Moment Polyptychs and the Equivariant Quantisation of Hypertoric Varieties

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Doctor of Philosophy

University of Edinburgh

2022

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Für Lise

Acknowledgements

It is with no doubt that this thesis would not have been possible without the support and guidance of my supervisor, Johan Martens, whose patience when listening my unintelligible ideas would leave the saints in envy. Whether it be a question that I had, or a concept that required clarification, he would ensure that they were addressed clearly and unpretentiously.

I would also like to thank both of my examiners, Andrew Dancer and Ana Rita Pires, for their encouraging remarks and supportive feedback on this thesis. It was also thanks to Andrew that I could visit the Mathematics Institute at The University of Oxford for the summer of 2022, which was a privilege that I greatly appreciated.

I am indebted to my parents, Rosemary and Geoffrey, as well as my sister, Lucinda, for their faith in me at each stage of my academic journey. To my friends: James, Ege, Jordan, Rory, Isak, Matt, Rob, and Mercedes, thank you for making the past several years greatly enjoyable, and I am sure for making the the future ones even more so.

And to Dr. Lise Cazzoli, thank you for your love and understanding – you are the push-forward to my pull-back.

Abstract

In this thesis, we develop a method to investigate the geometric quantisation of a hypertoric variety from an equivariant viewpoint, in analogy with the equivariant Verlinde formula for Higgs bundles. We do this by first using the residual circle action on a hypertoric variety to construct its symplectic cut that results in a compact cut space, which is needed for the localisation formulae to be well-defined and for the quantisation to be finite-dimensional. The hyperplane arrangement corresponding to the hypertoric variety is also affected by the symplectic cut, and to describe its effect we introduce the notion of a moment polyptych that is associated to the cut space. Also, we see that the prerequisite isotropy data that is needed for the localisation formulae can be read off from the combinatorial features of the moment polyptych. The equivariant Kawasaki-Riemann-Roch formula is then applied to the pre-quantum line bundle over each cut space, producing a formula for the equivariant character for the torus action on the quantisation of the cut space. Finally, using the quantisation of each cut space, we derive a formula expressing the dimension of each circle weight subspace of the quantisation of the hypertoric variety.

Contents

Chapter 1

Introduction

1.1 The Lay Summary

Given a symplectic manifold M, provided that it satisfies some constraints, can be associated with a quantisation $\mathcal{Q}(M)$ whose elements are reminiscent of "wave functions". Our work here involves a special type of symplectic manifold called a "hypertoric manifold" M, and this work focusses on studying its quantisation $\mathcal{Q}(M)$ and its properties. The space $\mathcal{Q}(M)$ will be infinite-dimensional since M is non-compact which, in an informal way, means that it is indefinitely expansive as an object. Instead of studying $\mathcal{Q}(M)$ however, what we consider instead in this thesis will be subspaces of $\mathcal{Q}(M)$ that are actually finite-dimensional.

To do this, we can break up $\mathcal{Q}(M)$ into these finite-dimensional subspaces, called weight spaces, by exploiting a symmetry inherent to M which, with regards to our own intents and purposes, are much more tractable to study given that they are finite dimensional. To be precise, this symmetry is that of a circle U_1 acting on M, which rotates half of the coordinates used to define a point of M. Moreover, this U_1-symmetry lets us perform a procedure called "symplectic cutting", which effectively corresponds to trimming off most of M but keeping just some finite part of it. How much we trim off depends on a parameter $\delta \in \mathbb{R}_{\geq 0}$, and the new space which we get from trimming down M is what we call the "cut space", $M^{\leq \delta}$.

A consequence of $M^{\leq \delta}$ being compact is that the quantisation $\mathcal{Q}(M^{\leq \delta})$ associated to it is a finite-dimensional vector space. Whilst we are interested in subspaces of $\mathcal{Q}(M)$, what we do is we use $\mathcal{Q}(M^{\leq \delta})$ as an auxiliary, allowing us to calculate the dimension of the weight spaces of $\mathcal{Q}(M)$; this is possible since the weight spaces of $\mathcal{Q}(M)$ and the cut spaces $M^{\leq \delta}$ are defined via the same U_1-symmetry, and a connection between their corresponding quantisations can be shown. With this in hand, then we can calculate the dimension of $\mathcal{Q}(M^{\leq \delta})$ via localisation formulae which cannot be applied to $\mathcal{Q}(M)$ itself; this again is due to the non-compactness of the M.

An interesting property about a hypertoric manifold M is that we can associate to it an arrangement of hyperplanes, $\mathcal{A}$. There are correspondences between the combinatorics of the hyperplane arrangement $\mathcal{A}$ and the geometry of the hypertoric manifold M. We find that the cut space $M^{\leq\delta}$ also shares a combinatorial configuration that stems from $\mathcal{A}$, that comes from truncating it. The result is an arrangement of convex and bounded polytopes, separated by where a hyperplane of $\mathcal{A}$ used to be; we call this arrangement a "moment polyptych", since it depends on both the arrangement $\mathcal{A}$ in addition to a consistent choice of sign attached to each hyperplane of $\mathcal{A}$. We hope that there is a combinatorial formula to be found, which expresses the dimension of each subspace of $\mathcal{Q}(M)$ in terms of the data associated to the polyptych of $M^{\leq\delta}$, without the need to perform any analytical computations. Such a correspondence exists already in the framework of toric varieties and convex bounded polytopes; if X is a toric variety then there exists a polytope that corresponds to it. It then follows that the dimension of the quantisation $\mathcal{Q}(X)$ coincides with the number of integral lattice points within its corresponding polytope.

1.2 The Story so far

The modern-day understanding of geometric quantisation was first developed by Kostant [Kos70] and Souriau [Sou66] in the 1960s. Given a symplectic manifold (X, ω), it is a framework in which one attempts to associate to X a Hilbert space $\mathcal{Q}(X)$, along with a correspondence between the real-valued functions on X with quantum mechanical operators of $\mathcal{Q}(X)$. The existence of such a quantisation $\mathcal{Q}(X)$ depends on two types of prerequisite data: the first is that of a Hermitian line bundle $\mathcal{L} \to X$ with a Hermitian connection ∇ whose curvature coincides with the symplectic two-form ω on X as $\mathcal{R}(\mathcal{L}) = (\sqrt{-1}/2\pi)\omega$. Such a line bundle $\mathcal{L}$ is called a *pre-quantum line bundle* over X which exists if, and only if, the class of ω is integral, i.e., that $[\omega] \in H^2(X; \mathbb{Z})$. Furthermore, $\mathcal{L}$ is unique up to gauge equivalence provided that X is a simply-connected manifold.

The second datum is that of a choice of *polarisation* on X, which prescribes how the coordinates of X should be effectively sorted into canonical position and momenta. The fact that a choice of polarisation must be made means that there is no canonical way of geometrically quantising each symplectic manifold X via a pre-quantum line bundle $\mathcal{L}$ over it, and, in general, a different choice of polarisation for X gives rise to distinct quantisations $\mathcal{Q}(X)$. A further condition on the quantisation $\mathcal{Q}(X)$ is that, should a compact Lie group G act on X in a Hamiltonian way, then this action should carry over to $\mathcal{Q}(X)$ as a unitary representation of G.

Fortunately, the quantisation procedure becomes much simpler when (X, ω, I) is additionally Kähler with complex structure I, and when $\mathcal{L}$ is furthermore a holomorphic pre-quantum line bundle over X. When these conditions hold, a natural choice of polarisation on X is obtained by considering the *holomorphic sections* of $\mathcal{L}$; such a choice of polarisation is called the *complex*, or *Kähler*, *polarisation*. We may then take the quantisation of X to be the $\mathbb{C}$-vector space $\mathcal{Q}(X) = H^0(X; \mathcal{L})$ of *holomorphic sections* of the line bundle $\mathcal{L}$ over X.

When X is a symplectic toric variety, that is to say when $\dim_{\mathbb{R}} X = 2n$ and X is equipped with an effective Hamiltonian action of a torus T^n, along with a moment map $\mu : X \to (\mathfrak{t}^n)^*$, then a classical result – usually attributed to Danilov [Dan78] – states that the dimension of $H^0(X; \mathcal{L}^{\otimes k})$ coincides with the number of integral lattice points inscribed within the moment polytope $\Delta = \mu(X)$ of X. I.e., that $\dim_{\mathbb{C}} H^0(X; \mathcal{L}^{\otimes k}) = \#\{k \cdot \Delta \cap (\mathfrak{t}^n_{\mathbb{Z}})^*\}$. More generally, if (X, ω, I) is a compact complex-analytic manifold and if $\mathcal{L} \to X$ is a holomorphic pre-quantum line bundle over it, then the eponymously-named index theorem of Atiyah and Singer [AS63] can be used, and which itself generalises the Hirzebruch-Riemann-Roch formula, [Hir66]. These theorems state that the quantisation $\mathcal{Q}(X)$ is isomorphic to the index of the Spin-$\mathbb{C}$ Dirac operator $\not{\partial}_{\mathbb{C}}$ on X:

$$\mathrm{Ind}_{\not{\partial}_{\mathbb{C}}}(X; \mathcal{L}) \cong H^0(X; \mathcal{L}),$$

and that their dimensions can be expressed as an integral over X of two specific characteristic classes, namely the *Todd class* $\mathrm{Td}(TX)$ of the tangent bundle TX, and the *Chern character* $\mathrm{Ch}(\mathcal{L})$ of $\mathcal{L}$:

$$\dim_{\mathbb{C}} \mathrm{Ind}_{\not{\partial}_{\mathbb{C}}}(X; \mathcal{L}) = \dim_{\mathbb{C}} H^0(X; \mathcal{L}) = \int_X \mathrm{Td}(TX) \wedge \mathrm{Ch}(\mathcal{L}).$$

So far, we have only considered (X, I) to be at least a *compact* complex-analytic manifold which, by a theorem of Serre and Cartan [CS53], guarantees that the sheaf cohomology $H^q(X; \mathcal{F})$ is a finite-dimensional $\mathbb{C}$-vector space for any coherent analytic sheaf $\mathcal{F}$ over X, and where $q \in \mathbb{Z}_{\geq 0}$. If, instead, X is *non-compact*, then the quantisation $\mathcal{Q}(X)$ ends up being infinite-dimensional and so, in this case, a suitably adjusted question has to be asked instead if we are to work with something concrete. Further complications arise when X is no longer a smooth manifold. If X is still "reasonably smooth" however, in the sense that it has at worst *orbifold singularities*, then the Kawasaki-Riemann-Roch theorem must be used in the place of the Hirzebruch-Riemann-Roch theorem, [Kaw79].

1.3 On this Research

This research presented in this thesis concerns itself with developing a suitable framework of quantising hypertoric varies which are non-compact, and therefore the traditional geometric quantisation must be adapted to take this into account. Hypertoric varieties can be thought of as the quaternionic cousins to symplectic toric varieties. They were first introduced by Bielawski and Dancer in [BD00], as the hyperkähler quotients of flat quaternionic vector spaces of an effective torus action, analogously to Delzant's construction of symplectic toric manifolds [Del88].

Just as the geometry of a symplectic toric variety is intimately related to the combinatorics of a convex polytope, an interplay exists between the geometry of a hypertoric variety and the combinatorics of a real hyperplane arrangement. In particular, the hyperplane arrangement involves the entirety of its ambient space, reflecting the fact that a hypertoric variety is non-compact. So, if one wished to study the quantisation $\mathcal{Q}(M)$ of a hypertoric variety M, they would soon find that it is infinite dimensional. This is not necessarily an issue per se, since infinite-dimensional quantum systems appear often in the mathematical physics literature. However, in the case of a hypertoric variety M, we want to say more about the quantisation. We should remark here that, to consider geometric quantisation in its fullest of forms, one should further consider how to endow $\mathcal{Q}(M)$ with a Hilbert space structure. This would involve the introduction of an inner-product on $\mathcal{Q}(M)$, say by the metaplectic correction. We do not concern ourselves with this question, and instead just consider $\mathcal{Q}(M)$ as a $\mathbb{C}$-vector space.

Unlike toric varieties, a hypertoric variety M is endowed with a Hamiltonian action of the circle U_1, which acts on it by rotating its cotangent coordinates via scalar multiplication. This U_1-action has a moment map $\Phi : M \to \mathbb{R}_{\geq 0}$, which is proper. Hence we are in the situation where we can use Lerman's symplectic cutting procedure to form the *symplectic cut* of M, [Ler95]. This produces a compact *cut space* $M_\nu^{\leq\delta} = (M \times \mathbb{C}) /\!\!/_\delta U_1$, for some value $\delta \in \mathbb{R}_{\geq 0}$. Since $M_\nu^{\leq\delta}$ is compact, the conventional geometric quantisation procedures can now be applied to the cut space $M_\nu^{\leq\delta}$ instead of its hypertoric variety M_ν.

Some comments are in order here: first, the cut space $M_\nu^{\leq\delta}$ is no longer hyperkähler but just Kähler, since the U_1-action preserves only one of the three hyperkähler two-forms. Hence the complex polarisation that we consider on $M_\nu^{\leq\delta}$ is determined by the complex structure associated to the Kähler two-form on M_ν that survives under the U_1-action. The second comment is that the U_1-action on M_ν can be described combinatorially, by restricting our attention to specific half-dimensional subvarieties of M_ν that make up its so-called *extended core* $\mathcal{E}$. The components $\mathcal{E}_A$ of the extended core $\mathcal{E}$ are indexed by finite subsets $A \subseteq \{1, 2, \ldots\}$, with the U_1-action on each $\mathcal{E}_A$ depending on the subset A. This is reflected in the cutting procedure and results in a "truncation" of the hyperplane arrangement $\mathcal{A}$. We coined the term *moment polyptych* to refer the resulting polytopal arrangement, that we denote by $\Delta_\nu^{\leq\delta}$, emphasising its dependence on not just $\mathcal{A}$ itself but also on the *poset of regions* $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$. The partial order of $\mathcal{P}(\mathcal{A})$ is given by "how far away" each region is from a pre-determined distinguished base region of $\mathcal{A}$. In particular, for the same hyperplane arrangement, distinct base regions give rise to non-isomorphic posets of regions and thus non-equivalent moment polyptychs.

Our approach to forming this cut space shares analogies with Hausel's thesis [Hau98], in which he uses the U_1-action, or the "Hitchin action" after [Hit87], on the moduli spaces of Higgs bundles $\mathcal{M}$ over a Riemann surface. The Hitchin action acts by rotating the Higgs field associated to a holomorphic vector bundle, akin to how our U_1-action rotates the cotangent fibre coordinates over a point in the base space and since both actions are Hamiltonian. Another way in which research on the moduli spaces of Higgs bundles has inspired the work presented here, is by that of the *equivariant Verlinde formula*, which was introduced by Pei, Gukov, and Andersen in [GP17; AGP16], and also by Halpern-Leistner in [HL16]. Since the moduli spaces of Higgs bundles are non-compact, their idea to circumvent this issue was to decompose the infinite-dimensional space $H^0(\mathcal{M}; \mathcal{L}^{\otimes k})$ of holomorphic sections of a pre-quantum line bundle $\mathcal{L} \to \mathcal{M}$ over $\mathcal{M}$ into a $\mathbb{Z}$-graded direct sum of finite-dimensional weight subspaces $H^0(\mathcal{M}; \mathcal{L}^{\otimes k})_d$, where $d \in \mathbb{Z}$ denotes the weight of Hitchin's U_1-action on $H^0(\mathcal{M}; \mathcal{L}^{\otimes k})$.

That is to say:

$$H^0(\mathcal{M};\mathcal{L}^{\otimes k}) \cong \bigoplus_{d\geq 0} H^0(\mathcal{M};\mathcal{L}^{\otimes k})_d.$$

On the other hand – and backtracking slightly – the non-equivariant Verlinde formula prescribes a simple recipe to calculate the dimension of the vector space $H^0(\mathcal{X};\mathcal{L}^{\otimes k})$ of holomorphic sections of a pre-quantum line bundle $\mathcal{L} \to \mathcal{X}$ over $\mathcal{X}$, the moduli space of flat G-connections over a Riemann surface. This is sometimes phrased in the physics literature as being the number of conformal blocks in a two-dimensional conformal field theory on a Riemann surface, [Ver88]. When $G = SU_2$, it was shown: that $\mathcal{X}$ can quantised using a real polarisation by Weitsman, in [Wei92]; that the quantisation dimension for a real polarisation agrees with that for a complex polarisation and hence the Verlinde formula, in [JW92]; and that a lattice-point count exists for the Verlinde point by the means of a moment polytope, in [JW94]. This thesis is motivated by the question on whether a similar phenomenon exists between hypertoric varieties and the equivariant Verlinde formula. In Tables 1.1 and 1.2, further analogies between symplectic toric varieties and the moduli space of flat connections are presented, as well as those between hypertoric varieties and the moduli spaces of Higgs bundles.

Symplectic Toric Variety X	Moduli Space of Flat G-Connections $\mathcal{X}$
Symplectic quotient: $X_\nu = \mathbb{C}^N /\!\!/_\nu K$	Symplectic quotient: $\mathcal{X} = \mathcal{A} /\!\!/_0 \mathcal{G}$
Compact Kähler	Compact Kähler
$\dim_{\mathbb{C}} H^0(X_\nu;\mathcal{L}^{\otimes k}) = \#(k\cdot\mu(X)\cap\mathbb{Z}^n)$, lattice points in moment polytope	$\dim_{\mathbb{C}} H^0(\mathcal{X};\mathcal{L}^{\otimes k}) =$ Verlinde formula at level-k, [Ver88]

Table 1.1: Table detailing the non-equivariant analogies.

Hypertoric Variety M	Moduli Space of G-Higgs Bundles $\mathcal{M}$
Hyperkähler quotient: $M_\nu = T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_{(\nu,0)} K$	Hyperkähler quotient: $\mathcal{M} = \mathcal{A}^H /\!\!/\!\!/\!\!/_{(0,0)} \mathcal{G}$
Non-compact hyperkähler	Non-compact hyperkähler
Residual Hamiltonian U_1-action with proper moment map $\Phi : M_\nu \to \mathbb{R}_{\geq 0}$	Hitchin's Hamiltonian U_1-action with proper moment map $\Phi : \mathcal{M} \to \mathbb{R}$
$T^*X_\nu \subseteq M_\nu$ with $\omega_{\mathbb{R}}\vert_{X_\nu} = \omega_{X_\nu}$ and $\Phi^{-1}(0) = X_\nu$	$T^*\mathcal{X} \cong \mathcal{M}$ with $\omega_{J_1}\vert_{\mathcal{X}} = \omega_{\mathcal{X}}$ and $\Phi^{-1}(0) = \mathcal{X}$
Cut space: $M_\nu^{\leq\delta} = (M_\nu \times \mathbb{C}) /\!\!/_\delta U_1$	Hausel's compactification of $\mathcal{M}$, [Hau98]
U_1-weight space decomposition: $H^0(M_\nu;\mathcal{L}) \cong \oplus_{d\geq 0} H^0(M_\nu;\mathcal{L}_\nu)_d$	U_1-weight space decomposition: $H^0(\mathcal{M};\mathcal{L}) \cong \oplus_{d\geq 0} H^0(\mathcal{M};\mathcal{L})_d$
$\dim_{\mathbb{C}} H^0(M_\nu;\mathcal{L}_\nu)_d =$ equation (7.15) from Corollary 7.9	$\dim_{\mathbb{C}} H^0(\mathcal{M};\mathcal{L})_d =$ equivariant Verlinde formula of [GP17; AGP16; HL16]

Table 1.2: Table detailing the equivariant analogies.

1.4 Thesis Outline

The outline of this thesis is as follows:

- In the first half of this thesis we focus mostly on introducing hypertoric varieties and their symplectic cuts. We begin first of all with Chapter 1, in which we introduce the geometric approach to defining hypertoric varieties, denoted throughout by M_ν as hyperkähler quotients of the flat N-dimensional quaternionic vector space $\mathbb{H}^N$ by subtori K of the N-dimensional real torus T^N, where $\nu \in \mathfrak{k}^*$ is an element in the dual space the Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$. We do this by discussing hyperkähler manifolds in general in Section 1.1, before moving onto hyperkähler quotients in Section 1.2, which is a fruitious source of *"more-interesting"* hyperkähler orbifolds that are presented as quotients of $\mathbb{H}^N$ by a Lie group G. Of course, a family of these *"more-interesting"* hyperkähler orbifolds include hypertoric varieties which are first defined in Section 1.3.

  For the latter half of Chapter 1, we shift to the algebraic way of defining hypertoric varieties which is via Geometric Invariant Theory (GIT), which we discuss generally in Section 1.4 at first, before honing in to hypertoric varieties in Section 1.5. Each point of view provides its own benefits when studying a hypertoric variety. On the one hand for example, the geometric lens reveals the combinatorial links between hypertoric varieties and hyperplane arrangements, whereas the GIT lens provides us with a firmer grasp over the space of global sections of some line bundle $\mathcal{L}_\nu \to M_\nu$ over the hypertoric variety M_ν;

- Next, in Chapter 2, we discuss how one can associate a hypertoric variety M_ν with a hyperplane arrangement $\mathcal{A}$. The hyperkähler moment map $\mu_{\mathrm{HK}} : M_\nu \to (\mathfrak{t}^n)^* \otimes_{\mathbb{R}} \mathrm{Im}(\mathbb{H})$ can be split into its real and complex components, as $\mu_{\mathrm{HK}} = \mu_{\mathbb{R}} + i\mu_{\mathbb{C}}$, where $\mu_{\mathbb{R}} : M_\nu \to (\mathfrak{t}^n)^*$ is said to be the real moment map and $\mu_{\mathbb{C}} : M_\nu \to (\mathfrak{t}^n)^* \otimes_{\mathbb{R}} \mathbb{C}$ the complex moment map. The hyperplane arrangement $\mathcal{A}$ is then determined by the image of the hypertoric variety M_ν under the real moment map $\mu_{\mathbb{R}}$ in $(\mathfrak{t}^n)^*)$, and this correspondence is studied in Section 2.1. One such property, that we subsequently look into in Section 2.2, is that the regions of $\mathcal{A}$ are given by the Kähler subvarieties of M_ν under $\mu_{\mathbb{R}}$. We denote the union of these subvarieties by $\mathcal{E}$ and call it the *extended core*, whereas we denote the union of the *compact* subvarieties by $\mathcal{C}$, which we call the *core*.

  Then in Section 2.4, we introduce what are known as *flats* of the hyperplane arrangement, which are expressed as non-empty intersections of a selection of hyperplanes. For each flat, we then obtain a decomposition of the torus T^n into two components – one "tangential" to the flat and the other one "normal" to it. The significance of this is made clear in Section 2.5, which is novel, as each flat of $\mathcal{A}$ determines a hypertoric subvariety of M_ν, whose torus is the one that acts tangentially in the T^n-decomposition. Whilst the concept of a hypertoric subvariety may appear to be a *non sequitur* at first, it becomes instrumental when proving the more complicated propositions and theorems that appear later on, since then they can be reduced down to one-or two-dimensional problems via an inductive argument that are much easier to prove, for example as in the proofs of Theorem 3.17, Lemma 6.1, Theorem 6.2, and Proposition 6.3.

- We begin Chapter 3 by recalling in general Lerman's symplectic cut [Ler95] in Section 3.1, which requires the Hamiltonian action of the circle U_1, and for which we need to specify a value $\delta \in \mathbb{R}_{\geq 0}$ to cut at. For a hypertoric variety M_ν, we introduce such a U_1-action in Section 3.2 before then, in Section 3.3, investigating its combinatorially behaviour when restricted to different extended core.

  We now move onto work that is original in Section 3.4, when we form the symplectic cut of M_ν, referring aptly to it as being the *cut space* $M_\nu^{\leq\delta}$ of M_ν. By construction, $M_\nu^{\leq\delta}$ is a compact variety, and the torus T^n that acted originally on M_ν descends to $M_\nu^{\leq\delta}$, as does the real moment map $\mu_\mathbb{R}$. Doing this, we see in Section 3.5 that we obtain a polytopal arrangement from the image of the cut space $M_\nu^{\leq\delta}$ under $\mu_\mathbb{R}$, that is essentially obtained by bounding the unbounded regions of the hyperplane arrangement $\mathcal{A}$, which we call the *moment polyptych* and denote it by $\Delta_\nu^{\leq\delta} = \mu_\mathbb{R}(M_\nu^{\leq\delta})$. The coinage of the term *polyptych* is to emphasise its dependence on the way that the hyperplanes are cooriented. Finally, we wrap up the chapter by providing some examples in Section 3.6, and establishing some properties possessed by the cut spaces in Section 3.7. In particular, we find that a generic choice of hypertoric variety will result in its cut space being an orbifold;

- For the latter half of this thesis, we shift our attention onto the *"quantisation"* of hypertoric varieties. Chapter 4 is a review chapter, and we start by first introducing the notion of a holomorphic *pre-quantum* line bundle $\mathcal{L} \to M$ over a general Kähler manifold M, in addition to its Dolbeault cohomology group $H^0(M;\mathcal{L})$, in Section 4.1. We also introduce the *Dolbeault-Dirac operator* $\not{\partial}_\mathcal{L}$ whose index, in the case particular to us, coincides with the Dolbeault cohomology group. In Section 4.2, we introduce a specific characteristic number called the *Riemann-Roch number* $\chi(M;\mathcal{L})$ that also equals $\dim_\mathbb{C} H^0(M;\mathcal{L})$. The *Riemann-Roch-Hirzebruch* theorem is also provided, which provides us with an equation to calculate $\chi(M;\mathcal{L})$. The Riemann-Roch-Hirzebruch theorem only applies to manifolds however, so, in Section 4.3, the *Kawasaki-Riemann-Roch* theorem is also provided, which calculates $\chi(M;\mathcal{L})$ even if M is an orbifold;

- Despite having now acquired a way of expressing the Riemann-Roch number $\chi(M;\mathcal{L})$, actually evaluating it by using the Riemann-Roch-Hirzebruch theorem proves to be another matter. Luckily, when the manifold M is acted upon by a torus T such that its action has only a finite number of fixed points that are isolated, then there is an easier way to obtain $\chi(M;\mathcal{L})$. Such methods are detailed in Chapter 5 which serves more as a review of equivariant cohomology and localisation formulae. Indeed, Section 5.1 introduces equivariant cohomology groups from the ground up in terms of the Borel construction, and then Section 5.2 introduces both the *Weil* and *Cartan models* of equivariant cohomology. Elements of the Cartan model are called *equivariant differential forms*, which we talk about in Section 5.3, and which possess the notions of closedness and exactness, leading us in Section 5.4 to discuss *equivariant characteristic classes*.

The chapter culminates with the *Atiyah-Bott-Berline-Vergne* localisation formula in Section 5.5, which is the key to transforming the integral over M from the Riemann-Roch-Hirzebruch formula into a finite sum whose terms involve local isotropy data for the T-action on M. Of course, this only holds when M is only a manifold since the Hirzebruch-Riemann-Roch formula is used. However, the localisation formula may also be applied to the Kawasaki-Riemann-Roch formula to obtain an analogous expression in the case when M is an orbifold, which is the content of Section 5.6;

- In order to apply the equivariant localisation formulae to our cut spaces $M_\nu^{\leq\delta}$, we must determine what the fixed points along with their isotropy data. This is the objective of Chapter 6, and the content from here onwards contains completely new results unless cited. In Section 6.1, we determine that the fixed points are finite in number and that each is isolated, with some being located in the interior of $M_\nu^{\leq\delta}$, whereas the rest are located along its boundary, and are produced when performing the symplectic cut. For each fixed point p, we then determine the isotropy weights for the T^n-action on its tangent space $T_pM_\nu^{\leq\delta}$, and find that they coincide with the edge vectors emanating out from the vertex v that p is mapped onto under $\mu_{\mathbb{R}}$, so that $v = \mu_{\mathbb{R}}(p)$. If $M_\nu^{\leq\delta}$ is an orbifold then each orbifold point possesses additional isotropy data, which we deal with in Section 6.2;

- The main body of this thesis culminates in Chapter 7, where we derive an expression for the dimension for the *"equivariant quantisation"* of a hypertoric variety M_ν. That is to say, if $\mathcal{L}_\nu \to M_\nu$ denotes a holomorphic pre-quantum line bundle over the hypertoric variety M_ν, then we determine an equation for $\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)$, where $d \in \mathbb{Z}_{\geq 0}$ is the weight of the representation of U_1 on the $\mathbb{C}$-vector space, $H^0(M_\nu; \mathcal{L}_\nu)$.

  To accomplish this, we first fix an integer $d \in \mathbb{Z}_{\geq 0}$, before then looking at how the pre-quantum line bundle $\mathcal{L}_\nu \to M_\nu$ descends to the cut space $M_\nu^{\leq\delta}$ as the pre-quantum line bundle $\mathcal{L}_\nu^{\leq d} \to M_\nu^{\leq d}$ in Section 7.1. Then, since we located the fixed points for the T^n-action on $M_\nu^{\leq\delta}$, as well as their isotropy data, in Section 7.5 we obtain a formula for the dimension $\dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$ using the Atiyah-Bott-Berline-Vergne equivariant localisation formula applied either to the Hirzebruch-Riemann-Roch theorem, if $M_\nu^{\leq d}$ is a manifold, or to the Kawasaki-Riemann-Roch theorem if $M_\nu^{\leq d}$ is otherwise an orbifold. In either case, we need to find a way to connect $H^0(M_\nu; \mathcal{L}_\nu)_d$ with $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$. This is the goal of Section 7.3 in which the *algebraic cut* is introduced – first mentioned by Edidin and William in [EG98] – and is the algebraic analogue of the symplectic cut. Here, we introduce the algebraic cut for semi-projective normal varieties is introduced, before using it to derive the formula:

  $$\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\leq(d-1)}; \mathcal{L}_\nu^{\leq(d-1)}).$$

  We then finish the chapter, and indeed the main body of this thesis, by seeing the formula in action with some examples in Section 7.5.

- Lastly, Appendix A contains some more general results pertaining to the theory of orbifolds, which we quote at times throughout this thesis.

Part I

Hypertoric Varieties, Cut Spaces, and Moment Polyptchs

Chapter 1

Hypertoric Varieties

We begin by introducing the background theory and results on the hypertoric varieties of Bielawski and Dancer [BD00], beginning first with the hyperkähler quotient.

1.1 Hyperkähler Manifolds

First defined by Calabi in 1979 [Cal79], a hyperkähler manifold is a Riemannian manifold (M, g) with three complex structures, J_1, J_2, and J_3, which satisfy the quaternionic identities and are *compatible* with the Riemannian metric g on M. By compatibility with, g, we mean that there exist three symplectic two-forms, ω_1, ω_2, and ω_3, on M, that satisfy the identities:

$$\omega_1(v, w) = g(J_1 v, w), \quad \omega_2(v, w) = g(J_2 v, w), \quad \omega_3(v, w) = g(J_3 v, w).$$

This condition is equivalent to each symplectic two-form, ω_1, ω_2, and ω_3, additionally being Kähler for the complex structures, J_1, J_2, and J_3, respectively.

If we fix one of the complex structures, J_1 say, then we may define a complex-valued two-form $\omega_{\mathbb{C}} := \omega_2 + i\omega_3$ on M, which is holomorphic with respect to the complex structure J_1. Hence, any hyperkähler manifold, (M, J_i, ω_i), can be thought of as a *holomorphic-symplectic* manifold, $(M, J_1, \omega_{\mathbb{C}})$.

Example 1.1. A basic yet fundamental example of a hyperkähler manifold is that of the four-dimensional flat quaternionic vector space, $\mathbb{H} \cong \mathbb{R}^4$. Denoting the coordinates of $\mathbb{H}$ by $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, its three Kähler two-forms, ω_1, ω_2, and ω_3, can be written as:

$$\begin{aligned}
\omega_1 &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3, \\
\omega_2 &= dx_0 \wedge dx_2 + dx_3 \wedge dx_1, \\
\omega_3 &= dx_0 \wedge dx_3 + dx_1 \wedge dx_2.
\end{aligned}$$

As mentioned previously, we may view $\mathbb{H}$ as a holomorphic-symplectic manifold by fixing the complex structure $J_1 = i$, and by setting $z = x_0 + ix_1$ and $w = x_2 + ix_3$. Then the holomorphic-symplectic two-form $\omega_{\mathbb{C}}$ is:

$$\omega_{\mathbb{C}} = \omega_2 + i\omega_3 = (dx_0 + idx_1) \wedge (dx_2 + idx_3) = dz \wedge dw = -d(wdz) = d\theta,$$

which one may identify with the Poincaré two-form on $T^*\mathbb{C}$, given by the exterior derivative of the Liouville one-form $\theta = -wdz$ on $T^*\mathbb{C}$. Analogously, we introduce a "real" Kähler two-form on M by just relabelling:

$$\begin{aligned} \omega_{\mathbb{R}} := \omega_1 &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3 \\ &= (i/2)\left[(dx_0 + idx_1) \wedge (dx_0 - idx_1) + (dx_2 + idx_3) \wedge (dx_2 - idx_3)\right] \\ &= (i/2)\left[dz \wedge d\bar{z} + dw \wedge d\bar{w}\right]. \end{aligned} \tag{1.1}$$

The above example naturally generalises to the $4N$-dimensional vector space $\mathbb{H}^N$, and we will make heavy use of the identification $\mathbb{H}^N \cong T^*\mathbb{C}^N$ throughout this thesis.

1.2 Hyperkähler Reduction

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ denote its dual Lie algebra. We say that an action of G on a hyperkähler manifold M is **hyperhamiltonian** if it is independently Hamiltonian for each Kähler two-form, ω_1, ω_2, and ω_3, on M. Precisely, this means that for each Kähler two-form, there exist three corresponding G-equivariant maps $\mu_1, \mu_2, \mu_3 : M \to \mathfrak{g}^*$ which, for any element $X \in \mathfrak{g}$, satisfy:

$$\langle d\mu_i,\, X\rangle = \imath_{\underline{X}}\omega_i,$$

where $\underline{X}$ is the fundamental vector field on M associated to the element $X \in \mathfrak{g}$. We may combine these three moment maps into a single **hyperkähler moment map**, as:

$$\phi_{\mathrm{HK}} := \phi_1 \oplus \phi_2 \oplus \phi_3 : M \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H}). \tag{1.2}$$

Similarly to how we introduced the real and holomorphic-symplectic two-forms, $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$, we may relabel the first moment map ϕ_1 as a "real" moment map:

$$\phi_{\mathbb{R}} := \phi_1 : M \to \mathfrak{g}^*,$$

and additionally combine the latter two moment maps, ϕ_2 and ϕ_3, into a single complex-valued moment map:

$$\phi_{\mathbb{C}} := \phi_2 \oplus i\phi_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}.$$

Given a central element $\nu \in \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$ for the coadjoint action of G on $\mathfrak{g}^*$, the level-set $\phi_{\mathrm{HK}}^{-1}(\nu)$ is a G-invariant submanifold of M. The **hyperkähler quotient** is then defined to be the quotient:

$$M_\nu = M /\!\!/\!\!/\!\!/_\nu G := \phi_{\mathrm{HK}}^{-1}(\nu)/G,$$

and which also has the structure of a hyperkähler manifold by the following theorem of Hitchin, Karlhede, Lindström, and Rovček in [HKLR].

Theorem 1.2 ([HKLR]). *Let M be a hyperkähler manifold equipped with a hyperhamiltonian action of a Lie group G, with corresponding hyperkähler moment map $\phi_{\mathrm{HK}} : M \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$. Suppose that $\nu \in \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$ is a regular value for ϕ_{HK} and invariant under the coadjoint action of G. Then, if G acts freely on $\phi_{\mathrm{HK}}^{-1}(\nu)$, the hyperkähler quotient M_ν is a hyperkähler manifold. Moreover, if G is compact and M is complete, then M_ν is a complete hyperkähler manifold.*

As stated in Theorem 1.2, the hyperkähler quotient $M_\nu = T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_\nu G$ is a manifold, provided that the G-action on the level-set $\phi_{\mathrm{HK}}^{-1}(\nu)$ is **free**, which means that the stabiliser subgroup for every point $p \in \phi_{\mathrm{HK}}^{-1}(\nu)$ is trivial. More generally, the hyperkähler quotient M_ν is an orbifold if the G-action on $\phi_{\mathrm{HK}}^{-1}(\nu)$ is **locally free**, which means that the stabiliser subgroup for every point $p \in \phi_{\mathrm{HK}}^{-1}(\nu)$ is at worst finite.

Let us focus now on the case when G is a compact Lie group acting linearly on $\mathbb{C}^N$, which is Hamiltonian with corresponding moment map:

$$\phi : \mathbb{C}^N \to \mathfrak{g}^*. \tag{1.3}$$

This induces a linear G-action on the cotangent bundle $T^*\mathbb{C}^N$, and which we subsequently identify as $T^*\mathbb{C}^N \cong \mathbb{H}^N$. In doing so, $T^*\mathbb{C}^N$ inherits the hyperkähler structure from $\mathbb{H}^N$, where the real Kähler two-form $\omega_{\mathbb{R}} \in \Omega^2(T^*\mathbb{C}^N)$ and the holomorphic-symplectic two-form $\omega_{\mathbb{C}} \in \omega^2(T^*\mathbb{C}^N)$ were defined in Example 1.1. That is to say, $\omega_{\mathbb{R}}$ is given by the sum of the pull-backs of the standard symplectic two-forms on $\mathbb{C}^N$ and on $(\mathbb{C}^N)^*$ respectively, whereas $\omega_{\mathbb{C}}$ is given by $\omega_{\mathbb{C}} = d\theta$, where $\theta \in \Omega^1(T^*\mathbb{C}^N)$ is the Liouville one-form on the complex cotangent bundle $T^*\mathbb{C}^N$.

As the G-action on $T^*\mathbb{C}^N$ is $\mathbb{H}$-linear, it is hyperhamiltonian with corresponding hyperkähler moment map $\phi_{\mathrm{HK}} = \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} : T^*\mathbb{C}^N \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$, where the real and complex moment maps are respectively:

$$\phi_{\mathbb{R}}(z,w) = \phi(z) - \phi(w), \quad \text{and} \quad \phi_{\mathbb{C}}(z,w)(X) = w(\underline{X}),$$

where $w \in T_z^*\mathbb{C}^N$, $X \in \mathfrak{g}_{\mathbb{C}}$, and $\underline{X} \in T_z\mathbb{C}^N$ is the vector field induced by the Lie algebra element X. If $\nu \in \mathfrak{g}^*$ is a central regular value for the real moment map $\phi_{\mathbb{R}}$, and if $(\nu, 0) \in \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$ is a central regular value for the hyperkähler moment map ϕ_{HK} then, following [HP04, §1], we call the hyperkähler quotient $M = T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_{(\nu,0)} G$ the **hyperkähler analogue for the Kähler quotient** $X = \mathbb{C}^N /\!\!/_\nu G$.

1.3 Hypertoric Varieties as Hyperkähler Quotients

Let us restrict our focus further to the case when G is an N-dimensional compact connected abelian Lie group, i.e., when $G = T^N \cong U(1)^N$ is the N-dimensional real torus with Lie algebra $\mathfrak{t}^N$. The **character lattice** $\mathfrak{t}^N_{\mathbb{Z}}$ of $\mathfrak{t}^N$ is the kernel of the exponential map $\exp : \mathfrak{t}^N \to T^N$, so:

$$\mathfrak{t}^N_{\mathbb{Z}} := \ker\left(\exp : \mathfrak{t}^N \to T^N\right).$$

Denote the dual space to the Lie algebra $\mathfrak{t}^N$ by $(\mathfrak{t}^N)^*$. Then by using a T^N-invariant bilinear form $\langle -,\, - \rangle$ on $\mathfrak{t}^N$, we define the **weight lattice** $(\mathfrak{t}^N_{\mathbb{Z}})^*$ of $(\mathfrak{t}^N)^*$ by

$$(\mathfrak{t}^N_{\mathbb{Z}})^* := \left\{\, \alpha \in (\mathfrak{t}^N)^* \;\middle|\; \langle \alpha,\, X \rangle \in 2\pi\mathbb{Z} \text{ for all } X \in \mathfrak{t}^N_{\mathbb{Z}} \,\right\}.$$

By choosing a basis $e_1, \ldots, e_N$ for the character lattice $(2\pi\mathfrak{t}_{\mathbb{Z}})^N$, we may identify $\mathfrak{t}^N_{\mathbb{Z}} \cong \mathbb{Z}^N$ and $\mathfrak{t}^N \cong \mathbb{R}^N$. Finally, let $\epsilon_1, \ldots, \epsilon_N$ be the basis for the weight lattice $(\mathfrak{t}^N_{\mathbb{Z}})^*$, dual to that of $e_1, \ldots, e_N$.

The N-dimensional real torus T^N acts linearly on the N-dimensional flat complex vector space $\mathbb{C}^N$, namely:

$$(t_1, \ldots, t_N) \cdot (z_1, \ldots, z_N) = (t_1 z_1, \ldots, t_N z_N), \tag{1.4}$$

and this action is Hamiltonian and its corresponding moment map is

$$\tilde{\phi} : \mathbb{C}^N \to (\mathfrak{t}^N)^*, \qquad \phi(z) = \sum_{i=1}^{N} |z_i|^2 \epsilon_i = \left(|z_1|^2, \ldots, |z_N|^2\right). \tag{1.5}$$

We extend this action to an induced one $T^*\mathbb{C}^N$ by:

$$(t_1, \ldots, t_N) \cdot (z_1, \ldots, z_N, w_1, \ldots, w_N) = (t_1 z_1, \ldots, t_N z_N, t_1^{-1} w_1, \ldots, t_N^{-1} w_N), \tag{1.6}$$

which is hyperhamiltonian with corresponding hyperkähler moment map:

$$\tilde{\phi}_{\mathrm{HK}} = \tilde{\phi}_{\mathbb{R}} \oplus \tilde{\phi}_{\mathbb{C}} : T^*\mathbb{C}^N \to (\mathfrak{t}^N)^* \otimes \mathrm{Im}(\mathbb{H}), \tag{1.7}$$

where $\tilde{\phi}_{\mathbb{R}}$ and $\tilde{\phi}_{\mathbb{C}}$ are the real and complex moment maps for $\tilde{\phi}_{\mathrm{HK}}$, respectively. They can be written explicitly as:

$$\begin{aligned} \tilde{\phi}_{\mathbb{R}}(z, w) &= \sum_{i=1}^{N} (|z_i|^2 - |w_i|^2)\epsilon_i \in (\mathfrak{t}^N)^*, \\ \tilde{\phi}_{\mathbb{C}}(z, w) &= \sum_{i=1}^{N} (z_i w_i)\epsilon_i \in (\mathfrak{t}^N)^* \otimes \mathbb{C}. \end{aligned} \tag{1.8}$$

We define a subtorus $K \subset T^N$ as follows: consider a collection of N distinct non-zero integral vectors $\{u_1, \dots, u_N\}$ in $\mathfrak{t}^n_{\mathbb{Z}}$, whose real span equals $\mathfrak{t}^n$. Define the map:

$$\pi : \mathfrak{t}^N \to \mathfrak{t}^n, \qquad \text{by} \qquad \pi(e_i) := u_i.$$

Set $\mathfrak{k} := \ker \pi$ and let $\imath : \mathfrak{k} \hookrightarrow \mathfrak{t}^N$ denote the inclusion. This yields the short exact sequence:

$$\{0\} \longrightarrow \mathfrak{k} \xhookrightarrow{\ \imath\ } \mathfrak{t}^N \xrightarrow{\ \pi\ } \mathfrak{t}^n \longrightarrow \{0\}. \tag{1.9}$$

Then, on the one hand, by exponentiating (1.9) we obtain a short exact sequence of tori:

$$\{1\} \longrightarrow K \xhookrightarrow{\ \imath\ } T^N \xrightarrow{\ \pi\ } T^n \longrightarrow \{1\}. \tag{1.10}$$

Whereas, on the other hand, we may dualise (1.9) to obtain a short exact sequence of dual spaces, along with their respective lattices:

$$\begin{tikzcd}
\{0\} & \mathfrak{k}^* \arrow[l] & (\mathfrak{t}^N)^* \arrow[l, "\imath^*"'] & (\mathfrak{t}^n)^* \arrow[l, "\pi^*"'] & \{0\} \arrow[l] \\
\{0\} & \mathfrak{k}^*_{\mathbb{Z}} \arrow[l] \arrow[u, hook] & (\mathfrak{t}^N_{\mathbb{Z}})^* \arrow[l, "\imath^*"'] \arrow[u, hook] & (\mathfrak{t}^n_{\mathbb{Z}})^* \arrow[l, "\pi^*"'] \arrow[u, hook] & \{0\}. \arrow[l]
\end{tikzcd} \tag{1.11}$$

Denote the dimension of K by $k = \dim_{\mathbb{R}} K$, and set:

$$\alpha_i = (a_{1i}, \dots, a_{ki}) := \imath^*(\epsilon_i) \in \mathfrak{k}^*_{\mathbb{Z}}, \qquad \text{for each } i = 1, \dots, N.$$

Then, if we denote the image of $(t_1, \dots, t_k) \in K$ under $\imath$ by:

$$\imath(t_1, \dots, t_k) = (\mathbf{t}^{\alpha_1}, \dots, \mathbf{t}^{\alpha_N}) \in T^N, \qquad \text{where} \qquad \mathbf{t}^{\alpha_i} := t_1^{a_{1i}} \dots t_k^{a_{ki}},$$

then the $\alpha_1, \dots, \alpha_N \in \mathfrak{k}^*_{\mathbb{Z}}$ are the weights for the K-action on $T^*\mathbb{C}^N$. That is:

$$\mathbf{t} \cdot (z_1, \dots, z_N, w_1, \dots, w_N) = (\mathbf{t}^{\alpha_1} z_1, \dots, \mathbf{t}^{\alpha_N} z_N, \mathbf{t}^{-\alpha_1} w_1, \dots, \mathbf{t}^{-\alpha_N} w_N). \tag{1.12}$$

The weights $\alpha_1, \dots, \alpha_N$ can be arranged into an $(k \times N)$-matrix

$$A := \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \ddots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ a_{k1} & \cdots & \cdots & a_{kN} \end{bmatrix} : (\mathfrak{t}^N_{\mathbb{Z}})^* \to \mathfrak{k}^*_{\mathbb{Z}}, \tag{1.13}$$

which represents that linear map $\imath^*$ relative to the $\epsilon_1, \dots, \epsilon_N$ basis of $(\mathfrak{t}^*_{\mathbb{Z}})^*$. That is to say, $A = [\imath^*]$.

The subtorus K acts on $T^*\mathbb{C}^N$ via the inclusion homomorphism $\imath$ from K into T^N. As K is a subtorus of T^N which itself acts on $T^*\mathbb{C}^N$ in a hyperhamiltonian way, the action of K is also hyperhamiltonian, whose corresponding hyperkähler moment map ϕ_{HK} is:

$$\phi_{\mathrm{HK}} := \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} := (\imath^* \circ \tilde{\phi}_{\mathbb{R}}) \oplus (\imath_{\mathbb{C}}^* \circ \tilde{\phi}_{\mathbb{C}}) \to \mathfrak{k}^* \otimes \mathrm{Im}(\mathbb{H}).$$

In coordinates, ϕ_{HK} is given by the equations:

$$\begin{aligned} \phi_{\mathbb{R}}(z, w) &= \left(\imath^* \circ \tilde{\phi}_{\mathbb{R}}\right)(z, w) = \frac{1}{2}\sum_{i=1}^{N}(|z_i|^2 - |w_i|^2)\alpha_i \in \mathfrak{k}^*, \\ \phi_{\mathbb{C}}(z, w) &= \left(\imath_{\mathbb{C}}^* \circ \tilde{\phi}_{\mathbb{C}}\right)(z, w) = \sum_{i=1}^{N}(z_i w_i)\alpha_i \in \mathfrak{k}_{\mathbb{C}}^*. \end{aligned} \tag{1.14}$$

The following proposition states the conditions for an element $(\nu, 0, 0) \in \mathfrak{k}^* \otimes \mathrm{Im}(\mathbb{H})$ to be a regular value for the hyperkähler moment map ϕ_{HK}, and is due to Konno [Kon00].

Proposition 1.3. *Fix an element $\nu \in \mathfrak{k}^*$. Then the following are equivalent:*

(i) $(\nu, 0, 0) \in \mathfrak{k}^* \otimes \mathrm{Im}(\mathbb{H})$ *is a regular value for the hyperkähler moment map* ϕ_{HK}*;*

(ii) for any $J \subset \{ 1, \ldots, N \}$, whose cardinality $|J|$ is strictly less than $\dim \mathfrak{k}^* = k$*, the element ν is not contained in the subspace of $\mathfrak{k}^*$ spanned by $\{\alpha_j \mid j \in J\}$.*

A combinatorial geometric interpretation of Proposition 1.3 for an element $\nu \in \mathfrak{k}^*$ to be a regular value of the real moment map $\phi_{\mathbb{R}}$, is that ν must not be contained in any proper subspace generated by any combination of the K-weights α_i, where $i = 1, \ldots, N$.

With this established, we can define the main objects of interest, namely that of *hypertoric varieties*. They were first introduced by Bielawski and Dancer in [BD00] who considered them from the differential-geometric angle, whereas Hausel and Sturmfels in [HS02] considered them from the algebro-geometric angle not too long afterwards.

Definition 1.4. Let $K \trianglelefteq T^N$ be the subtorus defined by $K := \ker \pi$ as in the short exact sequence (1.10), and let $(\nu, 0, 0) \in \mathfrak{k}^* \otimes \phi_{\mathrm{HK}}$ be a regular value for the hyperkähler moment map ϕ_{HK}. A **hypertoric variety** M_ν is the hyperkähler quotient of the complex cotangent space $T^*\mathbb{C}^N$, with respect to the action of the subtorus $K \subset T^N$ at the regular value $(\nu, 0, 0) \in \mathfrak{k}^* \otimes \mathrm{Im}(\mathbb{H})$, so:

$$M_\nu := T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_{(\nu,0)} K := \phi_{\mathrm{HK}}^{-1}(\nu, 0)/K = (\phi_{\mathbb{R}}^{-1}(\nu) \cap \phi_{\mathbb{C}}^{-1}(0))/K. \tag{1.15}$$

There is a residual quotient torus $T^n = T^N/K$, that acts on the hypertoric variety M_ν in a hyperhamiltonian way and induces the hyperkähler moment map $\mu_{\mathrm{HK}} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : M_\nu \to$

$(\mathfrak{t}^n)^* \otimes \mathrm{Im}(\mathbb{H})$. With respect to coordinates:

$$\begin{aligned} \mu_{\mathbb{R}}[z,w] &= \frac{1}{2}\sum_{i=1}^{N}(|z_i|^2 - |w_i|^2 - \lambda_i)\epsilon_i \in (\mathfrak{t}^n)^* \cong \ker \imath^*, \\ \mu_{\mathbb{C}}[z,w] &= \sum_{i=1}^{N}(z_i w_i)\epsilon_i \in (\mathfrak{t}^n_{\mathbb{C}})^* \cong \ker \imath^*_{\mathbb{C}}. \end{aligned} \tag{1.16}$$

Here, $[z,w] \in M_\nu$ denotes the K-equivalence class of the point $(z,w) \in \phi_{\mathrm{HK}}^{-1}(\nu, 0)$.

1.4 Hypertoric Varieties as Algebro-Geometric Quotients

As mentioned in the paragraph before Definition 1.4, there is an alternative algebro-geometric way to construct a hypertoric variety using the Proj construction and Geometric Invariant Theory (GIT), and was first considered in [HS02]. The differential- and algebro-geometric methods result in equivalent hypertoric varieties due to the *Kempf-Ness theorem*, with the latter method being essential to us when studying the equivariant quantisation of a hypertoric variety in Chapter 7. Several results in this section come from [Har77, Chapter II], [Dol03], [MFK94], and [Muk03], which themselves provide more wider-ranging discussions of the Proj construction and also GIT.

To start, suppose that X is a complex normal quasi-projective variety, that G is a linear algebraic group that acting linearly over X, and let $\pi : \mathcal{L} \to X$ be a line bundle on X. Then, the process of lifting the G-action on X up to one on $\mathcal{L}$ boils down to choosing a *linearisation* of the G-action on X.

Definition 1.5. Let G be a linear algebraic group acting on an affine variety X. A **linearisation** of the action is a line bundle $\pi : \mathcal{L} \to X$ together with a choice of G-action on $\mathcal{L}$, such that:

(i) the bundle projection $\pi : \mathcal{L} \to X$ is G-equivariant;

(ii) for every $g \in G$ and $p \in X$, the induced map between the fibres:

$$\mathcal{L}|_p \to \mathcal{L}|_{g\cdot p}, \qquad l \mapsto g \cdot l,$$

is linear, where $l \in \mathcal{L}|_p$ is an element in the fibre over the point $p \in X$.

A linearisation of a G-action on a line bundle $\pi : \mathcal{L} \to X$ is equivalently called a ***G*-equivariant line bundle**. It is common to refer to the lifting of the G-action on X to one on $\mathcal{L}$ implicitly, by simply stating that the line bundle $\pi : \mathcal{L} \to X$ is a ***G*-linearised line bundle**, i.e., we do not distinguish between a line bundle and its linearisation.

Example 1.6. Suppose that $X = \mathrm{Spec}\, A$ is an affine variety for some integral $\mathbb{C}$-algebra A, and that $\pi : \mathcal{L} = X \times \mathbb{C} \to X$ is the trivial line bundle over X. If G acts on X, then each G-linearisation of $\mathcal{L}$

corresponds uniquely to a choice of character $\chi : G \to \mathbb{C}^*$, see [Dol03, Theorem 7.1 & Corollary 7.1]. This correspondence is given via:

$$G \times (X \times \mathbb{C}) \to X \times \mathbb{C}, \quad \text{where} \quad (g, p, \xi) \mapsto (g \cdot p, \chi(g)\xi).$$

Example 1.7. Given a G-linearised line bundle $\mathcal{L} \to X$, not necessarily trivial, and a character $\chi : G \to \mathbb{C}^*$, which does not necessarily define the G-action on $\mathcal{L}$, then we can form a new G-linearisation $\mathcal{L}_\chi \to X$ by **twisting $\mathcal{L}$ by the character χ**, [B+18, §1]. More explicitly, we set $\mathcal{L}_\chi := \mathcal{L} \otimes \mathcal{O}_X^{(\chi)}$, where $\mathcal{O}_X^{(\chi)} = X \times \mathbb{C}$ is the trivial bundle over X equipped with a G-linearisation defined by the character χ, as in Example 1.6.

It turns out that, when a connected linear algebraic group G acts on a *normal* variety X over a field $\mathbb{F}$, with $\pi : \mathcal{L} \to X$ a line bundle over X, then some positive tensor power $\mathcal{L}^{\otimes m}$ admits a G-linearisation provided that X is proper, as proven in [MFK94, Corollary 1.6].

Now we shall consider how a linearisation affects the sections of a linearised line bundle. Given $\pi : \mathcal{L} \to X$ over X, we let $H^0(X; \mathcal{L})$ denote its space of global sections; if $\pi : \mathcal{L} \to X$ is the projection, then sections are the maps $\sigma : X \to \mathcal{L}$ such that $\pi \circ \sigma = \mathrm{Id}_X$. If furthermore $\mathcal{L} \cong X \times \mathbb{C}$ is trivial then, as a regular function $s \in \mathbb{C}[X]$ is simply a morphism $s : X \to \mathbb{C}$, we may identify the ring of global sections $H^0(X; \mathcal{L})$ on X with its ring of regular functions $\mathbb{C}[X]$. Explicitly, for a section σ there exists a unique regular function s such that:

$$\sigma(p) = \big(p, s(p)\big) \in X \times \mathbb{C} = \mathcal{L}, \qquad \text{for any } p \in X. \tag{1.17}$$

When the line bundle $\mathcal{L}$ on X is G-linearised, then there is an additional induced action of G on $H^0(X; \mathcal{L})$, given by:

$$(g \cdot \sigma)(p) := g \cdot \big(\sigma(g^{-1} \cdot p)\big), \qquad \text{for any } g \in G, \text{ and } p \in X. \tag{1.18}$$

Definition 1.8. We define the **subspace $H^0(X; \mathcal{L})^G$ of G-invariant sections**, by:

$$H^0(X; \mathcal{L})^G := \left\{ \sigma \in H^0(X; \mathcal{L}) \mid g \cdot \sigma = \sigma, \text{ for any } g \in G \right\}. \tag{1.19}$$

Similarly, we define the **subspace $\mathbb{C}[X]_\chi$ of semi-invariants of weight χ**, by:

$$\mathbb{C}[X]_\chi := \left\{ s \in \mathbb{C}[X] \;\middle|\; \begin{array}{c} s(g \cdot p) = \chi(g)s(p), \\ \text{for any } g \in G, \text{ and } p \in X \end{array} \right\}. \tag{1.20}$$

When X is affine then, from Example 1.6, every line bundle $\pi : \mathcal{L} = X \times \mathbb{C} \to X$ over X is trivial and their G-linearisations are defined by a character $\chi : G \to \mathbb{C}^*$. In this case, we have the following lemma from [CLS11a, Lemma 14.1.1].

Lemma 1.9. *Let $\pi : \mathcal{L}_\chi \to X$ be the G-linearised line bundle on X determined by the character $\chi : G \to \mathbb{C}^*$. Then:*

(i) if σ is the global section of $\mathcal{L}_\chi$ corresponding to s, then $g \cdot \sigma$ is the global section defined by:

$$(g \cdot \sigma)(p) = \big(p, \chi(g)s(g^{-1} \cdot p)\big), \tag{1.21}$$

for any $g \in G$ and $p \in X$;

(ii) the space of G-invariant global sections is isomorphic to the space of semi-invariants of weight χ:

$$H^0(X; \mathcal{L}_\chi)^G \cong \mathbb{C}[X]_\chi. \tag{1.22}$$

Proof. To prove (i), by (1.18), an element $g \in G$ acts on a global section $\sigma \in H^0(X; \mathcal{L}_\chi)$ as:

$$\begin{aligned}
(g \cdot \sigma)(p) &= g \cdot \big(\sigma(g^{-1} \cdot p)\big) && \big(\text{from (1.18)}\big)\\
&= g \cdot \big(g^{-1} \cdot p, s(g^{-1} \cdot p)\big) && \big(\text{as } \sigma(p) = (p, s(p))\big)\\
&= \big(g \cdot g^{-1} \cdot p, g \cdot s(g^{-1} \cdot p)\big)\\
&= \big(p, g \cdot s(g^{-1} \cdot p)\big).
\end{aligned}$$

We therefore see that:

$$p \xmapsto{\ \sigma\ } \big(p, g \cdot s(g^{-1} \cdot p)\big) \xmapsto{\ \pi\ } p,$$

i.e., that $\pi \circ (g \cdot \sigma) = \mathrm{Id}_X$. Thus $g \cdot \sigma$ is a global section for the line bundle $\pi : \mathcal{L}_\chi \to X$.

For (ii), recall that the G-action on $\mathcal{L}_\chi$ is via the character, $\chi : G \to \mathbb{C}^*$. From part (i), for every $g \in G$ and $p \in X$:

$$(g \cdot \sigma)(p) = \big(p, g \cdot s(g^{-1} \cdot p)\big) = \big(p, \chi(g)s(g^{-1} \cdot p)\big). \tag{1.23}$$

Hence, for any $p \in X$, we have that $(g \cdot \sigma)(p) = \sigma(p)$ if, and only if, $\chi(g)s(g^{-1} \cdot p) = s(p)$ from (1.23) if, and only if, $s(g \cdot p) = \chi(g)s(p)$, which implies that $s \in \mathbb{C}[X]_\chi$. Hence, the lemma follows. □

For each $m \geq 0$, define the $\mathbb{C}$-algebra of

$$R_m := H^0(X; \mathcal{L}_\chi^{\otimes m})^G \cong \left\{ s \in \mathbb{C}[X] \;\middle|\; \begin{array}{c} s(g \cdot p) = \chi(g)^m s(p), \\ \text{for all } g \in G, \text{ and } p \in X \end{array} \right\},$$

where the isomorphism comes from Lemma 1.9 (ii). Hence each R_m is the subring of semi-invariants for the character $\chi^{\otimes m}$, and these subrings can be assembled together as the graded components of a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra that is defined by:

$$R := \bigoplus_{m \geq 0} H^0(X; \mathcal{L}_\chi^{\otimes m})^G. \tag{1.24}$$

We call R in (1.24) the **invariant subring**.

When X is an affine variety and G is a reductive linear algebraic group that acts X, then the invariant subring R is a finitely-generated $\mathbb{C}$-algebra, due to the positive-affirmation of Hilbert's Fourteenth Problem [Hil02, Problem 14] when X is *normal*. The same holds for each R_m with $m \geq 0$, in addition to being R_0-algebras too.

Definition 1.10. Let X be an affine variety, G be a reductive linear algebraic group that acts on X, and $\mathcal{L}_\chi \to X$ be a G-linearised line bundle on X defined by a character $\chi : G \to \mathbb{C}^*$. Then the **GIT quotient** $X /\!\!/_\chi G$ of X by G is

$$X /\!\!/_\chi G := \operatorname{Proj} R^G = \operatorname{Proj} \left(\bigoplus_{m \geq 0} H^0(X; \mathcal{L}_\chi^{\otimes m})^G \right). \tag{1.25}$$

Whilst easy to state, it is not quite so easy to see what the GIT quotient $X /\!\!/_\chi G$ in Definition 1.10 represents geometrically. In order to do so, we introduce the following notion of stability for a G-linearised line bundle.

Definition 1.11. Let $\pi : \mathcal{L}_\chi \to X$ be a G-linearised line bundle over X, defined by a character $\chi : G \to \mathbb{C}^*$, and let $p \in X$ be a point. Then:

(i) p is said to be **semi-stable** with respect to $\mathcal{L}_\chi$ if there exists an $m \geq 1$, and a G-invariant section $s \in H^0(X; \mathcal{L}_\chi^{\otimes m})^G$, such that the **semi-stable locus**:

$$X^{\chi-\mathrm{ss}} := \{\, x \in X \mid s(x) \neq 0 \,\}$$

is affine and contains p;

(ii) p is said to be **stable** with respect to $\mathcal{L}_\chi$ if there exists a section s as in (i), and additionally the stabiliser subgroup $\operatorname{Stab}_G(p)$ is finite and every orbit of G in $X^{\chi-\mathrm{ss}}$ is closed. The **stable locus** is then defined to be:

$$X^{\chi-\mathrm{st}} := \left\{\, y \in X^{\chi-\mathrm{ss}} \;\middle|\; \operatorname{Stab}_G(p) \text{ is finite, and } G \cdot y = \overline{G \cdot y} \,\right\};$$

(iii) p is said to be **unstable** with respect to $\mathcal{L}_\chi$, if it is not semi-stable. The **unstable locus** is defined to be:

$$X^{\chi-\mathrm{us}} := X - X^{\chi-\mathrm{ss}}.$$

We may introduce an equivalence relation on the χ-semi-stable locus $X^{\chi-\mathrm{ss}}$ by defining:

$$x \sim y, \text{ for every } x, y \in X^{\chi-\mathrm{ss}} \quad \iff \quad \overline{G \cdot x} \cap \overline{G \cdot y} \cap X^{\chi-\mathrm{ss}} \neq \emptyset. \tag{1.26}$$

Then one of the fundamental results of geometric invariant theory is that $X /\!\!/_\chi G$ is a geometric quotient of $X^{\chi-\mathrm{ss}}$ by G.

Theorem 1.12. *There exists a good categorical quotient:*

$$\psi : X^{\chi-\mathrm{ss}} \longrightarrow X^{\chi-\mathrm{ss}}/\sim,$$

with $\psi(x) = \psi(y)$ if, and only if, the closures of the orbits $G \cdot x$ and $G \cdot y$ intersect in X. Furthermore:

(i) $X^{\chi-\mathrm{ss}}/\sim$ is a quasi-projective variety;

(ii) there exists an open subset U in $X^{\chi-\mathrm{ss}}/\sim$ such that $\psi^{-1}(U) = X^{\chi-\mathrm{st}}$, and the restriction of ψ to $X^{\chi-\mathrm{st}}$ is a geometric quotient of $X^{\chi-\mathrm{st}}$ by G;

(iii) there exists an ample line bundle $\mathcal{F}$ on $X^{\chi-\mathrm{ss}}/\sim$, such that $\psi^*(\mathcal{F}) \cong \mathcal{L}_\chi^{\otimes m}$ when restricted to $X^{\chi-\mathrm{ss}}/\sim$, for some $m \geq 0$;

(iv) we may identify the categorical quotient with the GIT quotient:

$$X^{\chi-\mathrm{ss}} /\!\!/_{\chi} G \cong \operatorname{Proj} R^G \cong X^{\chi-\mathrm{ss}}/\sim . \tag{1.27}$$

For the proofs of (i), (ii), and (iii), one may consult [MFK94], [New78], or [Dol03], for example. For a further details regarding the identification stated in (iv), see [Nak99] or [Pro05].

1.5 Hypertoric Varieties as GIT Quotients

If we apply the $\mathrm{Hom}_{\mathbb{Z}}(-;\mathbb{C}^*)$ functor to the short exact sequence of lattices in (1.11), then we obtain the following short exact sequence of complex algebraic tori:

$$\{1\} \longrightarrow K_{\mathbb{C}} \xhookrightarrow{\imath_{\mathbb{C}}} T^N_{\mathbb{C}} \xrightarrow{\pi_{\mathbb{C}}} T^n_{\mathbb{C}} \longrightarrow \{1\}. \tag{1.28}$$

In the same way as when we were dealing with real tori in Section 1.3, we let the complexified torus $T^N_{\mathbb{C}}$ act linearly on $T^*\mathbb{C}^N$, yielding the complex moment map:

$$\tilde{\phi}_{\mathbb{C}} : T^*\mathbb{C}^N \to (\mathfrak{t}^N_{\mathbb{C}})^*, \qquad \phi_{\mathbb{C}}(z,w) = \sum_{j=1}^{N} z_j w_j \epsilon_j.$$

Here, recall that $\epsilon_1, \ldots, \epsilon_N$ is a basis for $(\mathfrak{t}^N)^*$, dual to that of $e_1, \ldots, e_N$ for the Lie algebra $\mathfrak{t}^N$. Given an integral weight $\lambda \in (\mathfrak{t}^N_{\mathbb{Z}})^*$ of $T^N_{\mathbb{C}}$, we obtain a character $\chi_\lambda : T^N_{\mathbb{C}} \to \mathbb{C}^*$ via:

$$\chi_\lambda : T^N_{\mathbb{C}} \to \mathbb{C}^*, \qquad \chi_\lambda(\exp(X)) := e^{\lambda(X)}, \quad \text{for any } X \in \mathfrak{t}^N.$$

Let $\mathcal{O}_{T^*\mathbb{C}^N} = T^*\mathbb{C}^N \times \mathbb{C}$ be the holomorphic trivial line bundle on the complex cotangent space $T^*\mathbb{C}^N$. The integral weight $\lambda \in (\mathfrak{t}^N_{\mathbb{Z}})^*$ defines a character $\chi_\lambda : T^N_{\mathbb{C}} \to \mathbb{C}^*$, and hence a lift of the $T^N_{\mathbb{C}}$-action on $T^*\mathbb{C}^N$ up to on $\mathcal{O}_{T^*\mathbb{C}^N}$ from Example 1.6, via:

$$t \cdot \big((z,w),\xi\big) = \big(t \cdot (z,w), t \cdot \xi\big) = \big((tz, t^{-1}w), \chi_\lambda(t)\xi\big). \tag{1.29}$$

As $\phi_{\mathbb{C}}^{-1}(0)$ is a submanifold of $T^*\mathbb{C}^N$, we can restrict the trivial line bundle $\pi : \mathcal{O}_{T^*\mathbb{C}^N} \to T^*\mathbb{C}^N$ to obtain one on $\phi_{\mathbb{C}}^{-1}(0)$. If $j : \phi_{\mathbb{C}}^{-1}(0) \hookrightarrow T^*\mathbb{C}^N$ is the inclusion, then we get a holomorphic trivial line bundle $\pi : \mathcal{L} \to \phi_{\mathbb{C}}^{-1}(0)$ by:

$$\mathcal{O}_{\phi_{\mathbb{C}}^{-1}(0)} := \phi_{\mathbb{C}}^{-1}(0) \times \mathbb{C} \cong j^*\mathcal{O}_{T^*\mathbb{C}^N} \to \phi_{\mathbb{C}}^{-1}(0).$$

Furthermore, since $K_{\mathbb{C}}$ is a subtorus of $T_{\mathbb{C}}^N$ and since $T_{\mathbb{C}}^N$ acts linearly on $T^*\mathbb{C}^N$, we have an induced action of $K_{\mathbb{C}}$ on $\phi_{\mathbb{C}}^{-1}(0)$. The following lemma describes the relationship between the weights of $T_{\mathbb{C}}^N$ with those of its subtorus $K_{\mathbb{C}}$.

Lemma 1.13. *An element $\nu := \imath^*(\lambda) \in \mathfrak{k}_{\mathbb{Z}}^*$ occurs as a weight of $K_{\mathbb{C}}$ if, and only if, $\lambda + \mu \in (\mathfrak{t}_{\mathbb{Z}}^N)^*$ occurs as a weight of $T_{\mathbb{C}}^N$ for some $\mu \in (\mathfrak{t}_{\mathbb{Z}}^N)^* \cap \ker \imath^*$.*

Proof. Let us fix some notation: suppose $a = e^X \in K_{\mathbb{C}}$ for some $X \in \mathfrak{k}$. Denote their respective images under $\imath$ and its derivative $\imath_*$ by $b := \imath(a) \in T_{\mathbb{C}}^N$ and $Y = \imath_*(X) \in \mathfrak{t}^N$ respectively, so that $b = e^Y$. Then we have the diagram:

$$\begin{tikzcd}
X \arrow[rrr, mapsto] \arrow[ddd] & & & Y = \imath_*(X) \arrow[ddd] \\
& \mathfrak{k} \arrow[r, "\imath_*"] \arrow[d, "\exp"] & \mathfrak{t}^N \arrow[d, "\exp"] & \\
& K_{\mathbb{C}} \arrow[r, "\imath"] & T_{\mathbb{C}}^N & \\
a = e^X \arrow[rrr, mapsto] & & & b = e^{\imath_*(Y) = \imath(e^X)}
\end{tikzcd}$$

If $\lambda + \mu \in (\mathfrak{t}_{\mathbb{Z}}^N)^*$, where $\mu \in (\mathfrak{t}_{\mathbb{Z}}^N)^* \cap \ker \imath^*$, occurs as a weight of $T_{\mathbb{C}}^N$, then:

$$\begin{aligned}
\chi_{\lambda+\mu}(b) &= \chi_{\lambda+\mu}(e^Y) = e^{\langle \lambda+\mu,\, Y\rangle} = e^{\langle \lambda+\mu,\, \imath_*(X)\rangle} \\
&= e^{\langle \imath^*(\lambda+\mu),\, X\rangle} = e^{\langle \imath^*(\lambda),\, X\rangle} = e^{\langle \nu,\, X\rangle} = \chi_\nu(e^X) \\
&= \chi_\nu(a),
\end{aligned}$$

whence if $\lambda + \mu \in (\mathfrak{t}_{\mathbb{Z}}^N)^*$ occurs as a weight of $T_{\mathbb{C}}^N$, then $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$ occurs as a weight of $K_{\mathbb{C}}$. $\square$

We see from Lemma 1.13 that there are infinitely many weights $\nu + \lambda \in (\mathfrak{t}_{\mathbb{Z}}^N)^*$ that get projected onto the same weight $\nu \in \mathfrak{k}^*$, provided that $\lambda \in \ker \imath^*$. In Section 2.1, we shall see that the weight ν determines a hyperplane arrangement $\mathcal{A}$ in $(\mathfrak{t}^n)^*$, and a choice of $\lambda \in \ker \imath^*$ corresponds to a translation of $\mathcal{A}$.

Definition 1.14. An integral weight $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$ is called **generic**, if it is not contained in any proper subspace generated by the $K_{\mathbb{C}}$-weights α_i, where $i = 1, \ldots, N$.

Thus let us fix an integral weight $\nu \in \mathfrak{k}^*_{\mathbb{Z}}$ and denote its corresponding character by $\chi_\nu : K_{\mathbb{C}} \to \mathbb{C}^*$. Consequently, we obtain a $K_{\mathbb{C}}$-linearised line bundle $\pi : \mathcal{L}_{\chi_\nu} \to \phi_{\mathbb{C}}^{-1}(0)$ obtained by twisting the trivial line bundle $\pi : \mathcal{L} \to \phi_{\mathbb{C}}^{-1}(0)$ over $\phi_{\mathbb{C}}^{-1}(0)$ by χ_ν.

The next technical lemma characterises the stable and semi-stable loci in $\phi_{\mathbb{C}}^{-1}(0)$ with respect to the $K_{\mathbb{C}}$-linearised line bundle, $\pi : \mathcal{L}_{\chi_\nu} \to \phi_{\mathbb{C}}^{-1}(0)$, and was proven by Konno in [Kon08, Lemma 3.4].

Lemma 1.15. *Fix an integral weight $\nu \in \mathfrak{k}^*_{\mathbb{Z}}$. Then:*

(i) a point $(z, w) \in \phi_{\mathbb{C}}^{-1}(0)$ is χ_ν-semi-stable if, and only if:

$$\nu \in \sum_{\{i \,|\, z_i \neq 0\}} (\mathbb{R}_{\geq 0} \cdot \alpha_i) - \sum_{\{i \,|\, w_i \neq 0\}} (\mathbb{R}_{\geq 0} \cdot \alpha_i) \, ; \tag{1.30}$$

(ii) suppose that $(z, w) \in \phi_{\mathbb{C}}^{-1}(0)^{\chi_\nu - \mathrm{ss}}$. Then the $K_{\mathbb{C}}$-orbit through (z, w) is closed in $\phi_{\mathbb{C}}^{-1}(0)^{\chi_\nu - \mathrm{ss}}$ if, and only if:

$$\nu \in \sum_{\{i \,|\, z_i \neq 0\}} (\mathbb{R}_{> 0} \cdot \alpha_i) - \sum_{\{i \,|\, w_i \neq 0\}} (\mathbb{R}_{> 0} \cdot \alpha_i) \, . \tag{1.31}$$

Finally, we can state the definition of a hypertoric variety when it is defined using GIT.

Definition 1.16. Suppose that the integral weight $\nu \in \mathfrak{k}^*_{\mathbb{Z}}$ is generic, and let:

$$\mathcal{L}_\nu := \phi_{\mathbb{C}}^{-1}(0)^{\nu - \mathrm{ss}} \times_{K_{\mathbb{C}}} \mathbb{C}_{\chi_\nu} \to M_\nu$$

be the $K_{\mathbb{C}}$-linearised line bundle over $\phi_{\mathbb{C}}^{-1}(0)$. Then the **(algebro-geometric) hypertoric variety** M_ν is defined to be the projective quotient:

$$M_\nu = \phi_{\mathbb{C}}^{-1}(0)^{\nu - \mathrm{ss}} /\!\!/_{\chi_\nu} K_{\mathbb{C}} \cong \operatorname{Proj}\left(\bigoplus_{m \geq 0} H^0(\phi_{\mathbb{C}}^{-1}(0); \mathcal{L}_\chi^{\otimes m})^{K_{\mathbb{C}}} \right).$$

The last part of this section is to show that both of the two quotient constructions presented thus far, i.e., that of Definitions 1.4 and 1.16 for a hypertoric variety M_ν, coincide. To do so, following Konno [Kon08], let us first define a fibre-wise Hermitian metric on the holomorphic trivial line bundle $\mathcal{L} \to \phi_{\mathbb{C}}^{-1}(0)$ over the level-set $\phi_{\mathbb{C}}^{-1}(0)$, as:

$$\big\| \big((z, w), \zeta \big) \big\| := |\zeta| e^{-\frac{1}{2}\left(\|z\|^2 + \|w\|^2 \right)}. \tag{1.32}$$

This metric induces the Chern connection ∇ on the holomorphic trivial line bundle $\pi : \mathcal{L} \to \phi_{\mathbb{C}}^{-1}(0)$, whose first Chern form is $c_1(\nabla) = \omega_{\mathbb{R}}|_{\phi_{\mathbb{C}}^{-1}(0}$, where $\omega_{\mathbb{R}}$ is the real Kähler two-form from Example 1.1. The action of the subtorus $K_{\mathbb{C}}$ on $\mathcal{L}$ preserves this holomorphic structure, as well as its Hermitian metric (1.32) and hence its Chern connection, ∇.

The fundamental result that lets us identify the hypertoric variety M_ν presented as a hyperkähler quotient $T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_{(\nu,0)} K$ with that from the analogous GIT quotient $\phi_{\mathbb{C}}^{-1}(0) /\!\!/_{\chi_\nu} K_{\mathbb{C}}$ is the following fundamental result. Its proof requires that the $K_{\mathbb{C}}$-linearised line bundle to have a Hermitian structure defined on it, hence our preamble above.

Theorem 1.17 (Kempf-Ness-King). *Consider the restricted real moment map, $\phi_{\mathbb{R}} : \phi_{\mathbb{C}}^{-1}(0) \to \mathfrak{k}^*$. Then, the level-set $\phi_{\mathbb{R}}^{-1}(\nu) \subseteq \phi_{\mathbb{C}}^{-1}(0)$ meets each $K_{\mathbb{C}}$-orbit in precisely one K-orbit, and meets no other $K_{\mathbb{C}}$-orbit. Furthermore, each $K_{\mathbb{C}}$-orbit is closed in $\phi_{\mathbb{C}}^{-1}(0)^{\nu-\mathrm{ss}}$. In particular, the natural map $\phi_{\mathbb{R}}^{-1}(\nu)/K = \phi_{\mathrm{HK}}^{-1}(\nu, 0)/K \to \phi_{\mathbb{C}}^{-1}(0)^{\nu-\mathrm{ss}} /\!\!/_{\chi_\nu} K_{\mathbb{C}}$ is a bijection.*

The above theorem is actually a generalisation of the Kempf-Ness theorem by King [Kin94, Theorem 6.1 & Corollary 6.2], since the original Kempf-Ness theorem [KN79] only considers the case when $\nu = 0$.

Corollary 1.18. *The differential-geometric definition of a hypertoric variety in Definition 1.4, and the algebro-geometric definition of a hypertoric variety in Definition 1.16, coincide.*

To finish this section, suppose that M is a normal quasi-projective variety that can be described using the Proj construction in the form of $M \cong \operatorname{Proj} R$, where $R \cong \oplus_{j \in \mathbb{Z}_{\geq 0}} R_j$ is a $\mathbb{C}$-algebra that is finitely-generated as an R_0-algebra by R_1, and suppose that the canonical structure morphism $M \to M_0 \cong \operatorname{Spec} R_0$ is projective. In this case, then we say that M is **projective over the affine variety** M_0. Hypertoric varieties make up an example of a variety that is projective over an affine one, as a hypertoric variety M_ν is projective over its affinisation, M_0.

To be more precise, if the structure morphism

$$\pi_S : M = \operatorname{Proj} \bigoplus_{m=0}^{\infty} R_m \longrightarrow M_0 = \operatorname{Spec} R_0 \tag{1.33}$$

is projective, then we have the following diagram:

$$\begin{tikzcd} M_\nu \arrow[r, hook, "i"] \arrow[dr, "\pi_S"'] & M_0 \times \mathbb{P}(R_1^*) \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ & M_0 & \mathbb{P}(R_1^*) \end{tikzcd} \tag{1.34}$$

where $i : M_\nu \hookrightarrow M_0 \times \mathbb{P}(R_1^*)$ is a closed embedding, where $\mathrm{pr}_1 : M_0 \times \mathbb{P}(R_1^*) \to M_0$ and $\mathrm{pr}_2 : M_0 \times \mathbb{P}(R_1^*) \to \mathbb{P}(R_1^*)$ are the projections from $M_0 \times \mathbb{P}(R_1^*)$ onto its respective first, M_0, and second, $\mathbb{P}(R_1^*)$, factors, and where the structure morphism $\pi_S : M_\nu \to M_0$ factors as $\pi_S = \mathrm{pr}_1 \circ i$, [Har77, Chapter II.5].

Since i in (1.34) is an immersion, the variety M is isomorphic to a closed subscheme of $M_0 \times \mathbb{P}(R_1^*)$. It follows then, that there exists an ample line bundle $\mathcal{L} \to M$ over M which is said to be **very ample**

relative to M_0. It is obtained via $\mathcal{L}_\nu \cong i^*\mathcal{O}(1)$, where $\mathcal{O}(1) = \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}(R_1^*)}(1)$ is the pull-back of the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}(R_1^*)$, [Har77, Remark 5.16.1]. These properties are summarised in the following definition, first applied in [HS02, §1] to toric varieties that are projective over their affinisations, in addition to having at least one torus fixed point.

Definition 1.19. We say that a normal quasi-projective variety M is **semi-projective**, if it is projective over an affine variety M_0. That is to say, that there exists a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra $R = \oplus_{j \in \mathbb{Z}_{\geq 0}} R_j$, finitely-generated as an R_0-algebra by R_1, and such that the structure morphism $\pi_S : M \cong \operatorname{Proj} R \to M_0 := \operatorname{Spec} R_0$ is projective.

Examples of semi-projective varieties include hypertoric varieties [HS02] of course, but also quiver varieties [Rei03]. The GIT quotient construction, and more generally this algebraic approach, reveals several properties possessed by a hypertoric variety which were not immediately apparent from the hyperkähler quotient construction. For example, the following lemma is proven in [BK12, Lemmas 4.7 & 4.10] using the GIT method of constructing hypertoric varieties.

Lemma 1.20. *The moment map $\phi_{\mathbb{C}} : T^*\mathbb{C}^N \longrightarrow (\mathfrak{t}_{\mathbb{C}})^*$ is flat and $\phi_{\mathbb{C}}^{-1}(0)$ is a reduced complete intersection in $T^*\mathbb{C}^N$. Furthermore, if $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$ is generic, then M_ν is arithmetically Cohen-Macaulay.*

Chapter 2

Hyperplane Arrangements and Hypertoric Subvarieties

In [Del88], Delzant established a classification result which associates to each compact symplectic toric manifold a corresponding smooth closed convex polytope that equals the image of the manifold under the moment map. His result was further generalised by Lerman and Tolman in [LT97], who extended it to compact symplectic toric orbifolds by relating them to *simple* closed convex polytopes.

In this chapter, we wish to investigate an analogous phenomenon, which relates a hypertoric variety to a hyperplane arrangement. Geometric properties such as the smoothness of a hypertoric variety is then represented in the combinatorics of a hyperplane arrangement, and vice versa. Hyperplane arrangements will allow us to express the results regarding hypertoric varieties visually, and thus make for instrumental tools when proving results that concern hypertoric varieties.

2.1 Hyperplane Arrangements

The data used to construct a hypertoric variety M_ν can be compactly encoded within a hyperplane arrangement in the dual space $(\mathfrak{t}^n)^*$, where $(\mathfrak{t}^n)^*$ is the Lie algebra of the residual torus T^n acting on M_ν. Consider an integral vector $u \in \mathfrak{t}^n_{\mathbb{Z}}$, along with an element $\lambda \in (\mathfrak{t}^N)^*$. From this data, a **hyperplane** $H \subset (\mathfrak{t}^n)^*$ can be expressed as:

$$H = \{\, x \in (\mathfrak{t}^n)^* \mid \langle x,\, u \rangle + \lambda_H = 0 \,\} . \tag{2.1}$$

Thus, the integral vector u corresponds to the normal vector of H, whereas λ_H determines the position of H in the vector space $(\mathfrak{t}^n)^*$. We say that a hyperplane H is **weighted** if u is not a primitive vector in $\mathfrak{t}^n_{\mathbb{Z}}$, i.e., that for any other vector $v \in \mathfrak{t}^n_{\mathbb{Z}}$ such that $u = kv$ for some $k \in \mathbb{Z}_{\geq 0}$, then we must necessarily have that $k = 1$. We also say that the hyperplane H is **affine** if it does not pass through the origin in $\mathfrak{t}^n$.

Going one step further, given a set $\{u_1, \dots, u_N\}$ of vectors in $\mathfrak{t}^n$, we say that an **arrangement of hyperplanes**, or a **hyperplane arrangement**, is a set $\mathcal{A} := \{H_1, \dots, H_N\}$ of hyperplanes $H_1, \dots, H_N$ in $(\mathfrak{t}^n)^*$, each being of the form (2.1). We say that a hyperplane arrangement $\mathcal{A}$ is **simple** if, for every non-empty intersection of hyperplanes $\cap_{j=1}^k H_{i_j} \neq \emptyset$, the set of normal vectors $\{u_{i_j}\}_{j=1}^k$ is linearly independent. Furthermore, we say that a hyperplane arrangement $\mathcal{A}$ is **smooth**, if each set $\{u_{i_j}\}_{j=1}^k$ as above additionally forms a $\mathbb{Z}$-basis for $\mathfrak{t}^n_{\mathbb{Z}}$. In the sequel, we shall assume that each hyperplane H_i in a hyperplane arrangement $\mathcal{A}$ is distinct.

As remarked in the paragraph after Lemma 1.13, the element $\lambda \in (\mathfrak{t}^N)^*$ is a lift of the element $\nu \in \mathfrak{k}^*$ along the projection $\imath^*$, i.e., $\lambda = \lambda + \mu = \imath^*\nu$ for any $\mu \in \ker \imath^*$. Hence, the map $\lambda \mapsto \lambda + \mu$ corresponds to translating each hyperplane H_i in the hyperplane arrangement $\mathcal{A}$ by the vector μ, provided that $\mu \in \ker \imath^*$.

Each hyperplane H_i determines the two following half-spaces in $(\mathfrak{t}^n)^*$:

$$H_i^+ = \{ \, x \in (\mathfrak{t}^n)^* \mid \langle x, \, u_i \rangle + \lambda_i \geq 0 \, \} \, , \quad H_i^- = \{ \, x \in (\mathfrak{t}^n)^* \mid \langle x, \, u_i \rangle + \lambda_i \leq 0 \, \} \, , \tag{2.2}$$

so that $H_i = H_i^+ \cap H_i^-$. An arrangement $\mathcal{A}$ divides $(\mathfrak{t}^n)^*$ into a finite family of simple closed and convex polyhedra, not necessarily bounded, which we call the **regions** of $\mathcal{A}$. Each region of $\mathcal{A}$ can be expressed as a finite intersection of the half-spaces:

$$\Delta_A := \left(\cap_{i \notin A} H_i^+\right) \cap \left(\cap_{i \in A} H_i^-\right) , \tag{2.3}$$

which we index by subsets $A \subseteq \{1, \dots, N\}$. We denote by $\mathcal{R}(\mathcal{A}) = \{ \, \Delta_A \mid A \subseteq \{1, \dots, N\} \, \}$ the **set of regions** of an arrangement $\mathcal{A}$, and say that $\Delta_\emptyset$ is the **base**, or the **distinguished**, **region** of $\mathcal{A}$.

Remark 2.1. The set of regions $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ can be further equipped with a partial order $\preceq$, defined by the relation:

$$\Delta_A \preceq \Delta_B, \quad \text{if and only if} \quad A \subseteq B.$$

This makes $\mathcal{R}(\mathcal{A})$ into a poset $\mathcal{P}(\mathcal{A}) := (\mathcal{R}(\mathcal{A}), \preceq)$, called the **poset of regions** of the arrangement $\mathcal{A}$, [Ede84]. Implicit in this construction is that $\mathcal{P}(\mathcal{A})$ depends on both the arrangement $\mathcal{A}$ and on the choice of base region $\Delta_\emptyset$, since the partial order records how many hyperplanes separate the region Δ_A from $\Delta_\emptyset$. Different choices of $\Delta_\emptyset$ for the same arrangement $\mathcal{A}$ gives rise to non-isomorphic posets $\mathcal{P}(\mathcal{A})$, since $\Delta_\emptyset$ is the unique **infimum** or **meet** of the poset $\mathcal{P}(\mathcal{A})$. The choice of $\Delta_\emptyset$ is equivalent to a choice of coorientation for $\mathcal{A}$.

2.2 The Core and Extended Core of a Hypertoric Variety

Following [HP04], we define the **extended core** $\mathcal{E}$ of M_ν to be the zero level-set of the complex moment map $\mu_{\mathbb{C}}$:

$$\mathcal{E} := \mu_{\mathbb{C}}^{-1}(0) = \{ \, [z, w] \in M_\nu \mid z_i w_i = 0 \text{ for each } i \, \} \, , \tag{2.4}$$

which is a $2n$-dimensional reducible subvariety of M_ν. For each subset $A \subseteq \{1, \dots, N\}$, the subvariety:

$$\mathcal{E}_A = \mu_{\mathbb{R}}^{-1}(\Delta_A) \cap \mathcal{E}$$

is then an irreducible component of the extended core $\mathcal{E}$, and one should observe that $\mathcal{E} = \cup_A \mathcal{E}_A$.

The following lemma expresses how the image $\mu_{\mathbb{R}}(\mathcal{E})$ is partitioned by a hyperplane arrangement into the regions Δ_A, and was proven in [HP04, Lemma 3.1]:

Lemma 2.2. *Consider a point $[z, w] \in \mathcal{E} \subseteq M_\nu$. Then, if $w_i = 0$ then $\mu_{\mathbb{R}}[z, w] \in H_i^+$, whereas if $z_i = 0$ then $\mu_{\mathbb{R}}[z, w] \in H_i^-$. Then:*

$$\mathcal{E}_A \cong \{ \, [z, w] \in M_\nu \mid w_i = 0 \text{ if } i \notin A \text{ and } z_i = 0 \text{ if } i \in A \, \}.$$

For the set $I = \{ \, A \subseteq \{1, \dots, N\} \mid \Delta_A \text{ is bounded} \, \}$, we define the **core** of M_ν to be:

$$\mathcal{C} = \cup_{A \in I} \mathcal{E}_A, \tag{2.5}$$

and observe that $\mathcal{C} \subset \mathcal{E}$. We say that the core $\mathcal{C}$ is **reducible** if there exist at least two distinct and proper components $\mathcal{E}_A, \mathcal{E}_B \subsetneq \mathcal{C}$, such that $\mathcal{C} = \mathcal{E}_A \cup \mathcal{E}_B$. Otherwise, we say that the core $\mathcal{C}$ is **irreducible**. Each core component $\mathcal{E}_A \subseteq \mathcal{C}$ is a $2n$-dimensional $\omega_{\mathbb{C}}$-Lagrangian subvariety of M_ν, and can be identified with the $\omega_{\mathbb{R}}$-Kähler toric variety corresponding to the bounded polytope Δ_A. In particular, the variety $M_\emptyset$ corresponds to the Kähler quotient of the zero-section $\mathbb{C}^N \subset T^*\mathbb{C}^N$, since then $X_\nu := M_\emptyset = \mathbb{C}^N /\!\!/_\nu K$ with $T^*X_\nu \subseteq M_\nu$ as an open subset [BD00].

Remark 2.3. As mentioned in Remark 2.1, different coorientations of $\mathcal{A}$ give rise to non-isomorphic posets of regions $\mathcal{P}(\mathcal{A})$. Therefore if the core $\mathcal{C} \subset M_\nu$ is reducible, a different coorientation of $\mathcal{A}$ will result in a different base region $\Delta_\emptyset$ of the same arrangement. The Kähler toric variety $\mathcal{E}_\emptyset$ is independent of the coorientation [HP04, Lemma 2.2], but the Kähler quotient $X_\nu = \mathcal{E}_\emptyset$ that M_ν is the hyperkähler analogue to, *does depend* on the coorientation of $\mathcal{A}$. To emphasise this nuanced behaviour, in [Pro04; HP04], Proudfoot and Harada refer to hypertoric varieties of this form as **hyperkähler analogues of a given presentation** of the Kähler quotient, which reflects the dependency on the poset of regions $\mathcal{P}(\mathcal{A})$.

2.3 Examples

Here, we introduce a few examples that we will develop upon further during the course of this thesis.

Example 2.4. Let $u_1 = \epsilon_1$ and $u_2 = -\epsilon_1$, so then $K \cong \{(t, t) \in T^2 \mid t \in U_1\} \cong U_1$. Choose $\lambda = (\lambda_1, \lambda_2) \in (\mathfrak{t}^2)^*$ with $\lambda_1 < \lambda_2$ and $\imath^*(\lambda) = \lambda_1 + \lambda_2 = \nu$, then the Kähler quotient is $X_\nu \cong \mathbb{CP}^1$, and whose hyperkähler analogue is $M_\nu \cong T^*\mathbb{CP}^1$. The hyperplane arrangement $\mathcal{A}$ for the hypertoric variety $M_\nu = T^*\mathbb{CP}^1$ is displayed in Figure 2.1.

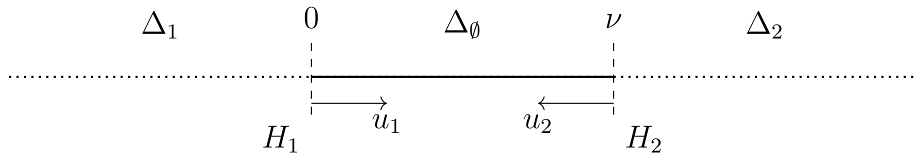


Figure 2.1: Hyperplane arrangement when $M_\nu = T^*\mathbb{CP}^1$ is the hyperkähler analogue to $X_\nu = \mathbb{CP}^1$.

Example 2.5 ([BD00]). Let $u_1 = \epsilon_1$ and $u_2 = u_3 = -\epsilon_1$, so $K \cong \{(t_1t_2, t_1, t_2) \mid (t_1, t_2) \in T^2\}$. Choose $\lambda = (0, \lambda_2, \lambda_3) \in (\mathfrak{t}^3)^*$, with $0 < \lambda_2 < \lambda_3$. Then the Kähler quotient is the resolution of the Kleinian singularity, $X_\nu = \mathbb{C}^2/\mathbb{Z}^3$, and M_ν is a hyperkähler analogue of it. Its hyperplane arrangement $\mathcal{A}$ is displayed in Figure 2.2.

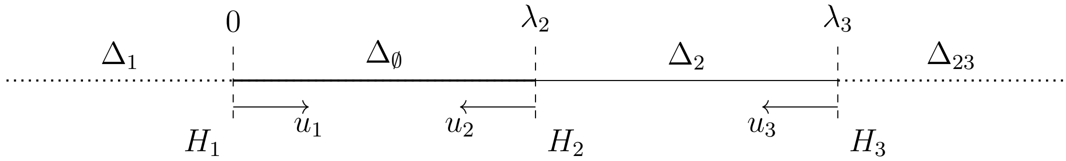


Figure 2.2: Hyperplane arrangement when M_ν is the hyperkähler analogue of the Kleinian singularity resolution, $X_\nu = \mathbb{C}^2/\mathbb{Z}_3$.

Example 2.6. Generalising Example 2.4 to the case when $N = n+1$, we let $u_i = e_i$ for $1 \leq i \leq n$, and $u_{n+1} = -e_1 - \ldots - e_n$. Then $K \cong \{\ (t, t, \ldots, t, t) \in T^{n+1} \mid t \in U_1\ \} \cong U_1$ is the diagonal circle subgroup in T^N. Choose $\lambda \in (\mathfrak{t}^N)^* \setminus \{0\}$ such that $\imath^*(\lambda) = \nu$ for some regular value $\nu \in \mathfrak{k}^*$. Analogously to Example 2.4, the Kähler quotient is $X_\nu \cong \mathbb{CP}^n$ and the hyperkähler quotient is $M_\nu \cong T^*\mathbb{CP}^n$. The hyperplane arrangement in the $N = 3$ case (i.e., when $n = 2$) is displayed in Figure 2.3.

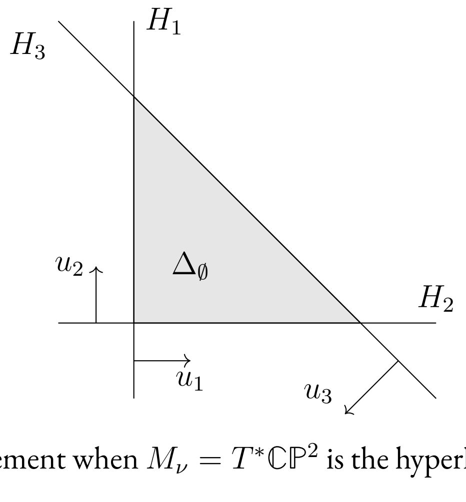


Figure 2.3: Hyperplane arrangement when $M_\nu = T^*\mathbb{CP}^2$ is the hyperkähler analogue to $X_\nu = \mathbb{CP}^2$.

Example 2.7. Now let $N = 4$ and $n = 2$, and consider $u_1 = -u_3 = e_1$, and $u_2 = -u_4 = e_2$. Then $K \cong \{ (t_1, t_2, t_1, t_2) \mid (t_1, t_2) \in T^2 \} \cong T^2$. Choose $\lambda = (0, 0, \lambda_3, \lambda_4) \in (\mathfrak{t}^4)^*$ with $0 < \lambda_3$ and $0 < \lambda_4$, such that $\imath^*(\lambda) = \nu \in \mathfrak{k}^*$. Then the Kähler quotient is $X_\nu \cong \mathbb{CP}^1 \times \mathbb{CP}^1$, and its hyperkähler analogue is $M_\nu \cong T^*(\mathbb{CP}^1 \times \mathbb{C}^1)$. The hyperplane arrangement $\mathcal{A}$ for $M_\nu = T^*(\mathbb{CP}^1 \times \mathbb{C}^1)$ is presented in Figure 2.4.

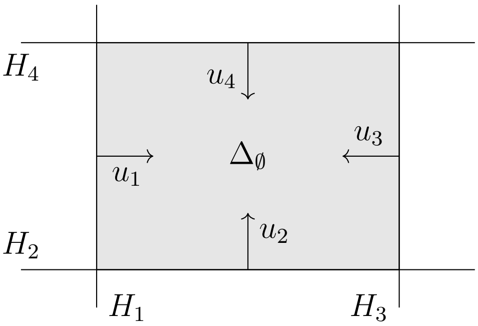
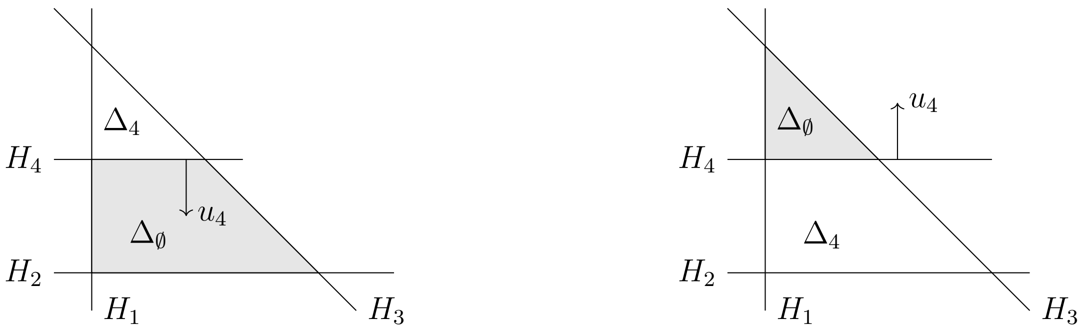


Figure 2.4: Hyperplane arrangement when $M_\nu = T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is the hyperkähler analogue to $X_\nu = \mathbb{CP}^1 \times \mathbb{CP}^1$.

Example 2.8. When $N = 4$ and $n = 2$, set $u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2$ and $u_4 = -u_2 = -e_2$. Choose $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathfrak{t}^4)^*$ with $\lambda_1 < \lambda_3$ and $\lambda_2 < \lambda_4 < \lambda_3$, where $\imath^*(\lambda) = \nu \in \mathfrak{k}^*$. Then $K \cong \{ (t_1, t_1 t_2, t_1, t_2) \mid (t_1, t_2) \in T^2 \} \cong T^2$. Then the Kähler quotient is the first Hirzebruch surface $X_\nu = \mathcal{H}_1$, with M_ν its hyperkähler analogue. The hyperplane arrangement is presented in Figure 2.5a. However, if we now let invert the sign of the normal vector, $u_4 \mapsto e_2 = u_2$, but keep everything else the same, then $K \cong \left\{ (t_1, t_1 t_2, t_1, t_2^{-1}) \;\middle|\; (t_1, t_2) \in T^2 \right\} \cong T^2$. Now the Kähler quotient is the complex projective plane $X_\nu = \mathbb{CP}^2$, with M_ν its hyperkähler analogue whose hyperplane arrangement is presented in Figure 2.5b.



(a) Arrangement when $X_\nu \cong \mathcal{H}_1$.

(b) Arrangement when $X_\nu \cong \mathbb{CP}^1$.

Figure 2.5: Different coorientations for the same hyperplane arrangement give rise to distinct hyperkähler analogues.

So, just by swapping $u_4 \mapsto -u_4$, we have obtained two different hypertoric varieties from the same arrangement $\mathcal{A}$, albeit now with a different coorientation. Both are T^2-equivariantly diffeomorphic however, see [HP04, Lemma 2.2].

2.4 Flats of an Arrangement

In Chapter 6, we will encounter several issues for which a direct approach would be difficult and tedious. In low dimensions however, the solutions to these problems appear suddenly to be obvious and essentially look as if they are trivial. Therefore, if we were able to reduce a complicated high-dimensional problem to a lower-dimensional one whose solution we can brute-force our way to, then the high-dimensional case is also dealt with via an inductive argument. To convert these problems from a high-dimensional setting to a low-dimensional one, we will devote the rest of this chapter to introducing *hypertoric subvarieties*, in addition to studying their properties.

Thus, let $\mathcal{A} = \{H_1, \ldots, H_N\}$ be a simple hyperplane arrangement in $(\mathfrak{t}^n)^*$ whose corresponding set of normal vectors is $\{u_1, \ldots, u_N\}$, where $u_j \in \mathfrak{t}^n_{\mathbb{Z}}$ for each $j = 1, \ldots, N$.

Definition 2.9. Given a subset $\mathcal{F} \subseteq \{1, \ldots, N\}$, we say that the intersection $H_{\mathcal{F}} = \cap_{i \in \mathcal{F}} H_i$ of hyperplanes, whose indices are the elements of $\mathcal{F}$, is a **flat** of the hyperplane arrangement $\mathcal{A}$, provided that $H_{\mathcal{F}}$ is non-empty. For convenience, let us refer to the subset $\mathcal{F} \subseteq \{1, \ldots, N\}$ as the **subset of the flat** $H_{\mathcal{F}}$, or the **flat subset**.

Definition 2.10. Denote by $L(\mathcal{A})$ the set of all flats $H_{\mathcal{F}}$ of $\mathcal{A}$, along with $(\mathfrak{t}^n)^*$ which we consider to be the trivial flat. Equip $L(\mathcal{A})$ with the partial order given by the reverse inclusion of subsets:

$$H_{\mathcal{F}} \leq H_{\mathcal{G}}, \quad \text{for each} \quad H_{\mathcal{F}}, H_{\mathcal{G}} \in L(\mathcal{A}) \qquad \text{if, and only if,} \qquad H_{\mathcal{G}} \subseteq H_{\mathcal{F}}. \tag{2.6}$$

Then we say that $L(\mathcal{A})$ is the **intersection poset** of the hyperplane arrangement $\mathcal{A}$.

Given a flat $H_{\mathcal{F}} \in L(\mathcal{A})$, we define the **arrangement under (the flat)** $H_{\mathcal{F}}$, the **restricted arrangement**, or just the **restriction**, to be the hyperplane arrangement:

$$\mathcal{A}^{\mathcal{F}} := \{\, H_i \cap H_{\mathcal{F}} \mid i \notin \mathcal{F} \,\}\,.$$

Intuitively, if $\mathcal{A}$ is a simple hyperplane arrangement, then the restricted arrangement $\mathcal{A}^{\mathcal{F}}$ is made up of the intersections of the $|\mathcal{F}^c| = N - |\mathcal{F}|$ hyperplanes whose indices do not belong to the flat subset $\mathcal{F}$ with the flat $H_{\mathcal{F}}$ itself.

For a flat $H_{\mathcal{F}} \in L(\mathcal{A})$, let us define the following $\mathbb{R}$-vector spaces:

$$\langle \mathcal{F} \rangle := \bigoplus_{i \in \mathcal{F}} \mathbb{R} u_i, \quad \text{and} \quad \langle \mathcal{F} \rangle^{\perp} := \mathfrak{t}^n / \langle \mathcal{F} \rangle, \tag{2.7}$$

which one can regard as being subspaces of $\mathfrak{t}^n$. Then we define the **rank** and the **corank** of the flat $H_{\mathcal{F}}$ to be:

$$\operatorname{rk} H_{\mathcal{F}} := |\mathcal{F}| = \dim_{\mathbb{R}}\langle \mathcal{F}\rangle, \quad \text{and} \quad \operatorname{crk} H_{\mathcal{F}} := n - |\mathcal{F}| = \dim_{\mathbb{R}}\langle \mathcal{F}\rangle^{\perp},$$

respectively. That is to say, the dimension and the codimension of the subspace $\langle \mathcal{F}\rangle$ from (2.7) in $\mathfrak{t}^n$, respectively.

Now let us define the lattice:

$$U_{\mathcal{F}} := \bigoplus_{i\in\mathcal{F}} \mathbb{Z}u_i \subsetneq \langle \mathcal{F}\rangle, \tag{2.8}$$

which has rank $\operatorname{rk} U_{\mathcal{F}} = |\mathcal{F}|$. Then we see that $U_{\mathcal{F}}$ is a sublattice of $\mathfrak{t}^n_{\mathbb{Z}}$ which is not necessarily saturated. To $U_{\mathcal{F}}$ we may associate another lattice:

$$V_{\mathcal{F}} := \langle \mathcal{F}\rangle \cap \mathfrak{t}^n_{\mathbb{Z}}, \tag{2.9}$$

of $\langle \mathcal{F}\rangle$ with rank $\operatorname{rk} V_{\mathcal{F}} = |\mathcal{F}|$ as well.

The sublattice $V_{\mathcal{F}}$ of $\mathfrak{t}^n_{\mathbb{Z}}$ is necessarily saturated by construction, and is a superlattice of $U_{\mathcal{F}}$. They are distinct if $U_{\mathcal{F}}$ is an unsaturated sublattice in $\mathfrak{t}^n_{\mathbb{Z}}$, and their quotient is the finite abelian group:

$$\Gamma_{\mathcal{F}} := V_{\mathcal{F}}/U_{\mathcal{F}}, \tag{2.10}$$

whose order is the index $|\Gamma_{\mathcal{F}}| = [V_{\mathcal{F}} : U_{\mathcal{F}}]$.

The quotients of $\langle \mathcal{F}\rangle$ by the lattices, $U_{\mathcal{F}}$ and $V_{\mathcal{F}}$, are the $|\mathcal{F}|$-dimensional real tori:

$$T_U^{\operatorname{rk}\mathcal{F}} := \langle \mathcal{F}\rangle/U_{\mathcal{F}}, \quad \text{and} \quad T_V^{\operatorname{rk}\mathcal{F}} := \langle \mathcal{F}\rangle/V_{\mathcal{F}}, \tag{2.11}$$

and, by using $T_V^{\operatorname{rk}\mathcal{F}}$, we may also define the quotient $(\operatorname{crk}\mathcal{F})$-dimensional real torus:

$$T_V^{\operatorname{crk}\mathcal{F}} := T^n/T_V^{\operatorname{rk}\mathcal{F}} \cong \left(\mathfrak{t}^n/\mathfrak{t}^n_{\mathbb{Z}}\right)/\left(\langle \mathcal{F}\rangle/V_{\mathcal{F}}\right) \cong \langle \mathcal{F}\rangle^{\perp}/V_{\mathcal{F}}^{\perp}. \tag{2.12}$$

Here, we have used subscripts to keep track of which lattice has been used to define each torus respectively.

Proposition 2.11. *Let $\mathcal{A} = \{H_1, \dots, H_N\}$ be a simple hyperplane arrangement in $(\mathfrak{t}^n)^*$, and let $H_{\mathcal{F}} \in L(\mathcal{A})$ be a flat of $\mathcal{A}$ for a given subset $\mathcal{F} \subseteq \{1, \dots, N\}$. Then:*

$$T_V^{\operatorname{rk}\mathcal{F}} \cong T_U^{\operatorname{rk}\mathcal{F}}/\Gamma_{\mathcal{F}},$$

and moreover there exists a non-canonical decomposition of the n-dimensional real torus T^n:

$$T^n \cong T_V^{\operatorname{rk}\mathcal{F}} \times T_V^{\operatorname{crk}\mathcal{F}} \cong \left(T_U^{\operatorname{rk}\mathcal{F}}/\Gamma_{\mathcal{F}}\right) \times T_V^{\operatorname{crk}\mathcal{F}},$$

where $T_V^{\operatorname{crk}\mathcal{F}}$ is defined in (2.12).

Proof. Since $V_{\mathcal{F}}$ is a saturated sublattice of $\mathfrak{t}^n_{\mathbb{Z}}$, the quotient $\mathfrak{t}^n_{\mathbb{Z}}/V_{\mathcal{F}}$ is torsion-free. Hence there exists a complementary sublattice $V_{\mathcal{F}}^{\perp}$ of $V_{\mathcal{F}}$ in $\mathfrak{t}^n_{\mathbb{Z}}$, and a non-canonical splitting [CLS11b, Exercise 1.3.5]:

$$\mathfrak{t}^n_{\mathbb{Z}} \cong V_{\mathcal{F}} \oplus V_{\mathcal{F}}^{\perp}. \tag{2.13}$$

The inclusion $U_{\mathcal{F}} \hookrightarrow V_{\mathcal{F}}$ induces the dualised short exact sequence:

$$\{0\} \longrightarrow V_{\mathcal{F}}^* \longrightarrow U_{\mathcal{F}}^* \longrightarrow \Gamma_{\mathcal{F}}^* \cong U_{\mathcal{F}}^*/V_{\mathcal{F}}^* \longrightarrow \{0\}.$$

Applying the contravariant and left-exact functor $\mathrm{Hom}_{\mathbb{Z}}(-;U_1)$ to this sequence, and by noting that U_1 is a divisible group [CLS11a, Proposition 1.3.18], we obtain the following short exact sequence:

$$\begin{tikzcd}
\{0\} \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\Gamma_{\mathcal{F}}^*;U_1) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_{\mathbb{Z}}(U_{\mathcal{F}}^*;U_1) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_{\mathbb{Z}}(V_{\mathcal{F}}^*;U_1) \arrow[r] \arrow[d, phantom, "\wr\|"] & \{0\}, \\
\{0\} \arrow[r] & \Gamma_{\mathcal{F}} \arrow[r] & T_U^{\mathrm{rk}\,\mathcal{F}} \arrow[r] & T_V^{\mathrm{rk}\,\mathcal{F}} \arrow[r] & \{0\}.
\end{tikzcd}$$

This implies that $T_V^{\mathrm{rk}\,\mathcal{F}} \cong T_U^{\mathrm{rk}\,\mathcal{F}}/\Gamma_{\mathcal{F}}$. This result, along with the splitting (2.13), then provides the desired splitting of tori:

$$T^n \cong (\mathfrak{t}^{\mathrm{rk}\,\mathcal{F}}/V_{\mathcal{F}}) \oplus (\mathfrak{t}^{\mathrm{crk}\,\mathcal{F}}/V_{\mathcal{F}}^{\perp}) \cong T_V^{\mathrm{rk}} \times T_V^{\mathrm{crk}\,\mathcal{F}} \cong \left(T_U^{\mathrm{rk}\,\mathcal{F}}/\Gamma_{\mathcal{F}}\right) \times T_V^{\mathrm{crk}\,\mathcal{F}} \tag{2.14}$$

for us. □

2.5 Hypertoric Subvarieties

The aim of this section is to formalise the notion of a *hypertoric subvariety*, the reason for this being that we will use hypertoric subvarieties in the proofs of Theorem 3.17, Lemma 6.1, Theorem 6.2, and Proposition 6.3

Despite the term *"hypertoric subvariety"* having been used in the literature previously, see [GH08] and [RSZV22], a formal definition has not yet been proposed – so let us start by doing so.

Definition 2.12. Let (M, ω_i^M, T^m) and (N, ω_i^N, T^n) be hypertoric varieties, where the tori T^m and T^n act on M and N in an effective and hyperhamiltonian fashion. Furthermore, suppose that N is a *hyperkähler* subvariety of M. Then we say that N is a **hypertoric subvariety** of M if there exists a T^n-equivariant embedding:

$$\iota : N \hookrightarrow M, \quad \text{such that} \quad \iota^*(\omega_i^M) = \omega_i^N, \quad \text{for each } i = 1,2,3.$$

The type of hypertoric subvariety that we shall be interested in are those which will correspond to the *flats* of the hyperplane arrangement.

So, let $\mathcal{A} = \{H_1, \dots, H_N\}$ be a simple hyperplane arrangement in $(\mathfrak{t}^n)^*$, and let $H_\mathcal{F} \in L(\mathcal{A})$ be a flat of $\mathcal{A}$ for a given subset $\mathcal{F}1, \dots, N$. Denote by $\mathbb{C}^{\mathcal{F}^c}$ the coordinate subspace of $\mathbb{C}^N$ supported on the flat subset $\mathcal{F}$, that is:

$$\mathbb{C}^{\mathcal{F}^c} := \left\{ \, z \in \mathbb{C}^N \;\middle|\; \text{if } i \in \mathcal{F}, \text{ then } z_i = 0 \, \right\},$$

and let $T^*\mathbb{C}^{\mathcal{F}^c}$ be the cotangent space to $\mathbb{C}^{\mathcal{F}^c}$ which, as a subspace of $T^*\mathbb{C}^N$, can be expressed as:

$$T^*\mathbb{C}^{\mathcal{F}^c} := \left\{ \, (z, w) \in T^*\mathbb{C}^N \;\middle|\; \text{if } i \in \mathcal{F}, \text{ then } z_i = w_i = 0 \, \right\}.$$

Analogously, denote by $\mathfrak{t}^\mathcal{F}$ and $\mathfrak{t}^{\mathcal{F}^c}$ the subspaces of $\mathfrak{t}^N$ that are supported on the flat subsets $\mathcal{F}$ and $\mathcal{F}^c$ respectively. That is to say:

$$\begin{aligned} \mathfrak{t}^\mathcal{F} &:= \left\{ \, x \in \mathfrak{t}^N \;\middle|\; \text{if } i \notin \mathcal{F}, \text{ then } x_i = 0 \, \right\}; \\ \mathfrak{t}^{\mathcal{F}^c} &:= \left\{ \, x \in \mathfrak{t}^N \;\middle|\; \text{if } i \in \mathcal{F}, \text{ then } x_i = 0 \, \right\}. \end{aligned} \tag{2.15}$$

Let us introduce the projection map:

$$p : \mathfrak{t}^N \to \mathfrak{t}^{\mathcal{F}^c}, \quad p(e_i) := \begin{cases} e_i, & \text{if } i \in \mathcal{F}^c, \\ 0, & \text{if } i \in \mathcal{F}, \end{cases} \tag{2.16}$$

that projects from $\mathfrak{t}^N$ onto the subspace $\mathfrak{t}^{\mathcal{F}^c}$. Furthermore, let us denote the image of the restriction of p to $\mathfrak{k} \subsetneq \mathfrak{t}^N$ by:

$$\mathfrak{k}^{\mathcal{F}^c} := p(\mathfrak{k}). \tag{2.17}$$

As shown in [Konoo, Section 7], since $\pi(e_i) \neq 0$ for each $i \in \mathcal{F}$, the restriction $p|_\mathfrak{k} : \mathfrak{k} \to \mathfrak{k}^{\mathcal{F}^c}$ is an isomorphism. We therefore obtain the following diagram:

$$\begin{tikzcd} \{0\} \arrow[r] & \mathfrak{k} \arrow[r, hook, "\imath"] \arrow[d, "p|_\mathfrak{k}", "\wr"'] & \mathfrak{t}^N \arrow[r, "\pi"] \arrow[d, "p"] & \mathfrak{t}^n \arrow[r] \arrow[d, "\bar{p}"] & \{0\}, \\ \{0\} \arrow[r] & \mathfrak{k}^{\mathcal{F}^c} \arrow[r, hook, "\imath_{\mathcal{F}^c}"] & \mathfrak{t}^{\mathcal{F}^c} \arrow[r, "\pi_{\mathcal{F}^c}"] & \mathfrak{t}^{\operatorname{crk} \mathcal{F}} \arrow[r] & \{0\}. \end{tikzcd} \tag{2.18}$$

We may then dualise the diagram (2.18) to obtain:

$$\begin{tikzcd} \{0\} & \mathfrak{k}^* \arrow[l] & (\mathfrak{t}^N)^* \arrow[l, "\imath^*"'] & (\mathfrak{t}^n)^* \arrow[l, "\pi^*"'] & \{0\}, \arrow[l] \\ \{0\} & (\mathfrak{k}^{\mathcal{F}^c})^* \arrow[l] \arrow[u, "(p|_\mathfrak{k})^*"', "\wr"] & (\mathfrak{t}^{\mathcal{F}^c})^* \arrow[l, "\imath^*_{\mathcal{F}^c}"'] \arrow[u, "p^*"'] & (\mathfrak{t}^{\operatorname{crk} \mathcal{F}})^* \arrow[l, "\pi^*_{\mathcal{F}^c}"'] \arrow[u, "\bar{p}^*"'] & \{0\}. \arrow[l] \end{tikzcd} \tag{2.19}$$

Finally, using the coordinate subspaces introduced in (2.15), we obtain the following tori:

$$\begin{aligned} T^{\mathcal{F}} &= \left\{ t \in T^N \;\middle|\; \text{if } i \notin \mathcal{F} \text{ then } t_i = 1 \right\}, \\ T^{\mathcal{F}^c} &= \left\{ t \in T^N \;\middle|\; \text{if } i \in \mathcal{F} \text{ then } t_i = 1 \right\}, \end{aligned} \tag{2.20}$$

which are subtori of T^N since coordinate-wise, we have that $T^{\mathcal{F}^c} \cong T^N/T^{\mathcal{F}}$ and that $T^{\mathcal{F}} \cong T^N/T^{\mathcal{F}^c}$. Their images in T^n under π are denoted respectively by $T^{\operatorname{rk}\mathcal{F}}$ and $T^{\operatorname{crk}\mathcal{F}}$, with $T^{\operatorname{crk}\mathcal{F}} \cong T^n/T^{\operatorname{rk}\mathcal{F}}$ and similarly with $T^{\operatorname{rk}\mathcal{F}} \cong T^n/T^{\operatorname{crk}\mathcal{F}}$.

Now, if we assume that the simple hyperplane arrangement $\mathcal{A}$ corresponds to a hypertoric variety M_ν, then the next theorem shows us that each flat $H_{\mathcal{F}} \in L(\mathcal{A})$, with $\mathcal{F} \subseteq \{1, \dots, N\}$, will determine a respective hypertoric subvariety that we will denote by $M_\nu^{\mathcal{F}}$. We also will see that the hypertoric subvariety $M_{\mathcal{F}}$ is cut out from the hypertoric variety M_ν, by the equation $z_i = w_i = 0$ for which $i \in \mathcal{F}$.

Theorem 2.13. *Let M_ν be a $4n$-dimensional hypertoric variety with the corresponding simple hyperplane arrangement $\mathcal{A}$ in $(\mathfrak{t}^n)^*$. Let $H_{\mathcal{F}} \in L(\mathcal{A})$ be a flat of $\mathcal{A}$ for some flat subset $\mathcal{F} \subseteq \{1, \dots, N\}$, and let $\mathcal{A}^{\mathcal{F}}$ be its restricted arrangement in $H_{\mathcal{F}}$. Then there exists a unique element $\nu_{\mathcal{F}} \in \mathfrak{k}^{\mathcal{F}^c}$ such that $\nu = (p|_{\mathfrak{k}})^*(\nu_{\mathcal{F}})$, for which the hypertoric variety $M_{\mathcal{F}}$ is a $(\operatorname{rk}\mathcal{F})$-codimensional hypertoric subvariety of M_ν.*

Proof. As $\mathfrak{k} \cong \mathfrak{k}^{\mathcal{F}^c}$, from the diagram in (2.19), there exists a unique $\nu_{\mathcal{F}} \in (\mathfrak{k}^{\mathcal{F}^c})^*$ such that $\nu = (p|_{\mathfrak{k}})^*(\nu_{\mathcal{F}})$. The torus $T^{\mathcal{F}^c}$ acts on $T^*\mathbb{C}^{\mathcal{F}^c}$ in a hyperhamiltonian way with hyperkähler moment map:

$$\phi_{\mathrm{HK}}^{\mathcal{F}^c} : T^*\mathbb{C}^{\mathcal{F}^c} \longrightarrow (\mathfrak{t}^{\mathcal{F}^c})^* \otimes \operatorname{Im}(\mathbb{H}), \tag{2.21}$$

and the subtorus $K^{\mathcal{F}^c}$ acts on $T^*\mathbb{C}^{\mathcal{F}^c}$ via the inclusion $\imath_{\mathcal{F}^c} : K^{\mathcal{F}^c} \hookrightarrow T^{\mathcal{F}^c}$. This $K^{\mathcal{F}^c}$-action is also hyperhamiltonian with hyperkähler moment map:

$$\mu_{\mathrm{HK}}^{\mathcal{F}^c} = (\imath_{\mathcal{F}^c}^* \otimes \operatorname{Id}) \circ \phi_{\mathrm{HK}}^{\mathcal{F}^c} : T^*\mathbb{C}^{\mathcal{F}^c} \longrightarrow (\mathfrak{k}^{\mathcal{F}^c})^* \otimes \operatorname{Im}(\mathbb{H}). \tag{2.22}$$

Now let:

$$\iota : T^*\mathbb{C}^{\mathcal{F}^c} \hookrightarrow T^*\mathbb{C}^N, \quad \text{where} \quad \iota : (z_i, w_i) \mapsto \begin{cases} (0,0), & \text{if } i \in \mathcal{F}, \\ (z_i, w_i), & \text{if } i \in \mathcal{F}^c, \end{cases} \tag{2.23}$$

define a closed embedding. Then the maps (2.21), (2.22), and (2.23) fit into the following diagram:

$$
\begin{tikzcd}
T^*\mathbb{C}^{\mathcal{F}^c} \arrow[r, "\phi_{\mathrm{HK}}^{\mathcal{F}^c}"] \arrow[d, "\iota"] \arrow[rr, bend left, "\mu_{\mathrm{HK}}^{\mathcal{F}^c}"] & (\mathfrak{t}^{\mathcal{F}^c})^* \otimes_{\mathbb{R}} \mathrm{Im}(\mathbb{H}) \arrow[r, "\imath_{\mathcal{F}^c}^*\otimes \mathrm{Id}"] \arrow[d, "p^*\otimes\mathrm{Id}"] & (\mathfrak{k}^{\mathcal{F}^c})^* \otimes_{\mathbb{R}} \mathrm{Im}(\mathbb{H}), \arrow[d, "(p|_{\mathfrak{k}})^*\otimes\mathrm{Id}"] \\
T^*\mathbb{C}^N \arrow[r, "\phi_{\mathrm{HK}}"] \arrow[rr, bend right, "\mu_{\mathrm{HK}}"'] & (\mathfrak{t}^N)^* \otimes_{\mathbb{R}} \mathrm{Im}(\mathbb{H}) \arrow[r, "\imath^*\otimes\mathrm{Id}"] & \mathfrak{k}^* \otimes_{\mathbb{R}} \mathrm{Im}(\mathbb{H}).
\end{tikzcd} \tag{2.24}
$$

The isomorphism $(p|_{\mathfrak{k}})^* : (\mathfrak{k}^{\mathcal{F}^c})^* \cong \mathfrak{k}^*$ allows us to identify the elements $(p|_{\mathfrak{k}})^*(\nu_{\mathcal{F}}) = \nu$, and implies that:

$$M_{\mathcal{F}} = (\mu_{\mathrm{HK}}^{\mathcal{F}})^{-1}(\nu_{\mathcal{F}}, 0)/K^{\mathcal{F}^c} = \left(\mu_{\mathrm{HK}}^{-1}(\nu, 0) \cap T^*\mathbb{C}^{\mathcal{F}^c}\right)/K \subset \mu_{\mathrm{HK}}^{-1}(\nu, 0)/K = M_\nu. \tag{2.25}$$

Hence from (2.25), we see that $M_{\mathcal{F}}$ can be seen as the subvariety of the hypertoric variety M_ν, that has been carved out by the closed subvarieties, $\{[z, w] \in M_\nu(\mathcal{A}) \mid \text{if } i \in \mathcal{F} \text{ then } z_i = w_i = 0\}$. We also obtain an expression for $M_{\mathcal{F}}$ as the hyperkähler quotient $(M_{\mathcal{F}}, T^{\mathrm{crk}\,\mathcal{F}}, \mu_{\mathrm{HK}}^{\mathcal{F}})$, where $T^{\mathrm{crk}\,\mathcal{F}} \cong T^n/T^{\mathrm{rk}\,\mathcal{F}}$. Hence $M_{\mathcal{F}}$ is itself hypertoric variety that is also a closed subvariety of M_ν.

For $M_{\mathcal{F}}$ to satisfy Definition 2.12, it remains to show that there exists a $T^{\mathrm{crk}\,\mathcal{F}}$-equivariant embedding of $M_{\mathcal{F}}$ into M_ν. The closed embedding $\iota : T^*\mathbb{C}^{\mathcal{F}^c} \hookrightarrow T^*\mathbb{C}^N$ from (2.23) is T^N-equivariant, and hence the induced embedding $\bar{\iota} : M_{\mathcal{F}} \hookrightarrow M_\nu$ is T^n-equivariant since $K \subseteq T^N$. From Proposition 2.11, there exists a non-canonical splitting $T^n \cong T^{\mathrm{rk}\,\mathcal{F}} \times T^{\mathrm{crk}\,\mathcal{F}}$, and this implies that $\bar{\iota}$ is $T^{\mathrm{crk}\,\mathcal{F}}$-equivariant as $T^{\mathrm{crk}\,\mathcal{F}}$ acts on M_ν via the inclusion $T^{\mathrm{crk}\,\mathcal{F}} \hookrightarrow T^{\mathrm{rk}\,\mathcal{F}} \times T^{\mathrm{crk}\,\mathcal{F}}$.

Lastly, $\bar{\iota}$ is a holomorphic-symplectic embedding since the Kähler two-forms, $\omega_{\mathbb{R}}^{\mathcal{F}}$ and $\omega_{\mathbb{C}}^{\mathcal{F}}$, on $M_{\mathcal{F}}$ are just obtained by restricting those from M_ν:

$$\bar{\iota}^*\omega_{\mathbb{R}} = \omega_{\mathbb{R}}\big|_{M_{\mathcal{F}}} = \omega_{\mathbb{R}}^{\mathcal{F}}, \quad \text{and} \quad \bar{\iota}^*\omega_{\mathbb{C}} = \omega_{\mathbb{C}}\big|_{M_{\mathcal{F}}} = \omega_{\mathbb{C}}^{\mathcal{F}}.$$

In summary, $\bar{\iota} : M_{\mathcal{F}} \hookrightarrow M_\nu$ is the required closed $T^{\mathrm{crk}\,\mathcal{F}}$-equivariant embedding in Definition 2.12 for $M_{\mathcal{F}}$ to be a hypertoric subvariety of M_ν. □

Constructions reminiscent to the statement in Theorem 2.13 have been made before, for example in the proof of Theorem 6.7 in [BD00], in Claim 7.1 of [Kono00], and also in Proposition 2.1 of [PW07]. Our contribution generalises them, in that we prove the hyperkähler subvariety, which itself is a hypertoric variety, is then a hypertoric subvariety in the sense of our proposed Definition 2.12, by using Proposition 2.11 to show that the inclusion is an equivariant embedding.

Given a hypertoric variety M_ν with a simple hyperplane arrangement $\mathcal{A}$ in $(\mathfrak{t}^n)^*$, Theorem 2.13 tells us that each flat $H_\mathcal{F} \in L(\mathcal{A})$ determines a hypertoric subvariety $M_\mathcal{F}$ of M_ν. The image of $\mu_\mathbb{R}^\mathcal{F}$ surjects onto the affine space $H_\mathcal{F} = \cap_{i\in\mathcal{F}} H_i$, and the restricted hyperplane arrangement $\mathcal{A}^\mathcal{F}$ in $H_\mathcal{F}$ is subsequently the hyperplane arrangement for the hypertoric variety $M_\mathcal{F}$.

As it is a hypertoric variety in its own right, $M_\mathcal{F}$ comes equipped with a hyperhamiltonian action of $T^{\operatorname{crk}\mathcal{F}}$ on $M_\mathcal{F}$, and therefore possesses a hyperkähler moment map:

$$\mu_{\mathrm{HK}}^\mathcal{F} = \mu_\mathbb{R}^\mathcal{F} \oplus \mu_\mathbb{C}^\mathcal{F} : M_\mathcal{F} \longrightarrow (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \otimes \operatorname{Im}(\mathbb{H}).$$

The real moment map component $\mu_\mathbb{R}^\mathcal{F}$ will therefore determine a hyperplane arrangement of its own for $M_\mathcal{F}$, which we denote by:

$$\mathcal{A}^{\operatorname{crk}\mathcal{F}} := \{\, F_i \mid i \in \mathcal{F}^c \,\}, \tag{2.26}$$

with each hyperplane F_i lying in $(\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$, for each $i \in \mathcal{F}^c$. Let us express each hyperplane of $\mathcal{A}^{\operatorname{crk}\mathcal{F}}$ as:

$$F_i := \left\{\, x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \;\middle|\; \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c} = 0 \,\right\}, \tag{2.27}$$

where $\lambda^{\mathcal{F}^c} \in (\mathfrak{t}^{\mathcal{F}^c})^*$ satisfies $p^*(\lambda^{\mathcal{F}^c}) = \lambda \in (\mathfrak{t}^N)^*$, and where $u_i^{\mathcal{F}^c} := \pi_{\mathcal{F}^c}(p(e_i)) \in \mathfrak{t}^{\operatorname{crk}\mathcal{F}}$.

The following proposition identifies the hyperplane arrangement $\mathcal{A}^{\operatorname{crk}\mathcal{F}}$ in $(\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$ with the restricted hyperplane arrangement $\mathcal{A}^\mathcal{F}$ under the flat $H_\mathcal{F}$, and is similar to [Konoo, Claim 7.1].

Proposition 2.14. *The restricted hyperplane arrangement $\mathcal{A}^\mathcal{F} = \{H_i \cap H_\mathcal{F} \mid i \notin \mathcal{F}\}$ in $H_\mathcal{F}$ can be identified with the arrangement $\mathcal{A}^{\operatorname{crk}\mathcal{F}}$ given by (2.26) in $(\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$, whose hyperplanes F_i are given by (2.27), where $\lambda^{\mathcal{F}^c} \in (\mathfrak{t}^{\mathcal{F}^c})^*$ satisfies $p^*(\lambda^{\mathcal{F}^c}) = \lambda \in (\mathfrak{t}^N)^*$, and $u_i^{\mathcal{F}^c} := \pi_{\mathcal{F}^c}(p(e_i)) \in \mathfrak{t}^{\operatorname{crk}\mathcal{F}}$.*

Proof. First of all, note that

$$(\mathfrak{t}^\mathcal{F})^* \cong (\mathfrak{t}^n/\mathfrak{t}^{\mathcal{F}^c})^* \cong \operatorname{Ann}_{\mathfrak{t}^n} \mathfrak{t}^{\mathcal{F}^c}, \quad \text{and} \quad (\mathfrak{t}^{\mathcal{F}^c})^* \cong (\mathfrak{t}^n/\mathfrak{t}^\mathcal{F})^* \cong \operatorname{Ann}_{(\mathfrak{t}^n)^*} \mathfrak{t}^\mathcal{F}. \tag{2.28}$$

Fix a point $y_0 \in H_\mathcal{F} \subseteq (\mathfrak{t}^n)^*$. This implies that $\langle y_0,\, u_i\rangle + \lambda_i = 0$ as $y_0 \in H_i$. But then:

$$\langle y_0,\, u_i\rangle + \lambda_i = \big\langle y_0,\, \pi(e_i)\big\rangle + \lambda_i = \big\langle \pi^*(y_0) + \lambda,\, e_i\big\rangle = 0, \tag{2.29}$$

for each $i \in \mathcal{F}$. Thus $\pi^*(y_0) + \lambda$ belongs to the annihilator of $\mathfrak{t}^\mathcal{F}$ in $(\mathfrak{t}^n)^*$, hence from (2.28) there exists an element $\lambda^{\mathcal{F}^c} \in (\mathfrak{t}^{\mathcal{F}^c})^*$ such that $p^*(\lambda^{\mathcal{F}^c}) = \pi^*(y_0) + \lambda$.

Since:

$$\begin{aligned}
\big((p|_\mathfrak{t})^* \circ \imath_{\mathcal{F}^c}^*\big)(\lambda^{\mathcal{F}^c}) &= (\imath^* \circ p^*)(\lambda^{\mathcal{F}^c}) && \big(\text{as } (p|_\mathfrak{t})^* \circ \imath_{\mathcal{F}^c}^* = \imath^* \circ p^* \text{ from (2.19)}\big)\\
&= \imath^*\big(\pi^*(y_0) + \lambda\big) \\
&= \imath^*(\lambda) && \big(\text{as } \imath^* \circ \pi^* = 0\big)\\
&= \nu,
\end{aligned}$$

it follows that $\imath_{\mathcal{F}^c}^*(\lambda^{\mathcal{F}^c}) = \nu^{\mathcal{F}^c}$ because $(p|_{\mathfrak{k}})^* : (\mathfrak{k}^{\mathcal{F}^c})^* \to \mathfrak{k}^*$ is an isomorphism, and therefore $(p|_{\mathfrak{k}})^*(\nu^{\mathcal{F}^c}) = \nu$.

Let us the following map:

$$\eta_{y_0} : (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \to (\mathfrak{t}^n)^*, \qquad \text{by} \qquad \eta_{y_0}(x) := \bar{p}^*(x) + y_0. \tag{2.30}$$

Then, by recalling (2.16), (2.29), and that $p^*(\lambda^{\mathcal{F}^c}) = \pi^*(y_0) + \lambda$, for a given $x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$, we have:

$$\begin{aligned}
\langle \eta_{y_0}(x),\, u_i\rangle + \lambda_i &= \Big\langle \bar{p}^*(x) + y_0,\, \pi_*(e_i)\Big\rangle + \lambda_i && \big(\text{from (2.30)}\big) \\
&= \Big\langle \big(\pi^* \circ \bar{p}^*\big)(x) + \pi^*(y_0) + \lambda,\, e_i\Big\rangle && \qquad (2.31) \\
&= \Big\langle \big(p^* \circ \pi_{\mathcal{F}^c}^*\big)(x),\, e_i\Big\rangle + \big\langle \pi^*(y_0) + \lambda,\, e_i\big\rangle && \big(\text{from (2.19)}\big).
\end{aligned}$$

On the one hand, if $i \in \mathcal{F}$, then $\langle \pi^*(y_0) + \lambda,\, e_i\rangle = 0$ since $y_0 \in H_{\mathcal{F}}$. So (2.31) becomes:

$$\Big\langle \big(p^* \circ \pi_{\mathcal{F}^c}^*\big)(x) + \pi^*(y_0) + \lambda,\, e_i\Big\rangle = \Big\langle x,\, \big(\pi_{\mathcal{F}^c} \circ p\big)(e_i)\Big\rangle = 0,$$

as $p(e_i) = 0$ from (2.16). Whereas, on the other hand, $p^*(\lambda^{\mathcal{F}^c}) = \pi^*(y_0) + \lambda$ and therefore, when $i \in \mathcal{F}^c$, we have:

$$\begin{aligned}
\Big\langle \big(p^* \circ \pi_{\mathcal{F}^c}^*\big)(x) + \pi^*(y_0) + \lambda,\, e_i\Big\rangle &= \Big\langle \big(p^* \circ \pi_{\mathcal{F}^c}^*\big)(x) + p^*(\lambda^{\mathcal{F}^c}),\, e_i\Big\rangle \\
&= \Big\langle \pi_{\mathcal{F}^c}^*(x) + \lambda^{\mathcal{F}^c},\, p(e_i)\Big\rangle \\
&= \Big\langle \pi_{\mathcal{F}^c}^*(x) + \lambda^{\mathcal{F}^c},\, e_i\Big\rangle && \big(\text{as } p(e_i) = e_i \text{ from (2.16)}\big) \\
&= \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c}.
\end{aligned}$$

Hence, for a given point $y_0 \in H_{\mathcal{F}}$, (2.31) shows that the image of η_{y_0} in $(\mathfrak{t}^n)^*$ is $H_{\mathcal{F}}$ and, for each hyperplane $F_i \subset (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$ defined in (2.27), we get $\eta(F_i) = H_i \cap H_{\mathcal{F}}$ for each $i \in \mathcal{F}^c$. $\square$

Example 2.15. As an example, Figure 2.6 displays the hyperkähler analogue of the resolution of the Kleinian singularity from Example 2.5, which we denote here by $M_{\mathcal{F}}$ with $\mathcal{F} = \{1\}$, as a hypertoric subvariety of the hyperkähler analogue to the first Hirzebruch surface from Example 2.8, which we denote here by M_ν.

If $\mathcal{A}$ is the hyperplane arrangement for M_ν, then the restricted hyperplane arrangement for the hypertoric subvariety $M_{\mathcal{F}}$ is given by $\mathcal{A}^{\mathcal{F}} = \{\, F_i = H_i \cap H_1 \mid i = 2, 3, 4\,\}$, where each $F_i \in \mathcal{A}^{\mathcal{F}}$ is a hyperplane in the flat $H_{\mathcal{F}} = H_1 \cong (\mathfrak{t}^1)^*$ of $\mathcal{A}$, since $\operatorname{crk}\mathcal{F} = n - \operatorname{rk}\mathcal{F} = 1$.

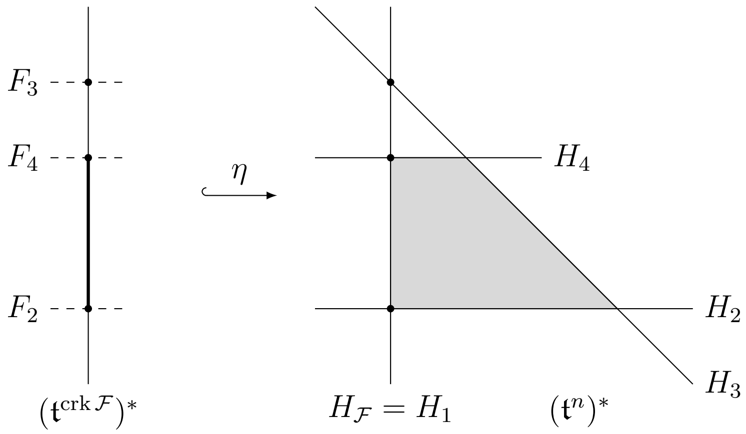


Figure 2.6: Restricted hyperplane arrangement $\mathcal{A}^{\mathcal{F}}$ for the hypertoric subvariety M_ν from Example and a subvariety $M_{\mathcal{F}}$, where $\mathcal{F} = \{1\}$.

Remark 2.16. We may further introduce the half-spaces

$$\begin{aligned} F_i^+ &:= \{x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \mid \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c} \geq 0\}, \\ F_i^- &:= \{x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \mid \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c} \leq 0\}, \end{aligned} \tag{2.32}$$

for each hyperplane $F_i \in \mathcal{A}^{\operatorname{crk}\mathcal{F}}$. Analogously to what was done in Proposition 2.14, one can show that $\eta_{y_0}(\Delta_{A^{\mathcal{F}^c}}) = \Delta_A \cap H_{\mathcal{F}}$, where we have defined the subset:

$$A^{\mathcal{F}^c} := \{\, i \in A \cap \mathcal{F}^c \mid \Delta_A \cap H_{\mathcal{F}} \neq \emptyset \,\} \subseteq \{1, \ldots, N\} \cap \mathcal{F}^c.$$

This observation naturally leads the notion of a *moment subpolyptych*, which we define in Definition 3.15 and will use when proving Theorem 6.2 and Proposition 6.3.

Chapter 3

The Symplectic Cut of a Hypertoric Variety

There exists a procedure called *symplectic cutting* which has the effect of slicing away part of a symplectic manifold, provided that it is equipped with a suitable U_1-action. Recall from the introduction of Chapter 2 that there exists a correspondence between symplectic toric varieties and simple convex polytopes. For these varieties, the symplectic cut can be arranged so that the corresponding polytope (or polyhedron) gets truncated, by intersecting it with a half-space whose normal vector depends on the U_1-action, as is mentioned in [Ler95, Remark 1.5].

In this chapter, we wish to construct the symplectic cut of a hypertoric variety M_ν in order to obtain something compact. In Lemma 3.4, we show that the moment map ρ, corresponding to the U_1-action that is used to define the symplectic cut of M_ν, is proper, provided that its hyperplane arrangement $\mathcal{A}$ contains at least one bounded region. Therefore we end up taking the quotient of a compact level-set $\rho^{-1}(\delta)$, where $\delta \in \mathbb{R}_{\geq 0}$, of ρ by the circle U_1, resulting in a compact orbifold which we call the *cut space*, $M_\nu^{\leq\delta}$, of the hypertoric variety M_ν.

The residual torus T^n that acted on M_ν descends to $M_\nu^{\leq\delta}$, as does its real moment map $\mu_\mathbb{R} : M_\nu^{\leq\delta} \to (\mathfrak{t}^n)^*$. The compactness of $M_\nu^{\leq\delta}$ is reflected in the hyperplane arrangement $\mathcal{A}$, in that it becomes a truncated arrangement of sorts, which we call the *moment polyptych* of $M_\nu^{\leq\delta}$, and is denoted by $\Delta_\nu^{\leq\delta}$. Each unbounded region of the original arrangement $\mathcal{A}$ is replaced by a bounded polytope which, from either Delzant's or Lerman and Tolman's classification schema, shows that $M_\nu^{\leq\delta}$ is made up from various toric Kähler subvarieties. We shall exploit the properties of the cut spaces $M_\nu^{\leq\delta}$ and their moment polyptychs $\Delta_\nu^{\leq\delta}$ in later chapters, and especially in Chapter 6 when we are able to associate specific isotropy data to the vertices of $\Delta_\nu^{\leq\delta}$.

3.1 Symplectic Cutting

Symplectic cutting is a technique first introduced by Lerman in [Ler95], which lets one construct new symplectic varieties from old ones, if they are equipped with a Hamiltonian U_1-action. The general procedure process is follows: let M be a symplectic orbifold with a Hamiltonian U_1-action and moment map $\Phi : M \to \mathbb{R}$. Consider the product $M \times \mathbb{C}$ and let U_1 act on it diagonally, so:

$$e^{i\theta} \cdot (p, \xi) = (e^{i\theta} \cdot p, e^{i\theta}\xi), \quad \text{for} \quad e^{i\theta} \in U_1, \text{ and } (p, \xi) \in M \times \mathbb{C}. \tag{3.1}$$

The diagonal U_1-action in (3.1) is also Hamiltonian with moment map:

$$\rho : M \times \mathbb{C} \to \mathbb{R}, \qquad \text{where} \qquad \rho(p, \xi) = \Phi(p) + \tfrac{1}{2}|\xi|^2.$$

Then the **symplectic cut**, $M^{\leq\delta}$, of M at δ is defined to be the symplectic quotient of $M \times \mathbb{C}$ with respect to the diagonal U_1-action:

$$M^{\leq\delta} := (M \times \mathbb{C}) /\!\!/_{\delta}\, U_1 = \rho^{-1}(\delta)/U_1,$$

where $\delta \in \mathbb{R}_{\geq 0}$ is a regular value of ρ.

The level-set:

$$\rho^{-1}(\delta) = \left\{ (p, \xi) \in M \times \mathbb{C} \;\middle|\; \Phi(p) + |\xi|^2 = \delta \right\} \subset M \times \mathbb{C} \tag{3.2}$$

fits into the following diagram:

$$\begin{tikzcd} & \rho^{-1}(\delta) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "q"] & \\ M & & M^{\leq\delta} = \rho^{-1}(\delta)/U_1 \end{tikzcd} \tag{3.3}$$

where $\mathrm{pr}_1 : \rho^{-1}(\delta) \to M$ is the projection $\mathrm{pr}_1(p, \xi) = p$ onto the first factor. Its image in M is $\mathrm{im}(\mathrm{pr}_1) = \{p \in M \mid \Phi(p) \leq \delta\}$. On the other hand, the map $q : \rho^{-1}(\delta) \to M^{\leq\delta}$ is the quotient map for the diagonal U_1-action on $\rho^{-1}(\delta)$.

The level-set $\rho^{-1}(\delta)$ in (3.2) decomposes into the disjoint union:

$$\rho^{-1}(\delta) \cong \Sigma_1 \sqcup \Sigma_2,$$

where:

$$\begin{aligned} \Sigma_1 &= \left\{ (p, \xi) \in M \times \mathbb{C} \;\middle|\; \Phi(p) + |\xi|^2 = \delta,\ \xi \neq 0 \right\}, \\ \Sigma_2 &= \{ (p, 0) \in M \times \mathbb{C} \mid \Phi(p) = \delta \}. \end{aligned}$$

On the suborbifold Σ_1, we have $|\xi| > 0$. It is possible therefore to find an element $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$ and $\mathrm{Im}(\xi) = 0$. This allows for a global section $\sigma : M \to \rho^{-1}(\delta)$ of $\mathrm{pr}_1 : \rho^{-1}(\delta) \to M$ to be defined, by:

$$\sigma(p) = (p, \xi), \qquad \text{where} \qquad \xi = \sqrt{\delta - \Phi(p)}.$$

Since any U_1-orbit contains a unique (p, ξ) when $\mathrm{Re}(\xi) > 0$ and $\mathrm{Im}(\xi) = 0$, the section σ is a U_1-equivariant diffeomorphism and identifies $\sigma : \{p \in M \mid \Phi(p) < \delta\} \xrightarrow{\sim} \Sigma_1/U_1$. On the other hand, the quotient of its complement Σ_2/U_1 is just the symplectic quotient $\Phi^{-1}(\delta)/U_1$.

For the linear U_1-action on $\mathbb{C}$, the only critical value of its moment map $\xi \mapsto |\xi|^2$ is zero. It follows then that the diagonal action on $M \times \mathbb{C}$ is locally free, except at the points which belong to the fixed-point set, $M^{U_1} \times \{0\}$. Therefore, to avoid quotients whose singular nature is worse than that of an orbifold, we will assume that $\delta \in \mathbb{R}_{\geq 0}$ is always a regular value for the moment map ρ by choosing δ to be large enough, as to avoid any critical points.

Example 3.1. Let U_1 act on $\mathbb{C}^N$ in the standard linear way, and extend this action to the product $\mathbb{C}^N \times \mathbb{C}$ as:

$$\tau \cdot \big((z_1, \dots, z_N), \xi\big) = \big((\tau z_1, \dots, \tau z_N), \tau\xi\big). \tag{3.4}$$

The moment map $\rho : \mathbb{C}^N \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ is $\rho(z, \xi) = \frac{1}{2}\|z\|^2 + \frac{1}{2}|\xi|^2$ and, for some $\delta \in \mathbb{R}_{\geq 0}$, the level-set $\rho^{-1}(\delta)$ is the $(2N+1)$-dimensional sphere:

$$\rho^{-1}(k) = \left\{ \ (z, \xi) \in \mathbb{C}^N \times \mathbb{C} \ \middle| \ \tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}|\xi|^2 = \delta \ \right\} \cong S^{2N+1}.$$

Hence the symplectic cut $(\mathbb{C}^N \times \mathbb{C}) /\!\!/_\delta\, U_1$ of $\mathbb{C}^N$ with respect to the U_1-action (3.4) is:

$$(\mathbb{C}^N \times \mathbb{C}) /\!\!/_\delta\, U_1 \cong S^{2N+1}/U_1 \cong \mathbb{CP}^N.$$

The next example comes from [GS89], whose work was a precursor to Lerman's [Ler95].

Example 3.2. In Example 3.1, if we instead let U_1 act on the product $\mathbb{C}^N \times \mathbb{C}$ as:

$$\tau \cdot \big((z_1, \dots, z_N), \xi\big) = \big((\tau z_1, \dots, \tau z_N), \tau^{-1}\xi\big), \tag{3.5}$$

then, taking the symplectic cut, we obtain the blow-up $M^{\geq\delta} = \mathrm{Bl}_0\, \mathbb{C}^N$ of $\mathbb{C}^N$ at the origin, instead. Indeed, the moment map $\rho : \mathbb{C}^N \times \mathbb{C} \to \mathbb{R}$ for the U_1-action in (3.5) now becomes $\rho(z, \xi) = \|z\|^2 - |\xi|^2$ and so, again for some $\delta > 0$, the level-set $\rho^{-1}(\delta)$ is the hypersurface:

$$\rho^{-1}(\delta) = \left\{ \ (z, \xi) \in \mathbb{C}^N \times \mathbb{C} \ \middle| \ |z_1|^2 + \ldots + |z_N|^2 = \delta + |\xi|^2 \ \right\} \subset \mathbb{C}^N \times \mathbb{C}.$$

Let us set:

$$v = \xi, \qquad \text{and} \qquad u_i = (\delta + |v|^2)^{-1/2} z_i, \qquad \text{for each } i = 1, \dots, N.$$

Then, in terms of the (u, v)-coordinates, $\rho^{-1}(\delta) \cong S^{2N-1} \times \mathbb{C}$, where:

$$S^{2N-1} \cong \left\{ w \in \mathbb{C}^N \;\middle|\; |u_1|^2 + \ldots + |u_N|^2 = 1 \right\} \subset \mathbb{C}^{2N-1},$$

whilst $v \in \mathbb{C}$ is unconstrained. The U_1-action in these coordinates sends $u_i \mapsto \tau u_i$, for $i = 1, \ldots, N$, and $v \mapsto \tau^{-1} v$.

The symplectic cut $M^{\geq \delta}$ (the $\geq \delta$ superscript is intentional) obtained by taking the quotient with respect to this U_1-action is $M^{\geq \delta} = (\mathbb{C}^N \times \mathbb{C}) /\!\!/_{\delta} U_1$. There exists an injection:

$$\begin{aligned} i : (\mathbb{C}^N \times \mathbb{C}) /\!\!/_{\delta} U_1 &\hookrightarrow \mathbb{CP}^{N-1} \times \mathbb{C}^N, \\ [u_1, \ldots, u_N, v] &\longmapsto \big([u_1, \ldots, u_N], (u_1 v, \ldots, u_N v)\big), \end{aligned}$$

whose image in $\mathbb{CP}^{N-1} \times \mathbb{C}^N$ identifies the symplectic cut $M^{\geq \delta}$ with the blow-up:

$$\mathrm{Bl}_0\, \mathbb{C}^N = \left\{ \big([U], v\big) \in \mathbb{CP}^{N-1} \times \mathbb{C}^N \;\middle|\; v_i U_j = v_j U_i, \text{ for each } i,\, j = 1, \ldots, N \right\}.$$

3.2 The Residual U_1-Action on a Hypertoric Variety

In the Section 3.1 we saw that, to take a symplectic cut of a symplectic orbifold, it suffices for it to be equipped with a Hamiltonian U_1-action. A suitable U_1-action on a hypertoric variety has been studied before in [HP04], which we recall here by first considering the complex cotangent space $T^*\mathbb{C}^N$. Then there is a U_1-action which "rotates" the cotangent fibre coordinates $T^*_z\mathbb{C}^N$ over a point $z \in \mathbb{C}^N$, which is to say:

$$\tau \cdot (z, w) = (z, \tau w). \tag{3.6}$$

This action is Hamiltonian with respect to the real Kähler two-form $\omega_{\mathbb{R}}$ on $T^*\mathbb{C}^N$, but it *does not* preserve the holomorphic-symplectic two-form $\omega_{\mathbb{C}}$ however, since $\tau^*\omega_{\mathbb{C}} = \tau\omega_{\mathbb{C}}$. With respect to $\omega_{\mathbb{R}}$, the moment map for this U_1-action is:

$$\Phi : T^*\mathbb{C}^N \to \mathbb{R}, \qquad \Phi(z, w) = \|w\|^2, \tag{3.7}$$

up to the addition of a constant. As this U_1-action $T^*\mathbb{C}^N$ commutes with the T^N-action, it descends to a residual Hamiltonian U_1-action on the hypertoric variety M_ν, whose moment map we shall continue to denote by Φ. It was proven in [HP04, Proposition 1.3] that if the original moment map $\phi : \mathbb{C}^N \to \mathfrak{k}^*$ for the K-action on $\phi_{\mathrm{HK}}^{-1}(\nu, 0)$ is proper, then so is the moment map $\Phi : M_\nu \to \mathbb{R}_{\geq 0}$ for the residual U_1-action.

Since $\Phi^{-1}(0) = X_\nu$, properness of μ is therefore equivalent to the compactness of the Kähler variety X_ν, or equivalently to the boundedness of $\Delta_\emptyset$. If we assume that the hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_N\}$ for the hypertoric variety M_ν is simple, then $\Delta_\emptyset$ will always be bounded, since any subcollection of n normal vectors from the $u_1, \ldots, u_N$, will form an $\mathbb{R}$-basis for $\mathfrak{t}^n$. Hence there exists a vertex that is equal to the intersection of n hyperplanes from the arrangement $\mathcal{A}$, which is trivially bounded [BD00, §6].

It will be interesting to know what the fixed-point locus $M_\nu^{U_1}$ is for the residual U_1-action. Moreover, knowing this fixed-point data is important when we take the symplectic cut of M_ν in Section 3.4, as we will want to avoid cutting into it. As the complex moment map $\mu_{\mathbb{C}} : M_\nu \to (\mathfrak{t}_{\mathbb{C}}^n)^*$ is U_1-equivariant, and as U_1 acts freely on $(\mathfrak{t}_{\mathbb{C}}^n)^* - \{0\}$, as discussed in [Pro04, §3.2] the U_1-fixed-point locus $M_\nu^{U_1}$ will be contained within the extended core $\mathcal{E}$ of M_ν, that is $M_\nu^{U_1} \subseteq \mu_{\mathbb{C}}^{-1}(0) = \mathcal{E}$.

Hence, for us to study the fixed-point locus M^{U_1}, it suffices to restrict our attention purely to the extended core $\mathcal{E}$ of M_ν. We shall revisit the question of what the fixed-point locus is in Section 3.3, once we have established a combinatorial description for the U_1-action on each extended core component, $\mathcal{E}_A$.

3.3 Combinatorics of the Residual U_1-Action

As the Hamiltonian U_1-action on $T^*\mathbb{C}^N$ descends to a residual one on the hypertoric variety M_ν, whose moment map $\Phi : M_\nu \to \mathbb{R}_{\geq 0}$ is proper if, and only if, the core $\mathcal{C}$ of M_ν is non-empty. This residual U_1-action does not act on M_ν as a circle subgroup of the torus T^n globally, but it does when restricted to an extended core component $\mathcal{E}$.

Given a subset $A \subseteq \{1, \ldots, N\}$, recall that the extended core component $\mathcal{E}_A = \mu_{\mathbb{R}}^{-1}(\Delta_A) \cap \mathcal{E}$ can be combinatorially expressed as:

$$\mathcal{E}_A = \{ \, [z, w] \in M_\nu \mid w_i = 0 \text{ if } i \notin A \text{ and } z_i = 0 \text{ if } i \in A \, \} \,.$$

Hence, for $\tau \in U_1$ and $[z, w] \in \mathcal{E}_A$, the circle U_1 acts as:

$$\tau \cdot [z, w] = [z, \tau w] = [\tau_1 z_1, \ldots, \tau_N z_N; \tau_1^{-1} w_1, \ldots, \tau_N^{-1} w_N], \text{ where } \tau_i = \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A. \end{cases}$$

With this observation, we may express the restricted U_1-action on the component $\mathcal{E}_A$ as that of a circle subgroup of T^N. To see this, we express this as the image of U_1 under the inclusion $\jmath_A : U_1 \hookrightarrow T^N$, defined by:

$$\jmath_A(\tau) := (\tau_1, \ldots, \tau_N), \qquad \text{where} \qquad \tau_i = \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A. \end{cases} \tag{3.8}$$

The composition of the inclusion $\jmath_A$ with the projection $\pi : T^N \to T^n$ prescribes how the circle fits inside of T^n, when its action is restricted to the subvariety $\mathcal{E}_A$. For conciseness, denote:

$$e_A := -\sum_{i \in A} e_i \in \mathfrak{t}^N, \quad u_A := -\sum_{i \in A} u_i \in \mathfrak{t}^n, \quad \text{and} \quad \lambda_A := \sum_{i \in A} \lambda_i \in \mathbb{R}. \tag{3.9}$$

On the Lie algebra level, the generator for the restricted U_1-action is just the image of $1 \in \mathbb{R}$ under the map

$$(\pi \circ \jmath_A)_* : \mathbb{R} \hookrightarrow \mathfrak{t}^N \longrightarrow \mathfrak{t}^n, \qquad 1 \xmapsto{\jmath_{A,*}} e_A \xmapsto{\pi} u_A. \tag{3.10}$$

We shall call the vector $u_A = (\pi \circ \jmath_A)_*(1) \in \mathfrak{t}^n$ the **restricted U_1-action generator** for the component $\mathcal{E}_A$. Since $\pi \circ \jmath_A : U_1 \hookrightarrow T^n$ is an inclusion and since T^n acts in a Hamiltonian way on M_ν, there is a moment map associated with the U_1-action on $\mathcal{E}_A$ obtained from the composition of $\mu_{\mathbb{R}}|_{\mathcal{E}_A} : \mathcal{E}_A \to (\mathfrak{t}^n)^*$ with the projection (3.10):

$$\begin{aligned} \Phi_A &:= (\pi \circ \jmath_A)^* \circ \mu_{\mathbb{R}}|_{\mathcal{E}_A} : \mathcal{E}_A \to \mathbb{R}, \\ \Phi_A[z,w] &= \big\langle \mu_{\mathbb{R}}[z,w],\, u_A \big\rangle = \tfrac{1}{2} \sum_{i \in A} |w_i|^2 + \lambda_A. \end{aligned} \tag{3.11}$$

3.4 The Cut Space of a Hypertoric Variety

Having described in Section 3.3 how the residual U_1-action on a hypertoric variety M_ν acts when restricted an extended core component $\mathcal{E}_A$, we may now form the symplectic cut of M_ν. As in Section 3.1, let U_1 act diagonally on the product $M_\nu \times \mathbb{C}$, as:

$$\tau \cdot ([z,w],\xi) = \big([z,\tau w],\tau\xi\big), \quad \text{where} \quad \tau \in U_1,\ \big([z,w],\xi\big) \in M_\nu \times \mathbb{C}. \tag{3.12}$$

This action is Hamiltonian with moment map:

$$\rho : M_\nu \times \mathbb{C} \to \mathbb{R}_{\geq 0}, \qquad \rho\big([z,w],\xi\big) = \Phi[z,w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Given a regular value $\delta \in \mathbb{R}_{\geq 0}$ of ρ, the circle U_1 acts locally freely on the level-set:

$$\rho^{-1}(\delta) = \left\{\ \big([z,w],\xi\big) \in M_\nu \times \mathbb{C} \ \middle|\ \tfrac{1}{2}\|w\|^2 + \tfrac{1}{2}|\xi|^2 = \delta\ \right\}.$$

Then, from Section 3.1, the symplectic cut of M_ν is the symplectic quotient:

$$\begin{aligned} \rho^{-1}(\delta)/U_1 &\cong \left\{\ [z,w] \in M_\nu \ \middle|\ \tfrac{1}{2}\|w\|^2 \leq \delta\ \right\} \\ &\cong \left\{\ [z,w] \in M_\nu \ \middle|\ \tfrac{1}{2}\|w\|^2 < \delta\ \right\} \sqcup \left[\left\{\ [z,w] \in M_\nu \ \middle|\ \tfrac{1}{2}\|w\|^2 = \delta\ \right\}/U_1\right]. \end{aligned} \tag{3.13}$$

Definition 3.3. Given a hypertoric variety M_ν and a regular value $\delta \geq 0$ for the moment map $\rho : M_\nu \times \mathbb{C} \to \mathbb{R}_{\geq 0}$, we define the **cut space** $M_\nu^{\leq\delta}$ of M_ν to be the symplectic cut of M_ν,

$$M_\nu^{\leq\delta} := (M_\nu \times \mathbb{C}) \mathbin{/\!\!/}_\delta U_1.$$

Our motivation for taking the symplectic cut of a hypertoric variety M_ν is that the cut space $M_\nu^{\leq\delta}$ is compact. This follows from the following lemma, the proof of which is a minor adaptation from that from [HP04, Proposition 1.3].

Lemma 3.4. *If the original moment map $\phi : \mathbb{C}^n \to \mathfrak{k}^*$ from (1.3) is proper, then so is the moment map $\rho : M_\nu \times \mathbb{C} \to \mathbb{R}_{\geq 0}$.*

Proof. Mutatus mutandi, by the same argument as in [HP04, Proposition 1.3], for some $R \geq 0$, we need to show that the set:

$$\rho^{-1}[0,\delta] = \left\{ (z,w,\xi) \mid \phi_{\mathbb{R}}(z,w) = \nu,\ \phi_{\mathbb{C}}(z,w) = 0,\ \tfrac{1}{2}\|w\|^2 + \tfrac{1}{2}|\xi|^2 \leq R \right\} / K$$

is compact. But this set is a closed subset of:

$$\phi^{-1}\left(\left\{ \nu + \phi(w) \mid \tfrac{1}{2}\|w\|^2 \leq R \right\} \right) \times \left\{ (w,\xi) \mid \tfrac{1}{2}\|w\|^2 + \tfrac{1}{2}|\xi|^2 \leq R \right\} \subset T^*\mathbb{C}^N \times \mathbb{C},$$

which is compact since ϕ is proper, and therefore so is ρ. □

Corollary 3.5. *The cut space $M_\nu^{\leq\delta}$ of a hypertoric variety M_ν is compact.*

Proof. Since ρ is proper, the level-set $\rho^{-1}(\delta) \subset M_\nu \times \mathbb{C}$ is compact. As the circle U_1 is a compact Lie group, the quotient $M_\nu^{\leq\delta} = \rho^{-1}(\delta)/U_1$ is also compact. □

The T^n- and U_1-actions on M_ν commute and thus descend to the respective actions on the cut space, $M_\nu^{\leq\delta}$, and we continue to denote their moment maps by $\mu_{\mathbb{R}} : M_\nu^{\leq\delta} \to (\mathfrak{t}^n)^*$ and $\Phi : M_\nu^{\leq\delta} \to \mathbb{R}_{\geq 0}$, respectively. As explained in Section 3.1, the cut space $M_\nu^{\leq\delta}$ can be decomposed into the disjoint union:

$$M_\nu^{\leq\delta} \cong M_\nu^{<\delta} \sqcup \mathcal{Z}_\nu^\delta, \tag{3.14}$$

where we have defined the **interior**:

$$M_\nu^{<\delta} := \left\{ [z,w] \in M_\nu \mid \tfrac{1}{2}\|w\|^2 < \delta \right\}, \tag{3.15}$$

and the **boundary**:

$$\mathcal{Z}_\nu^\delta := \Phi^{-1}(\delta)/U_1, \tag{3.16}$$

of the cut space $M_\nu^{\leq\delta}$. The interior $M_\nu^{<\delta}$ can be thought of as both a subvariety of the original hypertoric variety M_ν, or as a subvariety of the cut space $M_\nu^{\leq\delta}$.

As it stands so far, Definition 3.3 of the cut space $M_\nu^{\leq\delta}$ refers to the global U_1-action on M_ν, but it is more informative combinatorially to restrict the action to an extended core component $\mathcal{E}_A$, where $A \subseteq \{1,\ldots,N\}$. This is because the U_1-action can then be described combinatorially via the inclusion 3.8 introduced in Section 3.3, providing us with a more concrete grasp of the connection between the geometry of the cut space $M_\nu^{\leq\delta}$ and how the cutting procedure is reflected in the hyperplane arrangement $\mathcal{A}$

Definition 3.6. Let M_ν be a hypertoric variety and let $M_\nu^{\leq\delta}$ be its cut space for some $\delta \geq 0$. Given a subset $A \subseteq \{1, \dots, N\}$, we define the **cut component** $\mathcal{E}_A^{\leq\delta}$ of $M_\nu^{\leq\delta}$ to be the subvariety:

$$\mathcal{E}_A^{\leq\delta} := \left[\mathcal{E}_A \cap M_\nu^{<\delta}\right] \sqcup \left[\Phi_A^{-1}(\delta)/U_1\right] \subsetneq M_\nu^{\leq\delta}.$$

Equivalently, a cut component $\mathcal{E}_A^{\leq\delta}$ can be thought of as the symplectic cut of the corresponding extended core component, $\mathcal{E}_A^{\leq\delta} \cong (\mathcal{E}_A \times \mathbb{C}) /\!\!/_\delta \, U_1$, and thus it can be identified with the disjoint union, $\mathcal{E}_A^{\leq\delta} \cong \mathcal{E}_A^{<\delta} \sqcup \mathcal{Z}_A^\delta$, where:

$$\mathcal{E}_A^{<d} \cong \mathcal{E}_A \cap M_\nu^{<d}, \qquad \text{and} \qquad \mathcal{Z}_A^\delta := \Phi_A^{-1}(\delta)/U_1 \tag{3.17}$$

Analogously to the definitions of the cut space interior $M_\nu^{<\delta}$ (3.4) and the cut space boundary $\mathcal{Z}_\nu^\delta$ (3.16), we say that $\mathcal{E}_A^{<d}$ and $\mathcal{Z}_A^\delta$ are the **cut component interior** and the **cut component boundary** of $\mathcal{E}_A^{\leq\delta}$, respectively.

3.5 Moment Polyptychs

Recall, from the preamble to this chapter, that there is a correspondence between symplectic toric varieties and moment polyhedra from the work of Delzant [Del88] in the case of manifolds, and of Lerman and Tolman [LT97] in the case of orbifolds, and furthermore one between hypertoric varieties and hyperplane arrangements from the work of Bielawski and Dancer [BD00]. In both cases, the connection is formed via a moment map, which maps the geometric object onto the combinatorial object. This section is dedicated towards studying the image of a cut space $M_\nu^{\leq\delta}$ under the real moment map $\mu_\mathbb{R}$, and seeing what combinatorial results arise from this.

In our situation, each cut component $\mathcal{E}_A$ is a symplectic toric variety in its own right, since the residual torus T^n acts in a Hamiltonian and effective way on $\mathcal{E}_A$, whose Kähler structure comes from the real Kähler two-form $\omega_\mathbb{R}$ that descends from M_ν. Our choice of U_1-action in (3.12) guarantees that $\mathcal{E}_A^{\leq\delta}$ will be compact, since the circle moment map Φ is proper. On the combinatorial side, symplectic cutting has the effect of truncating the corresponding region Δ_A to $\mathcal{E}_A$, by intersecting it with a half-space whose normal vector is oriented inwards (i.e., directed towards $\Delta_\emptyset$) by the U_1-action, as mentioned in [Ler95, Remark 1.5].

Proposition 3.7. *The image $\mu_\mathbb{R}(\mathcal{E}_A^{\leq\delta})$ of the cut component $\mathcal{E}_A^{\leq\delta}$ in $(\mathfrak{t}^n)^*$ coincides with the convex polytope:*

$$\mu_\mathbb{R}(\mathcal{E}_A^{\leq\delta}) = \Delta_A \cap \{\, y \in (\mathfrak{t}^n)^* \mid \langle y,\, u_A\rangle + \delta + \lambda_A \geq 0 \,\}\,.$$

Proof. Recall, from (3.9), that:

$$u_A = -\sum_{i\in A} u_i, \quad e_A = -\sum_{i\in A} e_i, \quad \text{and} \quad \lambda_A = \sum_{i\in A} \lambda_i.$$

Given a point $\big([z,w],\xi\big) \in \rho^{-1}(\delta) \cap (\mathcal{E}_A \times \mathbb{C})$, observe that:

$$\begin{aligned}
\big\langle \mu_{\mathbb{R}}[z,w],\, u_A \big\rangle &= \big\langle \mu_{\mathbb{R}}[z,w],\, \pi_*(e_A) \big\rangle \\
&= \big\langle\, (\pi^* \circ \mu_{\mathbb{R}})\,[z,w],\, e_A \big\rangle \\
&= \big\langle \phi_{\mathbb{R}}(z,w) - \lambda,\, e_A \big\rangle \\
&= \big\langle \phi_{\mathbb{R}}(z,w),\, e_A \big\rangle - \langle \lambda,\, e_A \rangle \\
&= -\left\langle \sum_{i=1}^{N} \left(|z_i|^2 - |w_i|^2\right) \epsilon_i,\, \sum_{i\in A} e_i \right\rangle + \left\langle \lambda,\, \sum_{i\in A} e_i \right\rangle \\
&= \tfrac{1}{2} \sum_{i\in A} |w_i|^2 + \lambda_A \\
&\leq \delta + \lambda_A,
\end{aligned}$$

where the inequality $\frac{1}{2}\sum_{i\in A}|w_i|^2 \leq \delta$ comes from the symplectic cut. Hence:

$$[z,w] \in \mathcal{E}_A^{\leq\delta} \qquad \text{if, and only if} \qquad \big\langle \mu_{\mathbb{R}}[z,w],\, u_A \big\rangle + \delta + \lambda_A \geq 0. \qquad \square$$

Applying Proposition 3.7 to each cut component $\mathcal{E}_A^{\leq\delta}$ essentially *"truncates"* the arrangement $\mathcal{A}$, by trimming down any region Δ_A of $\mathcal{A}$ that is unbounded. As the half-spaces that appear in Proposition 3.7 are defined using the restricted U_1-action generator u_A, it is clear that the cut space $M_\nu^{\leq\delta}$ should depend on the coorientation of $\mathcal{A}$. Let us now being to formalise this construction by introducing some definitions.

Definition 3.8. Let $M_\nu^{\leq\delta}$ be a cut space and $\mu_{\mathbb{R}} : M_\nu^{\leq\delta} \to (\mathfrak{t}^n)^*$ its moment map for the T^n-action. We define its **moment polyptych**, denoted by $\Delta_\nu^{\leq\delta}$, to be the image of the cut space $M_\nu^{\leq\delta}$ under $\mu_{\mathbb{R}}$:

$$\Delta_\nu^{\leq\delta} := \mu_{\mathbb{R}}(M_\nu^{\leq\delta}) \subseteq (\mathfrak{t}^n)^*. \tag{3.18}$$

Similarly, we define the **polyptych boundary**, denoted by Π_ν^δ, to be the image of the cut space boundary $\mathcal{Z}_\nu^\delta$ under $\mu_{\mathbb{R}}$:

$$\Pi_\nu^\delta := \mu_{\mathbb{R}}(\mathcal{Z}_\nu^\delta) \subseteq \Delta_\nu^{\leq\delta}. \tag{3.19}$$

Likewise, for each subset $A \subseteq \{1,\ldots,N\}$, we define the **polyptych component**, denoted by $\Delta_A^{\leq\delta}$, to be the image of the cut space component $\mathcal{E}_A^{\leq\delta}$ under $\mu_{\mathbb{R}}$:

$$\Delta_A^{\leq\delta} := \mu_{\mathbb{R}}(\mathcal{E}_A^{\leq\delta}) = \Delta_A \cap \{\, x \in (\mathfrak{t}^n)^* \mid \langle x,\, u_A\rangle + \lambda_A + \delta \geq 0 \,\} \subseteq \Delta_\nu^{\leq\delta}, \tag{3.20}$$

and also we define the **polyptych boundary component**, denoted by Π_A^δ, to be:

$$\Pi_A^\delta := \mu_{\mathbb{R}}(\mathcal{Z}_A^\delta) = \Pi_\nu^\delta \cap \Delta_A^{\leq\delta} = \{\, x \in (\mathfrak{t}^n)^* \mid \langle x,\, u_A\rangle + \lambda_A + \delta = 0 \,\} \subseteq \Delta_A^{\leq\delta}.$$

Having introduced these definitions, the value $\delta \in \mathbb{R}_{\geq 0}$ will be a regular one for the moment map $\rho : M_\nu \times \to \mathbb{R}_{\geq 0}$, provided that the boundary Π_ν^δ of the moment polyptych $\Delta_\nu^{\leq\delta}$ avoids passing through any vertex of the hyperplane arrangement $\mathcal{A}$.

The term "moment polyptych" was chosen to reflect the fact that the cut space $M_\nu^{\leq\delta}$, and hence its moment polyptych $\Delta_\nu^{\leq\delta}$ in $(\mathfrak{t}^n)^*$, both depend on the hyperplane arrangement $\mathcal{A}$ *in addition to* a choice of distinguished base region, $\Delta_\emptyset$. This is due to the fact that changing the coorientation of just one hyperplane changes the residual U_1-action generator u_A on each cut component $\mathcal{E}_A^{\leq\delta}$. To say this in a more succinct manner, the moment polyptych $\Delta_\nu^{\leq\delta}$ depends on both the hyperplane arrangement *and* its poset of regions $\mathcal{P}(\mathcal{A})$, which is defined relative to a distinguished base region $\Delta_\emptyset$ as was discussed at the end of Section 2.1.

3.6 Examples

Finally, let us present some examples.

Example 3.9. Let $M_\nu = T^*\mathbb{CP}^1$ and $X_\nu = \mathbb{CP}^1$ be as in Example 2.4. For any $\delta \in \mathbb{R}_{\geq 0}$, we form the cut space $M_\nu^{\leq\delta} = (T^*\mathbb{CP})^{\leq\delta}$. We see that $\Delta_\emptyset^{\leq\delta} \cong \Delta_\emptyset$ since the core of M_ν is irreducible, so that $\mathcal{C} = X_\nu\mathbb{CP}^1$. Its moment polyptych $\Delta_\nu^{\leq\delta}$ is presented in Figure 3.1.

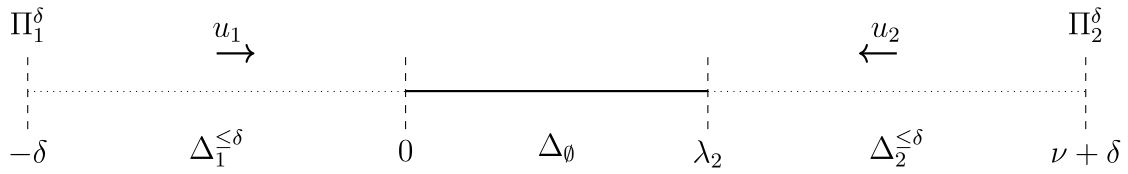


Figure 3.1: Moment polyptych $\Delta_\nu^{\leq\delta}$ of $M_\nu = T^*\mathbb{CP}^1$.

Example 3.10. Let X_ν be the resolution of $\mathbb{C}^2/\mathbb{Z}^3$ and M_ν is hyperkähler analogue, as in Example 2.5. Choose $\delta > \lambda_3$ to avoid cutting into the reducible core $\mathcal{C}$, so that $\Delta_\emptyset^{\leq\delta} \cong \Delta_\emptyset$ and $\Delta_2^{\leq\delta} \cong \Delta_2$. Then the moment polyptych $\Delta_\nu^{\leq d}$ of the cut space $M_\nu^{\leq\delta}$ is presented in Figure 3.2.

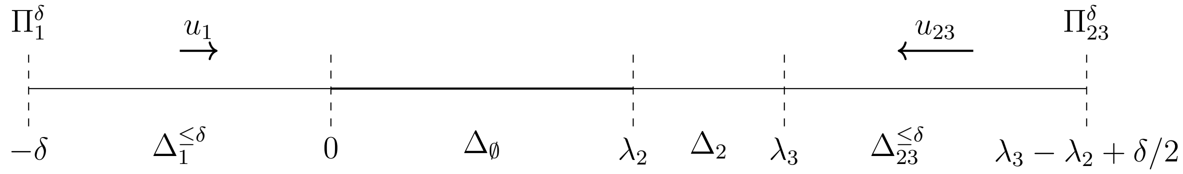


Figure 3.2: Moment polyptych $\Delta_\nu^{\leq\delta}$ of the cut space $(T^*M_\nu)^{\leq\delta}$ in $\mathfrak{t}^* \cong \mathbb{R}$, where X_ν is the resolution of $\mathbb{C}^2/\mathbb{Z}^3$.

Example 3.11. Let $M_\nu = T^*\mathbb{CP}^2$ and $X_\nu = \mathbb{CP}^2$ be as in Example 2.6. For any $\delta \in \mathbb{R}_{\geq 0}$, form the cut space $M_\nu^{\leq\delta} = (T^*\mathbb{CP}^2)^{\leq\delta}$. As $\mathcal{C}$ is irreducible again, we have $\mathcal{C} = X_\nu = \mathbb{CP}^2$. Its moment polyptych $\Delta_\nu^{\leq\delta}$ is presented in Figure 3.3.

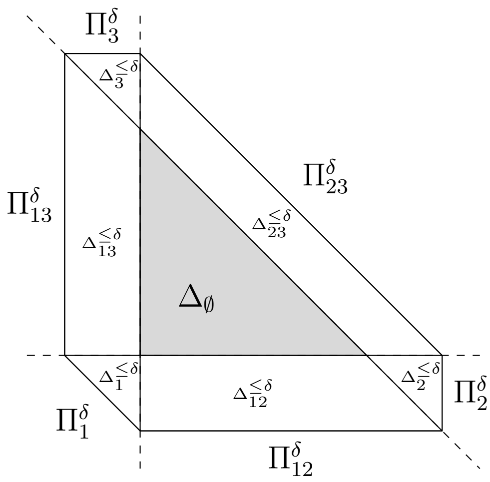


Figure 3.3: Moment polyptych $\Delta_\nu^{\leq\delta}$ of the cut space $M_\nu^{\leq\delta} = (T^*\mathbb{CP}^2)^{\leq\delta}$.

Example 3.12. Now consider $M_\nu = T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $X_\nu = \mathbb{CP}^1 \times \mathbb{CP}^1$ from Example 2.7. Choose any value $\delta \in \mathbb{R}_{\geq 0}$ to form the cut space $M_\nu^{\leq\delta}$, whose moment polyptych $\Delta_\nu^{\leq\delta}$ is presented in Figure 3.4.

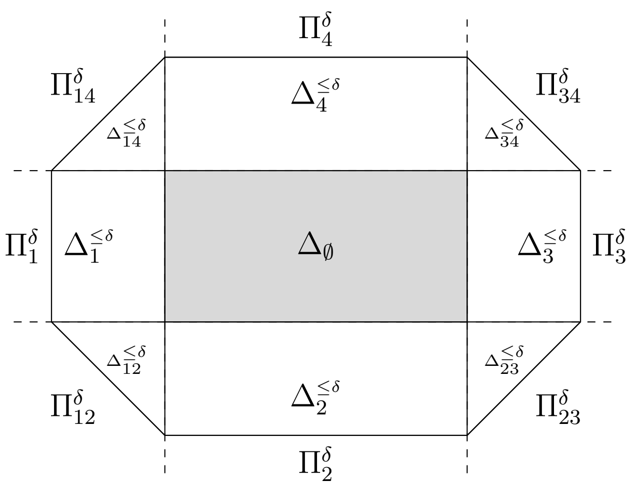
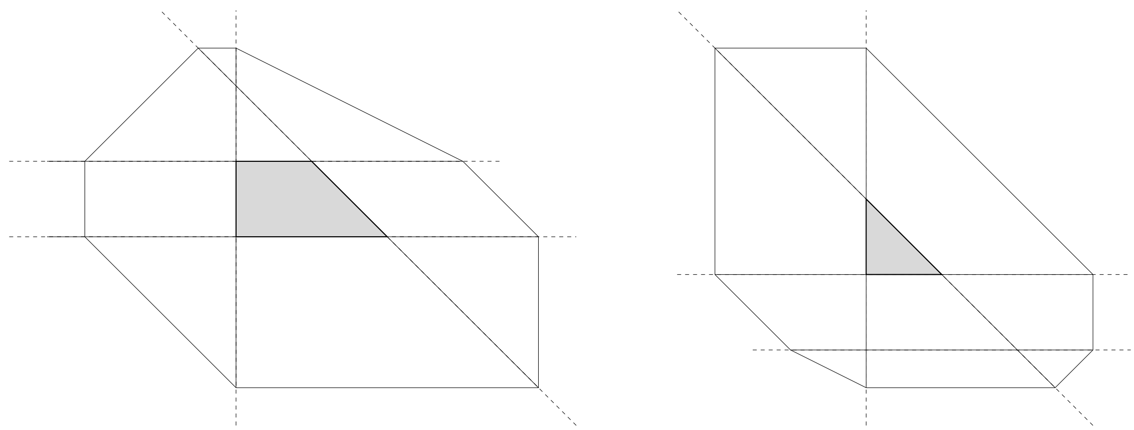


Figure 3.4: Moment polyptych $\Delta_\nu^{\leq\delta}$ in $(\mathfrak{t}^2)^* \cong \mathbb{R}^2$ of the cut space $M_\nu^{\leq\delta}$ when $M_\nu \cong T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

Example 3.13. Finally, consider the hyperkähler analogue M_ν to the first Hirzebruch surface $X_\nu = \mathcal{H}_1$ from Example 2.8. Choose $\delta \in \mathbb{R}_{\geq 0}$ large enough to avoid cutting into its core $\mathcal{C}$, since $\mathcal{C}$ is reducible now. In this case, the moment polyptych $\Delta_\nu^{\leq\delta}$ is presented in Figure 3.5a.

On the other hand, if we instead take M_ν to be the hyperkähler analogue to $X_\nu = \mathbb{CP}^2$, then cutting at δ we get obtain the moment polyptych $\Delta_\nu^{\leq\delta}$ in Figure 3.5b. We see then, that different hyperkähler analogues M_ν give rise to different cut spaces $M_\nu^{\leq\delta}$ and hence different moment polyptychs $\Delta_\nu^{\leq\delta}$. In particular, just changing the coorientation of a hyperplane in the arrangement $\mathcal{A}$ can drastically alter the cut space $M_\nu^{\leq\delta}$, since then the U_1-action generators are all different.



(a) Polyptych $\Delta_\nu^{\leq\delta}$ when $X_\nu \cong \mathcal{H}_1$.

(b) Polyptych $\Delta_\nu^{\leq\delta}$ when $X_\nu \cong \mathbb{CP}^1$.

Figure 3.5: Moment polyptychs $\Delta_\nu^{\leq\delta}$ arising from two different coorientations for the arrangement $\mathcal{A}$.

3.7 Properties of Cut Spaces

Let M_ν be a hypertoric variety and $\mathcal{A}$ be its hyperplane arrangement in $(\mathfrak{t}^n)^*$. In Section 2.5, we proposed a definition for a hypertoric subvariety of M_ν, and showed that each flat $H_\mathcal{F} \in L(\mathcal{A})$ of the arrangement corresponded to a hypertoric subvariety $M_\mathcal{F}$ in 2.13. In Proposition 2.14, we identified the restricted arrangement $\mathcal{A}^\mathcal{F}$ with its own corresponding hyperplane arrangement $\mathcal{A}^{\operatorname{crk}\mathcal{F}}$ in the affine space $H_\mathcal{F}$.

Let us continue in this manner by showing that, if $M_\mathcal{F}$ is a hypertoric subvariety of M_ν, then its cut space $M_\mathcal{F}^{\leq\delta}$ is a closed Kähler subvariety of $M_\nu^{\leq\delta}$. Furthermore, we shall show that its moment polyptych $\Delta_\mathcal{F}^{\leq\delta}$ in $(\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$ can be identified with its intersection $\Delta_\nu^{\leq\delta} \cap H_\mathcal{F}$ in $(\mathfrak{t}^n)^*$.

Proposition 3.14. *Let M_ν be a hypertoric variety and $\mathcal{A}$ be its simple hyperplane arrangement in $(\mathfrak{t}^n)^*$. Let $M_\mathcal{F}$ be the hypertoric subvariety of M_ν determined by the flat $H_\mathcal{F} \in L(\mathcal{A})$ for a given flat subset $\mathcal{F} \subseteq \{1, \ldots, N\}$, and let $\mathcal{A}^\mathcal{F}$ be its restricted arrangement in the affine space $H_\mathcal{F}$. Then, for a suitably large $\delta \in \mathbb{R}_{\geq 0}$, we have the following:*

(i) the cut space $M_{\mathcal{F}}^{\leq\delta}$ is a closed Kähler subvariety of $M_{\nu}^{\leq\delta}$;

(ii) the moment polyptych $\Delta_{\mathcal{F}}^{\leq\delta}$ of $M_{\mathcal{F}}^{\leq\delta}$ can be identified with the intersection $\Delta_{\nu}^{\leq\delta}\cap H_{\mathcal{F}}$ of the moment polyptych $\Delta_{\nu}^{\leq\delta}$ of $M_{\nu}^{\leq\delta}$ with the affine subspace $H_{\mathcal{F}}$.

Proof. For (i), note that the embedding $\iota : T^*\mathbb{C}^{\mathcal{F}^c} \hookrightarrow T^*\mathbb{C}^N$ defined in (2.23) is clearly U_1-equivariant. Thus the induced hyperkähler embedding $\bar{\iota} : M_{\mathcal{F}} \hookrightarrow M_{\nu}$ is additionally $\left(T^{\operatorname{crk}\mathcal{F}} \times U_1\right)$-equivariant. It is straightforward to see then, that:

$$M_{\mathcal{F}}^{\leq\delta} = M_{\nu}^{\leq\delta} \cap \{[z,w] \in M_{\nu}^{\leq\delta} \mid \text{if } i \in \mathcal{F} \text{ then } z_i = w_i = 0\}$$

is a Kähler subvariety of $M_{\nu}^{\leq\delta}$, whose Kähler two-form is $\omega_{\mathbb{R}}^{\mathcal{F}} = \omega_{\mathbb{R}}|_{M_{\mathcal{F}}^{\leq\delta}}$. It is closed since $M_{\mathcal{F}}^{\leq\delta}$ is cut out from $M_{\nu}^{\leq\delta}$ by the hypersurfaces $\{z_i = 0\}$ and $\{w_i = 0\}$ for each $i \in \mathcal{F}$.

For (ii), introduce the half-spaces:

$$\begin{aligned} F_i^+ &:= \{x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \mid \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c} \geq 0\}, \\ F_i^- &:= \{x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \mid \langle x,\, u_i^{\mathcal{F}^c}\rangle + \lambda_i^{\mathcal{F}^c} \leq 0\}, \end{aligned}$$

determined by the hyperplane arrangement $\mathcal{A}^{\operatorname{crk}\mathcal{F}} = \{F_i \mid i \in \mathcal{F}^c\}$ that was introduced in Proposition 2.14. For some $y_0 \in H_{\mathcal{F}}$, recall from (2.30) the map $\eta_{y_0} : (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \to (\mathfrak{t}^n)^*$, and also from (2.31) that its image is $\operatorname{im}(\eta_{y_0}) = H_{\mathcal{F}}$.

If we define:

$$A^{\mathcal{F}^c} := \{\, i \in A \cap \mathcal{F}^c \mid \Delta_A \cap H_{\mathcal{F}} \neq \emptyset \,\} \subseteq \{1, \ldots, N\} \cap \mathcal{F}^c,$$

then the arrangement $\mathcal{A}^{\operatorname{crk}\mathcal{F}}$ in $(\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^*$ corresponding $M_{\mathcal{F}}$ is defined by the regions:

$$\Delta_{A^{\mathcal{F}^c}} := \left(\cap_{i\notin A^{\mathcal{F}^c}} F_i^+\right) \cap \left(\cap_{i\in A^{\mathcal{F}^c}} F_i^-\right).$$

Furthermore, for any $x \in \Delta_{\mathcal{A}^{\mathcal{F}^c}}$ and for each $i \in \mathcal{F}^c$:

$$\begin{aligned} \left\langle \eta_{y_0}(x),\, u_i \right\rangle + \lambda_i &= \left\langle \bar{p}^*(x) + y_0,\, u_i \right\rangle + \lambda_i \\ &= \left\langle \bar{p}^*(x) + y_0,\, \pi_*(e_i) \right\rangle + \lambda_i \\ &= \left\langle (\pi^* \circ \bar{p}^*)(x) + \pi^*(y_0) + \lambda,\, e_i \right\rangle + \left\langle p^*(\pi_{\mathcal{F}^c}^*(x) + \lambda^{\mathcal{F}^c}),\, e_i \right\rangle \\ &= \begin{cases} \left\langle x,\, u_i^{\mathcal{F}^c} \right\rangle + \lambda_i^{\mathcal{F}^c} \geq 0, & \text{if } i \notin A \cap \mathcal{F}^c, \\ \left\langle x,\, u_i^{\mathcal{F}^c} \right\rangle + \lambda_i^{\mathcal{F}^c} \leq 0, & \text{if } i \in A \cap \mathcal{F}^c, \end{cases} \end{aligned}$$

hence $\eta_{y_0}(\Delta_{A^{\mathcal{F}^c}}) = \Delta_A \cap H_{\mathcal{F}}$.

Now suppose that $x \in \Pi^{\delta}_{A^{\mathcal{F}^c}} = \{x \in (\mathfrak{t}^{\operatorname{crk}\mathcal{F}})^* \mid \langle x,\, u^{\mathcal{F}^c}_{A^{\mathcal{F}^c}}\rangle + \delta + \lambda^{\mathcal{F}^c}_{A^{\mathcal{F}^c}} = 0\}$. Then:

$$\begin{aligned}\langle \eta_{y_0}(x),\, u_A\rangle &= \langle (\pi^* \circ \bar{p}^*)(x) + \pi^*(y_0),\, e_A\rangle \\ &= \langle (p^* \circ \pi^*_{\mathcal{F}^c})(x),\, e_A\rangle + \langle p^*(\lambda^{\mathcal{F}^c}) - \lambda, e_A\rangle \\ &= \langle x,\, u^{\mathcal{F}^c}_{A^{\mathcal{F}^c}}\rangle + \lambda^{\mathcal{F}^c}_{A^{\mathcal{F}^c}} - \lambda_A = -\delta - \lambda_A,\end{aligned}$$

that is to say:

$$\langle \eta_{y_0}(x),\, u_A\rangle + \delta + \lambda_A = 0, \qquad \text{for all } x \in \Pi^{\delta}_{A^{\mathcal{F}^c}}.$$

Therefore: $\eta_{y_0}(\Pi^{\delta}_{A^{\mathcal{F}^c}}) = \Pi^{\delta}_A$, proving (ii). □

In light of Proposition 3.14 and as briefly mentioned in Remark 2.16, we have the following definition.

Definition 3.15. Let $\mathcal{A}$ be a hyperplane arrangement in $(\mathfrak{t}^n)^*$ for a hypertoric variety M_ν, let $M_\nu^{\leq\delta}$ be its cut space with moment polyptych $\Delta_\nu^{\leq\delta}$ in $(\mathfrak{t}^n)^*$. Given a flat $H_{\mathcal{F}} \in L(\mathcal{A})$ with $\mathcal{F} \subseteq \{1, \dots, N\}$, we define a **moment subpolyptych**, denoted $\Delta_{\mathcal{F}}^{\leq\delta}$, to be the intersection:

$$\Delta_{\mathcal{F}}^{\leq\delta} := \Delta_\nu^{\leq\delta} \cap H_{\mathcal{F}}.$$

In calling M_ν a hypertoric *variety*, we have been intentionally ambiguous to whether M_ν is a manifold or an orbifold. The reason for this is that the symplectic cutting operation is closed within the category of symplectic toric orbifolds equipped with a Hamiltonian U_1-action, but this is not the case for manifolds when the circle only acts locally freely. When it comes to forming a cut space $M_\nu^{\leq\delta}$, the result of Theorem 3.17 is that, if the core $\mathcal{C}$ of the hypertoric variety M_ν is reducible, then the cut space $M_\nu^{\leq\delta}$ is an orbifold even – when if M_ν itself is smooth.

Given a point $v \in \Delta_\nu^{\leq\delta}$ of the moment polyptych $\Delta_\nu^{\leq\delta}$, denote by $\mathcal{I}_v$ the flat subset:

$$\mathcal{I}_v := \{\, i \mid v \in H_i \,\} \subseteq \{1, \dots, N\}. \tag{3.21}$$

That is, $\mathcal{I}_v$ tracks the indices of which hyperplanes contain the point v, if any. We will assume that each cut space $M_\nu^{\leq\delta}$ is constructed by choosing a $\delta \geq 0$ large enough, so that no part of the core $\mathcal{C}$ gets cut away, i.e., that $\mathcal{Z}_\nu^{\delta} \cap \mathcal{C} = \emptyset$.

Let us begin with the $n = 1$ case first, so that $\dim_{\mathbb{R}} M_\nu = 4$.

Lemma 3.16. *Let M_ν be a four-dimensional hypertoric manifold whose core $\mathcal{C}$ is reducible. Then, for a sufficiently large value $\delta \in \mathbb{R}_{\geq 0}$ such that $\mathcal{Z}_\nu^{\delta} \cap \mathcal{C} = \emptyset$, the cut space $M_\nu^{\leq\delta}$ is an orbifold.*

Proof. As M_ν is four-dimensional, let us identify $(\mathfrak{t}^1)^* \cong \mathbb{R}$. Then its arrangement $\mathcal{A}$ lies in $\mathbb{R}$, and $\mathcal{A}$ must have at least three hyperplanes to guarantee the existence of at least two bounded regions, which will be the images of components of the core, $\mathcal{C}$. Being in $\mathbb{R}$, these regions will therefore just be closed line intervals in $\mathbb{R} = \mu_{\mathbb{R}}(M_\nu)$, with each intersecting pair meeting a at a common vertex.

We shall write $\mathcal{A} = \{H_1, \ldots, H_N\}$ where $N \geq 3$. Assume that M_ν is the hyperkähler analogue to the Kähler quotient $X_\nu = \mathcal{E}_\emptyset$. Then $X_\nu \subsetneq \mathcal{C}$ since $\mathcal{C}$ is reducible, and its region $\Delta_\emptyset = \mu_{\mathbb{R}}(X_\nu)$ is a closed line interval in $\mathbb{R}$, it defines the distinguished base region for the poset of regions $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$.

There exist two proper chains within the poset $\mathcal{P}(\mathcal{A})$, each corresponding to the two endpoints of $\Delta_\emptyset$ in $\mathbb{R}$. As $\Delta_\emptyset \neq \mathcal{C}$, there exists an adjacent core component $\mathcal{E}_j \subsetneq \mathcal{C}$, where $1 \leq j \leq N$, such that $\mathcal{E}_j \cap X_\nu \neq \emptyset$ and $M_j \neq X_\nu$. Its corresponding region $\Delta_j = \mu_{\mathbb{R}}(\mathcal{E}_j)$ in $\mathbb{R}$ is another bounded line interval intersecting $\Delta_\emptyset$ in the hyperplane $H_j = \Delta_\emptyset \cap \Delta_j$ (which is just a vertex in $\mathbb{R}$). Considered as a poset element, Δ_j covers $\Delta_\emptyset$ in $\mathcal{P}(\mathcal{A})$, see Figure 3.6 for the simplest $N = 3$ case.

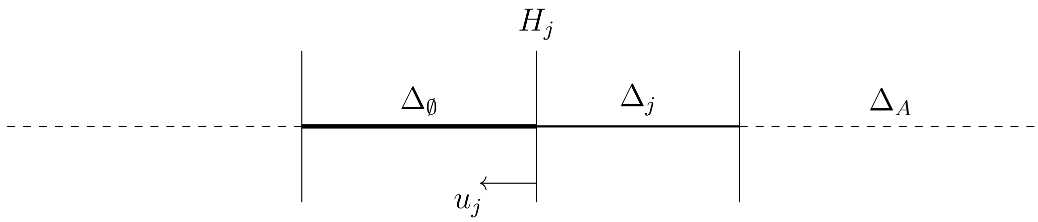


Figure 3.6: Example with $N = 3$. The distinguished region is $\Delta_\emptyset$, the other bounded region is Δ_j, and Δ_A is their lowest upper bound in $\mathcal{P}(\mathcal{A})$.

By continuing along the chain in $\mathcal{P}(\mathcal{A})$ containing Δ_j, one arrives at its lowest upper bound $\Delta_A = \Delta_\emptyset \vee \Delta_i$, where $A \subseteq \{1, \ldots, N\}$ is a subset with $j \in A$ and $|A| \geq 2$. As the real moment map $\mu_{\mathbb{R}}$ surjects onto $\mathbb{R}$, we see that Δ_A is an unbounded interval. Let $\mathcal{E}_A$ denote the non-compact Kähler subvariety corresponding to Δ_A.

Since $\mathcal{A}$ lies within $\mathbb{R}$, the normal vector to each hyperplane H_i must necessarily either $u_i = \pm 1$. Each of the two proper chains in $\mathcal{P}(\mathcal{A})$ consist solely of regions that are separated by hyperplanes whose normal vectors have the same sign. Hence, as $|A| \geq 2$, the restricted U_1-action generator on $\mathcal{E}_A$ is either $u_A = \pm \operatorname{rk} \Delta_A$, where $\operatorname{rk} \Delta_A = |A| \geq 2$. In particular, u_A is not primitive relative to the lattice $\mathfrak{t}^1_{\mathbb{Z}} \cong \mathbb{Z}$, as in Figure 3.7.

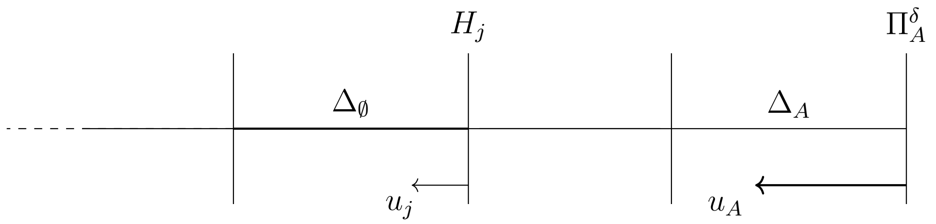


Figure 3.7: Example with $N = 3$. The U_1-action generator u_A is non-primitive, with $|u_A| \geq 2$.

Hence, provided that $\delta \geq 0$ is sufficiently large, then the intersection of the extended core component $\mathcal{E}_A$ with level-set $\Phi^{-1}(\delta)$ is just a point, $p_A := \Phi_A^{-1}(\delta)$, on which U_1 acts locally freely. Hence, after taking the symplectic cut to obtain the cut space $M_\nu^{\leq\delta}$, the point p_A is an orbifold point whose orbifold structure group Γ_{p_A} has order $m_{p_A} = |A|$ from A.5. $\square$

The generalisation of Lemma 3.16 to a higher-dimensional hypertoric variety is now quite straightforward. Indeed, if a $4n$-dimensional hypertoric variety M_ν, with hyperplane arrangement $\mathcal{A}$, has a reducible core $\mathcal{C}$ then we can find a four-dimensional hypertoric subvariety $M_{\mathcal{F}}$ of M_ν, for some flat $H_{\mathcal{F}} \in L(\mathcal{A})$ of $\mathcal{A}$ with rk $H_{\mathcal{F}} = n-1$, whose own core is reducible. Then, Lemma 3.16 implies that the cut subspace $M_{\mathcal{F}}^{\leq\delta}$ is a suborbifold of $M_\nu^{\leq\delta}$.

Theorem 3.17. *Let M_ν be a $4n$-dimensional hypertoric manifold whose core $\mathcal{C}$ is reducible. Then, for $\delta \geq 0$ sufficiently large so that $\mathcal{Z}_\nu^\delta \cap \mathcal{C} = \emptyset$, the cut space $M_\nu^{\leq\delta}$ is an orbifold.*

Proof. Let M_ν be the hyperkähler analogue to the Kähler quotient, $X_\nu = \mathcal{E}_\emptyset$. Then X_ν is a compact Kähler subvariety of M_ν that forms one of the irreducible components of the core, $X_\nu \subsetneq \mathcal{C}$. Its image is the bounded region $\Delta_\emptyset = \mu_\mathbb{R}(X_\nu)$ in $(\mathfrak{t}^n)^*$. There exists a subset $A \subseteq \{1, \ldots, N\}$ for which the bounded region Δ_A is adjacent to $\Delta_\emptyset$ in $(\mathfrak{t}^n)^*$, meaning that $\Delta_A \cap \Delta_\emptyset \neq \emptyset$ and $\Delta_A \neq \Delta_\emptyset$. As the two regions intersect, there exists a vertex $v \in \Delta_A \cap \Delta_\emptyset$ which equals the intersection of n hyperplanes as $\mathcal{A}$ is simple. We can consider v to be a flat itself, $\{v\} = H_{\mathcal{I}_v} = \cap_{j\in\mathcal{I}_v} H_j$ which we represent using the flat subset $\mathcal{I}_v \subseteq \{1, \ldots, N\}$ with $|\mathcal{I}_v| = n$. Furthermore, observe that $A \subseteq \mathcal{I}_v$.

There exists an element $j \in A$ such that the hyperplane H_j separates Δ_A from $\Delta_\emptyset$. Note that the choice of H_j may not necessarily be unique. Denote the flat subset obtained by removing j from $\mathcal{I}_v$ by $\mathcal{J}_{v,j} := \mathcal{I}_v \setminus \{j\}$, so that $|\mathcal{J}_{v,j}| = n-1$. Its flat $H_{\mathcal{J}_{v,j}} = \cap_{i\in\mathcal{J}_{v,j}} H_i$ is then an affine line in $(\mathfrak{t}^n)^*$. Since H_j separates Δ_A from $\Delta_\emptyset$, their intersections $\Delta_A \cap H_{\mathcal{J}_{v,j}}$ and $\Delta_\emptyset \cap H_{\mathcal{J}_{v,j}}$ are both edges of Δ_A and $\Delta_\emptyset$ respectively, meeting at v, as demonstrated in Figure 3.8.

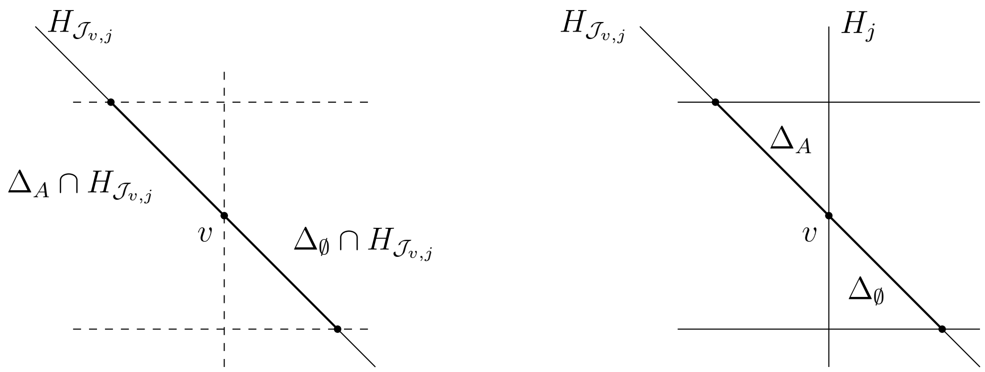


Figure 3.8: Restricted hyperplane arrangement $\mathcal{A}^{\mathcal{J}_{v,j}}$ of the hypertoric subvariety $M_{\mathcal{J}_{v,j}}$ and the hyperplane arrangement $\mathcal{A}$ for M_ν, respectively.

The four-dimensional hypertoric subvariety $M_{\mathcal{J}_{v,j}}$ of M_ν, that determines the flat $H_{\mathcal{J}_{v,j}}$ via its real moment map $H_{\mathcal{J}_{v,j}} = \mu_{\mathbb{R}}(M_{\mathcal{J}_{v,j}})$, satisfies the hypotheses of Lemma 3.16. Its four-dimensional cut space $M^{\leq \delta}_{\mathcal{J}_{v,j}}$ is therefore an orbifold, with at least one orbifold point belonging to its boundary $\mathcal{Z}^{\delta}_{\mathcal{J}_{v,j}}$, provided that we cut $M_{\mathcal{J}_{v,j}}$ at a sufficiently large value for $\delta \in \mathbb{R}_{\geq 0}$. $\square$

An equivalent statement of Theorem 3.17 is that the cut space $M^{\leq\delta}_\nu$ is a manifold only if $M_\nu \cong T^*X_\nu$, where the Kähler quotient X_ν is a product of projective spaces, i.e., $X_\nu = \mathbb{CP}^{k_1} \times \ldots \times \mathbb{CP}^{k_m}$ with $\sum_{i=1}^m k_i = n$, see Theorems 7.1 and 7.2 in [BD00].

A significant consequence of Theorem 3.17 is that the cut space $M^{\leq\delta}_\nu$ of a generic hypertoric variety M_ν will be a compact Kähler *orbifold*. Hence, in Chapter 4, we will have to use the Kawasaki-Riemann-Roch formula in the place of the Hirzebruch-Riemann-Roch formula, since the latter only applies to smooth manifolds.

Part II

Equivariant Localisation

Chapter 4

Riemann-Roch-Hirzebruch Theorem

In this Chapter, we introduce the Hirzebruch-Riemann-Roch formula in Theorem 4.8, which calculates the $\mathbb{C}$-vector space $H^0(M;\mathcal{L})$ of holomorphic sections of a suitable line bundle $\mathcal{L}\to M$ over a compact Kähler manifold M. It is this space $H^0(M;\mathcal{L})$ that we will base the quantisation of M upon, with the holomorphic sections playing the rôle analogous to the wave functions. General references for this chapter are [Huy05], [GH78], and [Dui11].

4.1 The Dolbeault-Dirac Operator

Suppose that M is a smooth manifold equipped with an almost-complex structure J, then we say that J is **$\boldsymbol{\omega}$-compatible** if, for every point $p\in M$, the bilinear form:

$$g_p(v,w)=\omega_p(Jv,w),\qquad \text{for all } v,w\in T_pM$$

is symmetric and positive-definite. With an ω-compatible almost-complex structure J on M, the exterior algebra of the cotangent bundle T^*M can be equipped with a Dolbeault structure. More precisely, J induces a splitting of the complexified tangent bundle $T_{\mathbb{C}}M$ into the $+\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J:

$$T_{\mathbb{C}}M := TM\otimes_{\mathbb{R}}\mathbb{C}\cong TM^{(1,0)}\oplus TM^{(0,1)},$$

and, similarly, for its complexified cotangent bundle:

$$T^*_{\mathbb{C}}M := T^*M\otimes_{\mathbb{R}}\mathbb{C}\cong\left[\Lambda(T^*M)^{(1,0)}\right]\otimes_{\mathbb{C}}\left[\Lambda(T^*M)^{(0,1)}\right].$$

This splitting of $T^*_{\mathbb{C}}M$ also extends to its various exterior powers, equipping them with the bigrading:

$$\Lambda^m\left(T^*_{\mathbb{C}}M\right)\cong\bigoplus_{i+j=m}\left[\Lambda^i(T^*M)^{(1,0)}\right]\otimes_{\mathbb{C}}\left[\Lambda^j(T^*M)^{(0,1)}\right],$$

and, similarly, the space of differential forms on M also decomposes into a direct sum, according to their bidegrees:

$$\Omega^m(M) := C^\infty\left(M; \Lambda^m(T^*_{\mathbb{C}}M)\right) \cong \bigoplus_{i+j=m} \Omega^{(i,j)}(M),$$

where:

$$\Omega^{(i,j)}(M) := C^\infty\big(M; \left[\Lambda^i(T^*_{\mathbb{C}}M)^{(1,0)}\right] \otimes_{\mathbb{C}} \left[\Lambda^j(T^*_{\mathbb{C}}M)^{(0,1)}\right]\big)$$

is the space of differential forms on M of bidegree (i, j).

We can define the following projection operators:

$$\pi^{(i,j)} : \Omega^\bullet(M) \longrightarrow \Omega^{(i,j)}(M),$$

which project a differential form onto its component of bidegree (i, j). For a differential form $\alpha \in \Omega^{(i,j)}(M)$, one sees that:

$$d\alpha \in \Omega^{(i+1,j)}(M) \oplus \Omega^{(i,j+1)}(M).$$

Using these $\pi^{(i,j)}$ operators, we can define the following differential operators:

$$\begin{aligned} \overline{\partial} &: \Omega^{(i,j)}(M) \longrightarrow \Omega^{(i,j+1)}(M), \\ \partial &: \Omega^{(i,j)}(M) \longrightarrow \Omega^{(i+1,j)}(M), \end{aligned} \qquad \text{by} \qquad \begin{aligned} \overline{\partial} &:= \pi^{(i,j+1)} \circ d, \\ \partial &:= \pi^{(i+1,j)} \circ d, \end{aligned}$$

respectively. From this, we arrive at the almost-complex analogue of the **(i, j)-Dolbeault complex**:

$$\{0\} \longrightarrow \Omega^{(i,0)}(M) \xrightarrow{\overline{\partial}} \Omega^{(i,1)}(M) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{(i,n)}(M). \tag{4.1}$$

The complex in (4.1) is not a genuine differential complex however, since $\overline{\partial}^2 \neq 0$, see, for example, [Dui11, Chapter 2] or [Gui94, Chapter 4]. But, if the almost-complex structure J on M is furthermore *integrable*, then J becomes a bonafide **complex structure** from the Newlander-Nirenberg theorem, [NN57, Theorem 1.1].

Definition 4.1. Let M be an almost-complex manifold with almost-complex structure J. Then J is said to be **integrable** if either one of the following two conditions holds:

(i) for any $\alpha \in \Omega^\bullet(M)$, one has that $d\alpha = \partial\alpha + \overline{\partial}\alpha$;

(ii) on $\Omega^{(1,0)}(M)$, one has that $\pi^{(0,2)} \circ d = 0$.

Both of the conditions, 4.1 and 4.1 in Definition 4.1, hold when M is a complex manifold [Huy05, Proposition 2.6.15]. So, in a sense, the notion of integrability determines whether an almost-complex structure J is an actual complex structure or not. The reason we are interested specifically in integrable almost-complex structures is due to the following lemma.

Lemma 4.2. *If J is an integrable almost-complex structure, then:*

$$\partial^2 = \overline{\partial}^2 = 0, \qquad \text{and} \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{4.2}$$

Conversely, if $\overline{\partial}^2 = 0$, then the almost-complex structure J is integrable.

As a corollary of Lemma 4.2, when J is integrable, the almost-complex Dolbeault complex in (4.1) becomes exact on the right, since now we can guarantee that $\overline{\partial}^2 = 0$:

$$\{0\} \longrightarrow \Omega^{(i,0)}(M) \xrightarrow{\overline{\partial}} \Omega^{(i,1)}(M) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{(i,n)}(M) \xrightarrow{\overline{\partial}} \{0\}, \tag{4.3}$$

where $n = \dim_{\mathbb{C}} M$. Thenceforth, we shall assume that J is a complex structure on M, as this is the scenario that concerns us. This leads us to the following fundamental theorem, proven by Dolbeault in [Dol53, Théorème 1]:

Theorem 4.3 (Dolbeault). *Let M be an almost-complex manifold, whose almost-complex structure J is integrable. Then the (i,j)-Dolbeault cohomology group $H^{(i,j)}(M)$ is the vector space:*

$$H^{(i,j)}(M) := H^j(M; \Omega^i(M)) \cong \frac{\ker\left(\overline{\partial} : \Omega^{(i,j)}(M) \to \Omega^{(i,j+1)}(M)\right)}{\operatorname{im}\left(\overline{\partial} : \Omega^{(i,j-1)}(M) \to \Omega^{(i,j)}(M)\right)}. \tag{4.4}$$

In Theorem 4.3, observe that, when $i = 0$, the isomorphism in (4.4) becomes:

$$H^{(0,j)}(M) = H^j(M; \mathcal{O}_M). \tag{4.5}$$

In other words, $H^{(0,j)}(M)$ coincides with the j-th cohomology of its sheaf of holomorphic sections on M and, more importantly to us, when $j = 0$ in (4.5), we have:

$$H^{(0,0)}(M) = H^0(M; \mathcal{O}_M) \tag{4.6}$$

is the $\mathbb{C}$-vector space of holomorphic sections of the sheaf $\mathcal{O}_M \to M$. When M is compact, then the cohomology groups $H^{(i,j)}(M)$ are guaranteed to be finite-dimensional by [CS53].

Now consider the case when M is a compact symplectic manifold with symplectic two-form $\omega \in \Omega^2(M)$, such that the cohomology class of ω is integral, $[\omega] \in H^2(M; \mathbb{Z})$. Suppose that there exists a Hermitian line bundle $\mathcal{L} \to M$ with a Hermitian connection ∇, whose first Chern class is $c_1(\mathcal{L}) = [\omega] \in H^2(M; \mathbb{Z})$ and whose curvature with respect to ∇ is $R(\mathcal{L}) = (2\pi/\sqrt{-1})\omega$. Such a line bundle $\pi : \mathcal{L} \to M$ that possesses these properties is called a **pre-quantum line bundle** over M.

Definition 4.4. Let M be a complex manifold and let $\pi : \mathcal{L} \to M$ be a holomorphic pre-quantum bundle over M. Then:

$$\Omega^{(i,j)}(M; \mathcal{L}) := C^\infty\big(M; \left[\Lambda^i(T^*_{\mathbb{C}}M)^{(1,0)}\right] \otimes_{\mathbb{C}} \left[\Lambda^j(T^*_{\mathbb{C}}M)^{(0,1)}\right] \otimes \mathcal{L}\big)$$

defines the $\mathbb{C}$-vector space of **$\mathcal{L}$-twisted (i,j)-forms** on M.

As mentioned in [Dui11, Chapter 2.2], a Hermitian metric on M induces Hermitian structures on both the tangent bundle TM and on the fibres of $\mathcal{L}$. Moreover, there exists a complex-linear isomorphism $T_{\mathbb{C}}M \cong T^*_{\mathbb{C}}M$ which transplants the Hermitian structure from $T_{\mathbb{C}}M$ onto one on its dual, $T^*_{\mathbb{C}}M$. For brevity, let us set:

$$E^j := T^*_{\mathbb{C}}M^{(0,j)} = \Lambda^j T^*_{\mathbb{C}}M^{(0,1)}, \quad \text{and} \quad E := \bigoplus_{j=0}^{n} E^j.$$

Then for each $0 \leq j \leq n$, the bundle E^j inherit a Hermitian structure from $T^*_{\mathbb{C}}$, and so does the direct sum E by requiring the summands to be pairwise orthogonal. Finally, the product bundles $E^j \otimes \mathcal{L}$ and $E \otimes \mathcal{L}$ each picks up a Hermitian structure from those of E^j and E, provided that $\mathcal{L}$ has a Hermitian inner product too [Dui11, §2.2].

Definition 4.5. If ∇ denotes the Hermitian connection on $\mathcal{L}$, then the **$\mathcal{L}$-twisted Dolbeault operator**, $\overline{\partial}_{\mathcal{L}}$, is defined to be the operator:

$$\overline{\partial}_{\mathcal{L}} := \overline{\partial} \otimes 1 + 1 \otimes \nabla : E^j \otimes \mathcal{L} \to E^{j+1} \otimes \mathcal{L}.$$

Many of the results above hold for the $\mathcal{L}$-twisted Dolbeault operator $\overline{\partial}_{\mathcal{L}}$ too [Sil96]. In particular, for (4.3) we get the **$\mathcal{L}$-twisted Dolbeault complex**:

$$0 \longrightarrow \Omega^{(i,0)}(M;\mathcal{L}) \xrightarrow{\overline{\partial}_{\mathcal{L}}} \Omega^{(i,1)}(M;\mathcal{L}) \xrightarrow{\overline{\partial}_{\mathcal{L}}} \dots \xrightarrow{\overline{\partial}_{\mathcal{L}}} \Omega^{(i,n)}(M;\mathcal{L}) \longrightarrow 0, \tag{4.7}$$

and, furthermore, we also obtain the $\mathcal{L}$-twisted version of Theorem 4.3.

Theorem 4.6 (**$\mathcal{L}$-twisted Dolbeault**). *Let M be an almost-complex manifold, whose almost-complex structure J is integrable. Suppose that $\mathcal{L} \to M$ is a holomorphic vector bundle over M. Then the **$\mathcal{L}$-twisted (i,j)-Dolbeault cohomology group**, $H^{(i,j)}(M;\mathcal{L})$, is the complex vector space:*

$$H^j(M;\Omega^i(M;\mathcal{L})) \cong \frac{\ker\left(\overline{\partial}_{\mathcal{L}} : \Omega^{(0,j)}(M;\mathcal{L}) \to \Omega^{(0,j+1)}(M;\mathcal{L})\right)}{\operatorname{im}\left(\overline{\partial}_{\mathcal{L}} : \Omega^{(0,j-1)}(M;\mathcal{L}) \to \Omega^{(0,j)}(M;\mathcal{L})\right)}. \tag{4.8}$$

By combining the Hermitian inner product on $E \otimes \mathcal{L}$ with the volume form $\mathrm{vol}(M)$, we can define the **adjoint** operator:

$$\overline{\partial}^*_{\mathcal{L}} : E^j \otimes \mathcal{L} \to E^{j-1} \otimes \mathcal{L},$$

to the $\mathcal{L}$-twisted Dolbeault operator $\overline{\partial}_{\mathcal{L}}$. If we set:

$$E^{\text{even}} := \bigoplus_{j \text{ even}} E^j, \quad \text{and} \quad E^{\text{odd}} := \bigoplus_{j \text{ odd}} E^j,$$

then we can furthermore define the **Dolbeault-Dirac operator**, $\partial\!\!\!/_{\mathbb{C}}$, to be the first-order elliptic differential operator $\partial\!\!\!/_{\mathcal{L}}$, which is given by:

$$\partial\!\!\!/_{\mathbb{C}} := \sqrt{2}\left(\overline{\partial}_{\mathcal{L}} + \overline{\partial}_{\mathcal{L}}^{*}\right) : E^{\text{even}} \otimes \mathcal{L} \to E^{\text{odd}} \otimes \mathcal{L}.$$

Its **index**, $\operatorname{Ind}_{\partial\!\!\!/_{\mathbb{C}}}(M;\mathcal{L})$, is the virtual vector space given by the formal difference:

$$\operatorname{Ind}_{\partial\!\!\!/_{\mathbb{C}}}(M;\mathcal{L}) := \ker(\partial\!\!\!/_{\mathbb{C}}) - \operatorname{coker}(\partial\!\!\!/_{\mathbb{C}}), \tag{4.9}$$

or equivalently as the alternating direct sum of virtual vector spaces:

$$\operatorname{Ind}_{\partial\!\!\!/_{\mathbb{C}}}(M;\mathcal{L}) = \bigoplus_{j\geq 0}(-1)^j H^{(0,j)}(M;\mathcal{L}). \tag{4.10}$$

4.2 The Riemann-Roch-Hirzebruch Theorem

Using (4.10), we can define an important symplectic invariant [Gui94, §3.1].

Definition 4.7. Let M be a Kähler manifold with Kähler two-form ω, and let $\pi : \mathcal{L} \to M$ be a holomorphic pre-quantum line bundle over M. Then the **Riemann-Roch number** of $\mathcal{L}$, denoted by $\chi(M;\mathcal{L})$, is defined to be:

$$\chi(M;\mathcal{L}) := \sum_{j=0}^{n}(-1)^j \dim_{\mathbb{C}} H^{(0,j)}(M;\mathcal{L}). \tag{4.11}$$

When ω is sufficiently positive then $H^{(0,j)}(M;\mathcal{L}) = 0$ for each $j \geq 1$ by Kodaira's vanishing theorem [Kod53]. When this holds, the index becomes:

$$\operatorname{Ind}_{\partial\!\!\!/_{\mathbb{C}}}(M;\mathcal{L}) = H^{(0,0)}(M;\mathcal{L}) \cong H^0(M;\mathcal{L})$$

from (4.10), whereas the Riemann-Roch number becomes:

$$\chi(M;\mathcal{L}) = \dim_{\mathbb{C}} H^0(M;\mathcal{L}). \tag{4.12}$$

However, calculating the Euler characteristic in (4.12) is an entirely different matter. Though in the instance where (M,ω) is a compact Kähler manifold and $\pi : \mathcal{L} \to M$ is a holomorphic pre-quantum line bundle with $c_1(\mathcal{L}) = [\omega]$ so that, by Hodge theory, the Riemann-Roch number $\chi(M;\mathcal{L})$, in (4.12) coincides with the dimension of the index $\operatorname{Ind}_{\partial\!\!\!/_{\mathcal{L}}}(M;\mathcal{L})$, in (4.9) for the Dolbeault-Dirac operator, $\overline{\partial}_{\mathcal{L}}$. That is to say:

$$\chi(M;\mathcal{L}) = \dim_{\mathbb{C}} \operatorname{Ind}_{\partial\!\!\!/_{\mathbb{C}}}(M;\mathcal{L}).$$

Thus, to calculate the Euler characteristic $\chi(M;\mathcal{L})$, it suffices to calculate the Dirac-Dolbeault index, $\operatorname{Ind}_{\partial\!\!\!/_{\mathcal{L}}}(M;\mathcal{L})$. An elegant way to do so is by using the **Atiyah-Singer index formula**, [AS68, Theorem 4.3]:

Theorem 4.8 (Atiyah-Singer index theorem). *Let (M, ω) be a compact Kähler manifold and let $\pi : \mathcal{L} \to M$ be a holomorphic pre-quantum line bundle over M. Then the Atiyah-Singer index formula in this case states that:*

$$\chi(M; \mathcal{L}) = \dim_{\mathbb{C}} \operatorname{Ind}_{\not{\partial}_{\mathbb{C}}}(M; \mathcal{L}) = \int_M \operatorname{Td}(TM) \wedge \operatorname{Ch}(\mathcal{L}), \tag{4.13}$$

*where $\operatorname{Td}(TM)$ and $\operatorname{Ch}(\mathcal{L})$ are the **Todd class** of the tangent bundle TM and the **Chern character** of $\mathcal{L}$, respectively.*

The statement in Theorem 4.8 of the Atiyah-Singer index theorem is actually a particular case of the actual index theorem. The statement of Theorem 4.8 is also known as the **Hirzebruch-Riemann-Roch theorem**, having been proven originally for complex projective algebraic varieties by Hirzebruch in [Hir66].

The integral (4.13) in Theorem 4.8 has the characteristic class $\operatorname{Td}(TM) \wedge \operatorname{Ch}(\mathcal{L})$ as its integrand, and is made up from the Todd class $\operatorname{Td}(TM)$ of the tangent bundle TM, and from the Chern character $\operatorname{Ch}(\mathcal{L})$ of the holomorphic pre-quantum line bundle $\pi : \mathcal{L} \to M$ over M. To introduce the Todd class $\operatorname{Td}(TM)$, we shall express it by the means of the **splitting principle** [BT82], which states verbatim that:

Theorem 4.9 (Splitting principle). *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

Theorem 4.9) permits us to assume that a vector bundle $\pi : E \to M$ decomposes as:

$$E \cong V_1 \oplus \ldots \oplus V_n, \qquad \text{where } n = \dim_{\mathbb{C}} M, \tag{4.14}$$

into a direct sum of $n = \dim_{\mathbb{C}} M$ complex line bundles, so $V_j \cong \mathbb{C}$ for each $j = 1, \ldots, n$.

Definition 4.10. Let M be a complex manifold of complex dimension $\dim_{\mathbb{C}} M = n$. Then, assuming that the tangent bundle $\pi : TM \to M$ splits as in Theorem 4.9, we define the **Todd class** $\operatorname{Td}(TM)$ to be the characteristic class:

$$\operatorname{Td}(TM) := \prod_{j=1}^{n} \frac{c_1(V_j)}{1 - e^{-c_1(V_j)}}. \tag{4.15}$$

The Todd class $\operatorname{Td}(TM)$ can be expressed explicitly as a formal power series in the Chern class, the first few terms of which are:

$$\operatorname{Td}(TM) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \ldots. \tag{4.16}$$

Here, $c_j := c_j(TM) \in H^{2j}(M; \mathbb{Z})$ is the j-th Chern class of the tangent bundle TM. One can derive (4.16) by writing (4.15) in terms of the Bernoulli numbers [Ver03, Lecture 1]. Moreover, if $n = \dim_{\mathbb{C}} M$, then the series (4.16) truncates after the n-th term.

The other characteristic class in the integrand (4.13) of the Hirzebruch-Riemann-Roch formula from Theorem 4.8 is that of the Chern character, $\mathrm{Ch}(\mathcal{L})$, of the holomorphic pre-quantum line bundle, $\pi : \mathcal{L} \to M$. Fortunately its definition is far less contrived than that of the Todd class $\mathrm{Td}(TM)$.

Definition 4.11. If M is an n-dimensional compact Kähler manifold with Kähler two-form ω, and if $\pi : \mathcal{L} \to M$ is a holomorphic pre-quantum line bundle over M, then the **Chern character** $\mathrm{Ch}(\mathcal{L})$ of the line bundle $\mathcal{L}$ is the characteristic class:

$$\mathrm{Ch}(\mathcal{L}) := e^{c_1(\mathcal{L})}. \tag{4.17}$$

To finish this section, let us go through an example in which the Riemann-Roch number is calculated when $\mathcal{L} = \mathcal{O}(m)$ is the twisted hyperplane line bundle over the complex projective plane, $M = \mathbb{CP}^2$.

Example 4.12. Let $M = \mathbb{CP}^2$ with the Fubini-Study metric $\omega = \omega_{\mathrm{FS}}$, and equip it with the hyperplane line bundle $\mathcal{L} = \mathcal{O}(m)$ for some non-negative integer, $m \in \mathbb{Z}_{\geq 0}$. Consider the **total Chern class**:

$$c(T\mathbb{CP}^2) := c_0(T\mathbb{CP}^2) + c_1(T\mathbb{CP}^2) + c_2(T\mathbb{CP}^2) + \dots,$$

of the tangent bundle $T\mathbb{CP}^2$, along with the dual of the Euler sequence:

$$\{0\} \longrightarrow \mathcal{O}_{\mathbb{CP}^2} \longrightarrow \mathcal{O}_{\mathbb{CP}^2}(1)^{\oplus 3} \longrightarrow T\mathbb{CP}^2 \longrightarrow \{0\}. \tag{4.18}$$

From the multiplicativity of the Chern class, the triviality of $\mathcal{O}_{\mathbb{CP}^2}$, and from (4.18), we have:

$$c(T\mathbb{CP}^2) = c(\mathcal{O}_{\mathbb{CP}^2}(1)^{\oplus 3}) \cdot c(\mathcal{O}_{\mathbb{CP}^2}) = c(\mathcal{O}_{\mathbb{CP}^2}(1)^{\oplus 3}) = (1 + [A])^3 \tag{4.19}$$

where $[A] \in H^2(\mathbb{CP}^2; \mathbb{Z})$ is the fundamental class of the hyperplane section. By equating the degrees in (4.19), we get:

$$c_0(T\mathbb{CP}^2) = 1, \quad c_1(T\mathbb{CP}^2) = 3[A], \quad \text{and} \quad c_2(T\mathbb{CP}^2) = 3[A]^2.$$

Using the series (4.16), the Todd class $\mathrm{Td}(T\mathbb{CP}^2)$ can be written as:

$$\mathrm{Td}(T\mathbb{CP}^2) = 1 + \tfrac{3}{2}[A] + [A]^2.$$

Now, turning to the Chern character $\mathrm{Ch}(\mathcal{O}(m))$, the first Chern class of $\mathcal{O}(m)$ is just $c_1(\mathcal{O}(m)) = m[A]$. Hence the Chern character $\mathrm{Ch}(\mathcal{O}(m))$ is:

$$\mathrm{Ch}(\mathcal{O}(m)) = e^{c_1(\mathcal{O}(k))} = e^{m[A]} = 1 + m[A] + \tfrac{m^2}{2}[A]^2.$$

So finally, by Theorem 4.8, the Riemann-Roch number for $\mathcal{O}(m) \to \mathbb{CP}^2$ is:

$$\begin{aligned}
\chi(\mathbb{CP}^2; \mathcal{O}(m)) &= \int_{\mathbb{CP}^2} \mathrm{Ch}(\mathcal{O}(m)) \wedge \mathrm{Td}(T\mathbb{CP}^2) \\
&= \int_{\mathbb{CP}^2} \left(1 + m[A] + \tfrac{m^2}{2}[A]^2\right) \wedge \left(1 + \tfrac{3}{2}[A] + [A]^2\right) \\
&= \int_{\mathbb{CP}^2} \left(\tfrac{m^2}{2} + \tfrac{3m}{2} + 1\right) [A]^2 + \dots \\
&= \frac{m^2}{2} + \frac{3m}{2} + 1 \\
&= \frac{(m+2)(m+1)}{2}.
\end{aligned}$$

Observe that this result coincides with the dimension of the space of degree m homogeneous polynomials on $\mathbb{CP}^2$, that is:

$$\tfrac{(m+2)(m+1)}{2} = \binom{m+2}{2} = \dim_{\mathbb{C}} \mathbb{C}[z_0, z_1, z_2]_m.$$

4.3 The Kawasaki-Riemann-Roch Formula for Orbifolds

So far in this chapter, we have considered M to be a smooth manifold only. However, in general, from Theorem 3.17 we will have to deal with hypertoric varieties whose cut spaces are orbifolds. The *"orbifold version"* of the Hirzebruch-Riemann-Roch theorem in Theorem 4.8 is the Kawasaki-Riemann-Roch theorem, [Kaw79], which applies to orbifolds. See Appendix A for a brief introduction to orbifold theory.

Let M be a $2n$-dimensional compact symplectic orbifold with symplectic two-form ω, then its inertia orbifold $\widehat{M}$ is also a compact symplectic orbifold. By choosing a compatible positive almost-complex structure J on M, the tangent bundles TM and $T\widehat{M}$ both become Hermitian vector orbibundles. The immersion $\tau : \widehat{M} \to M$ gives rise to a normal bundle $N_{\widehat{M}} \to \widehat{M}$ induced by the short exact sequence:

$$\{0\} \longrightarrow T\widehat{M} \longrightarrow TM|_{\tau(\widehat{M})} \longrightarrow N_{\widehat{M}} \longrightarrow \{0\},$$

and which can be equipped with a Hermitian structure.

Suppose now, that $F \subseteq M$ is a connected suborbifold of M so that its associated orbifold $\widehat{F}$ is a suborbifold of $\widehat{M}$. The inclusion $i_F : F \hookrightarrow M$ from F into M determines another normal bundle ν_F of F in M from the short exact sequence:

$$\{0\} \longrightarrow TF \longrightarrow TM|_{i_F(F)} \longrightarrow \nu_F \longrightarrow \{0\}.$$

Finally, by using the immersion $\tau : \widehat{F} \to F$ we can form the pull-back bundle $\widehat{\nu}_F = \tau^*\nu_F \to \widehat{F}$, which is a vector orbibundle over $\widehat{F}$.

Now, if $E \to M$ is a holomorphic Hermitian vector orbibundle over M with Hermitian connection ∇, by pulling E back to $\widehat{M}$ via τ we obtain the vector orbibundle, $\widehat{E} := \tau^*E \to \widehat{M}$, over the associated orbifold, $\widehat{M}$. Denote the curvature two-form associated to ∇ by $\mathcal{R}(\widehat{E}) \in \Omega^2(\widehat{M}; \mathrm{End}(\widehat{E}))$, and denote the canonical automorphism of $\widehat{E}$ by $A(\widehat{E}) \in \mathrm{Aut}(\widehat{E})$, which is defined in Appendix A.10.

Definition 4.13. We define the **twisted Chern class** $\mathrm{Ch}_{\widehat{M}}(\widehat{E})$ of the orbifold vector bundle $\widehat{E}$ by:

$$\mathrm{Ch}_{\widehat{M}}(\widehat{E}) := \mathrm{Tr}\left(A(\widehat{E})e^{\mathcal{R}(\widehat{E})}\right) \in \Omega^2(\widehat{M}), \tag{4.20}$$

and the **associated characteristic form** $D_{\widehat{M}}(\widehat{E})$ of the orbifold vector bundle $\widehat{E}$ by:

$$D_{\widehat{M}}(\widehat{E}) := \det\left(\mathrm{Id}_{\widehat{E}} - A(\widehat{E})^{-1}e^{-\mathcal{R}(\widehat{E})}\right) \in \Omega^2(\widehat{M}). \tag{4.21}$$

The Todd class $\mathrm{Td}(T\widehat{M})$ remains the same as in the manifold case, namely that if $\mathcal{R}(T\widehat{M}) \in \Omega^2(\widehat{M}; T\widehat{M})$ is the curvature of the tangent bundle $T\widehat{M}$, then:

$$\mathrm{Td}(T\widehat{M}) := \det\left[\frac{\mathcal{R}(T\widehat{M})}{\left(I - e^{-\mathcal{R}(T\widehat{M})}\right)}\right], \tag{4.22}$$

where I is the identity operator on $T\widehat{M}$.

We may finally present the Kawasaki-Riemann-Roch theorem for orbifolds, which is stated in [Mei96] and in [Sil96].

Theorem 4.14 (Kawasaki-Riemann-Roch). *Let M be a compact Kähler orbifold, and $E \to M$ be a holomorphic Hermitian orbifold vector bundle over M. Then the Riemann-Roch number $\chi(M; E)$ is given by the formula:*

$$\chi(M; E) = \int_{\widehat{M}} \frac{1}{d_{\widehat{M}}} \frac{\mathrm{Td}(T\widehat{M}) \cdot \mathrm{Ch}_{\widehat{M}}(\widehat{E})}{D_{\widehat{M}}(N_{\widehat{M}})}. \tag{4.23}$$

Here, $\mathrm{Ch}_{\widehat{M}}(\widehat{E})$ and $D_{\widehat{M}}(\widehat{E})$ are the twisted Chern class and the associated characteristic form of $\widehat{E}$ from (4.20) and from (4.21) respectively, $\mathrm{Td}(T\widehat{M})$ is the Todd class of $\widehat{M}$, and $d_{\widehat{M}}$ is the orbifold multiplicity of $\widehat{M}$ from A.

Chapter 5

Equivariant Cohomology and Integration

As one may have guessed from Example 4.12, evaluating the Riemann-Roch-Hirzebruch formula (4.13) in Theorem 4.8 becomes increasingly more cumbersome as the dimension of M increases, even with higher-dimensional complex projective spaces. Fortunately however, if M is equipped with an effective and Hamiltonian action of a torus T, then this provides us with a way of adding some powerful and elegant methods to our arsenal, that are otherwise not available to us in the non-equivariant setting.

These are known as *localisation formulae*, and they can reduce an integral over M, whose integrand involves characteristic classes, into a finite sum over the components of the fixed-point loci, M^T. In particular, when M has a T-action whose fixed-point locus is just a finite number of isolated-fixed points, so that $M^T = \{p_1, \ldots, p_k\}$, then the integrand is very easy to evaluate in comparison to the Hirzebruch-Riemann-Roch formula (4.13). What is more interesting, is that only the local isotropy data of M^T is required to evaluate the integral, as opposed to the global data of M – hence the term *localisation*. We shall see in Chapter 6 that the fixed-point locus $(M_\nu^{\leq\delta})^T$ of our cut space $M_\nu^{\leq\delta}$ consists solely of isolated fixed points, and, furthermore, that its moment polyptych $\Delta_\nu^{\leq\delta}$ encapsulates all of the fixed point data required to evaluate the Riemann-Roch-Hirzebruch (4.13) and the Kawasaki-Riemnn-Roch (4.23) formulae in Theorems 4.8 and 4.14, respectively.

This chapter is essentially just a review of equivariant cohomology and equivariant localisation, with none of it original, with most of the results being quoted from either [Tu20], [GS99], [AB84], and also [Bot99]. We begin this chapter by introducing the reader to equivariant cohomology, before then introducing equivariant integration and the famous fixed-point formulae.

5.1 Equivariant Cohomology and the Borel Construction

The idea of equivariant cohomology is motivated by the principle that, when M is a topological space and G is a compact Lie group that acts *freely* on M, then ideally the *equivariant* cohomology groups

$H^\bullet_G(M)$ should be just the *ordinary* cohomology groups $H^\bullet(M/G)$ for the quotient manifold M/G:

$$H^\bullet_G(M) \cong H^\bullet(M/G), \qquad \text{if } G \text{ acts freely.}$$

If G *does not* act freely on M however, then the quotient M/G may end up being difficult to work with in terms of an ordinary cohomology theory – equivariant cohomology strives, therefore, to find alternative cohomology groups, denoted by $H^\bullet_G(M)$, that generalise the notion of an ordinary cohomology group $H^\bullet(M/G)$ appropriately in such instances. This leads us to the idea of the *Borel construction* of $H^\bullet_G$.

Let EG be any contractible topological space, and assume that a compact Lie group G acts freely on EG. Denote its quotient by $\mathrm{BG} := \mathrm{EG}/G$. Since G acts freely on EG, we may consider $\mathrm{EG} \to \mathrm{BG}$ to be a G-principal fibre bundle over BG.

Definition 5.1. Let M be a topological manifold and G be a compact Lie group that acts on M. Let EG be any contractible space on which G acts freely. In this framework, we say that BG is the **classifying bundle** of G, and that $\mathrm{EG} \to \mathrm{BG}$ is the **universal bundle** of G.

Furthermore, the **Borel construction**, or the **homotopy quotient**, is defined to be the quotient

$$M_G := M \times_G \mathrm{EG} := (M \times \mathrm{EG})/G \tag{5.1}$$

with respect to the diagonal action of G on $M \times \mathrm{EG}$.

The Borel construction M_G is then the substitute space for M, in that the ordinary cohomology groups of M_G will be the equivariant cohomology groups of M.

Definition 5.2. Let M be a topological manifold and G be a compact Lie group acting on M. Let EG be any contractible space on which G acts freely. Then the G-**equivariant cohomology groups**, $H^\bullet_G(M)$, are defined to be the ordinary cohomology groups of the Borel construction:

$$H^\bullet_G(M) := H^\bullet(M \times_G \mathrm{EG}). \tag{5.2}$$

Example 5.3. If we assume that G acts freely on M, then the projection $M \times \mathrm{EG} \to M$ induces a fibration $M \times_G \mathrm{EG} \to M/G$, whose typical fibre is EG. Hence, since EG is assumed to be contractible:

$$H^\bullet_G(M) = H^\bullet(M \times_G \mathrm{EG}) \cong H^\bullet(M/G),$$

which is the result that we had been hoping for.

The reason that $\mathrm{EG} \to \mathrm{BG}$ is called the *"universal"* G-bundle is that, if $E \to B$ is *any* G-principal fibre bundle, whose total space E is contractible, then $E \to B$ is a universal G-bundle. This is the statement of the next theorem, and is proven in [GS99, Proposition 1.1.1, & Theorem 1.1.1].

Theorem 5.4. *The Borel construction $M_G = M \times_G \mathrm{EG}$, defined in (5.1), is independent of the choice of* EG.

So far, we have seen that *if* such a G-principal fibre bundle $E \to B$ exists, where E is a contractible space, then it is must necessarily be G-equivariantly homotopic to the universal G-bundle, $\mathrm{EG} \to \mathrm{BG}$. Of course, $\mathrm{EG} \to \mathrm{BG}$ has to exist first to be of use.

One construction of a universal G-bundle is via **Milnor's join construction**, [Mil56]. Milnor considered the infinite join of G with itself:

$$\mathrm{EG} = G \star G \star \ldots \star G \star \ldots = \varinjlim \mathrm{EG}(k), \qquad \text{where} \qquad \mathrm{EG}(k) := \bigstar_{i=1}^{k} G.$$

Intuitively, repeatedly taking joins of a topological group G gives us a space $\mathrm{EG}(k)$, which becomes more and more connected with each added join. In [Mil56], Milnor proved that his join spaces $\varinjlim \mathrm{EG}(n)$ are *weakly contractible* in the limit, and were further proven to be *contractible* by Dold later on in [Dol63, Theorem 8.1]. Hence:

Theorem 5.5. *The topological space* EG *is contractible.*

Let us see what the universal G-bundles $\mathrm{EG} \to \mathrm{BG}$ are in the cases when $G = U_1$ and $G = T^n$ are the circle and the n-dimensional torus respectively.

Example 5.6. Let $G = U_1$. There is an increasing sequence of complex vector spaces:

$$\mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \ldots,$$

and therefore an increasing sequence of odd-dimensional spheres:

$$S^1 \subset S^3 \subset S^5 \subset \ldots.$$

The U_1-action on each odd-dimensional sphere is compatible with each inclusion, thence giving rise to the following commutative diagram of U_1-principle fibre bundles:

$$\begin{tikzcd}
S^1 \arrow[d] & \subset & S^3 \arrow[d] & \subset & \cdots & \subset & S^{2k-1} \arrow[d] & \subset & S^{2k+1} \arrow[d] & \subset & \cdots, \\
\mathbb{CP}^0 & \subset & \mathbb{CP}^1 & \subset & \cdots & \subset & \mathbb{CP}^{k-1} & \subset & \mathbb{CP}^k & \subset & \cdots.
\end{tikzcd}$$

Then $\mathrm{EU}_1(k) \cong S^{2k+1}$ and $\mathrm{BU}_1(k) \cong \mathbb{CP}^k$ for each $k \geq 0$. There is therefore an induced U_1-action on the *infinite sphere*, $S^\infty = \cup_{k=0}^{\infty} S^{2k+1}$. Since U_1 acts freely on each S^{2k+1}, it acts freely on S^∞. The orbit space is the *infinite-dimensional complex projective space*, $\mathbb{CP}^\infty = S^\infty / U_1$.

While S^∞ is not a bonafide manifold due to its infinite-dimensionality, the projection $S^\infty \to \mathbb{CP}^\infty$ is topologically a U_1-principal fibre bundle, and it can be shown that $S^\infty \to \mathbb{CP}^\infty$ is topologically trivial, see [Tu20, Example 3.6]. The homotopy groups all vanish, $\pi_q(S^\infty) = 0$ for each $q \geq 0$, and thus S^∞ is *weakly contractible*. By Whitehead's theorem [Hat02, Theorem 4.5], S^∞ is actually *contractible*, unlike any finite-dimensional sphere. Hence, $\mathrm{EU}_1 \cong S^\infty$ and $\mathrm{BU}_1 \cong \mathbb{CP}^\infty$, and therefore $S^\infty \to \mathbb{CP}^\infty$ is the universal U_1-bundle.

Example 5.7. Now, when $G = T^n$ is the n-dimensional torus, Example 5.6 generalises to give $\mathrm{ET}^n(k) \cong (S^{2k+1})^n$ and $\mathrm{BT}^n(k) \cong (\mathbb{CP}^k)^n$. So, in taking the limit, $\mathrm{ET}^n \cong (S^\infty)^n$ and $\mathrm{BT}^n \cong (\mathbb{CP}^\infty)^n$. Hence the universal T^n-bundle is $(S^\infty)^n \to (\mathbb{CP}^\infty)^n$.

So far, we have see that the universal U_1-bundle is $\mathrm{EU}^1 \cong S^\infty$ in Example 5.6, and that the universal T^n-bundle is $\mathrm{ET}^n \cong (S^\infty)^n$ in Example 5.7, respectively. Let us see what $H^\bullet_G(M)$ is when M is just a point, i.e., when $M = \{\mathrm{pt}\}$.

As the G-action of *any* Lie group G on a point $M = \{\mathrm{pt}\}$ is trivial, we see that the Borel construction is:

$$\{\mathrm{pt}\}_G = \{\mathrm{pt}\} \times_G \mathrm{EG} = \mathrm{EG}\,/G = \mathrm{BG}\,.$$

Hence the G-equivariant cohomology group of any point $\{\mathrm{pt}\}$ is just the ordinary cohomology group of the classifying bundle:

$$H^\bullet_G(\{\mathrm{pt}\}) = H^\bullet(\{\mathrm{pt}\} \times_G \mathrm{EG}) \cong H^\bullet(\mathrm{BG}).$$

Example 5.8. From Example 5.6, we saw that $\mathrm{BU}_1(k) \cong \mathbb{CP}^k$ for each $k \geq 0$. Since $H^\bullet(\mathbb{CP}^k) \cong \mathbb{R}[u]/\langle u^{k+1}\rangle$, where $\deg(u) = 2$, after taking the limit we see that:

$$H^\bullet_{U_1}(\{\mathrm{pt}\}) = H^\bullet(\mathrm{BU}^1) \cong H^\bullet(\mathbb{CP}^\infty) \cong \mathbb{R}[u].$$

Example 5.9. In a similar vein to Example 5.8, when $G = T^n$, we as that $\mathrm{BT}^n(k) \cong (\mathbb{CP}^k)^n$ for each $k \geq 0$. Hence, by the Künneth formula and Example 5.8:

$$H^\bullet_{T^n}(\{\mathrm{pt}\}) = H^\bullet(\mathrm{BT}^n) \cong H^\bullet\left((\mathbb{CP}^\infty)^n\right) \cong \bigotimes_{i=1}^{n} \mathbb{R}[u_i] \cong \mathbb{R}[u_1, \ldots, u_n],$$

where $\deg(u_i) = 2$, for each $i = 1, \ldots, n$.

An important algebraic property found in equivariant cohomology, an which we shall refer back to in Section 5.5, is that it every ring $H^\bullet_G(-)$ is also a $H^\bullet(\mathrm{BG})$-algebra.

Lemma 5.10. *Let M be a topological manifold and let G be a compact Lie group that acts on M. Then the G-equivariant cohomology ring $H^\bullet_G(M)$ is an algebra over the ring $H^\bullet(\mathrm{BG})$.*

Proof. If M is a G-space, then the constant projection $\mathrm{pr} : M \to \{\mathrm{pt}\}$ is trivially G-equivariant. Hence it induces a map $\mathrm{pr}_G : M_G \to \{\mathrm{pt}\}_G$ between homotopy quotients, and moreover a ring homomorphism via the pull-back:

$$\begin{array}{ccc} \mathrm{pr}_G^* : H_G^\bullet(\{\mathrm{pt}\}) & \longrightarrow & H^\bullet(M_G) \\ \| & & \| \\ \mathrm{pr}_G^* : H^\bullet(\mathrm{BG}) & \longrightarrow & H_G^\bullet(M) \end{array} \tag{5.3}$$

Using the homomorphism in (5.3), we may define a scalar-multiplication operation in $H_G^\bullet(M)$ in the following manner:

$$u \cdot x := \mathrm{pr}_G^*(u)x, \qquad \text{where } u \in H^\bullet(\mathrm{BG}) \text{ and } x \in H_G^\bullet(M).$$

This scalar multiplication makes $H_G^\bullet(M)$ into an algebra over $H^\bullet(\mathrm{BG})$. □

Lemma 5.10 shows that $H_G^\bullet(M)$ is an algebra over the ring $H^\bullet(\mathrm{BG})$. However, unlike in ordinary cohomology, it is not necessarily the case that the coefficient ring $H^\bullet(\mathrm{BG})$ embeds into $H_G^\bullet(M)$ as a subring for any M. Fortunately, we have the following result when $M^G \neq \emptyset$, [Tu20, Proposition 9.8].

Proposition 5.11. *If $p \in M$ is a fixed-point for the G-action on M, then:*

(i) the inclusion $i : \{p\} \hookrightarrow M$ induces a section $i_G : \mathrm{BG} \to M_G$ of the G-principal fibre bundle, $M_G \to \mathrm{BG}$;

(ii) the constant projection $\mathrm{pr} : M \to \{p\}$ induces an injection, $\mathrm{pr}_G^* : H_G^\bullet(\{p\}) \hookrightarrow H_G^\bullet(M)$.

Proof. For (i), since p is a fixed point, the inclusion map $i : \{p\} \hookrightarrow M$ is a G-equivariant map such that $\mathrm{pr} \circ i = \mathrm{Id}_{\{p\}}$. Hence, there is an induced map of homotopy quotients such that $\mathrm{pr}_G \circ i_G = \mathrm{Id}$. Thus, $i_G : \mathrm{BG} \to M_G$ is a section of $M_G \to \mathrm{BG}$;

Next, for (ii), by functoriality [Tu20, Section 9.2]:

$$i_G^* \circ \mathrm{pr}_G^* = \mathrm{Id}_{H_G^\bullet(\{\mathrm{pt}\})}\,.$$

Therefore, $\mathrm{pr}_G^* : H^\bullet(\mathrm{BG}) \to H_G^\bullet(M)$ is injective. □

5.2 The Equivariant de Rham Theorem

Section 5.1 introduced equivariant cohomology groups and how to construct them topologically. However, in some scenarios, it is more convenient to use *equivariant de Rham theory*, if there is more geometry involved than than there is topology. When M is a smooth manifold acted upon by a compact Lie group, then one can form the Cartan model for equivariant cohomology, which is a differential complex whose elements are equivariant differential forms. Moreover, the localisation formulae, that we shall come across in Section 5.5, is expressed in terms of equivariant characteristic classes which naturally arise within the Cartan model.

So, let M be an n-dimensional smooth manifold and let G be a compact Lie group acting that acts on M. Then the quotient space M/G is also smooth manifold, and the projection $q : M \to M/G$ is a principal G-fibration, [Tu20, Chapter 12]. Our first step is to investigate which differential forms on M can be thought of those originating from the quotient M/G.

Definition 5.12. The subcomplex $q^*\Omega^\bullet(M/G) \subseteq \Omega^\bullet(M)$ of differential forms on M is called the complex of **basic forms**. It consists of the differential forms $q^*\omega$ on M which come from the differential forms ω on the quotient M/G under the injective pull-back $q^* : \Omega(M/G) \to \Omega(M)$.

For any point $p \in M$, let $q_{*,p} : T_pM \to T_{q(p)}M$ denote the differential of $q : M \to M/G$. Then we say that the **vertical tangent space** at $p \in M$, denoted by $\mathcal{V}_p$, is the kernel of the differential $q_{*,p} : T_qM \to T_{q(p)}M$. That is to say, $\mathcal{V}_p \cong \ker q_{*,p}$. The vectors that belong to $\mathcal{V}_p$ are said to be **vertical** to q_* at p. The key idea here, is that the vertical vectors should be "orthogonal" to the tangent space of the quotient under the differential $q_* : T_pM \to T_{q(p)}(M/G)$, and so any vertical vector should be killed off once M has been collapsed into its G-orbits after taking the quotient M/G.

For the G-principal fibre bundle $q : M \to M/G$, a differential form $\omega \in \Omega^\bullet(M)$ is said to be **horizontal** if, at any point $p \in M$, the form ω vanishes whenever one of its arguments is a vertical vector. That is to say, that $\imath_{X_p}\omega_p = 0$ for every $X_p \in \mathcal{V}_p$. Thus, horizontal differential forms on M are only "compatible" with non-vertical vectors. The crux of these notions is the following characterisation of basic differential forms, which has been taken from [Tu20, Theorem 12.5].

Theorem 5.13. *Let M be a smooth manifold and let G be a Lie group that acts freely on M. Let $q : M \to M/G$ be the G-principal fibre bundle induced by forming the quotient M/G. Then a differential form $\omega \in \Omega^\bullet(M)$ is basic if, and only if, it is G-invariant and horizontal.*

Corollary 5.14. *Suppose furthermore that G is connected with Lie algebra $\mathfrak{g}$. Then a differential form $\omega \in \Omega^\bullet(M)$ is basic if, and only if, $L_{X_\#}\omega = 0$ and $\imath_{X_\#}\omega = 0$ for every $X \in \mathfrak{g}$.*

Definition 5.15. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **Weil algebra**, $W(\mathfrak{g})$, of $\mathfrak{g}$ is defined to be the algebra:

$$W(\mathfrak{g}) := \wedge(\mathfrak{g}^*) \otimes_{\mathbb{C}} S(\mathfrak{g}^*).$$

Consider a G-principal fibre bundle $P \to M$, equipped with a $\mathfrak{g}$-valued connection one-form $\theta \in \Omega^1(P)$ on P, and with a $\mathfrak{g}$-valued curvature two-form $\Omega \in \Omega^2(P)$ on P, respectively. Given the dual Lie algebra elements, $\alpha_1, \ldots, \alpha_k \in \mathfrak{g}^*$, we define now two unique algebra homomorphisms; the first is:

$$f_\wedge : \wedge(\mathfrak{g}^*) \longrightarrow \Omega(P), \quad \text{where} \quad f_\wedge(\alpha_1 \wedge \ldots \wedge \alpha_k) := (\alpha_1 \circ \theta) \wedge \ldots \wedge (\alpha_k \circ \theta),$$

whereas the second is:

$$f_S : S(\mathfrak{g}^*) \longrightarrow \Omega(P), \quad \text{where} \quad f_S(\alpha_1 \wedge \ldots \wedge \alpha_k) := (\alpha_1 \circ \Theta) \wedge \ldots \wedge (\alpha_k \wedge \Theta).$$

We can then combine $f_\wedge$ and f_S together to form the bilinear mapping:

$$f_\wedge \times f_S : \wedge(\mathfrak{g}^*) \times S(\mathfrak{g}^*) \to \Omega(P), \quad \text{where} \quad f(\alpha, \beta) := f_\wedge(\alpha) \wedge f_S(\beta),$$

which in turn induces the linear mapping:

$$f_W : \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \to \Omega(P), \quad \text{where} \quad f(\alpha \otimes \beta) = f_\wedge(\alpha) \wedge f_S(\beta), \tag{5.4}$$

by the universal property of the tensor product.

Definition 5.16. The map $f_W : W(\mathfrak{g}^*) \to \Omega(P)$ defined in (5.4) is called the **Weil map**.

The Weil algebra $W(\mathfrak{g})$ can be made into a graded algebra, by assigning a degree of one to the elements of $\mathfrak{g}^*$ in $\wedge(\mathfrak{g}^*)$, and the elements of $\mathfrak{g}^*$ in $S(\mathfrak{g}^*)$ a degree of two. With these degrees, the Weil map f_W becomes a graded-algebra homomorphism.

Now let us a basis $X_1, \ldots, X_n$ for the Lie algebra $\mathfrak{g}$ with the dual basis $\alpha^1, \ldots, \alpha^n$ for $\mathfrak{g}^*$. Write:

$$\lambda_i := \alpha^i \otimes 1 \in \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*),$$
$$r_i := 1 \otimes \alpha^i \in \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*).$$

Then, in terms of these generators, the Weil algebra $W(\mathfrak{g})$ becomes:

$$W(\mathfrak{g}) = \wedge^\bullet(\lambda_1, \ldots, \lambda_n) \otimes_{\mathbb{C}} S^\bullet(r_1, \ldots, r_n),$$

where $\wedge(\mathfrak{g}^*)$ is the free exterior algebra generated by the $\lambda_1, \ldots, \lambda_n$, and $S(r_1, \ldots, r_n) \cong \mathbb{R}[r_1, \ldots, r_n]$ is the polynomial algebra generated by indeterminates $r_1, \ldots, r_n$. In terms of the grading, the Weil algebra $W(\mathfrak{g})$ can be written down explicitly as:

$$W(\mathfrak{g}) := \bigoplus_{k \geq 0} W^k(\mathfrak{g}) := \bigoplus_{k \geq 0} \bigoplus_{\substack{p,q \geq 0 \\ p+2q=k}} \wedge^p(\lambda_1, \ldots, \lambda_n) \otimes S^q(r_1, \ldots, r_n),$$

where $\wedge^p(\lambda_1, \ldots, \lambda_n)$ is the space of homogeneous elements of degree p in the $\lambda_1, \ldots, \lambda_n$, and $S^q(r_1, \ldots, r_n)$ is the space of homogeneous elements of degree q in the $r_1, \ldots, r_n$.

Example 5.17. Continuing from Example 5.7, let $G = T \cong U_1^n$ be the n-dimensional real torus, and let it act on M. Denote its Lie algebra by $\mathfrak{t}$ and let $\mathfrak{t}^*$ be its dual. Then the Weil algebra $W(\mathfrak{t})$ of $\mathfrak{t}$ is:

$$W(\mathfrak{t}) = \wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*) = \wedge(\lambda_1, \ldots, \lambda_n) \otimes \mathbb{R}[r_1, \ldots, r_n].$$

Since the connection one-form θ and the curvature two-form Θ are both $\mathfrak{g}$-valued differential forms, they can be written uniquely in terms of the basis $X_1, \ldots, X_n$ as the linear combinations:

$$\theta = \sum_{i=1}^{n} \theta^i X_i, \qquad \text{and} \qquad \Theta = \sum_{i=1}^{n} \Theta^i X_i,$$

where the θ^i and Θ^i are $\mathbb{R}$-valued one- and two-forms on P, respectively. Under this guise, the Weil map f_W from (5.4) becomes:

$$\begin{aligned} f_W(\lambda_k) &= \lambda_k \circ \theta = \lambda_k \circ \left(\sum \theta^j X_j\right) = \theta^k, \\ f_W(r_k) &= r_k \circ \Theta = r_k \circ \left(\sum \Theta^j X_j\right) = \Theta^j. \end{aligned} \tag{5.5}$$

As $\theta \in \Omega^1(P)$ is a connection one-form on P, it has to satisfy the second structural equation [KN96, Theorem 5.2], whereas since $\Theta \in \Omega^2(P)$ is a curvature two-form, it has to satisfy Bianchi's identity [KN96, Theorem 5.4]. These respectively are:

$$d\theta^k = \Theta^k - \frac{1}{2}\sum c_{ij}^k \theta^i \wedge \theta^j, \qquad \text{and} \qquad d\Theta^k = \sum c_{ij}^k \Theta^i \wedge \theta^j. \tag{5.6}$$

Here, the c_{ij}^k in (5.6) are the structure constants of the Lie algebra $\mathfrak{g}$.

Lemma 5.18. *For a differential δ on $W(\mathfrak{g})$ to commute with the Weil map f_W, it must satisfy:*

$$\delta\lambda_k = r_k - \sum_{i<j} c_{ij}^k \left[\lambda_i \wedge \lambda_j\right], \qquad \textit{and} \qquad \delta r_k = \sum_{i,j} c_{ij}^k \left[r_i \wedge \lambda_j\right]. \tag{5.7}$$

Proof. Recall from (5.5) that $f_W(\lambda_k) = \theta^k$ and $f_W(r_k) = \Theta^k$. Then:

$$\begin{aligned} df_W(\lambda_k) \overset{(5.5)}{=} d\theta^k \overset{(5.6)}{=} \Theta^k - \frac{1}{2}\sum c_{ij}^k\left[\theta^i \wedge \theta^j\right] &\overset{(5.5)}{=} f_W(r_k) - \frac{1}{2}\sum c_{ij}^k\left[f_W(\lambda_i) \wedge f_W(\lambda_j)\right] \\ &= f_W\left(r_k - \frac{1}{2}\sum c_{ij}^k \lambda_i \wedge \lambda_j\right), \end{aligned}$$

and:

$$\begin{aligned} df_W(r_k) \overset{(5.5)}{=} d\Theta^k \overset{(5.6)}{=} \sum c_{ij}^k \left[\Theta^i \wedge \theta^j\right] &\overset{(5.5)}{=} \sum c_{ij}^k \left[f_W(r_i) \wedge f_W(\lambda_j)\right] \\ &= f_W\left(\sum c_{ij}^k \left[r_k \wedge \lambda_j\right]\right). \end{aligned}$$

So, if we let δ act on both λ_k and r_k as in (5.7), then we see that $d \circ f_W = f_W \circ \delta$. □

We call the differential δ that operates on $W(\mathfrak{g})$ via (5.18) the **Weil differential**, and it can be shown that it is indeed a bonafide differential satisfying satisfies $\delta^2 = 0$, see [Tu20, Theorem 19.1]. If we fix a Lie algebra element $A \in \mathfrak{g}$, then we can also extend the *interior derivative* $\imath_A$ to the Weil algebra $W(\mathfrak{g})$ too. Since:

$$\imath_A \theta^k = \imath_{\underline{A}} \theta^k = \theta^k(\underline{A}), \qquad \text{and} \qquad \imath_A \Theta^k = 0,$$

then:

$$\sum \theta^k(\underline{A}) X_k = \theta(\underline{A}) = A = \sum \alpha^k(A) X_k,$$

and, recalling that $\alpha_1, \dots, \alpha_n$ is the basis of $\mathfrak{g}^*$, dual to the basis $X_1, \dots, X_n$ of $\mathfrak{g}$, we observe that:

$$\imath_A \theta^k = \theta^k(A_\#) = \alpha^k(A).$$

Hence, for the Weil map f_W to preserve the interior derivative $\imath_A$ on $\mathfrak{g}$, it should be defined on $W(\mathfrak{g})$ by:

$$\imath_A \lambda_k = \lambda_k(A) = \alpha^k(A), \qquad \text{and} \qquad \imath_A r_k = 0,$$

so that $\imath_A \circ f_W = f_W \circ \imath_A$.

Finally, one may combine both exterior and interior derivatives, δ and $\imath_A$, respectively on $W(\mathfrak{g})$, to define the *Lie derivative* on $W(\mathfrak{g})$:

$$L_A : W(\mathfrak{g}) \to W(\mathfrak{g}), \qquad \text{where} \qquad L_A := \delta \circ \imath_A + \imath_A \circ \delta. \tag{5.8}$$

As both the Weil map f_W and the interior derivative $\imath_A$ commute with the Weil derivative δ, we see that the Lie derivative L_A in (5.8) does too.

Example 5.19. Since the torus T is abelian, its structure constants c_{ij}^k are all zero. Hence, the Weil differential δ from (5.7) is the anti-derivation of degree 1 on $W(\mathfrak{t})$, that satisfies:

$$\delta \lambda_j = r_j, \qquad \delta r_j = 0, \qquad \text{for each } j = 1, \dots, n.$$

For any $A \in \mathfrak{t}$, the interior derivative $\imath_A$ on $W(\mathfrak{t})$ is likewise the anti-derivation of degree -1, which satisfies:

$$\imath_A \lambda_j = \lambda_j(A), \qquad \imath_A u_j = 0, \qquad \text{for each } j = 1, \dots, n.$$

Lastly, the Lie derivative L_A on $W(\mathfrak{t})$ is the derivation of degree 0, which satisfies:

$$\begin{aligned} L_A \lambda_i &= (\delta \circ \imath_A)(\lambda_i) + (\imath_A \circ \delta)(\lambda_i) = \delta \lambda_i(A) + \imath_A r_i = 0, \\ L_A r_i &= (\delta \circ \imath_A)(r_i) + (\imath_A \circ \delta)(r_i) = \delta(0) + \imath_A(0) = 0. \end{aligned}$$

The Weil algebra $W(\mathfrak{g})$, along with the derivations and anti-derivations δ, $\imath_A$, and L_A of the orders 1, -1, and 0, respectively, define what is known as a $\mathfrak{g}$-*differential graded algebra*.

Definition 5.20. A **$\mathfrak{g}$-differential graded algebra ($\mathfrak{g}$-dga)** is a commutative graded algebra $\Omega = \oplus_{k\geq 1}\Omega_k$ equipped with an anti-derivation $d : \Omega \to \Omega$ of degree 1 such that $d \circ d = 0$, and which has the two following $\mathfrak{g}$-actions:

$$\begin{aligned} \imath : \mathfrak{g} \times \Omega &\to \Omega, \\ \imath(A,\omega) &:= \imath_A \omega, \end{aligned} \qquad \text{and} \qquad \begin{aligned} L : \mathfrak{g} \times \Omega &\to \Omega, \\ L(A,\omega) &:= L_A \omega, \end{aligned}$$

where, for any $A \in \mathfrak{g}$, both $\imath_A$ and L_A are $\mathbb{R}$-linear in A; where $\imath_A$ acts on Ω as an anti-derivation of degree -1 such that $\imath_A \circ \imath_A = 0$; and where L_A acts on Ω as a derivation of degree 0; furthermore, d, $\imath_A$, and L_A satisfy Cartan's homotopy formula:

$$L_A = d \circ \imath_A + \imath_A \circ d.$$

Of course, another example of a $\mathfrak{g}$-differential graded algebras is that of the de Rham complex $(\Omega(P), d)$ for a smooth manifold P. It follows then that the Weil map $f_W : W(\mathfrak{g}) \to \Omega(P)$ is a morphism of $\mathfrak{g}$-differential graded algebras. Theorem 5.21, which follows below, shows that the Weil algebra $W(\mathfrak{g})$ is the algebraic analogue to the universal G-bundle, $\mathrm{EG} \to \mathrm{BG}$, as both of their cohomology complexes are *acyclic*, see [GS99, § 2.3.2].

Theorem 5.21. *Let $\mathfrak{g}$ be a Lie algebra. Then the Weil algebra $W(\mathfrak{g})$ is **acyclic**, i.e., that:*

$$H^j\left(W(\mathfrak{g}), \delta\right) \cong \begin{cases} \mathbb{R}, & \textit{if } j = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Since $(W(\mathfrak{g}), \delta)$ and $(\Omega(P), d)$ are both $\mathfrak{g}$-differential graded algebras, their tensor product $(W(\mathfrak{g}) \otimes \Omega(P), \delta \otimes 1 + 1 \otimes d)$ is also a $\mathfrak{g}$-differential graded algebra [Tu20, Section 18.2]. Furthermore, $H^\bullet(\mathrm{EG}) \cong H^\bullet(W(\mathfrak{g}))$ since they are both acyclic and, since $H^\bullet(M) = H^\bullet(\Omega(M))$ then, by the algebraic Künneth formula [Hat02, Theorem 3B.5] we observe that:

$$H^\bullet\left(M \times \mathrm{EG}\right) \cong H^\bullet(M) \otimes H^\bullet(\mathrm{EG}) = H^\bullet(M) \otimes H^\bullet(W(\mathfrak{g})).$$

Given that the homotopy quotient M_G makes up the base of the principal G-bundle $\mathrm{EG} \times M \to M$, and hence the basic forms on $\mathrm{EG} \times M$ are the pull-backs of those which already exist on the base space M_G, it makes sense for the basic subcomplex $\left(W(\mathfrak{g}) \otimes \Omega(M)\right)_{\text{basic}}$ to be a possible candidate for the cohomology of the homotopy quotient, M_G. This is indeed the case, thanks to the equivariant de Rham theorem [Car51b; Car51a].

Theorem 5.22 (Equivariant de Rham). *For a compact and connected Lie group G, with Lie algebra $\mathfrak{g}$, that acts on a manifold M, there exists a graded algebra isomorphism:*

$$H^\bullet_G(M) \cong H^\bullet\Big(\left(W(\mathfrak{g}) \otimes \Omega(M)\right)_{\textit{basic}}\Big).$$

The complex $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{basic}}$ when equipped with the Weil differential δ is called the **Weil model**.

Example 5.23. For the universal T-bundle $\mathrm{ET} \to \mathrm{BT}$, an algebraic model for $M \times \mathrm{ET}$ is:

$$W(\mathfrak{t}) \otimes \Omega(M) = \wedge(\lambda_1, \ldots, \lambda_n) \otimes \Omega(M)[r_1, \ldots, r_n].$$

Hence an element of $W(\mathfrak{t}) \otimes \Omega(M)$ can be written as a linear combination of monomials, as:

$$\lambda_I := \lambda_{i_1} \wedge \ldots \wedge \lambda_{i_k}, \tag{5.9}$$

where $1 \leq i_1 < \ldots < i_k \leq n$, whose coefficients are of the form:

$$a_I := a_{i_1 \ldots i_k} \in \Omega(M)[r_1, \ldots, r_n]. \tag{5.10}$$

That is to say:

$$\begin{aligned}
\alpha &= a + \sum_i a_i \lambda_i + \sum_{i<j} a_{ij}[\lambda_i \wedge \lambda_j] + \ldots + a_{1\ldots n}[\lambda_1 \wedge \ldots \wedge \lambda_n] \\
&= a + \sum a_I \lambda_I.
\end{aligned}$$

Let us continue to focus on the case when $G = T$ is the n-dimensional torus. Recall from Corollary 5.14 that a differential form $\alpha \in W(\mathfrak{t}) \otimes \Omega(M)$ is basic if, and only if, for every $X \in \mathfrak{t}$:

$$\imath_X \alpha = 0 \quad \text{(i.e., } \alpha \text{ is horizontal)}, \qquad \text{and} \qquad \imath_X \alpha = 0, \quad \text{(i.e., } \alpha \text{ is invariant)}.$$

Then we have the following lemma regarding the horizontal forms.

Lemma 5.24. *A differential form $\alpha = a + \sum a_I \lambda_I \in W(\mathfrak{t}) \otimes \Omega(M)$, where $I \subseteq \{1, \ldots, n\}$ is a subset and where λ_I and a_I are defined in (5.10) and (5.10) respectively, is horizontal if, and only if:*

$$\alpha = \left(\prod_{i=1}^{n} (1 - \lambda_i \imath_{X_i}) \right) a. \tag{5.11}$$

Proof. If $\alpha \in W(\mathfrak{t}) \otimes \Omega(M)$ is horizontal then, for every $X \in \mathfrak{t}$, we have that:

$$\imath_X \alpha = 0 \quad \iff \quad \begin{cases}
a_i = -\imath_{X_i} a \\
a_{ij} = \imath_{X_i} \imath_{X_j} a, \\
a_{ijk} = -\imath_{X_i} \imath_{X_j} \imath_{X_k} a, \\
\qquad \vdots \\
a_{i_1 \ldots i_k} = (-1)^k \imath_{X_{i_1}} \ldots \imath_{X_{i_k}} a, \\
\qquad \vdots \\
a_{1 \ldots n} = (-1)^n \imath_{X_1} \ldots \imath_{X_n} a,
\end{cases} \quad \iff \quad a_I = (-1)^{|I|} \imath_{X_I} a,$$

where $\imath_{X_I} := \imath_{X_{i_1}} \dots \imath_{X_{i_k}}$ and $a_I = a_{i_1 \dots i_k}$ for each subset $I = \{i_1, \dots, i_k\}$, analogously to (5.9) and (5.10). This can be seen by applying the interior derivative $\imath_{X_i}$ to α and then setting $\imath_{X_i}\alpha = 0$, before then comparing the coefficients to each basis vector $\lambda_{i_1} \wedge \dots \lambda_{i_k}$. To decompose α into the product as in (5.11), substitute in the expressions for the coefficients $a_{i_1 \dots i_k}$ derived above, and also noting that $\imath_{X_i}\lambda_j = \delta_{ij}$. □

Let us now deal with the invariance.

Lemma 5.25. *A differential form $\alpha = a + \sum a_I \lambda_I \in W(\mathfrak{t}) \otimes \Omega(M)$, where $I \subseteq \{1, \dots, n\}$ is a subset and where λ_I and a_I are defined in (5.10) and (5.10) respectively, is invariant if, and only if:*

$$L_X a = 0, \tag{5.12}$$

for every $X \in \mathfrak{t}$.

Proof. Since $L_X \lambda_I = 0$ for any $I \subseteq \{1, \dots, n\}$, and since $a_I = (-1)^{|I|}\imath_{X_I} a$ from Lemma 5.24, then:

$$\begin{aligned} L_X \alpha = 0 \iff & L_X a + (-1)^{|I|} \sum_I \lambda_I L_X(\imath_{X_I} a) = L_X a + (-1)^{|I|} \sum_I \lambda_I \imath_{X_I}(L_X a) = 0 \\ \iff & L_X a = 0, \end{aligned}$$

for every $X \in \mathfrak{t}$, since the Lie derivative L_X and the interior derivative $\imath_X$ commute from (5.8). □

Thence, Lemmas 5.24 and 5.25 provide the two conditions for a differential form $\alpha \in W(\mathfrak{t}) \otimes \Omega(M)$ to be basic.

Corollary 5.26. *An element $\alpha = a + \sum a_I \lambda_I \in W(\mathfrak{t}) \otimes \Omega(M)$, where $a_I \in \Omega(M)[u_1, \dots, u_n]$ with $I \subseteq \{1, \dots, n\}$, is basic if, and only if, the two conditions:*

$$a_I = (-1)^{|I|}\imath_{X_I} a, \qquad \textit{and} \qquad L_X a = 0, \tag{5.13}$$

are satisfied for every $X \in \mathfrak{t}$.

An element $a \in \Omega(M)[r_1, \dots, r_n]$ is a polynomial in the $r_1, \dots, r_n$, whose coefficients are the differential forms on M:

$$a = \sum a_I r_1^{k_1} \dots r_n^{k_n}, \qquad \text{where} \qquad a_I \in \Omega(M)[r_1, \dots, r_n].$$

Since $L_X u_i = 0$ for each $i = 0, \dots, n$:

$$L_X a = 0 \iff L_I a_I = 0, \qquad \text{for every } I \subseteq \{1, \dots, n\}.$$

As in (5.9) and (5.10), let us abbreviate $r_I = r_{i_1,\dots i_k}$ when $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. Whence $a = \sum a_I r_I \in \Omega(M)[r_1, \dots, r_n]$ is invariant if, and only if, each coefficient $a_I \in \Omega(M)$ is itself invariant. Let us write:

$$\begin{aligned}\Omega(M)^T &:= \{\, \omega \in \Omega(M) \mid L_X\omega = 0 \,\} \\ &= \{\, T\text{-invariant differential forms on } M \,\}\,.\end{aligned}$$

Our last task is express the Weil model in the terms of the Cartan model. This follows from the following result, proven by H. Cartan in [Car51a, Théorème 4].

Theorem 5.27 (Weil-Cartan Isomorphism). *For a compact and connected Lie group G, with Lie algebra $\mathfrak{g}$, that acts on a manifold M,*

Suppose that M is a smooth manifold and that G is a compact Lie group, with Lie algebra $\mathfrak{g}$, that acts on M. Then there exists a graded-algebra homomorphism:

$$\begin{gathered} F : \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{hor} \longrightarrow S(\mathfrak{g}^*) \otimes \Omega(M), \\ \alpha = a + \sum a_I \lambda_I \longmapsto a, \end{gathered} \tag{5.14}$$

along with the inverse homomorphism:

$$\begin{gathered} H : S(\mathfrak{g}^*) \otimes \Omega(M) \longrightarrow \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{hor}, \\ a \longmapsto \left(\prod_{i=1}^{n} (1 - \lambda_i \imath_{X_i})\right) a. \end{gathered} \tag{5.15}$$

Moreover, the graded-algebra homomorphism (5.14) induces a graded-algebra isomorphism between the basic subalgebras:

$$F : \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{basic} \xrightarrow{\sim} \big(S(\mathfrak{g}^*) \otimes \Omega(M)\big)^G. \tag{5.16}$$

The graded-algebra isomorphism F in (5.14) is known as the **Weil-Cartan isomorphism** [Tu20, Theorem 21.1], and essentially "forgets" any term that contains a λ_j factor.

Definition 5.28. The complex:

$$\Omega^\bullet_G(M) := \big(S(\mathfrak{g}^*) \otimes \Omega(M)\big)^G \tag{5.17}$$

is called the **Cartan model**. Elements that belong to the Cartan model $\Omega^\bullet_G(M)$ are called **equivariant differential forms**.

From Theorem 5.27, we get the following commutative diagram:

$$\begin{tikzcd}
\big(W(\mathfrak{g}) \otimes \Omega^\bullet(M)\big)_{\text{basic}} \arrow[d, "\delta"] & \Omega^\bullet_G(M) \arrow[l, "H"', "(5.15)"] \arrow[d, dashed, "d_{\mathfrak{g}}"] \\
\big(W(\mathfrak{g}) \otimes \Omega^\bullet(M)\big)_{\text{basic}} \arrow[r, "F", "(5.16)"'] & \Omega^\bullet_G(M)
\end{tikzcd} \tag{5.18}$$

where the map $d_{\mathfrak{g}} : \Omega^\bullet_G(M) \to \Omega^\bullet_G(M)$ is a differential that is defined in (5.19) below, and which we call the **Cartan differential**. In Proposition 5.29, we shall see that the Cartan differential $d_{\mathfrak{g}}$ equals the image of the Weil differential δ, under the Weil-Cartan isomorphism F in (5.16).

Proposition 5.29. *Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Then, given a basis, $X_1, \ldots, X_n$, of $\mathfrak{g}$, the Cartan differential $d_{\mathfrak{g}}$ of the Cartan complex $\Omega^\bullet_G(M)$ is:*

$$d_{\mathfrak{g}} : \Omega^\bullet_G(M) \to \Omega^\bullet_G(M), \qquad \textit{where} \qquad d_{\mathfrak{g}}\omega = \left(d - \sum u_i \imath_{X_i}\right)\omega. \tag{5.19}$$

Proof. Recall from the paragraph after Theorem 5.27, that the Weil-Cartan isomorphism F in (5.16) is the homomorphism that forgets any term that contains a λ_j factor, and whose inverse is $H = \prod(1 - \lambda_i \imath_{X_i})$ from (5.15).

Let $\alpha \in \Omega^\bullet_G(M)$ be an equivariant differential form, then:

$$\begin{aligned}
H(\alpha) &= \left(\prod(1 - \lambda_i \imath_{X_i})\right)\alpha \\
&= \alpha - \sum \lambda_i \imath_{X_i}\alpha + \sum(\lambda_i \imath_{X_i})(\lambda_j \imath_{X_j})\alpha - [\ldots],
\end{aligned}$$

and:

$$\delta H(\alpha) = \delta\alpha - \sum\left(r_i - \frac{1}{2}\sum c^i_{kl}\lambda_k \wedge \lambda_l\right)\imath_{X_i}\alpha + [\ldots], \tag{5.20}$$

where we have used the term "$[\ldots]$" to represent a sum that contains at least one λ_i factor. Next, since applying F to (5.20) is the same as dropping each term containing a λ_j factor, we get:

$$F\delta H(\alpha) = F(\delta\alpha) - \sum r_i \imath_{X_i}\alpha.$$

On the other hand, suppose that:

$$\alpha = \sum r_1^{i_1} \ldots r_n^{i_n} \tilde{\alpha}_{i_1 \ldots i_n} = \sum r^I \tilde{\alpha}_I, \qquad \text{where } \tilde{\alpha}_I \in \Omega(M).$$

Since $\delta r_i = \sum_{1 \le k,l \le n} c^i_{kl}[r_k \wedge \lambda_l]$ from Lemma 5.18, we observe that:

$$\delta\alpha = \sum(\delta r^I)\tilde{\alpha}_I + \sum r^I d\tilde{\alpha}_I = \sum[\ldots]\,\tilde{\alpha}_I + \sum r^I d\tilde{\alpha}_I,$$

and therefore:

$$\begin{aligned} F\delta H(\omega) &= F(\delta a) - \sum r_i \imath_{X_i} a \\ &= \sum r^I d\tilde{\alpha}_I - \sum r_i \imath_{X_i} \alpha. \end{aligned}$$

So, if we define the usual exterior derivative d on the Cartan model $\Omega_G^\bullet(M)$ by:

$$d\alpha := d\left(\sum r^I \tilde{\alpha}_I\right) := \sum r^I d\tilde{\alpha}_I,$$

then the Cartan differential $d_\mathfrak{g}$ is given by:

$$d_\mathfrak{g}\alpha = \left(d - \sum r_i \imath_{X_i}\right)\left(\sum r^I \tilde{\alpha}_I\right) = \sum r^I d\tilde{\alpha}_I - \sum r_i \imath_{X_i}\alpha = F\delta H(\alpha),$$

That is to say, that the diagram (5.18) commutes. $\square$

Corollary 5.30. *Let M be a smooth manifold and let G be a compact Lie group that acts on M with Lie algebra $\mathfrak{g}$. Then:*

(i) the Cartan differential $d_\mathfrak{g} : \Omega_G^\bullet(M) \to \Omega_G^\bullet(M)$ is an anti-derivation of degree -1;

(ii) the Cartan differential is zero on the component $S(\mathfrak{g}^*)^G$ of the Cartan model $\Omega_G^\bullet(M)$, where $\Omega_G^\bullet(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ from Definition 5.28.

Proof. For (i), as the Weil-Cartan isomorphism F and its inverse $H = F^{-1}$ are isomorphisms between graded algebras from Theorem 5.27 and as δ is an anti-derivation of degree -1, by using the commutative diagram (5.18) we see that $d_\mathfrak{g}$ is an anti-derivation of degree -1 too.

For (ii), if $\dim G = n$ then, since $S(\mathfrak{g}^*) \cong \mathbb{R}[r_1, \ldots, r_n]$ and as $d_\mathfrak{g}$ is an anti-derivation of degree 0, it suffices to show that $d_\mathfrak{g} r_k = 0$ for each indeterminate $r_1, \ldots, r_n$. As $\imath_{X_i} r_k = 0$, it follows from the definition of $d_\mathfrak{g}$ in (5.19), that:

$$\begin{aligned} d_\mathfrak{g} r_k &= dr_k - \sum\nolimits_{i=1}^n r_i \imath_{X_i} r_k \\ &= dr_k && \text{(since } \imath_{X_i} r_k = 0) \\ &= 0. \end{aligned}$$

Hence, $d_\mathfrak{g}$ annihilates any element belonging to $S(\mathfrak{g}^*)$. $\square$

5.3 Equivariant Differential Forms

Whilst we said in Definition 5.28 that any element belonging to the Cartan model, say $\alpha \in \Omega_G^\bullet(M)$, is an equivariant differential form, we would to use this section to study them some more. An element

$\alpha = \sum r^I \alpha_I = \sum r_1^{i_1} \dots r_n^{i_r} \alpha_{i_1 \dots i_r}$ in $S(\mathfrak{g}^*)$ can be thought of as a function from $\mathfrak{g}^*$ into $\mathbb{R}$, i.e., for any $X \in \mathfrak{g}$ and $\alpha_{i_1 \dots i_r} \in \mathbb{R}$:

$$\alpha(X) = \sum r_1(X)^{i_1} \dots r_n(X)^{i_n} \alpha_{i_1 \dots i_n} \in \mathbb{R}.$$

By tensoring $S(\mathfrak{g}^*)$ with the vector space $\Omega^\bullet(M)$, we see that $\alpha(X)$ becomes an $\Omega^\bullet(M)$-valued function on $\mathfrak{g}^*$, i.e. after tensoring, then $\alpha_{i_1 \dots i_n} \in \Omega(M)$, and thus $\alpha(X) \in \Omega(M)$. Furthermore, the Lie group G acts on $\mathfrak{g}^*$ via the coadjoint representation and so, if $\Omega(M)$ is a G-representation, an invariant element $\alpha = \sum r^I \alpha_I \in \Omega_G^\bullet(M)$ corresponds to a G-equivariant map $\alpha : \mathfrak{g}^* \to \Omega(M)$. Hence an alternative way of viewing the definition of an equivariant differential form from Definition 5.28 is the following:

Definition 5.31. Let M be a smooth manifold and let G be a compact Lie group acting on M, with Lie algebra $\mathfrak{g}$. Then we say that a ***G*-equivariant differential form** is a map $\alpha : \mathfrak{g} \to \Omega(M)$. That is to say, it is a differential form $\alpha = \sum r^I \alpha_I$ that is a polynomial in the $r_1, \dots, r_n$, with coefficients in $\Omega(M)$, and which is G-equivariant:

$$\alpha(\mathrm{Ad}_g(X)) = g \cdot \alpha(X), \qquad \text{for any } X \in \mathfrak{g} \text{ and } g \in G.$$

In terms of G-equivariant differential forms, the Cartan differential can be written as [BGV04, §7.1]:

Proposition 5.32. *The Cartan differential* $d_\mathfrak{g} : \Omega_G^\bullet(M) \to \Omega_G^\bullet(M)$ *is given by the formula:*

$$(d_\mathfrak{g}\alpha)(X) = d\left(\alpha(X)\right) - \imath_X\left(\alpha(X)\right) \tag{5.21}$$

where $\alpha \in \Omega_G^\bullet(M)$ *and* $X \in \mathfrak{g}$.

Example 5.33. When $G = T$ is a torus with Lie algebra $\mathfrak{t}$, the adjoint action of T on the symmetric algebra $S(\mathfrak{t})$ is trivial, so $S(\mathfrak{t}^*)^T = S(\mathfrak{t}^*)$ and hence, after choosing a basis $r_1, \dots, r_n$ of $\mathfrak{t}$:

$$\begin{aligned} \Omega_T^\bullet(M) = \left(S(\mathfrak{t}^*) \otimes \Omega^\bullet(M)\right)^T &= S(\mathfrak{t}^*) \otimes \Omega^\bullet(M)^T \\ &\cong \Omega^\bullet(M)^T[r_1, \dots, r_n]. \end{aligned}$$

Therefore, any T-equivariant differential form α must necessarily be a polynomial in the indeterminates $r_1, \dots, r_n$, whose coefficients are T-invariant differential forms on M:

$$\alpha = \sum r_1^{i_1} \dots r_n^{i_n} \alpha_{i_1 \dots i_n} = \sum r^I \alpha_I, \qquad \text{where } \alpha_I \in \Omega^\bullet(M)^T.$$

The Cartan differential $d_\mathfrak{t}$ is therefore given by:

$$d_\mathfrak{t}\alpha = d\alpha - \sum r_i \imath_{X_i} \alpha_i,$$

with $dr_i = 0$ and $\imath_{X_i} r_j = 0$ for each $i, j = 1, \dots, n$, and where $d\alpha$ is just the ordinary differential of α as an element of $\Omega^\bullet(M)$.

5.4 Equivariant Characteristic Classes

Let $\alpha \in \Omega^\bullet(M)$ be a differential form on M. Given an element $X \in \mathfrak{g}$, applying the Cartan differential $d_{\mathfrak{g}}$ from (5.21) twice to α, we see that:

$$\begin{aligned}(d_{\mathfrak{g}}^2\alpha)(X) &= \big(d - \imath_X\big)\big(d(\alpha(X)) - \imath_X(\alpha(X))\big) \\ &= d^2\big(\alpha(X)\big) - \big[(d \circ \imath_X)(\alpha(X)) + (\imath_X \circ d)(\alpha(X))\big] + \imath_X^2\big(\alpha(X)\big) \\ &= -L_X\alpha.\end{aligned} \tag{5.22}$$

Hence from (5.22), we see that α will be an exact differential form with respect to the Cartan differential $d_{\mathfrak{g}}$ if, and only if, $L_X\alpha = 0$. The condition that $L_X\alpha = 0$ is precisely the condition that α is an invariant differential form from Lemma 5.25. This implies that the Cartan model $\Omega_G^\bullet(M)$ can be made into a complex, by pairing it with the Cartan differential, $d_{\mathfrak{g}}$.

Lemma 5.34. *Let M be a smooth manifold and let G be a compact Lie group that acts on M with Lie algebra $\mathfrak{g}$. Then the Cartan model $\Omega_G^\bullet(M)$ equipped with the Cartan differential $d_{\mathfrak{g}}$ forms the* ***Cartan complex****, $(\Omega_G^\bullet(M), d_{\mathfrak{g}})$.*

From Lemma 5.34, the notions of closed and exact differential forms for the de Rham differential d carry over to that of closed and exact equivariant differential forms, in terms of the Cartan differential $d_{\mathfrak{g}}$. That is, an element $\alpha \in \Omega_G^\bullet(M)$ such that $d_{\mathfrak{g}}\alpha = 0$ are called equivariantly *closed* differential forms, whereas an element α such that $\alpha = d_{\mathfrak{g}}\beta$ for some $\beta \in \Omega_G^\bullet(M)$ are called equivariantly *exact* differential forms.

When M is a smooth manifold and when G is a compact Lie group acting on M, then M is a fibre of the homotopy quotient M_G when viewed as a G-principal fibre bundle $M_G \to \mathrm{BG}$. The inclusion $j : M \hookrightarrow M_G$ induces the restriction $j^* : H^\bullet(M_G) \cong H_G^\bullet(M) \to H^\bullet(M)$ in ordinary cohomology. Hence, j^* constitutes a canonical map from equivariant cohomology to ordinary cohomology.

A manifold M with a G-action is said to be **equivariantly formal**, if the canonical map $j^* : H_G^\bullet(M) \to H^\bullet(M)$ is surjective. In ordinary cohomology, a d-closed differential form $\omega \in \Omega(M)$ defines a cohomology class $[\omega] \in H^\bullet(M)$. Analogously, if there exists a $d_{\mathfrak{g}}$-closed equivariant differential form $\varpi \in H_G^\bullet(M)$ such that $j^*[\varpi] = [\omega] \in H^\bullet(M)$, then ϖ is said to be an **equivariantly closed extension** of the differential form ω.

Example 5.35. Let M be a smooth manifold and let G a compact Lie group that acts on M. A G-equivariant differential two-form $\varpi \in \Omega_G^2(M)$ must necessarily be of the form $\tilde{\omega} = \omega - \mu$, since:

$$\Omega_G^2(M) \cong \Big(\big(S^0(\mathfrak{g}^*) \otimes \Omega^2(M)\big) \oplus \big(S^1(\mathfrak{g}^*) \otimes \Omega^0(M)\big) \Big)^G,$$

where $\omega \in \Omega^2(M)$ is an ordinary differential two-form on M that is G-invariant and, from Section 5.3, the element $\mu \in \Omega_G^0(M)$ is a $\Omega^0(M)$-valued G-equivariant map from $\mathfrak{g}$ into the space of smooth functions on M:

$$\mu \in \Omega^0_G(M) \iff \mu : \mathfrak{g} \to \mathcal{C}^\infty(M;\mathbb{R}) \quad \text{and} \quad \mu(\mathrm{Ad}_g X) = g \cdot \mu(X),$$

for every $g \in G$ and $X \in \mathfrak{g}$.

Thus, on the one hand, as X varies smoothly throughout $\mathfrak{g}$, we obtain a smooth $\mathbb{R}$-valued function:

$$\mu(X) : M \to \mathbb{R}, \qquad \text{given by} \qquad \mu(X) \mapsto \mu(X)(p),$$

whereas, on the other hand, as μ is linear on $\mathfrak{g}$, it defines a map from M into the dual Lie algebra $\mathfrak{g}^*$:

$$\mu : M \to \mathfrak{g}^*, \qquad \text{given by} \qquad \langle \mu(m),\, X\rangle := \phi(X)(p).$$

Hence, for ϖ to be an equivariantly closed differential two-form, for every $X \in \mathfrak{g}$, it has to satisfy:

$$(d_\mathfrak{g}\varpi)(X) = (d - \imath_X)(\omega(X) - \mu(X)) = d\omega(X) - d\mu(X) - \imath_X\omega(X) + \imath_X\mu(X) = 0,$$

which, since $\imath_X\mu(X) = 0$, can be rephrased as the following two conditions:

$$d_\mathfrak{g}\varpi = 0 \qquad \text{if, and only if,} \qquad d\omega = 0 \quad \text{and} \quad \imath_X\omega = -d\mu^X. \tag{5.23}$$

Clearly, when G acts on a symplectic manifold M in a Hamiltonian way then, denoting by $\omega \in \Omega^2(M)$ the symplectic two-form of M, then the two conditions in (5.23) are satisfied by the moment map $\mu : M \to \mathfrak{g}^*$ corresponding to ω for the G-action.

We rephrase this result as:

Lemma 5.36. *Let M be a symplectic manifold with symplectic two-form ω, and suppose that a compact Lie group G acts on M with corresponding moment map $\mu : M \to \mathfrak{g}^*$. Then:*

$$\varpi(X) := \omega(X) + \langle \mu(-),\, X\rangle \in \Omega^2_G(M)$$

is an equivariantly closed extension of the symplectic two-form $\omega \in \Omega(M)$. Moreover, ϖ determines the equivariant cohomology class $[\varpi] \in H^2_G(M)$, since ϖ is $\mathrm{d}_\mathfrak{g}$-closed.

More generally, let $\pi : V \to M$ be a smooth G-equivariant vector bundle over M, that induces the map:

$$\pi_G : V_G := V \times_G \mathrm{EG} \longrightarrow M_G$$

between homotopy quotients. Then $\pi_G : V_G \to M_G$ is a vector bundle with the same rank as $\pi : V \to M$, and $\pi_G : V_G \to M_G$ is oriented, provided that $\pi : V \to M$ is too.

Definition 5.37. The ***G*-equivariant Euler class** $\mathrm{Eul}^G(V)$ of an oriented G-equivariant vector bundle $\pi : V \to M$, is just the ordinary Euler class $\mathrm{Eul}(V_G)$ of the homotopy quotient $\pi_G : V_G \to M_G$. That is to say:

$$\mathrm{Eul}^G(V) := \mathrm{Eul}(V_G) \equiv \mathrm{Eul}^G(V \times_G \mathrm{EG}) \in H^\bullet_G(M). \tag{5.24}$$

Similarly, if $\pi : V \to M$ is a complex G-equivariant vector bundle, then its ***G*-equivariant Chern classes** are just the ordinary Chern classes $c_i(V_G)$ of the homotopy quotient $\pi_G : V_G \to M_G$. That is to say:

$$c_i^G(V) := c_i(V_G) \equiv c_i(V \times_G \mathrm{EG}) \in H_G^\bullet(M). \tag{5.25}$$

If a compact Lie group G acts on M, then the correspondence between $\mathcal{L}$ and P can be made G-equivariant. If G acts trivially on M, then G acts on the fibre $\mathcal{L}_p$ for every point $p \in M$ and, provided that M is connected, this action is independent of the point p. Denote the weight of this G-action on the fibre $\mathcal{L}_p$ by $\alpha \in \mathfrak{g}^*$.

Lemma 5.38 (**G-equivariant first Chern classes**). *Let $\pi : \mathcal{L} \to M$ be a holomorphic pre-quantum line bundle over M. If G acts trivially on M and if M is connected, then the G-**equivariant first Chern class** of $\pi : \mathcal{L} \to M$ is:*

$$c_1^G(\mathcal{L}) = c_1(\mathcal{L}) - \alpha \in H_G^2(M), \tag{5.26}$$

where $c_1(\mathcal{L})$ is the ordinary first Chern class of $\pi : \mathcal{L} \to M$, and $\alpha \in \mathfrak{g}^*$ is the weight of the G-action on any fibre $\mathcal{L}_p$ with $p \in M$.

Proof. As $\alpha \in \mathfrak{g}^*$ is the weight of the G-action on $\mathcal{L}$ then, for any element $g = \exp(tX) \in G$ with $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, the G-action on the principal U_1-bundle P is just multiplication by $e^{\sqrt{-1}t\langle\alpha,\, X\rangle}$. Therefore:

$$X_P = \langle \alpha,\, X\rangle \frac{\partial}{\partial\phi}$$

generates the principal U_1-action on P. Denote by $\theta \in \Omega^1(P)$ the connection one-form on P and by $\Theta \in \Omega^2(M)$ the curvature two-form on M that satisfies $d\theta = \pi^*\Theta$, as in Appendix A.8. Then:

$$\begin{aligned}(d_{\mathfrak{g}}\theta)(X) &= (d\theta)(X) - \imath_{X_P}\theta \\ &= (\pi^*\Theta)(X) - \langle\alpha,\, X\rangle\theta(\partial/\partial\phi) \\ &= (\pi^*\Theta)(X) - \langle\alpha,\, X\rangle,\end{aligned}$$

which is to say, $d_{\mathfrak{g}}\theta = \pi^*\Theta - \alpha$. Finally, since:

$$c_1(\mathcal{L}) = [\pi^*\Theta] = [d\theta],$$

we see that:

$$[d_{\mathfrak{g}}\theta] = [\pi^*\Theta - \alpha] \iff c_1^G(\mathcal{L}) = c_1(\mathcal{L}) - \alpha,$$

and the result follows. ☐

In the case when $G = T$ is an n-dimensional torus, a fundamental result, that we shall very frequently employ, is the equivariant version of the splitting principle in Theorem 4.9.

Theorem 5.39 (Equivariant splitting principle). *If T is a torus that acts on a compact manifold M, and if $E \to M$ is a T-equivariant holomorphic vector bundle, then one may assume that E splits T-equivariantly into a direct sum of complex line bundles:*

$$E \cong V_1 \oplus \ldots \oplus V_n. \tag{5.27}$$

We are now able to apply Lemma 5.38 and the equivariant splitting principle from Theorem 5.39 to determine a few equivariant characteristic classes in the most useful cases to us.

Example 5.40 (*T*-equivariant Euler class). Let M be an n-dimensional complex manifold, and suppose that a torus T acts on M. Assume that the fixed-point locus M^T consists solely of isolated fixed-points. Then, for each fixed-point $p \in M^T$, its normal bundle $\nu_p \to \{p\}$ is a T-equivariant vector bundle since T acts trivially on $\{p\} \subseteq M^T$.

By Theorem 5.39, we may assume that the normal bundle ν_p splits as:

$$\nu_p \cong V_{\alpha_{p,1}} \oplus \ldots \oplus V_{\alpha_{p,n}},$$

where each $V_{\alpha_{p,j}}$ is a complex line bundle on which T acts with weight $\alpha_{p,j} \in \mathfrak{t}^*$. Then for any $\xi \in \mathfrak{t}$, the ***T*-equivariant Euler class**, denoted by $\mathrm{Eul}^T(\nu_p)$, of the normal bundle ν_p is:

$$\mathrm{Eul}^T(\nu_p; \xi) := \prod_{j=1}^{n} c_1^T(V_{p,j}; \xi) = \prod_{j=1}^{n} \left[(d\theta_j|_p)(\xi) - \langle \alpha_{p,j},\, \xi \rangle \right] = (-1)^n \prod_{j=1}^{n} \langle \alpha_{p,j},\, \xi \rangle,$$

where $d\theta_j|_p = 0$ since $\{p\}$ is zero-dimensional. To summarise, when $p \in M^T$ is an isolated fixed point, then:

$$\mathrm{Eul}^T(\nu_p) = (-1)^n \prod_{j=1}^{n} \alpha_{p,j}. \tag{5.28}$$

Example 5.41 (*T*-equivariant Todd class). Assuming the same hypotheses as in Example 5.40, let us now introduce the ***T*-equivariant Todd class**, denoted by $\mathrm{Td}^T(TM)$, of which the non-equivariant version $\mathrm{Td}(TM)$ was introduced in Definition 4.10.

Again, let us the equivariant splitting principle from Theorem 5.39, this time however applied to the tangent space T_pM for some isolated fixed point $p \in M^T$, so that:

$$T_pM \cong V_{\alpha_{p,1}} \oplus \ldots \oplus V_{\alpha_{p,n}},$$

where T acts on each summand $V_{\alpha_{p,j}}$ with weight $\alpha_{p,j} \in \mathfrak{t}^*$.

The T-equivariant Todd class $\mathrm{Td}^T(T_pM)$ is then obtain from the formula (4.15) for the non-equivariant Todd class $\mathrm{Td}(T_pM)$, by replacing the non-equivariant first Chern classes $c_1(V_{\alpha_{p,j}})$ with their T-equivariant counterparts, $c_1^T(V_{\alpha_{p,j}})$, from Lemma 5.38. If we choose some $\xi \in \mathfrak{t}$ such that $\langle \alpha_{p,j},\, \xi \rangle \neq 0$ for each $j = 1, \ldots, n$, then the T-equivariant Todd class for the tangent space T_pM over an isolated fixed point $p \in M^T$ is:

$$\mathrm{Td}^T(T_pM;\xi) = \prod_{j=1}^{n} \frac{c_1^T(V_{\alpha_{p,j}};\xi)}{\left[1 - e^{-c_1^T(V_{\alpha_{p,j}};\xi)}\right]} = (-1)^n \prod_{j=1}^{n} \frac{\langle \alpha_{p,j},\, \xi \rangle}{\left[1 - e^{\langle \alpha_{p,j},\, \xi \rangle}\right]}.$$

Since $\{p\}$ is zero-dimensional, observe that its tangent space in TM coincides with its normal bundle, $T_pM \cong \nu_p$. Hence, in summary:

$$\mathrm{Td}^T(\nu_p) = \mathrm{Td}^T(T_pM) = (-1)^n \prod_{j=1}^{n} \frac{\alpha_{p,j}}{[1 - e^{\alpha_{p,j}}]}. \tag{5.29}$$

5.5 Localisation and Equivariant Integration

In the category of smooth manifolds, the assignment $M \mapsto H^i(M)$ is contravariant, and then the de Rham theory of cohomology coincides with any other cohomology theory that satisfies the Eilenberg-Steenrod axioms [ES45], provided that $H^\bullet(\{\mathrm{pt}\}) \cong \mathbb{R}$ in dimension zero [Bot99, §3]. The de Rham cohomology group $H^\bullet(M)$ is finite-dimensional when M is compact and, furthermore, if $n = \dim_{\mathbb{C}} M$ and if M is oriented then, from the constant projection:

$$\pi : M \longrightarrow \{\mathrm{pt}\}, \tag{5.30}$$

we obtain the push-forward:

$$\pi_* : H^n(M) \longrightarrow H^0(\{\mathrm{pt}\}) \cong \mathbb{R}, \tag{5.31}$$

The push-forward in (5.31) is used to define fibre-wise integration:

$$\pi_*(\omega) = \int_M \omega, \tag{5.32}$$

since then the fibre of π equals M.

Lemma 5.42. *Let $\pi : E \to B$ be an orientable fibre bundle over a base manifold B whose fibres $\pi^{-1}(b)$ have codimension k in E for any $b \in B$. Then, by "integrating over the fibre variables", the push-forward:*

$$\pi_* : H^\bullet(E) \longrightarrow H^{\bullet - k}(B),$$

can be made into a $H^\bullet(B)$-homomorphism.

Proof. This is the non-equivariant analogue to Lemma 5.10, albeit with a more general base manifold. Let $e \in H^\bullet(E)$ and $b \in H^\bullet(B)$. Since:

$$\pi_*(e \cdot \pi^* b) = (\pi_* e) \cdot b,$$

let us consider $H^\bullet(E)$ to be a product over $H^\bullet(B)$ via $e \cdot b := e \cdot (\pi^* b)$. Then we see that:

$$\pi_*(e \cdot b) = \pi_*(e \cdot \pi^* b) = (\pi_* e) \cdot b \tag{5.33}$$

defines a binary multiplicative operation, thus making π_* into a $H^\bullet(B)$-module homomorphism. $\square$

Of course, Lemma 5.42 concerns itself with fibre-integration in terms of ordinary de Rham cohomology theory – for equivariant cohomology, we expect that the map:

$$\pi^G : M_G \longrightarrow \{\mathrm{pt}\}_G,$$

should give rise to the fibre-wise integral via the push-forward:

$$\pi^G_* : H^\bullet_G(M) \longrightarrow H^\bullet_G(\{\mathrm{pt}\}) \cong H^\bullet(\mathrm{BG}).$$

Recall from Lemma 5.10 that, for a G-space M, the equivariant cohomology $H^\bullet_G(M)$ is an algebra over the coefficient ring $H^\bullet(\mathrm{BG})$, yet the coefficient ring $H^\bullet(\mathrm{BG})$ is not necessarily a subring of $H^\bullet_G(M)$. However, in Proposition 5.11, we showed that when the G-fixed-point locus M^G is non-empty, then $H^\bullet(\mathrm{BG})$ can be embedded into $H^\bullet_G(M)$ as a subring.

To establish the equivariant analogue of fibre-wise integration, we must first cover some prerequisites. A **torsion submodule** of $H^\bullet_G(M)$ over $H^\bullet(\mathrm{BG})$, is the submodule of non-zero elements $\alpha \in H^\bullet_G(M)$ such that $\omega \cdot r = 0$ for some non-zero element $r \in H^\bullet(\mathrm{BG})$. When $G = T$ is an n-dimensional torus, denote $\mathbb{R}[r] := \mathbb{R}[r_1, \ldots, r_n]$, so that we have $H^\bullet(\mathrm{BT}) \cong \mathbb{R}[r]$ and thus $H^\bullet_T(M)$ becomes an $\mathbb{R}[r]$-module.

By considering now the bigger ring $\mathbb{R}[r, r^{-1}]$, whose elements are the Laurent series in the indeterminates $r_1, \ldots, r_n$, then it is possible to kill off the torsion as follows: note that an $\mathbb{R}[r]$-module A is a torsion module if, and only if, $A \otimes_{\mathbb{R}[r]} \mathbb{R}[r, r^{-1}]$ is the trivial module. Indeed, in the larger module $A \otimes_{\mathbb{R}[r]} \mathbb{R}[r, r^{-1}]$, any element $\alpha \in A$ can be written in the form $(\alpha r^k) r^{-k}$ for any $k \in \mathbb{Z}$, thus killing off any torsion by taking k to be a large enough integer.

Now suppose that the torus T acts smoothly on M, so that its fixed-point locus M^T is a regular submanifold [Tu20, Theorem 25.1] of M. Assume that $F \subseteq M^T$ is a connected component of the fixed-point locus, then as the fixed-point set F is necessarily T-invariant, the inclusion $\iota : F \hookrightarrow M$ is T-equivariant.

Both $H^\bullet_T(F)$ and $H^\bullet_T(M)$ are $H^\bullet(\mathrm{BT})$-modules and also rings – they are hence both $H^\bullet(\mathrm{BT})$-algebras. If there exists a *non-zero* element denoted by $\varphi \in H^\bullet(\mathrm{BT})$, that we shall assume and whose existence is proven by example (5.45), then one may localise $H_T^\bullet(F)$ and $H_T^\bullet(M)$ with respect to φ, and the inclusion $\iota : F \hookrightarrow M$ induces a $H^\bullet(\mathrm{BT})$-algebra homomorphism:

$$\iota_\varphi^* : H_T^\bullet(M) \longrightarrow H_T^\bullet(F)_\varphi.$$

The reason that we have introduced all of this is because of Borel's localisation theorem, that is stated in Theorem 5.43 next. What Borel's localisation theorem says, is that the equivariant cohomology of a T-manifold is concentrated on its fixed-point set M^T up to torsion, and that the isomorphism in *localised* equivariant cohomology of the manifold and its fixed-point set is in fact a *ring* isomorphism.

Theorem 5.43 (Borel localisation). *Suppose that a torus T acts on a manifold M with compact fixed-point set M^T. Let $F \subseteq M^T$ be a connected component of M^T, and let $\iota : F \hookrightarrow M$ be the inclusion. Then both the kernel and cokernel of the pull-back:*

$$\iota^* : H_T^\bullet(M) \longrightarrow H_T^\bullet(F)$$

are torsion $H^\bullet(\mathrm{BT})$-modules. Hence, after localising with respect to some non-zero element $\varphi \in H^\bullet(\mathrm{BT})$, the localised pull-back ι_φ^* becomes an isomorphism:

$$\iota_\varphi^* : H_T^\bullet(M)_\varphi \xrightarrow{\sim} H_T^\bullet(F)_\varphi. \tag{5.34}$$

See, for example, [Hsi75, Theorem (III.1)], for a proof of Theorem 5.43.

To avoid going too far afield, let us come back to the equivariant version of the fibre-wise integral in (5.32). As above, we specialise to the case when a torus T acts locally freely on a compact manifold M, and where $F \subseteq M^T$ is a connected component of the fixed-point set. Let:

$$\iota : F \hookrightarrow M \tag{5.35}$$

be the inclusion. Forgetting momentarily about any equivariance, the inclusion (5.35) induces a push-forward in homology:

$$\iota_* : H_\bullet(F) \longrightarrow H_\bullet(M),$$

as well as a pull-back in cohomology:

$$\iota^* : H^\bullet(M) \longrightarrow H^\bullet(F).$$

Since F and M are both compact and oriented manifolds, via Poincaré duality, we can obtain a push-forward in cohomology too [AB84, §2]. Namely, in denoting $m = \dim_{\mathbb{C}} M$ and $f = \dim_{\mathbb{C}} F$ then, for any $0 \leq q \leq m$, we have the commutative diagram:

$$\begin{tikzcd}
H_q(F) \arrow[rr, "\iota_*"] \arrow[d, "\wr\|\ \text{P.D.}"] & & H_q(M) \arrow[d, "\wr\|\ \text{P.D.}"] \\
H^{m-f}(F) \arrow[rr, dashed, "\iota_*"] & & H^{m-q}(M)
\end{tikzcd}$$

we obtain the **Gysin**, or ***umkehrungs***, **homomorphism**:

$$\iota_* : H^\bullet(F) \longrightarrow H^{\bullet+(m-f)}(M), \tag{5.36}$$

As (5.35) is the inclusion, its Gysin homomorphism $\iota_* : H^\bullet(F) \to H^\bullet(M)$ factors through the **Thom isomorphism** [AB84, §2]:

$$\Phi_F : H^\bullet(F) \xrightarrow{\sim} H_c^{\bullet+(m-f)}(\nu_F), \tag{5.37}$$

where $H_c^\bullet(\nu_F)$ is the compactly-supported cohomology of the normal bundle ν_F to F in M, as:

$$\iota_* : H^\bullet(F) \xrightarrow{\Phi_F} H_c^{\bullet+(m-f)}(\nu_F) \longrightarrow H^{\bullet+(m-f)}(M), \tag{5.38}$$

The image of the unit $1 \in H^\bullet(F)$ under the Thom isomorphism (5.37) defines the **Thom class** $\Phi_F(1) \in H_c^f(M)$, which is the cohomology class that is the Poincaré dual to the fundamental class of F in M, i.e., $[F] \in H_{m-f}(M)$. See, for example, [AB84] or [GS99] for more details.

One characteristic possessed by the Thom class $\Phi_F(1)$ which will be essential to us, is that its restriction to F coincides with the Euler class of the normal bundle ν_F to F in M, see [GS99, Theorem 10.5.1]. Therefore, together with (5.37), we get:

$$(\iota^* \circ \iota_*)(1) = \mathrm{Eul}(\nu_F). \tag{5.39}$$

The Gysin homomorphism (5.36), in addition to the results of (5.37) and (5.39), can be extended to the equivariant setting in a straightforward way, see [AB84, §2], as:

$$\iota_* : H_T^\bullet(F) \to H_T^\bullet(M), \quad \iota^* : H_T^\bullet(M) \to H_T^\bullet(F), \quad \text{and} \quad (\iota^* \circ \iota_*)(1) = \mathrm{Eul}^T(\nu_F). \tag{5.40}$$

As F is a connected component of M^T, it is T-invariant, so by the Künneth formula we find that:

$$H_T^\bullet(F) = H^\bullet(F \times_T \mathrm{ET}) \cong H^\bullet(F \times \mathrm{BT}) \cong H^\bullet(F) \otimes H^\bullet(\mathrm{BT}). \tag{5.41}$$

Therefore, since $\mathrm{Eul}^T(\nu_F) \in H_T^\bullet(F)$, from Theorem 5.43 we see that $\mathrm{Eul}(\nu_F)$ becomes invertible after passing to some suitably-localised module:

$$H_T^\bullet(F)_\varphi \cong H^\bullet(F) \otimes H^\bullet(\mathrm{BT})_\varphi,$$

where the non-zero element $\varphi \in H^\bullet(\mathrm{BT})$ appeared in the statement of Theorem 5.43.

To finally determine an example for the non-zero element $\varphi \in H^\bullet(\mathrm{BT})$, assume that the T-equivariant normal bundle ν_F to F decomposes as $\nu_F \cong \oplus_j V_{\alpha_{F,j}}$ via the equivariant splitting principle in Theorem 5.27, where T acts on each summand $V_{\alpha_{F,j}}$ with weight $\alpha_{F,j} \in \mathfrak{t}^*$. Observe that $\alpha_{F,j} \neq 0$ for each j, since otherwise the component $V_{\alpha_{F,j}}$ would be tangent to the fixed-point component F and consequently not normal to it.

From (5.28) in Example 5.40, the T-equivariant Euler class $\mathrm{Eul}^T(\nu_F) \in H^\bullet_T(F)$ can be written as the product:

$$\mathrm{Eul}^T(\nu_F) = \prod_j c_1^T(V_{\alpha_{F,j}}) = \prod_j \left(c_1(V_{\alpha_{F,j}}) - \alpha_{F,j}\right),$$

where the T-equivariant first Chern classes are $c_1^T(V_{\alpha_{F,j}}) = c_1(V_{\alpha_{F,j}}) - \alpha_{F,j}$ from Lemma 5.38. Let us define:

$$\varphi_F := \prod_j \alpha_{F,j}, \tag{5.42}$$

which is a non-zero element of $S(\mathfrak{t}^*)$. Therefore φ_F is a non-vanishing $\mathfrak{t}$-valued polynomial and so, by passing to the localisation $H^\bullet_T(F)_{\varphi_F}$, we can factor $\mathrm{Eul}^T(\nu_F)$ as:

$$\mathrm{Eul}^T(\nu_F) = \varphi_F \cdot \prod_j \left(1 - \frac{c_1(\nu_{F,j})}{\alpha_{F,j}}\right). \tag{5.43}$$

Observe that the $c_1(\nu_{F,j})/\alpha_{F,j}$ terms in (5.43) are nilpotent, since raising any one of them to a power greater than $m = \dim_{\mathbb{C}} M$ would annihilate it. Therefore, $\mathrm{Eul}^T(\nu_F)$ is invertible with inverse:

$$\frac{1}{\mathrm{Eul}^T(\nu_F)} = \frac{1}{\varphi_F} \cdot \prod_j \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{c_1(\nu_{F,j})}{\alpha_{F,j}}\right)^k\right] \tag{5.44}$$

since the infinite series truncates after $m = \dim_{\mathbb{C}} M$ terms.

Finally, let us an element $\varphi \in S(\mathfrak{t}^*)$ to be the product of the φ_F over each connected fixed-point component $F \subseteq M^T$, so that:

$$\varphi := \prod_{F \subseteq M^T} \varphi_F = \prod_{F \subseteq M^T} \prod_j \alpha_{F,j} \in S(\mathfrak{t}^*). \tag{5.45}$$

Then φ is also non-zero as each φ_F is, and so it follows that the localised isomorphism (5.40):

$$\iota_* \circ \iota^* = \mathrm{Eul}(\nu_F) : H^\bullet_T(F)_\varphi \xrightarrow{\sim} H^\bullet_T(M)_\varphi,$$

is can be explicitly inverted. Namely, if we denote:

$$Q = \sum_{F \subseteq M^T} \frac{\iota_F^*}{\mathrm{Eul}^T(\nu_F)} : H^\bullet_T(M)_\varphi \longrightarrow H^\bullet_T(F)_\varphi,$$

where $\iota_F : F \hookrightarrow M$, then Q defines the inverse to the ring homomorphism:

$$\iota_* : H^\bullet_T(F)_\varphi \longrightarrow H^\bullet_T(M)_\varphi.$$

Thence, for any T-equivariant differential form $\alpha \in H^\bullet_T(M)$, by considering the localisation $H^\bullet_T(M) \subseteq H^\bullet_T(M)_\varphi$, we see that:

$$\alpha = (\iota_* \circ Q)(\alpha) = \sum_{F \subseteq M^T} \frac{1}{\mathrm{Eul}(\nu_F)} (\iota_* \circ \iota^*)(\alpha). \tag{5.46}$$

Then, by finally applying the equivariant push-forward:

$$\pi^M_* : H^\bullet_T(M) \longrightarrow H^\bullet(\mathrm{BT})$$

to both sides of (5.46), and by using the functoriality of push-forwards, i.e., that $(\pi^M_* \circ \iota_*|_F) = \pi^F_*$, then we finally obtain the equivariant integration formula:

$$\pi^M_*(\omega) = \sum_{F \subseteq M^T} \pi^F_* \left\{ \frac{\iota^*_F(\omega)}{\mathrm{Eul}^T(\nu_F)} \right\} = \sum_{F \subseteq M^T} \int_F \frac{\iota^*_F(\omega)}{\mathrm{Eul}^T(\nu_F)} \in H^\bullet(\mathrm{BT}). \tag{5.47}$$

In terms of the dual basis $r_1, \ldots, r_N$ of $\mathfrak{t}^*$, the push-forward π^M_* is represented by the operation that sends an equivariant differential form:

$$\alpha = \sum_I \tilde{\alpha}_I r^I$$

to the integral:

$$\pi^M_*(\alpha) = \sum_I \left(\int_M \alpha_I \right) r^I,$$

giving us a polynomial with respect to the variables, $H^\bullet(\mathrm{BT}) \cong \mathbb{R}[r_1, \ldots, r_n]$.

The derivation of the equivariant integral formula in 5.48 is the one used by Atiyah and Bott in [AB84]. In [BV82], Berline and Vergne however adopted a more geometric stance in their proof of what is now called the *Atiyah-Bott-Berline-Vergne localisation formula*.

Theorem 5.44 (Atiyah-Bott-Berline-Vergne localisation formula). *Let T be an n-dimensional torus acting on a compact manifold M. For any T-equivariant characteristic class $\alpha \in H^\bullet_T(M)$,*

$$\pi^M_*(\alpha) = \int_M \alpha = \sum_{F \subseteq M^T} \int_F \frac{i^*_F \alpha}{\mathrm{Eul}^T(\nu_F)}, \tag{5.48}$$

where $i^*_F : H^\bullet_T(M)_\varphi \to H^\bullet_T(F)_\varphi$ is the restriction of α to the connected T-fixed-point component $F \subseteq M^T$, and where ν_F is the normal bundle to the F in TM.

Observe that in the Atiyah-Bott-Berline-Vergne localisation formula (5.48), the left-hand side belongs to $H_T^\bullet(M)$, whereas the right-hand side belongs to the localisation $H_T^\bullet(M)_\varphi$. Hence, one should expect there to be a remarkable amount of cancellation on the right-hand side of (5.48) with integrands whose terms involve several non-trivial denominators.

When M^T consists of a finite number isolated fixed-points, Theorem 5.44 simplifies to a finite sum of terms over the fixed points of M^T.

Corollary 5.45. *Assuming the hypotheses of Theorem 5.44, and that the fixed-point locus M^T consists of finitely-many isolated fixed-points. Then the equivariant integration formula in (5.48) becomes:*

$$\pi_*^M(\omega) = \int_M \omega = \sum_{p \in M^T} \frac{i_p^* \omega}{\mathrm{Eul}^T(\nu_p)} \tag{5.49}$$

When (M, ω) is a compact Kähler manifold that is furthermore acted upon by a torus T in an effective and Hamiltonian way with $n = \dim_{\mathbb{C}} M = \dim_{\mathbb{R}} T$ and corresponding the moment map $\mu : M \to \mathfrak{t}^*$. If the fixed-point locus M^T for the T-action is non-empty, then we may apply the localisation formula (5.48) from Theorem 5.44 to the Riemann-Roch-Hirzebruch formula from Theorem 4.8, which simplifies its evaluation significantly.

When $\mathcal{L} \to M$ is a holomorphic pre-quantum line bundle over M, so that its first Chern class is $c_1(\mathcal{L}) = [\omega]$, and moreover assume that the action of T on M lifts up to one on $\mathcal{L}$. Over a fixed-point $p \in M^T$, we have that $\nu_p \cong T_p M$ since $\{p\}$ is zero-dimensional, and moreover let us assume that ν_p decomposes via the equivariant splitting principle (5.27) of Theorem 5.39, as:

$$\nu_p \cong V_{\alpha_{p,1}} \oplus \dots V_{\alpha_{p,n}},$$

where T acts on each $V_{\alpha_{p,j}}$ with weight $\alpha_{p,j} \in \mathfrak{t}^*$ for each $j = 1, \dots, n$. Additionally from Appendix A.8, the T-weight on the line bundle $\mathcal{L}$ is given by the value $\mu(p) \in \mathfrak{t}^*$ of the moment map for the T-action. By applying the localisation formula (5.49) from Corollary 5.45 to the Hirzebruch-Riemann-Roch formula (4.13) from Theorem 4.8, we obtain the equivariant index formula for when M is smooth.

Theorem 5.46 (Equivariant index theorem). *For any element $X \in \mathfrak{t}$ such that $\langle \alpha_{p,j},\, X \rangle \neq 0$ for each $j = 1, \dots, n$, the equivariant Riemann-Roch number χ is given by the formula:*

$$(\chi \circ \exp)(X) = \sum_{p \in M^T} \frac{e^{\langle \mu(p),\, X \rangle}}{\prod_{j=1}^n \left(1 - e^{\langle \alpha_{p,j},\, X \rangle}\right)}, \tag{5.50}$$

where $\chi : T \to H^\bullet(\mathrm{BT})$ is the representation ring for the T-action on the $\mathcal{L}$-twisted Dolbeault cohomology group, $H^0(M; \mathcal{L})$.

5.6 The Equivariant Kawasaki-Riemann-Roch Theorem

There still remains a proverbial "elephant in the room" to address, namely that the the scope of Theorem 5.46 does not extend to orbifolds. Yet, in general, from Theorem 3.17 we know that most of our cut spaces will be orbifolds. This dilemma forces us to consider the equivariant analogue of Theorem 4.14, the *Kawasaki-Riemann-Roch theorem*.

Let M be a compact symplectic orbifold with symplectic two-form ω, and let $\mathcal{L} \to M$ be a holomorphic pre-quantum orbifold line bundle over M. Assume that a torus T acts on M effectively and in a Hamiltonian way with corresponding moment map $\mu : M \to \mathfrak{t}^*$, and assume that M^T consists of finitely-many isolated fixed points.

Denote by $\widehat{M}$ and $\widehat{p}$ the inertia orbifolds of M and $\{p\}$ respectively, defined in the Appendix A, and let $\tau : \widehat{M} \to M$ be the corresponding immersion. We can lift the action of T on M up to one on $\widehat{M}$ via its fundamental vector fields [Dui11, §15.4]. From the inclusion $\{p\} \hookrightarrow M$, we get the normal bundle ν_p and we assume that it splits T-equivariantly as:

$$\nu_p \cong \widehat{V}_{p,1} \oplus \ldots \oplus \widehat{V}_{p,n}.$$

Denote the pull-backs of each $V_{p,j}$ via τ by $\widehat{V}_{p,j} := \tau^* \nu_p$ for each $j = 1, \ldots, n$, in addition to the pull-back of the fibre $\mathcal{L}_p$ via τ as $\widehat{\mathcal{L}}_p := \tau^* \mathcal{L}_p$.

The components of the inertia orbifold $\widehat{p}$ are indexed by the conjugacy classes $\gamma \in \mathrm{Conj}(\Gamma_p)$. Denote the canonical automorphisms of $\widehat{\mathcal{L}}_p$ and of $\widehat{V}_{p,j}$ by $A(\widehat{\mathcal{L}})$ and $A(\widehat{V}_{p,j})$, induced by the action of Γ_p on $\widehat{p}_\gamma$. As Γ_p is a finite abelian group and since each $\widehat{\mathcal{L}}_p$ and $\widehat{V}_{p,j}$ is a line bundle, when restricted to the component $\widehat{p}_\gamma$, the automorphisms $A(\widehat{\mathcal{L}})$ and $A(\widehat{V}_{p,j})$ can be identified with elements of U_1, see [Sil96, Remark 10.10]. That is to say, they give us the following characters for the Γ_p-representation for $\widehat{\mathcal{L}}_p$ as:

$$\chi_{p,0}(\gamma) = A(\widehat{\mathcal{L}}_p)|_{\widehat{p}_\gamma} \in U_1,$$

and for $\widehat{V}_{p,j}$ as:

$$\chi_{p,j}(\gamma) = A(\widehat{V}_{p,j})|_{\widehat{p}_\gamma} \in U_1,$$

where $j = 1, \ldots, n$.

Fix an element $\xi \in \mathfrak{t}$. From Definition 4.4, restricting the T-equivariant twisted Chern class $\mathrm{Ch}^T_{\widehat{M}}(\widehat{\mathcal{L}}_p)$ to $\widehat{p}_\gamma$ we have:

$$\mathrm{Ch}^T_{\widehat{M}}(\widehat{\mathcal{L}}_p)|_{\widehat{p}_\gamma} = \chi_{p,0}(\gamma) \cdot e^{c_1^T(\mathcal{L}_p;\xi)} = \chi_{p,0}(\gamma) \cdot e^{\langle \mu(p), \xi \rangle}.$$

Similarly, we get for $\widehat{V}_{p,j}$:

$$\mathrm{Ch}^T_{\widehat{M}}(\widehat{V}_{p,j})|_{\widehat{p}_\gamma} = \chi_{p,j}(\gamma) \cdot e^{c_1^T(V_{p,j};\xi)} = \chi_{p,j}(\gamma) \cdot e^{\langle \alpha_{p,j}, \xi \rangle},$$

where each $\alpha_{p,j} \in \mathfrak{t}^*$ denotes the isotropy weight for the T-action on $V_{p,j}$ for $j = 1, \ldots, n$.

For the associated characteristic class $D_{\widehat{p}_\gamma}$ defined in (4.21) using the immersion $\tau|_{\widehat{p}_\gamma} : \{\widehat{p}_\gamma\} \to \{p\}$, since $\dim_{\mathbb{R}}\{\widehat{p}_\gamma\} = \dim_{\mathbb{R}}\{p\} = 0$ we have that $D_{\widehat{p}_\gamma}(N_{\widehat{p}_\gamma}) = 1$. Furthermore, using Frobenius' formula [GGK02, Appendix I], for any class function χ, we have that:

$$\sum_{\gamma \in \mathrm{Conj}(\Gamma_p)} \frac{\rho(\gamma)}{d_{m_{\widehat{p}}}} = \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \rho(g).$$

Here, it is the characters $\chi_{p,0}$ and $\chi_{p,j}$ that we consider as class functions. Then, by applying the Atiyah-Bott-Berline-Vergne localisation formula (5.48) to the Kawasaki-Riemann-Roch formula (4.23), we get:

Theorem 5.47 (Equivariant Kawasaki-Riemann-Roch formula). *Suppose that M^T consists of finitely-many isolated fixed points. Given an element $\xi \in \mathfrak{t}$ such that $\langle \alpha_{p,j}\, \xi \rangle \neq 0$ for each $j = 1, \ldots, n$, the equivariant character $\chi : T \to H^\bullet(\mathrm{BT})$ of $H^0(M; \mathcal{L})$ is given by the formula:*

$$(\chi \circ \exp)(\xi) = \sum_{p \in M^T} \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{\chi_{p,0}(g) \cdot e^{\langle \mu(p),\, \xi \rangle}}{\prod_{j=1}^n \left(1 - \chi_{p,j}(g) \cdot e^{\langle \alpha_{p,j},\, \xi \rangle}\right)}. \tag{5.51}$$

Chapter 6

Isotropy Data of the Cut Spaces

In order to apply the equivariant localisation formulae (5.48) of Atiyah-Bott-Berline-Vergne from Theorem 5.44 to a cut space $M_\nu^{\leq\delta}$, one must first know the prerequisite isotropy data. Namely, we need to know what points make up the fixed-point locus $(M_\nu^{\leq\delta})^T$ for the residual T-action on $M_\nu^{\leq\delta}$, in addition to the isotropy data associated to each connected fixed-point component $F \subseteq (M_\nu^{\leq\delta})^T$. If applicable, orbifold data will also have to be ascertained if $M_\nu^{\leq\delta}$ is an orbifold, so that we can then apply the Kawasaki-Riemann-Roch formula (4.23) from Theorem 4.14.

6.1 Fixed-Point Data of the Cut Spaces

Given a regular value $\nu \in \mathfrak{k}^*$, let M_ν be a hypertoric variety and $\mathcal{A} = \{H_1, \ldots, H_N\}$ be its a simple hyperplane arrangement in $\mathfrak{t}^*$. Given a suitable value of $\delta \in \mathbb{R}_{\geq 0}$ so that the cut space $M_\nu^{\leq\delta}$ is at least an orbifold, and denote by $\Delta_\nu^{\leq\delta} = \mu_\mathbb{R}(M_\nu^{\leq\delta})$ its moment polyptych. Given a point $v \in \Delta_\nu^{\leq\delta}$ of the polyptych, recall that $\mathcal{I}_v \subseteq \{1, \ldots, N\}$ is the subset defined by $\mathcal{I}_v = \{\, i \mid v \in H_i \,\}$, and determines the flat $H_{\mathcal{I}_v} \in L(\mathcal{A})$. If $\operatorname{rk} H_{\mathcal{I}_v} = k$ then, as the arrangement $\mathcal{A}$ is simple, we have $\dim_\mathbb{R} H_{\mathcal{I}_v} = n - k$ and so if $\operatorname{rk} H_{\mathcal{I}_v} = n$, then $v = H_{\mathcal{I}_v}$ is an interior vertex of $\Delta_\nu^{\leq\delta}$. On the other hand, if $v \in \Pi_\nu^\delta$ is a vertex of the polyptych boundary, then $\operatorname{rk} H_{\mathcal{I}_v} = n - 1$ and the flat $H_{\mathcal{I}_v}$ is the affine line passing through v in addition to another single boundary vertex on the opposite side of the polyptych boundary Π_ν^δ.

Lemma 6.1. *A point $p \in M_\nu^{\leq\delta}$ is a fixed point for the T-action on $M_\nu^{\leq\delta}$ if, and only if, $v = \mu_\mathbb{R}(p)$ is a vertex of the moment polyptych $\Delta_\nu^{\leq\delta}$. Furthermore, the fixed points of $M_\nu^{\leq\delta}$ come in one of two types:*

(i) points $p \in M_\nu^{<\delta}$ that belong to the interior of the cut space $M_\nu^{\leq\delta}$, each of which is mapped onto a vertex $v = H_{\mathcal{I}_v}$ in the polyptych interior $\Delta_\nu^{<\delta}$;

(ii) points $p \in \mathcal{Z}_\nu^\delta$ that belong to the boundary of the cut space $M_\nu^{\leq\delta}$, each of which is mapped onto a

vertex $v \in \Pi^\delta_\nu$ on the polyptych boundary Π^δ_ν.

Proof. For (i), from [BD00, Theorem 3.1], if a point $p \in M^{<\delta}_\nu$ lies in the interior of the cut space $M^{\leq\delta}_\nu$ whose image is in the moment polyptych $\Delta^{\leq\delta}_\nu$ is $v = \mu_{\mathbb{R}}(p) \in \Delta^{\leq\delta}_\nu$, then p will be fixed by T if, and only if, $v = H_{\mathcal{I}_v}$ with $\operatorname{rk} H_{\mathcal{I}_v} = n$ since $\mathcal{A}$ is simple.

Then for (ii), suppose that $p \in \mathcal{Z}^\delta_\nu$ is a point on the boundary of $M^{\leq\delta}_\nu$ and that $v = \mu_{\mathbb{R}}(p) \in \Pi^\delta_\nu$. Then p is fixed by the residual U_1-action on $M^{\leq\delta}_\nu$, since $\mathcal{Z}^\delta_\nu = \Phi^{-1}(\delta)/U_1$ by the definition of the symplectic cut. So, for p to be fixed by the residual torus T as well, it is necessary for v to belong to $n-1$ hyperplanes. That is to say, with $v \in \Pi^\delta_\nu \cap H_{\mathcal{I}_v}$ where $H_{\mathcal{I}_v} = \cap_{j \in \mathcal{I}_v} H_j$ with $H_{\mathcal{I}_v} \in L(\mathcal{A})$ a flat of rank $\operatorname{rk} H_{\mathcal{I}_v} = n-1$.

To prove the sufficiency, consider the four-dimensional hypertoric subvariety $M_{\mathcal{I}_v}$ of M_ν that is determined by the flat $H_{\mathcal{I}_v}$. From Proposition 3.14, $M^{\leq\delta}_{\mathcal{I}_v}$ is a closed Kähler subvariety of the cut space $M^{\leq\delta}_\nu$ and, since $v \in H_{\mathcal{I}_v}$, the boundary point $p \in \mathcal{Z}^\delta_\nu$ is also a point belonging to the boundary $\mathcal{Z}^\delta_{\mathcal{I}_v}$ of the cut subspace $M^{\leq\delta}_{\mathcal{I}_v}$. Since $M^{\leq\delta}_{\mathcal{I}_v} \subseteq M^{\leq\delta}_\nu$, the moment polyptych $\Delta^{\leq\delta}_{\mathcal{I}_v}$ coincides with the intersection $\Delta^{\leq\delta}_{\mathcal{I}_v} = \Delta^{\leq\delta}_\nu \cap H_{\mathcal{I}_v}$, and similarly for its polyptych boundary $\Pi^{\leq\delta}_{\mathcal{I}_v} = \Pi^\delta_\nu \cap H_{\mathcal{I}_v}$.

As $v \in \Pi^\delta_\nu \cap H_{\mathcal{I}_v}$, from the end of the last paragraph, it is also a boundary vertex of $\Delta^{\leq\delta}_{\mathcal{I}_v}$, hence $p \in \mathcal{Z}^\delta_{\mathcal{I}_v}$ from Proposition 3.14. However, as $\operatorname{rk} H_{\mathcal{I}_v} = n-1$, the residual torus $T^{\operatorname{crk}\mathcal{I}_v}$ that acts on $M^{\leq\delta}_{\mathcal{I}_v}$ is one-dimensional, and thus $T^{\operatorname{crk}\mathcal{I}_v} \cong U_1$ from Proposition 2.11. It follows then that the one-dimensional residual torus $T^{\operatorname{crk}\mathcal{I}_v}$ and the residual circle U_1 have the same fixed-point set, thus $T^{\operatorname{crk}\mathcal{I}_v}$ fixes $p \in \mathcal{Z}^\delta_{\mathcal{I}_v}$. Finally, as the embedding $\bar{\iota} : M^{\leq\delta}_{\mathcal{I}_v} \hookrightarrow M^{\leq\delta}_\nu$ is $T^{\operatorname{crk}\mathcal{I}_v}$-equivariant, the point $p \in \mathcal{Z}^\delta_{\mathcal{I}_v} \subset \mathcal{Z}^\delta_\nu$ is fixed by the T-action. $\square$

With the T-fixed-point locus $(M^{\leq\delta}_\nu)^T$ of the cut space $M^{\leq\delta}_\nu$ determined, the next step is to determine the isotropy representation of T on the tangent space $T_p M^{\leq\delta}_\nu$ to each fixed point $p \in (M^{\leq\delta}_\nu)^T$. Let $v = \mu_{\mathbb{R}}(p)$ denote its corresponding point in the moment polyptych $\Delta^{\leq\delta}_\nu$ so that, from Lemma 6.1, v must be either an interior vertex or a boundary vertex. Let $H_j \in \mathcal{A}$ be one of the hyperplanes that contains v, which implies that $j \in \mathcal{I}_v$.

We will denote the edge emanating out from v by $\varepsilon_{v,j}$, whose edge vector we will denote by $\varrho_{p,j} \in \mathfrak{t}^*$, and orient it by the condition that $\langle \varrho_{p,j},\, u_j \rangle = 1$. Similarly, we will denote the "opposite" edge emanating out from v by $\varphi_{v,j}$, whose edge vector we will denote by $\varsigma_{p,j} \in \mathfrak{t}^*$, and orient it by the condition that $\langle \varsigma_{p,j},\, u_j \rangle = -1$. Note that these two conditions imply that the edges, $\varepsilon_{v,j}$ and $\varphi_{v,j}$, do not lie along the hyperplane H_j in the polyptych $\Delta^{\leq\delta}_\nu$.

If $H_{\mathcal{I}_v} \in L(\mathcal{A})$ is a rank $\operatorname{rk} H_{\mathcal{I}_v} = n$ flat, then $v = H_{\mathcal{I}_v}$ is an interior vertex. In this case, for each $j \in \mathcal{I}_v$, the corresponding hyperplane H_j determines the edge pair $\{\varepsilon_{v,j}, \varphi_{v,j}\}$ emanating out from v as in Figure 6.1, since $\mathcal{A}$ is a simple hyperplane arrangement.

On the other hand, if $H_{\mathcal{I}_v}$ is a rank $\operatorname{rk} H_{\mathcal{I}_v} = n-1$ flat, then $v \in H_{\mathcal{I}_v} \cap \Pi^\delta_\nu$ is a boundary

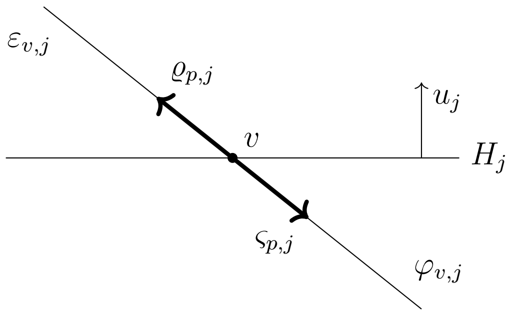


Figure 6.1: An edge pair $\{\varepsilon_{v,j}, \varphi_{v,j}\}$ emanating out from an interior vertex v, corresponding to the hyperplane H_j.

vertex. For each $j \in \mathcal{I}_v$, we shall continue to denote by $\{\varepsilon_{v,j}, \varphi_{v,j}\}$ the edge pair corresponding to the hyperplane H_j as before. Additionally however, we denote the edge emanating inwards from v by $\varkappa_v$, whose edge vector we will denote by $\vartheta_p \in \mathfrak{t}^*$ and, for any subset $A \subseteq \{1, \ldots, N\}$ such that $v \in \Pi^\delta_A$, orient it by the condition that $\langle \vartheta_p,\, u_A \rangle = 1$. Despite there being several possible choices for the subset A, the pairing between the edge vector ϑ_p and the residual U_1-action generator $u_A \in \mathfrak{t}$ is well-defined, since:

$$u_A \equiv u_B \bmod \mathrm{Span}_{\mathbb{R}}\{u_j \mid j \in \mathcal{I}_v\}, \text{for each } A, B \subseteq \{1, \ldots, N\} \text{ such that } v \in \Pi^\delta_A \cap \Pi^\delta_B,$$

and, as $\varkappa_p \subsetneq H_{\mathcal{I}_v}$:

$$\vartheta_p \in \bigcap_{j \in \mathcal{I}_v} \mathrm{Ann}_{\mathfrak{t}^*}\{u_j\},$$

which can be seen in Figure 6.2.

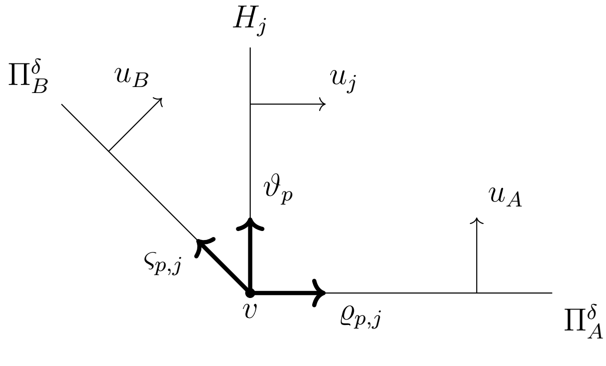


Figure 6.2: An edge triple $\{\vartheta_p, \varepsilon_{v,j}, \varphi_{v,j}\}$ emanating out from a boundary vertex v. Since $u_B = u_A + u_j$, and as ϑ_p annihilates u_j, we have that $\langle \vartheta_p,\, u_A \rangle = \langle \vartheta_p,\, u_B \rangle$.

Theorem 6.2. *Let M_ν be a hypertoric variety with $2n = \dim_{\mathbb{C}} M_\nu$, and let $\mathcal{A}$ be its simple hyperplane arrangement in $\mathfrak{t}^*$. Denote by $M^{\leq\delta}_\nu$ its cut space with corresponding moment polyptych $\Delta^{\leq\delta}_\nu$. Then:*

(i) if $v = H_{\mathcal{I}_v}$ is an interior vertex, where $\mathcal{I}_v \in L(\mathcal{A})$ is a rank $\operatorname{rk} H_{\mathcal{I}_v} = n$ *flat, then there exist $2n$ distinct edges, $\{\varepsilon_j, \varphi_j\}_{j \in \mathcal{I}_v}$, emanating out from v;*

(ii) if $v \in \Pi_A^\delta \cap H_{\mathcal{I}_v}$ is a boundary vertex, where $\mathcal{I}_v \in L(\mathcal{A})$ is a rank $\operatorname{rk} H_{\mathcal{I}_v} = n-1$ *flat and for some subset $A \subseteq \{1, \ldots, N\}$, then there exist $2n-1$ distinct edges, $\{\varkappa_v, \varepsilon_{v,j}, \varphi_{v,j}\}_{j \in \mathcal{I}_v}$, emanating out from v.*

Proof. For (i), this is proven in [HH05, Proposition]. For (ii), the result is clear when $n = 1$, since then $\mathfrak{t}^* \cong \mathbb{R}$ and so the moment polyptych $\Delta_\nu^{\leq\delta}$ is equal to a finite union of closed line intervals in $\mathbb{R}$. Therefore, one of the two endpoints of $\Delta_\nu^{\leq\delta}$ is the vertex v, and the edge $\varkappa_v$ is the polyptych component $\Delta_A^{\leq\delta}$ for which $v \in \Pi_A^\delta$.

When $n = 2$, so that now $\mathfrak{t}^* \cong \mathbb{R}^2$, we can write the boundary vertex as the intersection $v = \Pi_A^\delta \cap \Pi_B^\delta$ for some pair of subsets $A, B \subseteq \{1, \ldots, N\}$ such that $B = A \cup \{j\}$, where $j \in B \setminus A$. This implies that the hyperplane H_j contains v and that H_j separates the two regions, Δ_A and Δ_B, from each other as in Figure 6.3.

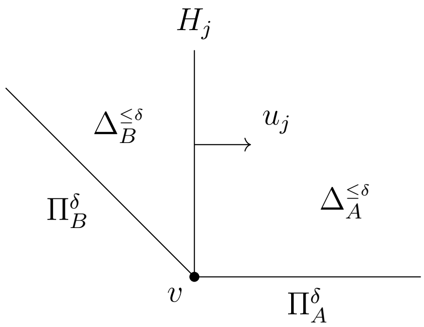


Figure 6.3: The vertex $v = \Pi_A^\delta \cap \Pi_B^\delta$ belongs to the hyperplane H_j, which separates $\Delta_A^{\leq\delta}$ from $\Delta_B^{\leq\delta}$.

The edge $\varepsilon_{v,j}$ emanates out from v along the boundary component Π_A^δ, since Π_A^δ is also cooriented positively with respect to the half-space H_j^+ since, for any $\alpha \in \Pi_A^\delta$, we have that $\langle \alpha,\ u_j \rangle \geq 0$. Similarly, the edge $\varphi_{v,j}$ emanates out from v along the boundary component Π_B^δ, albeit now with Π_B^δ cooriented negatively with respect to the half-space H_j^+ since, for any $\beta \in \Pi_B^\delta$, we have that $\langle \beta,\ u_j \rangle \leq 0$.

Therefore, we have the two edges, $\varepsilon_{v,j}$ and $\varphi_{v,j}$. The third and final edge κ_v is, of course, the edge emanating out from v along the interface $\Delta_A^{\leq\delta} \cap \Delta_B^{\leq\delta}$ between the polyptych components. That is to say, for any $\gamma \in \Delta_A^{\leq\delta} \cap \Delta_B^{\leq\delta}$, we have that $\langle \gamma,\ u_j \rangle = 0$. We therefore have $3 = 2n - 1$ edges, namely $\{\varkappa_v, \varepsilon_{v,j}, \varphi_{v,j}\}$, when $n = 2$.

Now, for the general $n \geq 3$ case, fix a boundary vertex $v \in \Pi_\nu^\delta \cap H_{\mathcal{I}_v}$ where the flat $H_{\mathcal{I}_v} \in L(\mathcal{A})$ has rank $\operatorname{rk} H_{\mathcal{I}_v} = n - 1$. Let y be the interior vertex that is adjacent to v, so that $y = H_{\mathcal{I}_y}$, where $\mathcal{I}_y \in L(\mathcal{A})$ is a flat of rank $\operatorname{rk} H_{\mathcal{I}_y} = n$. Observe that the edge $\varkappa_v$ connects v to y along the affine subspace $H_{\mathcal{I}_v}$ since $\operatorname{rk} H_{\mathcal{I}_v} = n-1 < n = \operatorname{rk} H_{\mathcal{I}_y}$, and as $y = H_{\mathcal{I}_y} \subsetneq H_{\mathcal{I}_v}$, we have that $\mathcal{I}_y = \mathcal{I}_v \cup \{j\}$, for some element $j \in \mathcal{I}_y$ since $\mathcal{A}$ is a simple arrangement.

Now, let us fix another element $k \in \mathcal{I}_v$, and define a new flat $H_{\mathcal{J}_{v,y,k}} \in L(\mathcal{A})$ using the subset:

$$\mathcal{J}_{v,y,k} := \mathcal{I}_v \setminus \{k\} = \mathcal{I}_y \setminus \{j,k\}.$$

Then clearly, $\mathcal{J}_{v,y,k}$ has cardinality $|\mathcal{J}_{v,y,k}| = \operatorname{rk} H_{\mathcal{J}_{v,y,k}} = n-2$ as $\mathcal{A}$ is simple, and also:

$$\mathcal{J}_{v,y,k} \subsetneq \mathcal{I}_v \subsetneq \mathcal{I}_y, \qquad \text{which implies that} \qquad H_{\mathcal{I}_y} \subsetneq H_{\mathcal{I}_v} \subsetneq H_{\mathcal{J}_{v,y,k}}. \tag{6.1}$$

As $\operatorname{rk} H_{\mathcal{J}_{v,y,k}} = n-2$ and as an affine space in $\mathfrak{t}^*$, $H_{\mathcal{J}_{v,y,k}}$ is two-dimensional and contains the flat $\mathcal{H}_{\mathcal{I}_v}$ as an affine line, as well as the two vertices v and y from (6.1). The intersection of $H_{\mathcal{J}_{v,y,k}}$ with the moment polyptych $\Delta_\nu^{\leq\delta}$ is the subpolyptych $\Delta_{\mathcal{J}_{v,y,k}}^{\leq\delta} = \Delta_\nu^{\leq\delta} \cap H_{\mathcal{J}_{v,y,k}}$ which contains the vertices v and y, since $v, y \in H_{\mathcal{I}_v} \subsetneq H_{\mathcal{J}_{v,y,k}}$, as demonstrated in Figure 6.4.

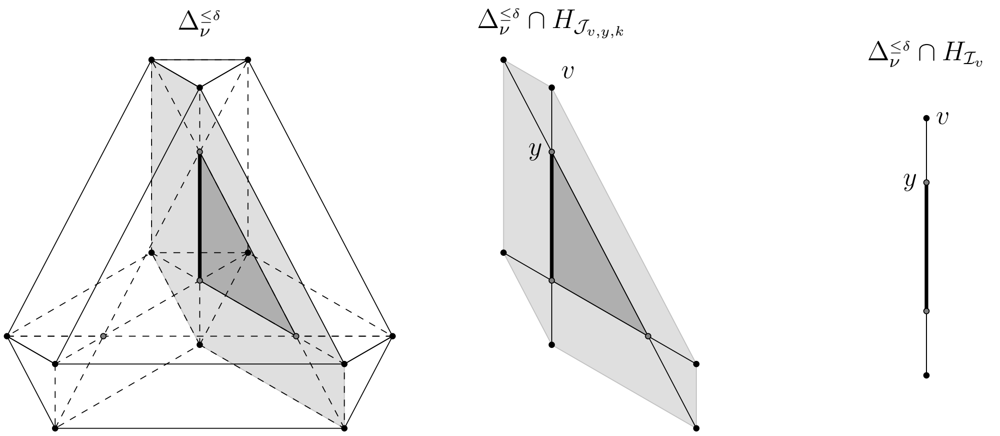


Figure 6.4: A moment polyptych $\Delta_\nu^{\leq\delta}$ and two of its subpolyptychs $\Delta_{\mathcal{J}_{v,y,k}}^{\leq\delta}$ and $\Delta_{\mathcal{I}_v}^{\leq\delta}$, with each corresponding to the intersection of $\Delta_\nu^{\leq\delta}$ with the flats, respectively $H_{\mathcal{J}_{v,y,k}}$ and $H_{\mathcal{I}_v}$, of $\mathcal{A}$.

The subpolyptych $\Delta_{\mathcal{J}_{v,y,k}}^{\leq\delta}$ is the moment polyptych of the closed Kähler subvariety by Proposition 3.14:

$$M_{\mathcal{J}_{v,y,k}}^{\leq\delta} \cong \left\{ [z,w] \in M_\nu^{\leq\delta} \mid z_l = w_l = 0 \text{ for each } l \in \mathcal{J}_{v,y,k} \right\}.$$

As $\operatorname{rk} H_{\mathcal{J}_{v,y,k}} = 2$, then $\dim_{\mathbb{R}} M_{\mathcal{J}_{v,y,k}}^{\leq\delta} = 4n - 4(n-2) = 8$. Therefore, the cut subspace $M_{\mathcal{J}_{v,y,k}}$ satisfies the hypotheses for the case when $n = 2$, since $v \in \Pi_{\mathcal{J}_{v,y,k}}^{\delta}$ is a boundary vertex of the subpolyptych $\Delta_{\mathcal{J}_{v,y,k}}^{\leq\delta}$, and $y \in \Delta_{\mathcal{J}_{v,y,k}}^{<\delta}$ is an interior vertex of the subpolyptych $\Delta_{\mathcal{J}_{v,y,k}}^{\leq\delta}$ as both are connected via the edge $\varkappa_v$ that lies along the flat $H_{\mathcal{I}_y}$, which itself is a subspace of the flat $H_{\mathcal{J}_{v,y,k}}$, so $H_{\mathcal{I}_v} \subsetneq H_{\mathcal{J}_{v,y,k}}$.

The edge $\varkappa_v$ emanates out from v along $H_{\mathcal{I}_v}$ with edge vector $\vartheta_p \in \mathfrak{t}^*$, whereas the edge $\varepsilon_{v,j}$ emanates out from v along $\Pi_A^\delta \cap H_{\mathcal{J}_{v,y,k}}$ with edge vector $\varrho_{p,j} \in \mathfrak{t}^*$, and lastly the edge $\varphi_{v,j}$ emanates out from v along $\Pi_B^\delta \cap H_{\mathcal{J}_{v,y,k}}$ with edge vector $\varsigma_{p,j} \in \mathfrak{t}^*$. Finally, as $\mathcal{I}_y = \mathcal{I}_v \cup \{j\} = \mathcal{J}_{v,y,k} \cup \{j,k\}$, we see that $H_{\mathcal{I}_v} \subsetneq H_{\mathcal{J}_{v,y,k}}$ for each $k \in \mathcal{I}_v$. There are $n-1$ choices of element for $k \in \mathcal{I}_v$ since $\operatorname{rk} H_{\mathcal{I}_v} = n-1$, with each choice producing an additional edge pair $\{\varepsilon_{v,j}, \varphi_{v,j}\}$ towards the final edge count, contributing $2n-2$ edges in total. As $\varkappa_v$ is a common subset in each affine plane $H_{\mathcal{J}_{v,y,k}}$, we only count $\varkappa_p$ once to get us $2n-1$ edges, $\{\varkappa_v, \varepsilon_{v,j}, \varphi_{v,j}\}_{j\in\mathcal{I}_v}$, overall. □

As argued in [HH05], let $\varepsilon_{v,j}$ be an edge in $\mathfrak{t}^*$ emanating out from a vertex $v \in \Delta_\nu^{\leq\delta}$, for any given $j \in \mathcal{I}_v$. Then $\varepsilon_{v,j}$ lies entirely outside of one of the hyperplanes H_j that contains v, and moreover $\varepsilon_{v,j}$ determines an edge of a polyptych component $\Delta_A^{\leq\delta}$ in $\mathfrak{t}^*$, which is equal to the image of a cut component $\Delta_A^{\leq\delta} = \mu_{\mathbb{R}}(\mathcal{E}_A^{\leq\delta})$. Let $p \in (\mathcal{E}_A^{\leq\delta})^T$ denote the fixed point in $\mathcal{E}_A^{\leq\delta}$ that corresponds to the vertex, i.e., such that $v = \mu_{\mathbb{R}}(p)$. As each cut component $\mathcal{E}_A^{\leq\delta}$ is itself a compact toric Kähler variety then, by the equivariant Darboux-Weinstein theorem [Wei77, Lecture 5], there exists a one-dimensional T-weight space $T_p\mathcal{E}_A^{\leq\delta}$ of $T_pM_\nu^{\leq\delta}$ corresponding to the edge $\varepsilon_{v,j}$, whose isotropy weight $\varrho_{p,j} \in \mathfrak{t}^*$ satisfies $\langle \varrho_p,\, u_j\rangle = 1$.

When $v = H_{\mathcal{I}_v}$ is an interior vertex of $\Delta_\nu^{\leq\delta}$ so that $\operatorname{rk} H_{\mathcal{I}_v} = n$ then, from Theorem 6.2, to each hyperplane H_j for which $j \in \mathcal{I}_v$, we can associate to it the edge pair, $\{\varepsilon_{v,j}, \varphi_{v,j}\}$. From the discussion in the previous paragraph, each edge has a corresponding T-weight which coincides with the corresponding edge vector, so $\varrho_{p,j} \in \mathfrak{t}^*$ for $\varepsilon_{v,j}$, and $\varsigma_{p,j} \in \mathfrak{t}^*$ for $\varphi_{v,j}$. On the other hand, when $v \in \Pi_\nu^\delta \cap H_{\mathcal{I}_v}$ is a boundary vertex, then it is not quite so obvious as to which edge should correspond to which T-weight. This is because, from Theorem 6.2, there exist only $2n-2$ edges that correspond to the $n-1$ hyperplanes H_j with $j \in \mathcal{I}_v$. However, we expect there to be $2n$ T-weights in total since $\dim_{\mathbb{C}} M_\nu^{\leq\delta} = 2n$, and the irreducible weight spaces of $T_pM_\nu^{\leq\delta}$ are all complex lines, i.e., of complex dimension one. So we have $2n - 2(n-1) = 2$ complex dimensions yet to be accounted for, and yet only one edge left, namely $\varkappa_v$ with edge vector $\vartheta_p \in \mathfrak{t}^*$.

Proposition 6.3. *Let M_ν be a hypertoric variety with $\dim_{\mathbb{C}} M_\nu = 2n$, and let $\mathcal{A}$ be its simple hyperplane arrangement in $\mathfrak{t}^*$. Denote by $M_\nu^{\leq\delta}$ its cut space with corresponding moment polyptych $\Delta_\nu^{\leq\delta}$. Then:*

(i) if $p \in (M_\nu^{<\delta})^T$ is an interior fixed point and $v = \mu_{\mathbb{R}}(p)$ is its interior vertex in $\Delta_\nu^{<\delta}$, then the isotropy representation of T on $T_pM_\nu^{\leq\delta}$ splits as:

$$T_pM_\nu^{\leq\delta} \cong \bigoplus_{j\in\mathcal{I}_v} \left(V_{\varrho_{p,j}} \oplus V_{\varsigma_{p,j}}\right). \tag{6.2}$$

Here, each summand $V_{\varrho_{p,j}}$ and $V_{\varsigma_{p,j}}$ in (6.2) is a weight space of the isotropy representation of T on $T_pM_\nu^{\leq\delta}$, with isotropy weights $\varrho_{p,j}$ and $\varsigma_{p,j}$ in $\mathfrak{t}^*$, respectively;

(ii) if $p \in (\mathcal{Z}_\nu^\delta)^T$ is a boundary fixed point and $v = \mu_{\mathbb{R}}(q)$ is its boundary vertex point in Π_ν^δ, then the isotropy representation of T on $T_pM_\nu^{\leq\delta}$ splits as:

$$T_pM_\nu^{\leq\delta} \cong \left(V_{\vartheta_p} \oplus V_{\vartheta_p}\right) \oplus \bigoplus_{j\in\mathcal{I}_v} \left(V_{\varrho_{p,j}} \oplus V_{\varsigma_{p,j}}\right), \tag{6.3}$$

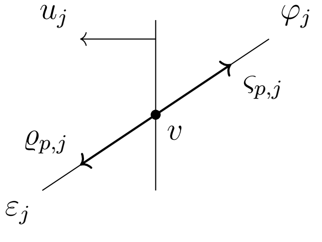
Here, each summand V_{ϑ_p}, $V_{\varrho_{p,j}}$, and $V_{\varsigma_{p,j}}$ in (6.3) is a weight space of the isotropy representation of T on $T_pM_\nu^{\leq\delta}$, with isotropy weights ϑ_p, $\varrho_{p,j}$, and $\varsigma_{p,j}$ in $\mathfrak{t}^*$, respectively.

Proof. For (i), this comes from the proof of [HH05, Proposition 3.2]. For (ii), fix a boundary vertex $v \in \Pi_\nu^\delta \cap H_{\mathcal{I}_v}$. From Lemma 6.2, there exist $2n-1$ edges, $\{\varkappa_v, \varepsilon_{v,j}, \varphi_{v,j}\}_{j\in\mathcal{I}_v}$, which emanate out from v where $2n-2$ of the edges, $\{\varepsilon_{v,j}, \varphi_{v,j}\}_{j\in\mathcal{I}_v}$, correspond to the $n-1$ hyperplanes H_j for which $j \in \mathcal{I}_v$. These contribute $2n-2$ of the isotropy weights in total, namely $\varrho_{p,j}$ and $\varsigma_{p,j}$ for each $j \in \mathcal{I}_v$.

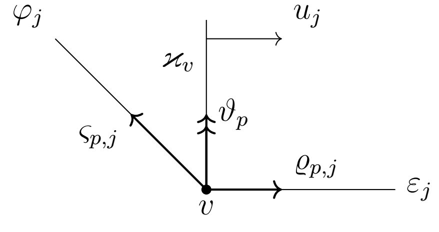
Now, for the remaining edge $\varkappa_v$, consider the hypertoric subvariety $M_{\mathcal{I}_v}$ of M_ν where $\dim_{\mathbb{C}} M_{\mathcal{I}_v} = 2$, along with its cut subspace, $M_{\mathcal{I}_v}^{\leq\delta}$. Since $v \in \Pi_\nu^\delta \cap H_{\mathcal{I}_v} \subsetneq \Pi_\nu^\delta$, from Proposition 3.14 we see that v is a boundary vertex of the subpolyptych $\Delta_{\mathcal{I}_v}^{\leq\delta} = \Delta_\nu^{\leq\delta} \cap H_{\mathcal{I}_v}$. The subpolyptych $\Delta_{\mathcal{I}_v}^{\leq\delta}$ is a subset of the real line as $(\mathfrak{t}^{\operatorname{crk}\mathcal{I}_v})^* \cong \mathbb{R}$, so the edge $\varkappa_v$ is the only possible edge which can emanate out from v. Hence, $\varkappa_v$ has at least one isotropy weight given by the edge vector $\vartheta_p \in (\mathfrak{t}^{\operatorname{crk}\mathcal{I}_b})^*$. However, since $\dim_{\mathbb{C}} M_{\mathcal{I}_v}^{\leq\delta}$, then $T_pM_{\mathcal{I}_v}^{\leq\delta} \cong \mathbb{C}^2$. Furthermore, as $M_{\mathcal{I}_b}^{\leq\delta}$ is a compact Kähler toric variety, by a dimensionality argument we must have that $T_pM_{\mathcal{I}_v}^{\leq\delta} \cong V_{\vartheta_p} \oplus V_{\vartheta_p}$. Hence the last two isotropy weights come from the edge vector ϑ_p counted multiplicity two. $\square$

To represent the isotropy data for a fixed point $p \in M_\nu^{\leq\delta}$, we superpose each corresponding isotropy weight as a vector pointing along its respective edge as in Figure 6.5. For the isotropy weights $\varrho_{p,j}$ and $\varsigma_{p,j}$ corresponding to the hyperplane H_j, recall that they are cooriented via the condition that $\langle \varrho_{p,j}, u_j\rangle = 1$ and $\langle \varsigma_{p,j}, u_j\rangle = -1$, for each $j \in \mathcal{I}_v$, and also recall that $\varrho_{p,j}$ and $\varsigma_{p,j}$ are the two weights in $\mathfrak{t}^*$ which do not lie along H_j.

Relative to the moment polyptych $\Delta_\nu^{\leq\delta}$, one sees that $\varrho_{p,j}$ points inwards towards $\Delta_\emptyset^{\leq\delta}$, whereas $\varsigma_{p,j}$ points outwards and away from the distinguished base region, $\Delta_\emptyset^{\leq\delta}$. Furthermore, $\varrho_{p,j}$ and $\varsigma_{p,j}$ are in some sense opposites to one another, since $\varrho_{p,j} = -\varsigma_{p,j}$ when $p \in (M_\nu^{\leq\delta})^T$ is an interior fixed point, and since $\varrho_{p,j} = -\varsigma_{p,j} \mod \operatorname{Ann}_{\mathfrak{t}^*}\{u_k \mid k \in \mathcal{I}_v\}$ when $p \in (\mathcal{Z}_\nu^\delta)^T$ is a boundary fixed point. Finally, for the isotropy weight ϑ_p of multiplicity two that points along the flat $H_{\mathcal{I}_v}$, we represent it using a double-headed arrow for purely illustrative purposes as in Figure 6.5.



(a) Isotropy weights of an interior fixed point.



(b) Isotropy weights of an interior fixed point.

Figure 6.5: Isotropy weights of a fixed point $p \in M_\nu^{\leq\delta}$, represented as edge vectors emanating out from its corresponding vertex $v = \mu_{\mathbb{R}}(p) \in \Delta_\nu^{\leq\delta}$.

6.2 The Finite Subgroup associated to a Flat

There is additional isotropy data yet to be specified in the case when the cut space $M_\nu^{\leq\delta}$ is an orbifold. To start, let $\mathcal{A} = \{H_1, \ldots, H_N\}$ be the simple hyperplane arrangement in $\mathfrak{t}^*$ that corresponds to the hypertoric variety M_ν, and let $H_\mathcal{F} \in L(\mathcal{A})$ be a flat of $\mathcal{A}$ with $\mathcal{F} \subseteq \{1, \ldots, N\}$. Recall from Section 2.4 the sublattices, $U_\mathcal{F} = \oplus_{j\in\mathcal{F}}\mathbb{R} \cdot u_j$ and $V_\mathcal{F} = \langle\mathcal{F}\rangle \cap \mathfrak{t}^n_\mathbb{Z}$, of $\mathfrak{t}^n_\mathbb{Z}$ in (2.8). Furthermore in (2.11), we obtained the quotient tori, $T_U^{\operatorname{rk}\mathcal{F}} = \langle\mathcal{F}\rangle/U_\mathcal{F}$ and $T_V^{\operatorname{rk}\mathcal{F}} = \langle\mathcal{F}\rangle/V_\mathcal{F}$. Finally, these lattices can be used to define the finite abelian group $\Gamma_\mathcal{F} = V_\mathcal{F}/U_\mathcal{F}$, which is trivial if and only if only $U_\mathcal{F}$ is a saturated sublattice of $V_\mathcal{F}$.

Let us focus on the instance where the flat $H_\mathcal{F}$ corresponds to an *interior* vertex $v \in \Delta_\nu^{<\delta}$ of the moment polyptych $\Delta_\nu^{\leq\delta}$. Then we have that $\mathcal{F} \equiv \mathcal{I}_v$ with $\operatorname{rk} H_{\mathcal{I}_v} = n$, that $V_{\mathcal{I}_v} \cong \mathfrak{t}^n_\mathbb{Z}$, and also that $v = H_{\mathcal{I}_v}$. If $p \in (M_\nu^{<\delta})^{T^n}$ denotes the interior fixed point such that $v = \mu_\mathbb{R}(p)$ then, from [LT97, Lemma 6.6], its orbifold structure group is:

$$\Gamma_p \cong \Gamma_{\mathcal{I}_v} = V_{\mathcal{I}_v}/U_{\mathcal{I}_v} \cong \mathfrak{t}^n_\mathbb{Z}/\operatorname{Span}_\mathbb{Z}\left\{\, u_j \mid j \in \mathcal{I}_v \,\right\}. \tag{6.4}$$

The same discussion essentially applies when v is a *boundary* vertex of $\Delta_\nu^{\leq\delta}$, with the only difference now being that $\mathcal{F} \equiv \mathcal{I}_v$ will have rank $\mathcal{I}_v = n-1$. Let $A \subseteq \{1, \ldots, N\}$ be a subset for which $v \in \Pi_A^\delta$. In this case, introduce the following sublattice:

$$U_{\mathcal{I}_v}^A := \operatorname{Span}_\mathbb{Z}\left\{\, u_j, u_A \;\middle|\; j \in \mathcal{I}_v \text{ and } v \in \Pi_A^\delta \,\right\},$$

and denote:

$$V_{\mathcal{I}_v}^A := \operatorname{Span}_\mathbb{R}\left\{\, u_j, u_A \;\middle|\; j \in \mathcal{I}_v \text{ and } v \in \Pi_A^\delta \,\right\} \cong \mathfrak{t}^n_\mathbb{Z}.$$

Since, for any other subset $B \subseteq \{1, \ldots, N\}$ for which $v \in \Pi_B^\delta$, the set difference between A and B consists only of elements in $\mathcal{I}_v$, we have that $U_{\mathcal{I}_v}^B \cong U_{\mathcal{I}_v}^A$. Hence the sublattice $U_{\mathcal{I}_v}^A$ is well-defined, regardless which subset $A \subseteq \{1, \ldots, N\}$ we choose to specify the boundary vertex $v \in \Pi_A^\delta$. If $p \in (\mathcal{Z}_\nu^\delta)^{T^n}$ is a boundary fixed point such that $v = \mu_\mathbb{R}(p) \in H_{\mathcal{I}_v} \cap \Pi_A^\delta$, then it follows from [LT97, Lemma 6.6] again that its orbifold structure group is given by:

$$\Gamma_p \cong \Gamma_{\mathcal{I}_v}^A := V_{\mathcal{I}_v}^A/U_{\mathcal{I}_v}^A \cong \mathfrak{t}^n_\mathbb{Z}/\operatorname{Span}_\mathbb{Z}\left\{\, u_j, u_A \;\middle|\; j \in \mathcal{I}_v \text{ and } v \in \Pi_A^\delta \,\right\}. \tag{6.5}$$

To summarise:

Lemma 6.4. *Let M_ν be a hypertoric variety and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be its corresponding simple hyperplane arrangement. If $p \in (M_\nu^{\leq\delta})^{T^n}$ is a fixed point in the cut space $M_\nu^{\leq\delta}$ for the residual T^n-action, and if $v = \mu_\mathbb{R}(p) \in \Delta_\nu^{\leq\delta}$ is its corresponding vertex of the moment polyptych $\Delta_\nu^{\leq\delta}$, then the orbifold structure group Γ_p is given by:*

$$\Gamma_p \cong \begin{cases} \Gamma_{\mathcal{I}_v} = \mathfrak{t}^n_\mathbb{Z}/\operatorname{Span}_\mathbb{Z}\left\{\, u_j \mid j \in \mathcal{I}_v \,\right\}, & \textit{if } v \in \Delta_\nu^{<\delta}, \\ \Gamma_{\mathcal{I}_v}^A = \mathfrak{t}^n_\mathbb{Z}/\operatorname{Span}_\mathbb{Z}\left\{\, u_j, u_A \mid j \in \mathcal{I}_v \textit{ and } v \in \Pi_A^\delta \,\right\}, & \textit{if } v \in \Pi_\nu^\delta. \end{cases}$$

Observe that either specification of the orbifold structure group in (6.4) or in (6.5) implies that Γ_p is a finite subgroup of the residual torus T^n. Indeed, since both $U_{\mathcal{I}_v}$ and $U^A_{\mathcal{I}_v}$ are sublattices of $\mathfrak{t}^n_{\mathbb{Z}}$, we see that:

$$\left.\begin{aligned} \Gamma_{\mathcal{I}_v} &\cong \mathfrak{t}^n_{\mathbb{Z}}/U_{\mathcal{I}_v} \\ \Gamma^A_{\mathcal{I}_v} &\cong \mathfrak{t}^n_{\mathbb{Z}}/U^A_{\mathcal{I}_v} \end{aligned}\right\} \subsetneq \mathfrak{t}^n/\mathfrak{t}^n_{\mathbb{Z}} \cong T^n. \tag{6.6}$$

Furthermore, if we denote the multiplicity of an orbifold point $p \in M^{\leq\delta}_\nu$ by m_p, then it coincides with the order of the orbifold structure group Γ_p:

$$m_p = \begin{cases} m_{\mathcal{I}_v} := |\Gamma_{\mathcal{I}_v}| = [V_{\mathcal{I}_v} : U_{\mathcal{I}_v}], & \text{if } v \in \Delta^{<\delta}_\nu, \\ m^A_{\mathcal{I}_v} := |\Gamma^A_{\mathcal{I}_v}| = [V^A_{\mathcal{I}_v} : U^A_{\mathcal{I}_v}], & \text{if } v \in \Pi^\delta_A, \end{cases} \tag{6.7}$$

and hence $\Gamma_p \cong \mathbb{Z}/m_p\mathbb{Z}$.

6.3 Canonical Automorphisms of the Cut Line Bundle $\mathcal{L}^{\leq\delta}_\nu$

For this section, assume that a holomorphic T^n-equivariant pre-quantum orbifold line bundle $\mathcal{L}$ exists over the cut space $M^{\leq\delta}_\nu$. For a fixed point $p \in (M^{\leq\delta}_\nu)^{T^n}$, we wish to determine what the characters:

$$\chi_{p,j} : \Gamma_p \to \begin{cases} \operatorname{Aut}(\widehat{\mathcal{L}}_p) \in U_1, & \text{for } j = 0; \\ \operatorname{Aut}(\widehat{V}_{p,j}) \in U_1, & \text{for } j = 1, \ldots, 2n, \end{cases}$$

are, for the representations $\widehat{\mathcal{L}}_p$ and $\widehat{V}_{p,j}$ of Γ_p that make an appearance in the equivariant Kawasaki-Riemann-Roch formula (5.51) of Theorem 5.47. As in Section 6.2 before, regardless of whether p belongs to the interior $M^{<\delta}_\nu$ or the boundary $\mathcal{Z}^\delta_\nu$ of the cut space, for clarity we shall denote:

$$U_p := \begin{cases} \operatorname{Span}_{\mathbb{Z}} \{ \, u_j \mid j \in \mathcal{I}_v \, \}, & \text{if } p \in M^{<\delta}_\nu; \\ \operatorname{Span}_{\mathbb{Z}} \{ \, u_j, u_A \mid j \in \mathcal{I}_v \text{ and } p \in \mathcal{Z}^\delta_A \, \}, & \text{if } p \in Z^\delta_\nu, \end{cases} \tag{6.8}$$

so that we can write its orbifold structure group Γ_p and multiplicity m_p respectively as:

$$\Gamma_p = \mathfrak{t}^n_{\mathbb{Z}}/U_p, \qquad \text{and} \qquad m_p = |\Gamma_p| = [\mathfrak{t}^n_{\mathbb{Z}} : U_p].$$

By definition, the dual lattice U^*_p to U_p is then isomorphic to:

$$U^*_p \cong \begin{cases} \{ \, \varrho_{p,j} \mid j \in \mathcal{I}_v \, \}, & \text{if } p \in M^{\leq\delta}_\nu; \\ \{ \, \varrho_{p,j}, \vartheta_p \mid j \in \mathcal{I}_v \text{ and } p \in \mathcal{Z}^\delta_A \, \}, & \text{if } p \in \mathcal{Z}^\delta_\nu, \end{cases} \tag{6.9}$$

since $\langle \varrho_{p,j}, \, u_j \rangle = 1$ for each $j = 1, \ldots, n$ whenever $\mu_{\mathbb{R}}(p) \in H_j$, and also because $\langle \vartheta_p, \, u_A \rangle = 1$ if additionally $p \in \mathcal{Z}^\delta_A$. Since $U_p \subseteq \mathfrak{t}^n_{\mathbb{Z}}$, we therefore see that $(\mathfrak{t}^n_{\mathbb{Z}})^* \subseteq U^*_p$. If we set:

$$\Gamma^*_p := U^*_p/(\mathfrak{t}^n_{\mathbb{Z}})^*,$$

and if $\gamma \in \mathfrak{t}_{\mathbb{Z}}^n$ represents the element $g \in \Gamma_p$ and similarly if $\alpha \in U_p^*$ represents the element $a \in \Gamma_p^*$, then from [CLS11a, Proposition 1.3.18], the pairing:

$$\Gamma_p^* \times \Gamma_p :\longrightarrow U_1, \qquad \text{where} \qquad (a, g) \longmapsto e^{2\pi\sqrt{-1}\langle \alpha, \gamma \rangle}, \tag{6.10}$$

is well-defined and induces an isomorphism $\Gamma_p \cong \mathrm{Hom}_{\mathbb{Z}}(\Gamma_p^*; U_1)$. From [KSW07], we can use (6.10) to express the character of the representation $\widehat{\mathcal{L}}_p$ of Γ_p as:

$$\chi_{p,0}(\gamma) = e^{2\pi\sqrt{-1}\langle \mu_{\mathbb{R}}(p), \gamma \rangle} \in U_1,$$

and also:

$$\chi_{p,j}(\gamma) = \begin{cases} \rho_{p,j}(\gamma) := e^{2\pi\sqrt{-1}\langle \varrho_{p,j}, \gamma \rangle}, & \text{on } \widehat{V}_{\varrho_{p,j}}; \\ \sigma_{p,j}(\gamma) := e^{2\pi\sqrt{-1}\langle \varsigma_{p,j}, \gamma \rangle}, & \text{on } \widehat{V}_{\varsigma_{p,j}}; \\ \theta_p(\gamma) := e^{2\pi\sqrt{-1}\langle \vartheta_p, \gamma \rangle}, & \text{on } \widehat{V}_{\vartheta_p}. \end{cases} \tag{6.11}$$

Chapter 7

Equivariant Quantisation of Hypertoric Varieties

Let us discuss our strategy; for a hypertoric variety M_ν, we formed its compact cut space $M_\nu^{\leq\delta}$ and determined the isotropy data for the T^n-action. Using the results of Chapter 6, this isotropy data can be superposed over the moment polyptych $\Delta_\nu^{\leq\delta}$ given by the image of the real moment map, $\Delta_\nu^{\leq\delta} = \mu_{\mathbb{R}}(M_\nu^{\leq\delta})$ in $(\mathfrak{t}^n)^*$. We now turn to deriving a formula for the subspace $H^0(M_\nu;\mathcal{L}_\nu)_d$ of weight-d holomorphic sections on the hypertoric variety M_ν, where $d \in \mathbb{Z}_{\geq 0}$ is some suitable non-negative integer. We accomplish this by first calculating the equivariant character $\chi : T^n \to H^\bullet(\mathrm{BT}^n)$ for the T^n-representation, $H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})$, in Theorem 7.3. We then derive an expression for the weight-d subspace $H^0(M_\nu;\mathcal{L}_\nu)_d$ as a quotient of the spaces $H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{d})$ in Theorem 7.8 and Corollary 7.9, and consequently its sought-after dimension formula.

7.1 Pre-Quantum Line Bundles on the Cut Spaces

To start, denote:

$$\mathcal{F}_\lambda := T^*\mathbb{C}^N \times \mathbb{C}_\lambda \longrightarrow T^*\mathbb{C}^N$$

the T^N-equivariant line bundle over $T^*\mathbb{C}^N$, which we consider to be holomorphic with respect to the complex-structure I_1 on $T^*\mathbb{C}^N$. Here, T^N acts on $\mathcal{F}_\nu$ with weight $\lambda \in (\mathfrak{t}_{\mathbb{Z}}^N)^*$. We can make $\mathcal{F}_\lambda$ into an I_1-holomorphic T^N-equivariant pre-quantum line bundle over $T^*\mathbb{C}^N$ by equipping $\mathcal{F}_\lambda$ with the Hermitian metric from (1.32), with Chern connection $\nabla_{\mathcal{L}}$ whose curvature is $R(\mathcal{F}_\lambda) = (2\pi/\sqrt{-1})\omega_{\mathbb{R}}$, similar to what was done before in Section 1.5 following [Konoo] and [DGMW, §3].

Let $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$ be the image of λ under $\imath^* : (\mathfrak{t}^N)^* \to \mathfrak{k}^*$ from (1.11), and restrict $\mathcal{F}_\lambda$ to the level-set $\phi_{\mathrm{HK}}^{-1}(\nu, 0)$. By an abuse of notation, relabel:

$$\mathcal{F}_\nu := \phi_{\mathrm{HK}}^{-1}(\nu, 0) \times \mathbb{C}_\nu \longrightarrow \phi_{\mathrm{HK}}^{-1}(\nu, 0)$$

to denote the resulting I_1-holomorphic T^N-equivariant pre-quantum line bundle over $\phi_{\mathrm{HK}}^{-1}(\nu, 0)$. The subtorus $K \trianglelefteq T^N$ preserves the I_1-holomorphic pre-quantum structure on $\mathcal{F}_\nu$ and so, if $(\nu, 0) \in \mathfrak{k}_{\mathbb{Z}}^* \otimes \mathrm{Im}(\mathbb{H})$ is a regular value of the hyperkähler moment map ϕ_{HK}, then $\mathcal{F}_\nu$ descends to the I_1-holomorphic pre-quantum line bundle:

$$\mathcal{L}_\nu := \mathcal{F}_\nu / K := \phi_{\mathrm{HK}}^{-1}(\nu, 0) \times_K \mathbb{C}_\nu \longrightarrow M_\nu$$

over the hypertoric variety M_ν. Since $\mathcal{F}_\nu$ is T^N-equivariant, it follows then that the I_1-holomorphic pre-quantum line bundle $\mathcal{L}_\nu$ is furthermore T^n-equivariant.

Similarly, we let:

$$\mathcal{L}_{\mathbb{C}} := \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

be the trivial holomorphic line bundle over $\mathbb{C}$ with respect to the standard complex structure, $I_{\mathbb{C}}$ say, and equip it with the Hermitian metric:

$$\left\|(\xi, \zeta)\right\| := |\zeta| e^{-\frac{1}{2}|\xi|^2},$$

thus inducing the Chern connection $\nabla_{\mathbb{C}}$ on $\mathcal{L}_{\mathbb{C}}$, whose curvature is $R(\mathcal{L}_{\mathbb{C}}) = (2\pi/\sqrt{-1}) d\xi \wedge d\bar{\xi}$ similar to before. Hence $\mathcal{L}_{\mathbb{C}}$ is also pre-quantisable in essentially the same way that $\mathcal{F}_\lambda$ is, though one can consult [DGMW, §3] for more details.

Now, let us consider the product orbifold $M_\nu \times \mathbb{C}$, along with the diagram:

$$\begin{tikzcd}
 & \mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}} \arrow[dl, bend right] \arrow[d] \arrow[dr, bend left] & \\
\mathcal{L}_\nu \arrow[d] & M_\nu \times \mathbb{C} \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \mathcal{L}_{\mathbb{C}} \arrow[d] \\
M_\nu & & \mathbb{C}
\end{tikzcd}$$

where:

$$\mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}} := \mathrm{pr}_1^* \mathcal{L}_\nu \otimes \mathrm{pr}_2^* \mathcal{L}_{\mathbb{C}} \longrightarrow M_\nu \times \mathbb{C}$$

is the external tensor product of $\mathcal{L}_\nu$ and $\mathcal{L}_{\mathbb{C}}$. Then $\mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}}$ is an $(I_1 \boxtimes I_{\mathbb{C}})$-holomorphic pre-quantum line bundle over $M_\nu \times \mathbb{C}$ via its product structure, namely that of the product Chern connection $\nabla_{\mathcal{L}} \boxtimes \nabla_{\mathbb{C}}$, see [DGMW, §3]. From Section 3.2, recall that the Hamiltonian actions of U_1 on M_ν and on the product $M_\nu \times \mathbb{C}$ gave rise to the following moment maps:

$$\Phi : M_\nu \to \mathbb{R}_{\geq 0}, \qquad \Phi[z, w] = \|w\|^2,$$

and:

$$\rho : M_\nu \times \mathbb{C} \to \mathbb{R}_{\geq 0}, \qquad \rho\left([z, w], \xi\right) = \Phi[z, w] + |\xi|^2,$$

respectively. Provided that we choose a "suitable", i.e. large enough and integral, $d \in \mathbb{Z}_{\geq 0}$, then we can lift the diagonal U_1-action on $M_\nu \times \mathbb{C}$ up to an action on $\mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}}$, thus obtaining an $(I_1 \boxtimes I_{\mathbb{C}})$-holomorphic $(T^n \times U_1)$-equivariant pre-quantum line bundle over $M_\nu \times \mathbb{C}$.

After taking the quotient with respect to the U_1-action to form the cut space $M_\nu^{\leq d}$, then $\mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}}$ descends $M_\nu^{\leq d}$ to become a T^n-equivariant pre-quantum line bundle over $M_\nu^{\leq d}$, which is now holomorphic with respect to the complex structure I_1 that descends to $M_\nu^{\leq d}$ [Mei98, Theorem 4.5], that we denote by:

$$\mathcal{L}_\nu^{\leq d} := \big((\mathcal{L}_\nu \boxtimes \mathcal{L}_{\mathbb{C}})\,|_{\rho^{-1}(d)}\big)/U_1 \longrightarrow M_\nu^{\leq d}. \tag{7.1}$$

Over the suborbifold $\{\Phi < d\}$ of $\rho^{-1}(d)$, there exist two candidate line bundles; one arising from the embedding $\{\Phi < d\} \hookrightarrow M_\nu$, and one arising from the embedding $\{\Phi < d\} \hookrightarrow M_\nu^{\leq d}$. Fortunately there is no cause of confusion between these line bundles, since they can be identified with one another thanks to the following lemma, proven in [Mei98, Theorem 4.5].

Lemma 7.1. *There exists the canonical isomorphisms of holomorphic T^n-equivariant pre-quantum line bundles over the cut space $M_\nu^{\leq d}$:*

$$\mathcal{L}_\nu^{\leq d}|_{M_\nu^{<d}} \cong \mathcal{L}_\nu|_{\{\Phi<d\}}, \qquad \textit{and} \qquad \mathcal{L}_\nu^{\leq d}|_{\mathcal{Z}_\nu^d} \cong \big(\mathcal{L}_\nu|_{\Phi^{-1}(d)}\big)/U_1.$$

Since $\mathcal{L}_\nu^{\leq d}$ is a Hermitian pre-quantum line bundle that is holomorphic with respect to the complex structure I_1, from Section 4.1 the $\mathcal{L}_\nu^{\leq d}$-twisted Dolbeault operator $\overline{\partial}_{\mathcal{L}}$ can be defined in this case. Therefore we can consider the $\mathcal{L}_\nu^{\leq d}$-twisted Dolbeault cohomology groups:

$$H^{(0,j)}(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}) = H^0(M_\nu^{\leq d};\Omega^j(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})), \qquad \text{for } j \geq 0,$$

for the sheaf of $\mathcal{L}_\nu^{\leq d}$-twisted differential forms over the cut space $M_\nu^{\leq d}$. However, since $\mathcal{L}_\nu^{\leq d}$ is both holomorphic and Hermitian however, by Kodaira's vanishing theorem, only the $j = 0$ cohomology group is non-trivial, and therefore:

$$H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}) \equiv H^0(M_\nu^{\leq d};\Gamma(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})) = \operatorname{Ind}_{\not{\partial}_{\mathcal{L}}}(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}).$$

7.2 U_1-Equivariant Quantisation of Hypertoric Varieties

Our aim now is to determine the dimension of $H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})$, which is finite dimensional since $M_\nu^{\leq d}$ is compact [CS53]. By Kodaira's vanishing theorem, the Riemann-Roch number $\chi(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})$ from (4.11) is equal to the index, $H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}) = \operatorname{Ind}_{\not{\partial}_{\mathcal{L}_\nu^{\leq d}}}(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})$. We therefore could theoretically calculate the dimension $\dim_{\mathbb{C}} H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d})$ either the Riemann-Roch theorem from Theorem 4.8, or the Kawasaki-Riemann-Roch theorem from Theorem 4.14.

In practice, using the non-equivariant formulae would be difficult. However, if we apply the Atiyah-Bott-Berline-Vergne localisation formula from Theorem 5.44 to the index formula instead, then we get an expression for the **equivariant character**, denoted by $\chi : T^n \to H^\bullet(\mathrm{BT}^n)$, for the $\mathbb{C}$-vector space $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$, considering it now as a representation of T^n. To obtain an expression for the dimension by means of the equivariant character, the following definition is required.

Definition 7.2. Consider a simple hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_N\}$ in $(\mathfrak{t}^n)^*$ where, for each $j = 1, \dots, N$, the hyperplane $H_j \in \mathcal{A}$ has the corresponding normal vector $u_j \in \mathfrak{t}^n$. Let $\alpha_j \in (\mathfrak{t}^n)$ be such that $\langle \alpha_j,\, u_k \rangle = \delta_{jk}$, where δ_{jk} is the Kronecker delta function. Then we say that an element $\xi \in \mathfrak{t}^n$ is **generic** if $\langle \alpha_j,\, \xi \rangle \neq 0$, for each $j = 1, \dots, N$.

In particular, if $\xi \in \mathfrak{t}^n$ is a generic element then, by definition [FH91, §2.1]:

$$\chi(e^\xi) := \mathrm{Tr}(e^\xi), \tag{7.2}$$

where e^ξ on the right-hand of (7.2) should be thought of as the respective automorphism of $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$. Therefore, if we take the limit as ξ tends towards 0:

$$\lim_{\xi \to 0} \chi(e^\xi) = \mathrm{Tr}(e^0) = \mathrm{Tr}(1) = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}),$$

then we recover the dimension $\dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$, thus circumventing any difficult evaluation of the non-equivariant index formulae.

Our plan therefore is to apply the localisation formula (5.48) to either the Hirzebruch-Riemann-Roch formula (4.13) if $M_\nu^{\leq d}$ is smooth, or the Kawasaki-Riemann-Roch formula (4.23) if $M_\nu^{\leq d}$ is an orbifold. Since the fixed-point locus $(M_\nu^{\leq d})^{T^n}$ consists of finitely-many isolated fixed points from Lemma 6.1, it will actually be feasible to obtain a formula for the equivariant character $\chi(e^\xi)$ from either Corollary 5.45 or from Theorem 5.47, before setting $\lim_{\xi \to 0} \chi(e^\xi) = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$.

This gets us an expression for the dimension of the zeroth cohomology of the sheaf of holomorphic section over the cut space $M_\nu^{\leq d}$, but nothing so far regarding the original hypertoric variety M_ν. We shall deal with this last part in Section 7.4.

Theorem 7.3. *Given a regular integral value $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$, let M_ν be a hypertoric variety with corresponding simple hyperplane arrangement $\mathcal{A}$ in $(\mathfrak{t}^n)^*$. Denote by $M_\nu^{\leq d}$ its cut space with moment polyptych $\Delta_\nu^{\leq d}$. For each fixed point $p \in (M_\nu^{\leq d})^T$, denote its orbifold structure group by Γ_p. Given a generic element $\xi \in \mathfrak{t}^n$, define a the equivariant character $\chi : T^n \to H^\bullet(\mathrm{BT}^n)$ for the representation $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$ of T^n is given by the formula:*

$$\begin{aligned}\chi(e^\xi) = &\sum_{p \in (M_\nu^{\leq d})^T} \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{e^{2\pi\sqrt{-1}\langle \mu_{\mathbb{R}}(p),\, \gamma\rangle} e^{\langle \mu_{\mathbb{R}}(p),\, \xi\rangle}}{\left[\prod_{j \in \mathcal{I}_v} (1 - e^{2\pi\sqrt{-1}\langle \varrho_{p,j},\, \gamma\rangle} e^{\langle \varrho_{p,j},\, \xi\rangle})(1 - e^{2\pi\sqrt{-1}\langle \varsigma_{p,j},\, \gamma\rangle} e^{\langle \varsigma_{p,j},\, \xi\rangle})\right]} \\ &+ \sum_{p \in (\mathcal{Z}_\nu^d)^T} \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{e^{2\pi\sqrt{-1}\langle \mu_{\mathbb{R}}(p),\, \gamma\rangle} e^{\langle \mu_{\mathbb{R}}(p),\, \xi\rangle}}{\left[\left(1 - e^{2\pi\sqrt{-1}\langle \vartheta_p,\, \gamma\rangle} e^{\langle \vartheta_p,\, \xi\rangle}\right)^2 \left(\prod_{j \in \mathcal{I}_v} (1 - e^{2\pi\sqrt{-1}\langle \varrho_{p,j},\, \gamma\rangle} e^{\langle \varrho_{p,j},\, \xi\rangle})(1 - e^{2\pi\sqrt{-1}\langle \varsigma_{p,j},\, \gamma\rangle} e^{\langle \varsigma_{p,j},\, \xi\rangle})\right)\right]},\end{aligned} \tag{7.3}$$

where $\varrho_{p,j}$, $\varsigma_{p,j}$, and ϑ_p, are the T^n-weights defined in Section 6.1.

Proof. The components that go into proving this theorem have essentially been proven already. From Lemma 6.1, the fixed point $(M_\nu^{\leq d})^{T^n}$ set of the cut space has been shown to consist of finitely-many isolated fixed points, and can be dichotomised into either fixed points in the cut space interior $M_\nu^{<\delta}$ that correspond to interior vertices of the moment polyptych $\Delta_\nu^{\leq d}$, or those on the boundary $\mathcal{Z}_\nu^\delta$ that correspond to boundary vertices of $\Delta_\nu^{\leq d}$. Hence by combining the equivariant Kawasaki-Riemann-Roch formula (5.51) from Theorem 4.14 with Lemma 6.1, we can split the localisation formula into two sums; one over the interior fixed points, and the other over the boundary ones.

First though, if $p \in (M_\nu^{\leq d})^{T^n}$ and for a generic element $\xi \in \mathfrak{t}^n$ then, for brevity, let us introduce the "local trace" χ_p [SG99], as:

$$\chi_p(e^\xi) := \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{\chi_{p,0}(g) \cdot e^{\langle \mu_{\mathbb{R}}(p),\, \xi \rangle}}{\prod_{j=1}^n \left(1 - \chi_{p,j}(g) \cdot e^{\langle \alpha_{p,j},\, \xi \rangle}\right)},$$

which is the contribution towards χ from each fixed point p. Then, since $M_\nu^{\leq d} \cong M_\nu^{<d} \sqcup \mathcal{Z}_\nu^d$, the equivariant Kawasaki-Riemann-Roch formula (5.51) can be decomposed as:

$$\chi(e^\xi) = \sum_{p \in (M_\nu^{<d})^{T^n}} \chi_p(e^\xi) + \sum_{p \in (\mathcal{Z}_\nu^d)^{T^n}} \chi_p(e^\xi). \tag{7.4}$$

Let us first deal with the first term on the right-hand side of (7.4). So consider an interior fixed point $p \in (M_\nu^{<d})^{T^n}$, and let $v = \mu_{\mathbb{R}}(p) \in \Delta_\nu^{\leq d}$ be its corresponding interior vertex, which is equal to the rank n flat $\{v\} = H_{\mathcal{I}_v}$, with $\mathcal{I}_v \subseteq \{1, \ldots, N\}$. In (6.2) from Proposition 6.3, its tangent space decomposes as:

$$T_p M_\nu^{\leq d} \cong \bigoplus_{j \in \mathcal{I}_v} \left(V_{\varrho_{p,j}} \oplus V_{\varsigma_{p,j}}\right),$$

with $\varrho_{p,j} \in (\mathfrak{t}^n)^*$ the isotropy weight for the representation $V_{\varrho_{p,j}}$ of T^n, and similarly with $\varsigma_{p,j} \in (\mathfrak{t}^n)^*$ for $V_{\varsigma_{p,j}}$. To summarise so far, the local trace χ_p for the interior fixed point $p \in (M_\nu^{<d})^{T^n}$ is given by:

$$\chi_p(e^\xi) = \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{\chi_{p,0}(g) \cdot e^{\langle \mu_{\mathbb{R}}(p),\, \xi \rangle}}{\left[\prod_{j \in \mathcal{I}_v} (1 - \chi_{p,j}(g) \cdot e^{\langle \varrho_{p,j},\, \xi \rangle})(1 - \chi_{p,j}(g) \cdot e^{\langle \varsigma_{p,j},\, \xi \rangle})\right]}. \tag{7.5}$$

If the interior fixed point p is an orbifold point so that its orbifold structure group Γ_p is non-trivial, then we have to take the orbifold structure of the cut space $M_\nu^{\leq d}$ into account, though this was covered in Section 6.3. From (6.11), if we let $\gamma \in \mathfrak{t}_{\mathbb{Z}}^n$ represent the element $g \in \Gamma_p$, we see that:

$$\chi_{p,j}(g) = \begin{cases} \chi_{p,0}(g) = e^{2\pi\sqrt{-1}\langle \mu_{\mathbb{R}}(p),\, \gamma \rangle}, & \text{on } \widehat{\mathcal{L}}_p; \\ \rho_{p,j}(g) = e^{2\pi\sqrt{-1}\langle \varrho_{p,j},\, \gamma \rangle}, & \text{on } \widehat{V}_{\varrho_{p,j}}; \\ \sigma_{p,j}(g) = e^{2\pi\sqrt{-1}\langle \varsigma_{p,j},\, \gamma \rangle}, & \text{on } \widehat{V}_{\varsigma_{p,j}}. \end{cases} \tag{7.6}$$

Substituting the characters (7.6) into the expression (7.5) for the local trace $\chi_p(e^{\xi})$ finally yields the first term on the right-hand side of (7.4).

As for the other case when $p \in (\mathcal{Z}_\nu^d)^{T^n}$ is a boundary fixed point, the reasoning is essentially the same as that for interior fixed points and so, given this, we shall cover it more quickly. Denote the corresponding boundary vertex by $v = \mu_{\mathbb{R}}(p) \in \Pi_\nu^d \cap H_{\mathcal{I}_v}$, where $H_{\mathcal{I}_v}$ is the rank $(n-1)$ flat. From Lemma 6.1, we have the decomposition:

$$T_p M_\nu^{\leq \delta} \cong \left(V_{\vartheta_p} \oplus V_{\vartheta_p}\right) \oplus \bigoplus_{j \in \mathcal{I}_v} \left(V_{\varrho_{p,j}} \oplus V_{\varsigma_{p,j}}\right),$$

with T^n acting on V_{ϑ_p} with weight $\vartheta_p \in (\mathfrak{t}^n)^*$, and on $V_{\varrho_{p,j}}$ and $V_{\varsigma_{p,j}}$ as in the interior point case. For the boundary fixed point p, we thus have the following local trace:

$$\chi_p(e^{\xi}) = \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \frac{\chi_{p,0}(g) \cdot e^{\langle \mu_{\mathbb{R}}(p),\, \xi \rangle}}{\left[\left(1 - \chi_{p,j}(g) e^{\langle \vartheta_p,\, \xi \rangle}\right)^2 \left(\prod_{j \in \mathcal{I}_v} \left(1 - \chi_{p,j}(g) e^{\langle \varrho_{p,j},\, \xi \rangle}\right)\left(1 - \chi_{p,j}(g) e^{\langle \varsigma_{p,j},\, \xi \rangle}\right)\right)\right]}. \tag{7.7}$$

If p is additionally an orbifold point then, from (6.11) and via the same discussion as in the interior fixed-point case, we have:

$$\chi_{p,j}(g) = \begin{cases} \theta_p(g) = e^{2\pi\sqrt{-1}\langle \vartheta_p,\, \gamma \rangle}, & \text{on } \widehat{V}_{\vartheta_p}; \\ \text{Interior point case, (7.6)}, & \text{otherwise.} \end{cases} \tag{7.8}$$

Thus, by substituting into (7.7) the characters from (7.6) and (7.8), we obtain the second term on the right-hand side of (7.4), and hence the equivariant Kawasaki-Riemann-Roch formula (7.3) for the cut space $M_\nu^{\leq d}$. $\square$

7.3 Algebraic Cutting

Recall, from Definition 1.19, that a complex normal quasi-projective variety M is said to be semi-projective if it is projective over an affine variety M_0. In this section, we will introduce an algebraic analogue to Lerman's symplectic cut $M^{\leq d}$ in the instance when M is semi-projective. An algebraic analogue to the symplectic cut was introduced by Edidin and William in [EG98], called the *"algebraic cut"*, and considered projective algebraic varieties with a linearised action of the one-dimensional split torus $\mathbb{G}_m \cong \mathbb{C}^*$. Our method here is different to that presented in [EG98], since our method forms the algebraic cut by the means of the Proj-construction being applied to semi-projective varieties, as described in Section 1.4.

Since M is semi-projective, there exist the isomorphisms, $M \cong \mathrm{Proj}\, R$ and $M_0 \cong \mathrm{Spec}\, R_0$, where R is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, that is furthermore finitely-generated by R_1 as an R_0-algebra. Moreover, from the projective structure morphism $\pi : M \to M_0$, we get an ample line bundle $\mathcal{L}_M$ over

M that is very ample relative to π. Now suppose that M is acted upon by the algebraic circle $\mathbb{C}^*$, and that this $\mathbb{C}^*$-action lifts to $\mathcal{L}_M$ a $\mathbb{C}^*$-linearised ample line bundle.

Similarly, consider the linear $\mathbb{C}^*$-action on $\mathbb{C}$. Then, on the one hand, as a $\mathbb{C}^*$-action on an affine variety is equivalent to a $\mathbb{Z}$-grading on its ring of regular functions, we obtain a $\mathbb{Z}$-grading on the affine coordinate ring $\mathcal{O}_{\mathbb{C}}(\mathbb{C})$ of $\mathbb{C}$, where the grading is given by the degree of the homogeneous polynomials. However, on the other hand, we can define another $\mathbb{Z}_{\geq 0}$-grading on the coordinate ring $\mathcal{O}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}[\xi]$ by adjoining a dummy variable as $\mathbb{C}[\xi][Y]$, and asserting that $\deg(\xi) = 0$ and that $\deg(Y) = 1$. With respect to this $\mathbb{Z}_{\geq 0}$-grading, $\mathbb{C}$ becomes a semi-projective variety as follows: for each $q \in \mathbb{Z}_{\geq 0}$, define the following $\mathbb{C}$-algebras, $S_q := \mathbb{C}[\xi] \cdot Y^q$, where $\deg(\xi) = 0$ and $\deg(Y) = 1$. Then:

$$S := \bigoplus_{q \geq 0} S_q = \bigoplus_{q \geq 0} \mathbb{C}[\xi] \cdot Y^q \cong \mathbb{C}[\xi][Y]$$

is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra, graded with respect to the variable Y, and is finitely-generated as an S_0-algebra by S_1. Therefore, the structure morphism is:

$$\mathbb{C} \cong \operatorname{Proj} S \cong \operatorname{Proj} \mathbb{C}[\xi][Y] \xrightarrow{\sim} \operatorname{Spec} S_0 = \mathbb{C}[\xi] \cong \mathbb{C},$$

showing, in particular, that $\mathbb{C}$ is projective over itself.

Given now that $M \cong \operatorname{Proj} R$ and $\mathbb{C} \cong \operatorname{Proj} S$, where R and S are both $\mathbb{C}$-algebras, then define the **Segre product** of R and S to be:

$$R \times_{\mathbb{C}} S := \bigoplus_{m \in \mathbb{Z}} R_m \otimes_{\mathbb{C}} S_m. \tag{7.9}$$

The following lemma is from [Har77, Exercise 5.11].

Lemma 7.4. *Let A be a ring, and let R and S be two $\mathbb{Z}$-graded A-algebras. If R is finitely-generated as an R_0-algebra by R_1, and if S is finitely-generated as an S_0-algebra by S_1, then $R \times_A S$ is finitely-generated by $(R \otimes_A S)_1$ as an $(R \otimes_A S)_0$-algebra.*

From Lemma 7.4, we see that $R \times_{\mathbb{C}} S$ is again a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, finitely-generated by $(R_1 \otimes_{\mathbb{C}} S_1)$ as an $(R_0 \otimes S_0)$-algebra. Hence the product $M \times_{\mathbb{C}} \mathbb{C}$ is isomorphic to the projective spectrum of the respective Segre product:

$$M \times_{\mathbb{C}} \mathbb{C} = \operatorname{Proj} R \times_{\mathbb{C}} \operatorname{Proj} S \cong \operatorname{Proj} (R \times_{\mathbb{C}} S).$$

The exterior tensor product $\mathcal{L} = \mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}$ is also ample over $M \times_{\mathbb{C}} \mathbb{C}$, and the $\mathbb{C}^*$-action makes $\mathcal{L}$ into an ample $\mathbb{C}^*$-linearised line bundle.

Definition 7.5. Let $M \cong R$ be a normal semi-projective variety, where $R \cong \oplus_{p \geq 0} R_p$ is a $\mathbb{C}$-algebra, finitely-generated as an R_0-module by R_1. Let $\mathcal{L}_M \to M$ be the ample line bundle over M that is very ample relative to the structure morphism, $\pi : M \to \operatorname{Spec} R_0$. Further suppose that $\mathbb{C}^*$ acts linearly on M. Then, for an integer $d \in \mathbb{Z}$, we define the **algebraic cut** of M at the level d, to be the projective GIT quotient:

$$M^{\leq d} := (M \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \, \mathbb{C}^* = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(M \times_{\mathbb{C}} \mathbb{C}; \mathcal{L}^{\otimes md})^{\mathbb{C}^*}, \tag{7.10}$$

where $\mathcal{L} = \mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}$ is the external tensor product over the Cartesian product $M \times_{\mathbb{C}} \mathbb{C}$.

The Proj-construction approach makes it particularly straightforward to come up with some examples of algebraic cuts.

Example 7.6. Consider the $\mathbb{Z}_{\geq 0}$-graded rings, R and S, given by:

$$R = R_0[X], \qquad \text{where} \qquad R_0 = \mathbb{C}[z_1, \ldots, z_N],$$

with $\deg(z_i) = 0$, for each $i = 1, \ldots, N$, and $\deg(X) = 1$, and also:

$$S = S_0[Y], \qquad \text{where} \qquad S_0 = \mathbb{C}[\xi],$$

with $\deg(\xi) = 0$ and $\deg(Y) = 1$. Then $\mathbb{C}^N = \operatorname{Spec} R_0 \cong \operatorname{Proj} R$, and $\mathbb{C} = \operatorname{Spec} S_0 \cong \operatorname{Proj} R$. Their Segre product $R \times_{\mathbb{C}} S$ is then:

$$R \times_{\mathbb{C}} S = \mathbb{C}[z_1, \ldots, z_n, \xi][Z], \qquad \text{where} \qquad Z := X \otimes Y.$$

To construct the algebraic cut of $\mathbb{C}^N$, let $\mathbb{C}^*$ act on R_0 and on S_0 as:

$$\tau \cdot z_i = \tau^{-1} z_i, \quad \text{and} \quad \tau \cdot \xi = \tau^{-1} \xi,$$

respectively, which carries over to $R \times_{\mathbb{C}} S$. Furthermore, for some integer $d \in \mathbb{Z}$, set:

$$\tau \cdot Z = \tau^d Z.$$

Hence the Cartesian product $\mathbb{C}^N \times_{\mathbb{C}} \mathbb{C}$ of $\mathbb{C}^N$ with $\mathbb{C}$ is:

$$\mathbb{C}^N \times_{\mathbb{C}} \mathbb{C} \cong \operatorname{Proj} R \otimes_{\mathbb{C}} S \cong \operatorname{Proj} \mathbb{C}[z_1, \ldots, z_N, \xi; Z].$$

There are various different outcomes for the algebraic cut $M^{\leq d} = (\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \, \mathbb{C}^*$, depending on which value for d is chosen.

Case 1: ($d \leq 0$). In this case, $(R \times_{\mathbb{C}} S)^{\mathbb{C}^*} = \mathbb{C}$, and so:

$$(\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \, \mathbb{C}^* = \operatorname{Proj}(R \times_{\mathbb{C}} S)^{\mathbb{C}^*} = \operatorname{Proj} \mathbb{C} = \emptyset.$$

Case 2: ($d = 1$). Now in this case:

$$(R \times_{\mathbb{C}} S)^{\mathbb{C}^*} = \mathbb{C}[z_1 Z, \dots, z_N Z, \xi Z] \cong \mathbb{C}[X_0, \dots, X_n],$$

after relabelling the indeterminates as $Z_0 := \xi X$ and $X_i := z_i Z$, for $i = 1, \dots, N$. As each X_i has degree one, they are all homogeneous; hence the algebraic cut is:

$$(\mathbb{C}^N \times_{\mathbb{C}} \mathbb{C}) /\!\!/_1 \, \mathbb{C}^* = \operatorname{Proj}(R \otimes_{\mathbb{C}} S)^{\mathbb{C}^*} = \operatorname{Proj} \mathbb{C}[X_0, \dots, X_n] \cong \mathbb{CP}^n,$$

and equipped with the line bundle: $\mathcal{O}_{\mathbb{CP}^n}(1)$.

Case 3: ($d \geq 2$). In this case, we still get:

$$(\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \, \mathbb{C}^* \cong \mathbb{CP}^n,$$

but now it comes with the d-twisted line bundle $\mathcal{O}_{\mathbb{CP}^n}(d)$.

Example 7.7. Let R and S be the same as in Example 7.6, so that:

$$R \times_{\mathbb{C}} S = \mathbb{C}[z_1, \dots, z_n, \xi][Z].$$

However, now let $\mathbb{C}^*$ act on $R \times_{\mathbb{C}} S$ as:

$$\tau \cdot z_i = \tau^{-1} z_i, \quad \text{and} \quad \tau \cdot \xi = \tau \xi,$$

and again with:

$$\tau \cdot Z = \tau^d Z,$$

for some $d \in \mathbb{Z}$. Let us see what happens when we let the value of d vary:

Case 1: ($d \leq 0$). In this case:

$$(R \times_{\mathbb{C}} S)^{\mathbb{C}^*} \cong \mathbb{C}[z_1 \xi, \dots, z_n \xi][\xi^d Z] \cong \mathbb{C}[x_1, \dots, x_n][X],$$

after relabelling $x_i := z_i \xi$, for $i = 1, \dots, N$, and $X := \xi^d Z$. Hence:

$$(\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \, \mathbb{C}^* = \operatorname{Proj} \mathbb{C}[x_1, \dots, x_n][X] \cong \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n] = \mathbb{C}^n.$$

Case 2: ($d = 1$). Now we have that:

$$\begin{aligned} (R \times_{\mathbb{C}} S)^{\mathbb{C}^*} &= \mathbb{C}[z_1 \xi, \dots, z_n \xi][z_1 Z, \dots, z_n Z] \\ &\cong \frac{\mathbb{C}[y_1, \dots, y_n][X_1, \dots, X_n]}{\langle y_i X_j - y_j X_i \,|\, i, j = 1, \dots, n \rangle}, \end{aligned}$$

where $x_i := z_i \xi$ and $X_i = z_1 Z$, for $i = 1, \dots, n$. Hence, in this case, the algebraic cut of $\mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ at the origin:

$$\begin{aligned} (\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_1 \, \mathbb{C}^* &= \operatorname{Proj}(R \times_{\mathbb{C}} S)^{\mathbb{C}^*} \\ &= \operatorname{Proj} \frac{\mathbb{C}[x_1, \dots, x_N; Y_1, \dots, Y_N]}{\langle y_i X_j - y_j X_i \,|\, i, j = 1, \dots, N \rangle} \\ &\cong \operatorname{Bl}_0 \mathbb{C}^n. \end{aligned}$$

Case 3: ($d \geq 2$). In the last case, we still get that $(\mathbb{C}^n \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \mathbb{C}^* \cong \mathrm{Bl}_0 \mathbb{C}^n$, but $\mathbb{CP}^{N-1}$ should be considered as being in its d-th Veronese embedding.

Consider a normal semi-projective variety M with an ample line bundle $\mathcal{L}_M \to M$ over it. If the ample line bundle $\mathcal{L}$ is not very ample, then we may replace it with the very ample line bundle $\mathcal{L}^{\otimes m_1}$, where $m_1 \in \mathbb{Z}_{\geq 0}$ is sufficiently large. For each $p \in \mathbb{Z}_{\geq 0}$, set:

$$R_p := H^0(M; \mathcal{L}_M^{\otimes p}) \qquad \text{and} \qquad R := \bigoplus_{p \geq 0} R_p. \tag{7.11}$$

Then R is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra. By replacing $\mathcal{L}_M$ with $\mathcal{L}_M^{\otimes m_2}$ for some sufficiently large $m_2 \in \mathbb{Z}_{\geq 0}$ in (7.11) if necessary, we may assume that R is finitely-generated by $R_1 = H^0(M; \mathcal{L}_M)$ as an R_0-algebra [Har77, Exercise 5.9, Exercise 5.13, & Exercise 5.14].

7.4 U_1-Equivariant Quantisation

Theorem 7.3 yields a formulae for the equivariant character $\chi : T^n \to H^\bullet(\mathrm{B}T^n)$ for the representation of T^n on $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$, and thus a way to obtain its dimension. We shall now use the finite-dimensional spaces $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$ to find an expression for the weight d subspace $H^0(M_\nu; \mathcal{L}_\nu)_d$, and consequently a formula for its dimension too. We first begin by proving a theorem that can be applied more general varieties than just hypertoric ones.

Theorem 7.8. *Let M be a complex semi-projective normal variety, and let $\mathcal{L}_M$ be an ample line bundle over M, that is very ample relative to the structure morphism $\pi : M \to \operatorname{Spec} R_0$. Suppose that $\mathbb{C}^*$ acts on M, and that this action lifts to make $\mathcal{L}_M$ into a $\mathbb{C}^*$-linearised ample line bundle.*

For $d \in \mathbb{Z}_{\geq 0}$ large enough, let $H^0(M; \mathcal{L}_M)_d$ represent the weight d subspace of $H^0(M; \mathcal{L}_M)$ for the $\mathbb{C}^*$-action. Then we have the isomorphism:

$$H^0(M; \mathcal{L}_M)_d \cong H^0(M^{\leq d}; \mathcal{L}_M^{\leq d}) / H^0(M^{\leq d-1}; \mathcal{L}_M^{\leq d-1}), \tag{7.12}$$

of $\mathbb{C}$-vector spaces.

Proof. Recall from Definition 7.5 that the the algebraic cut of M is the projective GIT quotient, $M^{\leq d} = (M \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \mathbb{C}^*$, with the ample line bundle $\mathcal{L}_M^{\leq d} = (\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}) /\!\!/_d \mathbb{C}^*$ over it. The $\mathbb{C}$-vector space of holomorphic sections on $M^{\leq d}$ is then:

$$H^0(M^{\leq d}; \mathcal{L}_M^{\leq d}) = H^0\big((M \times_{\mathbb{C}} \mathbb{C}) /\!\!/_\delta \mathbb{C}^*; (\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}) /\!\!/_d \mathbb{C}^*\big). \tag{7.13}$$

Denote by $(\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})(d)$ the twist of $\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}$ by the $\mathbb{C}^*$-character, $\chi_d(\tau) = \tau^d$. A section $\sigma \in H^0(M \times_{\mathbb{C}} \mathbb{C}; \mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})$ descends to a section $\tilde{\sigma} \in H^0(M^{\leq d}; \mathcal{L}^{\leq d})$ if, and only if, σ is $\mathbb{C}^*$-invariant with respect to the action induced by χ_d. In other words, there exists a bijection:

$$H^0\big((M \times_{\mathbb{C}} \mathbb{C}) /\!\!/_d \mathbb{C}^*; (\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}}) /\!\!/_d \mathbb{C}^*\big) \cong H^0\big(M \times_{\mathbb{C}} \mathbb{C}; (\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})(d)\big)^{\mathbb{C}^*}. \tag{7.14}$$

The term on the right-hand side of (7.14) is the space of χ_d-twisted $\mathbb{C}^*$-invariant global sections, which is the same as the space global sections of $\mathbb{C}^*$-weight d. That is:

$$H^0\big(M \times_{\mathbb{C}} \mathbb{C}; (\mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})(d)\big)^{\mathbb{C}^*} \cong H^0(M \times_{\mathbb{C}} \mathbb{C}; \mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})_d\,,$$

which, by the Künneth formula [Kem93, Proposition 9.2.4], is isomorphic to:

$$H^0(M \times_{\mathbb{C}} \mathbb{C}; \mathcal{L}_M \boxtimes \mathcal{L}_{\mathbb{C}})_d \cong \big[H^0(M; \mathcal{L}_M) \otimes_{\mathbb{C}} H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})\big]_d\,.$$

As M and $\mathbb{C}$ are both normal varieties, the line bundles $\mathcal{L}_M$ and $\mathcal{L}_{\mathbb{C}}$ both admit $\mathbb{C}^*$-linearisations. Their individual spaces of global sections then decompose into their respective direct sums of $\mathbb{C}^*$-weight spaces:

$$\begin{aligned}\big[H^0(M; \mathcal{L}_M) \otimes_{\mathbb{C}} H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})\big]_d &\cong \left[\left(\bigoplus_{i\in\mathbb{Z}} H^0(M; \mathcal{L}_M)_i\right) \otimes_{\mathbb{C}} \left(\bigoplus_{j\geq 0} H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})_j\right)\right]_d \\ &\cong \bigoplus_{\substack{i+j=d\\ j\geq 0}} \big(H^0(M; \mathcal{L}_M)_i \otimes_{\mathbb{C}} H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})_j\big)\,.\end{aligned}$$

Since $H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}}) \cong \mathbb{C}[\xi]$ where $\deg(\xi) = 0$, each $\mathbb{C}^*$-weight space has complex dimension one. Hence, as $\mathbb{C}$-vector spaces:

$$H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})_j \cong \mathbb{C}, \quad \text{for each } j \geq 0.$$

Therefore:

$$\bigoplus_{\substack{i+j=d\\ j\geq 0}} \big(H^0(M; \mathcal{L}_M)_i \otimes_{\mathbb{C}} H^0(\mathbb{C}; \mathcal{L}_{\mathbb{C}})_j\big) \cong \bigoplus_{i\leq d} H^0(M; \mathcal{L}_M)_i,$$

and so, after tracing back through the isomorphisms:

$$H^0(M^{\leq d}; \mathcal{L}_M^{\leq d}) \cong \bigoplus_{i\leq d} H^0(M; \mathcal{L}_M)_i$$

as $\mathbb{C}$-vector spaces. Thus we may extract the subspace of $\mathbb{C}^*$-weight *d*from $H^0(M; \mathcal{L}_M)$ by taking the quotient:

$$\begin{aligned}H^0(M^{\leq d}; \mathcal{L}_M^{\leq d}) / H^0(M^{\leq d}; \mathcal{L}_M^{\leq d-1}) &\cong \bigoplus_{i\leq d} H^0(M; \mathcal{L}_M)_i \bigg/ \bigoplus_{i\leq d-1} H^0(M; \mathcal{L}_M)_i \\ &\cong H^0(M; \mathcal{L}_M)_d,\end{aligned}$$

which strips away the lower $\mathbb{C}^*$-weight subspaces, yielding the desired result. □

Whilst Theorem 7.8 applies to any normal semi-projective variety, we are interested in applying it to a hypertoric variety M_ν. From Theorem 7.3, we can calculate the equivariant character $\chi(e^\xi)$, where $\xi \in \mathfrak{t}$ is generic relative to each hyperplane of the arrangement $\mathcal{A}$, for the T^n-action on the space $H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$ of holomorphic sections over $M_\nu^{\leq d}$ and whose dimension we calculate by taking the limit, $\lim_{\xi \to 0} \chi(e^\xi) = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d})$. Then, from Theorem 7.8, we see that:

$$H^0(M_\nu; \mathcal{L}_\nu)_d \cong H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) \big/ H^0(M_\nu^{\leq (d-1)}; \mathcal{L}_\nu^{\leq (d-1)}),$$

and so by considering their dimensions:

$$\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\leq (d-1)}; \mathcal{L}_\nu^{\leq (d-1)}).$$

Therefore, despite the dimension of $H^0(M_\nu; \mathcal{L}_\nu)$ being *infinite*, the residual $\mathbb{C}^*$-action on the hypertoric variety M_ν causes $H^0(M_\nu; \mathcal{L}_\nu)$ to decompose into its *finite-dimensional* $\mathbb{C}^*$-weights spaces, $H^0(M_\nu; \mathcal{L}_\nu)_d$, for each $d \in \mathbb{Z}_{\geq 0}$, as:

$$H^0(M_\nu; \mathcal{L}_\nu) \cong \bigoplus_{d \geq 0} H^0(M_\nu; \mathcal{L}_\nu)_d.$$

Let us summarise this result.

Corollary 7.9. *For a regular value $\nu \in \mathfrak{k}_{\mathbb{Z}}^*$, let M_ν be the corresponding hypertoric variety, and let $\mathcal{L}_\nu = \phi_{\mathrm{HK}}^{-1}(\nu, 0) \times_K \mathbb{C}_\nu \to M_\nu$ be the holomorphic pre-quantum ample line bundle over M_ν. For an integer $d \in \mathbb{Z}_{\geq 0}$, let $H^0(M_\nu; \mathcal{L}_\nu)_d$ denote the subspace of the $\mathbb{C}$-vector space $H^0(M_\nu; \mathcal{L}_\nu)$ of $\mathbb{C}^*$-weight d, induced from the residual $\mathbb{C}^*$-action on M_ν. Then the complex dimension of $H^0(M_\nu; \mathcal{L}_\nu)_d$ is given by the formula:*

$$\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d = \dim_{\mathbb{C}} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\leq (d-1)}; \mathcal{L}_\nu^{\leq (d-1)}), \tag{7.15}$$

where $M_\nu^{\leq d}$ is the cut space of M_ν at the level d, and where $\mathcal{L}_\nu^{\leq d}$ is the I_1-holomorphic T^n-linearised ample line bundle over M_ν defined in (7.1).

7.5 Examples of U_1-Equivariant Quantisations

Despite looking imposing, it is not too difficult to see how to put the formulae presented in Theorem 7.3 and Corollary 7.9 to use, once we have seen them in action with some examples. The calculations involving manifolds can be done by hand, whereas those involving orbifolds were calculated numerically using SymPy, [SymPy].

7.5.1 Equivariant Quantisation of $T^*\mathbb{CP}^1$

Here, we set $m = \nu \in \mathfrak{k}_{\mathbb{Z}}^* \cong \mathbb{Z}$ and $d = \delta \in \mathbb{Z}_{\geq 0}$. Then, continuing on from Examples 2.4 and 3.9, the cut space is now denoted by $M_m^{\leq d} = (T^*\mathbb{CP}^1)^{\leq d}$ with moment polyptych $\Delta_\nu^{\leq d}$.

The required isotropy data to calculate the character $\chi_{m,d}$ for the representation of T^1 on $H^0(M_\nu^{\leq\delta}; \mathcal{L}_\nu^{\leq\delta})$ is displayed in Figure 7.1, which the isotropy data presented below. Here, we let $v_i = \mu_\mathbb{R}(p_i)$ and $b_i = \mu_\mathbb{R}(q_i)$ for $i = 1, 2$, where $p_i \in (M_\nu^{<\delta})$ are the two interior fixed points, and $q_i \in (\mathcal{Z}_\nu^\delta)$ are the two boundary fixed points.

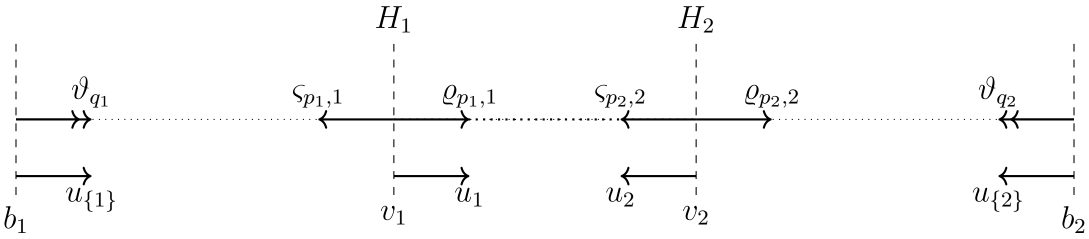


Figure 7.1: Labelling of the moment polyptych $\Delta_\nu^{\leq\delta}$ for $M_\nu = T^*\mathbb{CP}^1$.

$$
\begin{aligned}
v_1 &= 0 \quad \begin{cases} \varrho_{p_1,1} &= +1, \\ \varsigma_{p_1,1} &= -1, \end{cases} & v_2 &= m \quad \begin{cases} \varrho_{p_2,2} &= -1, \\ \varsigma_{p_2,2} &= +1 \end{cases} \\
b_1 &= -d \quad \begin{cases} \vartheta_{q_1} &= +1, \end{cases} & b_2 &= m+d \quad \begin{cases} \vartheta_{q_2} &= -1. \end{cases}
\end{aligned}
$$

Now let $e^\xi = t \in T^1$. Then, from (7.3), we obtain the following expression for the equivariant character:

$$
\begin{aligned}
\chi_{m,d}(t) &= \frac{1}{(1-t)(1-t^{-1})} + \frac{t^m}{(1-t)(1-t^{-1})} + \frac{t^{-d}}{(1-t)^2} + \frac{t^{m+d}}{(1-t^{-1})^2} \\
&= \left[\frac{1}{1-t} + \frac{t^{m+d}}{1-t^{-1}}\right] \cdot \left[\frac{1}{1-t^{-1}} + \frac{t^{-d}}{1-t}\right] \\
&= \left[\sum_{k=0}^{m+d} t^k\right] \cdot \left[\sum_{l=0}^{d} t^{-l}\right].
\end{aligned}
$$

By taking the limit $\xi \to 0$ so that $t \to 1$, we obtain:

$$
\dim_\mathbb{C} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) = (m+d+1)(d+1),
$$

and hence, from Corollary 7.9, we finally get:

$$
\begin{aligned}
\dim_\mathbb{C} H^0(M_\nu; \mathcal{L}_\nu)_d &= \dim_\mathbb{C} H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) - \dim_\mathbb{C} H^0(M_\nu^{\leq(d-1)}; \mathcal{L}_\nu^{\leq(d-1)}) \\
&= m + 2d + 1.
\end{aligned} \tag{7.16}
$$

7.5.2 Equivariant Quantisation of $T^*\mathbb{CP}^2$

Again, let $m, d \in \mathbb{Z}_{\geq 0}$, and now consider $M_\nu = T^*\mathbb{CP}^2$ from Example 2.6. The cut space $M_m^{\leq d} = (T^*\mathbb{CP}^2)^{\leq d}$ has the moment polyptych $\Delta_\nu^{\leq \delta}$ that was presented in Figure 3.3 and which we reproduce below in Figure 7.2, along with the superposed isotropy weights.

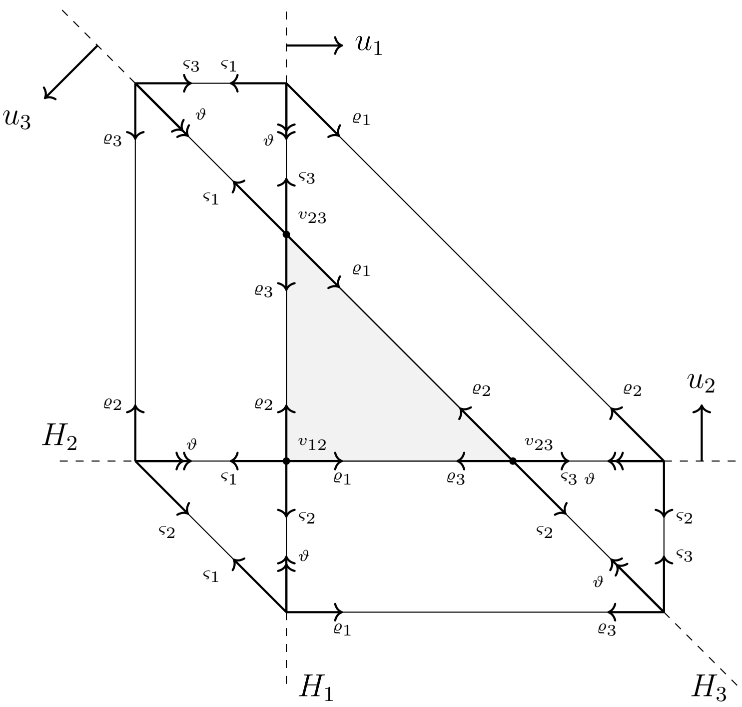


Figure 7.2: Moment polyptych $\Delta_\nu^{\leq \delta}$ of the cut space $M_\nu^{\leq \delta} = (T^*\mathbb{CP}^2)^{\leq d}$.

Below, we list the vertices of the polyptych $\Delta_\nu^{\leq d}$ and the corresponding isotropy weights. Here, $v_{ij} = \mu_\mathbb{R}(p_{ij}) \in \Delta_\nu^{<d}$ represent the interior vertices whereas $b_{ij}^{(k)} = \mu_\mathbb{R}(q_{ij}^{(k)}) \in \Pi_\nu^d$ represent the boundary vertices.

$$v_{12} = (0,0) \begin{cases} \varrho_{p_{12},1} & = (1,0), \\ \varsigma_{p_{12},1} & = (-1,0), \\ \varrho_{p_{12},2} & = (0,1), \\ \varsigma_{p_{12},2} & = (0,-1), \end{cases} \qquad v_{23} = (m,0) \begin{cases} \varrho_{p_{23},2} & = (-1,1), \\ \varsigma_{p_{23},2} & = (1,-1), \\ \varrho_{p_{23},3} & = (-1,0), \\ \varsigma_{p_{23},3} & = (1,0), \end{cases}$$

$$v_{13} = (0,m) \begin{cases} \varrho_{p_{13},1} & = (1,-1), \\ \varsigma_{p_{13},1} & = (-1,1), \\ \varrho_{p_{13},3} & = (0,1), \\ \varsigma_{p_{13},3} & = (0,-1), \end{cases}$$

$$b_{12}^{(1)} = (0,-d) \begin{cases} \varrho_{q_{12}^{(1)},1} & = (1,0), \\ \varsigma_{q_{12}^{(1)},1} & = (-1,1), \\ \vartheta_{q_{12}^{(1)}} & = (0,1), \end{cases} \qquad b_{12}^{(2)} = (-d,0) \begin{cases} \varrho_{q_{12}^{(2)},2} & = (0,1), \\ \varsigma_{q_{12}^{(2)},2} & = (1,-1), \\ \vartheta_{q_{12}^{(2)}} & = (1,0), \end{cases}$$

$$b_{23}^{(2)} = (m+d,0) \quad \begin{cases} \varrho_{q_{23}^{(2)},2} & = (-1,1), \\ \varsigma_{q_{23}^{(2)},2} & = (0,-1), \\ \vartheta_{q_{23}^{(2)}} & = (-1,0), \end{cases} \qquad b_{23}^{(3)} = (m+d,0) \quad \begin{cases} \varrho_{q_{23}^{(3)},3} & = (-1,0), \\ \varsigma_{q_{23}^{(3)},3} & = (0,1), \\ \vartheta_{q_{23}^{(3)}} & = (-1,1), \end{cases}$$

$$b_{13}^{(1)} = (0,m+d) \quad \begin{cases} \varrho_{q_{13}^{(1)},1} & = (1,-1), \\ \varsigma_{q_{13}^{(1)},1} & = (-1,0), \\ \vartheta_{q_{13}^{(1)}} & = (0,-1), \end{cases} \qquad b_{13}^{(3)} = (-d,m+d) \quad \begin{cases} \varrho_{q_{13}^{(3)},3} & = (0,-1), \\ \varsigma_{q_{13}^{(3)},3} & = (1,0), \\ \vartheta_{q_{13}^{(3)}} & = (1,-1), \end{cases}$$

Let us set $e^{\xi} = (t_1, t_2)$. Then, by using (7.3), we obtain:

$$\begin{aligned} \chi(e^{\xi}) &= \left[\frac{1}{(1-t_1)(1-t_2)} + \frac{t_1^{m+d}}{(1-t_1^{-1})(1-t_1^{-1}t_2)} + \frac{t_2^{m+d}}{(1-t_2^{-1})(1-t_1t_2^{-1})} \right] \\ &\cdot \left[\frac{1}{(1-t_1^{-1})(1-t_2^{-1})} + \frac{t_1^{-d}}{(1-t_1)(1-t_1t_2^{-1})} + \frac{t_2^{-d}}{(1-t_2)(1-t_1^{-1}t_2)} \right] \\ &= \left[\sum_{l_1+l_2 \leq m+d} t_1^{l_1} t_2^{l_2} \right] \cdot \left[\sum_{d_1+d_2 \leq d} t_1^{-d_1} t_2^{-d_2} \right]. \end{aligned}$$

As before, letting $(t_1,t_2) \to (1,1)$, we obtain the dimension of $H^0(M_m^{\leq d}; \mathcal{L}_m^{\leq d})$, yielding:

$$\dim_{\mathbb{C}} H^0(M_m^{\leq d}; \mathcal{L}_m^{\leq d}) = \frac{(m+d+1)(m+d+2)}{2} \frac{(d+1)(d+2)}{2}.$$

Hence, from Corollary 7.9:

$$\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d = (m+2d+1)(m+d+1)(d+1). \tag{7.17}$$

7.5.3 Equivariant Quantisation of $T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$

The hypertoric manifold $M_\nu = T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and its hyperplane arrangement $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ were introduced in Example 2.7. We set $\lambda = (0,0,m,n) \in (\mathfrak{t}^4_{\mathbb{Z}})^*$ so that $\nu = (m,n) = \imath^*(\lambda)$, and also set $d = \delta \in \mathbb{Z}_{\geq 0}$. We have denoted $v_{ij} = H_i \cap H_j$ for the interior vertices, and $b_{ij}^{(k)}$ for the boundary vertex, whose adjacent interior vertex is v_{ij} and satisfies $b_{ij}^{(k)} \in H_k$. The moment polyptych $\Delta_\nu^{\leq\delta}$ is recreated in Figure 7.3 with the corresponding isotropy data.

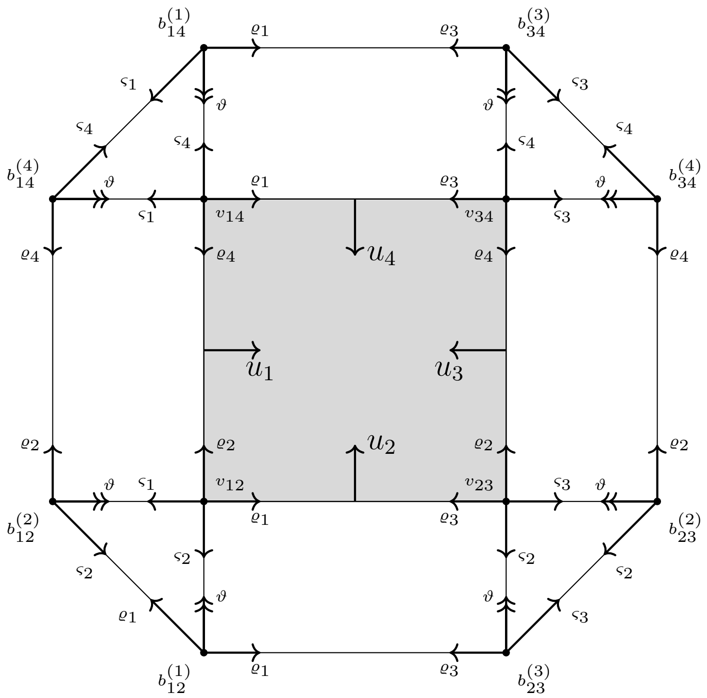


Figure 7.3: Moment polyptych for $M_\nu^{\leq\delta}$ when $M_\nu \cong T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

$$v_{12} = (0,0) \quad \begin{cases} \varrho_{p_{12},1} &= (1,0), \\ \varsigma_{p_{12},1} &= (-1,0), \\ \varrho_{p_{12},2} &= (0,1), \\ \varsigma_{p_{12},2} &= (0,-1), \end{cases} \qquad v_{23} = (m,0) \quad \begin{cases} \varrho_{p_{23},2} &= (0,1), \\ \varsigma_{p_{23},2} &= (0,-1), \\ \varrho_{p_{23},3} &= (-1,0), \\ \varsigma_{p_{23},3} &= (1,0), \end{cases}$$

$$v_{34} = (m,m) \quad \begin{cases} \varrho_{p_{34},3} &= (-1,0), \\ \varsigma_{p_{34},3} &= (1,0), \\ \varrho_{p_{34},4} &= (0,-1), \\ \varsigma_{p_{34},4} &= (0,1), \end{cases} \qquad v_{14} = (0,m) \quad \begin{cases} \varrho_{p_{14},1} &= (1,0), \\ \varsigma_{p_{14},1} &= (-1,0), \\ \varrho_{p_{14},4} &= (0,-1), \\ \varsigma_{p_{14},4} &= (0,1), \end{cases}$$

$$b_{12}^{(1)} = (0,-d) \quad \begin{cases} \varrho_{q_{12}^{(1)},1} &= (1,0), \\ \varsigma_{q_{12}^{(1)},1} &= (-1,1), \\ \vartheta_{q_{12}^{(1)}} &= (0,1), \end{cases} \qquad b_{12}^{(2)} = (-d,0) \quad \begin{cases} \varrho_{q_{12}^{(2)},2} &= (0,1), \\ \varsigma_{q_{12}^{(2)},2} &= (1,-1), \\ \vartheta_{q_{12}^{(2)}} &= (1,0), \end{cases}$$

$$b_{23}^{(2)} = (m+d,0) \quad \begin{cases} \varrho_{q_{23}^{(2)},2} &= (0,1), \\ \varsigma_{q_{23}^{(2)},2} &= (-1,1), \\ \vartheta_{q_{23}^{(2)}} &= (-1,0), \end{cases} \qquad b_{23}^{(3)} = (m,-d) \quad \begin{cases} \varrho_{q_{23}^{(3)},2} &= (-1,0), \\ \varsigma_{q_{23}^{(3)},2} &= (1,-1), \\ \vartheta_{q_{23}^{(3)}} &= (0,1), \end{cases}$$

$$b_{34}^{(3)} = (m,m+d) \quad \begin{cases} \varrho_{q_{34}^{(3)},3} &= (-1,0), \\ \varsigma_{q_{34}^{(3)},3} &= (1,-1), \\ \vartheta_{q_{34}^{(3)}} &= (0,-1), \end{cases} \qquad b_{34}^{(4)} = (m+d,m) \quad \begin{cases} \varrho_{q_{34}^{(4)},4} &= (0,-1), \\ \varsigma_{q_{34}^{(4)},4} &= (-1,1), \\ \vartheta_{q_{34}^{(4)}} &= (-1,0), \end{cases}$$

$$b_{14}^{(1)} = (m,m+d) \quad \begin{cases} \varrho_{q_{14}^{(1)},1} &= (1,0), \\ \varsigma_{q_{14}^{(1)},1} &= (-1,-1), \\ \vartheta_{q_{14}^{(1)}} &= (0,-1), \end{cases} \qquad b_{14}^{(4)} = (m+d,m) \quad \begin{cases} \varrho_{q_{14}^{(4)},4} &= (0,-1), \\ \varsigma_{q_{14}^{(4)},4} &= (1,1), \\ \vartheta_{q_{14}^{(4)}} &= (1,0), \end{cases}$$

Setting $e^{\xi} = (t_1, t_2) \in T^2$, we then can calculate the equivariant character $\chi(t_1, t_2)$ using (7.3):

$$\begin{aligned}
\chi_{\nu,d}(t_1,t_2) = &\frac{1}{(1-t_1)(1-t_2)}\left[\frac{1}{(1-t_1^{-1})(1-t_2^{-1})} + \frac{t_1^{-d}}{(1-t_1)(1-t_1t_2^{-1})} + \frac{t_2^{-d}}{(1-t_1^{-1}t_2)(1-t_2)}\right] \\
&+ \frac{t_1^m}{(1-t_1^{-1})(1-t_2)}\left[\frac{1}{(1-t_1)(1-t_2^{-1})} + \frac{t_1^{d}}{(1-t_1^{-1})(1-t_1^{-1}t_2^{-1})} + \frac{t_2^{-d}}{(1-t_1t_2)(1-t_1^{-1})}\right] \\
&+ \frac{t_2^n}{(1-t_1)(1-t_2^{-1})}\left[\frac{1}{(1-t_1^{-1})(1-t_2)} + \frac{t_1^{-d}}{(1-t_1)(1-t_1t_2^{-1})} + \frac{t_2^{-d}}{(1-t_1^{-1}t_2^{-1})(1-t_2^{-1})}\right] \\
&+ \frac{t_1^m t_2^n}{(1-t_1^{-1})(1-t_2^{-1})}\left[\frac{1}{(1-t_1)(1-t_2)} + \frac{t_1^{d}}{(1-t_1^{-1})(1-t_1^{-1}t_2)} + \frac{t_2^{d}}{(1-t_2^{-1})(1-t_1t_2^{-1})}\right] \\
= &\frac{1}{(1-t_1)(1-t_2)}\left[\sum_{d_1+d_2\le d} t_1^{-d_1}t_2^{-d_2}\right] + \frac{t_1^m}{(1-t_1^{-1})(1-t_2)}\left[\sum_{d_1+d_2\le d} t_1^{d_1}t_2^{-d_2}\right] \\
&+ \frac{t_2^n}{(1-t_1)(1-t_2^{-1})}\left[\sum_{d_1+d_2\le d} t_1^{-d_1}t_2^{d_2}\right] + \frac{t_1^m t_2^n}{(1-t_1^{-1})(1-t_2^{-1})}\left[\sum_{d_1+d_2\le d} t_1^{d_1}t_2^{d_2}\right] \\
= &\sum_{d_1+d_2\le d} t_1^{-d_1}t_2^{-d_2}\left(\frac{1}{1-t_2}\left[\frac{1}{1-t_1} + \frac{t_1^{m+2d_1}}{1-t_1^{-1}}\right] + \frac{t_2^{n+2d_2}}{1-t_2^{-1}}\left[\frac{1}{1-t_1} + \frac{t_1^{m+2d_1}}{1-t_1^{-1}}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d_1+d_2\le d} t_1^{-d_1}t_2^{-d_2}\left(\left[\frac{1}{1-t_1}+\frac{t_1^{m+2d_1}}{1-t_2^{-1}}\right]\cdot\left[\frac{1}{1-t_2}+\frac{t_2^{n+2d_2}}{1-t_2^{-1}}\right]\right)\\
&= \sum_{d_1+d_2\le d} t_1^{-d_1}t_2^{-d_2}\left(\left[\sum_{l_1=0}^{m+2d_1} t_1^{l_1}\right]\cdot\left[\sum_{l_2=0}^{n+2d_2} t_2^{l_2}\right]\right)\\
&= \sum_{d_1=0}^{d} t_1^{-d_1}\left(\sum_{d_2=0}^{d-d_1} t_2^{-d_2}\left(\sum_{\substack{0\le l_1\le m+2d_1\\ 0\le l_2\le n+2d_2}} t_1^{l_1}t_2^{l_2}\right)\right).
\end{aligned}
$$

In taking the limit $(t_1,t_2)\longrightarrow(1,1)$, we get:

$$
\begin{aligned}
\dim H^0(M_\nu^{\le d};\mathcal{L}_\nu^{\le d}) &= \sum_{d_1=0}^{d}\left(\sum_{d_2=0}^{d-d_1}(m+2d_1+1)(n+2d_2+1)\right)\\
&= \sum_{d_1=0}^{d}(d-d_1+1)(2d_1+m+1)(d-d_1+n+1)\\
&= \frac{(d+1)(d+2)(d^2+2dm+2dn+3d+3mn+3m+3n+3)}{6},
\end{aligned}
$$

so that by (7.14), the dimension of $H^0(M_\nu;\mathcal{L}_\nu)_d$ is:

$$
\begin{aligned}
\dim_{\mathbb{C}} H^0(M_\nu;\mathcal{L}_\nu)_d &= \dim_{\mathbb{C}} H^0(M_\nu^{\le d};\mathcal{L}_\nu^{\le d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\le (d-1)};\mathcal{L}_\nu^{\le (d-1)})\\
&= \frac{(d+1)(2d^2+3dm+3dn+4d+3mn+3m+3n+3)}{3}.
\end{aligned}
$$

7.5.4 Equivariant Quantisation of M_ν with a Reducible Core

The previous examples have all involved a hypertoric variety whose cut space ended up being a manifold, or equivalently, that its core was irreducible. But from Theorem 3.17, any hypertoric variety whose core is reducible will have an orbifold for its cut space, thus requiring the equivariant Kawasaki-Riemann-Roch formula to express the equivariant character.

Let us continue with the tradition of going through the examples from Section 2.3, by continuing onto Example 2.8 in which the core $\mathcal{C}$ of the hypertoric variety M_ν consisted of the first Hirzebruch surface and the complex projective plane, $\mathcal{C} = \mathcal{H}_1 \cup \mathbb{CP}^2$. There are two cases depending on whether the Kähler quotient X_ν is $\mathcal{H}_1$ or $\mathbb{CP}^2$. For both cases however, let us fix an integral element $(m+n,n) = \nu \in \mathfrak{k}^*_{\mathbb{Z}}$, and let $d = \delta \in 2\mathbb{Z}_{\ge 0}$ be an *even* positive integer since, otherwise, no pre-quantum line bundle $\mathcal{L}_\nu^{\le d} \longrightarrow M_\nu^{\le d}$ over $M_\nu^{\le d}$, that would additionally be compatible with the orbifold structure, would exist. This is due to the presence of orbifold points appearing along the boundaries $\mathcal{Z}_\nu^d$ of the cut spaces in both examples, whose orbifold structure groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. See [Sil96, §11] for a more explicit example of this phenomenon.

Case: $X_\nu \cong \mathcal{H}_1$

In this case, the moment polyptych $\Delta_\nu^{\leq d}$ of the cut space $M_\nu^{\leq d}$ is presented in Figure 7.4, with its interior and boundary vertices labelled using v's and b's, respectively. Also labelled are the polyptych boundary components Π_A^d for the corresponding subset $A \subseteq \{1,2,3,4\}$.

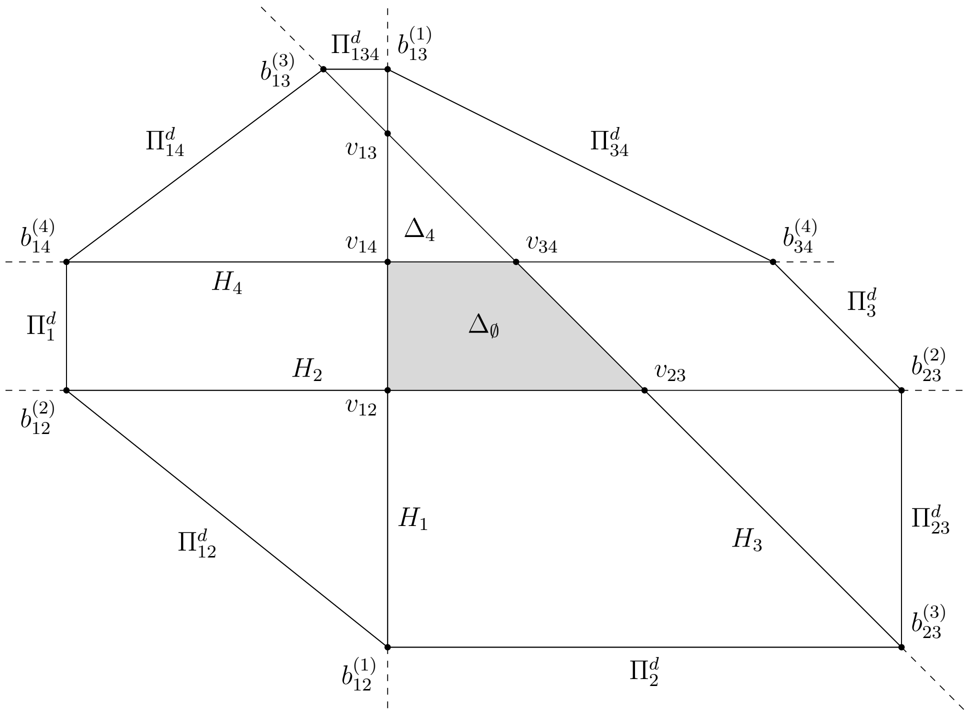


Figure 7.4: Moment polyptych $\Delta_\nu^{\leq d}$ when $X_\nu \cong \mathcal{H}_1$.

One can see that for the subset $A = \{1,3,4\}$, the U_1-action generator $u_{\{134\}} = u_1 + u_3 + u_4 = (0,-2)$ is non-primitive relative to $(\mathfrak{t}^2)^* \cong \mathbb{Z}^2$, and therefore the two boundary vertices, denoted $b_{13}^{(1)}$ and $b_{13}^{(3)}$ here, of the component Π_{134}^d correspond to two orbifold points which we denote by $q_{13}^{(1)}, q_{13}^{(3)} \in \mathcal{Z}_{134}^d$. The isotropy data for $q_{13}^{(1)}$ is:

$$b_{13}^{(1)} = \left(0, m+n+\tfrac{d}{2}\right) \begin{cases} \varrho_{q,1} & = \left(1, -\frac{1}{2}\right), \\ \varsigma_{q,1} & = (-1, 0), \\ \vartheta_q & = \left(0, \frac{1}{2}\right), \end{cases}$$

and, since $b_{13}^{(1)} \in H_1 \cap \Pi_{134}^d$, the orbifold structure group $\Gamma_{12^{(1)}}^{\{134\}}$ of $q_{13}^{(1)}$ is:

$$\Gamma_{12^{(1)}}^{\{134\}} \cong \mathfrak{t}_{\mathbb{Z}}^2 / \operatorname{Span}_{\mathbb{Z}} \left\{ \, u_1 = (1,0), \; u_{\{134\}} = u_1 + u_3 + u_4 = (0,-2) \, \right\} \cong \{0\} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Note that we could have also used Π^d_{34} instead of Π^d_{134}, since $u_{\{34\}} = u_3 + u_4 = (-1,-2)$ and thus:

$$\mathrm{Span}_{\mathbb{Z}}\left\{\, u_1 = (1,0),\ u_{\{34\}} = (-1,-2) \,\right\} \cong \mathrm{Span}_{\mathbb{Z}}\left\{\, u_1 = (1,0),\ u_{\{134\}} = (0,-2) \,\right\},$$

and therefore $\Gamma^{\{34\}}_{12^{(1)}} \cong \Gamma^{\{134\}}_{12^{(1)}}$, so it does not matter which polyptych boundary component Π^d_A that we use, just as long as the vertex does indeed belong to it.

Similarly, the isotropy data for $q^{(1)}_{13} \in \mathcal{Z}^d_{134}$ is:

$$b^{(3)}_{13} = \left(-\tfrac{d}{2}, m+n+\tfrac{d}{2}\right) \begin{cases} \varrho_{q,3} &= \left(-\frac{1}{2}, -\frac{1}{2}\right), \\ \varsigma_{q,3} &= (1,0), \\ \vartheta_q &= \left(\frac{1}{2}, \frac{1}{2}\right), \end{cases}$$

and so its orbifold structure group $\Gamma^{\{134\}}_{12^{(3)}}$ is:

$$\Gamma^{\{134\}}_{12^{(3)}} = \mathfrak{t}^2_{\mathbb{Z}} / \,\mathrm{Span}_{\mathbb{Z}}\left\{\, u_3 = (-1,-1),\ u_{\{134\}} = (0,-2) \,\right\} \cong \{0\} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The remaining vertices are not orbifold points, and the isotropy data regarding their corresponding fixed points is listed below:

$$v_{12} = (0,0) \begin{cases} \varrho_{p,1} &= (1,0), \\ \varsigma_{p,1} &= (-1,0), \\ \varrho_{p,2} &= (0,1), \\ \varsigma_{p,2} &= (0,-1), \end{cases} \qquad v_{23} = (m+n,0) \begin{cases} \varrho_{p,2} &= (-1,1), \\ \varsigma_{p,2} &= (1,-1), \\ \varrho_{p,3} &= (-1,0), \\ \varsigma_{p,3} &= (1,0), \end{cases}$$

$$v_{34} = (m,n) \begin{cases} \varrho_{p,3} &= (-1,0), \\ \varsigma_{p,3} &= (1,0), \\ \varrho_{p,4} &= (1,-1), \\ \varsigma_{p,4} &= (-1,1), \end{cases} \qquad v_{14} = (0,n) \begin{cases} \varrho_{p,1} &= (1,0), \\ \varsigma_{p,1} &= (-1,0), \\ \varrho_{p,4} &= (0,-1), \\ \varsigma_{p,4} &= (0,1), \end{cases}$$

$$v_{13} = (0,m+n) \begin{cases} \varrho_{p,1} &= (1,-1), \\ \varsigma_{p,1} &= (-1,1), \\ \varrho_{p,3} &= (0,-1), \\ \varsigma_{p,3} &= (0,1), \end{cases} \qquad b^{(1)}_{12} = (0,-m-2) \begin{cases} \varrho_{q,1} &= (1,0), \\ \varsigma_{q,1} &= (-1,1), \\ \vartheta_q &= (0,1), \end{cases}$$

$$b^{(2)}_{12} = (-m-d,0) \begin{cases} \varrho_{q,2} &= (0,1), \\ \varsigma_{q,2} &= (1,-1), \\ \vartheta_q &= (1,0), \end{cases} \qquad b^{(4)}_{34} = (2m+d,n) \begin{cases} \varrho_{q,4} &= (1,-1), \\ \varsigma_{q,4} &= \left(-1, \frac{1}{2}\right), \\ \vartheta_q &= (-1,0), \end{cases}$$

$$b_{23}^{(3)} = (2m+n+d, -m-d) \begin{cases} \varrho_{q,3} &= (-1,0), \\ \varsigma_{q,3} &= (0,1), \\ \vartheta_q &= (-1,1), \end{cases}$$

$$b_{23}^{(2)} = (2m+n+d, 0) \begin{cases} \varrho_{q,2} &= (-1,1), \\ \varsigma_{q,2} &= (0,-1), \\ \vartheta_q &= (-1,0), \end{cases}$$

$$b_{14}^{(4)} = (-m-d, n) \begin{cases} \varrho_{q,4} &= (0,-1), \\ \varsigma_{q,4} &= (1,1), \\ \vartheta_q &= (1,0). \end{cases}$$

With all of the isotropy data listed, we consider $\xi = (t, 3t) \in \mathfrak{t}^2$, where $t \in \mathbb{R}_{>0}$ is some positive real variable, such that ξ is a generic element. Then by using (7.3), we have the following expression for the equivariant character $\chi(e^\xi)$:

$$\chi(e^\xi) = \sum_{p \in (M_\nu^{<d})^{T^2}} \chi_p(e^\xi) + \sum_{\substack{p \in (\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p \cong \{1\}}} \chi_p(e^\xi) + \sum_{\substack{p \in (\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p \ncong \{1\}}} \chi_p(e^\xi),$$

where the contribution from the smooth interior fixed points is:

$$\begin{aligned} \sum_{p \in (M_\nu^{<d})^{T^2}} \chi_p(e^\xi) = {} & \frac{e^{t(m+n)}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} + \frac{e^{mt+3nt}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} \\ & + \frac{e^{3nt}}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} + \frac{1}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} \\ & + \frac{e^{3t(m+n)}}{(1-e^{-3t})(1-e^{-2t})(1-e^{2t})(1-e^{3t})}, \end{aligned}$$

the contribution from the smooth boundary fixed points is:

$$\begin{aligned} \sum_{\substack{p \in (\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p \cong \{1\}}} \chi_p(e^\xi) = {} & \frac{e^{3t(-d-m)}}{(1-e^{t})(1-e^{2t})(1-e^{3t})^2} + \frac{e^{3t(-d-m)+t(d+2m+n)}}{(1-e^{-t})(1-e^{2t})^2 \cdot (1-e^{3t})} \\ & + \frac{e^{t(-d-m)}}{(1-e^{-2t})(1-e^{t})^2 \cdot (1-e^{3t})} + \frac{e^{3nt+t(d+2m)}}{(1-e^{-2t})(1-e^{-t})^2 \cdot (1-e^{t})} \\ & + \frac{e^{3nt+t(-d-m)}}{(1-e^{-3t})(1-e^{t})^2 \cdot (1-e^{4t})} + \frac{e^{t(d+2m+n)}}{(1-e^{-3t})(1-e^{-t})^2 \cdot (1-e^{2t})}, \end{aligned}$$

and the contributions from each orbifold boundary fixed point:

$$q_{13}^{(1)} = \left(0, m+n+\tfrac{d}{2}\right), \qquad \text{and} \qquad q_{13}^{(3)} = \left(-\tfrac{d}{2}, m+n+\tfrac{d}{2}\right)$$

respectively, are:

$$\chi_{q_{13}^{(1)}}(e^{\xi}) = \frac{e^{3t\left(\frac{d}{2}+m+n\right)}}{2\left(1+e^{-\frac{3t}{2}}\right)^2 \cdot (1-e^{-t})\left(1+e^{-\frac{t}{2}}\right)} + \frac{e^{3t\left(\frac{d}{2}+m+n\right)}}{2\left(1-e^{-\frac{3t}{2}}\right)^2 \cdot (1-e^{-t})\left(1-e^{-\frac{t}{2}}\right)}$$

and:

$$\chi_{q_{13}^{(3)}}(e^{\xi}) = \frac{e^{-\frac{dt}{2}+3t\left(\frac{d}{2}+m+n\right)}}{2\cdot(1+e^{-2t})(1+e^{-t})^2\cdot(1-e^{t})} + \frac{e^{-\frac{dt}{2}+3t\left(\frac{d}{2}+m+n\right)}}{2\cdot(1-e^{-2t})(1-e^{-t})^2\cdot(1-e^{t})}.$$

Letting $t \to 0$, so that $e^{\xi} \to 1$, we obtain the following formula for the dimension of $H^0(M_\nu^{\le d}; \mathcal{L}_\nu^{\le d})$:

$$\begin{aligned}
\dim_{\mathbb{C}} H^0(M_\nu^{\le d}; \mathcal{L}_\nu^{\le d}) &= \lim_{\xi\to 0} \chi(e^{\xi}) \\
&= \frac{17d^4}{6} + 8d^3m + 4d^3n + \frac{17d^3}{2} + 8d^2m^2 + 8d^2mn + 18d^2m \\
&+ d^2n^2 + 9d^2n + \frac{29d^2}{3} + \frac{10dm^3}{3} + 5dm^2n + 12dm^2 + dmn^2 \\
&+ 12dmn + \frac{41dm}{3} + \frac{3dn^2}{2} + \frac{13dn}{2} + 5d + \frac{m^4}{2} + m^3n + \frac{5m^3}{2} \\
&+ \frac{m^2n^2}{4} + \frac{15m^2n}{4} + \frac{9m^2}{2} + \frac{3mn^2}{4} + \frac{17mn}{4} + \frac{7m}{2} + \frac{n^2}{2} + \frac{3n}{2} + 1,
\end{aligned}$$

and so from Corollary 7.9, recalling that $d \in 2\mathbb{Z}_{\ge 0}$, we calculate:

$$\begin{aligned}
\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d &= \dim_{\mathbb{C}} H^0(M_\nu^{\le d}; \mathcal{L}_\nu^{\le d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\le (d-2)}; \mathcal{L}_\nu^{\le (d-2)}) \\
&= \frac{17d^3}{12} + 6d^2m + 3d^2n + \frac{17d^2}{8} + 8dm^2 + 8dmn + 6dm + dn^2 \\
&+ 3dn + \frac{31d}{12} + \frac{10m^3}{3} + 5m^2n + 4m^2 + mn^2 + 4mn \\
&+ \frac{11m}{3} + \frac{n^2}{2} + \frac{3n}{2} + 1.
\end{aligned} \tag{7.18}$$

Case: $X_\nu \cong \mathbb{CP}_2$

Recall from Example 2.8 that this case is obtained by inverting the sign of the normal vector to the hyperplane H_4 as $u_4 = -e_2 \mapsto e_2 = (0,1)$. This changes the poset $\mathcal{P}(\mathcal{A})$ of regions of $\mathcal{A}$, so that now the hypertoric variety M_ν is the hyperkähler analogue to the complex projective plane, $X_\nu \cong \mathbb{CP}^2$.

As before in the previous case, we form the cut space $M_\nu^{\leq d}$ relative to a positive *even* integer $d \in 2\mathbb{Z}_{\geq 0}$, and we present the resulting moment polyptych $\Delta_\nu^{\leq d}$ in Figure 7.5.

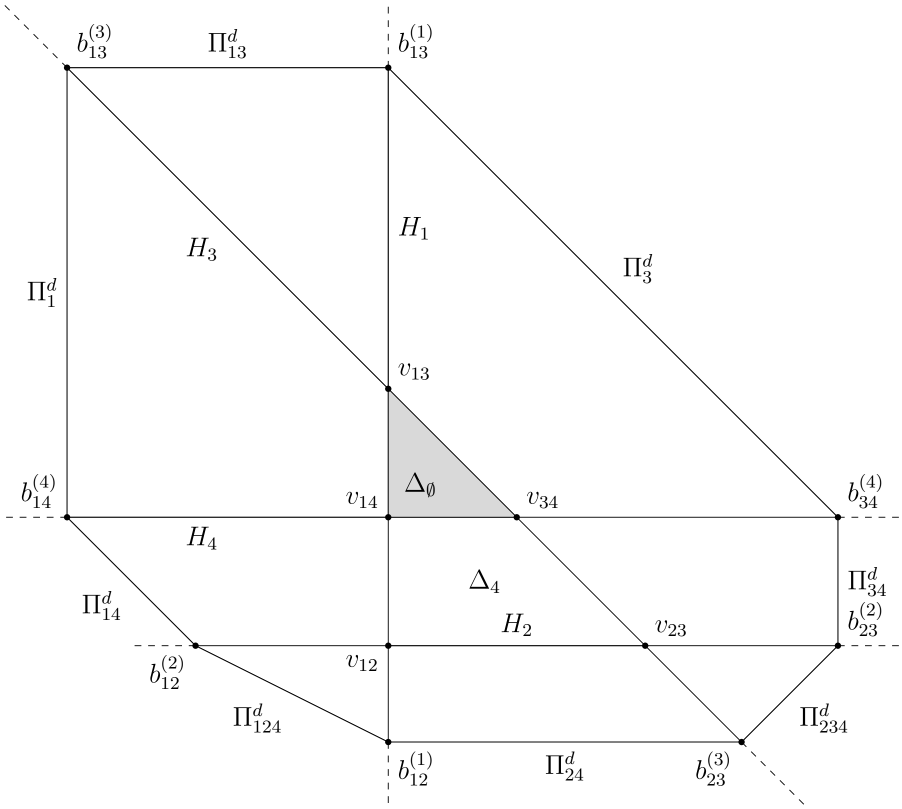


Figure 7.5: Moment polyptych $\Delta_\nu^{\leq d}$ when $X_\nu \cong \mathbb{CP}^2$.

For the subset $A = \{2,4\}$ now, it is the U_1-action generator $u_{\{24\}} = u_2 + u_4 = (0,2)$ that is non-primitive relative to $\mathbb{Z}^2$. Hence the boundary vertices $b_{12}^{(1)}$ and $b_{23}^{(3)}$ that lie on the boundary component Π_{24}^d are the ones corresponding to the orbifold points, $q_{12}^{(1)}, q_{23}^{(3)} \in \mathcal{Z}_{24}^d$. The isotropy data for $q_{12}^{(1)}$ is:

$$b_{12}^{(1)} = \left(0, -\tfrac{d}{2}\right) \begin{cases} \varrho_{q,1} &= (1,0), \\ \varsigma_{q,1} &= \left(-1, \tfrac{1}{2}\right), \\ \vartheta_q &= \left(0, \tfrac{1}{2}\right), \end{cases}$$

and from Lemma 6.4, the orbifold structure group $\Gamma_{12^{(1)}}^{\{134\}}$ of $q_{13}^{(1)}$ is:

$$\Gamma_{12^{(1)}}^{\{24\}} \cong \mathfrak{t}_{\mathbb{Z}}^2 / \operatorname{Span}_{\mathbb{Z}} \left\{ u_1 = (1,0),\ u_{\{24\}} = u_2 + u_4 = (0,2) \right\} \cong \{0\} \oplus \mathbb{Z}/2\mathbb{Z},$$

since $b_{12}^{(1)} \in H_1 \cap \Pi_{24}^d$.

Similarly for $q_{23}^{(3)} \in \mathcal{Z}_{24}^d$, we have:

$$b_{23}^{(3)} = \left(m+n+\tfrac{d}{2}, -\tfrac{d}{2}\right) \begin{cases} \varrho_{q,3} & = (-1,0), \\ \varsigma_{q,3} & = \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \\ \vartheta_q & = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \end{cases}$$

as $b_{23}^{(3)} \in H_3 \cap \Pi_{24}^d$, and so from Lemma 6.4, its orbifold structure group $\Gamma_{23^{(3)}}^{\{24\}}$ is:

$$\Gamma_{23^{(3)}}^{\{24\}} = \mathfrak{t}_{\mathbb{Z}}^2 / \operatorname{Span}_{\mathbb{Z}} \left\{ u_3 = (-1,-1),\ u_{\{24\}} = (0,2) \right\} \cong \{0\} \oplus \mathbb{Z}/2\mathbb{Z},$$

and the other vertices do not correspond to orbifold points, just like before. The isotropy data of their corresponding fixed points is listed below:

$$v_{12} = (0,0) \begin{cases} \varrho_{p,1} & = (1,0), \\ \varsigma_{p,1} & = (-1,0), \\ \varrho_{p,2} & = (0,1), \\ \varsigma_{p,2} & = (0,-1), \end{cases} \qquad v_{23} = (m+n,0) \begin{cases} \varrho_{p,2} & = (-1,1), \\ \varsigma_{p,2} & = (1,-1), \\ \varrho_{p,3} & = (-1,0), \\ \varsigma_{p,3} & = (1,0), \end{cases}$$

$$v_{34} = (m,n) \begin{cases} \varrho_{p,3} & = (-1,0), \\ \varsigma_{p,3} & = (1,0), \\ \varrho_{p,4} & = (1,-1), \\ \varsigma_{p,4} & = (-1,1), \end{cases} \qquad v_{14} = (0,n) \begin{cases} \varrho_{p,1} & = (1,0), \\ \varsigma_{p,1} & = (-1,0), \\ \varrho_{p,4} & = (0,-1), \\ \varsigma_{p,4} & = (0,1), \end{cases}$$

$$v_{13} = (0,m+n) \begin{cases} \varrho_{p,1} & = (1,-1), \\ \varsigma_{p,1} & = (-1,1), \\ \varrho_{p,3} & = (0,-1), \\ \varsigma_{p,3} & = (0,1), \end{cases} \qquad b_{12}^{(2)} = (-d,0) \begin{cases} \varrho_{q,2} & = (-1,1), \\ \varsigma_{q,2} & = \left(1, -\tfrac{1}{2}\right), \\ \vartheta_q & = (1,0), \end{cases}$$

$$b_{14}^{(4)} = (-n-d,n) \begin{cases} \varrho_{q,4} & = (0,1), \\ \varsigma_{q,4} & = (1,-1), \\ \vartheta_q & = (1,0), \end{cases} \qquad b_{34}^{(4)} = (m+n+d,n) \begin{cases} \varrho_{q,4} & = (-1,1), \\ \varsigma_{q,4} & = (0,-1), \\ \vartheta_q & = (-1,0), \end{cases}$$

$$b_{13}^{(1)} = (0,m+2n+d) \begin{cases} \varrho_{q,1} & = (1,-1), \\ \varsigma_{q,1} & = (-1,0), \\ \vartheta_q & = (0,-1), \end{cases}$$

$$b_{13}^{(3)} = (-n-d, m+2n+d) \begin{cases} \varrho_{q,3} & = (0,-1), \\ \varsigma_{q,3} & = (1,0), \\ \vartheta_q & = (1,-1), \end{cases}$$

As in the first case, we express the equivariant character $\xi(e^\xi)$ using (7.3) by choosing the generic element $\xi = (t, 3t) \in \mathfrak{t}^2$, and writing:

$$\chi(e^\xi) = \sum_{p\in(M_\nu^{<d})^{T^2}} \chi_p(e^\xi) + \sum_{\substack{p\in(\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p\cong\{1\}}} \chi_p(e^\xi) + \sum_{\substack{p\in(\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p\ncong\{1\}}} \chi_p(e^\xi),$$

where the contribution from the smooth interior fixed points is:

$$\begin{aligned}\sum_{p\in(M_\nu^{<d})^{T^2}} \chi_p(e^\xi) = & \frac{e^{t(m+n)}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} + \frac{e^{mt+3nt}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} \\ & + \frac{e^{3nt}}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} + \frac{1}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} \\ & + \frac{e^{3t(m+n)}}{(1-e^{-3t})(1-e^{-2t})(1-e^{2t})(1-e^{3t})},\end{aligned}$$

the contribution from the smooth boundary fixed points is:

$$\begin{aligned}\sum_{\substack{p\in(\mathcal{Z}_\nu^{<d})^{T^2} \\ \Gamma_p\cong\{1\}}} \chi_p(e^\xi) = & \frac{e^{-dt}}{(1-e^{-t})(1-e^{t})^2\cdot(1-e^{2t})} + \frac{e^{3nt+t(-d-n)}}{(1-e^{-2t})(1-e^{t})^2\cdot(1-e^{3t})} \\ & + \frac{e^{3nt+t(d+m+n)}}{(1-e^{-3t})(1-e^{-t})^2\cdot(1-e^{2t})} + \frac{e^{t(-d-n)+3t(d+m+2n)}}{(1-e^{-3t})(1-e^{-2t})^2\cdot(1-e^{t})} \\ & + \frac{e^{3t(d+m+2n)}}{(1-e^{-3t})^2\cdot(1-e^{-2t})(1-e^{-t})} + \frac{e^{t(d+m+n)}}{(1-e^{-4t})(1-e^{-t})^2\cdot(1-e^{3t})},\end{aligned}$$

and the contributions from each orbifold boundary fixed point:

$$q_{12}^{(1)} = \left(0, -\tfrac{d}{2}\right), \qquad \text{and} \qquad q_{23}^{(3)} = \left(m+n+\tfrac{d}{2}, -\tfrac{d}{2}\right)$$

respectively, are:

$$\chi_{q_{12}^{(1)}}(e^\xi) = \frac{e^{-\frac{3dt}{2}}}{2\cdot(1-e^{t})\left(e^{\frac{t}{2}}+1\right)\left(e^{\frac{3t}{2}}+1\right)^2} + \frac{e^{-\frac{3dt}{2}}}{2\cdot\left(1-e^{\frac{t}{2}}\right)(1-e^{t})\left(1-e^{\frac{3t}{2}}\right)^2},$$

and:

$$\chi_{q_{23}^{(3)}}(e^{\xi}) = \frac{e^{-\frac{3dt}{2}+t\left(\frac{d}{2}+m+n\right)}}{2\cdot(1-e^{-t})\,(e^t+1)^2\,(e^{2t}+1)} + \frac{e^{-\frac{3dt}{2}+t\left(\frac{d}{2}+m+n\right)}}{2\cdot(1-e^{-t})\,(1-e^t)^2\cdot(1-e^{2t})},$$

Taking the limit $t \to 0$ so that $e^{\xi} \to 1$, we get:

$$\begin{aligned}\dim_{\mathbb{C}} H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}) &= \lim_{\xi\to 0}\chi(e^{\xi})\\ &= \frac{17d^4}{96}+\frac{5d^3m}{12}+\frac{11d^3n}{12}+\frac{17d^3}{16}+\frac{d^2m^2}{4}+\frac{3d^2mn}{2}+\frac{15d^2m}{8}\\ &+\frac{3d^2n^2}{2}+\frac{33d^2n}{8}+\frac{29d^2}{12}+\frac{dm^2n}{2}+\frac{3dm^2}{4}+\frac{3dmn^2}{2}+\frac{9dmn}{2}\\ &+\frac{17dm}{6}+dn^3+\frac{9dn^2}{2}+\frac{73dn}{12}+\frac{5d}{2}+\frac{m^2n^2}{4}+\frac{3m^2n}{4}+\frac{m^2}{2}\\ &+\frac{mn^3}{2}+\frac{9mn^2}{4}+\frac{13mn}{4}+\frac{3m}{2}+\frac{n^4}{4}+\frac{3n^3}{2}+\frac{13n^2}{4}+3n+1,\end{aligned}$$

which, recalling again that $d \in 2\mathbb{Z}_{\geq 0}$, leads to:

$$\begin{aligned}\dim_{\mathbb{C}} H^0(M_\nu;\mathcal{L}_\nu)_d &= \dim_{\mathbb{C}} H^0(M_\nu^{\leq d};\mathcal{L}_\nu^{\leq d}) - \dim_{\mathbb{C}} H^0(M_\nu^{\leq(d-2)};\mathcal{L}_\nu^{\leq(d-2)})\\ &= \frac{17d^3}{12}+\frac{5d^2m}{2}+\frac{11d^2n}{2}+\frac{17d^2}{8}+dm^2+6dmn+\frac{5dm}{2}+6dn^2\\ &+\frac{11dn}{2}+\frac{31d}{12}+m^2n+\frac{m^2}{2}+3mn^2+3mn+\frac{3m}{2}\\ &+2n^3+3n^2+3n+1.\end{aligned} \tag{7.19}$$

Chapter 8

Conclusion

In this thesis, we have provided a formula in Corollary 7.9 that calculates the U_1-weight subspaces of the space of holomorphic sections $H^0(M_\nu; \mathcal{L}_\nu)$, when M_ν is a hypertoric variety and when $\mathcal{L}_\nu$ is a T^n-equivariant pre-quantum line bundle, holomorphic with respect to the complex structure I_1 on M_ν. We accomplished this by applying Lerman's symplectic cut to M_ν with respect to a residual U_1-action, which every hypertoric variety possesses. Doing so resulted in a cut space $M_\nu^{\leq\delta}$ which in particular was compact, since the moment map for the U_1-action was proper, and furthermore was Kähler with respect to the complex structure I_1 inherited from M_ν.

In a sense, these cut spaces $M_\nu^{\leq\delta}$ acted as auxiliary objects when it came to calculating the dimension of $H^0(M_\nu; \mathcal{L}_\nu)$, but their importance here should not be understated. To each cut space $M_\nu^{\leq\delta}$ there was a so-called "moment polyptych", which was denoted $\Delta_\nu^{\leq\delta}$, that can be seen combinatorially as coming the hyperplane arrangement $\mathcal{A}$ corresponding to M_ν by truncating $\mathcal{A}$ in a prescribed way that depended on the coorientation of $\mathcal{A}$. The moment polyptych $\Delta_\nu^{\leq\delta}$ allowed us to read off the necessary isotropy data for $M_\nu^{\leq\delta}$, allowing us then to use Theorem 5.44, that is the Atiyah-Bott-Berline-Vergne localisation theorem, and hence obtain an expression for the equivariant character $\chi : T^n \to H^\bullet(\mathrm{BT}^n)$ for the representation of T^n on $H^0(M_\nu^{\leq\delta}; \mathcal{L}_\nu^{\leq\delta})$. Provided that we chose suitable integral values $\nu \in \mathfrak{k}_\mathbb{Z}^*$ and $d \in \mathbb{Z}_{\geq 0}$, then the subspace $H^0(M_\nu; \mathcal{L}_\nu)_d$ of U_1-weight d was given by the formula:

$$H^0(M_\nu; \mathcal{L}_n)_d \cong H^0(M_\nu^{\leq d}; \mathcal{L}_\nu^{\leq d}) \big/ H^0(M_\nu^{\leq d-1}; \mathcal{L}_\nu^{\leq d-1}),$$

whose derivation formed the content of Corollary 7.9 for hypertoric variety M_ν, though we proved a more general result for normal semi-projective varieties in Theorem 7.8. Finally, we then went through some examples of calculating $H^0(M_\nu; \mathcal{L}_\nu)_d$.

It would be very interesting if one could simply read off the dimension $\dim_\mathbb{C} H^0(M_\nu; \mathcal{L}_\nu)_d$ from the moment polyptych $\Delta_\nu^{\leq\delta}$ of the cut space $M_\nu^{\leq\delta}$. For example, a combinatorial method such as a lattice-point count analogous to the case of a toric variety X, say, since the dimension of its space

$H^0(X; \mathcal{L}_X^{\otimes k})$ of holomorphic sections coincides with the number of lattice points that are inscribed within its moment polytope $\Delta_X = \mu(X)$, i.e., $\dim_{\mathbb{C}} H^0(X; \mathcal{L}_X^{\otimes k}) = \#\{(k \cdot \Delta) \cap \mathbb{R}^n\}$. However, since the moment polyptych $\Delta_\nu^{\leq d}$ consists of several polytopes all fitted together, their contributions appear to be conflated and it is not quite so straightforward to identify what counting algorithm should be implemented here, if at all. Furthermore, the asymptotic result that links the continuous volume of a lattice polytope Δ in $\mathbb{R}^n$ with its lattice-point count [BR15, §3.6], namely that of:

$$\mathrm{vol}(\Delta) = \lim_{d\to\infty} \frac{\#\,(d \cdot \Delta) \cap \mathbb{Z}^n}{d^n} = \lim_{d\to\infty} \frac{\dim_{\mathbb{C}} H^0(X; \mathcal{L}_X^{\otimes d})}{d^n},$$

yields, for example using the dimension of $\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d$ from (7.18):

$$\frac{\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d}{d^2} \sim \mathcal{O}(d) \longrightarrow \infty, \qquad \text{as } d \to \infty.$$

In fact, each of the two-dimensional examples (i.e., with $n = 2$) exhibit this phenomenon with $\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)_d$ behaving cubically with respect to d. On the other hand, for the one-dimensional example of $M_\nu \cong T^*\mathbb{CP}^1$ (i.e., with $n = 1$), we see from (7.17) that:

$$\frac{\dim_{\mathbb{C}} H^0(T^*\mathbb{CP}^1; \mathcal{L}_\nu)_d}{d} = 2.$$

On the other hand, the moment polyptych in Section 7.5.1 with $M_\nu \cong T^*\mathbb{CP}^1$ is made up of three closed intervals; $\Delta_\emptyset = [0, m]$, $\Delta_1^{\leq\delta} = [-d, 0]$, and $\Delta_2^{\leq\delta} = [m, m + d]$. Then:

$$\begin{aligned} \#\left(\Delta_\nu^{\leq d} \cap (\mathfrak{t}_{\mathbb{Z}}^1)^*\right) &= \#\left([0, m] \cap \mathbb{Z}\right) + \#\left([-d, 0] \cap \mathbb{Z}\right) + \#\left([m, m + d] \cap \mathbb{Z}\right) \\ &= (m + 1) + d + d \\ &= m + 2d + 1, \end{aligned}$$

which does actually coincide with the dimension of $H^0(M_\nu; \mathcal{L}_\nu)_d$ in (7.16). Therefore, some hope for a combinatorial description of $\dim_{\mathbb{C}} H^0(M_\nu; \mathcal{L}_\nu)$ does persist.

Appendix A

Orbifolds

We have delegated this appendix to be a brief introduction to orbifolds, hopefully so that this thesis is more self-contained. The notion of an orbifold was first introduced by Satake in [Sat56] when he first introduced them as *"V-manifolds"*. More general and sophisticated references for orbifolds include [Dui11, Chapter 14] and [BG08, Chapter 4], and this appendix is heavily influenced by [Sil96, Appendix A].

A.1 Orbifolds and their Charts

Let $|M|$ be a Hausdorff topological space.

Definition A.1. An **orbifold chart** for M is a triple $(\tilde{U}, \Gamma, \phi)$ that consists of:

(i) a connected and open subset $\tilde{U}$ of $\mathbb{R}^n$;

(ii) a finite group Γ acting linearly on $\tilde{U}$;

(iii) a continuous Γ-invariant map $\tilde{\phi} : \tilde{U} \to |M|$, that induces a homeomorphism:

$$\phi : \tilde{U}/\Gamma \to U := \phi(\tilde{U}) \subseteq |M|.$$

Definition A.2. An **orbifold atlas** for M is a collection of orbifold charts $(\tilde{U}_i, \Gamma_i, \phi_i)$, such that:

(i) the collection of open subsets $\tilde{U}_i$ forms a basis of $|M|$;

(ii) the collection of charts $(\tilde{U}_i, \Gamma_i, \phi_i)$ satisfy the following compatibility criteria: if $\tilde{U}_i \subseteq \tilde{U}_j$, then there exists a diffeomorphism $\iota : \tilde{U}_1 \to \tilde{U}_2$ and an isomorphism $J : \Gamma_1 \to \Gamma_2$, such that

$\phi_1 = \phi_2 \circ i$, and that ι is J-equivariant:

$$\iota \circ \gamma = J(\gamma) \circ i, \qquad \text{for all } \gamma \in \Gamma_1.$$

Definition A.3. An n-**dimensional orbifold** M is a Hausdorff topological space $|M|$ along with an atlas of orbifold charts, $(\tilde{U}_i, \Gamma_i, \phi_i)$.

Example A.4. Every ordinary manifold is a special case of an orbifold, obtained by considering each manifold chart as an orbifold chart with the trivial group for Γ.

Proposition A.5. *Let G be a compact Lie group that acts locally freely on a smooth manifold M. Then the orbit space M/G has a natural orbifold structure.*

Proof. This proof is from [Dui11, §14.1]. For any point $p \in M$, the stabiliser subgroup G_p of p in G is finite. The linearisation of the local G_p-action implies the existence of a *"slice"* through p for the G-action, i.e., a smooth G_p-invariant manifold S through p, such that $T_pM \cong T_pS \oplus T_p(G \cdot p)$, and such that each nearby G-orbit in M intersects S in a G_p-orbit in S. From this, the neighbourhoods U in M/G of the orbits $\mathcal{O}_p = G \cdot p$ are identified with the quotients S/G_p of smooth manifolds by finite groups G_p, thus yielding the sought-after orbifold charts. □

Let M be an n-dimensional orbifold, $p \in M$ a point, and $(\tilde{U}, \Gamma, \phi)$ an orbifold chart for a neighbourhood U of p.

Definition A.6. The **orbifold structure group**, Γ_p, of p, is the isotropy group of a pre-image of p under ϕ.

The orbifold structure group Γ_p is well-defined up to isomorphism, and one may choose an orbifold chart $(\tilde{U}, \Gamma, \phi)$ for which $\phi^{-1}(p)$ is a single point fixed by Γ. In this case, $\Gamma \cong \Gamma_p$, and $(\tilde{U}, \Gamma_p, \phi)$ is called a **structure chart** for the point p.

There is a natural stratification of the orbifold M into suborbifolds, according to their orbifold structure group types, which is called the **orbifold stratification**. On each connected component of M, there is an open and dense set of regular points in M, for which the order of the structure group is minimal. This is called the **principal stratum** of M. On each connected component of M, the abstract isotropy group of its principal stratum is called the **structure group** of that component, and its order of the group is said to be **multiplicity** of that component. By varying over each connected component of M, the multiplicities of each define a locally constant function $m_M : M \to \mathbb{N}$, called the **multiplicity function**.

In contrast to manifolds, orbifolds are allowed to have quotient-singularities, which fortunately are only mild ones.

Proposition A.7. *Let M be an orbifold. Then M is normal, Cohen-Macaulay, with only rational singularities.*

Proof. For the proof that M is normal, see [Car57, Théorème 4]; that M has only rational singularities, see [Vie77, Proposition 1]; and that M is Cohen-Macaulay, see [Sta79, Proposition 3.2]. $\square$

A.2 Suborbifolds

Suppose that M and N are two orbifolds with a continuous inclusion $|\imath| : |M| \hookrightarrow |N|$ between their underlying topological spaces. Assume that there exists an atlas of orbifold charts $(\tilde{U}, \Gamma, \phi)$ for N such that, for each chart $(\tilde{U}, \Gamma, \phi)$ which intersects M, by which we mean that $\phi(\tilde{U}) \cap |\imath|(|M|) \neq \emptyset$, then the pre-image of N is given by the intersection of $\tilde{U}$ with a linear subspace V of $\mathbb{R}^n$. Let Γ_V be the subgroup of Γ that consists of the elements whose action preserves V.

Definition A.8. We say that M is a **suborbifold** of N if the collection $(\tilde{U} \cap V, \Gamma_V, \phi|_{\tilde{U}\cap V})$ of triples, along with their induced injections, forms an atlas of orbifold charts for N.

For each orbifold chart $(\tilde{U}, \Gamma, \phi)$, we can find the subgroup of transformations in Γ which becomes the identity when restricted to $\tilde{U} \cap V$. The subgroup is well-defined as an abstract group for each connected component of N, since it is just the isotropy group of the respective principal stratum. In particular, when N is a connected component, then it is just the structure group of N.

A.3 Maps & Group Actions on Orbifolds

Definition A.9. We say that a **smooth map between orbifolds**, $f : M \to N$, is a continuous map between the underlying topological spaces that satisfies the following condition: given $p \in M$, let $(\tilde{V}, \Gamma_{f(p)}, \psi)$ be a structure chart for $f(p)$. Then there exists a structure chart $(\tilde{U}, \Gamma_p, \phi)$ for p, in addition to a smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$, such that $f \circ \phi = \psi \circ \tilde{f}$.

Definition A.10. A **smooth function** on M is a collection of smooth invariant functions on each orbifold chart $(\tilde{U}, \Gamma, \phi)$ that agree on the overlaps of the images $\phi(\tilde{U})$.

Definition A.11. A **smooth action** τ of a Lie group G on an orbifold M is a smooth orbifold map $\tau : G\times \to M$ that satisfies the ordinary group action axioms: for every $g, h \in G$ and $p \in M$, we have that:

$$\tau\big(g, \tau(h,p)\big) = \tau(gh, p), \qquad \text{and} \qquad \tau(e_G, p) = p.$$

A.4 Fibre & Vector Orbibundles

Definition A.12. An **orbifold fibre bundle** $\pi : E \to M$ is a collection of Γ-equivariant fibre bundles:

$$\begin{tikzcd} Z \arrow[r, hook] & E_{\tilde{U}} \arrow[d, "\pi_{\tilde{U}}"] \\ & \tilde{U} \end{tikzcd}$$

over each chart $(\tilde{U}, \Gamma, \phi)$, together with suitable compatibility criteria.

For each $p \in M$, in general the fibres $\pi^{-1}(p)$ are not diffeomorphic to Z. Rather they are only diffeomorphic to some quotient of Z by an action of the structure group Γ_p. If V is a vector space and $\Gamma \subseteq \mathrm{GL}(V)$ is a finite subgroup, quotients of the form V/Γ are called **vector orbispaces**. If each fibre $\pi^{-1}(p)$ is a vector orbispace, then $\pi : E \to M$ is a **orbifold vector bundle**. Denoting by $N(\Gamma)$ the normaliser group of Γ in $\mathrm{GL}(V)$, then the group $\mathrm{GL}(V/\Gamma)$ acts on the orbifold V/Γ.

A **Riemannian metric** on an orbifold vector bundle E is a Γ-invariant smooth type $(2, 0)$ tensor field of inner product:

$$\langle -, - \rangle \in H^0(M; E^* \otimes E^*)$$

on the fibres of $E_{\tilde{U}}$, for each orbifold chart $(\tilde{U}, \Gamma_p, \phi)$ and agreeing on their overlaps. A **complex orbifold vector bundle** is an orbifold vector bundle equipped with an almost-complex structure. A **complex structure** on an orbifold vector bundle E is a Γ-invariant smooth type $(1, 1)$ tensor field of linear operators:

$$J \in H^0(M; E \otimes E^*)$$

with $J \circ J = -\mathrm{Id}_E$, on the fibres of $E_{\tilde{U}}$ for each orbifold chart $(\tilde{U}, \Gamma_p, \phi)$ and agreeing on their overlaps.

A **Hermitian orbifold vector bundle** is a complex orbifold vector bundle $\pi : E \to M$ equipped with a **Hermitian structure**, which is a smooth type $(2, 0)$ tensor field:

$$\langle -, - \rangle \in H^0(M; E^* \otimes E^*)$$

of positive-definite Hermitian structures on the fibres of E. That is to say, for any smooth sections, $\sigma_1, \sigma_2, \sigma \in H^0(M; E)$, the inner-product $\langle \sigma_1, \sigma_2 \rangle$ is a $\mathbb{C}$-valued smooth function that is complex linear in its first argument, complex anti-linear in its second argument, and satisfies:

$$\overline{\langle \sigma_1, \sigma_2 \rangle} = \langle \sigma_2, \sigma_1 \rangle, \quad \text{and} \quad \langle \sigma, \sigma \rangle > 0, \quad \text{if} \quad \sigma \neq 0.$$

One may extend the familiar notions of *duals*, *tensor products*, *exterior products*, etc., to orbifold vector bundles too by forming these constructions over each orbifold chart and enforcing suitable compatibility criteria on their overlaps.

An **orbifold section** of an orbifold fibre bundle $\pi : E \to M$ is defined by a Γ-invariant section on the orbifold charts $(\tilde{U}, \Gamma, \phi)$ of M, that agrees on overlaps. If E is a complex orbifold vector bundle, then an **orbifold connection** on E is a differential operator:

$$\nabla : H^0(M; E) \longrightarrow H^0(M; T^*M \otimes E)$$

that satisfies the condition:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \qquad \text{for all } f \in C^\infty(M),\ \sigma \in H^0(M; E).$$

Given a Hermitian orbifold vector bundle $\pi : E \to M$ and orbifold connection ∇ on E, we say that ∇ is a **Hermitian connection** if, for any vector field $v \in H^0(M; TM)$ on M, we have that:

$$v\langle\sigma_1, \sigma_2\rangle = \langle\nabla_v\sigma_1,\ \sigma_2\rangle + \langle\sigma_1,\ \nabla_v\sigma_2\rangle, \qquad \text{for all } \sigma_1, \sigma_2 \in H^0(M; E).$$

A.5 Orbifold Characteristic Classes

If $\pi : E \to M$ is a Hermitian orbifold vector bundle with Hermitian connection ∇, then one may define the orbifold versions of the **curvature**, **characteristic classes**, and so on. If $F \in \Omega^2(M)$ is the curvature two-form with respect to ∇, then $\mathcal{R}(E) = (\sqrt{-1}/2\pi)F$ is a real-valued closed two-form on M. The **first Chern class** of E is then the cohomology class:

$$c_1(E) := [\mathrm{Tr}\, \mathcal{R}(E)].$$

Similarly, the **Chern character** of E is:

$$\mathrm{Ch}(E) = \left[\mathrm{Tr}\, e^{\mathcal{R}(E)}\right].$$

There also is an orbifold version of the splitting principle from Theorem 4.9, so a complex orbifold vector bundle E with $\dim_{\mathbb{C}} E = n$, decomposes as the formal direct sum:

$$E = V_1 \oplus V_n,$$

where $V_j \cong \mathbb{C}$ for $j = 1, \ldots, n$. Then the **orbifold Todd class** of E is given by:

$$\mathrm{Td}(E) = \prod_{j=1}^{n} \frac{c_1(V_j)}{\left[1 - e^{-c_1(V_j)}\right]}.$$

Given a point p of M and a structure chart $(\tilde{U}, \Gamma_p, \phi)$ for p, the **orbifold tangent space** to p is the quotient of the tangent space to $\tilde{p} := \phi^{-1}(p)$ in $\tilde{U}$ by the induced action of Γ:

$$T_pM := T_{\tilde{p}}\tilde{U}/\Gamma_p.$$

Taking the union of the orbifold tangent spaces as p ranges through M, along with the transition functions defined by the compatibility criteria, allows us to construct the **orbifold tangent bundle** $\pi : TM \to M$ of M. A **vector field** on M is a section v of the orbifold tangent bundle TM, so $v \in H^0(M; TM)$. A similar argument can be applied to define **orbifold differential forms** as sections of the exterior algebra of the orbifold cotangent bundle. A **Riemannian orbifold** is an orbifold equipped with a Riemannian metric on its orbifold tangent bundle. An **almost-complex orbifold** is an orbifold with an almost-complex structure on its tangent bundle. The **Todd class** of an almost-complex orbifold is just the Todd class of its orbifold tangent bundle. One may successively progress this way to define the orbifold analogues to de Rham theory and Dolbeault theory.

An orbifold M is **orientable** if we can assign an orientation to the subset $\tilde{U}$ of each orbifold chart $(\tilde{U}, \Gamma, \phi)$, and which agrees on their overlaps. If M is an n-dimensional orientable orbifold and if $\omega \in \Omega^n(M)$ is a differential top form of compact support on an open and connected set $\tilde{U}$, trivialised by an orbifold chart $(\tilde{U}, \Gamma, \phi)$, then the **integral** of ω is:

$$\int_M \omega = \frac{m_U}{|\Gamma|} \int_{\tilde{U}} \tilde{\omega},$$

where m_U is the multiplicity of the connected component of M containing U, and $\tilde{\omega}$ is the Γ-invariant form on U that represents ω. The integral of an arbitrary top-degree form on M is then defined via partitions of unity.

A.6 Connections on Line Bundles

Let $\mathcal{L} \to M$ be a holomorphic orbifold line bundle over M, and denote $\mathcal{L}^* := \mathcal{L} - \{\text{zero section}\}$. Given a connection ∇ on $\mathcal{L}$, there exists a unique one-form $\theta \in \Omega^1(\mathcal{L}^*)$ such that:

- θ is invariant under the $\mathbb{C}^*$-action;
- for any $p \in M$, we have that $\theta|_{\mathcal{L}_p^*} = \alpha_p$, where θ_p is the unique one-form on $\mathcal{L}_p^*$ such that $s^*(\alpha_p) = dz/z$ for any map $s : \mathbb{C}^* \to \mathcal{L}_p^*$;
- given a local section $\sigma : U \to \mathcal{L}^*|_U$ for each open subset $U \subseteq M$, we have that $\nabla\sigma/\sigma = s^*\theta$.

The one-form $\theta \in \Omega^1(\mathcal{L}^*)$ is called the **connection one-form** of $(\mathcal{L}, \nabla)$. Given a vector field $\underline{X}_M$ on M, there exists a unique horizontal vector field $\underline{X}_{\mathcal{L}}$ on $\mathcal{L}$ such that $\pi_*\underline{X}_{\mathcal{L}} = \underline{X}_M$, called the **horizontal lift** of $\underline{X}_M$ by ∇, and a **horizontal section** is a section $\sigma : U \to M$ such that $\nabla\sigma = 0$.

The exterior derivative $d\theta \in \Omega^2(\mathcal{L}^*)$ of the connection one-form θ is a $\mathbb{C}^*$-invariant horizontal two-form. Hence there exists a unique closed two-form $\Theta \in \Omega^2(M)$ on M, called the **curvature two-form** of $(\mathcal{L}, \nabla)$, such that $\pi^*\Theta = d\theta$, where $\pi : \mathcal{L}^* \to M$. Since any two connection one-forms on M differ by a one-form, the cohomology class $[\Theta]$ of Θ is independent of the choice of connection ∇ on $\mathcal{L}$.

Lemma A.13. *When* ∇ *is a Hermitian connection on* $\mathcal{L}$ *and* $\langle \sigma,\, \sigma\rangle = 1$*, then* ∇ *satisfies:*

$$\sigma^*\theta + \overline{\sigma^*\theta} = 0.$$

Proof. For any $X \in TM$:

$$0 = X\langle \sigma,\, \sigma\rangle = \langle \imath_X \nabla\sigma,\, \sigma\rangle + \langle \sigma,\, \imath_X \nabla\sigma\rangle = \imath_X \left(\langle \nabla\sigma,\, \sigma\rangle + \overline{\langle \sigma,\, \nabla\sigma\rangle}\right) = \imath_X \left(\sigma^*\theta + \overline{\sigma^*\theta}\right).$$

□

We therefore have $\Theta + \overline{\Theta} = 0$, hence $(\sqrt{-1}/2\pi)\Theta$ is a real-valued integral closed two-form on M, and hence $\omega := (\sqrt{-1}/2\pi)\Theta$ is a **Chern form** for $\mathcal{L}$, and the cohomology class that it represents is the **Chern class** $c_1(\mathcal{L}) = [(\sqrt{-1}/2\pi)\Theta]$ of $\mathcal{L}$.

A Hermitian orbifold line bundle $\mathcal{L}$ with a Hermitian connection ∇ is equivalent to a orbifold principal U_1-fibre bundle $P \to M$ with connection one-form $\theta \in \Omega^1(P)$, such that $\mathcal{L} = P \times_{U_1} \mathbb{C}$, such that the connection ∇ on $\mathcal{L}$ is induced by a connection one-form on P. The corresponding connection one-form θ on P satisfies $\theta(\partial/\partial\phi) = 1$, where $\partial/\partial\phi$ is the vector field generating the principal U_1-action on P.

A.7 Symplectic Orbifolds

A **symplectic orbifold** is an orbifold M equipped with a closed non-degenerate two-form $\omega \in \Omega^2(M)$. An orbifold almost-complex structure J on M is **compatible** with ω if, for every $p \in M$, the bilinear form:

$$g_p(v_1,\, v_2) := \omega_p(J_p v,\, w), \qquad \text{where } v_1, v_2 \in T_pM,$$

is symmetric and positive-definite. A group G acts **symplectically** on M if the G-action preserves ω. A **moment map** for a symplectic G-action is a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that:

$$\imath_X\omega = d\mu^X, \qquad \text{for all } X \in \mathfrak{g}.$$

If a moment map exists for a G-action on M, then we say that the action is **Hamiltonian**.

For a symplectic action of a *connected* Lie group G on a symplectic orbifold (M, ω), the fixed-point locus M^G is a suborbifold of M. For a fixed point $p \in M^G$ with structure chart $(\tilde{U}, \Gamma_p, \phi)$, then there is a local action of G on $\tilde{U}$. If G is furthermore *compact*, then this local action gives rise to an action of some finite cover $\tilde{G}$ of the identity component G^0 of G, commuting with the action of Γ_p. The group $\tilde{G}$ is an extension of G of degree no larger than the order of Γ_p, [Dui11, Proposition 15.4].

The $\tilde{G}$-action induces a linear representation of $\tilde{G}$ on $T_{\phi^{-1}(p)}U$ via its derivative, with weight $\alpha_{p,j} \in \mathfrak{g}^*$ for each $j = 1, \ldots, n$. The $\alpha_{p,j}$ are called the **isotropy weights** of the G-action on the fixed point p. Note that, unlike in the manifold case, only each $|\Gamma_p| \cdot \alpha_{p,j}$ needs to lie in the integral weight lattice $\mathfrak{g}^*_{\mathbb{Z}}$ of G, whereas the weights $\alpha_{p,j}$ themselves may be rational. The weights $\alpha_{p,j}$ are well-defined since they are independent of the choice of orbifold chart, and of the choice of compatible almost-complex structure.

A.8 Equivariant Pre-Quantisation

When $(\mathcal{L}, \nabla)$ is a Hermitian orbifold line bundle over M with Hermitian connection ∇, let θ be the corresponding connection one-form and Θ the curvature two-form. Suppose that an n-dimensional torus T acts on M, and that this T-action lifts to $\mathcal{L}$. We can assume that θ is a T-invariant one-form, since we can take its average over M if necessary, so that ∇ is a T-**invariant connection**.

Denote by $\underline{X}_{\mathcal{L}}$ the vector field on $\mathcal{L}^*$ that is generated by an element $X \in \mathfrak{t}$. Then $\theta(\underline{X})$ is constant along the fibres, and we can therefore define a map $\mu : M \to \mathfrak{t}^*$ by:

$$\pi^* \langle \mu, X \rangle = (\sqrt{-1}/2\pi)\theta(\underline{X}),$$

and μ is T-invariant since θ is. We then have that:

$$\langle d\mu, X \rangle = (\sqrt{-1}/2\pi)\imath_{\underline{X}_M}\Theta.$$

Hence μ is a moment map for the T-action on M with respect to $(\sqrt{-1}/2\pi)\Theta$.

The vector field $X_{\mathcal{L}} - \underline{X}_M$ is a vertical vector field on $\mathcal{L}$, where $\underline{X}_M$ is the horizontal lift of the vector field on M that is generated by $X \in \mathfrak{t}$. For some value $\alpha \in \mathfrak{t}^*$, on the level-set $\mu^{-1}(\alpha)$ we have:

$$X_{\mathcal{L}} = \underline{X}_M - 2\pi\sqrt{-1}\langle \alpha, X \rangle (\partial/\partial\phi),$$

where $\partial/\partial\phi$ is the vector field generating the principal U_1-action on P from A.6.

If $p \in M^T$ is a fixed point, then $\underline{X}_M = 0$, the vector field $X_{\mathcal{L}}$ is vertical for every $X \in \mathfrak{t}$, and the fibre $\mathcal{L}_p$ is a linear orbifold representation of T given by some character $\chi : T \to U_1$, where $e^X \mapsto e^{\langle \alpha_p, X \rangle}$ for every $X \in \mathfrak{t}$ and a fixed rational weight $\alpha_p \in \mathbb{Q}$. Then we have:

$$X_{\mathcal{L}}(\zeta) = \langle \alpha_p, X \rangle (\partial/\partial\phi),$$

where $\zeta \in \mathcal{L}_p$ is the fibre coordinate.

Proof. The Lie algebra representation $(d\chi)_e : \mathfrak{t} \to \mathbb{R}$ is given by $\xi \mapsto \langle \alpha_p, \xi \rangle$, so we have:

$$X_{\mathcal{L}}(\zeta) = \left(\frac{d}{dt}\chi(e^{tX})(\zeta) \right)\Big|_{t=0} = \chi'(\zeta) \left(\frac{\partial}{\partial\phi} \right) = \langle \alpha_p, X \rangle \left(\frac{\partial}{\partial\phi} \right).$$

□

Hence:

$$\langle \mu(p),\, X \rangle = \left(\frac{\sqrt{-1}}{2\pi}\right) \theta \left(\langle \alpha_p,\, X \rangle \left(\frac{\partial}{\partial \phi}\right)\right) = \left(\frac{\sqrt{-1}}{2\pi}\right) \langle \alpha_p,\, X \rangle,$$

and so:

$$\alpha_p = (2\pi/\sqrt{-1})\mu(p).$$

A.9 Inertia Orbifolds

Definition A.14. Given an orbifold M, the **inertia**, or the **associated**, **orbifold** $\widehat{M}$ to M is defined using the orbifold charts $(\tilde{\mathcal{V}}, \Gamma, \Psi)$ that are defined as follows: for each orbifold chart $(\tilde{U}, \Gamma, \phi)$ for M, define:

$$\tilde{V} := \left\{ (u, \gamma) \in \tilde{U} \times \Gamma \;\middle|\; \gamma \cdot u = u \right\}, \tag{A.1}$$

and let Γ act on each subset $\tilde{V}$ by:

$$g \cdot (u, \gamma) := (g \cdot u, g^{-1}\gamma g), \qquad \text{for all } (u, \gamma) \in \tilde{U} \times \Gamma,\ g \in \Gamma.$$

Lastly, we set:

$$V := \tilde{\mathcal{V}}/\Gamma.$$

The orbifold charts $(\tilde{\mathcal{V}}, \Gamma, \Psi)$ inherit the compatibility criteria from the orbifold charts $(\tilde{U}, \Gamma, \phi)$ for M. In general, the inertia orbifold $\widehat{M}$ has several connected components of varying dimension which can be described as follows: recall that for any point $p \in M$ there exists a structure chart $(\tilde{U}_p, \Gamma_p, \phi)$ with $p \in U_p := \phi(\tilde{U}_p)$. If $q \in U_p$ then, up to conjugation, there exists an injective homomorphism $\Gamma_q \hookrightarrow \Gamma_p$. Also for any $\gamma \in \Gamma_q$, the conjugacy class $(\gamma)_{\Gamma_p} \in \mathrm{Conj}(\Gamma_p)$ is well-defined and lets us define an equivalence relation $(\gamma)_{\Gamma_q} \sim (\gamma)_{\Gamma_p}$. We shall use (γ) to denote the equivalence class that $(\gamma)_{\Gamma_q}$ belongs to, and $\Gamma/\sim$ to refer to the set of all equivalence classes in Γ. Then underlying topological set $|\widehat{M}|$ is given by the disjoint union of connected components:

$$|\widehat{M}| = \bigsqcup_{(\gamma)\in\Gamma/\sim} M_{(\gamma)},$$

where:

$$M_{(\gamma)} := \left\{ \left(p, (\gamma)_{\Gamma_p}\right) \in M \times \mathrm{Conj}(\Gamma_p) \;\middle|\; \gamma \in \Gamma_p,\ (\gamma)_{\Gamma_p} \in (\gamma) \right\}.$$

Definition A.15. The component $M_{(e)} = |M|$ is called the **non-twisted sector** of M, whereas we call its complement suborbifold $M_{(\gamma)}$ with $\gamma \neq e$ the **twisted sector** of M.

An intuitive geometric description of an inertia orbifold $\widehat{M}$ is that they provide a method of parametrising the points p of an orbifold M along with their automorphisms in the form of the isotropy groups Γ_p. The non-twisted sector $M_{(e)}$ is just the original orbifold M, whereas the twisted sector $M_{(\gamma)}$ consists of the points $p \in M$ and their non-trivial structure groups Γ_p.

On each orbifold chart $(\tilde{V}, \Gamma, \Psi)$ of $\widehat{M}$ associated to the orbifold chart $(\tilde{U}, \Gamma, \phi)$ of M, there exists a Γ-equivariant immersion $\tilde{V} \to \tilde{U}$. By considering each orbifold chart for $\widehat{M}$ and for M, we may glue them together to give a Γ-equivariant immersion, $\rho : \widehat{M} \to M$. Let $\nu_{\widehat{M}} \to \widehat{M}$ denote the normal bundle to $\widehat{M}$ in TM that is induced by this immersion ρ. Locally, $\nu_{\widehat{M}}$ is obtained from each normal bundle $\nu_{\tilde{V}} \to \tilde{V}$ to the immersion $\tilde{V} \to \tilde{U}$ over the orbifold chart $(\tilde{V}, \Gamma, \Psi)$, and then by dividing out the Γ-action.

A.10 Canonical Automorphisms

For any orbifold vector bundle $\hat{\pi}_E : \widehat{E} \to \widehat{M}$ over the inertia orbifold $\widehat{M}$, there exists a **canonical automorphism**, $A(\widehat{E}) \in \mathrm{Aut}\, \widehat{E}$, which can be described as follows: given an orbifold chart $(\tilde{U}, \Gamma, \phi)$ of M, let $(\tilde{V}, \Gamma, \Psi)$ be the the associated chart of $\widehat{M}$. If $(p, \gamma) \in \tilde{V}$, then γ acts on this point as:

$$\gamma \cdot (p, \gamma) = (\gamma \cdot p, \mathrm{Ad}_\gamma(\gamma)) = (p, \gamma), \tag{A.2}$$

and therefore (p, γ) is fixed by γ. However, we may lift the action of γ on (p, γ) up to the fibre $\hat{\pi}^{-1}(p, \gamma) = \widehat{E}_{(p,\gamma)} \to \tilde{V}$ of the local Γ-equivariant orbifold vector bundle $\widehat{E}_{\tilde{V}} \to \tilde{V}$. This lifting defines an automorphism $A_{\tilde{V}} \in \mathrm{Aut}(\widehat{E}_{\tilde{V}})$. Gluing the automorphisms $A_{\tilde{V}}$ together over the orbifold charts $(\tilde{V}, \Gamma, \Psi)$ then gives rise to a canonical section A of the automorphism bundle $\mathrm{Aut}(\widehat{E})$ of $\widehat{E}$.

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