

A SUPERFIELD APPROACH TO THE SPONTANEOUS
BREAKDOWN OF LOCAL SUPERSYMMETRY

Thesis

Submitted by

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Declaration

The work in this thesis is entirely my own, except where otherwise indicated. Some of the results have been published elsewhere; these articles are appended. I have attempted to present the results in a simple and clear manner, in the hope that some of the calculations may be useful to others. If certain sections seem obscure, it is because (to steal a line from Pushkin) I couldn't make a better reckoning.

To my teachers

especially the two who taught me to cook

and to write the Greek alphabet.

ABSTRACT

The goal of this thesis is to obtain a superfield formulation of local supersymmetry, and to construct via this formalism a model of spontaneous local supersymmetry breakdown.

In the first chapter, the superfield method and some globally supersymmetric models are reviewed. These include Lagrangians for massive interacting chiral multiplets, and models for both massive and massless vector multiplets. In particular, the globally supersymmetric extension of the Higgs mechanism, due to Fayet, is described in detail. This model will form the basis of a locally supersymmetric model incorporating spontaneous supersymmetry breakdown in the third chapter. None of this work is original.

The second chapter is devoted to gauging supersymmetry without superfields. The earliest supergravity theories (those not involving matter coupling) are reviewed. The fiber bundle approach is described, and shown to be ambiguous. An alternative algebraic scheme for dealing with gravitational symmetries is given.

Superfield supergravity in two dimensions forms the subject matter of the third chapter. A brief glimpse of a one-dimensional locally supersymmetric theory (the spinning particle) is given. Its two-dimensional analogue, the spinning string, is obtained first without recourse to superfields, and then via an elegant superfield Ansatz due to Howe. It is shown how to derive this Ansatz and its transformation. Finally, a locally supersymmetric version of the Fayet model is given. The generalised Higgs mechanism works to remove the Goldstone spinor, but via a gauge field (the gravitino) which is forced to be non-dynamical in two dimensions.

The methods of the third chapter are extended to four dimensions in the fourth chapter. The corresponding vielbein is derived, and shown not

to transform covariantly without the addition of new terms. An attempt is made to find these terms, and it is argued that no additions can render the vielbein covariant. Consequently the approach of the third chapter proves inapplicable to four dimensions, and no matter-supergravity coupling can be obtained in this way.

Three appendices on the history of anticommuting variables, the use of differential forms, and on some useful identities, complete the thesis.

TABLE OF CONTENTS

	Page
<u>INTRODUCTION</u>	
§1. Motivation	1
§2. Methods and Strategy	7
<u>CHAPTER I GLOBAL SUPERSYMMETRY</u>	
§3. The Earliest Models	9
§4. The Calculus of Grassmann Variables	13
.1 Differentiation and Integration	13
.2 The Majorana Representation	18
.3 Fierz Rearrangements	20
§5. The Superfield Formalism	21
.1 Representations of the Supersymmetry Group	21
.2 Functions of Superfields	25
§6. Irreducible Representations	27
§7. A Lagrangian for the Chiral Multiplet	30
§8. Lagrangians Involving the Vector Supermultiplet	32
.1 The Free Vector Multiplet	32
.2 The Supersymmetric Extension of Quantum Electrodynamics	36
.3 A Supersymmetric Goldstone Model	40
§9. The Supersymmetric Extension of the Higgs Model (The Fayet Model).	43
<u>CHAPTER II. GAUGING SUPERSYMMETRY WITHOUT SUPERFIELDS</u>	
10. The Earliest Supergravity Theories	52
.1 The Deser-Zumino Model	52
.2 The Freedman, Ferrara and van Nieuwenhuizen Model	58

TABLE OF CONTENTS (Contd.)

	Page
§11. Fiber Bundle Methods	61
.1 The SU(2) Yang-Mills Theory via Fiber Bundles	61
.2 Fiber Bundles and Supergravity	64
.3 Troubles with the Translations	67
§12. An Alternative Approach to the Vierbein	68
.1 Gravitational Symmetries	68
.2 GL(4,R)	70
.3 The Lorentz Group	74
.4 The Role of the Vierbein	75
<u>CHAPTER III. SUPERFIELD SUPERGRAVITY: TWO DIMENSIONS</u>	
§13. The Locally Supersymmetric Spinning Particle	79
§14. A First Look at the Spinning String	84
§15. A Superspace Approach to the Spinning String	87
.1 An Ansatz for the (1+1) Vielbein	87
.2 The Matter Fields	91
.3 Constructing a Lagrangian for the Spinning String	92
.4 Proof of Invariance. Comparison with the Previous Lagrangian	94
§16. Deriving the Form of the (1+1) Vielbein	98
.1 The Failure of Minimal Coupling	98
.2 The Ehresmann Formulae and an Auxiliary Field	108
§17. The Higgs Mechanism in (1+1) Supergravity	117
<u>CHAPTER IV. SUPERFIELD SUPERGRAVITY: FOUR DIMENSIONS</u>	
§18. Constructing the Vielbein	124
.1 Covariance in χ_μ	124
.2 Covariance in $\omega_\mu^{\alpha\beta}$	128

TABLE OF CONTENTS (Contd.)

	Page
§18.3 Covariance in Both χ and ω . First form of E_M^A	129
.4 The Ehresmann Laws	130
§19. Auxiliary Fields	134
.1 A First Attempt	134
.2 A Clebsch-Gordan Argument	138
§20. Summary and Conclusions	140
Acknowledgements	141
References	143
Appendix A. An Abbreviated History of Anticommuting Variables	148
Appendix B. The Use of Differential Forms	159
Appendix C. Some Useful Identities	171

INTRODUCTION

§1. Motivation

In the past twenty years one of the most fruitful ideas in theoretical physics has been the concept of local symmetry; that is, the covariance of physical laws with respect to transformation groups whose parameters are functions of position. Typically a Lagrangian which is invariant under global transformations may be made invariant under the larger local symmetry only through the introduction of new fields, which compensate for the difference in transformation character between terms involving derivatives and those which do not. These fields are known in the mathematical literature as "connections" and are called "gauge fields"* by physicists. It now appears that all of the fundamental forces can be interpreted as gauge fields, allowing for one or another local symmetry. Usually these gauge fields will be associated with vector particles, but in general their spin content will be equal to that of the derivative of the local group parameter. On first inspection, it might seem that only a theory of long-range forces could be built on a gauge principle foundation. Consider for example the standard Maxwell Lagrangian. If it is to remain invariant under the gauge transformation

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \lambda ,$$

* The name is misleading and an accident of history. Weyl (1918) attempted to derive Maxwell's theory by extending the principle of general covariance to a "relativity of magnitude", which requires the introduction of a vector field whose transformation under a change of scale, or "gauge", is the usual one associated with the electromagnetic potential. Einstein (1919) immediately pointed out that the relevant scale-factor had to be unobservable. The correct relation between Maxwell's theory and local symmetry was not apparent before the discovery of quantum mechanics. Weyl later recanted (1944).

there can be no mass term present, of the form $-\frac{1}{2}m^2 A_\mu A^\mu$. The force associated with the exchange of a massless boson must be long-range, as the Fourier transform of its propagator will be proportional to the Coulomb potential. Nevertheless, two crucial ingredients of gauge models allow for theories of short-range forces to be established on grounds similar to electromagnetism: extension of the Abelian U(1) symmetry to non-Abelian groups, and the spontaneous breakdown of a gauge symmetry. The former generalisation alone as pioneered by Yang and Mills (1954) may prove sufficient for a theory of the strong interactions. In the case of the weak interactions, the non-Abelian gauge invariance must be realised as a spontaneously broken symmetry.

For several years before the spontaneous breakdown of a gauge symmetry was considered, it had been known through the work of Nambu (1960), Goldstone (1961), and others, that breaking a global internal symmetry (i.e., one not based on a space-time group) could generate mass differences between members of a symmetry multiplet. At the same time, this phenomenon has the undesirable property of requiring the presence of massless scalar particles, and a proof was soon presented (Goldstone, Salam and Weinberg, 1962) which suggested that, at least in relativistic field theories, spontaneous breakdown was inevitably accompanied by these massless "Goldstone bosons". On the other hand, that spontaneous breaking of a symmetry can occur without the emergence of massless scalar excitations was first pointed out by Anderson (1963). In the non-relativistic BCS model of the superconductor, spontaneous breakdown of the ground state's U(1) invariance leads to massive "plasmon" excitations (rather than massless modes, whose energy tends to zero with vanishing momentum) as a consequence of the long-range Coulomb interactions. An early relativistic model involving both long-range forces and spontaneous symmetry breakdown was presented by Englert and

Brout (1964), who showed that non-Abelian gauge particles coupled to a (not necessarily elementary) field ϕ acquired a non-vanishing mass if $\langle\phi\rangle \neq 0$, but the relevance of this model to the Goldstone theorem was unclear. The first explicit example of a relativistic Anderson mechanism was given by Higgs (1964a, b), who pointed out that the Goldstone et al. theorem did not allow for the kind of transformation associated with a gauge field, and presented an explicit Abelian model in which the gauge particle's longitudinal polarization came from (and replaced) the would-be Goldstone boson. The remaining scalar behaved as an ordinary massive zero-spin particle (the "Higgs boson"). In a related development, Guralnik, Hagen and Kibble (1964) showed that the Proca Lagrangian for a non-interacting vector field could be regarded as a gauge field coupled to a Goldstone-Nambu field ϕ with associated transformation law $\phi(x) \rightarrow \phi(x) + \alpha(x)$. As a consequence of the Higgs model, it became clear that when a local symmetry is broken, not only is the mass degeneracy between a pair of scalars removed, but also the gauge field necessary for the local symmetry completely absorbs the massless particle, and becomes massive itself. Subsequently, 't Hooft (1971) showed that a spontaneously broken gauge theory is a gauge theory nevertheless; the renormalisability of a gauge theory was in no way impaired if the local symmetry were spontaneously broken. Consequently the Higgs mechanism allows for the construction of a renormalisable theory which contains a set of massive vector particles - precisely what is required for a theory of weak interactions, as in the models of Salam (1968) and Weinberg (1967). One lesson to be learned: if a theory does not seem to be renormalizable, it may be because its observed symmetry group is too small. But perhaps in this theory there lies hidden what Coleman (1973) has called "secret symmetry". If the known symmetry is the subgroup of a larger group, then consider

a gauge theory of this larger group. By choosing an appropriate set of scalar fields, it may be possible to break the larger symmetry spontaneously down to the observed group. However, the broken theory will have all the enhanced renormalisation properties of the unbroken gauge theory. The motivation behind this thesis is to investigate in part whether these methods can be applied to gravity.

The limitations of the present theory of gravity are rather similar to those of the pre-1967 theory of weak interactions. The original Fermi theory of beta decay described the experimental data well, but as a field theory it could be applied only in lowest-order perturbation theory. Furthermore, the predictive power of the theory was restricted; it remained for experiment to determine the actual form of the four-fermion interaction. Likewise, Einstein's theory at the classical level accounts very accurately for observed orbits, the bending of light by the sun, the red shift, the precession of Mercury's perihelion, and explains the observed equivalence between gravitational and inertial mass. However, beyond lowest-order perturbation theory, it fails completely as a quantum theory. It may seem a bit foolish to quantise a field whose coupling is thirty-eight orders of magnitude less than the electromagnetic coupling, and whose quantum effects will be unobservable for many years yet. Nevertheless, if the uncertainty principle is not to be violated, then gravity, like all other forces, must have a fundamentally quantum nature. It is far from clear that a Weinberg-Salam approach to quantising gravity will work, but it seems to be the best method available. In this connection, it would be interesting, and maybe even useful, to find a gravitational analog of the Higgs mechanism. This requires both a group which contains the usual invariances of Einstein's theory as subgroups, as well

as a method of breaking the gauged symmetry down to the Einstein group of general coordinate transformations. One new group which may suit these requirements (and which has generated a great deal of interest in the past four years) is supersymmetry. This particular invariance has the additional benefit of reducing divergences even as a global symmetry. The goal of this thesis is to construct a Higgs-type model for the supersymmetric extension of gravity and in general to investigate the spontaneous breakdown of local supersymmetry at the semi-classical level.

There are some further indications that supersymmetry may be the right candidate for a new theory of gravity, based on the unusual features of this group. Its most striking property consists of transforming bosons into fermions, and vice versa. Both the group generators and the parameters transform as spinors, and anticommutators are used in place of the familiar commutators of Lie group theory. The associated gauge field must therefore be a Rarita-Schwinger (spin $3/2$) field, rather than the usual vector. These generators may be described as "square roots" of translations. It is possible to work out the irreducible representations of this group, and to write down Lagrangians invariant under global supersymmetry. Even at the global level, though, there are some problems with the application of supersymmetry to the physical world. First, the generator of Fermi-Bose transformations commutes with the translation generators, so that the irreducible supermultiplets describe a family of bosons and fermions with the same mass, as long as the symmetry is unbroken. These multiplets are unknown, except for the degeneracy of $m = 0$ for graviton, photon and neutrinos. This degeneracy can be removed by spontaneous supersymmetry breaking, but only at the cost of introducing a massless spin $\frac{1}{2}$ particle.

(This is a consequence of the supersymmetric extension of Goldstone's (1961) theorem, as proved by Salam and Strathdee (1975b)). Unfortunately, this fermion cannot be associated with either of the two usual neutrinos, for PCAC-type arguments show that beta decay and related processes involving a neutrino would be suppressed below the observed rate at low neutrino energies (de Wit and Freedman 1976, Mainland, Takasugi and Tanaka 1976). Of course, if the supersymmetry is gauged, the Higgs mechanism could remove the unwanted massless "Goldstone fermion". The significance of gauging supersymmetry lies in the fact that more than one gauge field is required. A consistent theory demands the presence of gravity. A simple reason for this is the fact that the supersymmetry group contains the Poincaré group as a subgroup. If gravity is regarded as the gauge theory of the Poincaré group, then local supersymmetry requires the presence of gravity. In fact, even if one does not adopt the point of view that gravity is the gauge theory of the Poincaré group, the algebra of super-covariant derivatives forces the introduction of the gravitational potential, the vierbein e_{μ}^{α} (see §16). In summary, there is a likely candidate for the gauge group in a generalised theory of gravity. This group, supersymmetry, is not without faults; but these faults are of precisely the right form as to require gravity itself as a cure. This situation is somewhat analogous to that encountered more than a decade ago; neither the Goldstone-Nambu hope of spontaneous generation of mass differences, nor Salam's programme of a gauge-theoretic foundation for all forces, could proceed alone. Together, however, each removed the other's weakness.

§2. Methods and Strategy

Supersymmetric field theories involve an enormous amount of algebra. There are two mathematical methods which lighten the burden a little, although they are somewhat abstract. The first involves the use of some differential geometry; fiber bundles, differential forms, and related apparatus. Appendix B is devoted to an elementary review of some tame manipulations involving this machinery; it will not be used until the second half of Chapter II, and then only obliquely. The second method, namely, the superfield formalism, is by now well-known to those who study supersymmetry, but only within the past year or so has it been applied to supergravity. The method employed here is, frankly, pedestrian in comparison to the two most recent formulations of superfield supergravity (Wess and Zumino 1978a, b, c; Grimm, Wess and Zumino 1978; Siegal 1978a, b, c, d; Siegal and Gates 1978; Gates 1978a, b). On the other hand, it seems closer to the usual description of gauge theories, and it has the advantage that everything is explicit.

In order to begin studying spontaneous breakdown of local supersymmetry, a certain familiarity with global supersymmetry is required. Chapter I is devoted to a review of the early models, starting from the first attempts to enlarge the Poincaré group to include spinor generators. Anticommutating variables and their calculus are introduced, culminating in the superfield formalism. The vector and chiral supermultiplets are described, and the irreducible particle multiplets are worked out. Several Lagrangians are discussed; the free chiral multiplet, the supersymmetric extension of quantum electrodynamics, the first example of global spontaneous supersymmetry breaking and the globally supersymmetric extension of the

Higgs mechanism. None of this work is original (but some results are described in a new way). In Chapter II, the first examples of gauging supersymmetry are reviewed. These models do not involve superfields. In each case, the coupling to matter is unclear. This is due in part to some difficulties involving the vierbein e_{μ}^{α} (the square-root of the metric tensor $g_{\mu\nu}$), and a new approach is devised. Chapter III begins with a brief survey of the first superfield formulations of supergravity, restricted to one and two dimensions. It is shown how to derive an elegant Ansatz for a super-vierbein, and the generalised Higgs mechanism is worked out explicitly with a polynomial Lagrangian for the first time (albeit in two dimensions). Finally in Chapter IV, following from the second half of Chapter III, a super-vierbein is obtained in four dimensions by considering the coupling to a scalar superfield. It is then shown that this super-vierbein fails to transform correctly. An attempt is made to repair the transformations by the inclusion of extra terms. This attempt fails, and two other remedies are shown to be ineffective as well. It is concluded reluctantly that a covariant super-vierbein in four dimensions cannot be constructed by the methods of Chapter III. Therefore this approach to supergravity-matter coupling cannot produce a model of the locally supersymmetric Higgs mechanism in four dimensions. Three appendices provide background material on the history of anti-commuting quantities ("Grassmann variables"), the calculus of differential forms, and certain identities in two and four dimensions.

CHAPTER I

GLOBAL SUPERSYMMETRY

§3. The Earliest Models

The idea of a physical transformation mixing bosons and fermions, and involving spinorial parameters and generators, was developed independently by two quite separate camps: field theorists interested in generalising some bosonic concepts (notably the Poincaré group) to fermionic equivalents, and phenomenologists who required new symmetries to remove "ghost" states from certain dual models. Among the first group, the main stimulus seems to have come from Berezin's (1966) work on fermionic path integrals. Later Berezin and Katz (1969) described Lie algebras in which some of the generators were spinorial; such an algebra is now known as a "graded Lie algebra", or GLA for short (Corwin, Ne'eman and Sternberg 1975). The first physical examples of such a group, together with some invariant Lagrangians were given by Gol'fand and Likhtman (1971), who argued that not every Poincaré invariant theory was realised in nature, and attempted to find a more stringent requirement by enlarging the Poincaré group to include a spinor generator W , obeying the algebra

$$\begin{aligned} [M_{\mu\nu}, W] &= \frac{1}{2} \sigma_{\mu\nu} W \\ [P_\mu, W] &= 0 \\ \{\bar{W}, W\} &= 2\mathcal{P} \end{aligned} \tag{1}$$

while the algebra of the Poincaré generators remained unchanged.

(Note $\mathcal{P} = \gamma^\mu P_\mu$; for conventions regarding the gamma matrices, see §4.2). Curiously, Gol'fand and Likhtman declined to give the

transformation laws for the various fields. Three years later, the algebra (1) was rediscovered and has since become the basis of all subsequent developments. The first explicit description of a spinorial translation seems to have been the attempt by Volkov and Akulov (1973) to explain the neutrino's zero mass as a consequence of the Goldstone theorem (Goldstone et al. 1962). The hallmark of a Goldstone particle is its inhomogeneous transformation law. Consider the neutrino's wave function, denoted ν , subjected to a "spinorial translation"

$$\nu \rightarrow \nu + \alpha \quad (2a)$$

where α is a spinor parameter, while x^μ also suffered a displacement;

$$x^\mu \rightarrow x^\mu - (\kappa/2i)(\bar{\alpha}\sigma^\mu\nu - \bar{\nu}\sigma^\mu\alpha) \quad (2b)$$

and κ is some constant. In §5, the action of linear operators on functions of the variables (x, ν) will translate the argument according to (2). It will be shown that these operators obey the algebra (1). It is easily verified that (2) leaves invariant the differential forms ω_μ ;

$$\omega_\mu = dx_\mu + (\kappa/2i)(\bar{\nu}\sigma_\mu d\nu - d\bar{\nu}\sigma_\mu\nu) \quad (3)$$

so that an invariant action was given by

$$\begin{aligned} L &= (1/\kappa) \int \omega_0 \times \omega_1 \times \omega_2 \times \omega_3 \\ &= (1/\kappa) \int \det W_{\mu\nu} d^4x \end{aligned}$$

where $W_{\mu\nu} = \eta_{\mu\nu} + (\kappa/2i)\nu \overset{\leftrightarrow}{\sigma}_\mu \partial_\nu \nu$.

At the same time, theorists concerned with dual resonance models were led to consider these extended Lie groups in order to rid the Neveu-Schwarz (1971) and Ramond (1971) models of unwanted fictitious states. It had been noted that conformal invariance provided enough gauge operators to remove many of the ghosts, analogous to the Gupta-Bleuler method in electrodynamics. Neveu and Schwarz were led to postulate the existence of "super-gauge" operators which would remove the remaining spurious states. Later, Gervais and Sakita (1971) provided a realisation of these operators through their action on the fields in the theory, mixing bosons with fermions. By focusing on the two key elements of these two-dimensional supergauge transformations, namely conformal invariance and spinor parameters, Weiss and Zumino (1974a) found analogous operators in four dimensions. Their reasoning may be described as follows.

The infinitesimal coordinate transformation

$$x^\mu \rightarrow x^\mu + \xi^\mu \quad (4)$$

belongs to the restricted conformal group (i.e. those rescalings of the metric tensor $\eta_{\mu\nu}$ which do not induce curvature) if the parameters ξ^μ satisfy the differential equation (Goldberg 1962)

$$\eta_{\mu\nu} \partial \cdot \xi = 2 \partial_{(\nu} \xi_{\mu)} \quad (5)$$

The solution to (5) is well-known;

$$\xi_\mu = c_\mu + \omega_{\mu\nu} x^\nu + x_\mu d + a_\mu x^2 - 2x_\mu a \cdot x \quad (6)$$

where the parameters (c, ω, d, a) are all constant and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Consider now a related spinor transformation, where $\alpha(x)$ is a Majorana spinor (see §4);

$$\eta_{\mu\nu} \not{\partial} \alpha = 2\partial_{(\nu} \gamma_{\mu)} \alpha \quad (7)$$

From (7) immediately follow two conditions:

- i) $\alpha(x)$ is at most linear in x ;
- ii) if $\alpha_1(x)$, $\alpha_2(x)$ are solutions to (7), then

$$\xi_\mu' = 2i\bar{\alpha}_1 \gamma_\mu \alpha_2 \text{ is a solution to (6).}$$

In analogy with the Gervais and Sakita transformations, Wess and Zumino introduced a "supermultiplet" $\{A, B, \psi, F, G\}$; A and F are scalars, B and G pseudoscalars and ψ a Majorana spinor. The postulated transformations are

$$\begin{aligned} \delta A &= i\bar{\alpha}\psi \\ \delta B &= i\bar{\alpha}\gamma^5\psi \\ \delta\psi &= \not{\partial}(A + \gamma^5 B)\alpha + n(A - \gamma^5 B)\not{\partial}\alpha + (F + \gamma^5 G)\alpha \\ \delta F &= i\bar{\alpha}\not{\partial}\psi + (i/2)(2n + 1)(\not{\partial}\bar{\alpha})\psi \\ \delta G &= i\bar{\alpha}\gamma^5\not{\partial}\psi - (i/2)(2n + 1)(\not{\partial}\bar{\alpha})\gamma^5\psi \end{aligned} \quad (8)$$

For ahistorical reasons, this will be called the "chiral multiplet" (originally it was the "scalar" multiplet). The number n is the conformal weight of the multiplet. If a Hermitean supergauge operator S is introduced, then (8) may be written

$$\delta A = [i\bar{\alpha}S, A]$$

where $A \rightarrow A' = \exp(i\bar{\alpha}S)A \exp(-i\bar{\alpha}S) = A + \delta A + \dots$

The commutator δ_3 of two such transformations ($\delta_3 = [\delta_1, \delta_2]$) acting on the various fields is

$$\begin{aligned}
 \delta_3 A_{\pm} &= \xi' \cdot \partial A_{\pm} + (n/2) \partial \cdot \xi' A_{\pm} \pm n \eta A_{\pm} \\
 \delta_3 F_{\pm} &= \xi' \cdot \partial F_{\pm} + \frac{1}{4} (2n+1) \partial \cdot \xi' F_{\pm} \pm \{(3/2) - n\} \eta F_{\pm} \\
 \delta_3 \psi &= \xi' \cdot \partial \psi + (1/8) (4n+1) \partial \cdot \xi' \psi + \{n - (3/4)\} \eta \gamma^5 \psi \\
 &\quad + \frac{1}{2} (\partial_{\mu} \xi_{\nu}') \sigma^{\mu\nu} \psi
 \end{aligned} \tag{10}$$

where $A_{+,-} = (A, B)$; $F_{+,-} = (F, G)$, ξ_{μ}' is as given above, and $\eta = i(\partial_{\mu} \bar{\alpha}_1 \gamma^5 \gamma^{\mu} \alpha_2 - \partial_{\mu} \bar{\alpha}_2 \gamma^5 \gamma^{\mu} \alpha_1)$. The commutator does not act in the same way on all fields in the multiplet. However, if the parameters $\alpha_i(x)$ are restricted to be constant, then $\partial_{\mu} \xi_{\nu}' = 0$, $\eta = 0$, and for all fields in the multiplet,

$$[i\bar{\alpha}_1 S, i\bar{\alpha}_2 S] = 2i\bar{\alpha}_1 \gamma^{\mu} \alpha_2 \partial_{\mu} = 2\xi^{\mu} P_{\mu} \tag{11}$$

It would seem that the restriction $\partial_{\nu} \xi_{\mu} = 0$ has collapsed the (extended) conformal group to the (extended) Poincaré group. This is indeed the case. From $\partial_{\mu} \xi_{\nu} = 0$ follows the weaker condition $\partial \cdot \xi = 0$; differentiating (6) and setting it equal to zero leads to the constraint $4d - 2a \cdot x = 0$, so that ξ_{μ} must be of the standard Poincaré form. To sum up: extending the ideas of Gervais and Sakita to four dimensions, Wess and Zumino were led to consider a spinorial conformal transformation. To obtain a simple commutator, these transformations were restricted to use only constant spinorial parameters, and the resulting algebra closed on the Poincaré group. From (11) it follows trivially for constant α

$$\{\bar{S}, S\} = 2i\mathbb{1} = 2P, \tag{12}$$

identical to the W generators of Gol'fand and Likhtman. Henceforth this group generated by the operators $(P_{\mu}, M_{\mu\nu}, S)$ will be called the supersymmetry group. Under this group, the chiral multiplet supergauge

transformations become

$$\begin{aligned}\delta A &= i\bar{\alpha}\psi; & \delta B &= i\bar{\alpha}\gamma^5\psi; & \delta F &= i\bar{\alpha}\not{\partial}\psi \\ \delta G &= i\bar{\alpha}\gamma^5\not{\partial}\psi; & \delta\psi &= \not{\partial}(A+\gamma^5B)\alpha + (F + \gamma^5G)\alpha\end{aligned}\tag{8}$$

Two features of the chiral multiplet suggest how to begin the search for other multiplets. Let $\phi(j_1, j_2)$ be a member of some multiplet which transforms according to the $D(j_1, j_2)$ representation of the Lorentz group. Then in general:

i) There must be as many Bose components as Fermi components in the multiplet (recall that the dimension of $\phi(j_1, j_2)$ is $(2j_1+1)(2j_2+1)$). A Majorana spinor counts as four Fermi components, while a vector counts as four Bose components. In general, not all of these will be true dynamical degrees of freedom.

ii) Let ϕ_1, ϕ_2 be bosonic fields and ψ_1, ψ_2 be fermionic fields in the multiplet. Then

$$\begin{aligned}\delta\phi_1 &= (\text{linear combination of } \bar{\alpha}\psi_1, \bar{\alpha}\psi_2, \bar{\alpha}\not{\partial}\psi_1, \bar{\alpha}\not{\partial}\psi_2), \\ \delta\psi_1 &= (\text{linear combination of } \phi_1\alpha, \phi_2\alpha, \partial^\mu\phi_{1,2}\gamma_\mu\alpha)\end{aligned}$$

and there should be a finite number of fields in the multiplet. The smallest multiplet containing a vector should thus require four scalar and two spinor fields as well. A consistent set of transformations is given by (Wess and Zumino 1974a)

$$\begin{aligned}\delta C &= i\bar{\alpha}\gamma^5\chi \\ \delta\chi &= (M + \gamma^5N - \gamma^5\not{\partial}C) \\ \delta M &= i\bar{\alpha}\mu + i\bar{\alpha}\not{\partial}\chi \\ \delta N &= i\bar{\alpha}\gamma^5\mu + i\bar{\alpha}\gamma^5\not{\partial}\chi \\ \delta V_\mu &= i\bar{\alpha}\gamma_\mu\lambda + i\bar{\alpha}\partial_\mu\chi \\ \delta\mu &= -F_{\mu\nu}\sigma^{\mu\nu}\alpha + D\gamma^5\alpha \\ \delta D &= i\bar{\alpha}\gamma^5\not{\partial}\mu\end{aligned}\tag{13}$$

where $F_{\mu\nu} = \partial_{[\mu} V_{\nu]}$. Further multiplets can be constructed with these guidelines, but there is a systematic method available, due to Salam and Strathdee (1975a): the superfield formalism.

§4. The Calculus of Grassmann Variables

In the usual theory of Lie groups, the generators G are defined on functions whose values lie in the same number field as the parameters of G . Reversing the argument, Salam and Strathdee (1975a) introduced functions of variables θ_i , which behave according to the nature of the generators S ; that is, they anticommute with each other and they transform as spinors under the Lorentz group. A set of n variables which obey the fundamental relation (Grassmann, 1878)

$$\theta_i \theta_j = -\theta_j \theta_i \quad (14)$$

is said to constitute a Grassmann algebra. These variables have been introduced repeatedly in mathematics and physics (Appendix A). As a consequence of (14), there are $\binom{n}{k}$ independent monomials

$\theta_{i_1} \theta_{i_2} \dots \theta_{i_k}$ of order k , and the vector space has a dimension equal to 2^n . In particular, all monomials of order $(n+1)$ and greater are identically zero. Any function of these θ 's is uniquely determined by its (necessarily finite) Taylor expansion. Henceforth n will be restricted to the values 1, 2 or 4.

§4.1 Differentiation and Integration

Let spinor indices be denoted $a, b, c \dots$. For consistency, the operator $\partial/\partial\theta^a$ must anticommute with all θ_b ;

$$\partial/\partial\theta^a (\theta_b \theta_c) = \delta_{ab} \theta_c - \theta_b \delta_{ac} \quad (15)$$

Note that in general left derivatives are not equal to right derivatives, for

$$(\theta_b \theta_c) \overleftarrow{\partial} / \partial \theta^a = \theta_b \delta_{ac} - \theta_c \delta_{ab} .$$

Unless otherwise specified, all differentiation will be from the left, according to (15). Integration over these variables has been defined through the functional formalism by Berezin (1966), who required that these integrals be translation invariant. For simplicity, consider the one-dimensional case. It is necessary that

$$\int f(\theta) d\theta = \int f(\theta + \alpha) d(\theta + \alpha) = \int f(\theta + \alpha) d\theta \quad (16)$$

Because $f(\theta) = a + b\theta$, (16) leads to

$$a \int d\theta = (a + b\alpha) \int d\theta$$

$$b \int \theta d\theta = b \int \theta d\theta$$

which forces the rules

$$\int d\theta = 0 \quad (17a)$$

$$\int \theta d\theta = (\text{constant}) . \quad (17b)$$

For convenience, the constant is arbitrarily set equal to one. By induction it then follows

$$\int f(\theta) d\theta_1 \dots d\theta_n = \frac{\partial}{\partial \theta^1} \dots \frac{\partial}{\partial \theta^n} f(\theta) ; \quad (18)$$

integration and differentiation are the same operation for functions of Grassmann variables. In the one-dimensional case, under a change of scale $\theta \rightarrow c\theta$

$$\int c\theta d(c\theta) = 1; \quad \text{thus } d(c\theta) = (1/c)d\theta . \quad (19)$$

In general, let $\theta'_a = \psi_a(\theta)$. It follows that

$$\int f(\theta) d^n \theta = \int f(\psi(\theta)) (1/J) d^n \psi \quad (20)$$

where J is the Jacobian determinant

$$J = \det |\partial \theta_a / \partial \psi_b|.$$

The proof proceeds analogously to the usual one for commuting variables. The quantities $\partial \theta_a / \partial \psi_b$ are bosonic, so that no difficulty is presented in forming a determinant. However, determinants of matrices whose elements include fermionic quantities will be needed in the sequel. Suppose that M_{AB} is a matrix whose elements $M_{\alpha\beta}$, M_{ab} are bosonic, while $M_{\alpha b}$ and $M_{a\beta}$ are fermionic. It has been established via an algebraic proof (Arnowitz, Nath and Zumino 1975) that

$$\det M_{AB} = (\det M_{\alpha\beta}) (\det (M^{-1})_{ab}). \quad (21)$$

An alternate, heuristic proof follows from (19). Consider an arbitrary function $f(x, \theta) = f(z)$. Let a constant linear transformation be given such that $z_A \rightarrow Z'_A = M_{AB} z_B$; where $z_\alpha = x_\alpha$, $z_a = \theta_a$. Then the determinant of M_{AB} may be defined by

$$\int f(z(z')) dz' = \det M \int f(z) dz \quad (22)$$

On the other hand, $(f^{(n)}) = d^n / d\theta^n f(x, \theta)$

$$\begin{aligned} \int f(z') dz' &= \int f^{(n)}(x') d^n x' (\theta)^n d^n \theta' \\ &= (\det M_{\alpha\beta}) \int f^{(n)}(x) d^n x \theta^n d^n \theta \det(\partial \theta_a / \partial \theta'_b) \end{aligned}$$

by (20). Consequently

$$\int f(z') dz' = (\det M_{\alpha\beta}) (\det (M)^{-1}_{ab}) \int f(z) dz$$

which was to be shown. Finally, it will be necessary to find a representation of the Grassmann delta-function. For one dimension, it is a trivial exercise to find $\delta(\theta)$ such that

$$\int \delta(\theta) d\theta = 1; \quad \int \delta(\theta' - \theta) f(\theta') d\theta' = f(\theta);$$

namely (Alvarez 1978, Ogievetski and Mezincescu 1974)

$$\delta(\theta' - \theta) = \theta' - \theta.$$

By induction it is easy to see that the general n -dimensional delta function is given by the n^{th} -order polynomial

$$\delta^{(n)}(\theta' - \theta) = \prod_{a=1}^n (\theta'_a - \theta_a). \quad (24)$$

§4.2 The Majorana Representation

In order that the generators S_a be restricted to linear combinations of four independent Hermitean components, it is necessary that they satisfy the Majorana constraint

$$S^C \equiv C(\bar{S}^T) = S \quad (25)$$

where C is the charge-conjugation matrix. Similarly, the parameters α and the Grassmann variables θ are also to satisfy (25). It is convenient to choose a representation of the Dirac matrices such that (25) becomes (Majorana 1937, Berestetski, Lifshitz and Pitaevski 1974)

$$\psi^C = \psi^* = \psi$$

for an arbitrary Majorana spinor. This representation also makes the

Dirac equation real. An example is given by (in four dimensions)

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} & \gamma^1 &= i \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} & \gamma^2 &= \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \\ \gamma^3 &= -i \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} & \gamma^5 &= i \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} & C &= \gamma^0 \end{aligned}$$

and may be obtained from the unitary transformation

$$\gamma_M^\mu = U \gamma_{BD}^\mu U^{-1}$$

where $U = U^{-1} = (1/\sqrt{2})(\gamma^0 + \gamma^0 \gamma^2)_{BD}$, and the suffix BD indicates the representation of Bjorken and Drell. Note that all of the γ 's are pure imaginary (except γ^5) in keeping with the "West coast" metric ($\eta_{\mu\nu} = \text{diag}(+---)$) form of the Dirac equation, $(i\not{D} - m)\psi = 0$. Also the set $(\gamma^0, \gamma^0 \gamma^5, \gamma^0 \gamma^\nu \gamma^5)$ are antisymmetric, while $(\gamma^0 \gamma^\mu, \gamma^0 \sigma^{\mu\nu})$ are symmetric ($\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$). There is a very useful relation for a bilinear of n gamma matrices in the Majorana representation: if ψ, χ are Majorana spinors, then by charge-conjugation* $(C\gamma^\mu C = -\gamma^{\mu T})$

$$\bar{\psi} \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_n} \chi = (-1)^n \bar{\chi} \gamma_{\mu_n} \gamma_{\mu_{n-1}} \dots \gamma_{\mu_1} \psi. \quad (26)$$

This relation holds in two dimensions as well, in which the Majorana representation is (Howe, 1977)

$$\gamma^0 = \sigma^2 \quad \gamma^1 = i\sigma^1 \quad \gamma^5 = \sigma^3 \quad (27)$$

* I wish to thank Dr. S.J. Gates, Jr., for suggesting this method of proof to me.

§4.3 Fierz Rearrangements

Given the product of two Dirac bilinears, it is always possible to rearrange the factors, owing to the completeness relation for the γ 's (Pauli 1936, Fierz 1937, Jauch and Rohrlich 1976). In four dimensions

$$\sum_x \Gamma_{ab}^x \Gamma_{cd}^x = 4\delta_{ad} \delta_{bc}$$

where x runs from 1 to 16, and $(\Gamma^x)^2 = 1$. If $\chi, \lambda, \psi, \eta$ are spinors then

$$\begin{aligned} \bar{\chi}\lambda\bar{\psi}\eta &= -\frac{1}{4}\bar{\chi}\eta\bar{\psi}\lambda + \frac{1}{4}\bar{\chi}\gamma^5\eta\bar{\psi}\gamma^5\lambda + \frac{1}{4}\bar{\chi}i\gamma^\nu\eta\bar{\psi}i\gamma_\nu\gamma^5\lambda \\ &\quad -\frac{1}{4}\bar{\chi}\gamma^\mu\eta\bar{\psi}\gamma_\mu\lambda + \bar{\chi}\sigma^{\mu\nu}\eta\bar{\psi}\sigma_{\mu\nu}\lambda. \end{aligned} \quad (28)$$

In two dimensions, the Fierz rearrangement formula is

$$\bar{\chi}\lambda\bar{\psi}\eta = -\frac{1}{2}\bar{\chi}\eta\bar{\psi}\lambda - \frac{1}{2}\bar{\chi}\gamma^5\eta\bar{\psi}\gamma^5\lambda - \frac{1}{2}\bar{\chi}\gamma^\mu\eta\bar{\psi}\gamma_\mu\lambda. \quad (29)$$

The Fierz formula is particularly useful whenever two or more of the spinors are the same, since several of the bilinears will vanish. For example, in two dimensions $\bar{\theta}\psi\bar{\lambda}\theta = -\frac{1}{2}\bar{\theta}\theta\bar{\lambda}\psi$, while in four dimensions $\bar{\epsilon}\gamma^\mu\bar{\theta}\theta\gamma^5\theta = -\bar{\epsilon}\gamma^\mu\gamma^5\theta\bar{\theta}\theta$, and $\bar{\epsilon}\gamma^\mu\theta\theta\gamma^\nu\gamma^5\theta = -\bar{\epsilon}\gamma^\mu\gamma^\nu\gamma^5\theta\bar{\theta}\theta$. There are an enormous number of these identities; a few of the most useful are listed in Appendix C. The properties of the Majorana representation determine convenient bases for the $\begin{bmatrix} n \\ k \end{bmatrix}$ monomials $\theta_{i_1} \dots \theta_{i_k}$:

$$n = 2: \quad 1, \theta_a, \bar{\theta}\theta \quad (30)$$

$$n = 4: \quad 1, \theta_a, \bar{\theta}\theta, \bar{\theta}\gamma^5\theta, \bar{\theta}i\gamma^\nu\gamma^5\theta, \bar{\theta}\theta\theta_a, (\bar{\theta}\theta)^2. \quad (31)$$

In particular, $\bar{\theta}\theta\bar{\theta}\gamma^5\theta = \bar{\theta}\theta\bar{\theta}i\gamma^\nu\gamma^5\theta = 0$, and $(\bar{\theta}\gamma^5\theta)^2 = (\bar{\theta}\theta)^2$.

That is, there is a unique factorisation of any number of θ 's in four dimensions except 2.

§5. The Superfield Formalism

§5.1 Representations of the Supersymmetry Group

Following from Volkov and Akulov's transformations (2), let θ_a denote a Grassmann variable (in particular, the four real components of a Majorana spinor) and consider the transformation of a function $\Phi(x, \theta)$ induced by the coordinate changes*

$$\begin{aligned}\theta &\rightarrow \theta + \epsilon \\ x_\mu &\rightarrow x_\mu - (i/2)\bar{\epsilon}\gamma_\mu\theta\end{aligned}\tag{33}$$

where now θ plays the role formerly assigned to the neutrino's field. Note that both ϵ and θ are taken to be independent of x . The parametrised generator $i\bar{\epsilon}S$ must satisfy, according to (33),

$$\begin{aligned}[i\bar{\epsilon}S, \theta] &= \delta\theta = \epsilon \\ [i\bar{\epsilon}S, x_\mu] &= \delta x_\mu = (-i/2)\bar{\epsilon}\gamma_\mu\theta\end{aligned}\tag{34}$$

and the unique realisation of this is

$$\begin{aligned}iS &= \partial/\partial\bar{\theta} - (i/2)\not{\theta} \\ -i\bar{S} &= \partial/\partial\theta + (i/2)\bar{\theta}\not{\partial}\end{aligned}\tag{35}$$

*N.B. Wess and Zumino (1974a) and Volkov and Akulov (1973) use the normalization $\theta \rightarrow \sqrt{2}\theta$ $\epsilon \rightarrow \sqrt{2}\alpha$, and becomes

$$\theta \rightarrow \theta + \alpha \quad x_\mu \rightarrow x_\mu - i\alpha\gamma_\mu\theta$$

Note that now

$$\{\bar{S}, S\} = i\delta = \not{1} \quad (12)$$

Using the basis (31), an arbitrary function $\Phi(x, \theta)$ may be expanded in terms of other fields. This arbitrary function, called a "superfield", can carry Lorentz indices corresponding to a spinor, vector, or whatever, or else it may be a scalar or pseudoscalar. In general $\Phi(x, \theta)$ may be expanded as

$$\begin{aligned} \Phi(x, \theta) = & \phi + i\bar{\theta}\psi + \frac{1}{4}\bar{\theta}\theta F + \frac{1}{4}\bar{\theta}\gamma^5\theta G \\ & + \frac{1}{4}\bar{\theta}i\gamma^\nu\gamma^5\theta A_\nu + \frac{1}{4}i\bar{\theta}\theta\bar{\theta}\lambda + (1/32)(\bar{\theta}\theta)^2 D \end{aligned} \quad (36)$$

where the fields (ϕ, F, G, D) carry the Lorentz indices of Φ (but note that the parity of G is opposite that of the other three), while λ and ψ carry extra Majorana spinor indices as well as the Lorentz indices of Φ , and A_ν carries an extra vector index and takes the same parity as G . The most useful superfield in the models derived to date is a scalar superfield, in which case (ϕ, F, G, D) all belong to different $D(0,0)$ representations of the Lorentz group while (ψ, λ) each belong to $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$, and A_μ belongs to $D(\frac{1}{2}, \frac{1}{2})$. Under reflections only A_μ and G change sign. Then

$$\begin{aligned} \delta\Phi &= [i\bar{\epsilon}S, \Phi] = \delta\phi + i\bar{\theta}\delta\psi + \dots \\ &= i\bar{\epsilon}\psi + \frac{1}{2}\bar{\epsilon}\theta F + \frac{1}{2}\bar{\epsilon}\gamma^5\theta G + \frac{1}{2}\bar{\epsilon}\gamma^\nu\gamma^5\theta A_\nu - \frac{1}{2}\bar{\epsilon}i\not{\lambda}\phi\theta \\ &\quad + \frac{1}{2}\bar{\epsilon}\gamma^\mu\bar{\theta}\partial_\mu\psi + \frac{1}{2}i\bar{\epsilon}\theta\bar{\theta}\lambda + \frac{1}{4}i\bar{\epsilon}\lambda\bar{\theta}\theta + (1/8)\bar{\theta}\theta\bar{\epsilon}\theta D \\ &\quad - (1/8)\bar{\theta}\theta\bar{\epsilon}\not{\lambda}F\theta - (1/8)\bar{\theta}\gamma^5\theta\bar{\epsilon}\not{\lambda}G\theta - (1/8)\bar{\theta}i\gamma^\nu\gamma^5\theta\bar{\epsilon}\not{\lambda}A_\nu\theta \\ &\quad + (1/8)\bar{\theta}\theta\bar{\theta}\partial_\mu\lambda\bar{\epsilon}\gamma^\mu\theta \end{aligned} \quad (37)$$

which implies, upon Fierz rearrangement,

$$\begin{aligned}
 \delta\phi &= i\bar{\epsilon}\psi \\
 \delta\psi &= \frac{1}{2}(-iF - i\gamma^5 G - \gamma^5 A + \not{d}\phi)\epsilon \\
 \delta F &= \frac{1}{2}(i\bar{\epsilon}\lambda - \bar{\epsilon}\not{d}\psi) \\
 \delta G &= \frac{1}{2}(i\bar{\epsilon}\gamma^5\lambda - \bar{\epsilon}\gamma^5\not{d}\psi) \\
 \delta A_\nu &= \frac{1}{2}(-\bar{\epsilon}\gamma_\nu\gamma^5\lambda + i\bar{\epsilon}\gamma^\mu\gamma_\nu\gamma^5\partial_\mu\psi) \\
 \delta\lambda &= \frac{1}{2}(\not{d}F + \not{d}\gamma^5 G - i\gamma^\nu\gamma^5\not{d}A_\nu - iD)\epsilon \\
 \delta D &= -\bar{\epsilon}\not{d}\lambda .
 \end{aligned} \tag{38}$$

These transformations are identical with those of the vector multiplet (13) if ϵ is replaced by $\sqrt{2}\alpha$, and if the set $(\phi, \psi, F, G, A_\nu, \lambda, D)$ is replaced by $(C, (1/\sqrt{2})\gamma^5\mu, -iN, iM, -V_\nu, i\sqrt{2}\mu + (i/\sqrt{2})\not{d}\gamma^5\chi, -2D - \partial^2 C)$. Note that the scalar and pseudoscalar fields have interchanged roles. It is also possible to derive the chiral multiplet as well, using the superfield formalism. First, however, an invariant constraint, removing half the fields, needs to be found.

To this end, Salam and Strathdee (1975a) introduced an invariant operator D :

$$D = \partial/\partial\bar{\theta} + (i/2)\not{d}\theta ; \tag{39}$$

$$\{D, S\} = 0 . \tag{40}$$

In addition to yielding the chiral multiplet, this operator will prove invaluable for the construction of Lagrangians. The invariant constraint to be solved is

$$P_+ D\Phi = 0 \tag{41}$$

where $P_\pm = \frac{1}{2}(1 \pm i\gamma^5)$ are the usual projection operators for left (+)

and right (-) handed spinors. Let $T = -\frac{1}{4}\bar{\theta}\not{\partial}\gamma^5\theta$. Then for Ψ an arbitrary superfield,

$$D(\exp T)\Psi = (\exp T)(\partial/\partial\bar{\theta} + iP_{-}\not{\partial}\theta)\Psi \quad (42)$$

so that

$$P_{+}\exp T\Psi = (\exp T)(P_{+}\partial/\partial\bar{\theta})\Psi \quad (43)$$

and the solutions to (41) are thus

$$\phi_{-} = \exp T \phi_{-}, \quad \text{where} \quad (44)$$

$$\phi_{-} = a + i\bar{\theta}P_{-}\beta + \frac{1}{2}\bar{\theta}P_{-}\not{\partial}f; \quad (45)$$

The superfield ϕ_{-} must terminate after the quadratic term, as $P_{-}\theta$ has but two independent components. Consequently, the Fierz formulae for these chiral spinors are basically those of the two-dimensional spinors. E.g., $\bar{\epsilon}\gamma^{\mu}P_{-}\theta\bar{\theta}P_{-}\not{\partial}_{\mu}\psi = -\frac{1}{2}\bar{\theta}P_{-}\theta\bar{\epsilon}\not{\partial}\psi$. It is also easy to construct right-handed superfields which satisfy the constraint

$$P_{-}D\Psi = 0.$$

These are given by

$$\phi_{+} = \exp(-T)\phi_{+}, \quad \text{where}$$

$$\phi_{+} = a + i\bar{\theta}P_{+}\beta - \frac{1}{2}\bar{\theta}P_{+}\not{\partial}f. \quad (46)$$

The product of n left-handed superfields is again a left-handed superfield, but the product of a left-handed superfield with one which is right-handed will be a scalar superfield. The transformations for a , β , and f are worked out as usual, to give

$$\begin{aligned}
 \delta a &= i\bar{\epsilon} P_{\beta} \\
 \delta P_{\beta} &= P_{\gamma}^{\mu} \epsilon \partial_{\mu} a - i P_{\epsilon} f \\
 \delta f &= -\bar{\epsilon} \not{P}_{\beta} .
 \end{aligned} \tag{47}$$

Agreement with the earlier chiral multiplet is obtained via the substitutions $\beta = \sqrt{2}\psi$, $\epsilon = \sqrt{2}\alpha$, $a = A - iB$, $f = iF - G$. Again, the scalar and pseudoscalar fields have changed places, owing to different conventions for the spinors. Writing out (47) explicitly gives (recall γ^{μ} are imaginary)

$$\begin{aligned}
 \delta A &= i\bar{\alpha}\psi \\
 \delta B &= i\bar{\alpha}\gamma^5\psi \\
 \delta\psi &= \not{\epsilon}(A + \gamma^5 B)\alpha + i(G - \gamma^5 F)\alpha \\
 \delta G &= -\bar{\alpha}\not{\epsilon}\psi \\
 \delta F &= -\bar{\alpha}\gamma^5\not{\epsilon}\psi
 \end{aligned} \tag{48}$$

(compare 8). Lagrangians for both the chiral multiplet and the vector multiplet have been constructed by Wess and Zumino (1974a, b) and Salam and Strathdee (1975a); these will be presented in §7.

§5.2 Functions of Superfields

Consider the case of $n = 1$, so that an arbitrary superfield Φ may be expanded as

$$\Phi = a + i\beta\theta.$$

By Taylor's theorem,

$$f(\Phi) = f(0) + \Phi f^{(1)}(0) + \frac{1}{2}\Phi^2 f^{(2)}(0) + \dots$$

On the other hand,

$$\Phi^n = a^n + nia^{n-1}\beta\theta,$$

so that

$$\begin{aligned} f(\phi) &= f(0) + (a + i\beta\theta)f^{(1)}(0) + \frac{1}{2}(a^2 + 2ia\beta\theta)f^{(2)}(0) + \dots \\ &= f(a) + i\beta\theta f^{(1)}(a). \end{aligned}$$

For a superfield which is an expansion in n θ 's it is only necessary to know the explicit form of ϕ^n in order to write down the formula for an arbitrary function of ϕ . For $n = 4$, the formula for ϕ^2 is

$$\phi^2 = (\phi^2)_\phi + i\bar{\theta}(\phi^2)_\psi + \frac{1}{2}\bar{\theta}\theta(\phi^2)_F + \dots$$

where

$$\begin{aligned} (\phi^2)_\phi &= \phi^2 \\ (\phi^2)_\psi &= 2\phi\psi \\ (\phi^2)_F &= 2F\phi + \bar{\psi}\psi \\ (\phi^2)_G &= 2G\phi - \bar{\psi}\gamma^5\psi \\ (\phi^2)_{A_\nu} &= 2A_\nu\phi - \bar{\psi}i\gamma_\nu\gamma^5\psi \\ (\phi^2)_\lambda &= 2\lambda\phi + 2\psi F - 2\gamma^5\psi G - 2i\gamma^\nu\gamma^5\psi A_\nu \\ (\phi^2)_D &= 2D\phi + 4\bar{\lambda}\psi + 2G^2 + 2F^2 + 2A_\mu A^\mu. \end{aligned} \tag{49}$$

Squaring this formula leads easily to the expression for ϕ^4 . An arbitrary function of ϕ is therefore given by

$$f(\phi) = f(\phi)_\phi + i\bar{\theta}f(\phi)_\psi + \dots + (1/32)(\bar{\theta}\theta)^2 f(\phi)_D$$

where

$$\begin{aligned} f(\phi)_\phi &= a_0 \\ f(\phi)_\psi &= a_1\psi \\ f(\phi)_F &= Fa_1 + \frac{1}{2}\bar{\psi}\psi a_2 \\ f(\phi)_G &= Ga_1 - \frac{1}{2}\bar{\psi}\gamma^5\psi a_2 \\ f(\phi)_{A_\nu} &= A_\nu a_1 - \frac{1}{2}\bar{\psi}i\gamma_\nu\gamma^5\psi a_2 \\ f(\phi)_\lambda &= \lambda a_1 + (F - \gamma^5 G - i\gamma^\nu\gamma^5 A_\nu)\psi a_2 + \bar{\psi}\psi a_3 \\ f(\phi)_D &= Da_1 + (F^2 + G^2 + (A_\nu)^2 + 2\bar{\psi}\lambda)a_2 + \frac{1}{2}(\bar{\psi}\psi)^2 a_4 \\ &\quad + (\bar{\psi}\psi F - \bar{\psi}\gamma^5\psi G - \bar{\psi}i\gamma^\nu\gamma^5\psi A_\nu)a_3 \end{aligned} \tag{50}$$

and where $a_n = d^n f(x)/dx^n|_{x=\phi}$. From (18) it immediately follows

$$\int f(\phi) d^4\theta d^4x = \int f(\phi)_D d^4x d^4\theta = (1/8) \int f(\phi)_D d^4x. \quad (51)$$

According to the representation of S acting on superfields

$$\delta f(\phi)_D = -\bar{\epsilon} \beta f(\phi)_\lambda. \quad (52)$$

Therefore, if ϵ is constant, the integral (51) is invariant and may serve as a Lagrangian. In the sequel, superfield Lagrangians will be denoted by a caret thus: $\hat{L}$. That is,

$$\hat{L} = (\bar{\theta}\theta)^n L + \dots$$

where the dots indicate terms of lower order in θ , and

$$\int d^4\theta \hat{L} = \int d^4\theta (\bar{\theta}\theta)^n L. \quad (53)$$

§6 Irreducible Representations

It is straightforward to work out the irreducible representations of the supersymmetry algebra. (Fayet and Ferrara 1977, Salam and Strathdee 1978). The crucial relation is (12), $\{S_a, S_b\} = \gamma_{ab}$. Consider first an irreducible multiplet in which at least one member has a non-vanishing mass. As long as supersymmetry is unbroken $S|\text{vac}\rangle = 0$, and in any event, $[S, P^2] = 0$. These two conditions are enough to ensure that all members of the irreducible multiplet have common mass. Let $\phi(p)$ create a bosonic state $|p, B\rangle$ and $\psi(p)$ create a fermionic state $|p, F\rangle$. Suppose $i[\bar{\alpha}S, \phi(p)] = i\bar{\alpha}\psi(p)$. Then

$$P^2|p, B\rangle = P^2\phi(p)|\text{vac}\rangle = [P^2, \phi(p)]|\text{vac}\rangle$$

because $P|\text{vac}\rangle = 0$. But because $\phi(p)$ satisfies the Klein-Gordon

equation, $[P^2, \phi(p)] = m_B^2 \phi(p)$, where m_B is the mass of the boson.

Hence

$$P^2 |p, B\rangle = m_B^2 |p, B\rangle .$$

On the other hand, if $S|vac\rangle = 0$,

$$i\bar{\alpha}S|p, B\rangle = [i\bar{\alpha}S, \phi(p)]|vac\rangle = i\bar{\alpha}\psi(p)|vac\rangle ,$$

and thus $i\bar{\alpha}S|p, B\rangle = i\bar{\alpha}|p, F\rangle$. Of course,

$$\begin{aligned} P^2 i\bar{\alpha}S|p, B\rangle &= i\bar{\alpha}S P^2 |p, B\rangle = i\bar{\alpha}S m_B^2 |p, B\rangle \\ &= m_B^2 i\bar{\alpha}S|p, B\rangle \end{aligned}$$

so that the state $i\bar{\alpha}S|p, B\rangle$ has mass m_B . If the generator S annihilates the vacuum, then the state $i\bar{\alpha}S|p, B\rangle$ is just (to within a constant) the state $i\bar{\alpha}|p, F\rangle$, and $P^2 |p, F\rangle = m_F^2 |p, F\rangle$. Therefore $m_F = m_B$, and the masses of the fermion and boson are equal. In an irreducible multiplet, each member ultimately may be transformed into the other by repeated application of the operator S . Thus all members of an irreducible multiplet have the same mass, so long as $S|vac\rangle = 0$. If $S|vac\rangle \neq 0$, the supersymmetry is broken, and it is no longer possible to identify the states $i\bar{\alpha}S|p, B\rangle$ and $i\bar{\alpha}|p, F\rangle$. Then in general $m_F \neq m_B$. In the rest frame, (12) may be written as

$$\{S_a, S_b\} = m\delta_{ab} . \quad (54)$$

It is more convenient to write (54) in terms of its chiral projections $S_{\pm} = P_{\pm} S$. Each projection has only two independent components, so that the relations (54) split into three equations ($i, j = 1, 2$)

$$\{S_{-i}, S_{-j}\} = 0 \quad (55a)$$

$$\{S_{+i}, S_{+j}\} = 0 \quad (55b)$$

$$\{S_{+i}, S_{-j}\} = (\sigma^{\mu})_{ij} P_{\mu} = \delta_{ij} m , \quad (55c)$$

Because of the projections, the antisymmetric product $S_{-i} S_{-j}$ has only one independent component while $S_{-i} S_{-j} S_{-k}$ vanishes identically. There are only sixteen independent products of the operators satisfying (55). This algebra is isomorphic to that of the Dirac matrices, namely the Clifford algebra of rank four, whose only finite-dimensional representation is in terms of four by four matrices (van der Waerden 1974). These representations split into four irreducible multiplets, which are each invariant under Wigner rotations. Therefore the massive irreducible (or time-like) representations of supersymmetry have dimension $4(2J+1)$. Let $|J\rangle$ be that member of the multiplet with spin J which is annihilated by S_{+i} : $S_{+i} |J\rangle = 0$. From this state, it is possible to construct only three more states, so that in all there are the four states $(S_{-1})^{n_1} (S_{-2})^{n_2} |J\rangle$, where $n_1, n_2 = 0, 1$. The spin content of this multiplet is $(J, J \pm \frac{1}{2}, J)$. Under spatial reflection, $S_{-} \rightarrow iS_{-}$, so that the two bosons have opposite parity; likewise the fermions. The smallest massive multiplet ($J = 0$) contains therefore a scalar, a pseudoscalar and a Majorana spinor, while $J = \frac{1}{2}$ corresponds to a multiplet whose members are a scalar (or pseudoscalar), two Majorana spinors, and a pseudovector (or vector). The massless (or lightlike) representations are found in the same manner. It is always possible to choose P_{μ} such that $P_{\mu} = (1, 0, 0, 1)$, in which case (55c) becomes

$$\{S_{+i}, S_{-j}\} = (\sigma^0 + \sigma^3)_{ij} \quad (56)$$

and the only non-vanishing commutator is $\{S_{+2}, S_{-2}\}$. Now from a state $|\lambda\rangle$ with helicity λ which is annihilated by S_{+i} may be built only one physical state $S_{-2} |\lambda\rangle$ (the others are zero norm), with helicity $\lambda + \frac{1}{2}$. These supermultiplets contain only two particles of spin, $J, J + \frac{1}{2}$ respectively. The massless multiplets of greatest interest to date are those of $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 2)$. Lagrangians involving the former

will be given in §§8-9. A dynamical theory of the latter provides the supersymmetric extension of general relativity, as will become clear in the sequel.

§7. A Lagrangian for the Chiral Multiplet

Although it is possible to construct a chiral multiplet from an arbitrary scalar superfield (see above), it is more economical to use a superfield from the start which depends only on $P_+\theta$ or $P_-\theta$. A consistent scheme is obtained by using in place of ϕ , D and iS the following operators:

$$\phi_- = a + i\bar{\theta}P_- \beta + \frac{1}{2}\bar{\theta}P_- \theta ig \quad (57a)$$

$$D_- = \partial/\partial\bar{\theta} + i P_- \not{\partial} \theta \quad (57b)$$

$$iS_- = \partial/\partial\bar{\theta} - i P_- \not{\partial} \theta \quad (57c)$$

where $a = A - iB$, $g = F + iG$ are complex scalar fields as before. The integral of any polynomial of right-handed superfields over $d^4x d^2P_- \theta$ is an invariant. It is easy to show that

$$\bar{D}_- D_- \phi_- = 2ig + 2\bar{\theta}P_+ \not{\partial} \beta + \bar{\theta}P_+ \theta \partial^2 a \quad (58)$$

and consequently $\bar{D}_- D_- \phi_-$ is a left-handed superfield. With the identifications

$$\phi_+ = a + i\bar{\theta}P_+ \beta - \frac{1}{2}\bar{\theta}P_+ \theta ig \quad (59a)$$

$$\phi_+^* = a^* - i\bar{\theta}P_+ \beta + \frac{1}{2}\bar{\theta}P_+ \theta ig^* \quad (59b)$$

it is obvious (by virtue of the closure of chirality) that

$\phi_+^* \bar{D}_- D_- \phi_-$ is a left-handed superfield, and thus

$$\int \phi_+^* \bar{D}_- D_- \phi_- d^4x d^2P_+ \theta = - \int (g^* g + i \bar{\beta} P_+ \not{\beta} + \partial^\mu a^* \partial_\mu a) d^4x \quad (60)$$

is an invariant. Similarly, powers of ϕ_+ are invariant when integrated over $d^2P_+ \theta$;

$$\phi_+^2 = \bar{\theta} P_+ \theta (-iaf + \frac{1}{2} \bar{\beta} P_+ \beta) + \dots \quad (61)$$

$$\phi_+^3 = (3/2) \bar{\theta} P_+ \theta (a \bar{\beta} P_+ \beta - ia^2 f) + \dots \quad (62)$$

where the dots indicate terms of lower order in $P_+ \theta$. Finally, let

$$\hat{L}_+ = - \frac{1}{4} \phi_+^* \bar{D}_- D_- \phi_- - \frac{1}{2} m \phi_+^2 + \frac{1}{3} g \phi_+^3$$

and L_- similarly. A suitable Lagrangian which is manifestly invariant is given by (recall $\delta(P_- \theta) = \bar{\theta} P_- \theta$)

$$\int (\delta(P_- \theta) \hat{L}_+ + \delta(P_+ \theta) \hat{L}_-) d^4x d^4\theta = \int (\bar{\theta} \theta)^2 d^4\theta d^4x L \quad (63)$$

where

$$\begin{aligned} L = & \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} \bar{\psi} i \not{\partial} \psi - \frac{1}{2} m \bar{\psi} \psi + \frac{1}{2} F^2 + \frac{1}{2} G^2 \\ & - m(AG - BF) + gG(A^2 - B^2) - 2gABF + g\bar{\psi}(A - \gamma^5 B)\psi . \end{aligned} \quad (64)$$

After elimination of the auxiliary fields F and G , the Lagrangian takes the more conventional form

$$\begin{aligned} L = & \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 (A^2 + B^2) + \frac{1}{2} \bar{\psi} (i \not{\partial} - m) \psi \\ & + mgA(A^2 + B^2) - \frac{1}{2} g^2 (A^2 + B^2)^2 + g\bar{\psi}(A - \gamma^5 B)\psi . \end{aligned} \quad (65)$$

This Lagrangian was proposed by Wess and Zumino (1974b) and found via superfield methods by Salam and Strathdee (1975a). The renormalisability of this model was found to be much enhanced over generic Yukawa and quartic couplings, requiring only one renormalisation constant Z

(Iliopoulos and Zumino 1975). A superfield formulation equivalent to this one (with the use of delta functions) was given by Ogievetski and Sokatchev (1977). The true dynamical degrees of freedom are the fields B , A and ψ , where as usual $m_B = m_A = m_\psi$, in accordance with the analysis in §6.

There is an alternate, less transparent form of the chiral Lagrangian which is of greater use in the formulation of supersymmetric gauge theories (§§8.2-9). If

$$\phi_+ = \exp(-T)(a^* + i\bar{\theta}P_+\beta - \frac{1}{2}\bar{\theta}P_+\theta ig^*)$$

$$\phi_-^* = \exp(T)(a^* - i\bar{\theta}P_-\beta - \frac{1}{2}\bar{\theta}P_-\theta ig^*)$$

are introduced in analogy to ϕ_- (as in §5), then the kinetic term may be written as (Salam and Strathdee 1975a)

$$4\phi_+\phi_- = (\bar{\theta}\theta)^2(\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}i\bar{\psi}\beta\psi + \frac{1}{2}(F^2+G^2)) + \dots \quad (66)$$

to within a divergence. The mass and interaction terms present no difficulties: merely replace $\phi_\pm$ by $\phi_\pm^*$ overall. The difference between $\phi_\pm$ and $\phi_\pm^*$ is $O(\theta^3)$, and the chiral delta functions eliminate these terms.

§8. Lagrangians Involving the Vector Supermultiplet

§8.1 The free vector multiplet

Although the field content of a superfield is a consequence of its Lorentz character and the Grassmann algebra, its particle content depends on what Lagrangian is chosen. Not all of the fields in the expansion will occur as dynamical degrees of freedom in the given Lagrangian. These non-dynamical, or auxiliary, fields may always be

solved in terms of the dynamical fields, and their solutions resubstituted back into the original Lagrangian. Thus the Lagrangian may be written without any appearance of these auxiliary fields. As an example, consider the scalar superfield Φ , whose components are the sixteen objects $(\phi, \psi, F, G, A, \lambda, D)$. It is possible to write the Lagrangian for Φ so that only the field components $(\partial_{[\mu} A_{\nu]}, \lambda, D)$ occur. Of the eight components $(A_{\mu}$ (omitting the gauge part), $\lambda, D)$ only six - λ and the two transverse polarisations of A_{μ} - are dynamical. (The numbers of degrees of freedom for the Bose variables A_{μ} and the Fermi variables λ are of course equal; two for each.) The components (A_{μ}, λ, D) together with a particular Lagrangian and a set of transformations, describe a massless vector multiplet. If a suitable mass term is added to this Lagrangian, ϕ, ψ and the longitudinal polarisation of A_{μ} are promoted to dynamical status. Now all sixteen components occur in the Lagrangian, but there are only eight degrees of freedom $(\phi, \lambda, \psi, A_1)$. This Lagrangian describes a massive vector multiplet with spin content $(0, \frac{1}{2}, \frac{1}{2}, 1)$. A convenient way of writing down Lagrangians (or at least their kinetic terms) which are manifestly supersymmetric is to use projection operators (Salam and Strathdee 1975a, Sokatchev 1975) which eliminate from the start those auxiliary fields not required for Lorentz invariance from massless theories. The projection operators for the superfields must be invariant under supersymmetry, and hence must be functions of D and ∂_{μ} . An arbitrary scalar superfield may be decomposed into two chiral multiplets and a vector multiplet as follows (Salam and Strathdee 1975a)

$$\Phi = \Phi_+ + \Phi_- + \Phi_V \quad (67)$$

where $E_V \Phi = \Phi_V$, $E_{\pm} \Phi = \Phi_{\pm}$, and

$$E_{\pm} = (1/p^2) \bar{D}P_{\pm} D \bar{D}P_{\mp} D \quad (68a)$$

$$E_V = 1 - (1/4p^2) (\bar{D}D)^2 \quad (68b)$$

In particular, the kinetic term of the chiral Lagrangian may be written as $-\phi \bar{D}P_+ D \bar{D}P_- D \phi$, which by an integration in parts (with respect to θ) becomes

$$L_{\text{kin, chiral}} = (\bar{D}P_+ D \phi) (\bar{D}P_- D \phi) \quad (69)$$

Expanding ϕ according to (36), this is equal to

$$\begin{aligned} (\bar{\theta}\theta)^2 \{ & (1/8) (D - \partial^2 \phi)^2 + \frac{1}{4} (\bar{\lambda} - i \bar{\psi} \overleftrightarrow{\not{D}}) i \not{D} (\lambda + i \not{D} \psi) \\ & + \frac{1}{2} (\partial \cdot A)^2 + \frac{1}{2} (\partial_{\mu} F)^2 + \frac{1}{2} (\partial_{\mu} G)^2 \} + \dots \end{aligned} \quad (70)$$

This expression is seen to be equal to the usual chiral kinetic terms upon the relabelling $\psi' = \lambda - i \not{D} \psi$, $F' = \frac{1}{2} (D - \partial^2 \phi)$, $G' = \partial \cdot A$, $A' = F$, $B' = G$. The equality between (70) and (66) suggests that $\bar{D}P_{\pm} D \phi$ are $\begin{bmatrix} \text{right} \\ \text{left} \end{bmatrix}$ -handed chiral superfields, and a calculation confirms that indeed $\bar{D}P_{\pm} D \phi$ depends only on $P_{\pm} \theta$. With the aid of the projection operator E_V , a kinetic term for the vector multiplet should be (the factor of 2 is for later convenience)

$$\begin{aligned} 2\phi \{ (\bar{D}D)^2 + 4\partial^2 \} \phi &= \hat{L}_{V, \text{kin}} \\ &= \frac{1}{4} (\bar{\theta}\theta)^2 \{ \frac{1}{2} (D + \partial^2 \phi)^2 + (\bar{\lambda} + i \bar{\psi} \overleftrightarrow{\not{D}}) i \not{D} (\lambda - i \not{D} \psi) - (F_{\mu\nu})^2 \}. \end{aligned} \quad (71)$$

If this is written in terms of the previous correspondence between the scalar superfield and the Wess-Zumino vector multiplet,

$$\begin{aligned}
 D + \partial^2 \phi &= -2H \\
 \lambda - i\partial\psi &= \sqrt{2} \gamma^5 \chi \\
 \sqrt{2} \psi &= \gamma^5 \mu \\
 A_\nu &= -V_\nu
 \end{aligned} \tag{72}$$

it becomes

$$\hat{L}_{V, \text{kin}} = (\bar{\theta}\theta)^2 \left\{ \frac{1}{2} H^2 + \frac{1}{2} \bar{\chi} i \partial \chi - \frac{1}{4} (F_{\mu\nu})^2 \right\} + \dots \tag{73}$$

The only dynamical degrees of freedom are those of a massless $(\frac{1}{2}, 1)$ multiplet. It is easy to add a mass term, which has the form $8m^2\phi^2$. Not only does this expression contain mass terms for V_μ and χ , but also kinetic terms for those additional degrees of freedom present in the massive case. A simple calculation gives

$$\begin{aligned}
 8m^2\phi^2 &= (\bar{\theta}\theta)^2 \left\{ -m^2\phi H + \frac{1}{2} m^2 (\partial_\mu \phi)^2 - m^2 \bar{\mu} \chi \right. \\
 &\quad \left. + \frac{1}{2} m^2 \bar{\mu} i \partial \mu + \frac{1}{2} m^2 F^2 + \frac{1}{2} m^2 G^2 + \frac{1}{2} m^2 (V_\mu)^2 \right\} + \dots
 \end{aligned} \tag{74}$$

Finally, let $m\phi = B$, and $m\mu = v$. Once the auxiliary fields are eliminated, the complete Lagrangian reads (in terms of $\psi = P_+ \chi + P_- v$)

$$\begin{aligned}
 &2\phi \{ (\bar{D}D)^2 + 4\partial^2 + 4m^2 \} \phi \\
 &= (\bar{\theta}\theta)^2 \left\{ -\frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} m^2 (V_\mu)^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 B^2 \right. \\
 &\quad \left. + \frac{1}{2} \bar{\psi} (i\partial - m) \psi \right\} + \dots
 \end{aligned} \tag{75}$$

which is that of a massive vector multiplet $(0, \frac{1}{2}, \frac{1}{2}, 1)$ with common mass m .

§8.2 The Supersymmetric Extension of Quantum Electrodynamics

The first example of an interaction for the vector multiplet was given by Wess and Zumino (1974c), who described its coupling to a charged massive chiral multiplet (or equivalently, to a pair of real massive chiral multiplets). The model for this interaction resembles scalar electrodynamics: the role of the vector potential is assigned to an entire vector multiplet; similarly that of the charged scalar field is taken by two chiral multiplets. Instead of the usual gauge function λ , it is necessary to introduce invariant derivatives of a scalar superfield Λ . That the Lagrangian for the massless vector multiplet admits a gauge invariance is already clear from the identity (Sokatchev 1975)

$$(\bar{D}D)^3 = -4\bar{D}D\partial^2 ;$$

the Lagrangian (71) is manifestly invariant under the gauge transformation (Higgs 1976, private communication)

$$\Phi \rightarrow \Phi + \bar{D}D\Lambda = \Phi' . \quad (76)$$

Let the components of Λ in the standard basis be denoted $(L, \tau, M, N, U_\nu, \pi, W)$. The gauge transformation (76) is given explicitly by

$$\begin{aligned} \phi' &= \phi + 2M \\ \psi' &= \psi + (\pi + i\beta\tau) \\ F' &= F + (\partial^2 L - W) \\ G' &= G + 2\partial \cdot U \\ A_\nu' &= A_\nu + 2\partial_\nu N \\ \lambda' &= \lambda + i\beta(\pi + i\beta\tau) \\ D' &= D + 2\partial^2 M . \end{aligned} \quad (77)$$

It is certainly possible to choose Λ in such a way that all of the components (ϕ', ψ', F', G') are zero. This choice is known as the Wess-Zumino gauge. Then $\phi = -2M$, $\psi = -(\pi + i\delta\psi)$, $\lambda' = \lambda - i\delta\psi = \sqrt{2} \gamma^5 \chi$, $D' = D + \partial^2 \phi = -2H$, and $A'_\nu = V_\nu$. In this Wess-Zumino gauge, Φ may be expanded as

$$\Phi_{WZ} = \frac{1}{4} \bar{\theta} i \gamma^\nu \gamma^5 \theta V_\nu + \frac{1}{2} (1/\sqrt{2}) i \bar{\theta} \theta \bar{\theta} \gamma^5 \chi - (1/16) (\bar{\theta} \theta)^2 H \quad (78)$$

In particular $(\Phi_{WZ})^2 = (1/16) (\bar{\theta} \theta)^2 (V_\mu)^2$, and all higher powers of Φ_{WZ} vanish.

To construct a supersymmetric model of a local $U(1)$ invariance, it is best to start from the Lagrangian for a pair of chiral superfields Φ_{-1}, Φ_{-2} which is globally invariant under an infinitesimal transformation of the form $\delta\Phi_{-1} = \epsilon\Phi_{-2}$, $\delta\Phi_{-2} = -\epsilon\Phi_{-1}$. Alternatively, the complex fields S and T may be introduced (Fayet and Ferrara 1977)

$$S = (1/\sqrt{2})(\Phi_{-1} + i\Phi_{-2}); \quad T = (1/\sqrt{2})(\Phi_{-1} - i\Phi_{-2})$$

which transform as $\delta S = i\epsilon S$, $\delta T = -i\epsilon T$. If

$$\bar{S} = (1/\sqrt{2})(\Phi_{-1}^* - i\Phi_{-2}^*); \quad \bar{T} = (1/\sqrt{2})(\Phi_{-1}^* + i\Phi_{-2}^*)$$

and $\delta\bar{S} = -i\epsilon\bar{S}$, $\delta\bar{T} = i\epsilon\bar{T}$, then an invariant Lagrangian is given by (for a global symmetry)

$$4(\bar{S}S + \bar{T}T) - \frac{1}{2} m S T \delta(P_+ \theta) - \frac{1}{2} m \bar{S} \bar{T} \delta(P_- \theta) \quad (79)$$

whose integral over $d^4\theta$ is merely the sum of two free massive chiral Lagrangians. In order to extend the invariance to the local symmetry, only the kinetic term need be remedied. First, however, the new transformation must be found for S and T . It must preserve chirality (in order not to introduce unwanted derivatives) and it must involve the

superfield Λ , - or at least its chiral projections. A consistent scheme is obtained if

$$\begin{aligned}\delta S &= 2g \bar{D}P_+ D\Lambda S \\ \delta \bar{S} &= 2g \bar{D}P_- D\Lambda \bar{S} \\ \delta T &= -2g \bar{D}P_+ D\Lambda T \\ \delta \bar{T} &= -2g \bar{D}P_- D\Lambda \bar{T}\end{aligned}\tag{80}$$

and if the kinetic term is rewritten as

$$\hat{L}_{\text{kin}}(S, T, \Phi) = 4\{\bar{S} \exp(-2g\Phi)S + \bar{T} \exp(2g\Phi)T\} .\tag{81}$$

Taking $\delta\Phi = \bar{D}D\Lambda$ as before, it is easy to show

$$\delta \hat{L}_{\text{kin}}(S, \Phi) = 2g(\bar{D}P_+ D\Lambda + \bar{D}P_- D\Lambda - \bar{D}D\Lambda)\hat{L}_{\text{kin}}(S, \Phi)$$

which vanishes; and similarly $\delta \hat{L}_{\text{kin}}(T, \Phi)$ vanishes. In terms of the fields Φ_{-1} and Φ_{-2} , the transformations (80) read

$$\delta\Phi_{-1} = 2ig \bar{D}P_+ D\Lambda \Phi_{-2}; \quad \delta\Phi_{-2} = -2ig \bar{D}P_+ D\Lambda \Phi_{-1} .\tag{82}$$

The complete Lagrangian is given by

$$\hat{L} = \hat{L}_{\text{kin}}(S, T, \Phi) + \hat{L}_V - \hat{L}_{\text{mass}}(S, T)\tag{83}$$

where $\hat{L}_{\text{kin}}$ is given by (81), $\hat{L}_V$ is the free vector Lagrangian (73), and

$$\hat{L}_{\text{mass}} = \frac{1}{2}m(\delta(P_+\theta)ST) + \delta(P_-\theta)\bar{S}\bar{T} .$$

For clarity, (83) may be rewritten as

$$\hat{L} = \hat{L}_V + \hat{L}_1 + \hat{L}_2 + \hat{L}_{\text{int}}\tag{83}$$

where $L_{1,2}$ are the Lagrangians for two free chiral multiplets

$$\hat{L}_1 = (\bar{\theta}\theta)^2\{\frac{1}{2}(\partial_\mu A_1)^2 + \frac{1}{2}(\partial_\mu B_1)^2 + \frac{1}{2}\psi(i\not{\partial}-m)\psi - \frac{1}{2}m^2(A_1^2 + B_1^2)\}$$

and similarly for $\hat{L}_2$. These coincide with (66) without cubic and quartic terms, and

$$\hat{L}_{int} = 4\{\bar{S}(\exp(-2g\phi) - 1)S + \bar{T}(\exp(2g\phi) - 1)T\}.$$

In principle the exponential should be expanded to all orders. Actually there is no need to do this. By virtue of the gauge invariance (77), Λ may be chosen such that $\phi \rightarrow \phi_{WZ}$, and the physics is unaltered by this choice. Then

$$\hat{L}_{int} = -8g\phi_{WZ}\{\bar{S}S - \bar{T}T\} + 8g^2\phi_{WZ}^2\{\bar{S}S + \bar{T}T\}.$$

In terms of the chiral superfields,

$$\{\bar{S}S + \bar{T}T\} = \phi_{-1}^* \phi_{-1} + \phi_{-2}^* \phi_{-2} \quad (84a)$$

$$= (A_1^2 + A_2^2 + B_1^2 + B_2^2) + (\text{terms of higher order in } \theta)$$

$$\begin{aligned} \{\bar{S}S - \bar{T}T\} &= i(\phi_{-1}^* \phi_{-2} - \phi_{-2}^* \phi_{-1}) \\ &= 2(B_1 A_2 - B_2 A_1) + i\bar{\theta}\{(B_2 + \gamma^5 A_2)\beta_1 - (B_1 + \gamma^5 A_1)\beta_2\} \\ &\quad + \frac{1}{2}\bar{\theta} i\gamma^\nu \gamma^5 \theta (\frac{1}{2}\bar{\beta}_1 \gamma_\nu \beta_2 - A_2 \overset{\leftrightarrow}{\partial}_\nu A_1 - B_2 \overset{\leftrightarrow}{\partial}_\nu B_1) + \dots \end{aligned} \quad (84b)$$

where the dots indicate terms which do not contribute to

$\int \hat{L}_{int} d^4x d^4\theta$. Rescaling $\beta_i = \sqrt{2} \psi_i$ as before, the interaction Lagrangian becomes

$$\begin{aligned} L_{int} &= \frac{1}{2}g^2(V_\mu)^2(A_1^2 + A_2^2 + B_1^2 + B_2^2) + gH(A_2 B_1 - A_1 B_2) \\ &\quad + g\bar{\chi}\{(A_1 - \gamma^5 B_1)\psi_2 - (A_2 - \gamma^5 B_2)\psi_1\} \\ &\quad - gV_\mu(\bar{\psi}\gamma^\mu\psi - A_2 \overset{\leftrightarrow}{\partial}^\mu A_1 - B_2 \overset{\leftrightarrow}{\partial}^\mu B_1) \end{aligned} \quad (85)$$

The only remaining auxiliary field is H . Its variation implies

$H = -g(A_2 B_1 - A_1 B_2)$. Inserting this solution for H , the Lagrangian

may be rewritten as

$$\begin{aligned}
 L = & L_1 + L_2 - \frac{1}{2}g^2(A_2B_1 - A_1B_2)^2 - \frac{1}{4}(F_{\mu\nu})^2 \\
 & + \frac{1}{2}g^2(V_\mu)^2(A_1^2 + A_2^2 + B_1^2 + B_2^2) + \frac{1}{2}\bar{\chi}i\not{\partial}\chi \\
 & + g\bar{\chi}\{(A_1 - \gamma^5 B_1)\psi_2 - (A_2 - \gamma^5 B_2)\psi_1\} \\
 & - gV_\mu\{\bar{\psi}_1\gamma^\mu\psi_2 - A_2\overset{\leftrightarrow}{\partial}^\mu A_1 - B_2\overset{\leftrightarrow}{\partial}^\mu B_1\} .
 \end{aligned} \tag{86}$$

This Lagrangian (or rather, the Lagrangian (83)) forms the basis of the first example of the spontaneous breakdown of global supersymmetry.

§8.3 A Supersymmetric Goldstone Model

Let the Lagrangian (83) be rewritten as

$$L = L_1 + L_2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi - \frac{1}{4}(F_{\mu\nu})^2 + L_{\text{int}}(A_i, B_i) + V(H) \tag{87}$$

where $V(H) = \frac{1}{2}H^2 + gH(A_2B_1 - A_1B_2)$. Although the generalised local $U(1)$ invariance is no longer manifest, nevertheless the Lagrangian remains invariant under the usual supersymmetry transformations. To (87) may be added any linear function of H without spoiling the supersymmetry invariance, due to its transformation law

$$\delta H = i\epsilon\gamma^5\not{\partial}\chi .$$

This term breaks parity (as H is a pseudoscalar) but it is otherwise acceptable. If the term $-\xi H$ is added to L , then the equation for H becomes (Fayet and Iliopoulos 1974)

$$H = g(A_1B_2 - A_2B_1) + \xi \equiv H_0 \tag{88}$$

so that

$$V(H_0) = -\frac{1}{2}\{g(A_1B_2 - A_2B_1) + \xi\}^2 \tag{89}$$

and new mass terms $-\xi g(A_1B_2 - A_2B_1)$ arise. Without loss of generality,

ξg may be assumed positive. The mass matrix for the remaining scalar fields is diagonalised by means of the substitutions

$$\begin{aligned} A_1 &= (1/\sqrt{2})(C_1 - D_2) & A_2 &= (1/\sqrt{2})(C_2 - D_1) \\ B_1 &= (1/\sqrt{2})(C_2 + D_1) & B_2 &= (1/\sqrt{2})(C_1 + D_2) \end{aligned}$$

so that the quadratic and quartic self-couplings become

$$\begin{aligned} L_{\text{int}}(C_i, D_i) &= -\frac{1}{2}(m^2 + \xi g)(C_1^2 + D_1^2) - \frac{1}{2}(m^2 - \xi g)(C_2^2 + D_2^2) \\ &\quad - (1/8)g^2(C_1^2 + D_1^2 - C_2^2 - D_2^2) \end{aligned} \quad (90)$$

Already supersymmetry has been spontaneously broken, because the scalar and spinor masses of the same chiral multiplet no longer share a common value. In accordance with a generalised Goldstone theorem, a massless particle with the quantum numbers of the broken generator (in this case, S) must be present. It is easy to see that χ is the Goldstone fermion. The distinguishing characteristic of all Goldstone particles is that the vacuum expectation value of their variation does not vanish. From (13),

$$\delta\chi = -F_{\mu\nu}\sigma^{\mu\nu}\alpha + H\gamma^5\alpha$$

and substituting (88) in for H ,

$$\langle\delta\chi\rangle = \xi\gamma^5\alpha \quad (91)$$

There may be a further would-be Goldstone particle present (depending on the sign of $(m^2 - \xi g)$), but inasmuch as all scalar fields are coupled to the vector V_μ , the Higgs mechanism will eliminate it. The second spontaneous breaking occurs if $m^2 < \xi g$. In this case, the minimum of the effective potential

$$V_{\text{eff}}(C_2) = \frac{1}{2}(m^2 - \xi g)(C_2^2 + D_2^2) + (1/8)g^2(C_1^2 + D_1^2 - C_2^2 - D_2^2) \quad (92)$$

no longer occurs at $\langle C_2 \rangle = \langle D_2 \rangle = 0$. In fact, the effective potential may be rewritten as

$$V_{\text{eff}}(\phi) = \frac{1}{2}(m^2 - \xi g)\phi^* \phi + (1/8)g^2(C_1^2 + D_1^2 - \phi^* \phi)^2$$

where $\phi = C_2 + iD_2$. Differentiating $\langle V_{\text{eff}}(\phi) \rangle$ with respect to ϕ^* , it is clear that the minimum occurs at $\langle \phi \rangle \neq 0$. Because the potential is invariant under rotations through an angle θ , in the (C_2, D_2) plane, it may be assumed that

$$\langle D_2 \rangle = 0; \quad \langle C_2 \rangle = v = \sqrt{2(\xi g - m^2)/g^2}. \quad (93)$$

According to the criterion, D_2 is the would-be Goldstone boson ($\langle \delta D_2 \rangle = -\langle \theta C_2 \rangle \neq 0$). It is possible to choose the ordinary gauge function λ such that $D_2 \rightarrow 0$, and $V_\mu \rightarrow W_\mu = V_\mu - \partial_\mu \lambda$. If the interesting pieces of the Lagrangian are rewritten (i.e., those involving $C = C_2 - v$ and W_μ), they become

$$\begin{aligned} L(C, W) = & -\frac{1}{4}(G_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu C)^2 + \frac{1}{2}v^2 g^2 (W_\mu)^2 - (1/8)(vgC)^2 \\ & + \frac{1}{2}g^2 C^2 (W_\mu^2) + g^2 v C (W_\mu^2) - \frac{1}{2}g^2 v C^3 - (1/8)g^2 C^4 \end{aligned} \quad (94)$$

where $G_{\mu\nu} = \partial_{[\mu} W_{\nu]}$. The fermion terms are also changed because of coupling to the scalar field C_2 . Nevertheless, a Goldstone spinor is still present (as a linear combination of ψ_1, ψ_2 , and χ). No essentially new information is gained by writing down these couplings. The Fayet-Iliopoulos model describes what may be called (in Fayet and Ferrara's (1977) phrase) "the supersymmetric extension of the Goldstone mechanism"; although it involves the Higgs mechanism, it does not do so in such a way that the Goldstone fermion disappears (which will be called "the generalised Higgs mechanism"). For "the supersymmetric extension of the Higgs mechanism", it is necessary that a massless vector multiplet should go over into a massive vector multiplet. In

the Fayet-Iliopoulos model, although the right degrees of freedom are present (a scalar, a vector and a pair of Majorana spinors), the masses of these particles are different: C has mass $\frac{1}{2}vg$, while the vector W_μ 's mass is twice this. Such an extended Higgs model has been constructed by Fayet (1976). As will become clear, this model is very important in the construction of a generalised Higgs mechanism. The Fayet model may be written down in either of two ways: as a self-interacting scalar superfield, or as a vector multiplet in the Wess-Zumino gauge interacting with a chiral multiplet.

§9. The Supersymmetric Extension of the Higgs Mechanism (The Fayet Model)

Before proceeding to the Fayet model, it is worthwhile to recall some features of the $U(1)$ Higgs model. This describes the interaction of a charged scalar field together with a massless neutral vector field representing the electromagnetic potential; (which is essentially the Goldstone model whose global $U(1)$ symmetry has been gauged). The degrees of freedom are four in number: one for each of a pair of real scalars, and two for the massless vector. Once the symmetry is spontaneously broken, one of the scalars is absorbed (via a gauge transformation) into the vector field, providing it with a longitudinal polarisation, while the second scalar gives the vector a mass and becomes massive itself. At the conclusion of this remarkable transmutation, there are still only four degrees of freedom: those of a massive vector and one massive scalar. In order to make this model fully supersymmetric, it is necessary to introduce supersymmetric partners for each of the three fields. A massless chiral multiplet provides the minimum enlargement of the set of two scalars (to two scalars plus a Majorana spinor) while the smallest supersymmetric family containing a massless vector is that

composed of a massless vector and a massless Majorana spinor. Clearly, the combined massless system of two scalars, two Majorana spinors and a vector, contains just the right number of degrees of freedom to represent a massive vector multiplet: the two Majorana spinors can combine to form a Dirac spinor, and the remaining pieces can reshuffle as in the conventional Higgs mechanism. It will turn out that this model describes two "phases", depending on the sign of the parameter ξ introduced previously in §8.3. When supersymmetry is preserved, gauge invariance is not, and vice-versa. Note that explicit mass terms for the chiral multiplet are forbidden by the global chiral invariance associated with the spinor member ψ_L . As long as supersymmetry remains unbroken, this additional symmetry ensures that the chiral multiplet is massless. One approach to the globally supersymmetric Higgs mechanism is simply to write down the most general Lagrangian of lowest degree in the fields which is invariant under both supersymmetry and the generalised U(1) invariance introduced in §8.2. This approach is closely related to the Fayet-Iliopoulos model, and will be described first. There is a second method, more suitable for an extended Higgs mechanism (in which supergravity is involved), which will be described in the sequel.

In terms of the vector multiplet V_ν , χ , and H , (as opposed to the scalar superfield components A_ν , λ , and D) the gauge transformation (76) is

$$\begin{aligned}\delta V_\nu &= -\delta A_\nu = 2\partial_\nu N \\ \delta \chi &= \delta(\lambda - i\partial\psi) = -i\partial(\pi + i\beta\tau) + i\partial(\pi + i\beta\tau) = 0 \quad (95) \\ \delta H &= -\frac{1}{2}\delta(D + \partial^2\phi) = -\frac{1}{2}(2\partial^2 M + \partial^2(-2M)) = 0.\end{aligned}$$

Consequently any term linear in H is separately invariant under both ordinary supersymmetry ($\delta H = i\epsilon\gamma^5\partial\chi$) and the gauge invariance (95). The most general Lagrangian of lowest degree in the chiral superfield

which is invariant under both these groups involving only a chiral superfield and a scalar superfield is thus

$$\hat{L} = \hat{L}_V + 4S^* \exp(2e\Phi)S - \xi\Phi \quad (96)$$

where $\hat{L}_V$ is the usual vector multiplet kinetic term. The interaction is just the Lagrangian (81) introduced previously, with $S = \Phi_{-1}$, $T = 0$ and $e = -g$. Again, (96) is most conveniently evaluated with Φ in the Wess-Zumino gauge. In terms of the component fields, (96) reads

$$\begin{aligned} \hat{L} = & (\bar{\theta}\theta)^2 \{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi + \frac{1}{2}H^2 - \xi H - \frac{1}{2}eV_\mu \bar{\psi}\gamma^\mu P_- \psi \\ & + \frac{1}{2}F^2 + \frac{1}{2}G^2 + \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}\bar{\psi}i\not{\partial}\psi + eV_\mu B\partial^\mu A \\ & + e\bar{\psi}(A\gamma^5 - B)\chi - \frac{1}{2}eH(A^2 + B^2) + \frac{1}{2}e^2(V_\mu)^2(A^2 + B^2) \} + \dots \quad (97) \end{aligned}$$

It is convenient to rewrite (97) in terms of the complex field

$\phi = (-i/\sqrt{2})(A - iB)$, and with the auxiliary fields F and G eliminated.

Setting

$$\begin{aligned} \nabla_\mu \phi &= \partial_\mu \phi + ieV_\mu \phi \\ \nabla_\mu \phi^* &= \partial_\mu \phi^* - ieV_\mu \phi^* \\ \nabla_\mu \psi &= \partial_\mu \psi + ieV_\mu \psi \end{aligned}$$

the Lagrangian (97) becomes

$$\begin{aligned} \hat{L} = & (\bar{\theta}\theta)^2 \{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi + \frac{1}{2}H^2 - \xi H + \nabla_\mu \phi^* \nabla^\mu \phi \\ & + \frac{1}{2}\bar{\psi}i\not{\partial}\psi + e\sqrt{2}(\bar{\psi}P_- \chi\phi + \bar{\psi}P_+ \chi\phi^*) - eH\phi^* \phi \} + \dots \quad (98) \end{aligned}$$

The equation of motion for H reads

$$H = \xi + e\phi^* \phi. \quad (99a)$$

Elimination of this auxiliary field leads to an effective potential for ϕ of the familiar Goldstone-Nambu type,

$$V_{\text{eff}}(\phi) = \frac{1}{2}(\xi + e\phi^*\phi)^2 \quad (99b)$$

so that (99a) implies

$$H = \sqrt{2V_{\text{eff}}(\phi)} \quad (99c)$$

Without loss of generality, e may be assumed positive. There are now two distinct theories described by the one Lagrangian (98), depending on the sign of ξ . If ξ is positive, the minimum of V_{eff} occurs for $\langle\phi\rangle = 0$. The scalar fields A and B have a mass equal to $\sqrt{e\xi}$, while their spinor partner ψ remains massless - supersymmetry is spontaneously broken. Again χ is the Goldstone fermion, for if $\langle\phi\rangle = 0$, then $\langle\delta\chi\rangle = \langle H\rangle\gamma^5\alpha = \xi\gamma^5\alpha$. If however ξ is negative, something more interesting happens. By minimising $\langle V_{\text{eff}}(\phi)\rangle$ for $\xi < 0$, it follows that V_{eff} attains its least value for $\langle\phi\rangle = \sqrt{-\xi/e} \exp(i\theta)$. Nothing is lost if θ is set equal to zero; that is,

$$\langle A\rangle = 0; \quad \langle B\rangle = \sqrt{-2\xi/e} \equiv v.$$

As in §8.3, it is convenient to introduce $C = B - v$, so that $\langle C\rangle = 0$. In terms of the translated field C , the Lagrangian reads

$$\begin{aligned} \hat{L} = & (\bar{\theta}\theta)^2 \{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi - \frac{1}{2}eV_{\mu}\bar{\psi}\gamma^{\mu}P_{-}\psi + \frac{1}{2}(\partial_{\mu}A)^2 \\ & + \frac{1}{2}(\partial_{\mu}C)^2 - \frac{1}{2}e^2v^2C^2 - \frac{1}{2}e^2vC^3 - (1/8)e^2C^4 - (1/8)e^2A^4 \quad (100) \\ & - \frac{1}{2}e^2vCA^2 + \frac{1}{2}\bar{\psi}i\not{\partial}P_{-}\psi - ev\bar{\psi}\chi - eAV^{\mu}\partial_{\mu}C \\ & + eV^{\mu}(C+v)\partial_{\mu}A + e\bar{\psi}(A\gamma^5 - C)\chi + \frac{1}{2}e^2v^2(V_{\mu})^2 \\ & + \frac{1}{2}e^2(V_{\mu})^2C^2 + e^2vC(V_{\mu})^2 + \frac{1}{2}e^2(V_{\mu})^2A^2 \} . \end{aligned}$$

Finally, V_{μ} may be transformed according to the remaining Maxwell invariance $V_{\mu} \rightarrow W_{\mu} = V_{\mu} - \partial_{\mu}\lambda$. If λ is chosen so as to eliminate A entirely, then the Lagrangian may be written in terms of the new fields

$$\begin{aligned}
 v &= P_- \psi + P_+ \chi, \quad W_\mu, \quad G_{\mu\nu} = \partial_{[\mu} W_{\nu]} \quad \text{as } (m = ev) \\
 \hat{L} &= (\bar{\theta}\theta)^2 \left\{ -\frac{1}{4} (G_{\mu\nu})^2 + \frac{1}{2} m^2 (W_\mu)^2 + \frac{1}{2} (\partial_\mu C)^2 - \frac{1}{2} m^2 C^2 \right. \\
 &\quad + \frac{1}{2} \bar{\psi} (i \not{\partial} - m) \psi - \frac{1}{2} e \bar{\psi} W_\mu P_- \psi - \frac{1}{2} e m C^3 - \left(\frac{1}{8} \right) e^2 C^4 \\
 &\quad \left. + \frac{1}{2} e^2 (W_\mu)^2 C^2 + e m C (W_\mu)^2 \right\} + \dots
 \end{aligned} \tag{101}$$

which clearly describes a self-interacting massive vector supermultiplet, and the chiral supermultiplet has been swallowed whole. There is an additional bonus unanticipated by the extension of the Higgs model to a globally supersymmetric version. The usual Goldstone model depends on two arbitrary parameters (the coefficients of ϕ^2 and ϕ^4), while the conventional Higgs model requires also the coupling constant e . It is evident from (101) that there are only two arbitrary parameters in this model; the higher degree of symmetry has constrained one of the parameters to be a fixed function of the other two. To summarise: a globally supersymmetric extension of the Higgs model was found by Fayet (1976) by extending the ingredients of the U(1) Higgs model to their most economical supersymmetric extensions. In addition, it was necessary to introduce the "trigger" term $-\xi\phi$, which has no counterpart in conventional Higgs models. Depending on the sign of ξ , two quite distinct theories emerged:

- i) $\xi > 0$: Supersymmetry is spontaneously broken, while the local U(1) Maxwell invariance is preserved. The theory describes a massive pair of scalars, two massless spinors and a massless vector. The spinor associated with the vector becomes a Goldstone fermion.
- ii) $\xi < 0$: Gauge invariance is spontaneously broken, but supersymmetry remains intact. The theory is that of an interacting massive vector multiplet, while the chiral supermultiplet disappears entirely via the extended Higgs mechanism.

From ii), it should be clear that an alternate approach will suffice

to produce the Lagrangian (97): consider the most general self-interaction of a vector multiplet which is invariant under supersymmetry (but not the generalised gauge invariance). The Grassmann integral of any function of Φ is such an invariant, so that the most general $\hat{L}$ for the vector supermultiplet is

$$\hat{L} = \hat{L}_V + g(\Phi), \quad (102)$$

where $g(x)$ is an arbitrary function of x , and $\hat{L}_V$ is the usual kinetic term for the massless vector multiplet

$$\begin{aligned} \hat{L}_V &= 2\Phi\{(\bar{D}D)^2 + 4\partial^2\}\Phi \\ &= (\bar{\theta}\theta)^2(-\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{D}\chi + \frac{1}{2}H^2) + \dots \end{aligned}$$

In general, of course, $g(\Phi)$ will contain terms involving other fields besides the three of $\hat{L}_V$. These new expressions will involve both new dynamic terms (and hence new degrees of freedom) as well as auxiliary terms, i.e. those involving fields which occur only algebraically. The presence of $\hat{L}_V$ in $\hat{L}$ is enough to ensure that (102) indeed describes a vector multiplet. For clarity, once more recall the relations between components of the scalar superfield and the vector multiplet:

$$\begin{aligned} H &= -\frac{1}{2}(D + \partial^2\phi) \\ \chi &= (1/\sqrt{2})(-\gamma^5\lambda + i\gamma^5\not{D}\psi) \\ V_\nu &= -A_\nu \quad \text{and} \quad \mu = -\sqrt{2}\gamma^5\psi. \end{aligned}$$

It will prove convenient to let $g(x) = 16f(x)$. The coefficient of $16f(\Phi)$ which contributes to L is given by, according to §5.2, (recall $a_n = f^{(n)}(\phi)$)

$$\begin{aligned} 16f(\Phi) &= \frac{1}{2}(\bar{\theta}\theta)^2\{Da_1 + (F^2 + G^2 + (A_\mu)^2 + 2\bar{\psi}\lambda)a_2 \\ &\quad + (\bar{\psi}\psi F - \bar{\psi}\gamma^5\psi G - \bar{\psi}i\gamma^\nu\gamma^5\psi A_\nu)a_3 + \frac{1}{2}a_4(\bar{\psi}\psi)^2\} + \dots \end{aligned} \quad (103)$$

which when expressed in terms of vector components becomes

$$16f(\phi) = \frac{1}{2}(\bar{\theta}\theta)^2\{(-2H - \partial^2\phi)a_1 + ((V_\mu)^2 - N^2 - M^2 + 2\bar{\mu}\chi + i\bar{\mu}\not{\partial}\mu)a_2 \\ + \frac{1}{2}(i\bar{\mu}\mu N - i\bar{\mu}\gamma^5\mu M + i\bar{\mu}\gamma^\nu\gamma^5\mu V_\nu)a_3 + (1/8)(\bar{\mu}\mu)^2a_4\}.$$

Discarding the divergences $\int(-\partial^2\phi)a_1d^4x = \int(\partial_\mu\phi)^2a_2d^4x$, and the sole coefficient of a_1 in L is now $-H$, while that of a_2 picks up the term $\frac{1}{2}(\partial_\mu\phi)^2$. The equations of motion for the auxiliary fields H , M and N are

$$N = \frac{1}{2}i\bar{\mu}\mu(a_3/a_2); \quad M = -\frac{1}{2}i\bar{\mu}\gamma^5\mu(a_3/a_2); \quad H = a_1.$$

Inserting these solutions, the Lagrangian (102) becomes

$$\hat{L} = (\bar{\theta}\theta)^2\{-\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi - \frac{1}{2}(a_1)^2 + \frac{1}{2}\bar{\mu}\gamma^\nu\gamma^5\mu V_\nu a_3 \\ + \frac{1}{2}(2\bar{\mu}\chi + (V_\mu)^2 + i\bar{\mu}\not{\partial}\mu + (\partial_\mu\phi)^2)a_2 \\ + (1/16)(\bar{\mu}\mu)^2(a_4 - (a_3^2/a_2))\}.$$
(104)

If the theory described by (104) is to be even quasi-renormalisable, the coefficient of $(\bar{\mu}\mu)$ must vanish. Then f must satisfy the differential equation

$$f^{(2)}f^{(4)} - (f^{(3)})^2 = 0.$$
(105)

There are apparently two solutions to (105):

$$a) \quad f(x) = b - \xi x + \kappa x^2$$
(106a)

$$b) \quad f(x) = b - \xi x + c \exp(\kappa x).$$
(106b)

In each case the inessential constant b may be dropped. Note that solution (106b) approaches (106a) in the limit $\kappa^2 \rightarrow 2\kappa_1$, $\kappa^2 \rightarrow 0$

$c \rightarrow \infty$, $\xi - c\kappa \rightarrow \xi'$. Solution (106a) implies a Lagrangian

$L = L_V + 16g\phi^2$; which is just that for a free massive vector multiplet of mass $m = 2\sqrt{g}$. The more interesting case is (b), when $\hat{L}$ becomes

$$\begin{aligned}
 \hat{L} = & (\bar{\theta}\theta)^2 \left\{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{D}\chi - \frac{1}{2}(c\kappa\exp(\kappa\phi) - \xi)^2 \right. \\
 & + \frac{1}{2}(2\bar{\mu}\chi + (V_\nu)^2 + \bar{\mu}i\not{D}\mu + (\partial_\mu\phi)^2)c\kappa^2\exp(\kappa\phi) \\
 & \left. + \frac{1}{4}c\kappa^3 \bar{\mu}\gamma^\nu\gamma^5\mu V_\nu \exp(\kappa\phi) \right\} + \dots
 \end{aligned} \tag{107}$$

Apparently the value of the constant c has no physical significance independent of the scalar field ϕ : c may be varied at will by a translation in ϕ . Let $\kappa = -2e$, and $c = \frac{1}{4}v^2$, where it is assumed $e > 0$. To facilitate the comparison with the previous Lagrangian (97), first set

$$v \exp(-e\phi) = B \tag{108a}$$

$$-ev \exp(-e\phi)\mu = v = -eB\mu \tag{108b}$$

in which case (107) becomes*

$$\begin{aligned}
 \hat{L} = & (\bar{\theta}\theta)^2 \left\{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{D}\chi - \frac{1}{2}(\xi + \frac{1}{2}eB^2)^2 \right. \\
 & - e\bar{v}\chi B + \frac{1}{2}\bar{v}i\not{D}v + \frac{1}{2}e^2B^2(V_\nu)^2 \\
 & \left. + \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}e\bar{v}\gamma^\mu\gamma^5v V_\mu \right\} + \dots
 \end{aligned} \tag{109}$$

As it stands, (109) is not identical to (97). To establish the identity it is necessary to introduce some new fields. Let

$$B + iA \equiv v \exp(-e\phi + iew) \tag{110a}$$

$$\psi \equiv -ev \exp(-e\phi + iew)\mu \tag{110b}$$

$$W_\mu \equiv V_\mu - \partial_\mu \omega \tag{110c}$$

* N.B. $\int (\bar{\mu}i\not{D}\mu)B^2d^4x = \int (\bar{v}i\not{D}v)d^4x + \int (\bar{\mu}\gamma^\lambda\mu)iB\partial_\lambda B d^4x$ but recall $\bar{\mu}\gamma^\lambda\mu \equiv 0$.

which is the generalisation of (108) to complex fields. If ω is now allowed to vary in the usual $U(1)$ manner

$$\omega \rightarrow \omega + \lambda$$

$$B + iA \rightarrow \exp(-ie\lambda)(B + iA)$$

$$\psi \rightarrow \exp(-ie\lambda)\psi$$

$$W_\mu \rightarrow W_\mu - \partial_\mu \lambda$$

the physics described by a Lagrangian in terms of these new fields will be invariant under a change in ω . Rewriting the Lagrangian (109) one last time, it becomes

$$\begin{aligned} \hat{L} = & (\theta\theta)^2 \{ -\frac{1}{4}(G_{\mu\nu})^2 + \frac{1}{2}\bar{\chi}i\not{\partial}\chi - \frac{1}{2}(\xi + \frac{1}{2}e(B^2 + A^2))^2 \\ & + e\bar{\psi}(A\gamma^5 - B)\chi + \frac{1}{2}\bar{\psi}i\not{\partial}\psi + \frac{1}{2}e^2(W_\nu)^2(A^2 + B^2) \\ & + \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}e\bar{\psi}\gamma^\mu P_- \psi W_\mu \\ & + e W_\mu \overset{\leftrightarrow}{B} \partial_\mu A \} + \dots \end{aligned}$$

which is clearly identical to (97) upon the elimination of the auxiliary fields F , G and H . Consequently the globally supersymmetric Higgs model may be derived from the most general Lagrangian describing a self-interaction of the vector multiplet which is compatible with renormalisation. In the case of $n = 4$, the interaction terms have the form $-\xi\phi + c \exp(\kappa\phi)$. It is remarkable that the "trigger term" $-\xi\phi$ is automatically included in the solutions to the differential equation (105).

This concludes the background in global supersymmetry necessary for proceeding to the study of local supersymmetry and its spontaneous breaking. A few other results from general relativity must also be marshalled. They will be called forth in the course of the next chapter.

II. GAUGING SUPERSYMMETRY WITHOUT SUPERFIELDS

§10 The Earliest Supergravity Theories

§10.1 The Deser-Zumino Model

The first Lagrangians invariant under local supersymmetry were found in a manner rather different from the usual Yang-Mills procedure, which emphasises matter-gauge field coupling from the start; and it is only this coupling which determines the gauge fields' transformations and Lagrangians. Instead, Freedman, Ferrara and van Nieuwenhuizen (1976) and independently Deser and Zumino (1976) started from the basic ingredients for a locally supersymmetric theory demanded only by the group theory: a massless spin $3/2$ field (Rarita and Schwinger 1941) which is to cancel the derivatives of the local spinor parameters, and a massless spin 2 field to act as the Rarita-Schwinger field's partner. (In principle, this spin $3/2$ field, hereafter denoted χ_μ , could be mated with a massless vector, rather than a tensor. Indeed, in an extended version of supergravity, such a multiplet is employed. However, it would not be possible to identify this spin $3/2$ field with the gauge field of local supersymmetry without the presence of an additional tensor field to compensate for the local translations required by local supersymmetry. Moreover, this tensor must have its own spin $3/2$ partner (this is the massless spin $(2, 3/2)$ multiplet). Consequently the smallest number of fields for a consistent, locally supersymmetric theory in which a spin $3/2$ field is partnered with a vector field is four: a vector, two spin $3/2$ fields, and a tensor field.) This tensor field is to be identified with the conventional graviton, which couples to the energy-momentum tensor (the Noether current associated with translations) produced by gravity and all matter fields, while the field χ_μ itself couples to the spinor

Noether current J_μ which is a consequence of global supersymmetric invariance. The Deser-Zumino formulation is far more elegant, so this version will be given first. That of Freedman et al. follows almost trivially from the Deser-Zumino theory with the help of some results from the Einstein-Cartan-Sciama-Kibble (or ECSK) theory of gravity (Kibble 1961, Hehl, von der Hyde, Kerlick and Nester, 1975).

The presence of both χ_μ and the graviton $g_{\mu\nu}$ suggest that the Lagrangian should be invariant under three local groups: local supersymmetry, and the two invariances of Einstein's theory, namely the local Lorentz group which acts on field components directly and the general coordinate group which acts through the underlying space-time. Furthermore, the presence of a fermion field χ_μ requires the use of a vierbein e_μ^α rather than a purely metric formulation (Deser and van Nieuwenhuizen 1974.) The vierbein formulation will be treated at some length in §§11-12. For the moment it is sufficient to point out that under both the local Lorentz group and the general coordinate group the vierbein transforms as a vector; α is a Lorentz index while μ is a coordinate index. Finally, the flat-space Lagrangian (to which the curved-space Lagrangian must reduce in the limit $e_\mu^\alpha \rightarrow \delta_\mu^\alpha$) must be invariant under the transformation

$$\chi_\mu \rightarrow \chi_\mu + \partial_\mu \alpha \quad (1)$$

where α is a spinor parameter. If the field χ_μ is to be identified with the gauge field of local supersymmetry, its transformation must include at least the term $\partial_\mu \alpha$. Moreover, the massless Rarita-Schwinger Lagrangian is known to possess such an invariance (Rarita and Schwinger 1941). The spinor nature of χ_μ suggests that the flat-space $L(\chi)$ should be linear in $\partial_\mu \chi_\nu$, while the gauge invariance (1) implies that $L(\chi)$ should take the form of a

curl, $\partial_{[\mu} \chi_{\nu]}$; and of course $L(\chi)$ must be a scalar under the Lorentz group. These considerations uniquely restrict $L(\chi)$ to the form

$$L(\chi) = i\kappa \epsilon^{\mu\nu\rho\sigma} \bar{\chi}_{\mu} \gamma^5 \gamma_{\nu} \partial_{\rho} \chi_{\sigma} \quad (2)$$

where κ is some numerical, dimensionless constant (the gravitational coupling constant being set equal to unity). How is this $L(\chi)$ to be written in a curved space? First, it is necessary that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}(x) \quad (3)$$

so henceforth (see e.g. Weinberg 1972)

$$\gamma^{\mu}(x) = e^{\mu}_{\alpha}(x) \gamma^{\alpha} \quad (4)$$

Then, suitable gauge fields must be introduced for the local Lorentz group. There is no need to introduce the affine connection $\Gamma^{\lambda}_{\mu\nu}$ associated with general covariance in (2); the curl of a vector is automatically covariant (Schrödinger 1950). The curved space Lagrangian and supersymmetry transformations take the trial forms

$$L(\chi) = i\kappa \epsilon^{\mu\nu\rho\sigma} \bar{\chi}_{\mu} \gamma^5 \gamma_{\nu} D_{\rho} \chi_{\sigma} \quad (5a)$$

$$D_{\rho} \chi_{\sigma} = \partial_{\rho} \chi_{\sigma} + \frac{1}{2} \omega_{\mu} \cdot \sigma \chi_{\sigma} \quad (5b)$$

$$\delta \chi_{\mu} = D_{\mu} \alpha = \partial_{\mu} \alpha + \frac{1}{2} \omega_{\mu} \cdot \sigma \alpha \quad (6)$$

where $\omega_{\mu} \cdot \sigma \equiv \omega_{\mu}^{\alpha\beta} \sigma_{\alpha\beta}$. To the Lagrangian (5a) (which is a density) must be added a kinetic term for the spin 2 partner, taken to be (to within a constant) the usual Einstein Lagrangian

$$L(e, \omega) = -\frac{1}{2} c e R(e, \omega)$$

where the Ricci scalar $R(e, \omega)$ is defined by $R(e, \omega) = e^{\mu}_{\alpha} e^{\nu}_{\beta} R^{\alpha\beta}_{\mu\nu}(\omega)$, $e = \det e^{\alpha}_{\mu}$, and the Riemann tensor $R^{\alpha\beta}_{\mu\nu}$ is defined by the relations

$$[D_\mu, D_\nu] \phi(j_1, j_2) = \frac{1}{2} R_{\mu\nu}^{\alpha\beta} T_{\alpha\beta}(j_1, j_2) \phi(j_1, j_2) \quad (14a)$$

where $\phi(j_1, j_2)$ is an irreducible representation of the Lorentz group and $T_{\alpha\beta}$ is the Lorentz generator corresponding to this representation. In general,

$$D_\mu \phi(j_1, j_2) = \partial_\mu \phi(j_1, j_2) + \frac{1}{2} \omega_\mu^{\alpha\beta} T_{\alpha\beta} \phi(j_1, j_2)$$

where $\omega_\mu^{\alpha\beta}$ is the gauge field corresponding to local Lorentz transformations, called the "spin connection", so that

$$R_{\mu\nu}^{\alpha\beta} = \partial_{[\mu} \omega_{\nu]}^{\alpha\beta} + \omega_{[\mu}^{\alpha\gamma} \omega_{\nu]}^{\beta\gamma} \quad (14b)$$

The supergravity Lagrangian L is constructed as the sum of $L(e, \omega)$ and $L(\chi)$;

$$L = L(e, \omega) + L(\chi) .$$

The equations of motion are

$$\frac{\delta L}{\delta \phi_i} = \frac{\partial L}{\partial \phi_i} - D_\nu \frac{\partial L}{\partial \phi_{i;\nu}} = 0$$

where $\phi_{i;\nu} = D_\nu \phi_i$, and ϕ_i are the set $(e_\mu^\alpha, \chi_\mu, \omega_\mu^{\alpha\beta})$.

Explicitly

$$\frac{\delta L}{\delta e_\mu^\beta} = c e G_\beta^\mu + i k \epsilon^{\mu\nu\rho\sigma} \bar{\chi}_\nu \gamma^5 \gamma_\beta D_\rho \chi_\sigma \quad (15)$$

$$\frac{\delta L}{\delta \chi_\mu} = 2 i k \gamma^5 \epsilon^{\mu\sigma\nu\rho} (\gamma_\sigma D_\nu \chi_\rho - \frac{1}{4} C_{\sigma\nu}^\alpha \gamma_\alpha \chi_\rho) \equiv 2 i k \gamma^5 R^\mu \quad (16)$$

$$\frac{\delta L}{\delta \omega_\mu^{\alpha\beta}} = \frac{1}{2} e c (C_{\alpha\beta}^\mu + e_\beta^\mu C_{\gamma\alpha}^\gamma - e_\alpha^\mu C_{\gamma\beta}^\gamma) - S_{\alpha\beta}^\mu \quad (17)$$

where G_β^μ is the Einstein tensor

$$G_\beta^\mu = R_{\beta\alpha}^{\mu\alpha} - \frac{1}{2} e_\beta^\mu R \equiv R_\beta^\mu - \frac{1}{2} e_\beta^\mu R \quad (18)$$

$S^\mu_{\alpha\beta}$ is the spin-density

$$S^\mu_{\alpha\beta} \equiv \frac{\partial L(\chi)}{\partial \omega_\mu^{\alpha\beta}} = (ike/2)(\bar{\chi} \cdot \gamma \chi_{[\beta} e_{\alpha]}^\mu - \bar{\chi}_\alpha \gamma^\mu \chi_\beta) \quad (19)$$

and $C^\alpha_{\mu\nu}$ is the torsion,

$$C^\alpha_{\mu\nu} \equiv D_{[\mu} e_{\nu]}^\alpha; \quad D_\mu e_\nu^\alpha = \partial_\mu e_\nu^\alpha - \omega_{\mu\beta}^\alpha e_\nu^\beta \quad (20)$$

For future reference, it is convenient to note that the vanishing of (17) implies

$$C^\mu_{\alpha\beta} = (ec)^{-1}(2S^\mu_{\alpha\beta} - e^\mu_\alpha S^\gamma_{\gamma\beta} + e^\mu_\beta S^\gamma_{\gamma\alpha}) \quad (21)$$

From the definitions of curvature and torsion follows the first Bianchi identity

$$\epsilon^{\lambda\mu\nu\rho}(R_{\lambda\mu\nu}^\alpha - D_\lambda C^\alpha_{\mu\nu}) = 0, \quad (22)$$

These equations will determine the form of $\delta\omega_\mu^{\alpha\beta}$, once a trial form for δe_μ^α is given. It is not difficult to guess an expression for δe_μ^α . The group theory presented in §6 suggests $\delta e \sim \bar{\alpha}\chi$, and the most straightforward way of achieving this is to set

$$\delta e_\mu^\beta = i\bar{\alpha}\gamma^\beta \chi_\mu. \quad (23)$$

Finally, $\delta\omega_\mu^{\alpha\beta}$ is found by requiring $\delta I = 0$ under local supersymmetry:

$$\delta I = \int \left[\frac{\delta L}{\delta e_\mu^\alpha} \delta e_\mu^\alpha + \delta \bar{\chi}_\mu \frac{\delta L}{\delta \bar{\chi}_\mu} + \frac{\delta L}{\delta \omega_\mu^{\alpha\beta}} \delta \omega_\mu^{\alpha\beta} \right] d^4x. \quad (24)$$

Inserting the explicit variations and the trial forms for $\delta \bar{\chi}_\mu$ and δe_μ^α , and performing an integration by parts, (24) becomes

$$\begin{aligned} \delta I = \int & \left[ceG_\beta^\mu + ike^{\mu\nu\rho\sigma} \bar{\chi}_\nu \gamma^5 \gamma_\beta D_\rho \chi_\sigma \right] i\bar{\alpha}\gamma^\beta \chi_\mu \\ & - 2ik\bar{\alpha}\gamma^5 D \cdot R + (\delta L / \delta \omega_\mu^{\alpha\beta}) \delta \omega_\mu^{\alpha\beta} \Big] d^4x. \end{aligned} \quad (25)$$

From the identity (22) and the identity

$$\epsilon^{\sigma\mu\nu\rho} \epsilon_{\mu\alpha\beta\tau} R_{\sigma\nu}^{\alpha\beta} = -4G_{\tau}^{\rho} \quad (26)$$

it follows (with a little Dirac algebra)

$$D \cdot R = \frac{1}{2} \epsilon^{\sigma\mu\nu\rho} C_{\sigma\mu}^{\alpha} \gamma_{\alpha} D_{\nu} \chi_{\rho} + \frac{1}{2} e G^{\tau\rho} \gamma^5 \gamma_{\tau} \chi_{\rho} \quad (27)$$

so that (25) becomes

$$\begin{aligned} \delta I = \int \{ & (ceG_{\beta}^{\mu} + i\kappa\epsilon^{\mu\nu\rho\sigma} \bar{\chi}_{\nu} \gamma^5 \gamma_{\beta} D_{\rho} \chi_{\sigma}) i\bar{\alpha} \gamma^{\beta} \chi_{\mu} \\ & + i\kappa G_{\tau}^{\rho} \bar{\alpha} \gamma^{\tau} \chi_{\rho} - \frac{1}{2} i\kappa\epsilon^{\sigma\mu\nu\rho} C_{\sigma\mu}^{\beta} \bar{\alpha} \gamma^5 \gamma_{\beta} D_{\nu} \chi_{\rho} \\ & + (\frac{1}{2} ec(C_{\alpha\beta}^{\mu} + e^{\mu}_{\beta} C^{\gamma}_{\gamma\alpha} - e^{\mu}_{\alpha} C^{\gamma}_{\gamma\beta}) - S^{\mu}_{\alpha\beta}) \delta\omega_{\mu}^{\alpha\beta} \} d^4x \end{aligned} \quad (28)$$

If $c = -\kappa$, the terms linear in the Einstein tensor vanish. Dividing by $-\kappa/2$, and using the Fierz rearrangement

$$\epsilon^{\mu\nu\rho\sigma} (\bar{\chi}_{\nu} \gamma^5 \gamma_{\beta} D_{\rho} \chi_{\sigma} \bar{\alpha} \gamma^{\beta} \chi_{\mu} - \frac{1}{2} \bar{\chi}_{\mu} \gamma^{\beta} \chi_{\nu} \bar{\alpha} \gamma^5 \gamma_{\beta} D_{\rho} \chi_{\sigma}) = 0, \quad (29)$$

the vanishing of (28) requires

$$0 = \int (C_{\alpha\beta}^{\mu} - i\bar{\chi}_{\alpha} \gamma^{\mu} \chi_{\beta}) A_{\mu}^{\alpha\beta} d^4x \quad (30)$$

where

$$A_{\mu}^{\alpha\beta} = i\epsilon^{\alpha\beta\nu\rho} \bar{\alpha} \gamma^5 \gamma_{\mu} D_{\nu} \chi_{\rho} + e(\delta\omega_{\mu}^{\alpha\beta} + e_{\gamma}^{\nu} e_{\mu}^{\alpha} \delta\omega_{\nu}^{\beta\gamma})$$

If (30) is to vanish off mass-shell, $A_{\mu}^{\alpha\beta}$ must vanish. By inspection, then,

$$\delta\omega_{\mu}^{\alpha\beta} = B_{\mu}^{\alpha\beta} + \frac{1}{2} e_{\mu}^{\beta} B_{\gamma}^{\gamma\alpha} - \frac{1}{2} e_{\mu}^{\alpha} B_{\gamma}^{\gamma\beta} \quad (31)$$

where

$$B_{\mu}^{\alpha\beta} = -ie^{-1} \epsilon^{\alpha\beta\rho\sigma} \bar{\alpha} \gamma^5 \gamma_{\mu} D_{\rho} \chi_{\sigma} \quad (32)$$

The choice of the constant c is dictated by the requirement that for

$\chi_\mu \equiv 0$, the Lagrangian (7) should reduce to the usual Einstein Lagrangian; i.e., $c = -1$. Then

$$L = \frac{1}{2}eR + i\epsilon^{\mu\nu\rho\sigma} \bar{\chi}_\mu \gamma^5 \gamma_\nu D_\rho \chi_\sigma \quad (7)$$

and the equations of motion become

$$G^{\nu\mu} = ie^{-1} \epsilon^{\mu\lambda\rho\sigma} \bar{\chi}_\lambda \gamma^5 \gamma^\nu D_\rho \chi_\sigma \quad (33)$$

$$R^\sigma = \epsilon^{\sigma\mu\nu\rho} (\gamma_\mu D_\nu \chi_\rho - \frac{1}{2} C^\alpha_{\mu\nu} \gamma_\alpha \chi_\rho) = 0 \quad (34)$$

$$C^\alpha_{\mu\nu} = i \bar{\chi}_\mu \gamma^\alpha \chi_\nu \quad (35)$$

In order that the theory be consistent, it is necessary that $D \cdot R = 0$ as a consequence of (33) and (35). Indeed, the vanishing of $D \cdot R$ follows from the Fierz rearrangement (30). Quite apart from supersymmetry, the Deser-Zumino model is interesting as the first consistent theory of an interacting Rarita-Schwinger field.

§10.2 The Freedman, Ferrara and van Nieuwenhuizen Model

As described, the Deser-Zumino theory is in "first order" form; so long as $\omega_\mu^{\alpha\beta}$ is regarded as an independent field, no terms appear which are higher than first-order in derivatives. On the other hand, (35) may be solved for $\omega_\mu^{\alpha\beta}$:

$$\omega_\mu^{\alpha\beta} = \omega_\mu^{\alpha\beta}(e) + (i/2) (\bar{\chi}^\alpha \gamma_\mu \chi^\beta + \bar{\chi}_\mu \gamma^\alpha \chi^\beta - \bar{\chi}_\mu \gamma^\beta \chi^\alpha) \quad (36)$$

where as usual (Veltman 1976)

$$\begin{aligned} \omega_\mu^{\alpha\beta}(e) &= \frac{1}{2} e^{\alpha\nu} (\partial_{[\mu} e_{\nu]}^\beta) + \frac{1}{2} e^{\alpha\rho} e^{\beta\sigma} (\partial_{[\sigma} e_{\rho]}^\gamma) e_{\gamma\mu} \\ &\quad - (\alpha \leftrightarrow \beta) \end{aligned} \quad (37)$$

The expression (36) is consistent with the forms adopted for the various

transformation laws, as it should be. If desired, the explicit form (37) may be substituted back into the original Lagrangian (7), to give the "second-order" form of L in terms of $\omega_{\mu}^{\alpha\beta}(e)$ and χ_{μ} ;

$$L_{(2)} = L(e, \chi) = L_E + L_{(3/2)} + L_4 \quad (38)$$

where L_E and $L_{(3/2)}$ are the Einstein and Rarita-Schwinger Lagrangians in which $\omega_{\mu}^{\alpha\beta}(e)$ replaces $\omega_{\mu}^{\alpha\beta}$ everywhere, and L_4 is the contact term

$$L_4 = (1/8) \{ \bar{\chi}^{\beta} \gamma^{\alpha} \chi^{\delta} (\bar{\chi}_{\beta} \gamma_{\alpha} \gamma_{\delta} + 2 \bar{\chi}_{\alpha} \gamma_{\beta} \chi_{\delta}) - 4 (\bar{\chi}_{\alpha} \gamma^{\alpha} \chi)^2 \} \quad (39)$$

The formulae for $\delta\chi_{\mu}$ and $\delta\omega_{\mu}^{\alpha\beta}$ are also considerably changed.

In the former case,

$$\delta\chi_{\mu} = D_{\mu}(\omega(e))\alpha + \frac{1}{4} (2 \bar{\chi}_{\mu} \gamma_{\delta} \chi_{\beta} + \bar{\chi}_{\delta} \gamma_{\mu} \chi_{\beta}) i \sigma^{\delta\beta} \alpha \quad (40)$$

while in the latter case $\delta\omega_{\mu}^{\alpha\beta}$ is given only through its dependence on e_{μ}^{α} , and depends now on derivatives of α . In general, (Kibble 1961) the second-order form of a Lagrangian depending on $\omega_{\mu}^{\alpha\beta}(e)$ may be obtained from the first-order expression by substituting for the original $\omega_{\mu}^{\alpha\beta}$ the following expression:

$$\omega_{\mu\alpha\beta} = \omega_{\mu\alpha\beta}(e) - \Sigma_{\mu\alpha\beta} - e_{\mu}[\alpha \Sigma^{\gamma}_{\beta}]_{\gamma} - \Sigma[\alpha]_{\mu}[\beta] \quad (41)$$

(where $\Sigma_{\mu\alpha\beta}$ is the spin-tensor; $e\Sigma_{\mu\alpha\beta} = S_{\mu\alpha\beta}$ as defined by (19)). This substitution results in the old Lagrangian with $\omega_{\mu}^{\alpha\beta}(e)$ replacing $\omega_{\mu}^{\alpha\beta}$ and supplemented by an additional term quadratic in the spin tensor;

$$L_{\text{contact}} = \frac{1}{2} e (2 \Sigma_{\alpha\beta\gamma} \Sigma^{\beta\gamma\alpha} - \Sigma_{\alpha\beta\gamma} \Sigma^{\alpha\beta\gamma} + 2 \Sigma^{\gamma}_{\gamma\alpha} \Sigma_{\beta}^{\beta\alpha}) \quad (42)$$

These general forms reduce to the specific (39), (40) when $S^{\alpha}_{\mu\nu}$ has

the form (19). Once the expression (37) has been used in the Lagrangian, the Ricci scalar becomes second-order in derivatives of the vierbein.

The supergravity theory of Freedman et al. (1976) may now be described in a very few words. It is just the second-order formulation of the Deser-Zumino theory, in which the trial Lagrangian is exactly (7) but with $\omega_{\mu}^{\alpha\beta}(e)$ replacing $\omega_{\mu}^{\alpha\beta}$. In order to make this trial Lagrangian invariant, the contact term (39) is then added, and the transformation law for χ_{μ} is altered to (40). The verification that the Lagrangian (38) is invariant under (40) and (24) at first proved impossible without the aid of a computer (owing to the complicated structure of terms quintic in χ_{μ}). Once the equivalent first-order theory was found, it became an elementary exercise to show the invariance by invoking the correspondence between first- and second-order formulations.

The two supergravity theories presented thus far (really, one theory in two equivalent formulations) give only the kinetic and self-interaction terms for the $(3/2, 2)$ massless supermultiplet. How is matter to be coupled into this $(3/2, 2)$ multiplet? A tedious algorithm has been used in the past. First, the kinetic term for a given multiplet (with some modification for the presence of a curved space) is added to the free supergravity Lagrangian (7). Then, using Noether's theorem, the supercurrent corresponding to this matter multiplet is found. For example, the supercurrent for the massive chiral multiplet is $J^{\mu} = \{(i\partial - m)(A - \gamma^5 B)\}\gamma^{\mu}\psi$. The lowest-order coupling is then given by $\bar{\chi}_{\mu}J^{\mu}$. Unfortunately this rarely suffices to give an invariant Lagrangian. Instead, additional terms in χ_{μ} are usually needed (to take care of the transformations of kinetic terms which will involve derivatives of the parameter α), and very frequently the transformation laws of the matter fields must be altered from their

globally supersymmetric forms. There is no ready prescription for the additional terms in the Lagrangian, or the new transformations; only trial and error serve to find them. It would seem highly advantageous to discover a systematic method which automatically determines the fields' behaviour under local supersymmetry. An approach which accomplishes this in part for the gauge fields is presented in the next section. Ultimately, the method fails; nevertheless it indicates a fruitful avenue which will be followed in the succeeding chapters.

§11. Fiber Bundle Methods

§11.1 The SU(2) Yang-Mills Theory via Fiber Bundles

In an elegant attempt to circumvent the trial and error methods of §10, Chamseddine and West (1977) employed the differential geometric methods of the fiber bundle formalism (Choquet-Bruhat, De Witt-Morette, and Dillard-Bleick 1977, Cho 1975). It is possible to use this machinery without a deep knowledge of the mathematics involved, and the spirit of the present approach is that of an engineer who wishes to compute an integral without delving into epsilon-delta arguments. Rather than plunging immediately into supersymmetry, these methods will be used to describe a more familiar gauge theory, namely that of SU(2). It will become evident that the fiber bundle analysis is nothing more than the usual Yang-Mills approach, but with a greater geometrical emphasis.

Consider an element of the Lie group SU(2). This element U may be parametrised by $U = \exp(i\lambda_j T_j)$ where the three numbers λ_j are the group parameters and T_j are the generators for a particular representation. Let the set λ_j be replaced by three continuous functions of x , $\lambda_j(x)$, when x is a point in Minkowski space, and let h_j be a point in λ_j 's range. Pick a set of functions σ_μ such that locally

$\lambda_j(\sigma(h_k)) = h_j$. The surface defined by $x=\sigma(h_k)$ for a given vector h_k is said to be a "cross-section". The "volume" formed from the Cartesian products of $|R^3$ (the domain of λ_j) and the "base space" of Minkowski space is said to be the "bundle space". The bundle space may be built up as stacks of these cross-sections. In this approach a change of gauge corresponds to a change in cross-section. The bundle space describes local invariance, while the ~~fiber~~ corresponds to global symmetry. The fundamental operators in these spaces are "tangent vectors" and their dual elements (see Appendix B). For the base space, the tangent vectors are just the ordinary derivatives ∂_μ , while for the fiber they are just the (abstract) group generators Q_j . These tangent vector operators satisfy the familiar algebra,

$$[Q_j, Q_k] = i\epsilon_{jkl} Q_l \quad (43a)$$

$$[Q_j, \partial_\mu] = 0 \quad (43b)$$

$$[\partial_\mu, \partial_\nu] = 0 \quad (43c)$$

The corresponding operators in the bundle space are the operators $(\hat{\partial}_\mu, Q_j^*)$ (Cho, 1975). The operators $\hat{\partial}_\mu$ are called "horizontal lifts". Just as σ "lifts" a point h_j to the point (h_j, x) , so does an associated mapping σ_* lift ∂_μ to a new operator $\sigma_*(\partial_\mu)$. The operators $\sigma_*(\partial_\mu)$ are just the form of ∂_μ required by the chain rule, $\sigma_*(\partial_\mu)f(\lambda) = \partial_\mu f(\sigma(\lambda))$. The inverse mapping λ_* sends $\hat{\partial}_\mu$ to ∂_μ , and the operators Q_j^* to zero. In general, $\hat{\partial}_\mu \neq \sigma_*(\partial_\mu)$. The mappings σ_* and λ_* are what the mathematicians call "functors". The functors are linear over functions

$$\lambda_*(g(x)\hat{\partial}_\mu) = g(x)\lambda_*(\hat{\partial}_\mu)$$

and distribute over Lie brackets:

$$\lambda_*([\hat{\partial}_\mu, \hat{\partial}_\nu]) = [\lambda_*(\hat{\partial}_\mu), \lambda_*(\hat{\partial}_\nu)] \quad (44)$$

The algebra to be obeyed is as far as possible that of (43), namely

$$[Q_j^*, Q_k^*] = i \varepsilon_{jkl} Q_l^* \quad (45a)$$

$$[Q_j^*, \hat{\partial}_\mu] = 0 \quad (45b)$$

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = i F_{\mu\nu k} Q_k^* \quad (45c)$$

The last equation ensures (44) is satisfied, for

$$\lambda_*(i F_{\mu\nu k} Q_k^*) = i F_{\mu\nu k} \lambda_*(Q_k^*) = 0.$$

Explicitly, $\hat{\partial}_\mu$ may be written as

$$\hat{\partial}_\mu = \partial_\mu + i A_{\mu k} Q_k^* \quad (46)$$

The change in gauge induces the usual transformation in the fields $A_{\mu k}$ (the gauge potentials) according to the Ehresmann law (Spivak 1970)

$$i T_k A'_{\mu k} = i U (T_k A_{\mu k}) U^{-1} + U \partial_\mu U^{-1} \quad (47)$$

where $U = \exp(i T_k(Q) \lambda_k)$ as before, and where $T_k(Q)$ corresponds to the adjoint representation of $SU(2)$. Explicitly, for an infinitesimal change

$$\delta A_{\mu l} = \varepsilon_{ljk} \lambda_j A_{\mu k} - \partial_\mu \lambda_l \quad (48)$$

while (45c) leads (with (46)) to the usual definition of $F_{\mu\nu k}$;

$$F_{\mu\nu k} = \partial_{[\mu} A_{\nu]k} + \varepsilon_{ijk} A_{\mu i} A_{\nu j} \quad (49)$$

From the Jacobi identity

$$[[\hat{\partial}_\mu, \hat{\partial}_\nu], \hat{\partial}_\lambda] + (\text{Cyclic permutations}) = 0$$

follows the second Bianchi identity

$$D_\lambda F_{\mu\nu k} + D_\mu F_{\nu\lambda k} + D_\nu F_{\lambda\mu k} = 0 \quad (50)$$

where

$$D_\lambda F_{\mu\nu k} = [\hat{\partial}_\lambda, F_{\mu\nu k}] .$$

It may not seem that anything has been gained over the usual approach. The key relation is (45b) whose content is that the potentials $A_{\mu k}$ transform according to the adjoint representation; .

$$[Q_j^*, A_{\mu k}] = i\varepsilon_{jkl} A_{\mu l} \quad (51)$$

(Note that this implies from (46) that $[\partial_\mu, Q_i^*] = 0$). Apparently, in order to use this approach, it is necessary that the group to be gauged should commute with the ordinary derivatives. It is on this last point that some difficulties are encountered in applying the fiber bundle methods to space-time groups.

§11.2 Fiber Bundles and Supergravity

Owing to (43), once the algebra of the group generators Q_j is known, the transformation laws for the gauge fields are determined. Chamseddine and West (1977) were thus able to derive the forms of δe_μ^α , $\delta \chi_\mu$, and $\delta \omega_\mu^{\alpha\beta}$.

Let the group generators be denoted $(P_\alpha, J_{\alpha\beta}, S_a)$, and let them satisfy the usual algebra (1.1) among themselves, as well as

$$\begin{aligned} [\partial_\mu, P_\alpha] &= 0 \\ [\partial_\mu, J_{\alpha\beta}] &= 0 \\ [\partial_\mu, S_a] &= 0 . \end{aligned}$$

This condition divorces the group space from Minkowski space. The gauge potentials associated with these operators are, respectively e_μ^α , $\omega_\mu^{\alpha\beta}$, and χ_μ^a ; while the lifts $\hat{\partial}_\mu$ are

$$\hat{\partial}_\mu = \partial_\mu - ie_\mu^\alpha P_\alpha^* + \frac{1}{2} i \omega_\mu^{\alpha\beta} J_{\alpha\beta}^* + i \chi_\mu^a S_a^* . \quad (52)$$

In order that (45b) be satisfied, $\hat{\partial}_\mu$ must commute with each of the

linearly independent operators P_α^* , $J_{\alpha\beta}^*$ and S_a^* . The vanishing of the commutators between $\hat{\partial}_\mu$ and each of the group generators implies that the coefficients of each of these generators on the right-hand side must vanish. For example,

$$[\hat{\partial}_\mu, P_\alpha^*] = -i [e_\mu^\beta, P_\alpha^*] P_\beta^* + \frac{1}{2} i \omega_\mu^{\beta\delta} [J_{\beta\delta}, P_\alpha^*] + \frac{1}{2} i [\omega_\mu^{\beta\delta}, P_\alpha^*] J_{\beta\delta}^* + i [\bar{\chi}_\mu^a, P_\alpha^*] S_a^* . \quad (53)$$

The vanishing of (53) implies three separate equations:

$$[-ie_\mu^\beta, P_\alpha^*] P_\beta^* + \frac{1}{2} i \omega_\mu^{\beta\delta} [J_{\beta\delta}, P_\alpha^*] = 0 \quad (54a)$$

$$[\omega_\mu^{\beta\gamma}, P_\alpha^*] J_{\beta\gamma}^* = 0 \quad (54b)$$

$$[\bar{\chi}_\mu^a, P_\alpha^*] S_a^* = 0 \quad (54c)$$

so that

$$[e_\mu^\beta, P_\alpha^*] = -i \omega_{\mu\alpha}^\beta \quad (55)$$

and the other gauge fields commute with P_α^* . Similarly some easy calculation leads to

$$[e_\mu^\alpha, J_{\beta\gamma}^*] = -i(e_{\mu\beta}^{\delta\alpha}{}_\gamma - e_{\mu\gamma}^{\delta\alpha}{}_\beta) \quad (56a)$$

$$[\omega_\mu^{\alpha\beta}, J_{\gamma\delta}^*] = 2i(\omega_{\mu\delta}^{\beta\delta\alpha}{}_\gamma - \omega_{\mu\gamma}^{\alpha\delta\beta}{}_\delta) \quad (56b)$$

$$[\bar{\chi}_\mu^a, J_{\gamma\delta}^*] = -\frac{1}{2} i (\bar{\chi}_\mu^\sigma \sigma_{\gamma\delta})^a \quad (56c)$$

and

$$[e_\mu^\alpha, S^*] = -i \gamma^\alpha \chi_\mu \quad (57a)$$

$$[\omega_\mu^{\alpha\beta}, S^*] = 0 \quad (57b)$$

$$\{\bar{\chi}_\mu, S^*\} = -\frac{1}{2} \omega_\mu \cdot \sigma . \quad (57c)$$

From the Ehresmann law (47), with $U = \exp(ia^\beta P_\beta^* - \frac{1}{2} i \lambda^{\alpha\beta} J_{\alpha\beta}^* - i \bar{\alpha} S^*)$,

the fields transform as

$$\delta e_{\mu}^{\beta} = D_{\mu} a^{\beta} + \lambda^{\beta}_{\alpha} e_{\mu}^{\alpha} + i \bar{\alpha} \gamma^{\beta} \chi_{\mu} \quad (58)$$

$$\delta \omega_{\mu}^{\alpha\beta} = D_{\mu} \lambda^{\alpha\beta} \quad (59)$$

$$\delta \chi_{\mu} = D_{\mu} \alpha + \frac{1}{2} \lambda \cdot \sigma \chi_{\mu} \quad (60)$$

where $a^{\beta}(x)$ and $\lambda^{\beta\alpha}(x)$ are the local parameters associated with P_{β}^{*} and $J_{\beta\alpha}^{*}$, respectively. The surprising aspect of these equations is that the connection $\omega_{\mu}^{\alpha\beta}$ is inert under supersymmetry. This has ambiguous consequences, as will become evident. The field strengths are given by

$$[\hat{\partial}_{\mu}, \hat{\partial}_{\nu}] = i C'_{\mu\nu}{}^{\alpha} P_{\alpha}^{*} + \frac{1}{2} i R_{\mu\nu}{}^{\alpha\beta} J_{\alpha\beta}^{*} + i \bar{D}_{\mu\nu}{}^a S_a^{*} \quad (61)$$

where (after a little computation)

$$C'_{\mu\nu}{}^{\alpha} = D_{[\mu} e_{\nu]}^{\alpha} - i \bar{\chi}_{\mu} \gamma^{\alpha} \chi_{\nu} \quad (62)$$

$$R_{\mu\nu}{}^{\alpha\beta} = \partial_{[\mu} \omega_{\nu]}^{\alpha\beta} + \omega_{[\mu}^{\alpha\gamma} \omega_{\nu]}{}_{\gamma}{}^{\beta} \quad (63)$$

$$\bar{D}_{\mu\nu}{}^a = D_{[\mu} \bar{\chi}_{\nu]}^a \quad (64)$$

The major difference between these field strengths and those of Deser and Zumino (1976) lies in the torsion $C'_{\mu\nu}{}^{\alpha}$. The Deser-Zumino equation of motion $C_{\mu\nu}^{\alpha} = i \bar{\chi}_{\mu} \gamma^{\alpha} \chi_{\nu}$ becomes for Chamseddine and West the constraint $C'_{\mu\nu}{}^{\alpha} = 0$. Remarkably, the same Lagrangian

$$L = \frac{1}{2} e R + i \epsilon^{\mu\nu\rho\sigma} \bar{\chi}_{\mu} \gamma^5 \gamma_{\nu} D_{\rho} \chi_{\sigma}$$

is invariant under the new transformations (58-60). Actually this should not be surprising, because it is only the form of $\delta \omega_{\mu}^{\alpha\beta}$ which differs from the Deser-Zumino forms; and $C'_{\mu\nu}{}^{\alpha}$ is just the variation of L with respect to $\omega_{\alpha}^{\mu\nu}$. This is apparent from (30); if $C_{\mu\nu}^{\alpha} = i \bar{\chi}_{\mu} \gamma^{\alpha} \chi_{\nu}$, the integrand vanishes identically. There are still ambiguities with this

approach. First, if $C'^{\alpha}_{\mu\nu}$ vanishes, then this equation may be used to solve for $\omega_{\mu}^{\alpha\beta}$, whose solution is just (36). Presumably, then, the variation of the spin connection should be consistent with (36). In fact, the consistent form of $\delta\omega_{\mu}^{\alpha\beta}$ is just that given by Deser and Zumino, (31), rather than $\delta\omega_{\mu}^{\alpha\beta} = 0$. The second problem has nothing to do with supersymmetry, but is perhaps related to the manner in which the Poincaré group is divorced from Minkowski space in the fiber bundle approach.

§11.3 Troubles with the Translations

Consider for a moment the lifts corresponding only to the gauged Poincaré group:

$$\hat{\partial}_{\mu} = \partial_{\mu} - ie_{\mu}^{\alpha} P_{\alpha}^{*} + \frac{1}{2} i \omega_{\mu}^{\alpha\beta} J_{\alpha\beta}^{*} \quad (65)$$

Let A^{α} be an arbitrary vector field, and consider

$$[\nabla_{\mu}, \nabla_{\nu}] A^{\alpha} = -i C_{\mu\nu}^{\beta} [P_{\beta}^{*}, A^{\alpha}] + \frac{1}{2} i R_{\mu\nu}^{\beta\gamma} [J_{\beta\gamma}^{*}, A^{\alpha}] \quad (66)$$

where the equality comes from (61) and the Jacobi identity. On general grounds, (Kibble 1961; §12) it should be that

$$[\nabla_{\mu}, \nabla_{\nu}] A^{\alpha} = -R_{\mu\nu}^{\alpha}{}_{\beta} A^{\beta} \quad (67)$$

which implies that

$$[J_{\beta\gamma}^{*}, A^{\alpha}] = -i(T(J)A)^{\alpha} = -i(\delta_{\gamma}^{\alpha} A_{\beta} - \delta_{\beta}^{\alpha} A_{\gamma}) \quad (68)$$

$$[P_{\beta}^{*}, A^{\alpha}] = 0 \quad \underline{\text{or}} \quad C_{\mu\nu}^{\beta} = 0 \quad (69)$$

for an arbitrary tangent vector field. The vierbein e_{μ}^{α} , despite its world index μ , should transform as a tangent vector, and yet it does not commute with P_{α}^{*} . The alternative is to assume that the torsion vanishes, in which case $C_{\mu\nu}^{\alpha} = 0$ yields the solution (37) for the spin connection.

According to (54b), the spin connection commutes with P_α^* , but also $[P_\alpha^*, \partial_\mu] = 0$. If the solution (37) and derivatives ∂_μ are to commute with P_α^* , then e_μ^α should also commute with P_α^* , which is in contradiction with (55). In short, the presence of the generators P_α^* leads to ambiguities (embodied in (69)). The simplest escape is that the vierbein should not be regarded as a gauge field, nor should P_α^* be regarded as part of the gauge group. It is possible to incorporate the vierbein in a fiber bundle style treatment of ordinary gravity without the ambiguities (Derbes, 1978a), and this will be done in the next section.

§12. An Alternative Approach to the Vierbein

§12.1 Gravitational Symmetries

In order to clarify the role of the vierbein, it is worthwhile to recall the invariances present in the standard theory of gravity (Weinberg 1972):

- i) freedom to relabel the coordinates which form the underlying space-time;
- ii) freedom to perform local homogeneous Lorentz transformations on the field components directly.

The first freedom is the Einstein group, the local $GL(4, R)$, while the second is the local Lorentz group (or $SL(2, C)$). Normally the affine connection $\Gamma_{\mu\nu}^\lambda$ serves as gauge field for the former, and the spin connection $\omega_\mu^{\alpha\beta}$ for the latter. Inasmuch as an infinitesimal general coordinate transformation is induced by

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu + \xi^\mu(x) \quad (70)$$

which is indistinguishable from a local translation, it might be thought that there are no differences between regarding the gauge group of gravity

as the Poincaré group, or as $SL(2,C)$ and $GL(4,R)$ taken together. Nevertheless, there are differences. Consider first $GL(4,R)$ whose infinitesimal parameters are $\varepsilon^\nu_\mu = \partial \xi^\nu / \partial x^\mu$. For the global theory ξ^ν is at most linear in x , while for the local symmetry, ξ^ν are any four sufficiently smooth functions of x . Let G^μ_ν denote the generators of the group, and $T(G)^\mu_\nu$ some matrix representation. Then the commutator of two such transformations is

$$[\varepsilon^\nu_\mu T(G)^\mu_\nu, \varepsilon^\rho_\sigma T(G)^\sigma_\rho] = i \varepsilon^\mu_\nu T(G)^\nu_\mu \quad (71)$$

where $\varepsilon^\mu_\nu = \varepsilon^\mu_\sigma \varepsilon^\sigma_\nu - \varepsilon^\mu_\sigma \varepsilon^\sigma_\nu$. This algebra is unchanged, whether the parameters are global or local. By contrast, if the representations of P_μ acting on arbitrary functions of x are to take the usual form, $P_\mu = i\partial_\mu$, then the commutator of two translations $[a^\mu_1 P_\mu, a^\nu_2 P_\nu]$ depends intimately on whether or not the parameters a^μ are local. In the global case, the commutator vanishes; but in the local case,

$$[a^\mu_1 P_\mu, a^\nu_2 P_\nu] = i a^\mu_3 P_\mu \quad (72)$$

where $a^\mu_3 = a^\mu_1 \cdot \partial a^\mu_2 - a^\mu_2 \cdot \partial a^\mu_1$. Perhaps the difficulty could be overcome by choosing a different representation of the translations, so that the commutator always vanished. If the usual correspondence is used, then in Veltman's (1976) phrase, "we are dealing with structure constants containing derivatives". It is far from clear what gauge field is to be introduced to compensate for the local translations; using the vierbein à la Chamseddine and West does not seem completely straightforward. It should be added, however, that several authors have presented the Einstein-Cartan-Sciama-Kibble theory of gravity as the gauge theory of the Poincaré group (Hehl et al. 1976, Trautman 1975). In general these approaches associate the vierbein not with the Abelian translations,

but with the Einstein covariant derivatives $\nabla_\mu = \partial_\mu + i\Gamma_{\mu\nu}^\lambda G^\nu_\lambda$. The algebra for these local transformations (i.e. parallel transport) is very different from the global case, for then $\nabla_\mu \rightarrow \partial_\mu$. If one simply drops the translations (whether regarded as ∂_μ or ∇_μ) from the set $\{G_i\}$ of gauge group generators, the onus of regarding e_μ^α as a gauge field is removed. Instead, it may be viewed as a bridge linking the two symmetries i) and ii). The presence of two local groups in the theory of general relativity suggests a synthesis of two Yang-Mills theories; one to deal with each symmetry. The divorce imposed by Chamseddine and West between the supersymmetry group and Minkowski space will be carried over in part by labelling all Lorentz objects with early Greek letters ($\alpha, \beta, \gamma \dots$, "tangent space" indices), and all $GL(4, R)$ quantities, including the derivatives ∂_μ , with later Greek letters ($\mu, \nu, \rho \dots$, "world" indices). Some quantities, notably the vierbein e_μ^α itself, carry both types of indices. In the treatment of local supersymmetry to be presented in the last two chapters, the general coordinate group will not explicitly appear; nevertheless for clarity it seems reasonable to present this method as it is used in a more familiar theory. First the gauge theories of the coordinate and Lorentz groups will be presented. Finally, a synthesis between these theories will be mediated through the vierbein, and Einstein's standard 1916 theory will emerge naturally.

§12.2 $GL(4, R)$

Essentially all of the salient details concerning the $GL(4, R)$ gauge theory may be found in the works of De Witt (1965), Friedman (1975) and Hayashi and Shirafuji (1977). The sixteen operators G^μ_ν which generate $GL(4, R)$ obey the algebra

$$[G^\mu_\nu, G^\rho_\sigma] = i\delta^\mu_\sigma G^\rho_\nu - i\delta^\rho_\nu G^\mu_\sigma; \quad (73)$$

that is, the structure constants are given by

$$f_{\nu\sigma\pi}^{\mu\rho\tau} = \delta_{\sigma}^{\mu} \delta_{\nu}^{\tau} \delta_{\pi}^{\rho} - \delta_{\nu}^{\rho} \delta_{\pi}^{\mu} \delta_{\sigma}^{\tau} . \quad (74)$$

Both scalar and spinor fields are to be regarded as invariants with respect to this group. For a contravariant vector A^{μ} , define the infinitesimal transformation

$$\delta A^{\mu}(x) = \bar{A}^{\mu}(\bar{x}) - A^{\mu}(x) = i[\epsilon_{\sigma}^{\rho} G_{\rho}^{\sigma}, A^{\mu}(x)] \quad (75)$$

where $\epsilon_{\sigma}^{\rho} = \partial \xi^{\rho} / \partial x^{\sigma}$, $\bar{x}^{\mu} = x^{\mu} + \xi^{\mu}$, as before. For a finite transformation,

$$\bar{A}^{\mu}(\bar{x}) = (\partial \bar{x}^{\mu} / \partial x^{\nu}) A^{\nu}(x) . \quad (76)$$

Similarly, for a covariant vector B_{μ} ,

$$\bar{B}_{\mu}(\bar{x}) = (\partial x^{\nu} / \partial \bar{x}^{\mu}) B_{\nu}(x) . \quad (77)$$

From (75), (76) and (77) it follows

$$[G_{\rho}^{\sigma}, A^{\mu}] = -iT(G)_{\rho}^{\sigma\mu\nu} A^{\nu} = -i\delta_{\rho}^{\mu} A^{\sigma} \quad (78a)$$

$$[G_{\rho}^{\sigma}, B_{\nu}] = iT(G)_{\rho}^{\sigma\mu\nu} B_{\mu} = i\delta_{\nu}^{\sigma} B_{\rho} \quad (78b)$$

$$[G_{\rho}^{\sigma}, A^{\mu} B_{\mu}] = 0 . \quad (78c)$$

The formula for a rank-N tensor $A_{\mu\nu\dots}^{\rho\sigma\dots\tau\dots}$ may be found by considering the special case

$$A_{\mu\nu\dots}^{\rho\sigma\dots\tau\dots} = B_{\mu} C_{\nu} \dots D^{\rho} E^{\sigma} \dots F_{\tau} \dots .$$

To construct a covariant derivative, introduce sixteen fields $A_{\mu\rho}^{\nu}$;

$\nabla_{\mu} = \partial_{\mu} + i\kappa A_{\mu\rho}^{\nu} G_{\nu}^{\rho}$. It is convenient to absorb the coupling constant

κ into a redefinition of the gauge fields, and call $\kappa A_{\mu\rho}^{\nu} \equiv \Gamma_{\mu\rho}^{\nu}$.

Then for example

$$\begin{aligned}\nabla_{\mu} A^{\nu} &= \partial_{\mu} A^{\nu} + i[\Gamma_{\mu\pi}^{\tau} G^{\pi}_{\tau}, A^{\nu}] \\ &= \partial_{\mu} A^{\nu} + \Gamma_{\mu\sigma}^{\nu} A^{\sigma}\end{aligned}\quad (79a)$$

and by the same token

$$\nabla_{\mu} B_{\nu} = \partial_{\mu} B_{\nu} - \Gamma_{\mu\nu}^{\lambda} A_{\lambda} \quad (79b)$$

and so forth for an arbitrary tensor $A_{\mu\nu\dots}^{\rho\sigma\dots}$. The Ehresmann law (47) becomes a little more complicated than usual in the case of $GL(4, R)$, because the index μ on the covariant derivative transforms under the group:

$$[\nabla_{\mu}, G^{\rho}_{\sigma}] = -i\delta^{\rho}_{\mu} \nabla_{\sigma} \quad (80)$$

This is quite contrary to the situation described by (45b). Instead of the expected

$$\bar{\Gamma}^{\mu}_{\nu\lambda} T(G)^{\lambda}_{\mu} = U\Gamma^{\mu}_{\nu\lambda} T(G)^{\lambda}_{\mu} U^{-1} + U\partial_{\nu} U^{-1} \quad (81)$$

(the additional matrix indices on T have been suppressed) where

$U = U^{\rho}_{\sigma} = \partial \bar{x}^{\rho} / \partial x^{\sigma}$, the Ehresmann law reads*

$$\bar{\Gamma}^{\mu}_{\nu\lambda} T(G)^{\lambda}_{\mu} = U\Gamma^{\mu}_{\nu\lambda} T(G)^{\lambda}_{\mu} U^{-1} U^{-1} + U\partial_{\nu} U^{-1} U^{-1} \quad (82)$$

or in terms of explicit components

$$\bar{\Gamma}^{\mu}_{\nu\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\tau}} \Gamma^{\tau}_{\rho\sigma} \frac{\partial x^{\rho}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\lambda}} + \frac{\partial \bar{x}^{\mu}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \bar{x}^{\lambda} \partial \bar{x}^{\nu}} \quad (83)$$

where the chain rule has been invoked,

$$\frac{\partial x^{\rho}}{\partial \bar{x}^{\nu}} \frac{\partial^2 \bar{x}^{\sigma}}{\partial x^{\rho} \partial \bar{x}^{\lambda}} = \frac{\partial^2 x^{\sigma}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\lambda}}$$

The field strengths $R^{\rho}_{\sigma\mu\nu}$ are defined as

* The apparent inconsistency does not arise if differential forms are employed. See Appendix B.

$$[\nabla_\mu, \nabla_\nu] A^\rho = -R^\rho_{\sigma\mu\nu} A^\sigma - \Gamma^\sigma_{[\mu\nu]} \nabla_\sigma A^\rho \quad (84)$$

where $R^\rho_{\sigma\mu\nu}$ have the usual Yang-Mills form

$$\begin{aligned} R^\rho_{\sigma\mu\nu} &= \partial_{[\mu} \Gamma^\rho_{\nu]\sigma} + f^{\pi\kappa\rho}_{\tau\lambda\nu} \Gamma^\tau_{\sigma\pi} \Gamma^\lambda_{\mu\kappa} \\ &= \partial_{[\mu} \Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{\mu\tau} \Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau} \Gamma^\tau_{\mu\sigma} . \end{aligned} \quad (85)$$

The antisymmetric part of $\Gamma^\mu_{\nu\rho}$ is a tensor (as is evident from (83)), called the torsion. Under a certain assumption, it is equal to the torsion $C^\mu_{\nu\rho}$ introduced previously, as will be shown in §12.4. Frequently the affine connection Γ is postulated to be symmetric, so that the torsion vanishes. If in addition, it is assumed that the covariant derivative of the metric tensor $g_{\mu\nu}$ vanishes,

$$\nabla_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^\tau_{\mu\rho} g_{\tau\sigma} - \Gamma^\tau_{\mu\sigma} g_{\rho\tau} = 0 , \quad (86)$$

then the affine connection satisfies the Christoffel relations

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau} (\partial_\nu g_{\tau\rho} + \partial_\rho g_{\tau\nu} - \partial_\tau g_{\nu\rho}) .$$

Note that the relation (84) bears a superficial resemblance to (66). Nevertheless, the content of these equations is very different, for in order to identify them, it would be necessary to interpret P_α^* as the generators of parallel transport, and these are not Abelian. Instead of postulating the symmetry of the affine connection, it seems more consistent to treat it as an independent variable when writing a Lagrangian. The equations of motion will then determine the symmetry properties of $\Gamma^\lambda_{\mu\nu}$.

§12.3 The Lorentz Group

The generators of the Lorentz group are the familiar operators

$M_{\alpha\beta} = -M_{\beta\alpha}$, satisfying the well-known algebra

$$\begin{aligned} [M_{\alpha\beta}, M_{\gamma\delta}] &= i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}) \\ &= \frac{1}{2} i f_{\alpha\beta\gamma\delta}^{\epsilon 1} M_{\epsilon 1} \end{aligned} \quad (88)$$

where the factor of $\frac{1}{2}$ allows for double counting, and $\eta_{\alpha\beta} = \text{diag}(+---)$. In general, a field ϕ^A transforms according to a particular representation of the Lorentz group:

$$i[M_{\alpha\beta}, \phi^A] = T_{\alpha\beta}^A{}_B \phi^B \quad (89)$$

where (A, B) are the appropriate indices, e.g.

$$\begin{aligned} \text{scalar} &: \phi^A = \phi, & T_{\alpha\beta} &= 0 \\ \text{spinor} &: \phi^A = \phi^a, & T_{\alpha\beta}^a{}_b &= \frac{1}{2} i (\sigma_{\alpha\beta})^a{}_b \\ \text{vector} &: \phi^A = \phi^\alpha, & T_{\alpha\beta}^\gamma{}_\delta &= i(\eta_{\alpha\delta}\delta^\gamma{}_\beta - \eta_{\beta\delta}\delta^\gamma{}_\alpha) \end{aligned}$$

and so forth. The covariant derivative for an arbitrary field ϕ is thus

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi + \frac{1}{2} i g A_\mu^{\alpha\beta} [M_{\alpha\beta}, \phi] \\ &= \partial_\mu \phi + \frac{1}{2} g A_\mu^{\alpha\beta} T_{\alpha\beta} \phi. \end{aligned} \quad (90)$$

Without loss of generality it may be assumed that $A_\mu^{\alpha\beta} = -A_\mu^{\beta\alpha}$, and

again let the coupling constant g be absorbed into the gauge field;

$g A_\mu^{\alpha\beta} = \omega_\mu^{\alpha\beta}$. The Ehresmann law required for covariance becomes

$$\frac{1}{2} \omega_\mu^{\alpha\beta} T_{\alpha\beta} = \frac{1}{2} U \omega_\mu^{\alpha\beta} T_{\alpha\beta} U^{-1} + U \partial_\mu U^{-1} \quad (91)$$

where U may be parametrised by

$$U = \exp(-\frac{1}{2} \lambda^{\alpha\beta} T_{\alpha\beta})$$

so that the infinitesimal form of (91) is

$$\begin{aligned}
 \delta \omega_{\mu}^{\alpha\beta} &= -\lambda^{\gamma\delta} f_{\gamma\delta}^{\alpha\beta} \omega_{\mu}^{\epsilon\iota} + \partial_{\mu} \lambda^{\alpha\beta} \\
 &= \lambda_{\gamma}^{\beta} \omega_{\mu}^{\gamma\alpha} - \lambda_{\gamma}^{\alpha} \omega_{\mu}^{\gamma\beta} + \partial_{\mu} \lambda^{\alpha\beta} \\
 &= D_{\mu} \lambda^{\alpha\beta}
 \end{aligned} \tag{92}$$

(compare (59)). The derivatives ∂_{μ} are taken to be scalars with respect to $M_{\alpha\beta}$, so that $[M_{\alpha\beta}, \partial_{\mu}] = 0$. Therefore the commutator of two covariant derivatives is

$$[D_{\mu}, D_{\nu}] = \frac{1}{2} F_{\mu\nu}^{\alpha\beta} T_{\alpha\beta} \tag{93}$$

where the field strengths are defined as usual (modulo the ubiquitous factors of $\frac{1}{2}$)

$$\begin{aligned}
 F_{\mu\nu}^{\alpha\beta} &= \partial_{[\mu} \omega_{\nu]}^{\alpha\beta} + \frac{1}{2} f_{\gamma\delta}^{\alpha\beta} \omega_{\mu}^{\gamma\delta} \omega_{\nu}^{\epsilon\iota} \\
 &= \partial_{[\mu} \omega_{\nu]}^{\alpha\beta} + \omega_{[\mu}^{\alpha\gamma} \omega_{\nu]\gamma}^{\beta}
 \end{aligned} \tag{94}$$

(compare (14)). The Lorentz operators $M_{\alpha\beta}$ will play a vital role in a new approach to gauging supersymmetry (§16 et seq.)

§12.4 The Role of the Vierbein

In order to unite the two Yang-Mills theories, it is necessary to interpret the vierbein not as a gauge field, but as a mapping between the integral spin representations of the Lorentz group and the tensors of $GL(4, R)$:

$$\begin{aligned}
 A_{\alpha} &\text{ is a Lorentz vector and a coordinate scalar} \\
 A_{\mu} &\text{ is a Lorentz scalar and a coordinate vector, and} \\
 A_{\mu}(x) &= e_{\mu}^{\alpha}(x) A_{\alpha}(x) .
 \end{aligned} \tag{95}$$

If the vierbein is defined such that

$$e_{\mu}^{\alpha} e_{\alpha}^{\nu} = \delta_{\mu}^{\nu} \quad (96a)$$

$$e_{\alpha}^{\nu} e_{\nu}^{\beta} = \delta_{\alpha}^{\beta} \quad (96b)$$

then in general

$$A_{\mu\alpha}^{\beta\nu\dots} = e_{\mu}^{\gamma} e_{\alpha}^{\rho} e_{\sigma}^{\beta} e_{\delta}^{\nu} \dots A_{\gamma\rho\dots}^{\sigma\delta\dots} \quad (97)$$

In particular, the invariant interval ds^2 leads to a link between the vierbein and the metric tensor $g_{\mu\nu}$. If

$$dx^{\alpha} = e_{\mu}^{\alpha} dx^{\mu}$$

(in general dx^{α} is not an exact differential of any function, see Landau and Lifshitz (1975) and Appendix B), the interval may be written as

$$ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} = g_{\mu\nu} dx^{\mu} dx^{\nu}; \text{ i.e.}$$

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} e_{\mu}^{\alpha}(x) e_{\nu}^{\beta}(x) \quad (98)$$

Because the two group manifolds are related only through the vierbein,

$$[G_{\sigma}^{\rho}, M_{\alpha\beta}] = 0 \quad (99)$$

Indeed, if ϕ has only early Greek and/or Latin indices, it commutes with G_{σ}^{ρ} , while if ϕ has only late Greek indices, it commutes with $M_{\alpha\beta}$. On the other hand,

$$[G_{\sigma}^{\rho}, e_{\mu}^{\alpha}] = -i\delta_{\mu}^{\rho} e_{\sigma}^{\alpha},$$

$$[M_{\alpha\beta}, e_{\mu}^{\gamma}] = ie_{\mu\alpha}\delta^{\gamma}_{\beta} - ie_{\mu\beta}\delta^{\gamma}_{\alpha}.$$

For an arbitrary field, the new covariant derivative is defined as (compare Veltman, 1976)

$$\nabla_{\mu}\phi = \partial_{\mu}\phi + i\Gamma_{\mu\lambda}^{\nu}[G_{\nu}^{\lambda}, \phi] + \frac{1}{2}i\omega_{\mu}^{\alpha\beta}[M_{\alpha\beta}, \phi] \quad (100)$$

Because the combination $\Gamma_{\mu\lambda}^{\nu} G_{\nu}^{\lambda} + \frac{1}{2}\omega_{\mu}^{\alpha\beta} M_{\alpha\beta}$ transforms under $GL(4, R)$

as a vector, the commutator of two covariant derivatives gives analogously to (84) (compare Kibble, 1961)

$$[\nabla_\mu, \nabla_\nu]\phi = R^\sigma_{\rho\mu\nu} T(G)^\rho_\sigma \phi + F_{\mu\nu}^{\alpha\beta} T(M)_{\alpha\beta} \phi - \Gamma^\lambda_{[\mu\nu]} \nabla_\lambda \phi. \quad (101)$$

Physically, the groups may be reconciled by the requirement

$$\nabla_\mu A^\alpha = e^\alpha_\nu \nabla_\mu A^\nu$$

or its equivalent

$$\nabla_\mu e^\alpha_\nu = \partial_\mu e^\alpha_\nu + \omega_\mu{}^\alpha{}_\beta e^\beta_\nu - \Gamma^\lambda_{\mu\nu} e^\alpha_\lambda = 0. \quad (102)$$

It is very easy to see that if (102) holds, then

$$\Gamma^\lambda_{[\mu\nu]} = e^\lambda_\alpha D_{[\mu} e_{\nu]}^\alpha \equiv C^\lambda_{\mu\nu}$$

so that the two definitions of torsion coincide. Because the Minkowski tensor $\eta_{\alpha\beta}$ commutes with $M_{\gamma\delta}$ (simply substitute η for M on the right-hand side of (88)) it follows immediately from (98) that the affine connection Γ is at least metric, i.e.

$$\nabla_\rho g_{\mu\nu} = 0.$$

(The only connection which is both metric and symmetric is the Christoffel connection (87)). If ϕ in (99) is allowed to be the vierbein itself, the last term vanishes identically, while the commutator itself is equal to zero. Therefore

$$0 = R^\rho_{\alpha\mu\nu} e^\alpha_\rho - F_{\mu\nu}^{\alpha\beta} e_{\sigma\beta}, \quad \text{or} \\ e_{\sigma\beta} e_{\rho\alpha} F_{\mu\nu}^{\alpha\beta} = R_{\rho\sigma\mu\nu} \quad (103)$$

consequently the covariant commutator may be rewritten solely in terms of $R^\rho_{\sigma\mu\nu}$ and the torsion. From the two Yang-Mills symmetries there is effectively one field strength. The rest of the argument is due entirely

to Kibble.

In order to find a Lagrangian, Kibble (1961) used the Lagrangian of lowest degree in $R^{\rho}_{\sigma\mu\nu}$ which is invariant under both groups; this is just (to within a constant) eR . The equations of motion obtained by variation with respect to e_{μ}^{α} and $\omega_{\mu}^{\alpha\beta}$ (or $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^{\lambda}$, respectively) are simply $R_{\mu\nu} = 0$ and $\Gamma_{[\mu\nu]}^{\lambda} = 0$. The latter implies that Γ is symmetric as well as metric, and therefore it is just the Christoffel connection. Equation (87) and the vanishing of the Ricci tensor $R_{\mu\nu}$ are the fundamental equations of the standard 1916 theory of gravity in the matter-free case.

To sum up: the second chapter began with a derivation of the first and simplest models of locally supersymmetric theories. The relation between these two models was discussed in terms of the first-order and second-order formulations. In the former, the spin-connection is regarded as an independent field, while in the latter it is regarded as a function of the supergravity variables e_{μ}^{α} and χ_{μ} . The fiber bundle techniques introduced to reduce some of the guesswork involved in the construction of locally supersymmetric theories were briefly outlined. A model involving these techniques led very elegantly to the variations of the supergravity fields, but seemed to be internally ambiguous. These ambiguities persisted even at the level of ordinary gravity, and a new scheme was presented which preserved some of the algebraic elegance while eliminating the ambiguities.

In the next chapter, the methods of superfields are brought in to alleviate some of the algebraic difficulties of gauging supersymmetry. First, some early models in 1 and (1+1) dimensions are reviewed. It is then shown how to derive systematically one of these models. Finally, a new coupling is presented which provides an example of the spontaneous breakdown of local supersymmetry.

CHAPTER III

SUPERFIELD SUPERGRAVITY: TWO DIMENSIONS

As an introduction to superfield methods in (1+1) supergravity, the one-dimensional model will be discussed first. This model provides the simplest example of a new superfield which is the key to all further developments, namely a super-vierbein, or "vielbein"*. The components of this super-vierbein are functions of the supergravity variables introduced previously in Chapter II. Historically, a preliminary version of the two-dimensional vielbein (Zumino 1976) preceded the one-dimensional model (Brink, Di Vecchia and Howe, 1976a) but many of the features of this first model were later discarded. Both the one- and two-dimensional models to be presented were originally derived without the use of superfields, but the verification of local invariance (in the latter case) is extremely complicated without superfields. The only non-trivial supergravity models which can be presented in one and two dimensions are those involving matter couplings, due to the nature of the Riemann tensor and spinor dynamics in a low number of dimensions. However, it is just this matter coupling which provides models incorporating spontaneous breakdown of local supersymmetry.

§13. The Locally Supersymmetric Spinning Particle

Let an arbitrary scalar superfield in one dimension be denoted $V(t, \theta)$;

$$V(t, \theta) = \phi(t) + i\theta\lambda(t) \quad (1)$$

*
"viel" (a. and adv.): much, a great deal; numerous; often; (as pref.) multi-, poly-." (The Pocket Oxford German-English Dictionary, eds. M.L. Barker and H. Homeyer, Oxford 1975). The name was apparently coined by M. Gell-Mann (P.K. Townsend, private communication).

In analogy with the four-dimensional case, let the supersymmetry generators S and invariant derivative D be written

$$S = (\partial/\partial\theta) - i\theta(\partial/\partial t) \quad (2a)$$

$$D = (\partial/\partial\theta) + i\theta(\partial/\partial t) \quad (2b)$$

so that

$$\delta\phi = i\varepsilon\lambda \quad (3a)$$

$$\delta\lambda = \varepsilon\dot{\phi} \quad (3b)$$

where the dot indicates differentiation with respect to t . A suitable Lagrangian is given by

$$\hat{L} = -(i/2)\dot{V}DV = \frac{1}{2}\dot{\phi}\lambda + \frac{1}{2}\theta(\dot{\phi}^2 + i\dot{\lambda}\lambda)$$

and thus

$$I = \int dt d\theta \hat{L} = \frac{1}{2} \int dt (\dot{\phi}^2 + i\dot{\lambda}\lambda) \quad (4)$$

is invariant under (3), so long as the parameter ε is constant.

Following a similar procedure to the usual vierbein formulation for coupling scalar fields ϕ to ordinary gravity,

$$\partial_\mu \phi \rightarrow e_\alpha^\mu \partial_\mu \phi$$

$$L(\phi, \partial_\mu \phi) \rightarrow eL(\phi, e_\alpha^\mu \partial_\mu \phi) ,$$

the action (4) may be made invariant under local supersymmetry by the replacements

$$\partial/\partial t \rightarrow E_\alpha^M \partial_M \equiv \nabla_\alpha \quad (5a)$$

$$D \rightarrow E_a^M \partial_M \equiv \nabla_a \quad (5b)$$

$$L(\dot{V}, DV) \rightarrow EL(\nabla_\alpha V, \nabla_a V) \quad (5c)$$

where E_M^A is a superfield vierbein, or vielbein for short, E_A^M its inverse and E its determinant (Brink et. al. 1976a, Zumino 1977). The indices $M = (\mu, m)$ stand for world tensors and world spinors, while $A = (\alpha, a)$ stand for tangent space tensors and spinors, respectively. Note that all indices take only the single value 1 for now, but the

same formalism will be used later in higher numbers of dimensions. The elements E_μ^α and E_m^a are taken as bosonic, while E_μ^a and E_m^α are fermionic. If the underlying manifold, denoted $z = (t, \theta)$ is subjected to the transformation

$$z^M \rightarrow z^M - \xi^M,$$

then this induces in the scalar superfield $V(z)$ the change

$$\delta V(z) \equiv V'(z) - V(z) = \xi^M \partial_M V \quad (6)$$

and in the world vector $E_M^A(z)$ the change

$$\delta E_M^A(z) \equiv E_M^{A'}(z) - E_M^A(z) = \xi^N \partial_N E_M^A + (\partial_M \xi^N) E_N^A. \quad (7)$$

Equations (6) and (7) mirror the corresponding relations in ordinary gravity. The action is to be invariant under the changes (6) and (7). However, there is a further invariance, the Weyl-like transformations

$$\delta E_\alpha^M = \pi E_a^M, \quad \delta E_a^M = 0 \quad (8a)$$

where π is an arbitrary spinor superfield. From (8a) it follows

$$\delta E_M^\alpha = 0 \quad \delta E_M^a = -E_M^\alpha \pi. \quad (8b)$$

Again, there is no problem with indices because α and a take only the value 1. Because

$$E_\alpha^M E_M^a = E_a^M E_M^\alpha = 0$$

it follows from the definition of δE (Arnowitt et al. 1976)

$$\delta E = E E_\alpha^M \delta E_M^\alpha - E E_a^M \delta E_M^a \quad (9a)$$

that

$$\delta E = -E E_a^M E_M^\alpha \pi = 0. \quad (9b)$$

Moreover, under (8),

$$\delta I = -(i/2) \int E \pi \nabla_a V \nabla_a V d\theta dt \quad (10)$$

which vanishes; $\nabla_a \nabla \nabla_a V = -\nabla_a \nabla \nabla_a V$, so $\nabla_a \nabla \nabla_a V = 0$. To work out both the explicit form of the action and the new transformations for the matter fields, it is necessary to use an Ansatz for the vielbein. The simplest is

$$E_M^\alpha = \Lambda \bar{E}_M^\alpha; \quad E_M^a = \Lambda^{\frac{1}{2}} \bar{E}_M^a \quad (11)$$

where $\bar{E}_M^A$ is the flat-space vielbein

$$\bar{E}_M^A = \begin{pmatrix} 1 & i\theta \\ 0 & 1 \end{pmatrix}; \quad \bar{E}_A^M = \begin{pmatrix} 1 & -i\theta \\ 0 & 1 \end{pmatrix} \quad (12)$$

i.e., $\bar{E}_\alpha^M \partial_M = \partial/\partial t$ and $\bar{E}_a^M \partial_M = D$, while Λ is a scalar superfield,

$$\Lambda = e + i\theta\mu. \quad (13)$$

To preserve the form (11) for the Ansatz, the combined Weyl and coordinate transformations must take the form

$$\begin{aligned} (\delta_C + \delta_W) E_M^\alpha &= \bar{E}_M^\alpha \delta\Lambda \\ (\delta_C + \delta_W) E_M^a &= \bar{E}_M^a \delta(\Lambda^{\frac{1}{2}}) \end{aligned} \quad (14)$$

where Λ is to transform as a scalar density of weight 2,

$$\delta\Lambda = \xi^M \partial_M \Lambda + 2(\partial_\mu \xi^\mu) \Lambda - 2(\partial_m \xi^m) \Lambda \quad (15)$$

The parameters ξ^M and π take the form

$$\xi^\mu = a + i\theta\beta; \quad \xi^m = \beta + \frac{1}{2}a\dot{\theta}; \quad \pi = \Lambda^{\frac{1}{2}} \dot{\xi}^m. \quad (16)$$

With (11) as the Ansatz, the new action takes the form

$$I = -(i/2) \int d^2z \bar{\nabla}_a V \bar{\nabla}_\alpha V (\Lambda^{-1}) \quad (17)$$

where $\bar{\nabla}_A = \nabla_A(\bar{E})$. If $\Lambda = e + i\theta\mu$, then

$$\Lambda^{-1} = e^{-1}(1 - ie^{-1}\mu\theta); \quad \text{and}$$

$$I = \frac{1}{2} \int dt (e^{-1} \dot{\phi}^2 + ie^{-1} \dot{\lambda}\lambda + ie^{-2} \dot{\phi}\lambda\mu). \quad (18)$$

As long as e^{-1} behaves as the inverse of the sole vierbein component, the first term is invariant:

$$\frac{1}{2} \int dt e^{-1} \dot{\phi}^2 = \frac{1}{2} \int e^{-1} \dot{\phi} d\phi \rightarrow \frac{1}{2} \int e^{-1} (dt'/dt) (d\phi/dt') d\phi .$$

However, if λ is a spinor, the second integral is not invariant; rather the integral

$$\int \lambda \dot{\lambda} dt = \int d\lambda \lambda$$

is invariant. Consequently $\lambda \dot{\lambda}$ must be a density, and thus

$$\lambda = \sqrt{e} \psi, \quad \psi \text{ a spinor.} \quad (19)$$

Further, because $\lambda^2 = \psi^2 = 0$,

$$e^{-1} \dot{\lambda} \lambda = e^{-\frac{1}{2}} \frac{d(e^{-\frac{1}{2}} \psi)}{dt} \psi = \dot{\psi} \psi . \quad (20)$$

A similar argument holds for the last integral, and μ must have the form

$$\mu = \sqrt{e} \chi, \quad \chi \text{ a spinor;}$$

in which case

$$I = \frac{1}{2} \int dt e^{-1} (\dot{\phi}^2 + i e \dot{\psi} \psi + i \dot{\phi} \psi \chi) . \quad (21)$$

The transformation law for Λ gives

$$\delta e = a \dot{e} + \dot{a} e + i \beta \sqrt{e} \chi \quad (22a)$$

$$\delta(\sqrt{e} \chi) = a(\sqrt{e} \chi) + (3/2) \dot{a} \sqrt{e} \chi + 2 \dot{\beta} e + \beta \dot{e} \quad (22b)$$

which may be cast into a clearer form by writing $e^{-\frac{1}{2}} \epsilon$ for β . Then

$$\delta e = a \dot{e} + \dot{a} e + i \epsilon \chi \quad (23a)$$

$$\delta \chi = a \dot{\chi} + \dot{a} \chi + 2 \dot{\epsilon} . \quad (23b)$$

Using the explicit forms (16) for the parameters ξ^M , the matter field transformations are given by

$$\delta \phi = a \dot{\phi} + i \epsilon \psi \quad (24a)$$

$$\delta \psi = a \dot{\psi} + e^{-1} (\epsilon \dot{\phi} - \frac{1}{2} i \epsilon \chi \psi) . \quad (24b)$$

The parameter a describes a one-dimensional coordinate transformation $t \rightarrow t - a(t)$, while ϵ is the usual supersymmetry parameter made local; $\epsilon = \epsilon(t)$. Under the coordinate group, e and χ are vectors, while ϕ and ψ are scalars. The dynamical significance of the new fields e and χ is restricted to the role of Lagrange multipliers. That is, the variations with respect to e and χ imply $\dot{\phi}^2 = 0$ and $\dot{\phi}\psi = 0$, which are the Noether currents of the Lagrangian (4) associated with translations and supersymmetry respectively. These currents are just the one-dimensional energy tensor and supercurrent. For example, from the flat-space action and associated transformations (3), the variation δI is just

$$\delta I = (i/2) \int dt \epsilon d(\dot{\phi}\psi)/dt = \int dt \epsilon \dot{J}.$$

An alternative (non-superspace) approach to the spinning particle would be to start from (4b), add the term $\chi J = (i/2)\dot{\phi}\chi\psi$, and incorporate e at the necessary places. However, the transformations for the fields which left this Lagrangian invariant would still have to be found by trial and error, and even if the transformations were guessed correctly, a direct proof that $\delta I = 0$ would be lacking. For example, if the two-dimensional analogs of this Lagrangian and these transformation laws were used in a new model, the resulting action would not be invariant. It is worthwhile to examine such a model very briefly. The correct model will be found - twice - via superfield methods.

§14. A First Look at the Spinning String

It is not difficult to follow the programme outlined at the end of the last section. The new variables $(\phi, \psi, e_\mu^\alpha, \chi_\mu)$ should transform as the two-dimensional analogs of (23) and (24) (Deser and Zumino 1976b, Brink et al. 1976b, Zumino 1977)

$$\delta e_{\mu}^{\alpha} = a \cdot \partial e_{\mu}^{\alpha} + e_{\lambda}^{\alpha} \partial_{\mu} a^{\lambda} + i \bar{\epsilon} \gamma^{\alpha} \chi_{\mu} \quad (25a)$$

$$\delta \chi_{\mu} = a \cdot \partial \chi_{\mu} + \chi_{\lambda} \partial_{\mu} a^{\lambda} + 2 D_{\mu} \epsilon \quad (25b)$$

$$\delta \phi = a \cdot \partial \phi + i \bar{\epsilon} \psi \quad (25c)$$

$$\delta \psi = a \cdot \partial \psi + (\partial_{\mu} \phi - (i/2) \bar{\chi}_{\mu} \psi) \gamma^{\mu} \epsilon \quad (25d)$$

where now (μ, α) run from 0 to 1, and in two dimensions

$$\begin{aligned} D_{\mu} \epsilon &= \partial_{\mu} \epsilon + (1/8) \omega_{\mu}^{\alpha\beta} [\gamma_{\alpha}, \gamma_{\beta}] \epsilon \\ &= \partial_{\mu} \epsilon - \frac{1}{2} \omega_{\mu} \gamma^5 \epsilon \end{aligned} \quad (26)$$

because the two-dimensional gamma matrices satisfy the identity

$$\gamma^{\mu} \gamma^{\nu} = g^{\mu\nu} + \epsilon^{-1} \epsilon^{\mu\nu} \gamma^5 \quad (27)$$

and where, owing to the Abelian nature of the Lorentz group in two dimensions,

$$\omega_{\mu}^{\alpha\beta} = \omega_{\mu} \epsilon^{\alpha\beta}; \quad \omega_{\mu} = -\frac{1}{2} \omega_{\mu}^{\alpha\beta} \epsilon_{\alpha\beta} \quad (28)$$

The connection ω_{μ} and its transformation law will be discussed shortly; for now it is enough that $\delta \omega_{\mu} = \partial_{\mu} \ell$, where ℓ is the Lorentz parameter. A trial Lagrangian takes the form

$$I_{\text{trial}} = \int L_{\text{trial}} d^2x \quad (29)$$

$$= \int (e/2) (g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + i \bar{\psi} \not{\partial} \psi + c \bar{\chi}^{\mu} J_{\mu}) d^2x$$

where c is some constant to be determined, and J_{μ} is the supercurrent associated with the flat-space spinning string Lagrangian

$$\begin{aligned} L &= \frac{1}{2} \int \{ (\partial_{\mu} \phi)^2 + i \bar{\psi} \not{\partial} \psi \} d^2x; \\ \delta L &= - \int \bar{\epsilon} \not{\partial}^{\mu} J_{\mu} d^2x \end{aligned}$$

where the flat-space variations of ϕ and ψ are given by

$$\delta\phi = i\bar{\psi}\epsilon ; \quad \delta\psi = (\not{\partial}\phi)\epsilon \quad \text{and thus}$$

$$J_\mu = i(\not{\partial}\phi)\gamma_\mu\epsilon \quad (30)$$

The spinor term $i\bar{\psi}\not{\partial}\psi$ in (29) appears to be non-covariant, but note

$$\bar{\psi}\not{\partial}\psi = \bar{\psi}\not{\partial}\psi$$

because $\bar{\psi}\gamma_\mu\gamma^5\psi = 0$. Let L_0 equal the Lagrangian (27) for $c = 0$; that is, $L_{\text{trial}} = L_0 + (ec/2)\bar{\chi}^\mu J_\mu$, and let $\bar{\delta}I_{\text{trial}}$ denote those parts of δI_{trial} which involve $\partial_\mu\epsilon$. If δI_{trial} is to vanish, then it must be that $\bar{\delta}I_{\text{trial}} = 0$. But

$$\begin{aligned} \bar{\delta}I_{\text{trial}} &= \int (e\bar{\delta}L_0 + ec(\partial_\mu\bar{\epsilon})J^\mu + (ec/2)\bar{\chi}_\mu\bar{\delta}J^\mu)d^2x \\ &= \int (e(\partial_\mu\bar{\epsilon})J^\mu + ec(\partial_\mu\bar{\epsilon})J^\mu + (ec/2)\bar{\chi}_\mu\bar{\delta}J^\mu)d^2x. \end{aligned}$$

Choosing $c = -1$, the first two terms vanish. Then

$$\begin{aligned} \bar{\delta}I_{\text{trial}} &= \int (e/2)\bar{\chi}_\mu\gamma^\nu\gamma^\mu\bar{\psi}\partial_\nu\epsilon \, d^2x \\ &= - \int (e/4)\bar{\psi}\psi\bar{\chi}_\mu\gamma^\nu\gamma^\mu\partial_\nu\epsilon \, d^2x \\ &= - \int (e/16)\bar{\delta}(\bar{\psi}\psi\bar{\chi}_\mu\gamma^\nu\gamma^\mu\chi_\nu)d^2x \end{aligned} \quad (31)$$

so that the trial action is not invariant. On the other hand,

$$\bar{\delta}(L_{\text{trial}} + (e/16)\bar{\psi}\psi\bar{\chi}_\mu\gamma^\nu\gamma^\mu\chi_\nu) = 0$$

so that the final Lagrangian should read

$$L = (e/2)(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + i\bar{\psi}\not{\partial}\psi - i\bar{\chi}_\mu\gamma^\nu\gamma^\mu\psi\partial_\nu\phi + (1/8)\bar{\psi}\psi\bar{\chi}_\mu\gamma^\nu\gamma^\mu\chi_\nu) . \quad (32)$$

It now remains to be shown that this action is invariant under the transformations (25). The proof is extremely lengthy, and will not be given in this section. Instead, the same Lagrangian (32) with transformations (25) had been derived via an elegant superspace approach (Howe, 1977), which

allows a direct proof of the invariance. It should be noted, however, that $\delta I = 0$ only if the torsion equation

$$C_{\mu\nu}^{\alpha} = (i/2) \bar{\chi}_{\mu} \gamma^{\alpha} \chi_{\nu} \equiv D_{[\mu} e_{\nu]}^{\alpha} \quad (33)$$

holds. The reason for this is due to the variation of χ_{μ} , which involves ω_{μ} . Even though the spin-connection does not explicitly appear in L , it surfaces in δL . The only way that these terms can vanish is if $\omega_{\mu}^{\alpha\beta}$ is given by the solution of (33), which is in two dimensions

$$\omega_{\mu\alpha\beta} = e_{\mu}^{\gamma} e_{[\alpha}^{\rho} e_{\beta]}^{\sigma} \partial_{\sigma} e_{\rho\gamma} + (i/2) \bar{\chi}_{\mu} \gamma_{[\alpha} \chi_{\beta]} \quad (34a)$$

so that

$$\omega_{\mu}^{\alpha\beta} = -\frac{1}{2} \omega_{\mu}^{\alpha\beta} \epsilon_{\alpha\beta} = e^{-1} e_{\mu}^{\gamma} \epsilon^{\rho\sigma} \partial_{\rho} e_{\sigma\gamma} + (i/2) \bar{\chi}_{\mu} \gamma^5 \gamma \cdot \chi \quad (34b)$$

This situation still obtains in the superspace approach, although for a slightly different reason. Also, it will become apparent (cf. (72) below) that the equation of motion for ψ following from (32) would not be co-variant except for ω_{μ} as given by (34). For future reference, note that under local supersymmetry

$$\delta\omega_{\mu} = ie^{-1} \epsilon^{\rho\sigma} \bar{\epsilon} \gamma_{\mu} D_{\rho} \chi_{\sigma} \quad (35)$$

and hence $\delta\omega_{\mu}^{\alpha\beta}$ in two dimensions does not involve the derivative of the spinor parameter ϵ ; $\bar{\delta}\omega_{\mu}^{\alpha\beta} = 0$ (compare the four-dimensional equivalent, (2.31)).

§15 A Superspace Approach to the Spinning String

§15.1 An Ansatz for the (1+1) Vielbein

Given a flat-space scalar superfield Lagrangian, it is very simple to couple in supergravity via the replacements (5). Before any physics emerges, however, it is necessary to know the functional dependence of the vielbein

on the familiar supergravity variables. Moreover, the transformation laws for the vielbein should reproduce the transformations for the various supergravity fields which serve as coefficients of the Grassmann-Taylor expansion of E_M^A . Finally, even though E_M^A and its transformation are at hand, some rigorous algebra is usually demanded in proving the entire scheme is consistent. Following on a first attempt by Zumino (1976) to establish the vielbein and its transformation, Howe (1977) succeeded in finding an Ansatz for E_M^A and constructed a Lagrangian manifestly invariant under a group of transformations closely related to those postulated by Zumino, and reducing to them upon the elimination of an auxiliary field. Instead of (7), let

$$\delta E_M^A = \xi^N \partial_N E_M^A + (\partial_M \xi^N) E_N^A - E_M^B L_B^A \quad (36)$$

where the structure of the matrix L_B^A is determined by the geometry of the z^M manifold. In order that global supersymmetry be recovered in the special case that the parameters ξ^M are constant, the matrices L_B^A must take the form (Zumino, 1976)

$$L_B^A = m(z) \begin{pmatrix} \epsilon_\beta^\alpha & 0 \\ 0 & \frac{1}{2}(\gamma_5)_b^a \end{pmatrix} \quad (37)$$

where $m(z)$ is some scalar superfield. That is, the Bose and Fermi fields of the theory are transformed according to one and the same local Lorentz transformation. The expansions for $m(z)$ and $\xi^M(z)$ are determined (as in the one-dimensional case) by the requirements that the components E_m^A are unchanged. The Ansatz for E_M^A themselves are (Howe, 1977)

$$E_\mu^\alpha = e_\mu^\alpha + i\bar{\theta}\gamma^\alpha \chi_\mu \quad (38a)$$

$$E_\mu^a = \frac{1}{2}\chi_\mu^a - \frac{1}{2}\omega_\mu(\gamma^5\theta)^a \quad (38b)$$

$$E_m^\alpha = -i(\bar{\theta}\gamma^\alpha)_m \quad (38c)$$

$$E_m^a = \delta_m^a \quad (38d)$$

Ultimately the inverse supervierbein E_A^M will be required. These components are given by

$$E_\alpha^\mu = e_\alpha^\mu + (i/2)\bar{\chi}_\alpha \gamma^\mu \theta - (1/8)\bar{\theta}\theta\bar{\chi}_\alpha \gamma^\nu \gamma^\mu \chi_\nu \quad (39a)$$

$$E_\alpha^m = -\frac{1}{2}\chi_\alpha^m - \frac{1}{4}i\bar{\chi}_\alpha \gamma^\nu \theta \chi_\nu^m + \frac{1}{2}\omega_\alpha(\gamma^5\theta)^m - (i/8)\bar{\theta}\theta(\gamma^5\gamma^\mu\chi_\alpha)^m\omega_\mu - (1/16)\bar{\theta}\theta\bar{\chi}_\alpha \gamma^\nu \gamma^\mu \chi_\nu \chi_\mu^m \quad (39b)$$

$$E_a^\mu = i(\bar{\theta}\gamma^\mu)_a + \frac{1}{4}(\bar{\chi}_\lambda \gamma^\mu \gamma^\lambda)_a \bar{\theta}\theta \quad (39c)$$

$$E_a^m = \delta_a^m - \frac{1}{2}i(\bar{\theta}\gamma^\mu)_a \chi_\mu^m + \frac{1}{4}i\bar{\theta}\theta(\gamma^5\gamma^\mu)^m{}_a \omega_\mu - (1/8)\bar{\theta}\theta(\bar{\chi}_\lambda \gamma^\mu \gamma^\lambda)_a \chi_\mu^m \quad (39d)$$

To find $m(z)$ and ξ^M , it is easiest to expand them:

$$m = -\ell - i\bar{\mu}\theta + \frac{1}{2}\bar{\theta}\theta n$$

$$\xi^\mu = a^\mu - i\bar{\beta}^\mu\theta + \frac{1}{2}\bar{\theta}\theta g^\mu$$

$$\xi^m = \epsilon^m + i(b\theta)^m + \frac{1}{2}\bar{\theta}\theta \pi^m$$

From the condition $\delta E_m^a = 0$ follow three equations, and similarly with δE_m^α . These six relations suffice to determine $(\mu, \pi, \beta, b, g, n)$ as functions of the supergravity fields and the parameters ℓ, a^μ , and ϵ^m .

One finds

$$m = -\ell + i\bar{\theta}\gamma^\mu \epsilon \omega_\mu + \frac{1}{4}\bar{\theta}\theta\bar{\epsilon}\gamma^\lambda \gamma^\nu \chi_\lambda \omega_\nu \quad (40a)$$

$$\xi^\mu = a^\mu - i\bar{\epsilon}\gamma^\mu \theta + \frac{1}{4}\bar{\theta}\theta\bar{\epsilon}\gamma^\lambda \gamma^\mu \chi_\lambda \quad (40b)$$

$$\xi^m = \epsilon^m - \frac{1}{2}\ell(\gamma^5\theta)^m + \frac{1}{4}\bar{\theta}\theta i\omega_\mu(\gamma^\mu\gamma^5\epsilon)^m - (1/8)\bar{\theta}\theta\bar{\epsilon}\gamma^\rho \gamma^\mu \chi_\rho \chi_\mu^m + \frac{1}{2}(i\bar{\epsilon}\gamma^\nu \theta)\chi_\nu^m \quad (40c)$$

With these parameters, the transformations (36) lead to the following variations in the supergravity fields:

$$\delta e_\mu^\alpha = a \cdot \partial e_\mu^\alpha + (\partial_\mu a^\lambda) e_\lambda^\alpha + \ell \epsilon_\beta^\alpha e_\mu^\beta + i\bar{\epsilon}\gamma^\alpha \chi_\mu \quad (41)$$

$$\delta\chi_\mu = a \cdot \partial\chi_\mu + (\partial_\mu a^\lambda)\chi_\lambda + \frac{1}{2}\ell\gamma^5\chi_\mu + 2D_\mu\epsilon \quad (42)$$

$$\delta\omega_\mu = a \cdot \partial\omega_\mu + (\partial_\mu a^\lambda)\omega_\lambda + \partial_\mu\ell \quad (43)$$

(compare (25)). Immediately a difficulty of the kind previously encountered in the Chamseddine-West approach arises. According to (43), ω_μ is inert under supersymmetry. If the dynamical content of the superspace approach is the same as that of the spinning string, then $\delta\omega_\mu$ is given by (35) and is not obviously zero. If (43) is to be consistent, then apparently the covariant curl of the Rarita-Schwinger field must vanish:

$$\epsilon^{\lambda\nu} D_\lambda \chi_\nu = 0. \quad (44)$$

This condition (44) imposes a further constraint on the parameter ϵ .

Under supersymmetry, (44) changes as follows:

$$\delta(\epsilon^{\lambda\nu} D_\lambda \chi_\nu) = 2\epsilon^{\lambda\nu} D_\lambda D_\nu \epsilon = eR\gamma^5\epsilon \quad (45)$$

where R is the two-dimensional Ricci scalar;

$$R = -\frac{1}{2}e^{-1}\epsilon_{\alpha\beta}\epsilon^{\mu\nu}R_{\mu\nu}^{\alpha\beta} \quad (46)$$

$$R_{\mu\nu}^{\alpha\beta} = \epsilon^{\alpha\beta}\partial_{[\mu}\omega_{\nu]}$$

and (45) should vanish. Rather than restrict ϵ , it is more in keeping with the trivial geometry of two dimensions to require the vanishing of the Ricci scalar. The restrictions (44) and $R = 0$ may be lifted by the suitable inclusion of an auxiliary scalar field (Howe, 1978) and this will be outlined in §16.2.

§15.2 The Matter Fields

From the spinning particle example, a scalar superfield $V(z)$ should transform as in (6);

$$\delta V = \xi^M \partial_M V$$

with ξ^M as given by (40). In two dimensions, an arbitrary scalar superfield may be expanded as

$$V = \phi + i\bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta F \quad (47)$$

so that using the parameters (40),

$$\delta\phi = i\bar{\epsilon}\psi + a \cdot \partial\phi \quad (48a)$$

$$\delta\psi = (\not{a}\phi)\epsilon - (i/2)\gamma^\rho \bar{\epsilon} \chi_\rho \psi - i\epsilon F + a \cdot \partial\psi + \frac{1}{2}\not{a}\gamma^5\psi \quad (48b)$$

$$\begin{aligned} \delta F = & -\bar{\epsilon} \not{a}\psi - (i/2)\bar{\epsilon}\gamma^\rho \chi_\rho F + \frac{1}{2}\bar{\epsilon}\gamma^\rho \gamma^\mu \chi_\rho \partial_\mu \phi \\ & - \frac{1}{2}i\bar{\epsilon}\gamma^\rho \gamma^\mu \chi_\rho \chi_\mu \psi + a \cdot \partial F \end{aligned} \quad (48c)$$

(again compare (25)). Earlier, in §14, there was only one scalar field, and it would have been very difficult to extrapolate the transformation law (48) from the spinning particle, whose corresponding superfield expansion does not contain such a field. However, if F is set equal to zero, the resulting transformations are just those found previously. It will turn out that F is an auxiliary field, (if the Lagrangian for the fields (ϕ, ψ) corresponds to the spinning string) and its presence is crucial whenever additional matter coupling is introduced. For example, the Fayet model could not have been constructed without the auxiliary field H associated with the massless vector multiplet.

§15.3 Constructing a Lagrangian for the Spinning String

The easiest way to construct the desired Lagrangian is to find a flat-space version and use (5). The simplest candidate for a flat-space kinetic term is

$$\overline{D}VDV = \{-\overline{\psi}\psi + 2i\overline{\psi}\theta F + 2\overline{\theta}\gamma^\mu\psi\partial_\mu\phi + i\overline{\theta}\theta\overline{\psi}\not{\partial}\psi + \overline{\theta}\theta(\partial_\mu\phi)^2 + \overline{\theta}\theta F^2\} \quad (49)$$

where

$$D = (\partial/\partial\overline{\theta}) + i\not{\partial}\theta. \quad \text{Then}$$

$$\frac{1}{4} \int \overline{D}VDV d^4z = \frac{1}{4} \int \{(\partial_\mu\phi)^2 + i\overline{\psi}\not{\partial}\psi + F^2\} d^2x \quad (50)$$

is an invariant under the global transformations

$$\begin{aligned} \delta\phi &= i\varepsilon\overline{\psi} \\ \delta\psi &= (\not{\partial}\phi)\varepsilon - i\varepsilon F \\ \delta F &= -\varepsilon\not{\partial}\psi \end{aligned} \quad (51)$$

which are just (48) with $e_\mu^\alpha = \delta_\mu^\alpha$ and $\chi_\mu = 0$. In place of the operator D , the covariant $\nabla_a = E_a^M \partial_M$ will be used:

$$\begin{aligned} \nabla_a V &= i(\overline{\theta}\not{\partial}\phi)_a - (\overline{\theta}\gamma^\mu)_a \overline{\theta}\partial_\mu\psi + \frac{1}{4}\overline{\theta}\theta(\overline{\chi}_\lambda\gamma^\mu\gamma^\lambda)_a \partial_\mu\phi \\ &\quad - i\overline{\psi}_a - i\overline{\theta}_a F + \frac{1}{2}(\overline{\theta}\gamma^\mu)_a \overline{\psi}\chi_\mu + \frac{1}{2}(\overline{\theta}\gamma^\mu)_a \overline{\theta}\chi_\mu F \\ &\quad + \frac{1}{4}\overline{\theta}\theta(\overline{\psi}\gamma^5\gamma^\mu\omega_\mu)_a - (i/8)\overline{\theta}\theta(\overline{\chi}_\lambda\gamma^\mu\gamma^\lambda)_a \overline{\psi}\chi_\mu \end{aligned} \quad (52)$$

so that

$$\begin{aligned} \eta^{ab}\nabla_a V \nabla_b V &= \overline{\nabla}V \nabla V \\ &= -\overline{\psi}\psi - i\overline{\theta}\gamma^\rho\psi\overline{\psi}\chi_\rho + 2\overline{\theta}\gamma^\mu\psi\partial_\mu\phi + 2\overline{\psi}\theta iF \\ &\quad + \overline{\theta}\theta\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + i\overline{\psi}\not{\partial}\psi - ig^{\mu\nu}\overline{\psi}\chi_\mu\partial_\nu\phi \\ &\quad - (i/2)\overline{\chi}_\lambda\gamma^\mu\gamma^\lambda\psi\partial_\mu\phi - (i/2)\overline{\psi}\gamma^\mu\chi_\mu F + F^2 \\ &\quad + (1/8)\overline{\chi}_\lambda\gamma^\mu\gamma^\lambda\chi_\mu\overline{\psi}\psi + (1/8)g^{\mu\nu}\overline{\psi}\psi\overline{\chi}_\mu\chi_\nu\} \end{aligned} \quad (53)$$

Finally, the determinant of E_M^A must be found. A useful stratagem* is to calculate $\det E_A^M = (\det E_M^A)^{-1}$, for

$$\det E_A^M = (\det E_\alpha^\mu)(\det E_m^a) = \det E_\alpha^\mu$$

if the Ansätze (38) are used. Consequently,

$$E = (\det E_\alpha^\mu)^{-1} \quad (54)$$

but from (38a) it is obvious that

$$E_\alpha^\mu = (e_\mu^\alpha - \frac{1}{2}i\bar{\chi}_\mu\gamma^\alpha\theta)^{-1} \quad (55)$$

so that

$$\begin{aligned} E &= e \det(\delta_\alpha^\beta - (i/2)\bar{\chi}_\alpha\gamma^\beta\theta) \\ &= e + (ie/2)\bar{\theta}\gamma\cdot\chi - \bar{\theta}\theta(\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu/8) \end{aligned} \quad (55)$$

As a check on (55), it must be that

$$\int E d^4z = (\text{invariant}).$$

In fact, under (42), performing an integration by parts,

$$\delta \int E d^4z = \int \epsilon^{\mu\nu}\bar{\epsilon}\gamma^5 D_\mu\chi_\nu d^2x$$

which vanishes if (44) is satisfied. The final action is thus

$$I = \frac{1}{4} \int E \bar{V} V V V d^4z = \frac{1}{2} \int d^2x \{L\}$$

where

$$\begin{aligned} L &= e \{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + i \bar{\psi} \not{\partial} \psi + F^2 \\ &\quad - i \bar{\chi}_\lambda \gamma^\mu \gamma^\lambda \psi \partial_\mu \phi + (1/8) \bar{\chi}_\mu \gamma^\nu \gamma^\mu \chi_\nu \bar{\psi} \psi \} \end{aligned} \quad (56)$$

(compare (32)). Except for the inclusion of the auxiliary field F , this L is the same as that found earlier.

* I wish to thank my advisor Dr. P.W. Higgs for suggesting this to me.

§15.4 Proof of Invariance. Comparison with the Previous Lagrangian

First, it will be shown that $\hat{E}L = E\bar{V}VVV$ transforms by a divergence under the group (36). It is simplest to consider (36) as the sum of two pieces:

$$\delta_1 E_M^A = L_B^A E_M^B \quad (57a)$$

$$\delta_2 E_M^A = \xi_N^N \partial_N E_M^A + (\partial_M \xi_N^N) E_N^A \quad (57b)$$

The first piece, corresponding to a local Lorentz rotation, obviously leaves $\bar{V}VVV$ invariant. Further

$$\begin{aligned} \delta_1 E &= E E_A^M \delta E_M^A (-)^m = E E_A^M L_B^A E_M^B (-)^m \\ &= E E_A^M E_M^B L_B^A (-)^a = E \text{Tr } L = 0 \end{aligned}$$

where $m = 0$ if M is a Bose index, and $m = 1$ if M is a Fermi index. (Recall that $\text{Tr } M_{AB} = (-)^a M_{AA}$). The second invariance demands only a little more work. The variation of E is given by (Arnowitt et al. 1975)

$$\begin{aligned} \delta_2 E &= E(-)^a E_A^M \xi_N^N \partial_N E_M^A + E E_A^M (-)^a (\partial_M \xi_N^N) E_N^A \\ &= E \{ (-)^a E_A^M \xi_N^N \partial_N E_M^A + (-)^n \delta_N^M \partial_M \xi_N^N \} \end{aligned} \quad (58)$$

But

$$\partial_N (E \xi_N^N) = E(-)^{a+n} E_A^M \xi_N^N \partial_N E_M^A + E \partial_N \xi_N^N \quad (59)$$

Therefore

$$\delta_2 E = (-)^n \partial_N (E \xi_N^N) \quad (60)$$

and $\int E d^4z$ is an invariant. For an arbitrary function $F(V)$ of a scalar superfield V ,

$$\begin{aligned} \delta_2 (E f(V)) &= (\delta E) f(V) + E \xi_N^N \partial_N f(V) \\ &= (-)^n \partial_N (E \xi_N^N f(V)). \end{aligned} \quad (61)$$

Hence any scalar function of V when multiplied by E transforms as a scalar density and $\int E f(V) d^4z$ is an invariant with respect to (57). Consequently the action (56) is invariant under the transformations (41) - (43) and (48), if (44) is satisfied.

It remains to be shown that the action is invariant if the auxiliary field F is eliminated, and the transformations (48) changed to (25). As usual, when terms involving a non-dynamical field are dropped, the invariance continues to hold only modulo an equation of motion. The procedure which will now be followed is the reverse of that used to find the form of $\delta \omega_\mu^{\alpha\beta}$ in §10.

The variation of the action under (48) and (41) - (43) may be written

$$\begin{aligned} \delta I = & \frac{1}{2} \int d^2x \left\{ \frac{\delta L}{\delta e_\mu^\alpha} \delta e_\mu^\alpha + \frac{\delta L}{\delta \chi_\mu} \delta \chi_\mu \right. \\ & \left. + \frac{\delta L}{\delta \psi} \delta \psi + \frac{\delta L}{\delta \phi} \delta \phi + \frac{\delta L}{\delta F} \delta F \right\}. \end{aligned} \quad (62)$$

According to (61), this vanishes. Therefore

$$\frac{1}{2} \int d^2x \delta L_0 = - \frac{1}{2} \int \left\{ \frac{\delta L}{\delta F} \right\} \delta F d^2x \quad (63)$$

where L_0 is the Lagrangian in (32). In particular, (63) holds when $F = 0$; in which case

$$\left. \frac{1}{2} \int d^2x \delta L_0 \right|_{F=0} = - \left. \int (F \delta F) \right|_{F=0} d^2x = 0.$$

Note that the variations included in $\delta L_0|_{F=0}$ reproduce (25). Therefore the trial Lagrangian (32) is invariant under the trial transformations, which was to be shown. For consistency, it is also necessary that the constraint $F = 0$ is compatible with supersymmetry, i.e.

$$\left. \delta F \right|_{F=0} = \delta(0) = 0. \quad (64)$$

From (48c), it follows

$$\left. \delta F \right|_{F=0} = -\bar{\epsilon} \not{D} \psi + \frac{1}{2} \bar{\epsilon} \gamma^\nu \gamma^\mu \chi_\nu \partial_\mu \phi - \frac{1}{4} i \bar{\epsilon} \gamma^\nu \gamma^\mu \chi_\nu \bar{\chi}_\mu \psi. \quad (65)$$

Fortunately, this variation for F is simply $(i\bar{\epsilon}/e)$ times the equation of motion for ψ , so that if ψ is on mass-shell, $\delta F|_{F=0}$ vanishes. That is,

$$\begin{aligned} (i\bar{\epsilon}/e) \left\{ \frac{\delta L}{\delta \bar{\psi}} \right\} \Big|_{F=0} &= (i\bar{\epsilon}/e) \left\{ \frac{\delta L_0}{\delta \bar{\psi}} \right\} \\ &= (i\bar{\epsilon}/e) \left\{ \frac{\partial L_0}{\partial \bar{\psi}} - \partial_\mu \frac{\partial L_0}{\partial \bar{\psi}_{,\mu}} \right\} = \delta F \Big|_{F=0} \end{aligned} \quad (66)$$

$L_0(\psi)$ is given by

$$L_0(\psi) = \frac{1}{2} \int e (i\bar{\psi} \not{\partial} \psi - i\bar{\chi}_\mu \gamma^\nu \gamma^\mu \psi \partial_\nu \phi + (1/8) \bar{\psi} \psi \bar{\chi}_\mu \gamma^\nu \gamma^\mu \chi_\nu) d^2 x \quad (67)$$

and the equation of motion obtained for $\bar{\psi}$ is

$$\begin{aligned} e \{ i \not{\partial} \psi - (i/2) \gamma^\lambda \gamma^\mu \chi_\lambda \partial_\mu \phi + (1/8) \bar{\chi}_\lambda \gamma^\mu \gamma^\lambda \chi_\mu \psi \\ + (i/2) (e^{-1} \partial_\mu e) \gamma^\mu \psi + (i/2) (\partial \cdot e_\alpha) \gamma^\alpha \psi \} = 0 \end{aligned} \quad (68)$$

The last two terms may be rewritten as $(-i/2) \gamma^\mu \omega_\mu(e) \gamma^5 \psi$. From the definition of $\omega_\mu(e)$ and the identity $\gamma_\mu \gamma^5 = e \epsilon_{\lambda\mu} \gamma^\lambda$, it follows that

$$\begin{aligned} \gamma^\mu \omega_\mu(e) \gamma^5 \psi &= -(e^\mu_\alpha \partial_\lambda e^\alpha_\nu) (\epsilon^{\lambda\nu} \epsilon_{\mu\rho}) \gamma^\rho \psi \\ &= -(e^{-1} \partial_\mu e) \gamma^\mu \psi - (\partial \cdot e_\alpha) \gamma^\alpha \psi \end{aligned} \quad (69)$$

The other part $\omega_\mu(\chi)$ of the connection comes from the quadratic χ term. Observe that

$$(1/8) \bar{\chi}_\lambda \gamma^\mu \gamma^\lambda \chi_\mu \psi = (-i/2) \gamma^\mu \omega_\mu(\chi) \gamma^5 \psi - \frac{1}{4} \gamma^\rho \gamma^\mu \chi_\rho \bar{\chi}_\mu \psi \quad (70)$$

This is a simple consequence of the identities

$$\begin{aligned} \gamma^\mu \gamma^5 \bar{\chi}_\mu \gamma^5 \gamma^\nu \chi_\nu &= \gamma^\mu \bar{\chi}_\mu \gamma^\nu \chi_\nu \\ \bar{\chi}_\mu \gamma_\nu \gamma^\lambda \gamma^\mu \chi_\lambda &= -\epsilon^{\lambda\mu} \epsilon_{\sigma\rho} \bar{\chi}_\mu \gamma^\rho \chi_\lambda \end{aligned}$$

The equation of motion for ψ thus becomes

$$e\{i\bar{\psi}\psi - (i/2)\gamma^\lambda\gamma^\mu\chi_\lambda\partial_\mu\phi - \frac{1}{4}\gamma^\rho\gamma^\mu\chi_\rho\bar{\chi}_\mu\psi\} = 0 ; \quad (71)$$

and indeed $(i\bar{\epsilon}/e)$ times (71) is just $\delta F|_{F=0}$. Thus the trial action (32) with transformations (25) constitute a consistent theory. In the last section of this chapter, a similar procedure will be employed to demonstrate the locally supersymmetric invariance of a Lagrangian suitable for investigating the generalised Higgs mechanism, in which a Goldstone fermion is absorbed by the Rarita-Schwinger field.

§16. Deriving the Form of the (1+1) Vielbein

§16.1 The failure of minimal coupling

Given the explicit form (38) of the Ansatz, it is not difficult to construct a two-dimensional model of the generalised Higgs mechanism, (§17). To extend these results, it is first necessary to obtain the form of E_M^A in four dimensions. In this section, the known form (38) of the two-dimensional vielbein is derived via the introduction of a "covariant derivative" modelled on the familiar gauge-theoretic prescription. Due to the unconventional features of the gauge group, it will be necessary to modify the prescription somewhat. It is hoped that this same method will yield the four-dimensional E_M^A as well. As a bonus, it will become obvious why a locally supersymmetric theory must be a supergravity theory, i.e. why the Rarita-Schwinger field introduced to gauge supersymmetry must necessarily be accompanied by the gravitational potential e_μ^α . A précis of the argument may be given as follows: Suppose only the one gauge field χ_μ were required for the local symmetry, and that its transformation were Abelian;

$$\delta\chi_\mu = 2\partial_\mu \epsilon^\dagger$$

The global variation of χ_μ is then zero, and apparently there is no need to introduce its supersymmetric partner. However, it soon becomes evident that the gamma matrices must be x-dependent, so that their anti-commutator must be the metric tensor $g_{\mu\nu}(x)$. Consequently the vierbein e_μ^α must be introduced, and all ordinary derivatives must be made covariant with respect to local Lorentz rotations. Then $\delta\chi_\mu = 2D_\mu \epsilon$, and the global variation of χ_μ is $-\omega_\mu \gamma^5 \epsilon$. This procedure will

[†] It would be more elegant to start from $\delta\chi_\mu = \partial_\mu \epsilon$, but this convention disagrees with that of the previous section and hinders comparison.

now be described in detail (Derbes, 1978b).

Consider the globally invariant Lagrangian introduced previously,

$$I = \frac{1}{4} \int d^2x d^2\theta \bar{D}V D V = \frac{1}{2} \int d^2x \{ (\partial^\mu \phi)^2 + i \bar{\psi} \not{\partial} \psi + F^2 \} \quad (49)$$

where as before V is given by (47) and the global variations of the matter fields are

$$\begin{aligned} \delta \phi &= i \epsilon \bar{\psi} \\ \delta \psi &= -i \epsilon F + (\not{\partial} \phi) \epsilon \\ \delta F &= -\epsilon \not{\partial} \psi \end{aligned} \quad (51)$$

Although the action (49) is invariant under (51), it changes if the parameters are local; the variations of terms such as $\partial_\mu \phi$ will involve derivatives on ϵ . That is, $\bar{\delta}(\partial_\mu \phi) \neq 0$. (Recall $\bar{\delta}f$ means: that part of δf containing $\partial_\mu \epsilon$; i.e. $\delta = \bar{\delta} + \delta_{\text{global}}$). For a "covariant" derivative ∇_μ it is a necessary condition (but alas not a sufficient one) that the variation of $\nabla_\mu f$ for an arbitrary quantity does not contain these $\partial_\mu \epsilon$ terms. That is, the necessary condition becomes $\bar{\delta}(\nabla_\mu f) = 0$. The standard prescription suggests the replacement

$$\partial_\mu \rightarrow \nabla_\mu \stackrel{?}{=} \partial_\mu - \frac{1}{2} \bar{\chi}_\mu S \quad (72)$$

where $\delta \chi_\mu = 2 \partial_\mu \epsilon + \dots$, the dots indicating terms which may become necessary. Then

$$\nabla_\mu \phi \stackrel{?}{=} \partial_\mu \phi - (i/2) \bar{\chi}_\mu \psi \quad (73a)$$

$$\nabla_\mu \psi \stackrel{?}{=} \partial_\mu \psi + (i/2) \chi_\mu F - \frac{1}{2} \gamma^\rho \chi_\mu \partial_\rho \phi \quad (73b)$$

$$\nabla_\mu F \stackrel{?}{=} \partial_\mu F + \frac{1}{2} \bar{\chi}_\mu \not{\partial} \psi \quad (73c)$$

The necessary condition $\bar{\delta}(\nabla_\mu f) = 0$ holds for only the first component of V , and not for the other two fields:

$$\bar{\delta}(\nabla_\mu \psi) = -\frac{1}{2}\gamma^\rho \chi_\mu \bar{\delta}(\partial_\rho \phi) = -(i/2)\gamma^\rho \chi_\mu (\partial_\rho \bar{\epsilon})\psi$$

$$\bar{\delta}(\nabla_\mu F) = \frac{1}{2}\bar{\chi}_\mu \bar{\delta}(\not{\partial}\psi) = -(i/2)\bar{\chi}_\mu (\not{\partial}\epsilon)F + \frac{1}{2}\bar{\chi}_\mu \gamma^\nu \not{\partial}\phi \partial_\nu \epsilon.$$

The standard procedure fails, evidently because there is an additional derivative present in both $\delta\psi$ and δF . The obvious manoeuvre to try next is the substitution of $\nabla_\rho \phi$ for $\partial_\rho \phi$ in $\nabla_\mu \psi$. Then

$$\begin{aligned} \nabla_\mu \psi &= \partial_\mu \psi + (i/2)\chi_\mu F - \frac{1}{2}\gamma^\rho \chi_\mu \nabla_\rho \phi \\ &= \partial_\mu \psi + (i/2)\chi_\mu F - \frac{1}{2}\gamma^\rho \chi_\mu \partial_\rho \phi + (i/4)\gamma^\rho \chi_\mu \bar{\chi}_\rho \psi. \end{aligned} \quad (74)$$

Unfortunately even now $\bar{\delta}(\nabla_\mu \psi) \neq 0$. Instead,

$$\bar{\delta}(\nabla_\mu \psi) = (i/2)\gamma^\rho \partial_\mu \epsilon \bar{\chi}_\rho \psi.$$

To render $\bar{\delta}(\nabla_\mu \psi) = 0$, there are apparently only two remedies available:

- i) introduce new gauge fields;
- ii) modify the original transformation laws.

The second approach is the more economical. Under local supersymmetry, then, ψ is required to transform in an essentially different way than under global supersymmetry:

$$\psi \rightarrow \psi + \delta' \psi$$

$$\delta' \psi = \delta\psi + \delta_G \psi = \delta\psi - (i/2)\gamma^\mu \epsilon \bar{\chi}_\mu \psi \quad (75)$$

and now in fact

$$\bar{\delta}'(\nabla_\mu \psi) = 0$$

as desired. The transformation δ_G is a new kind of "gauge transformation" on the matter fields. Thus the covariant derivative determines the transformation laws for the matter fields, rather than vice-versa.

This approach is strange, a little ugly, and successful. Note however that in the case of a global symmetry, χ_μ may be set equal to zero, and obviously $\delta'\psi$ reduces to $\delta\psi$.

Although not a dynamical field, F must be treated in the same way; in order to determine $\delta_G F$, it is necessary to consider $\nabla_\mu F$. By analogy with the field ψ ,

$$\begin{aligned} \nabla_\mu F &= \partial_\mu F + \frac{1}{2} \bar{\chi}_\mu \not{\partial} \psi \\ &= \partial_\mu F + \frac{1}{2} \bar{\chi}_\mu \not{\partial} \psi - \frac{1}{4} \bar{\chi}_\mu \gamma^\lambda \gamma^\nu \chi_\lambda \partial_\nu \phi \\ &\quad + (i/4) \bar{\chi}_\mu \gamma^\lambda \chi F + (i/8) \bar{\chi}_\mu \gamma^\lambda \gamma^\nu \chi_\lambda \bar{\chi}_\nu \psi . \end{aligned} \quad (76)$$

Indeed, $\bar{\delta}'(\nabla_\mu F) = 0$, so long as one adopts

$$\delta_G F = -(i/2) \bar{\epsilon} \gamma^\lambda \chi F + \frac{1}{2} \bar{\epsilon} \gamma^\lambda \gamma^\nu \chi_\lambda \partial_\nu \phi - (i/4) \bar{\epsilon} \gamma^\lambda \gamma^\nu \chi_\lambda \bar{\chi}_\nu \psi . \quad (77)$$

The new transformations may be written compactly as

$$\begin{aligned} \delta' V &= [\bar{\epsilon} \Sigma, V] \quad \text{where} \\ \Sigma &= S + (i/2) \gamma^\mu \bar{\theta} \chi_\mu S - (1/8) \bar{\theta} \theta \gamma^\lambda \gamma^\mu \chi_\lambda \bar{\chi}_\mu S \\ &= S + S_G . \end{aligned} \quad (78)$$

The substitution of a new operator Σ in place of S demonstrates what may be termed "the failure of minimal coupling": the passage from global to local supersymmetry is made not by the mere replacement of a local group parameter in place of a constant in the matter fields' transformations, but requires in addition an entirely new term which is at least linear in the gauge field. The resulting covariant derivatives are thus of quadratic and higher order in the gauge field. The origin of this new type of "gauge transformation" seems to be the mixing in global supersymmetry of fields with derivatives of other fields. After all,

minimal coupling works fine for ϕ , whose global variation does not involve derivatives. Such a mixing does not occur in Yang-Mills theories with an internal symmetry group.

To return to the action (49): what is required, according to (5), is not the Bose covariant derivative ∇_μ but the Fermi covariant derivative ∇_a (the local equivalent of D). Because

$$D_m = \frac{\partial}{\partial \theta^m} + i(\gamma^\nu \theta)_m \partial_\nu$$

it seems reasonable to guess that

$$\nabla_a = \frac{\partial}{\partial \theta^a} + i(\gamma^\nu \theta)_a \nabla_\nu \quad (79)$$

$$\text{where } \nabla_\nu = \partial_\nu - \frac{1}{2} \bar{\chi}_\nu \Sigma \quad (80)$$

Unfortunately, the operator (79) is not covariant, despite appearances to the contrary. Although the necessary condition has been satisfied, the sufficient condition is more stringent and requires that ∇_a transform according to the usual Ehresmann law (with allowances for Fermi statistics)

$$\delta \nabla_a \equiv (\delta E_a^M) \partial_M = + \bar{\epsilon} \{ \Sigma, \nabla_a \} \nabla_a \bar{\epsilon} \Sigma \quad (81)$$

One readily discovers that the Abelian law $\delta \chi_\nu = 2\partial_\nu \epsilon$ fails completely. Were it to succeed, the anticommutator in (81) would vanish. This does not occur. For example, equating in (81) the coefficients of ∂_μ which are linear in θ leads to the requirement

$$\delta(i\gamma^\mu \theta)_a = (\gamma^\nu \theta)_a \bar{\epsilon} \gamma^\mu \chi_\nu \quad (82)$$

which clearly implies $\gamma^\mu = \gamma^\mu(x)$; and hence

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x)$$

Thus the introduction of local supersymmetry forces the introduction of a

curved x-space, and new gauge fields for the local Lorentz group. This well-known fact is brought out very persuasively in the present approach. Now, somehow the Lorentz gauge fields must be incorporated into V_a and Σ .

In order to determine the gravitational parts of the covariant derivatives, it is instructive to consider a globally supersymmetric theory made invariant only with respect to local Lorentz rotations. Although V is a scalar, the quantity $\partial_\mu V$ is not covariant, because it contains the non-covariant piece $\partial_\mu \psi$. However, the quantity $D_\mu V$ is covariant, where

$$D_\mu V = \partial_\mu V + \omega_\mu [J, V] \quad (83)$$

$$\text{where} \quad J = M - \frac{1}{2} \bar{\theta} \gamma^5 \partial / \partial \bar{\theta} \quad (84)$$

The operator J is composed of the abstract Lorentz operator $M = -\frac{1}{2} \epsilon_{\alpha\beta} M^{\alpha\beta}$ (cf. §12.3; such an operator in this context was first introduced by Gates (1978b)) and an "internal" piece $-\frac{1}{2} \bar{\theta} \gamma^5 \partial / \partial \bar{\theta}$. The operator M acts only on the external indices of V , while the internal operator takes care of the non-scalar component of V ;

$$[M, V] = 0 \quad (85)$$

$$[-\frac{1}{2} \bar{\theta} \gamma^5 \partial / \partial \bar{\theta}, V] = -(i/2) \bar{\theta} \gamma^5 \psi; \text{ and hence} \quad (86)$$

$$D_\mu V = \partial_\mu \phi + i \bar{\theta} D_\mu \psi + \frac{1}{2} \bar{\theta} \theta \partial_\mu F \quad (87)$$

is covariant. It will turn out that in four dimensions, the operator $J_{\alpha\beta}$ is just the straightforward generalisation of J ; $J_{\alpha\beta} = M_{\alpha\beta} + \frac{1}{2} \bar{\theta} \sigma_{\alpha\beta} \partial / \partial \bar{\theta}$. This operator has been independently introduced into global supersymmetry by Salam and Strathdee (1978). As long as the action of D_μ is restricted to scalar quantities (such as V or $\bar{\epsilon} S$), the M part of D_μ may be dropped. However, the generator S must also be modified, to

$$\begin{aligned}
 S' &= (\partial/\partial\bar{\theta}) - i\gamma^\nu\theta D_\nu \\
 &= S - i\gamma^\nu\theta\omega_\nu M - (i/4)\bar{\theta}\theta\omega_\nu(\gamma^\nu\gamma^5)\partial/\partial\bar{\theta}
 \end{aligned} \tag{88}$$

where the Fierz formula has been invoked.

It is now an easy matter to write down the form of Σ ; merely replace S by S' in (78). This introduction of gravity is facilitated by noting that the last term of (78) may be written as

$$- \frac{1}{4}\gamma^\nu\theta\bar{\chi}_\nu\gamma^\mu\theta\bar{\chi}_\mu S, \quad \text{so that formally}$$

$$\Sigma = (1 - \frac{1}{2}i\gamma^\nu\theta\bar{\chi}_\nu)^{-1}S \tag{89}$$

and hence the full local supersymmetry generator is

$$\Sigma = (1 - \frac{1}{2}i\gamma^\nu\theta\bar{\chi}_\nu)^{-1}S' \tag{89}$$

Similarly the Fermi covariant derivative for a scalar superfield is

$$\nabla_a = \frac{\partial}{\partial\bar{\theta}} + i(\gamma^\nu\theta)_a (D_\nu - \frac{1}{2}\bar{\chi}_\nu\Sigma) \equiv E_a^M \partial_M \tag{90}$$

With these identifications, one finds

$$E_a^\mu = i(\gamma^\mu\theta)_a - \frac{1}{4}\bar{\theta}\theta(\gamma^\lambda\gamma^\mu\chi_\lambda)_a \tag{91a}$$

$$E_a^m = \delta_a^m - \frac{1}{2}(i\gamma^\mu\theta)_a\bar{\chi}_\mu^m + \frac{1}{4}\bar{\theta}\theta[i(\gamma^\mu\gamma^5)_a^m\omega_\mu + \frac{1}{2}(\gamma^\nu\gamma^\mu\chi_\nu)_a\bar{\chi}_\mu^m] \tag{91b}$$

which is in agreement with (38) (note that Howe (1977) writes

$\partial_m = \partial/\partial\theta^m$; here $\partial_m = \partial/\partial\bar{\theta}^m$). Further, if one assumes

$$\nabla_\alpha \equiv E_\alpha^M \partial_M = e_\alpha^\mu \nabla_\mu \tag{92}$$

then

$$E_\alpha^\mu = e_\alpha^\mu - \frac{1}{2}i\bar{\theta}\gamma^\mu\chi_\alpha - (1/8)\bar{\theta}\theta\bar{\chi}_\alpha\gamma^\nu\gamma^\mu\chi_\nu \tag{93a}$$

$$\begin{aligned}
 E_\alpha^m &= -\frac{1}{2}\bar{\chi}_\alpha^m - \frac{1}{4}i\bar{\chi}_\alpha\gamma^\nu\theta\bar{\chi}_\nu^m - \frac{1}{2}\omega_\alpha(\bar{\theta}\gamma^5)^m \\
 &\quad - (1/8)\bar{\theta}\theta[i(\bar{\chi}_\alpha\gamma^\mu\gamma^5)^m\omega_\mu + \frac{1}{2}\bar{\chi}_\alpha\gamma^\nu\gamma^\mu\chi_\nu\bar{\chi}_\mu^m]
 \end{aligned} \tag{93b}$$

again in agreement with (38). Using the explicit forms of E_A^M and Σ , one may verify that both the Bose

$$(\delta E_\alpha^M) \partial_M = [\bar{\epsilon} \Sigma, \nabla_\alpha]$$

and Fermi (81) Ehresmann formulae hold. This requires an incredible amount of algebra. It is far easier to work out the corresponding transformation laws for the inverse E_M^A , and verify that these hold. First, the Ehresmann formulae may be written as

$$\begin{aligned} (\delta E_A^M) \partial_M &= [\bar{\epsilon} \Sigma, E_A^M \partial_M] \\ &= [\xi^N \partial_N + mM, E_A^L \partial_L] \\ &= (\xi^N \partial_N E_A^M - E_A^N \partial_N \xi^M + mL_A^B E_B^M) \partial_M; \end{aligned} \quad (94)$$

and all that remains to be done is work out the form of the parameters.

The form of L_A^B is given by the known action of M (cf. 12.3);

$$\begin{aligned} [M, \phi] &= 0 \\ [M, \psi] &= -\frac{1}{2} \gamma^5 \psi \\ [M, A_\alpha] &= \epsilon_\alpha^\beta A_\beta \end{aligned}$$

so that

$$L_A^B = \begin{pmatrix} \epsilon_\alpha^\beta & 0 \\ 0 & -\frac{1}{2} (\gamma^5)_a^b \end{pmatrix}.$$

Note that there is no need to consider the term $-(E_A^M \partial_M m) M$ as the transformation is an operator statement defined only on scalar superfields, and $[M, V] = 0$. The transformations (94) follow naturally from the Ehresmann law and the approach to gravity described in §12. Comparing coefficients in the known form of $\bar{\epsilon} \Sigma$ leads to the following expressions for the parameters

$$\xi^\mu = -i\bar{\epsilon}\gamma^\mu\theta + \frac{1}{4}\bar{\epsilon}\gamma^\nu\gamma^\mu\chi_\nu\bar{\theta}\theta \quad (95a)$$

$$\xi^m = \bar{\epsilon}^m + \frac{1}{4}i\bar{\epsilon}\gamma^\nu\theta\chi_\nu^m - \frac{1}{4}\bar{\theta}\theta[i\omega_\mu(\bar{\epsilon}\gamma^\mu\gamma^5)^m + \frac{1}{4}\bar{\epsilon}\gamma^\mu\gamma^\nu\chi_\mu\chi_\nu^m] \quad (95b)$$

$$m = -i\bar{\epsilon}\gamma^\mu\theta\omega_\mu + \frac{1}{4}\bar{\epsilon}\gamma^\nu\gamma^\mu\chi_\nu\omega_\mu\bar{\theta}\theta$$

which are just the set (40) (ignoring the non-supersymmetric parts).

From the form of E_A^M , it is straightforward to work out E_M^A :

$$E_\mu^\alpha = e_\mu^\alpha + i\bar{\theta}\gamma^\alpha\chi_\mu \quad (96a)$$

$$E_\mu^\alpha = \frac{1}{2}\chi_\mu^\alpha - \frac{1}{2}\omega_\mu(\gamma^5\theta)^\alpha \quad (96b)$$

$$E_m^\alpha = -i(\bar{\theta}\gamma^\alpha)_m \quad (96c)$$

$$E_m^a = \delta_m^a \quad (96d)$$

which agree with the original Ansatz, (37). From the law (94) and the orthogonality relations follows the law for δE_M^A ;

$$\delta E_M^A = \xi^N \partial_N E_M^A + (\partial_M \xi^N) E_N^A - m L_B^A E_M^B. \quad (97)$$

This transformation is identical to that postulated by Howe (1977). The proof that the transformations for E_M^A are consistent obviously suffices to demonstrate the validity of the Ehresmann formulae. The direct calculation has also been performed and (owing to its length) will not be given here; the important result is that, so long as the constraints $D_{[\mu}\chi_{\nu]} = 0$, $R = 0$ hold, the transformations are consistent.

By the way, the peculiar form (89) of Σ may be obtained without any consideration of matter fields on the basis of two assumptions. First, on general grounds, ∇_μ must have the form

$$\nabla_\mu = D_\mu - \frac{1}{2}\bar{\chi}_\mu \Pi \quad (98)$$

where Π , the generator of local supersymmetry, is to be determined.

$$\xi^\mu = -i\bar{\epsilon}\gamma^\mu\theta + \frac{1}{4}\bar{\epsilon}\gamma^\nu\gamma^\mu\chi_\nu\bar{\theta}\theta \quad (95a)$$

$$\xi^m = \bar{\epsilon}^m + \frac{1}{2}i\bar{\epsilon}\gamma^\nu\theta\chi_\nu^m - \frac{1}{4}\bar{\theta}\theta[i\omega_\mu(\bar{\epsilon}\gamma^\mu\gamma^5)^m + \frac{1}{2}\bar{\epsilon}\gamma^\mu\gamma^\nu\chi_\mu\bar{\chi}_\nu^m] \quad (95b)$$

$$m = -i\bar{\epsilon}\gamma^\mu\theta\omega_\mu + \frac{1}{4}\bar{\epsilon}\gamma^\nu\gamma^\mu\chi_\nu\omega_\mu\bar{\theta}\theta$$

which are just the set (40) (ignoring the non-supersymmetric parts).

From the form of E_A^M , it is straightforward to work out E_M^A :

$$E_\mu^\alpha = e_\mu^\alpha + i\bar{\theta}\gamma^\alpha\chi_\mu \quad (96a)$$

$$E_\mu^\alpha = \frac{1}{2}\bar{\chi}_\mu^\alpha + \frac{1}{2}\omega_\mu^\alpha(\bar{\theta}\gamma^5)^a \quad (96b)$$

$$E_m^\alpha = -i(\gamma^\alpha\theta)_m \quad (96c)$$

$$E_m^a = \delta_m^a \quad (96d)$$

which agree with the original Ansatz, (37), From the law (94) and the orthogonality relations follows the law for δE_M^A ;

$$\delta E_M^A = \xi^N \partial_N E_M^A + (\partial_M \xi^N) E_N^A - m L_B^A E_M^B \quad (97)$$

This transformation is identical to that postulated by Howe (1977). The proof that the transformations for E_M^A are consistent obviously suffices to demonstrate the validity of the Ehresmann formulae. The direct calculation has also been performed and (owing to its length) will not be given here; the important result is that, so long as the constraints $D_{[\mu}\chi_{\nu]} = 0$, $R = 0$ hold, the transformations are consistent.

By the way, the peculiar form (89) of Σ may be obtained without any consideration of matter fields on the basis of two assumptions. First, on general grounds, ∇_μ must have the form

$$\nabla_\mu = D_\mu - \frac{1}{2}\bar{\chi}_\mu \Pi \quad (98)$$

where Π , the generator of local supersymmetry, is to be determined.

Second, it should be that Π is just the covariant form of S , i.e.

$$\Pi = \partial/\partial\bar{\theta} - i\gamma^\mu\bar{\theta}\nabla_\mu = \partial/\partial\bar{\theta} - i\gamma^\mu\bar{\theta}(D_\mu - \frac{1}{2}\bar{\chi}_\mu\Pi), \text{ whence}$$

$$\Pi = (1 - \frac{1}{2}i\gamma^\mu\bar{\chi}_\mu)^{-1}S' \equiv \Sigma.$$

Regrettably, this simple argument fails upon the introduction of an auxiliary scalar field.

§16.2 The Ehresmann Formulae and an Auxiliary Field

Thus far, the consistency of the superfield approach in two-dimensional supergravity is dependent on the constraints

$$\epsilon^{\lambda\mu} D_\lambda \chi_\mu = 0 \quad (99a)$$

$$\epsilon^{\lambda\mu} \partial_\mu \omega_\nu = 0 \quad (99b)$$

These constraints imply that the gauge fields χ_μ and ω_μ are "pure gauge"*, i.e. there are some parameters ϵ and ℓ such that

$$\chi_\mu = D_\mu \epsilon \quad (100a)$$

$$\omega_\mu = \partial_\mu \ell \quad (100b)$$

Moreover, these constraints are the two-dimensional analogues of the four-dimensional supergravity equations of motion. As Howe (1978) has shown, it is possible to remove these constraints by the suitable introduction of new terms involving an auxiliary scalar field. The method to be presented here used to find these terms differs from that of Howe, but the results are the same.

In general, the presence of a constraint in supersymmetry suggests that auxiliary fields have been eliminated. For example, in §15.4, it was shown that F could be dropped consistently from the spinning string Lagrangian without violating the invariance, because the transformation $\delta F|_{F=0} = 0$ was consistent with $F = 0$. In fact, the constraint following from $\delta F|_{F=0} = 0$ was shown to be the equation of motion for ψ , the supersymmetric partner of F . This circumstance may well be a general feature of supersymmetric theories: elimination of auxiliary fields

* I wish to thank Dr. P.K. Townsend for a valuable discussion on this point.

seems to invariably require that the equations of motion of their supersymmetric partner should hold. Here, the apparent equation of motion for χ_μ must be assumed to be satisfied, so perhaps the addition of a new field as a coefficient of the next higher power of θ can remove the necessity of imposing (99).

From the point of view expressed in §16.1, the constraints (99) arise as results of the Ehresmann formulae. In the Fermi case, δV_a , all goes well until one examines the quadratic (in θ) coefficient of $\partial/\partial\theta^m$, and requires

$$\frac{1}{4}i(\gamma^\mu\gamma^5)_a{}^m \delta\omega_\mu = \frac{1}{4}(\gamma^\mu\gamma^\nu\epsilon)_a{}^m D_{[\mu}\bar{\chi}_{\nu]}^m$$

which is impossible, unless (99a) is assumed to hold. Similarly, in the Bose case δV_a , problems arise at the linear level. Examining the linear coefficient of $\partial/\partial\theta^m$ leads to the statement

$$-\frac{1}{2}\bar{\theta}\gamma^5\delta\omega_\mu = \frac{1}{2}i\bar{\epsilon}\gamma^\nu\theta D_{[\mu}\bar{\chi}_{\nu]} \quad (101)$$

which again cannot be true unless the covariant curl of χ_μ vanishes. Much the same difficulty occurs for the quadratic (in θ) coefficient of $\partial/\partial\theta^m$; here in fact it is also necessary to require $R = 0$. Nevertheless, the variations implied by the Ehresmann formulae have the right structure to agree with $\delta\omega_\mu$ as given by (35); the problem is simply one of Dirac algebra. Were there some way of modifying (101) to the form

$$\begin{aligned} \delta(-\frac{1}{2}\bar{\theta}\gamma^5\omega_\mu + \frac{1}{2}\bar{\theta}\delta V_\mu) &= \frac{1}{2}i\bar{\epsilon}\gamma^\nu\theta D_{[\mu}\bar{\chi}_{\nu]} + \frac{1}{2}ia\bar{\epsilon}\gamma^5\theta D_{[\mu}\bar{\chi}_{\nu]} \gamma^\nu\gamma^5 \\ &\quad + \frac{1}{2}ib\epsilon\theta D_{[\mu}\bar{\chi}_{\nu]} \gamma^\nu \end{aligned} \quad (102)$$

then for particular choices of V_μ , a and b , it might be possible to obtain consistently

$$-\frac{1}{2}\bar{\theta}\gamma^5(\delta\omega_\mu) + \frac{1}{2}\bar{\theta}\delta V_\mu = -\frac{1}{2}i\bar{\theta}\gamma^5\epsilon^{-1\rho\sigma}\bar{\epsilon}\gamma_\mu D_\rho\chi_\sigma + \dots$$

where the additional terms do not involve the quantity $\bar{\theta}\gamma^5$. The new terms on the right-hand side of (102) are the only ones consistent with the transformation of a vector, while $\bar{\theta}V_\mu$ itself is chosen to take the form

$$\bar{\theta}V_\mu = \bar{\theta}C_\mu - i\bar{\theta}\gamma_\mu A, \quad (103)$$

where C_μ is a vector and A a scalar field. This choice is dictated by the requirement that there will be a complete set $\{1, \gamma^\mu, \gamma^5\}$ of Dirac matrices on the left-hand side of (102). After a Fierz rearrangement, the terms on the right-hand side of (102) become

$$\begin{aligned} \delta(-\frac{1}{2}\bar{\theta}\gamma^5\omega_\mu + \frac{1}{2}\bar{\theta}C_\mu - \frac{1}{2}i\bar{\theta}\gamma_\mu A) &= \frac{1}{4}i\bar{\theta} D_{[\mu}\bar{\chi}_{\nu]}\gamma^\nu\epsilon[1 - (b-a)] \\ &+ \frac{1}{4}i\bar{\theta}\gamma^5 D_{[\mu}\bar{\chi}_{\nu]}\gamma^5\gamma^\nu\epsilon[1 + (b-a)] \\ &+ \frac{1}{4}i\bar{\theta}\gamma^\sigma D_{[\mu}\bar{\chi}_{\nu]} [\gamma_\sigma\gamma^\nu - \gamma^\nu\gamma_\sigma(b+a)] + \dots \end{aligned} \quad (104)$$

In order to obtain the right transformation for ω_μ , it must be that $b - a = 1$. Consequently the variation of C_μ involving $D_{[\mu}\bar{\chi}_{\nu]}$ is zero, and thus there is no need to introduce C_μ at all. Henceforth this vector will be set equal to zero. For the transformation of A , there are apparently two solutions: $b + a = \pm 1$. The choice of -1 indicates $b = 0$, $a = -1$. However, this leads to the matrix coefficient of $\bar{\theta}$ being $\gamma_\mu\gamma^5$, rather than γ_μ . Hence the only allowed solution is $b = 1$, $a = 0$. As a result

$$\delta A = e^{-1}\epsilon^{\mu\nu} \bar{\epsilon}\gamma^5 D_\mu\bar{\chi}_\nu + \dots \quad (105)$$

Viewed in this way, the scalar field A acts as a Fierz partner of ω_μ , and thus a supersymmetric partner of χ_μ ; because $\delta\chi_\mu$ involves ω_μ , it must now also involve A .

The way to incorporate A into the vielbein should now be reasonably clear: in E_A^M , and in the parameters, insert the most general functions

linear in A wherever the spin connection occurs. But somewhere extra terms involving the covariant curl of χ_μ must also be included, so as to obtain the new terms on the right-hand side of (104). Therefore new fermionic pieces will be present besides those linear in A . Bitter experience teaches that it is far easier to work with E_M^A rather than E_A^M , so the programme proceeds as follows:

- (i) Add in the new functions of A and spinor terms, as yet unspecified, as described above, i.e. wherever ω_μ occurs in E_A^M . (Note that E_A^μ are unchanged.)
- (ii) Compute the inverse E_M^A .
- (iii) Work out δE_M^A , according to (97), and require that the entire scheme be consistent. This yields both the form of E_M^A and the parameters.
- (iv) Find E_A^M from the determined form of E_M^A .

According to (i), neither E_α^μ nor E_a^μ change, but

$$E_\alpha^m = -\frac{1}{2}\bar{\chi}_\alpha^m - \frac{1}{4}i\bar{\chi}_\alpha^\nu \gamma^\nu \theta \bar{\chi}_\nu^m - \frac{1}{2}(\bar{\theta}\gamma^5)^m \omega_\alpha - \frac{1}{2}i(\bar{\theta}\gamma_\alpha)^m A \quad (106a)$$

$$+ (1/16)\bar{\chi}_\alpha^\nu \gamma^\nu \chi_\nu^\mu \bar{\chi}_\mu^m \bar{\theta}\theta + (1/8)\bar{\theta}\theta (\bar{\chi}_\alpha^\mu \gamma^\mu \gamma^5)^m \omega_\mu + \frac{1}{4}\bar{\theta}\theta R_\alpha^m$$

$$E_a^m = \delta_a^m + \frac{1}{2}(i\gamma^\mu \theta)_a \bar{\chi}_\mu^m + \frac{1}{4}(\bar{\theta}\theta)(i\gamma^\mu \gamma^5)_a^m \omega_\mu \quad (106b)$$

$$+ \frac{1}{4}p \bar{\theta}\theta \delta_a^m A + (1/8)\bar{\theta}\theta (\gamma^\nu \gamma^\mu \chi_\nu)_a \bar{\chi}_\mu^m$$

where R_α^m is a field to be determined (but certainly involving both $D_{[\mu}\chi_{\nu]}$ and A) and p is a number to be found. These forms lead to the inverse components

$$E_\mu^\alpha = e_\mu^\alpha + i\bar{\theta}\gamma^\alpha \chi_\mu + \frac{1}{2}\bar{\theta}\theta e_\mu^\alpha A \quad (107a)$$

$$E_\mu^a = \frac{1}{2}\bar{\chi}_\mu^a + \frac{1}{2}\omega_\mu (\bar{\theta}\gamma^5)^a + \frac{1}{2}(i\bar{\theta}\gamma_\mu)^a A - \frac{1}{4}(2 + \frac{1}{2}p)\bar{\chi}_\mu^a \bar{\theta}\theta A - \frac{1}{4}\bar{\theta}\theta R_\mu^a \quad (107b)$$

$$E_m^\alpha = -i(\gamma^\alpha \theta)_m \quad (107c)$$

$$E_m^a = \delta_m^a - \frac{1}{4}\bar{\theta}\theta(2+p)\delta_m^a A. \quad (107d)$$

The parameters must be modified in the same way. Because ξ^μ does not contain ω_μ , it is unchanged. However, it should be that ξ^m and m are both modified, to

$$\xi^m = \bar{\epsilon}^m + \frac{1}{2}i\bar{\epsilon}\gamma^\mu\theta\bar{\chi}_\mu^m + \frac{1}{4}\bar{\theta}\theta[i(\bar{\epsilon}\gamma^5\gamma^\lambda)^m\omega_\lambda + d\bar{\epsilon}^mA - \frac{1}{2}\bar{\epsilon}\gamma^\lambda\gamma^\mu\chi_\lambda\bar{\chi}_\mu^m] \quad (108a)$$

$$m = -i\bar{\epsilon}\gamma^\mu\theta\omega_\mu - g\bar{\epsilon}\gamma^5\theta A + \frac{1}{2}\bar{\theta}\theta[\bar{\epsilon}\gamma^5\lambda + ik\bar{\epsilon}\gamma^5\gamma\cdot\chi A + \frac{1}{2}\bar{\epsilon}\gamma^\lambda\gamma^\nu\chi_\lambda\omega_\nu] \quad (108b)$$

where d , k , and g are constants, and the spinor λ is to be determined. It turns out that it is only necessary to consider the transformations of E_M^a in order to fix the unknowns $\{R_\mu, \lambda, p, k, g, d\}$.

(a) δE_m^a . The terms zeroth and linear in θ must vanish. Those zeroth in θ are unaltered and vanish as before. The linear terms involve either A or ω_μ ; the latter vanish. For the former, it is required that

$$(2+p)\bar{\theta}\epsilon\delta_m^a A - (\bar{\epsilon}\gamma^\nu)_m(\gamma_\nu\theta)^a A + d\bar{\theta}\epsilon^a A - g(\gamma^5)_m^a \bar{\epsilon}\gamma^5\theta A = 0. \quad (109)$$

The easiest way to solve this equation is to multiply both sides by the arbitrary spinors $\bar{\eta}^a$ and α^m , and use the Fierz formula. This procedure leads to three separate equations:

$$\bar{\theta}\alpha\bar{\eta}\epsilon\{\frac{1}{2}(2+p) + d + \frac{1}{2}g - 1\} = 0 \quad (110a)$$

$$\bar{\theta}\gamma^5\alpha\bar{\eta}\gamma^5\epsilon\{\frac{1}{2}(2+p) + \frac{1}{2}g - 1\} = 0 \quad (110b)$$

$$\bar{\theta}\gamma^\sigma\alpha\bar{\eta}\gamma_\sigma\epsilon\{\frac{1}{2}(2+p) - \frac{1}{2}g\} = 0. \quad (110c)$$

From these it follows $d = 0$, $g = 1$, $p = -1$. The quadratic terms may be expressed as

$$\begin{aligned}
 -\frac{1}{4}\bar{\theta}\theta\delta_m^a \delta A &= -\frac{1}{4}i\delta_m^a \bar{A}\epsilon\gamma^\lambda\bar{\theta}\theta\chi_\lambda - \frac{1}{4}i\bar{\theta}\theta A(\bar{\epsilon}\gamma^\nu)_m\chi_\nu^a \\
 &\quad -\frac{1}{4}i\bar{\theta}\theta(\bar{\epsilon}\gamma^\nu)_m R_\nu^a + \frac{1}{4}i\bar{\theta}_m \bar{\epsilon}\gamma^\nu\gamma^\lambda\chi_\nu(\gamma_\lambda\theta)^a A \\
 &\quad -\frac{1}{4}(\gamma^5)_m^a \bar{\theta}\theta\bar{\epsilon}\gamma^5\lambda - \frac{1}{4}ik(\gamma^5)_m^a \bar{\theta}\theta \bar{\epsilon}\gamma^5\gamma\cdot\chi A .
 \end{aligned}$$

The terms in A vanish for the choice $k = -\frac{1}{2}$. Assuming that this choice may be made, the resulting equation may be expressed (again multiplying by the spinors $\bar{\eta}, \alpha$)

$$\bar{\eta}\alpha\delta A = \bar{\alpha}\gamma^5\eta\bar{\epsilon}\gamma^5\lambda + \frac{1}{2}i\bar{\epsilon}\gamma\cdot R\bar{\alpha}\eta + \frac{1}{2}i\bar{\epsilon}\gamma^\nu\gamma^5 R_\nu\bar{\alpha}\gamma^5\eta + \frac{1}{2}i\bar{\epsilon}\gamma^\nu\gamma^\sigma R_\nu\bar{\alpha}\gamma_\sigma\eta .$$

The terms which are the coefficients of $\bar{\eta}\gamma^5\alpha$ and $\bar{\eta}\gamma_\sigma\alpha$ must vanish. If one recalls the two-dimensional identity $\gamma^\nu\gamma_\sigma\gamma_\nu = 0$, it follows

$$R_\nu = -i\gamma_\nu\lambda$$

so that

$$\delta A = \bar{\epsilon}\lambda. \quad (111)$$

Obviously, from (105), part of λ is determined. However, the remaining pieces are obtained from δE_μ^a .

(b) δE_μ^a . The unknown function λ is completely determined by the terms linear in θ . However, the zeroth term contains something new: the transformation for χ_μ is modified from (42) to

$$\delta\chi_\mu = 2D_\mu\epsilon - iA\gamma_\mu\epsilon \quad (112)$$

Owing to the dependence of ω_μ on χ_μ (from (34b)), the transformation $\delta\omega_\mu$ is expected to go over to

$$\delta\omega_\mu = ie^{-1}\epsilon^{\rho\sigma}\bar{\epsilon}\gamma_\mu D_\rho\chi_\sigma - \frac{1}{2}\bar{\epsilon}\gamma^5\gamma^\lambda\gamma_\mu\chi_\lambda A \quad (113)$$

and this form is borne out by detailed calculation with the linear terms. Multiplying these by $\bar{\eta}^a$, the transformation becomes

$$\begin{aligned}
 & -\bar{\eta}\gamma^5\theta\delta\omega_\mu - i\bar{\eta}\gamma_\mu\theta\delta A - i\bar{\eta}\gamma_\alpha\theta A\delta e_\mu^\alpha \\
 & = -i\bar{\epsilon}\gamma^\nu\theta\bar{n}D_{[\nu}\chi_{\mu]} + \frac{1}{2}\bar{\epsilon}\gamma^\lambda\theta\bar{\eta}\gamma_\mu\chi_\lambda A - (3/2)\bar{\eta}\chi_\mu\bar{\epsilon}\theta A \\
 & \quad + \frac{1}{2}\bar{\epsilon}\gamma^5\theta\bar{\eta}\gamma^5\chi_\mu A + i\bar{\epsilon}\theta\bar{\eta}\gamma_\mu\lambda.
 \end{aligned}$$

The right-hand side becomes upon a Fierz rearrangement a sum of terms involving $\bar{\eta}\gamma_\sigma\theta$, $\bar{\eta}\gamma^5\theta$, and $\bar{\eta}\theta$; these last must vanish. This implies

$$i\bar{\epsilon}\gamma^\nu D_{[\nu}\chi_{\mu]} - \frac{1}{2}\bar{\epsilon}\gamma^\lambda\gamma_\mu\chi_\lambda A + \bar{\epsilon}\chi_\mu A - i\bar{\epsilon}\gamma_\mu\lambda = 0$$

in which case

$$\lambda = -e^{-1}\epsilon^{\rho\sigma}\gamma^5 D_\rho\chi_\sigma - \frac{1}{2}i\gamma\cdot\chi A. \quad (114)$$

The remaining terms give the variation of ω_μ in agreement with (113) and that of A in agreement with (105). The new components E_M^A are finally determined to be

$$\begin{aligned}
 E_\mu^\alpha &= e_\mu^\alpha + i\bar{\theta}\gamma^\alpha\chi_\mu + \frac{1}{2}\bar{\theta}\theta e_\mu^\alpha A \\
 E_\mu^a &= \frac{1}{2}\bar{\chi}_\mu^a + \frac{1}{2}\omega_\mu(\bar{\theta}\gamma^5)^a + \frac{1}{2}(i\bar{\theta}\gamma_\mu)^a A - (3/8)\bar{\chi}_\mu^a A\bar{\theta}\theta - \frac{1}{4}(i\bar{\lambda}\gamma_\mu)^a\bar{\theta}\theta \\
 E_m^\alpha &= -i(\gamma^\alpha\theta)_m \\
 E_m^a &= \delta_m^a - \frac{1}{4}\bar{\theta}\theta\delta_m^a A
 \end{aligned} \quad (115)$$

and thus the inverse components E_A^M are

$$E_\alpha^\mu = e_\alpha^\mu - \frac{1}{2}i\bar{\theta}\gamma^\mu\chi_\alpha - (1/8)\bar{\theta}\theta\bar{\chi}_\alpha\gamma^\nu\gamma^\mu\chi_\nu \quad (116a)$$

$$E_a^\mu = i(\bar{\theta}\gamma^\mu)_a + \frac{1}{4}\bar{\theta}\theta(\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda)_a \quad (116b)$$

$$\begin{aligned}
 E_\alpha^m &= -\frac{1}{2}\bar{\chi}_\alpha^m - \frac{1}{4}i\bar{\chi}_\alpha\gamma^\nu\theta\bar{\chi}_\nu^m - \frac{1}{2}(\bar{\theta}\gamma^5)^m\omega_\alpha - \frac{1}{2}(i\bar{\theta}\gamma_\alpha)^m A \\
 & \quad + (1/16)\bar{\theta}\theta\bar{\chi}_\alpha\gamma^\nu\gamma^\mu\chi_\nu\bar{\chi}_\mu^m + \frac{1}{4}i\bar{\theta}\theta[(\bar{\lambda}\gamma_\alpha)^m + \frac{1}{2}(\bar{\chi}_\alpha\gamma^\mu\gamma^5)^m\omega_\mu] \quad (116c)
 \end{aligned}$$

$$E_a^m = \delta_a^m + \frac{1}{2}i(\gamma^\mu\theta)_a \bar{\chi}_\mu^m + \frac{1}{4}\bar{\theta}\theta[i(\gamma^\mu\gamma^5)_a^m \omega_\mu - \delta_a^m A - \frac{1}{2}i(\gamma^\nu\gamma^\mu\chi_\nu)_a \bar{\chi}_\mu^m] \quad (116d)$$

The new parameters (actually, only m is changed) are

$$\xi^\mu = -i\bar{\epsilon}\gamma^\mu\theta + \frac{1}{4}\bar{\theta}\theta\bar{\epsilon}\gamma^\nu\gamma^\mu\chi_\nu \quad (117a)$$

$$\xi^m = \bar{\epsilon}^m + \frac{1}{2}i\bar{\epsilon}\gamma^\nu\theta\bar{\chi}_\nu^m + \frac{1}{4}\bar{\theta}\theta[i(\bar{\epsilon}\gamma^5\gamma^\lambda)^m \omega_\lambda - \frac{1}{2}\bar{\epsilon}\gamma^\lambda\gamma^\nu\chi_\lambda\bar{\chi}_\nu^m] \quad (117b)$$

$$m = -i\bar{\epsilon}\gamma^\mu\theta\omega_\mu - \bar{\epsilon}\gamma^5\theta A + \frac{1}{4}\bar{\theta}\theta[\bar{\epsilon}\gamma^5\lambda - \frac{1}{2}i\bar{\epsilon}\gamma^5\gamma^\nu\chi_\nu A + \frac{1}{2}\bar{\epsilon}\gamma^\lambda\gamma^\nu\chi_\lambda\omega_\nu] \quad (117c)$$

The new transformations may be summarised as

$$\delta e_\mu^\alpha = i\bar{\epsilon}\gamma^\alpha\chi_\mu \quad (25)$$

$$\delta\chi_\mu = 2D_\mu\epsilon - iA\gamma_\mu\epsilon \quad (112)$$

$$\delta\omega_\mu = ie^{-1}\epsilon^{\rho\sigma}\bar{\epsilon}\gamma_\mu D_\rho\chi_\sigma - \frac{1}{2}\bar{\epsilon}\gamma^5\gamma^\nu\gamma_\mu\chi_\nu A \quad (113)$$

$$\delta A = \bar{\epsilon}\lambda = -e^{-1}\epsilon^{\rho\sigma}\bar{\epsilon}\gamma^5 D_\rho\chi_\sigma - \frac{1}{2}i\bar{\epsilon}\gamma^\nu\chi_\nu A. \quad (111)$$

The only field whose transformation has not been given explicitly is λ .

Its change may be found from the Ehresmann law, but it is easier to use its dependence on the basic fields $\{e, \chi, \omega, A\}$. After some algebra, $\delta\lambda$ may be expressed in the suggestive form

$$\delta\lambda = C\epsilon - \frac{1}{2}i\gamma^\mu\bar{\epsilon}\lambda\chi_\mu + i\gamma^\mu\epsilon\partial_\mu A \quad (118)$$

where the scalar C is given by

$$C = -\frac{1}{2}A^2 - R - \frac{1}{2}i\bar{\chi}\cdot\gamma\lambda + \frac{1}{4}e^{-1}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu A \quad (119)$$

The transformations of λ and A should be compared with those of the scalar superfield V 's components in §15.2; apparently $(A, -i\lambda, C)$ form the components of a scalar superfield T (Howe, 1978)

$$T = A + \bar{\theta}\lambda + \frac{1}{2}\bar{\theta}\theta C \quad (120)$$

in which case the transformation of C should be

$$\delta C = i\bar{\varepsilon}\not{\partial}\lambda - \frac{1}{2}i\bar{\varepsilon}\gamma\cdot\chi C + \frac{1}{2}\bar{\varepsilon}\gamma^\rho\gamma^\mu\chi_\rho\partial_\mu A - \frac{1}{4}\bar{\varepsilon}\gamma^\rho\gamma^\mu\chi_\rho\bar{\chi}_\mu\lambda. \quad (121)$$

The existence of the scalar superfield T allows for the construction of a Lagrangian for the supergravity variables themselves. The answer one obtains is a little disappointing, but it is presented for completeness.

First, note that the determinant of E_M^A changes with the addition of A :

$$\begin{aligned} E &= (\det E_\alpha^\mu)^{-1} (\det E_m^a)^{-1} \\ &= (e + \frac{1}{2}i\bar{\theta}\gamma\cdot\chi - (1/8)\varepsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu\bar{\theta}\theta)(1 - \frac{1}{4}\bar{\theta}\theta A)^{-1} \\ &= e + \frac{1}{2}i\bar{\theta}\gamma\cdot\chi - (1/8)\varepsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu\bar{\theta}\theta + \frac{1}{4}eA\bar{\theta}\theta. \end{aligned} \quad (122)$$

A suitable Lagrangian is given by $-ET$;

$$\begin{aligned} -\int d^2\theta d^2x ET &= -\int d^2x \{C + \frac{1}{2}A^2 + \frac{1}{2}i\bar{\chi}\cdot\gamma\lambda - (1/4)\varepsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu A\} \\ &= \int d^2x \quad = \int d^2x (\varepsilon^{\mu\nu}\omega_{\mu\nu}) \end{aligned} \quad (123)$$

which is a total divergence! Although it is certainly invariant, this Lagrangian is trivial. Perhaps even more remarkable is the utter insensitivity of the spinning string Lagrangian to the inclusion of A . The only change in $\nabla_a V$ is the additional term $-(i/4)\bar{\theta}\theta A\psi$, with the result that $\bar{V}VVV$ increases by the term $\frac{1}{2}\bar{\theta}\theta A\bar{\psi}\psi$. However the only term zeroth in θ in $\bar{V}VVV$ is $-\bar{\psi}\psi$, which when multiplied by the change in E , namely $\frac{1}{4}eA\bar{\theta}\theta$, exactly cancels the other increment. It should also be noted that the field A is something of a "spoiler" as regards the construction of Lagrangians. For example, the mass term $-\frac{1}{2}mEV^2$ may be added to the spinning string Lagrangian only if E does not contain A . However,

if A is present, the Lagrangian $\hat{L}$ includes A only in the term $-\frac{1}{4}m\phi^2 A\bar{\theta}\theta$. Variation of L with respect to A thus implies $\phi = 0$. Consequently, because A enters only linearly in the absence of any dynamics for the supergravity variables themselves, the scalar field A condemns to zero any hapless quantity of which it is the coefficient. In two dimensions, A 's role in the Lagrangian is limited to that of a murderous Lagrange multiplier.

§17. The Higgs Mechanism in (1+1) Supergravity

Once the vielbein (38) is available, the construction of a locally supersymmetric version of the Fayet model is almost effortless (Derbes, 1978c). Recall that in §9 the addition of interaction terms of the form $\xi V + c \exp(\kappa V)$ to a free Lagrangian for the scalar superfield led to spontaneous supersymmetry breakdown. The heart of this model was the coupling of the auxiliary field H to the other scalar fields A and B ; cf. (I.97). By analogy with this model, interaction terms which include similar couplings should be added to the Lagrangian (56), which is that of a scalar superfield "minimally" coupled to supergravity. These new terms will have the general appearance

$$- \xi e \bar{\theta}\theta F - g e \bar{\theta}\theta \phi^2 F$$

so that upon elimination of the field F , an effective potential of the Goldstone-Nambu form results for ϕ . The obvious way to achieve this is to add the Fayet-Higgs terms^{*}

$$\hat{L}_1 = -(2\xi V + (2/3)gV^3)E. \quad (124)$$

* Once again I wish to thank my advisor for suggesting the explicit form $aV + bV^3$ to me.

Note that E does not contain the field A here; that would unfortunately be fatal. (A minor technical note seems in order here. Unlike the string model proper, the components of the scalar superfield V do not carry an internal Minkowski index, so that the addition of functions odd in V is not prohibited. Hitherto it was not necessary to distinguish between the Lagrangian (56) and that of the spinning string.) It may be assumed without loss of generality that the coupling constant g is positive. Dropping terms zeroth and linear in θ , the interaction terms are

$$2\xi VE = \bar{\theta}\theta(e/2)[2\xi F + \xi\bar{\psi}\gamma\cdot\chi - \frac{1}{2}\xi e^{-1}\phi\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu]$$

$$(2/3)gV^3E = \bar{\theta}\theta(e/2)[2g\phi^2F + 2g\bar{\psi}\psi\phi + g\phi^2\bar{\psi}\gamma\cdot\chi - (1/6)e^{-1}g\phi^3\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu] .$$

The new Lagrangian is

$$\hat{L} = \frac{1}{4}E\bar{\nabla}VVV + \hat{L}_1 ;$$

$$\int \hat{L} d^2x d^2\theta = \int L d^2x, \quad \text{where}$$

$$\begin{aligned} L = & (e/2)[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + i\bar{\psi}\not{\partial}\psi + F^2 - i\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda\psi\partial_\mu\phi \\ & + (1/8)\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda\chi_\mu\bar{\psi}\psi - 2\xi F - \xi\bar{\psi}\gamma\cdot\chi - 2g\phi^2F \\ & + \frac{1}{2}\xi e^{-1}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu\phi - 2g\bar{\psi}\psi\phi - g\phi^2\bar{\psi}\gamma\cdot\chi \\ & + (1/6)g\phi^3e^{-1}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu] . \end{aligned} \tag{125}$$

The variation of L with respect to F leads to

$$F = \xi + g\phi^2 . \tag{126}$$

Substituting this value of F back into L gives

$$L = L_0 + L'$$

where L_0 is given by (30), and

$$L' = (e/2)[-(\xi + g\phi^2)^2 - (\xi + g\phi^2)\bar{\psi}\gamma\cdot\chi + 2g\bar{\psi}\psi\phi + \frac{1}{2}\phi(\xi + (1/3)g\phi^2)e^{-1}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu] .$$

As before, manifest invariance under local supersymmetry is lost once F is eliminated. All that is necessary for L to be invariant is that the variation of F be consistent with (126); cf. §15.4. The transformation of F is given by (47c);

$$\begin{aligned}\delta F &= -\bar{\epsilon}\not{\partial}\psi - \frac{1}{2}i\bar{\epsilon}\gamma\cdot\chi F + \frac{1}{2}\bar{\epsilon}\gamma^\rho\gamma^\mu\chi_\rho\partial_\mu\phi - \frac{1}{2}i\bar{\epsilon}\gamma^\rho\gamma^\mu\chi_\rho\bar{\chi}_\mu\psi \\ &= (i\bar{\epsilon}/e)\{\partial L_0/\partial\bar{\psi} - \partial_\mu(\partial L_0/\partial\bar{\psi}_{,\mu})\} - (i/2)\bar{\epsilon}\gamma\cdot\chi F \\ &= (i\bar{\epsilon}/e)\{\delta L_0/\delta\bar{\psi}\} - (i/2)\bar{\epsilon}\gamma\cdot\chi F .\end{aligned}$$

The "reduced" Lagrangian will be invariant if

$$\begin{aligned}\delta F|_{F=\xi+g\phi^2} &= (i\bar{\epsilon}/e)\{\delta L_0/\delta\bar{\psi}\} - (i/2)\bar{\epsilon}\gamma\cdot\chi(\xi + g\phi^2) \\ &= 2g\phi\delta\phi = 2ig\phi\bar{\epsilon}\psi ,\end{aligned}\tag{128}$$

or, what amounts to the same thing, if

$$(i\bar{\epsilon}/e)\{\delta L_0/\delta\bar{\psi}\} - (i/2)\bar{\epsilon}\gamma\cdot\chi(\xi + g\phi^2) - 2ig\phi\bar{\epsilon}\psi = 0 .\tag{129}$$

The last two terms may be identified with $\delta L'/\delta\bar{\psi}$;

$$(i\bar{\epsilon}/e)\{\delta L'/\delta\bar{\psi}\} = (i\bar{\epsilon}/e)\{-\frac{1}{2}(\xi + g\phi^2)\gamma\cdot\chi - 2g\psi\phi\} .$$

The condition for invariance then reduces to

$$0 = (i\bar{\epsilon}/e)\{\delta(L_0 + L')/\delta\bar{\psi}\} = (i\bar{\epsilon}/e)\{\delta L/\delta\bar{\psi}\}\tag{130}$$

which is met if ψ satisfies its equation of motion. As before, salvation comes from F 's supersymmetric partner ψ , once F itself has been eliminated.

Consider now the effective potential for ϕ ;

$$V_{\text{eff}}(\phi) = \frac{1}{2}(\xi + g\phi^2)^2. \quad (131)$$

Analogously to Fayet's (1976) model, there are two separate cases, distinguished by the sign of ξ :

(i) $\xi < 0$: $\langle V_{\text{eff}} \rangle$ is minimised for non-vanishing $\langle \phi \rangle$. Both ϕ and ψ become massive, with $m_\phi = m_\psi = 2\sqrt{-\xi}g$; supersymmetry is unbroken.

(ii) $\xi > 0$: $\langle V_{\text{eff}} \rangle$ is minimised for vanishing $\langle \phi \rangle$. The scalar field becomes massive with $m_\phi = \sqrt{2\xi}g$, while ψ is a would-be Goldstone fermion associated with the spontaneous breakdown of supersymmetry.

It is eliminated by a choice of gauge for the field ψ , but now all Fermi degrees of freedom are eliminated. Moreover, a "cosmological term" - $e\xi^2$ arises.

Independent of the sign of ξ , the vacuum satisfies

$$\begin{aligned} \partial \langle V_{\text{eff}} \rangle / \partial \phi &= 0 = 2g(\xi + g\phi^2)\phi \\ &\cong 2g\langle \phi \rangle \langle (\xi + g\phi^2) \rangle \end{aligned} \quad (132)$$

If ξ is positive, this equation admits only the solution $\langle \phi \rangle = 0$, in which case $V_{\text{eff},\text{min}} = \frac{1}{2}\xi^2$. On the other hand, if ξ is negative, there are two solutions to (132);

$$\langle \phi \rangle = 0 \quad \text{or} \quad \langle \phi \rangle = \pm \sqrt{-\xi/g}. \quad \text{However,}$$

$$V_{\text{eff}}(\langle \phi \rangle = 0) > V_{\text{eff}}(\langle \phi \rangle = \pm \sqrt{-\xi/g}),$$

and the vacuum corresponding to $\langle \phi \rangle = 0$ is unstable. The case $\xi < 0$ is easiest, so it will be considered first.

Let the positive value for $\langle \phi \rangle$ be chosen;

$$\langle \phi \rangle = + \sqrt{-\xi/g}. \quad (133)$$

To write L in terms of fields with vanishing expectation values, let

$$\phi = \sigma + \sqrt{-\xi/g} ; \quad \langle \sigma \rangle = 0 , \quad (134)$$

in which case (recall $F = 2\sqrt{V_{\text{eff}}(\phi)}$)

$$\langle \delta\psi \rangle = -i\langle F \rangle \epsilon = 0 . \quad (135)$$

Rewriting the Lagrangian in terms of σ leads to

$$L_{(i)} = (e/2) [g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - m^2 \sigma^2 + i\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi + \dots] \quad (136)$$

where the dots indicate terms of cubic and higher order in the fields, and

$$m = 2\sqrt{-\xi g}.$$

When the second situation holds, $\xi > 0$, there is no need to translate ϕ .

Then

$$L_{(ii)} = (e/2) [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2g\xi\phi^2 - \xi^2 + i\bar{\psi} \not{\partial} \psi - \xi \bar{\psi} \gamma \cdot \chi + \dots] \quad (137)$$

Now, there is no mass term for ψ , although

$$m_\phi = \sqrt{2\xi g}$$

and there is also a "cosmological term" $-\frac{1}{2}e\xi^2$. Further,

$$\langle \delta\psi \rangle = -i\langle F \rangle \epsilon = -i\xi \epsilon \quad (138)$$

which is characteristic of a Goldstone mode. Recall from §16.2 that the constraints (99) indicated that χ_μ was "pure gauge"; its only degree of freedom is the choice of a parameter α such that $\chi_\mu = D_\mu \alpha$. In view of the coupling of χ_μ to ψ , it should be possible to choose this parameter in such a way that the field ψ disappears entirely; this corresponds to the unitary gauge[†]. The Lagrangian $L_{(ii)}$ becomes in this gauge

†

I am greatly indebted to Dr. W.E. Leithhead for an illuminating discussion on the unitary gauge.

$$L_{(ii),\text{unitary}} = (e/2)[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2g\xi\phi^2 - \xi^2 - g^2\phi^4] \\ + \varepsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu[\xi\phi - (1/3)g\phi^3] \quad (139)$$

In two dimensions, neither e_μ^α nor χ_μ are dynamical fields, but merely Lagrange multipliers. Variation of L with respect to these leads to two new constraints;

$$T^\mu_\alpha = \delta L / \delta e_\mu^\alpha = 0; \quad J^\mu = \delta L / \delta \bar{\chi}_\mu = 0.$$

These quantities are the energy tensor and supercurrent, respectively.

Before choosing a gauge,

$$J^\mu_{(ii)} = \gamma^\lambda\gamma^\mu\psi\partial_\lambda\phi + \frac{1}{2}i\gamma^\lambda\gamma^\mu\chi_\lambda\bar{\psi}\psi + i(\xi + g\phi^2)\gamma^\mu\psi \\ - i(\xi + (g/3)\phi^2)\phi\varepsilon^{\mu\nu}e^{-1}\gamma^5\chi_\nu \quad (140)$$

The equation $J^\mu_{(ii)} = 0$ may in principle be solved for χ_μ . On the other hand, in the unitary gauge,

$$J^\mu_{(ii),\text{unitary}} = -i(\xi + (g/3)\phi^2)\phi\varepsilon^{\mu\nu}e^{-1}\gamma^5\chi_\nu = 0 \quad (141)$$

In case (ii), ξ is positive, and hence apart from the solution $\phi \equiv 0$, the only solution is $\chi_\mu \equiv 0$. The generalised Higgs mechanism removes the fermionic Goldstone mode, but via a gauge field which is forced to be trivial in two dimensions. These conclusions are in accord with earlier work by Deser and Zumino (1977) and Dereli and Deser (1977a,b). These models were based on the non-linear realisation of supergravity due to Volkov and Akulov (1973), and did not emerge from a superfield approach.

This chapter began with the superspace approach to a one-dimensional scalar multiplet's coupling to "supergravity" in one dimension. It was shown that a superspace approach to the equivalent two-dimensional model

removed most of the difficulties associated with the non-superspace algorithm, $L \rightarrow L + \frac{1}{2} \bar{\chi}_\mu J^\mu$. This approach was based on an Ansatz for a generalised vierbein, or vielbein, postulated by Howe (1977). A later version (Howe 1978) allowed the spin connection and Rarita-Schwinger field to obey non-trivial dynamics, but no Lagrangian could be found for the supergravity variables. How to extend this Ansatz to four dimensions remained unclear, and to deal with this problem a method of deriving the (1+1) vielbein was given. The inclusion of an auxiliary scalar field in the later version of the vielbein served to remove the constraints on χ_μ and ω_μ ; unfortunately this same scalar field proved fatal to the construction of any Lagrangian except that of the spinning string. Lastly, a two-dimensional model based on a Lagrangian related to that of the spinning string and incorporating the spontaneous breakdown of local supersymmetry was constructed. The model was shown to possess two phases, depending on the sign of ξ in the "trigger term" - $2\xi EV$: for $\xi < 0$, supersymmetry remained intact, while both ϕ and ψ became massive with common mass; for $\xi > 0$, supersymmetry was spontaneously broken, resulting in an induced cosmological constant - $\frac{1}{2}\xi^2$, a massive scalar ϕ and a would-be Goldstone fermion ψ which is absorbed by the gravitino field χ_μ . Due to the trivial nature of χ_μ in two dimensions, the gauge freedom is its only dynamical significance, and both it and ψ vanished in the unitary gauge.

In the fourth and final chapter, the methods of §§16-17 will be used in an attempt to extend these results to four dimensions.

CHAPTER IV

SUPERFIELD SUPERGRAVITY: FOUR DIMENSIONS

§18. Constructing the Vielbein

In this last chapter, the methods of §16 will be applied in an attempt to construct a vielbein for the scalar superfield $\Phi(x, \theta)$ introduced in §5.1. As in §16, the key to this construction lies in the form of the covariant derivatives for the component fields of Φ . These derivatives are to be covariant with respect to local supersymmetry and with respect to local Lorentz transformations. It is easiest to construct at first derivatives satisfying only the "necessary condition" $\bar{\delta}(\nabla_\mu A) = 0$ for an arbitrary field A ; these derivatives involve only the field χ_μ . The spin connection is most easily incorporated into the operators ∇_μ via the substitution $\partial_\mu \rightarrow D_\mu$, where D_μ is covariant with respect to the local Lorentz group. Nevertheless, these derivatives fail to satisfy the "sufficient condition", i.e. that they should transform according to the Ehresmann laws. Recall that in two dimensions, the derivatives were covariant only modulo the two constraints (III.99). This difficulty was overcome by the introduction of additional terms in the vielbein, involving either an auxiliary scalar field or the covariant curl of χ_μ . As will be demonstrated, similar difficulties arise in the four-dimensional case. Unfortunately, no combination of new terms succeeds in righting the transformation of the "covariant" derivatives. Consequently it seems that a four-dimensional vielbein which transforms covariantly does not exist; at least if one posits the transformation

$$\delta E_M^A = \xi^N \partial_N E_M^A + (\partial_M \xi^N) E_N^A - L_B^A E_M^B \quad (1)$$

with

$$L_B^A = \begin{pmatrix} \lambda_\beta^\alpha & 0 \\ 0 & \frac{1}{2}(\sigma \cdot \lambda) b^a \end{pmatrix} \quad (2)$$

If a four-dimensional vielbein were at hand, it would not be difficult to construct the locally supersymmetric Fayet model. Instead, this thesis concludes with a demonstration that the required vielbein cannot be constructed via the present methods, which is a strong argument against the existence of such an object. The vielbein's construction in four dimensions exactly parallels that in two dimensions; it is only at the stage of the constraints equivalent to (III.99) that the scheme fails.

§18.1 Covariance in χ_μ .

The components of Φ are the fields $(\phi, \psi, F, G, A_\nu, \lambda, D)$ obeying the global transformations (I.38). Each field will at first be considered separately. Initially the canonical prescription

$$\partial_\mu \rightarrow \nabla_\mu = \partial_\mu - \bar{\chi}_\mu S$$

will be attempted, with $\delta\chi_\mu = \partial_\mu \varepsilon + \dots$. Of course, in the light of §16, this approach will obviously be seen to fail: S must be replaced by Σ , which it is imagined will take the form

$$\Sigma = (1 - \frac{1}{2} i \gamma^\nu \theta \bar{\chi}_\nu)^{-1} S$$

in exact analogy to two dimensions (cf. (III.89)). The factor

$(1 - \frac{1}{2} i \gamma^\nu \theta \bar{\chi}_\nu)^{-1}$ will frequently occur, and it will henceforth be abbreviated Q . In this section, it will be shown that the operator

$$\nabla_\mu = \partial_\mu - \bar{\chi}_\mu Q S$$

indeed satisfies the "necessary condition", so long as

$$\delta' \Phi = [\bar{\varepsilon} \Sigma, \Phi] = \delta \Phi + \delta_G \Phi$$

i.e., the transformation law for Φ is changed. Now consider the superfield component by component:

$$(i) \quad \phi \cdot \nabla_\mu \phi \stackrel{?}{=} \partial_\mu \phi - i \bar{\chi}_\mu \psi ; \quad (3a)$$

this satisfies the necessary condition, so that

$$\delta_G \phi = 0 \quad (3b)$$

$$(ii) \quad \psi \cdot \nabla_\mu \psi \stackrel{?}{=} \partial_\mu \psi + \frac{1}{2} [iF + i\gamma^5 G - \gamma^5 A + \not{V}\phi] \chi_\mu \quad (4a)$$

(note that immediately $\nabla_\mu \phi$ has been substituted for $\partial_\mu \phi$; cf. §16).

Now however

$$\bar{\delta}(\nabla_\mu \psi) = \frac{1}{2} i \gamma^\nu (\partial_\mu \epsilon) \bar{\chi}_\nu \psi \quad (4b)$$

and consequently it is necessary to set

$$\delta_G \psi = -\frac{1}{2} i \gamma^\nu \epsilon \bar{\chi}_\nu \psi \quad (4c)$$

The algebra is on the verge of becoming cumbersome without a few abbreviations.

Let

$$\nabla_\mu \psi = \partial_\mu \psi - \Omega(\psi) \chi_\mu, \quad \text{where} \quad (5a)$$

$$\Omega(\psi) = \Omega_1(\psi) + \Omega_2(\psi), \quad \text{and} \quad (5b)$$

$$\Omega_1(\psi) = (\partial \delta \psi / \partial \epsilon); \quad \Omega_2(\psi) = (\partial \delta_G \psi / \partial \epsilon). \quad (5c)$$

Then

$$\delta_G \psi = \Omega_2(\psi) \epsilon.$$

Continuing in this way, one finds

$$(iii) \quad F \cdot \nabla_\mu F \stackrel{?}{=} \partial_\mu F - \frac{1}{2} i \bar{\chi}_\mu \lambda + \frac{1}{2} \bar{\chi}_\mu \not{V} \psi \quad (6a)$$

$$= \partial_\mu F - \frac{1}{2} i \bar{\chi}_\mu \lambda + \frac{1}{2} \bar{\chi}_\mu \not{V} \psi - \frac{1}{2} \bar{\chi}_\mu \gamma^\nu \Omega(\psi) \chi_\nu$$

$$\bar{\delta}(\nabla_\mu F) = -\frac{1}{2} (\partial_\mu \bar{\epsilon}) \gamma^\nu \Omega(\psi) \chi_\nu ; \quad \text{so} \quad (6b)$$

$$\delta_G F = \frac{1}{2} \bar{\epsilon} \gamma^\nu \Omega(\psi) \chi_\nu \equiv \bar{\epsilon} \Omega_2(F) \quad (6c)$$

The expressions for G and A_ν are very similar:

$$(iv) \quad G. \quad \nabla_{\mu} G \stackrel{?}{=} \partial_{\mu} G - \frac{1}{2} i \bar{\chi}_{\mu} \gamma^5 \lambda + \frac{1}{2} \bar{\chi}_{\mu} \gamma^5 \not{\nabla} \psi \quad (7a)$$

which requires

$$\delta_G G = \frac{1}{2} \bar{\epsilon} \gamma^5 \gamma^{\nu} \Omega(\psi) \chi_{\nu} \equiv \bar{\epsilon} \Omega_2(G) \quad (7b)$$

$$(v) \quad A_{\nu}. \quad \nabla_{\mu} A_{\nu} \stackrel{?}{=} \partial_{\mu} A_{\nu} + \frac{1}{2} \bar{\chi}_{\mu} \gamma^{\nu} \gamma^5 \lambda - \frac{1}{2} i \bar{\chi}_{\mu} \gamma^{\lambda} \gamma_{\nu} \gamma^5 \nabla_{\lambda} \psi, \quad \text{so} \quad (8a)$$

$$\delta_G A_{\nu} = -\frac{1}{2} i \bar{\epsilon} \gamma^{\lambda} \gamma^5 \Omega(\psi) \chi_{\lambda} \equiv \bar{\epsilon} \Omega_{\nu 2}(A) . \quad (8b)$$

$$(vi) \quad \lambda. \quad \nabla_{\mu} \lambda \stackrel{?}{=} \partial_{\mu} \lambda - \frac{1}{2} [\not{\nabla} F - \gamma^5 \not{\nabla} G + i \gamma^{\nu} \gamma^5 \not{\nabla} A_{\nu} - i D] \chi_{\mu}, \quad (9a)$$

$$\Omega_2(\lambda) = -\frac{1}{2} \gamma^{\nu} [\bar{\chi}_{\nu} \Omega(F) + \gamma^5 \bar{\chi}_{\nu} \Omega(G) - i \gamma^5 \gamma^{\lambda} \bar{\chi}_{\nu} \Omega_{\lambda}(A)] , \quad (9b)$$

$$\delta_G \lambda = \Omega_2(\lambda) \epsilon \quad (9c)$$

and finally

$$(vii) \quad D. \quad \nabla_{\mu} D \stackrel{?}{=} \partial_{\mu} D + \bar{\chi}_{\mu} \not{\nabla} \lambda ; \quad (10a)$$

$$\delta_G D = \bar{\epsilon} \gamma^{\nu} \Omega(\lambda) \chi_{\nu} \equiv \bar{\epsilon} \Omega_2(D) . \quad (10b)$$

Given these forms for the transformations of the various fields, it is not hard to guess the form of Σ . For an arbitrary field A , look at $\delta_G A$ and find the term of highest order in χ_{μ} . This term determines the coefficients $\partial^n f / \partial \theta^n$ in the expansion

$$\begin{aligned} \Sigma &= f(\theta, \chi) S \\ &= S + (\partial f / \partial \theta) \theta S + \dots \end{aligned}$$

For example, from $\delta_G \psi = -\frac{1}{2} i \gamma^{\mu} \bar{\epsilon} \chi_{\mu} \psi$, it seems reasonable that

$$\Sigma = S + \frac{1}{2} i \gamma^{\mu} \theta \bar{\chi}_{\mu} S + \dots$$

and in fact this is consistent with all terms linear in χ_{μ} in the various δ_G expressions. Proceeding in this way, Σ is found to be

$$\begin{aligned}
 \Sigma &= S + \frac{1}{2} i \gamma^\nu \bar{\theta} \chi_\nu S - \frac{1}{2} \gamma^\mu \bar{\theta} \chi_\mu \gamma^\nu \bar{\theta} \chi_\nu S \\
 &\quad - (i/8) \gamma^\mu \bar{\theta} \chi_\mu \gamma^\nu \bar{\theta} \chi_\nu \gamma^\lambda \bar{\theta} \chi_\lambda S + (1/16) \gamma^\mu \bar{\theta} \chi_\mu \gamma^\nu \bar{\theta} \chi_\nu \gamma^\lambda \bar{\theta} \chi_\lambda \gamma^\rho \bar{\theta} \chi_\rho S \\
 &= QS
 \end{aligned} \tag{11}$$

which is just as anticipated. The only hard part of this calculation is to confirm, via various Fierz rearrangements, that the expression for $\delta'\phi$ as given by $[\bar{\epsilon}\Sigma, \phi]$ concurs with that required by the "necessary condition".

§18.2 Covariance in $\omega_\mu^{\alpha\beta}$

Suppose that the global variations (I.38) were to be modified to allow only for the presence of gravity. Obviously the variations involving derivatives of the scalar fields will be unchanged, but those involving the spinor or vector fields must be altered. The new transformations are to be

$$\begin{aligned}
 \delta F &= \frac{1}{2} (i \bar{\epsilon} \lambda - \bar{\epsilon} \not{D} \psi) \\
 \delta G &= \frac{1}{2} (i \bar{\epsilon} \gamma^5 \lambda - \bar{\epsilon} \gamma^5 \not{D} \psi) \\
 \delta A_\nu &= \frac{1}{2} (-\bar{\epsilon} \gamma_\nu \gamma^5 \lambda + i \bar{\epsilon} \gamma^\mu \gamma_\nu \gamma^5 D_\mu \psi) \\
 \delta \lambda &= \frac{1}{2} (\not{D} F + \not{D} \gamma^5 G - i \gamma^\alpha \gamma^5 \not{D} A - i D) \epsilon \\
 \delta D &= -\bar{\epsilon} \not{D} \lambda
 \end{aligned} \tag{12}$$

where as before

$$\begin{aligned}
 D_\mu \psi &= \partial_\mu \psi + \frac{1}{2} \omega_\mu^{\alpha\beta} \sigma_{\alpha\beta} \psi \quad (\text{same for } \lambda) \\
 D_\mu A_\alpha &= \partial_\mu A_\alpha - \omega_\mu^{\alpha\beta} A_\beta
 \end{aligned}$$

Again, the generator S must be modified; this time to

$$S \rightarrow S' = S + \frac{1}{2} i \gamma^\mu \bar{\theta} \omega_\mu^{\alpha\beta} \sigma_{\alpha\beta} / \partial \bar{\theta} . \tag{13}$$

Put another way, the ∂_μ in S must go to D_μ ;

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - \frac{1}{2} \bar{\theta} \omega_\mu \cdot \sigma \partial / \partial \bar{\theta} . \quad (14)$$

Actually, this only incorporates the "internal" spin degrees of freedom.

In general (cf. the discussion in §16.1)

$$D_\mu = \partial_\mu + \frac{1}{2} (\bar{\theta} \omega_\mu \cdot \sigma \partial / \partial \bar{\theta}) + \frac{1}{2} \omega_\mu \cdot M$$

where $M_{\alpha\beta}$ is the generator introduced in §12.3. However, acting on a scalar quantity such as Φ ,

$$[M_{\alpha\beta}, \Phi] = 0 .$$

So long as only scalar superfields are being differentiated, the form (14) is all that is required. It is perhaps worth remarking that (14) suffices to give the new transformation for A_α only as a result of the identity

$$\bar{\theta} \sigma_{\alpha\beta} \gamma^\delta \gamma^5 \theta = \frac{1}{2} \bar{\theta} [\sigma_{\alpha\beta}, \gamma^\delta \gamma^5] \theta .$$

§18.3 Covariance in Both χ and ω . First form of E_M^A .

From the earlier work in two dimensions, the complete generator Σ should be given by

$$\Sigma = QS' = Q[(\partial/\partial\bar{\theta}) - \frac{1}{2} i \gamma^\mu \theta \{ \partial_\mu + \frac{1}{2} (\bar{\theta} \omega_\mu \cdot \sigma \partial / \partial \bar{\theta}) + \frac{1}{2} \omega_\mu \cdot M \}] . \quad (15)$$

As a first guess, the covariant derivative ∇_α should be given by

$$\begin{aligned} \nabla_\alpha &= e_\alpha^\mu \nabla_\mu = e_\alpha^\mu (D_\mu - \bar{\chi}_\mu \Sigma) \\ &\equiv E_\alpha^M \partial_M . \end{aligned} \quad (16)$$

In fuller detail,

$$E_\alpha^\mu = e_\alpha^\mu + \frac{1}{2} i \bar{\chi}_\alpha Q \gamma^\mu \theta = (e_\alpha^\mu - \frac{1}{2} i \bar{\chi}_\mu \gamma^\alpha \theta)^{-1} \quad (17a)$$

$$\begin{aligned}
 E_{\alpha}^m &= -(\bar{\chi}_{\alpha} Q)^m + \frac{1}{2}(\bar{\theta}\omega_{\alpha} \cdot \sigma)^m + \frac{1}{4}i\bar{\chi}_{\alpha} Q \gamma^{\nu} \theta (\bar{\theta}\omega_{\nu} \cdot \sigma)^m \\
 &= -E_{\alpha}^{\mu} (\bar{\chi}_{\mu} - \frac{1}{2}\bar{\theta}\sigma \cdot \omega_{\mu})^m
 \end{aligned} \tag{17b}$$

Assuming that a reasonable first guess for ∇_a is

$$\begin{aligned}
 \nabla_a &= (\partial/\partial \bar{\theta}^a) + \frac{1}{2}i(\gamma^{\alpha}\theta) \nabla_{\alpha} \\
 &\equiv E_a^M \partial_M,
 \end{aligned} \tag{18}$$

it follows

$$E_a^{\mu} = \frac{1}{2}iE_{\alpha}^{\mu}(\gamma^{\alpha}\theta)_a \tag{19a}$$

$$E_a^m = \delta_a^m + \frac{1}{2}i(\gamma^{\alpha}\theta)_a E_{\alpha}^m \tag{19b}$$

These are obviously the four-dimensional analogues of (III.91) and (III.93).

From the orthogonality relations, the forms E_M^A are (cf. (III.96))

$$E_{\mu}^{\alpha} = e_{\mu}^{\alpha} + i\bar{\theta}\gamma^{\alpha}\chi_{\mu} + \frac{1}{4}i\bar{\theta}\omega_{\mu} \cdot \sigma \gamma^{\alpha}\theta \tag{20a}$$

$$E_{\mu}^a = \bar{\chi}_{\mu}^a - \frac{1}{2}(\bar{\theta}\sigma \cdot \omega_{\mu})^a \tag{20b}$$

$$E_m^{\alpha} = -\frac{1}{2}(i\gamma^{\alpha}\theta)_m \tag{20c}$$

$$E_m^a = \delta_m^a. \tag{20d}$$

§18.4 The Ehresmann Laws

Given the forms (20) of E_M^A , do they provide a covariant derivative? The "sufficient condition" that $\nabla_A = E_A^M \partial_M$ transform covariantly is that they satisfy the Ehresmann laws

$$\delta \nabla_A = [\bar{\epsilon} \Sigma, \nabla_A] \equiv (\delta E_A^M) \partial_M. \tag{21}$$

As shown in §16.1, the laws (21) are equivalent to the transformations (1), with the various parameters given by

$$\xi^\mu = -\frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta \quad (22a)$$

$$\xi^m = (\bar{\epsilon}Q)^m - \frac{1}{4}(i\bar{\epsilon}Q\gamma^\mu\theta\bar{\theta}\omega_\mu\cdot\sigma)^m \quad (22b)$$

$$\lambda_{\alpha\beta} = -\frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta\omega_{\mu\alpha\beta} \quad (22c)$$

(Although (21) deals with E_A^M , it is much easier to work in terms of E_M^A ; if E_M^A transforms correctly, so will its inverse.) The requirement of covariance becomes: do the forms (20) transform under (1) in a manner consistent with (II.6,23,31)? The answer is no, or more precisely, not quite. In the two-dimensional case, the transformations were shown to hold only modulo the constraints (III.99), before introducing new terms into the vielbein which involved an auxiliary field and the covariant curl of χ_μ . A similar situation obtains here, as will now be shown in detail.

In the calculations which follow, frequent use is made of the identities

$$\bar{\epsilon}Q = \bar{\epsilon} + \frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta\bar{\chi}_\mu \quad (23a)$$

$$\bar{\epsilon}Q\gamma^\nu\theta\bar{\chi}_\nu = \bar{\epsilon}\gamma^\nu\theta\bar{\chi}_\nu Q \quad (23b)$$

(i) δE_m^a . For consistency, it must be that

$$\begin{aligned} \delta E_m^a &= \delta(\delta_m^a) = 0 \\ &= \xi_N^N \partial_N E_m^a + (\partial_m \xi_N^N) E_N^a - \frac{1}{2}(\sigma\cdot\lambda)_b^a E_m^b. \end{aligned} \quad (24)$$

The first piece vanishes identically. The remaining pieces are

$$\begin{aligned} \delta E_m^a &= \partial_m (\bar{\epsilon}Q - \frac{1}{4}i\bar{\epsilon}Q\gamma^\mu\theta\bar{\theta}\omega_\mu\cdot\sigma)^n \delta_n^a \\ &\quad + \partial_m (-\frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta) (\bar{\chi}_\mu - \frac{1}{2}\bar{\theta}\omega_\mu\cdot\sigma)^a - \frac{1}{2}(\sigma\cdot\lambda)_m^a. \end{aligned}$$

Those pieces not involving the spin connection obviously cancel;

$$\partial_m(\bar{\epsilon}Q) = \partial_m(\bar{\epsilon} + \frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta\chi_\mu).$$

The remaining terms are

$$\begin{aligned}\delta E_m^a &= -\frac{1}{4}i\partial_m(\bar{\epsilon}Q\gamma^\mu\theta\bar{\omega}_\mu\cdot\sigma)^a + \frac{1}{4}i[\partial_m(\bar{\epsilon}Q\gamma^\mu\theta)]\bar{\omega}_\mu\cdot\sigma^a \\ &\quad + \frac{1}{4}i(\bar{\epsilon}Q\gamma^\mu\theta\sigma\cdot\omega_\mu)_m^a \\ &= 0 ;\end{aligned}$$

and thus E_m^a transforms correctly.

(ii) δE_m^α . Again, it is necessary that this vanish:

$$\begin{aligned}\delta E_m^\alpha &= \delta(-\frac{1}{2}(\gamma^\alpha\theta)_m), = 0 \\ &= \xi^n\partial_n(-\frac{1}{2}i(\gamma^\alpha\theta)_m) + (\partial_m\xi^n)(-\frac{1}{2}i(\gamma^\alpha\theta)_n) \\ &\quad + (\partial_m\xi^\mu)E_\mu^\alpha + \frac{1}{2}i\lambda_\beta^\alpha(\gamma^\beta\theta)_m.\end{aligned}\tag{25}$$

To ease the manipulation of anticommuting quantities, multiply both sides by an arbitrary constant spinor $\bar{\eta}_m$. Then it is required that

$$\begin{aligned}0 &= \xi^n\partial_n(-\frac{1}{2}i\bar{\eta}\gamma^\alpha\theta) + (\bar{\eta}\cdot\partial\xi^n)(-\frac{1}{2}i\gamma^\alpha\theta)_n \\ &\quad + (\bar{\eta}\cdot\partial\xi^\mu)(e_\mu^\alpha + i\bar{\theta}\gamma^\alpha\chi_\mu + \frac{1}{2}i\bar{\theta}\sigma\cdot\omega_\mu\gamma^\alpha\theta) + \frac{1}{2}i\lambda_\beta^\alpha\bar{\eta}\gamma^\beta\theta\end{aligned}$$

where $\bar{\eta}\cdot\partial = \bar{\eta}\partial/\partial\bar{\theta}$. Those pieces independent of the spin connection cancel:

$$\begin{aligned}\frac{Q}{2}i\bar{\epsilon}\gamma^\alpha\eta - \frac{1}{2}i[\bar{\eta}\cdot\partial(\bar{\epsilon}Q)]\gamma^\alpha\theta + \bar{\eta}\cdot\partial(-\frac{1}{2}i\bar{\epsilon}Q\gamma^\alpha\theta) \\ + [\bar{\eta}\cdot\partial(-\frac{1}{2}i\bar{\epsilon}Q\gamma^\mu\theta)]i\bar{\theta}\gamma^\alpha\chi_\mu = 0.\end{aligned}$$

Now E_m^α transforms correctly only if the last pieces containing the spin connection cancel. After some minor algebra, one finds

$$\delta E_m^\alpha = \frac{1}{4} \bar{\epsilon} Q \gamma^\mu \theta (\frac{1}{2} \omega_\mu^{\rho\sigma} \bar{\theta} [\sigma_{\rho\sigma}, \gamma^\alpha] \eta + \omega_{\mu\beta}^{\alpha\gamma} \bar{\eta} \gamma^\beta \theta)$$

which vanishes. Thus $\delta E_m^\alpha = 0$, as desired. Consequently, the trivial pieces of E_M^A are well-behaved. Alas, the more interesting members are not so accommodating.

(iii) δE_μ^a . According to the transformation laws given in Chapter II, it should be that

$$\begin{aligned} \delta E_\mu^a &= \delta(\bar{\chi}_\mu^a - \frac{1}{2} \bar{\theta} \omega_\mu \cdot \sigma^a) \\ &= D_\mu \bar{\epsilon}^a - \frac{1}{2} (\bar{\theta} \sigma^{\alpha\beta})^a \delta \omega_{\mu\alpha\beta} \end{aligned}$$

with $\delta \omega_{\mu\alpha\beta}$ as given by (II.31). Instead, using

$$\delta E_\mu^a = \xi^N \partial_N E_\mu^a + (\partial_\mu \xi^N) E_N^a - \frac{1}{2} (E_\mu \sigma \cdot \lambda)^a \quad (26)$$

one finds after a fairly painless calculation that, as given by (26),

$$\delta E_\mu^a = D_\mu \bar{\epsilon}^a + \frac{1}{2} i \bar{\epsilon} Q \gamma^\nu \theta D_{[\mu} \bar{\chi}_{\nu]}^a - \frac{1}{4} i \bar{\epsilon} Q \gamma^\nu \theta (\bar{\theta} R_{\mu\nu}^{\alpha\beta} \sigma_{\alpha\beta})^a. \quad (27)$$

This result is strongly reminiscent of the two-dimensional transformation. The term zeroth order in θ is as it should be, but beyond that things go badly awry. The structure of the linear term suggests that again it may be possible to repair the damage by the inclusion of new terms in the covariant curl of χ_μ and some auxiliary fields. Finally, consider the last part of the vielbein:

(iv) δE_μ^α . The earlier transformations suggest

$$\delta E_\mu^\alpha = i \bar{\epsilon} \gamma^\alpha \chi_\mu + i \bar{\theta} \gamma^\alpha D_\mu \epsilon + \frac{1}{4} i \bar{\theta} \sigma^{\alpha\beta} (\delta \omega_\mu) \gamma^\beta \theta. \quad (28)$$

Unfortunately, after a lengthy calculation, one discovers that (1) implies

$$\begin{aligned} \delta E_\mu^\alpha &= i \bar{\epsilon} \gamma^\alpha \chi_\mu + i \bar{\theta} \gamma^\alpha D_\mu \epsilon - \frac{1}{2} \bar{\epsilon} Q \gamma^\nu \theta \bar{\theta} \gamma^\alpha D_{[\mu} \chi_{\nu]} \\ &\quad - (1/8) \bar{\epsilon} Q \gamma^\nu \theta \bar{\theta} \sigma_{\rho\sigma} \gamma^\alpha \theta R_{\mu\nu}^{\rho\sigma}. \end{aligned} \quad (29)$$

In order to obtain this result, it is necessary to use the identity

$$\omega_{\nu\beta} [\delta \epsilon^\alpha]_{\rho\sigma} = -\epsilon^{\delta\alpha}_{\rho\beta} \omega_{\nu\sigma} + \epsilon^{\delta\alpha}_{\sigma\beta} \omega_{\nu\rho}$$

as well as the equation of motion (II.35)

$$D_{[\mu} e_{\nu]}^\alpha = i \bar{\chi}_\mu \gamma^\alpha \chi_\nu.$$

The transformation (29) is correct to $O(\theta)$, but not beyond. Of course, if one were willing to demand the vanishing of both the covariant curl and the Riemann tensor, the transformations would be correct. But such constraints force the supergravity fields into non-dynamical roles, and this avenue will not be pursued.

§19. Auxiliary Fields

§19.1 A First Attempt

The first place where the transformation δE_M^A fails is at the linear term (in θ) in δE_μ^a . To $O(\theta)$, this reads

$$\delta(-\tfrac{1}{2}\bar{\theta}\omega_\mu \cdot \sigma)^a = \tfrac{1}{2}i\bar{\epsilon}\gamma^\nu \theta D_{[\mu} \bar{\chi}_{\nu]}^a \quad (30)$$

which is precisely equivalent to the two-dimensional result (III.101). As it stands, (30) is impossible, because the Fierz rearrangement of the right-hand side involves all sixteen Dirac matrices as coefficients of θ , and not just the tensor. To overcome this difficulty, one could introduce auxiliary fields linear in θ into E_μ^a ; e.g. set

$$\begin{aligned} E_\mu^a &= \bar{\chi}_\mu^a - \tfrac{1}{2}\bar{\theta}\omega_\mu \cdot \sigma^a - \tfrac{1}{2}i\bar{\theta}\gamma_\mu^a A - \tfrac{1}{2}i\bar{\theta}\gamma_\mu^5 a_B \\ &\quad - \tfrac{1}{2}\bar{\theta}^a F_\mu - \tfrac{1}{2}\bar{\theta}\gamma^5 a_C^\mu + \dots \end{aligned}$$

However, there is a far more serious difficulty. The relevant part of the Fierz rearrangement of the right-hand side of (30), namely that part

proportional to $\bar{\theta}\sigma^{\alpha\beta}$, does not give the variation of the spin connection in agreement with (II.31). Somehow, then, new terms linear in the covariant curl must enter in to δE_μ^a , in order to get the right form for $\delta\omega_{\mu\alpha\beta}$. This was also the situation in two-dimensions. The only place where these new pieces, linear in θ and in the covariant curl, can arise is from the presence in E_μ^a of terms quadratic in θ . There are only three independent expressions quadratic in θ , and any set of quadratics may be Fierz rearranged into a linear combination of these three (see §4.3). Consequently the most general expression for these new terms in E_μ^a has the form

$$\frac{1}{2}i\bar{\theta}\theta L_\mu^a + \frac{1}{2}i\bar{\theta}\gamma^5\theta M_\mu^a + \frac{1}{2}i\bar{\theta}\gamma^\nu\gamma^5\theta N_{\nu\mu}^a \quad (31)$$

where L_μ^a transforms as a vector-spinor, M_μ^a as a pseudovector-spinor, and $N_{\nu\mu}^a$ as a pseudotensor-spinor, and all three are presumed to be linear in the covariant curl of χ_λ . These new terms (31) change the linear expression in E_μ^a by the addition of the expression

$$\frac{1}{2}i\bar{\epsilon}\theta L_\mu^a + \frac{1}{2}i\bar{\epsilon}\gamma^5\theta M_\mu^a + \frac{1}{2}i\bar{\epsilon}\gamma^\nu\gamma^5\theta N_{\nu\mu}^a \quad (32)$$

The requirement for covariance now becomes

$$\bar{\theta}\sigma^{\alpha\beta}\delta\omega_{\mu\alpha\beta} = \text{F.R.}_T\{i\bar{\theta}\gamma^\nu\epsilon D_{[\mu}\bar{\chi}_{\nu]} - i\bar{\epsilon}\theta L_\mu - i\bar{\epsilon}\gamma^5\theta M_\mu - i\bar{\epsilon}\gamma^\nu\gamma^5\theta N_{\nu\mu}\} \quad (33)$$

where F.R._T means: only that part of the Fierz rearrangement which involves the tensor coefficient of $\bar{\theta}$. For example,

$$\begin{aligned} \text{F.R.}_T(i\bar{\theta}\gamma^\nu\epsilon D_{[\mu}\bar{\chi}_{\nu]}) &= \bar{\theta}\sigma^{\alpha\beta} D_{\mu\alpha\beta}, \quad \text{where} \\ D_{\mu\alpha\beta} &= -\frac{1}{2}i\bar{\epsilon}_{[\beta}\gamma^\nu_{\alpha]} D_{[\mu}\bar{\chi}_{\nu]} - \frac{1}{2}ie^{-1}\epsilon_{\alpha\beta}^{\rho\sigma}\bar{\epsilon}\gamma^5\gamma_\mu D_\rho\bar{\chi}_\sigma \\ &\quad + \frac{1}{2}ie^{-1}e_{\mu[\beta}\epsilon_{\alpha]}^{\nu\rho\sigma}\bar{\epsilon}\gamma^5\gamma_\nu D_\rho\bar{\chi}_\sigma. \end{aligned}$$

Using the explicit form (II.31) of $\delta\omega_\mu^{\alpha\beta}$,

$$\delta\omega_{\mu}^{\alpha\beta} = -ie^{-1}\epsilon^{\alpha\beta\rho\sigma}\bar{\epsilon}\gamma^5\gamma_{\mu}D_{\rho}\chi_{\sigma} - \frac{1}{2}ie^{-1}e_{\mu}^{\alpha}[\epsilon^{\beta}]^{\nu\rho\sigma}\bar{\epsilon}\gamma^5\gamma_{\nu}D_{\rho}\chi_{\sigma},$$

the equation to be satisfied by L , M and N becomes

$$\begin{aligned} & -\frac{1}{2}e^{-1}\epsilon_{\alpha\beta}^{\rho\sigma}\bar{\epsilon}\gamma^5\gamma_{\mu}D_{\rho}\chi_{\sigma} + \frac{1}{2}e_{[\beta}^{\nu}\bar{\epsilon}\gamma_{\alpha]}^D[\mu\chi_{\nu]} \\ & = L_{\mu\sigma\alpha\beta}\epsilon + M_{\mu\sigma\alpha\beta}\gamma^5\epsilon + N_{\nu\mu\sigma\alpha\beta}\gamma^{\nu}\gamma^5\epsilon. \end{aligned} \quad (34)$$

If this condition is met, $\delta\omega_{\mu}^{\alpha\beta}$ will be correct. It is now necessary to write out the most general forms of L_{μ} , M_{μ} , and $N_{\nu\mu}$, plug into (34), and work out the specific solution. The simple-minded approach is to write out every possible combination of γ 's and the covariant curl which transforms in the right way, and then to discover how many of these are linearly independent. For example, any of the following could enter into L_{μ} :

$$\begin{aligned} L_{\mu} \sim & D_{[\mu}\bar{\chi}_{\nu]}\gamma^{\nu}, \quad \bar{R}_{\mu}, \quad \bar{R}\cdot\gamma\gamma_{\mu}, \quad \bar{R}_{\nu}\gamma_{\mu}\gamma^{\nu}, \quad D_{\rho}\bar{\chi}_{\sigma}\sigma^{\rho\sigma}\gamma_{\mu} \\ & D_{\rho}\bar{\chi}_{\sigma}\gamma_{\mu}\sigma^{\rho\sigma}, \quad \bar{R}_{\nu}\sigma_{\mu}^{\nu}, \quad D_{[\rho}\bar{\chi}_{\sigma]}\sigma^{\rho}_{\mu}\gamma^{\sigma}, \quad D_{[\rho}\bar{\chi}_{\sigma]}\gamma^{\sigma}\sigma_{\mu}^{\rho}. \end{aligned}$$

After some algebraic manipulations, it becomes obvious that only two of these are independent. Hence L_{μ} may be written in all generality as

$$L_{\mu} = a D_{[\mu}\bar{\chi}_{\nu]}\gamma^{\nu} + b\bar{R}_{\mu} \quad (35)$$

where a and b are arbitrary constants. A similar analysis holds for M_{μ} , so that its form is

$$M_{\mu} = c D_{[\mu}\bar{\chi}_{\nu]}\gamma^{\nu}\gamma^5 + d\bar{R}_{\mu}\gamma^5 \quad (36)$$

where c and d are also arbitrary constants. The pseudo-tensor term $N_{\nu\mu}$ is necessarily more complicated. Rather than write down the fourteen trial forms, only the result will be stated: in full generality,

$$\begin{aligned}
 N_{\nu\mu} = & 2f \bar{R} \cdot \gamma \sigma_{\nu\mu} \gamma^5 + g \bar{R} \cdot \gamma g_{\nu\mu} \gamma^5 + h D_{[\mu} \bar{X}_{\nu]} \gamma^5 \\
 & + 2k D_{[\mu} \bar{X}_{\lambda]} \sigma_{\nu}^{\lambda} \gamma^5 + 2\ell D_{[\mu} \bar{X}_{\lambda]} \sigma_{\mu}^{\lambda} \gamma^5.
 \end{aligned}
 \tag{37}$$

Using the identity

$$2\sigma_{\nu\mu} \sigma_{\alpha\beta} \gamma^{\nu} = \sigma_{\alpha\beta} \gamma_{\mu}$$

the equation (34) becomes

$$\begin{aligned}
 & -\frac{1}{2} e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} \bar{\epsilon} \gamma^5 \gamma_{\mu} D_{\rho} X_{\sigma} + \frac{1}{2} e_{[\beta}^{\nu} \bar{\epsilon} \gamma_{\alpha]} D_{[\mu} X_{\nu]} \\
 = & (a - c) D_{[\mu} \bar{X}_{\nu]} \gamma^{\nu} \sigma_{\alpha\beta} \epsilon + (b - d) \bar{R}_{\mu} \sigma_{\alpha\beta} \epsilon \\
 & + (f+g) \bar{R} \cdot \gamma \sigma_{\alpha\beta} \gamma_{\mu} \epsilon + (h-k) D_{[\mu} \bar{X}_{\nu]} \sigma_{\alpha\beta} \gamma^{\nu} \epsilon \\
 & + 2\ell D_{[\nu} \bar{X}_{\lambda]} \sigma_{\mu}^{\lambda} \sigma_{\alpha\beta} \gamma^{\nu} \epsilon
 \end{aligned}
 \tag{38}$$

Letting $(a-c) = \alpha$, $(b-d) = \beta$, $(f+g) = \gamma$, $(h-k) = \kappa$, and using the identities in Appendix C, (38) becomes

$$\begin{aligned}
 & - e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} \bar{\epsilon} \gamma^5 \gamma_{\mu} D_{\rho} X_{\sigma} + e_{[\beta}^{\nu} \bar{\epsilon} \gamma_{\alpha]} D_{[\mu} X_{\nu]} \\
 = & - e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} \bar{\epsilon} \gamma^5 \gamma_{\mu} D_{\rho} X_{\sigma} (\alpha + \kappa - 2\gamma - \ell) \frac{1}{2} \\
 & + e_{[\beta}^{\nu} \bar{\epsilon} \gamma_{\alpha]} D_{[\mu} X_{\nu]} (\alpha - \kappa - 2\gamma - \ell) \\
 & - e^{-1} e_{\mu[\alpha} \epsilon_{\beta]}^{\lambda\rho\sigma} D_{\rho} \bar{X}_{\sigma} \gamma^5 \gamma_{\lambda} \epsilon (\alpha + \kappa + 2\gamma + \ell) \\
 & + e_{[\alpha}^{\sigma} e_{\beta]\mu} D_{[\rho} \bar{X}_{\sigma]} \gamma^{\rho} \epsilon (2\gamma - \beta) \\
 & + e_{[\alpha}^{\lambda} e_{\beta]\mu}^{-1} \epsilon_{\beta]}^{\rho\sigma} D_{\rho} \bar{X}_{\sigma} \gamma^5 \gamma_{\lambda} \epsilon (2\gamma + \beta) \\
 & - e_{\alpha}^{\rho} e_{\beta}^{\sigma} D_{[\rho} \bar{X}_{\sigma]} \gamma_{\mu} \epsilon (2\gamma + \beta - \ell) .
 \end{aligned}
 \tag{39}$$

If (39) is true, the following six simultaneous equations must hold:

$$\begin{aligned}
 \alpha - \kappa - 2\gamma - \ell &= 1 \\
 \alpha + \kappa - 2\gamma - \ell &= 1 \\
 \alpha + \kappa + 2\gamma + \ell &= 0 \\
 2\gamma - \beta &= 0 \\
 2\gamma + \beta &= 0 \\
 2\gamma + \beta - \ell &= 0
 \end{aligned}
 \tag{40}$$

Unfortunately, these six do not admit a solution; the five constants are over-determined. The attempt to build a covariant derivative is therefore stymied by the inability to obtain a vielbein which transforms in such a way that its components transform correctly. In particular, it does not seem possible, in four dimensions, to construct a vielbein transforming under (1) in such a way that $\delta\omega_{\mu}^{\alpha\beta}$ is given by (II.31). There are at least two possible escapes out of this dilemma:

- (a) the forms for L, M and N are not the most general;
- (b) a new transformation, a generalisation of (1), could perhaps be found.

A brief answer to (b) is that the structure of the group (1) emerged very naturally from a gauge-theoretic approach to local supersymmetry; and that alternative forms of the matrices L_B^A might not lead back to global supersymmetry transformations in the limit of constant spinor parameters and vanishing spin-connection. (See the discussion following (III.36).) Escape (a) will be foiled in the next section. The depressing conclusion is that the approach to supergravity-matter coupling described in Chapter III cannot be implemented in four dimensions.

§19.2. A Clebsch-Gordan Argument[†]

Perhaps the difficulty with the extra terms stems from a failure

[†] This argument is due entirely to my advisor, who also taught me the elegant techniques used.

to use the most general expressions for L , M and N . In all, nine terms were found. Are there others? According to a Clebsch-Gordan argument, the answer is no. The question may be rephrased as follows. Given the quantities $D_{[\mu}^{\lambda} \chi_{\nu]}$, an arbitrary spinor η , and the set $\{\bar{\epsilon}\theta, \bar{\epsilon}\gamma^5\theta, \bar{\epsilon}\gamma^\nu\gamma^5\theta\}$ how many quantities transforming as a vector can be constructed? (Note that here the quantities corresponding to $L_\mu\eta$, etc. rather than L_μ , are being considered. This does not change the count, of course, but it simplifies the algebra.) The covariant curl transforms both as an antisymmetric tensor and as a spinor; it belongs to a $[D(0,1) \oplus D(1,0)] \otimes [D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})]$ representation of the Lorentz group. The spinor η belongs to a $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$ representation, while the set $\{\bar{\epsilon}\theta, \bar{\epsilon}\gamma^5\theta, \bar{\epsilon}\gamma^\nu\gamma^5\theta\}$ are respectively $\{D(0,0), D(0,0), D(\frac{1}{2},\frac{1}{2})\}$ (but two $D(0,0)$ differ under parity transformations). The question is now: how many times does the representation $D(\frac{1}{2},\frac{1}{2})$ occur in the direct product of the above ingredients? These are

$$[D(0,1) \oplus D(1,0)] \otimes [D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})] \otimes [D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})] \\ \otimes [D(0,0) \oplus D(0,0) \oplus D(\frac{1}{2},\frac{1}{2})] ;$$

multiplying out all but the first, the relevant pieces come to:

$D(0,1) \otimes D(\frac{1}{2},\frac{1}{2})$	eight times
$D(0,1) \otimes D(\frac{1}{2},\frac{3}{2})$	once
$D(1,0) \otimes D(\frac{1}{2},\frac{1}{2})$	eight times
$D(1,0) \otimes D(\frac{3}{2},\frac{1}{2})$	once

so that the representation $D(\frac{1}{2},\frac{1}{2})$ occurs eighteen times. Half of these are even under parity, and half are odd. Consequently there are only nine independent expressions transforming as a true vector, as was earlier believed, and escape (a) is foiled.

§20. Summary and Conclusions

In this final chapter, the methods of §16 were applied in an ultimately fruitless attempt to obtain the four-dimensional vielbein. The vielbein was initially found by considering the "covariant" derivatives of a scalar superfield. The Ehresmann laws required for covariance were found to hold for the trivial members of the vielbein, but the "dynamical" pieces containing the gravitino, the vierbein and the spin connection failed to transform correctly. An attempt was made to alter the vielbein, so that the resulting pieces obeyed the Ehresmann laws, but it became apparent that no such alteration was possible. Consequently the superfield approach of Chapter III proved inapplicable to four dimensions.

What do the calculations in this last chapter imply for the vielbein formalism? If these are correct, and if the assumptions underlying them are not ill-founded, then the vielbein approach may be in trouble. That is to say, it may be that a covariant derivative transforming according to (1) does not exist, at least in four dimensions. This conclusion is hard to believe, but it is supported by calculation.

The original goal of this thesis was to obtain, via a superfield approach, the locally supersymmetric Fayet model. This goal was not reached. Without a suitable vielbein, the construction of this model could not be attempted. (Given the vielbein, it was an easy matter to work out the model in two dimensions, as in Chapter III). It is possible that an alternate superfield approach would serve to give the model, or perhaps it will ultimately be found, like so many of the early supersymmetry models, without recourse to superfields at all.

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APPENDIX A. An Abbreviated History of Anticommuting Variables

A number will find
fulfilment enough
in knowing its mind
and doing its stuff.

P. Hein.

Apparently the first person to introduce the multiplication rule

$$ab = -ba \quad (A.1)$$

was Hermann Grassmann in his Ausdehnungslehre (or "calculus of extensive magnitude") in 1844, which was later reprinted in an updated form in his Gesammelte Werke (1878). An account of his work is given by Coolidge (1944), whose presentation is followed here.

Grassmann's "extensive magnitudes" are objects e_i which form an n -dimensional vector space, but are otherwise kept as vague as possible. If two are multiplied, the quantity $[e_i e_j]$ results. These are also "extensive magnitudes", and also form a vector space. Moreover, they are to satisfy certain identities (using here and throughout the summation convention)

$$m_{ij} [e_i e_j] = 0.$$

It is postulated that these are to be invariant under the general linear group;

$$e_i = U_{ij} e_j',$$

whereupon it is necessary that

$$m_{ij} U_{ir} U_{js} [e_r' e_s'] = 0.$$

This is to be independent of U . After some index gymnastics, the following equation results:

$$(m_{ij} + m_{ji})([e_r' e_s'] + [e_s' e_r']) = 0.$$

If the original matrices m_{ij} are not required to be antisymmetric, then necessarily

$$[e_r' e_s'] = -[e_s' e_r']. \quad (\text{A.2})$$

Finally, writing $a = a_r e_r$, $b = b_s e_s$, it follows

$$ab = a_r b_s [e_r e_s] = -ba.$$

The most familiar example of this is an ordinary cross-product. In fact, Hamilton's quaternions are a special case of Grassmann's extensive magnitudes.

The next occasion upon which variables obeying (A.1) entered physics was not to be for another seventy-eight years, during the birth of quantum mechanics. In a fundamental paper, Dirac (1926) considered the assembly of two identical systems, and noted that the wave functions of the complete system were entirely specified either by the symmetric or by the antisymmetric combinations of the subsystem's wave functions. Further, if the antisymmetric set were chosen, Pauli's principle was found to hold: two or more particles could not occupy a given quantum state. On the basis of the exclusion principle, as Fermi had found some months earlier, it was easy to derive the energy distribution now bearing the names of both Fermi and Dirac. Some fifteen months later, a basis for the exclusion principle was found by Jordan and Wigner (1927).

Although Dirac had postulated the antisymmetric combinations for the two-electron wave functions, of the form

$$a_{mn}(\psi_m(x_1)\psi_n(x_2) - \psi_m(x_2)\psi_n(x_1)),$$

he did not suggest that the product $\psi_m \psi_n$ was itself antisymmetric. In the context of quantum field theory, such an idea arises almost naturally.

For a scalar field, one introduces creation and annihilation operators a_m^+ , a_n satisfying

$$[a_m, a_n] = 0; \quad [a_m^+, a_n^+] = 0; \quad [a_m, a_n^+] = \delta_{mn}.$$

Suppose however there is some field whose operators b_m, b_n^+ satisfy anticommutation relations:

$$\{b_m, b_n\} = 0; \quad \{b_m^+, b_n^+\} = 0; \quad \{b_m^+, b_n\} = \delta_{mn}. \quad (A.3)$$

As in the scalar case, the operator $N_n = b_n^+ b_n$ counts the number of modes in the n^{th} eigenstate. Because $(b_n^+)^2 = (b_n)^2 = 0$, it follows

$$\begin{aligned} (N_n)^2 &= b_n^+ b_n b_n^+ b_n \\ &= b_n^+ b_n (1 - b_n b_n^+) \\ &= b_n^+ b_n = N_n \end{aligned} \quad (A.4)$$

so that the eigenvalues λ_n of N_n must also satisfy $(\lambda_n)^2 = \lambda_n$. Thus there are but two values which λ_n can assume: zero and one. This is just Pauli's principle. Since electrons are found to satisfy the exclusion principle, their field operators must be quantised with anticommutators. Put another way, the anticommutators form the basis of the exclusion principle.

A generation later, anticommuting "numbers" were again found to be of use in quantum field theory, first by Feynman (1949) and Schwinger (1953), and later by several others. In his first paper on the theory of positrons, Feynman was led to introduce anticommuting operators obeying (A.3) in order to exhibit the equivalence between his calculational rules and the methods of hole theory. His techniques with these operators will be discussed below in connection with the results of Matthews and Salam (1956). Schwinger's

approach centred on a variational principle which allowed him to construct the Green's functions of interacting quantum fields as solutions to certain differential equations obtained from Dirac's (1947) "transformation functions" $\langle q'', t'' | q', t' \rangle$. With these objects, a state or an operator may be carried from a time t' to a later time t'' . In ordinary quantum mechanics, the states $|q, t\rangle$ obey the equations

$$\hat{q}|q, t\rangle = q(t)|q, t\rangle \quad (\text{A.5a})$$

$$\hat{p}|q, t\rangle = -i(d/dq)|q, t\rangle \quad \text{and} \quad (\text{A.5b})$$

$$[\hat{q}, \hat{p}] = i$$

as usual. In a scalar field theory, scalar fields ϕ and their conjugate momenta take the place of p and q . The corresponding transformation functions are $\langle \lambda'', \sigma'' | \lambda', \sigma' \rangle$ where λ is a complete set of observables and σ some hypersurface. The states $|\lambda, \sigma\rangle$ satisfy

$$\hat{\phi}|\lambda', \sigma'\rangle = \phi(\lambda', \sigma')|\lambda', \sigma'\rangle$$

where $\phi(\lambda', \sigma')$ is an ordinary function. However, for a spinor field, the states $|\mu, \sigma\rangle$ which form the corresponding transformation functions must satisfy

$$\hat{\psi}|\mu, \sigma\rangle = \chi(\mu, \sigma)|\mu, \sigma\rangle$$

where now $\chi(\mu, \sigma)$ is an anticommuting c-number. Schwinger introduced these anticommuting eigenvalues in order to work out the spinor field's Green's functions.

In an alternate, and closely related approach to field theory, Feynman had established that the transformation functions could be written (Abers and Lee, 1973)

$$\langle q'', t'' | q', t' \rangle \sim \int d[q] d[p] \exp[i \int_{t'}^{t''} (pq - H(p, q)) dt] \quad (\text{A.6})$$

(the constant of proportionality is inessential). The integral on the right-hand side is to be understood as the limit, as $n \rightarrow \infty$, of an n -fold multiple integral. Similar relations hold in field theory. However, as the spinor field requires expressions like

$$\int d[\psi] d[\bar{\psi}] \exp[i \int (\bar{\psi} \dot{\psi} - H) d^4x] , \quad (A.7)$$

how were the integrations over anticommuting fields to be realised? This was first answered by Matthews and Salam (1956). Following the basic idea of Feynman, the spinor fields were first split into two real fields (this is always possible in the Majorana representation)

$$\psi = \psi_a + i\psi_b$$

and then expanded into normal modes

$$\psi_a = \sum_n a_n \psi_n \quad \psi_b = \sum_n b_n \psi_n$$

whose eigenfunctions ψ_n anticommute among themselves and are normalized according to

$$\int \bar{\psi}_j (i\partial - m) \psi_k d^4x = \delta_{jk}.$$

The expansion coefficients a_n, b_n on the other hand are commuting. Finally, the integration is defined by

$$\begin{aligned} d[\psi] d[\bar{\psi}] &= \lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} \prod_n \psi_n \psi_m da_n db_m \\ &\sim \lim \prod da_n db_m \end{aligned}$$

where the inessential factor $\prod \psi_n \psi_m$ may be dropped. In this way, the anticommuting integrals are circumvented. The Green's functions themselves are given by (Abers and Lee, 1973)

$$\langle 0 | T(\bar{\psi}(1) \dots \bar{\psi}(n) \psi(1') \dots \psi(n')) | 0 \rangle$$

$$\sim \int d[\psi] d[\bar{\psi}] \bar{\psi}(1) \dots \psi(n') \exp(i \int L d^4 x).$$

The chief result of Matthews and Salam is that

$$\langle 0 | T(\bar{\psi}(1) \dots \psi(n')) | 0 \rangle \sim \det^{(n)} S_F(i, j') \quad (A.8)$$

where e.g.

$$\det^{(2)} S_F(i, j') = \begin{vmatrix} S_F(1, 1') & S_F(2, 1') \\ S_F(1, 2') & S_F(2, 2') \end{vmatrix}.$$

Let the free spinor fields now be subjected to a perturbation of the form

$$L \rightarrow L' = L + \bar{\psi} \psi V.$$

The vacuum to vacuum amplitude $\langle 0 | S | 0 \rangle = \exp(iW)$ may be written

$$\exp(iW) \sim \int d[\bar{\psi}] d[\psi] \exp(i \int L' d^4 x).$$

By expanding the exponential which contains V , it follows at once from (A.8) that

$$\exp(iW) \sim \sum \frac{1}{n!} \int dx_1 \dots \int dx_n \det^{(n)} S_F(x_i, x_j) V(x_j).$$

The right-hand side of this equation is just the definition of the Fredholm determinant $\det_F$ of the kernel $S_F V$ (Goursat, 1964);

$$\exp(iW) \sim \det_F(S_F V).$$

However, it is well-known from Fredholm theory that*

$$\det_F A = \exp \text{Tr} \log(1 - A)$$

* The Tr with capital T indicates integration over all space-time variables; the 1 is a δ -function.

In keeping with the standard conventions, let a generalised determinant Det be defined by $\text{Det } A = \exp \text{Tr } \log A$ for any kernel $A(x,y)$ (here considered as a "continuous matrix"). Then finally,

$$\exp(iW) \sim \text{Det}(1 - S_F V).$$

On the other hand, the result equivalent to (A.8) for the scalar field is

$$\langle 0 | T(\phi(1) \dots \phi(n')) | 0 \rangle \sim \text{perm}^{(n)} \Delta_F(i, j') \quad (\text{A.9})$$

where the "permanent" is the sum of all permutations of the products $\Delta_F(i_1, j_1') \dots \Delta_F(i_n, j_n')$ without the sign changes associated with the determinant. If the amplitude $\langle 0 | S | 0 \rangle$ is calculated in this instance, then

$$\exp(iW) \sim \sum \frac{1}{n!} \int dx_1 \dots \int dx_n \text{perm}^{(n)} \Delta_F(x_i, x_j) V(x_j).$$

Now the right-hand side is not equal to the Fredholm determinant, but rather to its inverse. Consequently for the Bose case,

$$\exp(iW) \sim (\text{Det}(1 - \Delta_F V))^{-1};$$

the overall difference is the sign of the determinant's exponent.

The relations (A.8) and (A.9) are consequences of the relation (A.6). In order to obtain Feynman's expression (A.6), it is necessary to construct "resolutions of the identity". In the simple quantum mechanical example, these are provided by

$$\int |q\rangle dq \langle q| = 1; \quad \int |p\rangle dp \langle p| = 1.$$

If the expression (A.7) is to be derived in a manner analogous to the method of obtaining (A.6), similar resolutions must be found in terms of canonical anticommuting variables. Consider again the system described by (A.3), and let there be eigenstates $|\beta\rangle, \langle\beta^+|$ satisfying

$$b|\beta\rangle = \beta|\beta\rangle \quad (\text{A.10a})$$

$$b^+|\beta^+\rangle = \beta^+|\beta^+\rangle, \quad \text{where} \quad (\text{A.10b})$$

$$\{\beta, \beta\} = \{\beta^+, \beta^+\} = \{\beta, \beta^+\} = 0. \quad (\text{A.10c})$$

In his paper of 1956, Candlin argued that the resolution

$$\sum |\beta\rangle\langle\beta| = 1 \quad (\text{A.11})$$

is impossible. On the other hand, it is possible to set

$$\sum |\beta\rangle\langle\beta| = \rho \quad (\text{A.12})$$

where ρ is another Grassmann variable (called by Candlin an "a-number").

This, he pointed out, was essentially the realisation chosen (in the continuous case) by Matthews and Salam; the role of ρ being there assumed by the anticommuting eigenfunctions ψ_n . As was later shown by Martin (1959), the relation (A.11) may be obtained in a modified form. Let λ be a nilpotent quantity satisfying

$$\lambda^{n+1} = 0.$$

To any column vector $|a\rangle$ with components $(a_0, \dots, a_n)$ associate the polynomial $(\lambda|a\rangle$;

$$(\lambda|a\rangle = \sum \lambda^l a_l. \quad (\text{A.14})$$

That is, $(\lambda|$ is the row vector

$$(\lambda| = (1, \lambda, \lambda^2, \dots, \lambda^n) \quad (\text{A.15})$$

The dual of $(\lambda|$ is the column vector $|\lambda\rangle$ with components

$(\lambda^n, \lambda^{n-1}, \dots, 1)$. It is easy to see that $(\lambda|$ is an eigenvector, with eigenvalue λ , of the matrix Λ whose only non-vanishing elements occur on the subdiagonal, and are equal to one;

$$(\lambda|\Lambda = (\lambda, \lambda^2, \dots, \lambda^n, 0) = (\lambda|\lambda.$$

Moreover, if the inner product is taken as usual

$$\langle a|b \rangle = a_m b_m \quad (m = 1, \dots, n)$$

then it is possible to set

$$\langle a|b \rangle = S_\lambda \langle a|\lambda \rangle (\lambda|b \rangle$$

where S_λ is an operator which chooses the coefficient of λ^n ;

$$S_\lambda \lambda^m = \delta^{mn}.$$

Consequently, one has the relation

$$S_\lambda |\lambda \rangle (\lambda| = 1, \quad (A.16)$$

which is the generalisation of (A.11).

An obvious realisation of the operator S_λ is

$$S_\lambda = (1/n!) d^n / d\lambda^n.$$

The systematic exposition of the calculus of Grassmann variables was finally given by Berezin (1966), again in connection with functional methods. In §4.1 there was given a derivation of his rule that integration and differentiation are, for these variables, the same operation. Harking back for a moment to Martin's work, the completeness relation (A.16) may now be written, with the help of Berezin's rule, in the appealing form

$$(1/n!) \int d^n \lambda |\lambda \rangle (\lambda| = 1. \quad (A.17)$$

As the last example, consider Berezin's version of the Fermi Gaussian integral

$$I = \int d^{2n} \theta \exp(\frac{1}{2} \theta^T A \theta) \quad (A.18)$$

where θ^T is the $2n$ -vector $(\theta_1, \theta_2, \dots, \theta_{2n})$, and A is an anti-symmetric matrix. It is well-known for the corresponding Bose integral

$$\int d^n x \exp(\frac{1}{2} x^T S x) = (\det S)^{-\frac{1}{2}}.$$

In order to evaluate (A.18), note that it is always possible to reduce A to the canonical form

$$A \rightarrow R A R^T = A_\lambda = \begin{pmatrix} 0 & \lambda_1 & 0 & . & . \\ -\lambda_1 & 0 & \lambda_2 & . & . \\ 0 & -\lambda_2 & 0 & . & . \\ . & . & . & . & . \end{pmatrix}$$

$$\theta \rightarrow R\theta = \epsilon$$

by a similarity transformation whose Jacobian is equal to one. Also, it is evident that

$$\det A = \det A_\lambda = \prod_{i=1}^n \lambda_i^2.$$

Accordingly,

$$I = \int d^{2n} \epsilon \exp(\frac{1}{2} \epsilon^T A_\lambda \epsilon)$$

where $d^{2n} \epsilon = \prod d\epsilon_i$, (similarly for derivatives), which by Berezin's rule is equal to

$$\begin{aligned} I &= \frac{\partial^{2n}}{\partial \epsilon^{2n}} \exp(\frac{1}{2} \epsilon^T A_\lambda \epsilon) \\ &= \frac{\partial^{2n}}{\partial \epsilon^{2n}} \left(\frac{1}{n!} (\lambda_1 \epsilon_1 \epsilon_2 + \lambda_2 \epsilon_3 \epsilon_4 + \dots + \lambda_n \epsilon_{2n-1} \epsilon_{2n})^n \right) \\ &= \prod_{i=1}^n \lambda_i = (\det A)^{\frac{1}{2}} \end{aligned} \tag{A.19}$$

which differs from the Bose result only by the sign of the exponent, a result presaged by Matthews and Salam. This famous sign difference has been exploited with great success in the past ten years for the quantisation

of non-Abelian gauge fields. Lagrangians involving these gauge fields are invariably singular, owing to gauge invariance; not all of the p 's and q 's are independent. To determine a set of canonical variables, it is first necessary to break the invariance by adding a "gauge-fixing" term. In addition, the passage from (A.6) to the functional integral involving only the canonical coordinates q usually introduces a Jacobian which depends non-trivially on the q 's themselves. In this form of the functional integral, the Feynman rules are unclear. By virtue of (A.19), however, this determinant may be rewritten as the functional integral of a Lagrangian describing anticommuting (scalar) fields (usually called "ghosts"). Once this determinant has been moved up from the integrand into the exponential of the action, the Feynman rules become transparent. This procedure was worked out by Faddeev and Popov (1969); the reader is referred to Abers and Lee (1973) for details. It is not unreasonable to suppose that there are other useful schemes waiting to be found, quite apart from supersymmetry, involving these intriguing "numbers".

APPENDIX B

The Use of Differential Forms

The differential geometric formulation of general relativity (as presented in, e.g., Misner, Thorne and Wheeler 1973) relies on the manipulation of operators in two different vector spaces, which are dual to each other in the same sense as Dirac's bra and ket vectors. The basis vectors of the first space (the "tangent space") are the differential operators ∂_μ ; those in the dual space are the differentials dx^ν themselves. These differentials are also regarded as operators, which act only on the set. An inner product $(\cdot, \cdot)$ between these two spaces is defined as follows:

$$(dx^\mu, \partial_\nu) = \delta^\mu_\nu. \quad (B.1)$$

This strange looking equation is merely the statement

$$\partial_\nu \int dx^\mu = \delta^\mu_\nu.$$

The single greatest advantage of the modern notation is that all tensors (except scalars) are replaced by coordinate-independent objects. (It turns out, in addition, that the modern approach frequently allows for easier calculation.) For example, with a contravariant vector field $u^\mu(x)$ may be associated the vector operator

$$\underline{u}(x) = u^\mu(x) \partial_\mu$$

in the tangent space. Obviously the operator $\underline{u}$ is invariant under the general coordinate group. Similarly, a covariant vector $\pi_\mu(x)$ may be associated with an operator in the dual space

$$\underline{\pi}(x) = \pi_\mu(x) dx^\mu \quad (B.2)$$

called a "differential form". Of course, any other basis

$e_\alpha(x) = e_\alpha^\mu(x) \partial_\mu$ of derivatives, or $\omega^\beta(x) = e_\mu^\beta dx^\mu$ of forms, would

serve equally well, so long as they were properly normalised:

$$(\omega^\alpha, e_\beta) = \delta^\alpha_\beta.$$

The choice of bases corresponding to $e_\mu = \partial_\mu$ and $\omega^\nu = dx^\nu$ is sometimes called the "natural basis"; this is not always the most convenient choice. The inner product between a "vector" $\underline{u}$ in the tangent space and a "form" $\underline{\pi}$ in the dual space may be written

$$\begin{aligned} (\underline{\pi}, \underline{u}) &= (\pi_\mu \omega^\mu, u^\nu e_\nu) = \pi_\mu u^\nu (\omega^\mu, e_\nu) \\ &= \pi_\mu u^\mu. \end{aligned} \quad (B.3)$$

That is, the inner product is linear with respect to arbitrary functions of x . Sometimes the inner product $(\underline{\pi}, \underline{u})$ is written $\underline{\pi}(\underline{u})$; forms may be regarded as operators which map "vectors" to functions (while "vectors" map functions to functions). Unlike basis vectors, basis forms are linear with respect to functions;

$$\omega^\mu(\underline{u}) = \omega^\mu(u^\nu e_\nu) = u^\nu \omega^\mu(e_\nu) = u^\mu. \quad (B.4)$$

Covariant derivatives ∇_μ are introduced as follows. Acting on ordinary functions (such as ordinary tensors $A_{\mu\nu} \dots^{\rho\sigma\dots}$), the covariant derivative is just the ordinary derivative. However, acting on basis vectors e_ν ,

$$\nabla_\mu e_\nu = \Gamma_{\mu\nu}^\lambda e_\lambda \quad (B.5)$$

where $\Gamma_{\mu\nu}^\lambda$ is the connection. The covariant derivative of a "vector" $\underline{u}$ is therefore given by

$$\begin{aligned} \nabla_\mu \underline{u} &= \nabla_\mu (u^\nu e_\nu) = (\partial_\mu u^\nu) e_\nu + u^\nu (\nabla_\mu e_\nu) \\ &= (\partial_\mu u^\nu + \Gamma_{\mu\nu}^\lambda u^\lambda) e_\nu \\ &\equiv u^\nu{}_{;\mu} e_\nu \end{aligned} \quad (B.6)$$

because ∇_μ satisfies the Leibniz rule, and also

$$\nabla_{\mu} u^{\nu} = e_{\mu}(u^{\nu}) = \partial_{\mu} u^{\nu}.$$

By differentiating the orthogonality relation (B.1), it follows

$$(\nabla_{\mu} \omega^{\nu}, e_{\lambda}) = -(\omega^{\nu}, \nabla_{\mu} e_{\lambda})$$

whence

$$\nabla_{\mu} \omega^{\nu} = -\Gamma_{\mu\lambda}^{\nu} \omega_{\lambda} \quad (B.7)$$

and obviously for a form π ,

$$\nabla_{\mu} \pi = (\partial_{\mu} \pi_{\lambda} - \Gamma_{\mu\lambda}^{\nu} \pi_{\nu}) \omega^{\lambda} = \pi_{\lambda;\mu} \omega^{\lambda} \quad (B.8)$$

(Sometimes it is convenient to employ a "directional derivative"

$$\nabla_u = \nabla_{u^{\nu} e_{\nu}} \equiv u^{\nu} \nabla_{\nu}.)$$

Additional vector spaces may be constructed through the use of a tensor product $\otimes$. This tensor product is completely analogous to that which occurs in the addition of angular momenta. For example, given two spin systems described by the bases $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$, one builds eigenstates $|j, m\rangle$ of the operator $J_z = J_{1z} + J_{2z}$ as tensor products of the eigenstates of J_{1z} and J_{2z} ;

$$\begin{aligned} & (J_{1z} + J_{2z})(|j_1, m_1\rangle \otimes |j_2, m_2\rangle) \\ &= (J_{1z}|j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (J_{2z}|j_2, m_2\rangle) . \end{aligned}$$

Then, for example, the tensor products $\omega^{\mu} \otimes \omega^{\nu}$ and $e_{\rho} \otimes e_{\sigma}$ obey the relation

$$(\omega^{\mu} \otimes \omega^{\nu}, e_{\rho} \otimes e_{\sigma}) = \delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} . \quad (B.9)$$

The tensor product allows any rank-N tensor to be described as a coordinate-independent operator. An example is the arbitrary tensor

$A^{\rho}_{\mu\nu\sigma}$, which may be associated with the operator $\underline{A}$:

$$\underline{A} = A_{\mu\nu}^{\rho} \omega^{\mu} \otimes \omega^{\nu} \otimes e_{\rho} \otimes \omega^{\sigma} .$$

Particularly valuable for calculations are the anti-symmetric tensor products, denoted by $\times$:

$$\omega^{\mu} \times \omega^{\nu} = \frac{1}{2}(\omega^{\mu} \otimes \omega^{\nu} - \omega^{\nu} \otimes \omega^{\mu}) \quad (B.10)$$

$$\omega^{\mu} \times \omega^{\nu} \times \omega^{\rho} = \left(\frac{1}{3!}\right) \delta_{\alpha\beta\gamma}^{\mu\nu\rho} \omega^{\alpha} \otimes \omega^{\beta} \otimes \omega^{\gamma} \quad (B.11)$$

and so on. The cross-product (sometimes called a "wedge product") of n differentials is called an n -form; the differentials themselves are one-forms. In general, if $\underline{\pi}$ is a p -form, and $\underline{\omega}$ a q -form, then $\underline{\pi} \times \underline{\omega}$ is a $(p+q)$ -form. Note however that

$$\underline{\pi} \times \underline{\omega} = (-)^{pq} \underline{\omega} \times \underline{\pi} . \quad (B.12)$$

There is another extremely important method of generating forms of higher degree from one-forms: Cartan's "exterior differentiation".

Consider the function $f(x) = x^{\mu}$. If the new operator "d" is applied to this function, it becomes the one-form dx^{μ} , satisfying $dx^{\mu}(e_{\nu}) = \delta^{\mu}_{\nu}$. Clearly this procedure can be generalised to an arbitrary function $f(x)$. Given $f(x)$, define the new one-form $\underline{df}$ by

$$(\underline{df}, \partial_{\nu}) \equiv \underline{df}(\partial_{\nu}) \equiv \partial_{\nu} f . \quad (B.13)$$

But if $\underline{df}$ is regarded as a one-form, it may be expanded as

$$\underline{df} = a_{\mu} dx^{\mu} \quad (B.14)$$

where $a_{\mu}(x)$ are some functions to be determined. By linearity,

$$(\underline{df}, \partial_{\nu}) = (a_{\mu}(x) dx^{\mu}, \partial_{\nu}) = a_{\mu} (dx^{\mu}, \partial_{\nu}) = a_{\nu}$$

but by definition (B.13), it follows $a_{\nu} = \partial_{\nu} f$, whence the classical formula

$$\underline{df} = \partial_v f dx^v$$

is rederived. Further, for a constant c , $\underline{dc} \equiv 0$. The operator $\underline{d}$ may profitably be defined to act on an arbitrary one-form $\underline{\pi}$ as

$$\underline{d\pi} = d(\pi_v dx^v) \equiv \partial_\lambda \pi_v dx^\lambda \times dx^v. \quad (B.16)$$

The resultant object is a two-form. In general, the exterior derivative of a p -form is a $(p+1)$ -form. For an arbitrary p -form

$$\begin{aligned} \underline{\pi} &= \pi_{\mu_1 \mu_2 \dots \mu_n} dx^{\mu_1} \times \dots \times dx^{\mu_n}, \\ \underline{d\pi} &= \partial_v \pi_{\mu_1 \mu_2 \dots \mu_n} dx^v \times dx^{\mu_1} \times \dots \times dx^{\mu_n} \end{aligned} \quad (B.17)$$

From (B.17) follows the Poincaré lemma: given an arbitrary p -form $\underline{\pi}$, then

$$\underline{d^2\pi} = 0. \quad (B.18)$$

Finally for $\underline{\pi}$ a p -form and $\underline{\omega}$ a q -form,

$$d(\underline{\pi} \times \underline{\omega}) = \underline{d\pi} \times \underline{\omega} + (-)^p \underline{\pi} \times \underline{d\omega}. \quad (B.19)$$

Now consider the action of these p -forms on p -fold tensor products of "vectors". Inasmuch as inner products were formerly linear with respect to functions, it should be that for $\underline{\pi}$ a one-form.

$$(\underline{d\pi}, \underline{u} \otimes \underline{v}) = u^\mu v^\nu (\underline{d\pi}, e_\mu \otimes e_\nu). \quad (B.20)$$

But

$$\begin{aligned} (\underline{d\pi}, e_\mu \otimes e_\nu) &= \partial_\rho \pi_\sigma (dx^\rho \times dx^\sigma, e_\mu \times e_\nu) \\ &= \frac{1}{2} \partial_\rho \pi_\sigma \delta_{\mu\nu}^{\rho\sigma} \\ &= \frac{1}{2} [e_\mu((\underline{\pi}, e_\nu)) - e_\nu((\underline{\pi}, e_\mu))] . \end{aligned} \quad (B.21)$$

Then, if $\underline{u}, \underline{v}$ are substituted for e_μ and e_ν ,

$$(\underline{d}\pi, \underline{u} \otimes \underline{v}) = \frac{1}{2} [\underline{u}((\pi, \underline{v})) - \underline{v}((\pi, \underline{u}))] \quad (B.22)$$

However,

$$\begin{aligned} \underline{u}((\pi, \underline{v})) &= u^\mu e_\mu(\pi_\rho v^\rho) \\ &= u^\mu v^\rho \partial_\mu \pi_\rho + u^\mu \pi_\rho \partial_\mu v^\rho \end{aligned}$$

so that as given by (B.22), $\underline{d}\pi$ is not linear over functions;

$$\begin{aligned} (\underline{d}\pi, \underline{u} \otimes \underline{v}) &= u^\mu v^\nu (\underline{d}\pi, e_\mu \otimes e_\nu) \\ &\quad + \frac{1}{2} [(\pi, e_\mu) \underline{u}(v^\mu) - (\pi, e_\nu) \underline{v}(u^\mu)] \end{aligned}$$

To repair the defect, consider the quantity

$$\begin{aligned} [\underline{u}, \underline{v}] &= \underline{u}(\underline{v}) - \underline{v}(\underline{u}) \\ &= u^\mu v^\nu [e_\mu, e_\nu] + \underline{u}(v^\mu) e_\mu - \underline{v}(u^\mu) e_\mu \end{aligned}$$

In the natural basis, of course,

$$[e_\mu, e_\nu] = 0$$

but in general

$$[e_\mu, e_\nu] = c^\lambda_{\mu\nu} e_\lambda$$

Subtracting the quantity $\frac{1}{2}(\pi, [\underline{u}, \underline{v}])$ from the first definition (B.22) leads to a linear inner product. Therefore the action of a two-form $\underline{d}\pi$ on a bivector $\underline{u} \times \underline{v}$ is defined to be

$$(\underline{d}\pi, \underline{u} \times \underline{v}) = \underline{u}((\pi, \underline{v})) - \underline{v}((\pi, \underline{u})) - (\pi, [\underline{u}, \underline{v}]) \quad (B.23)$$

which is obviously linear by construction:

$$(\underline{d}\pi, \underline{u} \times \underline{v}) = u^\mu v^\nu (\underline{d}\pi, e_\mu \times e_\nu) \quad (B.24)$$

The operator $\underline{d}$ may also be applied to vectors, in which case it acts

like the covariant derivative ∇ . The defining relation is (cf. (B.13))

$$\begin{aligned} (\underline{de}_\mu, e_\nu) &= \nabla_\nu e_\mu \\ &= \Gamma_{\nu\mu}^\lambda e_\lambda \quad \text{by definition (B.5)} \end{aligned} \quad (\text{B.25})$$

which means that

$$\underline{de}_\mu = \Gamma_{\mu\lambda}^\nu \omega^\lambda \otimes e_\nu. \quad (\text{B.26})$$

It will be convenient to define the connection one-forms ω^ν_μ as

$$\omega^\nu_\mu = \Gamma_{\lambda\mu}^\nu \omega^\lambda. \quad (\text{B.27})$$

The Bianchi identities arise as consequences of the Poincaré lemma applied to the basis and connection one-forms, as will now be shown.

First, it is necessary to define the torsion and curvature tensors;

$$2\underline{T}(\underline{u}, \underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u}, \underline{v}] \quad (\text{B.28a})$$

$$2\underline{R}(\underline{u}, \underline{v}) = \nabla_{\underline{u}} \nabla_{\underline{v}} - \nabla_{\underline{v}} \nabla_{\underline{u}} - \nabla_{[\underline{u}, \underline{v}]} \quad (\text{B.28b})$$

Using the identities

$$\begin{aligned} v^\mu &= \omega^\mu(\underline{v}) \\ \nabla_{\underline{u}} \underline{v} &= \{\underline{u}(v^\nu) + v^\mu \omega^\nu_\mu(\underline{u})\} e_\nu \end{aligned}$$

and the definition (B.24), it is straightforward to compute that

$$\underline{T}(\underline{u}, \underline{v}) = (\underline{d\omega}^\mu + \omega^\mu_\nu \times \omega^\nu, \underline{u} \otimes \underline{v}).$$

From the identification $\underline{T} = \underline{T}^\mu \otimes e_\mu$ comes Cartan's "first structure equation";

$$\underline{T}^\mu = \underline{d\omega}^\mu + \omega^\mu_\nu \times \omega^\nu \quad (\text{B.29})$$

Similarly the action of the curvature tensor $\underline{R}(\underline{u}, \underline{v})$ on some basis vector may be computed:

$$2\underline{R}(\underline{u}, \underline{v})e_\mu = (\nabla_{\underline{u}} \nabla_{\underline{v}} - \nabla_{\underline{v}} \nabla_{\underline{u}} - \nabla_{[\underline{u}, \underline{v}]})e_\mu, \quad \text{where}$$

$$\begin{aligned}
 \nabla_{\underline{u}} \nabla_{\underline{v}} e_{\underline{\mu}} &= \nabla_{\underline{u}} \omega_{\underline{\mu}}^{\underline{v}}(\underline{v}) e_{\underline{v}} \\
 &= \underline{u}(\omega_{\underline{\mu}}^{\underline{v}}(\underline{v})) e_{\underline{v}} + \omega_{\underline{\mu}}^{\underline{v}}(\underline{v}) \nabla_{\underline{u}} e_{\underline{v}} \\
 &= \underline{u}(\omega_{\underline{\mu}}^{\underline{v}}(\underline{v})) e_{\underline{v}} + \omega_{\underline{\mu}}^{\underline{v}}(\underline{v}) \omega_{\underline{\mu}}^{\lambda}(\underline{u}) e_{\underline{v}} .
 \end{aligned}$$

Using the expansion $[\underline{u}, \underline{v}] = (\omega^{\underline{\mu}}, [\underline{u}, \underline{v}]) e_{\underline{\mu}}$, it follows

$$\begin{aligned}
 \nabla_{[\underline{u}, \underline{v}]} &= (\omega^{\underline{\mu}}, [\underline{u}, \underline{v}]) \nabla_{\underline{\mu}} \quad \text{so that} \\
 \nabla_{[\underline{u}, \underline{v}]} e_{\underline{v}} &= (\omega^{\underline{\mu}}, [\underline{u}, \underline{v}]) \Gamma_{\underline{\mu} \underline{v}}^{\lambda} e_{\lambda} = (\omega_{\underline{v}}^{\lambda}, [\underline{u}, \underline{v}]) e_{\lambda} .
 \end{aligned}$$

Then

$$\underline{R}(\underline{u}, \underline{v}) e_{\underline{\mu}} = (\underline{d}\omega_{\underline{\mu}}^{\underline{v}} + \omega_{\lambda}^{\underline{v}} \times \omega_{\underline{\mu}}^{\lambda}, \underline{u} \otimes \underline{v}) e_{\underline{v}} .$$

Writing $\underline{R} = \underline{R}_{\underline{v}}^{\underline{\mu}} \otimes e_{\underline{\mu}} \otimes \omega^{\underline{v}}$ leads to Cartan's "second structure equation"

$$\underline{R}_{\underline{v}}^{\underline{\mu}} = \underline{d}\omega_{\underline{v}}^{\underline{\mu}} + \omega_{\lambda}^{\underline{\mu}} \times \omega_{\underline{v}}^{\lambda} \tag{B.30}$$

The tensor $\underline{R}_{\underline{v}}^{\underline{\mu}}$ is called the curvature two-form. For the natural basis, it follows after some trivial algebra that

$$\underline{R}_{\underline{v}}^{\underline{\mu}} = -\frac{1}{2} R_{\underline{v} \rho \sigma}^{\underline{\mu}} dx^{\rho} \times dx^{\sigma} \tag{B.31}$$

$$\underline{T}^{\underline{\mu}} = \frac{1}{2} \Gamma_{[\underline{v} \rho]}^{\underline{\mu}} dx^{\underline{v}} \times dx^{\rho} \tag{B.32}$$

where the Riemann tensor is defined as usual. The Bianchi identities are now proved as follows. Differentiating the first Cartan equation (letting the connection be symmetric, i.e. $T^{\underline{\mu}} = 0$) leads to

$$\begin{aligned}
 0 &= \underline{d}(\underline{d}\omega_{\underline{v}}^{\underline{\mu}} + \omega_{\underline{v}}^{\underline{\mu}} \times \omega^{\underline{v}}) = \underline{d}\omega_{\underline{v}}^{\underline{\mu}} \times \omega^{\underline{v}} - \omega_{\underline{v}}^{\underline{\mu}} \times \underline{d}\omega^{\underline{v}} \\
 &= \underline{d}\omega_{\underline{v}}^{\underline{\mu}} \times \omega^{\underline{v}} - \omega_{\underline{v}}^{\underline{\mu}} \times (-\omega_{\lambda}^{\underline{v}} \times \omega^{\lambda}) \\
 &= \underline{R}_{\underline{v}}^{\underline{\mu}} \times \omega^{\underline{v}} = -\frac{1}{2} R_{\underline{v} \rho \sigma}^{\underline{\mu}} \omega^{\underline{v}} \times \omega^{\rho} \times \omega^{\sigma}, \quad \text{whence}
 \end{aligned}$$

$R^\mu_{[\nu\rho\sigma]} = 0$, the first identity.

From the second Cartan equation,

$$\underline{d}\omega^\mu_\nu = \underline{R}^\mu_\nu - \omega^\mu_\lambda \times \omega^\lambda_\nu.$$

Recall the Poincaré lemma; $\underline{d}^2$ on a form vanishes. Then

$$\begin{aligned} \underline{d}(\underline{R}^\mu_\nu - \omega^\mu_\lambda \times \omega^\lambda_\nu) &= 0 = \underline{d}^2 \omega^\mu_\nu \\ &= \underline{d}\underline{R}^\mu_\nu - \underline{d}\omega^\mu_\lambda \times \omega^\lambda_\nu + \omega^\mu_\lambda \times \underline{d}\omega^\lambda_\nu \\ &= \underline{d}\underline{R}^\mu_\nu - (\underline{R}^\mu_\lambda - \omega^\mu_\sigma \times \omega^\sigma_\lambda) \times \omega^\lambda_\nu \\ &\quad + \omega^\mu_\lambda \times (\underline{R}^\lambda_\nu - \omega^\lambda_\sigma \times \omega^\sigma_\nu) \\ &= \underline{d}\underline{R}^\mu_\nu - \underline{R}^\mu_\lambda \times \omega^\lambda_\nu + \underline{R}^\lambda_\nu \times \omega^\mu_\lambda \\ &= -\frac{1}{2} R^\mu_{\nu[\rho\sigma;\tau]} \omega^\rho \times \omega^\sigma \times \omega^\tau \quad \text{so that} \end{aligned}$$

$R^\mu_{\nu[\rho\sigma;\tau]} = 0$, the second identity.

Finally the Ehresmann formula may be proved easily. By definition,

$$\underline{d}e_\mu = \omega^\nu_\mu \otimes e_\nu.$$

Let $e_\mu \rightarrow e_{\mu'} = U_{\mu'}^\lambda e_\lambda$ under the coordinate group. Because $\underline{d}$ is coordinate-independent,

$$\underline{d}e_{\mu'} = \omega^{\nu'}_{\mu'} \otimes e_{\nu'} = \omega^{\nu'}_{\mu'} \otimes U_{\nu'}^\lambda e_\lambda; \text{ but also}$$

$$\begin{aligned} \underline{d}e_{\mu'} &= \underline{d}(U_{\mu'}^\lambda e_\lambda) = \underline{d}U_{\mu'}^\lambda \otimes e_\lambda + U_{\mu'}^\lambda \underline{d}e_\lambda \\ &= (\underline{d}U_{\mu'}^\lambda + U_{\mu'}^\nu \omega^\lambda_\nu) \otimes e_\lambda; \quad \text{i.e.} \end{aligned}$$

$$\omega^{\nu'}_{\mu'} U_{\nu'}^\lambda = \underline{d}U_{\mu'}^\lambda + U_{\mu'}^\nu \omega^\lambda_\nu, \text{ or}$$

$$\omega^{\nu'}_{\mu'} = (dU_{\mu'}^\lambda)(U^{-1})^\lambda_{\nu'} + U_{\mu'}^\sigma \omega^\lambda_\sigma (U^{-1})^\lambda_{\nu'} \quad (\text{B.33})$$

which is just the Ehresmann formula. In terms of the connection, this reads

$$\Gamma_{\lambda', \mu'}^{\nu'} dx^{\lambda'} = (dU_{\mu'}^{\lambda'}) (U^{-1})_{\lambda'}^{\nu'} + U_{\mu'}^{\sigma} \Gamma_{\rho\sigma}^{\lambda} dx^{\rho} (U^{-1})_{\lambda'}^{\nu'}.$$

Recall that

$$(dx^{\rho}, e_{\tau'}) = U_{\tau'}^{\lambda} (dx^{\rho}, e_{\lambda}) = U_{\tau'}^{\rho}; \quad \text{so}$$

taking the inner product of (B.33) with $e_{\tau'}$ leads to

$$\Gamma_{\lambda', \mu'}^{\nu'} = (\partial_{\lambda'} U_{\mu'}^{\rho}) (U^{-1})_{\rho}^{\nu'} + U_{\mu'}^{\sigma} U_{\lambda'}^{\rho} \Gamma_{\rho\sigma}^{\tau} (U^{-1})_{\tau}^{\nu'}$$

which is the usual law. Consequently as claimed in §12.2, there is no contradiction between the Ehresmann law and the transformation of $\Gamma_{\mu\nu}^{\lambda}$.

It would be a shame, having gone into the details of this formulation, not to show its wonderful calculational facility, especially inasmuch as this is intimately related to one of the "stars" of this thesis, namely the vierbein e_{μ}^{α} . To that end a brief outline of Cartan's method of curvature calculation is given for the exceptionally easy case of the Kasner metric,

$$g = dt^2 - t^{2p} dx^2 - t^{2q} dy^2 - t^{2r} dz^2$$

where p, q and r are constants.

Instead of the natural basis it will be more convenient to use the basis

$$\begin{aligned} \omega^0 &= dt & e_0 &= \partial/\partial t \\ \omega^1 &= t^p dx & e_1 &= t^{-p} \partial/\partial x \\ \omega^2 &= t^q dy & e_2 &= t^{-q} \partial/\partial y \\ \omega^3 &= t^r dz & e_3 &= t^{-r} \partial/\partial z. \end{aligned} \tag{B.34}$$

This basis is related to the natural basis through the vierbein e_{μ}^{α} ;

$$e_{\mu}^{\alpha} = \text{diag. } (1, t^p, t^q, t^r)$$

and consequently the metric $g_{\mu\nu} = \eta_{\mu\nu}$. Introducing a dot product between basis vectors by

$$e_\mu \cdot e_\nu = g_{\mu\nu}$$

it is not hard to show (from B.5) that

$$\underline{d}g_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}. \quad (\text{B.35})$$

Therefore, by choosing the "moving frame" corresponding to (B.34), it follows that the connection forms $\omega_{\mu\nu}$ are antisymmetric:

$$\underline{d}\eta_{\mu\nu} = 0 = \omega_{\mu\nu} + \omega_{\nu\mu}.$$

These six objects, once found, lead via the second structure equation to $\underline{R}^\mu_\nu$, and thence to $R^\mu_{\nu\rho\sigma}$. There is a systematic approach to finding the $\omega_{\mu\nu}$, but it is usually easier to guess them from the first structure equation

$$0 = d\omega^\mu + \omega^\mu_\nu \times \omega^\nu.$$

From $\omega^0 = dt$, it follows

$$0 = d^2t + \omega^0_i \times \omega^i = \omega^0_i \times \omega^i$$

which is most easily satisfied by assuming

$$\omega^0_i \sim \omega^i.$$

Next, differentiating ω^1 leads to

$$\begin{aligned} d(t^p dx) + \omega^1_\mu \times \omega^\mu &= 0 \\ &= p t^{p-1} dt \times dx + \omega^1_\mu \times \omega^\mu \\ &= -(p/t) \omega^1 \times \omega^0 + \omega^1_\mu \times \omega^\mu \end{aligned}$$

which is most easily solved by assuming

$$\omega^1_0 = (p/t) \omega^1; \quad \omega^1_i = 0$$

and this is in harmony with the first assumption. Consequently there are only three non-vanishing connection one-forms:

$$\omega^1_0 = (p/t)\omega^1; \quad \omega^2_0 = (q/t)\omega^2; \quad \omega^3_0 = (r/t)\omega^3.$$

The second structure equation leads to R^μ_ν ; for example

$$\begin{aligned} R^0_1 &= d\omega^0_1 + \omega^0_\mu \times \omega^\mu_1 \\ &= d\omega^0_1 = d(pt^{p-1}dx) = (p(p-1)/t^2)dt \times \omega^1. \end{aligned}$$

From (B.31) it then follows

$$R^0_{101} = -p(p-1)/t^2$$

and similarly

$$R^0_{202} = -q(q-1)/t^2$$

$$R^0_{303} = -r(r-1)/t^2.$$

For the other components,

$$\begin{aligned} R^1_2 &= d\omega^1_2 + \omega^1_\mu \times \omega^\mu_2 \\ &= \omega^1_0 \times \omega^0_2 = (pq/t^2)\omega^1 \times \omega^2 \end{aligned}$$

and hence

$$R^1_{212} = -(pq/t^2)$$

and the only other independent components are

$$R^2_{323} = -(qr/t^2)$$

$$R^3_{131} = -(rp/t^2).$$

If guessing fails, a systematic method exists for calculating the $\omega_{\mu\nu}$; for this see Misner et al. (1973). The beauty of the vierbein lies in its reducing the number of these unknowns from sixteen to six.

APPENDIX C

Some Useful Identities

All identities are given in terms of curved space gamma matrices. To convert to the flat space equivalents, merely let $e_{\mu}^{\alpha} \rightarrow \delta_{\mu}^{\alpha}$, $e = \det e_{\mu}^{\alpha} \rightarrow 1$.

Two-dimensional identities:-----

$$\gamma^{\mu} \gamma^{\nu} = g^{\mu\nu} + e^{-1} \epsilon^{\mu\nu} \gamma^5$$

$$\gamma_{\mu} \gamma_{\nu} = g_{\mu\nu} + e \epsilon_{\mu\nu} \gamma^5$$

$$e^{\mu}_{\alpha} e^{\nu}_{\beta} \epsilon^{\alpha\beta} = e^{-1} \epsilon^{\mu\nu}$$

$$\epsilon_{\sigma\mu} \epsilon^{\rho\lambda} = \delta_{\sigma}^{\lambda} \delta_{\mu}^{\rho} - \delta_{\sigma}^{\rho} \delta_{\mu}^{\lambda}$$

$$\epsilon_{\sigma\mu} \epsilon^{\mu\lambda} = \delta_{\sigma}^{\lambda}$$

$$\gamma^{\lambda} = e^{-1} \epsilon^{\lambda\mu} \gamma^5 \gamma_{\mu}$$

$$\bar{\psi} \chi = \bar{\chi} \psi$$

$$\bar{\psi} \gamma^5 \chi = - \bar{\chi} \gamma^5 \psi$$

$$\bar{\psi} \gamma^{\mu} \chi = - \bar{\chi} \gamma^{\mu} \psi$$

$$\bar{\psi} \gamma^{\mu} \gamma^{\nu} \chi = \bar{\chi} \gamma^{\nu} \gamma^{\mu} \psi$$

$$\gamma^{\mu} \gamma^5 \psi \chi_{\mu} \gamma^5 \gamma^{\nu} \chi = \gamma^{\mu} \psi \chi_{\mu} \gamma^{\nu} \chi$$

$$\bar{\chi}_{\mu} \gamma_{\nu} \gamma^{\lambda} \gamma^{\mu} \chi_{\lambda} = - \epsilon^{\lambda\mu} \epsilon_{\sigma\rho} \bar{\chi}_{\mu} \gamma^{\rho} \chi_{\lambda}$$

$$\epsilon^{\mu\sigma} \bar{\chi}_{\mu} \gamma_{\sigma} \gamma^{\nu} \chi = 0$$

$$\bar{\chi} \cdot \gamma \gamma^{\sigma} \gamma^{\nu} \chi = 0$$

$$\bar{\chi} \cdot \gamma \gamma^5 \gamma^{\nu} \chi = 0$$

$$\gamma_{\mu} \gamma^{\sigma} \gamma^{\mu} = 0$$

$$\epsilon^{\lambda\sigma} [(\partial_{\lambda} \bar{\epsilon}) \gamma_{\mu} \chi_{\sigma} + (\partial_{\sigma} \bar{\epsilon}) \gamma_{\lambda} \chi_{\mu} + (\partial_{\mu} \bar{\epsilon}) \gamma_{\sigma} \chi_{\lambda}] = 0$$

$$\omega_{\mu}(e) = -\frac{1}{2} \epsilon^{\alpha\beta} \omega_{\mu\alpha\beta}(e) = e^{-1} \epsilon^{\lambda\nu} e_{\mu\alpha} \partial_{\lambda} e_{\nu}^{\alpha} = e^{-1} \epsilon^{\lambda\nu} e_{\nu\alpha} \partial_{\lambda} [e_{\mu}^{\alpha}]$$

Four-dimensional identities:

$$\gamma_{\mu}^{\sigma} \rho_{\sigma} = \frac{1}{2} g_{\mu\rho} \gamma_{\sigma} - \frac{1}{2} g_{\mu\sigma} \gamma_{\rho} - \frac{1}{2} \epsilon \epsilon_{\mu\rho\sigma\tau} \gamma^{\tau} \gamma^5$$

$$\sigma_{\rho\sigma} \gamma_{\mu} = \frac{1}{2} g_{\mu\sigma} \gamma_{\rho} - \frac{1}{2} g_{\mu\rho} \gamma_{\sigma} - \frac{1}{2} \epsilon \epsilon_{\mu\rho\sigma\tau} \gamma^{\tau} \gamma^5$$

$$\gamma^5 \sigma_{\mu\nu} = \frac{1}{2} \epsilon \epsilon_{\mu\nu\rho\tau} \sigma^{\rho\tau}$$

$$\begin{aligned} \sigma_{\lambda\mu}^{\sigma} \nu_{\pi} &= \frac{1}{2} g_{\lambda\pi} g_{\mu\nu} - \frac{1}{2} g_{\lambda\nu} g_{\mu\pi} - \frac{1}{2} g_{\mu\pi}^{\sigma} \lambda_{\nu} + \frac{1}{2} g_{\mu\nu}^{\sigma} \lambda_{\pi} \\ &\quad - \frac{1}{2} g_{\lambda\nu}^{\sigma} \mu_{\pi} + \frac{1}{2} g_{\lambda\pi}^{\sigma} \mu_{\nu} + \frac{1}{2} \epsilon \epsilon_{\lambda\mu\nu\pi} \gamma^5 \end{aligned}$$

$$\gamma^{\nu} \sigma_{\mu\nu} = -(3/2) \gamma_{\mu}$$

$$2\sigma_{\nu\mu}^{\sigma} \alpha_{\beta} \gamma^{\nu} = \sigma_{\alpha\beta} \gamma_{\mu}$$

$$e^{-1} \epsilon^{\lambda\mu\nu\rho} \gamma^5 \gamma_{\mu} = g^{\lambda\rho} \gamma^{\nu} - g^{\lambda\nu} \gamma^{\rho} - g^{\nu\rho} \gamma^{\lambda} + \gamma^{\lambda} \gamma^{\nu} \gamma^{\rho}$$

$$\omega_{\lambda\nu}^{\mu} - \omega_{\nu\lambda}^{\mu} = e_{[\nu}^{\alpha} \partial_{\lambda]} e_{\alpha}^{\mu} - i \bar{\chi}_{\nu} \gamma^{\mu} \chi_{\lambda}$$

$$R^{\lambda} = e^{-1} \epsilon^{\lambda\mu\nu\rho} \gamma^5 \gamma_{\mu} D_{\nu} \chi_{\rho}$$

$$\gamma \cdot R = 4\sigma^{\rho\sigma} D_{\rho} \chi_{\sigma}$$

$$\gamma^{\nu} D_{[\mu} \chi_{\nu]} = -R_{\mu} + \frac{1}{2} \gamma_{\mu} \gamma \cdot R = \gamma^{\nu} \gamma_{\mu} R_{\nu}$$

$$\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} = -\delta_{\nu\rho\sigma}^{\alpha\beta\gamma}$$

$$D_{[\mu} \chi_{\nu]} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\rho\sigma\kappa\lambda} D_{\kappa} \chi_{\lambda}$$

$$\begin{aligned} D_{[\mu} \bar{\chi}_{\nu]} \gamma^{\nu} \sigma_{\alpha\beta\epsilon} &= \frac{1}{2} D_{[\mu} \bar{\chi}_{\nu]} e_{[\alpha}^{\nu} \gamma_{\beta]} \epsilon - \frac{1}{2} e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\mu} \epsilon \\ &\quad + \frac{1}{2} e^{-1} e_{\mu[\alpha} \epsilon_{\beta]}^{\lambda\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^{\lambda} \gamma^5 \epsilon \end{aligned}$$

$$\begin{aligned} D_{[\mu} \bar{\chi}_{\nu]} \sigma_{\alpha\beta} \gamma^{\nu} \epsilon &= \frac{1}{2} D_{[\mu} \bar{\chi}_{\nu]} e_{[\beta}^{\nu} \gamma_{\alpha]} \epsilon - \frac{1}{2} e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\mu} \epsilon \\ &\quad + \frac{1}{2} e^{-1} e_{\mu[\alpha} \epsilon_{\beta]}^{\lambda\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^{\lambda} \gamma^5 \epsilon \end{aligned}$$

$$\begin{aligned} \bar{R}_{\mu}^{\sigma} \alpha_{\beta} \epsilon &= \frac{1}{2} e^{-1} \epsilon_{\mu}^{\nu\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{[\beta} e_{\alpha]}^{\nu} \epsilon + \frac{1}{2} e_{\mu[\beta} e_{\alpha]}^{\nu} D_{[\nu} \bar{\chi}_{\lambda]} \gamma^{\lambda} \epsilon \\ &\quad - \frac{1}{2} e_{\alpha}^{\nu} e_{\beta}^{\lambda} D_{[\nu} \bar{\chi}_{\lambda]} \gamma_{\mu} \epsilon \end{aligned}$$

$$\begin{aligned} \bar{R} \cdot \gamma \sigma_{\alpha\beta} \gamma_{\mu} \epsilon &= -e_{\alpha}^{\nu} e_{\beta}^{\lambda} D_{[\nu} \bar{\chi}_{\lambda]} \gamma_{\mu} \epsilon + e^{-1} \epsilon_{\alpha\beta}^{\rho\sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\mu} \epsilon \\ &\quad + e_{\mu[\alpha} e_{\beta]}^{\rho} D_{[\rho} \chi_{\sigma]} \gamma^{\sigma} \epsilon \end{aligned}$$

$$\begin{aligned}
 & + e^{-1} e_{\mu[\alpha \varepsilon \beta]}^{\nu \rho \sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma_{\nu} \gamma^5 \varepsilon \\
 & + D_{[\mu} \bar{\chi}_{\nu]} \gamma_{[\alpha} e_{\beta]}^{\nu \varepsilon} + e^{-1} e_{[\alpha}^{\lambda} \varepsilon_{\beta]}^{\rho \sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\lambda} \varepsilon
 \end{aligned}$$

$$\begin{aligned}
 D_{[\nu} \bar{\chi}_{\lambda]}^{\sigma} \varepsilon_{\sigma \alpha \beta}^{\lambda} \gamma^{\nu} \varepsilon & = \frac{1}{4} e^{\lambda}_{[\alpha} e_{\beta]}^{\nu} D_{[\nu} \bar{\chi}_{\lambda]} \gamma_{\mu} \varepsilon - \frac{1}{4} D_{[\mu} \bar{\chi}_{\lambda]} \gamma_{\nu} \varepsilon e^{\lambda}_{[\alpha} e_{\beta]}^{\nu} \\
 & - \frac{1}{4} e^{-1} e_{\mu[\alpha \varepsilon \beta]}^{\nu \rho \sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\nu} \varepsilon \\
 & + \frac{1}{4} e^{-1} \varepsilon_{\alpha \beta}^{\rho \sigma} D_{\rho} \bar{\chi}_{\sigma} \gamma^5 \gamma_{\mu} \varepsilon
 \end{aligned}$$

Poor Man's Gravity.

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Summary. — Einstein's standard matter-free theory of gravitation is constructed as the synthesis of two Yang-Mills theories, whose gauge groups are $GL_{4,R}$ and $SL_{2,C}$. In this approach the vierbein is not regarded as a Yang-Mills field, but rather as a mapping between the two-group manifolds.

Introduction.

As confidence increases in a Yang-Mills structure for the weak and electromagnetic forces ⁽¹⁾, more attempts ⁽²⁾ are being made to cast the two remaining forces into a similar mold. That gravitation is a gauge theory in this Yang-Mills sense has become part of the folklore, although the exact details of this correspondence have been disputed ⁽³⁾. This article presents another attempt

⁽¹⁾ C. N. YANG and R. L. MILLS: *Phys. Rev.*, **96**, 191 (1954); A. SALAM: in *Elementary Particle Theory*, edited by N. SVARTHOLM (Stockholm, 1968), p. 367; S. WEINBERG: *Phys. Rev. Lett.*, **19**, 1264 (1967).

⁽²⁾ See, for instance, E. S. ABERS and B. W. LEE: *Phys. Lett.*, **9 C**, 1 (1973); J. BERNSTEIN: *Rev. Mod. Phys.*, **46**, 7 (1974); F. W. HEHL, P. VON DER HYDE, G. D. KERLICK and J. M. NESTER: *Rev. Mod. Phys.*, **48**, 393 (1976); H. D. POLITZER: *Phys. Lett.*, **14 C**, 129 (1974); J. C. TAYLOR: *Gauge Theories of Weak Interactions* (Cambridge, 1976).

⁽³⁾ F. W. HEHL, P. VON DER HYDE, G. D. KERLICK and J. M. NESTER: *Rev. Mod. Phys.*, **48**, 393 (1976), and also, for instance, M. CARMELI: *Group Theory and General Relativity*, Chap. 9 (New York, N. Y., 1977); A. H. CHAMSEDDINE and P. C. WEST: Imperial College preprint ICTP/75/22 (September 1976); Y. M. CHO: *Journ. Math. Phys.*, **16**, 2029 (1975); *Phys. Rev. D*, **14**, 2521, 3335, 3341 (1976); S. W. MACDOWALL and F. MANSOURI: *Phys. Rev. Lett.*, **38**, 739 (1977); A. SALAM: in *Fundamental Interactions in Physics*, edited by B. KURSUNOGLU and A. PERLMUTTER (New York, N. Y., 1973), p. 55.

to display Einstein's matter-free theory as a classical Yang-Mills theory, but in such a way that parallels between the general Yang-Mills characteristics and the particulars of gravity are manifest. To be sure gravity possesses some special features not present in more familiar Yang-Mills theories; nevertheless the defining formulae of these non-Abelian gauge theories may be carried over virtually without modification in this « poor man's approach ». The first section of this article attempts to define in a general way the Yang-Mills characteristics. In the next three sections, the invariances of Einstein's theory are discussed and two groups are gauged. Finally, a gravitational theory identical to Einstein's is constructed as a sum of these two gauge theories. As the two component parts do, the sum satisfies the Yang-Mills characteristics. This approach has the advantage of trivial simplicity. In no sense, however, is this article intended to be anything but a synthesis of several well-known results.

1. — The Yang-Mills characteristics.

Suppose a Lagrangian $L(\varphi^A, \partial_\mu \varphi^A)$ is invariant under the action of the group Γ . Let the fields φ^A transform under Γ as

$$(1) \quad \varphi^A(x) \xrightarrow{\Gamma} \tilde{\varphi}^A(x) = U^A_B(\Gamma) \varphi^B(x).$$

The infinitesimal form of this transformation is

$$(2) \quad \delta \varphi^A(x) = \tilde{\varphi}^A(x) - \varphi^A(x) = i[\hat{G}, \varphi^A(x)] = T^A_B \varphi^B(x),$$

where $\hat{G} = \varepsilon^j \hat{G}_j$, with the set $\{\hat{G}_j\}$ being the generators of Γ , $T^A_B = \varepsilon^j T^A_{jB}$, while the set $\{T^A_{jB}\}$ are matrices which generate a particular representation U^A_B of Γ and ε^j are parameters. The group algebra is

$$(3) \quad [\hat{G}_k, \hat{G}_j] = i f_{kj}{}^i \hat{G}_i.$$

The matrix $U^A_B(\varepsilon)$ may be defined by the finite transformation

$$(4) \quad U^A_B(\varepsilon) \varphi^B(x) = \exp[i\hat{G}] \varphi^A \exp[-i\hat{G}] = [\exp[i\varepsilon^i T_i]]^A_B \varphi^B(x).$$

If

$$(5) \quad [\hat{G}, \partial^\mu] = 0$$

and

$$(6) \quad \partial^\mu \varepsilon^i = 0,$$

then

$$\begin{aligned} \exp[i\hat{G}]\partial^\mu\varphi^A\exp[-i\hat{G}] &= \partial^\mu(\exp[i\hat{G}]\varphi^A\exp[-i\hat{G}]) = \\ &= \partial^\mu(U^A_{B}(\varepsilon)\varphi^B) = U^A_{B}(\varepsilon)\partial^\mu\varphi^B, \end{aligned}$$

so that $\partial^\mu\varphi^A(x)$ transforms exactly as $\varphi^A(x)$ itself. If L has the form

$$L = (\partial^\mu\varphi^A)(\partial_\mu\varphi_A) - V(\varphi^A\varphi_A),$$

where φ_A transforms contragradiently to φ^A ,

$$(7) \quad \bar{\varphi}_A(x) = (U^{-1})^B_{A}\varphi_B(x);$$

then

$$\delta L = i[\hat{G}, L] = 0.$$

If, however, condition (5) does not hold, then δL will not vanish. There are three separate ways this can happen:

$$\begin{aligned} (5a) \quad [\varepsilon^i, \partial^\mu] &= 0, & \text{but } [\hat{G}_i, \partial^\mu] &\neq 0, \\ (5b) \quad [\hat{G}_i, \partial^\mu] &= 0, & \text{but } [\varepsilon^i, \partial^\mu] &\neq 0, \\ (5c) \quad [\hat{G}_i, \partial^\mu] &\neq 0 \text{ and } [\varepsilon^i, \partial^\mu] \neq 0, & \text{but } \varepsilon^i[\hat{G}_i, \partial^\mu] &= -[\varepsilon^i, \partial^\mu]\hat{G}_i. \end{aligned}$$

The standard Yang-Mills theory considers the case (5b) where the group parameters ε^i are «local», *i.e.* functions of x . For the particular case of gravity the relevant parameters will be local, but ∂^μ will no longer commute with the group generators $\hat{G}_i$. Then (5c) will be appropriate. Concerning situation (5b) there is a well-known remedy available to render $\delta L = 0$ even when the parameters are local. For each generator $\hat{G}_i$, one introduces a «gauge field» $A^i_\mu(x)$ whose transformation will compensate for the difference between $U^A_{B}(\varepsilon)\partial^\mu\varphi^B$ and $\partial^\mu(U^A_{B}(\varepsilon)\varphi^B)$. With these gauge fields an «extended derivative» ∇^μ may be defined (g is a coupling constant):

$$(8) \quad \nabla_\mu\varphi^A = \partial_\mu\varphi^A + ig[A^i_\mu\hat{G}_i, \varphi^A] = \partial_\mu\varphi^A + gA^i_\mu T_i{}^A_{B}\varphi^B$$

(this is the non-Abelian generalization of the minimal-coupling prescription in mechanics to an electromagnetic field; $p_\mu \rightarrow p_\mu - ieA_\mu$). The transformation law for A^i_μ is obtained by requiring that this extended derivative be *covariant*, *i.e.* that

$$(9) \quad U^A_{B}(\varepsilon)\nabla_\mu\varphi^B = \bar{\nabla}_\mu(U^A_{B}(\varepsilon)\varphi^B) = \partial_\mu(U^A_{B}(\varepsilon)\varphi^B) + ig\bar{A}^i_\mu[\hat{G}_i, U^A_{B}\varphi^B].$$

By the group property, $U^A_{B}\varphi^B$ satisfies the same algebra as φ^A , so that the

right-hand side of (9) is

$$(10) \quad U^A_B(\varepsilon) \partial_\mu \varphi^B + (\partial_\mu U^A_B(\varepsilon)) \varphi^B + g \bar{A}_\mu^j T_j^A{}_C U^C_B(\varepsilon) \varphi^B.$$

Setting (10) equal to (9) leads to

$$(11) \quad g \bar{A}_\mu^j T_j^A{}_B = U^A{}_C A_\mu^j T_j^C{}_D (U^{-1})^D{}_B + U^A{}_C \partial_\mu (U^{-1})^C{}_B,$$

which is associated with the name of Ehresmann⁽⁴⁾. Henceforth (11) will be referred to as « Ehresmann's formula », and frequently the matrix indices will be suppressed. The field strength $F_{\mu\nu}^j$, analogous to the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \partial_{[\mu} A_{\nu]}$ is defined as

$$(12) \quad [\nabla_\mu, \nabla_\nu] \varphi^A = g F_{\mu\nu}^j T_j^A{}_B \varphi^B,$$

where

$$(13) \quad F_{\mu\nu}^j = \partial_{[\mu} A_{\nu]}^j + g f_{kl}^j A_\mu^k A_\nu^l.$$

Note that (12) holds only in the case of (5b), when $\hat{G}_i$ and ∂_μ commute. After these extremely familiar preliminaries, the Yang-Mills characteristics may be defined by three not unreasonable requirements:

i) The covariant derivative is bilinear in the gauge fields A_μ^i and the group generators $\hat{G}_i$, as in (8); a « minimal coupling » prescription holds.

ii) The Ehresmann formula (11) governs the behaviour of the gauge fields under a group transformation.

iii) The commutators of two covariant derivatives acting on a field determine the field strengths which in turn determine gauge fields' dynamics.

2. - Gravitational symmetries.

If we turn now to gravity, there is a couple of points to be emphasized. The first concerns the invariance groups of Einstein's theory, and the second

⁽⁴⁾ C. EHRESMANN: *Colloque de topologie* (Brussels, 1950), p. 29. Strictly speaking Ehresmann's formula applies to differential forms. See, for instance, Y. M. CHO: *Journ. Math. Phys.*, **16**, 2029 (1975); C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, Cal., 1973), hereafter MTW; M. SPIVAK: *Differential Geometry*, Vol. 2, Chap. 8 (Boston, Mass., 1970). The fancy footwork between (24a) and (24c) is unnecessary when differential forms are employed. In MTW terminology (24a) reads (see their eq. (14.40))

$$\bar{\omega}^\nu{}_\lambda = U^\nu{}_\alpha \omega^\alpha{}_\beta (U^{-1})^\beta{}_\lambda + U^\nu{}_\alpha d(U^{-1})^\alpha{}_\lambda.$$

Dotting this with the MTW basis vector $e_\alpha = \partial/\partial x^\alpha$ yields (24c) immediately.

the role of the locally Lorentzian system of co-ordinates, the *vierbein* $e_\mu^a(x)$ ⁽⁵⁾. As for the groups themselves, there are two distinct invariances built into the standard 1916 theory of gravity:

a) freedom to relabel the co-ordinates which form the underlying manifold, and

b) freedom to perform local homogeneous Lorentz transformations on the field components directly.

This second invariance deserves some further discussion. After all, it is the inhomogeneous Lorentz, or Poincaré, group which figures so prominently in particle physics. In fact, the gauge theory of the Poincaré group has been successfully presented as the Einstein-Cartan-Sciama-Kibble theory of gravitation, notably by the Warsaw ⁽⁶⁾ and Princeton-Cologne ⁽⁷⁾ groups, and in a slightly different way by CHAMSEDDINE and WEST ⁽⁸⁾. In as much as an infinitesimal general co-ordinate transformation

$$(14) \quad x^\mu \rightarrow \bar{x}^\mu = x^\mu + \xi^\mu(x)$$

is indistinguishable from a local translation, it should make no difference whether the gauge group is regarded as a) and b) taken together (the local groups of $GL_{4,R}$ and $SL_{2,C}$, respectively), or as the Poincaré group. Nevertheless, there are differences. For the Poincaré approach, in order to satisfy i) above, it becomes necessary to associate the vierbein with the generators of translations. There are two possibilities now: regard these generators either as the Abelian derivatives ∂_μ ⁽⁸⁾, or as the standard Einstein covariant derivatives ∇_μ ^(6,7). In the latter case, the group generators no longer satisfy the Poincaré algebra and, as a consequence, the gauge fields do not transform according to the Ehresmann formula (11). In the former situation, one must associate the trivial representation to the translations due to their Abelian nature. In each situation the $GL_{4,R}$ invariance is carried only by the vierbein. This leads to the second

⁽⁵⁾ H. WEYL: *Zeits. Phys.*, **56**, 330 (1929); R. UTIYAMA: *Phys. Rev.*, **101**, 1597 (1956); T. W. B. KIBBLE: *Journ. Math. Phys.*, **2**, 212 (1961); D. W. SCIAMA: *Journ. Math. Phys.*, **2**, 472 (1961) and in the *Infeld Festschrift* (New York, N. Y., 1962); L. D. LANDAU and E. M. LIFSHITZ: *The Classical Theory of Fields* (Oxford, 1976), p. 291; M. VELTMAN: in *Methods in Field Theory*, edited by D. Balian and J. ZINN-JUSTIN (Amsterdam, 1977), p. 320; S. WEINBERG: *Gravitation and Cosmology*, subsect. 12.5 (New York, N. Y., 1972).

⁽⁶⁾ R. KERNER: *Ann. Inst. Henri Poincaré*, **9**, 143 (1968); A. TRAUTMAN: *Rep. Math. Phys. (Warsaw)*, **1**, 29 (1970); *Ann. N.Y. Acad. Sci.*, **262**, 241 (1975); *Bull. Pol. Acad. Sci.*, **20**, 185, 503, 895 (1972); **21**, 345 (1973).

⁽⁷⁾ F. W. HEHL, P. VON DER HYDE, G. D. KERLICK and J. M. NESTER: *Rev. Mod. Phys.*, **48**, 393 (1976).

⁽⁸⁾ A. H. CHAMSEDDINE and P. C. WEST: Imperial College preprint ICTP/75/22 (September 1976).

point of emphasis: the present approach requires the vierbein to behave *not* as a gauge field with its associated inhomogeneous transformation law (11), but as a *bridge* between the two symmetries *a*) and *b*). Both groups must enter into the extended derivative for gravity. Without $SL_{2,\sigma}$, there could be no minimal coupling between gravity and fermions⁽⁹⁾; without $GL_{4,R}$, the vierbein e_μ^a required by $SL_{2,\sigma}$ loses a large part of its freedom, and all other fields must be regarded as co-ordinate scalars. The presence of two symmetries suggests a synthesis of two Yang-Mills theories, one to deal with each gauge group. The dynamical fields may be conveniently divorced from their co-ordinate-induced behaviour under $GL_{4,R}$ by banishing them to the $SL_{2,\sigma}$ manifold, where fermions and bosons are treated on an equal footing. The co-ordinates themselves are left behind in the manifold of $GL_{4,R}$. To keep the invariances separate, all $SL_{2,\sigma}$ objects will be labelled by Latin indices $a, b, \dots$, while those defined with respect to $GL_{4,R}$ will bear Greek indices $\mu, \nu, \dots$. Some quantities, notably the vierbeins e_μ^a themselves, carry both types of indices. At last these two invariances will be reconciled by the «mediator», the vierbein. For clarity, the Yang-Mills theory of each group will be considered separately. In the final section the synthesis of the two will, with one additional compatibility condition, reproduce Einstein's theory.

3. - $GL_{4,R}$ (10).

The generators $\hat{G}_i$ are the sixteen operators $\hat{G}_\mu^\lambda$ whose structure constants f_{jk}^i are

$$(15) \quad f_{\beta\gamma\epsilon}^{\alpha} = (\delta_\epsilon^\alpha \delta_\beta^\epsilon \delta_\gamma^\sigma - \delta_\beta^\sigma \delta_\epsilon^\alpha \delta_\gamma^\sigma) .$$

Because the transformations on the fields are induced by the co-ordinate transformations, (1) must be modified to ($\Gamma: x^\mu \rightarrow \bar{x}^\mu$)

$$(16) \quad \varphi(x) \xrightarrow{\Gamma} \bar{\varphi}(\bar{x}) = U(\Gamma)\varphi(x) .$$

For example, if φ is a contravariant vector field B^μ ,

$$(17) \quad B^\mu(x) \rightarrow \bar{B}^\mu(\bar{x}) = U^\mu_\lambda B^\lambda(x) ,$$

where

$$(18) \quad U^\mu_\lambda = \partial \bar{x}^\mu / \partial x^\lambda .$$

⁽⁹⁾ S. DESER and P. VAN NIEUWENHUIZEN: *Phys. Rev. D*, **10**, 411 (1974).

⁽¹⁰⁾ B. S. DE WITT: *Dynamical Theory of Groups and Fields* (London, 1965), p. 95; M. H. FRIEDMAN: *Phys. Rev. D*, **12**, 2528 (1975); K. HAYASHI and T. SHIRAFUJI: *Prog. Theor. Phys.*, **58**, 353 (1977).

Similarly for a covariant vector field C_μ ,

$$(19) \quad C_\mu(x) \rightarrow \bar{C}_\mu(\bar{x}) = (U^{-1})^\lambda_\mu C_\lambda(x).$$

From (14), the infinitesimal form of (17) is

$$(20) \quad \bar{B}^\mu(\bar{x}) - B^\mu(x) = \delta B^\mu(x) = i[\hat{G}, B^\mu],$$

where

$$(21) \quad \hat{G} = \varepsilon^\alpha_\beta \hat{G}^\beta_\alpha \quad \text{and} \quad \varepsilon^\alpha_\beta = \partial \xi^\alpha / \partial x^\beta.$$

From (17), (20) and (21) it follows that

$$(22a) \quad [\hat{G}^\alpha_\beta, B^\mu] = -i\delta^\mu_\beta B^\alpha,$$

$$(22b) \quad [\hat{G}^\alpha_\beta, C_\mu] = i\delta^\alpha_\mu C_\beta,$$

$$(22c) \quad [\hat{G}^\alpha_\beta, B^\mu C_\mu] = 0.$$

Similarly the formula for a rank- N tensor $A_{\mu\nu}{}^{\rho\sigma} \dots$ may be found by considering the special case

$$A_{\mu\nu}{}^{\rho\sigma} \dots = B_\mu C_\nu D^\rho E^\sigma F_\tau \dots$$

To construct an extended derivative, introduce sixteen fields $A_\mu^i = A_\mu^{\alpha\beta}$, one for each generator $\hat{G}^\beta_\alpha$. Then, for example,

$$(23a) \quad \nabla_\mu B^\nu = \partial_\mu B^\nu + i\kappa[A_\mu^{\alpha\beta} \hat{G}^\beta_\alpha, B^\nu] = \partial_\mu B^\nu + \kappa A_\mu^{\nu\beta} B^\beta$$

and

$$(23b) \quad \nabla_\mu C_\nu = \partial_\mu C_\nu - \kappa A_\mu^{\beta\nu} C_\beta$$

and so forth for an arbitrary field $A_{\mu\nu}{}^{\rho\sigma} \dots$. The requirement of covariance again leads to the Ehresmann law (by suppressing the indices)

$$(24a) \quad \kappa \bar{A}_\mu^{\nu\lambda} T^{\lambda}_\nu = \kappa U A_\mu^{\nu\lambda} T^{\lambda}_\nu U^{-1} + U \partial_\mu U^{-1}.$$

By taking T^λ_ν to be the vector representation,

$$(T^\lambda_\nu)^\alpha_\beta = \delta^\lambda_\beta \delta^\alpha_\nu$$

with U as before, the transformation of $A_\mu^{\alpha\beta}$ becomes

$$(24b) \quad \kappa \bar{A}_\mu^{\alpha\beta} = \kappa \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} A_\mu^{\sigma\epsilon} \frac{\partial x^\epsilon}{\partial \bar{x}^\beta} + \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \partial_\mu \left(\frac{\partial x^\sigma}{\partial \bar{x}^\beta} \right).$$

For future comparisons, note that the Ehresmann law is tensorial in the μ -index. The freedom to transform this index remains. If $v^{\bar{\alpha}}$ means the components of the vector v^{α} with respect to the co-ordinate system $\bar{x}^{\beta}$, then

$$(25) \quad \bar{A}_{\mu}^{\alpha\beta} = A_{\mu}^{\bar{\alpha}\bar{\beta}}.$$

To rewrite (24b) in terms of $A_{\mu}^{\bar{\alpha}\bar{\beta}}$, it is sufficient to note

$$(26) \quad A_{\mu}^{\bar{\alpha}\bar{\beta}} = \frac{\partial x^e}{\partial \bar{x}^{\mu}} \bar{A}_e^{\alpha\beta},$$

whence multiplying both sides of (24b) by $\partial x^e / \partial \bar{x}^{\mu}$,

$$(24c) \quad \kappa A_{\mu}^{\bar{\alpha}\bar{\beta}} = \kappa \frac{\partial x^e}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\lambda}}{\partial \bar{x}^{\beta}} A_e^{\sigma\lambda} + \frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\beta}}.$$

Thus far everything for the $GL_{4,n}$ gauge theory has followed the standard Yang-Mills description precisely, with the slight generalization noted in (24c). Now, however, case (5b) is no longer appropriate, because

$$[\hat{G}^{\alpha\beta}, \partial^{\mu}] = -i\delta^{\mu}_{\beta} \partial^{\alpha} \neq 0.$$

The commutation of two covariant derivatives leads in this case to an additional term besides the usual $F_{\mu\nu}{}^i T_i$ contribution. Because ∇_{μ} obeys the same algebra as ∂_{μ} ,

$$(27) \quad [\hat{G}^{\alpha\beta}, \nabla_{\mu}] = i\delta^{\alpha}_{\mu} \nabla_{\beta}.$$

By explicit calculation, by using (15) and (27),

$$(28) \quad [\nabla_{\mu}, \nabla_{\nu}] B^{\lambda} = \kappa F_{\mu\nu}{}^{\alpha\beta} (T^{\beta}_{\alpha})^{\lambda}_{\mu} B^{\mu} - \kappa A_{[\mu}{}^{\alpha}{}_{\nu]} \nabla_{\alpha} B^{\lambda},$$

where as before

$$(29) \quad F_{\mu\nu}{}^{\alpha\beta} = \partial_{[\mu} A_{\nu]}{}^{\alpha\beta} + \kappa f^{\gamma}{}_{\tau}{}^e{}_{\sigma}{}^{\alpha\beta} A_{\mu}{}^{\tau}{}_{\gamma} A_{\nu}{}^{\sigma}{}_{\epsilon} = \partial_{[\mu} A_{\nu]}{}^{\alpha\beta} + \kappa A_{\mu}{}^{\alpha}{}_{\sigma} A_{\nu}{}^{\sigma}{}_{\beta} - \kappa A_{\nu}{}^{\alpha}{}_{\sigma} A_{\mu}{}^{\sigma}{}_{\beta}.$$

If the fields and field strengths are redefined as

$$(30a) \quad \kappa A_{\mu}{}^{\alpha}{}_{\nu} = \Gamma_{\mu\nu}^{\alpha},$$

$$(30b) \quad \kappa F_{\mu\nu}{}^{\alpha\beta} = R^{\alpha\beta}{}_{\mu\nu},$$

then (24c) and (29) become, respectively, the standard formulae⁽¹¹⁾ for the transformation of the affine connection and the definition of the Riemann tensor. The antisymmetric tensor $\Gamma^{\alpha}_{[\mu\nu]}$ is called the torsion. In Einstein's original theory the affine connection is postulated to be symmetric, so that the torsion vanishes identically. Most textbooks follow suit⁽¹²⁾. In the first instance, its presence in (28) derives from (27), but dynamics may constrain it to vanish.

4. - $SL_{2,C}$.

The generators $\hat{G}_i$ are the six operators $\hat{s}_{ab} = -\hat{s}_{ba}$, whose algebra is

$$(31) \quad [\hat{s}_{ab}, \hat{s}_{cd}] = i(\eta_{ac}\hat{s}_{bd} - \eta_{ad}\hat{s}_{bc} - \eta_{bc}\hat{s}_{ad} + \eta_{bd}\hat{s}_{ac}) = \frac{i}{2}f_{abcd}{}^{ef}\hat{s}_{ef},$$

where the factor of $\frac{1}{2}$ is to avoid double counting and η_{ab} is the Minkowski metric with signature $(+---)$. In general, for a field $\varphi(s)$ of spin s , the transformation law under $SL_{2,C}$ is

$$(32) \quad i[\hat{s}_{ab}, \varphi^A(s)] = T_{ab}(s)A_B \varphi^B(s),$$

where (A, B) is the appropriate kind of indices, *e.g.*

$$\begin{aligned} \text{scalar: } \varphi^A &= \varphi, & (T_{ab})^A{}_B &= 0, \\ \text{spinor: } \varphi^A &= \varphi^{\dot{a}}, & (T_{ab})^{\dot{a}}{}_{\dot{d}} &= (i/4)[\gamma_a, \gamma_b]^{\dot{a}}{}_{\dot{d}}, \\ \text{vector: } \varphi^A &= \varphi^a, & (T_{ab})^c{}_d &= i(\eta_{ad}\delta^c{}_b - \eta_{bd}\delta^c{}_a) \end{aligned}$$

and so on. The extended derivative becomes for an arbitrary field of spin s

$$(33) \quad \nabla_\mu \varphi(x; s) = \partial_\mu \varphi(x; s) + \frac{1}{2}igA_\mu{}^{ab}[\hat{s}_{ab}, \varphi(x; s)] = \\ = \partial_\mu \varphi(x; s) + \frac{1}{2}gA_\mu{}^{ab}T_{ab}(s)\varphi(x; s),$$

where again the factor of $\frac{1}{2}$ allows for double counting. Without loss of generality it may be assumed

$$A_\mu{}^{ab} = -A_\mu{}^{ba}.$$

⁽¹¹⁾ S. WEINBERG: *Gravitation and Cosmology* (New York, N. Y., 1972), p. 100, 133; L. D. LANDAU and E. M. LIFSHITZ: *The Classical Theory of Fields* (Oxford, 1976), p. 241, 260; C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, Cal., 1973), p. 262, 219.

⁽¹²⁾ An exception is E. SCHRÖDINGER: *Space-Time Structure* (Cambridge, 1950).

The Ehresmann law required by covariance leads to

$$(34) \quad \frac{1}{2} g \bar{A}_\mu^{ab} T_{ab}(s) = \frac{1}{2} U(s) g A_\mu^{ab} T_{ab}(s) U^{-1}(s) + U(s) \partial_\mu U^{-1}(s),$$

where U may be parametrized as

$$U(s) = \exp \left[\frac{1}{2} \lambda^{ab}(x) T_{ab}(s) \right]$$

with $\lambda^{ab}(x)$ being the local Lorentz parameters. Then, for an infinitesimal transformation,

$$(35) \quad \begin{aligned} \delta g \bar{A}_\mu^{ab} &= i g \lambda^{cd} [\hat{s}_{cd}, A_\mu^{ab}] - \partial_\mu \lambda^{ab} = \\ &= g \lambda^{cd} f_{cd}^{ab} A_\mu^{eh} - \partial_\mu \lambda^{ab} = g \lambda^a_c A_\mu^{cb} - g \lambda^b_c A_\mu^{ac} - \partial_\mu \lambda^{ab}. \end{aligned}$$

As far as the $SL_{2,0}$ generators $\hat{s}_{ab}$ are concerned, ∂^μ is a scalar:

$$[\hat{s}_{ab}, \partial^\mu] = 0,$$

so that now case (5b) obtains, and

$$(36) \quad [\nabla_\mu, \nabla_\nu] \varphi(x; s) = \frac{1}{2} g F_{\mu\nu}^{ab} T_{ab}(s) \varphi(x; s),$$

where, as usual (modulo the ubiquitous $\frac{1}{2}$),

$$(37) \quad F_{\mu\nu}^{ab} = \partial_{[\mu} A_{\nu]}^{ab} + \frac{1}{2} g f_{cd}^{ab} A_\mu^{cd} A_\nu^{eh} = \partial_{[\mu} A_{\nu]}^{ab} + g A_{[\mu}^{ac} A_{\nu]}^{cb}.$$

5. - Gravitation: a table of two manifolds.

In order to unite the two Yang-Mills theories, it is necessary to interpret the vierbein $e_\mu^a(x)$ not as a gauge field A_μ^i , but as a map between the integral spin representations of $SL_{2,0}$ and $GL_{4,0}$ tensors:

$A_a(x)$ is a Lorentz vector and a co-ordinate scalar,

$A_\mu(x)$ is a Lorentz scalar and a co-ordinate vector, and

$$(38a) \quad A_\mu(x) = e_\mu^a(x) A_a(x).$$

If the vierbein is defined such that

$$(39a) \quad e_\mu^a(x) e_\nu^b(x) = \delta_{ab},$$

$$(39b) \quad e_\mu^a(x) e^\nu_a(x) = \delta_\mu^\nu,$$

then

$$(38b) \quad A^\mu(x) = e^\mu_a(x) A^a(x),$$

$$(39c) \quad A^a(x) = e^a_\mu(x) A^\mu(x)$$

and, in general,

$$(40) \quad A_{\mu\nu\dots} e^{\sigma\dots} = e^\sigma_\mu e^\rho_\nu \dots e^\sigma_c e^\sigma_d \dots A_{ab\dots}{}^{cd}.$$

In particular, the invariant interval ds^2 leads to a link between the vierbein and the metric $g_{\mu\nu}$:

If we let $dx^a = e^\mu_a dx^\mu$ (in general dx^a is not an exact differential of any function, see LANDAU and LIFSHITZ⁽⁵⁾), the interval may be expressed as

$$ds^2 = \eta_{ab} dx^a dx^b = \eta_{ab} (e^\mu_a dx^\mu) (e^\nu_b dx^\nu) = g_{\mu\nu}(x) dx^\mu dx^\nu,$$

whence

$$(41) \quad g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x).$$

Because the two manifolds are related only through the mediator e^μ_a ,

$$(42) \quad [\hat{G}^\alpha_\beta, \hat{s}_{ab}] = 0.$$

Indeed, if φ has only Latin or Greek indices, it commutes with $\hat{G}^\alpha_\beta$ or $\hat{s}_{ab}$, respectively. On the other hand,

$$[\hat{G}^\alpha_\beta, e^\mu_a] = -i\delta^\alpha_\mu e^\mu_\beta,$$

$$[\hat{s}_{ab}, e^\mu_c] = ie_{\mu a} \delta^c_b - ie_{\mu b} \delta^c_a.$$

For an arbitrary field φ , define the covariant derivative (compare VELTMAN⁽⁶⁾)

$$(43) \quad \nabla_\mu \varphi = \partial_\mu \varphi + ig_1 A_\mu^\nu [\hat{G}_\nu, \varphi] + \frac{1}{2} ig_2 A_\mu^{ab} [\hat{s}_{ab}, \varphi] = \partial_\mu \varphi + i[A_\mu^i \hat{G}_i, \varphi],$$

where

$$A_\mu^i \hat{G}_i = g_1 A_\mu^\nu \hat{G}_\nu + \frac{1}{2} g_2 A_\mu^{ab} \hat{s}_{ab}.$$

As before, the coupling constants may be absorbed into a definition of the gauge fields:

$$(44) \quad g_1 A_\mu^\nu = \Gamma_{\mu\lambda}^\nu, \quad g_2 A_\mu^{ab} = \omega_{\mu}^{ab}.$$

Because $A_\mu^i = (\Gamma_{\mu\lambda}^\nu, \omega_{\mu}^{ab})$ transforms as a co-ordinate vector, again the commutator of two covariant derivatives gives analogously to (28) (compare

KIBBLE ⁽⁵⁾)

$$(45) \quad [\nabla_\mu, \nabla_\nu]\varphi = R^\alpha_{\beta\mu\nu} T^\beta_\alpha \varphi + G_{\mu\nu}{}^{ab} T_{ab} \varphi - \Gamma^\lambda_{[\mu\nu]} \nabla_\lambda \varphi,$$

where

$$(46) \quad G_{\mu\nu}{}^{ab} = \partial_{[\mu} \omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ac} \omega_{\nu]}{}^b{}_c.$$

Physically the manifolds may be reconciled by the requirement

$$\nabla_\mu A^a = e^a_\nu \nabla_\mu A^\nu,$$

or its equivalent

$$(47) \quad \nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_c e^c_\nu = 0.$$

Note that from

$$[\mathcal{S}_{ab}, \eta_{cd}] = -iT_{abcd}{}^{eh} \eta_{eh} = 0$$

it follows immediately from (47) that

$$(48) \quad \nabla_\mu g_{\rho\sigma} = 0.$$

Equation (48), together with the assumption of vanishing torsion, leads uniquely to the Christoffel relations for $\Gamma^\nu_{\mu\lambda}$ ⁽¹²⁾. If φ in (45) is allowed to be the vierbein itself, the last term vanishes identically, while the commutator itself is zero. This leads to

$$-R^\alpha_{\beta\mu\nu} e^a_\alpha + G_{\mu\nu}{}^{ab} e_{ab} = 0,$$

or

$$e^a_\alpha e_b{}^\beta R^\alpha_{\beta\mu\nu} = R^a_{b\mu\nu} = G_{\mu\nu}{}^a{}_b.$$

Consequently the covariant commutator may be rewritten solely in terms of $R^\alpha_{\beta\mu\nu}$ and the torsion; from the *two* Yang-Mills symmetries there is effectively *one* $F_{\mu\nu}{}^i$ field strength. The Ehresmann formula for the (Lie-algebra-valued) connection

$$A_\mu{}^i T_i = \Gamma^\lambda_{\mu\nu} T^\nu_\lambda + \frac{1}{2} \omega_\mu{}^{ab} T_{ab}$$

with

$$U = \exp [\varepsilon^\alpha_\beta T^\beta_\alpha + \frac{1}{2} \lambda^{ab} T_{ab}]$$

takes the same old form and merely reproduces the earlier results (24b) and (34). However, by recalling the discussion leading to (24c), the freedom to relabel the μ -index remains. By performing such a change, (24b) goes over into (24c),

while (35) goes over to

$$(49) \quad \delta \bar{\omega}_{\mu}{}^{ab} = \frac{\partial x^e}{\partial \bar{x}^{\mu}} \lambda^{ac} \omega_{e c}{}^b - \frac{\partial x^e}{\partial \bar{x}^{\mu}} \lambda^{bc} \omega_{e c}{}^a - \frac{\partial x^e}{\partial \bar{x}^{\mu}} \partial_e \lambda^{ab}.$$

The rest of the argument is due entirely to KIBBLE⁽⁵⁾. The Lagrangian of lowest degree invariant under *both* groups is

$$(50) \quad L = eR,$$

where

$$e = \det e_{\mu}{}^a = \sqrt{-\det g_{\mu\nu}}$$

and

$$R = e_{\mu}{}^a e_{\nu}{}^b R^{\mu\nu}{}_{ab}(\omega) = R^{\mu\nu}{}_{\mu\nu}(\Gamma).$$

Variation with respect to the vierbein leads to

$$(51) \quad R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = 0$$

and with respect to the spin connection $\omega_{\mu}{}^{ab}$ leads to

$$(52) \quad \Gamma^{\lambda}{}_{[\mu\nu]} = 0.$$

Equations (51) and (52) are the fundamental equations of Einstein's 1916 theory (although strictly speaking (52) enters the original as a postulate). In this approach, the postulate is a compatibility condition (47), from which (48) necessarily follows. From (52) and (48) the connection is the usual Christoffel symbol. For the matter-free theory, *both* connections $\Gamma^{\nu}_{\mu\lambda}$ and $\omega_{\mu}{}^{ab}$ are dependent *only* on the vierbein, so that all the gauge fields are obtained from the potential $e_{\mu}{}^a$; nevertheless it is not a gauge field in the Yang-Mills sense given above. As promised, this version of gravity reproduces the original Einstein theory and satisfies the three characteristics of Yang-Mills structure; (43) embodies condition i), (49) and (24c) guarantee that ii) is satisfied and (45) is a statement of condition iii).

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● RIASSUNTO (*)

La teoria gravitazionale normale di Einstein in assenza di materia si costruisce come la sintesi di due teorie di Yang-Mills, i cui gruppi di gauge sono $GL_{4,R}$ e $SL_{2,C}$. In questo approccio il vierbein non è considerato come un campo di Yang-Mills, ma piuttosto come una mappatura tra le molteplicità a due gruppi.

(*) Traduzione a cura della Redazione.

Сила тяжести.

Резюме (*). — Конструируется стандартная теория гравитации Эйнштейна в отсутствии вещества, как синтез двух теорий Янга-Миллса, группы калибровки которых представляют $GL_{4,R}$ и $SL_{2,C}$.

(*) *Переведено редакцией.*

$$\begin{aligned}
L &= L_0(F=0) + L_1(F=\xi - \frac{1}{4}g\phi^2) \\
&= \frac{1}{2}e[\eta^{ab}e^\mu_a e^\nu_b \partial_\mu \phi \partial_\nu \phi + i\bar{\psi}\gamma \cdot \partial \psi \\
&\quad - i\bar{\chi}_\lambda \gamma^\mu \gamma^\lambda \psi \partial_\mu \phi + \frac{1}{8}\bar{\chi}_\lambda \gamma^\mu \gamma^\lambda \chi_\mu \bar{\psi}\psi] \\
&\quad + [\frac{1}{4}eg\bar{\psi}\psi\phi - \frac{1}{2}e\xi\bar{\psi} \cdot \chi + \frac{1}{4}\xi\epsilon^{\mu\nu}\bar{\chi}_\mu \gamma^5 \chi_\nu \phi \\
&\quad + \frac{1}{8}eg\phi^2\bar{\psi}\gamma \cdot \chi - \frac{1}{48}g\epsilon^{\mu\nu}\bar{\chi}_\mu \gamma^5 \chi_\nu \phi^3 \\
&\quad - \frac{1}{2}e(\xi - \frac{1}{4}g\phi^2)^2] .
\end{aligned} \quad (5)$$

Again we verify the invariance of the "reduced" lagrangian. Now we require, instead of eq. (3), the new condition

$$\delta F|_{F=\xi-(1/4)g\phi^2} = -\frac{1}{2}g\phi\delta\phi = -\frac{1}{2}ig\phi\bar{\alpha}\psi, \quad (6)$$

and again salvation comes from the ψ equation of motion; for now

$$\begin{aligned}
[\delta L_0/\delta\bar{\psi} - \partial_\mu(\delta L_0/\delta\bar{\psi}, \mu)]_{F=0} &= -\delta L_1/\delta\bar{\psi}|_{F=\xi-(1/4)g\phi^2} \\
&= \frac{1}{2}e(\xi - \frac{1}{4}g\phi^2)\gamma \cdot \chi + \frac{1}{2}eg\psi\phi,
\end{aligned} \quad (7)$$

which is just the right form to guarantee eq. (6).

Next we investigate the effective potential

$$V_{\text{eff}}(\phi) = \frac{1}{2}(\xi - \frac{1}{4}g\phi^2)^2.$$

Analogously to Fayet's model [5], there are two distinguished cases:

(i) $\xi > 0$: $\langle V_{\text{eff}} \rangle$ is minimised for non-vanishing $\langle \phi \rangle$. Both ϕ and ψ become massive, where $m_\phi = m_\psi = \sqrt{\xi g}$; supersymmetry is unbroken.

(ii) $\xi < 0$: $\langle V_{\text{eff}} \rangle$ is minimized for vanishing $\langle \phi \rangle$. The scalar field becomes massive with $m_\phi = \sqrt{-\xi g/2}$, while ψ is a would-be Goldstone spinor associated with spontaneous supersymmetry breakdown. It is eliminated by a choice of gauge for the field χ_μ , except that now *all* Fermi degrees of freedom are eliminated, and a "cosmological term" $-e\xi^2$ arises.

In the former situation, we translate the field ϕ by writing

$$\phi = -2\sqrt{\xi/g} + \sigma, \quad (8)$$

where now $\langle \sigma \rangle = 0$ and $\langle \delta\psi \rangle = \langle F \rangle \alpha = 0$. The lagrangian becomes

$$\begin{aligned}
L_{(i)} &= \frac{1}{2}e[\eta^{ab}e^\mu_a e^\nu_b \partial_\mu \sigma \partial_\nu \sigma - g\xi\sigma^2 \\
&\quad + i\bar{\psi}\gamma \cdot \partial \psi - \sqrt{g\xi}\bar{\psi}\psi + \dots],
\end{aligned} \quad (9)$$

where the dots indicate terms of cubic and higher order in the fields. When case (ii) holds, however, there is no need to translate ϕ and

$$\begin{aligned}
L_{(ii)} &= \frac{1}{2}e[\eta^{ab}e^\mu_a e^\nu_b \partial_\mu \phi \partial_\nu \phi + \frac{1}{2}g\xi\phi^2 - \xi^2 \\
&\quad + i\bar{\psi}\gamma \cdot \partial \psi - \xi\bar{\psi}\gamma \cdot \chi + \dots].
\end{aligned} \quad (10)$$

Further, $\langle \delta\psi \rangle = \langle F \rangle \alpha = \xi\alpha$, which is characteristic of a Goldstone mode. If we use the gauge freedom to eliminate ψ , corresponding to the unitary gauge, the lagrangian becomes

$$\begin{aligned}
L_{(ii), \text{unitary}} &= \frac{1}{2}e[\eta^{ab}e^\mu_a e^\nu_b \partial_\mu \phi \partial_\nu \phi + \frac{1}{2}g\xi\phi^2 - \xi^2 \\
&\quad - \frac{1}{16}g^2\phi^4] + \frac{1}{4}\epsilon^{\mu\nu}\bar{\chi}_\mu \gamma^5 \chi_\nu [\xi\phi - \frac{1}{12}g\phi^3].
\end{aligned}$$

In two dimensions, neither e^μ_a nor χ^μ are dynamical fields, but merely Lagrange multipliers. Variation with respect to them leads to the two constraints:

$$T^\mu_a = \delta L/\delta e^\mu_a = 0, \quad J^\mu = \delta L/\delta \bar{\chi}_\mu = 0,$$

where T^μ_a and J^μ are the energy tensor and the supercurrent, respectively. Before choosing a gauge we have

$$\begin{aligned}
J^\mu_{(ii)} &= J^\mu_{0(ii)} + J^\mu_{1(ii)} = (\gamma^\lambda \gamma^\mu \psi \partial_\lambda \phi + \frac{1}{4}i\gamma^\lambda \gamma^\mu \chi_\lambda \bar{\psi}\psi) \\
&\quad + i(\xi - \frac{1}{4}g\phi^2)\gamma^\mu \psi - i(\xi - \frac{1}{12}g\phi^2)\phi\epsilon^{\mu\nu}\gamma^5 \chi_\nu,
\end{aligned}$$

which may in principle be solved for χ^μ . On the other hand, in the unitary gauge, we have

$$J^\mu_{(ii), \text{unitary}} = 0 = -i(\xi - \frac{1}{12}g\phi^2)\phi\epsilon^{\mu\nu}\gamma^5 \chi_\nu,$$

so that a nontrivial solution for ϕ demands $\chi_\mu \equiv 0$; and the unitary gauge corresponds to a vanishing Rarita-Schwinger field. The Higgs mechanism works to eliminate the spinor Goldstone mode, but via a gauge field which is forced to be trivial in two dimensions.

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THE HIGGS MECHANISM IN (1+1) SUPERGRAVITY

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New matter couplings are added to a superfield lagrangian related to the spinning string, resulting in a simple model for spontaneous local supersymmetry breaking. The Higgs mechanism eliminates a Goldstone spinor, but a "cosmological term" is induced.

Recently, Howe [1] has given a new ansatz for a superfield formulation of supergravity in two space-time dimensions. This formulation as pioneered by Wess and Zumino incorporates a vierbein which is itself a superfield [2], and provides an extremely elegant demonstration that the spinning string lagrangian [3] is locally supersymmetric. Further, the addition of other supergravity-matter couplings is greatly facilitated in the supervierbein formulation. In this letter, new terms are added to Howe's original lagrangian in order to obtain a simple model for spontaneous breaking of local supersymmetry. The conclusions reached from studying this model are in accord with other studies of the Higgs mechanism in supergravity [4]. The new features of this model are that both a linear representation of the group algebra and a polynomial interaction of the fields are employed.

Our starting point is Howe's lagrangian

$$\mathcal{L}_0 = \frac{1}{2} E E_a^M \partial_M V E^{aN} \partial_N V = \bar{\theta} \theta L_0 + \dots, \quad (1)$$

where $V = \phi + i\bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta F$, the dots indicate terms linear and zeroth in θ , and otherwise the notation follows that of ref. [1] (note that there is a typographical error in the sign of E_m^α). For completeness we give the expressions for the fermionic part of the inverse of the supervierbein E_M^A and its determinant:

$$\begin{aligned} E_a^\mu &= i(\bar{\theta}\gamma^\mu)_a + \frac{1}{4}\bar{\theta}\theta(\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda)_a, \\ E_a^m &= \delta_a^m - \frac{1}{2}i(\bar{\theta}\gamma^\mu)_a\chi_\mu^m \\ &\quad + \frac{1}{4}i\bar{\theta}\theta[(\gamma^5\gamma_\mu)^m{}_a\omega^\mu + \frac{1}{2}i(\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda)_a\chi_\mu^m], \\ E &= \det E_M^A = e + \frac{1}{2}ie\bar{\theta}\gamma\cdot\chi - \frac{1}{8}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu\bar{\theta}\theta. \end{aligned}$$

The lagrangian (1) differs from that given for the spinning string by inclusion of the term eF^2 , but the equation of motion for F is trivial and we may eliminate it entirely. However, invariance under supersymmetry is no longer manifest. Nevertheless, the transformation law for F ,

$$\begin{aligned} \delta F &= \bar{\alpha}\gamma\cdot D\psi - \frac{1}{2}\bar{\alpha}\gamma\cdot\chi F - \frac{1}{2}\bar{\alpha}\gamma^\rho\gamma^\mu\chi_\rho\partial_\mu\phi \\ &\quad + \frac{1}{4}i\bar{\alpha}\gamma^\rho\gamma^\mu\chi_\rho\bar{\chi}_\mu\psi, \end{aligned} \quad (2)$$

guarantees that $I = \int L_0(F=0)d^2x$ remains invariant, for

$$\delta F|_{F=0} = -(i\bar{\alpha}/e)\{\delta L_0/\delta\bar{\psi} - \partial_\mu(\delta L_0/\delta\bar{\psi}, \mu)\} = 0, \quad (3)$$

as may be easily verified (see below). In order to study spontaneous supersymmetry breaking we add the Fayet-Higgs terms^{†1} [5]

$$\mathcal{L}_1 = \frac{1}{2}(-4\xi V + \frac{1}{3}gV^3)E = \bar{\theta}\theta L_1 + \dots \quad (4)$$

It may be assumed that $g > 0$ without loss of generality. Upon elimination of the auxiliary field F , the lagrangian becomes (performing the integration over θ):

^{†1} Fayet's additional terms were of the form $aV + bf(V)$. In order not to spoil renormalizability in (3+1) dimensions $f(V)$ was restricted to be either a linear or an exponential function. In two dimensions, $f(V)$ may be polynomial and the case $f(V) = V^3$ is sufficient to trigger spontaneous supersymmetry breakdown. Note that unlike the string model, the fields here do not carry an internal Minkowski index, so that addition of odd functions of V is permitted.

THE FAILURE OF MINIMAL COUPLING IN SUPERGRAVITY

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A systematic method of deriving a superfield covariant derivative is heuristically presented. The example chosen is two-dimensional, but the method generalises easily. Agreement with an earlier ansatz is obtained.

One of the outstanding problems in supergravity [1] is to construct a unified form of covariant derivative acting on supersymmetric matter. Perhaps the most encouraging progress has been made through the use of superfield [2] techniques. In particular, one may introduce a generalised vierbein, or "vielbein" [3], which is itself a superfield, containing the usual spin 2 vierbein e_μ^α and its supersymmetric spin- $\frac{3}{2}$ partner, a Rarita-Schwinger "gravitino" field χ_μ . From this vielbein one may construct a super connection [4] which provides a covariant derivative for superfields (either matter or gravity) which behave as "vectors" under a particular type of local Lorentz rotations (so that the superspace geometry is non-riemannian in this approach). Recently the equations of supergravity [1] have been recast into superfield equations [5], and lagrangians have been presented whose variations lead to these equations even in the presence of coupling to a massless vector multiplet [6]. In the less ambitious framework of two space-time dimensions, a vielbein ansatz has been presented which provides a covariant derivative for a scalar superfield [7]. This ansatz greatly simplifies both the verification of local supersymmetric invariance of a lagrangian related to that of the spinning string, and also the construction of additional supergravity-matter couplings [8]. In this letter, we demonstrate how to derive the form of the ansatz in two dimensions by explicitly constructing a superfield covariant derivative. The method presented generalises easily to any number of dimensions which admits global supersymmetry. En passant, we discover precisely why a locally supersymmetric theory must be a supergravity theory; that

is, why introduction of a Rarita-Schwinger gauge field χ_μ must necessarily be accompanied by the introduction of a gravitational potential, the vierbein e_μ^α .

The fundamental operators in flat-space supersymmetry are the ordinary derivatives ∂_μ and the linear combinations [2]

$$S_m = \partial/\partial\bar{\theta}^m - i(\gamma^\mu\theta)_m\partial_\mu$$

(the supersymmetry generator),

$$D_m = \partial/\partial\bar{\theta}^m + i(\gamma^\mu\theta)_m\partial_\mu$$

(the invariant derivative), which obey the familiar algebra $\{S, \bar{S}\} = 2i\gamma^\mu\partial_\mu$, $\{S, D\} = 0$. Now let $V = \phi + i\bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta F$ be a scalar superfield. For $f(x)$ an arbitrary function, the action

$$I = \frac{1}{4} \int (f(V) + \bar{D}V D V) d^2\theta d^2x \quad (1)$$

is invariant under a global supersymmetry transformation, where $\delta V = [\bar{\epsilon}S, V]$. For the moment, we suppose that only *one* gauge field χ_μ need be introduced in order to compensate for local parameters $\epsilon(x)$. If the locally invariant lagrangian and transformations were known, it would be an easy matter to recover the (equivalent) globally invariant theory: set $\chi_\mu = 0$ everywhere. We expect, then, that the covariant derivative ∇_a should collapse to the globally invariant derivative $D_a = \delta_a^m D_m$, and that the action should reduce to (1). It is not so clear how to effect the passage the other way, from global to local invariance, in this particular instance. Unlike familiar Yang-Mills theories [9], the supersymmetry transformations mix fields

with *derivatives* of other fields. For example,

$$\delta V = i\bar{\epsilon}\psi + \bar{\epsilon}\theta F - i\bar{\epsilon}\gamma^\mu\theta\partial_\mu\phi + \bar{\epsilon}\gamma^\mu\theta\bar{\theta}\partial_\mu\psi,$$

or

$$\delta\phi = i\bar{\epsilon}\psi, \quad \delta\psi = -i\epsilon F + \gamma^\mu\epsilon\partial_\mu\phi, \quad \delta F = -\bar{\epsilon}\gamma\cdot\partial\psi.$$

As a first step towards construction of a covariant derivative, we introduce the supervierbein E_M^A as follows [3]:

$$\nabla_a = E_a^M \partial_M = E_a^\mu \partial_\mu + E_a^m \partial_m. \quad (2)$$

Let $E_A^M(f)$ denote a "flat" space, when it is possible to set χ_μ equal to zero everywhere. Then

$$\nabla_a(f) = E_a^M(f) \partial_M = D_a = \delta_a^m D_m. \quad (3)$$

With the assumption $E_\mu^\alpha(f) = \delta_\mu^\alpha$, it is possible to read off the forms of $E_M^A(f)$ and its inverse $E_A^N(f)$. In the spirit of ref. [7], we postulate a "flat" θ -space, but leave the x -geometry undetermined:

$$E_m^A(x, \theta) = E_m^A(f). \quad (4)$$

If we now consider the variation of terms such as $\partial_\mu\phi$, it is clear that these will involve derivatives on ϵ . For a "covariant" derivative, it is a necessary (but alas not a sufficient) condition that the variation of $\nabla_\mu A$ for an arbitrary field A does not contain these $\partial_\mu\epsilon$ terms. As a first guess we try the usual prescription

$$\partial_\mu \rightarrow \nabla_\mu \stackrel{?}{=} \partial_\mu - \bar{\chi}_\mu S,$$

where $\delta\chi_\mu = \partial_\mu\epsilon + \dots$, the dots indicating terms which may become necessary. Then

$$\nabla_\mu\phi \stackrel{?}{=} \partial_\mu\phi - i\bar{\chi}_\mu\psi, \quad \nabla_\mu\psi \stackrel{?}{=} \partial_\mu\psi + i\chi_\mu F - \gamma^\rho\chi_\mu\partial_\rho\phi,$$

$$\nabla_\mu F \stackrel{?}{=} \partial_\mu F + \bar{\chi}_\mu\gamma\cdot\partial\psi.$$

Denote the part of $\delta\nabla_\mu A$ which depends on $\partial_\mu\epsilon$ by $\bar{\delta}\nabla_\mu A$. Then although $\bar{\delta}\nabla_\mu\phi = 0$, this is not true for the other two fields. Instead, try

$$\begin{aligned} \nabla_\mu\psi &\stackrel{?}{=} \partial_\mu\psi + i\chi_\mu F - \gamma^\rho\chi_\mu\partial_\rho\phi \\ &= \partial_\mu\psi + i\chi_\mu F - \gamma^\rho\chi_\mu\partial_\rho\phi + i\gamma^\rho\chi_\mu\bar{\chi}_\rho\psi. \end{aligned}$$

Unfortunately, even now

$$\bar{\delta}\nabla_\mu\psi = i\gamma^\rho\partial_\mu\epsilon\bar{\chi}_\rho\psi.$$

To make $\delta\nabla_\mu\psi = 0$, there would seem to be only two available remedies:

(i) introduce new gauge fields,

(ii) modify the original transformation laws. We follow the second approach, and write

$$\delta'\psi = \delta\psi + \delta_G\psi = \delta\psi - i\gamma^\mu\epsilon\bar{\chi}_\mu\psi;$$

then $\bar{\delta}'\nabla_\mu\psi = 0$, as desired. Thus the covariant derivative determines the transformation law for the matter fields, rather than vice versa. This method is strange, a little ugly, and successful.

In order to find $\delta'F$, we need to consider $\nabla_\mu F$, and guess

$$\begin{aligned} \nabla_\mu F &\stackrel{?}{=} \partial_\mu F + \bar{\chi}_\mu\gamma\cdot\nabla\psi \\ &= \partial_\mu F + \bar{\chi}_\mu\gamma\cdot\partial\psi + i\bar{\chi}_\mu\gamma\cdot\chi F \\ &\quad - \bar{\chi}_\mu\gamma^\lambda\gamma^\nu\chi_\lambda\partial_\nu\phi + i\bar{\chi}_\mu\gamma^\lambda\gamma^\nu\chi_\lambda\bar{\chi}_\nu\psi. \end{aligned}$$

Indeed, $\bar{\delta}'\nabla_\mu F = 0$, so long as we adopt

$$\delta'F = \delta F + \delta_GF,$$

where

$$\delta_GF = -i\bar{\epsilon}\gamma\cdot\chi F + \bar{\epsilon}\gamma^\lambda\gamma^\nu\chi_\lambda\partial_\nu\phi - i\bar{\epsilon}\gamma^\lambda\gamma^\nu\chi_\lambda\bar{\chi}_\nu\psi.$$

The new transformation laws may be written compactly as

$$\delta'V = [\bar{\epsilon}\Sigma, V]$$

where

$$\Sigma = S + i\gamma^\mu\theta\bar{\chi}_\mu S - \frac{1}{2}\gamma^\lambda\gamma^\mu\chi_\lambda\bar{\theta}\bar{\chi}_\mu S. \quad (5)$$

This demonstrates what may be termed "the failure of minimal coupling": the passage from global to local supersymmetry is made *not* by the mere replacement of a local group parameter in place of a constant, but an entirely new transformation law which is at least linear in the gauge field. The resulting covariant derivatives are thus of quadratic and higher order in the gauge field.

Apparently we can now follow the standard prescription, but with Σ replacing S . That is, for $D_a \rightarrow \nabla_a$ it should be

$$\nabla_a = \partial/\partial\bar{\theta}^a + i(\gamma^\mu\theta)_a(\partial_\mu - \bar{\chi}_\mu\Sigma) \stackrel{?}{=} E_a^M\partial_M.$$

But is this derivative really covariant? For that we require ("sufficient" condition) that the operator ∇_a transform according to the usual law [9] (with allowances for Fermi statistics),

$$\delta\nabla_a \equiv (\delta E_a^M)\partial_M = -\bar{\epsilon}\{\Sigma, \nabla_a\} + (\nabla_a\bar{\epsilon})\Sigma. \quad (6)$$

One readily discovers that the abelian law $\delta\chi_\mu = \partial_\mu\epsilon$ fails completely. If it were to succeed, we would need $\{\Sigma, \nabla_a\} = 0$ which does not occur. Appearances are deceiving! Equating the coefficients of ∂_μ which are linear in θ leads to the requirement

$$\delta(i\gamma^\mu\theta)_a = 2(\gamma^\nu\theta)_a\bar{\epsilon}\gamma^\mu\chi_\nu,$$

which clearly implies $\gamma^\mu = \gamma^\mu(x)$; and hence $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x)$. Thus the introduction of local supersymmetry forces the introduction of a curved x-space. This well-known fact is brought out very forcefully in the present approach. Consequently we must retrace our steps and replace $\partial_\mu\psi$ in δF by $D_\mu\psi = \partial_\mu - \frac{1}{2}\omega_\mu\gamma^5\psi$, where ω_μ must be taken to be

$$\omega_\mu = e^{-1}e_\mu^\alpha\epsilon^{\rho\sigma}\partial_\rho e_{\sigma\alpha} + 2i\bar{\chi}_\mu\gamma^5\gamma^\lambda\chi_\lambda, \quad (7)$$

in order to get $\bar{\delta}\omega_\mu = 0$. Hence both Σ and ∇ must be modified, to

$$\begin{aligned} \Sigma_a &= S_a + i(\gamma^\mu\theta)_a\bar{\chi}_\mu S - \frac{1}{2}\bar{\theta}\theta(\gamma^\lambda\gamma^\mu\chi_\lambda)_a\bar{\chi}_\mu S \\ &\quad - \frac{1}{4}i\bar{\theta}\theta(\gamma^\nu\gamma^5\omega_\nu S)_a, \end{aligned} \quad (8)$$

$$\nabla_a = \partial/\partial\bar{\theta}^a + i(\gamma^\mu\theta)_a(\partial_\mu - \bar{\chi}_\mu\Sigma - \frac{1}{2}\omega_\mu\bar{\theta}\gamma^5\Sigma) \quad (9a)$$

$$= \partial/\partial\bar{\theta}^a + i(\gamma^\mu\theta)_a(\partial_\mu - E_\mu^b\Sigma_b) \quad (9b)$$

$$= E_a^M\partial_M, \quad (9c)$$

where now

$$E_a^\mu = i(\gamma^\mu\theta)_a - \frac{1}{2}\bar{\theta}\theta(\gamma^\lambda\gamma^\mu\chi_\lambda)_a, \quad (10a)$$

$$\begin{aligned} E_a^m &= \delta_a^m - i(\gamma^\mu\theta)_a\bar{\chi}_\mu^m \\ &\quad + \frac{1}{2}\bar{\theta}\theta[(\gamma^\mu\gamma^\nu\chi_\nu)_a\bar{\chi}_\mu^m + \frac{1}{2}i(\gamma^\mu\gamma^5)_a^m\omega_\mu], \end{aligned} \quad (10b)$$

which agrees with the ansatz of ref. [7] modulo different conventions for χ_μ . The variation of ∇_a is now given consistently by eq. (6): performing the lengthy calculation we find with $\gamma^\mu(x) = e^\mu_\alpha(x)\gamma^\alpha$, that

$$\delta e_\mu^\alpha = 2i\bar{\epsilon}\gamma^\alpha\chi_\mu, \quad \delta\chi_\mu = \partial_\mu\epsilon - \frac{1}{2}\omega_\mu\gamma^5\epsilon, \quad (11a,b)$$

$$\delta\omega_\mu = 2ie^{-1}\epsilon^{\lambda\rho}\bar{\epsilon}\gamma_\mu D_\lambda\chi_\rho = 0. \quad (11c)$$

Note from eq. (9) it follows that neither E_μ^α nor E_μ^a contain powers of θ higher than the first. Using the

orthogonality properties we obtain

$$E_\mu^\alpha = e_\mu^\alpha + 2i\bar{\theta}\gamma^\alpha\chi_\mu, \quad E_\mu^a = \chi_\mu^a - \frac{1}{2}\omega_\mu(\gamma^5\theta)^a. \quad (12a,b)$$

The action is given by

$$I = -\frac{1}{4}\int E\bar{\nabla}V\nabla V d^2\theta d^2x, \quad (13)$$

where

$$\begin{aligned} E &= (\det E_\mu^\alpha)(\det E_a^m) \\ &= e + i\bar{\theta}\gamma^\lambda\chi_\lambda e - \frac{1}{2}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma^5\chi_\nu\bar{\theta}\theta; \end{aligned}$$

or

$$\begin{aligned} I &= \frac{1}{2}\int [g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + i\bar{\psi}\gamma^\lambda D_\lambda\psi + F^2 \\ &\quad - 2i\bar{\chi}_\mu\gamma^\lambda\gamma^\mu\partial_\lambda\phi + \frac{1}{2}\bar{\chi}_\lambda\gamma^\mu\gamma^\lambda\chi_\mu\bar{\psi}\psi] e d^2x, \end{aligned} \quad (14)$$

which is that of ref. [7], as all the transformation laws embodied in eq. (5) with Σ given by eq. (8).

After this letter was substantially completed, I learned of similar results obtained in slightly different fashions by Brink et al. [10] and Gates Shapiro [11]. I wish to thank Drs. Gates and Shapiro for communicating their results to me prior to publication and Dr. P.W. Higgs for many valuable discussions.

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