# Stochastic partial differential and integro-differential equations

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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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*To the memory of my beloved grandmother,*

*Françoise*

# Abstract

In this work we present some new results concerning stochastic partial differential and integro-differential equations (SPDEs and SPIDEs) that appear in non-linear filtering. We prove existence and uniqueness of solutions of SPIDEs, we give a comparison principle and we suggest an approximation scheme for the non-local integral operators. Regarding SPDEs, we use techniques motivated by the work of De Giorgi, Nash, and Moser, in order to derive global and local supremum estimates, and a weak Harnack inequality.

# Contents

# Chapter 1

# Introduction

## 1.1 Motivation

The present thesis is focused on qualitative properties of solutions of stochastic partial differential equations (SPDEs), and stochastic partial integro-differential equations (SPIDEs), as well as on developing numerical methods for solving them.

The theory of SPDEs finds applications in many scientific fields, such as physics, biology, chemistry and finance. The main motivation for studying this type of equations is the filtering of partially observable processes. There, under consideration is a system whose evolution is modeled by a stochastic differential equation (signal process). One has access only to a noisy partial observation of the signal (observation process), and the classic problem is to estimate the probability density function of the signal at time $t$ by using the information obtained from the observation process up to time $t$. It turns out that the distribution of the best estimate (in mean square sense) of the signal at $t$ given the observation until $t$ satisfies a nonlinear SPDE, called Kushner-Shiryaev equation, which can be transformed to a linear SPDE, called the Zakai equation. The Zakai and the Kushner-Shiryaev equations have been extensively studied in the past decades in the case when the signal and observation are diffusion processes. In applications, however, one should often deal with more general signal and observation models where jumps in the signal or/and in the observation jumps may occur. This motivates recent interest in models with Itô-Lévy processes. In this case, the corresponding Zakai equation is a linear SPIDE.

## 1.2 Outline of the results and structure of the thesis

The results of this thesis are organized in two chapters, Chapter 2 and 3. Each chapter is divided into sections. Each section is then divided into subsections. The main results of each section are stated in the first subsection and their proofs are given in the last one. In the intermediate subsections, technical lemmas (which are of their own interest) are proved, in order to be used for the proofs of the main results.

In more detail:

In Chapter 2, we deal with two types of stochastic partial integro-differential equations driven by Lévy noise. In Section 2.1 we prove existence and uniqueness of solutions in $L_2$-spaces for these two classes of equations. We also prove an Itô formula for the square of the $L_2$-norm of the positive part, which is then applied in order to obtain a comparison principle. In Section 2.2 we suggest a discretization scheme (in space) for the non-local integral operators, which is combined with a finite difference scheme for the differential operators, to obtain that the rate of convergence of the scheme to the solution is one.

In Chapter 3 we turn our attention to stochastic partial differential equations driven by Wiener processes. Although the notion of SPDEs is less general than SPIDEs, we have decided to present our results concerning SPDEs in Chapter 3, since they seem to be more interesting from analytical point of view, and their proofs are more technical. The results in Chapter 3 rely on the De-Giorgi-Nash-Moser theory of elliptic and parabolic PDEs. In Section 3.1 we present global $L_\infty$-estimates for the solution of the Cauchy problem with zero boundary condition. We apply then these estimates in Section 3.2 in order to construct solutions for a certain class of semilinear SPDEs in Section 3.2. In Section 3.3 we obtain local $L_\infty$-estimates for convex functions of solutions of equations with no boundary conditions and then we use these estimates to obtain a weak Harnack inequality for solutions of SPDEs.

## 1.3 Notation and useful lemmas

Let us introduce some basic notation that will be used through the rest of this thesis. Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$, such that $\mathscr{F}_0$ contains all $P$-zero sets. We consider a $\sigma$-finite measure space $(Z, \mathcal{Z}, \nu)$. Let $N(dt, dz)$ be an $\mathscr{F}_t-$ Poisson random measure on $[0,\infty) \times Z$. We assume that its compensator is $dt\nu(dz)$ and we use the notation

$$\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz).$$

We also consider a sequence of independent real valued $\mathscr{F}_t$-Wiener processes $\{w_t^k\}_{k=1}^\infty$. The notation $\mathscr{P}$ is used for the predictable $\sigma$-algebra on $\Omega \times [0, T]$.

A process $\xi = (\xi_t)_{t\in[0,T]}$ with values in a topological space $X$ will be called *càdlàg* if with probability one the trajectories of $\xi$ are continuous from the right in $t \in [0, T)$

and have limits from the left at every $t \in (0,T]$ in the topology of $X$. If a process $\xi$ takes values in a Banach space $Y$, it will be called *strongly càdlàg* if it is càdlàg with respect to the norm topology in $Y$. For a càdlàg process $\xi = (\xi_t)_{t\in[0,T]}$ we will write $\xi_{t-} = \lim_{s\uparrow t} \xi_s$, for $t \in (0,T]$ and $\xi_{t-} = \xi_0$ for $t = 0$. Also when we deal with (stochastic) integrals that are càdlàg in the time variable $t$, with the notation $\int_0^t \cdot$, we mean $\int_{(0,t]} \cdot$.

If $X$ is a topological space then $\mathscr{B}(X)$ is the Borel $\sigma$-algebra on $X$. If $X$ is a normed linear space then $|x|_X$ denotes the norm of $x \in X$, $X^*$ is the dual of $X$, and $\langle x^*, x\rangle$ denotes the action of $x^* \in X^*$ on $x \in X$. If $A$ is a set, then $I_A$ will denote the indicator function of $A$. The notation $Q$ stands for the whole space $\mathbb{R}^d$, for an integer $d \geq 1$, or for a bounded Lipschitz domain in $\mathbb{R}^d$. We write

$$\partial_i u := \frac{\partial u}{\partial x_i},\ \partial_{ij} u := \frac{\partial^2 u}{\partial x_i \partial x_j},\ \text{for } i, j = 1, ..., d,$$

for the first and second order partial derivatives of a function $u$ defined on $Q$. Let us also denote by $\partial_0$ the identity operator and $\partial_{-i} = -\partial_i$ for $i = 1, ..., d$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ of order $|\alpha| := \alpha_1 + ... + \alpha_d$, we write

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_d} x_d}.$$

We write $C_c^\infty(Q)$ for the set of all smooth functions that are compactly supported in $Q$. As usual, for an integer $m \geq 0$ and a real $p \geq 1$, we denote by $W_p^m(Q)$ the space of functions $u \in L_p(Q)$, whose generalized derivatives up to order $m$ lie in $L_p(Q)$. We set $H^m(Q) := W_2^m(Q)$ and we write $H_0^1(Q)$ for the closure of $C_c^\infty(Q)$ in $H^1(Q)$ under the norm

$$|u|_{H^1} = \Big(\sum_{i=1}^d |\partial_i u|_{L_2}^2 + |u|_{L_2}^2\Big)^{1/2}.$$

The inner product in $L_2(Q)$ will be denoted by $(\cdot,\cdot)$. We will use the notation $H^{-1}(Q)$ for the dual of $H_0^1(Q)$. The space of all square-summable sequences will be denoted by $\ell_2$.

Lest us now introduce the concept of a Gel'fand triple. Let $(H, (\cdot,\cdot)_H)$ be a separable Hilbert space and let $(V, |\cdot|_V)$ be a separable reflexive Banach space such that $V$ is continuously and densely embedded in $H$. By identifying $H$ with its dual $H^*$ by the help of the inner product $(\cdot,\cdot)_H$ in $H$, we get

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where $H^* \hookrightarrow V^*$ is the adjoint embedding of $V \hookrightarrow H$. It also follows that the embedding $H^* \hookrightarrow V^*$ is continuous and dense. Notice that $\langle v^*, v\rangle = (v^*, v)_H$ when $v^* \in H$. The triad $(V, H, V^*)$ will be called a Gel'fand triple.

Finally, we note that unless otherwise indicated, the summation convention is used with respect to repeated integer-valued indices throughout this thesis.

## 1.4 Stochastic Evolution Equations

The SPDEs and SPIDEs under consideration, will be understood as *stochastic evolution equations* on the Gel'fand triple $(H_0^1(Q), L_2(Q), H^{-1}(Q))$. A variational approach for stochastic evolution equations driven by continuous martingales where introduced in [43], and by semimartingales, in [22]. The ones that we will present here are a particular case of [22]. Let $(V, H, V^*)$ be a Gel'fand triple. Let us also consider the mappings $\mathscr{A} : \Omega \times [0,T] \times V \to V^*$, $\mathscr{B}^k : \Omega \times [0,T] \times V \to H$, for $k \in \mathbb{N}_+$, and $\mathscr{C}$ : $\Omega \times [0,T] \times V \to L_2(Z, \nu; H)$ that are $\mathscr{P} \times \mathscr{B}(V)$-measurable, and an $\mathscr{F}_0$−measurable random variable $\psi$ with values in $H$. We pose the following conditions:

**Assumption 1.4.1.** There exist constants $L \geq 0$, $\gamma > 0$, and a predictable process $g = (g_t)_{t\in[0,T]}$, such that

I) "Semicontinuity of $\mathscr{A}$": For any $v$, $v_1$, $v_2 \in V$, and $(\omega, t) \in \Omega \times [0,T]$, the function $\langle v, \mathscr{A}_t(v_1 + \lambda v_2)\rangle$ is continuous in $\lambda$ on $\mathbb{R}$ ,

II) "Monotonicity ": For any $v_1$, $v_2 \in V$, and $(\omega, t) \in \Omega \times [0,T]$,

$$2\langle \mathscr{A}_t(v_1) - \mathscr{A}_t(v_2), v_1 - v_2\rangle + \sum_{k=1}^{\infty} |\mathscr{B}_t^k(v_1) - \mathscr{B}_t^k(v_2)|_H^2 \\ + \int_Z |\mathscr{C}_t(z, v_1) - \mathscr{C}_t(z, v_2)|_H^2 \nu(dz) \leq L|v_1 - v_2|_H^2 ,$$

III) "Coercivity ": For any $v \in V$, and $(\omega, t) \in \Omega \times [0,T]$,

$$2\langle \mathscr{A}_t(v), v\rangle + \sum_{k=1}^{\infty} |\mathscr{B}_t^k(v)|_H^2 + \int_Z |\mathscr{C}_t(z, v)|_H^2 \nu(dz) \leq g_t + L|v|_H^2 - \gamma|v|_V^2 ,$$

IV) " Restriction of growth on $\mathscr{A}$" : For any $v \in V$, and $(\omega, t) \in \Omega \times [0,T]$,

$$|\mathscr{A}_t(v)|_{V_*}^2 \leq g_t + L|v|_V^2 \ .$$

V) $E|\psi|_H^2 + E\int_0^T g_t dt < \infty$.

Let us consider now the problem

$$du_t = \mathscr{A}_t(u_t)dt + \mathscr{B}^k_t(u_t)dw^k_t + \int_Z \mathscr{C}_t(z, u_{t-})\tilde{N}(dt, dz) \tag{1.4.1}$$

$$u_0 = \psi \tag{1.4.2}$$

We will say that $u$ is a solution of (1.4.1)-(1.4.2) if

- $u = (u_t)_{t\in[0,T]}$ is an adapted, strongly càdlàg $H$-valued process,
- $u_t \in V$ for $dP \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, and $E\int_0^T |u_t|_V^2 dt < \infty$,
- for each $\phi \in V$, almost surely

$$(u_t, \phi) = \psi + \int_0^t \langle \mathscr{A}_t(u_s), \phi\rangle ds + \int_0^t (\mathscr{B}^k_t(u_s), \phi) dw^k_s + \int_0^t \int_Z (\mathscr{C}_t(z, u_{s-}), \phi)\tilde{N}(ds, dz),$$

for any $t \in [0, T]$.

Two solutions $u$ and $v$ will be considered identical if with probability one, $u_t = v_t$ for all $t \in [0, T]$ (as elements of $H$).

The following theorem is a simple consequence of Theorems 2.9 and 2.10 from [22].

**Theorem 1.4.1.** *Under Assumption 1.4.1, there exists a unique solution of* (1.4.1)-(1.4.2). *Moreover, the following estimate holds:*

$$E \sup_{t\in[0,T]} |u_t|_H^2 + E\int_0^T |u_t|_V^2 dt \le C\left(E|\psi|_H^2 + E\int_0^T g_t dt\right),$$

*where $C$ is a constant depending only on $L$ and $\gamma$.*

# Chapter 2

# Stochastic partial integro-differential equations

As already mentioned, stochastic partial integro-differential equations play an important role in non-linear filtering of jump-diffusion processes. For more information on the subject, and in particular, derivation of the Zakai equation, we refer the reader to [20] and [21]. In this chapter, we introduce the notion of SPIDE, we prove existence and uniqueness of the solutions, we derive a comparison principle, and we give a numerical approximation scheme.

## 2.1 Solvability and Comparison Principle

Our goal in this section is to prove existence and uniqueness of solutions, as well as comparison principles, for stochastic partial integro-differential equations driven by Lévy processes. The existence and uniqueness of solutions, is a simple consequences of the results concerning general stochastic evolutions equations driven by semi-martingales that exist in [22]. For the comparison principles, we need to obtain an Itô formula for the square of the $L_2$-norm of the positive part of (possibly) discontinuous semimartingales with values in $H^{-1}$ that have $dP \otimes dt$-versions in $H_0^1$. Our formula extends an Itô formula from [44] proved for continuous semimartingales.

Comparison principles are powerful tools and play important role in PDE theory. Comparison theorems for SPDEs are known in various generalities in the literature. To the best of our knowledge, the first results on comparison of solutions of SPDEs appear in [38] and [15]. Recent results appear in [44], [13], [11] and [12]. In [11] and [13] quasi linear SPDEs, and in [12] quasi-linear SPDEs with obstacle are studied. In the above publications, the equations under consideration are driven by

Wiener processes, or cylindrical Wiener processes, and only differential operators are present. Here, in Theorems 2.1.2 and 2.1.4, we present comparison theorems for two classes of quasilinear SPIDEs, linear versions of which, arise in non-linear filtering.

The results in this section are from [7], a joint work with István Gyöngy.

### 2.1.1 Existence, uniqueness and Comparison Theorems

In this section we will be dealing with two types of equations. In order to introduce these equations, together with the space $(Z,\mathscr{Z})$ and the random measure $N(dt,dz)$ of Chapter 1, let us consider another measurable space $(F,\mathfrak{F})$, an $\mathscr{F}_t$−Poisson random measure $M(dt,d\zeta)$ on $[0,\infty)\times F$ and two $\sigma$−finite measures $\pi^{(1)}$, $\pi^{(2)}$ on $F$. We assume that the compensator of $M(dt,d\zeta)$ is $dt\pi^{(2)}(d\zeta)$ and we write

$$\tilde{M}(dt,d\zeta)=M(dt,d\zeta)-dt\pi^{(2)}(d\zeta).$$

First we consider the equation

$$\begin{aligned}du_t(x)=&\Big(L_tu_t(x)+f_t(x,u_t(x),\nabla u_t(x))+\partial_i f_t^i(x)\Big)\,dt\\&+G_t^k(u)(x)dw_t^k+\int_Z g_t(x,z,u_{t-}(x))\tilde{N}(dt,dz),\end{aligned}\tag{2.1.1}$$

for $(t,x)\in[0,T]\times Q$, with initial condition

$$u_0(x)=\psi(x),\quad x\in Q,\tag{2.1.2}$$

where

$$\begin{aligned}L_tu(x)&=\partial_j(a_t^{ij}(x)\partial_iu(x))+\mathscr{I}_t^{(1)}u_t(x),\\\mathscr{I}_t^{(1)}u(x)&=\int_F[u(x+c_t(x,\zeta))-u(x)-c_t(x,\zeta)\cdot\nabla u(x)]m_t(x,\zeta)\pi^{(1)}(d\zeta),\\G_t^k(u)(x)&=\phi_t^{ik}(x)\partial_iu(x)+\sigma_t^k(x,u(x)).\end{aligned}$$

In the formulas above as well as in (2.1.1), the summation takes place over $i,j\in\{1,...,d\}$ and $k\in\mathbb{N}$. We make the following assumptions. Let $K>0$ denote a constant.

**Assumption 2.1.1.**

i) The coefficients $a^{ij}$, are real-valued $\mathscr{P}\otimes\mathscr{B}(Q)$ measurable functions on $\Omega\times[0,T]\times Q$ and are bounded by $K$ for every $i,j=1,...,d$. The coefficient $\phi^i=(\phi^{ik})_{k=1}^{\infty}$

is an $l_2$-valued $\mathscr{P}\otimes\mathscr{B}(Q)$-measurable function on $\Omega\times[0,T]\times Q$ for every $i=1,2,...,d$, such that

$$\sum_i\sum_k|\phi_t^{ik}(x)|^2\le K \quad \text{for all } \omega,\ t \text{ and } x.$$

ii) $f$ is a real valued $\mathscr{P}\otimes\mathscr{B}(Q)\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)$ -measurable function on $\Omega\times[0,T]\times Q\times\mathbb{R}\times\mathbb{R}^d$, and $\sigma=(\sigma^k)_{k=1}^{\infty}$ is a $\mathscr{P}\otimes\mathscr{B}(Q)\otimes\mathscr{B}(\mathbb{R})$-measurable function on $\Omega\times[0,T]\times Q\times\mathbb{R}$, with values in $l_2$. The function $g$ is defined on $\Omega\times[0,T]\times Q\times Z\times\mathbb{R}$ with values in $\mathbb{R}$ and it is $\mathscr{P}\otimes\mathscr{B}(Q)\otimes\mathscr{Z}\otimes\mathscr{B}(\mathbb{R})$-measurable. We assume that there exists a predictable process $\bar{h}_t$ with values in $L_2(Q)$, such that for all $\omega,t,x,z,r,r'$

$$|f_t(x,r,r')|^2+\sum_k|\sigma_t^k(x,r)|^2+\int_Z|g_t(x,z,r)|^2\nu(dz)$$

$$\le K|r|^2+K|r'|^2+|\bar{h}_t(x)|^2,$$

and

$$E\int_0^T|\bar{h}_t|^2_{L_2(Q)}dt<\infty.$$

iii) $\psi$ is an $\mathscr{F}_0$-measurable random variable in $L_2(Q)$ with $E|\psi|^2_{L_2}<\infty$.

iv) There exists a constant $\kappa>0$ such that for all $\omega,t,x$ and for all $\xi=(\xi_1,...\xi_d)\in\mathbb{R}^d$ we have

$$a_t^{ij}(x)\xi_i\xi_j-\frac{1}{2}\phi_t^{ik}(x)\phi_t^{jk}(x)\xi_i\xi_j\ge\kappa|\xi|^2.$$

v) For all $\omega,t,x,z,r_1,r_2$

$$\sum_k|\sigma_t^k(x,r_1)-\sigma_t^k(x,r_2)|^2\le K|r_1-r_2|^2.$$

vi) For all $i\in\{1,...,d\}$, $f^i$ are $L_2(Q)$-valued $\mathscr{P}$-measurable functions on $\Omega\times[0,T]$ such that

$$E\int_0^T\sum_{i=1}^d|f_t^i|^2_{L_2(Q)}\,dt<\infty\,.$$

We will refer to the constant $\kappa$ as the *parabolicity constant*. If $\kappa$, as in our case, is strictly positive, the corresponding equation will be called *non-degenerate*, while if it is zero, then the equation will be called *degenerate*. For solvability of degenerate SPIDEs we refer the reader to [49] and [4].

**Assumption 2.1.2.** The function $f_t(x,r,r')$ is continuous in $r$, for each $\omega,t,x$ and $r'$.

**Assumption 2.1.3.** For all $\omega,t,x,r_1,r_2,r_1',r_2'$

$$2(r_1-r_2)(f_t(x,r_1,r_1')-f_t(x,r_2,r_1'))$$

$$+\int_Z |g_t(x,z,r_1) - g_t(x,z,r_2)|^2 \nu(dz) \le K|r_1 - r_2|^2,$$

and

$$|f_t(x,r_1,r_1') - f_t(x,r_1,r_2')| \le K|r_1' - r_2'|.$$

**Assumption 2.1.4.** The function $r + g_t(x,z,r)$ is non-decreasing in $r$ for all $\omega, t, x, z$.

**Assumption 2.1.5.** The function $c$ maps $\Omega \times [0,T] \times \mathbb{R}^d \times F$ into $\mathbb{R}^d$, it is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathfrak{F}$-measurable, and there exists an $\mathfrak{F}$-measurable real function $\bar{c}$ on $F$ such that

(i) $|c_t(x,\zeta)| \le \bar{c}(\zeta)$, for all $\omega, t, x, \zeta$,

(ii) $\int_F \bar{c}^2(\zeta) \wedge \bar{c}(\zeta)\, \pi^{(1)}(d\zeta) \le K$,

(iii) $|c_t(x,\zeta) - c_t(y,\zeta)| \le \bar{c}(\zeta)|x-y|$, for all $\omega, t, x, y, \zeta$.

**Assumption 2.1.6.** The function $m$ maps $\Omega \times [0,T] \times \mathbb{R}^d \times F$ into $\mathbb{R}$, it is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathfrak{F}$-measurable, and we have

(i) $0 \le m_t(x,\zeta) \le K$, for all $\omega, t, x, \zeta$,

(ii) $|m_t(x,\zeta) - m_t(y,\zeta)| \le K|x-y|$, for all $\omega, t, x, y, \zeta$.

**Assumption 2.1.7.** The functions $c_t^l(x,\zeta)$, $l = 1,\dots,d$, are twice continuously differentiable in $x$, for each $\omega, t, \zeta$, and

(i) $|\partial_i c_t^l(x,\zeta)| \le K$, $|\partial_{ij} c_t^l(x,\zeta)| \le K$, for all $i, j, l = 1,\dots,d$,

(ii) $K^{-1} \le |det(\mathbb{I} + \theta \nabla c_t(x,\zeta))|$

for all $\omega, t, x, \zeta$ and $\theta \in [0,1]$, where $\mathbb{I}$ denotes the identity matrix.

*Remark* 2.1.1. Denote by $T_{\theta,t,\zeta}$ the mapping $x \mapsto x + \theta c_t(x,\zeta)$, for fixed $\omega, t, \theta$ and $\zeta$. By virtue of the inverse function theorem, it follows from (ii) of Assumption 2.1.7 that $T_{\theta,t,\zeta}$ is a $local$ diffeomorphism. In addition, by the first inequality in (i) and by (ii) of Assumption 2.1.7, there exists a constant $\gamma > 0$, such that the norm of the matrix $(\mathbb{I} + \theta \nabla c_t(x,\zeta))^{-1}$ is uniformly bounded by $\gamma$. Hence, by Hadamard's theorem (see, eg, Theorem 5.1.5 in [1]), $T_{\theta,t,\zeta}$ is a $global$ diffeomorphism, for fixed $\omega, t, \theta$ and $\zeta$. We denote by $J_{\theta,t,\zeta}$ the inverse of $T_{\theta,t,\zeta}$. Notice that for fixed $\omega, t, \theta, \zeta$ and for all $j = 1,\dots,d$, the functions $J^j_{\theta,t,\zeta}(x)$ are twice continuously in $x$, and their first and second order derivatives are uniformly bounded.

We want to give a meaning to equation (2.1.1) as a stochastic evolution equation on the Gel'fand triple $(H_0^1(Q), L_2(Q), H^{-1}(Q))$, where the embedding $H_0^1(Q) \hookrightarrow L_2(Q)$ is the identity.

To this end, let us see how the operator $\mathscr{I}_t^{(1)}$ acts on functions $u \in H_0^1(Q)$: Under Assumptions 2.1.5 through 2.1.7, $\mathscr{I}_t^{(1)}$ is a bounded linear operator from $H_0^1(Q)$ into $H^{-1}(Q)$ for fixed $(\omega, t)$, and for all $u, v \in H_0^1(Q)$ the process $\langle \mathscr{I}_t^{(1)} u, v\rangle$ is predictable. To see this, consider first the case $Q = \mathbb{R}^d$. For $u \in C_c^\infty(\mathbb{R}^d)$(even for $u \in W_2^2(\mathbb{R}^d)$) one can easily see that $\mathscr{I}_t^{(1)}u(x)$ is a function in $L_2(\mathbb{R}^d)$. For $\delta \in (0,1)$, let us also denote by $\mathscr{I}^{(1\delta)}$ and $\bar{\mathscr{I}}^{(1\delta)}$ the operators defined as $\mathscr{I}^{(1)}$ but with $F$ replaced by $F_\delta = \{\xi \in F : \bar{c}(\xi) < \delta\}$ and by $F_\delta^c = F \setminus F_\delta$ respectively. Then for $v \in C_c^\infty(\mathbb{R}^d)$ we have by Taylor's formula

$$
\begin{aligned}
(\mathscr{I}_t^{(1)}u, v) &= (\mathscr{I}_t^{(1\delta)}u, v) + (\bar{\mathscr{I}}_t^{(1\delta)}u, v) \\
&= \int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} \partial_{ki} u(T_{\theta,t,\zeta}) c_t^i(x,\zeta) c_t^k(x,\zeta) m_t(x,\zeta) v(x)\, dx \pi^{(1)}(d\zeta)\, d\theta \\
&+ \int_{F_\delta^c} [u(x + c_t(x,\zeta)) - u(x) - c_t(x,\zeta) \cdot \nabla u(x)] m_t(x,\zeta) v(x)\; dx\, \pi^{(1)}(d\zeta) \\
&= \int_0^1 (\theta - 1) \int_{F_\delta} \int_{\mathbb{R}^d} \partial_i u(x + \theta c_t(x,\zeta)) \partial_j (q_t^{ij}(x,\zeta,\theta) v(x))\, dx \pi^{(1)}(d\zeta)\, d\theta \\
&+ \int_{F_\delta^c} [u(x + c_t(x,\zeta)) - u(x) - c_t(x,\zeta) \cdot \nabla u(x)] m_t(x,\zeta) v(x)\; dx\, \pi^{(1)}(d\zeta),
\end{aligned}
\tag{2.1.3}
$$

where the last equality is obtained by integration by parts, and $q^{ij}$ is given by

$$
q_t^{ij}(x,\zeta,\theta) := \sum_{l=1}^d c_t^l(x,\zeta) c_t^i(x,\zeta) m_t(x,\zeta) \partial_l J_{\theta,t,\zeta}^j(T_{\theta,t,\zeta}(x)).
$$

Due to Assumptions 2.1.5 through 2.1.7 for a constant $N = N(d,K)$,

$$
(\mathscr{I}_t^{(1)}u, v) \le N |u|_{H^1(\mathbb{R}^d)} |v|_{H^1(\mathbb{R}^d)},
$$

which shows that $\mathscr{I}_t^{(1)}$ extends uniquely to a bounded linear operator from $H^1$ to $H^{-1}$, and the duality product $\langle \mathscr{I}_t^{(1)}u, v\rangle$ is given by the right-hand side of (2.1.3). In case $Q$ is a bounded Lipschitz domain, one can define the action of $\mathscr{I}_t^{(1)}u$ on $v \in H_0^1(Q)$ again by (2.1.3), where $u$ and $v$ this time are extended to zero outside of $Q$. For further study of these operators we refer to [17].

**Definition 2.1.1.** A strongly càdlàg (continuous if $\nu \equiv 0$) adapted process $u$ with values in $L_2(Q)$ is called a solution of the problem (2.1.1)-(2.1.2) if

i) $u_t \in H_0^1(Q)$ for $dP \otimes dt$ almost every $(\omega, t) \in \Omega \times [0,T]$,

ii) $E\int_0^T |u_t|^2_{H_0^1(Q)}\, dt < \infty$,

iii) for all $\varphi \in H_0^1(Q)$ we have almost surely

$$
\begin{aligned}
(u_t,\varphi) =&(\psi,\varphi) + \int_0^t \Big( (a_s^{ij}\partial_i u_s + f_t^j, \partial_{-j}\varphi) + (f_s(u_s,\nabla u_s),\varphi) + \langle \mathscr{I}_s^{(1)} u_s,\varphi\rangle \Big)\, ds \\
&+ \int_0^t (\phi_s^{ik}\partial_i u_s + \sigma_s^k(u_s),\varphi)\, dw_s^k \\
&+ \int_0^t \int_Z (g_s(z,u_{s-}),\varphi)\tilde N(dz,ds)
\end{aligned}
$$

for all $t\in[0,T]$, where recall that $(\cdot,\cdot)$ is the inner product in $L_2(Q)$.

**Theorem 2.1.1.** *Let Assumptions 2.1.1 through 2.1.3 and 2.1.5 through 2.1.7 hold. Then there exists a unique solution of the problem* (2.1.1)-(2.1.2). *Moreover the following estimate holds*

$$
E\sup_{t\in[0,T]}|u_t|^2_{L_2(Q)} + E\int_0^T |u|^2_{H_0^1(Q)}\,dt \le N\left( E|\phi|^2_{L_2(Q)} + E\int_0^T (|\bar h_t|^2_{L_2(Q)} + \sum_{i=1}^d |f_t^i|^2_{L_2(Q)})dt \right),
$$

*where $N$ is a constant depending only on $d,\kappa,K$ and $T$.*

After some preliminaries we will see that Theorem 2.1.1 follows easily from Theorem 1.4.1 (i.e. Theorems 2.9 and 2.10 from [22]).

Together with (2.1.1)-(2.1.2) let us also consider the problem

$$
\begin{aligned}
dv_t(x) =&\Big( L_t v_t(x) + F_t(x,v_t(x),\nabla v_t(x)) + \partial_i f_t^i(x)\Big)\, dt \\
&+ G_t^k(v)(x)dw_t^k + \int_Z g_t(x,z,v_{t-}(x))\tilde N(dt,dz), \qquad (2.1.4)\\
v_0(x) =&\Psi(x), \qquad (2.1.5)
\end{aligned}
$$

where $F$ satisfies ii) from Assumption 2.1.1 and $\Psi$ is an $\mathscr{F}_0$-measurable random variable in $L_2(Q)$.

**Theorem 2.1.2.** *Suppose that Assumptions 2.1.1 and 2.1.4 through 2.1.7 hold. Let $u$ and $v$ be solutions of the problems* (2.1.1)-(2.1.2) *and* (2.1.4)-(2.1.5) *respectively. Suppose that either $f$ or $F$ satisfy Assumption 2.1.3. Let $f\le F$ and $\psi\le\Psi$. Then almost surely, for all $t\in[0,T]$ we have $u_t(x)\le v_t(x)$ for almost every $x\in Q$.*

*Remark* 2.1.2. For equations driven by continuous noise, under the assumptions posed for existence-uniqueness of the solutions, one gets the comparison principle with no extra conditions. Notice that this is not the case when jump-type noise is present. Namely, Assumption 2.1.4 needs to hold. Notice also that this assumption is not satisfied even by linear functions of the form $f(r) = Cr$, when $C < -1$. However, Assumption 2.1.4 cannot be omitted in Theorem 2.1.2. Consider for example the SDE

$$u_t = 1 - \int_0^t 2u_{s-}\,d\tilde{N}_s,$$

where $N_t$ is a Poisson process with intensity one. Let $\tau$ be the time that the first jump of $N$ occurs. Then $P(\tau \le T) > 0$. Since $u_t = e^{-2t}$ on $[0,\tau)$, one can see that on the set $\{\tau \le T\}$ we have $u(\tau) = -e^{-2\tau} < 0$.

The second equation that we will deal with is

$$\begin{aligned} du_t(x) = &\Big(\mathscr{L}_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x)) + \partial_i f_t^i(x)\Big)\,dt \\ &+ G_t^k(u_t)(x)\,dw_t^k + \int_F S_{t,\zeta} u_{t-}(x)\ \tilde{M}(ds, d\zeta) \end{aligned} \tag{2.1.6}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, with initial condition

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}^d, \tag{2.1.7}$$

where

$$\begin{aligned} \mathscr{L}_t u(x) &= L_t u(x) + \mathscr{I}_t^{(2)} u(x), \\ \mathscr{I}_t^{(2)} u(x) &= \int_F [\lambda_t(x + b_t(\zeta), \zeta)\, u(x + b_t(\zeta)) - \lambda_t(x,\zeta)\, u(x) \\ &\quad - b_t(\zeta) \cdot \nabla(\lambda_t(x,\zeta)\, u(x))]\,\pi^{(2)}(d\zeta), \qquad (2.1.8) \\ S_{t,\zeta} u(x) &= \lambda_t(x + b_t(\zeta), \zeta)\, u(x + b_t(\zeta)) - \lambda_t(x,\zeta)\, u(x) \qquad (2.1.9) \\ &\quad + (\lambda_t(x,\zeta) - 1)\, u(x). \end{aligned}$$

Obviously, if we ask later for some of the previous assumptions to hold for equation (2.1.6), we mean with $g \equiv 0$.

**Assumption 2.1.8.** The function $b$ maps $\Omega \times [0,T] \times F$ into $\mathbb{R}^d$, it is $\mathscr{P} \otimes \mathfrak{F}$-measurable, and there exists an $\mathfrak{F}$-measurable real function $\bar{b}$ on $F$, such that for all $\omega$, $t$ and $\zeta$ we have

$$|b_t(\zeta)| \le \bar{b}(\zeta), \quad \int_F \bar{b}^2(\zeta) \wedge \bar{b}(\zeta)\ \pi^{(2)}(d\zeta) \le K.$$

The function $\lambda$ maps $\Omega \times [0,T] \times \mathbb{R}^d \times F$ to $[0,\infty)$, is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathfrak{F}$-measurable, it is twice continuously differentiable in $x$ for all $\omega, t, \zeta$, and we have

$$|\lambda_t(x,\zeta)| + |\nabla \lambda_t(x,\zeta)| + |\nabla^2 \lambda_t(x,\zeta)| \le K,$$

$$|1-\lambda_t(x,\zeta)| \le \bar{b}(\zeta), \text{ for all } \omega, t, x, \zeta.$$

It is easy to see that due to Assumption 2.1.8 for every $t \in [0,T]$ and $\omega \in \Omega$ the mapping $\mathscr{I}_t^{(2)}$, defined in the same way as $\mathscr{I}_t^{(1)}$, is a bounded linear operator from $H_0^1$ to $H^{-1}$, and $\langle \mathscr{I}^{(2)}u, v\rangle$ is a predictable process for any $u, v \in H^1$.

The solution of the problem (2.1.6)-(2.1.7) is understood in the same sense as that of (2.1.1)-(2.1.2), and we have the following existence and uniqueness result.

**Theorem 2.1.3.** *Let Assumptions 2.1.1 through 2.1.3 and 2.1.5 through 2.1.8 hold. Then there exists a unique solution of the problem* (2.1.6)-(2.1.7). *Moreover the following estimate holds*

$$E \sup_{t\in[0,T]} |u_t|^2_{L_2(Q)} + E\int_0^T |u|^2_{H_0^1(Q)}\,dt \le N\left(E|\phi|^2_{L_2(Q)} + E\int_0^T (|\bar{h}_t|^2_{L_2(Q)} + \sum_{i=1}^d |f_t^i|^2_{L_2(Q)})dt\right),$$

*where $N$ is a constant depending only on $d, \kappa, K$ and $T$.*

We also consider the problem

$$\begin{aligned} dv_t(x) = &\Big(\mathscr{L}_t v_t(x) + F_t(x, v_t(x), \nabla v_t(x)) + \partial_i f_t^i(x)\Big)dt \\ &+ G_t^k(v)(x)dw_t^k + \int_F S_{t,\zeta}v(x)\ \tilde{M}(ds,d\zeta), \end{aligned} \tag{2.1.10}$$

$$v_0(x) = \Psi(x), \tag{2.1.11}$$

where $F$ and $\Psi$ are as in (2.1.4)-(2.1.5).

**Theorem 2.1.4.** *Suppose that Assumptions 2.1.1, and 2.1.5 through 2.1.8 hold. Let $u$ and $v$ solve* (2.1.6)-(2.1.7) *and* (2.1.10)- (2.1.11) *respectively. Suppose that either $f$ or $F$ satisfy Assumption 2.1.3. Let $f \le F$ and $\psi \le \Psi$. Then almost surely, for all $t \in [0,T]$ we have $u_t(x) \le v_t(x)$ for almost every $x \in \mathbb{R}^d$.*

### 2.1.2 Itô's formula for the square of the norm of the positive part

In order to prove our comparison principles, we want to obtain an Itô's formula for $|u_t^+|^2_{L_2(Q)}$, where $u_t$ is an $H^{-1}(Q)$-valued semimartingale taking values in $H_0^1(Q)$ for $dP\otimes dt$ almost every $(\omega, t) \in \Omega\times[0,T]$. Our approach to obtain it is similar to that in [44]. To state the formula we set

$$V := H_0^1(Q), \quad H := L_2(Q), \quad V^* := H^{-1}(Q),$$

and we consider the following processes

$$v : \Omega\times[0,T] \to V,\ v^* : \Omega\times[0,T] \to V^*,\ h^k : \Omega\times[0,T] \to H,$$

$$K : \Omega \times [0,T] \times Z \to H,$$

for integers $k \geq 1$, where $v$ is progressively measurable, $v^*$ and $h^k$ are $\mathscr{F}_t$-adapted, measurable in $(\omega, t)$, and $K$ is $\mathscr{P} \otimes \mathscr{Z}$ measurable. We consider also $\psi$, an $\mathscr{F}_0$-measurable random variable in $H$.

We make the following assumption.

**Assumption 2.1.9.**

(i) Almost surely

$$\int_0^T \Big( |v_t|_V^2 + |v_t^*|_{V^*}^2 + \sum_k |h_t^k|_H^2 + \int_Z |K_t(z)|_H^2 \nu(dz) \Big) dt < \infty,$$

(ii) for each $\phi \in V$ and for $dP \otimes dt$-almost every $(\omega, t)$, we have

$$(v_t, \phi) = (\psi, \phi) + \int_0^t \langle v_s^*, \phi \rangle ds + \int_0^t (h_s^k, \phi) dw_s^k$$

$$+ \int_0^t \int_Z (K_s(z), \phi) \tilde{N}(ds, dz).$$

**Theorem 2.1.5.** *Suppose that Assumption 2.1.9 is satisfied. Then there exists a set $\tilde{\Omega} \subset \Omega$ of probability one, and an $H$-valued strongly càdlàg adapted process $u_t$ such that $u_t = v_t$ for $dP \otimes dt$-almost every $(\omega, t)$. Moreover for $\omega \in \tilde{\Omega}$, $t \in [0,T]$ we have*

$$i)\ u_t = \psi + \int_0^t v_s^* ds + \int_0^t h_s^k dw_s^k + \int_0^t \int_Z K_s(z) \tilde{N}(ds, dz), \tag{2.1.12}$$

$$ii)\ |u_t^+|_H^2 = |\psi^+|_H^2 + 2\int_0^t \langle v_s^*, u_s^+ \rangle ds + 2\int_0^t (h_s^k, u_s^+) dw_s^k$$

$$+2\int_0^t \int_Z (K_s(z), u_{s-}^+) \tilde{N}(dz, ds) + \int_{(0,t]} \sum_k |I_{u_s > 0} h_s^k|_H^2 ds$$

$$+ \int_0^t \int_Z |(u_{s-} + K_s(z))^+|_H^2 - |u_{s-}^+|_H^2 - 2(K_s(z), u_{s-}^+)_H \, N(dz, ds).$$

To prove Theorem 2.1.5 we need two lemmas.

**Lemma 2.1.6.** *Let $(X, \Sigma, \mu)$ be a measure space, and let $u_n, u \in L_1(X)$ such that $u_n \to u$ in $L_1(X)$ as $n \to \infty$. Then there exists a subsequence $\{u_{n(k)}\}_{k=1}^{\infty}$ and a function $v \in L_1(X)$ such that for all $k \geq 1$ we have $|u_{n(k)}(x)| \leq v(x)$ for all $x \in X$, and $u_{n(k)}(x) \to u(x)$ for $\mu$-almost every $x$ as $k \to \infty$.*

*Proof.* There exists a subsequence $u_{n(k)}$ such that

$$|u_{n(k)} - u|_{L_1(X)} \leq 1/2^k \quad \text{for } k \geq 1.$$

Set $v(x) = |u(x)| + \sum_k |u_{n(k)}(x) - u(x)|$. Then $v$ has the desired properties. Moreover, $\sum_k |u_{n(k+1)} - u_{n(k)}|_{L_1(X)} < \infty$, which implies that $u_{n(k)}$ converges $\mu$-almost everywhere. □

The next lemma is from [13].

**Lemma 2.1.7.** *Let $\mathcal{Q}$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Take $\phi_n \in C_c^\infty(\mathcal{Q})$, $n \in \mathbb{N}$, with*

*i)* $0 \le \phi_n \le 1$

*ii)* $\phi_n = 1$ *on* $\{x \in \mathcal{Q}, r(x) \ge 1/n\}$

*iii)* $\phi_n = 0$ *on* $\{x \in \mathcal{Q}, r(x) \le 1/2n\}$

*iv)* $|(\phi_n)_{x_i}| \le Cn$,

*where $C$ is a constant and $r(x) = dist(x, \partial\mathcal{Q})$. Then $\phi_n v \to v$ in $H_0^1(\mathcal{Q})$ for all $v \in H_0^1(\mathcal{Q})$, and for some constant $C$ we have*

$$\sup_n |\phi_n v|_{H_0^1(\mathcal{Q})} \le C|v|_{H_0^1(\mathcal{Q})}, \qquad \forall v \in H_0^1(\mathcal{Q}).$$

*Remark* 2.1.3. One can easily see the existence of a sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfying the conditions of the previous lemma. Then note that $\phi_n^2$ also satisfies i)-iv). Hence, $\phi_n^2 v \to v$ in $H_0^1(\mathcal{Q})$, for all $v \in H_0^1(\mathcal{Q})$, and for some constant $C$ we have

$$\sup_n |\phi_n^2 v|_{H_0^1(\mathcal{Q})} \le C|v|_{H_0^1(\mathcal{Q})}, \qquad \forall v \in H_0^1(\mathcal{Q}).$$

We introduce now the functions $\alpha_\delta(r)$, $\beta_\delta(r)$ and $\gamma_\delta(r)$ on $\mathbb{R}$, for $\delta > 0$, given by

$$a_\delta(r) = \begin{cases} 1 & \text{if } r > \delta \\ \frac{r}{\delta} & \text{if } 0 \le r \le \delta \\ 0 & \text{if } r < 0, \end{cases}$$

$$\beta_\delta(r) = \int_0^r a_\delta(s)\,ds, \qquad \gamma_\delta(r) = \int_0^r \beta_\delta(s)\,ds.$$

For all $r \in \mathbb{R}$ we have $\alpha_\delta(r) \to I_{r>0}$, $\beta_\delta(r) \to r^+$ and $\gamma_\delta(r) \to (r^+)^2/2$ as $\delta \to 0$. Also, for all $r, r_1, r_2$ and $\delta$, the following inequalities hold

$$|\alpha_\delta(r)| \le 1,\ |\beta_\delta(r)| \le |r|,\ |\gamma_\delta(r)| \le \frac{r^2}{2},$$

$$|\gamma_\delta(r_1 + r_2) - \gamma_\delta(r_1) - \beta_\delta(r_1) r_2| \le |r_2|^2. \tag{2.1.13}$$

We are now ready to prove Theorem 2.1.5.

*Proof of Theorem 2.1.5.* We only prove ii) since the rest of the assertions are proved in [26], in greater generality. First we prove the statement when $Q = \mathbb{R}^d$. We have that equality (2.1.12) is satisfied if and only if, almost surely, for all $\varphi \in V$ and $t$ we have

$$(u_t, \varphi) = (\psi, \varphi) + \int_{(0,t]} \langle v_s^*, \varphi \rangle ds + \int_0^t (h_s^k, \varphi) dw_s^k$$
$$+ \int_0^t \int_Z (K_s(z), \varphi) \tilde{N}(ds, dz). \tag{2.1.14}$$

Let $\phi$ be a mollifier with compact support and set $\phi_\epsilon(x) := \epsilon^{-d}\phi(x/\epsilon)$. For fixed $x$, the function $\phi_\epsilon(x - \cdot)$ is in $V$, so we can plug it in (2.1.14) instead of $\varphi$, to get that almost surely, for all $t \in [0, T]$

$$u_t^\epsilon(x) = u_0^\epsilon(x) + \int_0^t v_s^{*\epsilon}(x) ds + \int_0^t h_s^{k\epsilon}(x) dw_s^k$$

$$+ \int_0^t \int_Z K_s^\epsilon(z, x) \tilde{N}(ds, dz),$$

where for $g \in V^*$ we use the notation $g^\epsilon(x) := \langle g, \phi_\epsilon(x - \cdot) \rangle$. Note that $u_0^\epsilon$ is $\mathscr{F}_0 \otimes \mathscr{B}(\mathbb{R}^d)$ measurable. Also $u^\epsilon, v^{*\epsilon}$ and $h^{k\epsilon}$ are jointly measurable in $(t, \omega, x)$, $\mathscr{F}_t \otimes \mathscr{B}(\mathbb{R}^d)$ measurable for each $t$, and $K^\epsilon$ is $\mathscr{P} \otimes \mathscr{Z} \otimes \mathscr{B}(\mathbb{R}^d)$ measurable. It is also easy to see that there exists a constant $C_\epsilon$, depending on $\epsilon$, such that for all $t, \omega, x, z$

$$|u_t^\epsilon(x)| \le C_\epsilon |u_t|_H,\ |u_0^\epsilon(x)| \le C_\epsilon |u_0|_H, |v_t^{*\epsilon}|_H \le C_\epsilon |v_t^*|_{V^*}$$

$$|v_t^{*\epsilon}(x)| \le C_\epsilon |v_t^*|_{V^*},\ |h_t^{k\epsilon}(x)| \le C_\epsilon |h_s^k|_H,$$

$$|K_t^\epsilon(x, z)| \le C_\epsilon |K_t(z)|_H.$$

One can also check that for a constant $C$, for all $\epsilon$

$$|u_t^\epsilon|_H \le C|u_t|_H,\ |u_0^\epsilon|_H \le C|u_0|_H,\ |K_t^\epsilon(z)|_H \le C|K_t(z)|_H$$

$$|h_t^{k\epsilon}|_H \le C|h_s^k|_H, |v_t^{*\epsilon}|_{V^*} \le C|v_t^*|_{V^*},\ |u_t^\epsilon|_V \le C|u_t|_V.$$

Now let $\alpha_\delta, \beta_\delta, \gamma_\delta$ be as before, and fix $x$. By Itô's formula (see for example [35] or [60]), for each $x$ there exists a set $\Omega_x$ of full probability, such that for all $\omega \in \Omega_x$ and

$t \in [0,T]$ we have

$$\begin{aligned}\gamma_\delta(u_t^\epsilon(x)) =& \gamma_\delta(u_0^\epsilon(x)) + \int_0^t \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x)ds \\ &+ \sum_k \int_0^t \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dw_s^k \\ &+ \frac{1}{2}\sum_k \int_0^t \alpha_\delta(u_s^\epsilon(x))|h_s^{k\epsilon}(x)|^2 ds \\ &+ \int_0^t \int_Z \beta_\delta(u_{s-}^\epsilon(x)) K_s^\epsilon(z,x)\tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \gamma_\delta(u_s^\epsilon(x) + K_s^\epsilon(z,x)) \\ &- \gamma_\delta(u_{s-}^\epsilon(x)) - \beta_\delta(u_{s-}^\epsilon(x)) K_s^\epsilon(z,x) N(dz,ds).\end{aligned} \tag{2.1.15}$$

One can redefine the stochastic integrals such that (2.1.15) holds for all $(\omega, t, x)$. Integrating (2.1.15) over $\mathbb{R}^d$, taking appropriate versions of the stochastic integrals and using the Fubini and the stochastic Fubini theorems we get for each $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma_\delta(u_t^\epsilon(x))\,dx =& \int_{\mathbb{R}^d} \gamma_\delta(u_0^\epsilon(x))\,dx + \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x)\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx\,dw_s^k \\ &+ \frac{1}{2}\sum_k \int_{(0,t]} \int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x))|h_t^{k\epsilon}(x)|^2\,dx\,ds \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} \beta_\delta(u_{s-}^\epsilon(x)) K_s^\epsilon(z,x)\,dx\tilde{N}(ds,dz) \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z,x)) \\ &- \gamma_\delta(u_{s-}^\epsilon(x)) - \beta_\delta(u_{s-}^\epsilon(x)) K_s^\epsilon(z,x)\,dx N(dz,ds)\ (a.s.).\end{aligned} \tag{2.1.16}$$

For a stochastic Fubini theorem we refer to [45], noting that the Fubini theorem there can be extended easily, by obvious changes in its proof, to our situation. Since each term in the above equation is a càdlàg process in $t$, we see that (2.1.16) holds almost surely, for all $t \in [0,T]$. We claim that for each $t \in [0,T]$ both sides of (2.1.16) converges in probability as $\epsilon \to 0$ to give that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \gamma_\delta(u_t(x))dx = &\int_{\mathbb{R}^d} \gamma_\delta(u_0(x))dx + \int_0^t \langle v_s^*, \beta_\delta(u_s)\rangle ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x) dx dw_s^k \\
&+ \frac{1}{2}\sum_k \int_0^t \int_{\mathbb{R}^d} \alpha_\delta(u_s(x))|h_s^k(x)|^2 dx ds \\
&+ \int_0^t \int_Z \int_{\mathbb{R}^d} \beta_\delta(u_{s-}(x)) K_s(z,x) dx \tilde{N}(ds,dz) \\
&+ \int_0^t \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_{s-}(x) + K_s(z,x)) \\
&- \gamma_\delta(u_{s-}(x)) - \beta_\delta(u_{s-}(x)) K_s(z,x) dx N(dz,ds). \qquad (2.1.17)
\end{aligned}
$$

holds almost surely for each $t \in [0,T]$. We are going to show that each term in (2.1.16) converges in probability to the corresponding one in (2.1.17). Since for any sequence $\epsilon_k \downarrow 0$ we have $u_t^{\epsilon_k} \to u_t$ in $L_2(\mathbb{R}^d)$ for every $\omega \in \Omega$, by the equality $a^2 - b^2 = (a-b)(a+b)$ we have $(u_t^{\epsilon_k})^2 \to u_t^2$ in $L_1(\mathbb{R}^d)$. Thus for every $\omega \in \Omega$ by Lemma 2.1.6 there exist $g \in L_1(\mathbb{R}^d)$ and a subsequence, denoted again by $\epsilon_k$, such that for all $k \geq 1$

$$
|\gamma_\delta(u_t^{\epsilon_k}(x))| \leq \frac{(u_t^{\epsilon_k}(x))^2}{2} \leq \frac{g(x)}{2} \text{ for all } x.
$$

Since $\gamma_\delta(u_t^{\epsilon_k}(x)) \to \gamma_\delta(u_t(x))$ for almost every $x$, as $k \to \infty$, by Lebesgue's theorem on dominated convergence we obtain

$$
\int_{\mathbb{R}^d} \gamma_\delta(u_t^{\epsilon_k}(x))dx \to \int_{\mathbb{R}^d} \gamma_\delta(u_t(x))dx \quad \text{as } k \to \infty.
$$

Thus, for $\epsilon \downarrow 0$ the left-hand side of (2.1.16) converges to the left-hand side of (2.1.17) for every $\omega \in \Omega$, and hence also in probability, for each $t \in [0,T]$. To see the convergence of the second term in the right-hand side of (2.1.16) we fix $(s,\omega)$ such that $u_s \in V$. Then it is straightforward to check that

$$
|\beta_\delta(u_s^\epsilon) - \beta_\delta(u_s)|_V \to 0, \text{ as } \epsilon \to 0.
$$

Taking into account the well-known fact that there exist $f_s^0$ and $f_s^i \in L^2(\mathbb{R}^d)$ for $i = 1,\dots,d$ such that

$$
v_s^* = f_s^0 + \partial_i f_s^i,
$$

we have

$$
v_s^{*\epsilon} = f_s^{0\epsilon} + \partial_i f_s^{i\epsilon},
$$

which gives

$$|v_s^* - v_s^{*\epsilon}|_{V^*} \le \sum_{i=0}^{d} |f_s^{i\epsilon} - f_s^i|_H \to 0, \text{ as } \epsilon \to 0.$$

Hence we conclude

$$\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x))\, v_s^{*\epsilon}(x)\,dx = \langle v_s^{*\epsilon}, \beta_\delta(u_s^\epsilon)\rangle \to \langle v_s^*, \beta_\delta(u_s)\rangle.$$

Notice that there is a constant $C$ such that

$$\Big|\int_{\mathbb{R}^n} \beta_\delta(u_s^\epsilon(x))\, v_s^{*\epsilon}(x)\,dx\Big| \le C(|u_s|_V^2 + |v_s^*|_{V^*}^2)$$

for all $\epsilon > 0$, $\omega \in \Omega$ and $s \in [0,T]$. Therefore, almost surely

$$\int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x))\, v_s^{*\epsilon}(x)\,dx\,ds \to \int_0^t \langle v_s^*, \beta_\delta(u_s)\rangle\,ds \quad \text{for all } t.$$

For the sum of the stochastic integrals against the Wiener processes we just note that almost surely for all $s \in [0,T]$

$$\sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)\,dx\Big|^2 \to 0 \quad \text{as } \epsilon \downarrow 0,$$

and

$$\sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)\,dx\Big|^2$$
$$\le 4 \sup_{t\le T} |u_t|_{L^2}^2 \sum_k |h_s^k|_{L^2}^2 \quad \text{for all } \epsilon > 0.$$

Hence almost surely

$$\int_0^T \sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)\,dx\Big|^2 ds \to 0,$$

which implies that for $\epsilon \downarrow 0$

$$\int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx\,dw_s^k \to \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)\,dx\,dw_s^k$$

in probability, uniformly in $t \in [0,T]$. Note that for each $k$ we have

$$\Big|\int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x))|h_s^{k\epsilon}(x)|^2 - \alpha_\delta(u_s(x))|h_s^k(x)|^2\,dx\Big|$$

$$\le \int_{\mathbb{R}^d} |(h_s^{k\epsilon}(x))^2 - (h_s^k(x))^2|\,dx$$

$$+\int_{\mathbb{R}^d} |h_s^k(x)|^2 |\alpha_\delta(u_s^\epsilon(x)) - \alpha_\delta(u_s(x))| dx \to 0.$$

for each $\omega \in \Omega$ and $s \in [0,T]$. Moreover,

$$\Big| \int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x)) |h_s^{k\epsilon}(x)|^2 dx \Big| \le |h_s^k|_H^2,$$

where the right-hand side is almost surely integrable on $[0,T]$. Hence the almost sure convergence of the fourth term in the right-hand side of (2.1.16) follows. By the inequalities in (2.1.13), similar arguments show the convergence of the last two terms in probability. We conclude that for each $t \in [0,T]$ equation (2.1.17) holds almost surely. Since the stochastic processes in both sides of (2.1.17) are càdlàg processes, equation (2.1.17) holds almost surely for all $t \in [0,T]$.

Now by letting $\delta \to 0$ in (2.1.17), using arguments similar to the previous ones, and keeping in mind the inequalities (2.1.13) and the fact that for all $v \in V$

$$|\beta_\delta(v) - v^+|_V \to 0, \ |\beta_\delta(v)|_V \le |v|_V,$$

we can finish the proof of the theorem for $Q = \mathbb{R}^d$.

We reduce the case of a bounded Lipschitz domain $Q$ to that of the whole space by using the sequence $\phi_n$ from Lemma 2.1.7. Remember that $\phi_n$ has compact support in $Q$. Thus for a function $\eta$ on $Q$ we denote by $\phi_n\eta$, not only the function defined on $Q$ by the multiplication of $\phi_n$ and $\eta$, but also its extension to zero outside of $Q$. Notice that when $u$ satisfies (2.1.12) on $Q$, then $\phi_n u$ satisfies

$$\phi_n u_t = \phi_n u_0 + \int_0^t \phi_n v_s^* ds + \int_0^t \phi_n h_s^k dw_s^k$$

$$+ \int_0^t \int_Y \phi_n K_s(z) \tilde{N}(ds,dz)$$

on the whole $\mathbb{R}^d$, where the functional $\phi_n v^*$ is defined by

$$\langle \phi_n v_s^*, g \rangle := \langle v_s^*, \phi_n g \rangle_Q$$

for $g \in H^1(\mathbb{R}^d)$. The notation $\langle \cdot,\cdot \rangle_Q$ means the duality product between $H_0^1(Q)$ and $H^{-1}(Q)$. Notice that $\langle v_s^*, \phi_n g \rangle_Q$ is well defined, since the restriction of $\phi_n g$ to $Q$

belongs to $H^1_0(Q)$. Then by the result in the case of the whole space, we have

$$\begin{aligned}\int_Q \phi_n^2 |u_t^+|^2 dx = &\int_Q |\phi_n u_0^+|^2 dx + 2\int_0^t \langle v_s^*, \phi_n^2 u_s^+\rangle_Q ds \\ &+ 2\int_0^t \int_Q \phi_n^2 h_s^k u_s^+ \, dx dw_s^k \\ &+ \int_0^t \int_Q \sum_k |I_{\{\phi_n u_s > 0\}} \phi_n h_s^k|^2 dx ds \\ &+ \int_0^t \int_Z \int_Q 2K_s(z)\phi_n^2 u_{s-}^+ \, dx \tilde{N}(ds, dz) \\ &+ \int_0^t \int_Z \int_Q |\phi_n (u_{s-} + K_s(z))^+|^2 - |\phi_n u_{s-}^+|^2 \\ &- 2K_s(z)\phi_n^2 u_{s-}^+ \, dx N(dz, ds),\end{aligned} \tag{2.1.18}$$

since $\phi_n$ is supported in $Q$. It is now easy to take $n \to \infty$ here to finish the proof of the theorem. We only note that for the second term on the right-hand side we have by Lemma 2.1.7 and Remark 2.1.3

$$\langle v_s^*, \phi_n^2 u_s^+\rangle_Q \to \langle v_s^*, \; u_s^+\rangle_Q \quad \text{for all } \omega, s,$$

and for a constant $C$,

$$\langle v_s^*, \phi_n^2 u_s^+\rangle_Q \leq C|v_s^*|_{V^*}|u_s|_V \quad \text{for all } n.$$

□

### 2.1.3 The main estimates

In this section we present some lemmas that we will need for the proofs of Theorems 2.1.1 through 2.1.4. The following is well known (see, e.g., [50], or exercise 1.3.19 in [41], or some more general results in [58]).

**Lemma 2.1.8.** *Let $u \in W^1_p(Q)$. Let $u_n \in W^1_p(Q)$ such that $|u_n - u|_{W^1_p} \to 0$ as $n \to \infty$. Then we have $|u_n^+ - u^+|_{W^1_p} \to 0$.*

For the next three lemmas, we assume that Assumptions 2.1.5 through 2.1.8 hold. For $u \in C_c^\infty(\mathbb{R}^d)$, let us define the quantities,

$$\varrho_t(u) := \int_{\mathbb{R}^d} \int_F (\lambda_t(x + b_t(\zeta))\, u(x + b_t(\zeta)))^2 - (\lambda_t(x,\zeta)u(x))^2$$

$$-2b_t(z) \cdot \nabla(\lambda_t(x,\zeta)\, u(x))\, \lambda_t(x,\zeta)\, u(x)\, \pi^{(2)}(d\zeta) dx,$$

$$\tilde{\varrho}_t(u) := \int_{\mathbb{R}^d}\int_F [(\lambda_t(x+b_t(\zeta))\,u(x+b_t(\zeta)))^+]^2 - [(\lambda_t(x,\zeta)u(x))^+]^2$$
$$-2b_t(z)\cdot\nabla(\lambda_t(x,\zeta)\,u(x))\,\lambda_t(x,\zeta)\,u^+(x)\,\pi^{(2)}(d\zeta)\,dx.$$

**Lemma 2.1.9.** *For any $u \in C_c^\infty(\mathbb{R}^d)$, $\omega \in \Omega$, $t \in [0,T]$ and $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} \mathscr{I}_t^{(1)} u^2(x)\,dx \le \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \tag{2.1.19}$$
$$\int_{\mathbb{R}^d} \mathscr{I}_t^{(1)} (u^+)^2(x)\,dx \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}, \tag{2.1.20}$$
$$\varrho_t(u) \le \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \tag{2.1.21}$$
$$\tilde{\varrho}_t(u) \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}, \tag{2.1.22}$$

*where the constant $N(\varepsilon)$ depends only on $\varepsilon$, $K$ and $d$.*

*Proof.* We prove (2.1.20). Clearly,

$$\int_{\mathbb{R}^d} \mathscr{I}_t^{(1)} (u^+)^2(x)dx = \int_{\mathbb{R}^d} \mathscr{I}_t^{(1\delta)} (u^+)^2(x)\,dx + \int_{\mathbb{R}^d} \bar{\mathscr{I}}_t^{(1\delta)} (u^+)^2(x)\,dx. \tag{2.1.23}$$

The first term on the right-hand side is equal to

$$\int_0^1 (1-\theta)\int_{F_\delta}\int_{\mathbb{R}^d} \partial_{ij}(u^+)^2(x+\theta c_t(x,\zeta))$$
$$\times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)\,dx\pi^{(1)}(d\zeta)d\theta$$
$$= E_1(t,\delta) + E_2(t,\delta),$$

where

$$E_1(t,\delta) = \int_0^1 (1-\theta)\int_{F_\delta}\int_{\mathbb{R}^d} 2\partial_i u^+(x+\theta c_t(x,\zeta))\partial_j u^+(x+\theta c_t(x,\zeta))$$
$$\times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)\,dx\pi^{(1)}(d\zeta)d\theta,$$
$$E_2(t,\delta) = \int_0^1 (1-\theta)\int_{F_\delta}\int_{\mathbb{R}^d} 2u^+(x+\theta c_t(x,\zeta))\partial_{ij}u(x+\theta c_t(x,\zeta))$$
$$\times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)\,dx\pi^{(1)}(d\zeta)d\theta.$$

Using Assumptions 2.1.5, 2.1.6 and 2.1.7, we see after a change of variables that

$$|E_1(t,\delta)| \le C(\delta)C|u^+|^2_{H^1(\mathbb{R}^d)},$$

where $C(\delta) = \int_{F_\delta} \bar{c}^2(\zeta)\pi(dz)$ and $C$ is a constant depending only on $K$ and $d$. For $E_2$

we have

$$E_2(t,\delta) = \int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} 2\partial_j (\partial_i u(x+\theta c_t(x,\zeta))$$

$$\times q_t^{ij}(x,\zeta,\theta) u^+(x+\theta c_t(x,\zeta))\, dx \pi^{(1)}(d\zeta)\, d\theta.$$

By integration by parts and using the Assumptions 2.1.5, 2.1.6 and 2.1.7 again we see that

$$|E_2(t,\delta)| \le C(\delta) C |u^+|^2_{H^1(\mathbb{R}^d)}.$$

For the second term in (2.1.23) by Young's inequality and Assumptions 2.1.6, 2.1.5, we have

$$\int_{\mathbb{R}^d} \bar{\mathscr{I}}_t^{(1\delta)} (u^+)^2(x)\ dx \le \gamma |u|^2_{H^1(\mathbb{R}^d)} + C(\gamma)|u|^2_{L_2(\mathbb{R}^d)},$$

for all $\gamma > 0$, where $C(\gamma)$ depends only on $\gamma$ and $K$. Putting these estimates together and choosing $\delta$ and $\gamma$ sufficiently small, we finish the proof of (2.1.20). One can repeat the same calculation with $c$ replaced by $b$, $m = 1$ and $\lambda u$ in place of $u$ to get (2.1.22). Also (2.1.19) and (2.1.21) can be proved in the same way. □

**Lemma 2.1.10.** *For any $u \in H_0^1(Q)$, $\omega \in \Omega$, $t \in [0,T]$ and $\varepsilon > 0$ we have*

$$2\langle \mathscr{I}_t^{(1)} u, u\rangle \le \varepsilon |u|^2_{H_0^1(Q)} + N(\varepsilon)|u|^2_{L_2(Q)}, \tag{2.1.24}$$

$$2\langle \mathscr{I}_t^{(1)} u, u^+\rangle \le \varepsilon |u^+|^2_{H_0^1(Q)} + N(\varepsilon)|u^+|^2_{L_2(Q)}, \tag{2.1.25}$$

*where the constant $N(\varepsilon)$ depends only on $\varepsilon$ and $K$ and $d$.*

*Proof.* We prove (2.1.25). It suffices to prove it for $Q = \mathbb{R}^d$. Due to Lemma 2.1.8 and the continuity of the operator $\mathscr{I}_t^{(1)} : H^1 \to H^{-1}$, we may and will also assume that $u \in C_c^\infty(\mathbb{R}^d)$. Notice that for any $\alpha, \beta \in \mathbb{R}$

$$2(\beta-\alpha)\alpha^+ \le (\beta^+)^2 - (\alpha^+)^2 - (\beta^+ - \alpha^+)^2 \le (\beta^+)^2 - (\alpha^+)^2. \tag{2.1.26}$$

Consequently, for any $\alpha, \beta, \gamma \in \mathbb{R}$

$$2(\beta-\alpha-\gamma)\alpha^+ \le (\beta^+)^2 - (\alpha^+)^2 - 2\gamma\alpha^+.$$

Using this with $\alpha = u(x)$, $\beta = u(x + c_t(x,\zeta))$ and $\gamma = c_t(x,\zeta)\nabla u(x)$, and taking into account that $2\nabla u u^+ = \nabla (u^+)^2$, we can easily see that

$$2\langle \mathscr{I}_t^{(1)} u, u^+\rangle = 2(\mathscr{I}_t^{(1)} u, u^+) \le \int_{\mathbb{R}^d} \mathscr{I}_t^{(1)} (u^+)^2(x)\ dx.$$

Hence (2.1.25) follows from Lemma 2.1.9. One can prove (2.1.24) in the same way, by using the inequality $2(\beta-\alpha)\alpha \le \beta^2 - \alpha^2$, instead of (2.1.26). □

For $u \in H^1(\mathbb{R}^d)$ we set

$$\begin{aligned}
\mu_t(u) :=& \int_F \int_{\mathbb{R}^d} [(\lambda_t(x+b_t(\zeta),\zeta)u(x+b_t(\zeta)))]^2 - [u(x)]^2 \\
& - 2u(x)[\lambda_t(x+b_t(\zeta),\zeta)u(x+b_t(\zeta)) - u(x)]\, dx\pi^{(2)}(d\zeta), \\
\rho_t(u) :=& 2\langle \mathscr{I}_t^{(2)} u, u\rangle + \mu_t(u), \qquad (2.1.27) \\
\tilde{\mu}_t(u) :=& \int_F \int_{\mathbb{R}^d} [(\lambda_t(x+b_t(\zeta),\zeta)u(x+b_t(\zeta)))^+]^2 - [u^+(x)]^2 \\
& - 2u^+(x)[\lambda_t(x+b_t(\zeta),\zeta)u(x+b_t(\zeta)) - u(x)]\, dx\pi^{(2)}(d\zeta), \\
\tilde{\rho}_t(u) :=& 2\langle \mathscr{I}_t^{(2)} u, u^+\rangle + \tilde{\mu}_t(u). \qquad (2.1.28)
\end{aligned}$$

Using the simple inequality $|[(x+y)^+]^2 - [x^+]^2 - 2x^+y| \le 2|y|^2$, and Assumption 2.1.8 one can see that $\tilde{\mu}_t(u)$ is continuous in $u \in H^1(\mathbb{R}^d)$. It can be shown similarly that $\mu_t(u)$ is continuous in $u \in H^1(\mathbb{R}^d)$.

**Lemma 2.1.11.** *For any $u \in H^1(\mathbb{R}^d)$, $(\omega, t) \in \Omega \times \mathbb{R}^+$ and $\varepsilon > 0$ we have*

$$\rho_t(u) \le \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \qquad (2.1.29)$$

$$\tilde{\rho}_t(u) \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}. \qquad (2.1.30)$$

*Proof.* Since (2.1.29) can be shown in the same way as (2.1.30), we only prove the latter one. Clearly it suffices to prove it for $u \in C_c^\infty(\mathbb{R}^d)$. A simple calculation shows that

$$\tilde{\rho}_t(u) = \tilde{\varrho}_t(u) + \int_F \int_{\mathbb{R}^d} (\lambda_t(x,\zeta) - 1)^2 [u^+(x)]^2 \, dx\pi^{(2)}(d\zeta)$$

$$+ \int_F \int_{\mathbb{R}^d} 2b_t(\zeta) \cdot \nabla(u(x)\lambda_t(x,\zeta)) u^+(x)(\lambda_t(x,\zeta) - 1)\, dx\pi^{(2)}(d\zeta)$$

By Young's inequality, Assumption 2.1.8 and (2.1.22) we get that

$$\tilde{\rho}_t(u) \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}.$$

□

**Lemma 2.1.12.** *Let Assumption 2.1.3 hold. Then for all $(\omega, t) \in \Omega \times [0, T]$, $u \in H_0^1(Q)$ and $\varepsilon > 0$ we have*

$$2(f_t(u, \nabla u) - f_t(v, \nabla v), u - v) + \int_Z |g_t(z, u) - g_t(z, v)|^2_{L_2(Q)} \nu(dz)$$

$$\le \varepsilon |u - v|^2_{H_0^1(Q)} + N(\varepsilon)|u - v|^2_{L_2(Q)}, \qquad (2.1.31)$$

$$2(f_t(u, \nabla u) - f_t(v, \nabla v), (u - v)^+) + \int_Z |I_{u>v}(g_t(z, u) - g_t(z, v))|^2_{L_2(Q)} \nu(dz)$$

$$\leq \varepsilon|(u-v)^+|^2_{H^1_0(Q)} + N(\varepsilon)|(u-v)^+|^2_{L_2(Q)}, \qquad (2.1.32)$$

*where $N(\varepsilon)$ depends only on $\varepsilon$ and $K$.*

*Proof.* We show (2.1.32). Using the second part of Assumption 2.1.3 and Young's inequality we have

$$2(f_t(u,\nabla u) - f_t(v,\nabla v),(u-v)^+) \leq \frac{K}{\varepsilon}|(u-v)^+|^2_{L_2(Q)} + \varepsilon|\nabla(u-v)^+|^2_{L_2(Q)}$$

$$+\int_Q (f_t(x,u,\nabla u) - f_t(x,v,\nabla u))(u(x)-v(x))^+\,dx.$$

This combined with Assumption 2.1.3 gives (2.1.32). Inequality (2.1.31) can be shown in the same way. □

### 2.1.4 Proof of Theorems 2.1.1, 2.1.2, 2.1.3 and 2.1.4

We are now ready to proceed with the proofs of the main theorems.

*Proof of Theorems 2.1.1 and 2.1.3.* We prove Theorem 2.1.1. It suffices to show that conditions I) through IV) from Assumption 1.4.1 are satisfied. The growth condition of the operator

$$u \stackrel{A}{\longmapsto} L_t u + f_t(u,\nabla u) + \partial_i f^i$$

can be verified easily. Notice that for every $\omega, t$ and $x$, the function $f_t(x,r,r')$ is continuous in $(r,r')$. Using this, ii) from Assumption 2.1.1 and the fact that $L_t$ is a bounded linear operator from $H^1_0(Q)$ into $H^{-1}(Q)$, we see that $A$ is semicontinuous. Now, by ii) and iv) from Assumption 2.1.1, the boundedness of $\phi$ and (2.1.24) we see that for a $\theta > 0$ and a constant $C$ we have

$$2\langle L_t u, u\rangle + 2(u, f_t(u,\nabla u)) - 2(f^i_t, D_i u) + \sum_k |G^k_t(u)|^2_{L_2(Q)} + \int_Z |g_t(u)|^2_{L_2(Q)}\nu(dz)$$

$$\leq -\theta|u|^2_{H^1_0(Q)} + C|u|^2_{L_2(Q)} + C|\bar{h}_t|^2_{L_2(Q)} + C\sum_{i=1}^d |f^i_t|^2_{L_2(Q)}.$$

for all $t$, $\omega$ and $u \in H^1_0(Q)$. This shows that the coercivity condition is satisfied. Using i), iv), v) from Assumption 2.1.1 and (2.1.24) we see that for all $(t,\omega)$ and $\gamma > 0$

$$2\langle L_t u - L_t v, u-v\rangle + \sum_k |G^k(u) - G^k(v)|^2_{L_2(Q)}$$

$$\leq (\gamma-\kappa)|u-v|^2_{H^1_0(Q)} + C(\gamma)|u-v|^2_{L_2(Q)},$$

for all $u, v \in H^1_0(Q)$, where $\kappa$ is the ellipticity constant form part (iv) of Assumption 2.1.1. Combining this with (2.1.31) we have that the monotonicity condition is also satisfied. The proof of Theorem 2.1.3 goes in the same way. We omit the details, we only note that one also has to use (2.1.29). □

*Proof of Theorem 2.1.2.* Without loss of generality we can assume that Assumption 2.1.3 is satisfied by $f$. For the difference $h = u - v$ we have

$$\begin{aligned} h_t =& h_0 + \int_0^t L_s h_s + f_s(u_s, \nabla u_s) - F_s(v_s, \nabla v_s)\, ds \\ &+ \int_0^t \phi_s^{ki} \partial_i h_s + \sigma_s^k(u_s) - \sigma_s^k(v_s)\, dw_s^k \\ &+ \int_0^t \int_Z g(s,z,u_{s-}) - g(s,z,v_{s-}))\tilde{N}(ds,dz). \end{aligned}$$

By Theorem 2.1.5 we have

$$|h_t^+|^2_{L^2} = \int_0^t A_s^{(1)} + A_s^{(2)} + A_s^{(3)} + 2\langle \mathscr{I}_s^{(1)} h_s, h_s^+\rangle\; ds + m_t$$

for a martingale $m_t$, where

$$A_s^{(1)} = \int_Q \Big\{ -2a_s^{ij}(x)\partial_i h_s^+(x)\partial_j h_s^+(x)$$

$$+ \sum_k \Big| I_{h_s>0} \sum_i \phi_s^{ki}(x)\partial_i h_s(x) + I_{h_s>0}(\sigma_s^k(x,u_s(x)) - \sigma_s^k(x,v_s(x)))\Big|^2 \Big\} dx \qquad (2.1.33)$$

$$A_s^{(2)} = 2\int_Q (f_s(x,u_s,\nabla u_s) - F_s(x,v_s,\nabla v_s))h_s^+(x)dx \qquad (2.1.34)$$

$$A_s^{(3)} = \int_Z \int_Q \{[h_s(x) + g_s(x,z,u_{s-}(x)) - g_s(x,z,v_{s-}(x))]^+\}^2 - |h_s(x)^+|^2$$

$$-2h_s^+(x)[g_s(x,z,u_s(x)) - g_s(x,z,v_s(x))]\,dx\nu(dz).$$

One can easily see that for every $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ depending only on $\varepsilon$, $K$ and $d$, such that

$$A_s^{(1)} \le (-\kappa + \varepsilon)|h_s^+|^2_{H^1_0(Q)} + C(\varepsilon)|h_s^+|^2_{L_2(Q)}.$$

By Assumption 2.1.4 we obtain

$$A_s^{(3)} = \int_Z \int_Q I_{h_s>0}|g_s(x,z,u_s) - g_s(x,z,v_s)|^2 dx\nu(dz).$$

Hence, by (2.1.32) we have

$$A_s^{(2)} + A_s^{(3)} \le \varepsilon |h_s^+|^2_{H^1_0(Q)} + C(\varepsilon)|h_s^+|^2_{L_2(Q)}.$$

Combining these estimates and using (2.1.25) we have a constant $C$ such that, almost surely

$$|h_t^+|^2_{L^2(Q)} \le C \int_0^t |h_s^+|^2_{L^2(Q)} \, ds + m_t \quad \text{for all } t \in [0,T].$$

Then we have,

$$E|h_t^+|^2_{L^2(Q)} \le C \int_0^t E|h_s^+|^2_{L^2(Q)} \, ds < \infty \quad \text{for all } t \in [0,T],$$

and the result follows by Gronwall's lemma. □

*Proof of Theorem 2.1.4.* We assume again that Assumption 2.1.3 is satisfied by $f$. For the difference $h = u - v$ we have

$$\begin{aligned} h_t = & h_0 + \int_0^t \{\mathscr{L}_s h_s + f_s(u_s, \nabla u_s) - F_s(v_s, \nabla v_s)\} \, ds \\ & + \int_0^t \{\phi_s^{ki} \partial_i h_s + \sigma_s^k(u_s) - \sigma_s^k(v_s)\} \, dw_s^k + \int_0^t \int_F S_{s,\zeta} h_{s-} \tilde{M}(ds, d\zeta) \end{aligned}$$

By Theorem 2.1.5 we have

$$|h_t^+|^2_{L^2(\mathbb{R}^d)} = \int_0^t A_s^{(1)} + A_s^{(2)} + \tilde{\rho}_s(h_s) + \langle \mathscr{I}_s^{(1)} h_s, h_s^+ \rangle \, ds + m_t$$

for a martingale $m_t$. Here $A^{(1)}, A^{(2)}$ are as in (2.1.33), (2.1.34) (with the integration over $\mathbb{R}^d$ instead of $Q$), and $\tilde{\rho}$ is defined in (2.1.28). By using the same arguments as in the previous proof, this time also using (2.1.30), we bring the proof to an end. □

## 2.2 Numerical approximation

In the previous section we showed that the equations under consideration are solvable. However, to obtain an explicit solution is usually impossible, and therefore we are interested in approximating them numerically.

Various methods have been developed to solve SPDEs numerically (see, for example, [19], [24], [23] and [29]). Among the various methods considered in the literature is the method of finite differences. For second order linear SPDEs driven by continuous martingale noise it is well-known that the error of approximation in space is proportional to the parameter $h$ of the finite difference (see, e.g., [34]). In [29], I. Gyöngy and A. Millet consider abstract discretization schemes for stochastic evolution equations driven by continuous martingale noise in the variational framework and, as a particular example, show that the rate of convergence of an Euler-Maruyuma (explicit and implicit) finite difference scheme is of order one in space and one-half in time. More recently, it was shown by I. Gyöngy and N.V. Krylov that under certain regularity conditions, the rate of convergence in space of a semi-discretized finite difference approximation of a linear second order SPDE can be accelerated to any order by Richardson's extrapolation method. For the non-degenerate case, we refer to [27] and [28], and for the degenerate case, we refer to [25]. In [30] and [31], was proved that the same method of acceleration can be applied to time-discretized SPDEs.

Concerning equations involving non-local operators, in spatial dimension one, a finite difference scheme for degenerate integro-differential equations (deterministic) has been studied in [3]. The authors in [3] first approximate the integral operator near the origin with a second derivative operator. The resulting PDE is then non-degenerate and has an integral operator of order zero. The error of this approximation is obtained by means of the probabilistic representation of the solution of both the original equation and the non-degenerate equation. In the second step of their approximation, the authors consider a finite difference scheme and obtain pointwise error estimates of order one in space. As a consequence of the two-step approximation scheme, there are two separate errors for the approximation

In this section, we consider a non-degenerate linear SPIDE and we give a spatial approximation scheme whose error is proportional to the parameter of the discretization. The approximations of the non-local integral operators that we suggest are natural and they fit in the finite-difference framework. We are able to treat the singularity of the integral operators near the origin directly, and therefore we only have to control one error. To obtain error estimates for our approximations, we use the approach of [34], where the discretized equations are first solved as SDEs in Sobolev spaces over $\mathbb{R}^d$ and an error estimate is obtained in Sobolev norms. After obtaining error estimates in Sobolev norms, the Sobolev embedding theorem is used to obtain pointwise error estimates.

For a space-time discretization of these equations we refer to [8], where using the spatial approximation presented here, in combination with an Euler scheme in time we fully discretize the SPIDE under consideration, in a joint work with James-Michael Leahy.

### 2.2.1 A spatial discretization scheme

In this section, $N(dt,dz)$ and $\nu(dz)$ are as in the previous section, with $Z=\mathbb{R}^d$, and in addition we consider a $\sigma$-finite measure $\mu$ on $\mathbb{R}^d$. On the cylinder $[0,T]\times\mathbb{R}^d$, we consider the stochastic integro-differential equation

$$du_t(x) = \left(\mathscr{L}_t u_t(x) + \mathscr{I} u_t(x) + \sum_{i=0}^{d} \partial_i f_t^i(x)\right) dt + \left(\mathscr{M}_t^\varrho u_t(x) + g_t^\varrho(x)\right) dw_t^\varrho$$
$$+ \int_{\mathbb{R}^d} \mathscr{J}(z) u_{t-}(x)\, \tilde{N}(dz, dt) \tag{2.2.35}$$

with initial condition

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}^d,$$

where

$$\mathscr{L}_t \phi(x) = a_t^{ij}(x)\partial_{ij}\phi(x),\ \mathscr{M}_t^\varrho \phi(x) = b_t^{i\varrho}(x)\partial_i \phi, \qquad \text{(sum over } i, j \in \{0, ..., d\}\text{)}$$
$$\mathscr{I}\phi(x) = \int_{\mathbb{R}^d} \left(\phi(x+z) - \phi(x) - \partial_i \phi(x) z^i\right) \mu(dz) \qquad \text{(sum over } i \in \{1, ..., d\}\text{)}, \tag{2.2.36}$$
$$\mathscr{J}(z)\phi(x) = \phi(x+z) - \phi(x),$$

and where for $i, j \in \{0, ..., d\}$ and $\varrho \in \mathbb{N}_+$, $a_t^{ij}$, $f_t$, $b_t^{i\varrho}$, and $g_t^\varrho$ are random functions depending on $(t, x) \in [0, T] \times \mathbb{R}^d$.

Through this section we will assume that

$$\int_{\mathbb{R}^d} |z|^2 \wedge |z|\ \mu(dz) < \infty, \int_{\mathbb{R}^d} |z|^2 \wedge 1\ \nu(dz) < \infty. \tag{2.2.37}$$

Let $m \geq 0$ be an integer.

**Assumption 2.2.1.** For $i, j \in \{0, \ldots, d\}$, $a_t^{ij} = a_t^{ij}(x)$ are real-valued functions that are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable and $b_t^i = (b_t^{i\varrho}(x))_{\varrho=1}^\infty$ are $\ell_2$-valued functions that are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable. Moreover,

(i) for each $(\omega, t) \in \Omega \times [0, T]$, the functions $a^{ij}$ are $\max(m, 1)$-times continuously differentiable in $x$ for $i, j \in \{1, ..., d\}$ and $m$-times continuously differentiable in $x$ for $i = 0$ or $j = 0$. For each $(\omega, t) \in \Omega \times [0, T]$, the functions $b^i$ are $m$-times continuously differentiable in $x$ for $i \in \{0, ..., d\}$. Moreover, there exists a constant $K_m \geq 0$ with such that for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$|\partial^\alpha a_t^{ij}| \leq K_m,\ \forall |\alpha| \leq \max(m, 1),\ \forall i, j \in \{1, ..., d\}$$

$$|\partial^\alpha a_t^{0i}| \vee |\partial^\alpha a_t^{i0}| \vee |\partial^\alpha b_t^i|_{\ell_2} \vee |\partial^\alpha b_t|_{\ell_2} \leq K_m,\ \forall |\alpha| \leq m,\ \forall i\{0, ..., d\}.$$

(ii) there exists a positive constant $\kappa > 0$ such that for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$

and $\xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^{d} \left(2a_t^{ij} - b_t^{i\varrho} b_t^{j\varrho}\right) \xi_i \xi_j \geq \kappa |\xi|^2.$$

We define the following spaces:

$$\mathbb{H}^m := L_2(\Omega \times [0,T], \mathscr{P}; H^m), \ \ \mathbb{H}^m(\ell_2) := L_2(\Omega \times [0,T], \mathscr{P}; H^m(\ell_2))$$

$$\mathbb{H}_2^m(\nu) := L_2(\Omega \times [0,T] \times \mathbb{R}^d, \mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d), dP \otimes dt \otimes d\nu; H^m).$$

**Assumption 2.2.2.** The random function $\psi$ is a $H^m$-valued $\mathscr{F}_0$-measurable random variable, $f^i \in \mathbb{H}^m$ for $i \in \{0, ..., d\}$, and $g \in \mathbb{H}^m(\ell_2)$. Furthermore,

$$\kappa_m^2 := E|\psi|_{H^m}^2 + E\int_0^T \left( \sum_{i=0}^{d} |f_s^i|_{H^m}^2 + |g_s|_{H^m(\ell_2)}^2 \right) ds < \infty.$$

For a real-valued twice continuous differentiable function $u$ on $\mathbb{R}^d$, it is easily seen that

$$u(x+z) - u(x) - \sum_{i=1}^{d} \partial_i u(x) z^i = \int_0^1 \sum_{i,j=1}^{d} \partial_{ij} u(x+\theta z) z^i z^j (1-\theta)\, d\theta, \ \ x \in \mathbb{R}^d.$$

Due to (2.2.37), there exists $\delta > 0$ such that

$$\int_{|z|\leq\delta} |z|^2 \nu(dz) < \frac{\kappa}{12}, \ \int_{|z|\leq\delta} |z|^2 \mu(dz) < \frac{\kappa}{12},$$

$$\int_{|z|>\delta} |z| \nu(dz) + \mu(\{|z| \geq \delta\}) + \nu(\{|z| \geq \delta\}) < \infty. \tag{2.2.38}$$

Let us fix such $\delta$ and notice that $\mathscr{I} = \mathscr{I}_\delta + \mathscr{I}_{\delta^c}$, where

$$\mathscr{I}_\delta \phi(x) = \int_{|z|\leq\delta} \int_0^1 \sum_{i,j=1}^{d} \partial_{ij} \phi(x+\theta z) z^i z^j (1-\theta)\, d\theta$$

and where $\mathscr{I}_{\delta^c}$ is defined as in (2.2.36) with integration over $\{|z| > \delta\}$ instead of $\mathbb{R}^d$.

The solution of equation (2.2.35) is understood as in the previous section, that is, an $L_2(\mathbb{R}^d)$-valued strongly càdlàg adapted process $u$ such that

(i) $u_t \in H^1$ for $dP \otimes dt$-almost every $(\omega, t) \in \Omega \times [0,T]$;

(ii) $E\int_0^T |u_s|_{H^1}^2 ds < \infty$;

(iii) there exists a set $\tilde{\Omega} \subset \Omega$ of probability one such that for all $(\omega, t) \in [0,T] \times \tilde{\Omega}$ and

$\phi \in C_c^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
(u_t,\phi) = (\psi,\phi) + \int_0^t \left( \left( \partial_j u_s, \partial_{-i}(a_s^{ij}\phi) \right) + \langle \mathscr{I} u_s, \phi \rangle + (f_i, \partial_{-i}\phi) \right) \, ds \\
+ \int_0^t \left( b_s^{i\varrho} \partial_i u_s + g_s^\varrho, \phi \right) \, dw_s^\varrho + \int_0^t \int_{\mathbb{R}^d} \left( u_{s-}(\cdot + z) - u_{s-}, \phi \right) \tilde{N}(dz, ds).
\end{aligned}
\tag{2.2.39}
$$

The proof of the following theorem is a simple consequence of Theorem 1.4.1, and can be found in [8].

**Theorem 2.2.1.** *If Assumptions 2.2.1 and 2.2.2 hold with $m \geq 0$, then there exist a unique solution $u$ of* (2.2.35). *Furthermore, $u$ is a strongly càdlàg $H^m$-valued process with probability one and there exists a constant $N$ depending only on $d, m, \kappa, K_m$, and $\nu$ such that*

$$
E \sup_{t \leq T} |u_t|_{H^m}^2 + E \int_0^T |u_s|_{H^{m+1}}^2 \, ds \leq N \kappa_m^2. \tag{2.2.40}
$$

We turn our attention to the discretisation of equation (2.2.35). For each $h \in \mathbb{R} \setminus \{0\}$ and standard basis vector $e_i$, $i \in \{1, \ldots, d\}$, of $\mathbb{R}^d$ we define the first-order difference operator $\delta_{h,i}$ by

$$
\delta_{h,i}\phi(x) = \frac{\phi(x + he_i) - \phi(x)}{h},
$$

for all real-valued functions $\phi$ on $\mathbb{R}^d$. We define $\delta_{h,0}$ to be the identity operator. Notice that for all $\psi, \phi \in L_2(\mathbb{R}^d)$ and $i \in \{1, ..., d\}$, we have

$$
(\phi, \delta_{-h,i}\psi) = -(\delta_{h,i}\phi, \psi).
$$

We introduce the grid $\mathbb{G}^h := \{hz_k : z_k \in \mathbb{Z}^d, k \in \mathbb{N}\}$ with meshsize $|h|$. Let $\ell_2(\mathbb{G}_h)$ be the set of real-valued functions $\phi$ on $\mathbb{G}_h$ such that

$$
|\phi|_{\ell_2(\mathbb{G}_h)}^2 := |h|^d \sum_{x \in \mathbb{G}^h} |\phi(x)|^2 < \infty.
$$

We approximate the operators $\mathscr{L}$ and $\mathscr{M}^\varrho$ by

$$
\begin{aligned}
\mathscr{L}_t^h \phi(x) &= a_t^{ij}(x) \delta_{h,i} \delta_{-h,j} \phi(x), \\
\mathscr{M}_t^{h,\varrho} \phi(x) &= b_t^{i\varrho}(x) \delta_{h,i} \phi(x).
\end{aligned}
$$

In order to approximate the operator $\mathscr{I}$, we approximate $\mathscr{I}_\delta$ and $\mathscr{I}_{\delta^c}$ separately. For

$k \in \mathbb{N}$ and $h \neq 0$, let $A_k^h$ be the rectangle

$$\left(z_k^1 h - \frac{|h|}{2}, z_k^1 h + \frac{|h|}{2}\right] \times \ldots \times \left(z_k^d h - \frac{|h|}{2}, z_k^d h + \frac{|h|}{2}\right],$$

where $z_k^i$, $i = 1, \ldots, d$ are the coordinates of $z_k \in \mathbb{Z}^d$, and set

$$B_k^h := A_k^h \cap \{|z| \leq \delta\}, \quad \bar{B}_k^h := A_k^h \cap \{|z| \geq \delta\}.$$

We approximate $\mathscr{I}_{\delta^c}$ by

$$\mathscr{I}_{\delta^c}^h u(x) := \sum_{k=0}^{\infty} \left( (u(x + hz_k) - u(x)) \bar{\zeta}_{hk} - \sum_{i=1}^{d} \bar{\xi}_{hk}^i \delta_{h,i} u(x) \right), \tag{2.2.41}$$

where

$$\bar{\zeta}_{hk} := \mu(\bar{B}_k^h) \text{ and } \bar{\xi}_{hk}^i := \int_{\bar{B}_k^h} z^i \mu(dz).$$

We continue with the approximation of the operator $\mathscr{I}_\delta$. Notice that

$$\mathscr{I}_\delta \phi(x) = \sum_{k=0}^{\infty} \int_{B_k^h} \int_0^1 \sum_{i,j=1}^{d} z^i z^j \partial_{ij} \phi(x + \theta z)(1 - \theta) d\theta \mu(dz),$$

where there are only a finite number of non-zero terms in the infinite sum over $k$. The closest point in $\mathbb{G}_h$ to any point $z \in B_k^h$ is clearly $hz_k$. This simple observation leads us to the following (intermediate) approximation of $\mathscr{I}_\delta \phi(x)$:

$$\sum_{k=0}^{\infty} \int_0^1 \sum_{i,j=1}^{d} \int_{B_k^h} z^i z^j \partial_{ij} \phi(x + \theta hz_k)(1 - \theta) d\theta \mu(dz).$$

However, in order to ensure that our approximation is well-defined for functions $\phi \in \ell_2(\mathbb{G}_h)$, we need to approximate the integral over $\theta \in [0,1]$. For fixed $h \neq 0$ and $k \in \mathbb{N} \cup \{0\}$, there exist $\sigma(h,k) \in \mathbb{N}$, $r_l^{h,k} \in \mathbb{N} \cup \{0\}$, for $l \in \{0, \ldots, \sigma(h,k) - 1\}$, and real numbers $(\theta_l^{h,k})_{l=0}^{\sigma(h,k)}$ satisfying $0 = \theta_0^{h,k} \leq \cdots \leq \theta_{\sigma(h,k)}^{h,k} = 1$, such that the line segment $\{\theta hz_k\}_{\theta \in [0,1]}$ is contained in the set $\cup_{l=0}^{\sigma(h,k)-1} A_{r_l^{h,k}}^h$, and for $\theta \in (\theta_l^{h,k}, \theta_{l+1}^{h,k})$, we have $\theta hz_k \in A_{r_l^{h,k}}^h$. In particular, for $k = 0$, we have $\sigma(h,0) = 1$, $r_0^{h,0} = 0$, $\theta_0^{h,0} = 0$, and $\theta_0^{h,1} = 1$. Since the diagonal of a $d$-dimensional hypercube with side length $|h|$ has length $\sqrt{d}|h|$, for each $k \in \mathbb{N} \cup \{0\}$, $z \in B_k^h$, and $l \in \{0, \ldots, \sigma(h,k) - 1\}$ we have,

$$|\theta z - hz_{r_l^{h,k}}| \leq |\theta z - \theta hz_k| + |\theta hz_k - hz_{r_l^{h,k}}| \leq \sqrt{d}|h|, \tag{2.2.42}$$

for all $\theta \in (\theta_l^{h,k}, \theta_{l+1}^{h,k})$. Set

$$\zeta_{h,k}^{ij} = \int_{B_k^h} z^i z^j \mu(dz), \quad \bar{\theta}_l^{h,k} = \int_{\theta_l^{h,k}}^{\theta_{l+1}^{h,k}} (1-\theta) d\theta$$

and define the operator

$$\mathscr{I}_\delta^h \phi(x) =: \sum_{k=0}^{\infty} \sum_{l=0}^{\sigma(h,k)-1} \bar{\theta}_l^{h,k} \sum_{i,j=1}^{d} \zeta_{h,k}^{ij} \delta_{h,i} \delta_{-h,j} \phi(x + hz_{r_l^{h,k}}). \tag{2.2.43}$$

There are only finitely many non-zero terms in the infinite sum over $k$. We set $\mathscr{I}^h = \mathscr{I}_\delta^h + \mathscr{I}_{\delta^c}^h$. Let us now introduce the following martingales:

$$m_t^{k,i,h} = \int_0^t \int_{B_k^h} z^i \tilde{N}(dz, dt), \quad \bar{m}_t^{k,h} = \tilde{N}(\bar{B}_k^h, ]0,t]), \tag{2.2.44}$$

and let us consider the equation

$$d\hat{u}_t^h(x) = \left( \mathscr{L}_t^h \hat{u}_t^h(x) + \mathscr{I}^h \hat{u}_t(x) + \sum_{i=0}^{d} \delta_{h,i} f_t^i(x) \right) dt + \sum_{\varrho=1}^{\infty} \left( \mathscr{M}_t^{h,\varrho} \hat{u}_t^h(x) + g_t^\varrho(x) \right) dw_t^\varrho$$

$$+ \sum_{k=1}^{\infty} \sum_{i=1}^{d} \Big( \sum_{l=0}^{\sigma(h,k)-1} \bar{\theta}_l^{h,k} \delta_{h,i} \hat{u}_t^h(x + z_{r_l^{h,k}}) \Big) dm_t^{k,i,h} + \sum_{k=1}^{\infty} \left( \hat{u}_t^h(x + hz_k) - \hat{u}_t^h(x) \right) d\bar{m}_t^{k,h}, \tag{2.2.45}$$

with initial condition

$$\hat{u}_0^h(x) = \psi(x), \tag{2.2.46}$$

for $(t,x) \in [0,T] \times \mathbb{G}^h$.

*Remark* 2.2.1. It is known that for $n > d/2$, $H^n$ is embedded into $C_b(\mathbb{R}^d)$, the space of continuous bounded functions on $\mathbb{R}^d$. Whenever a function is in $H^n$ and $n > d/2$, we always take its continuous version. Moreover , for $|h| < 1$, there exists a constant $N$ depending only on $d$ and $n$ such that for all $\phi \in H^n$,

$$|\phi|_{\ell_2(\mathbb{G}^h)} \leq N|\phi|_{H^n}^2. \tag{2.2.47}$$

For a proof of this embedding we refer to [27]. By (2.2.47), under Assumptions 2.2.1 (i) and 2.2.2 with $m > d/2$, (2.2.45) is an Itô equation in $\ell_2(\mathbb{G}^h)-$ with Lipschitz continuous coefficients, and therefore, it has unique càdlàg $\ell_2(\mathbb{G}^h)-$valued solution.

**Theorem 2.2.2.** *Let Assumptions 2.2.1 and 2.2.2 hold with $m > 2 + d/2$, and let $u$ and $\hat{u}^h$ be the solutions of* (2.2.35) *and* (2.2.45)*, respectively. Then the following estimate* *holds*

$$E\sup_{t\le T}\sup_{x\in\mathbb{G}^h}|u_t(x)-\hat{u}^h_t(x)|^2+E\sup_{t\le T}|u_t-\hat{u}^h|_{\ell_2(\mathbb{G}^h)}\le N|h|^2, \tag{2.2.48}$$

*where $N$ is a constant depending only $d, m, \kappa, K_m, \kappa^2_m$, and $\nu$.*

## 2.2.2 Estimates for the approximation of the operators

In this section, we present some results that will be needed for the proof of Theorem 2.2.2. Let us introduce the operators

$$\mathscr{I}^h_\delta(z)\phi(x) := \sum_{k=0}^{\infty} I_{B^h_k}(z)\sum_{l=0}^{\sigma(h,k)-1}\sum_{i=1}^{d}\bar{\theta}^{h,k}_l z^i\delta_{h,i}\phi(x+hz_{r^{h,k}_l}), \tag{2.2.49}$$

$$\mathscr{I}^h_{\delta^c}(z)\phi(x) := \sum_{k=0}^{\infty} I_{\bar{B}^h_k}(z)[\phi(x+hz_k)-\phi(x)], \tag{2.2.50}$$

$$\mathscr{I}^h(z)\phi(x) := \mathscr{I}^h_\delta(z)\phi(x)+\mathscr{I}^h_{\delta^c}(z)\phi(x). \tag{2.2.51}$$

where $\tilde{\theta}^{k,h}_l := \theta^{h,k}_{l+1}-\theta^{h,k}_l$ , and consider the following Itô equation in $L_2(\mathbb{R}^d)$,

$$du^h_t(x) = (\mathscr{L}^h_t u^h_t(x)+\mathscr{I}^h u^h_t(x)+f_t(x))dt+(\mathscr{M}^{h,\varrho}_t u^h_t(x)+g^\varrho_t)dw^\varrho_t$$
$$+\int_{\mathbb{R}^d}\mathscr{I}^h(z)u^h_t(x)\tilde{N}(dz,dt). \tag{2.2.52}$$

We will show later that if $u^h$ is a solution of (2.2.52) and $u^h\in H^m$ with $m>d/2$, then it can be restricted to the grid $\mathbb{G}^h$, and it agrees with $\hat{u}^h$.

The following is very well known and a proof for a more general statement can be found in [27].

**Lemma 2.2.3.** *Let $m\ge 0$ be an integer. Then for all $u\in H^{m+2}$, $v\in H^{m+3}$ and $i,j\in\{1,..,d\}$ we have*

$$|\delta_{h,i}u-\partial_i u|_{H^m}\le\frac{1}{2}|h||u|_{H^{m+2}},$$

$$|\delta_{h,i}\delta_{-h,j}u-\partial_{ij}u|_{H^m}\le N|h||u|_{H^{m+3}},$$

*where $N$ depends only on $d$ and $n$.*

**Lemma 2.2.4.** *Let $m\ge 0$ be an integer. For all $u\in H^{m+3}$ we have*

$$|\mathscr{I}u-\mathscr{I}^h u|_{H^m}\le N|h||u|_{H^{m+3}}, \tag{2.2.53}$$

*where the constant $N$ depends only on $d$, $n$ and $\nu$.*

*Proof.* It suffices to show (2.2.53) for $u\in C^\infty_c(\mathbb{R}^d)$. We begin with $m=0$. A simple calculation shows that

$$\mathscr{I}_{\delta^c} u(x) - \mathscr{I}^h_{\delta^c} u(x)$$

$$= \sum_{k=0}^{\infty} \int_{\bar{B}^h_k} \left( u(x+z) - u(x+hz_k) - \sum_{i=1}^{d} z^i (\partial_i u(x) - \delta^h_i u(x)) \right) \mu(dz)$$

$$= \sum_{k=0}^{\infty} \int_{\bar{B}^h_k} \int_0^1 \sum_{i=1}^{d} (z^i - hz^i_k) \partial_i u(x + hz_k + \theta(z - hz_k)) d\theta \mu(dz)$$

$$- \sum_{k=0}^{\infty} \int_{\bar{B}^h_k} \sum_{i=1}^{d} z^i (\partial_i u(x) - \delta^h_i u(x)) \mu(dz).$$

By Minkowski's inequality, we get

$$\begin{aligned} |\mathscr{I}_{\delta^c} u - \mathscr{I}^h_{\delta^c} u|_{L_2} \le & \sum_{k=0}^{\infty} \int_{\bar{B}^h_k} \sum_{i=1}^{d} |z^i - hz^i_k| |\partial_i u|_{L_2} \mu(dz) \\ & + \sum_{k=0}^{\infty} \int_{\bar{B}^h_k} \sum_{i=1}^{d} |z^i| |\partial_i u(x) - \delta^h_i u(x)|_{L_2} \mu(dz) \\ & \le N|h||u|_{H^3} + N \sum_{i=1}^{d} |\partial_i u(x) - \delta^h_i u(x)|_{L_2}, \end{aligned}$$

since $|z - hz_k| \le |h|\sqrt{d}/2$. Thus, by Lemma 2.2.3, we have

$$|\mathscr{I}_{\delta^c} u - \mathscr{I}^h_{\delta^c} u|_{L_2} \le N|h||u|_{H^3}. \tag{2.2.54}$$

We also have

$$\begin{aligned} & \mathscr{I}_{\delta} u(x) - \mathscr{I}^h_{\delta} u(x) = \\ & \sum_{k=0}^{\infty} \int_{B^h_k} \sum_{l=0}^{\sigma(h,k)-1} \int_{\theta^{h,k}_l}^{\theta^{h,k}_{l+1}} \sum_{i,j=1}^{d} z^i z^j [\partial_i \partial_j u(x + \theta z) \\ & - \delta_{h,i} \delta_{-h,j} u(x + hz_{r^{h,k}_l})](1-\theta) d\theta \mu(dz). \end{aligned} \tag{2.2.55}$$

Notice that for each $i, j \in \{1, \dots d\}$,

$$\begin{aligned} & \partial_i \partial_j u(x + \theta z) - \delta_{h,i} \delta_{-h,j} u(x + hz_{r^{h,k}_l}) \\ = & \partial_i \partial_j u(x + \theta z) - \partial_i \partial_j u(x + hz_{r^{h,k}_l}) \\ & + \partial_i \partial_j u(x + hz_{r^{h,k}_l}) - \delta_{h,i} \delta_{-h,j} u(x + hz_{r^{h,k}_l}) \\ = & \int_0^1 \sum_{q=1}^{d} (\theta z^q - hz^q_{r^{h,k}_l}) \partial_q \partial_i \partial_j u \left( x + hz_{r^{h,k}_l} + \rho(\theta z - hz_{r^{h,k}_l}) \right) d\rho \\ & + \partial_{ij} u(x + hz_{r^{h,k}_l}) - \delta_{h,i} \delta_{-h,j} u(x + hz_{r^{h,k}_l}). \end{aligned}$$

and also by 2.2.42, we have $|\theta z^q - hz^q_{r_l^{h,k}}| \le N|h|$. Hence, substituting the above relation in (2.2.55), using Minkowski's inequality, (2.2.38), and Lemma 2.2.3, we obtain

$$|\mathscr{I}_\delta u - \mathscr{I}_\delta^h u|_{L_2} \le |h|N|u|_{H^3}. \tag{2.2.56}$$

Combining (2.2.54) and (2.2.56), we have (2.2.53) for $m = 0$. The case $m > 0$ follows from the case $m = 0$, since for a multi-index $\alpha$, we have

$$\partial^\alpha(\mathscr{I}u - \mathscr{I}^h u) = \mathscr{I}\partial^\alpha u - \mathscr{I}^h \partial^\alpha u.$$

□

**Lemma 2.2.5.** *Let $m \ge 0$ be an integer. For all $u \in H^{m+2}$ we have*

$$\int_{\mathbb{R}^d} |\mathscr{J}^h(z)u - \mathscr{J}(z)u|^2_{H^m} \nu(dz) \le N|h|^2|u|^2_{H^{m+2}}, \tag{2.2.57}$$

*where the constant $N$ depends only on $n, d$ and $\nu$.*

*Proof.* It suffices to prove the lemma for $u \in C_c^\infty(\mathbb{R}^d)$ and $m = 0$. We have

$$\mathscr{J}_\delta(z)u(x) - \mathscr{J}_\delta^h(z)u(x) =$$

$$\sum_{k=0}^\infty I_{B_k^h}(z) \sum_{l=0}^{\sigma(h,k)-1} \int_{\theta_l^{h,k}}^{\theta_{l+1}^{h,k}} \sum_{i=1}^d z^i(\partial_i u(x+\theta z) - \delta_{h,i} u(x + hz_{r_l^{h,k}}))d\theta.$$

Notice that

$$\partial_i u(x+\theta z) - \delta_{h,i} u(x + hz_{r_l^{h,k}})$$

$$= \int_0^1 \sum_{j=1}^d \partial_i\partial_j u(x + \theta z + \rho(\theta z - hz_{r_l^{h,k}}))(\theta z^j - hz^j_{r_l^{h,k}})d\rho$$

$$+\partial_i u(x + hz_{r_l^{h,k}}) - \delta_{h,i} u(x + hz_{r_l^{h,k}}).$$

Thus, by Remark 2.2.42 and Lemma 2.2.3, we get

$$|\mathscr{J}_\delta^h(z)u - \mathscr{J}_\delta(z)u|^2_{L_2} \le I_{|z|\le\delta}|z|^2 N|h|^2|u|^2_{L_2},$$

and hence by (2.2.38), we obtain

$$\int_{\mathbb{R}^d} |J_\delta^h(z)u - \mathscr{J}_\delta(z)u|^2_{L_2} \nu(dz) \le N|h|^2|u|^2_{L_2}. \tag{2.2.58}$$

We also have

$$|\mathscr{J}_{\delta^c}(z)u(x) - \mathscr{J}^h_{\delta^c}(z)u(x)| = \sum_{k=0}^{\infty} I_{\bar{B}^h_k}(z)|u(x+z) - u(x+hz_k)|$$

$$\leq \sum_{k=0}^{\infty} I_{\bar{B}^h_k}(z)\int_0^1 \sum_{i=1}^{d} |\partial_i u(x+hz_k+\rho(z-hz_k))||z^i - hz^i_k|d\rho.$$

Consequently,

$$|\mathscr{J}^h_{\delta^c}(z)u - \mathscr{J}_{\delta^c}(z)u|^2_{L_2} \leq I_{|z|>\delta}N|h|^2|u|^2_{H^1},$$

which implies that

$$\int_{\mathbb{R}^d} |\mathscr{J}^h_{\delta^c}(z)u - \mathscr{J}_{\delta^c}(z)u|^2_{L_2}\nu(dz) \leq N|h|^2|u|^2_{H^1}. \tag{2.2.59}$$

Combining (2.2.59) and (2.2.58), we have (2.2.57) for $m=0$. The case $m>0$ follows from the case $m=0$, since for a multi-index $\alpha$, we have

$$\partial^\alpha(Ju - \mathscr{J}^h u) = J\partial^\alpha u - \mathscr{J}^h\partial^\alpha u.$$

□

**Lemma 2.2.6.** *Let Assumption 2.2.1 hold. Then for any $\phi \in L_2(\mathbb{R}^d)$ we have*

$$2(\phi, \mathscr{L}^h_t\phi) + \sum_{\varrho=1}^{\infty} |\mathscr{M}^{h,\varrho}_t\phi|^2_{L_2} \leq -\frac{\kappa}{2}\sum_{i=1}^{d}|\delta_{h,i}\phi|^2_{L_2} + N|\phi|^2_{L_2}, \tag{2.2.60}$$

$$\mathbb{Q}_t(\phi) := 2(\phi, \mathscr{L}^h_t\phi) + 2(\phi, \mathscr{I}^h\phi) + \sum_{\varrho=1}^{\infty} |\mathscr{M}^{h,\varrho}_t\phi|^2_{L_2}$$

$$+ \int_{\mathbb{R}^d} |\mathscr{J}^h(z)\phi|^2_{L_2}\nu(dz) \leq -\frac{\kappa}{4}\sum_{i=1}^{d}|\delta_{h,i}\phi|^2_{L_2} + N|\phi|^2_{L_2}, \tag{2.2.61}$$

$$\int_{\mathbb{R}^d} |\mathscr{J}^h(z)\phi|^2_{L_2}\nu(dz) \leq \frac{\kappa}{12}\sum_{i=1}^{d}|\delta_{h,i}\phi|^2_{L_2} + N|\phi|^2_{L_2}, \tag{2.2.62}$$

*where $N$ depends only on $\kappa, K_1, d$, and $\nu$.*

*Proof.* Inequality (2.2.60) is proved in [27]. For (2.2.61) it suffices to show that

$$2(\phi, \mathscr{I}^h\phi) + \int_{\mathbb{R}^d} |\mathscr{J}^h(z)\phi|^2_{L_2}\nu(dz) \leq \frac{\kappa}{4}\sum_{i=1}^{d}|\delta_{h,i}\phi|^2_{L_2} + N|\phi|^2_{L_2}.$$

We have

$$2(\phi, \mathscr{I}^h_\delta\phi) = \sum_{k=0}^{\infty}\int_{B^h_k}\sum_{l=1}^{\sigma(k,h)-1}\bar{\theta}^{h,k}_l z^i z^j \int_{\mathbb{R}^d}\delta_{h,i}\delta_{-h,j}\phi(x+hz_{r^{h,k}_l})\phi(x)\,dx\,\mu(dz).$$

Since

$$\int_{\mathbb{R}^d} \delta_{h,i}\delta_{-h,j}\phi(x+hz_{r_l^{h,k}})\phi(x)\,dx = \int_{\mathbb{R}^d} \delta_{h,i}\phi(x+hz_{r_l^{h,k}})\delta_{h,j}\phi(x)\,dx,$$

we get by Hölder's inequality

$$2(\phi, \mathscr{I}_\delta^h\phi) \le \int_{|z|\le\delta} |z|^2 \sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2 \mu(dz) \le \frac{\kappa}{12}\sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2. \tag{2.2.63}$$

We calculate

$$2(\phi, \mathscr{I}_{\delta^c}^h\phi) = 2\sum_{k=0}^\infty \int_{\bar{B}_k^h}\int_{\mathbb{R}^d} [\phi(x+hz_k) - \phi(x) - z^i\delta_{h,i}\phi(x)]\phi(x)\,dx\mu(dz)$$

$$\le 2\int_{|z|\ge\delta} z^i |\delta_{h,i}\phi|_{L_2}|\phi|_{L_2}\mu(dz) \le 2\int_{|z|\ge\delta}|z|\mu(dz)\Big(\sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2\Big)^{1/2}|\phi|_{L_2}$$

$$\le \frac{\kappa}{12}\sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2 + N|\phi|_{L_2}^2, \tag{2.2.64}$$

where the last follows from Young's inequality. We also have

$$|\mathscr{J}_\delta^h(z)\phi|_{L_2}^2 = \int_{\mathbb{R}^d}\Big(\sum_{k=0}^\infty I_{B_k^h}(z)\sum_{l=0}^{\sigma(k,h)-1}\bar{\theta}_l^{h,k} z^i\delta_{h,i}\phi(x+z_{r_l^{h,k}})\Big)^2 dx$$
$$\le \sum_{k=0}^\infty I_{B_k^h}(z)|z|^2\sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2,$$

and

$$|\mathscr{J}_{\delta^c}^h(z)\phi|_{L_2}^2 \le 4\sum_{k=0}^\infty I_{\bar{B}_k^h}(z)|\phi|_{L_2}^2.$$

Hence,

$$\int_{\mathbb{R}^d} |\mathscr{J}^h(z)\phi|_{L_2}^2 \nu(dz) \le \frac{\kappa}{12}\sum_{i=1}^d |\delta_{h,i}\phi|_{L_2}^2 + N|\phi|_{L_2}^2,$$

which, proves (2.2.62) and also combined with (2.2.63) and (2.2.64) proves (2.2.61).

□

The proof of the following theorem goes in the same way as the proof of Theorem 3.2 from [27], but we include it for the convenience of the reader.

**Theorem 2.2.7.** *Let $f^i \in \mathbb{H}^m$ for $i \in \{0,...,d\}$, $g \in \mathbb{H}^m(l_2)$ and $r \in \mathbb{H}^m(\nu)$. Suppose that Assumption 2.2.1 (i) holds. For any $h \neq 0$, there exists a unique $L_2(\mathbb{R}^d)-$valued* *solution of*

$$du_t^h(x) = (\mathscr{L}_t^h u_t^h(x) + \mathscr{I}^h u_t^h(x) + \delta_{h,i} f_t^i(x))dt + (\mathscr{M}_t^{h,\varrho} u_t^h(x) + g_t^\varrho(x))dw_t^\varrho$$
$$+ \int_{\mathbb{R}^d} \mathscr{J}^h(z) u_{t-}^h(x) + r_t(x,z)\tilde{N}(dz,dt), \qquad (2.2.65)$$

*for any $H^m$−valued $\mathscr{F}_0$−measurable initial condition $\psi$. This solution is $H^m$−valued càdlàg process. Moreover, if Assumption 2.2.1 (ii) holds, then*

$$E\sup_{t\le T} |u_t^h|_{H^m}^2 + E\int_0^T \sum_{i=1}^d |\delta_{h,i} u_t^h|_{H^m}^2 \le NE|\psi|_{H^m}^2$$
$$+ NE\int_0^T \Big( \sum_{i=0}^d |f_t^i|_{H^m}^2 + |g_t|_{H^m(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{H^m}^2 \nu(dz)\Big)dt, \qquad (2.2.66)$$

*where $N$ depends only on $d, m, \kappa, K_m, T$ and $\nu$.*

*Proof.* Note that (2.2.65) is an SDE in $L_2(\mathbb{R}^d)$ with Lipschitz continuous coefficients, and consequently there exists a unique solution. Similarly, it has a unique solution with values in $H^m$, and since $H^m \subset L_2(\mathbb{R}^d)$, the first assertion of the theorem follows. It is also easy to see that estimate (2.2.66) holds with a constant depending on $h$. By Ito's formula we have

$$d|u_t^h|_{L_2}^2 = \{\mathbb{Q}_t(u_t^h) + |g_t|_{L_2(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz)$$
$$+ 2(\delta_{h,i} u_t^h, f_t^i) + 2(b_t^{i\varrho}\delta_{h,i} u_t^h, g_t^\varrho) + 2\int_{\mathbb{R}^d} (\mathscr{J}_t^h(z) u_t^h, r_t(z))\nu(dz)\}dt$$
$$+ 2(u_t^h, b^{i\varrho}\delta_{h,i} u_t^h + g_t^\varrho)dw_t^\varrho + 2\int_{\mathbb{R}^d} (\mathscr{J}^h(z) u_{t-}^h + r_t(z), u_{t-}^h)\tilde{N}(dz,dt)$$
$$+ \int_{\mathbb{R}^d} |\mathscr{J}^h(z) u_{t-}^h + r_t(z)|_{L_2}^2 \tilde{N}(dz,dt) \qquad (2.2.67)$$

Using Young's inequality, (2.2.61) and (2.2.62) we obtain

$$E|u_t^h|_{L_2}^2 + \frac{\kappa}{8} E\int_0^t \sum_{i\neq 0} |\delta_{h,i} u_s^h|_{L_2}^2 ds \le E|\psi|_{L_2}^2$$

$$+ NE\int_0^T \Big( |u_s^h|_{L_2}^2 + \sum_{i=0}^d |f_t^i|_{L_2}^2 + |g_t|_{L_2(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz)\Big)dt < \infty. \qquad (2.2.68)$$

By Gronwall's lemma we get

$$E\int_0^t \sum_{i=0}^d |\delta_{h,i} u_s^h|_{L_2}^2 \, ds \le NE|\psi|_{L_2}^2 + NE\int_0^T \Big( \sum_{i=0}^d |f_t^i|_{L_2}^2 + |g_t|_{L_2(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz) \Big) dt. \quad (2.2.69)$$

Going back to (2.2.67) and applying Davis' inequality (Chapter IV, Theorem 4.1 in [56]) where appropriate, we get

$$\begin{aligned} E\sup_{t\le T} |u_t|_{L_2}^2 \le N' A_1 + N' A_2 + NE|\psi|_{L_2}^2 & \\ + NE\int_0^T \Big( \sum_{i=0}^d |f_t^i|_{L_2}^2 + |g_t|_{L_2(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz) \Big) dt & \\ + 2\int_0^T \int_{\mathbb{R}^d} |\mathscr{I}^h(z) u_t^h + r_t(z)|_{L_2}^2 \nu(dz) dt & \end{aligned}$$

where

$$\begin{aligned} A_1 &= E\Big( \int_0^T \sum_{\varrho=1}^\infty (u_t^h, b^{i\varrho}\delta_{h,i} u_t^h + g_t^\varrho)^2 dt \Big)^{1/2} \\ &\le (4N')^{-1} E\sup_{t\le T} |u_t^h|_{L_2}^2 + NE\int_0^T (\sum_{i=0}^d |\delta_{h,i} u_t^h|_{L_2}^2 + |g_t|_{L_2(\ell_2)}^2) dt, \\ A_2 &= E\Big( \int_0^T \int_{\mathbb{R}^d} (\mathscr{I}^h(z) u_t^h + r_t(z), u_t^h)^2 \nu(dz) dt \Big)^{1/2} \\ &\le (4N')^{-1} E\sup_{t\le T} |u_t^h|_{L_2}^2 + NE\int_0^T (\sum_{i=0}^d |\delta_{h,i} u_t^h|_{L_2}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz)) dt. \end{aligned}$$

Using this, (2.2.62) and (2.2.69) we see that

$$E\sup_{t\le T} |u_t|_{L_2}^2 + E\int_0^T \sum_{i=0}^d |\delta_{h,i} u_t^h|_{L_2}^2 \, dt \le NE|\psi|_{L_2}^2 + NE\int_0^T \Big( \sum_{i=0}^d |f_t^i|_{L_2}^2 + |g_t|_{L_2(\ell_2)}^2 + \int_{\mathbb{R}^d} |r_t(z)|_{L_2}^2 \nu(dz) \Big) dt. \quad (2.2.70)$$

This proves the theorem for $m = 0$. Now suppose we have (2.2.66) with $n$ instead of $m$, for some $n \in \{0, ..., m-1\}$. Let $\alpha$ be a multi-index, with $|\alpha| = n+1$. By differentiating (2.2.65) and using the notation $\tilde{\phi} = \partial^\alpha \phi$, we have

$$\begin{aligned} d\tilde{u}_t^h(x) = (\mathscr{L}_t^h \tilde{u}_t^h(x) + \mathscr{I}^h \tilde{u}_t^h(x) + \delta_{h,i} \bar{f}_t^i(x)) dt + (\mathscr{M}_t^{h,\varrho} \tilde{u}_t^h(x) + \bar{g}_t^\varrho(x)) dw_t^\varrho & \\ + \int_{\mathbb{R}^d} \mathscr{I}^h(z) \tilde{u}_{t-}^h(x) + \tilde{r}_t(x,z) \tilde{N}(dz, dt), & \end{aligned}$$

where

$$\bar{f}^i = \tilde{f}^i,\ i \neq 0,\ \bar{f}^0 = \tilde{f}^0 + \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} a^{ij} \delta_{h,i}\delta_{-h,j}\partial^\beta u^h,$$

$$\bar{g}^\varrho = \tilde{g}^\varrho + \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} b^{i\varrho} \delta_{h,i}\partial^\beta u^h.$$

We proceed as before, and for $|\beta| = n$ and $j \neq 0$ we use the estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |\tilde{u}^h_s \delta_{h,i}\delta_{-h,j}\partial^\beta u^h_s|\,dx &\leq \gamma |\partial_j \partial^\beta \delta_{h,i} u^h_s|^2_{L_2} + \gamma^{-1}|\tilde{u}^h_s|^2_{L_2} \\ &\leq \gamma \sum_{|\zeta|=n+1} |\delta_{h,i}\partial^\zeta u^h_s|^2_{L_2} + \gamma^{-1}|\tilde{u}^h_s|^2_{L_2},\end{aligned}$$

where $\gamma > 0$ is arbitrary, to obtain

$$\begin{aligned}E|\tilde{u}^h_t|^2_{L_2} + E\int_0^t \sum_{i=0}^d |\delta_{h,i}\tilde{u}^h_s|^2_{L_2}\,ds &\leq NE|\psi|^2_{L_2} \\ &+ (2d^n)^{-1} E\int_0^t \sum_{\substack{|\zeta|=n+1 \\ i\in\{0,..,d\}}} |\delta_{h,i}\partial^\zeta u^h_s|^2_{L_2}\,ds \\ + NE\int_0^t \Big(|\tilde{u}^h_s|^2_{L_2} + \sum_{i=0}^d |f^i_t|^2_{H^{n+1}} &+ |g_t|^2_{H^{n+1}(\ell_2)} + \int_{\mathbb{R}^d} |r_t(z)|^2_{H^{n+1}}\nu(dz)\Big)\,dt. \qquad (2.2.71)\end{aligned}$$

By writing the above relation for all multi-indices $\alpha$, with $|\alpha| = n+1$, and summing them up, we see that the term with the factor $(2d^n)^{-1}$ can be dropped. After that, by proceeding as before we obtain

$$\begin{aligned}E\sup_{t\leq T} |\tilde{u}^h_t|^2_{L_2} + E\int_0^T \sum_{i=0}^d |\delta_{h,i}\tilde{u}^h_s|^2_{L_2}\,ds &\leq NE|\psi|^2_{L_2} \\ + NE\int_0^T \Big(\sum_{i=0}^d |f^i_t|^2_{H^{n+1}} + |g_t|^2_{H^{n+1}(\ell_2)} &+ \int_{\mathbb{R}^d} |r_t(z)|^2_{H^{n+1}}\nu(dz)\Big)\,dt, \qquad (2.2.72)\end{aligned}$$

which combined with the induction hypothesis, gives estimate (2.2.66) with $n+1$ instead of $m$. This brings the proof to an end. □

### 2.2.3 Proof of Theorem 2.2.2

**Theorem 2.2.8.** *Let Assumptions 2.2.1 and 2.2.2 hold with $m \geq 2$, and let $u$ and $u^h$ be the solutions of* (2.2.35) *and* (2.2.52)*, respectively, with initial condition $\psi$. Then*

$$E\sup_{t\leq T} |u_t - u^h_t|^2_{H^{m-2}} + E\int_0^T \sum_{i=1}^d |\delta_{h,i}u_t - \delta_{h,i}u^h_t|^2_{H^{m-2}}\,ds \leq N|h|^2\kappa^2_m, \qquad (2.2.73)$$

*where N is a constant depending only on* $d, m, \kappa, K_m$, *and* $\nu$.

*Proof.* Set $r^h := u_t^h - u_t$ and set

$$F_t^h := (\mathscr{L}_t^h - \mathscr{L}_t + \mathscr{I}^h - \mathscr{I})u_t + \sum_{i=1}^d (\delta_{h,i} - \partial_i) f_t^i,$$
$$G_t^{h,\varrho} := \left(\mathscr{M}_t^{h,\varrho} - M_t^\varrho\right) u_t,$$
$$H_t^h(z) := \left(\mathscr{J}^h(z) - \mathscr{J}(z)\right) u_{t-}.$$

By virtue of Assumption 2.2.1 (i), Lemmas 2.2.3, 2.2.4, 2.2.5, and estimate (2.2.40), we have

$$E\int_0^T |F_t|^2_{H^{m-2}} dt \le N|h|^2 E\int_0^T \left(|u_t|^2_{H^{m+1}} + \sum_{i=1}^d |f_t^i|_{H^m}\right) dt \le N|h|^2\kappa_m^2$$

and

$$E\int_0^T \left(|G_t^h|_{\ell_2(H^{m-2})} + \int_{\mathbb{R}^d} |H_t^h(z)|^2_{H^{m-2}} \nu(dz)\right) dt$$
$$\le N|h|^2 E\int_0^T |u_t|^2_{H^m} dt \le N|h|^2\kappa_m^2,$$

where $N$ is a constant depending only on $d, m, K_m$, and $\nu$. Therefore, $F^h \in \mathbb{H}^{m-2}$, $G^h \in \mathbb{H}^{m-2}(\ell_2)$, and $H^h \in \mathbb{H}^{m-2}(\nu)$. Note that $r^h$ is a $H^{m-2}$-valued strongly càdlàg process, satisfying the equation

$$dr_t^h = \left((\mathscr{L}_t^h + \mathscr{I}^h)r_t^h + F_t^h\right) dt + \left(\mathscr{M}_t^{h,\varrho} r_t^h + G_t^{h,\varrho}\right) dw_t^\varrho$$
$$+ \int_{\mathbb{R}^d} \left(\mathscr{J}^h(z) r_{t-}^h + H_t^h(z)\right) \tilde{N}(dz, dt), \quad t \in (0, T], \tag{2.2.74}$$

with $r_0^h = 0$. Hence, by Theorem 2.2.7,

$$E\sup_{t\le T} |r_t^h|^2_{H^{m-2}} + E\int_0^T \sum_{i=1}^d |\delta_{h,i} r_t^h|^2_{H^{m-2}} ds \le N|h|^2\kappa_m^2, \tag{2.2.75}$$

where $N$ is a constant depending only on $d, m, \kappa, K_m$, and $\nu$. □

By virtue of Sobolev's embedding theorem and (2.2.47), as in [27], we obtain the following direct corollary of Theorem 2.2.8, with the convention that $\{1, ..., d\}^0 = \{0, ..., 0\}$.

**Corollary 2.2.9.** *Suppose the assumptions of Theorem 2.2.8 hold with* $m > n + 2 + d/2$, *where* $n \ge 0$ *is an integer. Then for all* $\lambda = (\lambda^1, \dots, \lambda^n) \in \{1, \dots, d\}^n$ *and* $\delta_{h,\lambda} =$

$\delta_{h,\lambda^1}\cdots\delta_{h,\lambda^n}$, *we have*

$$E\sup_{t\le T}\sup_{x\in\mathbb{R}^d}|\delta_{h,\lambda}u_t(x)-\delta_{h,\lambda}u^h_t(x)|^2 + E\sup_{t\le T}|\delta_{h,\lambda}u_t-\delta_{h,\lambda}u^h_t|^2_{\ell_2(\mathbb{G}_h)} \le N|h|^2\kappa^2_m,$$

*where $N$ is a constant depending only on $d, m, \kappa, K_m, \nu$ and $n$ .*

*Proof of Theorem 2.2.2.* Let $u^h$ be the unique solution of (2.2.52) with initial condition $\psi$. By virtue of Theorems 2.2.1 and 2.2.7, $u$, $u^h$ are strongly càdlàg $H^m$-valued process. Notice that $u$, $u^h$ are càdlàg processes with values in $C_b(\mathbb{R}^d)$. We only need to show that almost surely $\hat{u}^h_t(x) = u^h_t(x)$, for all $t \in [0,T]$ and $x \in \mathbb{G}_h$, and then Theorem 2.2.2 follows from Corollary 2.2.9. To this end, let $\mathfrak{J}$ be the embedding operator form $H^m$ to $\ell_2(\mathbb{G}_h)$. By applying $\mathfrak{J}$ to both sides of (2.2.65) we see that $\mathfrak{J}u^h$ satisfies (2.2.45) and therefore the result follows from the uniqueness of the solution of (2.2.45). □

# Chapter 3

# Stochastic Partial differential equations

In this chapter we turn our attention to Stochastic PDEs. The equation under consideration is

$$du_t = (\mathscr{L}_t u_t + \partial_i f^i_t + f^0_t)dt + (\mathscr{M}^k_t u_t + g^k_t)dw^k_t,\ u_0 = \psi, \tag{3.0.1}$$

for $(t,x) \in [0,T] \times Q$, where the operators $\mathscr{L}_t$, and $\mathscr{M}^k_t$ are given by

$$\mathscr{L}_t u = \partial_j(a^{ij}_t \partial_i u) + b^i_t \partial_i u + c_t u,\ \mathscr{M}^k_t u = \sigma^{ik}_t \partial_i u_t + \mu^k_t u. \tag{3.0.2}$$

Here, $Q$ is a bounded Lipschitz domain in $\mathbb{R}^d$ and he summation for the parameters $i$ and $j$ takes place over the set $\{1,...,d\}$, and for $k$ over the positive integers.

Solvability of these type of equations in Sobolev spaces has been extensively studied, mainly through a variational approach ($L_2$-theory, see for example [43], [57], and [54]) and an analytic approach ($L_p$-theory, see [40], [37]). By the results in these two approaches, one can obtain regularity (integrability or/and differentiability) of the solution under the assumption that the data of the equation are regular enough. In this chapter we deal with another question. Under the minimal conditions that guarantee the existence of a solution, can we derive some further information for the solution? As we will see in the next sections, the answer is affirmative.

The results of this chapter are from [6] and [5], two joint works with Máté Gerencsér.

In order to ease the notation followed in the previous chapter, for $p,r,q \in [1,\infty]$, let us set

$$L_{r,q} := L_r([0,T];L_q(Q))$$
$$\mathbf{L}_p := L_p(\Omega,\mathscr{F}_0;L_p(Q))$$
$$\mathbb{L}_p := L_p(\Omega\times[0,T],\mathscr{P};L_p(Q))$$
$$\mathbb{L}_p(l_2) := L_p(\Omega\times[0,T],\mathscr{P};L_p(Q;l_2))$$

and also

$$|\cdot|_p = |\cdot|_{L_p(Q)},\ \|\cdot\|_{r,q} := |\cdot|_{L_{r,q}},\ \|\cdot\|_r := \|\cdot\|_{r,r}.$$

The assumptions posed are the following.

**Assumption 3.0.3.** i) The coefficients $a^{ij}$, $b^i$ and $c$ are real-valued $\mathscr{P}\times\mathscr{B}(Q)$ measurable functions on $\Omega\times[0,T]\times Q$ and are bounded by a constant $K\geq 0$, for any $i,j=1,...,d$. The coefficients $\sigma^i=(\sigma^{ik})_{k=1}^{\infty}$ and $\mu=(\mu^k)_{k=1}^{\infty}$ are $l_2$-valued $\mathscr{P}\times Q$-measurable functions on $\Omega\times[0,T]\times Q$ such that

$$\sum_i\sum_k|\sigma_t^{ik}(x)|^2+\sum_k|\mu_t^k(x)|^2\leq K \quad \text{for all } \omega,\ t \text{ and } x,$$

ii) $f^l$, for $l\in\{0,...,d\}$, and $g=(g^k)_{k=1}^{\infty}$ are $\mathscr{P}\times\mathscr{B}(Q)$-measurable functions on $\Omega\times[0,T]\times Q$ with values in $\mathbb{R}$ and $l_2$, respectively, such that

$$E(\sum_{l=0}^{d}\|f^l\|_2^2+\||g|_{l_2}\|_2^2)<\infty$$

iii) $\psi$ is an $\mathscr{F}_0$-measurable random variable in $L_2(Q)$ such that $E|\psi|_2^2<\infty$

**Assumption 3.0.4** (Parabolicity)**.** There exists a constant $\lambda>0$ such that for all $\omega,t,x$ and for all $\xi=(\xi_1,...\xi_d)\in\mathbb{R}^d$ we have

$$a_t^{ij}(x)\xi_i\xi_j-\frac{1}{2}\sigma_t^{ik}(x)\sigma_t^{jk}(x)\xi_i\xi_j\geq\lambda|\xi|^2,$$

We will refer to the constants $K,T,\lambda,d$ and $|Q|$, where the latter is the Lebesgue measure of $Q$, as structure constants.

The solution of equation (3.0.1) is again understood to be an $L_2(Q)$-valued, $\mathscr{F}_t-$adapted, strongly continuous process $(u_t)_{t\in[0,T]}$, such that

i) $u_t\in H_0^1(Q)$, for $dP\times dt$ almost every $(\omega,t)\ \in\Omega\times[0,T]$

ii) $E\int_0^T(|u_t|_2^2+|\nabla u_t|_2^2)dt<\infty$

iii) for all $\phi \in C_c^\infty(Q)$ we have with probability one

$$(u_t, \phi) = (\psi, \phi) + \int_0^t (a_s^{ij} \partial_i u_s + f_s^j, \partial_{-j}\phi) + (b_s^i \partial_i u_s + c_s u_s + f_s^0, \phi) ds$$

$$+ \int_0^t (\mathcal{M}_s^k u_s + g_s^k, \phi) dw_s^k,$$

for all $t \in [0, T]$.

## 3.1 Global Boundedness

We are interested in boundedness properties of solutions of (3.0.1) under Assumptions 3.0.3 and 3.0.4. The corresponding problem in the deterministic case, has been extensively studied. The first results for non-degenerate equations in divergence form are due to [18] and [51] for the elliptic case and [53] for both elliptic and parabolic equations. Later, the techniques of [51] were extended to the parabolic case in [52]. The approach of [18] was also applied for parabolic equations (see for example [48]). In these articles, boundedness is obtained as an intermediate step in order to prove Hölder continuity and Harnack inequalities. Another proof of the parabolic Harnack inequality was given in [16]. Hölder estimates and Harnack inequality were also obtained in [59] and [47], for elliptic and parabolic equations in non-divergence form. More recently, these results were also proved for a wider class of parabolic equations, including, for example, the $p$-Laplacian as the driving operator (see [14] and references therein).

Boundedness of solutions of SPDEs can be proved through embedding theorems of Sobolev spaces. Such results can be obtained from $L_p-$theory, see e.g. [40], for equations considered on the whole space. This approach, however, requires some regularity of the coefficients. For SPDEs where these regularity assumptions are dropped or weakened, the literature has been expanding recently. In [55] a maximum principle is obtained for a class of backward SPDEs. Under the additional assumption $\sigma = 0$, variants of the problem are treated in [9], [33], and [36], with methods that strongly rely on the absence of derivatives of $u$ in the noise term. In [10], through the technique of Moser's iteration, introduced in [51], boundedness results are derived without posing regularity assumptions on the coefficients, for a class of quasilinear equations, by staying in the $L_2-$framework. This served as a main motivation to our work. However, in [10], it is assumed that there exist constants $\lambda > \beta > 0$, such that for any $\xi \in \mathbb{R}^d$, one has $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ and $(72 + 1/2)\sigma^{ik}\sigma^{jk}\xi_i\xi_j \leq \beta|\xi|^2$, which seems a rather strong condition. In the work presented here, only the classical stochastic parabolicity condition will be assumed in order to get estimates for the uniform bound of the solution of equation (3.0.1). We note that the results of this section can also be extended to quasilinear equations under suitable conditions. Having accessibility in mind, such generalizations are not included here.

### 3.1.1 The global $L_\infty$-estimates

Notice that under Assumptions 3.0.3 and 3.0.4, by virtue of Theorem 2.1.1 for example, equation (3.0.1) admits a unique solution $u$, and the following estimate holds

$$E\sup_{0\le t\le T}|u_t|_2^2 \le NE(|\psi|_2^2+\sum_{l=0}^{d}\|f^l\|_2^2+\||g|_{l_2}\|_2^2), \tag{3.1.3}$$

where $N = N(d,K,\lambda,T)$.

Let

$$\Gamma_d = \left\{(r,q)\in(1,\infty]^2 \,\middle|\, \frac{1}{r}+\frac{d}{2q}<1\right\}.$$

The following is our main result.

**Theorem 3.1.1.** *Suppose that Assumptions 3.0.3 and 3.0.4 hold, and let u be the unique solution of equation* (3.0.1). *Then for any* $(r,q)\in\Gamma_d$ *and* $\eta>0$,

$$E\|u\|_\infty^\eta \le NE(|\psi|_\infty^\eta+\|f^0\|_{r,q}^\eta+\sum_{i=1}^{d}\|f^i\|_{2r,2q}^\eta+\||g|_{l_2}\|_{2r,2q}^\eta), \tag{3.1.4}$$

*where* $N = N(\eta,r,q,d,K,\lambda,|Q|,T)$.

*Remark* 3.1.1. Notice that in particular we obtain

$$E\|u\|_\infty^2 \le NE(|\psi|_\infty^2+\sum_{l=0}^{d}\|f^l\|_\infty^2+\||g|_{l_2}\|_\infty^2), \tag{3.1.5}$$

and by interpolating between (3.1.3) and (3.1.5), for any $p\ge 2$, one obtains

$$E\sup_{0\le t\le T}|u_t|_p^2 \le NE(|\psi|_p^2+\sum_{l=0}^{d}\|f^l\|_p^2+\||g|_{l_2}\|_p^2)$$

where $N$ can be chosen to be independent of $p$. In fact, such a uniform estimate for the $L_p$-norms of the solutions is equivalent to (3.1.5).

Theorem 3.1.1 will be proved in Section 3.1.3. We will adapt the technique of Moser from [51] and [52]. The strategy, in short, and for the moment ignoring the contributions from the initial and free data, is the following: with a suitable intermediate norm $[u]_n$ we obtain estimates of the form $E\|u\|_{r_{n+1},q_{n+1}}^\eta \le N(n)E[u]_n^\eta$,

$E[u]_n^\eta \le N(n)E\|u\|_{r_n,q_n}^\eta$, with $r_n, q_n \nearrow \infty$. The constants $N(n)$ in these estimates are controlled so that one can iterate this procedure, take limits, and finally obtain estimates for the supremum norm.

### 3.1.2 Itô's formula for the $L_p$-norm and energy-type estimates

In this section we gather some results that we will need for the proof of Theorem 3.1.1. First let us invoke (II.3.4) from [48].

**Lemma 3.1.2.** *Suppose that $v \in L_2([0,T], H_0^1(Q)) \cap L_\infty([0,T], L_2(Q))$. Let $r, q \in (2,\infty)$, satisfying $1/r + d/2q = d/4$. Then $v$ belongs to $L_r([0,T], L_q(Q))$, and*

$$\left(\int_0^T \left(\int_Q |v_t|^q dx\right)^{r/q} dt\right)^{2/r} \le N\left(\sup_{0\le t\le T}\int_Q |v_t|^2 dx + \int_0^T \int_Q |\nabla v_t|^2 dx dt\right)$$

*with $N = N(d, |Q|, T)$.*

The right hand side of the inequality in the above lemma plays the role of the "suitable norm" (for $n = 2$), which was discussed at the end of the previous section. We are also going to use the following result (see Proposition IV.4.7 and Exercise IV.4.31/1, [56]).

**Proposition 3.1.3.** *Let $X$ be a non-negative, adapted, right-continuous process, and let $A$ be a non-decreasing, continuous process such that*

$$E(X_\tau|\mathscr{F}_0) \le E(A_\tau|\mathscr{F}_0)$$

*for any bounded stopping time $\tau$. Then for any $\sigma \in (0,1)$*

$$E\sup_{t\le T} X_t^\sigma \le \sigma^{-\sigma}(1-\sigma)^{-1} E A_T^\sigma.$$

In order to obtain our estimates, we will need an Itô formula for $|u_t|_p^p$. The difference between the next lemma and Lemma 8 in [10], is that we obtain supremum (in time) estimates, that are essential for having (3.1.7) almost surely, for all $t \in [0,T]$. Therefore, we give a whole proof for the sake of completeness.

**Lemma 3.1.4.** *Suppose that $u$ satisfies equation (3.0.1), $f^l \in \mathbb{L}_p$, for $l \in \{0,...,d\}$, $g \in \mathbb{L}_p(l_2)$, and $\psi \in \mathbf{L}_p$ for some $p \ge 2$. Then there exists a constant $N = N(d,K,\lambda,p)$, such that*

$$E\sup_{t\le T}|u_t|_p^p + E\int_0^T\int_Q |\nabla u_s|^2 |u_s|^{p-2} dx ds \le NE(|\psi|_p^p + \sum_{l=0}^d \|f^l\|_p^p + \||g|_{l_2}\|_p^p). \quad (3.1.6)$$

*Moreover, almost surely*

$$
\begin{aligned}
\int_Q |u_t|^p dx = \int_Q |u_0|^p dx + p\int_0^t \int_Q (\sigma_s^{ik}\partial_i u_s + \mu^k u_s + g^k) u_s |u_s|^{p-2} dx dw_s^k \\
+ \int_0^t \int_Q -p(p-1) a_s^{ij} \partial_i u_s |u_s|^{p-2} \partial_j u_s - p(p-1) f_s^i \partial_i u_s |u_s|^{p-2} dx ds \\
+ \int_0^t \int_Q p(b_s^i \partial_i u_s + c_s u_s + f_s^0) u_s |u_s|^{p-2} dx ds \\
+ \frac{1}{2} p(p-1) \int_0^t \int_Q \sum_{k=1}^{\infty} |\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g_s^k|^2 |u_s|^{p-2} dx ds,
\end{aligned}
\tag{3.1.7}
$$

*for any $t \le T$.*

*Proof.* Consider the functions

$$
\phi_n(r) = \begin{cases} |r|^p & \text{if } |r| < n \\ n^{p-2} \frac{p(p-1)}{2} (|r| - n)^2 + p n^{p-1}(|r| - n) + n^p & \text{if } |r| \ge n. \end{cases}
$$

Then one can see that $\phi_n$ are twice continuously differentiable, and satisfy

$$
|\phi_n(x)| \le N|x|^2, \ |\phi_n'(x)| \le N|x|, \ |\phi_n''(x)| \le N,
$$

where $N$ depends only on $p$ and $n \in \mathbb{N}$. We also have that for any $r \in \mathbb{R}$, $\phi_n(r) \to |r|^p$, $\phi_n'(r) \to p|r|^{p-2} r$, $\phi_n''(r) \to p(p-1)|r|^{p-2}$, as $n \to \infty$, and

$$
\phi_n(r) \le N|r|^p, \ \phi_n'(r) \le N|r|^{p-1}, \ \phi_n''(r) \le N|r|^{p-2}, \tag{3.1.8}
$$

where $N$ depends only on $p$. Then for each $n \in \mathbb{N}$ we have almost surely

$$
\begin{aligned}
\int_Q \phi_n(u_t) dx = \int_Q \phi_n(u_0) dx + \int_0^t \int_Q (\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g^k) \phi_n'(u_s) dx dw_s^k \\
+ \int_0^t \int_Q -a_s^{ij} \partial_i u_s \phi_n''(u_s) \partial_j u_s - f^i \phi_n''(u_s) \partial_i u_s dx ds \\
+ \int_0^t \int_Q b_s^i \partial_i u_s \phi_n'(u_s) + c_s u_s \phi_n'(u_s) + f_s^0 \phi_n'(u_s) dx ds \\
+ \frac{1}{2} \int_0^t \int_Q \sum_{k=1}^{\infty} |\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g_s^k|^2 \phi_n''(u_s) dx ds,
\end{aligned}
\tag{3.1.9}
$$

for any $t \in [0, T]$ (see for example, Section 3 in [42]). By Young's inequality, and the parabolicity condition we have for any $\varepsilon > 0$,

$$\begin{aligned}\int_Q \phi_n(u_t)\,dx \le m_t^{(n)} &+ \int_Q \phi_n(u_0)\,dx \\ &+ \int_0^t \int_Q (-\lambda|\nabla u_s|^2 + \varepsilon|\nabla u_s|^2 + N\sum_{i=1}^d |f_s^i|^2)\phi_n''(u_s)\,dx\,ds \\ &+ \int_0^t \int_Q (\epsilon|\nabla u_s|^2 + N|u_s|^2 + N\sum_{k=1}^\infty |g_s^k|^2)\phi_n''(u_s)\,dx\,ds \\ &+ \int_0^t \int_Q (b_s^i \partial_i u_s + c_s u_s + f_s^0)\phi_n'(u_s)\,dx\,ds, \end{aligned} \tag{3.1.10}$$

where $N = N(d, K, \epsilon)$, and $m_t^{(n)}$ is the martingale from (3.1.9). One can check that the following inequalities hold,

i) $|r\phi_n'(r)| \le p\phi_n(r)$

ii) $|r^2\phi''(r)| \le p(p-1)\phi_n(r)$

iii) $|\phi_n'(r)|^2 \le 4p\, \phi_n''(r)\phi_n(r)$

iv) $[\phi_n''(r)]^{p/(p-2)} \le [p(p-1)]^{p/(p-2)}\phi_n(r)$,

which combined with Young's inequality imply,

i) $\partial_i u_s \phi_n'(u_s) \le \epsilon\phi_n''(u_s)|\partial_i u_s|^2 + N\phi_n(u_s)$

ii) $|u_s\phi_n'(u_s)| \le p\phi_n(u_s)$

iii) $|f_s^0\phi_n'(u_s)| \le |f_s^0||\phi_n''(u_s)|^{1/2}|\phi_n(u_s)|^{1/2} \le N|f_s^0|^p + N\phi_n(u_s)$

iv) $|u_s|^2\phi_n''(u_s) \le N\phi_n(u_s)$

v) $\sum_k |g_s^k|^2\phi_n''(u_s) \le N\phi_n(u_s) + N\Big(\sum_k |g_s^k|^2\Big)^{p/2}$

vi) $\sum_{i=1}^d |f_s^i|^2\phi_n''(u_s) \le N\phi_n(u_s) + N\sum_{i=1}^d |f_s^i|^p$,

where $N$ depends only on $p$ and $\epsilon$.

By choosing $\epsilon$ sufficiently small, and taking expectations we obtain

$$E\int_Q \phi_n(u_t)\,dx + EI_A\int_0^t \int_Q |\nabla u_s|^2\phi_n''(u_s)\,dx\,ds \le NE\mathcal{K}_t + N\int_0^t E\int_Q \phi_n(u_s)\,dx\,ds,$$

where $N = N(d, p, K, \lambda)$ and

$$\mathcal{K}_t = |\psi|_p^p + \int_0^t \sum_{l=0}^d |f_s^l|_p^p + |g_s|_p^p\,ds.$$

By Gronwall's lemma we get

$$E\int_Q \phi_n(u_t)dx + E\int_0^t\int_Q |\nabla u_s|^2\phi_n''(u_s)dxds \le NE\mathcal{K}_t$$

for any $t \in [0,T]$, with $N = N(T,d,p,K,\lambda)$. Going back to (3.1.10), using the same estimates, and the above relation, by taking suprema up to $T$ we have

$$E\sup_{t\le T}\int_Q \phi_n(u_t)dx \le NEI_A\mathcal{K}_t + E\sup_{t\le T}|m_t^{(n)}|.$$

$$\le NE\mathcal{K}_T + NE\left(\int_0^T\sum_k\left(\int_Q|\sigma^{ik}\partial_i u_s + \mu^k u_s + g_s^k||\phi_n''(u_s)\phi_n(u_s)|^{1/2}dx\right)^2 ds\right)^{1/2}$$

$$\le NE\mathcal{K}_T + NE\left(\int_0^T\int_Q(|\nabla u_s|^2 + |u_s|^2 + \sum_{k=1}^\infty |g_s^k|^2)\phi_n''(u_s)dx\int_Q\phi_n(u_s)dxds\right)^{1/2}$$

$$\le NE\mathcal{K}_T + \frac{1}{2}E\sup_{t\le T}\int_Q\phi_n(u_t)dx < \infty,$$

where $N = N(T,d,p,K,\lambda)$. Hence,

$$E\sup_{t\le T}\int_Q\phi_n(u_t)dx + E\int_0^T\int_Q|\nabla u_s|^2\phi_n''(u_s)dxds \le NE\mathcal{K}_T,$$

and by Fatou's lemma we get (3.1.6). For (3.1.7), we go back to (3.1.9), and by letting a subsequence $n(k)\to\infty$ and using the dominated convergence theorem, we see that each term converges to the corresponding one in (3.1.7) almost surely, for all $t \le T$. This finishes the proof.

□

**Corollary 3.1.5.** *Let $\gamma > 1$ and denote $\kappa = 4\gamma/(\gamma-1)$. Suppose furthermore that $r, r', q, q' \in (1,\infty)$, satisfying $1/r + 2/r' = 1$ and $1/q + 2/q' = 1$. Suppose that $u$ satisfies the conditions of Lemma 3.1.4 for any $p \in \{2\gamma^n, n \in \mathbb{N}\}$. Then, for any $p \in \{2\gamma^n, n\in\mathbb{N}\}$, almost surely, for all $t \le T$*

$$\int_Q |u_t|^p dx + \frac{p^2}{4}\int_0^t\int_Q|\nabla u_t|^2|u_t|^{p-2}dxds \le N'm_t$$

$$+N\left[|\psi|_p^p + p^\kappa\|u\|^p_{r'p/2,q'p/2} + p^{-p}(\|f^0\|^p_{r,q} + \sum_{i=1}^d\|f^i\|^p_{2r,2q} + \||g|_{l_2}\|^p_{2r,2q})\right], \quad (3.1.11)$$

*where $m_t$ is the martingale from* (3.1.7)*, and $N, N'$ are constants depending only on $K, d, T, \lambda, |Q|, r, q$.*

*Proof.* By Lemma 3.1.4, the parabolicity condition, and Young's inequality we have

$$\int_Q |u_t|^p dx + \frac{p^2}{4}\int_0^t \int_Q |\nabla u_s|^2 |u_s|^{p-2} dx ds \le N' m_t + N_1 \Bigg( \int_Q |\psi|^p dx$$

$$+ \int_0^t \left[ \int_Q p^2 |u_s|^p + p|f_s^0||u_s|^{p-1} + p^2 \sum_{i=1}^d |f_s^i|^2 |u_s|^{p-2} + p^2 |g_s|_{l_2}^2 |u_s|^{p-2} dx \right] ds \Bigg).$$

Then by Hölder's inequality we have

$$\int_0^t \int_Q |f_s^0||u_s|^{p-1} dx ds \le \|f^0\|_{r,q} \|u\|^{p-1}_{q'(p-1)/2, r'(p-1)/2},$$

and by Young's inequality we obtain

$$\begin{aligned} p\|f^0\|_{r,q}\|u\|^{p-1}_{q'(p-1)/2,r'(p-1)/2} &\le p^{-p}\|f^0\|^p_{r,q} + p^{\kappa}\|u\|^p_{r'(p-1)/2,q'(p-1)/2} \\ &\le p^{-p}\|f^0\|^p_{r,q} + N_2 p^{\kappa}\|u\|^p_{r'p/2,q'p/2}. \end{aligned}$$

Similarly, for $n \ge 1$,

$$\begin{aligned} p^2 \int_0^t \int_Q |f_s^i|^2 |u_s|^{p-2} dx ds &\le p^2 \|f^i\|^2_{2r,2q} \|u\|^{p-2}_{r'(p-2)/2, q'(p-2)/2} \\ &\le p^{-p}\|f^i\|^p_{2r,2q} + p^{\kappa}\|u\|^p_{r'(p-2)/2,q'(p-2)/2} \\ &\le p^{-p}\|f^i\|^p_{2r,2q} + N_3 p^{\kappa}\|u\|^p_{r'p/2,q'p/2}. \end{aligned}$$

The same holds for $g$ in place of $f^i$. The case $n = 0$ can be covered separately with another constant $N_4$, and then $N$ can be chosen to be $\max\{N_1(N_2 + N_3), N_4\}$. This finishes the proof.

□

**Lemma 3.1.6.** *Suppose that $u$ satisfies equation* (3.0.1)*, $f^l \in \mathbb{L}_p$, for $l \in \{0, ..., d\}$, $g \in \mathbb{L}_p(l_2)$, and $\psi \in \mathbf{L}_p$ for some $p \ge 2$. Then for any $0 < \eta < p$, and for any $\epsilon > 0$,*

$$E\left( \sup_{t \le T} |u_t|_p^p + \frac{p^2}{4} E \int_0^T \int_Q |\nabla u_s|^2 |u_s|^{p-2} dx ds \right)^{\eta/p}$$

$$\le \epsilon E\|u\|^{\eta}_{\infty} + N(\epsilon, p) E \left[ |\psi|_p^{\eta} + \|f^0\|_1^{\eta} + \sum_{i=1}^d \|f^i\|_2^{\eta} + \||g|_{l_2}\|_2^{\eta} \right]$$

*where $N(\epsilon, p)$ is a constant depending only on $\epsilon, \eta, K, d, T, \lambda, |Q|$, and $p$.*

*Proof.* As in the proof of corollary 3.1.5, for any $\mathscr{F}_0-$measurable set $B$, we have almost surely

$$I_B \int_Q |u_t|^p dx + \frac{p^2}{4} I_B \int_0^t \int_Q |\nabla u_s|^2 |u_s|^{p-2} dx ds \le N' I_B m_t + N_1 I_B \Bigg( \int_Q |\psi|^p dx$$

$$+\int_0^t \left[\int_Q p^2|u_s|^p + p|f_s^0||u_s|^{p-1} + p^2\sum_{i=1}^d |f_s^i|^2|u_s|^{p-2} + p^2|g_s|_{l_2}^2|u_s|^{p-2}dx\right]ds\Bigg), \tag{3.1.12}$$

for any $t \in [0,T]$. The above relation, by virtue of Gronwal's lemma implies that for any stopping time $\tau \le T$

$$\sup_{t\le T} EI_B\int_Q |u_{t\wedge\tau}|^p dx + EI_B\int_0^\tau\int_Q |\nabla u_s|^2|u_s|^{p-2}dxds \le NEI_B\mathscr{V}_\tau, \tag{3.1.13}$$

where

$$\mathscr{V}_t := \int_Q |\psi|^p dx + \int_0^t\int_Q |f_s^0||u_s|^{p-1} + \sum_{i=1}^d |f_s^i|^2|u_s|^{p-2} + |g_s|_{l_2}^2|u_s|^{p-2}dxds.$$

Going back to (3.1.12), and taking suprema up to $\tau$ and expectations, and having in mind (3.1.13), gives

$$E\sup_{t\le\tau} I_B\int_Q |u_t|^p dx \le NE\sup_{t\le\tau} I_B|m_t| + NEI_B\mathscr{V}_\tau.$$

By the Burkholder-Gundy-Davis inequality and (3.1.13) we have

$$\begin{aligned}
E\sup_{t\le\tau} I_B|m_t| &\le NEI_B\left(\int_0^\tau\left(\int_Q |u_t|^{p-2}\left(|\nabla u_t| + |u_t| + |g|_{l_2}\right)dx\right)^2 dt\right)^{1/2}\\
&\le NEI_B\left(\int_0^\tau\int_Q |u_t|^p dx\int_Q (|\nabla u_t|^2 + |u_t|^2 + |g|_{l_2}^2)|u|^{p-2}dxdt\right)^{1/2}\\
&\le \frac{1}{2}E\sup_{t\le\tau} I_B\int_Q |u_t|^p dx + NEI_B\mathscr{V}_\tau.
\end{aligned}$$

Hence,

$$E\sup_{t\le\tau} I_B\int_Q |u_t|^p dx \le NEI_B\mathscr{V}_\tau,$$

which combined with (3.1.13), by virtue of Lemma 3.1.3 gives

$$E\left(\sup_{t\le T}|u_t|_p^p + \frac{p^2}{4}E\int_0^T\int_Q |\nabla u_s|^2|u_s|^{p-2}dxds\right)^{\eta/p} \le NE\mathscr{V}_T^{\eta/p}$$

$$\begin{aligned}
&\le NE\left[|\psi|_p^p + \|u\|_\infty^{p-1}\|f^0\|_1 + \|u\|_\infty^{p-2}\left(\sum_{i=1}^d \|f^i\|_2^2 + \||g|_{l_2}\|_2^2\right)\right]^{\eta/p}\\
&\le \epsilon E\|u\|_\infty^\eta + NE\left[|\psi|_p^\eta + \|f^0\|_1^\eta + \sum_{i=1}^d \|f^i\|_2^\eta + \||g|_{l_2}\|_2^\eta\right],
\end{aligned}$$

which brings the proof to an end. □

### 3.1.3 Proof of Theorem 3.1.1

*Proof.* Throughout the proof, the constants $N$ in our calculations will be allowed to depend on $\eta, r, q$ as well as on the structure constants. Notice that we may, and we will assume that $r, q < \infty$. Without loss of generality we assume that the right hand side in (3.1.4) is finite. Also, in the first part of the proof we make the assumption that $\psi$, $f^l$, $l = 0, \ldots, d$, and $g$ are bounded by a constant $M$. in particular, by (3.1.6), $u \in L_\eta(\Omega, L_{r,q})$ for any $\eta, r, q$.

Let us introduce the notation

$$\mathscr{M}_{r,q,p}(t) = \|\mathbf{1}_{[0,t]} f^0\|_{r,q}^p + \sum_{i=1}^d \|\mathbf{1}_{[0,t]} f^i\|_{2r,2q}^p + \|\mathbf{1}_{[0,t]} |g|_{l_2}\|_{2r,2q}^p.$$

Since $(r,q) \in \Gamma_d$, if we define $r'$ and $q'$ by $1/r + 2/r' = 1$, $1/q + 2/q' = 1$, we have

$$\frac{d}{4} < \frac{1}{r'} + \frac{d}{2q'} =: \gamma \frac{d}{4}$$

for some $\gamma > 1$. Then $\hat{r} = \gamma r'$ and $\hat{q} = \gamma q'$ satisfy

$$\frac{1}{\hat{r}} + \frac{d}{2\hat{q}} = \frac{d}{4}.$$

By applying Lemma 3.1.2 to $\hat{r}, \hat{q}$, and $\bar{v} = |v|^{p/2}$, we have, for any $p \geq 2$

$$E\left[|\psi|_\infty^\eta \vee \left(\int_0^T \left(\int_Q |v_t|^{\hat{q}p/2} dx\right)^{\hat{r}/\hat{q}} dt\right)^{2\eta/\hat{r}p}\right]$$

$$\leq \left[E|\psi|_\infty^\eta \vee N^{\eta/p}\left(\sup_{0\leq t\leq T}\int_Q |v_t|^p dx + \frac{p^2}{4}\int_0^T\int_Q |\nabla v_t|^2 |v_t|^{p-2} dx dt\right)^{\eta/p}\right]. \quad (3.1.14)$$

To estimate the right-hand side above, first notice that, if $p = 2\gamma^n$ for some $n$, then by taking supremum in (3.1.11), we have for any stopping time $\tau \leq T$, and any $\mathscr{F}_0$-measurable set $B$,

$$I_B \sup_{0\leq s\leq\tau}\int_Q |v_s|^p dx$$

$$\leq N I_B\left(|\psi|_\infty^p + p^\kappa \|\mathbf{1}_{[0,\tau]} v\|_{r'p/2, q'p/2}^p + p^{-p}\mathscr{M}_{r,q,p}(\tau)\right) + N' I_B \sup_{0\leq s\leq\tau} |m_s|, \quad (3.1.15)$$

By the Davis inequality we can write

$$E I_B \sup_{0\leq s\leq\tau} |m_s| \leq N E I_B\left(\int_0^\tau \sum_k \left(\int_Q p(\sigma_s^{ik}\partial_i v_s + \mu^k v_s + g^k) v_s |v_s|^{p-2} dx\right)^2 ds\right)^{1/2}$$

$$\le NEI_B\left(\sup_{0\le s\le\tau}\int_Q|v_s|^p\,dx\right)^{1/2}\left(\int_0^\tau\int_Q p^2\sum_k|\sigma_s^{ik}\partial_i v_s+\mu^k v_s+g^k|^2|v_s|^{p-2}\,dx\,ds\right)^{1/2}.$$

Applying Young's inequality and recalling the already seen estimates in the proof of Corollary 3.1.5 (i) for the second term yields

$$EI_B\sup_{0\le s\le\tau}|m_s|\le\varepsilon EI_B\sup_{0\le s\le\tau}\int_Q|v_s|^p\,dx$$

$$+\frac{N}{\varepsilon}EI_B\left(p^2\int_0^\tau\int_Q|\nabla v_s|^2|v_s|^{p-2}\,dx\,ds+p^\kappa\|\mathbf{1}_{[0,\tau]}v\|^p_{r'p/2,q'p/2}+p^{-p}\|\mathbf{1}_{[0,\tau]}|g|_{l_2}\|^p_{2r,2q}\right)$$

for any $\varepsilon>0$. With the appropriate choice of $\varepsilon$, combining this with (3.1.15) and using (3.1.11) once again, now without taking supremum, we get

$$EI_B\left(\sup_{0\le s\le\tau}\int_Q|v_s|^p\,dx+\frac{p^2}{4}\int_0^\tau\int_Q|\nabla v_s|^2|v_s|^{p-2}\,dx\,ds\right)$$

$$\le NEI_B\left(|\psi|^p_\infty+p^2\int_0^\tau\int_Q|\nabla v_s|^2|v_s|^{p-2}\,dx\,ds+p^\kappa\|\mathbf{1}_{[0,\tau]}v\|^p_{r'p/2,q'p/2}+p^{-p}\mathcal{M}_{r,q,p}(\tau)\right)$$

$$\le NEI_B\left(|\psi|^p_\infty+p^\kappa\|\mathbf{1}_{[0,\tau]}v\|^p_{r'p/2,q'p/2}+p^{-p}\mathcal{M}_{r,q,p}(\tau)\right)+N'EI_Bm_\tau,$$

and the last expectation vanishes. Now consider

$$X_t=|\psi|^p_\infty\vee\left(\sup_{0\le s\le t}\int_Q|v_s|^p\,dx+\frac{p^2}{4}\int_0^t\int_Q|\nabla v_s|^2|v_s|^{p-2}\,dx\,ds\right)$$

and

$$A_t=Cp^\kappa\left(|\psi|^p_\infty\vee\|\mathbf{1}_{[0,t]}v\|^p_{r'p/2,q'p/2}+p^{-p}\mathcal{M}_{r,q,p}(t)\right)$$

for a large enough, but fixed $C$. The argument above gives that

$$EI_BX_\tau\le EI_B\left(|\psi|^p_\infty+\sup_{0\le s\le\tau}\int_Q|v_s|^p\,dx+\frac{p^2}{4}\int_0^\tau\int_Q|\nabla v_s|^2|v_s|^{p-2}\,dx\,ds\right)$$

$$\le NEI_B\left(|\psi|^p_\infty+p^\kappa\|\mathbf{1}_{[0,\tau]}v\|^p_{r'p/2,q'p/2}+p^{-p}\mathcal{M}_{r,q,p}(\tau)\right)\le EI_BA_\tau.$$

Therefore the condition of Proposition 3.1.3 is satisfied, and thus for $\eta<p$ we obtain

$$E\left(|\psi|^p_\infty\vee\left(\sup_{0\le t\le T}\int_Q|v_t|^p\,dx+\frac{p^2}{4}\int_0^T\int_Q|\nabla v_t|^2|v_t|^{p-2}\,dx\,dt\right)\right)^{\eta/p}$$

$$\le (Np^{\kappa+1})^{\eta/p}\frac{p}{p-\eta}E\left(|\psi|_\infty^p \vee \|v\|^p_{r'p/2,q'p/2}+p^{-p}\mathcal{M}_{r,q,p}(T)\right)^{\eta/p}$$
$$\le (Np^{\kappa+1})^{\eta/p}\frac{p}{p-\eta}E\left(|\psi|_\infty^\eta \vee \|v\|^\eta_{r'p/2,q'p/2}+p^{-\eta}\mathcal{M}_{r,q,\eta}(T)\right). \tag{3.1.16}$$

Let us choose $p=p_n=2\gamma^n$ for $n\ge 0$, and use the notation $c_n=(Np_n^{\kappa+1})^{\eta/p_n}\frac{p_n}{p_n-\eta}$. Upon combining (3.1.14) and (3.1.16), for $p_n>\eta$ we can write the following inequality, reminiscent of Moser's iteration:

$$E|\psi|_\infty^\eta \vee \|v\|^\eta_{r'p_{n+1}/2,q'p_{n+1}/2}\le c_nE\left[|\psi|_\infty^\eta \vee \|v\|^\eta_{r'p_n/2,q'p_n/2}+Np_n^{-\eta}\mathcal{M}_{r,q,\eta}(T)\right]. \tag{3.1.17}$$

Consider the minimal $n_0=n_0(d,\eta)$ such that $p_{n_0}>2\eta$. Taking any integer $m\ge n_0$ we have

$$\prod_{n=n_0}^m c_n\le \prod_{n=n_0}^m (N\gamma^{\kappa+1})^{\eta n/2\gamma^n}e^{2\eta/2\gamma^n}$$
$$=\exp\left[\log(N\gamma^{\kappa+1})\sum_{n=n_0}^m\frac{\eta n}{2\gamma^n}+\sum_{n=n_0}^m\frac{\eta}{\gamma^n}\right]\le N_0,$$

where $N_0$ does not depend on $m$. Also,

$$N\sum_{n=n_0}^m p_n^{-\eta}\le N_1,$$

where $N_1$ does not depend on $m$. Therefore, by iterating (3.1.17) we get

$$\liminf_{m\to\infty}E|\psi|_\infty^\eta \vee \|v\|^\eta_{r'p_m/2,q'p_m/2}\le N_0N_1E\mathcal{M}_{r,q,\eta}(T)$$
$$+N_0E|\psi|_\infty^\eta \vee \|v\|^\eta_{r'(p_{n_0}+1)/2,q'(p_{n_0}+1)/2},$$

and thus by Fatou's lemma

$$E\|v\|_\infty^\eta\le NE(|\psi|_\infty^\eta \vee \|v\|^\eta_{r'(p_{n_0}+1)/2,q'(p_{n_0}+1)/2}+\mathcal{M}_{r,q,\eta}(T)), \tag{3.1.18}$$

in particular, the left-hand side is finite. By Lemma 3.1.6 we get

$$E\left(|\psi|_\infty^p\vee\left(\sup_{0\le t\le T}\int_Q|v_t|^p\,dx+\frac{p^2}{4}\int_0^T\int_Q|\nabla v_t|^2|v_t|^{p-2}\,dx\,dt\right)\right)^{\eta/p}$$
$$\le \epsilon E\|v\|_\infty^\eta+N(\epsilon,p)E\left(|\psi|_\infty^\eta+\mathcal{M}_{1,1,\eta}(T)\right) \tag{3.1.19}$$

for any $\epsilon>0$. Combining (3.1.14) and (3.1.19) for $p=p_{n_0}$ gives

$$E|\psi|_\infty^\eta \vee \|v\|^\eta_{r'(p_{n_0}+1)/2,q'(p_{n_0}+1)/2}=E|\psi|_\infty^\eta \vee \|v\|^\eta_{\hat{r}p_{n_0}/2,q'p_{n_0}/2}$$

$$\leq \epsilon E\|v\|_\infty^\eta + N(\epsilon, p_{n_0})E\left(|\psi|_\infty^\eta + \mathcal{M}_{1,1,\eta}(T)\right). \tag{3.1.20}$$

Choosing $\epsilon$ sufficiently small, plugging (3.1.20) into (3.1.18), and rearranging yields the desired inequality

$$E\|v\|_\infty^\eta \leq NE(|\psi|_\infty^\eta + \mathcal{M}_{r,q,\eta}(T)). \tag{3.1.21}$$

As for the general case, set

$$\psi^{(n)} = (-n) \vee (\psi \wedge n), \quad f^{l,(n)} = (-n) \vee (f^l \wedge n), \quad g^{k,(n)} = (-n/k) \vee (g^k \wedge (n/k)),$$

define $\mathcal{M}_{r,q,p}^{(n)}$ correspondingly, and let $v^n$ be the solution of the corresponding equation. This new data is now bounded by a constant, so the previous argument applies, and thus

$$E\|v^n\|_\infty^\eta \leq NE(|\psi^{(n)}|_\infty^\eta + \mathcal{M}_{r,q,\eta}^{(n)}(T) \leq NE(|\psi|_\infty^\eta + \mathcal{M}_{r,q,\eta}(T)).$$

Since $v^n \to v$ in $\mathbb{L}_2$, for a subsequence $k(n)$, $v^{k(n)} \to v$ for almost every $\omega, t, x$. In particular, almost surely $\|v\|_\infty \leq \liminf_{n\to\infty} \|v^{k(n)}\|_\infty$, and by Fatou's lemma

$$E\|v\|_\infty^\eta \leq \liminf_{n\to\infty} E\|v^{k(n)}\|_\infty^\eta \leq NE(|\psi|_\infty^\eta + \mathcal{M}_{r,q,\eta}(T)).$$

□

## 3.2 Application to semilinear equations

In this section, we will use the uniform norm estimates obtained in the previous section, to construct solutions for the following equation

$$du_t = (\mathcal{L}_t u_t + f_t(u_t))dt + (\mathcal{M}_t^k u_t + g_t^k)dw_t^k, u_0 = \psi \tag{3.2.22}$$

for $(t, x) \in [0, T] \times Q$, where $f$ is a real function defined on $\Omega \times [0, T] \times Q \times \mathbb{R}$ and is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R})-$measurable.

**Assumption 3.2.1.** The function $f$ satisfies the following

i) for all $r, r' \in \mathbb{R}$ and for all $(\omega, t, x)$ we have

$$(r - r')(f_t(x, r) - f_t(x, r')) \leq K|r - r'|^2$$

ii) For all $(\omega, t, x)$, $f_t(x, r)$ is continuous in $r$

iii) for all $N>0$, there exists a function $h^N \in \mathbb{L}_2$ with $E\|h^N\|_\infty < \infty$, such that for any $(\omega, t, x)$

$$|f_t(x,r)| \leq |h_t^N(x)|,$$

whenever $|r| \leq N$.

iv) $E|\psi|_\infty + E\||g|_{l_2}\|_\infty < \infty$

**Definition 3.2.1.** An $B$-solution of equation (3.2.22) is an $\mathscr{F}_t$−adapted, strongly continuous process $(u_t)_{t\in[0,T]}$ with values in $L_2(Q)$ such that

i) $u_t \in H_0^1(Q)$, for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0,T]$

ii) $\int_0^T |u_t|^2_{H_0^1(Q)} dt < \infty$ (a.s.)

iii) almost surely, $u$ is essentially bounded in $(t,x)$

iv) for all $\phi \in C_c^\infty(Q)$ we have with probability one

$$(u_t, \phi) = (\psi, \phi) + \int_0^t (a_s^{ij}\partial_i u_s, \partial_{-j}\phi) + (b_s^i \partial_i u_s + c_s u_s, \phi) + (f_s(u_s), \phi) ds$$

$$+ \int_0^t (M_s^k u_s + g_s^k, \phi) dw_s^k,$$

for all $t \in [0,T]$.

Notice that by Assumption 3.2.1 iii), and (iii) from Definition 3.2.1, the term $\int_0^t (f_s(u_s), \phi) ds$ is meaningful.

**Theorem 3.2.1.** *Under Assumptions 3.0.3, 3.0.4, and 3.2.1, there exists a unique $B$-solution of equation (3.2.22).*

*Remark* 3.2.1. From now on we can and we will assume that the function $f$ is decreasing in $r$ or else, by virtue of Assumption 3.2.1, we can replace $f_t(x,r)$ by $\tilde{f}_t(x,r) := f_t(x,r) - Kr$ and $c_t(x)$ with $\tilde{c}_t(x) := c_t(x) + K$.

*Proof of Theorem 3.2.1.* We truncate the function $f$ by setting

$$f_t^{n,m}(x,r) = \begin{cases} f_t(x,m) & \text{if } r > m \\ f_t(x,r) & \text{if } -n \leq r \leq m \\ f_t(x,-n) & \text{if } r < -n, \end{cases}$$

for $n,\ m \in \mathbb{N}$ we consider the equation

$$\begin{aligned} du_t^{n,m} &= (\mathscr{L}_t u_t^{n,m} + f_t^{n,m}(u_t^{n,m}))dt + (\mathscr{M}_t^k u_t^{n,m} + g_t^k) dw_t^k, \\ u_0^{n,m} &= \psi \end{aligned} \tag{3.2.23}$$

We first fix $m \in \mathbb{N}$. One can easily check that under Assumptions 3.0.3, 3.0.4 and 3.2.1, by virtue of Theorem 2.1.1, equation (3.2.23) has a unique solution (in the sense of definition 2.1.1) $(u_t^{n,m})_{t\in[0,T]}$. We also have that for $n' \geq n$, $f^{n',m} \geq f^{n,m}$. By Theorem 2.1.2 we get that almost surely, for all $t \in [0,T]$

$$u_t^{n',m}(x) \geq u_t^{n,m}(x), \text{ for almost every } x. \tag{3.2.24}$$

We define now the stopping time

$$\tau^{R,m} := \inf\{t \geq 0 : \int_Q (u_t^{1,m} + R)_-^2 \, dx > 0\} \wedge T.$$

We claim that for each $R \in \mathbb{N}$, there exists a set $\Omega_R$ of full probability, such that for each $\omega \in \Omega_R$, and for all $n \geq R$ we have that

$$u_t^{n,m} = u_t^{R,m}, \text{ for } t \in [0, \tau^{R,m}]. \tag{3.2.25}$$

Notice that by (3.2.24) and the definition of $\tau^{R,m}$, for all $n \geq R$

$$f_t^{n,m}(x, u_t^{n,m}(x)) = f_t^{R,m}(x, u_t^{n,m}(x)), \text{ for } t \in [0, \tau^{R,m}].$$

This means that for all $n \geq R$ the processes $u_t^{n,m}$ satisfies

$$\begin{aligned} dv_t &= (\mathscr{L}_t v_t + f_t^{R,m}(v_t))dt + (\mathscr{M}_t^k v_t + g_t^k)dw_t^k, \\ v_0 &= \psi, \end{aligned} \tag{3.2.26}$$

on $[0, \tau^{R,m}]$. The uniqueness of the $L_2-$solution of the above equation shows (3.2.25). Notice that by Assumption 3.2.1 (iii) and (iv), Theorem 3.1.1 guarantees that $u^{1,m}$ is almost surely essentially bounded in $(t,x)$. Therefore, for almost every $\omega \in \Omega$, $\tau^{R,m} = T$ for all $R$ large enough. On the set $\tilde{\Omega} := \cap_{R\in\mathbb{N}} \Omega_R$ we define $u_t^{\infty,m} = \lim_{n\to\infty} u_t^{n,m}$, where the limit is in the sense of $L_2(Q)$. Since for each $\omega \in \tilde{\Omega}$, we have $u_t^{\infty,m} = u_t^{n,m}$ for all $t \leq \tau^{R,m}$, and for any $n \geq R$, it follows that the process $(u_t^{\infty,m})_{t\in[0,T]}$ is an adapted continuous $L_2(Q)-$valued process such that

i) $u_t^{\infty,m} \in H_0^1(Q)$, for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0,T]$

ii) $\int_0^T |u_t^{\infty,m}|_{H_0^1(Q)}^2 dt < \infty$(a.s.)

iii) $u_t^{\infty,m}$ is almost surely essentially bounded in $(t,x)$

iv) for all $\phi \in C_c^\infty(Q)$ we have with probability one

$$(u_t^{\infty,m}, \phi) = \int_0^t (a_s^{ij} \partial_i u_s^{\infty,m}, \partial_{-j}\phi) + (b_s^i \partial_i u_s^{\infty,m} + c_s u_s^m, \phi) + (f_s^m(u_s^{\infty,m}), \phi) ds$$

$$+\int_0^t (\sigma_s^{ik}\partial_i u_s^{\infty,m} + \nu_s^k u_s^{\infty,m} + g_s^k, \phi)\, dw_s^k + (\psi, \phi),$$

for all $t \in [0, T]$, where

$$f_t^m(x,r) = \begin{cases} f_t(x,m) & \text{if } r > m \\ f_t(x,r) & \text{if } r \le m. \end{cases}$$

Now we will let $m \to \infty$. Let us define the stopping time

$$\tau^R := \inf\{t \ge 0 : \int_Q (u_t^{\infty,1} - R)_+^2\, dx > 0\} \wedge T.$$

As before we claim that for any $R > 0$, there exists a set $\Omega'_R$ of full probability, such that for any $\omega \in \Omega'_R$ and any $m, m' \ge R$,

$$u_t^{\infty,m'} = u_t^{\infty,m} \text{ on } [0, \tau^R]. \tag{3.2.27}$$

To show this it suffices to show that for each $R \in \mathbb{N}$, almost surely, for all $m \ge R$, we have $u_t^{n,m} = u_t^{n,R}$ on $[0, \tau^R]$ for all $n \in \mathbb{N}$. To show this we set

$$\tau_n^R := \inf\{t \ge 0 : \int_Q (u_t^{n,1} - R)_+^2\, dx > 0\} \wedge T.$$

For all $m \ge R$ we have that the processes $u_t^{n,m}$ satisfy the equation

$$\begin{aligned} dv_t &= (\mathscr{L}_t v_t + f_t^{n,R}(v_t))\, dt + (\mathscr{M}_t^k v_t + g_t^k\}\, dw_t^k, \\ v_0(x) &= \psi(x), \end{aligned} \tag{3.2.28}$$

for $t \le \tau_n^R$. It follows that almost surely, $u_t^{n,m} = u_t^{n,R}$ for $t \le \tau_n^R$, for all $n$. We just note here that by the comparison principle again, we have $\tau^R \le \tau_n^R$ and this shows (3.2.27). Also for almost every $\omega \in \Omega$, we have $\tau^R = T$ for $R$ large enough. Hence we can define $u_t = \lim_{m\to\infty} u_t^{\infty,m}$, and then one can easily see that $u_t$ has the desired properties.

For the uniqueness, let $u^{(1)}$ and $u^{(2)}$ be $B$-solutions of (3.2.22). Then one can define the stopping time

$$\tau_N = \inf\{t \ge 0 : \int_Q (|u_t^{(1)}| - N)_+^2\, dx \vee \int_Q (|u_t^{(2)}| - N)_+^2\, dx > 0\},$$

to see that for $t \le \tau_N$, the two solutions satisfy equation (3.2.23) with $n = m = N$, and the claim follows, since $\tau_N = T$ almost surely, for large enough $N$.

□

## 3.3 Weak Harnack inequality

Harnack inequalities, introduced by [32], provide a comparison of values at different points of nonnegative functions which satisfy a partial differential equation (PDE). This type of inequalities have a vast number of applications, in particular, they played a significant role in the study of PDEs with discontinuous coefficients in divergence form. This is the celebrated De Giorgi-Nash-Moser theory ([18], [53], [51]), in which Hölder continuity of the solutions is established. Later, by using a weaker version of Harnack's inequality, a simpler proof in the parabolic case was given in [39]. Harnack inequality and Hölder estimate for equations in non divergence form, also known as the Krylov-Safonov estimate, was proved in [47] and [59]. Since then, similar results have been proved for more general equations, including for example integro-differential operators of Lévy type (see [2]) and singular equations (see [14] and references therein).

It is well known (see e.g. [43], [40]) that the stochastic partial differential equations (SPDEs)

$$du_t = \mathscr{L}_t u_t \, dt + \mathscr{M}_t^k u_t \, dw_t^k, \tag{3.3.29}$$

are in many ways the natural stochastic extensions of parabolic equations $du_t = \mathscr{L}_t u_t \, dt$. It is therefore also natural to ask whether the above mentioned results, fundamental in deterministic PDE theory, have stochastic counterparts.

Here, following the approach of [39], we derive a stochastic version of the aforementioned weak Harnack inequality in Theorem 3.3.1. Here "weak" stands for that in order to estimate the minimum of a nonnegative solution $u$, not only the maximum of $u$ is required to be bounded from below by 1, but $u$ itself on at least half of the domain. In the deterministic case such an inequality yields the Hölder continuity of the solutions almost immediately. This last step appears to be more elusive in the stochastic setting, but a weaker type of continuity is nevertheless obtained in Theorem 3.3.2. We note that Harnack inequalities for solutions of SPDEs - not to be confused with Harnack inequalities for the transition semigroup of SPDEs, for which we refer the reader to [61] and the references therein - have not been previously established even for equations with smooth coefficients. We also remark that the supremum estimates needed in the proofs are local, as opposed to the global ones established in the previous Section. The latter is proved through a stochastic Moser's iteration, while here, for the sake of novelty we use a stochastic version of De Giorgi's iteration.

Let us introduce some notation used in this section. For $R \geq 0$, let $B_R = \{x \in \mathbb{R}^d : |x| < R\}$, $G_R = [4 - R^2, 4] \times B_R$, and $G := G_2$. We will also assume that for simplicity that

$T=4$ and $Q=B_2$. Subsets of $\mathbb{R}^{d+1}$ of the form $J\times(B_R+x)$, where $J$ is a closed interval in $[0,4]$ and $x\in\mathbb{R}^d$, will be referred to as cylinders. For $p\in[1,\infty]$ and a subset $A$ of $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$, the norm in $L_p(A)$ will be denoted by $|\cdot|_{p,A}$ and $\|\cdot\|_{p,A}$, respectively. For a real function defined on a set $A\subset\mathbb{R}^d$ we write $\operatorname{osc}_A u:=\sup_A u-\inf_A u$. By inf, sup we always mean essential ones.

### 3.3.1 Formulation and main results

Notice that neither boundary nor initial condition is posed on (3.3.29). We will denote by $\mathscr{H}$ the set of all strongly continuous $L_2(B_2)$-valued predictable processes $u=(u_t)_{t\in[0,4]}$ on $\Omega\times[0,4]$ such that $u_t\in H^1(B_2)$ for $dP\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,4]$, and

$$E\sup_{t\leq 4}|u_t|_2^2+E\int_0^4\int_{B_2}|\nabla u_t|^2\,dx\,dt<\infty.$$

**Definition 3.3.1.** We will say that $u$ satisfies (or is a solution of) (3.3.29), if $u\in\mathscr{H}$ and for each $\phi\in C_c^\infty(B_2)$, with probability one,

$$(u_t,\phi)=(u_0,\phi)+\int_0^t(a_t^{ij}\partial_i u_t,\partial_{-j}\phi)\,dt+\int_0^t(\sigma_t^{ik}\partial_i u_t,\phi)\,dw_t^k,$$

for all $t\in[0,4]$.

Denote by $\Lambda$ the set of functions $v$ on $[0,4]\times B_2$ such that $v\geq 0$ and

$$|\{x\in B_2|\,v_0(x)\geq 1\}|\geq\frac{1}{2}|B_2|.$$

Let us recall the Harnack inequality that is essentially proved in [39] : *If $u$ is a solution of $du=\partial_i(a^{ij}\partial_j u)dt$ and $u\in\Lambda$, then*

$$\inf_{G_1}u\geq h$$

*with $h=h(d,\lambda,K)>0$.* In the stochastic case clearly it can not be expected that such a lower estimate holds uniformly in $\omega$. It does hold, however, with $h$ above replaced with a strictly positive random variable, this is the assertion of our main theorem.

**Theorem 3.3.1.** *Let $u$ be a solution of* (3.3.29) *such that on an event $A\in\mathscr{F}$, $u\in\Lambda$. Then for any $N>0$ there exists a set $D\in\mathscr{F}$, with $P(D)\leq CN^{-\delta}$, such that on $A\cap D^c$,*

$$\inf_{(t,x)\in G_1}u_t(x)\geq e^{-N}.$$

*where $C$ and $\delta$, are positive constants depending only on $d,\lambda$ and $K$.*

Later on we will refer to the quantity $e^{-N}$ above as the lower bound corresponding to the probability $CN^{-\delta}$. With the help of Theorem 3.3.1, we obtain the following continuity result.

**Theorem 3.3.2.** *Let $u$ be a solution of* (3.3.29) *and* $(t_0, x_0) \in (0,4) \times B_2$. *Then $u$ is almost surely continuous at* $(t_0, x_0)$.

*Remark* 3.3.1. Notice that Theorems 3.3.1 and 3.3.2 remain valid if $\mathscr{L}u$ and $\mathscr{M}^k u$ are replaced by $\tilde{\mathscr{L}}u = \partial_i(a^{ij}(u)\partial_j u)$, and $\tilde{\mathscr{M}}^k u = \sigma^{ik}(u)\partial_i u$ where the function

$$(\alpha^{ij}(\cdot))_{i,j=1}^d = (2\alpha^{ij}(\cdot) - \sigma^{ik}(\cdot)\sigma^{jk}(\cdot))_{i,j=1}^d$$

takes values in the set $\{(\beta^{ij})_{i,j=1}^d : \forall z \in \mathbb{R}^d, \lambda^{-1}|z|^2 \geq \beta^{ij} z_i z_j \geq \lambda |z|^2\}$ for some $\lambda > 0$. Also, we only consider equations with higher order terms, in the same spirit as in [52], since the lower order terms with bounded and measurable coefficients can be easily treated.

### 3.3.2 Martingale growth estimates and Itô's formula

The first three lemmas might be considered standard in the context of stochastic processes and parabolic PDEs, respectively. For the sake of completeness we provide short proofs.

**Lemma 3.3.3.** *Let $T > 0$ and let $(m_t)_{t\in[0,T]}$ be a continuous local martingale. Then for any $\alpha > 0$, and $\kappa \geq 2$*

$$P(\inf_{t\in[0,T]} m_t \geq -\alpha, \sup_{t\in[0,T]} m_t \geq \kappa\alpha) \leq C\frac{\sqrt{\log\kappa}}{\kappa} \leq C\kappa^{-1/2}$$

*with an absolute constant $C$.*

*Proof.* Without loss of generality, we can assume that our probability space can support a Wiener process $B$ for which $B_{\langle m\rangle_t} = m_t$. Then for any $\alpha, \beta \geq 0$,

$$P(\inf_{t\in[0,T]} m_t \geq -\alpha, \langle m\rangle_T \geq \beta) \leq P(\inf_{s\in[0,\beta]} B_s \geq -\alpha).$$

Using the well-known distribution of the minimum of $B$,

$$P(\inf_{s\in[0,\beta]} B_s \geq -\alpha) = \sqrt{\frac{2}{\pi\beta}} \int_{-\alpha}^{0} e^{-x^2/2\beta}\, dx \leq C\frac{\alpha}{\sqrt{\beta}}.$$

On the other hand, for any $\beta, \gamma$

$$P(\langle m\rangle_T \le \beta, \sup_{t\in[0,T]} m_t \ge \gamma) \le P(\sup_{s\in[0,\beta]} B_s \ge \gamma),$$

and

$$P(\sup_{s\in[0,\beta]} B_s \ge \gamma) \le \sqrt{\frac{2}{\pi\beta}}\frac{\beta}{\gamma}\int_\gamma^\infty \frac{x}{\beta} e^{-x^2/2\beta}\,dx = C\sqrt{\frac{\beta}{\gamma^2}}e^{-\gamma^2/2\beta}. \tag{3.3.30}$$

Substituting $\beta = (\kappa\alpha)^2/\log\kappa^2$ and $\gamma = \alpha\kappa$ yields the first inequality, while the second one is obvious. □

**Lemma 3.3.4.** *For any $c > 0$, there exists $N_0(c) > 0$, such that for any continuous local martingale $m_t$, and for any $N \ge N_0$,*

$$P\left(\sup_{t\ge 0}(m_t - c\langle m\rangle_t) > N\right) \le Ce^{-Nc/4},$$

*with an absolute constant $C$.*

*Proof.* Again, we can assume that our probability space can support a Wiener process $B$ for which $B_{\langle m\rangle_t} = m_t$. Then for any $\beta > 0$

$$P\left(\sup_{t\ge 0}(m_t - c\langle m\rangle_t) > N\right) \le P\left(\sup_{s\ge 0}(B_s - cs) > N\right)$$

$$\le P(\sup_{s\in[0,\beta]} B_s > N) + \sum_{i=1}^\infty P(\sup_{s\in[0,(i+1)\beta]} B_s > ic\beta).$$

By (3.3.30),

$$P\left(\sup_{t\ge 0}(m_t - c\langle m\rangle_t) > N\right) \le C\sqrt{\frac{\beta}{N^2}}e^{-N^2/2\beta} + \sum_{i=1}^\infty C\sqrt{\frac{(i+1)}{c^2 i^2\beta}}e^{-c^2 i^2\beta/2(i+1)}.$$

Choosing $\beta = N/c$ yields the claim. □

**Lemma 3.3.5.** *Suppose that $u \in L_2([0,4], H^1(B_2)) \cap L_\infty([0,4], L_2(B_2))$. Let $J \subset [0,4]$ be a subinterval, $Q = B_\rho$ for some $0 < \rho < 2$, $\varphi \in C_c^\infty(Q)$, and $\alpha > \beta \ge 0$. Then*

$$\|(u-\alpha)^+\varphi\|^2_{2,J\times Q} \le C(|\varphi|^2_\infty + |\nabla\varphi|^2_\infty)\left[\frac{\|(u-\beta)^+\|_{2,J\times Q}}{\alpha-\beta}\right]^{\frac{4}{d+2}}$$

$$\times\left[\sup_{t\in J}|(u-\alpha)^+|^2_{2,Q} + \|I_{u>\alpha}\nabla u\|^2_{2,J\times Q}\right],$$

*with $C = C(d)$.*

*Proof.* By Hölder's inequality,

$$\|(u-\alpha)^+\varphi\|_{2,J\times Q}^2 \le \|(u-\alpha)^+\varphi\|_{2(d+2)/d,J\times Q}^2 \|I_{u>\alpha}\|_{2,J\times Q}^{4/d+2}.$$

Noticing that

$$I_{u>\alpha} \le \frac{(u-\beta)^+}{\alpha-\beta},$$

and using the embedding inequality

$$\|v\|_{2(d+2)/d,G}^2 \le C(d)\left(\sup_{t\in[0,4]} |v|_{2,B_2}^2 + \|\nabla v\|_{2,G}^2\right)$$

for $v \in L_2([0,4], H_0^1(B_2)) \cap L_\infty([0,4], L_2(B_2))$ (see, e.g. Lemma 3.2, [52]), applied to the function $(u-\alpha)^+ I_J\varphi$, we get the required inequality. □

Finally, let us formulate the version of Itô's formula we will use later. We denote by $\mathscr{D}$ the set of twice continuously differentiable functions $f$ from $\mathbb{R}$ to $\mathbb{R}$, such that $f''$ is bounded. Notice that if $f \in \mathscr{D}$, then there exists a constant $\hat{K}$ such that for all $r \in \mathbb{R}$

$$|f(r)| \le \hat{K}(1+|r|^2), \ |f'(r)| \le \hat{K}(1+|r|).$$

We denote by $\mathscr{C}$ the set of twice continuously differentiable functions $f$ from $\mathbb{R}$ to $\mathbb{R}$, such that both $f'$ and $ff''$ are bounded.

**Lemma 3.3.6.** *Let $u$ satisfy* (3.3.29)*, and let $g \in \mathscr{D}$, $\varphi \in C_c^\infty(B_2)$, and $\psi \in C^\infty[0,4]$. Then almost surely,*

$$\begin{aligned}\int_{B_2} \varphi^2\psi_t^2 g(u_t)\,dx &= \int_{B_2} \varphi^2\psi_0^2 g(u_0)\,dx + \int_0^t\int_{B_2} 2\psi_s\psi_s'\varphi^2 g(u_s)\,dx\,ds \\ &- \int_0^t\int_{B_2} 2\psi_s^2\varphi\partial_j\varphi g'(u_s)a_s^{ij}\partial_i u_s\,dx\,ds + \int_0^t\int_{B_2}\psi_s^2\varphi^2 g'(u_s)\sigma^{ik}\partial_i u_s\,dw_s^k \\ &- \int_0^t\int_{B_2}\psi_s^2\varphi^2 g''(u_s)[a_s^{ij}\partial_i u_s\partial_j u_s - \frac{1}{2}\sigma^{ik}\sigma^{jk}\partial_i u_s\partial_j u_s]\,dx\,ds, \qquad (3.3.31)\end{aligned}$$

*for all $t \in [0,4]$.*

*Proof.* Let $\kappa$ be nonnegative a $C^\infty$ function on $\mathbb{R}^d$, bounded by 1, supported on $\{|x| < 1\}$, and having unit integral. We denote $\kappa_\varepsilon(x) = \varepsilon^{-d}\kappa(x/\varepsilon)$, for $\varepsilon > 0$ and for $v \in L_2(B_2)$ we write

$$v^\varepsilon(x) = (v)^\varepsilon(x) = \int_{B_2} \kappa_\varepsilon(x-y)v(y)\,dy, \text{ for } x \in \mathbb{R}^d.$$

Let us choose $\varepsilon > 0$ small enough such that $\varphi$ is supported in $B_{2-\varepsilon}$. Then for

$x \in B_{2-\varepsilon}$ we have

$$u_t^\varepsilon(x) = u_0^\varepsilon(x) + \int_0^t (a_s^{ij}\partial_j u_s, \partial_i \kappa_\varepsilon(x-\cdot))dt + \int_0^t (\sigma_s^{ik}\partial_i u_s)^\varepsilon(x)dw_s^k.$$

Then one can write Itô's formula for the processes $\varphi^2(x)\psi_t^2 g(u_t^\varepsilon(x))$ for $x \in B_2$, use Fubini and stochastic Fubini theorems (for the latter, see [46]), and integrate by parts to obtain that almost surely,

$$\int_{B_2} \varphi^2\psi_t^2 g(u_t^\varepsilon)dx = \int_{B_2} \varphi^2\psi_0 g(u_0^\varepsilon)dx + \int_0^t \int_{B_2} 2\psi_s\psi_s'\varphi^2 g(u_s^\varepsilon)dxds$$

$$-\int_0^t \int_{B_2} 2\psi_s^2\varphi\partial_j\varphi g'(u_s^\varepsilon)(a_s^{ij}\partial_i u_s)^\varepsilon dxds + \int_0^t \int_{B_2} \psi_s^2\varphi^2 g'(u_s^\varepsilon)(\sigma_s^{ik}\partial_i u_s)^\varepsilon dxdw_s^k$$

$$-\int_0^t \int_{B_2} \psi_s^2\varphi^2 g''(u_s^\varepsilon)[(a_s^{ij}\partial_j u_s)^\varepsilon\partial_i u_s^\varepsilon - \frac{1}{2}(\sigma_s^{ik}\partial_i u_s)^\varepsilon(\sigma_s^{jk}\partial_j u_s)^\varepsilon]dxds,$$

for all $t \in [0,4]$. Then for fixed $t$ one lets $\varepsilon \to 0$ to obtain that (3.3.31) holds almost surely, and the result follows since both sides of (3.3.31) are continuous in $t$. □

**Lemma 3.3.7.** *Let $u$ satisfy* (3.3.29) *, and let $g \in \mathscr{C}$, $\varphi \in C_c^\infty(B_2)$, and $\psi \in C^\infty[0,4]$. Set $v_t = (g(u_t))^+$. Then $v \in \mathscr{H}$, and almost surely,*

$$\begin{aligned}
|\varphi\psi_t v_t|_2^2 = |\varphi\psi_0 v_0|_2^2 &+ \int_0^t \int_{B_2} 2\psi_s^2\varphi^2 v_s\sigma^{ik}\partial_i v_s dxdw_s^k + \int_0^t \int_{B_2} 2\psi_s\psi_s'\varphi^2 v_s^2 dxds \\
&- \int_0^t \int_{B_2} \psi_s^2\varphi^2[2a_s^{ij}\partial_i v_s\partial_j v_s - \sigma^{ik}\sigma^{jk}\partial_i v_s\partial_j v_s]dxds \\
&- \int_0^t \int_{B_2} \psi_s^2\varphi^2 v_s g''(u_s)[2a_s^{ij}\partial_i u_s\partial_j u_s - \sigma^{ik}\sigma^{jk}\partial_i u_s\partial_j u_s]dxds \\
&- \int_0^t \int_{B_2} 4\psi_s^2\varphi\partial_j\varphi v_s a_s^{ij}\partial_i v_s dxds, \qquad (3.3.32)
\end{aligned}$$

*for all $t \in [0,4]$.*

*Proof.* Since $g$ has bounded first derivative, it follows easily that $v \in \mathscr{H}$. We introduce now the functions $\alpha_\delta(r)$, $\beta_\delta(r)$ and $\gamma_\delta(r)$ on $\mathbb{R}$, for $\delta > 0$, given by

$$\alpha_\delta(r) = \begin{cases} 2 & \text{if } r > \delta \\ \frac{2r}{\delta} & \text{if } 0 \le r \le \delta \\ 0 & \text{if } r < 0, \end{cases}$$

$$\beta_\delta(r) = \int_0^r a_\delta(s)ds, \qquad \gamma_\delta(r) = \int_0^r \beta_\delta(s)ds.$$

For all $r \in \mathbb{R}$ we have $\alpha_\delta(r) \to 2I_{r>0}$, $\beta_\delta(r) \to 2r^+$ and $\gamma_\delta(r) \to (r^+)^2$ as $\delta \to 0$. Also, for all $r, r_1, r_2$ and $\delta$, the following inequalities hold

$$|\alpha_\delta(r)| \leq 2, \; |\beta_\delta(r)| \leq 2|r|, \; |\gamma_\delta(r)| \leq r^2.$$

It follow then that since $g \in \mathscr{C}$, the function $\zeta_\delta(r) := \gamma_\delta(g(r))$ lies in $\mathscr{D}$. Hence, by virtue of Lemma 3.3.6 one can write Itô's formula for $|\psi\varphi\zeta_\delta(u_t)|_2^2$, i.e. (3.3.31) with $g, g'$ and $g''$ replaced by $\gamma_\delta(g), \beta_\delta(g)g'$ and $\alpha_\delta(g)|g'|^2 + \beta_\delta(g)g''$ respectively. Then we let $\delta \to 0$ to obtain (3.3.32).

□

### 3.3.3 Local supremum estimates

While supremum estimate have been obtained in Section 3.1.1, there are two differences to that in the following version. First, the estimate presented here is local, and therefore no initial or boundary condition needs to be specified. Second, we need the estimates to hold not only for the solutions, but for certain functions of solutions as well.

**Theorem 3.3.8.** *Let $f \in \mathscr{C}$ such that $ff'' \geq 0$, and let $u$ be a solution of* (3.3.29). *Then there exist positive constants $\delta, C, \hat{C}$, depending only on $d, \lambda, K$, such that for any $\alpha > 0$ and $\kappa \geq 1$*

*(i)* $P(\|f(u)^+\|^2_{\infty,G_1} \geq \hat{C}\kappa\alpha, \|f(u)^+\|^2_{2,G_{3/2}} \leq \alpha) \leq C\kappa^{-\delta}$,

*(ii)* $P(\|f(u)\|^2_{\infty,G_1} \geq \hat{C}\kappa\alpha, \|f(u)\|^2_{2,G_{3/2}} \leq \alpha) \leq C\kappa^{-\delta}$.

*Proof.* First we prove $(i)$. It is easy to see that it suffices to show the existence of $\hat{\gamma}, \delta_1, \delta_2, C > 0$ such that

$$P(\|f(u)^+\|^2_{\infty,G_1} \geq 1, \|f(u)^+\|^2_{2,G_{3/2}} \leq \kappa^{-\delta_1}\hat{\gamma}) \leq C\kappa^{-\delta_2}, \tag{3.3.33}$$

since by plugging in (3.3.33) $\tilde{f} = (\gamma/\kappa^{\delta_1}\alpha)^{1/2}$ in place of $f$, we obtain the desired inequality with $\delta = \delta_2/\delta_1$ and $\hat{C} = 1/\hat{\gamma}$.

To this end, take $r \in [0,4]$, $\rho \in [1,2]$, $\psi \in C^\infty([0,4])$ with $\psi = 0$ on $[0,r]$, and $\varphi \in C_c^\infty(B_\rho)$. For $j = 0,1,\ldots$ let $g^j(u) := f(u) - (1-2^{-j})$, $v^j = (g^j(u))^+$, and let us apply Lemma 3.3.7 with $g^{j+1}$. Using the parabolicity condition and Young's inequality, as well as the nonnegativity of $v^{j+1}(g^{j+1})''$, we get for any $\varepsilon > 0$

$$\int_{B_\rho} \varphi^2\psi_t^2|v_t^{j+1}|^2\,dx$$

$$\leq m_t^{j+1} + \int_r^t\int_{B_\rho} 2\psi_s\psi_s'\varphi^2|v_s^{j+1}|^2\,dx\,ds - \int_r^t\int_{B_\rho} \lambda\psi_s^2\varphi^2|\nabla v_s^{j+1}|^2\,dx\,ds$$

$$+\int_r^t\int_{B_\rho}[\varepsilon\psi_s^2\varphi^2K^2|\nabla v_s^{j+1}|^2+16/\varepsilon\psi_s^2|\nabla\varphi|^2|v_s^{j+1}|^2]dxds$$

almost surely for all for $t\in[r,4]$, where

$$m_t^{j+1}=\int_r^t\int_{B_\rho}2\varphi^2\psi_s^2v_s^{j+1}M^kv_s^{j+1}dxdw_s^k.$$

Choosing $\varepsilon$ sufficiently small, we arrive at

$$\int_{B_\rho}\varphi^2\psi_t^2|v_t^{j+1}|^2dx+\int_r^t\int_{B_\rho}\varphi^2\psi^2|\nabla v_s^{j+1}|^2dxds$$

$$\leq C'm_t^{j+1}+C\int_r^t\int_{B_\rho}\varphi^2\psi_s\psi_s'|v_s^{j+1}|^2+|\nabla\varphi|^2\psi_s^2|v_s^{j+1}|^2dxds. \tag{3.3.34}$$

Now let us choose $r=r_j=3-(5/4)2^{-j}$ and $\rho=\rho_j=1+(1/2)2^{-j}$, that is, $[r_0,4]\times B_{\rho_0}=G_{3/2}$. Also we introduce the notation $F_j=[r_j,4]\times B_{\rho_j}$. Furthermore, choose $\psi=\psi^j$ and $\varphi=\varphi^j$ such that

(i) $0\leq\psi^j\leq1$, $\psi^j|_{[0,r_j]}=0$, $\psi^j|_{[r_{j+1},4]}=1$;

(ii) $0\leq\varphi^j\leq1$, $\varphi^j\in C_0^\infty(B_{\rho_j})$, $\varphi^j|_{B_{\rho_{j+1}}}=1$;

(iii) $|\partial_t\psi^j|+|\nabla\varphi^j|^2<C4^j$.

Then by running $t$ over $[r_j,4]$, by (3.3.34) we obtain

$$\sup_{t\in[r_{j+1},4]}|v_t^{j+1}|^2_{2,B_{\rho_{j+1}}}+\|\nabla v^{j+1}\|^2_{2,F_{j+1}}\leq C4^j\|v^{j+1}\|^2_{2,F_j}+C\sup_{t\in[r_j,4]}m_t^{j+1}. \tag{3.3.35}$$

Notice that, since the left-hand side of (3.3.34) is nonnegative, running $t$ over $I_j$ gives

$$\inf_{t\in[r_j,4]}m_t^{j+1}\geq-C4^j\|v^{j+1}\|^2_{2,F_j}. \tag{3.3.36}$$

Applying Lemma 3.3.5 with $\alpha=1-2^{-(j+1)}$, $\beta=1-2^{-j}$, and $\varphi=\varphi^{j+1}$, we get

$$\begin{aligned}\|v^{j+1}\|_{2,F_{j+2}}&\leq\|\varphi^{j+1}v^{j+1}\|_{2,F_{j+1}}\\&\leq C^j\|v^j\|^{4/d+2}_{2,F_{j+1}}\left[\sup_{t\in[r_{j+1},4]}|v_t^{j+1}|^2_{2,B_{\rho_{j+1}}}+\|\nabla v^{j+1}\|_{2,F_{j+1}}\right].\end{aligned}$$

Combining this with (3.3.35) yields

$$\|v^{j+1}\|^2_{2,F_{j+2}}\leq C^j\|v^j\|^{4/d+2}_{2,F_{j+1}}\left[\|v^{j+1}\|^2_{2,F_j}+\sup_{t\in[r_j,4]}m_t^{j+1}\right].$$

Since for $j > i$, we have $v^j \le v^i$ and $F_j \subset F_i$, we obtain for $V_j = \|v^j\|^2_{2,F_j}$

$$V_{j+2} \le C^j V_j^{2/(d+2)} \left[ 4^j V_j + \sup_{t\in[r_j,4]} m_t^{j+1} \right].$$

Let $\gamma_0, \gamma \in (0,1)$ and suppose that $V_j \le \gamma_0 \gamma^j$ on a set $\Omega_j \subset \Omega$. By (3.3.36) we have

$$\inf_{t\in[r_j,4]} m_t^{j+1} \ge -C4^j \|v^{j+1}\|^2_{2,F_j} \ge -C4^j V_j,$$

and Lemma 3.3.3 can be applied with $\alpha = C4^j \gamma_0 \gamma^j$, and $\kappa$ replaced by $\kappa 4^j$. That is we obtain a subset $\Omega_{j+2} \subset \Omega_j$ such that $P(\Omega_j \setminus \Omega_{j+2}) \le C\kappa^{-1/2} 2^{-j}$ and on $\Omega_{j+2}$

$$\sup_{t\in[r_j,4]} m_t^{j+1} \le \kappa C 16^j \gamma_0 \gamma^j.$$

Consequently, on $\Omega_{j+2}$,

$$V_{j+2} \le C^j \gamma_0^{2/(d+2)} \gamma^{2j/(d+2)} \gamma_0 \gamma^j (1+\kappa) \le \gamma_0 \gamma^{j+2},$$

provided that

$$\gamma = C^{-(d+2)/2},\ \gamma_0 \le (\gamma^2/(1+\kappa))^{(d+2)/2}.$$

Proceeding iteratively, we can conclude that on $\cap_{j\ge 0}\Omega_{2j}$, $V_j \to 0$, and therefore

$$\|f(u)^+\|^2_{\infty,G_1} \le 1,$$

and moreover,

$$P\left(\Omega_0 \setminus \cap_{j\ge 0}\Omega_{2j}\right) = \sum_{j\ge 0} P(\Omega_{2j} \setminus \Omega_{2j+2}) \le 2C\kappa^{-1/2}.$$

This proves (3.3.33), since $\Omega_0 = \{\|f(u)^+\|^2_{2,G_{3/2}} = V_0 \le \gamma_0 = \kappa^{-(d+2)/2}\hat{\gamma}\}$ with a constant $\hat{\gamma} = \hat{\gamma}(d, \lambda, K)$.

For part $(ii)$, we have

$$P(\|f(u)\|^2_{\infty,G_1} \ge \hat{C}\kappa\alpha, \|f(u)\|^2_{2,G_{3/2}} \le \alpha)$$

$$\le P(\|f(u)^+\|^2_{\infty,G_1} \ge \hat{C}\kappa\alpha, \|f(u)^+\|^2_{2,G_{3/2}} \le \alpha)$$

$$+P(\|f(u)^-\|^2_{\infty,G_1} \ge \hat{C}\kappa\alpha, \|f(u)^-\|^2_{2,G_{3/2}} \le \alpha),$$

which by virtue of $(i)$ and the fact that $-f$ satisfies the conditions of the lemma, finishes the proof. □

Let us consider the case when the initial value is 0. Note that in this case in the proof of Theorem 3.3.8 the time-cutoff function $\psi$ can be omitted. Doing so and repeating the same steps afterwards, we get the following.

**Theorem 3.3.9.** *Let $f \in \mathscr{C}$ such that $ff'' \geq 0$, $u$ be a solution of* (3.3.29) *on $[s,r] \times B_2$, where $0 \leq s < r \leq 4$, and suppose that $f(v)(s,\cdot) \equiv 0$. Then there exist positive constants $\delta, C, \hat{C}$, depending only on $d, \lambda, K$, such that for any $\alpha > 0$ and $\kappa \geq 1$*

*(i)* $P(\|f(u)^+\|^2_{\infty,[s,r]\times B_1} \geq \hat{C}\kappa\alpha, \|f(u)^+\|^2_{2,[s,r]\times B_2} \leq \alpha) \leq C\kappa^{-\delta}$,

*(ii)* $P(\|f(u)\|^2_{\infty,[s,r]\times B_1} \geq \hat{C}\kappa\alpha, \|f(u)\|^2_{2,[s,r]\times B_2} \leq \alpha) \leq C\kappa^{-\delta}$.

Recall that from [57] it is known that solutions of (3.3.29) with 0 boundary and $L_p$ initial conditions are weakly continuous in $L_p$ for any $p \in (0,\infty)$. A simple consequence of Theorem 3.3.8 is that in fact strong continuity holds, away from $t = 0$.

**Corollary 3.3.10.** *Let $u$ be a solution of* (3.3.29) *and $p \in (0,\infty)$. Then*

*(a)* $(u_t)_{t\in[3,4]}$ *is strongly continuous in $L_p(B_1)$;*

*(b) If furthermore $u|_{\partial B_2} = 0$, then $(u_t)_{t\in[3,4]}$ is strongly continuous in $L_p(B_2)$.*

*Proof.* (a) First notice that the supremum in time can be taken to be real (and not only essential) supremum: the function $|(u - \|u\|_{\infty,G_1})^+|_{2,B_1}$ is 0 for almost all $t$, hence by the continuity of $u$ in $L_2$ it is 0 for all $t$, and therefore, for all $t$, almost every $x$, $u_t(x) \leq \|u\|_{\infty,G_1}$. Now fix $t \in [3,4]$, and take a sequence $t_n \to t$. Then $u_{t_n} \to u_t$ in $L_2(B_2)$, hence for a subsequence $t_{n_k}$, for almost every $x$. On the other hand, $|u_{t_{n_k}} I_{B_1}| \leq \|u\|_{\infty,G_1} < \infty$, therefore by the dominated convergence theorem, $u_{t_{n_k}} \to u_t$ in $L_p$.

For part (b), notice that when $u \in H^1_0(B_2)$ for almost all $\omega, t$, then in the special case $f(r) = r$ the space-cutoff function $\varphi$ in the proof of Theorem 3.3.8 can be omitted. We then obtain that $\|u\|_{\infty,[3,4]\times B_2} < \infty$ with probability 1, and by the same argument as above we get the claim. □

### 3.3.4 Proof of Theorem 3.3.1

Before turning to the proof we need one more lemma, which can be considered as a weak version of Theorem 3.3.1.

**Lemma 3.3.11.** *Let $u$ be a solution of* (3.3.29)*, such that on $A \in \mathscr{F}$, $u \in \Lambda$. Then for any $N > 0$, there exists a set $D_1 \in \mathscr{F}$, with $P(D_1) \leq Ce^{-cN}$, such that on $A \cap D_1^c$, for all $t \in [0,4]$,*

$$|\{(x \in B_\rho |\, v(t,x) \geq e^{-N}\}| \geq \frac{1}{8}|B_\rho|,$$

*where ρ is defined by*

$$|B_\rho| = \frac{3}{4}|B_2|,$$

*and the constants $c, C > 0$, depend only on $d, \lambda, K$.*

*Proof.* Clearly it is sufficient to prove the statement for $N > N_0$ for some $N_0$. Introduce the functions

$$f_h(x) = \begin{cases} a_h x + b_h & \text{if } x < -h/2 \\ \log^+ \frac{1}{x+h} & \text{if } x \geq -h/2, \end{cases}$$

for $h > 0$ where $a_h$ and $b_h$ is chosen such that $f_h$ and $f_h'$ are continuous. Let $\kappa$ be nonnegative a $C^\infty$ function on $\mathbb{R}$, bounded by 1, supported on $\{|x| < 1\}$, and having unit integral. Denote $\kappa_h(x) = h^{-1}\kappa(x/h)$ and

$$F_h = f_h * \kappa_{h/4}.$$

We claim that $F_h$ has the following properties:

(i) $F_h(x) = 0$ for $x \geq 1$;

(ii) $F_h(x) \leq \log(2/h)$ for $x \geq 0$;

(iii) $F_h(x) \geq \log(1/2h)$ for $x \leq h/2$;

(iv) $F_h \in \mathcal{D}$ and $F_h''(x) \geq (F_h'(x))^2$ for $x \geq 0$.

The first three properties are obvious, while for the last one notice that $F_h$ has bounded second derivative, $f_h''(x) \geq (f_h'(x))^2$ for $x \geq -h/2$, and therefore, for $x \geq 0$

$$\begin{aligned} (F_h'(x))^2 &= \left( \int f_h'(x-z)\kappa_{h/4}^{1/2}(z)\kappa_{h/4}^{1/2}(z)\,dz \right)^2 \\ &\leq \int (f_h'(x-z))^2 \kappa_{h/4}(z)\,dz \\ &\leq \int f_h''(x-z)\kappa_{h/4}(z)\,dz = F_h''(x). \end{aligned}$$

Let us denote $v = F_h(u)$. Applying Lemma 3.3.6 and using the parabolicity condition, we get

$$\begin{aligned} \int_{B_2} \varphi^2 v_t\,dx - \int_{B_2} \varphi^2 v_0\,dx \leq \int_0^t \int_{B_2} C\varphi \nabla\varphi \nabla v - (\lambda/2)\varphi^2 F_h''(u)(\nabla u)^2\,dx\,ds \\ + \int_0^t \int_{B_2} \varphi^2 \mathcal{M}^k v\,dx\,dw_s^k \end{aligned} \tag{3.3.37}$$

for any $\varphi \in C_c^\infty$. Let us denote the stochastic integral above by $m_t$, and notice that provided $|\varphi| \le 1$,

$$\langle m \rangle_t \le C \int_0^t \int_{B_2} \varphi^2 (\nabla v)^2 \, dx \, ds.$$

Let $c$ be such that $cC \le \lambda/4$. From Lemma 3.3.4, there exists a set $D_1$ with $P(D_1) \le Ce^{-Nc/4}$, such that on $D_1^c$ we have

$$\int_{B_2} \varphi^2 v_t \, dx - \int_{B_2} \varphi^2 v_0 \, dx$$

$$\le N + \int_0^t \int_{B_2} C\varphi \nabla \varphi \nabla v - (\lambda/2)\varphi^2 F_h''(u)(\nabla u)^2 + cC\varphi^2 (\nabla v)^2 \, dx \, ds. \qquad (3.3.38)$$

On $A \cap D_1^c$, by the property (iv) above, we have $F_h''(u)(\nabla u)^2 \ge (\nabla v)^2$, and therefore

$$\int_{B_2} \varphi^2 v_t \, dx \le N + C \int_{B_2} |\nabla \varphi|^2 \, dx + \int_{B_2} \varphi^2 v_0 \, dx. \qquad (3.3.39)$$

Let us denote

$$\mathcal{O}_t(h) = \{x \in B_\rho : u(t,x) \ge h\}.$$

Choosing $\varphi$ to be 1 on $B_\rho$, by properties (i), (ii), and (iii) of $F_h$ and (3.3.39), on $A \cap D_1^c$, for all $t \in [0,4]$

$$|B_\rho \setminus \mathcal{O}_t(h/2)| \log(1/2h) \le C + N + \frac{1}{2}\log(2/h)|B_2| = C + N + \frac{2}{3}\log(2/h)|B_\rho|.$$

Hence

$$|\mathcal{O}_t(h/2)| \ge |B_\rho| - \frac{C+N}{\log(1/2h)} - \frac{2}{3}\frac{\log(2/h)}{\log(1/2h)}|B_\rho|,$$

and choosing $N_0 = C$ and $h = 2e^{-C'N}$ for a sufficiently large $C'$ finishes the proof of the lemma. □

*Proof of Theorem 3.3.1* By Lemma 3.3.11, there exists a set $D_1$ with $P(D_1) \le Ce^{-cN}$ such that on $A \cap D_1^c$ we have

$$|\{(x \in B_\rho | \, v(t,x) \ge e^{-N}| \ge \frac{1}{8}|B_\rho|, \qquad (3.3.40)$$

for all $t \in [0,4]$. Let us denote $h := e^{-N}$. For $0 < \epsilon \le h/2$, we introduce the function

$$f_\epsilon(x) = \begin{cases} a_\epsilon x + b_\epsilon & \text{if } x < -\epsilon/2 \\ \log^+ \frac{h}{x+\epsilon} & \text{if } x \ge -\epsilon/2, \end{cases}$$

where $a_\epsilon$ and $b_\epsilon$ is chosen such that $f_\epsilon$ and $f_\epsilon'$ are continuous. Let $\kappa$ be a nonnegative $C^\infty$ function on $\mathbb{R}$, bounded by 1, supported on $\{|x| < 1\}$, and having unit integral.

Denote $\kappa_\varepsilon(x) = \varepsilon^{-1}\kappa(x/\varepsilon)$ and

$$F_\epsilon = f_\epsilon * \kappa_{\epsilon/4}.$$

Similarly to $F_h$ in the proof of Lemma 3.3.11, $F_\epsilon$ has the following properties:

(i) $F_\epsilon(x) = 0$ for $x \geq h$;

(ii) $F_\epsilon(x) \leq \log(2h/\epsilon)$ for $x \geq 0$;

(iii) $F_\epsilon(x) \geq \log(h/(x+\epsilon)) - 1$ for $x \geq 0$;

(iv) $F_\epsilon \in \mathscr{D}$ and $F''_\epsilon(x) \geq (F'_\epsilon(x))^2$ for $x \geq 0$.

Let us denote $v = F_\epsilon(u)$. Similarly to (3.3.38), there exists a set $D_2$ with $P(D_2) \leq Ce^{-Nc}$, such that on $D_2^c$ we have

$$\int_{B_2} \varphi^2 v_t \, dx - \int_{B_2} \varphi^2 v_2 \, dx$$
$$\leq N + \int_0^t \int_{B_2} C\varphi \nabla\varphi \nabla v - (\lambda/2)\varphi^2 F''_\epsilon(u)(\nabla u)^2 + (\lambda/4)\varphi^2(\nabla v)^2 \, dx \, ds.$$

On $A \cap D_2^c$, by property (iv), we have,

$$\int_0^4 \int_{B_2} \varphi^2 |\nabla v_t|^2 \, dx dt \leq C(N + \int_{B_2} |\nabla\varphi|^2 \, dx + \int_{B_2} \varphi^2 v_2 \, dx). \tag{3.3.41}$$

By choosing $\varphi \in C_c^\infty(B_2)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $B_\rho$ we get,

$$\int_0^4 \int_{B_\rho} |\nabla v_t|^2 \, dx dt \leq C(N + \int_{B_2} |\nabla\varphi|^2 \, dx + \int_{B_2} \varphi^2 v_2 \, dx).$$

Hence, by property (ii),

$$\int_0^4 \int_{B_\rho} |\nabla v_t|^2 \, dx dt \leq CN + C + C\log\frac{2h}{\epsilon}. \tag{3.3.42}$$

Using property (i), by a version of Poincaré's inequality (see, e.g., Lemma II.5.1, [48]) we get for all $t$

$$\int_{B_\rho} |v_t|^2 dx \leq C\frac{\rho^{2(d+1)}}{|O_t(h)|^2} \int_{B_\rho} |\nabla v_t|^2 dx,$$

which, by virtue of (3.3.40) and (3.3.42) implies

$$\int_0^4 \int_{B_\rho} |v_t|^2 dx \leq C + CN + C\log\frac{2h}{\epsilon}.$$

on $A \cap D_1^c \cap D_2^c$. By Theorem 3.3.8 and noting that $G_{3/2} \subset [0,4] \times B_\rho$ we get that there exists a set $D_3 \in \mathscr{F}$ with $P(D_3) \le CN^{-\delta}$, such that on $A \cap D_1^c \cap D_2^c \cap D_3^c$ we have

$$\sup_{(t,x)\in G_1} v_t(x) \le [N(C + CN + C\log\frac{2h}{\epsilon})]^{1/2}.$$

By applying property (iii), we get

$$\sup_{(t,x)\in G_1} \log\frac{h}{u_t(x)+\epsilon} \le [N(C + CN + C\log\frac{2h}{\epsilon})]^{1/2} + 1,$$

and therefore,

$$\inf_{(t,x)\in G_1} u_t(x) \ge he^{-[N(C+CN+C\log 2h - C\log\epsilon)]^{1/2}-1} - \epsilon.$$

Letting $\epsilon = e^{-c'N}$ with a sufficiently large $c'$, it is easy to see that the right-hand side above is bounded from below by $\epsilon$, finishing the proof. □

### 3.3.5 Proof of Theorem 3.3.2

*Proof.* Consider the parabolic transformations $\mathfrak{P}_{\alpha,t',x'}$:

$$t \to \alpha^2 t + t',$$

$$x \to \alpha x + x'.$$

It is easy to see that if $v$ is a solution of (3.3.29) on a cylinder $V$, then $v \circ \mathfrak{P}^{-1}_{\alpha,t',x'}$ is also solution of (3.3.29), on the cylinder $\mathfrak{P}_{\alpha,t',x'}V$, with another sequence of Wiener martingales on another filtration, and with different coefficients that still satisfy Assumptions 3.0.3-3.0.4 with the same bounds. To ease notation, for a cylinder $V$ let $\mathfrak{P}_V$ denote the unique parabolic transformation that maps $V$ to $G$, if such exists. Also, for an interval $[s,r] \subset [0,4]$ let $\mathfrak{P}_{[s,r]} = \mathfrak{P}_{2/\sqrt{r-s},-4s/(r-s),0}$. That is, $\mathfrak{P}_{[s,r]}[s,r] \times B_1 = [0,4] \times B_{2/\sqrt{r-s}}$, which, when $r - s \le 1$, contains $G$.

Without loss of generality $x_0 = 0$ can and will be assumed, as will the almost sure boundedness of $u$ on $G$, since these can be achieved with appropriate parabolic transformations, using the boundedness obtained on sub-cylinders in 3.3.8. Also let us fix a probability $\delta > 0$, denote the corresponding lower bound $3\epsilon_2$ obtained from the Harnack inequality, and take an arbitrary $0 < \epsilon_1 < \epsilon_2/2$.

Apply Theorem 3.3.9 $(i)$ twice, with the function $f(r) = r$, with the interval $[t_0 - 4s, t_0 + s]$, and with solutions $v = u - \sup_{\{t_0-4s\}\times B_2} u$ and $v = -u + \inf_{\{t_0-4s\}\times B_2} u$. Also notice that (for both choices of $v$)

$$\| v^+ \|^2_{2,[t_0-4s,t_0+s]\times B_2} \le Cs \| u\|^2_{\infty,G} \to 0$$

as $s \to 0$ for almost every $\omega$, and thus in probability as well (recall that the fact that the functions $v$ are well-defined and that the above - seemingly trivial - inequality holds, is justified in the proof of Corollary 3.3.10). In other words,

$$P(\| v^+ \|^2_{2,[t_0-4s,t_0+s]\times B_2} > \alpha)$$

can be made arbitrarily small by choosing $s$ sufficiently small. Therefore, we obtain an $s > 0$ and an event $\Omega_0$, with $P(\Omega_0) > 1-\delta$, such that on $\Omega_0$,

$$\sup_{[t_0-4s,t_0+s]\times B_1} u - \sup_{\{t_0-4s\}\times B_2} u < \epsilon_1^2/6$$

$$\inf_{[t_0-4s,t_0+s]\times B_1} u - \inf_{\{t_0-4s\}\times B_2} u > -\epsilon_1^2/6.$$

Let us rescale $u$ at the starting time:

$$u'_\pm(t,x) = \pm\left(2\frac{u(t,x) - \sup_{\{t_0-4s\}\times B_2} u}{\sup_{\{t_0-4s\}\times B_2} u - \inf_{\{t_0-4s\}\times B_2} u} + 1\right),$$

that is, $\sup_{B_2} u'_\pm(t_0-4s,\cdot) = 1, \inf_{B_2} u'_\pm(t_0-4s,\cdot) = -1$. Now we can write $\Omega_0 = \Omega_A \cup \Omega_B$, where

- On $\Omega_A$, $\operatorname{osc}_{\{t_0-4s\}\times B_2} u < \epsilon_1/3$, and therefore, $\operatorname{osc}_{[t_0-4s,t_0+s]\times B_1} u < \epsilon_1/3 + 2\epsilon_1^2/6 < \epsilon_1$;
- On $\Omega_B$, $|u'_\pm| < 1 + 2(\epsilon_1^2/6)/(\epsilon_1/3) = 1+\epsilon_1$, on $[t_0-4s, t_0+s] \times B_1$.

Notice that in the event $\Omega_B$, on the cylinder $[t_0-4s, t_0+s] \times B_1$, the functions $u'_\pm/(1+\epsilon_1)+1$ take values between 0 and 2. Therefore one of $(u'_\pm/(1+\epsilon_1)+1)\circ \mathfrak{P}^{-1}_{[t_0-4s,t_0+s]}\Big|_G$ (see Figure 3.1), denoted for the moment by $u''$, satisfies the conditions of Theorem 3.3.1 with $A = \Omega_B$.

We obtain that on an event $\Omega'_B$

$$\inf_{G_1} u'' > 3\epsilon_2,$$

and thus

$$\operatorname{osc}_V u < \frac{(2-3\epsilon_2)(1+\epsilon_1)}{2}\operatorname{osc}_{\{t_0-4s\}\times B_2} u < (1-\epsilon_2)\operatorname{osc}_{\{t_0-4s\}\times B_2} u,$$

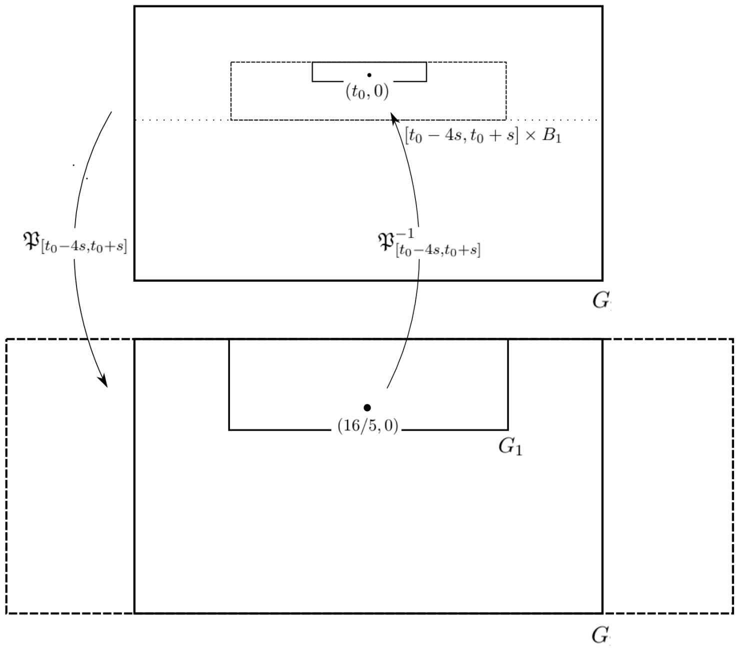


Figure 3.1:

where $V = \mathfrak{P}^{-1}_{[t_0-4s,t_0+s]} G_1$. Moreover, $P(\Omega_B \setminus \Omega'_B) < \delta$. Also, notice that $(t_0, 0) \in V$. Let us denote $\Omega_1 = \Omega_A \cup \Omega'_B$. We have shown the following lemma:

**Lemma 3.3.12.** *Let $\delta > 0$ and let $3\epsilon_2$ be the lower bound corresponding to the probability $\delta$ obtained from the Harnack inequality. For any $u$ that is a solution of* (3.3.29) *on $G$, $t_0 > 0$, and for any sufficiently small $\epsilon_1 > 0$ there exists an $s > 0$ and an event $\Omega_1$ such that*

*(i)* $P(\Omega_1) > 1 - 2\delta$;

*(ii) On $\Omega_1$, at least one of the following is satisfied:*

*(a)* $\operatorname{osc}_V u < \epsilon_1$;

*(b)* $\operatorname{osc}_V u < (1 - \epsilon_2) \operatorname{osc}_G u$,

*where* $V = \mathfrak{P}^{-1}_{[t_0-4s,t_0+s]}(G_1)$.

Now take $u = u^{(0)}$ and $t_0 = t_0^{(0)}$ from the statement of the theorem and a sequence $(\epsilon_1^{(n)})_{n=0}^{\infty} \downarrow 0$, and for $n \geq 0$ proceed inductively as follows:

- Apply Lemma 3.3.12 with $u^{(n)}$, $t_0^{(n)}$, and $\epsilon_1^{(n)}$, and take the resulting $\Omega_1^{(n)}$ and $V^{(n)}$;

- Let $u^{(n+1)} = u^{(n)} \circ \mathfrak{P}^{-1}_{V^{(n)}}$ and $(t_0^{(n+1)}, 0) = \mathfrak{P}_{V^{(n)}}(t_0^{(n)}, 0)$.

On $\limsup_{n\to\infty} \Omega_1^{(n)}$ the function $u$ is continuous at the point $(t_0, 0)$. Indeed, the sequence of cylinders $V^{(0)}, \mathfrak{P}_{V^{(0)}}^{-1} V^{(1)}, \mathfrak{P}_{V^{(0)}}^{-1} \mathfrak{P}_{V^{(1)}}^{-1} V^{(2)}, \dots$ contain $(t_0, 0)$, and the oscillation of $u$ on these cylinders tends to 0. However, $P(\limsup_{n\to\infty} \Omega_1^{(n)}) \geq 1 - 2\delta$, and since $\delta$ can be chosen arbitrarily small, $u$ is continuous at $(t_0, 0)$ with probability 1, and the proof is finished.

$\square$

*Remark* 3.3.2. It is natural to attempt to modify the above argument to bound expectations and higher moments of the oscillations, in the hope to apply Kolmogorov's continuity criterion and obtain Hölder estimates. The main obstacle appears to be to establish a uniform integrability property to a family of (normalized) oscillations. Indeed, the present Harnack inequality can bring down the oscillation by a given factor outside of a small event, and therefore one would like to exclude the possibility that the majority of the oscillation's mass is concentrated on that exceptional event.

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