37]) to show that the hyperfinite II_1 factor is unique up to conjugacy. We have already stated the technical lemmas we need in subsection 1.2.2.

Lemma 3.2.11. *Let A be a masa in R and $(\epsilon_n)_{n=1}^{\infty}$ be a given sequence of positive reals. Fix an increasing sequence of finite dimensional subspaces $(V_n)_{n=1}^{\infty}$ of R with $(\bigcup_{n=1}^{\infty} V_n)'' = R$. Let $(\delta_n)_{n=1}^{\infty}$ be a sequence of strictly positive reals converging to zero.*

There exist $N_1 \subset N_2 \subset \cdots \subset N_n \subset \ldots$, unitaries $(u_n)_{n=1}^{\infty}$ in R, and $B_1 \subset B_2 \subset \cdots \subset B_n \subset \ldots$ such that, for each r, we have:

1. *N_r is a type $\mathrm{I}_{2^{p_r}}$ subfactor of R containing B_r, with*
$$\|(I - \mathbb{E}_{N_r})\,\mathbb{E}_{V_r}\|_{\infty,2} \leq \delta_r;$$
2. *B_r is a 2^{q_r}-dimensional abelian *-subalgebra of $u_r \ldots u_1 A u_1^* \ldots u_r^*$ with equivalent minimal projections in R;*
3. *$u_r \in B'_{r-1}$;*
4. *$\|u_n - 1\|_2 \leq 2^{-n}$;*
5. $$\left\| \left(\mathbb{E}_{B_{r+1}} - \mathbb{E}_{B'_{r+1}\cap R} \right) \mathbb{E}_{N_r} \right\|_{\infty,2} \leq \epsilon_n.$$

Proof. Define $B_1 = \mathbb{C}1$ and $u_1 = 1$. By Lemma 1.2.6, we can find a type $\mathrm{I}_{2^{p_1}}$ subfactor $N_1 \subset R$ with

$$\|(I - \mathbb{E}_{N_1})\,\mathbb{E}_{V_1}\|_{\infty,2} \leq \delta_1.$$

Suppose inductively that we have found all the required objects upto the n-th stage. We start the construction of N_{n+1}, B_{n+1} and u_{n+1} by finding some $2^{q_{n+1}}$-dimensional abelain subalgebra C with equivalent minimal projections, such that $B_n \subsetneq C \subset u_n \dots u_1 A u_1^* \dots u_n^*$. By Proposition 1.4.3, applied to the masa $u_n \dots u_1 A u_1^* \dots u_n^*$, we can demand that

$$\|(\mathbb{E}_C - \mathbb{E}_{C' \cap R}) \mathbb{E}_{N_n}\|_{\infty,2} \leq \frac{\epsilon_n}{3}. \tag{3.2.5}$$

Let $l = q_{n+1} - q_n \geq 1$, set $D_0 = B_n$ and $M_0 = N_n$. We shall find $D_0 \subset D_1 \subset D_2 \dots \subset D_l = C$, $M_0 \subset M_1 \subset M_2 \subset \dots \subset M_l$ and unitaries $v_1, \dots, v_l \in R$ with:

i D_i is a 2^{q_n+i} dimensional *-algebra with equivalent minimal projections;

ii M_i is a type $\mathrm{I}_{2^{r_i}}$ subfactor of R;

iii v_i lies in $(v_{i-1} \dots v_1 D_{i-1} v_1^* \dots v_{i-1}^*)'$ and has $\|v_i - 1\|_2 \leq \min\{2^{-(n+1)}, \epsilon_n/12\}/l$;

iv $v_i v_{i-1} \dots v_1 D_i v_1^* \dots v_{i-1}^* v_i^* \subset M_i$.

Suppose this has been achieved, define $u_{n+1} = v_1^* \dots v_l^*$ which commutes with $D_0 = B_n$, and has

$$\|u_{n+1} - 1\|_2 \leq \sum_{i=1}^{l} \|v_i - 1\|_2 \leq \min\{2^{-(n+1)}, \epsilon_n/12\},$$

so conditions 3 and 4 hold. Let $B_{n+1} = u_{n+1} C u_{n+1}^*$, which contains B_n and is contained in $u_{n+1} \dots u_1 A u_1^* \dots u_{n+1}^*$ so satisfies condition 2. Observe that

$$\begin{aligned}
&\left\| \left(\mathbb{E}_{B_{n+1}} - \mathbb{E}_{B'_{n+1} \cap R}\right) \mathbb{E}_{N_n} \right\|_{\infty,2} \\
&\leq \|(\mathbb{E}_C - \mathbb{E}_{C' \cap R}) \mathbb{E}_{N_n}\|_{\infty,2} + \|\mathbb{E}_{B_{n+1}} - \mathbb{E}_C\|_{\infty,2} + \left\|\mathbb{E}_{B'_{n+1} \cap R} - \mathbb{E}_{C' \cap R}\right\|_{\infty,2} \\
&\leq \frac{1}{3}\epsilon_n + 4\|u_{n+1} - 1\|_2 + 4\|u_{n+1} - 1\|_2 \leq \epsilon_n,
\end{aligned}$$

verifying condition 5. Finally, we use hyperfiniteness in the form of Lemma 1.2.6 to find $N_{n+1} \supset M_l \supset u_{n+1}^* C u_{n+1}$ which satisfies condition 1.

Now, we must perform the second induction. Suppose we have completed the $(i-1)$-th stage for some $1 \leq i \leq l$. Temporarily fix a minimal projection $e \in v_{i-1} \dots v_1 D_{i-1} v_1^* \dots v_{i-1}^*$. Choose projections e_1 and e_2 in $v_{i-1} \dots v_1 C v_1^* \dots v_{i-1}^*$ with $e = e_1 + e_2$ and $\mathrm{tr}(e_1) = \mathrm{tr}(e_2)$. For $\eta > 0$, to be specified later, use Lemma 1.2.6 to find $M_i \supset M_{i-1}$ such that

$$\left\|(I - \mathbb{E}_{M_i}) \mathbb{E}_{v_{i-1} \dots v_1 C v_1^* \dots v_{i-1}^*}\right\|_{\infty,2} \leq \eta.$$

Set $h = \mathbb{E}_{M_i}(e_1)$ which then has $\|e_1 - h\|_2 \leq \eta$. We can crudely estimate

$$\begin{aligned}
\|h - h^2\|_2 &\leq \|h - e_1\|_2 + \|(e_1)^2 - h^2\|_2 \\
&\leq (1 + \|e_1\| + \|h\|)\|h - e_1\|_2 \leq 3\eta.
\end{aligned}$$

Provided $3\eta < 1/4$, which we can certainly insist upon, Lemma 1.2.5 shows that

$$\|h - g\|_2 \leq 2\sqrt{3\eta},$$

when g is the spectral projection of h corresponding to the interval $[1 - \sqrt{3\eta}, 1]$, which lies in M_i.

We now adjust g so that it has the same trace as e_1. Let f_1 be a projection in M_i with $\mathrm{tr}(f_1) = \mathrm{tr}(e_1)$ and such that, either $f_1 \leq g$ or $f_1 \geq g$, so that $\|f_1 - g\|_2 = \sqrt{|\mathrm{tr}(g - e_1)|}$. We have

$$\begin{aligned} \big\| |e_1 - f_1| \big\|_2 &= \|e_1 - f_1\|_2 \\ &\leq \|e_1 - g\|_2 + \|g - f_1\|_2 \\ &= \|e_1 - g\|_2 + \sqrt{|\mathrm{tr}(g - e_1)|} \\ &\leq \|e_1 - g\|_2 + \sqrt{\|e_1 - g\|_2} \leq \eta', \end{aligned}$$

where

$$\eta' = (2\sqrt{3\eta} + \eta) + \sqrt{2\sqrt{3\eta} + \eta}.$$

Work in the II_1 factor eRe and find, by Lemma 1.2.4, a unitary $w_e \in eRe$ with $w_e e_1 w_e^* = f_1$ and $|w_e - e| \leq \sqrt{2}\,|e_1 - f_1|$. Set $f_2 = e - f_1$ so that $w_e e_2 w_e^* = f_2$.

Do this for each minimal projection $e \in v_{i-1} \dots v_1 D_{i-1} v_1^* \dots v_{i-1}{}^*$. The chosen projections in $v_{i-1} \dots v_1 C v_1^* \dots v_{i-1}^*$ are used to define $v_{i-1} \dots v_1 D_i v_1^* \dots v_{i-1}^*$ and so D_i. By construction, all the minimal projections of D_i have the same trace. Let $v_i = \sum_e w_e$, the sum of all the unitaries produced above. By construction, v_i commutes with $v_{i-1} \dots v_1 D_{i-1} v_1^* \dots v_{i-1}^*$, and we have $v_i \dots v_1 D_i v_1^* \dots v_i{}^* \subset M_i$. Finally

$$\|v_i - 1\|_2 \leq \sum_e \big\| |w_e - e| \big\|_2 \leq 2^{q_n + i - 1}\sqrt{2}\eta',$$

so we are done if we choose η so that this last quantity is small enough, that is less than $\min\{2^{-(n+1)}, \epsilon_n/12\}/l$. □

Proof of Theorem 3.2.10. Let A be a masa in R. Given a sequence $(\epsilon_n)_{n=1}^\infty$ of strictly positive reals, fix an increasing sequence of finite dimensional subspaces $(V_n)_{n=1}^\infty$ of R with $(\bigcup_{n=1}^\infty V_n)'' = R$. Let $(\delta_n)_{n=1}^\infty$ be a sequence of strictly positive reals converging to zero.

Find $(N_n)_{n=1}^\infty$, $(u_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ as in Lemma 3.2.11. Condition 1 ensures that $(\bigcup_{n=1}^\infty N_n)'' = R$. Let $B = (\bigcup_{n=1}^\infty B_n)''$, which is a masa in R by condition 5 and Proposition 1.4.3. Furthermore, we have $B_n \subset N_n$, so condition 5 shows that the masa B is expressed exactly in the form with respect to the N_n that we wish our A to be. It will then be enough to show that B is conjugate to A via an automorphism of R. For suppose that $\theta \in \text{Aut}(R)$ has $\theta(B) = A$, then take $M_n = \theta(N_n)$, and $A_n = A \cap M_n$.

Our automorphism θ arises as the limit of Ad $u_1^* \dots u_n^*$ in the u-topology on automorphisms. For each $x \in R$, the bounded sequence $(u_1^* \dots u_n^* x u_n \dots u_1)_{n=1}^\infty$ is $\|.\|_2$-Cauchy, for

$$\begin{aligned}\left\|u_1^* \dots u_n^* x u_n \dots u_1 - u_1^* \dots u_{n+1}^* x u_{n+1} \dots u_1\right\|_2 &= \|x - u_{n+1}{}^* x u_{n+1}\|_2 \\ &\leq 2\|x\| \|u_{n+1} - 1\|_2 \leq 2^{-n}\|x\|.\end{aligned}$$

The sequence then converges in $L^2(R)$ to some $\theta(x)$. This $\theta(x)$ actually lies in R, as the unit ball in a separable II_1 factor is $\|.\|_2$-closed. This θ then defines an injective *-homomorphism of R into R. To see that it is an automorphism, we must check that it is invertible. This is immediate, as for each $x \in R$, $(u_n \dots u_1 x u_1^* \dots u_n^*)_{n=1}^\infty$ converges to some $\phi(x) \in R$ exactly as before. This ϕ is also a *-homomorphism which is obviously θ^{-1}.

Observe that

$$\theta(B_n) = u_1^* \dots u_n^* B_n u_n \dots u_1 \subset A,$$

by condition 2, as u_m commutes with B_n for $m > n$, by condition 3. Hence $\theta(B) \subset A$, but as both A and B are masas, we have $\theta(B) = A$, exactly as required. □

The matrices produced by this proof have $\text{Glimm}\left((N_n)_{n=1}^\infty\right) = 2^\infty$. Only minor modifications would have been required to obtain any other legal Glimm invariant, so Theorem 3.2.10 might also be useful in investigating the range of Glimm-invariants of a Tauer masa, as discussed earlier.

3.3 Uncountably many singular masas with the same Pukánszky invariant

Corollary 2.4.1, giving us uncountably many semi-regular Tauer masas no pair of which are conjugate by an automorphism of R, combines with Theorem 3.2.1 to ensure that each of these have Pukánszky invariant $\{1\}$. We will not be able to generalise this idea to produce uncountably many pairwise non-conjugate singular masas with the same Pukánszky invariant, as the distinguishing feature of the masas of Corollary 2.4.1 was the difference in the structure of the normalising algebras $\mathcal{N}(A)''$. Parts of the next three subsections, including Theorem 3.3.5, will appear in [63].

3.3.1 The plan

We look instead for a different idea. Folklore gives another method of obtaining uncountably many pairwise non-conjugate masas in the hyperfinite II_1 factor. If we take a projection $p \neq 0, 1$ in R then both pRp and $(1-p)R(1-p)$ are hyperfinite II_1 factors so isomorphic to R.[12] Take a Cartan masa A_1 inside pRp and a singular masa A_2 inside $(1-p)R(1-p)$ and consider the abelian von Neumann subalgebra A in R generated by A_1 and A_2. This is necessarily a masa in R, which when $\mathrm{tr}(p) = 1/2$ is the direct sum of A_1 and A_2 as mentioned at the end of section 3.1. We regard A as the $\mathrm{tr}(p)$-direct sum of A_1 and A_2 in R, which we view pictorially in Figure 3.2, and occasionally write as $A_1 \oplus_{\mathrm{tr}(p)} A_2$.

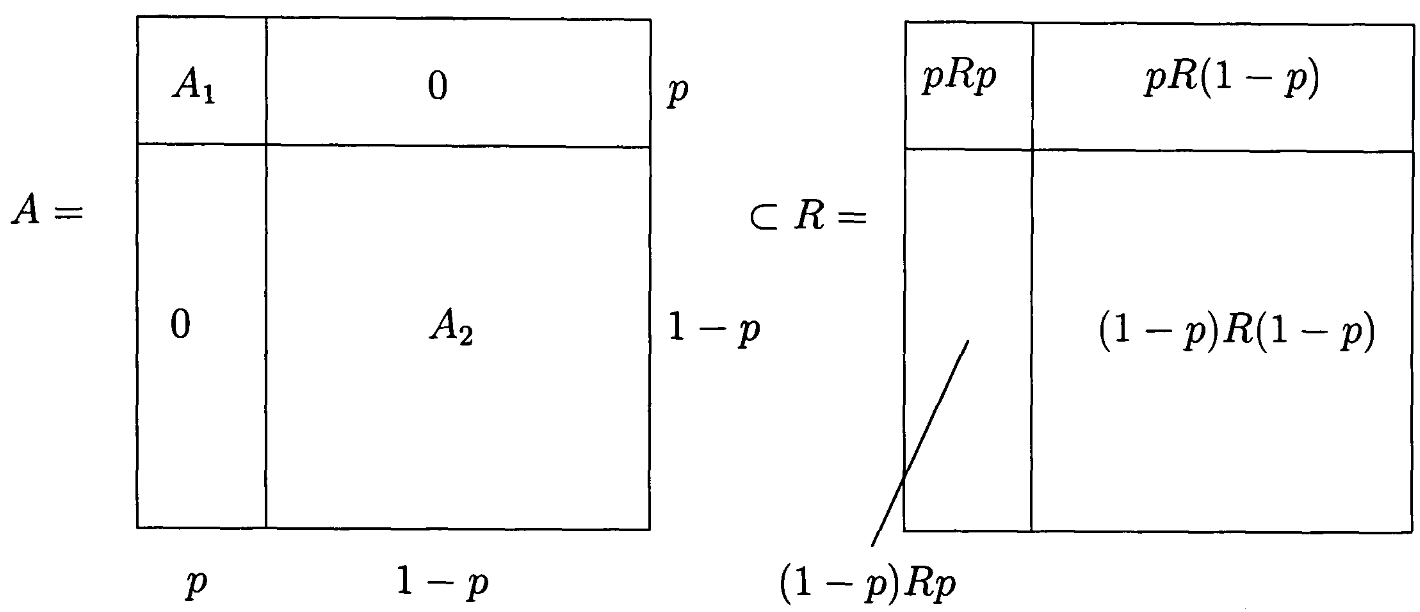


Figure 3.2: The masa A inside R.

We can immediately compute $\mathcal{N}(A)''$ as expressed in Figure 3.3. Observe that p is the only non-trivial minimal projection in the centre of $\mathcal{N}(A)''$. Take another projection $q \neq 0, 1$ in R and construct a masa B as the direct sum of a Cartan masa B_1 in qRq and a singular masa B_2 in $(1-q)R(1-q)$. If $\mathrm{tr}(p) \neq \mathrm{tr}(q)$ then A can not be conjugate to B via an automorphism of R, as $\mathcal{N}(A)''$ is certainly not conjugate to $\mathcal{N}(B)''$ via any such automorphism.

It is possible to introduce an invariant demonstrating this lack of automorphic equivalence, namely for a masa A in a II_1 factor N

$$\sup\{\, \mathrm{tr}(p) \mid p \text{ is a projection in } A \text{ with } Ap \text{ a Cartan masa in } pNp \,\} \qquad (3.3.1)$$

is a conjugacy invariant of A lying in $[0, 1]$.[13] It is immediate that when A is the $\mathrm{tr}(p)$-direct sum of A_1 and A_2 as described above, then this invariant is $\mathrm{tr}(p)$. In

[12] This is the well known result that the fundamental group of R is all of $\mathbb{R}_+$, but we have no plans to involve ourselves in a potentially lengthy discussion of fundamental groups of von Neumann algebras here.

[13] This gives a genuinely continuous invariant of masas, in the sense of subsection 1.4.6, but of course not one for the singular masas - where it is identically zero.

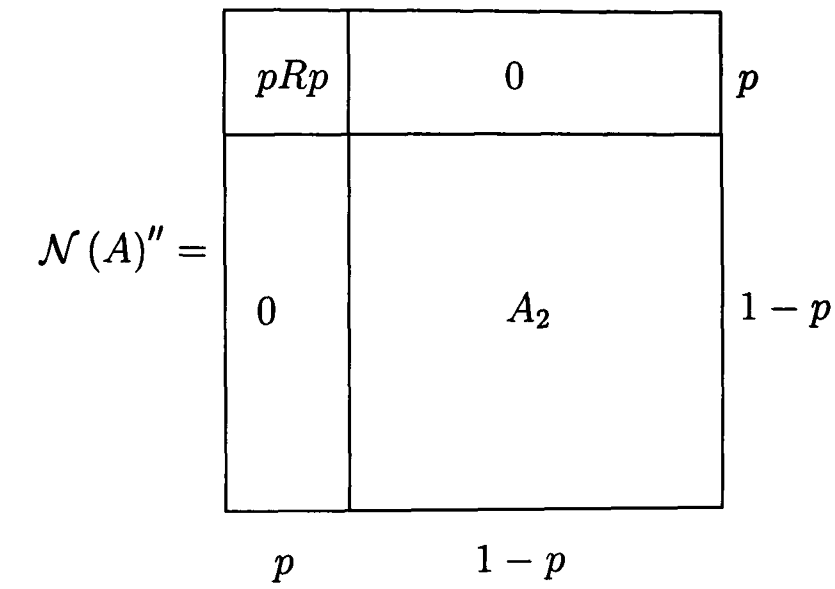


Figure 3.3: Pictorial form of $\mathcal{N}(A)''$

this way we obtain uncountably many masas in R no pair of which is conjugate via an automorphism of R.

Of course, there are two problems here in using this method to produce uncountably many pairwise non-conjugate singular masas in R with Pukánszky invariant $\{1\}$, namely:

1. these masas are not singular;

2. we can not in general control the Pukánszky invariant of a direct sum of masas.

The second difficulty is outflanked by using Tauer masas in the manner suggested by Remark 3.2.6. The first problem would seem more fundamental, but the method above allows us to produce an uncountable infinity of pairwise non-conjugate masas in R whenever we can construct two masas A_1 and A_2 in R such that A_1 has a certain property in R and for all projections $e \in A_2$, A_2e does not have this property in eRe.[14] Following Størmer and Neshveyev's use ([38, Corollary 4.12]) of centralising sequences in masas to give two non-conjugate singular masas with Pukánszky invariant $\{1\}$, it should come as no surprise that the property we shall use here is the existence in A of non-trivial centralising sequences for R, as discussed in subsection 1.2.4. We make a formal definition of the invariant for masas which arises in the same manner as (3.3.1) using centralising sequences.

Definition 3.3.1. Let A be a masa in a II_1 factor N. Define $\Gamma(A)$ to be the supremum of $\mathrm{tr}(p)$ over all projections $p \in A$ such that Ap contains non-trivial centralising sequences for pNp.

[14]In the proceeding discussion the property is being Cartan, and it is immediate that when A_2 is singular in R, then so is A_2e in eRe.

It is immediate that this $\Gamma(A)$ is a conjugacy invariant of A, in the sense that for an automorphism θ of N, we have $\Gamma(\theta(A)) = \Gamma(A)$. Conveniently, the masas A we have been occasionally calling Γ masas, are precisely those with $\Gamma(A) = 1$. As we have suggested, if we form a $\mathrm{tr}(p)$-direct sum of two masas A_1 and A_2 such that A_1 contains non-trivial centralising sequences and no A_2e does, then we should obtain $\Gamma(A_1 \oplus_{\mathrm{tr}(p)} A_2) = \mathrm{tr}(p)$. This is essentially obvious, but we record it formally, as it will be a calculational tool in the sequel.

Proposition 3.3.2. *Let A be a masa in a II_1 factor N. Suppose that there is a projection $p \in A$ such that:*

- *Ap contains non-trivial centralising sequences for pNp;*
- *For every projection $e \leq 1 - p$ in A, Ae does not contain non-trivial centralising sequences for eNe.*

Then $\Gamma(A) = tr(p)$.

Proof. Take a projection $r \in A$ such that Ar contains non-trivial centralising sequences for rNr. To obtain a contradiction, suppose that $r \not\leq p$. Let $(x_n)_{n=1}^{\infty}$ be a non-trivial centralising sequence for rNr in Ar, then write $y_n = x_n pr$ and $z_n = x_n r(1-p)$, so that $x_n = y_n + z_n$ for all n. The sequence $(z_n)_{n=1}^{\infty}$ is a centralising sequence of $r(1-p)Nr(1-p)$ and so is trivial by hypothesis. Without losing generality, we may assume that $z_n = r(1-p)$ for all n.

Take a partial isometry $v \in N$ with $v^*v \leq r(1-p)$ and $vv^* = p_0 \leq pr$, so that $y_n v = x_n v$ and $v = vz_n = vx_n$. Now

$$\|(y_n - 1)p_0\|_2 = \|y_n v - v\|_2 = \|x_n v - vx_n\|_2 \to 0, \tag{3.3.2}$$

as $n \to \infty$. Find orthogonal projections $(p_m)_{m=1}^{m_0}$ in A, with $p_m \leq pr$ and $\mathrm{tr}(p_m) \leq \mathrm{tr}(r(1-p))$, for each m, so that $\sum_{m=1}^{m_0} p_m = pr$. By (3.3.2), we have

$$\|y_n - 1\|_2 \leq \sum_{m=1}^{m_0} \|(y_n - 1)p_m\|_2 \to 0,$$

so that $(x_n)_{n=1}^{\infty}$ is a trivial centralising sequence. This contradiction ensures that $r \leq p$ and so

$$\Gamma(A) = \mathrm{tr}(p),$$

as required. □

The second condition in the preceding proposition will appear again, so we introduce some nomenclature for this situation.

Definition 3.3.3. The inclusion $A \subset N$ of a masa in a II_1 factor is called *completely non* Γ if, for every projection $e \in A$, all the centralising sequences in Ae for eNe are trivial. Equivalently this is when $\Gamma(A) = 0$.

In this way, the hypothesis of Proposition 3.3.2 becomes that A is the $\mathrm{tr}(p)$-direct sum of a Γ masa and a completely non Γ masa.

3.3.2 The execution

Our objective is to demonstrate that Γ gives us, what we called in subsection 1.4.6 a genuinely continuous invariant for singular masas. We first show that Γ is infinity-two norm continuous.

Proposition 3.3.4. *There exist constants* $0 < \delta < 1$ *and* $K > 0$ *such that if* A *and* B *are masas in a separable* II_1 *factor* N *with*

$$\|\mathbb{E}_A - \mathbb{E}_B\|_{\infty,2} = \epsilon < \delta, \tag{3.3.3}$$

then

$$|\Gamma(A) - \Gamma(B)| \leq K\epsilon. \tag{3.3.4}$$

In particular, if A_n *is a sequence of masas in* N *with*

$$\lim_{n\to\infty} \|\mathbb{E}_{A_n} - \mathbb{E}_B\|_{\infty,2} = 0,$$

for some masa B *in* N, *then* $\Gamma(A_n) \to \Gamma(B)$ *as* $n \to \infty$.

Proof. Let δ and K be the δ and K_1 of Theorem 1.4.15 respectively. Given masas A and B in N satisfying (3.3.3), Theorem 1.4.15 gives us projections $p \in A$ and $q \in B$ and a unitary $u \in N$ with $u(Ap)u^* = Bq$ and

$$\|1-p\|_2, \|1-q\|_2 \leq K\epsilon.$$

Given a projection $e \in A$ such that Ae has non-trivial central sequences for eNe, take $f = uepu^*$ - a projection in B under q. Since $u(Aep)u^* = Bf$, we can use u to conjugate the centralising sequences in Ape for $peRpe$ into centralising sequences for fRf lying in Bf. Therefore,

$$\Gamma(B) \geq \mathrm{tr}(ep) = \mathrm{tr}(e) - \mathrm{tr}(e(1-p)) \geq \mathrm{tr}(e) - \|e\|_2 \|1-p\|_2 \geq \mathrm{tr}(e) - K\epsilon,$$

for every such e, and hence

$$\Gamma(B) \geq \Gamma(A) - K\epsilon.$$

By interchanging the roles of A and B we have

$$\Gamma(A) \geq \Gamma(B) - K\epsilon,$$

and these two inequalities combine to give (3.3.4), as required. □

To complete our objective, we now exhibit a continuous path through the singular masas from $\Gamma = 0$ to $\Gamma = 1$. Since this path can be found in the set of masas with Pukánszky invariant $\{1\}$, this gives us our desired uncountably many different singular masas with the same Pukánszky invariant.

Theorem 3.3.5. *There is a map $t \mapsto A(t)$, taking each $t \in [0,1]$ to a masa $A(t)$ in the hyperfinite* II_1 *factor R such that:*

- $d_{\infty,2}(A(s), A(t)) \to 0$ *as* $|s-t| \to 0$;
- *Every $A(t)$ has Pukánszky invariant $\{1\}$;*
- *Each $A(t)$ is strongly singular;*
- $\Gamma(A(t)) = t$, *for each t;*

Corollary 3.3.6. *There exist uncountably many singular masas in the hyperfinite* II_1 *factor R, each with Pukánszky invariant $\{1\}$, such that no pair of these masas is conjugate by an automorphism of R.*

The plan is fairly simple, we shall construct Tauer masas $A(t)$ for a dense set of t in $[0,1]$ satisfying the conditions of Theorem 3.3.5. We will then use the $d_{\infty,2}$-completeness of the set of masas in R, Proposition 1.4.11, to extend the domain of definition of A to all of $[0,1]$. That these $A(t)$ satisfy the last three conditions in Theorem 3.3.5 will then follow by continuity arguments. Let us start with the construction of Tauer masas for a dense collection of t.

Construction 3.3.7. Let $k_1 = 2$ and, for each $n \geq 2$, take k_n to be a prime exceeding $k_1 \dots k_{n-1}$. Let M_r be the algebra of $k_r \times k_r$ matrices. By Proposition 1.2.10, there is a family $({}^rD^{(m)})_{m=0}^{k_1\dots k_{r-1}}$ of pairwise orthogonal masas in M_r. Let ${}^re_l^{(m)}$ be the minimal projections of ${}^rD^{(m)}$ indexed by $l = 0, 1 \dots, k_r - 1$. Take N_n to be the tensor product $\bigotimes_{r=1}^n M_r$. We have the natural unital inclusion $x \mapsto x \otimes 1$ of N_n inside N_{n+1}, and we work in the hyperfinite II_1 factor R, obtained as the direct limit of these N_n with respect to normalised trace.

For each $n \in \mathbb{N}$ write

$$I_n = \left\{ \frac{m}{k_1 \dots k_n} \,\middle|\, m = 0, 1, 2, \dots, k_1 \dots k_n \right\},$$

which has $I_n \subset I_{n+1}$, for each n. Let $I = \bigcup_{n=1}^\infty I_n$ - a dense set of rationals in $[0,1]$. For $t \in I$, let $n_0(t)$ be the minimal n for which $t \in I_n$. For each $t \in I$, we will define a Tauer masa $A(t)$ in R, with respect to the chain $(N_n)_{n=n_0(t)}^\infty$, denoting the approximates by $A_n(t)$ for $n \geq n_0(t)$. The minimal projections of $A_n(t)$ with be enumerated as ${}^nf_m(t)$ for $0 \leq m < k_1 \dots k_n$.

The process begins by defining $A_0(0) = A_0(1/2) = A_0(1) = {}^1D^{(0)}$, with the minimal projections ${}^1f_m(0) = {}^1f_m(1/2) = {}^1f_m(1) = {}^1e_m^{(0)}$ coinciding for $m = 0, 1$. Suppose that, for some n_1, $A_n(t)$ has been defined for all $t \in I_{n_1}$ and $n_0(t) \leq n \leq n_1$. For $t \in I_{n_1}$, the definition of $A_{n_1+1}(t)$ is split into two cases, depending on whether n is even or odd. The even case is designed so that the resulting Tauer masa will be strongly singular, whereas the odd case ensures that it will have the desired value of the Γ-invariant.

1. n_1 is *even*: Set

$$A_{n_1+1}(t) = \bigoplus_{m=0}^{k_1 \dots k_{n_1}-1} {}^{n_1}f_m(t) \otimes {}^{n_1+1}D^{(m)}, \tag{3.3.5}$$

with the enumeration of the minimal projections ${}^{n_1+1}f_{m'}(t)$ given, by writing $m' = k_{n_1+1}m + l$ for some $0 \leq l < k_{n'+1}$, and taking

$$ {}^{n_1+1}f_{m'}(t) = {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(m)}. \tag{3.3.6}$$

2. n_1 is *odd*: Here, we take

$$A_{n_1+1}(t) = \bigoplus_{m=0}^{tk_1 \dots k_{n_1}-1} {}^{(n_1)}f_m(t) \otimes {}^{n_1+1}D^{(k_1 \dots k_{n_1})} \oplus \bigoplus_{m=tk_1 \dots k_{n_1}}^{k_1 \dots k_{n_1}-1} {}^{n_1}f_m(t) \otimes {}^{n_1+1}D^{(m)}. \tag{3.3.7}$$

The enumeration of the minimal projections happens in the same way as previously. Namely, given $0 \leq m' < k_1 \dots k_{n_1+1}$, write $m' = mk_{n_1+1} + l$ for some $0 \leq l < k_{n_1+1}$ and set

$$ {}^{n_1+1}f_{m'}(t) = \begin{cases} {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(k_1 \dots k_{n_1})} & 0 \leq m < tk_1 \dots k_{n_1} \\ {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(m)} & tk_1 \dots k_{n_1} \leq m < k_1 \dots k_{n_1} \end{cases}. \tag{3.3.8}$$

It remains to define $A_{n_1+1}(t)$ for $t \in I_{n_1+1} \setminus I_{n_1}$ - the first approximate for these $A(t)$. Let $m_0 = \lfloor tk_1 \dots k_{n_1} \rfloor$. The plan for the construction of $A_{n_1+1}(t)$ is indicated in the 'number line' of Figure 3.4. More formally, we take $A_{n_1+1}(t)$ to have minimal projections ${}^{n_1+1}f_m(t)$ given by

$$ {}^{n_1+1}f_m(t) = \begin{cases} {}^{n_1+1}f_m((m_0+1)/k_1 \dots k_{n_1}) & 0 \leq m < tk_1 \dots k_{n_1+1} \\ {}^{n_1+1}f_m(m_0/k_1 \dots k_{n_1}) & tk_1 \dots k_{n_1+1} \leq m < k_1 \dots k_{n_1+1} \end{cases}.$$

Theorem 3.2.1 shows that the Tauer masas constructed above have Puk $(A(t)) = \{1\}$, which is the second condition of Theorem 3.3.5. We now check that the Tauer masas produced in Construction 3.3.7 satisfy the last two conditions of Theorem

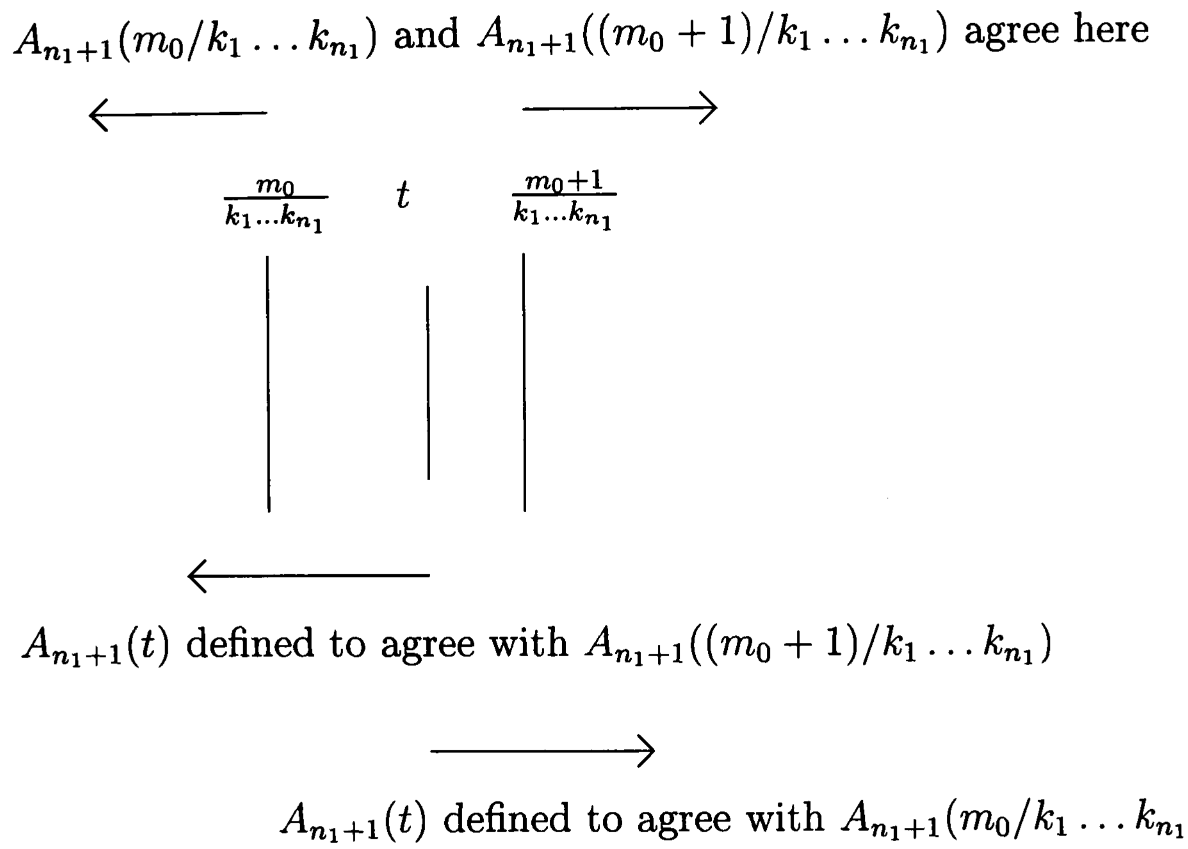


Figure 3.4: Construction of $A_{n_1+1}(t)$.

3.3.5. The strong singularity of $A(t)$ comes from the even n_1 construction in exactly the same way that singularity was demonstrated for Tauer's original example in Theorem 2.2.3.

Proposition 3.3.8. *The Tauer masas $A(t)$ of Construction 3.3.7 are strongly singular.*

Proof. Fix $t \in I$ and let $n \geq n_0(t)$ be even. With the notation of (2.1.3) in section 2.1, the even stage of Construction 3.3.7 gives

$$A^{({}^n f_m(t))}_{n+1,n}(t) = {}^{n+1}D^{(m)}.$$

Take a unitary $w \in {}^{n+1}D^{(m)}$ with $\mathrm{tr}(w) = 0$. If $m' \neq m$, then $\mathbb{E}_{{}^{n+1}D^{(m')}}(w) = 0$ by the orthogonality of ${}^{n+1}D^{(m)}$ and ${}^{n+1}D^{(m')}$. Proposition 2.2.2 then shows that $A(t)$ has the weak asymptotic homomorphism property. Strong singularity of $A(t)$ follows from Lemma 1.4.24. □

To calculate $\Gamma(A(t))$ for the masas of Construction 3.3.7 we shall regard them as appropriate direct sums and use Proposition 3.3.2. We shall also see how the odd stage of Construction 3.3.7 allows us to control the location of non-trivial centralising sequences inside cutdowns of $A(t)$.

Proposition 3.3.9. *For each $t \in I$, there exists a projection $p \in A(t)$ with $tr(p) = t$ such that:*

1. *$A(t)p$ contains non-trivial centralising sequences for pRp;*

2. $A(t)(1-p)$ is completely non Γ in $(1-p)R(1-p)$.

For fixed $t \in I$, write n_0 for $n_0(t)$, then the required projection is given by

$$p = \sum_{m=0}^{tk_1 \dots k_{n_0}-1} {}^{n_0}f_m(t). \tag{3.3.9}$$

Proof of 1: Note that

$$p = \sum_{m=0}^{tk_1 \dots k_n - 1} {}^{n}f_m(t),$$

for all $n \geq n_0$. Fix $n \geq n_0$ odd and consider $x_1, \dots, x_r \in N_n$. Let $v \in {}^{n+1}D^{(k_1 \dots k_n)}$ be a unitary with $\mathrm{tr}(v) = 0$, so that, examining the odd n form of Construction 3.3.7,

$$u = \sum_{m=0}^{tk_1 \dots k_n - 1} {}^{n}f_m(t) \otimes v = p \otimes v \in N_n \otimes M_{n+1} = N_{n+1}$$

is a trace free unitary in $A_{n+1}(t)p$. It is then immediate that u commutes with each px_ip, and so $A(t)p$ contains non-trivial centralising sequences for pRp by the $\|.\|_2$-density of $\cup_{n=1}^{\infty} N_n$ in R. $\square$

We prove part 2 of Proposition 3.3.9 in two stages, using the Popa's orthogonality idea found in Proposition 1.2.12. We first establish the hypothesis of this Proposition when $e \leq 1-p$ is a minimal projection of some $A_n(t)$. A density argument, which contains the proof of Proposition 1.2.12, then establishes the result.

Proposition 3.3.10. *With the notation of Construction 3.3.7, fix $t \in I$, $n \geq n_0(t)$ and m, m' with $tk_1 \dots k_n \leq m < m' < k_1 \dots k_n$. Let v be a partial isometry in N_n with $vv^* = {}^{n}f_m(t)$ and $v^*v = {}^{n}f_{m'}(t)$.*[15] *Then $v(A(t)\, {}^{n}f_{m'}(t))v^*$ is orthogonal to $A(t)\, {}^{n}f_m(t)$ in ${}^{n}f_m(t)R\, {}^{n}f_m(t)$.*

Proof. Fix $n \geq n_0(t)$ and regard R as $N_n \overline{\otimes} R_1$, where R_1 is generated as the infinite von Neumann tensor product $(\bigotimes_{r=n+1}^{\infty} M_r)''$ with respect to the unique normalised trace. Using the notation of (2.1.3), for $n_1 > n$ we have

$$A_{n_1}(t) = \bigoplus_{m=0}^{k_1 \dots k_n - 1} {}^{n}f_m(t) \otimes A_{n_1,n}^{({}^{n}f_m(t))}(t),$$

for masas $A_{n_1,n}^{({}^{n}f_m(t))}(t)$ in $\bigotimes_{r=n+1}^{n_1} M_r$, giving rise to Tauer masas

$$A_{\infty,n}^{({}^{n}f_m(t))}(t) = \left(\bigcup_{n_1=n+1}^{\infty} A_{n_1,n}^{({}^{n}f_m(t))}(t) \right)''$$

[15] Such a v is uniquely determined up to multiplication by a scalar of unit modulus.

in R_1, so that

$$A(t) = \bigoplus_{m=0}^{k_1\dots k_n-1} {}^n f_m(t) \otimes A_{\infty,n}^{({}^n f_m(t))}(t).$$

Now take m, m' and v as in the statement, since

$$v(A(t)\ {}^n f_{m'}(t))v^* = {}^n f_m(t) \otimes A_{\infty,n}^{({}^n f_{m'}(t))}(t),$$

it suffices to show that $A_{\infty,n}^{({}^n f_m(t))}(t)$ and $A_{\infty,n}^{({}^n f_{m'}(t))}(t)$ are orthogonal masas in R_1. We shall show that $A_{n_1,n}^{({}^n f_m(t))}(t)$ and $A_{n_1,n}^{({}^n f_{m'}(t))}(t)$ are orthogonal in $\bigotimes_{r=n+1}^{n_1} M_r$, for all $n_1 > n$, from which the result immediately follows by density.

To this end note that Construction 3.3.7 gives $A_{n+1,n}^{({}^n f_m(t))}(t) = {}^{n+1}D^{(m)}$ and $A_{n+1,n}^{({}^n f_{m'}(t))}(t) = {}^{n+1}D^{(m')}$, from (3.3.5) when n is even and from (3.3.7) when n is odd, noting that in this case we use the hypothesis that $tk_1 \dots k_n \leq m < m'$. As $D^{(m)}$ and $D^{(m')}$ are orthogonal masas in M_{n+1}, we certainly have the claim when $n_1 = n+1$.

Suppose inductively, that the claim holds for some $n_1 > n$. Write

$$A_{n_1+1,n}^{({}^n f_m(t))}(t) = \bigoplus_{g \in \mathcal{P}_{\min}\left(A_{n_1,n}^{({}^n f_m(t))}(t)\right)} g \otimes B^{(g,m)},$$

and

$$A_{n_1+1,n}^{({}^n f_{m'}(t))}(t) = \bigoplus_{h \in \mathcal{P}_{\min}\left(A_{n_1,n}^{\left({}^n f_{m'}(t)\right)}(t)\right)} h \otimes B^{(h,m')},$$

for masas $B^{(g,m)}$ and $B^{(h,m')}$ in M_{n_1+1}. Again, Construction 3.3.7 ensures that all these masas are pairwise orthogonal. This is immediate from (3.3.5) for even n_1, when n_1 is odd we again use the hypothesis $tk_1 \dots k_n \leq m < m'$ in our examination of (3.3.7). The orthogonality of $A_{n_1+1,n}^{({}^n f_m(t))}(t)$ and $A_{n_1+1,n}^{({}^n f_{m'}(t))}(t)$ follows immediately, yielding the result. □

Proof of part 2 of Proposition 3.3.9: Take $t \in I$ and fix some projection $0 \neq e \leq 1-p$ in $A(t)$. For each $n \in \mathbb{N}$, find $l_n \geq n_0(t)$ and a family $P_n \subset \mathcal{P}_{\min}(A_{l_n}(t))$ of minimal projections in $A_{l_n}(t)$ lying under $1-p$, such that, writing $q_n = \sum_{q \in P_n} q$, we have

$$\|q_n - e\|_2^2 < 1/n.$$

Now let x_n be a unitary in $q_n N_{l_n} q_n$ which, when acting by conjugation, permutes the elements of P_n, in a fixed point free fashion.[16] Proposition 3.3.10, ensures that

$$x_n(Aq_n)x_n^* \perp Aq_n \tag{3.3.10}$$

[16]Formally, we have $x_n q x_n^* = \sigma_l(q) \in P_n$, for every $q \in P_n$, where σ_l is a permutation of P_n such that $\sigma_l(q) \neq q$ for every q.

in q_nRq_n, as such an x_n must be a sum of partial isometries satisfying the hypothesis of this proposition.[17]

Suppose that Ae contains non-trivial centralising sequences for eRe. Find a sequence of unitaries $u_n \in A$, with $\mathrm{tr}(u_n e) = 0$ and

$$\|eu_nex_ne - ex_neu_ne\|_2 < \|e - q_n\|_2\,, \tag{3.3.11}$$

for each n. We have the following simple estimate, showing that the u_nq_n asymptotically commute with the $q_nx_nq_n$,

$$\begin{aligned}
&\|q_nu_nq_nx_nq_n - q_nx_nq_nu_nq_n\|_2\\
\leq &\|(q_n - e)u_nq_nx_nq_n\|_2 + \|eu_n(q_n - e)x_nq_n\|_2 + \|eu_nex_n(q_n - e)\|_2\\
&+ \|eu_nex_ne - ex_neu_ne\|_2 + \|ex_neu_n(e - q_n)\|_2 + \|ex_n(e - q_n)u_nq_nm\|_2\\
&\quad + \|(e - q_n)x_nq_nu_nq_n\|_2\\
\leq &7\,\|e - q_n\|_2 \to 0.
\end{aligned}$$

On the other hand, using the orthogonality of (3.3.10), we have

$$\begin{aligned}
&\|q_nx_nq_nu_nq_n - q_nu_nq_nx_nq_n\|_2^2\\
= &\|q_nx_nu_nq_nx_n^*q_n - u_nq_n\|_2^2\\
= &\|q_nx_nu_nq_nx_n^*q_n\|_2^2 + \|u_nq_n\|_2^2 - 2\Re\mathrm{tr}(x_nu_nq_nx_n^*u_nq_n)\\
= &2\,\|q_n\|_2^2 - 2\Re\mathrm{tr}(x_nu_nq_nx_n^*)\mathrm{tr}(u_nq_n)/\mathrm{tr}(q_n) \to 2\,\|e\|_2^2 \neq 0,
\end{aligned}$$

where the last line comes from the orthogonality relationship (3.3.10).[18] The convergence is a simple calculation, as

$$|\mathrm{tr}(u_nq_n)| \leq |\mathrm{tr}(u_ne)| + |\mathrm{tr}(u_n(q_n - e))| \leq 0 + \|u_n\|_2\,\|q_n - e\|_2 \to 0.$$

This contradiction completes the proof. □

To summarise, currently we have a collection $A(t)$ of strongly singular Tauer masas, defined for t in the dense set $I \subset [0,1]$ with $\Gamma(A(t)) = t$. We wish to use completeness to define $A(t)$ for $t \in [0,1]\setminus I$ and so we need to control the distance between the $A(t)$'s we have already defined. It is here that the form of $A_{n_0(t)}(t)$ specified in Construction 3.3.7 becomes important.

[17] The cross terms $v(A)w^*$ vanish, when v and w are partial isometries, with orthogonal initial projections in A. Any pair of distinct partial isometries v, w in the sum making up x_n have this property.

[18] The quotient of $\mathrm{tr}(q_n)$ is a normalisation constant.

Proposition 3.3.11. *Fix $s, t \in I$ with $s < t$. Let n_0 be the maximum of $n_0(s)$ and $n_0(t)$ and take*

$$q = \sum_{m=0}^{sk_1 \dots k_{n_0}-1} {}^{n_0}f_m(s) + \sum_{m=tk_1 \dots k_{n_0}}^{k_1 \dots k_{n_0}-1} {}^{n_0}f_m(s)$$

a projection of trace $1-(t-s)$. Then q lies in $A(s) \cap A(t)$ and $A(s)q = A(t)q$.

Proof. We shall demonstrate that Construction 3.3.7 ensures that whenever we have $s, t \in I_n$, then

$${}^n f_m(s) = {}^n f_m(t), \tag{3.3.12}$$

for all m with

$$0 \le m < sk_1 \dots k_n \text{ or } tk_1 \dots k_n \le m < k_1 \dots k_n. \tag{3.3.13}$$

This will immediately show that q lies in $A(t)$, as well as $A(s)$. Furthermore, as $A(s)q$ and $A(t)q$ are generated by all the ${}^n f_m(s)$ and ${}^n f_m(t)$ respectively, with $n \ge \max\{n_0(s), n_0(t)\}$ and m satisfying (3.3.13), this claim also implies that $A(s)q = A(t)q$, as required.

The claim is established by induction on n. When $n = 1$, the result is certainly true, as Construction 3.3.7 began by defining $A_0(0) = A_0(1/2) = A_0(1)$ with the minimal projections also coinciding. Suppose that we have established the claim for all $n \le n_1$. We investigate the $n_1 + 1$ situation, starting with the case when s and t both lie in I_{n_1}.

Take $s, t \in I_{n_1}$ with $s < t$. Take m' with either $0 \le m' < sk_1 \dots k_{n_1+1}$ or $tk_1 \dots k_{n_1+1} \le m' < k_1 \dots k_{n_1+1}$, and divide by k_{n_1+1} to obtain $m' = mk_{n_1+1} + l$ with $0 \le l < k_{n_1+1}$. This m must have $0 \le m < sk_1 \dots k_{n_1}$ in the first case, or $tk_1 \dots k_{n_1} \le m < k_1 \dots k_{n_1}$ in the second. In either event, the inductive hypothesis ensures that ${}^{n_1}f_m(s) = {}^{n_1}f_m(t)$. When n_1 is even, the definition (3.3.6) of ${}^{n_1+1}f_{m'}(s)$ and ${}^{n_1+1}f_{m'}(t)$ immediately gives ${}^{(n_1+1)}f_{m'}(s) = {}^{n_1+1}f_{m'}(t)$. When n_1 is odd, this is also true, as we have excluded the possibility that $sk_1 \dots k_{n_1} \le m < tk_1 \dots j_{n_1}$, so both these minimal projections must come from the same case in the definition (3.3.8). Therefore, in this case the minimal projections ${}^{n_1+1}f_{m'}(s)$ and ${}^{n_1+1}f_{m'}(t)$ coincide, whenever they are required to do so.

We now examine what happens when only one of s and t lies in I_{n_1}. Take s in I_{n_1} and $t \in I_{n_1+1} \setminus I_{n_1}$ with $s < t$. As in the definition of $A_{n_1+1}(t)$, given pictorially in Figure 3.4, we write $m_0 = \lfloor tk_1 \dots k_n \rfloor$ so that $s \le m_0/k_1 \dots k_{n_1}$. For $0 \le m < sk_1 \dots k_{n_1+1}$, we have

$${}^{n_1+1}f_m(s) = {}^{n_1+1}f_m((m_0+1)/k_1 \dots k_{n_1}) = {}^{n_1+1}f_m(t),$$

where the second equality is the definition of the minimal projections ${}^{n_1+1}f_m(t)$, and the first equality follows as these minimal projections for $A_{n_1+1}(s)$ and $A_{n_1+1}((m_0+1)k_1 \dots k_{n_1})$ coincide, by the case we have just analysed. When $tk_1 \dots k_{n_1+1} \leq m < k_1 \dots k_{n_1+1}$, we have

$$ {}^{n_1+1}f_m(t) = {}^{n_1+1}f_m(m_0/k_1 \dots k_{n_1}) = {}^{n_1+1}f_m(s), $$

the first equality being the definition of ${}^{n_1+1}f_m(t)$, and the second equality is the (3.3.12) for appropriate minimal projections of $A_{n_1+1}(s)$ and $A_{n_1+1}(m_0/k_1 \dots k_{n_1})$ as $m \geq m_0 k_{n_1+1}$.[19] Interchanging the roles of s and t above ensures that the claim is satisfied for n_1+1 whenever at least one of s or t lies in I_{n_1}.

We complete the proof by examining the case when $s, t \in I_{n_1+1} \setminus I_{n_1}$. We shall do this pictorially by examining more 'number lines' - this could be made precise in exactly the same way as the calculation above, but this would add little, if any, extra understanding. Take $s < t$ with $s, t \in I_{n_1+1} \setminus I_{n_1}$. Suppose first that $\lfloor sk_1 \dots k_{n_1} \rfloor = \lfloor tk_1 \dots k_{n_1} \rfloor = m_0$. The situation is as in Figure 3.5 which demonstrates the agreement of the required minimal projections in $A_{n_1+1}(s)$ and $A_{n_1+1}(t)$.

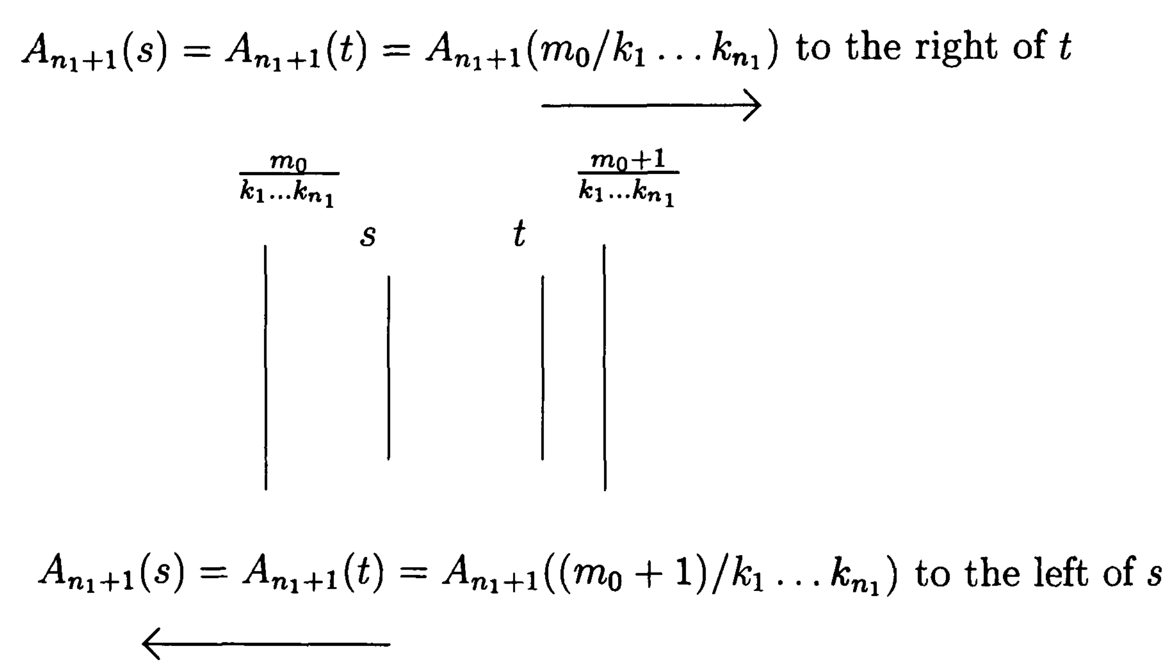


Figure 3.5: Number line describing the case $\lfloor sk_1 \dots k_{n_1} \rfloor = \lfloor tk_1 \dots k_{n_1} \rfloor$.

Finally, suppose that $\lfloor sk_1 \dots k_{n_1} \rfloor = m_0 \neq m_1 = \lfloor tk_1 \dots k_{n_1} \rfloor$, as indicated in Figure 3.6. To the 'left' of s, $A_{n_1+1}(s)$ is defined to be $A_{n_1+1}((m_0+1)/k_1 \dots k_{n_1})$ and $A_{n_1+1}(t)$ is defined to be $A_{n_1+1}((m_1+1)/k_1 \dots k_{n_1})$. This is all happening 'left' of $(m_0+1)/k_1 \dots k_{n_1}$ where the minimal projections of $A_{n_1+1}((m_0+1)/k_1 \dots k_{n_1})$ and $A_{n_1+1}((m_1+1)/k_1 \dots k_{n_1})$ coincide. Hence, the minimal projections of $A_{n_1+1}(s)$ and $A_{n_1+1}(t)$ to the 'left' of s agree. A similar argument shows

[19]These two algebras may turn out to be the same, but then the minimal projections will certainly coincide!

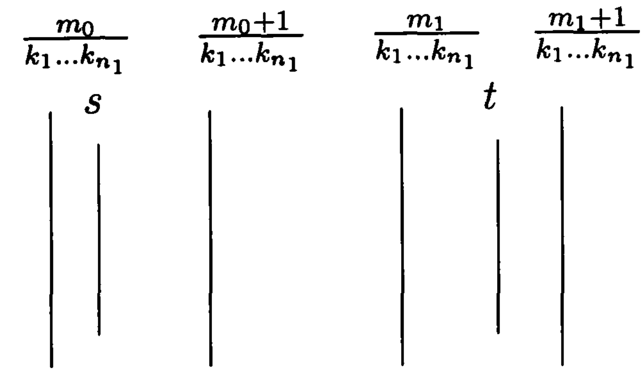


Figure 3.6: Number line describing the case when $\lfloor sk_1 \dots k_{n_1} \rfloor \neq \lfloor tk_1 \dots k_{n_1} \rfloor$

that the minimal projections of $A_{n_1+1}(s)$ and $A_{n_1+1}(t)$ agree to the 'right' of t, establishing the claim for all $n \leq n_1 + 1$. □

Corollary 3.3.12. *Use the notation of Construction 3.3.7. For $s, t \in I$ we have*

$$\left\| \mathbb{E}_{A(s)} - \mathbb{E}_{A(t)} \right\|_{\infty,2} \leq 2\sqrt{|s-t|}$$

Proof. We may assume that $s < t$. Let n_0 be the maximum of $n_0(s)$ and $n_0(t)$. Let q be the projection of Proposition 3.3.11, so that $A(s)q = A(t)q$. The estimate

$$\left\| \mathbb{E}_{A(s)} - \mathbb{E}_{A(t)} \right\|_{\infty,2} \leq 2 \left\| 1 - q \right\|_2,$$

follows from Proposition 1.4.14, and, as $\mathrm{tr}(1-q) = t - s$, is exactly what was claimed. □

It is now easy to complete the proof of Theorem 3.3.5. Namely, for $t \in [0,1] \backslash I$, we define $A(t)$ by taking a sequence $t_n \to t$ with each t_n in the dense set of rationals I. The resulting sequence of masas $(A(t_n))_{n=1}^{\infty}$ is then $d_{\infty,2}$-Cauchy by Corollary 3.3.12, and so converges to a masa $A(t)$ by Proposition 1.4.11. This masa is well defined, in that $A(t)$ is independent of the choice of sequence $(t_n)_{n=1}^{\infty}$ in I converging to t.

Furthermore, $A(t)$ is strongly singular, as each $A(t_n)$ is, and the set of all strongly singular masas is closed - see Proposition 1.4.13. It has Pukánszky invariant $\{1\}$, again by closure - the necessary result here being Corollary 3.1.6. That $\Gamma(A(t)) = t$ for every $t \in [0,1]$ follows first by observing that Proposition 3.3.9 combines with Proposition 3.3.2 to give the result for $t \in I$. Continuity gives the result for all t, this time in the form of Proposition 3.3.4.

3.3.3 Transitive masas

In a way, our construction of uncountably many singular masas with the same Pukánszky invariant is somewhat undesirable. The technique of gluing two masas together in a direct sum of two hyperfinite factors in different proportions leads to masas which do not arise naturally. It would be preferable to obtain uncountably many such singular masas which do not decompose in this way. This motivates the following definition.

Definition 3.3.13. A masa A in a II_1 factor N is said to be *transitive* if for any two projections $p, q \in A$ with $\mathrm{tr}(p) = \mathrm{tr}(q)$ there is a *-isomorphism Θ of pNp onto qNq with $\Theta(Ap) = Aq$. If, under the same hypotheses, an automorphism Θ of N can be found with $\Theta(p) = q$ and $\Theta(A) = A$ then A will be called *strongly transitive.*

In a transitive masa, any two cut downs are conjugate as masas in the cutdown factors, hence the nomenclature. In particular a transitive masa can not arise as a (non-trivial) direct sum of two non-conjugate masas. We view strongly transitive masas as being suitably natural, in that all parts of them 'look the same'.

Question 3.3.14. Does there exist an uncountable family of pairwise non-conjugate strongly transitive, strongly singular masas in the hyperfinite II_1 factor, each with Pukánszky invariant $\{1\}$?

There is a weaker concept, which also prevents masas from arising as a direct sum.

Definition 3.3.15. A masa A in a II_1 factor N is said to be *weakly transitive* if, given two non-zero projections $p, q \in A$, we can find non-zero subprojections $p_0 \leq p$ and $q_0 \leq q$ in A with $\mathrm{tr}(p_0) = \mathrm{tr}(q_0)$, and a *-isomorphism Θ from p_0Np_0 onto q_0Nq_0 conjugating Ap_0 onto Aq_0.

A simple maximality argument shows that weak transitivity of $A \subset N$ is equivalent to the statement that given projections p and q in A with $\mathrm{tr}(p) = \mathrm{tr}(q)$, then countable families $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$ and $(\Theta_n)_{n=1}^{\infty}$ can be found such that:

- The p_n are orthogonal projections in A with sum p, and likewise the q_n are orthogonal projections in A with sum q;
- $\mathrm{tr}(p_n) = \mathrm{tr}(q_n)$ for each n;
- Each Θ_n is a *-isomorphism from p_nNp_n onto q_nNq_n with $\Theta(Ap_n) = Aq_n$.

These concepts would be meaningless without examples. Firstly note that, if A is a Cartan masa in N, then A is strongly-transitive, with the required automorphisms coming from adjunction by normalising unitaries, see [46]. The nature of these automorphisms, shows that any semi-regular masa is also strongly transitive. For a strongly transitive singular masa one only needs to look at the generator masa in a free group factor. Indeed, work in $\mathcal{L}(\mathbb{F}_2)$, thought of as the von Neumann free product $L^\infty[0,1] * L^\infty[0,1]$, with A being the generator masa $L^\infty[0,1] * 1 \cong L^\infty[0,1]$ in $\mathcal{L}(\mathbb{F}_2)$. Any automorphism Θ of A extends to an automorphism $\overline{\Theta} = \Theta * I$ of $\mathcal{L}(\mathbb{F}_2)$, and in this way we get automorphisms of $\mathcal{L}(\mathbb{F}_2)$ which interchange Ap and Aq whenever p and q are projections in A with the same trace.

At present, we do not have examples of masas showing that the three transitivity properties defined are genuinely different, although it seems difficult to see how the automorphisms guaranteed by weak transitivity could be extended to give transitivity in general. We give some terminology which will help us to formulate statements about transitivity in the future.

Definition 3.3.16. Let A be a masa in the II_1 factor N. Two projections p and q in A with $\mathrm{tr}(p) = \mathrm{tr}(q)$ are called *weakly A-conjugate* if the cutdowns Ap and Aq are conjugate masas, via a *-isomorphism from pNp onto qNq. They are called *A-conjugate*, or conjugate by an automorphism preserving A, if there is an automorphism Θ of N with $\Theta(A) = A$ such that $\Theta(p) = q$.

Observe that A is transitive if all projections with the same trace in A are weakly A-conjugate and strongly transitive if all such projections are A-conjugate. For a transitive masa A in N it makes sense to talk about a relative fundamental group, as a compression of $A \subset N$ by a projection $p \in A$ of fixed trace is well defined up to conjugacy.

Definition 3.3.17. Let A be a transitive masa in a II_1 factor N. Define the fundamental group, $\mathcal{F}(A \subset N)$ to be the subgroup of $\mathbb{R}_+$ under multiplication, generated by all those $t > 0$ for which there is a projection $p \in A$ of trace t such that $Ap \subset pNp$ is conjugate to $A \subset N$; that is, there exists a *-isomorphism Θ from N onto pNp, with $\Theta(A) = Ap$.

It seems fruitless to ask 'what is the fundamental group of the generator masa in the free group factor on finitely many generators?' - the question of whether the fundamental group of $\mathcal{L}(\mathbb{F}_2)$ is $\{1\}$ or $\mathbb{R}_+$ is equivalent to the notoriously hard isomorphism problem for the free group factors.[20] When we have infinitely many generators things are better. Rădulescu has shown in [53], that $\mathcal{F}(\mathcal{L}(F_\infty)) = \mathbb{R}_+$, so it is reasonable to ask about the transitive singular generator masa here.

Question 3.3.18. Let A be the generator masa in $\mathcal{L}(\mathbb{F}_\infty)$. Is it the case that $\mathcal{F}(A \subset \mathcal{L}(\mathbb{F}_\infty)) = \mathbb{R}_+$?

[20]It is known that one of these cases must occur.

3.4 Masas in R_ω

Throughout this section $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ will denote a fixed non-principal ultrafilter. As discussed in subsection 1.4.2, Popa has shown that when A is a masa in N, then A^ω is a masa in the ultraproduct A^ω, [44]. Furthermore, [48] can be used to show that if A is singular then so is A^ω. In this section, we address the same question in the central sequence algebra $N_\omega = N^\omega \cap N'$. We shall work in the hyperfinite II_1 factor, R. Given a property Γ masa A in R, we know that $A^\omega \cap R'$ is a non-trivial subalgebra of $R_\omega = R^\omega \cap R'$. When is $A^\omega \cap R'$ a masa in R_ω?

To perform our calculations, we shall need to recall some facts about conditional expectations. Let M be a von Neumann subalgebra of the hyperfinite II_1 factor R. Proposition 1.3.1, due to Christensen, tells us that $\mathbb{E}_{M'\cap R}(x)$ is the $\|.\|_2$-minimal element of the weakly closed convex hull of $\{\, uxu^* \mid u \in \mathcal{U}(M)\,\}$. This process of averaging over the unitaries in N is much easier when M is finite dimensional. In this case $\mathcal{U}(N)$ is a compact topological group, and we have

$$\mathbb{E}_{M'\cap R}(x) = \int_{\mathcal{U}(M)} uxu^* \mathrm{d}u, \tag{3.4.1}$$

where $\mathrm{d}u$ is the normalised left Haar measure on $\mathcal{U}(M)$. The situation we shall need is easier still. Suppose we have the factorisation $R = M\overline{\otimes}S$, where M is a type $\mathrm{I_n}$ factor and S is a hyperfinite II_1 factor. We regard M as embedded in R by identifying it with $M \otimes 1$, then we have $M' \cap R = 1 \otimes S$. We can write down an explicit formula for the conditional expectation onto this relative commutant. Take matrix units $(e_{i,j})_{i,j=1}^n$ for M, and write a typical element $x \in R$ as $x = \sum_{i,j=1}^n e_{i,j} \otimes x_{i,j}$. When $i \neq j$, $e_{i,j} \otimes x_{i,j}$ is orthogonal to $1 \otimes S$. When $i = j$, we have

$$e_{i,i} \otimes x_{i,i} = 1_M \otimes \frac{1}{n}x_{i,i} + \left(e_{i,i} - \frac{1}{n}1_M\right) \otimes x_{i,i}.$$

The first term on the right hand side above lies in $M' \cap R$, while the second is orthogonal to it. In conclusion, we have

$$\mathbb{E}_{M'\cap R}(x) = 1_M \otimes \left(\frac{1}{n}\sum_{i=1}^n x_{i,i}\right).$$

On the other hand, we know that $\mathbb{E}_{M'\cap R}(x)$ can be approximated by a convex combination of elements uxu^*, for some unitaries $u \in M$. The good news here, is that we can take this combination to be finite and the unitaries involved will not depend on x. Given a permutation σ of $\{1, 2, \ldots, n\}$ and $\lambda_i = \pm 1$, consider the unitary

$$u = \sum_{i=1}^n \lambda_i e_{\sigma(i),i}$$

in M, which has

$$ue_{j,k}u^* = \lambda_j\lambda_k e_{\sigma(j),\sigma(k)}.$$

There are finitely many, in fact $2^n n!$, such unitaries in M. For $x \in R$ we take the average of uxu^* over these unitaries. Writing $x = \sum_{i,j=1}^n e_{i,j} \otimes x_{i,j}$ as before, it is easy to check that

$$\frac{1}{2^n n!}\sum_u uxu^* = 1_M \otimes \left(\frac{1}{n}\sum_{i=1}^n x_{i,i}\right) = \mathbb{E}_{M'\cap R}(x). \tag{3.4.2}$$

We now reveal the point of all this, the proof of which follows directly from (3.4.2).

Proposition 3.4.1. *Let $R = M\overline{\otimes}S$ be a factorisation of the hyperfinite* II_1 *factor as the tensor product of a type* $\mathrm{I_n}$ *factor M and a* II_1 *factor S. Regard M as being identified with $M \otimes 1$ in R. If $(x_r)_{r=1}^\infty$ is an ω-centralising sequence in R, then*

$$\lim_{r\to\omega} \|x_r - \mathbb{E}_{M'\cap R}(x_r)\|_2 = 0.$$

3.4.1 $D^\omega \cap R'$ is a masa in $R^\omega \cap R'$

In this subsection we show that the Cartan masa D inside R does give rise to a masa $D^\omega \cap R'$ inside the central sequence algebra R_ω.

We shall again work with R in the form of an infinite von Neumann tensor product of 2×2 matrices, that is $R = (\bigotimes_{n=1}^\infty \mathrm{Mat}_2(\mathbb{C}))''$, where the infinite tensor product is taken with respect to the unique normalised trace. Let C denote the diagonal matrices in $\mathrm{Mat}_2(\mathbb{C})$, so that the infinite tensor product $(C^{\otimes\infty})''$ is a copy of the Cartan masa D in R. Write N_n for the finite tensor product, $\mathrm{Mat}_2(\mathbb{C})^{\otimes n}$, so that R factorises as $N_n\overline{\otimes}R_n$, where R_n is the tail II_1 factor $\left(\bigotimes_{r=n+1}^\infty \mathrm{Mat}_2(\mathbb{C})\right)''$. The Cartan masas D respects this factorisation. We write $D_n = C^{\otimes n}$ and $E_n = \left(\bigotimes_{r=n+1}^\infty C\right)''$ so that $D = D_n\overline{\otimes}E_n$.

The plan is to take a unitary x in $R_\omega \setminus (D^\omega \cap R')$ and construct some element in $D^\omega \cap R'$ which fails to commute with x. To this end, lift such an x to an ω-centralising sequence $(x_n)_{n=1}^\infty$ of unitaries, using Proposition 1.2.13. Define sets $I_n \subset \mathbb{N}$ by

$$I_n = \left\{\, r\in\mathbb{N} \,\middle|\, \left\|x_r - \mathbb{E}_{N_n'}(x_r)\right\|_2 < 1/n \,\right\}, \tag{3.4.3}$$

which, as $N_n' \supset N_{n+1}'$, satisfy $I_n \supset I_{n+1}$. Proposition 3.4.1 ensures that these sets lie in the ultrafilter ω. If $r \in \bigcap_{n\geq n_0} I_n$, then $x_r \in N_n'$ for all $n \geq n_0$ and so $x_r \in \mathbb{C}1$. Since $x \notin D^\omega$, we deduce that $\bigcap_{n\geq n_0} I_n \notin \omega$ for each n_0. An equivalent centralising sequence $(\tilde{x}_r)_{r=1}^\infty$ can be defined, by taking $\tilde{x}_r = 1$ for $r \notin I_1$ or $r \in \bigcap_{n=1}^\infty I_n$ and, for $r \in I_n \setminus I_{n+1}$, take $\tilde{x}_r = \mathbb{E}_{N_n'}(x_r)$. We write $\tilde{x}_r = 1^{\otimes n} \otimes z_r$

for some $z_r \in R_n$. The assumption that x does not lie in D_ω now manifests itself in the statement that

$$\lim_{r\to\omega} \|z_r - \mathbb{E}_{E_n}(z_r)\|_2 = K > 0,$$

for some non-zero constant K. Recall here that E_n is the 'tail' Cartan masa $\left(\bigotimes_{r=n+1}^{\infty} C\right)''$ in R_n.

The sets I_n are also used to define a sequence $y = (y_r)_{r=1}^{\infty}$. Take $y_r = 1$ whenever $r \notin I_1$ or $r \in \bigcap_{n=1}^{\infty} I_n$, and for $r \in I_n \setminus I_{n+1}$, use Proposition 1.3.1 to find a unitary $u_r \in E_n$ with

$$\|z_r - u_r z_r u_r^*\|_2 \geq \|z_r - \mathbb{E}_{E_n}(z_r)\|_2,$$

and set $y_r = 1^{\otimes n} \otimes u_r$.

The very definition of the $(y_r)_{r=1}^{\infty}$ ensures that it is an ω-centralising sequence. Indeed, for $z \in N_n$, y_r commutes with z whenever $r \in I_n$, and so as $I_n \in \omega$ we have

$$\lim_{r\to\omega} \|y_r z - z y_r\|_2 = 0.$$

The $\|.\|_2$-density of the algebraic tensor product $\bigotimes_{n=1}^{\infty} M_n$ in R ensures that $(y_r)_{r=1}^{\infty}$ is ω-centralising. On the other hand, for $r \in I_1 \setminus \bigcap_{n=1}^{\infty} I_n$, we have

$$\|y_r x_r - x_r y_r\|_2 = \|z_r - u_r z_r u_r^*\|_2 \geq \|z_r - \mathbb{E}_{E_n}(z_r)\|_2 \to K > 0,$$

as $r \to \omega$. As $I_1 \setminus \bigcap_{n=1}^{\infty} I_n \in \omega$, this is

$$\|yx - xy\|_2 = \lim_{r\to\omega} \|y_r x_r - x_r y_r\|_2 \geq K > 0,$$

and so x and y do not commute in R_ω. In conclusion the only elements of R_ω commuting with every operator in D_ω must in fact lie in D_ω. We state this formally.

Theorem 3.4.2. *Let D be the Cartan masa in the hyperfinite* II_1 *factor R. For each $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, D_ω is a masa in R_ω.*

The fact that the R was written as a tensor product of *finite dimensional* factors was crucial here. If we fix a masa A in R not containing centralising sequences, then we can form the infinite tensor product $R^{\overline{\otimes}\infty} = S$ which is again hyperfinite and contains the masa $A^{\overline{\otimes}\infty} = B$. This B certainly contains non-trivial centralising sequences for S, but on the other hand $B^\omega \cap S'$ is never a masa in S_ω. To see this we factorise $S = R\overline{\otimes}R^{\overline{\otimes}\infty}$ by removing the first copy of R from the infinite tensor product. Similarly, B factorises as $A\overline{\otimes}A^{\overline{\otimes}\infty}$. Theorem 1.2.15 ensures that any ω-centralising sequence $(x_n)_{n=1}^{\infty}$ in B is equivalent to one in $1 \otimes A^{\overline{\otimes}\infty}$. In particular elements of $B^\omega \cap S'$ commute with the ω-centralising sequences of the form $(y_n \otimes 1_{R^{\overline{\otimes}\infty}})_{n=1}^\infty$, when $(y_n)_{n=1}^\infty$ lies in R_ω.

As a masa in R_ω, we could ask what is the Pukánszky invariant of $D^\omega \cap R'$?[21] There are obviously non-trivial normalisers of $D^\omega \cap R'$ in R_ω, for example $(1^{\otimes n} \otimes u_n)_{n=1}^\infty$, where $u_n \in R_n$ is a normaliser of the Cartan masa E_n with $\mathrm{tr}(u_n) = 0$. Hence, $1 \in \mathrm{Puk}\,(D_\omega)$. On the other hand, D_ω is not Cartan as there are no Cartan masas in R_ω, [18, Corollary 6.2]. It is possible that some other value might occur in this Pukánskzy invariant.

Question 3.4.3. What is $\mathrm{Puk}\,(D_\omega)$?

3.4.2 The singular Tauer masa of section 3.3 with $\Gamma(A) = 1$

Here we examine a singular masa A in the hyperfinite II_1 factor containing non-trivial centralising sequences. More specifically, we shall consider the singular Tauer masa A with $\Gamma(A) = 1$ that we constructed in section 3.3. Let us start with the somewhat lengthy technical lemma which is the key to our analysis.

Lemma 3.4.4. *Let A be a Tauer masa in the hyperfinite II_1 factor with respect to the chain $(N_m)_{m=1}^\infty$ of finite type I subfactors which arise as the tensor product $N_m = \bigotimes_{r=1}^n M_r$ as discussed in section 2.1. Fix natural numbers $k < n$ and suppose that, when we write*

$$A_n = \bigoplus_{e \in \mathcal{P}_{min}(A_{n-1})} e \otimes A_{n,n-1}^{(e)}$$

with respect to the decomposition $N_n = N_{n-1} \otimes M_n$, the $(A_{n,n-1}^{(e)})_{e \in \mathcal{P}_{min}(A_{n-1})}$ form a family of pairwise orthogonal masas in M_n. Write S for the tail II_1 factor $\left(\bigotimes_{m=n+1}^\infty M_m\right)''$, so that $R = N_n \overline{\otimes} S$, and set $B = A_{n-1} \otimes \mathbb{C}1 \overline{\otimes} S$. For any $x \in A_n \overline{\otimes} S$, we have

$$\left\| \left(I - \mathbb{E}_{N_k' \cap R}\right)(x - \mathbb{E}_B(x)) \right\|_2 \geq \left(1 - \frac{1}{d_k}\right) \|x - \mathbb{E}_B(x)\|_2, \tag{3.4.4}$$

where N_k is type I_{d_k}.

Proof. Take $x \in A_n \overline{\otimes} S$ which we write as

$$x = \sum_{e \in \mathcal{P}_{\min}(A_k)} e \otimes \left(\sum_{f \in \mathcal{P}_{\min}(A_{n-1,k}^{(e)})} f \otimes x_{e,f} \right),$$

[21]This object should now be redefined for non-separable Hilbert spaces, and will possibly involve uncountable cardinals - a good reason, if one was needed, to ensure our discussion of this subject ends promptly.

for some $x_{e,f} \in A_{n,n-1}^{(e\otimes f)}\overline{\otimes}S$. To calculate $\mathbb{E}_B(x)$ we project each $x_{e,f}$ from $A_{n,n-1}^{(e\otimes f)}\overline{\otimes}S$ onto $\mathbb{C}1 \otimes S$. Since the desired estimate, (3.4.4), involves $x - \mathbb{E}_B(x)$, we assume that each $x_{e,f} \in \left(A_{n,n-1}^{(e\otimes f)} \ominus \mathbb{C}1\right) \otimes S$. With this assumption, we are required to show that

$$\left\|x - \mathbb{E}_{N_k' \cap R}(x)\right\|_2 \geq \left(1 - \frac{1}{d_k}\right) \|x\|_2 . \tag{3.4.5}$$

Fortunately, calculation of $\mathbb{E}_{N_k' \cap R}(x)$ is easy. Indeed, (3.4.2) gives us

$$\mathbb{E}_{N_k' \cap R}(x) = 1_{N_k} \otimes \left(\frac{1}{d_k} \sum_{\substack{g \in \mathcal{P}_{\min}(A_k) \\ h \in \mathcal{P}_{\min}(A_{n-1,k}^{(g)})}} h \otimes x_{g,h} \right),$$

so that

$$\begin{aligned}
&\left\|x - \mathbb{E}_{N_k' \cap R}(x)\right\|_2^2 \\
&= \frac{1}{d_k} \sum_{e \in \mathcal{P}_{\min}(A_k)} \left\| \sum_{f \in \mathcal{P}_{\min}(A_{n-1,k}^{(e)})} f \otimes x_{e,f} - \frac{1}{d_k} \sum_{\substack{g \in \mathcal{P}_{\min}(A_k) \\ h \in \mathcal{P}_{\min}(A_{n-1,k}^{(g)})}} h \otimes x_{g,h} \right\|_2^2 \\
&= \frac{1}{d_k} \sum_{e \in \mathcal{P}_{\min}(A_k)} \left\| \left(1 - \frac{1}{d_k}\right) \sum_{f \in \mathcal{P}_{\min}(A_{n-1,k}^{(e)})} f \otimes x_{e,f} - \frac{1}{d_k} \sum_{\substack{g \in \mathcal{P}_{\min}(A_k) \\ g \neq e \\ h \in \mathcal{P}_{\min}(A_{n-1,k}^{(g)})}} h \otimes x_{g,h} \right\|_2^2 .
\end{aligned}$$

As $x_{e,f} \in \left(A_{n,n-1}^{(e\otimes f)} \ominus \mathbb{C}1\right) \otimes S$, distinct $x_{e,f}$ are orthogonal in $L^2(M_n \overline{\otimes} S)$. In particular, the two terms inside the norm in the last line above are orthogonal, giving us the estimate

$$\left\|x - \mathbb{E}_{N_k' \cap R}(x)\right\|_2^2 \geq \left(1 - \frac{1}{d_k}\right)^2 \frac{1}{d_k} \sum_{e \in \mathcal{P}_{\min}(A_k)} \left\| \sum_{f \in \mathcal{P}_{\min}(A_{n-1,k}^{(e)})} f \otimes x_{e,f} \right\|_2^2 . \tag{3.4.6}$$

This is exactly what we need, as

$$\|x\|_2^2 = \frac{1}{d_k} \sum_{e \in \mathcal{P}_{\min}(A_k)} \left\| \sum_{f \in \mathcal{P}_{\min}(A_{n-1,k}^{(e)})} f \otimes x_{e,f} \right\|_2^2 ,$$

which substitutes into (3.4.6) to give the desired (3.4.5). □

The singular Tauer masa with property Γ in R of section 3.3 is constructed relative to the tensor product $R = (\bigotimes_{r=1}^{\infty} M_r)''$ of matrix algebras of quickly growing size.[22] A Tauer masa A is inductively constructed inside the chain of finite tensor products $N_n = \bigotimes_{r=1}^{n} M_r$, by fixing any masa A_1 in M_1 and defining A_{n+1} in terms of A_n according to whether n is even or odd:

1. For odd n, we take $A_{n+1} = A_n \otimes D_{n+1}$ for some masa D_{n+1} in M_{n+1};

2. For even n, we ensure that $(A^{(e)}_{n+1,n})_{e\in\mathcal{P}_{\min}A_n}$ is a family of pairwise orthogonal masas in M_{n+1}, and define

$$A_{n+1} = \bigoplus_{e\in\mathcal{P}_{\min}A_n} e \otimes A^{(e)}_{n+1,n}.$$

The singularity of the resulting Tauer masa A follows, Proposition 3.3.8, from the fact that stage 2 occurs infinitely often. That A is a Γ masa in R is a consequence of the infinite occurrence of stage 1, see Proposition 3.3.9.

We regard this process as an infinite game of blackjack: the odd stage is thought of as *sticking*, A_{n+1} is designed to contain non-trivial unitaries commuting with $\bigotimes_{r=1}^{n} M_r$; whereas the even stage involves *twisting* the masa A_n so as to ensure that the only unitaries in $\bigotimes_{r=1}^{n} M_r$ that come close to normalising A_{n+1} are those near A_n, hence guaranteeing the singularity of A. It is then clear that given any subset $I \subset \mathbb{N}$ such that I and $\mathbb{N}\setminus I$ are both infinite, we can construct a singular Tauer masa A_I in the hyperfinite II_1 factor with $\Gamma(A_I) = 1$, by 'twisting' the $(n+1)$-th stage of the construction if $n \in I$ and 'sticking' otherwise.

This gives a collection of singular Γ masas. Time constraints have prevented us from investigating the question of conjugacy for this family. In particular, we should ask whether varying the asymptotic density of I in $\mathbb{N}$ allows us to produce non-conjugate singular masas with property Γ, or whether all these masas are conjugate in R. At present, it is still unknown whether there are non-conjugate singular masas in the hyperfinite II_1 factor (or any other II_1 factor for that matter) with property Γ and the same Pukánszky invariant.

Question 3.4.5. Are the masas A_I in R conjugate via an automorphism of R? Can we find two non-conjugate singular Γ masas with the same Pukánszky invariant in R?

This suggests the question of what Pukánszky invariants are possible for Γ masas in R. So as to avoid a diversion, we shall return to this at the end of the

[22] More precisely, Construction 3.3.7 required that the M_r are $k_r \times k_r$ matrices, where the k_r are primes such that $k_{r+1} > k_1 \dots k_r$.

subsection. Returning to the main point of this subsection, we can now show exactly where the centralising sequences in the masas A_I actually lie. Define a von Neumann subalgebra B_I of A_I by

$$B_I = \left(\bigotimes_{n=1}^{\infty} B_I(n)\right)'',$$

where

$$B_I(n) = \begin{cases} A_1 & n = 1 \\ D_n & n \geq 2 \text{ and } n-1 \notin I \\ \mathbb{C}1 & n \geq 2 \text{ and } n-1 \in I \end{cases}.^{23} \tag{3.4.7}$$

This B_I is abelian, non-atomic and designed so that it contains all the 'obvious' centralising sequences of R in A_I. In fact all the centralising sequences lying in A_I are found in B_I.

Theorem 3.4.6. *Let $I \subset \mathbb{N}$ be such that both I and $I^C = \mathbb{N} \setminus I$ are both infinite. Let $A_I \subset R$ be the singular Tauer masa in the hyperfinite II_1 factor, constructed above which contains non-trivial centralising sequences for R and B_I be the sub von Neumann algebra of A_I defined above. For every $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ we have $(A_I)^\omega \cap R' = (B_I)^\omega \cap R'$ and so $(A_I)^\omega \cap R'$ is not a masa in $R^\omega \cap R'$.*

Proof. The conclusion that $(A_I)^\omega \cap R'$ is not a masa in $R^\omega \cap R'$ is immediate from the first part of the Theorem. Indeed, if we let I be enumerated as $n_1 < n_2 < n_3 < \dots$, and take $x_r = 1^{\otimes n_r - 1} \otimes y_r$ for some unitary $y_r \in M_{n_r}$ with $\mathrm{tr}(y_r) = 0$, then $(x_r)_{r=1}^{\infty}$ is a centralising sequence for R. This sequence is not ω-equivalent to any sequence lying in B_I and evidentally commutes with any sequence lying in B_I. In this way $(x_r)_{r=1}^{\infty}$ lies in the commutant of $(A_I)^\omega \cap R'$ in $R^\omega \cap R'$, but is not in $(A_I)^\omega \cap R'$.

For the main statement, we fix $n_0 > k \geq 1$, with $n_0 \notin I$. Enumerate those n in the range $k < n < n_0$ with $n - 1 \in I$ as $n_1 > n_2 > \dots > n_l$. Let us define algebras B_s for $0 \leq s \leq l$, by

$$B_s = A_{n_s - 1} \otimes \bigotimes_{r=n_s}^{n_0} B_I(r),$$

where the $B_I(r)$ are as given in (3.4.7). As $n_0 \notin I$, this definition ensures that $B_0 = A_{n_0}$. Take $x \in B_s$ for some $0 \leq s < l$. Decompose $x = \mathbb{E}_{B_{s+1}}(x) + (x - \mathbb{E}_{B_{s+1}}(x))$, and note that

$$\mathbb{E}_{B_{s+1}}(x) \in N_{n_{s+1}-1} \otimes \mathbb{C}1 \otimes \bigotimes_{r=n_{s+1}+1}^{n_0} M_r = X_1,$$

[23] Recall that the D_n are the fixed masas in M_n used at the 'sticking' stages of the construction of A_I previously.

while

$$x - \mathbb{E}_{B_{s+1}}(x) \in N_{n_{s+1}-1} \otimes (M_{n_{s+1}} \ominus \mathbb{C}1) \otimes \bigotimes_{r=n_{s+1}+1}^{n_0} M_r = X_2.$$

The linear spaces X_1 and X_2, defined above, are orthogonal in $L^2(N_{n_0})$. Furthermore, the form, (3.4.2), of the conditional expectation onto $N_k' \cap R$ ensures that

$$\mathbb{E}_{N_k' \cap R}(X_1) \subset X_1,$$

and

$$\mathbb{E}_{N_k' \cap R}(X_2) \subset X_2.$$

In this way we have

$$\left\|x - \mathbb{E}_{N_k' \cap R}(x)\right\|_2^2 = \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right) \mathbb{E}_{B_{s+1}}(x)\right\|_2^2 + \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right)\left(x - \mathbb{E}_{B_{s+1}}(x)\right)\right\|_2^2. \tag{3.4.8}$$

The algebra B_s is equal to $A_{n_{s+1}} \otimes \bigotimes_{r=n_{s+1}+1}^{n_0} B_I(r)$, as

$$A_{n_s-1} = A_{n_{s+1}} \otimes \bigotimes_{r=n_{s+1}+1}^{n_s-1} B_I(r).$$

The assumption that $n_{s+1} - 1 \in I$ ensures that the $(A^{(e)}_{n_{s+1}, n_{s+1}-1})_{e \in \mathcal{P}_{\min}(A_{n_{s+1}-1})}$ are pairwise orthogonal masas in $M_{n_{s+1}}$. We are then able to hit the last term in (3.4.8) with Lemma 3.4.4, yielding

$$\left\|\left(I - \mathbb{E}_{N_k' \cap R}\right)\left(x - \mathbb{E}_{B_{s+1}}(x)\right)\right\|_2 \geq \left(1 - \frac{1}{d_k}\right)\left\|x - \mathbb{E}_{B_{s+1}}(x)\right\|_2, \tag{3.4.9}$$

where N_k is type I_{d_k}.

We now assemble all of this. Fix $y \in A_{n_0}$ which we have already observed is B_0. The preceding argument gives

$$\left\|y - \mathbb{E}_{N_k' \cap R}(y)\right\|_2^2 \geq \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right) \mathbb{E}_{B_1}(y)\right\|_2^2 + \left(1 - \frac{1}{d_k}\right)^2 \left\|y - \mathbb{E}_{B_1}(y)\right\|_2^2.$$

Suppose that we have

$$\left\|y - \mathbb{E}_{N_k' \cap R}(y)\right\|_2^2 \geq \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right) \mathbb{E}_{B_s}(y)\right\|_2^2 + \left(1 - \frac{1}{d_k}\right)^2 \left\|y - \mathbb{E}_{B_s}(y)\right\|_2^2, \tag{3.4.10}$$

for some $s < l$. Apply (3.4.9) again to the first term on the right hand side above, with $x = \mathbb{E}_{B_s}(y)$. In this way we have

$$\begin{aligned}\left\|y - \mathbb{E}_{N_k' \cap R}(y)\right\|_2^2 &\geq \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right) \mathbb{E}_{B_{s+1}}(y)\right\|_2^2 + \left(1 - \frac{1}{d_k}\right)^2 \left\|\mathbb{E}_{B_s}(y) - \mathbb{E}_{B_{s+1}}(y)\right\|_2^2 \\ &\quad + \left(1 - \frac{1}{d_k}\right)^2 \left\|y - \mathbb{E}_{B_s}(y)\right\|_2^2 \\ &= \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right) \mathbb{E}_{B_{s+1}}(y)\right\|_2^2 + \left(1 - \frac{1}{d_k}\right)^2 \left\|y - \mathbb{E}_{B_{s+1}}(y)\right\|_2^2,\end{aligned}$$

which is (3.4.10) with s replaced by $s+1$. Inductively, we have (3.4.10) for $s = l$.

Why is this what we need? Write $C_k = 1 - 1/d_k$, and use the fact that $\mathbb{E}_{B_l}(y) - \mathbb{E}_{N_k' \cap R}\mathbb{E}_{B_l}(y)$ is orthogonal to $y - \mathbb{E}_{B_l}(y)$ again to rewrite our conclusion as

$$\begin{aligned} \left\|y - \mathbb{E}_{N_k' \cap R}(y)\right\|_2^2 &\geq \left\|\left(I - \mathbb{E}_{N_k' \cap R}\right)\mathbb{E}_{B_l}(y)\right\|_2^2 + C_k^2 \left\|y - \mathbb{E}_{B_l}(y)\right\|_2^2 \\ &= (1 - C_k^2)\left\|\left(I - \mathbb{E}_{N_k' \cap R}\right)\mathbb{E}_{B_l}(y)\right\|_2^2 + C_k^2 \left\|y - \mathbb{E}_{N_k' \cap R}\mathbb{E}_{B_l}(y)\right\|_2^2 \\ &\geq C_k^2 \left\|y - \mathbb{E}_{N_k' \cap R}\mathbb{E}_{B_l}(y)\right\|_2^2. \end{aligned} \tag{3.4.11}$$

Finally, $\mathbb{E}_{N_k' \cap R}(B_l) \subset B_I$. This holds as, for $k < n < n_l$, we have $n - 1 \notin I$, so that

$$A_{n_l - 1} = A_k \otimes \bigotimes_{r=k+1}^{n_l - 1} B_I(r),$$

and

$$B_l = A_k \otimes \bigotimes_{r=k+1}^{n_0} B_I(r).$$

Ergo,

$$\mathbb{E}_{N_l \cap R}(B_l) \subset 1_{N_k} \otimes \bigotimes_{r=k+1}^{n_0} B_I(r) \subset B_I.$$

In this way, the estimate (3.4.11) gives

$$\left\|y - \mathbb{E}_{N_k' \cap R}(y)\right\|_2 \geq C_k \left\|y - \mathbb{E}_{B_I}(y)\right\|_2, \tag{3.4.12}$$

for all $y \in A_{n_0}$, as

$$\left\|y - \mathbb{E}_{B_I}(y)\right\|_2 \leq \left\|y - \mathbb{E}_{N_k' \cap R}\mathbb{E}_{B_l}(y)\right\|_2.$$

In this final estimate n_0 is arbitrary. Given an ω-centralising sequence $(y_r)_{r=1}^\infty$ in A_I, by density we may assume that each $y_r \in \bigcup_{n=1}^\infty A_n$. Fix some k, which could be 1. Proposition 3.4.1 ensures that

$$\lim_{r \to \omega} \left\|y_r - \mathbb{E}_{N_k' \cap R}(y_r)\right\|_2 = 0,$$

when (3.4.12) gives

$$\lim_{r \to \omega} \left\|y_r - \mathbb{E}_{B_I}(y_r)\right\|_2 = 0,$$

as $C_k > 0$. This is precisely the statement that $(y_r)_{r=1}^\infty$ is ω-equivalent to some centralising sequence in B_I, exactly as required. □

This example was presented as I believe the behaviour of this masa should be typical. It is not easy to see how one could have a singular Γ masa A in R with $A^\omega \cap R'$ a masa in the central sequence algebra R_ω. These two conditions pull in different directions. The singularity requires that different cut downs of A of the same size look sufficiently different.[24] For $A^\omega \cap R'$ to be a masa in $R^\omega \cap R'$, we should need 'large subalgebras' of these cutdowns to be close. At present though, I have been unable to formalise these ideas into a proof.

Question 3.4.7. Does every singular masa A in the hyperfinite II_1 factor R fail to give rise to a masa $A^\omega \cap R'$ in the ultraproduct $R^\omega \cap R'$? If a masa A in R does induce a masa $A^\omega \cap R'$ in $R^\omega \cap R'$, is A necessarily Cartan?

We now briefly indicate, a masa which fails to decompose as a direct sum of a Γ masa and a completely non-Γ masa. The plan is to take a direct sum of two the masas in this section corresponding to the same I, but with different D_n. For each even n, we take two orthogonal masas D_n and E_n in M_n. We define two Tauer masas A and B in R, by twisting as usual when n is even, and when n is odd we take

$$A_{n+1} = A_n \otimes D_n, \quad B_{n+1} = B_n \otimes E_n.$$

These masas are Γ masas, and elements of $A^\omega \cap R'$ are equivalent to ω centralising sequences in the infinite tensor product $(\mathbb{C}1 \otimes D_2 \otimes \mathbb{C}1 \otimes D_4 \otimes \dots)''$ which we call D, while $B^\omega \cap R'$ is identified with the ω-centralising sequences lying in $E = (\mathbb{C}1 \otimes E_2 \otimes \mathbb{C}1 \otimes E_4 \otimes \dots)''$.

Consider $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, a masa in $\mathrm{Mat}_2(R)$. Let $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ be a non-zero projection in $A \oplus B$. If $e \neq 0$ then $\begin{pmatrix} Ae & 0 \\ 0 & 0 \end{pmatrix}$ is Γ in $\begin{pmatrix} eRe & 0 \\ 0 & 0 \end{pmatrix}$, whereas if $f \neq 0$ then $\begin{pmatrix} 0 & 0 \\ 0 & Bf \end{pmatrix}$ is a Γ masa in $\begin{pmatrix} 0 & 0 \\ 0 & fRf \end{pmatrix}$. In any event, we deduce that there is no non-zero cut down of $A \oplus B$ which is completely non-Γ.

On the other hand, all ω-centralising sequences of $A \oplus B$ are trivial. Let $\left(\begin{pmatrix} x_n & 0 \\ 0 & y_n \end{pmatrix}\right)_{n=1}^\infty$ be an ω-centralising sequence. For this sequence to asymptotically commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we must have

$$\lim_{n\to\omega} \|x_n - y_n\|_2 = 0.$$

The preceding discussion allows us to assume that each x_n lies in D while the y_n lie in E. These two algebras are orthogonal, as each D_n and E_n are. Therefore,

$$\|x_n - \mathrm{tr}(x_n)1\|_2 = \|x_n - \mathbb{E}_{E_n}(x_n)\|_2 \leq \|x_n - y_n\|_2 \to 0,$$

as $n \to \omega$. In this way $(x_n)_{n=1}^\infty$ is seen to be trivial, whence so too is $(y_n)_{n=1}^\infty$, exactly as claimed.

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[24]after conjugation by a partial isometry which interchanges the equivalent reducing projections

A more refined analysis would have shown that $\Gamma(A \oplus B) = 1/2$, with a projection $p \in A \oplus B$ giving rise to a Γ masa $(A \oplus B)p$ in the cut-down, if and only if, either $p \leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $p \leq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Now back to possible Pukánszky invariants of singular Γ-masas. Note that tensoring a Γ-masa A, with $\mathrm{Puk}\,(A) = \{1\}$, by one of Størmer and Neshveyev's examples, [38], gives Γ-masas in R whose Pukánskzy invariant can be any subset of $\mathbb{N}_\infty$ containing 1. Størmer and Neshveyev's examples all satisfy the weak asymptotic homomorphism property,[25] and we can demand that A does too. In this way we ensure, in the absence of a positive answer to Question 1.4.29, that the masas resulting from these tensor products are singular. Alternatively, take an infinite von Neumann tensor product of a singular masa A in the hyperfinite II_1 factor, with $\mathrm{Puk}\,(A) = \{\infty\}$. This gives a singular[26] Γ-masa, B, with $\mathrm{Puk}\,(B) = \{\infty\}$. Very recent examples of Dykema, which will hopefully be found in [17], show that any subset of $\mathbb{N}_\infty$ containing ∞ can be attained as the Pukánszky invariant of a singular masa in the hyperfinite II_1 factor. Take the von Neumann tensor product of one of these examples and B, to see that we can find Γ-masas whose Pukánszky invariant can be any subset of $\mathbb{N}_\infty$ containing ∞. If 1 is not in this Pukánszky invariant, then the resulting masa is automatically singular (by Popa's result, Theorem 3.1.3), and we have already dealt with the case when 1 is in the Pukánszky invariant previously. Let us summarise this diversion formally.

Theorem 3.4.8. *Let X be a subset of $\mathbb{N}_\infty$ such that either $1 \in X$, or $\infty \in X$. There exists a singular Γ-masa A in the hyperfinite II_1 factor R, with $\mathrm{Puk}(A) = X$.*

We do not have examples yet of singular Γ masas for the other known values of the Pukánszky invariant.

Question 3.4.9. Let X be a subset of $\mathbb{N}_\infty$, with $1, \infty \notin X$, such that there is some masa in R with Pukánskzy invariant X. Does there exist a singular Γ-masa A in R with $\mathrm{Puk}\,(A) = X$?

3.5 Some thoughts about automorphisms fixing masas

Here we investigate some questions that arise when considering Connes' work on automorphisms in the context of masas. Firstly, as noted in [38], it is known that

[25] They all come from weakly mixing ergodic actions, and Lemma 3.2 of [54], shows that in this case we obtain the weak asymptotic homomorphism property.

[26] This time by an infinite version of Proposition 3.1.2 and then part 2 of Theorem 3.1.3.

Connes' connection between the closure of the inner automorphism group and the failure of property Γ, [6], holds in this situation. A proof is readily obtained by following the original ([6]) or Takesaki's account, [69, Theorem XIV.3.8].

Proposition 3.5.1. *Let A be a masa in a separable II_1 factor N. The following conditions are equivalent:*

1. *A does not contain non-trivial centralising sequences for N;*

2. *$\{ Ad\, u \,|\, u \in \mathcal{U}(A) \}$ is closed in $Aut(N)$.*

3.5.1 A relative automorphism group

We shall define a relative automorphism group for the inclusion of a masa A in a II_1 factor. We want the inner automorphisms in this group to be adjunction by unitaries in A, leading us to the definition.

Definition 3.5.2. Given a masa A in a II_1 factor N. Write $\mathrm{Aut}\,(A \subset N)$ for

$$\{ \theta \in \mathrm{Aut}\,(N) \,|\, \theta(a) = a \text{ for all } a \in A \},$$

the group of *relative automorphisms* of A in N, which inherits the u-topology from $\mathrm{Aut}\,(N)$. Write $\mathrm{Inn}\,(A \subset N) = \mathrm{Inn}\,(N) \cap \mathrm{Aut}\,(A \subset N)$, the *relative inner automorphisms* of A in N, which consists of the automorphisms $\mathrm{Ad}\ u$ for unitaries $u \in A$.

It is immediate that $\mathrm{Inn}\,(A \subset N)$ is an abelian subgroup, so normal in $\mathrm{Aut}\,(A \subset N)$ - unsurprisingly we write $\mathrm{Out}\,(A \subset N)$ for the quotient. In fact $\mathrm{Inn}\,(A \subset N)$ lies in the centre, $\mathcal{Z}(\mathrm{Aut}\,(A \subset N))$, of $\mathrm{Aut}\,(A \subset N)$, as the next simple calculation confirms. Take $\theta \in \mathrm{Aut}\,(A \subset N)$ and consider $u \in \mathcal{U}(A)$. For $x \in N$, we have

$$\left(\theta \circ \mathrm{Ad}\ u \circ \theta^{-1}\right)(x) = \theta(u\theta^{-1}(x)u^*) = \theta(u)x\theta(u^*) = uxu^*,$$

so $\theta \circ \mathrm{Ad}\ u = \mathrm{Ad}\ u \circ \theta$. To what extent the converse to this is true is not easy to determine. For example, a lengthy calculation[27] can be used to demonstrate that when A is the generator masa in a free group factor $\mathcal{L}\,(\mathbb{F}_k)$, then $\mathrm{Out}\,(A \subset \mathcal{L}\,(\mathbb{F}_k))$ is an I.C.C. group, and so here we have $\mathcal{Z}(\mathrm{Aut}\,(A \subset \mathcal{L}\,(\mathbb{F}_k))) = \mathrm{Inn}\,(A \subset \mathcal{L}\,(\mathbb{F}_k))$. When B is the Laplacian masa in $\mathcal{L}\,(\mathbb{F}_k)$,[28] we have been unable to make progress in this area. Suppose that $a_1, \ldots, a_k$ are the generators of $\mathbb{F}_k$, then the automorphism taking each a_i to a_i^{-1} certainly lies in $\mathrm{Aut}\,(B \subset \mathcal{L}\,(\mathbb{F}_k))$, which commutes

[27] Sufficiently long that the reader will be pleased to learn that it has been omitted!

[28] If $a_1, \ldots, a_k$ are the generators of $\mathbb{F}_k$, let $w = \sum_{i=1}^{k}(a_i + a_i^{-1})$ - an element of the group algebra $\mathbb{C}\mathbb{F}_k$. The Laplacian masa B is the von Neumann subalgebra of $\mathcal{L}\,(\mathbb{F}_k)$ generated by w. See [60] for further background.

with all the elements of $\mathrm{Aut}\,(B \subset \mathcal{L}(\mathbb{F}_k))$ that I am aware of.[29] This is really all part of a limited attempt to deal with a conjugacy question which, despite the considerable interest in free group factors, seems not to have been asked.

Question 3.5.3. Let $k \geq 2$ be a fixed integer. Is the generator masa in $\mathcal{L}(\mathbb{F}_k)$ conjugate to the Laplacian masa in $\mathcal{L}(\mathbb{F}_k)$ via an automorphism of $\mathcal{L}(\mathbb{F}_k)$?

It is easy to see that we can not have $\mathcal{Z}(\mathrm{Aut}\,(A \subset N)) = \mathrm{Inn}\,(A \subset N)$ in general - simply take a Γ masa A in the hyperfinite II_1 factor R, when Proposition 3.5.1 ensures that $\mathrm{Inn}\,(A \subset R)$ is not closed, while the centre certainly is.

3.5.2 Density of the relative inner automorphism group

Recall that all automorphisms of the hyperfinite II_1 factor are approximately inner. We define the approximately inner relative automorphism group of a masa A in N to be the closure of $\mathrm{Inn}\,(A \subset N)$ in $\mathrm{Aut}\,(A \subset N)$.

Question 3.5.4. Let A be a masa in the hyperfinite II_1 factor R. When is $\mathrm{Inn}\,(A \subset R)$ dense in $\mathrm{Aut}\,(A \subset R)$?

In the case of the Cartan masa, we can give a quick positive answer.

Proposition 3.5.5. *Let D be the Cartan masa in the hyperfinite* II_1 *factor R. Then $\mathrm{Inn}(D \subset R)$ is dense in $\mathrm{Aut}(D \subset R)$.*

Proof. We work, as in subsection 3.4.1, with R as an infinite von Neumann tensor product $(\bigotimes_{n=1}^{\infty} \mathbb{M}\mathrm{at}_2(\mathbb{C}))''$ of matrix algebras. By choosing a masa E in $\mathbb{M}\mathrm{at}_2(\mathbb{C})$ we obtain the Cartan masa D in R as the tensor product $(\bigotimes_{n=1}^{\infty} E)''$. Fix an automorphism θ in $\mathrm{Aut}\,(D \subset R)$.

Fix n, and write $N = \bigotimes_{r=1}^{n} \mathbb{M}\mathrm{at}_2(\mathbb{C})$ and choose matrix units $(e_{i,j})_{i,j=1}^{k}$ for N such that the $e_{i,i}$ are the minimal projections of $E^{\otimes n}$. It is enough to show that we can find a unitary $u \in D$ such that $\theta(x) = uxu^*$ for all $x \in N$. Define

$$u = \sum_{i=1}^{k} \theta(e_{i,1})e_{1,i},$$

and note that

$$uu^* = \sum_{i,j=1}^{k} \theta(e_{i,1})e_{1,i}e_{j,1}\theta(e_{1,j}) = \sum_{i=1}^{k} \theta(e_{i,i}) = 1,$$

[29]Unfortunately there are not very many of these! I have been unable to determine whether the 'swap automorphism' described in the text is central for $\mathrm{Aut}\,(B \subset \mathcal{L}(\mathbb{F}_k))$.

so that u is a unitary in R. Furthermore, for fixed i, j we have

$$ue_{i,j}u^* = \theta(e_{i,1})e_{1,i}e_{i,j}e_{j,1}\theta(e_{1,j}) = \theta(e_{i,1}e_{1,1}e_{1,j}) = \theta(e_{i,j}),$$

so that on N, θ agrees with Ad u.

We check that u is an element of D. Take some $d \in \left(\bigotimes_{r=n+1}^{\infty} E\right)''$ and consider $e_{i,i} \otimes d$. We have

$$\begin{aligned}(e_{i,i} \otimes d)u &= (e_{i,i} \otimes d)\theta(e_{i,1})e_{1,i} = \theta(e_{i,1} \otimes d)e_{1,i} = \theta(e_{i,1})(e_{1,1} \otimes d)e_{1,i}\\ &= \theta(e_{i,1})e_{1,i}(e_{i,i} \otimes d) = u(e_{i,i} \otimes d),\end{aligned}$$

so that u commutes with each $e_{i,i} \otimes d$. Hence, u commutes with every element of D. Since D is a masa, we deduce that u must lie in D. □

The answer to question 3.5.4 is not always positive. To see this we examine Pukánskzy's original examples, [51] of masas in R, in the form presented in [61, Example 5.1]. Fix $n \geq 2$, and set

$$F_n = \left\{ \frac{p}{q}2^{kn} \,\middle|\, p, q, k \in \mathbb{Z} \text{ with } p, q \text{ odd} \right\},$$

an index n subgroup of $\mathbb{Q}^{\times}$, the non-zero rationals under multiplication. Take

$$G = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \,\middle|\, f \in F_n, x \in \mathbb{Q} \right\},$$

and

$$H = \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, f \in F_n \right\},$$

so that, as noted in [61], $\mathcal{L}(G)$ is the hyperfinite II_1 factor and $\mathcal{L}(\mathcal{H})$ is a strongly singular masa in $\mathcal{L}(G)$ with Pukánszky invariant $\{n\}$.[30] Sinclair and Smith use Corollary 1.4.27 to demonstrate the strong singularity of this masa. We wish to deduce slightly more, namely that $\mathcal{L}(H)$ is not a Γ masa and so that $\mathrm{Inn}(\mathcal{L}(H) \subset \mathcal{L}(G))$ is closed - for which we use Proposition 1.2.12. We must check then that for some $g \in G \setminus H$ we have $gHg^{-1} \cap H = \{1\}$. It is no hardship to do this for an arbitrary $g = \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}$ with $x \neq 0$. For then we have

$$\begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} h & x(1-h) \\ 0 & 1 \end{pmatrix},$$

so that this can only lie in H when $h = 1$ as claimed.

[30] Pukánskzy used a countable union of finite fields to produce his examples, as this obviously gave rise to the hyperfinite II_1 factor. The construction above uses Connes' uniqueness of the injective II_1 factor, [7], to see that $\mathcal{L}(G)$ is hyperfinite, as the group G is amenable.

Take some $y \in \mathbb{Q}^{\times} \setminus F_n$ and define ϕ_y on G by

$$\phi_y\left(\begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} f & xy \\ 0 & 1 \end{pmatrix}. \tag{3.5.1}$$

This ϕ_y maps G into G and then the first form of the definition ensures that it is actually a automorphism of G. Furthermore, H is fixed pointwise so ϕ_y extends to an automorphism of $\mathcal{L}(G)$ lying in Aut $(\mathcal{L}(H) \subset \mathcal{L}(G))$ which we also denote by ϕ_y. We shall show that ϕ_y is not an inner automorphism of $\mathcal{L}(G)$, when it provides an element of Aut $(\mathcal{L}(H) \subset \mathcal{L}(G))$ not in the closure of the relative inner automorphisms.[31]

Something more general is going on here. Whenever we have an automorphism of a discrete I.C.C. group which is not inner as an automorphism of the group it can not be inner as an automorphism of the group II_1 factor, [32]. Surprisingly this result is not well known - we provide a proof in Appendix C for completeness, where it appears as Theorem C.1. To complete our argument then, it suffices to show that the ϕ_y above is not an inner automorphism of G. This is immediate, for suppose there was some $g \in G$ with $\phi_y = \mathrm{Ad}\ g$, then as H is maximal abelian in G,[32] our g must lie in H; take $g = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ for some $h \in F_n$. Equation (3.5.1) then demonstrates that $xy = xh$ for all $x \in \mathbb{Q}$. Just taking $x = 1$ though, is enough to see that $h = y$ - a contradiction as $y \notin F_n$. Let us summarise this discussion.

Proposition 3.5.6. *The example of a strongly singular masa A in the hyperfinite* II_1 *factor, R, found in Example 5.1 of [61] with Pukánszy invariant $\{n\}$ for $n = 2, 3, \ldots, \infty$,*[33] *fails to have property Γ. For this masa Inn$(A \subset R) \neq$ Aut$(A \subset R)$ and, as the inner automorphism group is closed, it can not be dense in Aut$(A \subset R)$.*

When $n = 1$, we have $F_1 = \mathbb{Q}^{\times}$, and the resulting inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$ gives a masa in the hyperfinite II_1 factor with Pukánszky invariant $\{1\}$. This masa also fails to contain non-trivial centralising sequences for R for exactly the same reason as the $n \geq 2$ case, but we do not obtain an non-inner automorphism in Aut $(\mathcal{L}(H) \subset \mathcal{L}(G))$ in the same way as the $n \geq 2$ case. Here $F_1 = \mathbb{Q}^{\times}$, so there are no y to choose in $\mathbb{Q}^{\times} \setminus F_1$. At present we do not know whether, in this $n = 1$ case, the relative inner automorphisms are all of Aut $(\mathcal{L}(H) \subset \mathcal{L}(G))$.

[31] Which we have already noted are closed in this case.

[32] $\mathcal{L}(H)$ is a masa in $\mathcal{L}(G)$.

[33] Actually we have only established this for n finite here. The infinite case works also with exactly the same proof. Here F_∞ is the infinite index subgroup of $\mathbb{Q}^{\times}$ of all rationals p/q with p and q odd.

To see that some possible connection between the relative automorphism groups and the Pukánszky invariant is not all that far fetched we need some folklore. Given an automorphism θ of a II_1 factor N, we may extend θ to an operator U on $L^2(N)$ by $\|.\|_2$-continuity. This U is a unitary in $\mathbb{B}(L^2(N))$, which implements θ. By this we mean that, as operators on $L^2(N)$ we have $UxU^* = \theta(x)$. This follows as, for $y \in N \subset L^2(N)$, we have

$$(U \circ x)(y) = U(xy) = \theta(x)\theta(y) = (\theta(x) \circ U)(y),$$

so that $Ux = \theta(x)U$. Suppose now that we have a masa A in N, and that θ fixes the operators in A pointwise. Then for each $a \in A$ we have $UaU^* = \theta(a) = a$, so that $U \in A'$. Furthermore, for $a \in A$ and $y \in N \subset L^2(N)$ we have

$$(UJaJU^*)(y) = (UJaJ^*)\theta^{-1}(y) = U(\theta^{-1}(y)a^*) = y\theta(a^*) = ya^* = (JaJ)(y),$$

so that U commutes with JAJ too.

Proposition 3.5.7. *Let A be a masa in a II_1 factor N. Given an automorphism $\theta \in Aut(A \subset N)$, we can extend θ to a unitary operator U on $L^2(N)$ which has $UxU^* = \theta(x)$ for each $x \in N$. This unitary U lies in $\mathcal{A}'$, where $\mathcal{A} = (A \cup JAJ)''$ is the augmented algebra used to compute the Pukánszky invariant of A.*

When A is a masa in N with Pukánszky invariant $\{1\}$, Proposition 3.5.7 ensures that the unitaries implementing automorphisms in $\mathrm{Aut}\,(A \subset N)$ actually lie in $\mathcal{A}$. In the hyperfinite case, is this enough to show that the relative inner automorphisms are dense?

Question 3.5.8. Do all masas A in R with $\mathrm{Puk}\,(A) = \{1\}$ have $\overline{\mathrm{Inn}\,(A \subset R)} = \mathrm{Aut}\,(A \subset R)$?

We can say something positive when A is a masa in N with $\mathrm{Puk}\,(A) = \{1\}$. Under this assumption, $\mathcal{A}' = \mathcal{A}$ - an abelian algebra, therefore $\mathrm{Aut}\,(A \subset N)$ is also abelian.

3.5.3 Characterising approximately inner relative automorphisms

We can generalise Connes' characterisation of approximately inner automorphisms of a II_1 factor, ([7, Theorem 3.1], Theorem 1.2.16) to the relative situation. Actually more is true. In the next result, we can replace the masa A be any von Neumann subalgebra M such that its relative commutant $M' \cap N = M^{\mathrm{C}}$ is diffuse. Condition 1 remains unchanged, that is θ should lie in the closure of the inner automorphisms that fix M pointwise - here adjunction by unitaries in $M^{\mathbb{C}}$. The references to A in conditions 2, 3 and 4 should be replaced by $M^{\mathbb{C}}$. For masas, we have $A^{\mathbb{C}} = A$ which is certainly a diffuse von Neumann algebra.

Theorem 3.5.9. *Let A be a masa in a separable II_1 factor N. Given $\theta \in Aut(A \subset N)$ the following conditions are equivalent.*

1. *θ lies in $\overline{Inn(A \subset N)}$, the closure of the relative inner automorphisms;*
2. *There exists an automorphism Θ of $C^*(N, JNJ, e_A) \subset \mathbb{B}(L^2(N))$, such that $\Theta|_N = \theta$, $\Theta|_{JNJ} = I$ and $\Theta(e_A) = e_A$;*
3. *For unitaries $u_1, \ldots, u_n \in N$ and $\epsilon > 0$ there exists $\xi \in L^2(A) \subset L^2(N)$ with $\|\xi\|_2 = 1$ and*
$$\|\theta(u_i)Ju_iJ\xi - \xi\|_2 < \epsilon,$$
for each $i = 1, 2, \ldots, n$;
4. *There exists a bounded sequence $(a_n)_{n=1}^{\infty}$ in A, not converging strongly to 0, such that $a_n y - \theta(y)a_n$ converges strongly to 0, for each $y \in N$.*

To prove this result, one should work through Connes' original proof and carefully follow the consequences of the additional hypotheses in each deduction. For example, Connes original proof of $1 \Rightarrow 2$ takes an element $\theta = \lim_{n\to\infty} \mathrm{Ad}\ u_n$ of $\overline{\mathrm{Inn}\,(N)}$, and considers the inner automorphisms α_n of $C^*(N, N')$ implemented by u_n. He continues[34]:

> *For any element $x = \sum_{i=1}^{k} x_i y_i$ of the algebraic tensor product of N and N', the sequence $(\alpha_n(x))_{n=1}^{\infty}$ of elements of $\mathbb{B}(L^2(N))$ converges strongly to $\sum_{i=1}^{k} \theta(x_i)y_i$. So the automorphism $\theta \otimes 1$ of the algebraic tensor product $N \otimes N'$ is norm preserving for the norm of $\mathbb{B}(L^2(N))$ and thus extends uniquely to the norm closure $C^*(N, N')$ as an automorphism of $C^*(N, N')$ satisfying the required conditions.*

Instead of tensor products, we could have taken x in $\mathrm{Alg}^*(N, N')$, the *-algebra generated by N and N'.

Now let us see what happens when each of our initial unitaries u_n lies in the masa A. In this case $u_n e_A u_n^* = e_A$,[35] so that $\alpha_n = \mathrm{Ad}\ u_n$ is an automorphism of $\mathrm{Alg}^*(N, N', e_A)$. An element x of this *-algebra can be written as

$$\sum_{i=1}^{k} x_{i,1} x_{i,2} \ldots x_{i,m(i)},$$

[34]See page 93 of [7], from which we have only made minor notational changes.
[35]Recall that A commutes with e_A.

where $x_{i,j}$ lie in $N \cup N' \cup \{e_A\}$. Multiplication is jointly strong-operator continuous on bounded sets so that

$$\lim_{n\to\infty} \alpha_n(x) = \sum_{i=1}^{k} y_{i,1} y_{i,2} \cdots y_{j,m(i)},$$

exists in strong topology, where

$$y_{i,j} = \begin{cases} \theta(x_{i,j}) & x_{i,j} \in N \\ x_{i,j} & x_{i,j} \in N' \cup \{e_A\} \end{cases}.$$

Exactly as before, we must have $\|\lim_{n\to\infty} \alpha_n(x)\| = \|x\|$ in $\mathbb{B}(L^2(N))$ so that this automorphism extends to the C^*-closure $C^*(N, N', e_A)$ in $\mathbb{B}(L^2(N))$, where it obviously has the required properties.

The original proofs of the remaining implications, particularly $3 \Rightarrow 4$, are more involved. The principle remains though - follow the original proof, and check that the extra hypotheses available produce the required extra conclusions. We shall not give any further details, as we have no wish to copy out vast tracks of [7] here.

Chapter 4

Singularity and the θ-masas

In this chapter we provide positive answers to Questions 1.4.18 and 1.4.20 in the case of Tauer masas. As a consequence, we will develop a convergence criterion for the singularity of Tauer masas which we use in section 4.2 to establish Theorem 2.2.6, regarding the singularity of the θ-masas of Construction 2.2.4. We continue our study of these θ-masas in section 4.3, demonstrating that they all contain non-trivial centralising sequences.

4.1 Singularity for Tauer masas

In this section we aim to prove the following result, showing that for Tauer masas the concepts of singularity and strong singularity are identical. This theorem, and the whole of section 4.1.1, is to appear in [77].

Theorem 4.1.1. *Let A be a singular Tauer masa in R, then A has the weak asymptotic homomorphism property and so is strongly singular.*

We can use this theorem to address Question 1.4.29 in the class of Tauer masas, which we have already noted in Proposition 2.1.3, is closed under taking tensor products. Proposition 1.4.28 shows that the tensor product of two masas with the weak asymptotic homomorphism property again has the weak asymptotic homomorphism property, so the next Corollary follows directly. A similar statement for infinite tensor products also holds.

Corollary 4.1.2. *Let A and B be singular Tauer masas in the hyperfinite* II_1 *factor R. The Tauer masa $A\overline{\otimes}B$ inside $R\overline{\otimes}R$ is then necessarily singular.*

4.1.1 A convergence criterion for the singularity of Tauer masas

We shall work in the tensor product situation, so we take R to be the infinite tensor product of finite type I factors M_r and write $N_n = \bigotimes_{r=1}^{n} M_r$. For the remainder of this section we consider these N_n fixed and consider masas A which are Tauer with respect to the chain $(N_n)_{n=1}^{\infty}$. Occasionally we shall need the size of the M_r as a normalising constant, we define integers d_r so that M_r is a I_{d_r} factor. We shall use the notation of (2.1.3) to consider the inclusion $A_n \subset A_m$, which defines the algebras $A_{m,n}^{(e)}$ used repeatedly.

We will prove a stronger statement than Theorem 4.1.1 and aim to characterise singularity of Tauer masas by a convergence criterion involving the approximating chain of masas (A_n). This Theorem will give an effective test for the singularity of a Tauer masa.

Theorem 4.1.3. *Let A be a Tauer masa in R with respect to the subfactors N_n. The following statements are equivalent:*

1. *A has the weak asymptotic homomorphism property;*
2. *A is singular;*
3. *For each $n \geq 1$ and minimal projections e_1, e_2 for A_n with $e_1 \neq e_2$ we have*

$$\lim_{m\to\infty} \sum_{f\in\mathcal{P}_{min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2 = 0. \tag{4.1.1}$$

Before establishing Theorem 4.1.3, we record the key properties about the sequence in (4.1.1), in particular we note that the limit exists.

Proposition 4.1.4. *Let $n \geq 1$, and e_1 and e_2 be distinct minimal projections in A_n. We have:*

1.

$$\sum_{f\in\mathcal{P}_{min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2 = \left(\prod_{r=n+1}^{m} d_r\right) \sum_{\substack{f_1\in\mathcal{P}_{min}(A_{m,n}^{(e_1)})\\ f_2\in\mathcal{P}_{min}(A_{m,n}^{(e_2)})}} tr(f_1 f_2)^2; \tag{4.1.2}$$

2.

$$0 \leq \sum_{f\in\mathcal{P}_{min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2 \leq 1,$$

for each $m > n$;

3.

$$\sum_{\substack{f_1\in\mathcal{P}_{min}(A_{m_2,n}^{(e_1)})\\ f_2\in\mathcal{P}_{min}(A_{m_2,n}^{(e_1)})}} tr(f_1 f_2)^2 = \sum_{\substack{\tilde{f}_1\in\mathcal{P}_{min}(A_{m_1,n}^{(e_1)})\\ \tilde{f}_2\in\mathcal{P}_{min}(A_{m_2,n}^{(e_2)})}} \left(tr(\tilde{f}_1\tilde{f}_2)^2 \sum_{\substack{g_1\in\mathcal{P}_{min}(A_{m_2,m_1}^{(e_1\otimes\tilde{f}_1)})\\ g_1\in\mathcal{P}_{min}(A_{m_2,m_1}^{(e_1\otimes\tilde{f}_2)})}} tr(g_1 g_2)^2 \right), \tag{4.1.3}$$

for $m_2 > m_1 > n$;

4. *The sequence*

$$\left(\sum_{f \in \mathcal{P}_{min}(A^{(e_1)}_{m,n})} \left\| \mathbb{E}_{A^{(e_2)}_{m,n}}(f) \right\|_2^2 \right)_{m=n+1}^{\infty}$$

is decreasing and so converges.

Proof. Equation (4.1.2) is just Parseval's formula, using the orthogonal basis $(f_2\sqrt{\prod_{r=n+1}^{m} d_r})_{f_2 \in \mathcal{P}_{\min}(A^{(e_2)}_{m,n})}$ of $L^2(A^{(e_2)}_{m,n})$ to calculate $\left\| \mathbb{E}_{A^{(e_2)}_{m,n}}(f_1) \right\|_2^2$. Given two minimal projections $f, f' \in \mathcal{P}_{\min}(A^{(e_1)}_{m,n})$, we have $\mathrm{tr}(\mathbb{E}_{A^{(e_2)}_{m,n}}(f)\, \mathbb{E}_{A^{(e_2)}_{m,n}}(f')) \geq 0$. Hence, we have

$$0 \leq \sum_{f \in \mathcal{P}_{\min}(A^{(e_1)}_{m,n})} \left\| \mathbb{E}_{A^{(e_2)}_{m,n}}(f) \right\|_2^2 \leq \left\| \sum_{f \in \mathcal{P}_{\min}(A^{(e_1)}_{m,n})} \mathbb{E}_{A^{(e_2)}_{m,n}}(f) \right\|_2^2 = 1,$$

which is statement 2.

To obtain the third statement, note that for $m_2 > m_1 > n$ and a minimal projection e in A_n, we have

$$A^{(e)}_{m_2,n} = \bigoplus_{\tilde{f} \in \mathcal{P}_{\min}(A^{(e)}_{m_1,n})} \tilde{f} \otimes A^{(e \otimes \tilde{f})}_{m_2,m_1}.$$

A minimal projection f in $A^{(e)}_{m_2,n}$ can then be written as $f = \tilde{f} \otimes g$ for a minimal projection $\tilde{f}$ of $A^{(e)}_{m_1,n}$ and a minimal projection g of $A^{(e\otimes\tilde{f})}_{m_2,m_1}$. Decomposing the f_1 and f_2 on the left hand side of (4.1.3) in this way gives the result.

The last statement, follows from applying the Parseval identity (4.1.2), using the factorisation (4.1.3) and then the upper bound of statement 2 (again in combination with the Parseval identity). □

4.1.2 Proof of Theorem 4.1.3: $2 \Rightarrow 3$

Since Lemma 1.4.24, originally found in [54, Lemma 2.1], demonstrates that 1 implies 2 we have only two implications remaining to establish. Objectively the more elementary direction is $3 \Rightarrow 1$, as no heavy machinery is required here; in contrast $2 \Rightarrow 3$ will require the deep perturbation work, Theorem 1.4.17, of Popa, Sinclair and Smith, found in [50]. From the point of view of the details required here though, this later implication is more straight forward and so it is with this that we start. The next lemma gives the estimate we shall need to successfully apply the perturbation theorem.

Lemma 4.1.5. *Let $n \geq 1$ and e_1, e_2 be distinct minimal projections in A_n. Fix a partial isometry $v \in N_n$, with $v^*v = e_1$ and $vv^* = e_2$. If x is the unitary $v + v^* + 1 - (e_1 + e_2) \in N_n$, then*

$$\|(\mathbb{E}_{xAx^*} - I)\mathbb{E}_A\|_{\infty,2}^2 \leq \frac{4}{\prod_{r=1}^n d_r} \left(1 - \lim_{m\to\infty} \sum_{f \in \mathcal{P}_{min}(A_{m,n}^{(e_1)})} \left\| \mathbb{E}_{A_{m,n}^{(e_2)}}(f) \right\|_2^2 \right).$$

Proof. First observe, using (1.3.1), that

$$\begin{aligned} \|(\mathbb{E}_{xAx^*} - I)\mathbb{E}_A\|_{\infty,2}^2 &= \lim_{m\to\infty} \sup_{w \in \mathcal{U}(A_m)} \left(1 - \|\mathbb{E}_{xAx^*}(w)\|_2^2\right) \\ &= \lim_{m\to\infty} \sup_{w \in \mathcal{U}(A_m)} \left(1 - \|\mathbb{E}_{A_m}(x^*wx)\|_2^2\right). \end{aligned}$$

Now, for $m > n$ and a unitary $w \in A_m$, use the decomposition (2.1.3) to write

$$w = \sum_{e \in \mathcal{P}_{\min}(A_n)} e \otimes w_e,$$

for some unitaries w_e in $A_{m,n}^{(e)}$. We have

$$\|\mathbb{E}_{A_m}(x^*wx)\|_2^2 = \|1 - e_1 - e_2\|_2^2 + \left\| e_1 \otimes \mathbb{E}_{A_{m,n}^{(e_1)}}(w_{e_2}) \right\|_2^2 + \left\| e_2 \otimes \mathbb{E}_{A_{m,n}^{(e_2)}}(w_{e_1}) \right\|_2^2.$$

Now write w_{e_2} as the sum $\sum_{f_2} \lambda_{f_2} f_2$ over all minimal projections f_2 in $A_{m,n}^{(e_2)}$, for some constants λ_{f_2} with $|\lambda_{f_2}| = 1$. We can then compute

$$\begin{aligned} \left\| \mathbb{E}_{A_{m,n}^{(e_1)}}(w_{e_2}) \right\|_2^2 &= \sum_{f_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)})} \left\| \sum_{f_2 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2)})} \lambda_{f_2} \frac{\operatorname{tr}(f_1 f_2) f_1}{\|f_1\|_2^2} \right\|_2^2 \\ &\geq \left(\sum_{\substack{f_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)}) \\ f_2 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2)})}} \operatorname{tr}(f_1 f_2)^2 - \sum_{\substack{f_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)}) \\ f_2, f_3 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2)}) \\ f_2 \neq f_3}} \operatorname{tr}(f_1 f_2)\operatorname{tr}(f_1 f_3) \right) \prod_{r=n+1}^m d_r \\ &= \left(2 \sum_{\substack{f_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)}) \\ f_2 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2)})}} \operatorname{tr}(f_1 f_2)^2 \right) \prod_{r=n+1}^m d_r - 1 \\ &= 2 \sum_{f \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)})} \left\| \mathbb{E}_{A_{m,n}^{(e_2)}}(f) \right\|_2^2 - 1. \end{aligned}$$

We also obtain an identical estimate for $\left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(w_{e_1})\right\|_2^2$, and so, noting that $\|e_1\|_2^2 = \|e_2\|_2^2 = 1/\prod_{r=1}^n d_r$, we have the bound

$$1 - \|\mathbb{E}_{A_m}(xwx^*)\|_2^2 \leq \frac{4}{\prod_{r=1}^n d_r}\left(1 - \sum_{f \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2\right).$$

The result now follows by letting m increase to infinity. □

We can now complete the proof of 2 ⇒ 3. Suppose condition 3 of Theorem 4.1.3 fails, then we can find n_0 and distinct minimal projections e_1 and e_2 in A_{n_0} such that

$$\lim_{m\to\infty} \sum_{f \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2 = C > 0.$$

Let α_0 be the constant given in Theorem 1.4.17 and choose δ such that $1 > \delta > 1 - \alpha_0^2/2$. Write

$$C_m = \sum_{f \in \mathcal{P}_{\min}(A_{m,n}^{(e_1)})} \left\|\mathbb{E}_{A_{m,n}^{(e_2)}}(f)\right\|_2^2 = \left(\prod_{r=n_0+1}^{m} d_r\right) \sum_{\substack{f_1 \in \mathcal{P}_{\min}(A_{m,n_0}^{(e_1)}) \\ f_2 \in \mathcal{P}_{\min}(A_{m,n_0}^{(e_2)})}} \operatorname{tr}(f_1 f_2)^2,$$

where the second equality comes from the Parseval formula (4.1.2). Part 4 of Proposition 4.1.4 shows that $(C_m)_{m>n}$ is a decreasing sequence, so we can find n such that $\delta^{-1}C > C_m \geq C > 0$, whenever $m \geq n$. If $\tilde{f}_1$ and $\tilde{f}_2$ are minimal projections of $A_{n,n_0}^{(e_1)}$ and $A_{n,n_0}^{(e_2)}$ respectively, write

$$C_m^{(\tilde{f}_1,\tilde{f}_2)} = \left(\prod_{r=n_0+1}^{n} d_r\right) \operatorname{tr}(\tilde{f}_1\tilde{f}_2)^2 \left(\prod_{r=n+1}^{m} d_r\right) \sum_{\substack{g_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1 \otimes \tilde{f}_1)}) \\ g_2 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2 \otimes \tilde{f}_2)})}} \operatorname{tr}(g_1 g_2)^2,$$

and $C^{(\tilde{f}_1,\tilde{f}_2)} = \lim_{m\to\infty} C_m^{(\tilde{f}_1,\tilde{f}_2)}$. Part 3 of Proposition 4.1.4 ensures that

$$C_m = \sum_{\substack{\tilde{f}_1 \in \mathcal{P}_{\min}(A_{n,n_0}^{(e_1)}) \\ \tilde{f}_2 \in \mathcal{P}_{\min}(A_{n,n_0}^{(e_2)})}} C_m^{(\tilde{f}_1,\tilde{f}_2)},$$

and so we can find some $\tilde{f}_1, \tilde{f}_2$ with $\delta^{-1}C^{(\tilde{f}_1,\tilde{f}_2)} > C_n^{(\tilde{f}_1,\tilde{f}_2)} \geq C^{(\tilde{f}_1,\tilde{f}_2)}$. Now divide through to obtain

$$\frac{C^{(\tilde{f}_1,\tilde{f}_2)}}{C_n^{(\tilde{f}_1,\tilde{f}_2)}} = \lim_{m\to\infty}\left(\prod_{r=n+1}^{m} d_r\right) \sum_{\substack{g_1 \in \mathcal{P}_{\min}(A_{m,n}^{(e_1 \otimes \tilde{f}_1)}) \\ g_2 \in \mathcal{P}_{\min}(A_{m,n}^{(e_2 \otimes \tilde{f}_2)})}} \operatorname{tr}(g_1 g_2)^2 > \delta > 1 - \alpha_0^2/2. \tag{4.1.4}$$

Let v be a partial isometry in N_n, with $v^*v = \tilde{f}_1$ and $vv^* = \tilde{f}_2$. Define x to be the unitary $v + v^* + 1 - (\tilde{f}_1 + \tilde{f}_2)$ in A_n. Lemma 4.1.5 and (4.1.4) give the estimate

$$\|(\mathbb{E}_{xAx^*} - I)\mathbb{E}_A\|_{\infty,2}^2 < \frac{2}{\prod_{r=1}^n d_r}\alpha_0^2,$$

and so the perturbation inequality (Theorem 1.4.17, [50, Theorem 6.4]), shows that

$$d_2(x, N(A))^2 < \frac{2}{\prod_{r=1}^n d_r}.$$

On the other hand,

$$d_2(x, A)^2 = \frac{2}{\prod_{r=1}^n d_r},$$

and so $N(A)$ can not be contained in A. Therefore, A is not singular, and we have established the implication $2 \Rightarrow 3$ of Theorem 4.1.3.

4.1.3 Proof of Theorem 4.1.3: $3 \Rightarrow 1$

We have already done some of the work towards proving the implication 3 implies 1 in section 2.2. The next result shows that hypotheses of Proposition 2.2.2 hold for a Tauer masa satisfying the convergence criterion (4.1.1). A proof of Lemma 4.1.6 will then conclude the proof of Theorem 4.1.3.

Lemma 4.1.6. *Suppose that condition 3 of Theorem 4.1.3 is satisfied. Fix n_1 and a minimal projection e_1 in A_{n_1}. For $\epsilon > 0$ there exists $n > n_1$ and a unitary w_{e_1} in $A_{n,n}^{(e_1)}$ with*

$$\left\|\mathbb{E}_{A_{n,n_1}^{(e)}}(w_{e_1})\right\|_2 < \epsilon,$$

for every minimal projection $e \neq e_1$ in A_{n_1}.

The following obvious piece of complex arithmetic will be required in the proof of Lemma 4.1.6.

Proposition 4.1.7. *Given complex numbers C_{n_1,n_2} for $1 \leq n_1, n_2 \leq n_0$ with $\overline{C_{n_1,n_2}} = C_{n_2,n_1}$, we can find $(\lambda_n)_{n=1}^{n_0}$ on the unit circle such that*

$$\sum_{n_1 \neq n_2} \lambda_{n_1}\overline{\lambda_{n_2}} C_{n_1,n_2} \leq 0.$$

Proof of Lemma 4.1.6. Suppose A satisfies the condition 3 of Theorem 4.1.3. Fix n_1 a minimal projection e_1 for A_{n_1} and $\epsilon > 0$. Let $t_0 = \prod_{r=1}^n d_r$ and enumerate the set $\mathcal{P}_{\min}(A_{n_1}) \setminus \{e_1\}$ as $e_2, \ldots e_{t_0}$. We use condition 3 and Parseval's formula (4.1.2) repeatedly, to find $n_1 < n_2 < \cdots < n_{t_0}$ with the property that

$$\left(\prod_{r=n_{t-1}+1}^{n_t} d_r\right) \sum_{\substack{g_1 \in \mathcal{P}_{\min}(A_{n_t,n_{t-1}}^{(e_1\otimes f_1)}) \\ g_2 \in \mathcal{P}_{\min}(A_{n_t,n_{t-1}}^{(e_t\otimes f_2)})}} \operatorname{tr}(g_1 g_2)^2 < \frac{\epsilon^2}{\prod_{r=n_1+1}^{n_{t-1}} d_r}, \tag{4.1.5}$$

whenever $f_1 \in \mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})$ and $f_2 \in \mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1})$, for each $t = 2, \ldots, t_0$.

We shall construct our required w_{e_1} in $A^{(e_1)}_{n_{t_0},n_1}$ by means of a downward induction on t. Suppose for $t = t_0, \ldots, 2$, we can find unitaries $v_g^{(t)} \in A^{(e_1 \otimes g)}_{n_{t_0},n_t}$ for each minimal projection $g \in \mathcal{P}_{\min}(A^{(e)}_{n_t,n_1})$, such that any unitary $u^{(t)}$ of the form

$$u^{(t)} = \sum_{g \in \mathcal{P}_{\min}(A^{(e_1)}_{n_t,n_1})} \mu_g g \otimes v_g^{(t)},^{1} \tag{4.1.6}$$

satisfies

$$\left\| \mathbb{E}_{A^{(e_s)}_{n_T,n_1}} \left(u^{(t)}\right) \right\|_2 < \epsilon,$$

whenever $t < s \leq t_0$. For $t = t_0$ we may start the induction by taking each v_g equal to 1 - there is nothing to show here. If we can obtain this statement for $t = 1$, then we have established the lemma as can we take $w_{e_1} = v_1^{(1)}$.

For each minimal projection $f \in \mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})$, we shall set

$$v_f^{(t-1)} = \sum_{g \in \mathcal{P}_{\min}(A^{(e_1 \otimes f)}_{n_t,n_{t-1}})} \lambda_{f,g} g \otimes v^{(t)}_{f \otimes g},$$

for some scalars $\lambda_{f,g}$ to be chosen later with $|\lambda_{f,g}| = 1$. As any unitary

$$u^{(t-1)} = \sum_{f \in \mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})} \mu_f f \otimes v_f^{(t-1)}, \tag{4.1.7}$$

can also be expressed in the form of equation (4.1.6), the inductive hypothesis ensures that we have

$$\left\| \mathbb{E}_{A^{(e_s)}_{n_{t_0},n_1}} \left(u^{(t-1)}\right) \right\|_2 < \epsilon, \tag{4.1.8}$$

whenever $s > t$. It remains to choose the $\lambda_{f,g}$ so that (4.1.8) holds for $s = t$.

Now,

$$\begin{aligned}
&\left\| \mathbb{E}_{A^{(e_t)}_{n_{t_0},n_1}} \left(u^{(t-1)}\right) \right\|_2^2 \\
&= \sum_{h \in \mathcal{P}_{\min}(A^{(e_t)}_{n_{t_0},n_1})} \left| \operatorname{tr}(u^{(t-1)} h) \right|^2 / \|h\|_2^2 \\
&= \Big(\prod_{r=n_1+1}^{n_{t_0}} d_r\Big) \sum_{\substack{h_1 \in \mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1}) \\ h_2 \in \mathcal{P}_{\min}(A^{(e_t \otimes h_1)}_{n_t,n_{t-1}}) \\ h_3 \in \mathcal{P}_{\min}(A^{(e_t \otimes h_1 \otimes h_2)}_{n_{t_0},n_t})}} \left| \sum_{\substack{f \in \mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1}) \\ g \in \mathcal{P}_{\min}(A^{(e_1 \otimes f)}_{n_t,n_{t-1}})}} \mu_f \lambda_{f,g} \operatorname{tr}(f h_1) \operatorname{tr}(g h_2) \operatorname{tr}(v^{(t)}_{f \otimes g} h_3) \right|^2,
\end{aligned}$$

[1] where of course each $|\mu_g| = 1$

so we are able to use the Cauchy-Schwartz inequality for the sum over f, to obtain

$$\left\|\mathbb{E}_{A^{(e_t)}_{n_{t_0},n_1}}\left(u^{(t-1)}\right)\right\|_2^2 \leq \left(\prod_{r=n_1+1}^{n_{t_0}} d_r\right) \sum_{h_1\in\mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1})} \left(\left(\sum_{f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})} \mathrm{tr}(fh_1)^2\right)\right.$$

$$\left.\left(\sum_{\substack{h_2\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1)}_{n_t,n_{t-1}})\\ h_3\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t})}} \sum_{f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})} \left|\sum_{g\in\mathcal{P}_{\min}(A^{(e_1\otimes f)}_{n_t,n_{t-1}})} \lambda_{f,g}\mathrm{tr}(gh_2)\mathrm{tr}(v^{(t)}_{f\otimes g}h_3)\right|^2\right)\right). \tag{4.1.9}$$

For each h_1, we have

$$\sum_{f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})} \mathrm{tr}(fh_1)^2 \leq \mathrm{tr}(h_1)^2 = \frac{1}{\prod_{r=n_1+1}^{n_{t-1}} d_r^2}.$$

We use this estimate, and expand the second component (4.1.9) of the product above, using $|z|^2 = z\overline{z}$, to obtain the upper bound

$$\left(\frac{\prod_{r=n_{t-1}+1}^{n_{t_0}} d_r}{\prod_{r=n_1+1}^{n_{t-1}} d_r}\right)\left(\sum_{\substack{h_1\in\mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1})\\ h_2\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1)}_{n_t,n_{t-1}})\\ h_3\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t})}} \sum_{\substack{f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})\\ g\in\mathcal{P}_{\min}(A^{(e_1\otimes f)}_{n_t,n_{t-1}})}} \mathrm{tr}(gh_2)^2\left|\mathrm{tr}(v^{(t)}_{f\otimes g}h_3)\right|^2\right.$$

$$\left.+\sum_{\substack{f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})\\ g_1,g_2\in\mathcal{P}_{\min}(A^{(e_1\otimes f)}_{n_t,n_{t-1}})\\ g_1\neq g_2}} \sum_{\substack{h_1\in\mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1})\\ h_2\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1)}_{n_t,n_{t-1}})\\ h_3\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t})}} \lambda_{f,g_1}\overline{\lambda_{f,g_2}}\mathrm{tr}(g_1h_2)\mathrm{tr}(g_2h_2)\mathrm{tr}(v^{(t)}_{f\otimes g_1}h_3)\overline{\mathrm{tr}(v^{(t)}_{f\otimes g_2})}\right),$$

for $\left\|\mathbb{E}_{A^{(e_t)}_{n_T,n_1}}\left(u^{(t-1)}\right)\right\|_2^2$. For each f, we appeal to Proposition 4.1.7 and choose $\lambda_{f,g}$ so that the second term above is negative. The first term can be written as

$$\left(\frac{\prod_{r=n_{t-1}+1}^{n_t} d_r}{\prod_{r=n_1}^{n_{t-1}} d_r}\right) \cdot \sum_{\substack{h_1\in\mathcal{P}_{\min}(A^{(e_t)}_{n_{t-1},n_1})\\ h_2\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1)}_{n_t,n_{t-1}}))\\ f\in\mathcal{P}_{\min}(A^{(e_1)}_{n_{t-1},n_1})\\ g\in\mathcal{P}_{\min}(A^{(e_1\otimes f)}_{n_t,n_{t-1}}}} \left(\mathrm{tr}(gh_2)^2\left(\prod_{r=n_t+1}^{n_{t_0}} d_r \sum_{h_3\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t})} \left|\mathrm{tr}(v^{(t)}_{f\otimes g}h_3)\right|^2\right)\right).$$

As

$$\left(\prod_{r=n_t+1}^{n_{t_0}} d_r\right) \sum_{h_3\in\mathcal{P}_{\min}(A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t})} \left|\mathrm{tr}(v^{(t)}_{f\otimes g}h_3)\right|^2 = \left\|\mathbb{E}_{A^{(e_t\otimes h_1\otimes h_2)}_{n_{t_0},n_t}}\left(v^{(t)}_{f\otimes g}\right)\right\|_2^2 \leq 1,$$

for all f, g, h_1 and h_2, we can use the initial estimate (4.1.5), to see that for this choice of the $\lambda_{f,g}$, we have

$$\left\| \mathbb{E}_{A^{(e_t)}_{n_T,n_1}} \left(u^{(t-1)}\right)\right\|_2 < \epsilon,$$

for all unitaries $u^{(t-1)}$ of the form (4.1.7). $\square$

4.2 Singularity of θ-masas of Construction 2.2.4

We now return to the masas of Construction 2.2.4. Our objective is to apply Theorem 4.1.3 to establish Theorem 2.2.6. We shall use the second formulation of the θ-masas given in Construction 2.2.4. We shall make some slight changes to the notation, and make no apologies for recalling the definition of these masas here for the readers convenience.

Let M denote the algebra of 2×2 matrices, and $D^{(0)}$ the masa consisting of all diagonal elements of M. The minimal projections of $D^{(0)}$ are given by

$$e_0^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } e_1^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Given an angle $\theta \in [0, \pi/4]$ we define a masa $D^{(\theta)}$ in M by conjugating $D^{(0)}$ by the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. The minimal projections $e_0^{(\theta)}$ and $e_1^{(\theta)}$ are given by conjugating those for $D^{(0)}$ and are found to be

$$e_0^{(\theta)} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \text{ and } e_1^{(\theta)} = \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}.$$

The θ-masas are given by a sequence of angles $(\theta_n)_{n=2}^{\infty}$ in the range $[0, \pi/4]$. We shall regularly identify $n \geq 2$ with the pair (t, r), where $n = 2^t + r$ with $1 \leq r \leq 2^t$ and $t \geq 0$, and write $\theta_{t,r}$ for θ_n in this case. We construct a Tauer masa $A^{(\theta)}$, corresponding to these $\theta_{t,r}$, with respect to the subfactors $(N_t)_{t=0}^{\infty}$ given by

$$N_t = M^{\otimes 2^t},$$

with the natural inclusion $x \mapsto x \otimes 1^{\otimes 2^t}$ of N_t inside N_{t+1}. The zero-th approximation $A_0^{(\theta)}$ is taken to be the masa $D^{(0)}$ in N_0. We write ${}^0f_0 = e_0^{(0)}$ and ${}^0f_1 = e_1^{(0)}$ for the minimal projections of $A_0^{(\theta)}$. Suppose inductively that we have constructed the t-th approximate $A_t^{(\theta)}$ with minimal projections ${}^tf_i$ indexed by $i \in \{0,1\}^{2^t}$, then we define the $(t+1)$-th approximate by

$$A_{t+1}^{(\theta)} = \bigoplus_{i \in \{0,1\}^{2^t}} {}^tf_i \otimes \bigotimes_{r=1}^{2^t} D^{(\delta_{i_r=1}\theta_{t,r})},$$

so that the masa in the r-th component of the tensor product above is $D^{(0)}$ when $i_r = 0$ and $D^{(\theta_{t,r})}$ when $i_r = 1$. We will write, for $i_1, \ldots, i_{2^{t+1}} \in \{0,1\}$,

$$ {}^{t+1,t}g^{(i_1,\ldots,i_{2^t})}_{(i_{2^t+1},\ldots,i_{2^{t+1}})} = \bigotimes_{r=1}^{2^t} e^{(\delta_{i_r=1}\theta_{t,r})}_{i_{2^t+r}}, $$

and now index the minimal projections for $A^{(\theta)}_{t+1}$ in the natural way, namely by

$$ {}^{t+1}f_{(i_1,\ldots,i_{2^{t+1}})} = {}^{t}f_{(i_1,\ldots,i_{2^t})} \otimes {}^{t+1,t}g^{(i_1,\ldots,i_{2^t})}_{(i_{2^t+1},\ldots,i_{2^{t+1}})}. $$

When $s > t$, we define ${}^{s,t}g^{(i_{2^t+1},\ldots,i_{2^s})}_{(i_1,\ldots,i_{2^t})}$ inductively by

$$ {}^{s,t}g^{(i_{2^t+1},\ldots,i_{2^s})}_{(i_1,\ldots,i_{2^t})} = {}^{s-1,t}g^{(i_{2^t+1},\ldots,i_{2^{s-1}})}_{(i_1,\ldots,i_{2^t})} \otimes \bigotimes_{r=1}^{2^{s-1}} e^{(\delta_{i_r=1}\theta_{s-1,r})}_{i_{2^{s-1}+r}}. $$

With this notation, and a minimal projection ${}^tf_i$ of A_t given by $i \in \{0,1\}^{2^t}$, the minimal projections of $A^{({}^tf_i)}_{s,t}$ are given by ${}^{s,t}g^{(i_{2^t+1},\ldots,i_{2^s})}_{(i_1,\ldots,i_{2^t})}$ for $(i_{2^t+1},\ldots,i_{2^s}) \in \{0,1\}^{2^s-2^t}$.

We shall examine the sequence of (4.1.1) for $n = t$ and $m = s$, relating this sequence to the angles $\theta_{t,r}$ by the next two estimates. Finally, some simple asymptotics will yield Theorem 2.2.6.

Proposition 4.2.1. *For distinct minimal projections e_1 and e_2 in A_t and $s > t$, we have*

$$ \sum_{f \in \mathcal{P}_{min}(A^{(e_2)}_{s,t})} \left\| \mathbb{E}_{A^{(e_1)}_{s,t}}(f) \right\|_2^2 \leq \max_{r=1,\ldots,2^t} \prod_{v=t}^{s-1} (\cos^4\theta_{v,r} + \sin^4\theta_{v,r}). $$

Proposition 4.2.2. *For each $1 \leq r \leq 2^t$, we can find distinct minimal projections e_1 and e_2 in A_t with*

$$ \prod_{v=t}^{s-1} \cos^4\theta_{v,r} \leq \sum_{f \in \mathcal{P}_{min}(A^{(e_2)}_{s,t})} \left\| \mathbb{E}_{A^{(e_1)}_{s,t}}(f) \right\|_2^2. $$

Suppose we have two indices $i, j \in \{0,1\}^{2^t}$ with $i \neq j$, part 1 of Proposition 4.1.4 gives

$$ \begin{aligned} \sum_{g \in \mathcal{P}_{\min}(A^{({}^tf_i)}_{s,t})} \left\| \mathbb{E}_{A^{({}^tf_j)}_{s,t}}(g) \right\|_2^2 &= 2^{2^s-2^t} \sum_{\substack{g_1 \in \mathcal{P}_{\min}(A^{({}^tf_i)}_{s,t}) \\ g_2 \in \mathcal{P}_{\min}(A^{({}^tf_j)}_{s,t})}} \operatorname{tr}(g_1g_2)^2 \\ &= 2^{2^s-2^t} \sum_{\substack{i_{2^t+1},\ldots,i_{2^s}=0,1 \\ j_{2^t+1},\ldots,j_{2^s}=0,1}} \operatorname{tr}\left({}^{s,t}g^{(i_{2^t+1},\ldots,i_{2^s})}_{(i_1,\ldots,i_{2^t})} \, {}^{s,t}g^{(j_{2^t+1},\ldots,j_{2^s})}_{(j_1,\ldots,j_{2^t})} \right)^2. \end{aligned} \tag{4.2.1} $$

We can crudely obtain a lower bound for (4.2.1) by only summing over those indices $i_r = j_r$, $(2^t+1 \leq r \leq 2^s)$. Under this hypothesis we have, for $t \leq v \leq s-1$,

$$\operatorname{tr}\left(e_{i_{2^v+r}}^{(\delta_{i_r=1}\theta_{v,r})} e_{i_{2^v+r}}^{(\delta_{j_r=1}\theta_{v,r})}\right) = \begin{cases} 1/2 & r > 2^t \\ 1/2 & 1 \leq r \leq 2^t \text{ and } i_r = j_r \\ \cos^2\theta_{v,r}/2 & 1 \leq r \leq 2^t \text{ and } i_r \neq j_r \end{cases}, \tag{4.2.2}$$

giving the estimate

$$\sum_{g \in \mathcal{P}_{\min}(A_{s,t}^{({}^t f_i)})} \left\| \mathbb{E}_{A_{2^s,2^t}^{({}^t f_j)}}(g) \right\|_2^2 \geq \prod_{v=t}^{s-1} \prod_{\substack{1 \leq r \leq 2^t \\ i_r \neq j_r}} \cos^4\theta_{v,r}.$$

Proposition 4.2.2 now follows; given some r in the range $1, \ldots, 2^t$, take indices $i, j \in \{0,1\}^{2^t}$ with $i_r \neq j_r$ and $i_{r'} = j_{r'}$, for $r \neq r'$, then the minimal projections $e_1 = {}^t f_i$ and $e_2 = {}^t f_j$ give the required estimate.

To obtain the upper bound of Proposition 4.2.1, we must work slightly harder. Let $i, j \in \{0,1\}^{2^t}$ be fixed. Fix values for $i_{2^t+1}, \ldots, i_{2^{s-1}+2^t}, j_{2^t+1}, \ldots, j_{2^{s-1}+2^t}$ and observe that

$$2^{2^{s-1}-2^t} \sum_{\substack{i_{2^{s-1}+2^t+1}, \ldots, i_{2^s}=0,1 \\ j_{2^{s-1}+2^t+1}, \ldots, j_{2^s}=0,1}} \prod_{r=2^t+1}^{2^{s-1}} \operatorname{tr}\left(e_{i_{2^{s-1}+r}}^{(\delta_{i_r=1}\theta_{s-1,r})} e_{j_{2^{s-1}+r}}^{(\delta_{j_r=1}\theta_{s-1,r})}\right)^2 \leq 1, \tag{4.2.3}$$

exactly as in Proposition 4.1.4. For $r = 1, \ldots, 2^t$, we extend (4.2.2) to

$$\operatorname{tr}\left(e_{i_{2^{s-1}+r}}^{(\delta_{i_r=1}\theta_{s-1,r})} e_{j_{2^{s-1}+r}}^{(\delta_{j_r=1}\theta_{s-1,r})}\right) = \begin{cases} 1/2 & i_r = j_r \text{ and } i_{2^{s-1}+r} = j_{2^{s-1}+r} \\ 0 & i_r = j_r \text{ and } i_{2^{s-1}+r} \neq j_{2^{s-1}+r} \\ \cos^2\theta_{2^{s-1}+r}/2 & i_r \neq j_r \text{ and } i_{2^{s-1}+r} = j_{2^{s-1}+r} \\ \sin^2\theta_{2^{s-1}+r}/2 & i_r \neq j_r \text{ and } i_{2^{s-1}+r} \neq j_{2^{s-1}+r} \end{cases}.$$

We combine this with (4.2.3), to obtain the estimate for (4.2.1) of

$$\begin{aligned} &\sum_{g \in \mathcal{P}_{\min}(A_{s,t}^{({}^t f_i)})} \left\| \mathbb{E}_{A_{s,t}^{({}^t f_j)}}(g) \right\|_2^2 \\ &= 2^{2^s-2^t} \sum_{\substack{i_{2^t+1}, \ldots, i_{2^s}=0,1 \\ j_{2^t+1}, \ldots, j_{2^s}=0,1}} \operatorname{tr}\left({}^{s,t}g_{(i_1,\ldots,i_{2^t})}^{(i_{2^t+1},\ldots,i_{2^s})}\, {}^{s,t}g_{(j_1,\ldots,j_{2^t})}^{(j_{2^t+1},\ldots,j_{2^s})}\right)^2 \\ &\leq \left(2^{2^{s-1}-2^t} \sum_{\substack{i_{2^t+1}, \ldots, i_{2^{s-1}}=0,1 \\ j_{2^t+1}, \ldots, j_{2^{s-1}}=0,1}} \operatorname{tr}\left({}^{s-1,t}g_{(i_1,\ldots,i_{2^t})}^{(i_{2^t+1},\ldots,i_{2^{s-1}})}\, {}^{s-1,t}g_{(j_1,\ldots,j_{2^t})}^{(j_{2^t+1},\ldots,j_{2^{s-1}})}\right)^2 \right) \\ &\quad \left(\prod_{\substack{1 \leq r \leq 2^t \\ i_r \neq j_r}} (\cos^4\theta_{s-1,r} + \sin^4\theta_{s-1,r}) \right). \end{aligned}$$

We repeat this argument to yield

$$\sum_{g\in\mathcal{P}_{\min}(A_{s,t}^{(tf_i)})}\left\|\mathbb{E}_{A_{s,t}^{(tf_j)}}(g)\right\|_2^2 \leq \prod_{v=t}^{s-1}\prod_{\substack{r=1,\dots,2^t\\ i_r\neq j_r}}\left(\cos^4\theta_{v,r}+\sin^4\theta_{v,r}\right)$$

$$\leq \max_{r=1,\dots,2^t}\prod_{v=t}^{s-1}\left(\cos^4\theta_{v,r}+\sin^4\theta_{v,r}\right),$$

which is Proposition 4.2.1.

We ahve nowestablished the two Propositions 4.2.1 and 4.2.2 which allow us to relate the sequence $(\theta_n)_{n=2}^{\infty}$ tto the convergence criterion of Theorem 4.1.3. More precisely, we see that if

$$\prod_{t>\lceil\log_2 r\rceil}\cos^4\theta_{t,r}+\sin^4_{t,r}=0,$$

for every r, then Proposition 4.2.1 would ensure that condition 3 of Theorem 4.1.3 holds and so A will be singular. On the other hand, if we can find some r for which

$$\prod_{t>\lceil\log_2 r\rceil}\cos^4\theta_{t,r}>0,$$

then Proposition 4.2.2, would then furnish some minimal projections e_1 and e_2 in some A_t, for which

$$\lim_{s\to\infty}\sum_{f\in\mathcal{P}_{\min}(A_{s,t}^{(e_1)})}\left\|\mathbb{E}_{A_{s,t}^{(e_2)}}(f)\right\|_2^2>0$$

and so Theorem 4.1.3 demonstrates that A is not singular. We complete the proof of Theorem 2.2.6, with the next proposition showing that the convergence of these infinite products coincides precisely with the convergence of the sum (2.2.8) of Theorem 2.2.6

Proposition 4.2.3. *Given a sequence $(\phi_n)_{n=1}^{\infty}$ in $[0,\pi/4]$, we have*

$$\sum_{n=1}^{\infty}\phi_n^2=\infty\Leftrightarrow\prod_{n=1}^{\infty}\cos^4\phi_n=0\Leftrightarrow\prod_{n=1}^{\infty}\left(\cos^4\phi_n+\sin^4\phi_n\right)=0.$$

Proof. We proceed by simple minded estimates using Taylor expansions for log and cos as necessary. Before we start, note that we may assume that the sequence ϕ_n converges to zero allowing us to use these Taylor expansions with impunity.

We then have,

$$\prod_{n=1}^{\infty}\cos^4\phi_n = 0 \Leftrightarrow \sum_{n=1}^{\infty}\log\cos^4\theta_n = -\infty \tag{4.2.4}$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\log\cos\phi_n = -\infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty}(\cos\phi_n - 1) = -\infty \tag{4.2.5}$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\phi_n^2 = \infty, \tag{4.2.6}$$

where (4.2.4) is the definition of convergence for an infinite product, and (4.2.5) and (4.2.6) follow from the Taylor expansions for $\log(1+x)$ and $\cos x$ respectively. In a similar vain we have

$$\prod_{n=1}^{\infty}\left(\cos^4\phi_n + \sin^4\phi_n\right) = 0 \Leftrightarrow \sum_{n=1}^{\infty}\log\left((1+\cos^2 2\phi_n)/2\right) = -\infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty}(\cos^2 2\phi_n - 1) = -\infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\left((1-2\phi_n^2)^2 - 1\right) = -\infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\phi_n^2 = \infty,$$

as required. □

Before moving on, it is natural to ask whether all the singular masas we have produced in this way are conjugate. Unfortunately, all my attempts to resolve this matter completely have proved fruitless. Some observations will appear in the next section, but we leave the general problem for future investigation.

Question 4.2.4. Given two sequences $(\theta_n)_{n=2}^{\infty}$ and $(\phi_n)_{n=2}^{\infty}$ in $[0, \pi/4]$ with

$$\sum_{t\geq\lceil\log_2 r\rceil}\theta_{2^t+r}^2 = \sum_{t\geq\lceil\log_2 r\rceil}\phi_{2^t+r}^2 = \infty,$$

for each r, when are the singular Tauer masas A_θ and A_ϕ produced by Construction 2.2.4 corresponding to these sequences conjugate via an automorphism of R?

4.3 Is Tauer's singular masa transitive?

In this section, we investigate different cutdowns of Tauer's singular masa, from Construction 2.2.1. We would like to show that this masa, A, satisfies one, or more, of the transitivity properties introduced in subsection 3.3.3. At present we are unable to do this, but we can give a variety of automorphisms of this masa which suggest that transitivity is plausible here. Furthermore, this investigation will lead us to see that Tauer's singular masa factorises as an infinite von Neuamnn tensor product of copies of itself, and so contains non-trivial centralising sequences for R. Some of the results we produce also hold for the θ-masas, the construction of which generalises Tauer's singular masa. Where this is the case, we shall state versions for the θ-masas but the only proofs given will be for the Tauer's singular masa.

4.3.1 Weak A-conjugacy of minimal projections in A_t

In this section, we use yet another slightly different formulation of Tauer's singular masa to that given earlier. Let M denote the 2×2 matrices, with the usual minimal projections $e_0^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_1^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $b = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$, a self-adjoint unitary conjugating the diagonals in M onto an orthogonal masa and write $e_i^{(1)} = be_i^{(0)}b^*$ for $i = 0, 1$.

For $t \geq 0$, let $N_t = M^{\otimes 2^t}$ and let A_t be the masa in N_t with minimal projections

$$ {}^t f_i = e_{i_1}^{(0)} \otimes \bigotimes_{s=0}^{2^{t-1}} \bigotimes_{r=1}^{2^s} e_{i_{2^s+r}}^{(i_r)}, \tag{4.3.1} $$

indexed by $i = (i_1, \ldots, i_{2^t}) \in \{0,1\}^{2^t}$. We can immediately see these ${}^t f_i$ satisfy the recurrence relation (2.2.4), for the minimal projections of Tauer's singular masa, namely that

$$ {}^{t+1} f_i = {}^t f_{i|_{2^t}} \otimes \bigotimes_{r=1}^{2^t} e_{i_{2^t+r}}^{(i_r)} \tag{4.3.2} $$

for all $i = (i_1, \ldots, i_{2^{t+1}}) \in \{0,1\}^{2^{t+1}}$. In this way, we see that we have given an alternative construction of Tauer's singular masa A as the direct limit of these A_t, a Tauer masa in the hyperfinite II_1 factor R coming from the direct limit of the N_t. Before we begin, we need some notation for the θ-masas in this section. Ergo, when $(\theta_n)_{n=2}^{\infty}$ is a sequence in $[0, \pi/4]$, write $A^{(\theta_n)}$ for the Tauer masa corresponding to this sequence with approximates $A_t^{(\theta_n)}$ in N_t outlined in section 4.2 and formally defined in Construction 2.2.4.

Proposition 4.3.1. *For each $t \geq 0$, any two minimal projections in A_t are weakly A-conjugate.*

Proof. Fix $t \geq 0$ and take $i, j \in \{0,1\}^{2^t}$ with $i \neq j$. We shall exhibit an automorphism Θ of R with $\Theta(A^t f_i) = A^t f_j$, this is slightly stronger than the statement of the proposition, in that the automorphism extends to all of R. This Θ will not necessarily normalise A so we will not end up proving the A-conjugacy of ${}^t f_i$ and ${}^t f_j$ at this stage.[2]

Let u_t be a unitary in N_t with $u_t\, {}^t f_i u_t^* = {}^t f_j$. Let $v = \bigotimes_{r=1}^{2^t} v_r$ be the unitary in $M^{\otimes 2^t}$, where

$$v_r = \begin{cases} 1 & i_r = j_r \\ b & i_r \neq j_r \end{cases}.$$

For all $k \in \{0,1\}^{2^t}$, we have

$$v \left(\bigotimes_{r=1}^{2^t} e_{k_r}^{(i_r)} \right) v^* = \bigotimes_{r=1}^{2^t} e_{k_r}^{(j_r)}.$$

Set $u_{t+s} = v \otimes 1^{\otimes 2^{t+s-1}-2^t}$, a unitary in $M^{\otimes 2^{t+s-1}}$. The desired automorphism Θ is given by

$$\Theta = \bigotimes_{s=t}^{\infty} \mathrm{Ad}\; u_{t+s},$$

acting on R represented as the infinite von Neumann tensor product

$$(N_t \otimes M^{2^t} \otimes M^{\otimes 2^{t+1}} \otimes M^{\otimes 2^{t+2}} \otimes \dots)'',$$

with respect to the unique factor trace. That this does define an automorphism of R is well known, see for example [69, Theorem XIV.1.13], from which we can also see that this automorphism is not inner.[3] It is immediate that $\Theta({}^t f_i) = {}^t f_j$. Now consider a minimal projection ${}^{t+s} f_k$ for A_{t+s} lying under ${}^t f_i$ so that $k \in \{0,1\}^{2^{t+s}}$ has $k_1 = i_1, \dots, k_{2^t} = i_{2^t}$. Define $k' \in \{0,1\}^{2^{t+s}}$ by

$$k'_r = \begin{cases} j_r & 1 \leq r \leq 2^t \\ k_r & 2^t + 1 \leq r \leq 2^{t+s} \end{cases}$$

We shall show, by induction on s, that

$$\Theta({}^{t+s} f_k) = {}^{t+s} f_{k'},$$

from which we deduce that $\Theta(A\, {}^t f_i) = A\, {}^t f_j$ immediately. Assume the claim

[2]There are no prizes for guessing what might come later though.

[3]If Θ is to do what we want, then this should certainly be the case, as the singularity of A ensures that θ cannot be inner.

holds for $s-1$, and compute, using (4.3.2)

$$\begin{aligned}
\Theta({}^{t+s}f_k) &= \Theta\left({}^{t+s-1}f_{k|_{2^{t+s-1}}} \otimes \bigotimes_{r=1}^{2^{t+s-1}} e^{(k_r)}_{k_{2^{t+s-1}+r}}\right) \\
&= (\mathrm{Ad}\ u_t \otimes \cdots \otimes \mathrm{Ad}\ u_{t+s-1})\left({}^{t+s-1}f_k\right) \otimes u_{t+s}\left(\bigotimes_{r=1}^{2^{t+s-1}} e^{(k_r)}_{k_{2^{t+s-1}+r}}\right)u^*_{t+s} \\
&= (\mathrm{Ad}\ u_t \otimes \cdots \otimes \mathrm{Ad}\ u_{t+s-1})\left({}^{t+s-1}f_k\right) \otimes \bigotimes_{r=1}^{2^{t+s-1}} e^{(k'_r)}_{k_{2^{t+s-1}+r}} \\
&= {}^{t+s-1}f_{k'} \otimes \bigotimes_{r=1}^{2^{t+s-1}} e^{(k'_r)}_{k_{2^{t+s-1}+r}} = {}^{t+s}f_{k'},
\end{aligned}$$

as required. □

Observe that for $t=0$, the automorphism Θ of R taking $A\ {}^0f_0$ onto $A\ {}^0f_1$ is of order 2, so in fact interchanges these cutdowns. In particular, ${}^0f_0$ and ${}^0f_1$ are A-conjugate, not just weakly A-conjugate.

The idea of Proposition 4.3.1 works for any of the θ-masas. The details are the same, once the cost of the additional notation required for this situation has been paid.

Proposition 4.3.2. *Given a sequence $(\theta_n)_{n=2}^{\infty}$, any two minimal projections in the t-th approximate $A_t^{(\theta_n)}$ of $A^{(\theta_n)}$ are weakly $A^{(\theta_n)}$-conjugate.*

4.3.2 Finite tensor powers of A

In this section, we examine tensor powers of A. As a consequence, we will be able to improve on Proposition 4.3.1 to establish the A-conjugacy of minimal projections of each A_t. For a 2^t-tuple $i \in \{0,1\}^{2^t}$, write $(i)_{\mathrm{odd}}$ for the 2^{t-1}-tuple $(i_1, i_3, i_5, \ldots, i_{2^t-1})$ and $(i)_{\mathrm{even}}$ for $(i_2, i_4, \ldots, i_{2^t})$.

Proposition 4.3.3. *There exists an *-isomorphism Θ of R onto $R\overline{\otimes}R$ under which we have $\Theta(A) = A\overline{\otimes}A$. Furthermore, we can insist that this Θ has*

$$\Theta({}^tf_i) = {}^{t-1}f_{(i)_{odd}} \otimes {}^{t-1}f_{(i)_{even}}, \tag{4.3.3}$$

for all t, and $i \in \{0,1\}^{2^t}$.

Repeatedly applying Proposition 4.3.3, shows that A is conjugate to any finite von Neuamnn tensor power of itself.

Corollary 4.3.4. *For each $n \in \mathbb{N}$, there exists a *-isomorphism θ of R onto $R^{\overline{\otimes}n}$ under which we have $\Theta(A) = A^{\overline{\otimes}n}$.*

A slightly more subtle deduction from Proposition 4.3.3 is the hyped A-conjugacy of the minimal projections in each A_t.

Corollary 4.3.5. *For each $t \geq 0$, any two minimal projections of A_t are A-conjugate.*

Proof. The plan here is to repeatedly use the *-isomorphism of Proposition 4.3.3. We have already dealt with the $t = 0$ case in the remark following Proposition 4.3.1. Fix $t \geq 1$ and apply Proposition 4.3.3 to obtain a *-isomorphism from R onto $R^{\overline{\otimes}2}$ under which A maps to $A^{\overline{\otimes}2}$. If $t > 1$, then apply this Proposition again to each copy of R to obtain the composite - a *-isomorphism from R onto $R^{\overline{\otimes}4}$ taking A to $A^{\overline{\otimes}4}$. Continue in this way to obtain a *-isomorphism Θ from R onto $R^{\overline{\otimes}2^t}$ with $\Theta(A) = A^{\overline{\otimes}2^t}$.

We need to follow what happens to minimal projections ${}^sf_i$ under Θ. For $s > t$ and $i \in \{0,1\}^{2^s}$ write $(i)_r$ for the 2^{s-t}-tuple $(i_r, i_{2^t+r}, i_{2.2^t+r}, \ldots, i_{(2^{s-t}-1)2^t+r})$ whenever $1 \leq r \leq 2^t$. After an appropriate reordering[4] of the tensor powers, we have

$$\Theta(\ {}^sf_i) = \bigotimes_{r=1}^{2^t} {}^{s-t}f_{(i)_r}.$$

Hence, upto this reordering, which we have absorbed into the definition of Θ,

$$\Theta(A\ {}^tf_i) = \bigotimes_{r=1}^{2^t}(A\ {}^0f_{i_r}),$$

for any $i \in \{0,1\}^{2^t}$.

It is now easy to show the A-conjugacy of minimal projections in A_t. Indeed, let i and j be distinct elements in $\{0,1\}^{2^t}$. Write ϕ for the automorphism of R of Proposition 4.3.1 which makes ${}^0f_0$ and ${}^0f_1$ A-conjugate projections by interchanging $A\ {}^0f_0$ with $A\ {}^0f_1$. Now define Φ, an automorphism of $R^{\overline{\otimes}2^t}$, to be the tensor product $\Phi = \bigotimes_{r=1}^{2^t} \Phi_r$, where

$$\Phi_r = \begin{cases} I & i_r = j_r \\ \phi & i_r \neq j_r \end{cases}.$$

The composite $\Theta^{-1} \circ \Phi \circ \Theta$, is then an automorphism of R which fixes A and interchanges $A\ {}^tf_i$ with $A\ {}^tf_j$, exactly as required. □

We shall see composites of this sort again in the next subsection, for now we give the proof of Proposition 4.3.3.

[4]A reordering is desirable for notational purposes, as the Θ we have constructed actually has $\Theta(\ {}^sf_i) = {}^{s-t}f_{(i)_1} \otimes {}^{s-t}f_{(i)_{2^{t-1}+1}} \otimes {}^{s-t}f_{(i)_{2^{t-2}+1}} \otimes \cdots \otimes {}^{s-t}f_{(i)_{2^t}}$, which is certainly not as convenient.

Proof of Proposition 4.3.3. The desired *-isomorphism will arise as the composition of conjugation by a unitary, designed to 'untwist' A_1 in N_1, followed by an isomorphism of R with $R\overline{\otimes}R$ obtained by reordering the infinite tensor product $(M^{\otimes\infty})''$, which defines R. Let u be the unitary in N_1 given by

$$u = e_0^{(0)} \otimes 1 + e_1^{(0)} \otimes b^*,$$

so that $u(\,{}^1f_{i_1,i_2})u^* = {}^0f_{i_1} \otimes {}^0f_{i_2} \in M \otimes M$, for each $i = (i_1, i_2) \in \{0,1\}^2$.

Think of R as $(\bigotimes_{n=1}^{\infty} M_n)''$ with each M_n being a copy of M. Let $R_1 = (\bigotimes_{n \text{ odd}} M_n)''$ and $R_2 = (\bigotimes_{n \text{ even}} M_n)''$. We have the natural *-isomorphism from R onto $R_1\overline{\otimes}R_2$, given by reordering the tensor product, which on N_t is given by

$$N_t \to N_{t-1} \otimes N_{t-1}; \quad \bigotimes_{r=1}^{2^t} x_r \mapsto \bigotimes_{r=1}^{2^{t-1}} x_{2r-1} \otimes \bigotimes_{r=1}^{2^{t-1}} x_{2r}.$$

Denote this automorphism by Θ_1, and let $\Theta = \Theta_1 \circ \mathrm{Ad}\, u$.

Take $i \in \{0,1\}^{2^t}$, and use (4.3.1) to see that

$$u(\,{}^tf_i)u^* = e_{i_1}^{(0)} \otimes e_{i_2}^{(0)} \otimes \bigotimes_{s=1}^{2^{t-1}} \bigotimes_{r=1}^{2^s} e_{i_{2^s+r}}^{(i_r)}.$$

Apply the reordering Θ_1 to see that

$$\begin{aligned}
\Theta(\,{}^tf_i) &= \left(e_{(i_1)}^{(0)} \otimes \bigotimes_{s=1}^{2^{t-1}} \bigotimes_{r=1}^{2^{s-1}} e_{i_{2^s+2r-1}}^{(i_{2r-1})} \right) \otimes \left(e_{(i_2)}^{(0)} \otimes \bigotimes_{s=1}^{2^{t-1}} \bigotimes_{r=1}^{2^{s-1}} e_{i_{2^s+2r}}^{(i_{2r})} \right) \\
&= \left(e_{(i_1)}^{(0)} \otimes \bigotimes_{s=0}^{2^{t-2}} \bigotimes_{r=1}^{2^{s}} e_{i_{2(2^s+r)-1}}^{(i_{2r-1})} \right) \otimes \left(e_{(i_2)}^{(0)} \otimes \bigotimes_{s=0}^{2^{t-2}} \bigotimes_{r=1}^{2^{s}} e_{i_{2(2^s+r)}}^{(i_{2r})} \right) \\
&= {}^{t-1}f_{(i)_{\text{odd}}} \otimes {}^{t-1}f_{(i)_{\text{even}}}.
\end{aligned}$$

In conclusion, our automorphism Θ satisfies $\Theta(A_t) = A_{t-1} \otimes A_{t-1} \subset N_{t-1} \otimes N_{t-1}$ for all t, which completes the proof. □

What happens when we work with the θ-masas here? Given a sequence of angles $(\theta_n)_{n=2}^{\infty}$ in $[0, \pi/4]$, let $\theta_n^{(\text{odd})} = \theta_{2n+1}$ and $\theta_n^{(\text{even})} = \theta_{2n+2}$. We can construct an isomorphism Θ from R onto $R\overline{\otimes}R$ as the composite of adjunction by a unitary untwisting $A_1^{(\theta_n)}$ and a reordering of the infinite tensor product making up R by collecting the odd and even multiplicands separately, so that

$$\Theta\left(A^{(\theta_n)}\right) = A^{\left(\theta_n^{(\text{odd})}\right)} \overline{\otimes} A^{\left(\theta_n^{(\text{even})}\right)}.$$

More generally, for each $t \geq 0$, we have the factorisation of a θ-masa into a von Neumann tensor product of 2^t masas coming from subsequences of $(\theta_n)_{n=2}^{\infty}$.

Proposition 4.3.6. *Let $(\theta_n)_{n=2}^{\infty}$ be a sequence in $[0,\pi/4]$. Fix $t \geq 1$ and define sequences $(\theta_m^{(r)})_{m=2}^{\infty}$ by $\theta_m^{(r)} = \theta_{(m-1)2^t+r}$ for $1 \leq r \leq 2^t$. Then there is a *-isomorphism Θ from R onto $R^{\overline{\otimes}2^t}$, under which*

$$\Theta(A^{(\theta_n)}) = A^{\left(\theta_m^{(1)}\right)}\overline{\otimes}\ldots\overline{\otimes}A^{\left(\theta_m^{(2^t)}\right)}.$$

In the case when all the angles θ_n agree, we can deduce the analogous version of Corollary 4.3.5.

Corollary 4.3.7. *Let $(\theta_n)_{n=2}^{\infty}$ be the constant sequence taking some value in $[0,\pi/4]$. For each $t \geq 0$, any two minimal projections in $A_t^{(\theta_n)}$ are $A^{(\theta_n)}$-conjugate.*

4.3.3 Infinite tensor powers of A

Motivated by Proposition 4.3.3, we move on to look at the von Neumann tensor product of infinitely many copies of A. So as to avoid suspense, we announce the result immediately.

Theorem 4.3.8. *There exists a *-isomorphism Θ of R onto $R^{\overline{\otimes}\infty}$ under which we have $\Theta(A) = A^{\overline{\otimes}\infty}$.*

An immediate application of the factorisation of Theorem 4.3.8 is the existence of non-trivial centralising sequences for the hyperfinite II_1 factor inside Tauer's singular masa. We have seen this idea repeatedly in subsection 1.4.5 and section 3.3. A direct proof of this fact can be given, but the computation is messy - and the version of this in the analogous θ-masa situation is sufficiently ugly to not bear thinking about.

Corollary 4.3.9. *Tauer's singular masa A contains non-trivial centralising sequences for R.*

The proof of Theorem 4.3.8 works in exactly the same way as the finite tensor power case contained in Proposition 4.3.3, in that the desired automorphism Θ will be the composite of an untwisting followed by an identification of R with $R^{\overline{\otimes}\infty}$ obtained by reordering the infinite tensor product $M^{\overline{\otimes}\infty}$ making up R. Formally, we will have the commuting diagram of *-isomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\quad\Theta\quad} & R^{\overline{\otimes}\omega} \\ & {\searrow}^{\Theta_1} \quad {\nearrow}_{\Theta_2} & \\ & R & \end{array}$$

exhibiting this composition. Unlike the finite case, we start here by constructing Θ_2.

For notational reasons, enumerate $R^{\overline{\otimes}\infty}$ as $\left(\bigotimes_{m=1}^{\infty} R^{(m)}\right)''$, where each $R^{(m)}$ is a copy of the hyperfinite II_1 factor, and so is the direct limit of copies $N_t^{(m)}$ of N_t. We define Θ_2 on each $M^{\otimes 2^t-1}$ in turn, so that

$$\Theta_2(M^{\otimes 2^t-1}) = N_{t-1}^{(1)} \otimes N_{t-2}^{(2)} \otimes \cdots \otimes N_0^{(t)}.$$

This is done by defining a bijection ψ_t from $\{1, 2, \ldots, 2^t - 1\}$ onto the set of pairs (m, s) with $1 \leq m \leq t$ and $1 \leq s \leq 2^{t-m}$, so that the r-th copy of M in $M^{\otimes 2^t-1}$ is taken by Θ_2 to the s-th copy of M in $N_{t-m}^{(m)}$. Furthermore, we shall ensure that these bijections are compatible, in that the restriction of ψ_{t+1} to $\{1, 2, \ldots, 2^t - 1\}$ is ψ_t. In this way, Θ_2 gives a well defined *-isomorphism from R onto $\left(\bigotimes_{m=1}^{\infty} R^{(m)}\right)''$ as a reordering of the infinite tensor product coming from the bijection $\psi = \bigcup_{t=1}^{\infty} \psi_t$ between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.[5]

For $1 \leq n \leq 2^t - 1$, define $\psi_t(n) = (m, s)$ where,

- m is minimal such that $2^m \not| n$.
- $n = 2^{m-1} + (s - 1)2^m$.

This makes sense, as the minimality of the m first condition ensures that there is a unique s with $1 \leq s \leq 2^{t-m}$ such that the second condition holds. The formulation of the second condition provides an inverse to ψ_t, which is therefore a bijection, and as the definition is clearly independent of t, this completes our construction of Θ_2.

Before turning to Θ_1, we introduce yet more notation, namely for $i \in \{0, 1\}^{2^t-1}$, write

$$\begin{aligned}
(i)_1 &= (i_1, i_3, i_5, \ldots, i_{2^t} - 1) \\
(i)_2 &= (i_2, i_6, i_{10}, \ldots, i_{2^t} - 2) \\
&\ldots \\
(i)_m &= \left(i_{2^{m-1}}, i_{2^m+2^{m-1}}, i_{2.2^m+2^{m-1}}, i_{3.2^m+2^{m-1}}, \ldots, i_{2^t-2^{m-1}}\right) \\
&\ldots \\
(i)_t &= (i_{2^t-1}),
\end{aligned}$$

so that each $(i)_s$ lies in $\{0, 1\}^{2^{t-s}}$. The plan shall be to construct Θ so that the minimal projections of $\widetilde{A_t}$, defined to be $A \cap M^{\otimes 2^t-1}$, are taken to elements of the form

$${}^{t-1}f_{(i)_1} \otimes {}^{t-2}f_{(i)_2} \otimes \cdots \otimes {}^{0}f_{(i)_t},$$

[5] Think of these bijections as graphs for the purpose of taking this union.

for some $i \in \{0,1\}^{2^t-1}$. The form of Θ_2 ensures that

$$\begin{aligned}
&\Theta_2^{-1}\left({}^{t-1}f_{(i)_1} \otimes {}^{t-2}f_{(i)_2} \otimes \cdots \otimes {}^{0}f_{(i)_t}\right) \\
&= e^{(0)}_{i_1} \otimes \bigotimes_{s=0}^{t-2} \left(\bigotimes_{r=1}^{2^s-1} e^{(i_r)}_{i_{2^s+r}} \otimes e^{(0)}_{i_{2^s+1}} \right) \otimes \bigotimes_{r=1}^{2^{t-1}-1} e^{(i_r)}_{i_{2^{t-1}+r}},
\end{aligned}$$

whereas (4.3.1) shows that the minimal projections of $\widetilde{A_t}$ are of the form

$$^t\widetilde{f_i} = e^{(0)}_{i_1} \otimes \bigotimes_{s=0}^{t-2} \left(\bigotimes_{r=1}^{2^s-1} e^{(i_r)}_{i_{2^s+r}} \otimes e^{(i_{2^s})}_{i_{2^s+1}} \right) \otimes \bigotimes_{r=1}^{2^{t-1}-1} e^{(i_r)}_{i_{2^{t-1}+r}},$$

for $i \in \{0,1\}^{2^t-1}$. This leads us to conclude that the desired automorphism Θ_1 should 'untwist' the 2^s-th components of the tensor product, for every $s \geq 1$. More precisely, Theorem 4.3.8 will follow immediately from the next result.

Proposition 4.3.10. *There exists an automorphism Θ_1 of R with*

$$\begin{aligned}
&\Theta_1 \left(e^{(0)}_{i_1} \otimes \bigotimes_{s=0}^{t-2} \left(\bigotimes_{r=1}^{2^s-1} e^{(i_r)}_{i_{2^s+r}} \otimes e^{(i_{2^s})}_{i_{2^s+1}} \right) \otimes \bigotimes_{r=1}^{2^{t-1}-1} e^{(i_r)}_{i_{2^{t-1}+r}} \right) \\
&= e^{(0)}_{i_1} \otimes \bigotimes_{s=0}^{t-2} \left(\bigotimes_{r=1}^{2^s-1} e^{(i_r)}_{i_{2^s+r}} \otimes e^{(0)}_{i_{2^s+1}} \right) \otimes \bigotimes_{r=1}^{2^{t-1}-1} e^{(i_r)}_{i_{2^{t-1}+r}},
\end{aligned}$$

for every $t \geq 1$ and (2^t-1)-tuple $i \in \{0,1\}^{2^t-1}$.

Proof. Define unitaries $u_t \in N_t$ by

$$u_t = 1^{\otimes(2^{t-1}-1)} \otimes \left(e^{(0)}_0 \otimes 1^{\otimes 2^{t-1}} + e^{(0)}_1 \otimes 1^{\otimes(2^{t-1}-1)} \otimes b^* \right).$$

We will take

$$\Theta_1 = \lim_{t\to\infty} \mathrm{Ad}\ u_t u_{t-1} \dots u_1, \tag{4.3.4}$$

with convergence in pointwise weak*-topology as usual. Our first objective is to demonstrate that the limit in (4.3.4) exists, and defines an automorphism Θ_1 of R.

We claim that for each $x \in N_t$, the sequence

$$(u_{t+s}u_{t+s-1} \dots u_{t+1} x u^*_{t+1} \dots u^*_{t+s-1} u^*_{t+s})_{s=1}^{\infty} \tag{4.3.5}$$

is Cauchy in $\|.\|_2$. It will then converge to some $\Theta_1(x) \in L^2(R)$ in 2-norm and since the ball

$$\{\, y \in R \mid \|y\| \leq \|x\| \,\}$$

in R is closed in $\|.\|_2$, actually $\Theta_1(x)$ will lie in R. This will be enough to ensure that we have (4.3.4) with pointwise $\|.\|_2$-convergence for all $x \in R$. The map Θ_1

will be a *-homomorphism from R into R. It will be necessary to show Θ_1 is an automorphism, which we shall do by confirming the existence of an inverse.[6]

Turning to the sequence (4.3.5), it is enough by linearity to consider $x = {}^t e_{i,j} = \bigotimes_{r=1}^{2^t} e^{(0)}_{i_r,j_r}$, for $i,j \in \{0,1\}^{2^t}$. If $i_{2^t} = j_{2^t}$, then u_{t+1} commutes with x. Moreover, x is the linear combination of matrix units ${}^{t+1}e_{i',j'}$ for N_{t+1} with $i'_{2^{t+1}} = j'_{2^{t+1}}$ so that u_{t+2} also commutes with x. Proceeding in this fashion we see that in this case all u_{t+s} commutes with x, so that we have the convergence (4.3.4) with $\Theta_1(x) = x$. When $i_{2^t} \neq j_{2^t}$, recalling that conveniently $b = (e^{(0)}_{0,0} + e^{(0)}_{0,1} + e^{(0)}_{1,0} - e^{(0)}_{1,1})/\sqrt{2}$ is self-adjoint, we have

$$\begin{aligned} u_{t+1} x u^*_{t+1} &= x \otimes 1^{\otimes(2^{t-1}-1)} \otimes b \\ &= x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,0} - e^{(0)}_{1,1})\sqrt{2} \qquad (4.3.6) \\ &\quad + x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0})/\sqrt{2} \qquad (4.3.7) \end{aligned}$$

The term (4.3.6) commutes with all u_{t+s}, $(s \geq 2)$, as before, so we need only examine the term (4.3.7) to see that

$$\begin{aligned} u_{t+2}u_{t+1} x u^*_{t+1} u^*_{t+2} &= x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,0} - e^{(0)}_{1,1})\sqrt{2} \\ &\quad + x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^t-1)} \otimes b/\sqrt{2}. \qquad (4.3.8) \end{aligned}$$

This gives the identity

$$\left\| u_{t+2}u_{t+1} x u^*_{t+1} u^*_{t+2} - u_{t+1} x u^*_{t+1} \right\|_2 = \|x\|_2 \, \|1 - b\|_2 / \sqrt{2}.$$

We then split up (4.3.8) to obtain

$$\begin{aligned} u_{t+2}u_{t+1} x u^*_{t+1} u^*_{t+2} &= x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,0} - e^{(0)}_{1,1})\sqrt{2} \\ &\quad + x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^t-1)} \otimes (e^{(0)}_{0,0} - e^{(0)}_{1,1})/2 \\ &\quad + x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^t-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0})/2. \end{aligned}$$

The first two terms appearing on the right hand side commute with u_{t+s}, for $s \geq 3$, and the third term has

$$\begin{aligned} & u_{t+3} \left(x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^t-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0})/2 \right) u^*_{t+3} \\ =& x \otimes 1^{\otimes(2^{t-1}-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^t-1)} \otimes (e^{(0)}_{0,1} + e^{(0)}_{1,0}) \otimes 1^{\otimes(2^{t+1}-1)} \otimes b/2^2. \end{aligned}$$

Hence, we have

$$\left\| (\mathrm{Ad}\ u_{t+3} - I) \left(u_{t+2}u_{t+1} x u^*_{t+1} u^*_{t+2} \right) \right\|_2 = 2^{-1} \|x\|_2 \, \|1 - b\|_2 .$$

[6]This will then guarantee the convergence of (4.3.4) in pointwise weak*-topology as this agrees with pointwise $\|.\|_2$-topology on the automorphism group.

Continuing in this way, yields

$$\left\|(\text{Ad } u_{t+s+1} - I)\left(u_{t+s}u_{t+s-1}\dots u_{t+1}xu_{t+1}^*\dots u_{t+s-1}^*u_{t+s}^*\right)\right\|_2 = 2^{-(s-1)/2}\|x\|_2\|1-b\|_2,$$

from which we can immediately deduce that the sequence (4.3.5) is $\|.\|_2$-Cauchy as claimed.

We now check that Θ_1 is an automorphism of R, rather than merely an endomorphism. It is clear what the inverse to Θ_1 must be, if it exists, namely

$$\Theta_1^{-1} = \lim_{t\to\infty} \text{Ad } u_1^*u_2^*\dots u_t^*, \tag{4.3.9}$$

with convergence again in pointwise weak*-topology. Just as before, it is enough to check that

$$\lim_{t\to\infty}\left(\text{Ad } u_1^*u_2^*\dots u_t^*\right)(x)$$

is $\|.\|_2$-convergent for every $x \in \cup_{n=1}^{\infty} N_t$. This is immediate, as for $x \in N_t$, u_{t+s} commutes with x, for every $s \geq 2$.

We have now constructed an automorphism Θ_1 of R. It remains to check that it behaves as required, for which it is easier to work with Θ_1^{-1}. Indeed, take $i \in \{0,1\}^{2^t-1}$ for some t, and

$$x = e_{i_1}^{(0)} \otimes \bigotimes_{s=0}^{t-2}\left(\bigotimes_{r=2}^{2^s-1} e_{i_{2^s+r}}^{(i_r)} \otimes e_{i_{2^s+1}}^{(0)}\right) \otimes \bigotimes_{r=2}^{2^{t-1}-1} e_{i_{2^{t-1}+r}}^{(i_r)} \in M^{\otimes 2^t-1}.$$

Observe that $u_{t+1}, u_{t+2}, \dots$ all commute with x and, less obviously, so too does u_t, giving

$$\Theta_1^{-1}(x) = u_1^*\dots u_{t-1}^*xu_{t-1}\dots u_1.$$

Now

$$u_{t-1}^*xu_{t-1} = e_{i_1}^{(0)}\otimes\bigotimes_{s=0}^{t-3}\left(\bigotimes_{r=1}^{2^s-1} e_{i_{2^s+r}}^{(i_r)} \otimes e_{i_{2^s+1}}^{(0)}\right)\otimes\left(\bigotimes_{r=1}^{2^{t-2}-1} e_{i_{2^{t-2}+r}}^{(i_r)} \otimes e_{i_{2^{t-1}}}^{(i_{2^{t-2}})}\right)\otimes\bigotimes_{r=1}^{2^{t-1}-1} e_{i_{2^{t-1}+r}}^{(i_r)},$$

as, if $i_{2^{t-2}} = 1$ the 2^{t-1}-th tensor component of x is conjugated by b which takes $e_j^{(0)}$ to $e_j^{(1)}$, and if $i_{2^{t-2}} = 0$, then adjunction by u_{t-1} has no effect on x. We can then see that

$$\begin{aligned}
&u_{t-2}^*u_{t-1}^*xu_{t-1}u_{t-2}\\
&= u_{t-2}^*\left(e_{i_1}^{(0)} \otimes \bigotimes_{s=0}^{t-3}\left(\bigotimes_{r=1}^{2^s-1} e_{i_{2^s+r}}^{(i_r)} \otimes e_{i_{2^s+1}}^{(0)}\right)\right)u_{t-2}\\
&\quad\otimes\left(\bigotimes_{r=1}^{2^{t-2}-1} e_{i_{2^{t-2}+r}}^{(i_r)} \otimes e_{i_{2^{t-1}}}^{(i_{2^{t-2}})}\right)\otimes\bigotimes_{r=1}^{2^{t-1}-1} e_{i_{2^{t-1}+r}}^{(i_r)}\\
&= e_{i_1}^{(0)} \otimes \bigotimes_{s=0}^{t-4}\left(\bigotimes_{r=1}^{2^s-1} e_{i_{2^s+r}}^{(i_r)} \otimes e_{i_{2^s+1}}^{(0)}\right) \otimes \bigotimes_{r=1}^{2^{t-3}} e_{i_{2^{t-3}+r}}^{(i_r)} \otimes \bigotimes_{r=1}^{2^{t-2}} e_{i_{2^{t-2}+r}}^{(i_r)} \otimes \bigotimes_{r=1}^{2^{t-1}-1} e_{i_{2^{t-1}+r}}^{(i_r)}.
\end{aligned}$$

Continuing in this way gives

$$\Theta_1^{-1}(x) = e_{i_1}^{(0)} \otimes \bigotimes_{s=0}^{t-2} \left(\bigotimes_{r=1}^{2^s-1} e_{i_{2^s+r}}^{(i_r)} \otimes e_{i_{2^s+1}}^{(i_{2^s})} \right) \otimes \bigotimes_{r=1}^{2^{t-1}-1} e_{i_{2^{t-1}+r}}^{(i_r)},$$

exactly as we require. □

The analogous result for the θ masas, is that $A^{(\theta_n)}$ factorises inside R as an infinite von Neumann tensor product of θ-masas coming from subsequences of (θ_n). It is easy to see what these subsequences are, given $(\theta_n)_{n=2}^{\infty}$ in $[0,\pi/4]$, define $\theta_s^{(m)} = \theta_{\psi^{-1}(m,s)}$ where ψ is the bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ defined previously.

Theorem 4.3.11. *Given a sequence $(\theta)_{n=2}^{\infty}$ in $[0,\pi/4]$ there is a *-isomorphism Θ of R onto $R^{\overline{\otimes}\infty}$ with*

$$\Theta\left(A^{(\theta_n)}\right) = \left(A^{\left(\theta_s^{(1)}\right)}\overline{\otimes}A^{\left(\theta_s^{(2)}\right)}\overline{\otimes}\dots\right)''.$$

Corollary 4.3.12. *The θ-masas of Construction 2.2.4 are Γ-masas for their underlying hyperfinite* II$_1$ *factor.*

We now return to the examination of cutdowns of Tauer's singular masa by minimal projections coming from some A_t. It will turn out that these cutdowns look naturally like an infinite tensor power of A, and so Theorem 4.3.8 shows that they are conjugate as masas to A itself.[7]

Theorem 4.3.13. *Fix $t\in\mathbb{N}$ and take $i\in\{0,1\}^{2^t}$. There is a *-isomorphism Θ from R onto ${}^tf_iR\,{}^tf_i$ under which $\Theta(A) = A\,{}^tf_i$.*

Proof. We can make our lives easier here by assuming, using either Proposition 4.3.1 or the stronger result, Proposition 4.3.5, that $i=(0,0,\dots,0)$. Regard $R = N_t\overline{\otimes}\widetilde{R}$, where $\widetilde{R}$ is the infinite tensor product $(\bigotimes_{n=2^t+1}^{\infty} M_n)''$ of copies M_n of M. Following the formulation (2.1.3) of Tauer masas, we write $A_{\infty,t}^{({}^tf_i)}$ for the Tauer masa obtained in $\widetilde{R}$ so that $A^t\,f_i = {}^tf_i\otimes A_{\infty,t}^{({}^tf_i)}$. The desired isomorphism will be exhibited as the composite

$$\Theta:\ {}^tf_iR^t\ f_i \overset{{}^tf_i\otimes x\mapsto x}{\to} \widetilde{R} \overset{\text{reordering }\Theta_1}{\to} R^{\overline{\otimes}\infty} \to R,$$

where Θ_1 will take $A_{\infty,t}^{({}^tf_i)}$ onto $A^{\overline{\otimes}\infty}$ in $R^{\overline{\otimes}\infty}$. The final *-isomorphism in the composition will be that of Theorem 4.3.8, which will then ensure that $\Theta(A^tf_i) = A$.

[7]It could be argued that the objective here is, assuming A is transitive - a result we have been unable to achieve, to fail to determine the fundamental group of $A\subset R$ also.

We now give a formulation for the reordering Θ_1. This comes, as in Theorem 4.3.8, from a bijection $\psi : \{2^t+1, 2^t+2, 2^t+3, \dots\} \to \mathbb{N} \times \mathbb{N}$. The isomorphism Θ_1 will be given by taking the n-th component M_n in the tensor product making up $\widetilde{R}$ to the $\psi_2(n)$-th copy of M in the tensor product making up $R_{\psi_1(n)}$, where we write $\psi(n) = (\psi_1(n), \psi_2(n))$. We will actually find it easier to define $\psi^{-1}(m,s)$. Given m, write $m = k2^t + l$ for some $k \geq 0$ and $1 \leq l \leq 2^t$. For $s \geq 1$, define

$$\psi^{-1}(m,s) = (s-1)2^{t+k+1} + 2^{t+k} + l.$$

Let us check that ψ^{-1} is a bijection. Suppose $\psi^{-1}(m_1,s_1) = \psi^{-1}(m_2,s_2)$ then, on writing $m_j = k_j 2^t + l_j$ with $1 \leq l_j \leq 2^t$, and dividing $\psi^{-1}(m_1,s_1)$ and $\psi^{-1}(m_2,s_2)$ by 2^t, we obtain $l_1 = l_2$. We are left with the identity

$$(s_1-1)2^{t+k_1+1} + 2^{t+k_1} = (s_2-1)2^{t+k_2+1} + 2^{t+k_2}.$$

If $k_1 < k_2$, then we can divide by 2^{t+k_1+1} to see that $2^{t+k_1} = 0$ - a patent contradiction. Hence, by symmetry $k_1 = k_2$, and we can then immediately deduce that $s_1 = s_2$ also. In conclusion ψ^{-1} is injective.

Given $n > 2^t$, let $n \equiv l \pmod{2^t}$ for some $1 \leq l \leq 2^t$. Now $n - l$ is divisible by 2^t, let $k \geq 0$ be maximal such that $n-l$ is divisible by 2^{t+k}. Here $n - l - 2^{t+k}$ must be divisible by 2^{t+k+1}, as otherwise $n - l - 2^{t+k} = r2^{t+k+1} + 2^{t+k}$ for some r say, contradicting the non-divisibility of $n-l$ by 2^{t+k+1}. In particular, we can write $n - l - 2^{t+k}$ as $(s-1)2^{t+k+1}$ for some $s \geq 1$. Now taking $m = k2^t + l$, we have $n = \psi^{-1}(m,s)$ so that ψ^{-1} is also surjective.

It remains to check that the Θ_1 induced by ψ does have $\Theta_1(A_{\infty,t}^{({}^t f_i)}) = A^{\overline{\otimes}\infty}$. Write $A_{s,t}^{({}^t f_i)}$ for the approximation $A_{\infty,t}^{({}^t f_i)} \cap \bigotimes_{n=2^t+1}^{2^s} M$. Minimal projections for $A_{s,t}^{({}^t f_i)}$ are of the form

$${}^s g_i = \bigotimes_{u=t}^{s-1} \bigotimes_{r=1}^{2^u} e_{i_{2^u+r}}^{(i_r)},$$

indexed by $(i_{2^t+1}, i_{2^t+2}, \dots, i_{2^s}) \in \{0,1\}^{2^s-2^t}$. Recall that we have fixed $i_1 = i_2 = \cdots = i_{2^t} = 0$, so that

$${}^s g_i = \bigotimes_{u=t}^{s-1} \left(\bigotimes_{r=1}^{2^t} e_{i_{2^u+r}}^{(0)} \otimes \bigotimes_{r=2^t+1}^{2^u} e_{i_{2^u+r}}^{(i_r)} \right).$$

The map Θ_1 takes $\bigotimes_{n=2^t+1}^{2^{t+1}} M$ onto $N_0^{\otimes 2^t} \subset \overline{\bigotimes}_{m=1}^{2^t} R_m \subset R^{\overline{\otimes}\infty}$, and takes ${}^{t+1}g_i$ onto $\bigotimes_{m=1}^{2^t} {}^0 f_{(i_{2^t+m})}$, so that $A_{t+1,t}^{({}^t f_i)}$ is mapped to $A_0^{\otimes 2^t}$. We can also check that

$$\begin{aligned} \Theta_1({}^{t+2}g_i) &= \bigotimes_{n=2^t+1}^{2^{t+1}} \left(e_{i_n}^{(0)} \otimes e_{i_{2^{t+1}+n}}^{(i_n)} \right) \otimes \bigotimes_{n=2^{t+1}+1}^{2^{t+1}+2^t} e_{i_n}^{(0)} \\ &= \bigotimes_{m=1}^{2^t} {}^1 f_{(i_{2^t+m}, i_{2^{t+1}+2^t+m})} \otimes \bigotimes_{m=1}^{2^t} {}^0 f_{(i_{2^{t+1}+m})}. \end{aligned}$$

Hence,

$$\Theta_1(A_{t+2,t}^{({}^t f_i)}) = A_1^{\otimes 2^t} \otimes A_0^{\otimes 2^t} \subset N_1^{\otimes 2^t} \otimes N_0^{\otimes 2^t} \subset \overline{\bigotimes}_{m=1}^{2^{t+1}} R_m \subset R^{\overline{\otimes}\infty}.$$

We can continue in this way to see that

$$\Theta_1(A_{s,t}^{({}^t f_i)}) = A_{2^{s-t}-1}^{\otimes 2^t} \otimes A_{2^{s-t}-2}^{\otimes 2^t} \otimes \cdots \otimes A_0^{\otimes 2^t} \subset N_{2^{s-t}-1}^{\otimes 2^t} \otimes N_{2^{s-t}-2}^{\otimes 2^t} \otimes \cdots \otimes N_0^{\otimes 2^t}$$
$$\subset \overline{\bigotimes}_{m=1}^{(s-t)2^t} R_m \subset R^{\overline{\otimes}\infty},$$

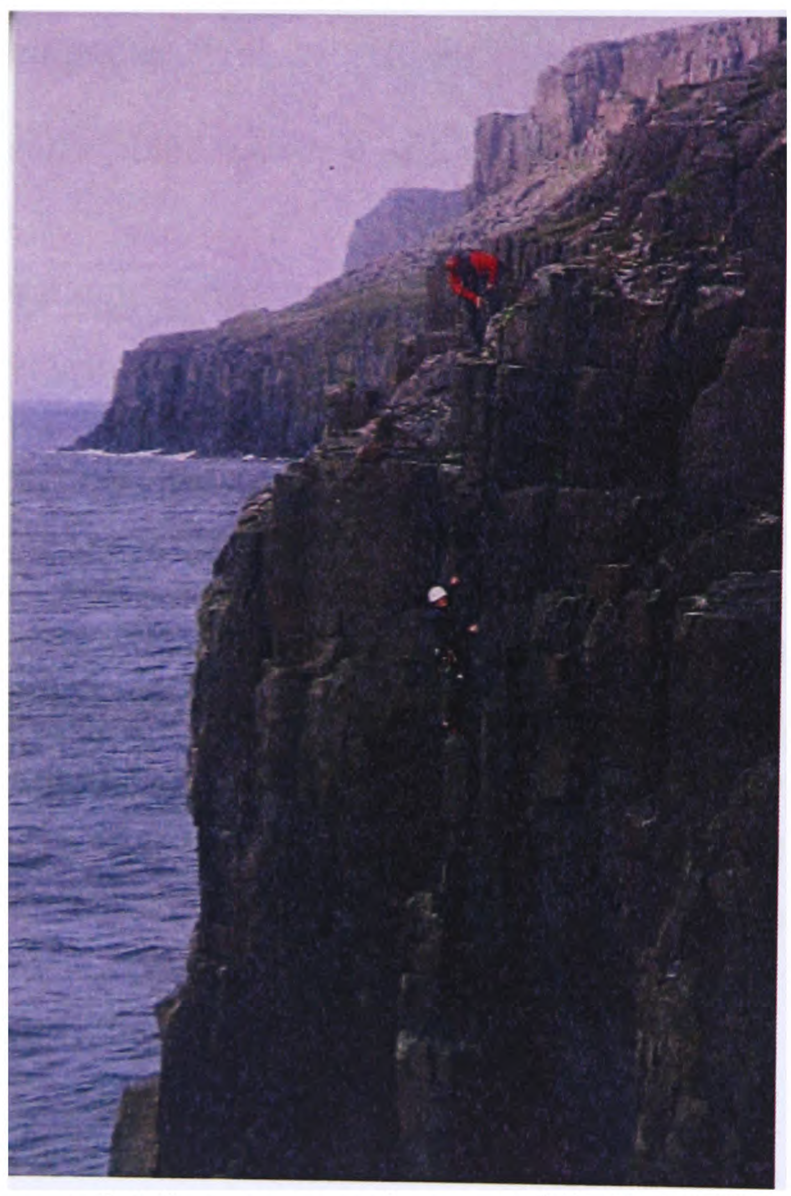
for each $s > t$. So $\Theta_1(A_{\infty,t}^{({}^t f_i)}) = A^{\overline{\otimes}\infty} \subset R^{\overline{\otimes}\infty}$, as required. □

The analogous result for the θ masas holds when all the angles θ_n are the same.

Theorem 4.3.14. *Let $\theta_n = \theta \in [0, \pi/4]$ for each $n \geq 2$. For each $t \geq 0$ and minimal projection f for $A_t^{(\theta_n)}$ there is a *-isomorphism from R onto fRf which maps A onto Af.*

Unfortunately, at the time of writing, we have not been able to find an argument to attack the A-conjugacy, weak or otherwise, of projections in A not minimal in some A_t. We can not even deal with all the cases of finite sums of such minimal projections. Instead, we leave the reader with the anitclimatic photo, overleaf.

In conclusion



Completing the successful first ascent of *copout* (S), Neist Point, Skye, May 2003.

Photo: Jon Powell

Appendix A

Failure of the weak asymptotic homomorphism property

In subsection 2.3.5 we asserted that, when A is Tauer's semi-regular masa in R of length 3 of Construction 2.3.5, $\mathcal{N}^1(A)$ does not have the weak asymptotic homomorphism property away from $\mathcal{N}^2(A)$. Here, some hundred or so pages later, we get round to justifying this claim. We begin by recalling the situation from section 2.3.

Let M be the algebra of 2×2 matrices, with standard matrix units $(e^{(0)}_{i,j})^1_{i,j=0}$. Let b be the self adjoint unitary $(e^{(0)}_{0,0}+e^{(0)}_{0,1}+e^{(0)}_{1,0}-e^{(0)}_{1,1})/\sqrt{2}$ in M, and write

$$e^{(i',j')}_{i,j}=b^{i'}e^{(0)}_{i,j}b^{j'},$$

for $i,i',j,j'\in\mathbb{Z}_2$. Define $N_n=M^{\otimes n}\otimes M^{\otimes n}$, which is included in N_{n+1} by $x\otimes y\mapsto(x\otimes 1)\otimes(y\otimes 1)$ on elementary tensors. Let R be the hyperfinite II_1 factor obtained as the direct limit of the chain $(N_n)^\infty_{n=1}$ with respect to the normalised trace on each N_n.

Given $i^{(1)}_r,i^{(2)}_r,j^{(1)}_r,j^{(2)}_r\in\mathbb{Z}_2$ for $1\leq r\leq n$, write

$${}^n f_{i,j}=\bigotimes_{r=1}^n e^{(0)}_{i^{(1)}_r,j^{(1)}_r}\otimes\bigotimes_{r=1}^n e^{\left(\sum_{t=1}^{r-1}i^{(1)}_t,\sum_{t=1}^{r-1}j^{(1)}_t\right)}_{i^{(2)}_r,j^{(2)}_r},$$

with the usual convention that the sums in the second tensor product are taken in $\mathbb{Z}_2$ and defined to be zero when they make no sense (that is when $r=1$). These ${}^n f_{i,j}$ are matrix units for N_n. Define equivalence relations on the indices i,j by $i\sim j$ if and only if $\sum_{r=1}^n(i^{(1)}_r-j^{(1)}_r)=0$ in $\mathbb{Z}_2$ and $i\approx j$ if and only if $i\sim j$ and $\sum_{r=1}^n(i^{(2)}_r-j^{(2)}_r)=0$ in $\mathbb{Z}_2$. Let T_n be the *-subalgebra of N_n generated by all ${}^n f_{i,j}$ with $i\sim j$, and S_n be the *-subalgebra of T_n generated by all ${}^n f_{i,j}$ with $i\approx j$. As each $T_n\subset T_{n+1}$ and $S_n\subset S_{n+1}$, we let T and S be the von Neumann subalgebras of R generated by the $(T_n)^\infty_{n=1}$ and $(S_n)^\infty_{n=1}$ respectively. These are subfactors of R with $\mathcal{N}(S)''=T$ and indeed, up to an automorphism of notation, this is the $l=3$ case of Proposition 2.3.11 and Lemma 2.3.12.

Our objective is to show that S does not have the weak asymptotic homomorphism property away from T, despite the fact that T is generated by the normalisers of S. This is not actually the claim made earlier, as to obtain normalising algebras of Tauer's original length 3 semi-regular masa, we must tensor by another hyperfinite II_1 factor. Ergo, let R_1 be another hyperfinite II_1 factor then Tauer's semi-regular masa A of length 3 can be realised in $R\overline{\otimes}R_1$ with $\mathcal{N}(A)'' = S\overline{\otimes}R_1$ and $\mathcal{N}_2(A) = T\overline{\otimes}R_1$. The proof we are about to give that S does not have the weak asymptotic homomorphism property away from T also shows that $S\overline{\otimes}R_1$ does not have the weak asymptotic homomorphism property away from $T\overline{\otimes}R_1$, but we see no need to deal with this situation further.

Theorem A.1. *With the notation above, S does not have the weak asymptotic homomorphism property away from T.*

The plan is to consider the finite set X in $N_1 \ominus T_1$ given by

$$X = \left\{ {}^1f_{i,j} \,\middle|\, i_1^{(1)}, i_1^{(2)}, j_1^{(1)}, j_1^{(2)} \in \mathbb{Z}_2 \text{ with } i_1^{(1)} \neq j_1^{(1)} \right\}, \tag{A.1}$$

and demonstrate that any unitary $u \in S$ has $\|\mathbb{E}_S(xuy)\|_2$ 'large', for some $x, y \in X$. More precisely, we shall establish the next result.

Lemma A.2. *Fix $n \geq 2$, and let u be a unitary in S_n. Then*

$$\sum_{x,y\in X} \|\mathbb{E}_{S_n}(xuy)\|_2^2 = \frac{1}{2}.$$

Lemma A.2, is enough to establish Theorem A.1, as each xuy lies in N_n so that $\mathbb{E}_{S_n}(xuy) = \mathbb{E}_S(xuy)$, by the commutative diagram in Figure 2.4 back in section 2.3. The set X has 8 elements, all of which have $\mathbb{E}_{T_1}(x) = \mathbb{E}_T(x) = 0$. Hence, for any unitary $u \in S_n$ it is possible to find some $x, y \in X$ with

$$\|\mathbb{E}_S(xuy)\|_2 \geq \frac{1}{8\sqrt{2}}. \tag{A.2}$$

Given a unitary $v \in S$, find a unitary $u \in S_n$ for some n such that $\|u - v\|_2 \leq 1/16\sqrt{2}$. Let x, y be elements in X so that (A.2) holds. Then

$$\begin{aligned} \|\mathbb{E}_S(xvy)\|_2 &\geq \|\mathbb{E}_S(xuy)\|_2 - \|x(u-v)y\|_2 \\ &\geq \frac{1}{8\sqrt{2}} - \|u-v\|_2 \\ &\geq \frac{1}{16\sqrt{2}}, \end{aligned}$$

since elements $x \in X$ certainly have operator norm $\|x\| \leq 1$. This completes the proof of Theorem A.1, as it is then not possible to find a unitary $v \in S$ for which

$$\|\mathbb{E}_S(xvy)\|_2 < \frac{1}{16\sqrt{2}},$$

for every x, y in the finite set X.

Proof of Lemma A.2. Fix n, and a unitary $u \in S_n$. Write

$$u = \sum_{\substack{i^{(1)} \sim j^{(1)} \\ i_1^{(2)}, j_1^{(2)} \in \mathbb{Z}_2}} \bigotimes_{r=1}^{n} e^{(0)}_{i_r^{(1)}, j_r^{(1)}} \otimes \left(e^{(0)}_{i_1^{(2)}, j_1^{(2)}} \otimes u_{i,j} \right),$$

for some $u_{i,j} \in M^{\otimes(n-1)}$.[1] We decompose S_n in the same way as we have decomposed u. Namely, write

$$S_n = \bigoplus_{\substack{i^{(1)} \sim j^{(1)} \\ i_1^{(2)}, j_1^{(2)} \in \mathbb{Z}_2}} \bigotimes_{r=1}^{n} e^{(0)}_{i_r^{(1)}, j_r^{(1)}} \otimes \left(e^{(0)}_{i_1^{(2)}, j_1^{(2)}} \otimes S_n(i,j) \right),$$

where, for $i^{(1)} \sim j^{(1)}$ and $i_1^{(2)}, j_1^{(2)} \in \mathbb{Z}_2$, $S_n(i,j)$ is the subspace of $M^{\otimes(n-1)}$ generated by elements of the form

$$^{n}g_{i^{(2)}, j^{(2)}} = \bigotimes_{r=2}^{n} e^{\left(\sum_{t=1}^{r-1} i_t^{(1)}, \sum_{t=1}^{r-1} j_t^{(1)}\right)}_{i_r^{(2)}, j_r^{(2)}}, \tag{A.3}$$

for some $i_r^{(2)}, j_r^{(2)} \in \mathbb{Z}_2$, $(2 \le r \le n)$, satisfying

$$i_1^{(2)} + \sum_{r=2}^{n} i_r^{(2)} = j_1^{(2)} + \sum_{r=2}^{n} j_r^{(2)}.$$

We also write $S_n(i,j)$ as $S_n(i^{(1)}, j^{(1)}, i_1^{(2)}, j_2^{(2)})$. Observe that we have

$$S_n(i^{(1)}, j^{(1)}, 0, 0) = S_n(i^{(1)}, j^{(1)}, 1, 1) \tag{A.4}$$

$$S_n(i^{(1)}, j^{(1)}, 0, 1) = S_n(i^{(1)}, j^{(1)}, 1, 0), \tag{A.5}$$

and furthermore a simple calculation shows that these two subspaces are orthogonal in $M^{\otimes(n-1)}$. The direct sum of (A.4) and (A.5) is all of $M^{\otimes(n-1)}$ as any $^{n}g_{i^{(2)}, j^{(2)}}$ lies in one of these two subspaces.

Now consider $x = {}^{1}f_{k,i}$ and $y = {}^{1}f_{j,l}$ in X for some $i_1^{(q)}, j_1^{(q)}, k_1^{(q)}, l_1^{(q)} \in \mathbb{Z}_2$ for $q = 1, 2$, and regard these indices as being fixed until further notice. We have $i_1^{(1)} \neq k_1^{(1)}$ and $j_1^{(1)} \neq l_1^{(1)}$, by (A.1), so

$$xuy = \sum_{\substack{i_r^{(1)}, j_r^{(1)} \in \mathbb{Z}_2 : r=2,\dots,n \\ \sum_{r=1}^{n} i_r^{(1)} = \sum_{r=1}^{n} j_r^{(1)}}} \left(e^{(0)}_{k_1^{(1)}, l_r^{(1)}} \otimes \bigotimes_{r=2}^{n} e^{(0)}_{i_r^{(1)}, j_r^{(1)}} \right) \otimes \left(e^{(0)}_{k_r^{(2)}, l_r^{(2)}} \otimes u_{i,j} \right).$$

We compute $\mathbb{E}_{S_n}(xuy)$ using the subspaces $S_n(i,j)$ introduced above. We have

$$\mathbb{E}_{S_n}(xuy) = \sum_{\substack{i_r^{(1)}, j_r^{(1)} \in \mathbb{Z}_2 : r=2,\dots,n \\ i^{(1)} \sim j^{(1)}}} \left(e^{(0)}_{k_1^{(1)}, l_1^{(1)}} \otimes \bigotimes_{r=2}^{n} e^{(0)}_{i_r^{(1)}, j_r^{(1)}} \right)$$

$$\otimes \left(e^{(0)}_{k_1^{(2)}, l_1^{(2)}} \otimes e_{S_n(i'^{(1)}, j'^{(1)}, k_1^{(2)}, l_1^{(2)})}(u_{i,j}) \right),$$

[1]There are, of course, additional restrictions on the $u_{i,j}$, so that u lies in S_n. In a minute, these will become that every $u_{i,j}$ lies in the corresponding $S_n(i,j)$.

where $i'^{(1)}$ and $j'^{(1)}$ are defined by

$$i'^{(1)}_r = \begin{cases} k_1^{(1)} & r = 1 \\ i_r^{(1)} & r = 2, \ldots, n \end{cases} \quad \text{and } j'^{(1)}_r = \begin{cases} l_1^{(1)} & r = 1 \\ j_r^{(1)} & r = 2, \ldots, n \end{cases} .$$

Hence,

$$\left\| \mathbb{E}_{S_n} \left({}^1 f_{k,i} u \, {}^1 f_{j,l} \right) \right\|_2^2 = \frac{1}{2^{n+1}} \sum_{\substack{i_r^{(1)}, j_r^{(1)} \in \mathbb{Z}_2 : r = 2, \ldots, n \\ i^{(1)} \sim j^{(1)}}} \left\| e_{S_n(i'^{(1)}, j'^{(1)}, k_1^{(2)}, l_1^{(2)})} \left(u_{i,j} \right) \right\|_2^2 . \quad \text{(A.6)}$$

We now unfix $l_1^{(2)}$. Sum over $l_1^{(2)} \in \mathbb{Z}_2$, and use the fact that (A.4) and (A.5) are orthogonal with direct sum all of $M^{\otimes(n-1)}$, to obtain by Pythagoras' theorem

$$\sum_{l_1^{(2)} \in \mathbb{Z}_2} \left\| \mathbb{E}_{S_n} \left({}^1 f_{k,i} u \, {}^1 f_{j,l} \right) \right\|_2^2 = \frac{1}{2^{n+1}} \sum_{\substack{i_r^{(1)}, j_r^{(1)} \in \mathbb{Z}_2 : r = 2, \ldots, n \\ i^{(1)} \sim j^{(1)}}} \left\| u_{i,j} \right\|_2^2 ,$$

for all $i_1^{(2)}, j_1^{(2)}, k_1^{(2)} \in \mathbb{Z}_2$ and all $i_1^{(1)}, j_1^{(1)}, k_1^{(1)}, l_1^{(1)} \in \mathbb{Z}_2$. Now unfix the values of $i_1^{(1)}, i_1^{(2)}, j_1^{(1)}$ and $j_1^{(2)}$, to obtain, recalling that $k_1^{(1)} = 1 - i_1^{(1)}$ and $l_1^{(1)} = 1 - j_1^{(1)}$,

$$\sum_{i_1^{(1)}, i_1^{(2)}, j_1^{(1)}, j_1^{(2)}, l_1^{(2)} \in \mathbb{Z}_2} \left\| \mathbb{E}_{S_n} \left({}^1 f_{k,i} u \, {}^1 f_{j,l} \right) \right\|_2^2 = \frac{1}{2^{n+1}} \sum_{\substack{i^{(1)} \sim j^{(1)} \\ i_1^{(2)}, j_1^{(2)} \in \mathbb{Z}_2}} \left\| u_{i,j} \right\|_2^2 = \left\| u \right\|_2^2 = 1.$$

The only index still remaining fixed is $k_1^{(2)}$. Summing over this index as well, gives

$$\sum_{x,y \in X} \left\| \mathbb{E}_{S_n} (xuy) \right\|_2^2 = 2,$$

as claimed. □

Appendix B

Radial masas in factors coming from free products of certain finite groups

In this appendix we examine some very different masas to those appearing earlier. Let $G = G_1 * \cdots * G_m$ be the free product of m (non-trivial) discrete groups. Provided $m \geq 2$, G is a discrete I.C.C. group and so gives rise to the II_1 factor $\mathcal{L}(G)$ in which we work. There is a natural length function on elements of G, obtained by writing any g in $G \setminus 1_G$ uniquely as $g = g_1 \dots g_l$ where each g_j is a member of some $G_{i_j} \setminus \{1_{G_{i_j}}\}$ and $i_1 \neq i_2 \neq i_3 \cdots \neq i_l$. We refer to this expression for g as a *reduced word*, with the $g_1, \dots, g_l$ being the *syllables*. Define the length of g to be l, and the length of the identity 1_G to be 0.

Assume further that $G_1, \dots, G_m$ are all finite and of the same order, say $k \geq 2$.[1] For $n \geq 1$, let w_n be the element of the group algebra $\mathbb{C}G$ consisting of the sum of all elements in G of length n, and $w_0 = 1_G$. An easy computation, see [5], gives

$$w_1^2 = w_2 - (k-2)w_1 - m(k-1)w_0,$$

and for $n \geq 2$,

$$w_n w_1 = w_1 w_n = w_{n+1} + (k-2)w_n + (m-1)(k-1)w_{n-1}.$$

Hence, the unital von Neumann algebra generated by w_1 is the span of the w_n's. We denote by A the abelian von Neumann algebra generated by w_1, called the *radial subalgebra* or *Laplacian subalgebra.*

By analysising the spectrum of w_1, A was shown in [72] to be a masa if and only if $m \geq k$ - a standing assumption henceforth. Boca and Rădulescu first considered the normalisers of these A in [3], where they demonstrated that when $m = k = 2$, A is Cartan, and if $m \geq \max(k, 3)$, then $\mathrm{Puk}(A) = \{\infty\}$ and so,

[1] We explicitly note that there is no assumption that these groups are isomorphic.

by Popa's result ([46]), Theorem 3.1.3 in this thesis, A is singular. Our objective here is to use the methods of [60], where the natural radial masa in $\mathcal{L}(\mathbb{F}_k)$ was examined, to show that A is strongly singular; this will yield a shorter proof of the singularity of A.

Theorem B.1. *With the proceeding notation, when $m \geq \max(k,3)$ the radial masa A has the asymptotic homomorphism property in $\mathcal{L}(G)$ and so is strongly singular.*

We begin by stating a sufficient condition due to Sinclair and Smith for conditional expectations to be asymptotic homomorphisms, designed for radial masas. This result is obtained using the Riemann-Lebesque lemma.

Theorem B.2 ([60, Theorem 3.1]). *Let A be an abelian von Neumann subalgebra of a type II_1 factor N, and suppose that there is a $*$-isomorphism $\pi : A \to L^\infty[0,1]$ which induces an isometry from $L^2(A, tr)$ onto $L^2[0,1]$. Let*

$$\{ v_n \mid n \geq 0, v_n \in A \}$$

be an orthonormal basis for $L^2(A, tr)$, and $Y \subset N$ be a set whose linear span is norm dense in $L^2(N, tr)$. If

$$\sum_{n=0}^{\infty} \left\|\mathbb{E}_A(xv_ny) - \mathbb{E}_A(x)\, v_n \mathbb{E}_A(y)\right\|_2^2 < \infty \tag{B.1}$$

for all $x, y \in Y$, then A has the asymptotic homomorphism property in N.

Since A is a masa in $\mathcal{L}(G)$, Proposition 1.4.1 gives us the required $*$-isomorphism between A and $L^\infty[0,1]$ for the first hypothesis of Theorem B.2. The set Y in the Theorem will be taken to be G. The main work required in computing (B.1) will be a careful analysis of cancelations in products of G, a similar, although slightly more complicated, process to the analysis of cancelations in $\mathbb{F}_k$ of [60].

Distinct elements of Γ are orthogonal in $\ell^2(G)$, so normalising the orthogonal sequence $(w_n)_{n=0}^\infty$ gives an orthonormal basis $v_n = (w_n/\left\|w_n\right\|_2)_{n=0}^\infty$ for $L^2(A)$. Hence the conditional expectation $\mathbb{E}_A$ is given by

$$\mathbb{E}_A(x) = \sum_{n=0}^{\infty} \frac{\mathrm{tr}(xw_n)w_n}{\left\|w_n\right\|_2^2}. \tag{B.2}$$

This orthogonality enables the normalising constants above to be computed as

$$\left\|w_n\right\|_2^2 = m(k-1)[(m-1)(k-1)]^{n-1},$$

the number of distinct reduced words of length n, when $n \geq 1$.

Write $S_j = G_j \setminus \{1_{G_j}\}$ and $S = \cup_{i=1}^m S_i$. For non-empty subsets σ, τ of S, let $w_n(\sigma, \tau)$ denote the sum of all reduced words of length n whose first syllable lies in σ and last syllable lies in τ. Again $\|w_n(\sigma,\tau)\|_2^2$ is equal to the number of these reduced words, we could compute this exactly, but the asymptotic estimate which follows will be sufficient for our needs.

Lemma B.3. *If σ, τ are non-empty subsets of S, then*

$$\left| \frac{\|w_n(\sigma,\tau)\|_2^2}{\|w_n\|_2^2} - \frac{|\sigma||\tau|}{(m(k-1))^2} \right| \le \frac{1}{(m-1)^{n-2}}, \tag{B.3}$$

for $n \ge 2$.

Proof. First we consider $w_n(S_j, S_j)$ for a fixed $j = 1, 2, \ldots, m$. For $n \ge 2$, let

$$\alpha_n = \frac{\|w_n(S_j, S_j)\|_2^2}{\|w_n\|_2^2}.$$

Counting the number of choices for the last syllable of a word in the sum $w_{n+1}(S_j, S_j)$ gives

$$\|w_{n+1}(S_j,S_j)\|_2^2 = (k-1)(\|w_n(S_j,S)\|_2^2 - \|w_n(S_j,S_j)\|_2^2).$$

As all the G_i are all of the same order, we have

$$\|w_n(S_j,S)\|_2^2 = \frac{\|w_n\|_2^2}{m} = (k-1)[(m-1)(k-1)]^{n-1},$$

which can be substituted in to the previous equation to give

$$\|w_{n+1}(S_j,S_j)\|_2^2 = (k-1)\left(\frac{\|w_n\|_2^2}{m} - \|w_n(S_j,S_j)\|_2^2\right).$$

Hence α_n satisfies the recursion relation,

$$\alpha_{n+1} = \frac{1}{m(m-1)} - \frac{\alpha_n}{m-1}$$

for all $n \ge 2$. Solving this difference equation with initial condition $\alpha_2 = 0$, gives

$$\alpha_n = \frac{1}{m^2}\left(1 - \left(\frac{-1}{m-1}\right)^{n-2}\right), \tag{B.4}$$

a closed expression for α_n.

If i, j, k are distinct elements of $\{1, 2, \ldots, m\}$, then $\|w_n(S_i,S_j)\|_2^2 = \|w_n(S_i,S_k)\|_2^2$. So, for $i \ne j$,

$$\|w_n(S_i,S_j)\|_2^2 = \frac{\|w_n(S_i,S)\|_2^2 - \|w_n(S_i,S_i)\|_2^2}{m-1}.$$

Therefore,

$$\frac{\|w_n(S_i,S_j)\|_2^2}{\|w_n\|_2^2} = \frac{1/m-\alpha_n}{m-1}$$
$$= \frac{1}{m^2}\left(1-\left(\frac{-1}{m-1}\right)^{n-1}\right),$$

for $i \neq j$ and $n \geq 2$. Combining this with (B.4), we see that

$$\left|\frac{\|w_n(S_i,S_j)\|_2^2}{\|w_n\|_2^2} - \frac{1}{m^2}\right| \leq \frac{1}{m^2(m-1)^{n-2}},$$

for any elements i, j of $\{1, 2, \ldots, m\}$ and $n \geq 2$.

For $x \in H_i$ and $y \in H_j$, we have

$$\|w_n(\{x\},\{y\})\|_2^2 = \frac{\|w_n(S_i,S_j)\|_2^2}{(k-1)^2},$$

so that

$$\left|\frac{\|w_n(\{x\},\{y\})\|_2^2}{\|w_n\|_2^2} - \frac{1}{(m(k-1))^2}\right| \leq \frac{1}{m^2(k-1)^2(m-1)^{n-2}}$$

for $n \geq 2$. Therefore, for σ and τ as in the hypothesis, summing over $x \in \sigma$ and $y \in \tau$ gives

$$\left|\frac{\|w_n(\sigma,\tau)\|_2^2}{\|w_n\|_2^2} - \frac{|\sigma||\tau|}{(m(k-1))^2}\right| \leq \frac{|\sigma||\tau|}{m^2(k-1)^2(m-1)^{n-2}} \leq \frac{1}{(m-1)^{n-2}},$$

for all $n \geq 2$ as required. □

Given reduced words $x, v \in G$ of lengths l_1 and l_2 respectively we say that there are r cancellations in the product xv if the length of xv as a reduced word is $l_1 + l_2 - r$. For example, if a, b, c, d lie in different groups G_i, then when $x = abc$, $v = cd$ we have $xv = abc^2d$ and, assuming c is not of order 2, precisely one cancellation has occurred. However, if $v = c^{-1}d$ then $xv = abd$, and we have two cancellations in xv. Further cancelations would occur if b and d where found in the same G_i. Next, we examine cancellation in the product xw_ny.

Lemma B.4. *Let $x, y \in G$ be non-trivial reduced words of length l_1 and l_2 respectively. There exist subsets $\sigma_r(x), \tau_s(y)$ of S, whose cardinalities depend only on l_1, l_2, r and s such that the number, $\mu(r, s, n; x, y)$, of reduced words occurring in xw_ny with r cancellations on the left (with x) and s on the right (with y), is given by*

$$\mu(r,s,n;x,y) = \left\|w_{n-\lfloor r/2\rfloor-\lfloor s/2\rfloor}(\sigma_r(x),\tau_s(y))\right\|_2^2$$

for $0 \leq r \leq 2l_1$, $0 \leq s \leq 2l_2$ and $n \geq l_1 + l_2 + 2$.

Proof. Write $x = x_{l_1}x_{l_1-1}\dots x_1$ and $y = y_1y_2\dots y_{l_2}$ as reduced words with $x_i \in S_{\alpha(i)}$ and $y_j \in S_{\beta(j)}$. Now define the $\sigma_r(x)$'s and $\tau_s(y)$'s by

$$\sigma_r(x) = \begin{cases} S \setminus S_{\alpha(1)} & r = 0 \\ S \setminus (S_{\alpha(t+1)} \cup S_{\alpha(t)}) & r = 2t, 0 < t < l_1 \\ S_{\alpha(t)} \setminus \{x_t^{-1}\} & r = 2t-1, 0 < t \le l_1 \\ S \setminus S_{\alpha(l_1)} & r = 2l_1 \end{cases},$$

and

$$\tau_s(y) = \begin{cases} S \setminus S_{\beta(1)} & s = 0 \\ S \setminus (S_{\beta(t+1)} \cup S_{\beta(t)}) & s = 2t, 0 < t < l_2 \\ S_{\beta(t)} \setminus \{y_t^{-1}\} & s = 2t-1, 0 < t \le l_2 \\ S \setminus S_{\alpha(l_2)} & r = 2l_2 \end{cases}.$$

Since we have $\alpha(i) \neq \alpha(i+1)$ for $i = 1, \dots, l_1 - 1$ and $\beta(j) \neq \beta(j+1)$ for $j = 1, \dots, l_2 - 1$ we see that the cardinalities of the sets defined above depend only on l_1, l_2, r and s (and explicitly not on x or y).

Let v be a word of length n in G. There are no cancellations in xv if and only if the first syllable of v lies outside $S_{\alpha(1)}$. There are precisely $2t$ $(0 < t \le l_1)$ cancellations in xv, if and only if, v begins as $x_1^{-1}x_2^{-1}\dots x_r^{-1}u$ where $u \in S \setminus (S_{\alpha(r+1)} \cup S_{\alpha(r)})$.[2] There are precisely $2t-1$ $(0 < t < l_1)$ cancellations in xv, if and only if, v begins as $x_1^{-1}x_2^{-1}\dots x_{r-1}^{-1}u$ for some $u \in S_{\alpha(r)} \setminus \{x_r^{-1}\}$.

As $n \ge l_1 + k_2 + 2$, there can never be a complete cancellation of v, so we can perform a similar analysis of cancellations in the product vy and this establishes the lemma. □

We are now in a position to estimate the terms in the sum (B.1).

Lemma B.5. *If $m \ge 3$, then*

$$\sum_{n=0}^{\infty} \|\mathbb{E}_A(xv_ny) - \mathbb{E}_A(x)v_n\mathbb{E}_A(y)\|_2^2 \tag{B.5}$$

is finite, for all $x, y \in G$, where we have written $v_n = w_n / \|w_n\|_2$.

Proof. Let x and y be elements in G of lengths l_1 and l_2 respectively. We may assume that $l_1, l_2 \ge 1$ otherwise (B.5) is zero. Let z be any element of G of length l_1 then, using the notation of Lemma B.4 and the expression (B.2) for the conditional expectation, we have

$$\mathbb{E}_A(xw_ny) - \mathbb{E}_A(zw_ny) = \sum_{r=0}^{2l_1}\sum_{s=0}^{2l_2}(\mu(r,s,n;x,y) - \mu(r,s,n;z,y))\frac{w_{n+l_1+l_2-r-s}}{\|w_{n+l_1+l_2-r-s}\|_2^2}. \tag{B.6}$$

[2] When $t = l_1$, this constraint involving u becomes $u \in S \setminus S_{\alpha(r)}$.

Now, by combining Lemmas B.3 and B.4, we see that

$$|\mu(r,s,n;x,y)-\mu(r,s,n;z,y)| \leq \frac{2\left\|w_{n-\lfloor r/2\rfloor-\lfloor s/2\rfloor}\right\|_2^2}{(m-1)^{n-2}},$$

for $0 \leq r \leq 2l$, $0 \leq s \leq 2m$ and $n \geq l+m+2$. Plugging this estimate into (B.6) gives

$$\begin{aligned}
&\|\mathbb{E}_A(xw_ny)-\mathbb{E}_A(zw_ny)\|_2 \\
&\leq \sum_{r=0}^{2l_1}\sum_{s=0}^{2l_2} \frac{2\left\|w_{n-\lfloor r/2\rfloor-\lfloor s/2\rfloor}\right\|_2^2}{\|w_{n+l_1+l_2-r-s}\|_2 (m-1)^{n-2}} \\
&= \sum_{r=0}^{2l_2}\sum_{s=0}^{2l_2} \left(\frac{2(m-1)^2 m(k-1)\left((m-1)(k-1)\right)^{-1-\lfloor r/2\rfloor-\lfloor s/2\rfloor}}{\sqrt{m(k-1)}\left((m-1)(k-1)\right)^{(l_1+l_2-r-s-1)/2}}\right) \frac{\left((m-1)(k-1)\right)^{n/2}}{(m-1)^n}.
\end{aligned}$$

Write $A_{l_1,l_2,r,s}$ for the first bracket in the sum above and B_{l_1,l_2} for $\sum_{r=0}^{2l_1}\sum_{s=0}^{2l_2} A_{l_1,l_2,r,s}$ so

$$\|\mathbb{E}_A(xw_ny)-\mathbb{E}_A(zw_ny)\|_2 \leq B_{l_1,l_2}\left(\frac{k-1}{m-1}\right)^{n/2}.$$

Now sum over all $\|w_{l_1}\|_2^2$ elements $z \in G$ of length l_1 to obtain

$$\left\|\|w_{l_1}\|_2^2\,\mathbb{E}_A(xw_ny)-\mathbb{E}_A(w_{l_1}w_ny)\right\|_2 \leq \|w_{l_1}\|_2^2\, B_{l,m}\left(\frac{k-1}{m-1}\right)^{n/2}.$$

As $\mathbb{E}_A(x) = w_{l_1}/\|w_{l_1}\|_2^2$ this is

$$\|\mathbb{E}_A(xw_ny)-\mathbb{E}_A(x)\,w_n\mathbb{E}_A(y)\|_2 \leq B_{l,m}\left(\frac{k-1}{m-1}\right)^{n/2},$$

which, upon substitution of $v_n = w_n/\|w_n\|_2$, gives

$$\|\mathbb{E}_A(xv_ny)-\mathbb{E}_A(x)\,v_n\mathbb{E}_A(y)\|_2^2 \leq \frac{B_{l,m}^2}{m(m-1)^{2n-1}},$$

for $n \geq l+m+2$. As $m \geq 3$, the sum (B.5) converges as required. □

This concludes the verification of the second hypothesis of Theorem B.2 and so completes the proof of Theorem B.1. Since the validity of the second hypothesis of Theorem B.2 depends only on whether $m \geq 3$, it is tempting to use the fact that A is a masa if and only if $m \geq k$ to deduce that, when $m < k$ there can be no *-isomorphism from A onto $L^\infty[0,1]$ inducing an isometry from $L^2(A,\mathrm{tr})$ onto $L^2[0,1]$. However, this result is already known to us; an examination of Trenholme's proof that A is a masa if and only if $m \geq k$ gives an isolated point in the spectrum when $m < k$. In this case certainly no such *-isomorphism can exist.

As with Sinclair and Smith's work on the Radial masa in $\mathcal{L}(\mathbb{F}_k)$, we can examine the situation of the free product of $m \geq 3$ countable discrete groups G_i each of which contains a finite subgroup H_i of order k. Write $G = G_1 * \ldots G_m$ and $H = H_1 * \ldots H_m$. Let B denote the abelian subalgebra of M generated by

$$w_1 = \sum_{i=1}^{m} \sum_{h \in H_i \setminus \{1_{H_i}\}} h.$$

It is not obvious that this is a masa in $\mathcal{L}(G)$, but when $m \geq k$, we can extend the weak asymptotic homomorphism property from $\mathcal{L}(H)$, where it most certainly is, to all of $\mathcal{L}(G)$.

Theorem B.6. *With the notation above, if $m \geq k$, then B has the asymptotic homomorphism property in $\mathcal{L}(G)$ and so is a strongly singular masa in $\mathcal{L}(G)$.*

Proof. Define $w_0 = e_H$ and, for $n \geq 1$, w_n to be the sum of all reduced words of length n in H. Then, as before, B is generated by w_1 and upon setting $v_n = w_n / \|w_n\|_2$ we obtain an orthonormal basis for $L^2(B)$. Since B is a masa in $\mathcal{L}(H)$ (as $m \geq k$), the first condition of Theorem B.2 holds and it then again enough to show that the sum (B.1) is finite for all $x, y \in G$. When $x, y \in H$ this is Lemma B.5.

Let $\mathbb{E}_{\mathcal{L}(H)}$ and $\mathbb{E}_B$ denote the trace preserving conditional expectations from $\mathcal{L}(G)$ onto $\mathcal{L}(H)$ and B respectively. As $B \subset \mathcal{L}(H)$, we have $\mathbb{E}_B = \mathbb{E}_B \mathbb{E}_{\mathcal{L}(H)}$. If $x \in H$ and $y \in G \setminus H$ then $\mathbb{E}_B(y) = 0$ and

$$\mathbb{E}_B(x v_n y) = \mathbb{E}_B \mathbb{E}_{\mathcal{L}(H)}(x v_n y) = \mathbb{E}_B\left(x v_n \mathbb{E}_{\mathcal{L}(H)}(y)\right) = 0,$$

when all terms in (B.1) vanish. Similarly when $y \in H$ and $x \in G \setminus H$ all terms of (B.1) again vanish.

Finally we consider $x, y \in G \setminus H$ so $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, and we must show that

$$\sum_{n=0}^{\infty} \|\mathbb{E}_B(x v_n y)\|_2^2 < \infty. \tag{B.7}$$

Let x, y have lengths l_1, l_2 as words in the G_i's. For $u \in H$ the product xuy can only lie in H if some cancelation occurs between some syllables of x and y (either as $x_i y_j$ or $x_i u_k y_j$ where all these syllables lie in the same G_r). Therefore, if u has length $n \geq l_1 + l_2 + 2$ in the $H_i' s$ then $xuy \in G \setminus H$ and hence the terms in (B.7) with $n \geq l_1 + l_2 + 2$ vanish. □

Appendix C

A note on automorphisms of group von Neumann algebras

The construction of group von Neumann algebras from groups is functorial, so that given an automorphism θ of a discrete group G we obtain an automorphism $\mathcal{L}(\theta)$ of the group von Neumann algebra $\mathcal{L}(G)$. This automorphism is of course obtained by extending θ first to the group ring $\mathbb{C}G$ then to $\mathcal{L}(G)$ by linearity and continuity. We immediately drop the notation $\mathcal{L}(\theta)$ in favour of using θ for both the automorphism of G and its extension to $\mathcal{L}(G)$.

The following observation dates back to Kallman ([32]), but does not seem to be as well known as it should be. We give a proof for completeness.

Theorem C.1. *Let G be a countable discrete I.C.C. group. If θ is an automorphism of G such that θ is an inner automorphism of $\mathcal{L}(G)$, then θ is an inner automorphism of G.*

Corollary C.2. *Let G be a countable discrete I.C.C. group. There is a natural injective map from $Out(G)$ into $Out(\mathcal{L}(G))$.*

Proof of Theorem C.1. Let θ be an automorphism of G and u a unitary in $\mathcal{L}(G)$ such that the extension of θ to $\mathcal{L}(G)$ has $\theta(x) = uxu^*$ for all $x \in \mathcal{L}(G)$. We must show that u can be taken to lie in $G \subset \mathcal{L}(G)$. Write

$$u = \sum_{h \in G} \alpha_h h,$$

for some constants $\alpha_h \in \mathbb{C}$. We have $\alpha_{h_0} \neq 0$ for some $h_0 \in G$, and so we may assume that $\alpha_e \neq 0$ by replacing θ with Ad $h_0^{-1} \circ \theta$ if necessary.

Now, for $g \in G$, we can write

$$\sum_{h,k \in G} \alpha_h \overline{\alpha_k} h g k^{-1} = \theta(g) \in G,$$

with convergence in 2-norm. Substitution of $k = \theta(g)^{-1}hg$ in the above gives

$$\sum_{h\in G} \alpha_h \overline{\alpha_{\theta(g)^{-1}hg}} = 1, \tag{C.1}$$

for all $g \in G$. Apply the Cauchy-Schwartz inequality to (C.1) to obtain

$$1 = \sum_{h\in G} \alpha_h \overline{\alpha_{\theta(g)^{-1}hg}} \leq (\sum_{h\in G} |\alpha_h|^2)^{1/2} (\sum_{h\in G} \left| \alpha_{\theta(g)^{-1}hg} \right|^2)^{1/2} = 1,$$

and so we have equality in this application of Cauchy-Schwartz. Therefore, for each $g \in G$ there is a constant C_g such that

$$\alpha_h = C_g \alpha_{\theta(g)^{-1}hg},$$

holds for all $h \in G$. In particular, since $(\alpha_h)_{h\in G}$ and $(\alpha_{\theta(g)^{-1}hg})_{h\in G}$ both have 2-norm equal to 1, we see that

$$|\alpha_h| = \left| \alpha_{\theta(g)^{-1}hg} \right|, \tag{C.2}$$

for all $g, h \in G$.

By taking $h = e$ in (C.2) we see that the set

$$X = \{\theta(g)^{-1}g | g \in G\},$$

is finite, as $\alpha_e \neq 0$. For each $g \in G$, write $x(g)$ for the element of X with $\theta(g) = x(g)g$. Now, for $g, k \in G$, we have

$$\theta(gk) = \theta(g)\theta(k) = x(g)gx(k)g^{-1}gk,$$

and hence

$$x(g)^{-1}x(gk) = gx(k)g^{-1}. \tag{C.3}$$

Now the left hand side of (C.3) lies in $X^{-1}X$ a finite set, whereas $\{gx(k)g^{-1} | g \in G\}$ is an infinite set unless $x(k) = e$ because G is an I.C.C. group. Hence $x(k) = e$, for all $k \in G$ and so θ is the identity automorphism, an inner automorphism of G □

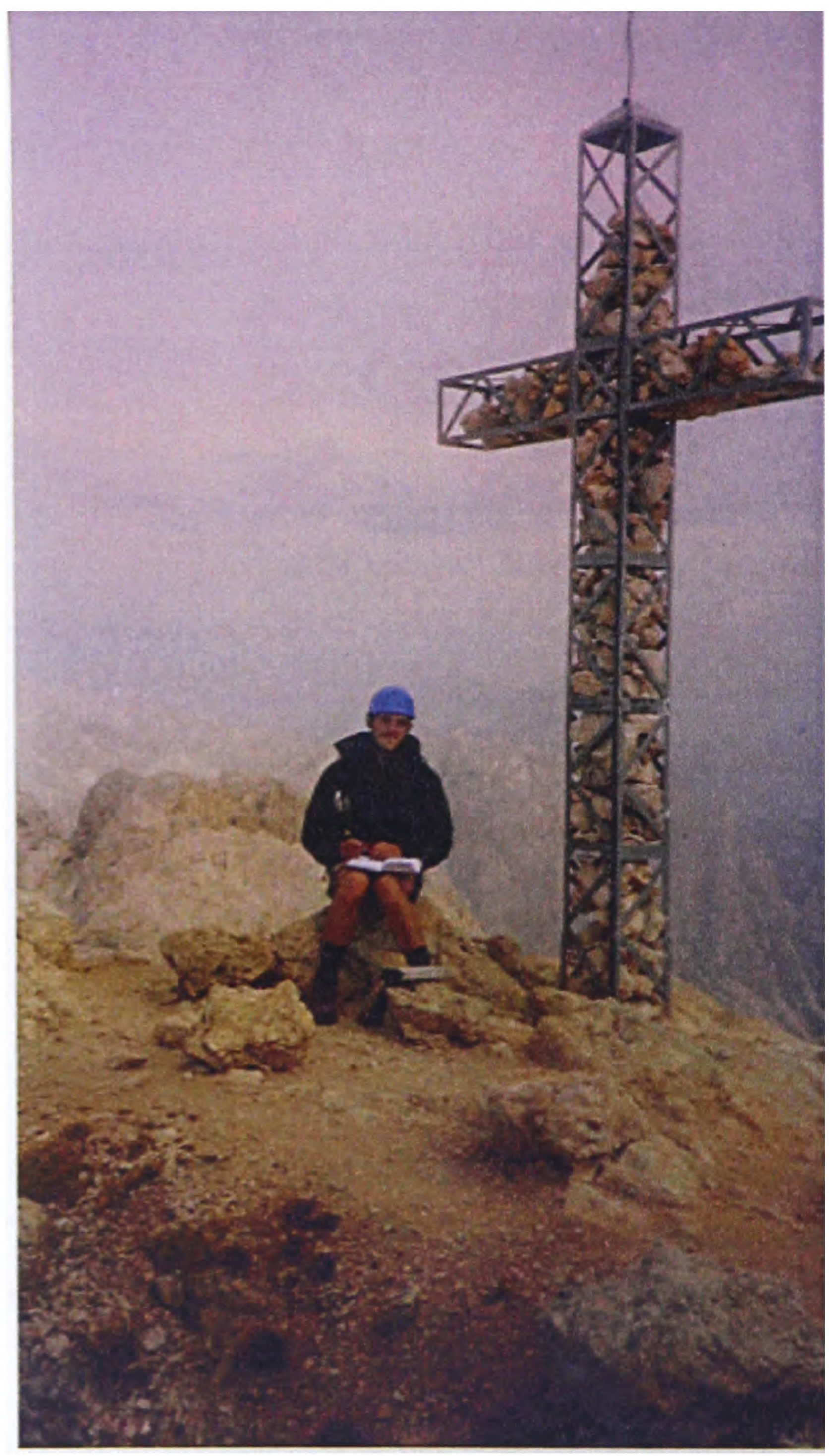
Very recently, Ioana, Peterson and Popa, [23], have solved an old problem of Connes and produced II_1 factors with trivial outer automorphism group. The examples in [23] do not arise as group von Neumann algebras.[1] Does there exist an I.C.C. group G such that $\mathrm{Out}(\mathcal{L}(G)) = \{1\}$? If such a G existed then, by Corollary C.2, it would have $\mathrm{Out}(G) = \{1\}$. I.C.C. groups with this last property

[1] At least the examples are not obviously group von Neumann algebras. We are certainly not claiming here that there is no I.C.C. group G which gives rise to these factors.

are known. In 1975 Dyer and Formanek, [14], showed that for the free group, $\mathbb{F}_k$, on $2 \leq k < \infty$ generators, Aut $(\mathbb{F}_k)$ is *complete*, terminology which means that all automorphisms of Aut $(\mathbb{F}_k)$ are inner. It is not hard to check that Aut $(\mathbb{F}_k)$ is a countable discrete I.C.C. group. This might then be a good place to start looking for a group II_1 factor with no outer automorphisms.

Question C.3. Are all the automorphisms of $\mathcal{L}\,(\mathrm{Aut}\,(\mathbb{F}_k))$ inner?

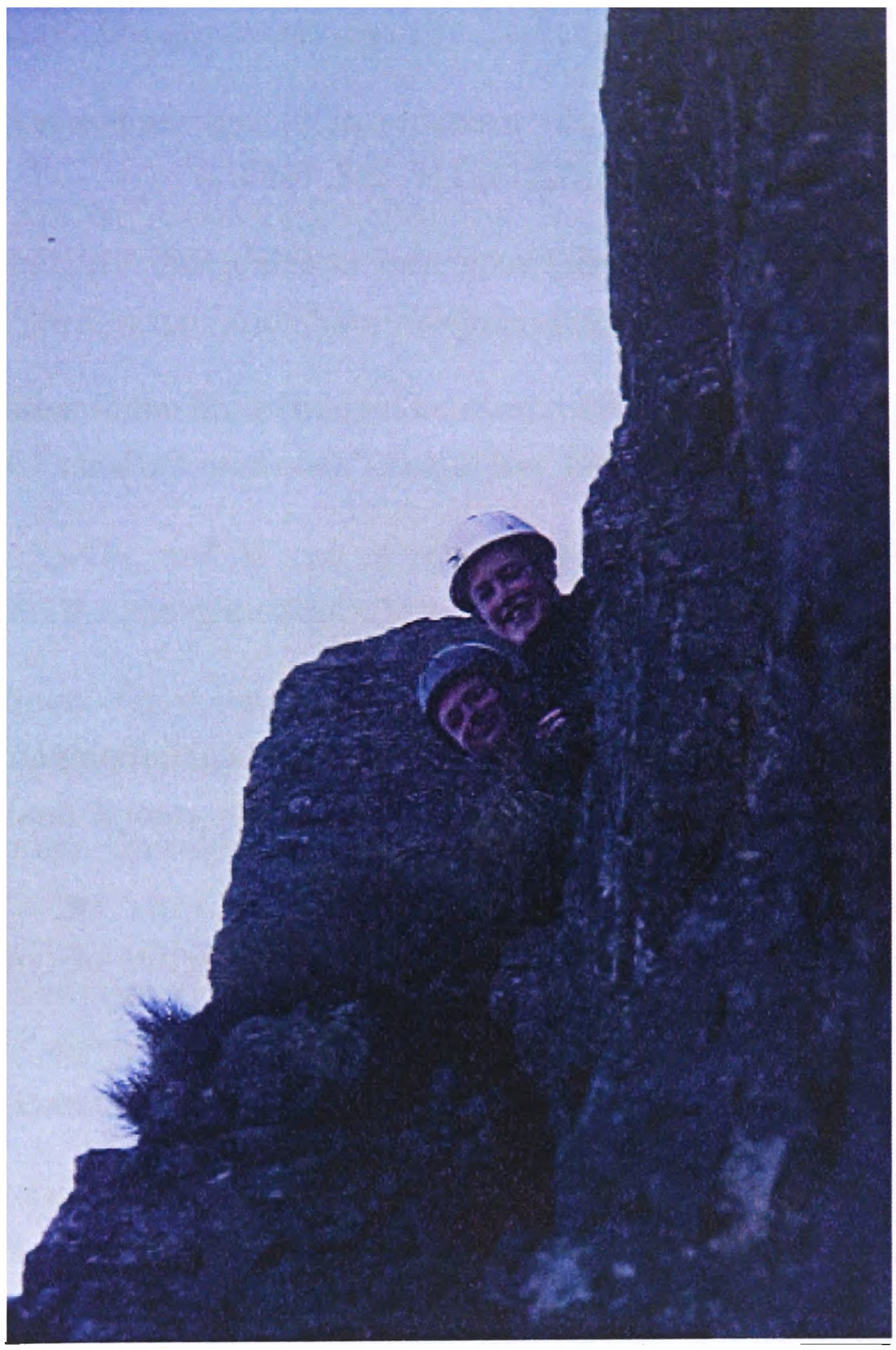
Religious studies



Pete Dodd, on a typical Dolomitic summit, 2003

Photo: Saw

Home to roost



Tom and Andy, perched on Agag's grove (VD), Buchaille Etive Mor, 2003.

Photo: Jon Powell

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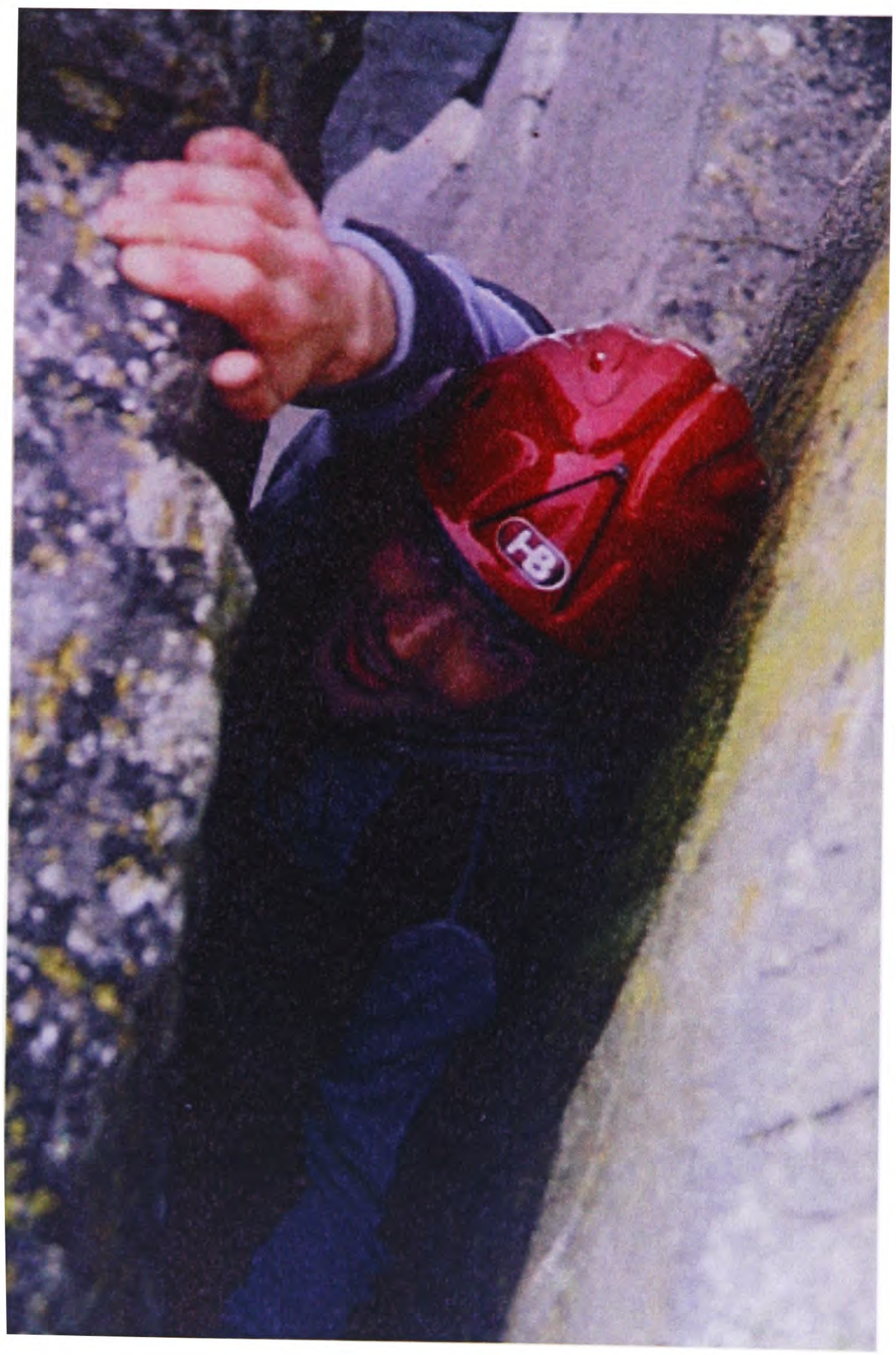
Getting there in the end



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